ESSENTIAL CENTRAL SPECTRUM AND RANGE IN A W*-ALGEBRA

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ABSTRACT

Halpern has defined a center valued essential spectrum, $\Sigma_I(A)$, and numerical range, $W_{\mathcal{Z}}(A)$, for operators A in a von Neumann algebra Φ. By restricting our attention to algebras Φ which act on a separable Hilbert space, we can use a direct integral decomposition of Φ to obtain simple characterizations of these quantities, and this in turn enables us to prove analogues of some classical results.

Since the essential central spectrum is defined relative to a central ideal, we first show that, under the separability assumption, every ideal, modulo the center, is an ideal generated by finite projections. This leads to the following decomposition theorem:

<u>Theorem:</u> $Z = \int_\Lambda \oplus\, c(\lambda)d\mu \in \Sigma_I(A)$ if and only if $c(\lambda) \in \sigma_e(A(\lambda))$ μ-a.e., where $A = \int_\Lambda \oplus\, A(\lambda)d\mu$ and σ_e is a suitable spectrum in the algebra $\Phi(\lambda)$.

Using mainly measure-theoretic arguments, we obtain a similar decomposition result for the norm closure of the central numerical range:

<u>Theorem:</u> $Z = \int_\Lambda \oplus\, c(\lambda)d\mu \in \overline{W_{\mathcal{Z}}(A)}$ if and only if $c(\lambda) \in \overline{W(A(\lambda))}$ μ-a.e.

By means of these theorems, questions about $\Sigma_I(A)$ and W (A) in Φ can be reduced to the factors $\Phi(\lambda)$. As examples, we

show that spectral mapping holds for Σ_I, namely $f(\Sigma_I(A)) = \Sigma_I(f(A))$, and that a generalization of the power inequality holds for $W_{\mathfrak{J}}(A)$.

Dropping the separability assumption, we show that central ideals can be defined in purely algebraic terms, and that the following perturbation result holds:

<u>Theorem</u>: $\Sigma_I(A + X) = \Sigma_I(A)$ for all $A \in \Phi$ if and only if $X \in I$.

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INTRODUCTION

In this chapter, we will state the basic results and terminology that will be used repeatedly in the sequel. Section I will collect various results on von Neumann algebras. In section 2, we will, following mainly the presentation of Schwartz [15], introduce the concept of a direct integral of Hilbert spaces, and we will state the important reduction theorem. The basic facts about operators can be found in [8] and [9], while [4] is the standard reference for von Neumann algebras. A discussion of reduction theory can also be found in [11] and [13].

1. Von Neumann Algebras

Let h be a Hilbert space, B(h) the algebra of all bounded linear operators on h; 1 will denote the identity operator. If Φ is a *-subalgebra of B(h) which is closed in the weak operator topology, Φ is called a <u>von Neumann algebra.</u> The set of all operators $Z \in \Phi$ which satisfy $ZA = AZ$ for all $A \in \Phi$ is called the <u>center</u> of Φ, and is denoted by $\mathfrak{z}(\Phi)$ or by $\mathfrak{z}$ when no confusion is possible. If $\mathfrak{z}$ consists of only the scalar multiples of the identity, Φ is called a <u>factor.</u> If $\Delta \subset B(h)$, the commutant of Δ, Δ', will be all operators which commute with all operators in Δ. A von Neumann algebra is characterized by the fact that $\Phi'' = (\Phi')' = \Phi$. $[\Delta \cup \Delta^*]''$ is the smallest von Neumann algebra containing Δ, and is called the algebra <u>generated by Δ</u>.

An operator $F \in B(h)$ is called a *projection* if $F^2 = F$ and $F^* = F$. Since no confusion is possible, we will often let F denote both the projection and the subspace F(h). If E and F are two projections, we say that F *dominates* E (or E is smaller than F) if FE = EF = E; we write $E \leq F$. If now $E, F \in \Phi$, we call F and E *equivalent* (relative to Φ), $F \sim E$, if there exists $U \in \Phi$ such that $UU^* = F$, $U^*U = E$; U is called a *partial isometry*. We write $E \precsim F$ if E is equivalent to a projection which is dominated by F. Finally, for $F \in \Phi$ a projection, we define the *central support* of F, cs(F), as the smallest projection P in $\mathcal{Z}$ such that P dominates F.

For projections E and F, we define *inf (E, F)* to be the projection onto $R(E) \cap R(F)$; similarly, *sup (E, F)* will be the projection onto the closed subspace generated by R(E) and R(F). If $E, F \in \Phi$, then both inf (E, F) and sup (E, F) are in Φ.

A projection $F \in \Phi$ is said to be *finite* if whenever $E \leq F$, $E \sim F$ implies $E = F$. Hence, an *infinite* projection is one that is equivalent to a proper sub-projection of itself. F is called *purely infinite* if F does not dominate any finite projection other than 0. F is called properly infinite if FQ is either 0 or infinite for all central projections Q. F is called *abelian* if the algebra $F \Phi F$ is abelian. Φ is called *finite* (*purely infinite*, *properly infinite*) if 1 is finite (respectively purely infinite, properly infinite).

For A Φ, it is important to know that particular operators related to A are also in Φ. We define the *range projection* of A, R(A), to be the projection onto the closure of the range of A, and

N(A) will denote the projection onto the null space of A; N(A) and R(A) are in Φ. If H is the positive square root of A*A, there is a unique partial isometry U such that N(U) = N(A) and A = UH. This representation is called the polar decomposition of A; we have U, H $\in \Phi$, and R(A) = UU** R(A*) = U*U. Furthermore, if $H = \int_0^M \lambda dE(\lambda)$ is the spectral representation of H, then Φ contains all of the spectral projections.

An ideal in Φ will always mean a norm closed two sided ideal. ζ will be reserved for a maximal ideal of $\mathcal{Y}$, and $[\zeta]$ will denote the ideal in Φ generated by ζ. K(h) will be the ideal of compact operators in B(h). There is a close connection between ideals and projections. If I is an ideal, and E is a projection in I, then $F \in I$ for all $F \lesssim E$. Also, if $E, F \in I$, then sup $(E,F) \in I$. Furthermore, if $A \in I$ and the range of A is closed, then $R(A) \in I$.

2. Direct Integral Decomposition

Let $h_1 \subset h_2 \subset \ldots \subset h_\infty$ be a sequence of Hilbert spaces, h_n having dimension n and h_∞ separable. μ will denote a finite, positive, regular Borel measure on a compact set $\Lambda \subset \mathbb{R}$. Finally, let e_n, $1 \leq n \leq \infty$, be a sequence of disjoint Borel sets, $\bigcup_1^\infty e_n = \Lambda$. e_n are called the dimension sets. Let h denote all functions $x: \Lambda \to h_\infty$ such that

(1) $x(\lambda) \in h_n$ if $\lambda \in e_n$

(2) $x(\cdot)$ is μ-measurable

(3) $\int_\Lambda ||x(\lambda)||^2 d\mu < \infty$

If we further define

$$(4)\quad (x, y) = \int_{\Lambda} (x(\lambda),\ y(\lambda))d\mu$$

then h becomes a Hilbert space if we identify functions which are equal μ-a.e. We write $h = \int_{\Lambda} \oplus\ h(\lambda)d\mu$, and call this a <u>direct integral decomposition</u> of h. If $\mu(e_k) = 0$ for all k except k = n, h is said to be of <u>pure dimension n.</u>

Suppose now that A: $\Lambda \to B(h(\lambda))$ is such that $A(\lambda)\ x\ (\lambda)$ is μ measurable for all $x \in h$ and $||\ A(\lambda)\ ||$ is bounded. Then

$$(*)\qquad x(\cdot) \to A(\cdot)x(\cdot)$$

defines a bounded linear operator in h. Any $A \in B(h)$ for which a function $A(\lambda)$ exists such that (*) represents A is said to be <u>decomposable</u>, and we write $A = \int_{\Lambda} \oplus\ A(\lambda)d\mu$, which is called the <u>direct integral decomposition</u> of A. If $A(\lambda) = c(\lambda)\ 1_{\lambda}$, where $c(\lambda)$ is a scalar valued function, A is called a <u>diagonal operator.</u>

Let $A = \int_{\Lambda} \oplus\ A(\lambda)d\mu$ and $B = \int_{\Lambda} \oplus\ B(\lambda)d\mu$ be two decomposable operators. The basic properties of direct integral decompositions are:

(1) A^* is decomposable and $A^* = \int_{\Lambda} \oplus\ A^*(\lambda)\ d\mu$

(2) $\alpha A + \beta B$ is decomposable, and $\alpha A + \beta B = \int_{\Lambda} \oplus [\ \alpha A(\lambda) + \beta B(\lambda)] d\mu$.

(3) AB is decomposable and $AB = \int_{\Lambda} \oplus\ A(\lambda)B(\lambda)d\mu$

(4) $||\ A\ || = \text{essential}\ \sup_{\lambda}\ ||\ A(\lambda)\ ||$

(5) A is invertible if and only if $A(\lambda)$ is

is invertible μ-a.e. and $\|A(\lambda)^{-1}\|$ is essentially bounded; in this case, $A^{-1} = \int_\Lambda \oplus A(\lambda)^{-1}\, d\mu$.

Suppose now that $A_n = \int_\Lambda \oplus A_n(\lambda) d\mu$ is a sequence of decomposable operators; let $\Phi(\lambda)$ be the von Neumann algebra in the Hilbert space $h(\lambda)$ which is generated by $\{A_n(\lambda)\}_1^\infty$, and let Φ be the algebra in h generated by $\{A_n\}$ and all the diagonal operators. Then $\{\Phi(\lambda)\}$ is called a measurable family of von Neumann algebras, and Φ is called the direct integral of $\Phi(\lambda)$: in symbols, $\Phi = \int_\Lambda \oplus \Phi(\lambda) d\mu$. If Φ is an algebra in $h = \int_\Lambda \oplus h(\lambda) d\mu$ for which a measurable family $\Phi(\lambda)$ exists with $\Phi = \int_\Lambda \oplus \Phi(\lambda) d\mu$, Φ is said to be decomposable relative to the direct integral decomposition of h. The main theoren, due to von Neumann, can now be stated (see [11], [14], [15]).

Theorem: Let Φ be a von Neumann algebra acting on the separable Hilbert space h. Then there exists a direct integral decomposition of h, $h = \int_\Lambda \oplus h(\lambda) d\mu$, relative to which Φ is decomposable, $\Phi = \int_\Lambda \oplus \Phi(\lambda) d\mu$, and $\Phi(\lambda)$ is a factor μ-a.e. Furthermore, $\mathfrak{z}(\Phi)$ is the set of diagonal operators.

CHAPTER 1

ESSENTIAL CENTRAL SPECTRUM

1. Central Ideals

Definition 1.1: An ideal I will be called central if for any bounded sequence $\{A_k\}_1^\infty \subset I$, and mutually orthogonal central projections $\{P_k\}_1^\infty$, we have $\sum_i^\infty A_k P_k \in I$.

Examples: (1) In a factor, any ideal is central.

(2) In any algebra, the ideal generated by the finite projections is central.

(3) If P is a central projection in the algebra Φ, then ΦP is a central ideal in Φ.

If Φ acts on a separable Hilbert space, the last two examples are basically the only central ideals. A precise formulation of this result, Theorem 1.5, is the main result of this section. This result will follow easily from a theorem of Halpern [7]; to state it, we need the following definition:

Definition 1.2: Let P be a central projection, E a properly infinite projection dominated by P. Denote by $I_P(E)$ the ideal generated by all projections F which satisfy

(i) $F \leq P$

(ii) if $EQ \precsim FQ$ for a central projection Q, then $EQ = 0$.

The theorem mentioned above is then:

Theorem 1.3 (Halpern): An ideal I is central if and only if it is of the form $I_P(E)$.

Lemma 1.4: Let F be an infinite projection. Then there exists $F_1 \leq F$ with F_1 properly infinite.

Proof: Let Δ be the collection of all sets $\{Q_k\}$, where the Q_k are mutually orthogonal central projections such that FQ_k is finite for all k. If $\Delta = \phi$, then F is properly infinite, and we are finished. If $\Delta \neq \phi$, then partially order Δ by set inclusion; the union of a chain is clearly a least upper bound. By Zorn, Δ contains a maximal element $\{Q'_k\}$. Let $Q' = \sum_k Q'_k$, and $F_1 = F(1-Q')$.

We will first show that FQ' is finite. If $G < FQ'$, $FQ' \sim G$, then $G = \sum_k GQ_k \sim \Sigma FQ_k$, or since the Q_k are orthogonal, $GQ_k \sim FQ_k$; but FQ_k is finite, and so $GQ_k = FQ_k$, and $G = FQ'$. Hence, $F_1 \neq 0$, and it is now easy to see that F_1 must be properly infinite, for if not, it would contradict the maximality of $\{Q_k\}$.

Theorem 1.5: Let Φ be a von Neumann algebra acting on a separable Hilbert space, I a central ideal. Then there exists central projections P_I, Q_I such that $P_I Q_I = 0$, and

(i) $I(1-(P_I+Q_I)) = 0$

(ii) $IQ_I = \Phi Q_I$; $I \cap \mathfrak{Z} = IQ_I \cap \mathfrak{Z}$

(iii) IP_I is the ideal generated by all finite projections $\leq P_I$, and ΦP_I is properly infinite.

Proof: By Theorem 1.3, $I = I_P(E)$, where E is properly infinite and P is a central projection. If we set $P_I = cs(E)$, and $Q_I = P - cs(E)$, then the first statement is clearly true. Further, one observes that any central projection $S \leq Q_I$ is in $I_P(E)$. That these are the only central projections in $I_P(E)$ follows from the fact that if $SP_I \neq 0$, S a central projection, then

$$(P_I S)E \precnsim (SP_I)S = SP_I$$

and $P_I SE \neq 0$. Hence, $S \notin I_P(E)$. This proves (ii).

To prove (iii), we first notice that P_I, being the central support of a properly infinite projection, is itself properly infinite, and hence, so is the algebra ΦP_I. Suppose now that F is a finite projection $\leq P_I$, and SE $\precnsim$ FS for some central projection S. But FS $\leq$ F is finite, while SE is either 0 or infinite since E is properly infinite. Hence SE = 0, or $F \in I$. To see that these are the only projections in IP_I, suppose now that $F \leq P_I$ is infinite; by Lemma 1.4, we may assume that F is properly infinite.

Let $\Phi = \int_\Lambda \oplus (\lambda) d\mu$ be the direct integral decomposition of Φ, and let $cs(F) = \int_\Lambda \oplus \chi_G d\mu$ where χ_G is the characteristic function of the measurable set G. We can then write $F = \int_G \oplus F(\lambda) d\mu$, where $F(\lambda)$ is infinite μ-a.e. since F is properly infinite. But then $F(\lambda) \sim 1(\lambda)$, because in a factor on a separable Hilbert space, all infinite projections are equivalent ([11]), and so, $F \sim cs(F)$ ([14]). If $F \in I$, then $cs(F) \in I$, but this contradicts the fact that $IP_I \cap \mathcal{Z} = 0$. This completes the proof.

<u>Remarks</u>: (a) Everything in the above theorem goes over to the non-separable case, as the proof shows, except for the statement that IP is generated only by the finite projections. If $\Phi = B(H)$ and dimension

$H = c$, then the ideal generated by all projections P which satisfy dimension $R(P) \leq \aleph_0$ is central, but is not generated by just the finite projections.

(b) Suppose that Φ is purely infinite, I a central ideal. Then, as was shown in the above proof, if $F \in I$, F a projection, then $cs(F) \in I$. Hence, $I = \Phi Q$ for a central projection Q. Therefore, we may also assume that the algebra ΦP_I has no pure infinite part.

(c) Theorem 1.5 can also be proved using the results of [2].

Part (ii) of the above theorem guarantees that every central ideal contains a maximal central projection, or, stating this in a slightly different form, there is a maximal central projection Q_I such that $1Q_I \in I$ (this fact can also be observed directly from the definition). If we now replace the identity by an arbitrary projection in Φ, we obtain an algebraic characterization of central ideals; the necessity of this condition was observed by Halpern. We separate out the following simple but useful lemmas:

Lemma 1.6: Let J be an ideal, $A \in J$. Then for $\epsilon > 0$, there exists $B \in J$ such that the range of B is closed, and $\|A - B\| < \epsilon$.

Proof: Let $A = UH$ be the polar decomposition of A. It is sufficient to prove the lemma for $A = H$, because $H = U^*A \in J$, and if H_0 satisfies the above conditions for H, then

$$\|A - UH_0\| = \|U(H - H_0)\| = \|H - H_0\| < \epsilon$$

and the range of UH_0 is closed.

Let $H = \int_0^M \lambda dE(\lambda)$ be the spectral decomposition of H, and let E_n denote the spectral projection associated with the interval $[1/n, M]$. Clearly, E_nH has closed range, $\|H - E_nH\| \leq 1/n$, and $E_nH \in J$ since $H \in J$, and the proof is complete.

Lemma 1.7: Let ζ be a maximal ideal of $\mathcal{Z}$, $[\zeta]$ the ideal generated in Φ by ζ. Then elements of the form $\sum_1^n A_kQ_k$, $A_k \in \Phi$, $Q_k \in \zeta$ and the range of Q_k is closed, are dense in $[\zeta]$. Furthermore, elements of the form $J + \sum_1^n A_kQ_k$, $J \in I$, A_k, Q_k as above, are dense in $I + [\zeta]$.

Proof: Clearly $\sum_1^n A_kQ_k$ form a dense set in $[\zeta]$; if $A \in [\zeta]$, there exist $A_k, Q_k, k = 1, \ldots, n$ $A_k \in \Phi$, $Q_k \in \zeta$ such that

$$\|A - \sum_1^n A_kQ_k\| < \epsilon/2 \quad .$$

However, by Lemma 1.6, we can find $Q_k' \in \zeta$, range of Q_k' is closed, and $\|Q_k - Q_k'\| < \epsilon/n \|A_k\|$. Hence,

$$\|A - \sum_1^n A_kQ_k'\| \leq \|A - \sum_1^n A_kQ_k\| + \|\sum_1^n A_k(Q_k - Q_k')\|$$

$$\leq \epsilon/2 + \sum_1^n \|A_k\| \|Q_k - Q_k'\| < \epsilon$$

which completes the proof, as the last statement follows immediately.

Theorem 1.8: An ideal $I \subset \Phi$ is central if and only if for any projection $F \in \Phi$, there exists a maximal central projection Q such that $QF \in I$.

Proof: The proof of the necessity of the condition is exactly the same as the proof of Lemma 1.4, and so will be omitted. Suppose that I satisfies the condition, and let $\{A_k\}_1^\infty \subset I$ be a bounded set, and let $\{P_k\}_1^\infty$ be a sequence of mutually orthogonal central projections; we wish to show $\sum_1^\infty A_k P_k \in I$. Now by Lemma 1.6, we can find $B_k \in I$, the range of B_k is closed, and $\|A_k - B_k\| < 1/n$. Then $\|\sum_1^\infty A_k P_k - \sum_1^\infty B_k P_k\| < 1/n$, and since n was arbitrary and I is closed, it suffices to assume that the range of A_k is closed.

Let $T = \sum_1^\infty A_k P_k$. If we show that $R(T) \in I$, then $T = R(T)T \in I$. Now because P_k are central projections,

$$R(T) = R(\sum_1^\infty A_k P_k) = \sum_1^\infty R(A_k) P_k \quad ,$$

and since the range of A_k is closed, $R(A_k) \in I$. Therefore, the problem has been reduced to showing that if $F_k \in I$ are projections, then $F = \sum_1^\infty F_k P_k \in I$. By assumption, there is a maximal central projection Q such that $QF \in I$. If $Q \neq 1$, there is a P_{k_0} such that $P_{k_0}(1 - Q) \neq 0$ (there is no harm in assuming that $\sum_1^\infty P_k = 1$). But then,

$$FP_{k_0}(1 - Q) = F_{k_0} P_{k_0}(1 - Q) \in I$$

and thus $Q + P_{k_0}(1 - Q)$ contradicts the choice of Q.

2. Decomposition Theorem

For this section, we will always assume that our von Neumann algebra Φ acts on a separable Hilbert space. Recall that ζ will denote a maximal ideal of the center $\mathfrak{z}$, and $[\zeta]$ (respectively $I + [\zeta]$) will denote the ideal generated by ζ (respectively I and ζ). The following definition is due to Halpern:

Definition 1.9: Let $A \in \Phi$, I a central ideal. The essential central spectrum of A relative to the ideal I is

$$\Sigma_I(A) = \{Z \in \mathfrak{z} \,|\, \hat{Z}(\zeta) \quad \sigma(A/I + [\zeta]) \ \forall\ \zeta\}$$

where $\hat{Z}$ denotes the Gelfand transform of Z, and σ is the usual spectrum in the C*-algebra $\Phi/I + [\zeta]$.

If $\Phi = B(h)$, and I is the ideal of compact operators, then this definition reduces to one of the usual definitions of the essential spectrum σ_e; namely, $\lambda \in \sigma_e(A)$ if and only if $A - \lambda$ is not invertible modulo the compacts (i.e., $A - \lambda$ is not Fredholm). The same is true if we take Φ to be a II_∞ factor and take for I the ideal generated by the finite projections. To simplify the terminology, we will, following Breuer [1], agree to call the ideal generated by the finite projections (in any algebra) the ideal of compact operators, and we will call any operator that is invertible modulo this ideal Fredholm.

The following simple example may aid in understanding the decomposition theorem. We choose Φ to be the von Neumann algebra acting on the Hilbert space $h \oplus h$ which consists of all 2×2 matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where $A, B \in B(h)$. If we take I to be the compact operators, then it is clear that

$$I = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} K_1, K_2 \in K(h)$$

and

$$\mathfrak{z}(\Phi) = \begin{bmatrix} \alpha 1 & 0 \\ 0 & \beta 1 \end{bmatrix} \alpha, \beta \in \mathbb{C}$$

and that there are only two maximal ideals in $\mathfrak{z}$, one generated by $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and the other by $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. With this, it is not difficult to see that 0 belongs to the essential central spectrum of $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ if and only if $0 \in \sigma_e(A) \cap \sigma_e(B)$, where σ_e is the essential spectrum mentioned above.

Suppose we now change I; let

$$I = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix} K_1 \in K(h) \quad .$$

Then all of the above remains the same except that now 0 belongs to the essential central spectrum of $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ if and only if $0 \in \sigma_e(A)$ and $0 \in \sigma(B)$, where σ is the ordinary spectrum.

Since a direct integral decomposition can be viewed as a continuous direct sum, it seems plausible that the above observations can be extended to an arbitrary algebra and central ideal. However, as the second example makes clear, in relating $\Sigma_I(A)$ to the spectrum of the components of A, the ideal I will determine whether the essential or ordinary spectrum is called for.

Suppose now that $\int_\Lambda \oplus \Phi(\lambda) d\mu$ is the direct integral decomposition of Φ. The following theorem of von Neumann [14] will help us prove the measurability of certain sets:

Theorem 1.10 (von Neumann): Let $E = \int_\Lambda \oplus E(\lambda) d\mu$ be a projection. Then the sets $\{\lambda \mid E(\lambda) = 0\}$ and $\{\lambda \mid E(\lambda) \text{ is finite}\}$ are measurable.

Lemma 1.11: Let $A = \int_\Lambda \oplus A(\lambda) d\mu \in \Phi$. Then $\Lambda_0 = \{\lambda \in \Lambda \mid A(\lambda) \text{ is invertible}\}$ is measurable.

Proof: Let $A = UH$ be the polar decomposition of A. It is easy to see that $U = \int_\Lambda \oplus U(\lambda) d\mu$, $H = \int_\Lambda \oplus H(\lambda) d\mu$ where $U(\lambda)H(\lambda)$ is the polar decomposition of $\Phi(\lambda)$. Since $A(\lambda)$ is invertible if and only if both $U(\lambda)$ and $H(\lambda)$ are, $\Lambda_0(A) = \Lambda_0(U) \cap \Lambda_0(H)$, and we can therefore treat these cases separately.

For U, we note that

$$\Lambda_0(U) = \{\lambda \mid N(U(\lambda)) = 0\} \cap \{\lambda \mid N(U^*(\lambda)) = 0\}$$

where N(T) denotes the projection on the null space of T. However, both of these sets are measurable by Theorem 1.10, and hence, so is $\Lambda_0(U)$. For H, we let E_n be the spectral projection associated with the interval $[0, 1/n]$. Clearly then, $\Lambda_0(H) = \bigcup_1^\infty \{\lambda \mid E_n(\lambda)H(\lambda) = 0\}$, and since each set in the union equals

$$\{\lambda \mid R((E_n H)(\lambda)) = 0\}$$

$\Lambda_0(H)$ is measurable, again by Theorem 1.10. This completes the proof.

A well-known theorem of Atkinson states that an operator $A \in B(h)$ is Fredholm if and only if the range of A is closed and $N(A)$ and $N(A^*)$ are finite. The following result, due to Breuer, generalizes this to an arbitrary algebra.

<u>Lemma 1.12 (Breuer)</u>: $A \in \Phi$ is Fredholm if and only if

(i) $N(A)$ is finite

(ii) there exists a projection $F \in \Phi$ such that $F \subset$ range of A and $1 - F$ is finite.

<u>Lemma 1.13</u>: Let $A = \int_\Lambda \oplus A(\lambda) d\mu \in \Phi$. Then the set $\Lambda_e = \{\lambda \mid A(\lambda)$ is Fredholm$\}$ is measurable.

<u>Proof</u>: As in Lemma 1.11 we use the polar decomposition $A = UH$, $A(\lambda) = U(\lambda)H(\lambda)$. Since the Fredholm operators are closed under multiplication, $A(\lambda)$ is Fredholm, $N(A(\lambda)) = N(U(\lambda))$ and $N(A^*(\lambda)) = N(U^*(\lambda))$ are finite, and so, $U(\lambda)$ and $U^*(\lambda)$ are Fredholm. But $H(\lambda) = U^*(\lambda)A(\lambda)$, and therefore, $\Lambda_e(A) = \Lambda_e(U) \cap \Lambda_e(H)$. However,

$$\Lambda_e(U) = \{\lambda \mid N(U(\lambda)) \text{ is finite}\} \cap \{\lambda \mid N(U^*(\lambda) \text{ is finite}\}$$

implies $\Lambda_e(U)$ is measurable by Theorem 1.10. If we could verify that

$$\Lambda_e(H) = \bigcup_1^\infty \{\lambda \mid E_n(\lambda) \text{ is finite}\}$$

where E_n is the spectral projection of H associated with $[0, 1/n]$, then $\Lambda_e(H)$ would be measurable, completing the proof. Clearly, if $E_n(\lambda)$ is finite for some n, then $H(\lambda)$ is Fredholm, for then $H(\lambda) = E_n(\lambda)H(\lambda) + (1 - E_n(\lambda))H(\lambda)$ and $H(\lambda)$ maps $1 - E_n(\lambda)$ onto $1 - E_n(\lambda)$. Therefore, it only remains to show that if $H \in \Phi$ is positive hermitian and Fredholm, then there exists a spectral projection $E([0, \alpha])$ which is finite.

Because H is Fredholm, we can find $F \in \Phi$, $F \subset$ range of H, and $1 - F$ finite. Now there is a closed subspace $V \subset R(H)$ such that $H : V \to F$ one-to-one and onto. Since $V = 1 - N(FH) = R((FH)^*)$, $V \in \Phi$, and furthermore, $1 - V = N(FH)$, and because both H and F are Fredholm, $1 - V$ is finite. $H : V \to F$ is an invertible mapping, and so there exists $\beta > 0$ such that

$$\|Hx\| \geqslant \beta \|x\| \quad \text{for all } x \in V \quad . \tag{1}$$

Let E be the spectral projection of $[0, \beta/2]$; for $x \in E$ we have

$$\|Hx\| \leqslant (\beta/2)\|x\| \quad . \tag{2}$$

Comparing (1) and (2) we see that $\inf(E, V) = 0$. But then, by parallelogram law [10],

$$E = E - \inf(E, V) \sim 1 - V - \inf(1 - V, 1 - E) \precsim 1 - V$$

which says that E is finite. This completes the proof.

Returning now to the essential central spectrum, we need the following two results due to Halpern:

<u>Lemma 1.14</u>: Let I be a central ideal, Q_I the largest central projection in I, and let S be an arbitrary central projection. Then

(i) IS is a central ideal

(ii) $Q_I \Sigma_I(A) = 0$ for all $A \in \Phi$

(iii) $\Sigma_I(A) = \Sigma_{IS}(AS) \oplus \Sigma_{IS_1}(AS_1)$

where $S_1 = 1 - S$, and the essential spectra on the right-hand side are taken in the algebras ΦS and ΦS_1, respectively.

Theorem 1.15 (Halpern): Suppose that Φ is properly infinite, and that $I = I_P(E)$. Then if $Z \in \Sigma_I(A)$, there exists two sequences of mutually orthogonal projections $\{E_n\}$, $\{F_n\}$ such that $cs(E_n) = cs(F_n) = P$, all are properly infinite, and $\|(A - Z)E_n\| \leq 1/n$, $\|F_n(A - Z)\| \leq 1/n$.

Part (ii) of Lemma 1.14 says that as far as the essential central spectrum is concerned, we might as well assume that $I \cap \mathcal{Z} = 0$, or $Q_I = 0$, and we shall do this from now on. Under this assumption, a central ideal will, by Theorem 1.5, divide the algebra Φ into two parts, $\Phi = \Phi P \oplus \Phi(1 - P)$, such that $I \cap \Phi(1 - P) = I(1 - P) = 0$ and IP is the ideal of compact operators in ΦP. Let $P = \int_\Lambda \oplus \chi_G d\mu$, where χ_G is the characteristic function of the measurable set G. We will, in order to simplify the statement of the decomposition theorem, accept the following convention: in the factor $\Phi(\lambda)$, $\lambda \in G$, σ_e will denote the essential spectrum of invertibility modulo the compacts whereas for $\Phi(\lambda)$, $\lambda \notin G$, σ_e will be the ordinary spectrum.

Theorem 1.16: Let $I = I_P(E)$ be a central ideal, $P = cs(E) = \int_\Lambda \oplus \chi_G d\mu$, and let $A = \int_\Lambda \oplus A(\lambda) d\mu \in \Phi$. Then $Z = \int_\Lambda \oplus c(\lambda) d\mu \in \Sigma_I(A)$ if and only if $c(\lambda) \in \sigma_e(A(\lambda))$ μ-a.e.

Proof: It suffices to treat the case $Z = 0$, so assume that $0 \in \Sigma_I(A)$. Then A is invertible modulo $I + [\zeta]$ for some ζ;

$$AB = 1 + K \quad \text{where} \quad B \in \Phi,\ K \in I + [\zeta] \quad . \tag{3}$$

Now, by Lemma 1.7, we can find $J \in I$, $Q_k \in \zeta$, and $D_k \in \Phi$, $k = 1, \ldots, n$, such that the range of Q_k is closed and

(4) $$\| K - (J + \sum_1^n D_k Q_k) \| \leq 1/2 \ .$$

From (3) we find that

$$AB - (J + \sum_1^\infty D_k Q_k) = 1 + [K - (J + \sum_1^\infty D_k Q_k)]$$

and from (4) we see that the right-hand side is invertible, and therefore there is a $B' \in \Phi$ such that

$$ABB' - (JB' + \sum_1^n D_k B' Q_k) = 1$$

or

(5) $$AB_0 = 1 + J_0 + \sum_1^n C_k Q_k \qquad J_0 \in I, B_0, C_k \in \Phi \ .$$

Now, $R(\Sigma_1^n C_k Q_k) \leq \sup_k R(C_k Q_k) \leq \sup_k R(Q_k) \in \zeta$ since the range of Q_k is closed. Hence, if $Q = \sup_k R(Q_k)$ we have $1 - Q \neq 0$ and $(1 - Q)(\Sigma_1^n C_k Q_k) = 0$. Hence, from (5) we get

$$AB_0(1 - Q) = 1 - Q + J(1 - Q)$$

which says that A, as an operator from $(1 - Q)$ into $(1 - Q)$, is invertible modulo I. Therefore if $1 - Q = \int_\Lambda \oplus \chi_Y d\mu$, we have that $A(\lambda)$, for

$\lambda \in Y$, is either Fredholm or invertible (depending upon whether $\lambda \in G$ or $\lambda \notin G$). Hence, $0 \in \sigma_e(A(\lambda))$ on a set of positive measure.

Conversely, suppose that $0 \in \Sigma_I(A)$. If $I = 0$, consider $\Lambda_0 = \{\lambda \mid A(\lambda) \text{ is invertible}\}$ which, by Lemma 1.11, is measurable. If $\mu(\Lambda_0) > 0$, then we can find $\Lambda_0' \subset \Lambda_0$, $\mu(\Lambda_0') > 0$, such that $\|A(\lambda)^{-1}\|$ is bounded for $\lambda \in \Lambda_0'$. Therefore, A is invertible as an operator on Q, where $Q = \int \oplus \chi_{\Lambda'} d\mu$. Hence, $0 \in \sigma(A/[\zeta])$, where ζ is any maximal ideal in $\mathcal{Z}$ containing $1 - Q$, which contradicts our assumption that $0 \in \Sigma_I(A)$. Hence, $\mu(\Lambda_0) = 0$ as required.

By Lemma 1.14 and Theorem 1.5, it remains to consider the case when Φ is properly infinite with no pure infinite part and I is the ideal of compact operators in Φ. In this case, by Theorem 1.15, we can find two sequences of mutually orthogonal projections $\{E_n\}$ and $\{F_n\}$ satisfying all the conditions in that theorem. Since all projections have central support 1, the above conditions hold for each $A(\lambda)$ except possibly on a μ-null set. However, by Lemma 1.12, this says that $A(\lambda)$ cannot be Fredholm, and so, $0 \in \sigma_e(A(\lambda))$ μ-a.e. This completes the proof.

As a corollary, we get the following converse to Theorem 1.15.

<u>Corollary 1.17</u>: Suppose that Φ is properly infinite, $A \in \Phi$, $\{E_n\}$ and $\{F_n\}$ as above. Then $0 \in \Sigma_I(A)$.

3. Spectral Mapping

Either directly from the definition, or from Theorem 1.16, it is clear that $\sigma(Z) \subset \sigma(A)$ if $Z \in \Sigma_I(A)$. Hence, if f is analytic on a neighborhood of $\sigma(A)$, f(Z) is also defined, and so it is natural to expect that a mapping theorem holds. The next lemma is directed towards showing that

$$f(A) = \int_\Lambda \oplus f(A(\lambda))d\mu \quad .$$

Lemma 1.18: Let $A = \int_\Lambda \oplus A(\lambda)d\mu \in \Phi$. Then $\sigma(A(\lambda)) \subset \sigma(A)$ μ-a.e.

Proof: Let α_0 be in the resolvent set of A. Then there is a $\beta > 0$ such that $\|(A - \alpha_0)x\| \geq \beta\|x\|$ for all x. Hence, except on a μ-null set Y_{α_0}, $A(\lambda) - \alpha_0$ is invertible, and $\|(A(\lambda) - \alpha_0)x_\lambda\| \geq \beta\|x_\lambda\|$ for all $x_\lambda \in h(\lambda)$. Therefore, there is an open disk W_{α_0} about α_0, such that $A(\lambda) - \alpha$ is invertible for all $\alpha \in W_{\alpha_0}$, $\lambda \in \Lambda \backslash Y_{\alpha_0}$. If we do this for all $\alpha_0 \in \rho(A)$, we get a collection of open disks covering $\rho(A)$, from which we can select a countable subcovering $\{W_{\alpha_k}\}_1^\infty$. If Y_{α_k} are the corresponding μ-null sets, then $\sigma(A(\lambda)) \subset \sigma(A)$ provided $\lambda \notin \bigcup_1^\infty Y_{\alpha_K}$ as required.

Corollary 1.19: If f is analytic on a neighborhood of $\sigma(A)$, then $f(A) = \int_\Lambda \oplus f(A(\lambda))d\mu$.

Proof: By the above, $f(A(\lambda))$ is well defined μ-a.e. Since the result is clearly true if f is a polynomial, a limiting process yields the full statement.

Theorem 1.20: Let $A = \int_\Lambda \oplus A(\lambda)d\mu \in \Phi$, f analytic on an open set $U \supset \sigma(A)$. Then $\Sigma_I(f(A)) = f(\Sigma_I(A))$.

Proof: We first show $f(\Sigma_I(A)) \subset \Sigma_I f(A)$. Let

$Z = \int_\Lambda \oplus c(\lambda)d\mu \in \Sigma_I((A))$. From Theorem 1.16, $c(\lambda) \in \sigma_e(A(\lambda))$ μ-a.e., and $f(c(\lambda)) \in f(\sigma_e(A(\lambda))$. However, by a result of Grammsch and Lay [5], $f(\sigma_e(A(\lambda)) = \sigma_e(f(A(\lambda)))$; and so $f(c(\lambda)) \in \sigma_e(f(A(\lambda)))$. Again by Theorem 1.16, $f(Z) = \int \oplus f(c(\lambda))d\mu \in \Sigma_I f(A)$.

To verify that $\Sigma_I f(A) \subset f(\Sigma_I(A))$, considerably more work is required. We will first establish that this is true 'locally': i.e., we will show that if $Z \in \Sigma_I f(A)$, then there is a non-zero central projection Q such that $ZQ \in f(\Sigma_I(AQ))$. We proceed to prove this by examining three cases:

I: Assume that $\sigma(A)$ has an isolated point a. Since $Z = \int_\Lambda \oplus c(\lambda)d\mu \in \Sigma_I f(A)$, we have $c(\lambda) \in \sigma_e\ f(A(\lambda)) = f[\sigma_e(A(\lambda))]$, and hence, $c(\lambda) \in$ range of f. Let $Y_a = f^{-1}(a)$. Now since a is isolated, there is a measurable set G, $\mu(G) > 0$, such that $a \in f[\sigma_e(A(\lambda))]$ for all $\lambda \in G$; in other words, $Y_a \cap \sigma_e(A(\lambda)) \neq \phi$ for $\lambda \in G$. If $Q = \int_\Lambda \oplus \chi_G d\mu$, we claim that $Y_a \cap \sigma(AQ/IQ + [\zeta]) \neq \phi$ for all ζ where ζ will now be a maximal ideal of $\mathfrak{z}Q$; if not there is a ζ such that for each $y \in Y_a$, $AQ - y$ is invertible modulo $IQ + [\zeta]$. As in the proof of the decomposition theorem, this means that for $y \in Y_a$ there is a central projection $Q_y \leq Q$, $\hat{Q}_y(\zeta) = 1$, and such that $AQ - y$ is invertible modulo IQ as an operator from Q_y into Q_y. By the openness of invertible operators, this last condition holds for a neighborhood W_y of y. Y_a is closed, and so compact, and we can therefore select a finite number $W_{y_1}, \ldots, W_{y_n}$ to cover Y_a. If $Q_{y_1}, \ldots, Q_{y_n}$ are the corresponding projections, set $Q' = \prod_1^n Q_{y_k}$; $\hat{Q}'(\zeta) = 1$, and so $Q' \neq 0$. Furthermore, by construction, $AQ' - y$, as an operator from Q' into Q', is invertible modulo IQ for all $y \in Y_a$. Consequently $Y_a \cap \sigma_e(A(\lambda)) = \phi$ for all $\lambda \in G'$, where $Q' = \int_\Lambda \oplus \chi_{G'} d\mu$, which is a contradiction. Therefore $Y_a \cap \sigma(AQ/IQ + [\zeta]) \neq \phi$ for all ζ. But then a theorem of Halpern ([7], Theorem 3.5), guarantees a $Z_0 \cap \mathfrak{z}Q$ such that $\hat{Z}_0(\zeta) \in Y_a$ for all ζ. Hence, $Z_0 = \int_\Lambda \oplus d(\lambda)\chi_G d\mu$ and $d(\lambda) \in Y_a$ for $\lambda \in G$. Recall that $ZQ = \int_G \oplus a(\lambda) d\mu$ where $a(\lambda) \equiv a$. Clearly, $ZQ = f(Z_0) \in f(\Sigma_I AQ)$.

II. We now assume $f' = \frac{d}{dz} f \equiv 0$ on $Y = f^{-1}(\sigma(Z))$. If $U = \bigcup_1^\infty U_k$, where U_k are the components of U, then either $U_k \cap Y$ is finite or f is constant on U_k. Hence, f assumes at most a countable number of values, and therefore, we must have an isolated point in $\sigma(Z)$. This puts us back in case I.

III. From I and II, we may now assume that we have a point $w_0 \in \sigma(Z)$ such that $Y_{w_0} = f^{-1}(w_0)$ is at most a finite set u_k, $k = 1, \ldots, n$, and $f'(u_k) \neq 0$ for all k. Hence, there is a neighborhood of w_0, W, and open sets U_k, $k = 1, \ldots, n$, $u_k \in U_k$, such that $f : U_k \to W$ has an analytic inverse g_k. Further, by choosing W small enough, we can assume that $f^{-1}(W) \subset \bigcup_1^n U_k$. If $c(\lambda_0) \in W$, we have

$$c(\lambda_0) = f[g_k(c(\lambda_0))] \in \sigma_e f(A(\lambda_0)) = f(\sigma_e(A(\lambda_0))) \quad ,$$

and so for k_0, $1 \leq k_0 \leq n$, $g_{k_0}(c(\lambda_0)) \in \sigma_e(A(\lambda_0))$. Let $G = \{\lambda \mid c(\lambda) \in W\}$; G is measurable since c is a measurable function, and $\mu(G) > 0$ since $w_0 \in \sigma(Z)$. By Lemmas 1.11 and 1.13, $G_k = \{\lambda \in G \mid g_j(c(\lambda)) \in \sigma_e(A(\lambda))\}$ is measurable, and by the above remark, $G = \bigcup_1^n G_k$. Hence, we can assume that $\mu(G_1) > 0$. Let

$Q = \int_{\Lambda} \oplus \chi_{G_1} d\mu$; then

$$ZQ = \int_{\Lambda} \oplus c(\lambda)\chi_{G_1} d\mu = f(\int_{\Lambda} \oplus g_1(c(\lambda))\chi_{G_1} d\mu)$$

and by Theorem 1.16, $\int_{\Lambda} \oplus g_1(c(\lambda))\chi_{G_1} d\mu \in \Sigma_I AQ$. This establishes our 'local' result.

To complete the proof of the theorem we will use a Zorn's lemma argument. Define

$$\Delta = \{Q \mid Q \text{ a central projection, } ZQ \in f(\Sigma_I AQ)\}$$

where $Z = \int_{\Lambda} \oplus c(\lambda) d\mu \in \Sigma_I f(A)$. Δ inherits the standard partial ordering of projections, and $\Delta \neq \phi$ by the local result. If $\{Q_\alpha\}$ is a chain, let $Q_0 = \sup Q_\alpha$. By the separability, we know that $\{Q_\alpha\}$ is countable, so write $\{Q_k\}_1^\infty$; $Q_k = \int_{\Lambda} \oplus \chi_{V_k} d\mu$. Now since $ZQ_k \in f(\Sigma_I AQ_k)$, there is a function $t_k(\lambda)$ such that $c(\lambda) = f(t_k(\lambda))$ for $\lambda \in V_k$. Let $t(\lambda) = \sum_0^\infty t_{k+1}(\lambda)[\chi_{V_{k+1}}(\lambda) - \chi_{V_k}(\lambda)]$. Clearly, $c(\lambda) = f(t(\lambda))$ for $\lambda \in V = \bigcup_1^\infty V_k$, and so $Q_0 \in \Delta$, and is the required upper bound.

Hence, Δ has a maximal element, but the local result forces this projection to be the identity. This completes the proof.

CHAPTER 2
PERTURBATION THEOREM

We have already mentioned that when $\Phi = B(h)$ and I is the ideal of compact operators, the essential central spectrum is the essential spectrum of invertibility modulo the compacts. In this case, Gustafson and Weidmann [6] have demonstrated the following converse to Weyl's Theorem: if $\sigma_e(A+X) = \sigma_e(A)$ for all $A \in B(h)$ (where σ_e is either the essential spectrum mentioned above, or the Weyl essential spectrum), then X is compact. The purpose of this section is to prove the corresponding statement for the essential central spectrum.

For this section, we will drop the assumption that the algebra Φ operates on a separable Hilbert space.

<u>Definition 2.1</u>: Let I be a central ideal; we define W_I by

$$W_I = \{X \in \Phi \mid \Sigma_I(A+X) = \Sigma_I(A) \ \forall\ A \in \Phi\} \quad .$$

Our objective is to show that $W_I = I$. Clearly, $I \subset W_I$, and so it only remains to verify the reverse inclusion. The next result will enable us to restrict our attention to the projections in W_I.

<u>Proposition 2.2</u>: W_I is an ideal.

<u>Proof</u>: That W_I is closed under addition and scalar multiplication is clear. Hence, let $X \in W_I$, $A, B \in \Phi$. We first note that

$$\Sigma_I(A+BX) = \Sigma_I(A+(B-\lambda)X+\lambda X) = \Sigma_I(A+(B-\lambda)X)$$

and therefore, we may assume that B is invertible. Now,

$$
\begin{aligned}
Z \quad \Sigma_I(A+BX) &\Leftrightarrow 0 \in \Sigma_I(A+BX-Z) \\
&\Leftrightarrow 0 \in \Sigma_I(B^{-1}A+X-B^{-1}Z) \\
&\Leftrightarrow 0 \in \Sigma_I(B^{-1}A-B^{-1}Z) \\
&\Leftrightarrow 0 \in \Sigma_I(A-Z) \\
&\Leftrightarrow Z \in \Sigma_I(A) \quad .
\end{aligned}
$$

Hence, $BX \in W_I$; XB is handled similarly, and thus it remains to show that W_I is norm closed. Let $\{X_n\} \subset W_I$, $X_n \to X$. We have,

$$
\begin{aligned}
z \quad \Sigma_I(A) &\Leftrightarrow Z \in \Sigma_I(A+X_n) \ \forall \ n \\
&\Leftrightarrow 0 \ \in \Sigma_I(A+X_n-Z) \ \forall \ n \\
&\Leftrightarrow A+X_n-Z \text{ not invertible}
\end{aligned}
$$

modulo $I + [\zeta] \ \forall \ n, \zeta$.

Since the invertible operators in any C*-algebra form an open set, the above implies that $A + X - Z$ is not invertible modulo $I + [\zeta] \ \forall \ \zeta$, and therefore, $Z \in \Sigma_I(A+X)$. Hence, $\Sigma_I(A) \subset \Sigma_I(A+X)$. If we now replace A by $A+X$, and X_n by $-X_n$, then the reverse inclusion is obtained, and the proof is complete.

<u>Remark</u>: The above method furnishes an alternate proof to the theorem of Gustafson and Weidmann, since in the event that $\Phi = B(h)$, the compacts are the only proper ideal; the same is true when Φ is a Π_∞ factor, for again the uniform closure of the finite elements is the only proper ideal.

By the above proposition, to show that $W_I = I$, it suffices to show that the projections in W_I also belong to I [16]. As was done previously, we will decompose Φ into several parts, and handle them separately. We will require the following lemmas:

Lemma 2.3: $X \in W_I$ iff $0 \in \Sigma_I(A) \Rightarrow 0 \in \Sigma_I(A \pm X)$.

Lemma 2.4: Let I be a central ideal, ζ a maximal ideal of $\mathfrak{z}$, and suppose P is a central projection with $P \notin \zeta$. Let $\zeta_1 = \zeta P$, $\Phi_1 = \Phi P$. Then

(i) ζ_1, is a maximal ideal of $\mathfrak{z}(\Phi_1) = \mathfrak{z}(\Phi)P$

(ii) $[\zeta]P = [\zeta_1]$, $(I+[\zeta])P = I_1 + [\zeta_1]$ when $I_1 = IP$ and $[\zeta_1]$ is the ideal generated in the algebra Φ_1.

(iii) If ζ' is a maximal ideal of $\mathfrak{z}(\Phi_1)$, then $\zeta = \zeta' \oplus (1-P)\mathfrak{z}(\Phi)$ is a maximal ideal of $\mathfrak{z}(\Phi)$.

Proof: The only statement that isn't immediate is that $[\zeta_1] = [\zeta]P$. $[\zeta]P \supset [\zeta_1]$. On the other hand, elements of the form AQ, $A \in \Phi$, $Q \in \zeta$, generate $[\zeta]$, and $(AQ)P = (AP)(QP) \in [\zeta_1]$, and so $[\zeta]P = [\zeta_1]$.

Since we will be working with more than one algebra, we will need a way of identifying in which algebra a certain essential central spectrum is taken. As in the above lemma, we will denote the algebras with subscripts, Φ_1, and use $\Sigma^1_{I_1}$ for the essential central spectrum relative to I_1 in the algebra Φ_1. If no subscript is present, it will always mean the algebra Φ.

Lemma 2.5: Let P be a central projection; $I_1 = IP$, $\Phi_1 = \Phi P$. Let $A_1 \in \Phi_1$. Then $0 \in \Sigma^1_{I_1}(A_1)$ iff $0 \in \Sigma_I(A_1)$. Also, if $A \in \Phi$ and $0 \in \Sigma_I(A)$, then $0 \in \Sigma^1_{I_1}(AP)$.

Proof: Suppose that $0 \in \Sigma^1_{I_1}(A_1)$ and $0 \notin \Sigma_I(A_1)$. Hence, there is a maximal ideal ζ and $B \in \Phi$ such that

$$(1) \qquad A_1B = 1 + Y \qquad Y \in I + [\zeta] \quad .$$

Note that $P \notin \zeta$, since that would imply that $A_1 \in [\zeta]$, and so by (1), $1 \in I + [\zeta]$. Hence, $1 - P \in \zeta$, and

$$A_1B = A_1BP = P + Y' \qquad Y' \in I + [\zeta] \quad .$$

This equation shows that $Y' = Y'P$, or that $Y' \in I_1 + [\zeta_1]$ by the previous lemma. But this contradicts our assumption that $0 \in \Sigma^1_{I_1}(A_1)$.

Conversely, assume that $0 \in \Sigma_I(A_1)$ and that $0 \notin \Sigma^1_{I_1}(A_1)$. Then

$$A_1B_1 = P + Y_1 \ , \quad B_1 \in \Phi_1 \ , \quad Y_1 \in I_1 + [\zeta_1] \quad .$$

Let $\zeta = \zeta_1 \oplus (1 - P)\mathfrak{z}(\Phi)$; then

$$A_1B_1 = P + Y_1 + (1 - P) - (1 - P) = 1 + Y \qquad Y \in I + [\zeta]$$

which says $0 \notin \Sigma_I(A_1)$. The proof of the last statement follows in a similar manner.

We are now in a position to show that W_I behaves nicely when Φ is decomposed.

Proposition 2.6: $W_I P = W_{IP}$, where P is a central projection and $IP = I_1$ is considered as a central ideal in the algebra $\Phi_1 = \Phi P$.

Proof: We first show that $W_I P \subset W_{I_1}$. Let $Q \in W_I P$, $A_1 \in \Phi_1$, and $0 \in \Sigma^1_{I_1}(A_1)$. Lemma 2.5 shows that $0 \in \Sigma_I(A_1)$, and hence, $0 \in \Sigma_1(A_1 \pm Q)$ since $Q \in W_I$. Again by the above lemma, $0 \in \Sigma^1_{I_1}(A_1 \pm Q)$ since $Q \in \Phi_1$. By Lemma 2.3, $Q \in W_{I_1}$.

Suppose now that $Q \in W_{I_1}$, and that $0 \in \Sigma_I(A)$ for $A \in \Phi$. By Lemma 2.5 we have $0 \in \Sigma^1_{I_1}(AP)$. If we assume that $0 \in \Sigma_I(A+Q)$, then

$$(2) \qquad (A+Q)B = 1 + Y \qquad B \in \Phi \qquad Y \in I + [\zeta]$$

for some ζ. Note that $P \notin \zeta$; for if not, $Q \in [\zeta]$, and the equation would say $0 \notin \Sigma_I(A)$. Hence, multiplying (2) by P, we get

$$(AP+Q)(BP) = P + Y' \qquad Y' \in I_1 + [\zeta]$$

or $0 \notin \Sigma^1_{I_1}(AP+Q)$, and since $Q \in W_{I_1}$, $0 \notin \Sigma^1_{I_1}(AP)$, which is the desired contraction.

As was discussed in Chapter 1, Halpern has shown that every central ideal is of the form $I_P(E)$, where P is a central projection, and E is a properly infinite projection which is dominated by P. Every central ideal therefore has a natural decomposition

$$I = I(1-P) \oplus I(P-Q_I) \oplus IQ_I$$

where $Q_I = P - cs(E)$ is the largest central projection in I. As before, what happens in the algebra ΦQ_I does not affect the essential central spectrum, and hence we may assume $Q_I = 0$. By Proposition 2.6, we are then left with considering the case $I = I(1 - P) = 0$ and the case $I = I_1(E)$, where $cs(E) = 1$. We will begin with the latter case.

Definition 2.7: A projection F will be called I-properly infinite if whenever $FQ \in I$ for some central projection Q, then $FQ = 0$.

The terminology is justified, for if we let I be the ideal generated by the finite projections, then we obtain one of the standard definitions of properly infinite. The following lemma is a generalization of the rather obvious statement that any projection (in $B(h)$) which does not dominate any infinite projections must be finite.

Lemma 2.8: Let $S \in \Phi$ be a projection which does not dominate any I-properly infinite projection. Then $S \in I$.

Proof: Consider the following sets of central projections:

$$\Omega = \{ \{Q_i\} | Q_i \leq cs(S),\ Q_iQ_j = 0,\ 0 \neq Q_iS \in I\} \quad .$$

Since S is not I-properly infinite, Ω is non-void; it is partially ordered by set inclusion, and clearly, the upper bound of any chain is the union over the chain. Hence, by Zorn, we can find a maximal element $\{Q_i\}$; we claim $\sum_i Q_i = cs(S)$. If not, then $S(cs(S) - \sum_i Q_i) \neq 0$, and is not I-properly infinite since it is dominated by S. Therefore $S(cs(S) - \sum_i Q_i)\, Q_0 \in I$ for some $Q_0 \in \mathcal{Z}$, but then $\{Q_i\} \cup (cs(S) - \sum_i Q_i)Q_0$ contradicts the maximality of $\{Q_i\}$. Since $\sum_i Q_i = cs(S)$,

$$S = cs(S)S = (\Sigma Q_i)S = \Sigma Q_iS \quad .$$

By construction, $Q_iS \in I$, $\|Q_iS\| \leq 1$, and since I is a central ideal, $S \in I$.

A basic tool that we will need is the following characterization of $\Sigma_I(A)$ when $A = A^*$ due to Halpern ([7] Corollary 3.15).

Theorem (Halpern) 2.9: Let Φ be properly infinite, $A \in \Phi$ hermitian. Then $Z \in \Sigma_I(A)$ iff there is a sequence of mutually orthogonal projections $\{E_n\}$ such that

(i) $cs(E_n) = 1 - Q_I$

(ii) E_n are I-properly infinite

(iii) $\|(A - Z)E_n\| \leq 1/n$.

Proposition 2.10: Let Φ be properly infinite, $I = I_1(E)$ where $cs(E) = 1$. Then $W_I = I$.

Proof: We have already shown that it is sufficient to show that every projection in W_I is in I; hence, let $S \in W$ be a projection. We will first show that S cannot be I-properly infinite.

If S is I-properly infinite, then so is $1 - cs(S) + S$; suppose $Q \in \mathcal{Z}$, and $Q(1 - cs(S)) + QS \in I$. But QS is orthogonal to $Q(1 - cs(S))$, and since I is an ideal, we have $QS \in I$ and $Q(1 - cs(Q)) \in I$. However the former term is zero since S was assumed to be I-properly infinite, while the latter vanished because $I \cap \mathcal{Z} = 0$. Hence, $1 - cs(S) + S$ is I-properly infinite, which also implies that it is properly infinite, since I contains all finite projections.

It is well-known that if F is properly infinite, then $F = \sum_1^\infty F_k$ where F_k are mutually orthogonal and $F_k \sim F$([4]). Hence,

$1 - cs(S) + S = \sum_1^\infty S_k$, $S_k \sim 1 - cs(S) + S$. Therefore, the $\{S_k\}$ are I-properly infinite, and $cs(S_k) = 1$, since equivalent projections have the same central support. Then by Halpern's Theorem,

$$0 \in \Sigma_I(1 - cs(S) + S)) = \Sigma_I(cs(S) - S)$$

or

$$0 \in \Sigma_I(cs(S))$$

since $S \in W_I$. This, however, is impossible unless $cs(S) = 0$ because $cs(S)$ is invertible modulo ζ, where ζ is any maximal ideal containing $1 - cs(S)$. Hence, we have succeeded in showing that $Q \in W_I \Rightarrow Q$ is not I-properly infinite. Further, Q cannot dominate any I-properly infinite projection, for if $Q > Q_0$, then $Q_0 \in W_I$, which we have just shown to be impossible. Hence, by Lemma 2.8, $Q \in I$. This completes the proof.

Proposition 2.11: Let Φ be a von Neumann algebra, $I = 0$. Then $W_I = 0$.

Proof: Let $S \in W_I$, S a projection. As was done above, it is sufficient to verify that $0 \in \Sigma_I(cs(S) - S)$, since this implies that $cs(S) = 0$.

We assume that $0 \notin \Sigma_I(cs(S) - S)$; i.e., there is a ζ for which $cs(S) - S$ is invertible modulo $[\zeta]$:

$$(cs(S) - S)B = 1 + Y \qquad Y \in [\zeta]$$

or

$$Y = 1 - (cs(S) - S)B \quad .$$

Hence, the range of $(1 - cs(S) + S)Y$ is equal to $1 - cs(S) + S$, and since $(1 - cs(S) + S)Y \in [\zeta]$, $R((1 - cs(S) + S)Y) = 1 - cs(S) + S \in \zeta$. Therefore, $S \in [\zeta]$.

By Lemma 1.7, we can find $A_k \in \Phi$, $Q_k \in \zeta$, $k = 1, \ldots, n$, such that the range of Q_k is closed, and

$$(3) \qquad \|S - \sum_{1}^{n} A_k Q_k\| < 1/2 \quad .$$

Now,

$$\|S - S(\sum_{1}^{n} A_k Q_k)S\| = \|S(S - \sum_{1}^{n} A_k Q_k)S\| \leq \|S - \sum_{1}^{n} A_k Q_k\|$$

and $S(\sum_{1}^{n} A_k Q_k)S = \sum_{1}^{n} (SA_k S)Q_k$, there is no harm in assuming that $\sum_{1}^{n} A_k Q_k$ maps S into S. However, (3) says that as an operator from S into S, $\sum_{1}^{n} A_k Q_k$ is invertible, and so, $R(\sum_{1}^{n} A_k Q_k) = S$. Then

$$S = R(\sum_{1}^{n} A_k Q_k) \leq \sup_k R(A_k Q_k) \leq \sup_k R(Q_k) = Q \quad .$$

But $R(Q_k) \in \zeta$ since the range of Q_k was chosen to be closed, and therefore, $Q \in \zeta$. But $S \leq A$, $Q \in \gamma$ implies $cs(S) \leq Q$, which in turn says that $cs(S) \in \zeta$, a contradiction.

Therefore, $0 \in \Sigma_I(cs(S) - S)$, and the proof is complete.

<u>Theorem 2.12</u>: $W_I = I$.

<u>Corollary 2.13</u>: $\bigcap_{\zeta} I + [\zeta] = I$.

<u>Proof</u>: $I \subset \bigcap_{\zeta} I + [\zeta] \subset W_I$.

CHAPTER 3
CENTRAL RANGE

In addition to the essential central spectrum, Halpern has also introduced an essential central numerical range. In this chapter, we will investigate this numerical range when $I = 0$, and thus shorten the terminology to just the central range. In the first section, we prove results aimed towards obtaining a decomposition theorem, as was accomplished for the central spectrum in Chapter 1. It turns out, however, that a complete decomposition theorem is obtained instead for the norm closure of the central range.

1. The Central Range

We again insist that our algebra Φ act on a separable Hilbert space. Recall that a projection $E \in \Phi$ is called Abelian if the algebra $E\Phi E$ is abelian. If we define $\Psi = \mathcal{Z}'$, there exist abelian projections in Ψ whose central supports equal 1 ([14]). Further, if E is such a projection, and $A \in \Psi \supset \Phi$, then $EAE = ZE$ for some $Z \in \mathcal{Z}$. With this in mind, we state the following definition due to Halpern:

<u>Definition 3.1</u>: Let $A \in \Phi$. We define the <u>central range of A, $W_{\mathcal{Z}}(A)$</u>, by

$$W_{\mathcal{Z}}(A) = \{Z \in \mathcal{Z} \mid EAE = ZE\}$$

where $E \in \Psi = \mathcal{Z}'$ is an abelian projection of central support 1.

This definition is not as strange as it might look at first. If $A \in B(h)$, then the numerical range, $W(A)$, is given by

$$W(A) = \{\lambda \in \mathbb{C} \mid \lambda = (Ax,x)\ x \in h,\ \|x\| = 1\} \quad .$$

On the other hand, every abelian projection in B(h) is given by a one-dimensional projection E_x whose range is the subspace spanned by x, $\|x\| = 1$. Furthermore,

$$E_x A E_x = (Ax,x)E_x$$

and so the definitions agree for B(h). In addition, when Φ is a factor, $W_{\mathfrak{z}}(A) = W(A)$.

<u>Remark</u>: It is easily observed that $W_{\mathfrak{z}}(A)$ satisfies the following linearity properties:

(a) $W_{\mathfrak{z}}(\alpha A) = \alpha W_{\mathfrak{z}}(A) \qquad \alpha \in \mathbb{C}$

(b) $W_{\mathfrak{z}}(A+Z) = W_{\mathfrak{z}}(A) + Z \in \mathfrak{z}$.

We will now begin to study the connection between $W_{\mathfrak{z}}(A)$ and $W(A(\lambda))$. We point out that, at first, it might appear that there is a problem in trying to use a direct integral decomposition to study $W_{\mathfrak{z}}(A)$, since the definition uses operators from outside the algebra Φ (namely, the abelian projections of Ψ). However, if $\Phi = \int_\Lambda \oplus\ \Phi(\lambda)d\mu$, we have $\Psi = \int_\Lambda \oplus\ \Psi(\lambda)d\mu$, same Λ and μ, because a direct integral decomposition is taken with respect to a commutative algebra [14] and $\Phi \cap \Phi' = \Psi \cap \Psi' = \mathfrak{z}$. Alternatively, one could just note that $W_{\mathfrak{z}}(A)$, $A \in \Phi$ is the same as $W_{\mathfrak{z}}(A)$ when A is looked at as an operator in Ψ, and therefore, we could assume $\Phi = \Psi$.

<u>Proposition 3.2</u>: Let $A = \int_\Lambda \oplus\ A(\lambda)d\mu \in \Phi$. Then $Z = \int_\Lambda \oplus\ c(\lambda)d\mu \in W_{\mathfrak{z}}(A)$

if and only if $c(\lambda) = (A(\lambda)x(\lambda),\ x(\lambda))$, where $x \in h = \int_\Lambda \oplus\ h(\lambda)d\mu$ and $\|x(\lambda)\| = 1$ μ-a.e.

<u>Proof</u>: Let $x \in h$ with $\|x(\lambda)\| = 1$ μ-a.e. Then $y \in h$, $(y(\cdot),x(\cdot))$ is a bounded measurable function, and therefore, $(y(\cdot),x(\cdot))\ x(\lambda) \in h$. Denote by E_x the projection which maps

$$y(\cdot) \to (y(\cdot),x(\cdot))\ x(\cdot) \quad .$$

If $Z_0 \in \mathcal{Z}$, $Z_0 = \int_\Lambda \oplus\ b(\lambda)d\mu$, then

$$\begin{aligned} E_x Z_0 y(\lambda) &= (Z_0 y(\lambda), x(\lambda))\ x(\lambda) \\ &= (b(\lambda)y(\lambda), x(\lambda))\ x(\lambda) \\ &= b(\lambda)\,(y(\lambda), x(\lambda))\ x(\lambda) \\ &= Z_0(E_x y(\lambda)) \end{aligned}$$

and hence, $E_x \in \mathcal{Z}'$. Since $\|x(\lambda)\| = 1$ μ-a.e., $cs(E_x) = 1$, and we have

$$\begin{aligned} E_x A E_x y(\lambda) &= E_x A(g(\lambda)x(\lambda)) \qquad g(\lambda) = (y(\lambda), x(\lambda)) \\ &= (Ag(\lambda)x(\lambda), x(\lambda))\ x(\lambda) \\ &= (Ax(\lambda), x(\lambda))\ y(\lambda)x(\lambda) \\ &= (Ax(\lambda), x(\lambda))\ E_x y(\lambda) \end{aligned}$$

which shows that E_x is abelian and that $Z = \int_\Lambda \oplus\ (Ax(\lambda), x(\lambda))d\mu \in W_{\mathcal{Z}}(A)$.

Conversely, suppose that $F \in \Psi$ is any abelian projection with $cs(F) = 1$. Choose $x \in R(F)$ with $\|x(\lambda)\| = 1$ μ-a.e., and let E_x denote the projection constructed above. Now $E_x \leq F$, and so $E_x F = E_x$. Therefore, if $FAF = ZF$, then

$$ZE_x = E_x AFE_x = E_x FAFE_x = E_x AE_x$$

which shows that Z is of the required form.

Corollary 3.3: $Z = \int_\Lambda \oplus c(\lambda)d\mu \in W_{\mathfrak{Z}}(A)$ implies $c(\lambda) \in W(A(\lambda))$ μ-a.e.

A complete decomposition theory would require the converse of Corollary 3.3 also, which we have been unable to do. However, we have been able to get satisfactory results for $\overline{W_{\mathfrak{Z}}(A)}$, the uniform closure of W(A). The next few lemmas are directed towards this theorem.

Lemma 3.4: Let $A = \int_\Lambda \oplus A(\lambda)d\mu \in \Phi$. For $\epsilon > 0$, there is a measurable set $X_\epsilon \subset \Lambda$, $\mu(X_\epsilon) < \epsilon$, such that $(A(\cdot)x(\cdot), x(\cdot))$ is a continuous function on $\Lambda \setminus X_\epsilon$ for all $x \in h$.

Proof: Let $x_1, x_2, \ldots,$ be an orthonormal basis for h. $g_{ij}(\lambda) = (A(\lambda)x_i(\lambda), x_j(\lambda))$ is a measurable function, and so by Lusin's Theorem ([12]), there is a set $X_{ij} \subset \Lambda$, $\mu(X_{ij}) < \epsilon/2^{i+j}$ such that g_{ij} is continuous on $\Lambda\ X_{ij}$. Put $X_\epsilon = \bigcup_{i,j} X_{ij}$; then $\mu(X_\epsilon) < \epsilon$ and for $\lambda \in \Lambda\setminus X$,

$$(A(\lambda)x(\lambda), x(\lambda)) = (A(\lambda)\sum_1^\infty \alpha_k x_k(\lambda), \sum_1^\infty \alpha_j x_i(\lambda))$$

$$= \sum_1^\infty \sum_1^\infty \alpha_k \bar{\alpha}_j (A(\lambda)x_k(\lambda), x_j(\lambda))$$

which is the uniform limit of continuous functions. This completes the proof.

To simplify things, we define

$$V = \{x \in h \mid \|x(\lambda)\| = 1 \ \mu\text{-a.e.}\} \quad .$$

Lemma 3.5: Suppose $0 \in \overline{W(A(\lambda))}$ μ-a.e. Then for $\epsilon, \delta > 0$, there exists a measurable set $X \subset \Lambda$, $\mu(X) < \delta$, and $x \in V$, such that $|(A(\lambda)x(\lambda), x(\lambda))| < \epsilon$ for $\lambda \in \Lambda \setminus X$.

Proof: We first assume that h is a direct integral Hilbert space of pure dimension. By the above lemma, we can find a set X $\mu(X) < \delta$, such that $(A(\lambda)x(\lambda), x(\lambda))$ is a continuous function on $\Lambda \setminus X$ for all $x \in h$. Fix $\lambda_0 \in \Lambda \setminus X$. Since $0 \in \overline{W(A(\lambda_0))}$, there is an $x_0 \in h(\lambda_0)$, $\|x_0\| = 1$, and $|(A(\lambda_0)x_0, x_0)| < \epsilon/2$. Because h is of pure dimension, the function $x_0(\lambda) \equiv x_0$ is in h. $(A(\lambda)x_0(\lambda), x_0(\lambda))$ is a continuous function, so we can find an open ball U_{λ_0} centered at λ_0 such that

$$|([A(\lambda) - A(\lambda_0)]x_0(\lambda), x_0(\lambda))| < \epsilon/2 \quad \text{for} \quad \lambda \in U_{\lambda_0} \quad .$$

The sets U_{λ_0}, $\lambda_0 \in \Lambda \setminus X$, cover $\Lambda \setminus X$, and so by the Lindelof property, $\Lambda \setminus X = \bigcup_1^\infty U_{\lambda_k}$. We change this into a disjoint union in the usual manner: $Y_1 = Y_1, \ldots, Y_k = U_k \setminus Y_{k-1}$. Recalling that each U_{λ_k} brought with it a vector $x(\lambda) \equiv x_k$, we define $\tilde{x}(\lambda) = \sum_1^\infty \chi_{Y_k}(\lambda)x_k(\lambda) \in h$. Now for $\lambda \in \Lambda \;\; X$, $\lambda \in Y_k$ for exactly one value of k, and so,

$$\begin{aligned} |(A(\lambda)x(\lambda), \tilde{x}(\lambda))| &= |(A(\lambda)x_k, x_k)| \\ &\leq |([A(\lambda) - A(\lambda_k)]x_k, x_k)| + |(A(\lambda_k)x_k, x_k)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \quad . \end{aligned}$$

The last inequality follows from the fact that $\lambda \in Y_k \subset U_k$ and from the choice of x_k.

To complete the proof, decompose h into a direct sum of spaces of pure dimension; $h = \sum_{1}^{\infty} \oplus h_k$. Apply the above result to each h_k with $\delta_k = \delta/2^k$, and let $U = \bigcup_{1}^{\infty} X_k$.

Lemma 3.6: Suppose $0 \in \overline{W(A(\lambda))}$ μ-a.e. Then for $\epsilon > 0$, there exists $x \in V$ such that $|(A(\lambda)x(\lambda), x(\lambda))| < \epsilon$ μ-a.e.

Proof: By Lemma 3.5, there exists a set X_1, $\mu(X_1) < (1/2)\ \mu(\Lambda)$, and $X_1 \in h$ such that $|(A(\lambda)x_1(\lambda), x_1(\lambda))| < \epsilon$ for $\lambda \in Y_1 = \Lambda \setminus X_1$. Repeating this process, we construct a sequence $\{X_n\}$ such that $X_n \subset X_{n-1}$, $\mu(X_n) < (1/2)\ \mu(X_{n-1})$, and a sequence $\{x_n\}$ such that

$$(\dagger) \qquad |(A(\lambda)x_n(\lambda), x_n(\lambda))| < \epsilon \qquad \lambda \in Y_n = X_{n-1} \setminus X_n \quad .$$

Now

$$\Lambda \setminus \bigcup_{1}^{\infty} Y_n = \bigcap_{1}^{\infty} X_n \quad .$$

Hence, $\mu(\Lambda \setminus \bigcup_{1}^{\infty} Y_n) \leq \mu(X_n) < (1/2^n)\ \mu(\Lambda) \to 0$ and so we may assume $\bigcup_{1}^{\infty} Y_n = \Lambda$.

Define $x_0 = \sum_{1}^{\infty} \chi_{Y_n}(\lambda) x_n(\lambda)$. The fact that $\{Y_n\}$ is a disjoint sequence and $(\dagger)$ combine to show that x_0 is the required vector.

We are now in a position to prove the announced result.

Theorem 3.7: $Z = \int_{\Lambda} \oplus\, c(\lambda) d\mu \in \overline{W_{\mathfrak{Z}}(A)}$ if and only if $c(\lambda) \in \overline{W(A(\lambda))}$ μ-a.e.

Proof: It suffices to examine $Z = 0$. If $0 \in \overline{W(A(\lambda))}$ μ-a.e., apply Lemma 3.6 with $\epsilon = 1/n$, $n = 1,2,\ldots$, to obtain $x_n \in V$, $|(A(\lambda)x_n(\lambda), x_n(\lambda))| < 1/n$. By Proposition 3.2, $Z_n = \int_\Lambda \oplus (A(\lambda)x_n(\lambda), x_n(\lambda))d\mu \in W_{\mathfrak{z}}(A)$, and clearly $Z_n \to 0$. Therefore $0 \in \overline{W_{\mathfrak{z}}(A)}$.

Conversely, suppose that $Z_n \in W_{\mathfrak{z}}(A)$, $Z_n \to 0$ uniformly. If $Z_n = \int_\Lambda \oplus c_n(\lambda)d\mu$, $c_n(\lambda) \to 0$ μ-a.e. and $c_n(\lambda) \in W(A(\lambda))$ by Proposition 3.2. Hence, $0 \in \overline{W(A(\lambda))}$ μ-a.e.

Theorem 3.7 yields the following generalization of a classical result:

Corollary 3.8: $\Sigma_0(A) \subset \overline{W_{\mathfrak{z}}(A)}$.

Proof: $Z = \int_\Lambda \oplus c(\lambda)d\mu \in \Sigma_0(A)$ implies, by Theorem 1.5, $c(\lambda) \in \sigma(A(\lambda)) \subset \overline{W(A(\lambda))}$. An application of Theorem 3.7 completes the proof.

2. The Power Inequality

As in the case of the essential central spectrum, the decomposition theorem (Theorem 3.7) is a useful tool in reducing problems in the algebra down to the factors, where they are easier to handle. As examples, we will prove a generalization of the power inequality, and a von Neumann algebra analogue of a theorem of DePrima and Richard [3]. As with the central spectrum, we need to know that certain sets are measurable, and this information will be provided by Lemmas 3.10 and 3.13.

Lemma 3.9: There exists a countable dense subset of V, $\{x_n\}_1^\infty$, such that for each $\lambda_0 \in \Lambda$, $\{x_n(\lambda_0)\}_1^\infty$ is dense in the unit sphere of $h(\lambda_0)$.

Proof: Let $x_1, x_2, \ldots,$ be a dense subset of the unit sphere of h_∞, and let P_n denote the projection of h_∞ onto h_n. Choose $y_n \in h_n$, $\|y_n\| = 1$. We define the following functions:

$$x_k(\lambda) = \begin{cases} P_n x_k / \|P_n x_k\| & \lambda \in e_n, \ \|P_n Y_k\| \neq 0 \\ \\ y_n & \lambda \in e_n, \ \|P_n Y_k\| = 0 \end{cases}$$

where e_n are the dimension sets of h. Clearly, $x_k \in V$, and by the continuity of the projections P_n, satisfy the condition of the lemma.

Lemma 3.10: Let $A = \int_\Lambda \oplus A(\lambda) d\mu \in \Phi$. Then $\Lambda_w = \{\lambda \mid 0 \notin \overline{W(A(\lambda))}\}$ is measurable.

Proof: Let $\{x_k\}_1^\infty$ be a dense subset of V guaranteed by Lemma 3.9. The set $G_{k,n}$

$$G_{k,n} = \{\lambda \mid \ |(A(\lambda)x_k(\lambda), x_k(\lambda))| > 1/n\}$$

is measurable, and we claim that, modulo a set of μ measure zero, $\Lambda_w = \bigcup_{n=1}^\infty \bigcap_{k=1}^\infty G_{k,n}$.

If $0 \notin \overline{W(A(\lambda_0))}$, then $|(A(\lambda_0)y, y)| > 1/n_0$ for all $y \in h(\lambda_0)$, $\|y\| = 1$, and n_0 large enough. Hence, $\lambda_0 \in \bigcap_{k=1}^\infty G_{k,n_0}$. Conversely, suppose that $\lambda_0 \in \bigcup_{n=1}^\infty \bigcap_{k=1}^\infty G_{n,k}$; then for some n, $|(A(\lambda_0)x_k(\lambda_0), x_k(\lambda_0)| \geq 1/n$ for all k. Now since $\{x_k(\lambda_0)\}$ is dense in the unit sphere of $h(\lambda_0)$, for $y \in h(\lambda_0)$, $\|y\| = 1$, we can find a subsequence $x_j(\lambda_0) \to y$, and therefore, if $z_j = y - x_j(\lambda_0)$,

$$|(A(\lambda_0)y, y)| \geq |(A(\lambda_0)x_j, x_j)| - |(A(\lambda_0)z_j, y) + (A(\lambda_0)x_j, z_j)|$$

and then by applying Cauchy Schwarz to the last term and letting $j \to \infty$, we get $|(A(\lambda_0)y, y)| \geq 1/n$, or $0 \notin \overline{W(A(\lambda_0))}$. This completes the proof.

The DePrima-Richard result mentioned above states that if $A \in B(h)$, $\mathrm{Re}(W(A^n)) \geq 0$ for all n (where Re denotes the real part), then A is non-negative Hermitian. We will demonstrate that this remains true of $W(A^n)$ is replaced by $W_{\mathfrak{z}}(A^n)$, and of course, $\mathrm{Re}(A) = (Z + Z^*)/2$. The following lemma, which will also be used to establish the power inequality, is all that we need to prove this generalization.

<u>Lemma 3.11</u>: Let $A = \int_\Lambda \oplus A(\lambda)d\mu \in \Phi$, and $\mathrm{Re}(\overline{W_{\mathfrak{z}}(A)}) \geq 0$. Then $\mathrm{Re}[W(A(\lambda))] \geq 0$ μ-a.e.

<u>Proof</u>: We first note that $\mathrm{Re}(Z) \geq 0$, $Z = \int_\Lambda \oplus c(\lambda)d\mu$, translates to $\mathrm{Re}(c(\lambda)) \geq 0$ μ-a.e. Second, we claim that $\mathrm{Re}[\sigma(A(\lambda))] \geq$ μ-a.e. If $\mathrm{Re}\,\alpha < 0$ and

$$\Lambda_0(\alpha) = \{\lambda \in \Lambda \,|\, A(\lambda) - \alpha \quad \text{not invertible}\}$$

then $\Lambda_0(\alpha)$ is measurable (Lemma 1.11) and we must have $\mu(\Lambda_0(\alpha)) = 0$; for if not, choose $Z = \int_\Lambda \oplus c(\lambda)d\mu \in \overline{W_{\mathfrak{z}}(A)}$ and define

$$g(\lambda) = \begin{cases} c(\lambda) & \lambda \notin \Lambda_0 \\ \alpha & \lambda \notin \Lambda_0 \end{cases}$$

By Theorem 3.7, $Z' = \int_\Lambda \oplus\, g(\lambda)d\mu \in \overline{W_{\mathfrak{z}}(A)}$, but clearly Re(Z′) does not have positive real part. Hence, $\mu(\Lambda_0(\alpha)) = 0$, and the argument used in the proof of Lemma 1.18 can now be applied, and thus, the claim is verified.

In precisely the same manner, with Lemma 3.10 used instead of Lemma 1.11, we have that $\alpha \in \overline{W(A(\lambda))}$, $\operatorname{Re}\alpha < 0$, only on a set of measure 0. Let $\{\alpha_k\}_1^\infty$ be a countable dense subset of the left half plane. By removing a set of measure zero, we can assume that $\alpha_k \notin \overline{W(A(\lambda))}$ for all k, λ. Suppose now that $\beta \in \overline{W(A(\lambda_0))}$, $\operatorname{Re}\beta < 0$; then, $\overline{W(A(\lambda_0))}$ can only be a straight line, for if it were any larger, there would have to be an interior point of $\overline{W(A(\lambda_0))}$ in the left-hand plane by the convexity of the numerical range [9], which has been ruled out by the deletion of the dense set $\{\alpha_k\}_1^\infty$. Hence, $A(\lambda_0)$ is a rotated and translated hermitian operator, and as such, the endpoints of $\overline{W(A(\lambda_0))}$ (one of which must lie in the left-hand plane) are also spectral points [9]. But, as we have seen above, this can only occur on a set of measure zero. This completes the proof.

<u>Corollary 3.12</u>: $\operatorname{Re}[\overline{W_{\mathfrak{z}}(A^n)}] \geq 0$ for all n implies that A is hermitian.

<u>Proof</u>: By the above lemma, the hypothesis holds for each A(λ) except on a μ-null set. By the DePrima-Richard result, A(λ) is hermitian, and therefore, so is A.

If $A \in B(H)$, the numerical radius of A, $\omega(A)$, is defined as $\sup|\alpha|$ where $\alpha \in W(A)$. The power inequality then states that $\omega(A^n) \leq \omega(A)^n$ [9]. For $W_{\mathfrak{z}}(A)$, we define the central radius, $\omega_{\mathfrak{z}}(A)$, in a similar fashion: $\omega_{\mathfrak{z}}(A) = \sup\|Z\|$, $Z \in W_{\mathfrak{z}}(A)$. In Lemma 3.14 we give a formula for $\omega_{\mathfrak{z}}(A)$ which will yield the power inequality as a consequence.

Lemma 3.13: $A = \int_\Lambda \oplus A(\lambda)d\mu \in \Phi$. Then $\omega(A(\lambda)) \in L_\infty(\Lambda)$.

Proof: As usual, we may assume that h is of pure dimension. It suffices to prove that

$$G_\beta = \{\lambda \,|\, \omega(A(\lambda)) > \beta\}_j, \quad \beta > 0 \quad ,$$

is a measurable set. By repeated applications of Lemma 3.4, we can find a sequence of sets $\{X_n\}_1^\infty$ satisfying the following properties:

(a) X_n are disjoint, measurable

(b) $\mu(\Lambda \setminus \bigcup_1^\infty X_n) = 0$

(c) $(A(\lambda)x(\lambda), x(\lambda))$ is a continuous function on X_n for all $x(\lambda)$.

Since $G_\beta = \bigcup_1^\infty (G \cap X_n)$, it suffices to show $G_\beta \cap X_n$ is measureable; suppose $\lambda_0 \in G_\beta \cap X_n$. Then $\omega(A(\lambda_0)) = \alpha > \beta$, and hence, there exists $x_0 \in h(\lambda_0)$, $\|x_0\| = 1$, with $|(A(\lambda_0)x_0, x_0)| > (\alpha + \beta)/2$. Then by continuity, we have $|(A(\lambda)x_0, x_0)| > \beta$ for $\lambda \in X_n$ and $|\lambda - \lambda_0| < \epsilon_0$. In other words, for each $\lambda_0 \in G_\beta \cap X_n$, there is an open set U_{λ_0} containing λ_0, such that $U_{\lambda_0} \cap X_n \subset G_\beta$. The sets $\{U_{\lambda_0}\}$ cover $G_\beta \cap X_n$; select a countable subcover $\{U_{\lambda_k}\}_1^\infty$. Then

$$G_\beta \cap X_n = \left(\bigcup_1^\infty U_{\lambda_k}\right) \cap (G_\beta \cap X_n)$$
$$= \left(\bigcup_1^\infty [U_{\lambda_k} \cap X_n]\right) \cap G_\beta$$
$$= \bigcup_1^\infty U_{\lambda_k} \cap X_n \quad .$$

Hence, $G_\beta \cap X_n$ is measurable, and the proof is complete.

Lemma 3.14: $\omega_{\mathfrak{Z}}(A) = \text{essential} \sup_\lambda \omega(A(\lambda))$.

Proof: By the above lemma, $\beta = \text{essential} \sup_\lambda \omega(A(\lambda))$ makes sense. We first show $\omega_{\mathfrak{Z}}(A) \leqslant \beta$. If $Z = \int_\Lambda \oplus\, c(\lambda) d\mu \in W_{\mathfrak{Z}}(A)$, $\|Z\| = \text{ess.} \sup_\lambda |c(\lambda)|$. But by Corollary 3.3, $c(\lambda) \in W(A(\lambda))$, and hence, $|c(\lambda)| \leqslant \omega(A(\lambda))$. Therefore $\|Z\| \leqslant \beta$, and thus $\omega_{\mathfrak{Z}}(A) \leqslant \beta$.

On the other hand, for $\epsilon > 0$ there is a set of positive measure G_ϵ such that

$$\beta \geqslant \omega(A(\lambda)) > \beta - \epsilon \qquad \lambda \in G_\epsilon \quad .$$

Hence, for $\lambda \in G_\epsilon$, there is a $d_\lambda \in W(A(\lambda))$ with $|d_\lambda| > \beta - \epsilon$; let $\Delta - \cup d_\lambda$. By multiplying A by $e^{i\theta}$ if necessary, we can insure that

$$d_\lambda \in \Delta \cap \{\alpha \mid |\alpha| \leqslant \beta,\ \mathrm{Re}\,\alpha < \epsilon - \beta\}$$

on a set of positive outer measure Γ. But this means that there exists $Z = \int_\Lambda \oplus\, c(\lambda) d\mu \in \overline{W_{\mathfrak{Z}}(A)}$ and $c(\lambda) \in \{\alpha \mid |\alpha| \leqslant \beta,\ \mathrm{Re}\,\alpha < \epsilon - \beta\}$ on a set of positive measure; for if not, then $\mathrm{Re}[W_{\mathfrak{Z}}(A + (\beta - \epsilon)1)] \geqslant 0$, and so by Lemma 3.11, $\mathrm{Re}[W(A(\lambda) + (\beta - \epsilon)1)] \geqslant 0$. But this contradicts the existence of the set Γ. Therefore, $Z \in W_{\mathfrak{Z}}(A)$ and $\|Z\| \geqslant \beta - \epsilon$, and so, $\omega_{\mathfrak{Z}}(A) \geqslant \beta$, and we are finished.

<u>Theorem 3.15</u>: $\omega_{\mathfrak{z}}(A^n) \leq \omega_{\mathfrak{z}}(A)^n$.

<u>Proof</u>:
$$\begin{aligned}
\omega_{\mathfrak{z}}(A^n) &= \text{ess.} \sup_{\lambda} \omega(A^n(\lambda)) \\
&= \text{ess.} \sup_{\lambda} \omega(A(\lambda))^n \\
&= [\text{ess.} \sup_{\lambda} \omega(A(\lambda))]^n \\
&= \omega_{\mathfrak{z}}(A)^n \quad .
\end{aligned}$$

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