ELECTROMAGNETIC WAVE PROPAGATION AND RADIATION IN A SUDDENLY CREATED PLASMA

Thesis by

Ching-Lin Jiang

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1973

(Submitted June 5, 1972)

ACKNOWLEDGMENTS

The author would like to express his deep indebtedness to his advisor, Professor Charles H. Papas for his guidance, inspiration, and encouragement during the course of this work and throughout his studies at the California Institute of Technology.

The author also would like to express his thanks to Dr. T. C. Mo for many helpful discussions. Special thanks are extended to Mr. T. W. Lee for his help in computer programming. The author is grateful to Mrs. Ruth Stratton for typing the manuscript, and to Miss Ingrid Vierheilig and Mrs. Kathy Ellison for their help.

ABSTRACT

Propagation and radiation of electromagnetic waves from oscillating sources in a suddenly created plasma are studied in this investigation. Field expressions are derived through the use of Laplace transformations. The spatial distribution of sources is taken to be arbitrary but confined.

Two cases are considered in detail: (1) plane wave propagation in a source-free region, and (2) electric point dipole radiation. In the case of plane wave propagation, various aspects such as wave split, frequency shift, phase and group velocities, amplitude changes, power flows and energy relations are discussed. In the case of electric dipole radiation, the electromagnetic fields and instantaneous radiated power are calculated and expressed in terms of Lommel functions of two variables. Asymptotic expressions and graphical results of numerical calculations of these quantities are presented. Many interesting properties of the spherical waves and power radiation are discussed.

TABLE OF CONTENTS

1. INTRODUCTION

The study of the interaction between the electromagnetic radiation and the plasmas having time-varying parameters (for example, free electron density) has been of interest recently [1-15]. This interaction plays an important role in radio communication in the disturbed ionosphere or in the vicinity of a nuclear explosion. It is significant in microwave plasma devices and in astrophysics. The employment of lasers to produce plasmas and the prospect of using lasers in controlling thermonuclear fusion has also led many investigators to this area of research.

One of the interesting subjects is to investigate the properties of the electromagnetic waves propagated in time-varying plasmas. The temporal variations of the plasma parameters might be produced by the solar flares, by a strong laser pulse, or by some other means, say, a nuclear explosion; however, the physical processes of the interaction between the medium and the ionozation agents are beyond the scope of this work. In this paper, we are mainly concerned with the small-amplitude waves in a suddenly-created plasma. The free electron density in the medium increases suddenly from zero to some constant value. The plasma is assumed to be isotropic, cold, lossless, homogeneous, and linear. This simple model is a useful theoretical case that could provide some insight into the basic features of the wave propagation and source radiation in the presence of a time-varying plasma.

The problem of wave propagation in a dielectric medium whose permittivity and permeability vary with time was studied by

Morgenthaler [16]. With device application in mind many authors have also investigated the properties of waves in time-varying magneto-elastic media and transmission lines [17-21]. Recently, using finite difference method Taylor, Lam, and Shumpert obtained some numerical results for pulse scattering from a perfectly conducting cylindrical rod in a cylindrical waveguide filled with time-varying inhomogeneous lossy media [13]. Felsen and Whitman solved the time-domain scalar wave equation with a pulsed excitation in time-varying nondispersive and dispersive media [14]; and Fante derived complicated expressions for the electric fields transmitted into a half-space with time-varying properties [ ]. In these previous works, however, the problem of waves generated from oscillating sources in a time-varying plasma has not been studied.

This report presents a systematic study on the electromagnetic wave propagation and radiation from sinusoidally oscillating sources in a suddenly-created plasma. In the second chapter, Maxwell's equations are formulated; a free current term is introduced to account for the interaction of the field with the plasma. The spatial distributions of sources are assumed to be arbitrary but confined. Originally, it is assumed that monochromatic waves are propagated and radiated. Through the use of Laplace transformation, the field equations in the suddenly-created plasma are solved. The solutions for the electromagnetic fields are expressed in terms of their initial values and the inverse Laplace transforms of the source currents and the Green's dyadic functions. Various aspects of the wave propagation in a source-free region are then discussed in detail in the third chapter. In the fourth chapter the electric and magnetic fields are calculated for an oscillating electric point dipole. The amount of power radiated by the dipole is also evaluated.

2. ELECTROMAGNETIC FIELD SOLUTIONS

This chapter will deal with the derivation of the field expressions. We consider a medium which is suddenly ionized at the time $t = 0$. This suddenly-created plasma is assumed to be isotropic, cold, lossless, homogeneous, and linear.

The field quantities obey Maxwell's equations:

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t} \mu_o \underline{H} \tag{2.1}$$

$$\nabla \times \underline{H} = \underline{J}_t + \frac{\partial}{\partial t} \varepsilon_o \underline{E} \tag{2.2}$$

$$\nabla \cdot \varepsilon_o \underline{E} = \rho_t \tag{2.3}$$

$$\nabla \cdot \mu_o \underline{H} = 0 \tag{2.4}$$

where $\underline{J}_t$ and ρ_t are the total macroscopic current density and total macroscopic charge density respectively. Neglecting the bound current and charge in the medium, the total current density $\underline{J}_t$ consists of the applied source current density $\underline{J}_s$ and free current density $\underline{J}_f$, and the total charge density ρ_t consists of the applied source charge density ρ_s and free charge density ρ_f, that is,

$$\underline{J}_t = \underline{J}_s + \underline{J}_f \tag{2.5}$$

$$\rho_t = \rho_s + \rho_f \tag{2.6}$$

where the free current and charge densities have been introduced to account for the interaction between the wave and the suddenly-created plasma.

The applied source current density will now be specified as

$$\underline{J}_s = \mathrm{Re}\{\underline{J}_o(\underline{r})\, e^{-i\omega_o t}\} \quad \text{for all } t \tag{2.7}$$

where "Re" is the shorthand for "real part of". $\underline{J}_o(\underline{r})$ is an arbitrary function of $\underline{r}$ but is confined to a finite part of space.

A. $t < 0$

For $t < 0$, i.e., before the plasma is created, $\underline{J}_f = 0$; and a monochromatic wave is propagated and radiated from the source region. The electric and magnetic fields take the following form:

$$\underline{E}(\underline{r},t) = \underline{E}_o(\underline{r})\, e^{-i\omega_o t} \tag{2.8}$$

$$\underline{H}(\underline{r},t) = H_o(\underline{r})\, e^{-i\omega_o t} \tag{2.9}$$

where we have dropped $\mathrm{Re}\{\ \}$ for the sake of simplicity. It is easily shown by using Maxwell's equations and Eq. (2.8) and (2.9) that the vector wave equation for $\underline{E}_o(\underline{r})$ is

$$\nabla \times \nabla \times \underline{E}_o(\underline{r}) - k_o^2\, \underline{E}_o(\underline{r}) = i\omega_o\mu_o\underline{J}_o(r) \tag{2.10}$$

where $k_o \equiv \omega_o\sqrt{\mu_o\varepsilon_o}$, and

$$\underline{H}_o(\underline{r}) = \frac{1}{i\omega_o\mu_o} \nabla \times \underline{E}_o(\underline{r}) \tag{2.11}$$

In an unbounded medium the solutions which satisfy the radiation conditions are [22]:

$$\underline{E}_o(\underline{r}) = i\omega_o\mu_o \int_V \underline{\underline{\Gamma}}_o(\underline{r},\underline{r}') \cdot \underline{J}_o(\underline{r}')dV' \tag{2.12}$$

$$\underline{H}_o(\underline{r}) = \int_V \nabla G_o(\underline{r},\underline{r}') \times \underline{J}_o(\underline{r}')dV' \tag{2.13}$$

where the integration with respect to the primed coordinates extends throughout the volume occupied by $\underline{J}_o$. The scalar and dyadic Green's functions $G_o(\underline{r},\underline{r}')$ and $\underline{\underline{\Gamma}}_o(\underline{r},\underline{r}')$ are defined respectively by

$$G_o(\underline{r},\underline{r}') = \frac{e^{ik_o|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} \tag{2.14}$$

and

$$\underline{\underline{\Gamma}}_o(\underline{r},\underline{r}') = (\underline{\underline{u}} + \frac{1}{k_o^2}\nabla\nabla)\, G_o(\underline{r},\underline{r}') \tag{2.15}$$

where $\underline{\underline{u}}$ is the unit dyad defined as

$$\underline{\underline{u}} \equiv \sum_{m=1}^{3} \sum_{n=1}^{3} \underline{e}_m\, \underline{e}_n\, \delta_{mn} \tag{2.16}$$

$\underline{e}_i$ $(i = 1,2,3)$ are the unit base vectors, and δ_{mn} is the Kronecker delta which is 1 for $m = n$ and 0 for $m \neq n$.

Then the electric and magnetic fields just before the plasma is created (i.e., at $t = 0^-$) are

$$\underline{E}(\underline{r},0^-) = \underline{E}_o(\underline{r}) \tag{2.17}$$

$$\underline{H}(\underline{r},0^-) = \underline{H}_o(\underline{r}) \tag{2.18}$$

B. $t = 0$

At $t = 0$ the plasma is created. It is assumed that the newly created free electrons and positive ions are stationary at the moment, that is, their velocity just after the creation $(t = 0^+)$ is zero. Thus the free current density in the medium which is zero for $t < 0$ is still zero at $t = 0^+$. The electric and magnetic fields at $t = 0^+$ are the same as at $t = 0^-$.

$$\underline{E}(\underline{r},0^+) = \underline{E}(\underline{r},0^-) = \underline{E}_o(\underline{r}) \tag{2.19}$$

$$\underline{H}(\underline{r},0^+) = \underline{H}(\underline{r},0^-) = \underline{H}_o(\underline{r}) \tag{2.20}$$

and it follows that

$$\left(\frac{\partial}{\partial t}\underline{E}\right)(\underline{r},0^+) = -i\omega_o\underline{E}_o(\underline{r}) \tag{2.21}$$

C. $t > 0$

After $t = 0$, however, the free electrons and ions in the created plasma are then set in motion by the field in the medium. Since the ions are much more massive than the electrons, the velocity imparted to the ions by the field is negligibly small compared to the velocity given to the electrons. Thus the free current density $\underline{J}_f$ can be written as

$$\underline{J}_f(\underline{r},t) = Ne\,\underline{v}(\underline{r},t) \tag{2.22}$$

where N is the free electron density in the plasma, e is the electronic charge, and $\underline{v}(\underline{r},t)$ is the average velocity at time t and position $\underline{r}$. From Newton's second law and the Lorentz force equation

$$Nm\frac{d}{dt}\underline{v} = Ne(\underline{E} + \underline{v}\times\mu_o\underline{H}) \tag{2.23}$$

where m is the electron mass. Linearizing Eq. (2.23), neglecting the $\underline{v}\times\mu_o\underline{H}$ term and multiplying it by e/m, we obtain the desired relation between $\underline{J}_f$ and $\underline{E}$:

$$\frac{\partial}{\partial t} \underline{J}_f = \omega_p^2 \varepsilon_o \underline{E} \tag{2.24}$$

where the plasma frequency ω_p is defined as

$$\omega_p \equiv \sqrt{\frac{Ne^2}{m\varepsilon_o}} \tag{2.25}$$

Taking the curl of Eq. (2.1) and using Eq. (2.2), (2.5), and (2.24) yields the vector wave equation for $\underline{E}$ for $t > 0$

$$\nabla \times \nabla \times \underline{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \underline{E} + \frac{\omega_p^2}{c^2} \underline{E} = -\mu_o \frac{\partial}{\partial t} \underline{J}_s \tag{2.26}$$

where $c^2 = 1/\mu_o \varepsilon_o$. Equations (2.26) and (2.1) now serve to determine $\underline{E}(\underline{r},t)$ and $\underline{H}(\underline{r},t)$ (for $t > 0$) which satisfy the initial and radiation conditions.

To solve for the field expressions we employ the method of Laplace transformation. Laplace transformation with respect to time is now performed. If $F(t)$ is the function of time under consideration, $\hat{F}(s)$ defined as

$$\hat{F}(s) = \int_0^\infty F(t)\, e^{-st}\, dt \tag{2.27}$$

is its Laplace transformation. $F(t)$ is recoverable by means of inversion formula

$$F(t) \equiv L^{-1}(\hat{F}) = \frac{1}{2\pi i} \int_C \hat{F}(s)\, e^{st}\, ds \tag{2.28}$$

where C is the straight-line path $\sigma - i\infty$ to $\sigma + i\infty$ and the path lies to the right of all the poles and branch points of $\hat{F}(s)$.

The result of the transformation of Eq. (2.26) is

$$\nabla \times \nabla \times \underline{\hat{E}}(\underline{r},s) + \frac{s^2+\omega_p^2}{c^2} \underline{\hat{E}}(\underline{r},s)$$

$$= \frac{i\omega_o\mu_o}{s+i\omega_o} \underline{J}_o(\underline{r}) + \frac{s-i\omega_o}{c^2} \underline{E}_o(\underline{r}) \qquad (2.29)$$

where we have used Eq. (2.19) and (2.21) for initial conditions. It is still difficult to obtain useful solutions because of the presence of the term $\frac{s-i\omega_o}{c^2} \underline{E}_o(\underline{r})$ acting as a source which is not confined to a finite part of space. However, we are able to get rid of it by making appropriate transformations. In order to do this, let us rewrite the $\underline{E}_o(\underline{r})$ in Eq. (2.29) as

$$\underline{E}_o(\underline{r}) = F_1\underline{E}_o(\underline{r}) + F_2\underline{E}_o(\underline{r}) \qquad (2.30)$$

where F_1 and F_2 are spatially constant and satisfy the relation

$$F_1 + F_2 = 1 \qquad (2.31)$$

From Eq. (2.10) we can write

$$\underline{E}_o(\underline{r}) = \frac{1}{k_o^2} (\nabla \times \nabla \times \underline{E}_o(\underline{r}) - i\omega_o\mu_o \underline{J}_o(\underline{r})) \qquad (2.32)$$

Thus Eq. (2.30) can be written as

$$\underline{E}_o(\underline{r}) = F_1\underline{E}_o(\underline{r}) + \frac{F_2}{k_o^2} (\nabla \times \nabla \times \underline{E}_o(\underline{r}) - i\omega_o\mu_o \underline{J}_o(\underline{r})) \qquad (2.33)$$

Substituting Eq. (2.33) into Eq. (2.29) and regrouping the similar terms, we obtain

$$\nabla \times \nabla \times (\hat{\underline{E}}(\underline{r},s) - \frac{s - i\omega_o}{\omega_o^2} F_2 \underline{E}_o(\underline{r}))$$

$$+ \frac{s^2 + \omega_p^2}{c^2} (\hat{\underline{E}}(\underline{r},s) - \frac{s - i\omega_o}{s^2 + \omega_p^2} F_1 \underline{E}_o(\underline{r}))$$

$$= (\frac{1}{s + i\omega_o} - \frac{s - i\omega_o}{\omega_o^2} F_2) \, i\omega_o \mu_o \underline{J}_o(\underline{r}) \qquad (2.34)$$

Now we choose F_1 and F_2 such that

$$\frac{1}{\omega_o^2} F_2 = \frac{1}{s^2 + \omega_p^2} F_1 \qquad (2.35)$$

Solving Eq. (2.31) and (2.35), we obtain F_1 and F_2 :

$$F_1 = \frac{s^2 + \omega_p^2}{s^2 + \omega_o^2 + \omega_p^2} \qquad (2.36)$$

$$F_2 = \frac{\omega_o^2}{s^2 + \omega_o^2 + \omega_p^2} \qquad (2.37)$$

Hence Eq. (2.34) becomes

$$\nabla \times \nabla \times (\hat{\underline{E}}(\underline{r},s) - \frac{s - i\omega_o}{s^2 + \omega_o^2 + \omega_p^2} \underline{E}_o(\underline{r}))$$

$$+ \frac{s^2 + \omega_p^2}{c^2} (\hat{\underline{E}}(\underline{r},s) - \frac{s - i\omega_o}{s^2 + \omega_o^2 + \omega_p^2} \underline{E}_o(r))$$

$$= \frac{i\omega_o \omega_p^2}{(s + i\omega_o)(s^2 + \omega_o^2 + \omega_p^2)} \mu_o \underline{J}_o(\underline{r}) \qquad (2.38)$$

Therefore by defining

$$\underline{\hat{E}}'(\underline{r},s) \equiv \underline{\hat{E}}(\underline{r},s) - \frac{s - i\omega_o}{s^2 + \omega_o^2 + \omega_p^2} \underline{E}_o(\underline{r}) \tag{2.39}$$

$$\underline{\hat{J}}'(\underline{r},s) \equiv - \frac{i\omega_o \omega_p^2}{s(s + i\omega_o)(s^2 + \omega_o^2 + \omega_p^2)} \underline{J}_o(\underline{r}) \tag{2.40}$$

the vector wave equation (2.29) is transformed into

$$\nabla \times \nabla \times \underline{\hat{E}}'(\underline{r},s) + \frac{s^2 + \omega_p^2}{c^2} \underline{\hat{E}}'(\underline{r},s) = -s\mu_o \underline{\hat{J}}'(\underline{r},s) \tag{2.41}$$

To solve for the magnetic field, we Laplace transform Eq. (2.1):

$$\nabla \times \underline{\hat{E}}(\underline{r},s) = -s\mu_o \underline{\hat{H}}(\underline{r},s) + \mu_o H_o(\underline{r}) \tag{2.42}$$

Making use of Eq. (2.39), we find

$$\nabla \times \underline{\hat{E}}'(\underline{r},s) = -s\mu_o \underline{\hat{H}}'(\underline{r},s) \tag{2.43}$$

where

$$\underline{\hat{H}}'(\underline{r},s) \equiv \underline{\hat{H}}(\underline{r},s) - \frac{s^2 - i\omega_o s + \omega_p^2}{s(s^2 + \omega_o^2 + \omega_p^2)} \underline{H}_o(\underline{r}) \tag{2.44}$$

Thus the retarded field solutions are (compare Eq. (2.41) and (2.43) with Eq. (2.10) and (2.11)) [22]:

$$\underline{\hat{E}}'(\underline{r},s) = -s\mu_o \int_V \underline{\underline{\Gamma}}(\underline{r},\underline{r}') \cdot \underline{\hat{J}}'(\underline{r}',s) \, dV' \tag{2.45}$$

$$\underline{\hat{H}}(\underline{r},s) = \int_V \nabla G(\underline{r},\underline{r}') \times \underline{\hat{J}}'(\underline{r}',s) \, dV' \tag{2.46}$$

where the integration with respect to the primed coordinates extends throughout the volume V occupied by $\hat{\underline{J}}'$. The Green's functions $G(\underline{r},\underline{r}')$ and $\underline{\underline{\Gamma}}(\underline{r},\underline{r}')$ are defined by

$$G(\underline{r},\underline{r}') = \frac{e^{-\frac{|\underline{r}-\underline{r}'|}{c}\sqrt{s^2+\omega_p^2}}}{4\pi|\underline{r}-\underline{r}'|} \tag{2.47}$$

and

$$\underline{\underline{\Gamma}}(\underline{r},\underline{r}') = (\underline{\underline{u}} - \frac{c^2}{s^2+\omega_p^2}\nabla\nabla)\; G(\underline{r},\underline{r}') \tag{2.48}$$

where $\underline{\underline{u}}$ is the unit dyad given by Eq. (2.16). The square root $\sqrt{s^2+\omega_p^2}$ is defined to be positive for real, positive s . For the purpose of the inversion technique the definition of the square root will be continued into the left half plane by Eq. (2.49) and Fig. 2.1.

$$\sqrt{s^2+\omega_p^2} = \sqrt{\rho_1\rho_2}\; e^{\frac{i}{2}(\phi_1+\phi_2)}$$

$$-\frac{\pi}{2} \leq \phi_{1,2} \leq \frac{3\pi}{2} \tag{2.49}$$

It should be noted that the advanced field solutions can be obtained by replacing $G(\underline{r},\underline{r}')$ by $G_a(\underline{r},\underline{r}')$ in Eq. (2.45) to (2.48), where

$$G_a(\underline{r},\underline{r}') = \frac{e^{\frac{|\underline{r}-\underline{r}'|}{c}\sqrt{s^2+\omega_p^2}}}{4\pi|\underline{r}-\underline{r}'|} \tag{2.50}$$

Substituting Eq. (2.45) and (2.46) into Eq. (2.39) and (2.44), we obtain the fields in Laplace transforms:

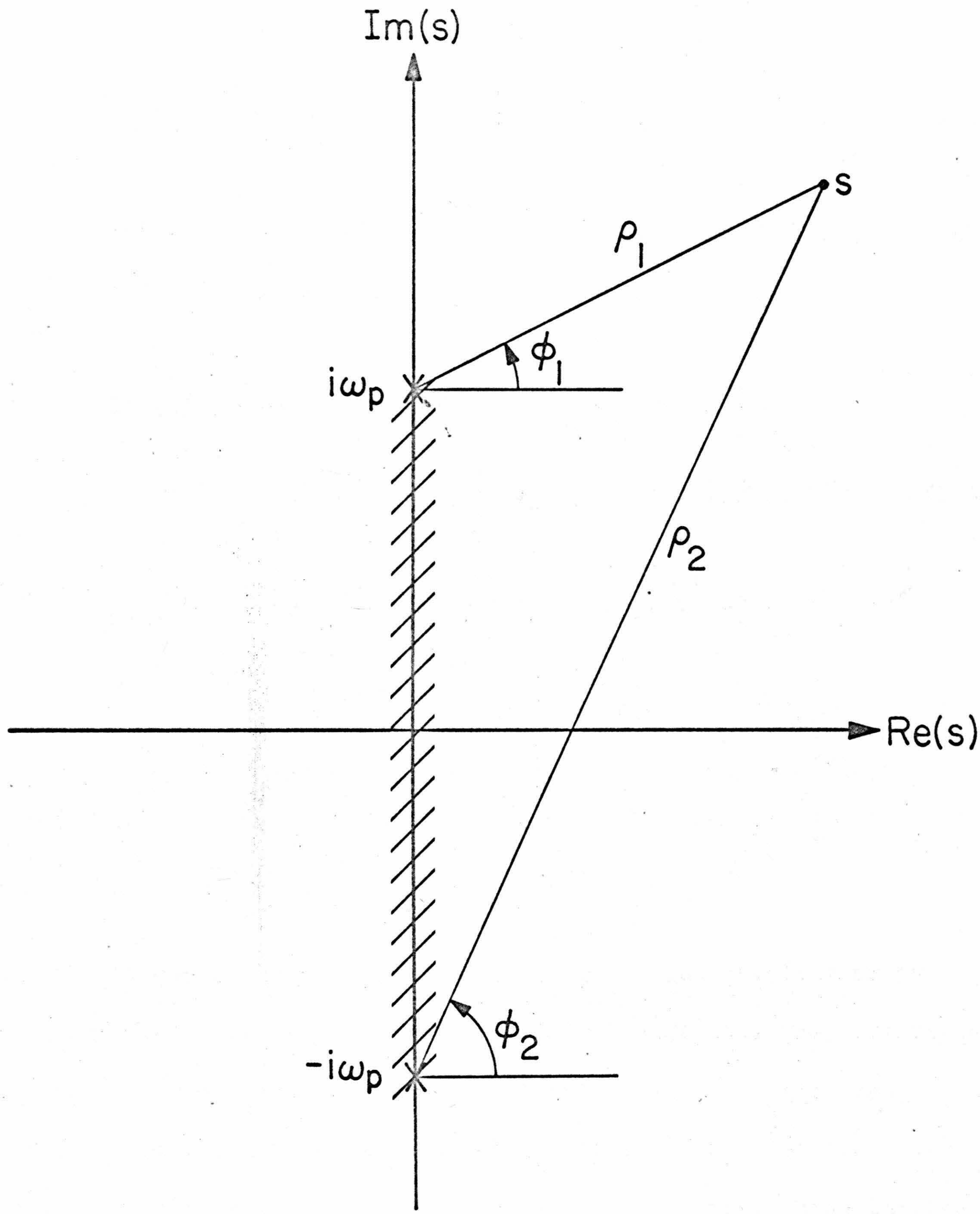


Fig. 2.1 A definition of $\sqrt{s^2+\omega_p^2}$

$$\hat{\underline{E}}(\underline{r},s) = \frac{s - i\omega_o}{s^2 + \omega_o^2 + \omega_p^2}\underline{E}_o(\underline{r}) - s\mu_o \int_V \underline{\underline{\Gamma}}(\underline{r},\underline{r}')\cdot\hat{\underline{J}}'(\underline{r}',s)dV' \quad (2.51)$$

$$\hat{\underline{H}}(\underline{r},s) = \frac{s^2 - i\omega_o s + \omega_p^2}{s(s^2 + \omega_o^2 + \omega_p^2)}\underline{H}_o(\underline{r}) + \int_V \nabla G(\underline{r},\underline{r}')\times\hat{\underline{J}}'(\underline{r}',s)dV' \quad (2.52)$$

Inverse Laplace transformation is now performed to obtain the field quantities as functions of time. Hence, for $t > 0$,

$$\underline{E}(\underline{r},t) = \frac{\omega + \omega_o}{2\omega}\underline{E}_o(\underline{r})e^{-i\omega t} + \frac{\omega - \omega_o}{2\omega}\underline{E}_o(\underline{r})e^{i\omega t} + L^{-1}\{-s\mu_o \int_V \underline{\underline{\Gamma}}(\underline{r},\underline{r}')\cdot\hat{\underline{J}}'(\underline{r}',s)dV'\} \quad (2.53)$$

$$\underline{H}(\underline{r},t) = \frac{\omega_o(\omega + \omega_o)}{2\omega^2}\underline{H}_o(\underline{r})e^{-i\omega t} - \frac{\omega_o(\omega - \omega_o)}{2\omega^2}\underline{H}_o(\underline{r})e^{i\omega t} + \frac{\omega_p^2}{\omega^2}\underline{H}_o(\underline{r}) + L^{-1}\{\int_V \nabla G(\underline{r},\underline{r}')\times\hat{\underline{J}}'(\underline{r}',s)dV'\} \quad (2.54)$$

where

$$\omega = \sqrt{\omega_o^2 + \omega_p^2} \quad (2.55)$$

It can be seen that there are waves which result from the original fields at $t = 0$. In Eq. (2.53) and (2.54) they are represented by those terms involving the initial values $\underline{E}_o(\underline{r})$ and $\underline{H}_o(\underline{r})$. There are also waves which result from the presence of sources in the medium. They are represented by the inverse Laplace transforms in Eq. (2.53) and (2.54). Their values are zero for $t < \frac{1}{c}$ times the minimum of $|\underline{r} - \underline{r}'|$ ($\underline{r}'$ within V) because of the presence of $\exp[st - \frac{|\underline{r}-\underline{r}'|}{c}\sqrt{s^2 + \omega_p^2}]$ in the integrands of the inversion integrals.

In summary, the electromagnetic field expressions have been derived for arbitrary but confined source distributions in a suddenly-created plasma. In the following chapters plane wave propagation and electric dipole radiation will be studied in detail based on these general field expressions.

3. PROPERTIES OF WAVE PROPAGATION

Suppose that a circularly polarized plane wave is propagated in the positive z-direction for $t < 0$. The fields are

$$\underline{E}(\underline{r},t) = \frac{1}{\sqrt{2}}(\underline{e}_x + i\underline{e}_y)\, E_o e^{ik_o z - i\omega_o t} \tag{3.1}$$

$$\underline{H}(\underline{r},t) = \frac{1}{\sqrt{2}}(\underline{e}_y - i\underline{e}_x)\, H_o e^{ik_o z - i\omega_o t} \tag{3.2}$$

where E_o is a constant and $H_o = \sqrt{\frac{\varepsilon_o}{\mu_o}}\, E_o$.

For $t > 0$, i.e., after the plasma is created, the fields are (from Eq. (2.53) and (2.54))

$$\underline{E}(\underline{r},t) = \frac{1}{\sqrt{2}}(\underline{e}_x + i\underline{e}_y)\frac{\omega+\omega_o}{2\omega} E_o e^{ik_o z - i\omega t} + \frac{1}{\sqrt{2}}(\underline{e}_x + i\underline{e}_y)\frac{\omega-\omega_o}{2\omega} E_o e^{ik_o z + i\omega t} \tag{3.3}$$

$$\underline{H}(\underline{r},t) = \frac{1}{\sqrt{2}}(\underline{e}_y - i\underline{e}_x)\frac{\omega_o(\omega+\omega_o)}{2\omega^2} H_o e^{ik_o z - i\omega t} - \frac{1}{\sqrt{2}}(\underline{e}_y - i\underline{e}_x)\frac{\omega_o(\omega-\omega_o)}{2\omega^2} H_o e^{ik_o z + i\omega t} + \frac{1}{\sqrt{2}}(\underline{e}_y - i\underline{e}_x)\frac{\omega_p^2}{\omega^2} H_o e^{ik_o z} \tag{3.4}$$

where $\omega = \sqrt{\omega_o^2 + \omega_p^2}$. Expressed in terms of the parameter

$$\alpha \equiv \frac{\omega_p}{\omega_o} \tag{3.5}$$

Eq. (3.3) and (3.4) become

$$\underline{E}(\underline{r},t) = \frac{1}{\sqrt{2}}(\underline{e}_x + i\underline{e}_y)\,\frac{\sqrt{1+\alpha^2}+1}{2\sqrt{1+\alpha^2}}\,E_o e^{ik_o z - i\omega_o\sqrt{1+\alpha^2}\,t}$$

$$+ \frac{1}{\sqrt{2}}(\underline{e}_x + i\underline{e}_y)\,\frac{\sqrt{1+\alpha^2}-1}{2\sqrt{1+\alpha^2}}\,E_o e^{ik_o z + i\omega_o\sqrt{1+\alpha^2}\,t} \tag{3.6}$$

$$\underline{H}(\underline{r},t) = \frac{1}{2}(\underline{e}_y - i\underline{e}_x)\,\frac{\sqrt{1+\alpha^2}+1}{2(1+\alpha^2)}\,H_o e^{ik_o z - i\omega_o\sqrt{1+\alpha^2}\,t}$$

$$- \frac{1}{\sqrt{2}}(\underline{e}_y - i\underline{e}_x)\,\frac{\sqrt{1+\alpha^2}-1}{2(1+\alpha^2)}\,H_o e^{ik_o z + i\omega_o\sqrt{1+\alpha^2}\,t}$$

$$+ \frac{1}{\sqrt{2}}(\underline{e}_y - i\underline{e}_x)\,\frac{\alpha^2}{1+\alpha^2}\,H_o e^{ik_o z} \tag{3.7}$$

It is clear that now there are two waves. One is a wave propagated in the positive z-direction, and the other is a wave propagated in the negative z-direction. The original field accelerates the electrons in the newly created plasma, which in turn radiate a wave in the negative direction as well as one in the positive z-direction. The sense of the circular polarization of the two waves remains the same as the original one.

The frequency of these two waves has been shifted from ω_o to $\omega_o\sqrt{1+\alpha^2}$. It can also be explained in the following way: From the initial conditions (2.19) and (2.20), the propagation constant k_o of the wave must be conserved in a suddenly-created plasma. Hence the following relations must hold:

$$k_o = \frac{\omega_o}{c} = \frac{\omega}{v_p} \tag{3.8}$$

where

$$v_p = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \tag{3.9}$$

is the phase velocity in the plasma. It follows from Eq. (3.8) that the new frequency must be blue-shifted, since the phase velocity in a plasma is greater than c . It is also noted that $\omega > \omega_p$. For high frequencies, i.e., $\alpha = \frac{\omega_p}{\omega_o} << 1$

$$\omega \simeq \omega_o(1 + \frac{1}{2}\alpha^2) = \omega_o + \frac{\omega_p^2}{2\omega_o} \tag{3.10}$$

which is slightly higher than the original frequency; the frequency shift is

$$\frac{\Delta\omega}{\omega_o} = \frac{\Delta f}{f_o} \simeq \frac{\alpha^2}{2} = 40.5 \frac{N}{f_o^2} \tag{3.11}$$

where $\Delta f = \Delta\omega/2\pi = (\omega - \omega_o)/2\pi$, $f_o = \omega_o/2\pi$ and N is the free electron density. For low frequencies, i.e., $\alpha = \frac{\omega_p}{\omega_o} >> 1$,

$$\omega \simeq \omega_p(1 + \frac{1}{2\alpha^2}) = \omega_p + \frac{\omega_o^2}{2\omega_p} \tag{3.12}$$

which is slightly higher than the plasma frequency; the frequency shift is

$$\frac{\Delta\omega}{\omega_o} = \frac{\Delta f}{f_o} \simeq \alpha = 9 \frac{\sqrt{N}}{f_o} . \tag{3.13}$$

In Fig. 3.1, ω/ω_o is plotted as a function of $\alpha = \omega_p/\omega_o$.

Expressed in terms of α , the phase velocity v_p is

$$v_p = c\sqrt{1+\alpha^2} \tag{3.14}$$

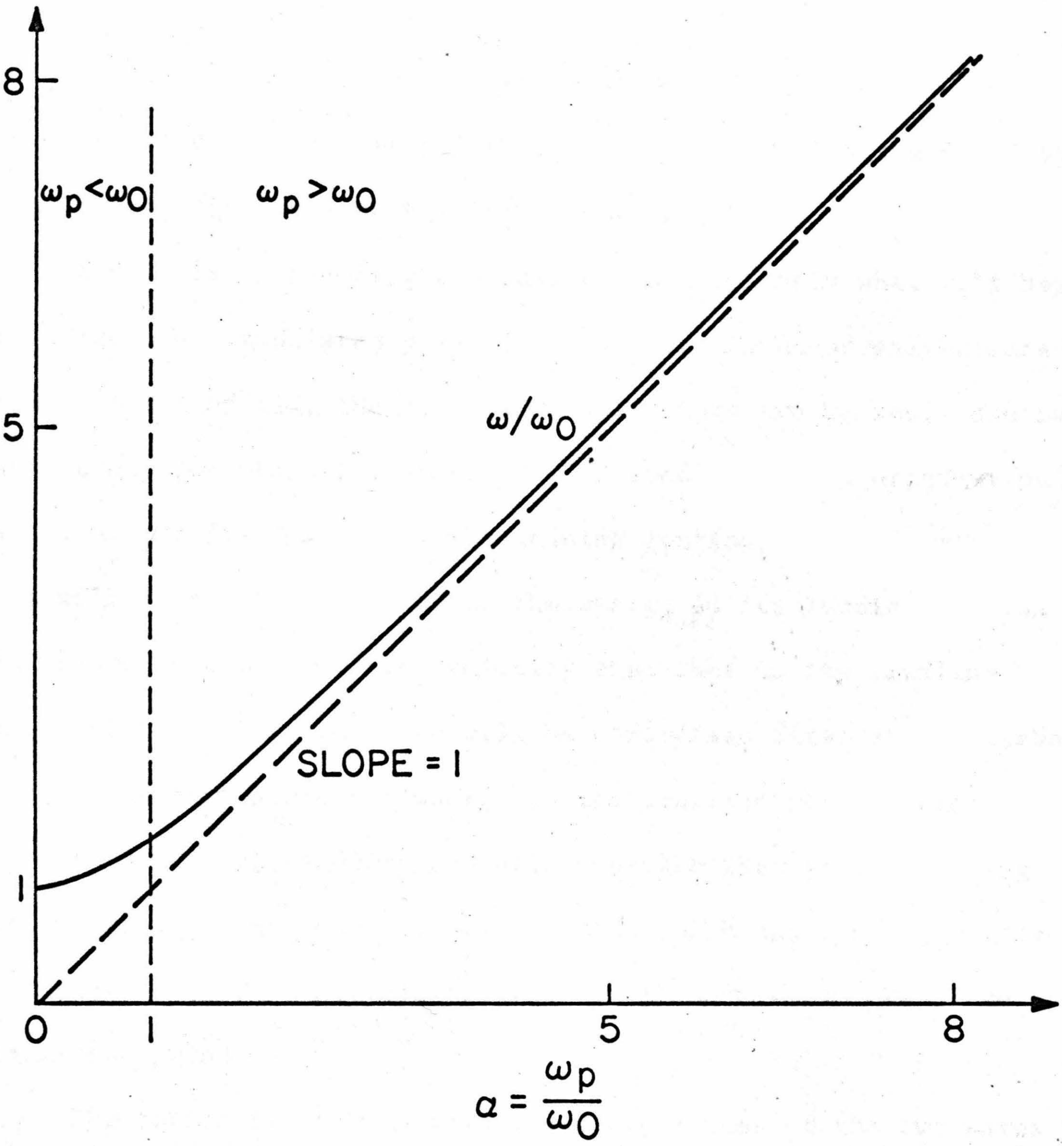


Fig. 3.1 Frequency shift

The group velocity is calculated according to the formula [23]

$$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{3.15}$$

Thus

$$v_g = \frac{c}{\sqrt{1 + \alpha^2}} \tag{3.16}$$

which is greater for higher frequencies. As $\alpha = \frac{\omega_p}{\omega_o} \to 0$, $v_g \sim c$; as $\alpha \to \infty$, $v_g \sim 0$. Group velocity is also interpreted as the velocity of energy propagation in lossless structures [24].

Now it is interesting to consider qualitatively what will happen when a frequency modulated pulse is propagated in a suddenly-created plasma. First of all, the pulse will split into two pulses; one is transmitted, the other is reflected. Secondly, if the original pulse has the higher frequencies in the leading portion, the transmitted pulse will be stretched out since the energy in its leading portion will propagate with a greater velocity than that in its trailing portion. But the reflected pulse will be compressed first and stretched out subsequently, since the energy in its trailing portion will propagate with a greater velocity and will overtake that in its leading portion [25,26]. However, if the original pulse has the higher frequencies in the trailing portion, the compression will take place in the transmitted pulse.

The ratios of the electric field amplitudes of the two waves to that of the original wave are

$$\frac{E_+}{E_o} = \frac{\sqrt{1+\alpha^2} + 1}{2\sqrt{1+\alpha^2}} \tag{3.17}$$

$$\frac{E_-}{E_o} = \frac{\sqrt{1+\alpha^2} - 1}{2\sqrt{1+\alpha^2}} \qquad (3.18)$$

where E_+ and E_- are the amplitude of the wave propagated in the positive z-direction and the amplitude of the wave propagated in the negative z-direction, respectively. Both ratios are plotted as functions of $\alpha = \omega_p/\omega_o$ in Fig. 3.2 . For high frequencies, $\alpha = \frac{\omega_p}{\omega_o} << 1$,

$$\frac{E_+}{E_o} \simeq 1 - \frac{\alpha^2}{4} \qquad (3.19)$$

$$\frac{E_-}{E_o} \simeq \frac{\alpha^2}{4} \qquad (3.20)$$

A very small amount will be propagated in the negative z-direction. For low frequencies, $\alpha = \frac{\omega_p}{\omega_o} >> 1$,

$$\frac{E_+}{E_o} \simeq \frac{1}{2} + \frac{1}{2\alpha^2} \qquad (3.21)$$

$$\frac{E_-}{E_o} \simeq \frac{1}{2} - \frac{1}{2\alpha^2} \qquad (3.22)$$

where nearly half will be propagated in the positive z-direction and half in the negative direction.

The ratios of the magnetic field amplitudes are

$$\frac{H_+}{H_o} = \frac{\sqrt{1+\alpha^2} + 1}{2(1+\alpha^2)} \qquad (3.23)$$

$$\frac{H_-}{H_o} = -\frac{\sqrt{1+\alpha^2} - 1}{2(1+\alpha^2)} \qquad (3.24)$$

see Fig. 3.3. Both will go to zero in the low frequency limit. $|H_-|$

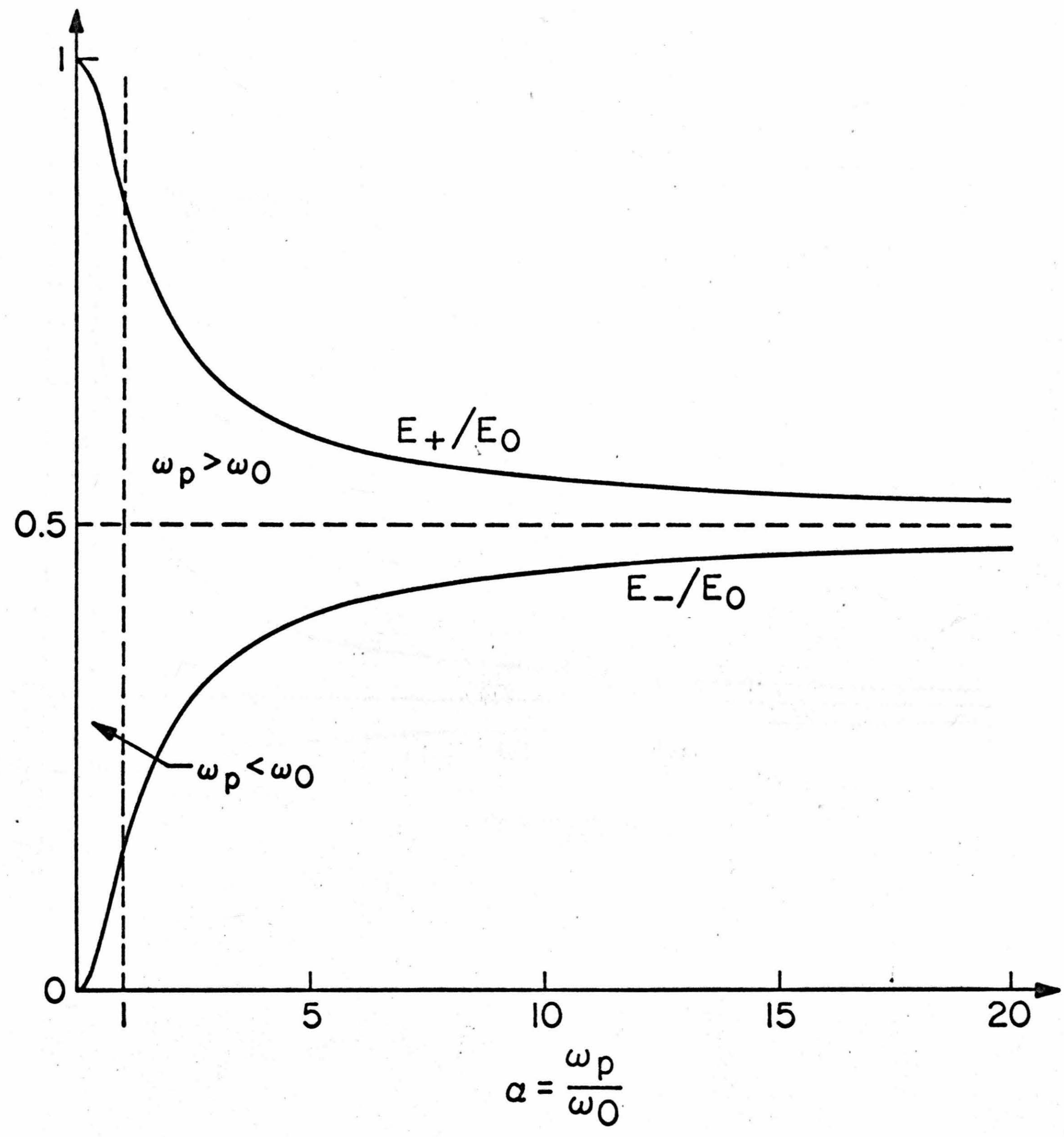


Fig. 3.2 Amplitudes of electric field

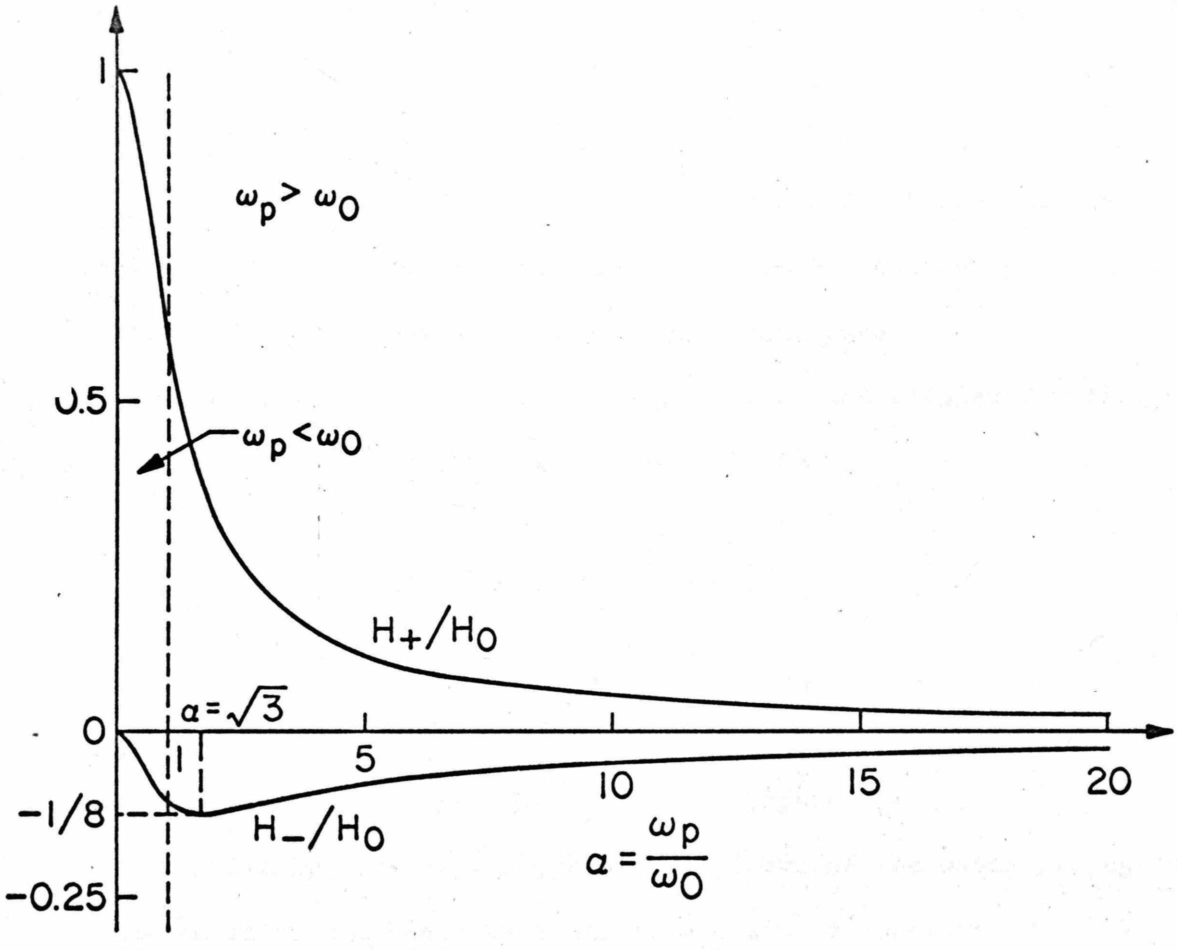


Fig. 3.3 Amplitudes of magnetic field

is less than 1/8 of the original magnitude $|H_o|$. It is also noted that there is a static but spatially varying magnetic field in Eq. (3.4) or (3.7). The slightest loss in the medium would damp out this residual static component. The collision effects will be further discussed in Appendix A.

The power relations will now be considered. Since the field amplitudes do not vary with time or distance, the average power is the instantaneous power for a circularly polarized wave.

For $t < 0$, by taking the real part of the complex Poynting vector [27], we obtain the original power flow:

$$\underline{S}_o \equiv \underline{e}_z S_o = \mathrm{Re}\{\frac{1}{2}(\frac{\underline{e}_x + i\underline{e}_y}{\sqrt{2}} E_o e^{ik_o z}) \times (\frac{\underline{e}_y - i\underline{e}_x}{\sqrt{2}} H_o e^{ik_o z})^*\}$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}\, |E_o|^2 \qquad (3.25)$$

where the star (*) denotes the complex conjugate.

Similarly, for $t > 0$, the power flows of the waves propagated in the positive and negative z-directions are, respectively,

$$\underline{S}_+ = \underline{e}_z \frac{(\sqrt{1+\alpha^2} + 1)^2}{4(1+\alpha^2)^{3/2}} S_o \qquad (3.26)$$

$$\underline{S}_- = -\underline{e}_z \frac{(\sqrt{1+\alpha^2} - 1)^2}{4(1+\alpha^2)^{3/2}} S_o \qquad (3.27)$$

$|S_-|$ is less than 1/27 of the original value S_o . In the low frequency limit the power flows for both waves are nearly zero, since in this limit the velocity of energy propagation which is v_g in Eq. (3.16) approaches zero, (see Eq. (3.35) and (3.36).)

The net power flow is then

$$\underline{S}_{net} = \underline{S}_+ + \underline{S}_- = \underline{e}_z \frac{1}{1+\alpha^2} S_o \tag{3.28}$$

Thus the net power flow is in the positive z-direction. The magnitude $|\underline{S}_{net}| < S_o$. For high frequencies, $\alpha = \frac{\omega_p}{\omega_o} << 1$,

$$\underline{S}_{net} \simeq \underline{e}_z (1 - \alpha^2)\, S_o \tag{3.29}$$

which is nearly equal to the original one. For low frequencies, $\alpha = \frac{\omega_p}{\omega_o} >> 1$,

$$\underline{S}_{net} \simeq \underline{e}_z \frac{1}{\alpha^2} S_o \tag{3.30}$$

which is very small. In Fig. 3.4, S_+/S_o, S_-/S_o, S_{net}/S_o are plotted as functions of $\alpha = \omega_p/\omega_o$.

The energy density of the wave for $t < 0$ is

$$w_o = \frac{1}{4} \varepsilon_o |E_o|^2 + \frac{1}{4} \mu_o |H_o|^2 = \frac{1}{2} \varepsilon_o |E_o|^2 \tag{3.31}$$

For $t > 0$, the energy density of the wave propagated in the positive z-direction is [28]

$$w_+ = \frac{1}{4} \{\frac{\partial}{\partial\omega}[\omega\varepsilon_o(1 - \frac{\omega_p^2}{\omega^2})]\} |E_+|^2 + \frac{1}{4} \mu_o |H_+|^2 \tag{3.32}$$

$$= \frac{1}{4} \frac{(\sqrt{1+\alpha^2} + 1)^2}{1 + \alpha^2} w_o \tag{3.33}$$

Similarly,

$$w_- = \frac{1}{4} \frac{(\sqrt{1+\alpha^2} - 1)^2}{1 + \alpha^2} w_o \tag{3.34}$$

as $\alpha = \frac{\omega_p}{\omega_o} \to 0$, $w_+ \sim w_o$, $w_- \sim 0$; as $\alpha \to \infty$, $w_+ \sim w_- \sim \frac{w_o}{4}$. It is

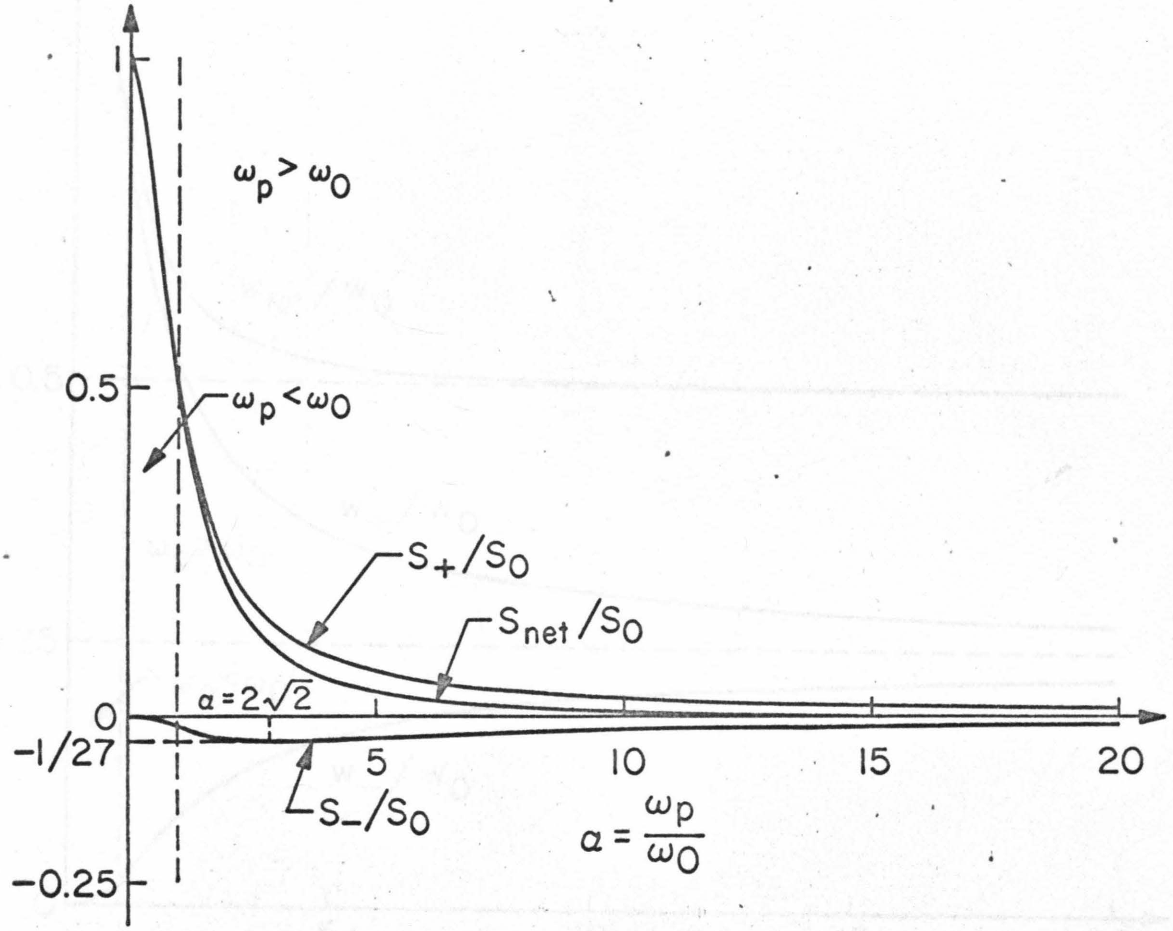


Fig. 3.4 Power flows

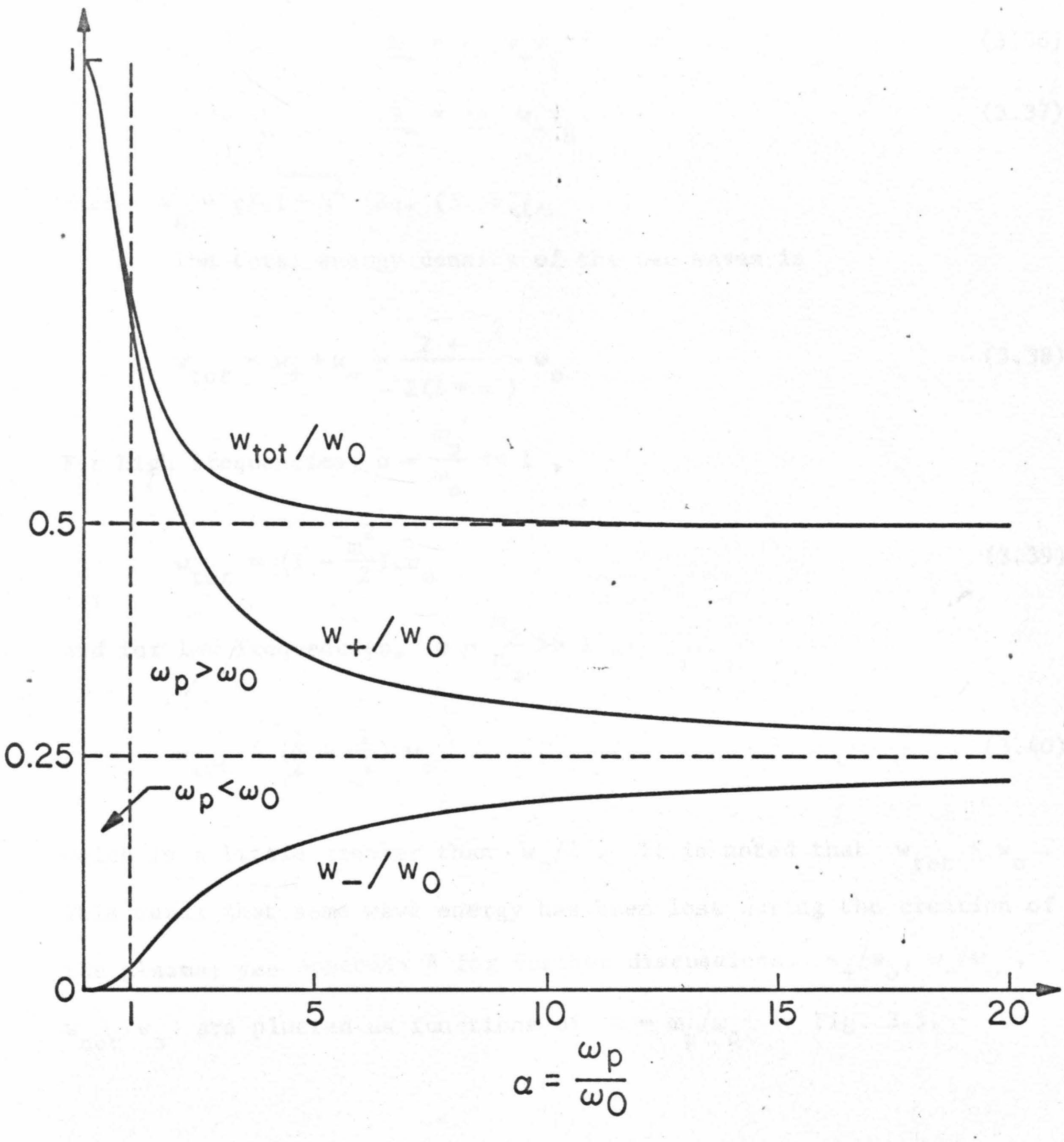


Fig. 3.5 Energy densities

noted that

$$\underline{S}_o = \underline{e}_z \, w_o c \tag{3.35}$$

$$\underline{S}_+ = \underline{e}_z \, w_+ v_g \tag{3.36}$$

$$\underline{S}_- = -\underline{e}_z \, w_- v_g \tag{3.37}$$

where $v_g = c/\sqrt{1+\alpha^2}$ (Eq. (3.16)).

The total energy density of the two waves is

$$w_{tot} = w_+ + w_- = \frac{2+\alpha^2}{2(1+\alpha^2)} w_o \tag{3.38}$$

For high frequencies, $\alpha = \frac{\omega_p}{\omega_o} << 1$,

$$w_{tot} \simeq (1 - \frac{\omega^2}{2}) \, w_o \tag{3.39}$$

and for low frequencies, $\alpha = \frac{\omega_p}{\omega_o} >> 1$,

$$w_{tot} \simeq (\frac{1}{2} + \frac{1}{\alpha^2}) \, w_o \tag{3.40}$$

which is a little greater than $w_o/2$. It is noted that $w_{tot} < w_o$. This means that some wave energy has been lost during the creation of the plasma; see Appendix B for further discussions. w_+/w_o, w_-/w_o , w_{tot}/w_o are plotted as functions of $\alpha = \omega_p/\omega_o$ in Fig. 3.5.

4. ELECTRIC DIPOLE RADIATION

In this chapter the problem of radiation from a sinusoidally oscillating electric point dipole in a suddenly-created plasma will be considered. The origin of a spherical coordinate system is chosen to be at the dipole. Rectangular coordinate system is also shown, see Fig. 4.1. The directions of the dipole and the positive z-axis are the same.

The dipole has a moment of amplitude p and sinusoidal frequency ω_o. It is represented mathematically by

$$\underline{M}(\underline{r},t) = \underline{e}_z p\ \delta(\underline{r}) \sin \omega_o t \tag{4.1}$$

Here $\delta(\underline{r})$ is the three-dimensional Dirac δ function. The applied source current density is related to $\underline{M}$ by

$$\underline{J}_s(\underline{r},t) = \frac{\partial}{\partial t} \underline{M}(\underline{r},t) = \underline{e}_z\ \omega_o p\ \delta(\underline{r}) \cos \omega_o t \tag{4.2}$$

For $t < 0$, the steady state electromagnetic fields radiated from this electric point dipole are well known [29]:

$$E_r(\underline{r},t) = \frac{\mu_o \omega_o^2 p}{4\pi r} 2 \cos \theta \left\{\frac{1}{k_o r} \cos(k_o r - \omega_o t) - \frac{1}{k_o^2 r^2} \sin(k_o r - \omega_o t)\right\} \tag{4.3}$$

$$E_\theta(\underline{r},t) = \frac{\mu_o \omega_o^2 p}{4\pi r} \sin \theta \left\{\sin(k_o r - \omega_o t) + \frac{1}{k_o r} \cos(k_o r - \omega_o t) - \frac{1}{k_o^2 r^2} \sin(k_o r - \omega_o t)\right\} \tag{4.4}$$

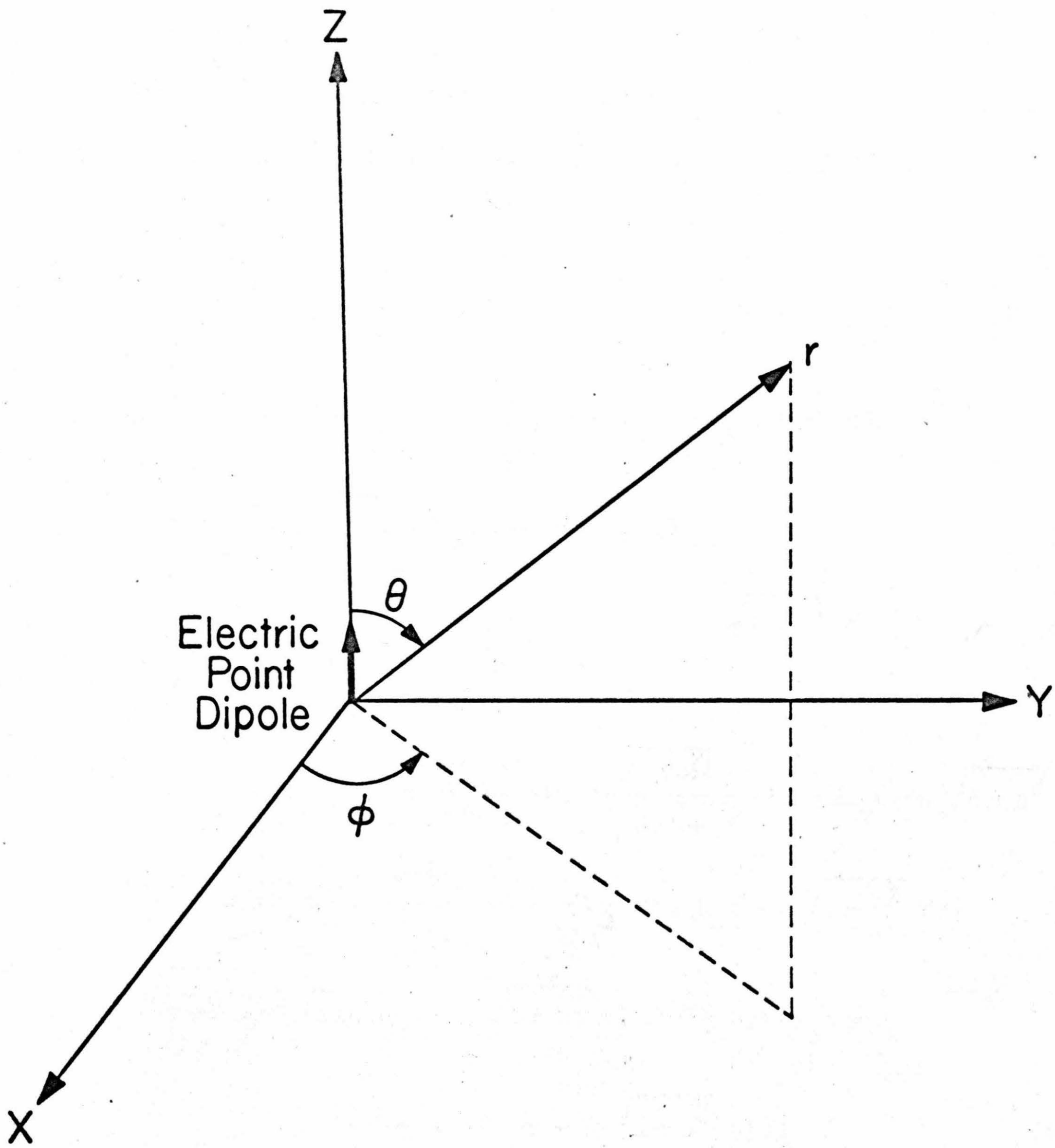


Fig. 4.1 Configuration and coordinate systems for the dipole radiation problem

$$H_\phi(\underline{r},t) = \frac{\omega_o^2 p}{4\pi rc} \sin\theta \{\sin(k_o r - \omega_o t) + \frac{1}{k_o r} \cos(k_o r - \omega_o t)\} \qquad (4.5)$$

All the other components are zero.

For $t > 0$, i.e., after the plasma is created, the fields can be obtained by taking the real part of Eq. (2.53) and (2.54):

$$E_r(\underline{r},t) = E_r'(\underline{r},t) + \frac{\mu_o \omega_o^2 p}{4\pi r} 2 \cos\theta \{\frac{\sqrt{1+\alpha^2}+1}{2\sqrt{1+\alpha^2}} [\frac{1}{k_o r} \cos(k_o r - \omega_o\sqrt{1+\alpha^2}\, t) - \frac{1}{k_o^2 r^2} \sin(k_o r - \omega_o\sqrt{1+\alpha^2}\, t)]$$
$$+ \frac{\sqrt{1+\alpha^2}-1}{2\sqrt{1+\alpha^2}}[\frac{1}{k_o r} \cos(k_o r + \omega_o\sqrt{1+\alpha^2}\, t) - \frac{1}{k_o^2 r^2} \sin(k_o r + \omega_o\sqrt{1+\alpha^2}\, t)]\} \qquad (4.6)$$

$$E_\theta(\underline{r},t) = E_\theta'(\underline{r},t) + \frac{\mu_o \omega_o^2 p}{4\pi r} \sin\theta \{\frac{\sqrt{1+\alpha^2}+1}{2\sqrt{1+\alpha^2}}[\sin(k_o r - \omega_o\sqrt{1+\alpha^2}\, t) + \frac{1}{k_o r} \cos(k_o r - \omega_o\sqrt{1+\alpha^2}\, t) - \frac{1}{k_o^2 r^2} \sin(k_o r - \omega_o\sqrt{1+\alpha^2}\, t)]$$
$$+ \frac{\sqrt{1+\alpha^2}-1}{2\sqrt{1+\alpha^2}} [\sin(k_o r + \omega_o\sqrt{1+\alpha^2}\, t) + \frac{1}{k_o r} \cos(k_o r + \omega_o\sqrt{1+\alpha^2}\, t) - \frac{1}{k_o^2 r^2} \sin(k_o r + \omega_o\sqrt{1+\alpha^2}\, t)]\} \qquad (4.7)$$

$$H_\phi(\underline{r},t) = H'(\underline{r},t) + \frac{\omega_o^2 p}{4\pi rc} \sin\theta\{\frac{\sqrt{1+\alpha^2}+1}{2(1+\alpha^2)}[\sin(k_o r - \omega_o\sqrt{1+\alpha^2}\, t)$$

$$+ \frac{1}{k_o r}\cos(k_o r - \omega_o\sqrt{1+\alpha^2}\, t)] - \frac{\sqrt{1+\alpha^2}-1}{2(1+\alpha^2)}[\sin(k_o r + \omega_o\sqrt{1+\alpha^2}\, t)$$

$$+ \frac{1}{k_o r}\cos(k_o r + \omega_o\sqrt{1+\alpha^2}\, t)] + \frac{\alpha^2}{1+\alpha^2}(\sin k_o r + \frac{1}{k_o r}\cos k_o r)\} \qquad (4.8)$$

where $\alpha = \omega_p/\omega_o$; $E'_r(\underline{r},t)$, $E'(\underline{r},t)$ and $H'(\underline{r},t)$ are the nonzero spherical components of

$$\underline{E}'(\underline{r},t) = L^{-1}\{\hat{\underline{E}}'(\underline{r},s)\} = L^{-1}\{ - \frac{\mu_o sJ'(s)}{4\pi} [\underline{e}_z \frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r}$$

$$- \frac{c^2}{s^2+\omega_p^2} \nabla \frac{\partial}{\partial z} (\frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r})] \} \qquad (4.9)$$

and

$$\underline{H}'(\underline{r},t) = L^{-1}\{- \frac{1}{\mu_o s} \nabla \times \hat{\underline{E}}'(\underline{r},s)\} \qquad (4.10)$$

where $J'(s)$ is calculated by using Eq. (2.40) and (4.2):

$$J'(s) = - \frac{\omega_o^3\, \omega_p^2\, p}{s(s^2+\omega_o^2)(s^2+\omega_o^2+\omega_p^2)} \qquad (4.11)$$

It is readily seen from Eq. (4.6) to (4.8) that, like the plane wave case, there are also two waves traveling in opposite directions. One is a spherical wave propagated outwardly from the origin, and the other is a spherical wave propagated inwardly into the origin. Their frequency is shifted from ω_o to $\omega_o\sqrt{1+\alpha^2}$. The amplitude and power relations are the same as in the plane wave case. Moreover, the outgoing wave will vanish at a finite observation point since there is no source at the origin to radiate this wave any more; but the incoming wave will focus into the origin and be reflected. These novel aspects of the wave propagation are expressed mathematically in the terms $E_r'(\underline{r},t)$, $E_\theta'(\underline{r},t)$ and $H_\phi'(\underline{r},t)$. In addition to these waves, there is the radiation by the dipole which is also included in the expressions for $E_r'(\underline{r},t)$, $E_\theta'(\underline{r},t)$ and $H_\phi'(\underline{r},t)$.

The inverse integration will now be performed; and $\underline{E}'(\underline{r},t)$ and $\underline{H}'(\underline{r},t)$ will be calculated. Performing the differentiations in Eq. (4.9) and expressing the components of $\hat{\underline{E}}'(\underline{r},s)$ in spherical coordinates, we have

$$\hat{E}_r'(\underline{r},s) = \frac{\mu_o s\, J'(s)}{4\pi r}\, 2\cos\theta \left(\frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}} + \frac{1}{\frac{r^2}{c^2}(s^2+\omega_p^2)} \right) e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}} \tag{4.12}$$

$$\hat{E}_\theta'(\underline{r},s) = \frac{\mu_o s\, J'(s)}{4\pi r} \sin\theta \left(1+\frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}} + \frac{1}{\frac{r^2}{c^2}(s^2+\omega_p^2)} \right) e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}} \tag{4.13}$$

$$\hat{E}_\phi'(\underline{r},s) = 0 \tag{4.14}$$

From Eq. (4.10),

$$\hat{H}_r'(\underline{r},s) = 0 \tag{4.15}$$

$$\hat{H}_\theta'(\underline{r},s) = 0 \tag{4.16}$$

$$\hat{H}_\phi'(\underline{r},s) = \frac{\sqrt{s^2+\omega_p^2}\, J'(s)}{4\pi r c} \sin\theta \left(1+\frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}} \right) e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}} \tag{4.17}$$

$J'(s)$ will now be expanded into partial fractions:

$$J'(s) = \omega_o p \left(\frac{s}{s^2 + \omega_o^2} - \frac{1}{1+\alpha^2} \frac{s}{s^2+\omega^2} - \frac{\alpha^2}{1+\alpha^2} \frac{1}{s} \right) \tag{4.18}$$

where $\omega = \sqrt{\omega_o^2 + \omega_p^2}$. Then $E_r'(\underline{r},t)$, $E_\theta'(\underline{r},t)$ and $H_\phi'(\underline{r},t)$ are

$$E_r'(\underline{r},t) = \frac{1}{2\pi i} \int_C \hat{E}_r'(\underline{r},s)\, e^{st}\, ds \tag{4.19}$$

$$= \frac{\mu_o \omega_o p}{4\pi r}\, 2\cos\theta\, \frac{1}{2\pi i} \int_C \left(\frac{s^2}{s^2+\omega_o^2} - \frac{1}{1+\alpha^2} \frac{s^2}{s^2+\omega^2} - \frac{\alpha^2}{1+\alpha^2} \right)$$

$$\times \left(\frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}} + \frac{1}{\frac{r^2}{c^2}(s^2+\omega_p^2)} \right) e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds \tag{4.20}$$

$$E_\theta'(\underline{r},t) = \frac{\mu_o \omega_o p}{4\pi r} \sin\theta\, \frac{1}{2\pi i} \int_C \left(\frac{s^2}{s^2+\omega_o^2} - \frac{1}{1+\alpha^2} \frac{s^2}{s^2+\omega^2} - \frac{\alpha^2}{1+\alpha^2} \right)$$

$$\times \left(1 + \frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}} + \frac{1}{\frac{r^2}{c^2}(s^2+\omega_p^2)} \right) e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds \tag{4.21}$$

$$H_\phi'(\underline{r},t) = \frac{\omega_o p}{4\pi r c} \sin\theta\, \frac{1}{2\pi i} \int_C \left(\frac{s\sqrt{s^2+\omega_p^2}}{s^2+\omega_o^2} - \frac{1}{1+\alpha^2} \frac{s\sqrt{s^2+\omega_p^2}}{s^2+\omega^2} \right.$$

$$\left. - \frac{\alpha^2}{1+\alpha^2} \frac{\sqrt{s^2+\omega_p^2}}{s} \right) \left(1 + \frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}} \right) e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds \tag{4.22}$$

To avoid duplication of calculations, we evaluate the following integrals:

$$I_1(\Omega) \equiv \frac{1}{2\pi i} \int_C \frac{s^2}{s^2+\Omega^2}\, e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds \tag{4.23}$$

$$I_2(\Omega) \equiv \frac{1}{2\pi i}\int_C \frac{s^2}{s^2+\Omega^2}\left(\frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}}+\frac{1}{\frac{r^2}{c^2}(s^2+\omega_p^2)}\right)e^{st-\frac{r}{c}\sqrt{s^2+\omega_p^2}}\,ds \tag{4.24}$$

$$I_3(\Omega) \equiv \frac{1}{2\pi i}\int_C \frac{s\sqrt{s^2+\omega_p^2}}{s^2+\Omega^2}\left(1+\frac{1}{\frac{r}{c}\sqrt{s^2+\omega_p^2}}\right)e^{st-\frac{r}{c}\sqrt{s^2+\omega_p^2}}\,ds \tag{4.25}$$

Hence,

$$E_r'(\underline{r},t) = \frac{\mu_o\omega_o p}{4\pi r}\,2\cos\theta\left(I_2(\omega_o)-\frac{1}{1+\alpha^2}I_2(\omega)-\frac{\alpha^2}{1+\alpha^2}I_2(0)\right) \tag{4.26}$$

$$E_\theta'(\underline{r},t) = \frac{\mu_o\omega_o p}{4\pi r}\sin\theta\left\{(I_1(\omega_o)+I_2(\omega_o)) - \frac{1}{1+\alpha^2}(I_1(\omega)+I_2(\omega)) - \frac{\alpha^2}{1+\alpha^2}(I_1(0)+I_2(0))\right\} \tag{4.27}$$

$$H_\phi'(\underline{r},t) = \frac{\omega_o p}{4\pi rc}\sin\theta\left(I_3(\omega_o)-\frac{1}{1+\alpha^2}I_3(\omega)-\frac{\alpha^2}{1+\alpha^2}I_3(0)\right) \tag{4.28}$$

Performing the integration technique described in Appendix D, we obtain

$$I_1(\Omega) = \frac{1}{2\pi i}\int_C \left(1-\frac{\Omega^2}{s^2+\Omega^2}\right)e^{st-\frac{r}{c}\sqrt{s^2+\omega_p^2}}\,ds \tag{4.29}$$

$$= \delta\left(t-\frac{r}{c}\right) - \frac{\omega_p^2}{r/c}\frac{J_1(q)}{q}H\left(t-\frac{r}{c}\right) - \frac{\Omega^2}{\omega_p\pi i}\int_0^{2\pi}\frac{x(1-x^2)\,e^{iq\cos\psi}}{(x^2-A^2(\Omega))(x^2-A^2(\Omega))}\,d\psi\,H\left(t-\frac{r}{c}\right) \tag{4.30}$$

where $\beta \equiv r/ct$, $q \equiv \omega_p t\sqrt{1-\beta^2}$, $\gamma \equiv \sqrt{(1-\beta)/(1+\beta)}$, $x \equiv \gamma e^{i\psi}$ and

$$A(\Omega) = \frac{\Omega + \sqrt{\Omega^2 - \omega_p^2}}{\omega_p} \qquad \text{when} \quad \Omega > \omega_p$$

$$= \frac{\Omega + i\sqrt{\omega_p^2 - \Omega^2}}{\omega_p} \qquad \text{when} \quad \Omega < i\omega_p \tag{4.31}$$

$H(t - \frac{r}{c})$, the Heaviside step function, is 0 when $t < \frac{r}{c}$, or 1 when $t > \frac{r}{c}$. Partial fractioning of the integrands yields

$$I_1(\Omega) = \delta(t - \frac{r}{c}) - \frac{\omega_p^2}{r/c}\frac{J_1(q)}{q} H(t - \frac{r}{c})$$

$$+ \frac{1}{2\pi i}\left\{\frac{A^2(\Omega)/\omega_p}{1 + A^2(\Omega)}\int_0^{2\pi} \frac{x\, e^{iq\cos\psi}}{x^2 - A^2(\Omega)}\, d\psi\right.$$

$$\left. + \frac{A^{-2}(\Omega)/\omega_p}{1 + A^{-2}(\Omega)}\int_0^{2\pi} \frac{x\, e^{iq\cos\psi}}{x^2 - A^{-2}(\Omega)}\, d\psi\right\} H(t - \frac{r}{c}) \tag{4.32}$$

Using the integration formulas in Appendix D, we find that

$$I_1(\Omega) = \delta(t - \frac{r}{c}) - \frac{\omega_p^2}{r/c}\frac{J_1(q)}{q} H(t - \frac{r}{c})$$

$$- \Omega[U_1(\gamma q A(\Omega), q) + U_1(\gamma q A^{-1}(\Omega), q)]\; H(t - \frac{r}{c}) \tag{4.33}$$

See Appendix C for some mathematical properties of Lommel functions of two variables $U_n(w,z)$.

Similarly, the integration technique described in Appendix D gives

$$I_2(\Omega) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(1 + x^2)^2}{(x^2 - A^2(\Omega))(x^2 - A^{-2}(\Omega))}\left(\frac{1}{r/c} + \frac{2i}{\frac{r^2}{c^2}\omega_p}\frac{x}{x^2 - 1}\right)$$

$$\times e^{iq\cos\psi}\, d\psi\; H(t - \frac{r}{c}) \tag{4.34}$$

Partial fractioning followed by the use of the integration formulas in Appendix D yields

$$I_2(\Omega) = \{\frac{1}{r/c} J_o(q) + \frac{c\,\Omega}{r\sqrt{\Omega^2 - \omega_p^2}} (U_o(\gamma q A(\Omega),q) - U_o(\gamma q A^{-1}(\Omega),q))$$

$$+ \frac{c^2}{r^2(\Omega^2 - \omega_p^2)} [\Omega(U_1(\gamma q A(\Omega),q) + U_1(\gamma q A^{-1},q))$$

$$- 2\omega_p U_1(\gamma q,q)]\} H(t - \frac{r}{c}) \tag{4.35}$$

And, for I_3 ,

$$I_3(\Omega) = \delta(t - \frac{r}{c}) - \frac{\omega_p^2}{r/c} \frac{J_1(q)}{q} H(t - \frac{r}{c})$$

$$- \frac{1}{2\pi i} \int_0^{2\pi} [\frac{\omega_p}{2} \frac{-2x^2 + A^2(\Omega) + \bar{A}^{-2}(\Omega)}{(x^2 - A^2(\Omega))(x^2 - \bar{A}^{-2}(\Omega))} x(1 - x^2)$$

$$+ \frac{1}{\frac{r}{c} i} \frac{1 - x^4}{(x^2 - A^2(\Omega))(x^2 - A^{-2}(\Omega))}] e^{iq \cos\psi} d\psi \, H(t - \frac{r}{c}) \tag{4.36}$$

then it gives

$$I_3(\Omega) = \delta(t - \frac{r}{c}) - \frac{\omega_p^2}{r/c} \frac{J_1(q)}{q} H(t - \frac{r}{c}) - \omega_p \gamma \, J_1(q) \, H(t - \frac{r}{c})$$

$$-\{\sqrt{\Omega^2 - \omega_p^2} \, [U_1(\gamma q A(\Omega),q) - U_1(\gamma q A^{-1}(\Omega),q)]$$

$$+ \frac{c}{r}[-J_o(q) + U_o(\gamma q A(\Omega),q) + U_o(\gamma q A^{-1}(\Omega),q)]\} \, H(t - \frac{r}{c}) \tag{4.37}$$

Therefore,

$$E'_r(\underline{r},t) = \frac{\mu_o \omega_o^2 p}{4\pi r} 2 \cos\theta \left\{ \frac{1}{\sqrt{1-\alpha^2}\, k_o r} (U_o(\gamma q \xi_o, q) - U_o(\gamma q \xi^{-1}, q)) \right.$$

$$+ \frac{1}{(1-\alpha^2) k_o^2 r^2} (U_1(\gamma q \xi_o, q) + U_1(\gamma q \xi_o^{-1}, q))$$

$$- \frac{2\alpha}{(1-\alpha^2) k_o^2 r^2} U_1(\gamma q, q) - \frac{1}{\sqrt{1+\alpha^2}\, k_o r} (U_o(\gamma q \xi, q) - U_o(\gamma q \xi^{-1}, q))$$

$$\left. - \frac{1}{\sqrt{1+\alpha^2}\, k_o^2 r^2} (U_1(\gamma q \xi, q) + U_1(\gamma q \xi^{-1}, q)) \right\} H(t - \frac{r}{c}) \qquad (4.38)$$

$$E'_\theta(\underline{r},t) = \frac{\mu_o \omega_o^2 p}{4\pi r} \sin\theta \left\{ -(U_1(\gamma q \xi_o, q) + U_1(\gamma q \xi_o^{-1}, q)) \right.$$

$$+ \frac{1}{\sqrt{1-\alpha^2}\, k_o r} (U_o(\gamma q \xi_o, q) - U_o(\gamma q \xi_o^{-1}, q))$$

$$+ \frac{1}{(1-\alpha^2) k_o^2 r^2} (U_1(\gamma q \xi_o, q) + U_1(\gamma q \xi_o^{-1}, q)) - \frac{2\alpha}{(1-\alpha^2) k_o^2 r^2} U_1(\gamma q, q)$$

$$+ \frac{1}{\sqrt{1+\alpha^2}} (U_1(\gamma q \xi, q) + U_1(\gamma q \xi^{-1}, q)) - \frac{1}{\sqrt{1+\alpha^2}\, k_o r} (U_o(\gamma q \xi, q)$$

$$\left. - U_o(\gamma q \xi^{-1}, q)) - \frac{1}{\sqrt{1+\alpha^2}\, k_o^2 r^2} (U_1(\gamma q \xi, q) + U_1(\gamma q \xi^{-1}, q)) \right\} H(t - \frac{r}{c})$$

$$(4.39)$$

$$H'_\phi(\underline{r},t) = \frac{\omega_o^2 p}{4\pi r c} \sin\theta \left\{ -\sqrt{1-\alpha^2} (U_1(\gamma q \xi_o, q) - U_1(\gamma q \xi_o^{-1}, q)) \right.$$

$$+ \frac{1}{k_o r} (U_o(\gamma q \xi_o, q) + U_o(\gamma q \xi_o^{-1}, q)) + \frac{1}{1+\alpha^2} (U_1(\gamma q \xi, q) - U_1(\gamma q \xi^{-1}, q))$$

$$- \frac{1}{(1+\alpha^2) k_o r} (U_o(\gamma q \xi, q) + U_o(\gamma q \xi^{-1}, q))$$

$$\left. + \frac{2\alpha^2}{1+\alpha^2} (i\alpha U_1(i\gamma q, q) - \frac{1}{k_o r} U_o(i\gamma q, q)) \right\} H(t - \frac{r}{c}) \qquad (4.40)$$

where $\xi_o \equiv A(\omega_o)$ and $\xi \equiv A(\omega)$.

When Eq. (4.38) to (4.40) are substituted into Eq. (4.6) to (4.8), the total electric and magnetic fields $E_r(\underline{r},t)$, $E_\theta(\underline{r},t)$ and $H_\phi(\underline{r},t)$ for $t > 0$ are obtained.

For small γ , corresponding to times just after the arrival of the first disturbance, the disturbing fields are

$$E'_r(\underline{r},t) = \frac{\mu_o \omega_o^2 p}{4\pi r} 2 \cos\theta \frac{4}{\alpha k_o^2 r^2} J_3(q)\, \gamma^3 H(t - \frac{r}{c}) + 0(\gamma^4) \qquad (4.41)$$

$$E'_\theta(\underline{r},t) = \frac{\mu_o \omega_o^2 p}{4\pi r} \sin\theta \,(-3 + \frac{1}{k_o^2 r^2}) \frac{2}{\alpha} J_3(q) \gamma^3 H(t - \frac{r}{c}) + 0(\gamma^4) \qquad (4.42)$$

$$H'_\phi(\underline{r},t) = \frac{\omega_o^2 p}{4\pi r c} \sin\theta \frac{-8}{\alpha} J_3(q) \gamma^3 H(t - \frac{r}{c}) + 0(\gamma^4) \qquad (4.43)$$

The amplitudes are small. The Bessel functions with arguments $q = \omega_p t \sqrt{1-\beta^2}$ represent oscillations having a frequency that is initially high but continuously decreases to ω_p .

We shall now obtain the asymptotic behavior of the total fields at a point $\underline{r}$ as $t \to \infty$. Through the use of the asymptotic formulas in Appendix C, we find, for $\omega_o > \omega_p$

$$E_r(\underline{r},t) \sim \frac{\mu_o \omega_o^2 p}{4\pi r} 2 \cos\theta \Big\{ \frac{1}{\sqrt{1-\alpha^2}\, k_o r} \cos(k_o \sqrt{1-\alpha^2}\, r - \omega_o t)$$

$$- \frac{1}{(1-\alpha^2) k_o^2 r^2} \sin(k_o \sqrt{1-\alpha^2}\, r - \omega_o t) - \frac{\alpha}{(1-\alpha^2) k_o^2 r^2} \sin \omega_p t$$

$$+ \frac{\sqrt{1+\alpha^2} - 1}{2\sqrt{1+\alpha^2}} [(\frac{1}{k_o r} \cos(k_o r + \omega_o \sqrt{1+\alpha^2}\, t) - \frac{1}{k_o^2 r^2} \sin(k_o r + \omega_o \sqrt{1+\alpha^2}\, t))$$

$$+ \left(\frac{1}{k_o r} \cos(k_o r - \omega_o\sqrt{1+\alpha^2}\ t) - \frac{1}{k_o^2 r^2} \sin(k_o r - \omega_o\sqrt{1+\alpha^2}\ t)\right)]\} \quad (4.44)$$

$$E_\theta(\underline{r},t) \sim \frac{\mu_o \omega_o^2 p}{4\pi r} \sin\theta \ \{\sin(k_o\sqrt{1-\alpha^2}\ r - \omega_o t) + \frac{1}{\sqrt{1-\alpha^2}\ k_o r}$$

$$\times \cos(k_o\sqrt{1-\alpha^2}\ r - \omega_o t) - \frac{1}{(1-\alpha^2)k_o^2 r^2} \sin(k_o\sqrt{1-\alpha^2} - \omega_o t)$$

$$- \frac{2\alpha}{(1-\alpha^2)k_o^2 r^2} \sin \omega_p t + \frac{\sqrt{1+\alpha^2}-1}{2\sqrt{1+\alpha^2}}[(\sin(k_o r + \omega_o\sqrt{1+\alpha^2}\ t)$$

$$+ \frac{1}{k_o r} \cos(k_o r + \omega_o\sqrt{1+\alpha^2}\ t) - \frac{1}{k_o^2 r^2} \sin(k_o r + \omega_o\sqrt{1+\alpha^2}\ t)$$

$$+ (\sin(k_o r - \omega_o\sqrt{1+\alpha^2}\ t) + \frac{1}{k_o r} \cos(k_o r - \omega_o\sqrt{1+\alpha^2}\ t)$$

$$- \frac{1}{k_o^2 r^2} \sin(k_o r - \omega_o\sqrt{1+\alpha^2}\ t))]\} \quad (4.45)$$

$$H_\phi(\underline{r},t) \sim \frac{\omega_o^2 p}{4\pi r c} \sin\theta \ \{\sqrt{1-\alpha^2}\ \sin(k_o\sqrt{1-\alpha^2}\ r - \omega_o t)$$

$$+ \frac{1}{k_o r} \cos(k_o\sqrt{1-\alpha^2} - \omega_o t) - \frac{\sqrt{1+\alpha^2}-1}{2(1+\alpha^2)}[(\sin(k_o r + \omega_o\sqrt{1+\alpha^2}\ t)$$

$$+ \frac{1}{k_o r} \cos(k_o r + \omega_o\sqrt{1+\alpha^2}\ t)) - (\sin(k_o r - \omega_o\sqrt{1+\alpha^2}\ t)$$

$$+ \frac{1}{k_o r} \cos(k_o r - \omega_o\sqrt{1+\alpha^2}\ t))] + \frac{\alpha^2}{1+\alpha^2}(\sin k_o r + \frac{1}{k_o r} \cos k_o r$$

$$- \alpha\, e^{-\omega_p r/c} - \frac{1}{k_o r} e^{-\omega_p r}) \} \quad (4.46)$$

and for $\omega_o < \omega_p$,

$$E_r(\underline{r},t) \sim \frac{\mu_o\omega_o^2 p}{4\pi r}\, 2\cos\theta \Big\{ \frac{1}{\sqrt{\alpha^2-1}\,k_o r}\, e^{-k_o\sqrt{\alpha^2-1}\,r} \sin\omega_o t$$

$$- \frac{1}{(\alpha^2-1)k_o^2 r^2}\, e^{-k_o\sqrt{\alpha^2-1}\,r} \sin\omega_o t + \frac{\alpha}{(\alpha^2-1)k_o^2 r^2} \sin\omega_p t$$

$$+ \frac{\sqrt{1+\alpha^2}-1}{2\sqrt{1+\alpha^2}} \text{[the same as the corresponding terms in Eq. (4.44)]}\Big\} \tag{4.47}$$

$$E_\theta(\underline{r},t) \sim \frac{\mu_o\omega_o^2 p}{4\pi r} \sin\theta \Big\{ -e^{-k_o\sqrt{\alpha^2-1}\,r} \sin\omega_o t$$

$$- \frac{1}{\sqrt{\alpha^2-1}\,k_o r}\, e^{-k_o\sqrt{\alpha^2-1}\,r} \sin\omega_o t - \frac{1}{(\alpha^2-1)k_o^2 r^2}\, e^{-k_o\sqrt{\alpha^2-1}\,r}$$

$$\times \sin\omega_o t + \frac{\alpha}{(\alpha^2-1)k_o^2 r^2} \sin\omega_p t + \frac{\sqrt{1+\alpha^2}-1}{2\sqrt{1+\alpha^2}} \text{[the same as the corresponding terms in Eq. (4.45)]}\Big\} \tag{4.48}$$

$$H_\phi(\underline{r},t) \sim \frac{\omega_o^2 p}{4\pi r c} \sin\theta \Big\{ \sqrt{\alpha^2-1}\, e^{-k_o\sqrt{\alpha^2-1}\,r} \cos\omega_o t$$

$$+ \frac{1}{k_o r}\, e^{-k_o\sqrt{\alpha^2-1}\,r} \cos\omega_o t - \frac{\sqrt{1+\alpha^2}-1}{2(1+\alpha^2)} \text{[the same as the corresponding terms in Eq. (4.46)]}$$

$$+ \frac{\alpha^2}{1+\alpha^2} \text{[the same as the corresponding terms in Eq. (4.46)]}\Big\} \tag{4.49}$$

The neglected terms have amplitudes vanishing faster than $t^{-1/2}$ as $t \to \infty$. It can be seen that radiation from the dipole has assumed its steady-state value. The outgoing wave has vanished but the incoming wave still exists because the plasma is assumed to be lossless and unbounded. The reflection of the incoming wave has also established its steady-state propagation. The incoming wave and its reflection carry the same energy but in opposite directions, hence there is no net power flow associated with these waves. It is noted that the magnetic field has a static but spatially varying component and that the electric field has a residual oscillation at the plasma frequency. Both do not contribute to the radiation and would be damped out by the slightest loss in the medium; see Appendix A for the effects of collisions.

Numerical calculations of the field quantities were performed. Since the medium is isotropic, the fields were only computed in the equatorial plane (z = 0). In this plane only the fields E_θ and H_ϕ are nonzero. Figures 4.2 through 4.4 show these fields as a function of the normalized time $\omega_o t$. The time-harmonic electric field's amplitude is denoted by horizontal dashed lines in the figures. Figure 4.2 shows the fields for $\omega_p/\omega_o = 0.5$ and $k_o r = 50$, corresponding to the observation at a large distance compared with the original wavelength of the radiation. Figure 4.3 corresponds to the case of dense plasma and long distance from the source with $\omega_p/\omega_o = 0.9$ and $k_o r = 50$. Figure 4.4 shows the fields in an overdense plasma in which $\omega_p/\omega_o = 1.2$ and $k_o r = 5$, corresponding to the observation near the source. The beating phenomena can be seen in the fields in Figs. 4.2 through 4.4. In Figs. 4.2 and 4.3 the effect of the group velocity on the propagation in a dispersive plasma can be seen. After $t = \frac{r}{c}$, i.e., $\omega_o t > k_o r = \omega$, a small disturbance comes, then the faster signal of frequency ω arrives at $\omega_o t \sim 50\sqrt{1+\alpha^2}$, and finally the signal of frequency ω_o builds up at $\omega_o t \sim 50/\sqrt{1-\alpha^2}$. The time lag between these two signals is

$\Delta t \sim \frac{1-\sqrt{1-\alpha^4}}{\sqrt{1-\alpha^2}}\, r$ or $\Delta(\omega_o t) \sim 50\,\frac{1-\sqrt{1-\alpha^4}}{\sqrt{1-\alpha^2}}$. This phenomenon is

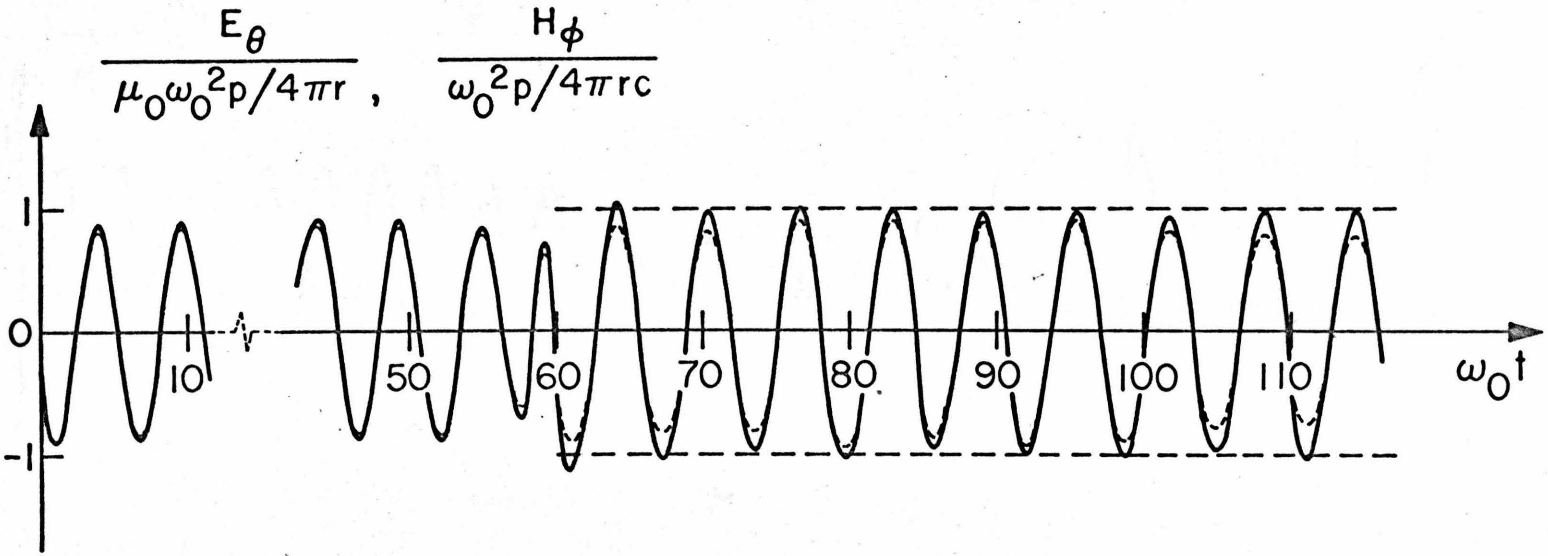


Fig. 4.2. E_θ(—) and H_ϕ(--) for $\alpha = \omega_p/\omega_o = 0.5$, $k_o r = 50$, $\theta = 90^o$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ .

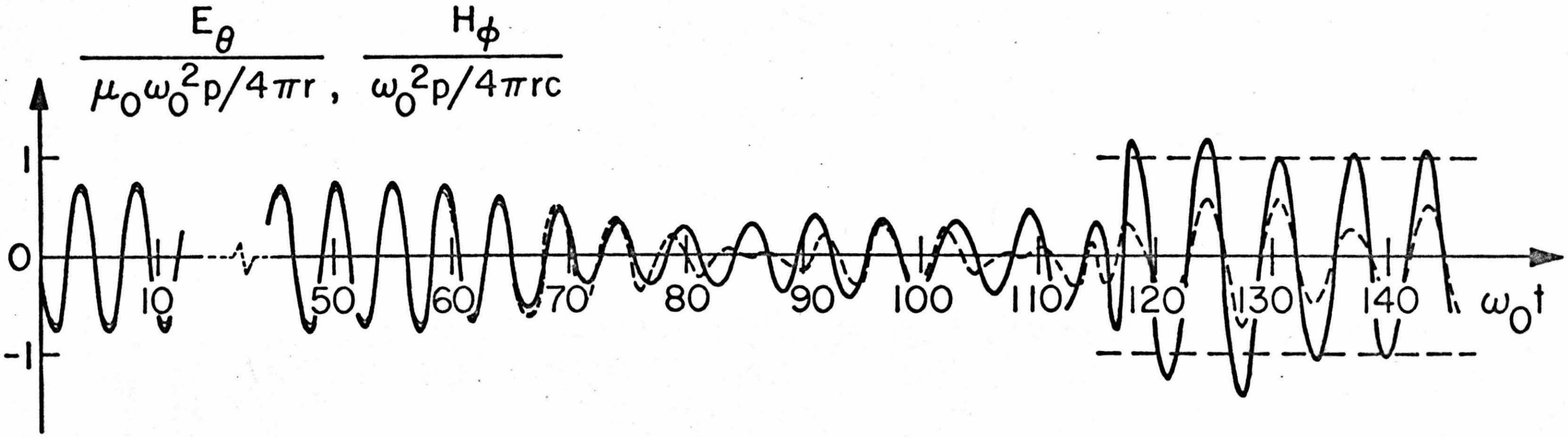


Fig. 4.3 E_θ(—) and H_ϕ(--) for $\alpha = \omega_p/\omega_o = 0.9$, $k_o r = 50$, $\theta = 90^o$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ .

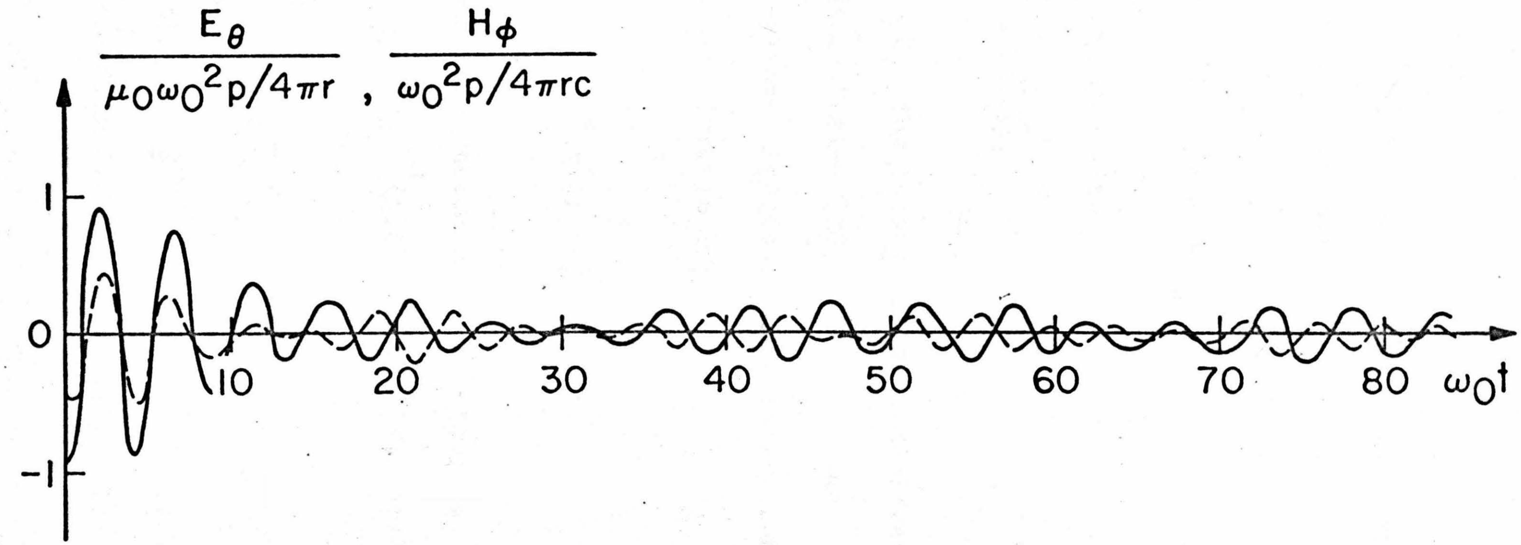


Fig. 4.4 E_θ(—) and H_ϕ(--) for $\alpha = \omega_p/\omega_o = 1.2$, $k_o r = 5$, $\theta = 90^o$.

clearly observed in Fig. 4.3 in which the ratio of the group velocity at ω to that at ω_o is $(1 - (0.9)^4)^{-1/2} \simeq 1.7$ and the time lag is large, $\Delta(\omega_o t) \sim 46$.

The radiated power by the dipole will now be calculated. For $t < 0$, it is well-known that

$$P^{rad}(t) = \frac{\mu_o \omega_o^4 p^2}{6\pi c} \cos^2 \omega_o t \tag{4.50}$$

which has the average value

$$P^{rad} = \frac{\mu_o \omega_o^4 p^2}{12\pi c} \tag{4.51}$$

For $t > 0$, the theory of the time-irreversible power radiated by a current distribution [30,31] will be employed here. A detailed discussion of various aspects of this theory can be found in Ref. 32. The power radiation formula is

$$P^{rad}(t) = - \int_V \underline{E}^{rad}(\underline{r},t) \cdot \underline{J}_s(\underline{r},t) \, dV \tag{4.52}$$

where the volume integration extends throughout the volume V occupied by the source $\underline{J}_s$. $\underline{E}^{rad}(\underline{r},t)$ is the Dirac radiation field given by

$$\underline{E}^{rad}(\underline{r},t) = \frac{1}{2}(\underline{E}^{ret}(\underline{r},t) - \underline{E}^{adv}(\underline{r},t)) \tag{4.53}$$

where $\underline{E}^{ret}(\underline{r},t)$ and $\underline{E}^{adv}(\underline{r},t)$ are the retarded and advanced solutions for the electric field respectively. For time-harmonic outgoing waves,

$$\underline{E}^{rad}(\underline{r},t) = \frac{1}{2} (\underline{E}^{out}(\underline{r},t) - \underline{E}^{in}(\underline{r},t)) \tag{4.54}$$

where $\underline{E}^{out}(\underline{r},t)$ and $\underline{E}^{in}(\underline{r},t)$ are the outgoing and incoming electric fields respectively. In the case of time-harmonic incoming waves, $\underline{E}^{out}(\underline{r},t)$ will be replaced by $\underline{E}^{in}(\underline{r},t)$ in Eq. (4.54), and vice versa since the incoming wave will bring energy into the source region. Now, carrying out the volume integration of Eq. (4.52) results in

$$P^{rad}(t) = -E_z^{rad}(\underline{0},t)\ \omega_o p \cos \omega_o t \tag{4.55}$$

It is easily shown that

$$E_z^{rad}(\underline{0},t) = -\frac{\mu_o \omega_o^3 p}{6\pi c} \frac{1}{\sqrt{1+\alpha^2}} \cos(\omega_o \sqrt{1+\alpha^2}\ t) + E_z^{rad'}(\underline{0},t) \tag{4.56}$$

where

$$E_z^{rad'}(\underline{0},t) = \frac{1}{2}(E_z^{ret'}(\underline{0},t) - E_z^{adv'}(\underline{0},t)) \tag{4.57}$$

$E_z^{ret'}(\underline{0},t)$ is the z-component of $\underline{E}'(\underline{r},t)$ in Eq. (4.9) at the origin; $E_z^{adv'}(\underline{0},t)$ is the z-component of the advanced field

$$\begin{aligned} \underline{E}^{adv'}(\underline{r},t) &= L^{-1}\{\hat{\underline{E}}^{adv'}(\underline{r},s)\} \\ &= L^{-1}\{\frac{-\mu_o s\ J'(s)}{4\pi}\ [\underline{e}_z \frac{e^{+\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r} \\ &\quad - \frac{c^2}{s^2+\omega_p^2} \nabla \frac{\partial}{\partial z} (\frac{e^{+\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r})] \end{aligned} \tag{4.58}$$

at the origin.

To calculate $E_z^{rad'}(\underline{0},t)$, we start with $\hat{E}_z^{rad'}(\underline{r},s)$.

$$\hat{E}_z^{rad'}(\underline{r},s) = \frac{1}{2}(\hat{E}'_z(\underline{r},s) - \hat{E}_z^{adv'}(\underline{r},s)) \tag{4.59}$$

$$= -\frac{\mu_o s\, J'(s)}{8\pi}\left[\frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}} - e^{\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r} - \frac{c^2}{s^2+\omega_p^2}\frac{\partial^2}{\partial z^2}\left(\frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}} - e^{\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r}\right)\right] \tag{4.60}$$

Carrying out the differentiation we find, as $r \to 0$,

$$\hat{E}_z^{rad'}(\underline{r},s) = \frac{\mu_o}{6\pi c}\, s\, J'(s)\sqrt{s^2+\omega_p^2} + O(r) \tag{4.61}$$

$$= \frac{\mu_o \omega_o p}{6\pi c}\left(\frac{s^2}{s^2+\omega_o^2} - \frac{1}{1+\alpha^2}\frac{s^2}{s^2+\omega^2} - \frac{\alpha^2}{1+\alpha^2}\right)\sqrt{s^2+\omega_p^2} + O(r) \tag{4.62}$$

where $\omega = \sqrt{\omega_o^2 + \omega_p^2}$. Defining

$$I_4(\Omega) \equiv \frac{1}{2\pi i}\int_C \frac{s^2\sqrt{s^2+\omega_p^2}}{s^2+\Omega^2}\, e^{st}\, ds \tag{4.63}$$

we obtain

$$E_z^{rad'}(\underline{0},t) = \frac{\mu_o \omega_o p}{6\pi c}\left(I_4(\omega_o) - \frac{1}{1+\alpha^2} I_4(\omega) - \frac{\alpha^2}{1+\alpha^2} I_4(0)\right) \tag{4.64}$$

Using the same technique (see Appendix D) as was used in evaluating $I_{1,2,3}$ we find

$$I_4(\Omega) = \delta'(t) - \frac{\omega_p^2}{8\pi}\int_0^{2\pi} \frac{(1-x^4)[1-x^2-(x^2-A^2(\Omega))(x^2-A^{-2}(\Omega))]}{x^2(x^2-A^2(\Omega))(x^2-A^{-2}(\Omega))} \times e^{i\omega_p t\cos\psi}\, d\psi \tag{4.65}$$

$$= \delta'(t) + \omega_p^2 \frac{J_1(\omega_p t)}{\omega_p t} - \Omega^2 J_o(\omega_p t)$$
$$-\Omega\sqrt{\Omega^2-\omega_p^2}[U_o(A(\Omega)\omega_p t,\omega_p t) - U_o(A^{-1}(\Omega)\omega_p t,\omega_p t)] \tag{4.66}$$

where $\delta'(t) \equiv \frac{d}{dt}(\delta(t))$ and where $A(\Omega)$ is defined by Eqs. (4.31). Hence,

$$E_z^{rad'}(\underline{0},t) = -\frac{\mu_o\omega_o^3 p}{6\pi c}\{\sqrt{1-\alpha^2}(U_o(\xi_o\omega_p t,\omega_p t) - U_o(\xi_o^{-1}\omega_p t,\omega_p t))$$
$$- \frac{1}{\sqrt{1+\alpha^2}}(U_o(\xi\omega_p t,\omega_p t) - U_o(\xi^{-1}\omega_p t,\omega_p t))\} \tag{4.67}$$

where $\xi_o = A(\omega_o)$ and $\xi = A(\omega)$.

Thus, for $t > 0$, the power radiated by the dipole is

$$P^{rad}(t) = \frac{\mu_o\omega_o^4 p^2}{6\pi c}\cos\omega_o t\{\frac{1}{\sqrt{1+\alpha^2}}[\cos(\omega_o\sqrt{1+\alpha^2}\,t)$$
$$- (U_o(\xi\omega_p t,\omega_p t) - U_o(\xi^{-1}\omega_p t,\omega_p t))]$$
$$+\sqrt{1-\alpha^2}(U_o(\xi_o\omega_p t,\omega_p t) - U_o(\xi_o^{-1}\omega_p t,\omega_p t))\} \tag{4.68}$$

For t small, i.e., $\omega_p t << 1$,

$$P^{rad}(t) = \frac{\mu_o \omega_o^4 p^2}{6\pi c} \cos \omega_o t \left\{\frac{1}{\sqrt{1+\alpha^2}} \cos(\omega_o \sqrt{1+\alpha^2}) - \frac{1}{2}\omega_p^2 t^2\right\} + 0(\omega_p^4 t^4) \tag{4.69}$$

As $t \to 0^+$, that is, just after the plasma is created,

$$P^{rad}(0^+) = \frac{\mu_o \omega_o^4 p^2}{6\pi c} \frac{1}{\sqrt{1+\alpha^2}} \tag{4.70}$$

But from Eq. (4.50), as $t \to 0^-$, that is, just before the plasma is created,

$$P^{rad}(0^-) = \frac{\mu_o \omega_o^4 p^2}{6\pi c} \tag{4.71}$$

The instantaneous radiated power at $t = 0^+$ decreases to $\frac{1}{\sqrt{1+\alpha^2}}$ of its value at $t = 0^-$ because work must be done by the driving source in order to keep the dipole moment constant during the creation of the plasma.

As $t \to \infty$, we find, for $\omega_o > \omega_p$,

$$P^{rad}(t) \sim \frac{\mu_o \omega_o^4 p^2}{6\pi c} \sqrt{1-\alpha^2} \cos^2 \omega_o t \tag{4.72}$$

for $\omega_o = \omega_p$,

$$P^{rad}(t) \sim \frac{\mu_o \omega_o^4 p^2}{6\pi c} \sqrt{\frac{2}{\pi \omega_p t}} \cos \omega_o t \cos(\omega_p t - \frac{\pi}{4}) \to 0 \tag{4.73}$$

and for $\omega_o < \omega_p$

$$P^{rad}(t) \sim \frac{\mu_o \omega_o^4 p^2}{6\pi c} \sqrt{\alpha^2 - 1} \cos \omega_o t \sin \omega_o t \tag{4.74}$$

These are the power radiation formulas for an oscillating electric point dipole immersed in an isotropic, cold, lossless, homogeneous and linear plasma. The average values are

$$\overline{P^{rad}} = \frac{\mu_o \omega_o^4 p^2}{12\pi c} \sqrt{1 - \alpha^2} \qquad \text{for} \quad \omega_o > \omega_p \tag{4.75}$$

$$= 0 \qquad \text{for} \quad \omega_o \leq \omega_p \tag{4.76}$$

Numerical calculations of the instantaneous radiated power were performed. Figures 4.5 through 4.7 show the power as a function of the normalized time $\omega_o t$. The value $2\overline{P^{rad}}$ where $\overline{P^{rad}}$ is given by Eq. (4.75) is denoted by horizontal dashed lines in the figures. Figures 4.5 and 4.6 correspond to cases with $\omega_p/\omega_o = 0.5$ and $\omega_p/\omega_o = 0.9$ respectively. Figure 4.7 shows the power oscillation in an overdense plasma. Beating phenomena can be seen in Figs. 4.5 through 4.7. In each case, a decrease in power radiation at $t = 0$ is clearly observed.

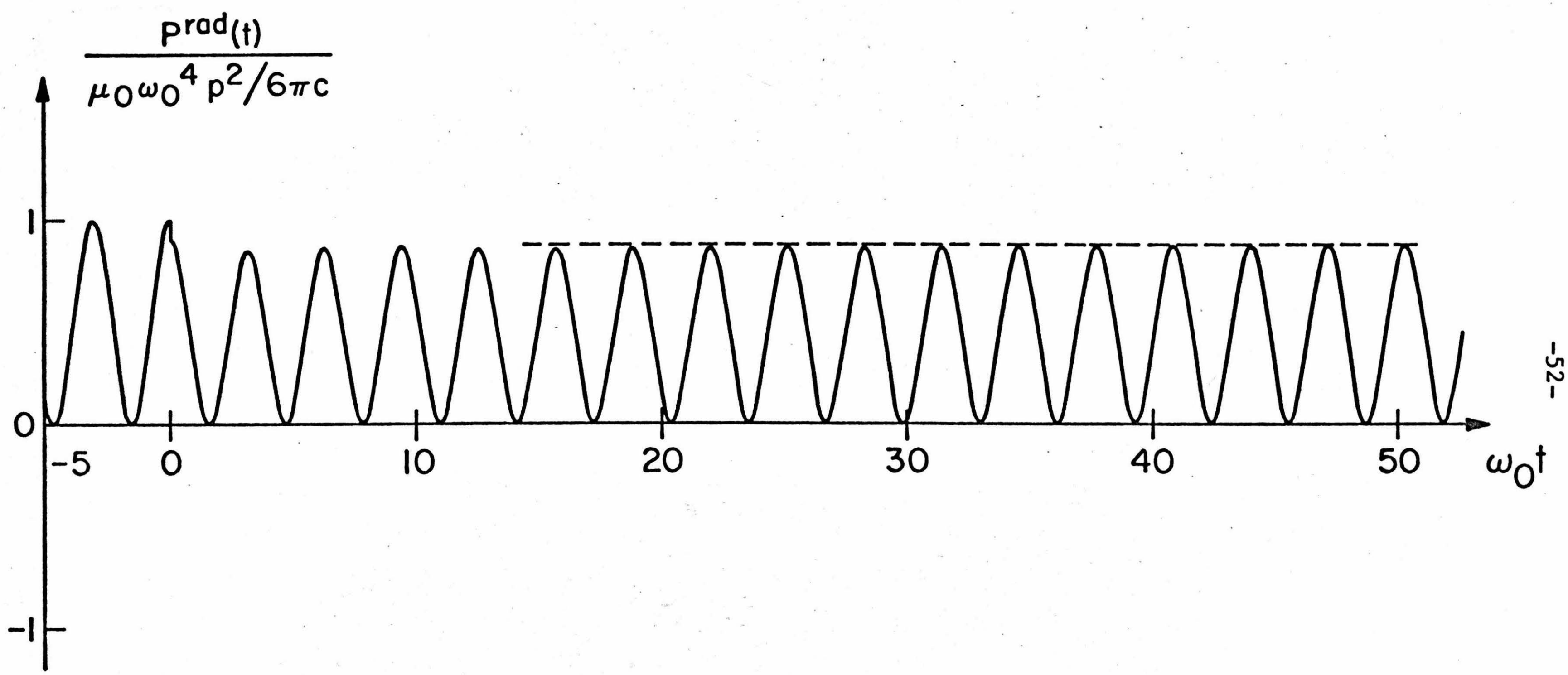


Fig. 4.5 $P^{rad}(t)$ for $\alpha = \omega_p/\omega_o = 0.5$. Horizontal dashed lines indicate the value $2\overline{P}^{rad}$

where $\overline{P}^{rad} = \frac{\mu_o \omega_o^4 p^2}{12\pi c}\sqrt{1 - \alpha^2}$

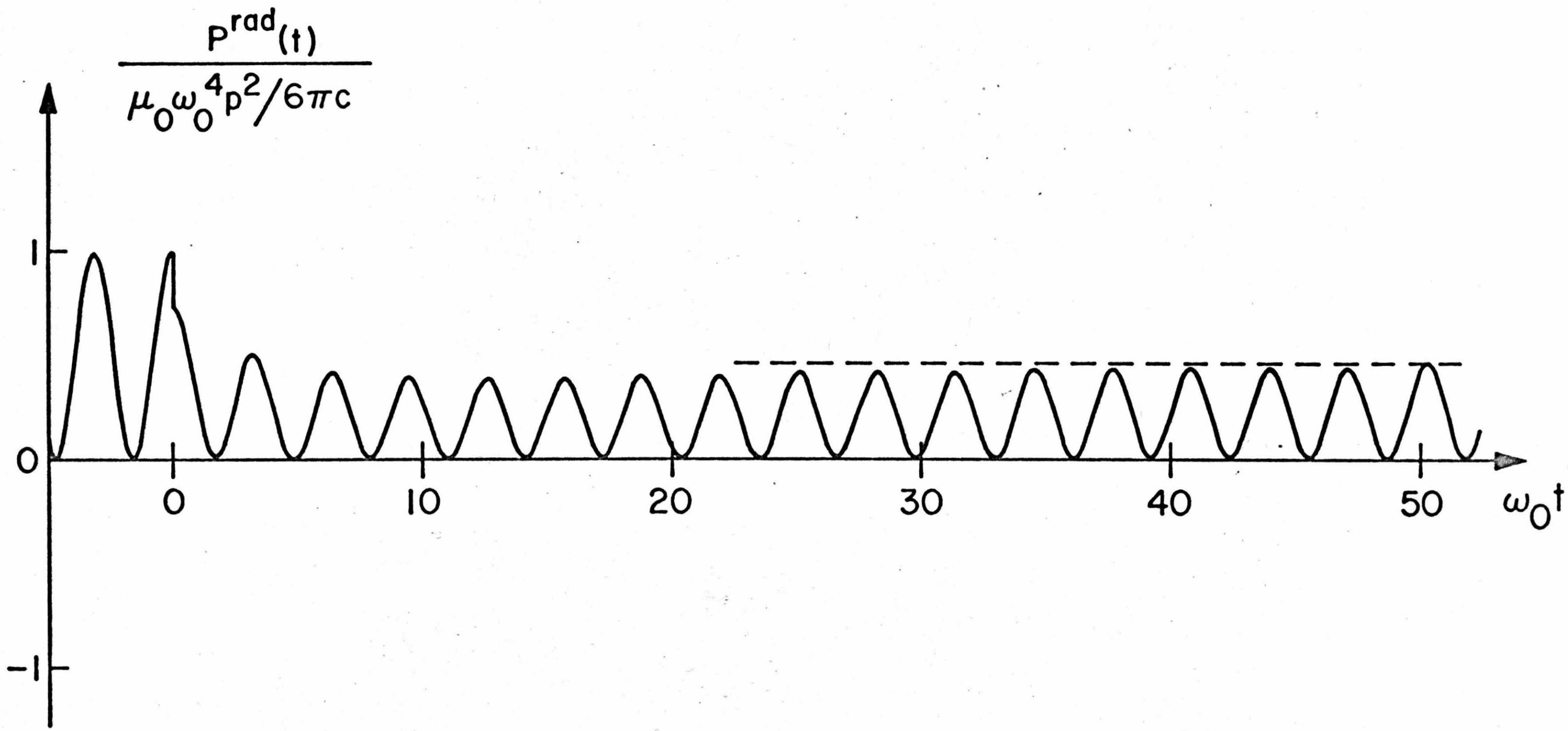


Fig. 4.6 $P^{rad}(t)$ for $\alpha = \omega_p/\omega_o = 0.9$. Horizontal dashed lines indicate the value $2P^{rad}$

where $P^{rad} = \frac{\mu_o^4\omega_o^2 p}{12\pi c}\sqrt{1-\alpha^2}$

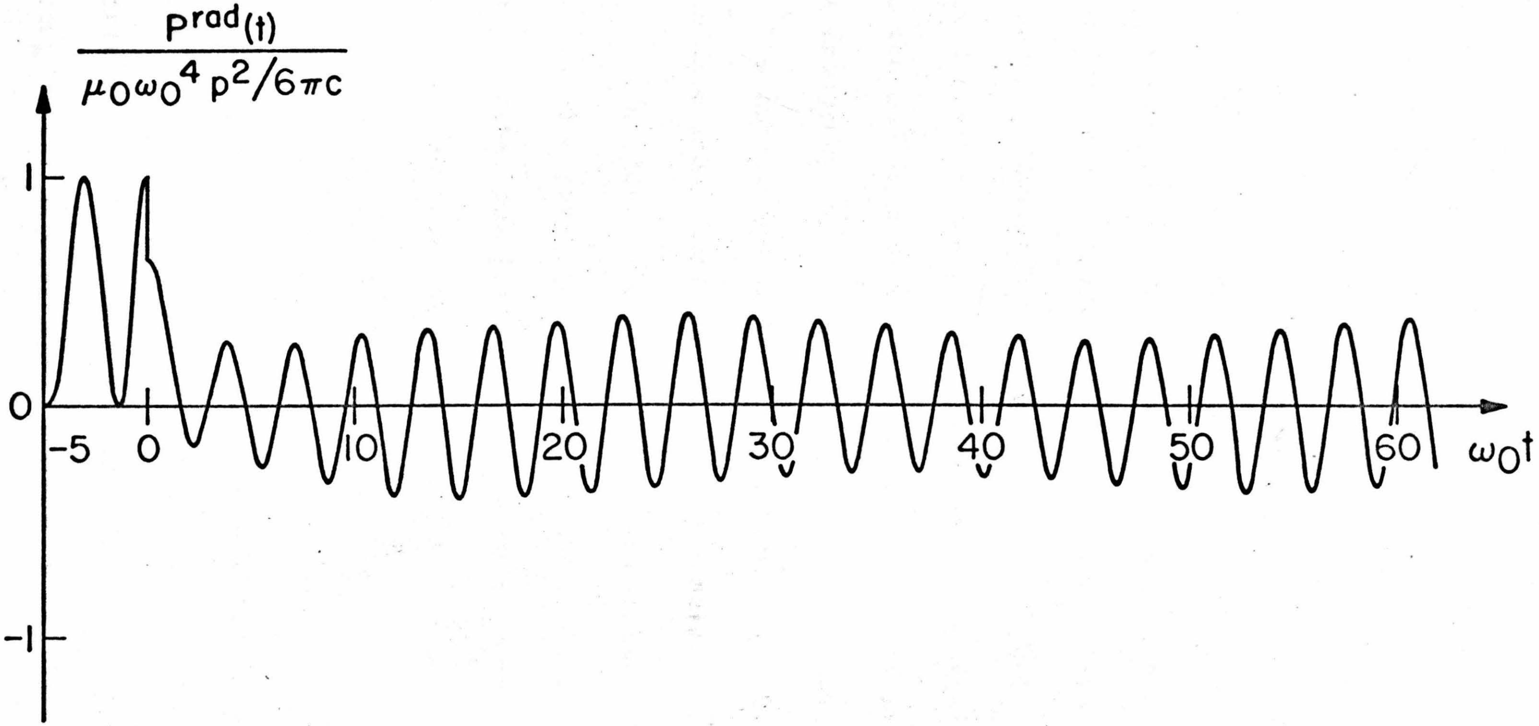


Fig. 4.7 $P^{rad}(t)$ for $\alpha = \omega_p/\omega_o = 1.2$

5. CONCLUSIONS

This work studies the problem of electromagnetic wave propagation and radiation from oscillating sources in a suddenly-created plasma. A free current density has been introduced in Maxwell's equations to account for the interaction of the wave with the plasma. Through the use of Laplace transformation, general expressions of the field quantities in the suddenly created plasma have been obtained.

In the case of plane wave propagation, it is found that there are two waves in the suddenly created plasma. One is a wave propagated in the direction of the original wave, and the other is a wave propagated in the opposite direction. Their frequency is shifted from the original frequency ω_o to $\sqrt{\omega_o^2 + \omega_p^2}$ where ω_p is the plasma frequency. The amplitudes of the electric and magnetic fields of these two waves have been found and plotted as functions of the parameter $\alpha = \omega_p/\omega_o$. The power flows and energy densities have also been calculated. The net power flow of the two waves is in the direction of the original wave, but its magnitude is only $\frac{1}{1+\alpha^2}$ of the original value. It is also noted that the total energy density of the two waves is less than the original one, that is, some wave energy has been lost during the creation of the plasma.

In the case of electric dipole radiation in a plasma suddenly created at $t = 0$, there are also two waves for $0 < t < r/c$ where r is the distance between the dipole and the observation point. One is an outgoing spherical wave and the other is an incoming spherical wave. Their frequency shift, amplitudes and power relations are the same as those in the plane wave case. At $t = r/c$, the first small disturbances arrive at r. Around $t \sim \frac{r}{c/\sqrt{1+\alpha^2}}$, the outgoing wave fades away, and a new wave arrives which has the same frequency and amplitude as the incoming wave but is propagated away from the origin. This new outgoing wave is a result of the reflection of the incoming wave at the origin. Later on, in a plasma with $\omega_p < \omega_o$, around $t \sim \frac{r}{c/\sqrt{1-\alpha^2}}$ the radiation from the dipole builds up its steady state value. However, in an overdense plasma, the fields due to the oscillating dipole decay exponentially with distance. The exact field solutions have been obtained; and numerical calculations have been performed. The radiated power by the dipole has also been evaluated. It is found that the instantaneous radiated power suffers a decrease at the moment of plasma creation since some work must be done by the driving source to keep the dipole moment unchanged during the creation of the plasma. The power radiation gradually reaches its steady state and has the average value $\frac{\mu_o \omega_o^4 p^2}{12\pi c}\sqrt{1-\alpha^2}$ for $\alpha = \omega_p/\omega_o < 1$ or 0 for $\alpha \geq 1$.

Finally, it should be noted that boundary value problems in a suddenly-created plasma can also be solved using the techniques developed in this work. However, the inversion integrations will be so complicated that numerical evaluation of integrals will be needed.

APPENDIX A. THE EFFECTS OF COLLISIONS

In this appendix the effects of collisions will be discussed. In the presence of collisions, the equation for the average velocity $\underline{v}(\underline{r},t)$ of free electrons in a plasma will be [33].

$$Nm\frac{d}{dt}\underline{v} = Ne(\underline{E} + \underline{v} \times \mu_o\underline{H}) - Nm\omega_{eff}\,\underline{v} \tag{A.1}$$

where the proportionality constant ω_{eff} is the collision frequency and measures the number of effective collisions an electron makes per unit time. Then the free current density is related to the electric field by

$$\frac{\partial}{\partial t}\underline{J}_f + \omega_{eff}\underline{J}_f = \omega_p^{\,2}\varepsilon_o\underline{E} \tag{A.2}$$

where the nonlinear term and the $\underline{v} \times \mu_o\underline{H}$ term have been neglected.

It can be easily shown that the result of Laplace transformation of the vector wave equation (for t<0) is

$$\nabla \times \nabla \times \hat{\underline{E}}(\underline{r},s) + \frac{s^2 + \frac{s}{s+\omega_{eff}}\omega_p^{\,2}}{c^2}\hat{\underline{E}}(\underline{r},s) = \frac{i\omega_o\mu_o}{s+i\omega_o}\underline{J}_o(\underline{r}) + \frac{s-i\omega_o}{c^2}\underline{E}_o(\underline{r}) \tag{A.3}$$

Comparing Eq. (A.3) with Eq. (2.29) the solutions can be obtained by simply replacing $\omega_p^{\,2}$ by $\frac{s}{s+\omega_{eff}}\,\omega_p^{\,2}$ in Eqs. (2.51) and (2.52).

Thus

$$\hat{\underline{E}}(\underline{r},s) = \frac{(s-i\omega_o)(s+\omega_{eff})}{s^3+\omega_{eff}s^2+\omega^2 s+\omega_{eff}\omega_o^{\,2}}\,\underline{E}_o(r) - s\mu\int_V \underline{\underline{\Gamma}}_c(\underline{r},\underline{r}')\cdot\hat{\underline{J}}_c'(\underline{r},s)\,dV' \tag{A.4}$$

$$\hat{\underline{H}}(\underline{r},s) = \frac{s-i\omega_o + \omega_p^2}{s^3+\omega_{eff}\ s^2 + \omega^2 s+\omega_{eff}\omega_o^2}\ \underline{E}_o(r) + \int_V \nabla G_c(\underline{r},\underline{r}') \times \hat{\underline{J}}_c(\underline{r}',s)dV' \tag{A.5}$$

where $\omega = \sqrt{\omega_o^2+\omega_p^2}$

$$\hat{\underline{J}}_c'(\underline{r}',s) = -\frac{i\omega_o\omega_p^2}{(s+i\omega_o)(s^3+\omega_{eff}s^2+\omega^2 s+\omega_{eff}\omega_o^2)}\ \underline{J}_o(r) \tag{A.6}$$

$$G_c(\underline{r},\underline{r}') = \frac{1}{4\pi|\underline{r}-\underline{r}'|}\ \exp\left\{-\frac{|\underline{r}-\underline{r}'|}{c}\sqrt{\frac{s(s^2+\omega_{eff}s+\omega_p^2)}{s+\omega_{eff}}}\right\} \tag{A.7}$$

$$\underline{\Gamma}_c\ (\underline{r},\underline{r}') = (\underline{\underline{u}} - \frac{c^2(s+\omega_{eff})}{s(s^2+\omega_{eff}s+\omega_p^2)}\ \nabla\nabla)\ G_c(\underline{r},\underline{r}') \tag{A.8}$$

A. Effects on plane wave propagation

In the lossless plasma, two poles appear at $s=\pm i\omega$ corresponding to two waves propagated in oppositive directions, and an additional pole appears at s=0 corresponding to the static but spatially varying magnetic field. See Eq. (2.51) and (2.52). In the presence of collisions, the poles now appear at the three roots of the cubic equation

$$s^3 + \omega_{eff}\ s^2 + \omega^2 s + \omega_{eff}\omega_o^2 = o \tag{A.9}$$

In the case of very slight loss, i.e. $\frac{\omega_{eff}}{\omega_o} << 1$, the roots are

$$s_1 \approx -\frac{\omega_{eff}\omega_o^2}{\omega^2} \tag{A.10}$$

$$s_2 \approx -\frac{\omega_{eff}\omega_p^2}{2\omega^2} - i\omega \tag{A.11}$$

$$s_3 \approx -\frac{\omega_{eff}\omega_p^2}{2\omega^2} + i\omega \tag{A.12}$$

Hence, there are two waves whose amplitudes decay with time as $e^{-\frac{\omega_{eff}\omega_p^2}{2\omega^2}t}$. There is also a nonoscillatory (in time) field decaying with time as $e^{-\frac{\omega_{eff}\omega_o^2}{\omega^2}t}$. If $\omega_o^2 > 2\omega_p^2$, the nonoscillatory field decays faster than the wave amplitudes, otherwise the waves will decay faster. Thus, a limitation on the formulation of the lossless problem is for the time small enough so that collision effects on waves can be neglected that is,

$$t \ll \frac{2\omega^2}{\omega_{eff}\omega_p^2} \tag{A.13}$$

B. Effects on dipole radiation

From Eq. (A.4) through (A.8), it can be seen that the collisions would displace the branch cut shown in Fig. 2.1 into the left half plane a distance $\frac{\omega_{eff}}{2}$. Additional branch points appear at $s=o, -\omega_{eff}$. Thus a limitation on the integration technique described in Appendix D is for the elliptical path (Eq. (D.7)) to lie well away from these singularities so that the influence of the collision frequency will be negligibly small. Then by Eq. (D.7),

$$\frac{\omega_{eff}}{2} \ll \frac{\omega_p\beta}{\sqrt{1-\beta^2}} \tag{A.14}$$

Thus

$$t \ll \frac{2\omega_p}{\omega_{eff}} \frac{r}{c} \sqrt{1 + \left(\frac{\omega_{eff}}{2\omega_p}\right)^2} \tag{A.15}$$

APPENDIX B. ENERGY CONSIDERATION

The difference in wave energy densities before and after the plasma creation, see Eq. (3.38), will now be discussed in terms of the plasma model used in this investigation.

It has been found that the total wave energy density for t>0 is

$$w_{tot} = w_+ + w_- = \frac{2+\alpha^2}{2(1+\alpha^2)} w_o \tag{3.38}$$

where $\alpha = \frac{\omega_p}{\omega_o}$. It is noted that, besides waves, there is a static but spatially varying magnetic field (from Eq. (3.7)):

$$\underline{H}_M = \frac{\underline{e}_y - i\underline{e}_x}{\sqrt{2}} \frac{\alpha^2}{1+\alpha^2} H_o e^{ik_o z} \tag{B.1}$$

where $H_o = \sqrt{\frac{\varepsilon_o}{\mu_o}} E_o$. And there is an electronic current density corresponding to this field:

$$\underline{J}_M = \nabla \times \underline{H}_M = \frac{\underline{e}_x + i\underline{e}_y}{\sqrt{2}} \frac{i\alpha^2}{1+\alpha^2} \varepsilon_o E_o e^{ik_o z} \tag{B.2}$$

Hence the average velocity of electrons is

$$\underline{v} = \frac{1}{Ne} \underline{J}_M = \frac{\underline{e}_x + i\underline{e}_y}{\sqrt{2}} \frac{i\alpha^2}{1+\alpha^2} \frac{\omega_o \varepsilon_o}{Ne} E_o e^{ik_o z} \tag{B.3}$$

The energy densities associated with the field and the motion can be calculated.

$$w_M = \frac{1}{2} \mu_o [\mathrm{Re}\{\underline{H}_M\}]^2 \tag{B.4}$$

$$= \frac{\alpha^4}{2(1+\alpha^2)^2} \cdot \frac{1}{2} \varepsilon_o |E_o| = \frac{\alpha^4}{2(1+\alpha^2)^2} w_o \tag{B.5}$$

and

$$w_K = \frac{1}{2} Nm \, [\mathrm{Re}\{\underline{v}\}]^2 \tag{B.6}$$

$$= \frac{\alpha^4}{2(1+\alpha^2)^2} \frac{\omega_o^{\,2} m\varepsilon_o}{Ne^2} \cdot \frac{1}{2} \varepsilon_o |E_o|^2 = \frac{\alpha^2}{2(1+\alpha^2)^2} w_o \tag{B.7}$$

since $w_o = \frac{1}{2} \varepsilon_o |E_o|^2$ and $\frac{\omega_o^{\,2} m\varepsilon_o}{Ne^2} = \frac{\omega_o^{\,2}}{\omega_p^{\,2}} = \frac{1}{\alpha^2}$.

It is clear that

$$w_o - w_{tot} = w_K + w_M \tag{B.8}$$

Thus a certain amount of wave energy has been transferred to the static magnetic energy and the kinetic energy of the electrons during the creation of the plasma.

APPENDIX C. PROPERTIES AND ASYMPTOTIC FORMULAS OF LOMMEL FUNCTIONS OF TWO VARIABLES

In this appendix some properties and asymptotic formulas of Lommel functions of two variables are summarized in the following [36,37]:

The Lommel functions of two variables for integral orders are defined by the series

$$U_n(w,z) = \sum_{m=o}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{n+2m} J_{n+2m}(z) \tag{C.1}$$

$$V_n(w,z) = \sum_{m=o}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{-n-2m} J_{-n-2m}(z) \tag{C.2}$$

The two types of Lommel functions are interrelated in the following way:

$$U_n(w,z) - V_{-n+2}(w,z) = \cos\left(\frac{w}{2} + \frac{z^2}{2w} - \frac{n\pi}{2}\right) \tag{C.3}$$

$$V_n(w,z) = (-1)^n U_n(z^2/w,z) \tag{C.4}$$

Then the recursion relations are

$$U_n(w,z) + U_{n+2}(w,z) = \left(\frac{w}{z}\right)^n J_n(z) \tag{C.5}$$

$$V_n(w,z) + V_{n+2}(w,z) = \left(\frac{w}{z}\right)^{-n} J_{-n}(z) \tag{C.6}$$

As $t \to \infty$

$$U_o(\gamma q,q) \sim \frac{1}{2}J_o(q) + \frac{1}{2}\cos\omega_p t + \frac{1}{2}\frac{\omega_p r}{c}\sqrt{\frac{2}{\pi\omega_p t}}\cos\left(\omega_p t - \frac{3\pi}{4}\right) \tag{C.7}$$

$$U_1(\gamma q,q) \sim \frac{1}{2}\sin\omega_p t - \frac{1}{2}\frac{\omega_p r}{c}\sqrt{\frac{2}{\pi\omega_p t}}\ \cos\left(\omega_p t - \frac{\pi}{4}\right) \qquad (C.8)$$

$$\frac{U_1(\gamma q\xi,q) - U_1(\gamma q\xi^{-1},q)}{\sqrt{\omega^2 - \omega_p^{\,2}}}$$

$$\sim \begin{cases} \dfrac{e^{-\frac{r}{c}\sqrt{\omega_p^{\,2} - \omega^2}}}{\sqrt{\omega_p^{\,2} - \omega^2}}\ \cos\omega t - \dfrac{\omega_p}{\omega^2 - \omega_p^{\,2}}\sqrt{\dfrac{2}{\pi\omega_p t}}\ \cos\left(\omega_p t - \dfrac{3\pi}{4}\right), \omega<\omega_p & (C.9) \\[2ex] \dfrac{\sin\left(\omega t - \frac{r}{c}\sqrt{\omega^2 - \omega_p^{\,2}}\right)}{\sqrt{\omega^2 - \omega_p^{\,2}}} - \dfrac{\omega_p}{\omega^2 - \omega_p^{\,2}}\sqrt{\dfrac{2}{\pi\omega_p t}}\ \cos\left(\omega_p t - \dfrac{3\pi}{4}\right), \omega>\omega_p & (C.10) \end{cases}$$

$$U_o(\gamma q\xi,q) + U_o(\gamma q\xi^{-1},q)$$

$$\sim \begin{cases} e^{-\frac{r}{c}\sqrt{\omega_p^{\,2} - \omega^2}}\ \cos\omega t + \sqrt{\dfrac{2}{\pi\omega_p t}}\ \cos\left(\omega_p t - \dfrac{\pi}{4}\right), \omega<\omega_p & (C.11) \\[2ex] \cos\left(\omega t - \frac{r}{c}\sqrt{\omega^2 - \omega_p^{\,2}}\right) + \sqrt{\dfrac{2}{\pi\omega_p t}}\ \cos\left(\omega_p t - \dfrac{\pi}{4}\right), \omega>\omega_p & (C.12) \end{cases}$$

$$\frac{U_o(\gamma q\xi,q) - U_o(\gamma q\xi^{-1},q)}{\sqrt{\omega^2 - \omega_p^{\,2}}}$$

$$\sim \begin{cases} -\dfrac{e^{-\frac{r}{c}\sqrt{\omega_p^{\,2} - \omega^2}}}{\sqrt{\omega_p^{\,2} - \omega^2}} \sin \omega t - \dfrac{\omega}{\omega^2 - \omega_p^{\,2}} \sqrt{\dfrac{2}{\pi \omega_p t}} \cos(\omega_p t - \dfrac{\pi}{4}), \\ \qquad \omega < \omega_p & \text{(C.13)} \\ \dfrac{\cos(\omega t - \frac{r}{c}\sqrt{\omega^2 - \omega_p^{\,2}})}{\sqrt{\omega^2 - \omega_p^{\,2}}} - \dfrac{\omega}{\omega^2 - \omega_p^{\,2}} \sqrt{\dfrac{2}{\pi\omega_p t}} \cos(\omega_p t - \dfrac{\pi}{4}), \\ \qquad \omega > \omega_p & \text{(C.14)} \end{cases}$$

$$U_1(\gamma q\xi,q) + U_1(\gamma q\xi^{-1},q)$$

$$\sim \begin{cases} e^{-\frac{r}{c}\sqrt{\omega_p^{\,2} - \omega^2}} \sin \omega t \quad , \quad \omega < \omega_p & \text{(C.15)} \\ \sin(\omega t - \frac{r}{c}\sqrt{\omega^2 - \omega_p^{\,2}}) \quad , \quad \omega > \omega_p & \text{(C.16)} \end{cases}$$

APPENDIX D. INTEGRATION TECHNIQUE AND FORMULAS [34,35]

The inversion integrals in this work are of the form

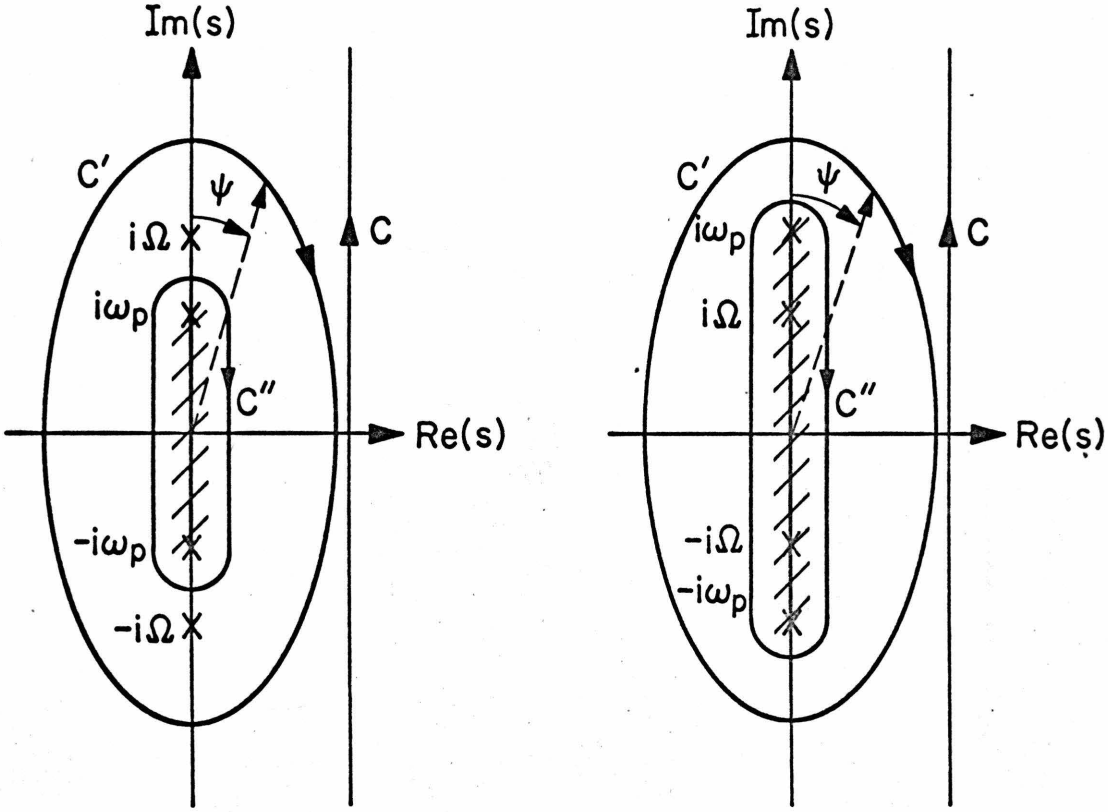
$$F(t) = \frac{1}{2\pi i} \int_C \frac{F_1(s,\sqrt{s^2+\omega_p^2})}{F_2(s,\sqrt{s^2+\omega_p^2})} e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}} ds \tag{D.1}$$

where F_1 and F_2 are polynomials in their arguments. As s goes to infinity, $F_1/F_2 \sim f_o + f_o/s + f_2/s^2 + \cdots$. If $f_o \neq 0$, there is a δ function in $F(t)$. This can be subtracted away and inverted separately, in view of the fact that the inverse of $\exp[-r/c\sqrt{s^2+\omega_p^2}]$ is $\delta(t - \frac{r}{c}) - \frac{r\omega_p^2}{c}\frac{J_1(q)}{q} H(t - \frac{r}{c})$ where $q = \omega_p t\sqrt{1-\beta^2}$ $(\beta \equiv r/ct)$. Then $F_1/F_2 - f_o = \tilde{F}_1/F_2$ which is $O(s^{-1})$ as s goes to infinity. It can be shown by closing the contour with a very large arc in the right-hand plane that $F(t)$ is zero for $t < r/c$.

The singularities in the integrand occur at $s = \pm i\Omega$ and $s = \pm i\omega_p$, see Fig. D.1. For the purpose of this method the complex s-plane is cut along the imaginary axis between $\pm i\omega_p$.

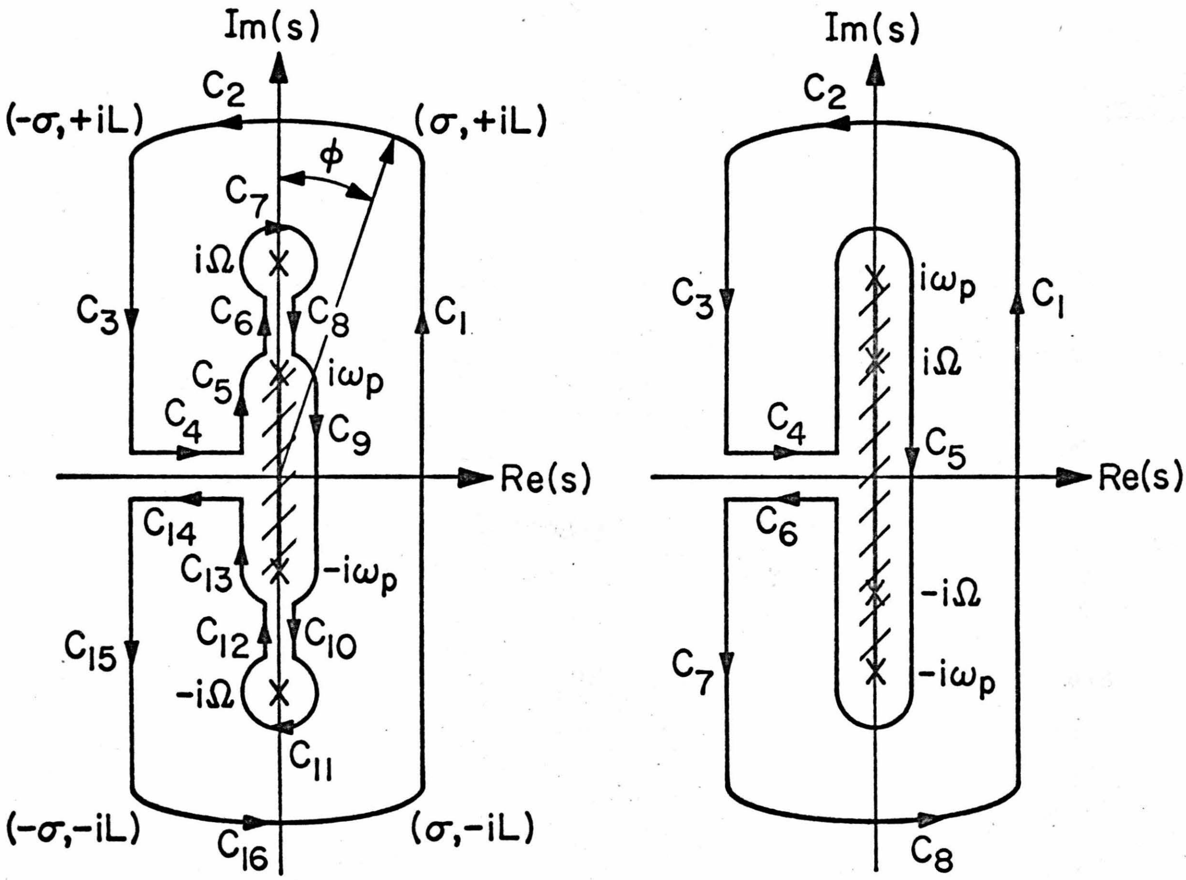
The integral along C can be shown to be equal to an integral along a path C' around the branch cut plus contributions from the residues (Fig. D.1). Figure D.2 indicates the path deformation for the proof.

In case 1, $\Omega > \omega_p$, the integrals along C_4 and C_{14} , C_6 and C_8, and C_{10} and C_{12} cancel one another, and the integrals C_7 and C_{11} have values equal to $-2\pi i$ times the residues of the integrand at $\pm i\Omega$,



Case 1. $\Omega > \omega_p$ Case 2. $\Omega < \omega_p$

Fig. D.1 Integration contours



Case 1. $\Omega > \omega_p$ Case 2. $\Omega < \omega_p$

Fig. D.2 Integration paths C_j

respectively. The value of the sum of the integrals along C_3 and C_{15} is zero when $L \to \infty$ for $t > r/c$ since there are no singularities to the left of $-\sigma$. Then

$$\lim_{L \to \infty} \left| \frac{1}{2\pi i} \int_{C_2} \frac{\tilde{F}_1}{F_2} e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}} ds \right|$$

$$\leq \lim_{L \to \infty} \frac{1}{2\pi} \int_{\frac{\pi}{2} - \psi'}^{\frac{\pi}{2} + \psi'} M e^{\sqrt{\sigma^2 + 4^2}\,(t - \frac{r}{c}) \cos\phi} d\phi \qquad \text{(D.2)}$$

where $\psi' = \tan^{-1} \frac{\sigma}{4}$ and M is a constant which arises from the asymptotic property of $\tilde{F}_1/F_2$ as $s \to \infty$. Changing variables in the integral on the right-hand side we obtain for that integral

$$\frac{M}{2\pi} \int_{-\psi'}^{\psi'} e^{(t - \frac{r}{c})\sqrt{\sigma^2 + L^2} \sin\phi} d\phi$$

Using $\frac{2}{\pi} \phi \leq \sin \phi \leq \phi$ for $0 \leq \phi \leq \frac{\pi}{2}$, it can be shown that this integral is less than

$$\frac{M[1 - e^{-\frac{2}{\pi}(t - \frac{r}{c})\sqrt{\sigma^2 + L^2}\tan^{-1}\frac{\sigma}{L}}]}{r(t - \frac{r}{c})\sqrt{\sigma^2 + L^2}} + \frac{M[e^{(t - \frac{r}{c})\sqrt{\sigma^2 + L^2}\tan^{-1}\frac{\sigma}{4}} - 1]}{2\pi(t - \frac{r}{c})\sqrt{\sigma^2 + L^2}}$$

which vanishes as $L \to \infty$. Therefore,

$$\lim_{L \to \infty} \frac{1}{2\pi i} \int_{C_2} \tilde{F}_1/F_2 \, e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}} \, ds = 0 \qquad \text{(D.3)}$$

Similarly the contribution from the integral on path C_{16} can be shown to go to zero as $L \to \infty$. It follows from Cauchy's integral formula that

$$\sum_{n=1}^{16} \frac{1}{2\pi i} \int_{C_n} \frac{\tilde{F}_1}{F_2} e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}} \, ds = 0 \qquad \text{(D.4)}$$

Since as $L \to \infty$, path C_1 becomes C,

$$F(t) + \frac{1}{2\pi i} \int_{C'' = C_5 + C_9 + C_{13}} \frac{\tilde{F}_1}{F_2} e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}} \, ds$$

$$- \sum_{\pm i\Omega} \mathrm{Res}\left(\frac{\tilde{F}_1}{F_2} e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}}\right) = 0 \qquad \text{(D.5)}$$

In the same way it can be shown in case 2 that

$$F(t) + \frac{1}{2\pi i} \int_{C''} \frac{\tilde{F}_1}{F_2} e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}} \, ds = 0 \qquad \text{(D.6)}$$

Now the path of integration about the branch cut is made to conform to the ellipse C' (Fig. D.1) described by

$$s = \frac{\omega_p}{\sqrt{1 - \beta^2}} [\beta \sin \psi + i \cos \psi] , \quad \beta \equiv \frac{r}{ct} < 1 ,$$

$$0 \le \psi \le 2\pi \qquad \text{(D.7)}$$

In case 1, if the path C' lies outside $\pm i\Omega$, i.e., $\omega_p/\sqrt{1 - \beta^2} > \Omega$, then,

$$F(t) + \frac{1}{2\pi i}\int_{C'} \frac{\tilde{F}_1}{F_2}\, e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds = 0 \tag{D.8}$$

If $\omega_p/\sqrt{1-\beta^2} < \Omega$, then

$$F(t) + \frac{1}{2\pi i}\int_{C'} \frac{\tilde{F}_1}{F_2}\, e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds - \sum_{\pm i\Omega} \mathrm{Res}\left(\frac{\tilde{F}_1}{F_2}\, e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\right) = 0 \tag{D.9}$$

On C'

$$\sqrt{s^2+\omega_p^2} = \frac{\omega_p}{\sqrt{1-\beta^2}}\,[\sin\psi + i\beta\cos\psi] \tag{D.10}$$

and

$$st - \frac{r}{c}\sqrt{s^2+\omega_p^2} = i\omega_p t\sqrt{1-\beta^2}\,\cos\psi = iq\cos\psi \tag{D.11}$$

If we let $x = \gamma e^{i\psi}$ where $\gamma = \sqrt{(1-\beta)/(1+\beta)}$,

$$s = \frac{i\omega_p}{2x}(1+x^2) \tag{D.12}$$

$$\sqrt{s^2+\omega_p^2} = \frac{i\omega_p}{2x}(1-x^2) \tag{D.13}$$

$$ds = \frac{\omega_p}{2x}(1-x^2)\, d\psi \tag{D.14}$$

Therefore the integral in Eq. (D.8) or (D.9) takes the form

$$\int_0^{2\pi} \frac{P(e^{i\psi})}{Q(e^{i\psi})}\, e^{iq\cos\psi}\, d\psi \tag{D.15}$$

P and Q are polynomials in $e^{i\psi}$. This ratio of polynomials can be expanded into partial fractions

$$\frac{P}{Q} = \sum_{n=0}^{N} a_n e^{in\psi} + \sum_{m=1}^{N} \sum_{\ell=1}^{L} b_{m,\ell} \frac{1}{(c_{m,\ell} + e^{\lambda\psi})^{\ell}} \tag{D.16}$$

Using the integral representation for the Bessel function

$$2\pi i^n J_n(q) = \int_0^{2\pi} e^{\pm in\psi} e^{iq\cos\psi} d\psi \tag{D.17}$$

the integral in Eq. (D.15) can be expressed in terms of Bessel functions and Lommel functions of two variables.

Two basic integrals in this work will now be evaluated.

$$F_\ell(q,K) = \int_0^{2\pi} \frac{x^\ell e^{iq\cos\psi}}{x^2 - K^2} d\psi \tag{D.18}$$

for $\ell = 0$ and 1 and $K = 1, A(\Omega)$ where $A(\Omega)$ is defined by Eq. (4.31).

For $\ell = 0$

$$F_o(q,K) = \int_0^{2\pi} \frac{e^{iq\cos\psi}}{x^2 - K^2} d\psi = \frac{-1}{K^2} \int_0^{2\pi} \frac{e^{iq\cos\psi}}{1 - \frac{\gamma^2}{K^2} e^{i2\psi}} d\psi \tag{D.19}$$

For $K > 1$ and $\beta = r/ct > 0$, the denominator can be expanded into a geometric series,

$$F_o(q,K) = -\frac{1}{K^2} \int_0^{\infty} \sum_{n=0}^{\infty} \left(\frac{\gamma}{K} e^{i\psi}\right)^{2n} e^{iq\cos\psi} d\psi \tag{D.20}$$

when the order of integration and summation is reversed and the integral representation for the Bessel function is used, we obtain

$$F_o(q,K) = -\frac{2\pi}{K^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\gamma}{K}\right)^{2n}J_{2n}(q) \tag{D.21}$$

In terms of Lommel functions of two variables (see Appendix C)

$$F_o(q,K) = -\frac{2\pi}{K^2}U_o\left(\frac{\gamma q}{K},q\right) \tag{D.22}$$

For $K < 1$, the cases $\gamma/K < 1$ and $\gamma/K > 1$ must be considered separately in evaluating $F_o(q,K)$. This occurs when $K^{-1} = A(\omega_o) = \xi_o$ for $\omega_o > \omega_p$ or when $K^{-1} = A(\omega) = \xi$. Taking, say, $K = \xi_o^{-1}$, we have

$$F_o(q,\xi_o^{-1}) = -2\pi\xi_o^2\,U_o(\gamma q\xi_o,q) \tag{D.23}$$

when $\gamma\xi_o < 1$. When $\gamma\xi_o > 1$,

$$F_o(q,\xi_o^{-1}) = \int_0^{2\pi}\frac{e^{iq\cos\psi}}{x^2-\xi_o^{-2}}\,d\psi$$

$$= \int_0^{2\pi}\gamma^{-2}e^{-i2\psi}\sum_{n=0}^{\infty}\left(\frac{e^{-i\psi}}{\gamma\xi_o}\right)^{2n}e^{iq\cos\psi}\,d\psi$$

$$= 2\pi\sum_{n=0}^{\infty}\gamma^{-2(n+1)}\xi_o^{-2n}(-1)^{n+1}J_{2n+2}(q)$$

$$= -2\pi\xi_o^2\,U_2\left(\frac{q}{\gamma\xi_o},q\right) \tag{D.24}$$

By means of Eq. (C.3) and (C.4) we obtain

$$U_2\left(\frac{q}{\gamma\xi_o},q\right) = U_o(\gamma q\xi_o,q) - \cos\left(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2-\omega_p^2}\right) \tag{D.25}$$

Hence,

$$F_o(q,\xi_o^{-1}) = -2\pi\xi_o^2 \, U_o(\gamma q\xi_o,q) + 2\pi\xi_o^2 \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2}) \, H(t - \frac{r}{c}\sqrt{1 - \alpha^2}) \quad \text{(D.26)}$$

But in the Laplace inversion there is also a contribution due to poles $\pm i\omega_o$ in the integrand. This term appears only for $t > \frac{r}{c}\sqrt{1-\alpha^2}$ and it exactly cancels the second term on the righthand side of Eq. (D.26). Therefore for the value of $F_o(q,\xi_o^{-1})$ we shall formally use Eq. (D.23) and the residue contribution will automatically be taken care of. The same thing applies for $F_o(q,\xi^{-1})$.

In summary,

$$F_o(q,K) = -\frac{2\pi}{K^2} U_o(\frac{\gamma q}{K},q) \quad \text{(D.27)}$$

Similarly,

$$F_1(q,K) = -\frac{2\pi i}{K} U_1(\frac{\gamma q}{K},q) \quad \text{(D.28)}$$

References

[1] S. A. Ramsden and W.E.R. Davies, "Radiation Scattered from the Plasma Produced by a Focused Ruby Laser Beam", Phys. Rev. Lett. 13, 227-229 (1964)

[2] S. F. Paik and O. Ben-Dov, "Laser-Induced Perturbation in a Plasma", Proc. IEEE 53, 1145-1146 (1965)

[3] F. Percorella and G. C. Vases, "The Production of Highly Ionized, Low Density Plasmas by Means of Strong Ultraviolet Radiation", Phys. Lett. 28A, 616-617 (1969)

[4] R. G. Tomlinson, "Scattering and Beam Trapping Laser-Produced Plasmas in Gases", IEEE J. Quantum Electronics QE-5, 591-595 (1969)

[5] R. L. Monroe, "Electromagnetic Radiation in a Time-Varying Plasma", J. Appl. Phys. 41, 560-562 (1970)

[6] R. L. Fante, "Propagation of Electromagnetic Waves in a Simplified Nonlinear Gas", J. Appl. Phys. 42, 4202-4207 (1971)

[7] M. J. Lubin and A. P. Fraas, "Fusion by Lasers", Sci. Am. 224, 21-33 (June 1971)

[8] F. W. Crawford, "Microwave Plasma Devices--Promise and Progress", Proc. IEEE 59, 4-19 (1971)

[9] A. D. MacDonald, Microwave Breakdown in Gases, Wiley, N. Y. (1966)

[10] K. Davis, Ionospheric Radio Waves, Blaisdell Co., Waltham, Mass. (1969)

[11] Yu. A. Kravtsov, "The Geometric Optics Approximation in the General Case of Inhomogeneous and Nonstationary Media with Frequency and Spatial Dispersion", Soviet Phys. JETP 28, 769-772 (1969)

[12] C. Elachi, "Electromagnetic Wave Propagation and Source Radiation in Space-Time Periodic Media", Tech. Rept. No. 61, Antenna Lab., California Institute of Technology (1971)

[13] C. D. Taylor, D. H. Lam, and T. H. Shumpert, "Electromagnetic Pulse Scattering in Time-Varying Inhomogeneous Media", IEEE Trans. Ant. Prop. AP-17, 585-589 (1969)

[14] L. B. Felson and G. M. Whitman, "Wave Propagation in Time-Varying Media", IEEE Trans. Ant. Prop. AP-18, 242-253 (1970)

[15] R. L. Fante, "Transmission of Electromagnetic Waves into Time-Varying Media", IEEE Trans. Ant. Prop. AP-19, 417-424 (1971)

[16] F. R. Morgenthaler, "Velocity Modulation of Electromagnetic Waves", IEEE Trans. Microwave Theory and Tech. MTT-6, 167-172 (1958)

[17] F. R. Morgenthaler, "Phase-Velocity-Modulated Magnetoelastic Waves", J. Appl. Phys. 37, 3326-3327 (1966)

[18] B. J. Elliott and J. B. Gunn, "Signal Processing with a Time Varying Transmission Line", Presented at IEEE International Solid-State Circuit Conference, Philadelphia, Pa. (1966)

[19] J. B. Gunn, "Transformation and Reversal of Time Scale by a Time-Varying Transmission Line", Electronics Lett. 2, 247-248 (1966)

[20] B. A. Auld, J. H. Collins, and H. R. Zapp, "Signal Processing in a Nonperiodically Time-Varying Magnetoelastic Medium", Proc. IEEE 56, 258-272 (1968)

[21] S. M. Rezende and F. R. Morgenthaler, "Magnetoelastic Waves in Time-Varying Magnetic Fields - I Theory; II Experiment", J. Appl. Phys. 40, 524-545 (1969)

[22] C. H. Papas, Theory of Electromagnetic Wave Propagation, McGraw-Hill, N. Y. (1965), pp. 27-28.

[23] J. A. Stratton, Electromagnetic Theory, McGraw-Hill, N. Y. (1941), pp. 330-331

[24] H. C. Bertoni and A. Hessel, "Surface Waves on a Uniaxial Plasma Slab--Their Group Velocity and Power Flow", IEEE Trans. Antenna Prop. AP-14, 352-359 (1966)

[25] R. E. McIntosh and S. E. El-Khang, "Compression of Transmitted Pulses in Plasmas", IEEE Trans. Antenna Prop. AP-18, 236-241 (1970)

[26] L. B. Felsen, "Asymptotic Theory of Pulse Compression in Dispersive Media", IEEE Trans. Antenna Prop. AP-19, 424-432 (1971)

[27] See Ref. [22], p. 16

[28] See Ref. [22], pp. 14-17

[29] See Ref. [22], pp. 89-93

[30] K.S.H. Lee and C. H. Papas, "Radiation Resistance and Irreversible Power of Antennas in Gyroelectric Media", IEEE Trans. Antenna Prop.,AP-13, 834-835 (1965)

[31] K.S.H. Lee and C. H. Papas, "Irreversible Power and Radiation Resistance of Antennas in Anisotropic Ionized Gases", J. Res. Natl. Bur. Stds. (U.S.) 69D, 1313-1320 (1965)

[32] J. J. Kenny, "Electric Dipole Radiation in Isotropic and Uniaxial Plasmas", Tech. Rept. No. 44, Antenna Lab., California Institute of Technology (1968)

[33] See Ref. [22], pp. 175-177

[34] P. I. Kuznetsov, "An Expression of a Contour Integral", Prikladnaia Matematika I Mekhanika 11, 267-270 (1947)

[35] See Ref. [32], pp. 61-72

[36] See Ref. [32], pp. 73-85

[37] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge Univ. Press (1966), pp. 537-550