# ELECTRODYNAMICS IN A STRONG MAGNETIC FIELD

Thesis by

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To the memory of my mother

Marjorie Mead Rassbach

and the inspiration she provided.

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# ABSTRACT

This thesis is a study of the electrodynamic phenomena which can occur in strong magnetic fields (on the order of $4.41 \times 10^{13}$ gauss). These phenomena are studied by means of a perturbation formalism, developed here to be exact to all orders in the field strength, and which is closely analogous to the diagrammatic formalism of empty-space electrodynamics. Using this method, the rate for pair production by single photons is calculated, as well as an approximation to it. The index of refraction of the strong field region is also calculated, as is the low-frequency photon splitting amplitude. In addition, this thesis studies some phenomena occurring at very high field strengths, in particular the energy of the ground state to first order in the fine-structure constant, and the counter-intuitive non-annihilating states. Numerical calculations are made where relevant.

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# 1. INTRODUCTION

In this thesis I analyze the consequences of conventional quantum electrodynamics in the presence of a strong magnetic field, on the order of $4 \times 10^{13}$ gauss. This work was originally stimulated by the suggestion that fields of this size may exist around pulsars, in models where the pulsar is taken to be a neutron star. Later, it became apparent that when the fields became extremely strong, there were some counter-intuitive phenomena, which were then studied for their own sake.

A number of authors(1) have developed models of neutron stars which postulate strong magnetic fields resulting from the compression of the star's original field during its collapse to nuclear density. Several electrodynamic processes, which are negligible under more normal circumstances, can occur in the postulated strong fields. Principally these are electron-positron pair production by single photons and the splitting of one photon into two or more. In addition, due to the absorption of photons by these processes, the strong-field region has an index of refraction greater than 1, reducing the photon velocity below c.

These problems have been partially analyzed previously. Toll(2) and Klepikov(3) have calculated the pair production rate by high-energy photons, and Toll has calculated the low-field index of refraction by a dispersion relation from this pair production rate. Skobov(4) has made an erroneous (not gauge-invariant) calculation of the photon splitting. Recently, Bialnicka-Berula and Bialnicki-Berula(5) and Adler, Bahcall, Callen and Rosenbluth(6) have calculated the photon splitting rate to lowest order in the external field, and the former also calculated the index of refraction to lowest order in the field. A more recent paper by Adler[7] contains exact calculations of the index of refraction and photon splitting for photons below the $e^+e^-$ production threshold.

In this thesis I develop a formalism for making strong-field calculations and evaluate the pair-production rate, index of refraction, and some very strong field effects. The pair calculation is given in Section 3 as well as two complementary approximations to the pair production rate, which are valid when many final states are kinematically allowed. That section also gives a numerical calculation and graph of the rate in a typical situation where these approximations are not good, and a numerical evaluation of an integral appearing in one of these approximations. Section 4 gives an exact calculation (except for radiative corrections) of the effects of vacuum polarization on the propagation of photons of arbitrary 4-momentum. When this is evaluated on the photon mass-shell, it gives the index of refraction of the strong-field region, which is then evaluated numerically and plotted. In Section 5, I give a summary of the work which has been done on photon splitting.

The pair-production rate has an exponential decrease in the quantity $\frac{m_e^3}{eBE}$ (where E is the photon energy and $\hbar=c=1$), and thus drops off sharply with the field. This makes it possible for photon-splitting to dominate the total photon absorption rate, even above the pair-production threshold, in spite of its smaller numerical coefficient. The range of variables where this occurs is where the splitting distance is on the order of a kilometer or so, and thus interesting astrophysically.

The only constant-field process which is currently observable experimentally is the well-known synchrotron radiation. There is extensive literature[8] on this process, both classically and quantum-mechanically, and I will not consider it, even though the methods and approximations developed here could easily be applied. Pair production and photon splitting are not quite experimentally observable, although these processes may be observable in the not-too-distant future.

Aside from the processes which may be important near neutron stars, there are some phenomena of intrinsic interest which occur in very strong fields, $B \gg 4x10^{13}$ gauss. First, due to its anomalous magnetic moment, an electron has an energy slightly <u>less</u> than $m_e$ in its ground state in a magnetic field. This has led to speculations[9] that the energy of an electron might drop to zero in a sufficiently strong field. However, the electron is sufficiently "bent" by the strong field that the energy drops and then rises as the field strength is increased. Section 6 contains a calculation and numerical evaluation of this ground-state energy.

Another interesting phenomenon in very strong fields is the existence of non-annihilating states. These are electron-positron states which overlap strongly in space (having nearly the same probability distribution), but which are absolutely forbidden from annihilating. These states, being electrically neutral, have well defined momenta perpendicular to the magnetic field. Section 6 contains an examination of these states and some of their properties.

There are a number of notations used throughout this thesis. First, the symbol ππ (pronounced "serk", short for "circle") is a generally useful abbreviation for $2\pi$: ππ $= 2\pi = 6.283185...$ This is the number which naturally arises in mathematics rather than the historically accidental $\pi$. I will use units so that $\hbar=c=1$, and will employ rationalized electromagnetic units, $e^2 = 2\pi\alpha = (=4\pi\alpha) = 2\pi/137.03...$ It is also useful to select units so that eB=1. This quantity has the dimensions of a mass squared, so that in this convention a strong field corresponds to a numerically small electron mass, and a small field to a numerically large mass. To avoid confusion with indices I will use the letter $\mu$ for the electron mass.

The space-time metric will have signature + - - -, and if $p_\nu$ is a four-vector, p standing alone will mean $p_\nu \gamma_\nu$. In many circumstances it is useful to separate the components of a vector perpendicular to the magnetic field (x and y) from those parallel to it (t and z), where the strong constant field is in the z-direction. When this is done, the letter for a vector standing alone will correspond only to the appropriate directions, i.e., $p_\parallel = p_t\gamma_t - p_z\gamma_z$ ; $p_\perp = -p_x\gamma_x - p_y\gamma_y$. $\partial_i$ will mean $\partial/\partial x_i$ ; $\partial_z = \partial/\partial z$, etc.

If we examine the simple non-relativistic spinless particle in a magnetic field we find much of the characteristic physics of the more complex Dirac particle. In this simple case we have the Hamiltonian

$$H\psi = \frac{1}{2\mu}(-i\vec{\nabla} - e\vec{A})^2 \psi \qquad (1.1)$$

where A is the 3-vector potential, satisfying $\vec{\nabla}\times\vec{A} = \vec{B} = (0,0,B)$, corresponding to a magnetic field in the z-direction. If we use an asymmetrical gauge, $\vec{A} = B\,(0,x,0)$, then the Hamiltonian becomes

$$H = \frac{1}{2\mu}(-\partial_z^2 - \partial_x^2 - \partial_y^2 + 2ieBx\partial_y + (eB)^2x^2) \qquad (1.2)$$

This operator commutes with $i\partial_z$ and $i\partial_y$, so that these will be constants of the motion. If we select eigenfunctions of the form

$$\psi(x,y,z) = \varphi(x)\ e^{+ip_y y + ip_z z}$$

then $$H\psi = \frac{1}{2\mu}(p_z^{\ 2} + p_y^{\ 2} - \partial_x^{\ 2} - 2eBp_y x + (eB)^2 x^2)\varphi(x)e^{+ip_y y+ip_z z} \qquad (1.3)$$

The equation for $\varphi$ is the familiar equation for a harmonic oscillator, so that we have

$$\varphi(x) = (eB)^{1/4} h_n(\sqrt{eB}\ x - \frac{p_y}{\sqrt{eB}})\ , \qquad (1.4)$$

$$H\psi = \frac{1}{2\mu}(p_z^{\ 2} + (2n+1)eB)\psi \qquad (1.5)$$

where $$h_n(x) = (\frac{2}{\pi})^{1/4} \frac{1}{\sqrt{n!}}(\frac{-\partial_x + x}{\sqrt{2}})^n\ e^{-x^2/2} \qquad (1.6)$$

are the normalized harmonic oscillator eigenfunctions.

The unusual feature of the solution is that the energy does not depend on the quantum number $p_y$: all states with given $p_z$ and n are degenerate. As a result of this, we may select any complete set of functions of $p_y$ as our basis set, instead of being restricted to the functions $\delta(p-p_y)$ alone, as would be the case with more ordinary Hamiltonians.

The structure of this infinite degeneracy is elucidated by working in the symmetrical gauge $\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B} = \frac{1}{2}B(-y,x,0)$. The eigenfunctions of the Hamiltonian in this gauge are $\exp(+ieBxy/2)$ times those for the gauge used above:

$$\psi = e^{+i(p_z z + p_y y - \frac{eBxy}{2})}\ (eB)^{1/4}\ h_n(\sqrt{eB}\ x - \frac{p_y}{\sqrt{eB}}) \qquad (1.7)$$

$$\frac{1}{2\mu}(i\nabla - eA)^2\ \psi = \frac{1}{2\mu}(p_z^{\ 2} + (2n+1)eB)\ \psi\ .$$

Using units where eB = 1, we have for n = 0 that

$$\psi_0 = \left(\frac{2}{\pi}\right)^{1/4} \exp\left(+ip_z z + ip_y y + \frac{ixy}{2} - \frac{(x-p_y)^2}{2}\right)$$

$$= \left(\frac{2}{\pi}\right)^{1/4} \exp\left(+ip_z z - \frac{(x+iy)(x-iy)}{4} - \frac{(x+iy-2p_y)^2}{4} + \frac{p_y^2}{2}\right). \qquad (1.8)$$

In this form it is clear that the quantum number $p_y$ affects the dependence of the wave function on x iy only: by superposing states of various $p_y$'s, one can get any arbitrary analytic function of (x+iy). A similar procedure can be carried out for harmonically excited states.

One can easily understand the harmonic motion of the particle intuitively: a classical non-relativistic particle circulating in a magnetic field moves in a circle with a period independent of the energy, while quantum-mechanically one would expect the energy to be restricted to multiples of the circulation frequency. The infinite degeneracy of the states is a result of the translational invariance of the system, with $p_y$ representing the x-position of the center of the orbit. The degeneracy is an expression of the translational invariance of the physics resulting from non-invariant equations. In fact, using the symmetrical gauge, if we wish to translate the center of coordinates by $\epsilon$ in the x-direction, $x \to x - \epsilon$, the Hamiltonian will return to its original form only if we also make the gauge transformation $\vec{A} \to \vec{A} + \frac{1}{2}eB(0,\epsilon,0)$, $\psi \to \exp(ieB\epsilon y/2)\psi$. Similarly, to move the origin in the y-direction, it is necessary to make a gauge transformation $\vec{A} \to \vec{A} + \frac{1}{2}eB(-\epsilon,0,0)$, $\psi \to \exp(-ieB\epsilon x/2)\psi$. Since these transformations change the x and y dependence of the eigenfunctions, it is clear that the states must be infinitely degenerate. In Section 2 this analysis of the eigenfunctions is made more complete and abstract.

One interesting aspect of this non-relativistic problem is that there is a simple Green's function in position space for this Hamiltonian. The Green's function is a function $G(\vec{x}, t; \vec{x}', t')$ which satisfies

$$G(\vec{x}, t; \vec{x}', t) = \delta^3(\vec{x} - \vec{x}') \qquad t = t' \tag{1.9}$$

and

$$i \frac{\partial G}{\partial t'} = H(\vec{x}', t')G(\vec{x}, t, \vec{x}', t') \qquad t' > t$$

If the eigenfunctions $\psi_\alpha$ of H are known, then

$$G(x,t,x',t') = \sum_\alpha \psi_\alpha(x')e^{i(t-t')E_\alpha} \psi^*_\alpha(x) \tag{1.10}$$

satisfies the above equations, where $\alpha$ runs over a complete orthonormal set of eigenstates of H.

Using the relation 10.13 (22) (Mehler's formula) from Erdelyi(10) one has

$$\sum_{n=0}^{\infty} z^n h_n(x)h_n(y) = \sqrt{\frac{2}{\pi}} \frac{1}{(1-z^2)^{1/2}} \exp \frac{4xyz - (x^2 + y^2)(1+z^2)}{2(1-z^2)} \tag{1.11}$$

This identity makes it possible to do the sums completely, giving the Green's function

$$G(x,t,x',t') = \frac{\sqrt{i\mu/T}}{2\pi^{3/2}\sin(T/2\mu)} \cdot$$

$$\exp\left[\frac{i}{2}(+ \frac{1}{2} \cot \frac{T}{2\mu}((x - x')^2 + (y - y')^2) + \frac{\mu}{T}(z - z')^2 + \vec{x}\,\vec{x}'\cdot\hat{k})\right]$$

$$(t' > t) \tag{1.12}$$

where $T = t'-t$ and $\hat{k}$ is a unit vector in the z direction. This Green's function and the identity leading to it can be derived more directly using the methods described in R. P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals(11). (They are actually given as problems there.) Schwinger(12) gives an integral representation for the Green's function of a Dirac electron in a magnetic field, which can be integrated to give the above (times a phase $\exp(i\mu T)$) for low field strengths, large times, and disregarding spin.

The Dirac particle in a magnetic field is quite similar to the spinless, non-relativistic particle described above. The states have the same infinite degeneracy and integer spacing as above; however, as might be expected, the uniform spacing is in $E^2$ rather than in E. The effect of spin is to displace the states one-half unit in either direction, so that $E^2 = m^2 + p_z{}^2 + 2keB$, where k is an integer. There is one state with $k = 0$, two with $k = 1$ (one in the harmonic oscillator ground state, spin against the field; the other with one harmonic oscillator excitation and spin with the field), two with $k = 2,3,4,\ldots$, etc. This has the interesting consequence, mentioned above, that due to the anomalous magnetic moment of the electron, the ground-state energy is slightly less than the electron mass.

## 2. FORMALISM

In this section I develop a formalism for working with a Dirac electron in a magnetic field. In the relativistic case, $eB/\mu^2$ is a pure number and gives a dimensionless parameter for the strength of the field: $eB/\mu^2 \ll 1$ is a small field, with different characteristics from a strong field, $eB/\mu^2 \gg 1$. The critical field strength $B_c$ ($eB_c/\mu^2 = 1$) is $4.4143 \times 10^{13}$ gauss ($4.4143 \times 10^9$ weber/m$^2$).

The formalism developed is similar to the normal perturbation formalism for electrodynamics; the principal difference is that I use the eigenstates in the magnetic field as my basis set of states. The propagator thus takes a simple form, but the vertex functions become complicated.

The addition of relativity and spin makes two changes from the non-relativistic situation: first, it is the square of the energy which has uniform spacing, rather than the energy; thus $E^2 = \mu^2 + 2eBk + p_z{}^2$ where k is an integer. Second, the magnetic moment makes an additional contribution of eB/2 to the energy, cancelling, for the lowest state, the zero-point energy of the harmonic oscillator.

The magnetic field will be taken in the -z direction: $B = (0,0,-B)$. This field can be produced by a vector potential $A_\mu = (0, \alpha y, -\beta x, 0)$, where $\alpha$ and $\beta$ are constants so that $\alpha + \beta = 1$. (The different values of $\alpha$ and $\beta$ satisfying this relation are related by a gauge transformation.) I will normally use $\alpha = \beta = \frac{1}{2}$.

The Dirac equation in this field is

$$(p_\mu - eA_\mu)\gamma_\mu \psi = \mu\psi \tag{2.1}$$

Writing

$$\pi_\mu = p_\mu - eA_\mu \tag{2.2}$$

and using

$$p_\mu = (i\partial/\partial t;\ -i\nabla), \tag{2.3}$$

we have

$$\left[\pi_\mu, \pi_\nu\right] = -ieF_{\mu\nu} . \tag{2.4}$$

If we write

$$\pi_+ = \frac{\pi_x + i\pi_y}{\sqrt{2}} \quad ; \quad \pi_- = \frac{\pi_x - i\pi_y}{\sqrt{2}} \tag{2.5}$$

we have

$$\left[\pi_+, \pi_-\right] = -eF_{xy} = -eB . \tag{2.6}$$

If we select our system of units so that $eB = 1$, (which will be done in the rest of this thesis unless otherwise stated) then

$$\left[\pi_+ , \pi_-\right] = -1 \tag{2.7}$$

which are the commutation relations of a harmonic oscillator, with $\pi_+$ the raising operator and $\pi_-$ the lowering operator. I will write

$$\gamma_+ = \frac{\gamma_x + i\gamma_y}{2} \quad ; \quad \gamma_- = \frac{\gamma_x - i\gamma_y}{2}$$

and

$$\Sigma = i\gamma_x\gamma_y = \frac{1}{2}\left[\gamma_-, \gamma_+\right] . \tag{2.8}$$

Vectors will be written in the notation

$$V = (V_t, V_z, V_+ = \frac{V_x + iV_y}{\sqrt{2}} , V_- = \frac{V_x - iV_y}{\sqrt{2}}) \tag{2.9}$$

with dot product

$$A \cdot B = A_t B_t - A_z B_z - A_+ B_- - A_- B_+ \; . \tag{2.10}$$

I will also write

$$\Sigma^- = \frac{1 - \Sigma}{2} \; ; \; \Sigma^+ = \frac{1 + \Sigma}{2} \; . \tag{2.11}$$

This gives the following useful algebraic table

| | | | |
|---|---|---|---|
| $\gamma_+\gamma_+ = 0$ | $\gamma_+\gamma_- = -2\Sigma^+$ | $\gamma_+\Sigma^+ = 0$ | $\gamma_+\Sigma^- = \gamma_+$ |
| $\gamma_-\gamma_+ = -2\Sigma^-$ | $\gamma_-\gamma_- = 0$ | $\gamma_-\Sigma^+ = \gamma_-$ | $\gamma_-\Sigma^- = 0$ |
| $\Sigma^+\gamma_+ = \gamma_+$ | $\Sigma^+\gamma_- = 0$ | $\Sigma^+\Sigma^+ = \Sigma^+$ | $\Sigma^+\Sigma^- = 0$ |
| $\Sigma^-\gamma_+ = 0$ | $\Sigma^-\gamma_- = \gamma$ | $\Sigma^-\Sigma^+ = 0$ | $\Sigma^-\Sigma^- = \Sigma^-$ |

TABLE 2.1

In terms of this notation, the Dirac equation is

$$\pi \cdot \gamma \psi = \mu \psi \tag{2.12}$$

Using this twice, we get:

$$\begin{aligned} (\pi \cdot \gamma)(\pi \cdot \gamma)\psi &= \mu^2 \psi \\ &= (\pi^2 - \Sigma)\psi \\ &= (\pi_t^{\,2} - \pi_z^{\,2} - 2\pi_+\pi_- - 2\Sigma^+)\psi \end{aligned} \tag{2.13}$$

where in this last equation the operators $\pi$ have been normal ordered.

The above equation depends on only two quantum numbers (besides spin): $\pi_z$ and the harmonic oscillator index $\pi_+\pi_-$. $\pi_z = p_z$, the momentum in the z-direction, and the harmonic oscillator index corresponds to the circulation of the electron in the field. This is as in the nonrelativistic case, where the levels are infinitely degenerate and where one of the three quantum numbers required to index a state makes no contribution to the Hamiltonian. This degeneracy is made most readily apparent on the ground state $|0\rangle$ defined by $\pi_-|0\rangle = 0$. Since $\left[\pi_-, x-iy\right] = 0$, we have that $\pi_- f(x-iy)|0\rangle = 0$, and thus the ground state is defined only up to an arbitrary analytic function of $(x-iy)$.

Using the symmetrical gauge, $eA_\mu = (0, y/2, -x/2, 0)$, the ground state functions can be written (disregarding z dependence) as

$$\psi_{a0}(x,y) = f_a(x - iy) \exp\left[-\frac{x^2}{4} - \frac{y^2}{4}\right] \tag{2.14}$$

and we have for any harmonic oscillator level,

$$\psi_{an}(x,y) = f_a(x - iy) \frac{(\pi^+)^n}{\sqrt{n!}} \exp\left[-\frac{x^2}{4} - \frac{y^2}{4}\right]. \tag{2.15}$$

There are two physically useful sets of functions $f_a(x - iy)$, corresponding to linear momentum and angular momentum representations. The easiest way to handle the angular momentum representation is to define the operators

$$\rho_x = p_x + eA_y \quad \text{and} \quad \rho_y = p_y + eA_y , \tag{2.16}$$

corresponding to the position of the orbit center in the x-y plane. These operators commute with $\pi_\pm$, and thus with the Hamiltonian, but not with each other:

$$\left[\rho_x, \rho_y\right] = i, \text{ or, } \left[\rho_+, \rho_-\right] = 1 \tag{2.17}$$

(Note that $\rho_-$ is the raising operator and $\rho_+$ the lowering operator.)

If we use the quantity

$$\rho^2 = 2\rho_-\rho_+ + 1 = \rho_x{}^2 + \rho_y{}^2 \tag{2.18}$$

as our additional quantum number, we may start from a ground state $|0\rangle$ such that $\rho_+|0\rangle = 0$. We may thus describe the (x,y) variation of the states in terms of a basis $|m,n\rangle$, where

$$\rho^2|m,n\rangle = (2n+1)|m,n\rangle \quad ; \quad 2\pi_+\pi_-|m,n\rangle = 2m|m,n\rangle$$

$$\rho_+|m,0\rangle = 0 \quad ; \quad \pi_-|0,n\rangle = 0 \tag{2.19}$$

The orbital angular momentum operator $L_z$ in this representation is

$$\begin{aligned} L_z &= p_x y - p_y x \\ &= i\left(p_+x_- - p_-x_+\right) \\ &= \pi_+\pi_- - \rho_-\rho_+ \end{aligned} \tag{2.20}$$

and
$$L_z|m,n\rangle = (m-n)|m,n\rangle \tag{2.21}$$

Thus these states are eigenstates of angular momentum as well as energy. The probability density for the ground state in this representation is concentrated in a circle with radius $\sqrt{2n+1}$ .

In the linear momentum representation, the additional quantum number is selected to be the momentum in some direction, here the x-direction. We thus wish

$$\psi_{a0}(x,y) = f(x - iy)\exp\left[-\frac{x^2}{4} - \frac{y^2}{4}\right]$$

to have x variation at y = 0 of the form $e^{ixp_x}$. This requires

$$f(p_x, x - iy) = \exp\left[\frac{(x - iy)^2}{4} + i\, p_x(x - iy) - \frac{{p_x}^2}{2}\right]$$

$$\psi_0(p_x) = \exp\left[i\, x\, p_x - \frac{(y - p_x)^2}{2} - \frac{ixy}{2}\right] \tag{2.22}$$

To quantize in a direction at an angle $\varphi$ from the x-axis we substitute $e^{i\varphi}(x - iy)$ for $(x - iy)$ in f above. The probability density of the ground state here is concentrated in lines running in the x-direction. Excited states can be formed by using the raising operator $\pi^+$, which commutes with the above quantum number $p_x$. The complete set of states in this representation are Hermite polynomials in the y-direction, centered around a point laterally displaced a distance $p_x$ from the origin, and with an x variation $e^{ix(p_x - y/2)}$. The electron in this case is harmonically oscillating in the y-direction around a point displaced by an amount $p_x$ from the origin. The term $e^{-ixy/2}$ drops out, if instead of the symmetrical gauge we use the gauge where $eA_\mu = (0, y, 0, 0)$. It is interesting that the x momentum, $p_x$, has no effect on the energy, yet, as will be shown below, must still be conserved. This leads to some surprising consequences, which will be elaborated in a later section.

As was noted in the non-relativistic case, the operator corresponding to a change in the origin of coordinates is not the normal $(p_x, p_y)$, but requires in addition a gauge transform. The operators which do this are the operators $(\rho_x, \rho_y)$.

Much of the abstract discussion of the states in this section has paralleled that of M. H. Johnson and B.A. Lippmann[13].

I will generally use the linear momentum as my additional quantum number, since the quantum number labelling the degenerate states normally has no effect, and the algebra is simpler in the linear momentum representation.

So far, we have calculated only the eigenstates of the square of the Dirac operator, with no spin effects included. If we consider any state $\chi$ satisfying

$$(\pi \cdot \gamma)^2 \chi = \mu^2 \chi$$

then the state

$$\psi = (\pi\cdot\gamma + \mu)\, \chi \tag{2.23}$$

satisfies the Dirac equation

$$(\pi\cdot\gamma - \mu)\, \psi = 0\ . \tag{2.24}$$

If we go to the original form of the Dirac equation, we have

$$\pi_t \psi = \gamma_t(\gamma_z \pi_z - \gamma_+ \pi_- + \gamma_- \pi_+ + \mu)\psi \tag{2.25}$$

which also is satisfied by the states $(\pi\cdot\gamma + \mu)\psi$.

The energy spectrum is determined by the equation

$$\pi_t{}^2 \psi = (\pi_z{}^2 + 2\pi_+\pi_- + 2\Sigma^+ + \mu^2)\psi \tag{2.26}$$

or, in more familiar variables, $E^2 = p_z{}^2 + \mu^2 + 2n + 2\Sigma^+$.
It is clear here that the operator $n + \Sigma^+$ is more significant than either of its constituents, and this excitation operator, usually called k, will normally be used. There is one state with $k = 0$, and two states with $k = 1$, $k = 2$, etc. This spectrum is plotted below in Fig.2.1.

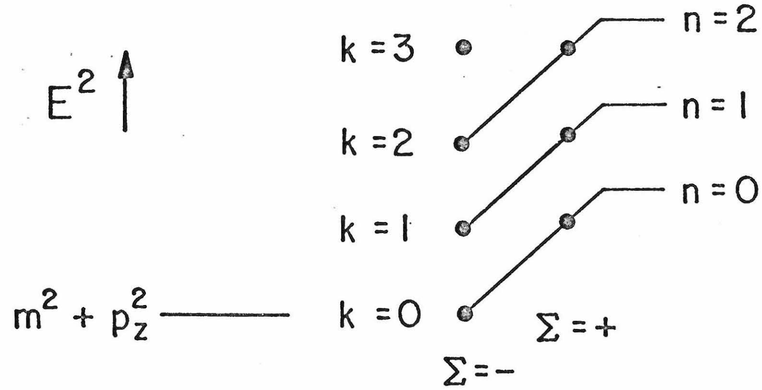



FIGURE 2.1

When there is an exact degeneracy of states, as above, it is interesting to see whether there is a symmetry operation relating them. In the case above, if we consider the operators

$$\frac{\Sigma}{2} \; ; \; \frac{1}{2}\,(\pi_+\gamma_- + \pi_-\gamma_+) = a \; ; \quad \frac{i}{2}\,(\pi_+\gamma_- - \pi_-\gamma_+) = b,$$

we get essentially the commutation relations of SU(2); (actually SO(2,1))

$$\left[\frac{\Sigma}{2}, a\right] = ib \; ; \; \left[b, \frac{\Sigma}{2}\right] = ia \; ; \left[a,b\right] = (-2k)\,\frac{i\Sigma}{2} \; .$$

In the ground state, $a|0\rangle = b|0\rangle = 0$ and $(\Sigma/2)|0\rangle = (-\frac{1}{2})|0\rangle$. The ground state is thus a singlet under this pseudo-spin, and thus in some sense has pseudo-spin 0, although displaced half a unit in $\frac{\Sigma}{2}$. In the higher states, if we work with $a/\sqrt{-2k}$ and $b/\sqrt{-2k}$, then the operators obey exactly the SU(2) commutation relations. The higher states thus have pseudo-spin $\frac{1}{2}$.

All of the above is exact only in the approximation that the absorption and emission of photons can be ignored: only the external magnetic field has been included in the vector potential. The effects due to the rest of the field are most easily handled by a perturbation formalism similar to the one conventionally used in quantum electrodynamics.

Several conventions are useful in developing a perturbation expansion. The additional degenerate quantum number will be taken as the momentum in the x-direction. By "momentum" will be meant $(p_t, p_z, p_x)$ only, and thus $d^3p/\pi^3$ and $\delta^3(p)$ will refer to these directions only. $p \cdot q$ will mean $p_t q_t - p_z q_z$. (This definition is useful because of the degeneracy in $p_x$.) The "excitation" of a state will mean the value of the operator $k = \pi^+\pi^- + \Sigma^+$ on it; a will be used for the operator $\pi^+\gamma_- + \pi^-\gamma_+$, and $a^2 = 2(\pi^+\pi^- + \Sigma^+)$, so that acting between two states a will normally reduce after the spinology calculation to $\sqrt{2k}$, where k is the excitation of the state. A particle state will be specified by the triple (p,k,r), denoting respectively the momentum, excitation and spin of the state. Finally, in writing $A_\mu$ for the vector potential, I will exclude the vector potential due to the magnetic field, i.e., $A_\mu$ refers only to the perturbation field.

The complete Dirac equation in the field then is:

$$(p + a - \mu + A)\,\psi = 0 \,. \tag{2.27}$$

It is relatively simple to develop a perturbation theory for this equation.

If we consider the external field acting in first order, then we need only the matrix element of A between the two states. If we agree to apply the projection operators $(p + a + \mu)$ on the external electrons, corresponding to the normalization $\bar{\psi}\,\psi = 2\mu$, then we need only the matrix elements of A between the states $\chi$ satisfying the squared Dirac equation, $(p^2 - 2n - 2\Sigma^+ - \mu^2)\,\chi = 0$. Since the spin and momentum of the photon can be treated independently, this matrix element $(p,k,r|A|p',\ell,s)$ between two (momentum, excitation, spin) states can be written as the sum of matrix elements with the various spin projections, between the harmonic oscillator states. For a photon with polarization $a_\nu$ and wave function $e^{iq_\mu x_\mu}$, we have

$$(p,k,r|A|p',\ell,s) = (p,k,r|ae^{iq_\mu x_\mu}|p',\ell,s)$$

$$= \pi^3\delta^3(p + q - p')e^{iq_y(p_x + q_x/2)}$$

$$\Big[(r|\Sigma^- a\Sigma^-|s)T(k,\ell,q_\perp) + (r|\Sigma^- a\Sigma^+|s)T(k,\ell-1,q_\perp)$$

$$+ (r|\Sigma^+ a\Sigma^-|s)T(k-1,\ell,q_\perp) + (r|\Sigma^+ a\Sigma^+|s)T(k-1,\ell-1,q_\perp)\Big] \tag{2.28}$$

where $$T(k,\ell,q_\perp) = \int_{-\infty}^{\infty} dy\; h_n\,(y - p_x)\; e^{iq_y y}\; h_n(y - p_x - q_x) \tag{2.29}$$

and is independent of $p_x$. The $\delta$ function shows that the 3-momenta $(p_t,p_z,p_x)$ are conserved, although all the various $p_x$ states have the same energy. Similarly, the T is the matrix element of the photon between the harmonic-oscillator part of the states. I will use the abbreviations $T^{++}$, $T^{+-}$, $T^{-+}$, $T^{--}$ for the T's with the respective arguments above. (Here k and $\ell$ are not written explicitly.) The evaluation of T is best done using creation and destruction operators (writing $q_y$ as the sum of raising and lowering operators, and translating in $p_x$ with the difference). The result is (in several equivalent forms)

$$T(k,\ell,q_\perp) = e^{-q_\perp^2/4}\sqrt{k!\,\ell!}\,\sum_r \frac{q_-^{k-r}(-q_+)^{\ell-r}}{(k-r)!(\ell-r)!r!}$$

$$= e^{-q_\perp^2/4}\sqrt{k!\,\ell!}\;e^{i(k-\ell)\varphi}(-1)^\ell\left(\frac{q_\perp^2}{2}\right)^{(k+\ell)/2}\cdot\sum_r\frac{(-q_\perp^2/2)^{-r}}{(k-r)!(\ell-r)!r!}$$

$$= e^{-q_\perp^2/4}\,q_-^{k-\ell}\sqrt{\frac{k!}{\ell!}}\;{}_1F_1(-k,k-\ell+1;\,q_\perp^2/2)$$

$$= e^{-q_\perp^2/4}\,q_-^{k-\ell}\sqrt{\frac{\ell!}{k!}}\;L_\ell^{k-\ell}(q_\perp^2/2) \qquad (2.30)$$

where $\varphi$ is the angle between the photon and the x-axis, ${}_1F_1$ is the confluent hypergeometric function, and $L_\ell^{k-\ell}$ are the Laguerre polynomials. Below, I will set $z = q_\perp^2/2$ and assume the photon is travelling in the x-direction. $T(k,\ell)$ satisfies the following identities:

$$\left[z\left(\frac{\partial}{\partial z}\right)^2 + \left(\frac{\partial}{\partial z}\right) - \frac{(k-\ell)^2}{4z} + \frac{k+\ell+1}{2} - \frac{z}{4}\right]T(k,\ell,z) = 0 \qquad (2.31)$$

$$T^{-+} = \frac{1}{2\sqrt{kz}}\left[k-\ell+2z\frac{\partial}{\partial z} + z\right]T^{++}$$

$$T^{+-} = \frac{1}{2\sqrt{\ell z}}\left[k-\ell-2z\frac{\partial}{\partial z} - z\right]T^{++} \qquad (2.32)$$

$$T^{--} = \frac{1}{2\sqrt{k\ell}}\left[k+\ell-2z\frac{\partial}{\partial z} - z\right]T^{++}$$

The expression for the matrix element can be rewritten

$$(p,k,r|A|p',\ell,s) = \pi^3\delta^3(p+q-p')\ e^{iq_x(p_x+q_y/2)}$$

$$(r|\Sigma^- aT^{++}\Sigma^- + \Sigma^- aT^{+-}\Sigma^+ + \Sigma^+ aT^{-+}\Sigma^- + \Sigma^+ aT^{--}\Sigma^+|s) \qquad (2.33)$$

This matrix element is for the squared Dirac equation solutions only; to work with actual particles, one must use the projection operators $(p + a + \mu)$ on the external lines.

For processes which operate in higher order in the photon field, we may put the Dirac equation into iterative form, $\psi = (p + a - \mu)^{-1} eA\psi$. It seems simplest to work with the harmonic oscillator states as the basis for $\psi$, in which case $(p + a - \mu)^{-1}$ is diagonal (except for spin). Then we have

$$\psi = \frac{p + a + \mu}{p^2 - 2k - \mu^2}\, eA\psi \qquad (2.34)$$

This new $\psi$ is of higher order in eA, so that substituting it back into the perturbation series will give us higher order terms. We can write the theory in terms of diagrams, like empty-space electrodynamics, using a propagator $(p + a + \mu)/(p^2 - 2k - \mu^2)$, a vertex as in equation (2.33) above, and $(p + a + \mu)$ on each external electron line. The relation of the relativistic matrix elements to the rates, cross-sections, etc., requires the normal (1/2E) and 1/2 from spin averaging; and the set of diagrams used (including the signs due to statistics) are the same as in empty-space electrodynamics. Positrons are represented, as usual, with negative momenta; however, they have positive excitations.

Schwinger[12] also has developed a formalism treating electrons in a constant field. The principle difference between the formalism he uses and that used here is that he uses a position basis for the wave functions, rather than using space-time eigenstates. The effect of this is that he has relatively simple vertices, but a more complicated propagator (involving irreducible integrals over proper time). Stated briefly, he puts the physics of the constant field into the propagator, whereas the formalism above puts it into the vertices.

Schwinger's formalism, when the various equations are put together, seems to me to be slightly more complicated than that used here. It has been applied to the problem of photon splitting in a weak field by Skobov[4] ; incorrectly, since his results are not gauge invariant, and disagree with results obtained by conventional electrodynamics. (Skobov's results are quoted by Erber[14]) . Schwinger's formalism seems to me to be considerably more abstract, since the properties of the electron motion are never apparent. For these two reasons, I decided at the beginning of this research to develop an independent formalism. It is not clear now whether this decision minimized the labor involved in calculations, at least for closed electron-loop processes, since a number of transformations used on the vertex functions in this case essentially put the system into proper-time notation.

Schwinger's formalism has wider validity than that used here, since it has no requirement that $E \cdot B = 0$. Adler[7] has recently obtained identical results for a few of the problems considered here, using the Schwinger formalism.

Klepikov[3] has developed a formalism for electrodynamics in a constant field which has some similarities to that used here. His method, however, is based on Hamiltonian rather than space-time methods, is only developed enough to work in first order, and is not very concise. He has applied it to synchrotron radiation, 1-photon pair annihilation, and 1-photon pair production in the region $E \gg \mu$.

Toll[2] has done some work in this area, for fields with $E^2 - B^2 = E \cdot B = 0$. He was interested in the dispersion relations for light in this situation, so he calculated the pair-production rate and index of refraction. He has also extended his results for pair production by physical reasoning to a pure magnetic field in the region $E \gg \mu$.

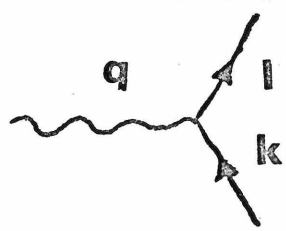
Somewhat more detailed discussions of parallel work are given in the appropriate sections.

## 3. PAIR PRODUCTION

In this section I calculate the rate for production of electron-positron pairs by single photons in a strong magnetic field ($\gamma \underset{\vec{B}}{\rightarrow} e^+e^-$). For the region where only a few electron states are kinematically allowed, $q_\perp^2 - 4\mu^2 \sim 1$, the calculations can usually be done exactly. For the more useful region $q_\perp^2 - 4\mu^2 \gg 1$, it is possible to approximate the vertex functions by a saddle-point method. For z, k, and $\ell$ uniformly large, the expression

$$q_-^{k-\ell}\sqrt{\frac{k!}{\ell!}}\,\frac{1}{(k-\ell)!}\;{}_1F_1\left(-\ell,\ k-\ell+1;\ z = -\frac{q_\perp^2}{2}\right)e^{-z}$$

for the vertex




may be rewritten using the integral representation

$$ {}_1F_1(-\ell, k-\ell+1,\ z) = \frac{(-)^\ell}{i}\,\frac{\ell!(k-\ell)!}{k!}\int_C \exp\left[zt + k\ln(1-t) - \ell\ln t\right]\frac{dt}{t} \tag{3.1}$$

where the contour C circles the origin in the positive direction.

The argument of the exponential has its maximum at

$$ t = t_0 \equiv \frac{1 - \left(\frac{k-\ell}{z}\right) \pm \sqrt{1 - 2\left(\frac{k+\ell}{z}\right) + \left(\frac{k-\ell}{z}\right)^2}}{2} \tag{3.2}$$

Useful variables here are s and d for the sum and difference of the harmonic oscillator energies, and R for the square root above:

$$ s = \frac{k+\ell}{z}\ ;\quad d = \frac{k-\ell}{z}\ ;\quad R = \sqrt{1 - 2s + d^2}\ . \tag{3.3}$$

This gives

$$t_o = \frac{1-d\pm R}{2} \quad . \tag{3.4}$$

The variable R is also useful kinematically: for external particles,

$$R = \sqrt{\frac{p_z{}^2+\mu^2}{z}}$$

The relative sizes of d and R, and the kinematic constraint, $R^2 > 4\mu^2/q_\perp{}^2$, require that the positive root be used for normal processes, such as $e^\pm \to e^\pm + \gamma$, and the negative root be used for the pair process being studied here, $(\gamma \to e^+e^-)$. Below I will use $t_o = (1-d-R)/2$; however, these results can be used for normal processes by substituting $R \to -R$.

Letting $t = t_o e^\tau$, the argument of the exponential becomes

$$zt + k\ln(1-t) - \ell\ln t = z(t_o + \frac{s+d}{2}\ln(1-t_o) - \frac{s-d}{2}\ln t_o)$$
$$+ \frac{za\,\tau^3}{2} + \frac{zb\,\tau^3}{6} + \frac{zc\,\tau^4}{24} + \ldots$$

where

$$a = t - \frac{s+d}{2}\,\alpha(1+\alpha) = R\alpha$$

$$b = t - \frac{s+d}{2}\,\alpha(1+\alpha)(1+2\alpha) = R\alpha - \alpha^2(1+d-R) \tag{3.5}$$

$$\alpha = \frac{t_o}{1-t_o} = \frac{1-d-R}{1+d+R}$$

The integration contour C runs t through 0 in the positive imaginary direction, or, in the saddle-point approximation, from $-i\infty$ to $i\infty$ . In such an approximation, the term in $\tau^3$ is normally ignored. The neglect of this term, however, requires that $(\frac{za}{2})^{1/2} \gg (\frac{zb}{6})^{1/3}$; this condition is not satisfied if R is small. If the approximation is to be generally valid, we must, therefore, include the $\tau^3$ term as well. This term will always dominate the $\tau^4$ term which will, therefore, be neglected. We may thus write

$$ {}_1F_1\,(-\ell,k-\ell+1,z) \approx \frac{(-1)^{\ell}}{i}\,\frac{\ell!\,(k-\ell)!}{k!}\,\exp\left[zt_o + k\ln(1-t_o) - \ell\ln t_o\right] \cdot $$

$$ \cdot \int_{-i\infty}^{i\infty} d\tau\,\exp\left[-\frac{az\tau^2}{2} - \frac{zb\tau^3}{6}\right] \qquad (3.6) $$

The above integral in $\tau$ can be calculated by displacing the origin to eliminate the $\tau^2$ term, resulting in an Airy integral, finally giving,

$$ \int_{-i\infty}^{i\infty} d\tau\, e^{-za\tau^2/2 - zb\tau^3/6} = \frac{2ia}{\sqrt{3}b}\,\exp\left[\frac{za^3}{3b^2}\right]K_{1/3}\left[\frac{za^3}{3b^2}\right] \qquad (3.7) $$

Altogether,

$$ {}_1F_1\,(-\ell,k+\ell-1;z) \approx \frac{(-1)^{\ell}\ell!\,(k-\ell)!}{k!}\,\exp\left[zt_o + \frac{s+d}{2}\ln(1-t_o) - \frac{s-d}{2}\ln t_o\right] $$

$$ \left\{\frac{2R\alpha}{\sqrt{3}\pi b}\,\exp\left[\frac{R^3\alpha^3}{3b^2}\right]K_{1/3}\left[\frac{zR^3\alpha^3}{3b^2}\right]\right\} \qquad (3.8) $$

This calculation is quite similar to that of Klepikov[3], who in addition gives a bound of order $\ell^{-2/3}$ to the error in the curly bracket term.

Using Stirling's approximation for the factorials, we get the final result for the vertex function,

$$T(k,\ell,z) \approx \frac{(-1)^{\ell} 2R\alpha}{\sqrt{3}\,\pi b} \left(\frac{\ell}{k}\right)^{1/4} (\exp\cdot K_{1/3})\left(\frac{zR^3\alpha^3}{3b^2}\right) e^{zX/2}$$

$$X = -R + \frac{s}{2}\ln\left[\frac{1-s+R}{1-s-R}\right] + \frac{d}{2}\ln\left[\frac{(1-d-R)(1+d+R)}{(1-d+R)(1+d-R)}\right] \tag{3.9}$$

$$b = R\alpha - \alpha^2(1+d-R)$$

$$\alpha = \frac{1-d-R}{1+d+R}$$

Since $R > 2\mu/q_{\perp}$, (as will be shown in the kinematic calculation), if $4\mu^3 \gg q_{\perp}$, the $K_{1/3}$ function has a large argument, permitting an asymptotic expansion. This yields

$$T^{++} \equiv T\ (k,\ell,z) = \frac{(-1)^{\ell}}{(\pi Rz)^{1/2}} \left(\frac{1-s+R}{1-s-R}\right)^{1/4} \exp\left(\frac{zX}{2}\right) \tag{3.10}$$

The derivative relations 2.32 and the relations

$$(1+d\pm R)(1-d\pm R) = 2(1-s\pm R)$$
$$(1\pm d-R)(1\pm d+R) = 2(s\pm d) \tag{3.11}$$
$$(1-s+R)(1-s-R) = s^2-d^2$$

give symmetrical forms for this second approximation:

$$T^{++} = \frac{(-)^{\ell}}{\sqrt{\pi Rz}} \left[\frac{(1+d+R)(1-d+R)}{(1+d-R)(1-d-R)}\right]^{1/4} \exp\left(\frac{zX}{2}\right)$$

$$T^{-+} = \frac{(-)^{\ell}}{\sqrt{\pi Rz}} \left[\frac{(1+d-R)(1-d+R)}{(1+d+R)(1-d-R)}\right]^{1/4} \exp\left(\frac{zX}{2}\right)$$

$$T^{+-} = \frac{(-)^{\ell}}{\sqrt{\pi Rz}} \left[\frac{(1+d+R)(1-d-R)}{(1+d-R)(1-d+R)}\right]^{1/4} \exp\left(\frac{zX}{2}\right) \tag{3.12}$$

$$T^{--} = \frac{(-)^{\ell}}{\sqrt{\pi Rz}} \left[\frac{(1+d-R)(1-d-R)}{(1+d+R)(1-d+R)}\right]^{1/4} \exp\left(\frac{zX}{2}\right)$$

The phase space for the process $\gamma \rightarrow e^+e^-$, with the diagram

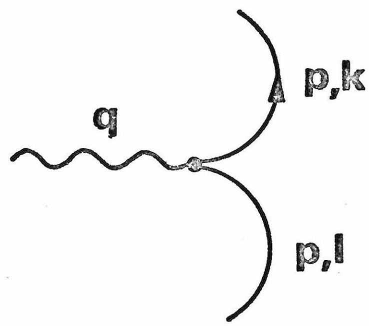



FIGURE 3.1

is $$\int \frac{d^2p}{\pi^2}\,\frac{d^2p'}{\pi^2}\;\pi\delta(p^2-2k-\mu^2)\;\pi\delta(p'^2-2\ell-\mu^2)\;\pi^2\delta^2(p-p'-q_{\parallel})$$

$$= 2\cdot\frac{1}{4}\cdot\left(\frac{q_{\parallel}^{2}}{4} - (k+\ell+\mu^2)q_{\parallel}^{2} + (k-\ell)^2\right)^{-1/2} \qquad (3.13)$$

where $q_{\parallel}$ is the $(t,z)$ portion of the photon momentum. The factor of 2 is necessary since either particle may have the positive momentum $p_z$. If the photon is on the empty-space mass-shell (which will be assumed from now on), then $q_{\parallel} = q_{\perp}$ and the phase space becomes

$$\int \frac{d^2p}{\pi^2}\,\frac{d^2p'}{\pi^2}\;\pi\delta(p^2-2k-\mu^2)\;\pi\delta(p'^2-2\ell-\mu^2)\;\pi^2\delta(p-p'-q_{\parallel})$$

$$= \frac{1}{2z(R^2-r^2)^{1/2}} = \frac{1}{2z}\left(R^2-\frac{2\mu^2}{z}\right)^{-1/2}, \qquad (3.14)$$

where I have defined $r = \frac{2\mu}{q_{\perp}}$ as the electron mass in terms of the photon's perpendicular momentum; $r = 1$ is threshold.

Since R depends on discrete final state variables k and $\ell$ as the photon energy increases across the threshold for a particular $(k,\ell)$

state, there is a square-root infinity in the rate. Physically, this is because just above threshold for a given state, the energy depends on the square of the z-momentum: a small energy interval can thus correspond to a large momentum interval. The area under each of these peaks is finite, however, so if we are averaging over a range of B or of q, we may consider k and $\ell$ to be continuous variables.

If plotted accurately, the total rate has many of these square-root infinities as the photon energy crosses various $(k,\ell)$ thresholds. The total number of edges below a given energy E is approximately

$$N(E) = \int_0^\infty dk \int_0^\infty d\ell \left[\frac{E^4}{4} - (k+\ell+\mu^2)E^2 + (k-\ell)^2 > 0\right]$$

$$= \frac{E^4}{24}\left[1 - 3\left(\frac{2\mu}{E}\right)^2 + 2\left(\frac{2\mu}{E}\right)^3\right], \qquad (3.15)$$

giving a density

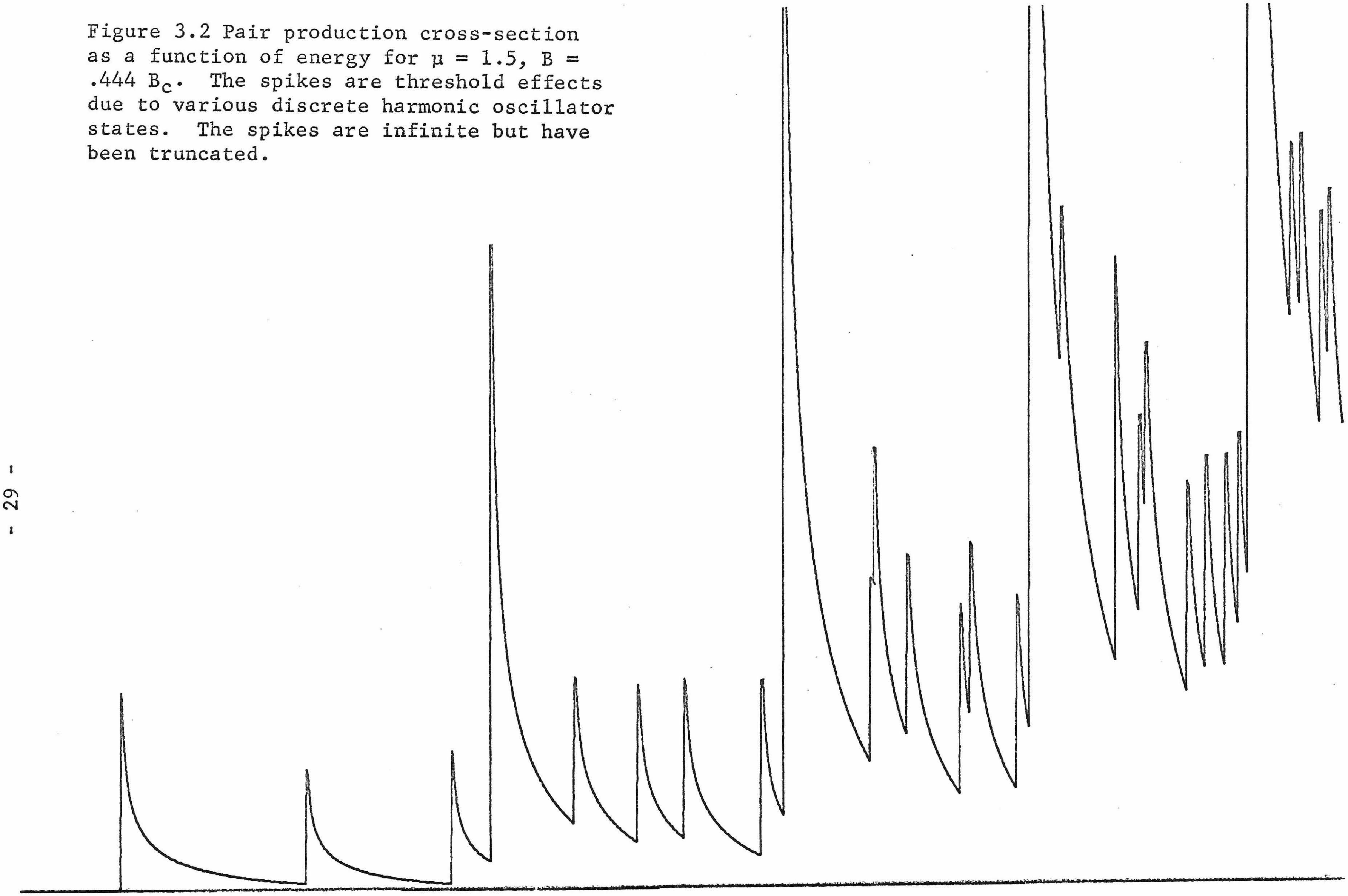
$$dN = \frac{1}{12}\left[2E^3 - 3E(2\mu)^2 + (2\mu)^3\right] dE \,. \qquad (3.16)$$

If $E^2 - 4\mu^2 \gg 1$, then the edges are very close together, and we can assume without much error that the states are continuous. This is particularly true if there is variation in the magnetic field or uncertain photon energies.

The exact pair production rate, as a function of energy, is shown in Fig. 3.2 for the case $\mu = 1.5$ ($B = 1.962\mathrm{x}10^{13}$ gauss = .4444 $B_c$).

A problem arises when we attempt to calculate the rate from the matrix element. Normally an S-matrix element has a $\pi^4\delta^4(p_{in}-p_{out})$ which is removed before squaring. In this case, however, this factor is

Figure 3.2 Pair production cross-section as a function of energy for $\mu = 1.5$, $B = .444\ B_c$. The spikes are threshold effects due to various discrete harmonic oscillator states. The spikes are infinite but have been truncated.



$$e^{iq_y(p_x+q_x/2)} \pi^3 \delta^3(p-p'-q)$$

instead. We cannot simply drop the extra term, since then the integral over the degenerate momentum variable $p_x$ will give $\infty$. This is not a real infinity because the variable $p_x$ also represents the position of the center of the orbit in the y-coordinate, and we are assuming the electron to be in a box of finite size, which cancels with the normalization of the state. We may use the relation between the pair rate and the imaginary forward scattering amplitude to remove the difficulty in the following manner. In the normal $\pi^4\delta^4(p_{in}-p_{out})$ case, we may alternatively use different variables for the momenta in the matrix element and in its conjugate. Integrating over intermediate states will reduce the pair of delta functions to a single one, which is then dropped to get the rate. In the situation above, if we again substitute different momenta and integrate, the exponential gives us the necessary fourth delta function, so that we may drop a factor of $\pi^4\delta^4(q_{in}-q'_{in})$ to get the actual rate.

The rate for the process $\gamma \to e^+e^-$, summed over the outoing pair spins, for an initial photon with polarization e and 4-momentum $q_\mu, q^2=0$, is then

$$\mathrm{Rate}(\gamma \to e^+e^-) = \frac{-e^2}{2q_t}\int\frac{d^3p}{\pi 3}\,\frac{d^3p'}{\pi 3}\,\pi\delta(p^2-2k-\mu^2)\pi\delta(p'^2-2\ell-\mu^2)\,\cdot$$

$$\pi^3\delta^3(p-p'+q)\pi^3\delta^3(p-p'-a')\; e^{iq_y(p_x+q_x/2)}e^{-iq'_y(p_x+a_x'/2)}\,\cdot$$

$$\sum_{k,\ell} \mathrm{Tr}\langle(p+a+\mu)(\Sigma^-T^{++}e\Sigma^- + \Sigma^-T^{+-}e\Sigma^+ + \Sigma^+T^{-+}e\Sigma^- + \Sigma^+T^{--}e\Sigma^+)\,\cdot$$

$$(p'+a+\mu)(\Sigma^-T^{++*}e\Sigma^- + \Sigma^-T^{+-*}e\Sigma^+ + \Sigma^+T^{-+*}e\Sigma^- + \Sigma^+T^{--*}e\Sigma^+)\rangle \qquad (3.17)$$

Calculating the integrals (but not the sum) using the above phase-space calculation, and dropping the delta functions as discussed above, gives

$$\text{Rate} = \sum_{k\ell} \frac{\alpha}{2q_t z (R^2-r^2)^{1/2}} \text{Tr}\langle \text{ etc.} \rangle \tag{3.18}$$

where the integral over the degenerate quantum number in the final state has already been done.

In calculating the total rate, it is also necessary to sum over the final k and $\ell$. In the same approximation used for the vertex functions, one can convert this sum to an integral, with the result (using R as the integration variable rather than s)

$$\sum_{k,\ell=0}^{\infty} f(k+\ell, k-\ell) = z^2 \int_r^1 R\,dR \int_0^{1-R} dd\; f(\tfrac{z}{2}(1+d^2-p^2), zd) \tag{3.19}$$

for any f.

All that remains to be done in the rate calculation is the spinology of the trace and the evaluation of the integrals. The complex angular factor $e^{i\varphi(k-\ell)}$ drops out on absolute squaring, so I will take $\varphi = 0$ (photon momentum in the x-direction). This gives for the trace

$$\langle e_z^2 \rangle = \text{Trace for z polarization} =$$

$$\text{Tr}\langle (p+a+\mu)(\Sigma^- T^{++} e\Sigma^- + \Sigma^- T^{+-} e\Sigma^+ + \Sigma^+ T^{-+} e\Sigma^- + \Sigma^+ T^{--} e\Sigma^+)$$

$$(p'+a+\mu)(\Sigma^- T^{++*} e\Sigma^- + \Sigma^- T^{+-*} e\Sigma^+ + \Sigma^+ T^{-+*} e\Sigma^- + \Sigma^+ T^{--*} e\Sigma^+)\rangle$$

$$= \frac{1}{2}(T^{++2} + T^{--2})\text{Tr}\langle (p+\mu)\gamma_z(p'+\mu)\gamma_z\rangle + 2\sqrt{k\ell}\, T^{++}T^{--}\text{Tr}\langle \gamma^+\gamma_z\gamma^-\gamma_z\rangle$$

$$\langle e_z^2 \rangle = 4\left\{\frac{1}{2}(T^{++2} + T^{--2})(p\cdot p' + 2p_z p'_z - \mu^2) - 2\sqrt{k\ell}\, T^{++}T^{--}\right\} \tag{3.20}$$

Similarly,

$$\langle e_y{}^2 \rangle = 4\left\{\frac{1}{2}(T^{+-2} + T^{-+2})(p\cdot p'-\mu^2) - 2\sqrt{k\ell}\, T^{+-}\, T^{-+}\right\}$$

The following kinematic relations are helpful in using the above:

$$p\cdot p' = -z(1-s-\mu^2/z)$$

$$p\cdot p' + 2p_z p'_z = -z(s-d^2+\mu^2/z)$$

$$\sqrt{k\ell} = \frac{z}{2}(s^2-d^2) \tag{3.21}$$

$$p_z p'_z = z\left[\frac{1-2s+d^2}{2} - \frac{4\mu^2}{q_\perp{}^2}\right] = z(R^2-r^2)$$

Assembling all these factors and dropping the photon momentum $\delta$ function, we have

$$\text{Rate (z pol.)} = \frac{-2\alpha z}{q_t}\int_r^1 \frac{RdR}{(R^2-r^2)^{1/2}}\int_0^{1-R} dd$$

$$\left\{\frac{1}{2}(T^{++2} + T^{--2})(-\frac{z}{2})(1-d^2-R^2+r^2) - z(s^2-d^2)T^{++}T^{--}\right\}$$

$$\text{Rate (y pol.)} = \frac{-2\alpha z}{q_t}\int_r^1 \frac{RdR}{(R^2-r^2)^{1/2}}\int_0^{1-R} dd$$

$$\left\{\frac{1}{2}(T^{+-2} + T^{-+2})(-\frac{z}{2})(1-d^2-R^2) - z(s^2-d^2)T^{+-}\, T^{-+}\right\} \tag{3.22}$$

where $s = (1+d^2-R^2)/2$, and the T's are as defined in section 2, using $k = z(s+d)/2$, $\ell = z(s-d)/2$.

If we attempt to use the vertex function approximation directly, the formulae become quite complicated. It is possible, however, to develop two overlapping approximations which cover the entire region of validity of the original vertex approximation. These two limits are $q_\perp^2 \sim 4\mu^2$ ($r \sim 1$) and $q_\parallel \gg \mu^2$ ($r \ll 1$).

(1) $q_\perp^2 \sim 4\mu^2$, ($r \sim 1$)

We still assume $q_\perp^2 - 4\mu^2 = q_\parallel^2 - 4\mu^2 \gg 1$.

Since $z$ and $\mu^2$ are both large and $r \sim 1$, $zr^3 = 4\mu^3/q_\perp$ and the argument of the Bessel function is everywhere large. We may thus use the second vertex approximation, Eq. 3.12. The result is

$$\text{Rate}\binom{z}{y} = \frac{2\alpha z}{q_t} \int_r^1 d(R^2-r^2)^{1/2} \int_0^{1-R} dd \begin{Bmatrix} sR^2+(1-s)r^2 \\ sR^2 \end{Bmatrix} \frac{\exp(zX)}{\pi R(s^2-d^2)^{1/2}} , \tag{3.23}$$

where $s = (1+d^2-R^2)/2$, and

$$X = \left(-r + \frac{1-r^2}{2} \ln \frac{1+r}{1-r}\right)\left(1 - \frac{q^2}{2} \cdot \frac{2}{1-r^2}\right) - (R-r)r \ln \left(\frac{1+r}{1-r}\right) .$$

On integration, this becomes

$$\text{Rate}\ \binom{z}{y} = \frac{\alpha r}{2q_t} \frac{\exp\left\{-z\left[r-\frac{1-r^2}{2}\ln\frac{1+r}{1-r}\right]\right\}}{\left(\ln\frac{1+r}{1-r}\right)^{1/2}\left(r-\frac{1-r^2}{2}\ln\frac{1+r}{1-r}\right)^{1/2}} \begin{pmatrix} \left(\frac{1-r^2}{2}\right)^{1/2} \\ \left(\frac{1-r^2}{2}\right)^{1/2} \end{pmatrix} \tag{3.24}$$

If $r$ is small, this simplifies to

$$\text{Rate} \begin{pmatrix} z \\ y \\ \text{unpol.} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3/2 \end{pmatrix} \frac{\sqrt{3}\,\alpha}{8\sqrt{2}\mu} e^{-2zr^3/3} \tag{3.25}$$

where $\dfrac{2zr^3}{3} = \dfrac{8\mu^3}{3q_\perp(eB)}$ .

(2) $q_\parallel \gg \mu^2$ , ($r \ll 1$).

If we expand the argument of the $K_{1/3}$ function in Eq. 3.9, for small R, we see that it becomes $zR^3/3(1-d^2)$. Since z is large, and $K_{1/3}$ drops exponentially, the integral will be dominated by the region where $z(R^3-r^3)$ is small. Since r is small, R must be also. It can be shown that the argument of the $K_{1/3}$ always increases with R, so there is no additional region of importance. Expanding for R small, we get:

$$T^{++} = \frac{(-)^{\ell}2}{\sqrt{3}\,\pi} \frac{R}{(1-d^2)^{1/2}} K_{1/3}\left[\frac{zR^3}{3(1-d^2)}\right]$$

$$\frac{\partial}{\partial z} T^{++} = \frac{(-)^{\ell+1}2}{\sqrt{3}\,\pi} \frac{R^2}{(1-d^2)^{1/2}} K_{2/3}\left[\frac{zR^3}{3(1-d^2)}\right] \tag{3.26}$$

$$\text{Rate}(z) = -\frac{8\alpha z^2 r}{3\pi^2 q_t} \int_{R=r}^{1} d(R^2-r^2)^{1/2} \int_{0}^{1} dd \cdot$$

$$\left(\frac{R^2}{1-d^2}\right)^2 \left[K_{2/3}^2 + ((1-d^2)\frac{r^2}{R^2} + d^2)\, K_{1/3}^2\right] \tag{3.27}$$

$$\text{Rate}(y) = -\frac{8\alpha z^2 r}{3\pi^2 q_t} \int_{R=r}^{1} d(R^2-r^2)^{1/2} \int_{0}^{1} dd \cdot$$

$$\left(\frac{R^2}{1-d^2}\right)^2 \left[K_{1/3}^2 + d^2 K_{2/3}^2\right],$$

where the K's have argument $-zR^3/3(1-d^2)$. These integrals can be made less singular by substituting $R=r\cosh x$ and $d = \tanh y$. In this form it becomes clear that the result depends only on $zr^3 = 4\mu^3/eBq_\perp$ and is thus (except for an overall factor of $1/\mu$) independent of the value of photon energy giving the particular value of $zr^3$. When the above substitution is made and the result averaged over initial polarizations, this is the form of the result given by Klepikov(3).

It is possible, by expanding the K's in Airy integrals and symmetrizing the variables, to reduce the above expressions to single integrals:

$$\text{Rate}\binom{z}{y} = \frac{\alpha z r^3}{2\sqrt{3}\pi\mu} \int_0^\infty dy \left[ \begin{pmatrix} 4\text{ch}^2 y - 1 \\ 4\text{ch}^2 y - 3 \end{pmatrix} \text{ch}^{-2} y \;\; K_{2/3} \left( \frac{2zr^3}{3} \text{ch}^2 y \right) + \right.$$

$$\left. + \frac{4zr^3}{3} \text{sh}^2 y \; K_{1/3} \left( \frac{2zr^3}{3} \text{ch}^2 y \right) \right] \tag{3.28}$$

Where $zr^3 = 4\mu^3/q_\perp = 4(B_c/B)(\mu/q_\perp)$. This result agrees with that of Toll[(2)], if one expresses his Airy functions in terms of Bessel functions and does the necessary algebra.

If $zr^3$ is small, we have

$$\text{Rate}\begin{pmatrix} z \\ y \\ \text{unpol.} \end{pmatrix} \approx \frac{\alpha \left(\frac{4zr^3}{3}\right)^{1/3}}{14\mu} \; \frac{\Gamma(5/6)}{\Gamma(7/6)} \begin{pmatrix} 3 \\ 2 \\ 5/2 \end{pmatrix}, \tag{3.29}$$

and in the limit $r \ll 1$, $zr^3 \gg 1$, the region of validity of this approximation overlaps that of the previous approximation, and gives the same rate. In the high-energy limit, if we shift from the large $zr^3$ approximation (Eq. 3.25) to the low $zr^3$ approximation (Eq. 3.29) at a value of $zr^3$ of .29 for y polarization or .47 for z polarization, then the approximations differ by only 35 and 39 percent respectively from the exact values.

If we use the expressions

$$\text{Rate}\binom{z}{y} = \frac{\alpha}{\mu} \left\{ \left[ \frac{\Gamma(5/6)}{14\Gamma(7/6)} \left(\frac{4zr^3}{3}\right)^{1/3} \binom{3}{2} \right]^{-2} + \frac{32}{3} e^{+4zr^3/3} \binom{1}{4} \right\}^{-1/2} \tag{3.30}$$

for the rate, then over the entire range the result is in error by at most 13 percent for y polarization and 20 percent for z polarization. Fig.3.3 shows the rate approximations, Equations 3.28 and 3.30.

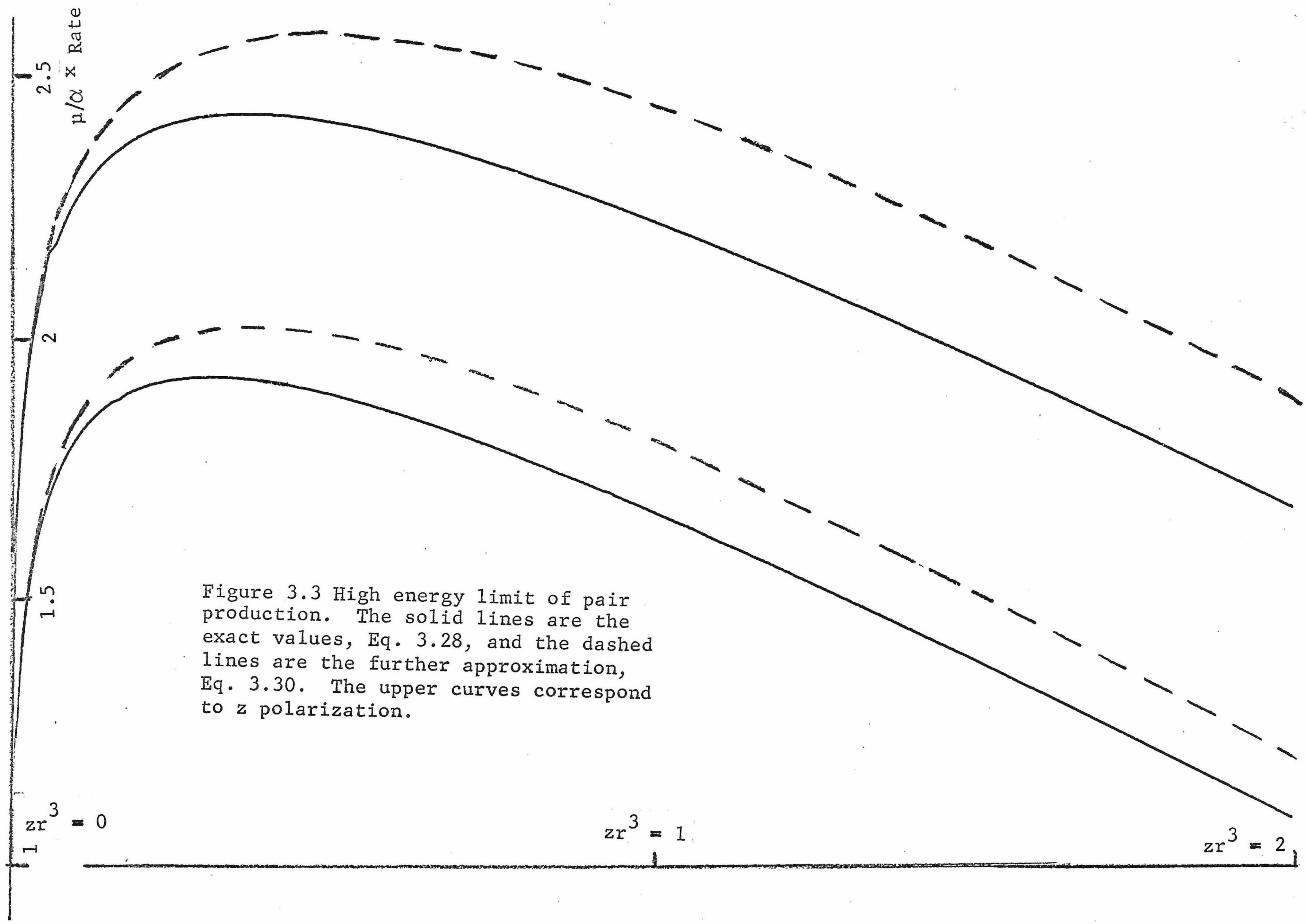



Figure 3.3 High energy limit of pair production. The solid lines are the exact values, Eq. 3.28, and the dashed lines are the further approximation, Eq. 3.30. The upper curves correspond to z polarization.

Using these two approximations allows us to calculate the pair production rate anywhere that $q_\perp^2 - 4\mu^2 \gg 1$. If we allow $q_\perp^2 - 4\mu^2 \sim 1$, then there are only a few states available, and it is necessary to use exact vertex functions and direct summation over final states. Fig. 3.2 above was calculated by this method.

The results calculated above can be extended to the case where the electric field is not zero, so long as $E \cdot B = 0$. In fact, it is possible to extend the result even to the case $E^2 - B^2 = 0$ and at the same time understand the dependence of the limit $E \gg 2\mu$ only on $zr^3 = 4\mu^3/q_\perp(eB)$.

The only relativistic invariants in this case are $F^2 = F_{\mu\nu}F_{\mu\nu}$ and $|q_\mu F_{\mu\nu}|^2$ . Now if $F^2\mu^2 \ll |q_\mu F_{\mu\nu}|^2$, then there is a Lorentz frame where the photon energy is high (compared to $\mu$), but where $B^2 \approx E^2$. The rate in this frame should not depend on slight fluctuations in E or in B, which correspond to large proportional changes in $E^2 - B^2 = F^2$; and thus the rate should depend strongly only on $|q_\mu F_{\mu\nu}| = (q_\perp B)$. If we now select the Lorentz frame where $F_{\mu\nu}$ is pure magnetic, the photon energy will be very high. Thus the high-energy limit will depend only on $q_\perp B$. If $E^2 - B^2 = 0$, then this limit should be exact, if we use $|eq_\mu F_{\mu\nu}|$ where Eq. 3.28 has $q_\perp$. If $E_y = B_z = B'$, then the argument of the exponential is $2\mu^3/q_x B'$, where in the pure magnetic case it was $4\mu^3/q_x B$.

Toll[(2)] has used essentially the above argument to get the high-energy limit of pair production in a pure magnetic field from his rates for the case $E^2 - B^2 = E \cdot B = 0$.

It is interesting in some circumstances to know the distribution of the pairs in $p_z$, the momentum along the field. According to Eq. 3.21,

$$p_z p_z' = \frac{z}{2}(R^2 - r^2) = \frac{zR^2}{2} - \mu^2 .$$

If the photon momentum has no z-component, then $p_z = p'_z$ (corresponding to opposite physical momenta by the convention for the positron momentum) and $R^2 = 2(p^2 + \mu^2)/z$; $RdR = 2pdp/z$. Making these substitutions and integrating over d will give the z-momentum distribution. In the simplest case, $E \gg 2\mu$ and $zr^3 = 4\mu^3/q_\perp \gg 1$, the distribution becomes approximately

$$\text{Rate}\ (p_z, (^z_y)) = (^2_1)\ \frac{\sqrt{3}\alpha}{8\sqrt{2}\mu}\left(\frac{q_\perp}{8\pi\mu}\right)^{1/2} e^{-\frac{2zr^3}{3}(1 + \frac{3p_z^2}{\mu^2})}$$

$$= (^2_1)\frac{\alpha\sqrt{3}\ e^{-2zr^3/3}}{8\sqrt{2}\ \mu}\left(\frac{q_\perp}{8\pi\mu}\right)^{1/2} e^{-(\frac{8\mu}{q_\perp}p_z^2)} \qquad (3.31)$$

It can be seen from this that the z-momentum distribution is very narrow, with the pair having much less than $\mu^2$ average z-momentum. This can be most easily seen from the strong dependence of the exponential on the mass, and from the intuitive fact that the z-momentum acts like additional mass, so far as the magnetic field effects are concerned.

It turns out that one-photon pair production in a magnetic field is not quite observable experimentally at present. With the magnetic field variable reinserted, we have for Eq. 3.25,

$$\text{Rate (unpol.)} = \frac{3\sqrt{3}\alpha\mu}{16\sqrt{2}}\ (\frac{B}{B_c})\ \exp\left[-\frac{8}{3}\frac{\mu}{q_\perp}(\frac{B_c}{B})\right] \qquad (3.32)$$

To get a conversion of about $10^{-6}$/cm. for fields near $10^7$ gauss, $\mu/q_\perp(B_c/B)$ can be no larger than about 8. With 200 GeV photons, this requires a field of $1.3\times10^7$ gauss.

Near a neutron star, for photons of a few MeV energy, high conversion probabilities again requires $\mu/q_\perp(B_c/B)$ to be less than about 8. In this region, a 3 percent fluctuation in either $q_\perp$ or in B will cause a factor of e difference in the rate of pair production. What this effectively means is that for a given photon energy, there will be a surface from inside of which no photons can emerge.

If we assume the field is somewhat less than the critical field, ($\mu^2 \gg 1$), and if there are many high-energy photons in this region, the object should emit a background of $\gamma$-rays with a large fraction of an MeV energy. This is because, as shown above, the transverse energy of the pair is much less than their mass. Since the field is relatively small, most annihilations will produce two photons. Also, the pair density should be small enough that most electrons (or positrons) will lose their orbital energy by synchrotron emission before any annihilation can take place. This should also be true, to a lesser extent, if the pairs escape from the strong-field region before annihilating, since the annihilation cross-section drops inversely with the center-of-mass energy, so that the positrons will normally annihilate only when they have dropped to an MeV or so energy. The gamma-ray spectrum around .5 MeV could thus test whether appropriate combinations of high-energy photons and field strengths exist around various astronomical objects.

## 4. INDEX OF REFRACTION

In this section I calculate the effect of vacuum polarization on the propagation of photons of arbitrary 4-momentum. The contribution of lowest order in $\alpha$ comes from the diagram

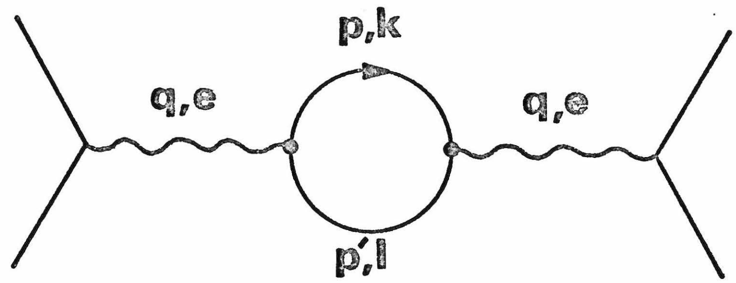



FIGURE 4.1

which acts as a correction to the diagram

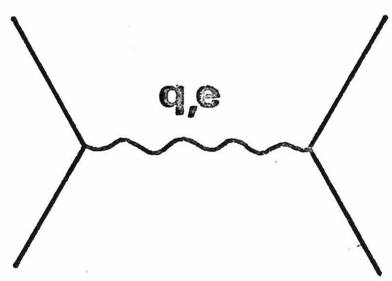



FIGURE 4.2

If we write $P_{\mu\nu}(q)$ for the bubble alone (without the - sign from statistics) the photon propagator becomes

$$\frac{-i}{q^2}\delta_{\mu\nu} + \frac{-i}{q^2}\left[-P_{\mu\nu}(q)\right]\frac{-i}{q^2} \tag{4.1}$$

The imaginary part of P corresponds to a change in electric charge, or, in principle, to a photon mass (actually excluded by gauge invariance). In the case here, where the field violates the Lorentz invariance of the rest of the system, the correction can change the speed of photon propagation by depending on the space-like parts of the momentum differently from the time-like parts. This means that the index of refraction of the high-field region can differ from 1.

If we take as our base polarization states y and z linear polarization, then there is no vacuum polarization term connecting the two. This is clear from the invariance of the system under reflection in the z plane (which leaves the field invariant.) Adler[7] has shown that this is formally CP invariance. For a photon polarized in the direction e, the propagator will then be

$$\frac{-i}{q^2}(-1) + \frac{-i(-1)}{q^2}P_{ee}(q)\frac{-i}{q^2} \approx \frac{-i(-1)}{q^2+iP_{ee}(q)} \tag{4.2}$$

If the vacuum polarization correction has the form $-i(aq_t^2 - bq_x^2)$, then the propagator becomes

$$\frac{i}{(1+a)q_t^2 - (1+b)q_x^2} \; .$$

This moves the photon pole to a "slower" position: free photons now travel at a velocity

$$V = \left(\frac{1+b}{1+a}\right)^{1/2} \approx 1 - \frac{1}{2}(a-b) \; . \tag{4.3}$$

The index of refraction n will then be $n = \frac{1}{V} \approx 1 + \frac{1}{2}(a-b)$. Since n is nearly 1, we can assume in calculating it that $q_t^2 = q_x^2$: the fact that the photon is not on mass-shell will be of higher order in the fine structure constant. Thus

$$n = 1 + \frac{i}{2}\frac{P_{ee}(q)}{q_t^2} \tag{4.4}$$

For low field strengths and off-shell photons, $P_{ee}(q)$ is the normal vacuum polarization correction to photon propagation[15].

At first glance, it might seem that P could be most easily calculated via a dispersion relation from the previously calculated pair-production rate. However, in that calculation a number of approximations were made, which would make the dispersion result only approximate. In addition, it turns out that there is an exact method which simplifies the results nearly as much as they were in the pair-production calculation.

We thus wish to calculate the value of the term corresponding to the diagram

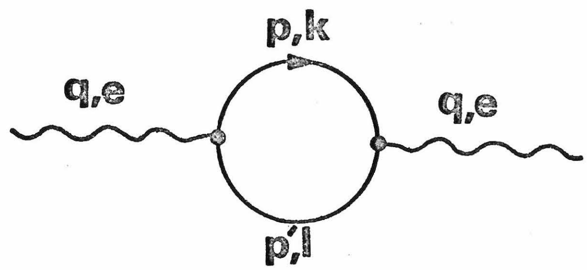



FIGURE 4.3

where the photon polarization e can be in the y or z directions. The invariant matrix elements for this process are

$$P_{\binom{zz}{yy}} = e^2\int\frac{d^3p}{\pi^3}\frac{d^3p'}{\pi^3}\sum_{k\ell} e^{iq\,(p_x+\frac{q_x}{2})}\ \pi^3\delta^3(p - p' + q)$$

$$e^{iq_y'(p_x'+q_x'/2)}\ \pi^3\delta^3(p' - p - q')\ \mathrm{Tr}(T(\gamma_{\substack{z\\y}},q)\ \frac{p+a+\mu}{p^2-2k-\mu^2}$$

$$T(\gamma_{\substack{z\\y}},-q')\ \frac{p'+a+\mu}{p^2-2\ell-\mu^2} \tag{4.5}$$

where T is as given in Equation 2.31. The calculation of the factor $\mathrm{Tr}((p+a+\mu)\cdot T\cdot(p'+a+\mu)\cdot T)$ was made in the pair-production problem, Equation 3.20. Doing the trivial integrations and removing the photon momentum conservation $\pi\delta$ functions, we have

$$P_{yy} = \frac{4e^2}{\pi}\int\frac{d^2p}{\pi^2}\sum_{k\ell}\frac{\frac{1}{2}(T^{+-2}+T^{-+2})\left[(p+\frac{q}{2})\cdot(p-\frac{q}{2})-\mu^2\right]-2\sqrt{k\ell}\ T^{+-}T^{-+}}{((p+\frac{q}{2})^2-2k-\mu^2)((p-\frac{q}{2})^2-2\ell-\mu^2)^2}$$

$$P_{zz} = \frac{4e^2}{\pi}\int\frac{d^2p}{\pi^2}\sum_{k\ell}\frac{\frac{1}{2}(T^{++2}+T^{--2})\left[(p+\frac{q}{2})\cdot(p-\frac{q}{2})+2(p+\frac{q}{2})_z(p-\frac{q}{2})_z-\mu^2\right]-2\sqrt{k\ell}T^{++}T^{--}}{((p+\frac{q}{2})^2-2k-\mu^2)((p-\frac{q}{2})^2-2\ell-\mu^2)} \tag{4.6}$$

We can do the two-momentum integrals by assembling the denominators with a Feynman parameter $\eta$, and using the basic 2-dimensional integrals

$$\int\frac{d^2k}{\pi^2}\ \frac{1}{(k^2-2p\cdot k-\Delta+i\epsilon)} = \frac{-1}{2\pi i}\ (\ln L+C)$$

$$\int\frac{d^2k}{\pi^2}\ \frac{(1;k_j;k_jk_\ell)}{(k^2-2p\cdot k-\Delta+i\epsilon)^2} = \frac{-1}{2\pi i}\ \left(\frac{1}{L};\ \frac{p_j}{L};\ \frac{p_jp_\ell}{L}+\frac{j\cdot\ell}{2}(\ln L+C+1)\right) \tag{4.7}$$

where $L = p^2 + \Delta$ and C is an infinite constant (consistent between the two expressions). We can further put the denominators into an exponential using the relation $1/D = \int du \exp(-uD)$, where the parameter u is conjugate to the mass, and thus represents the proper time along the electron's trajectory. This gives the result

$$P_{yy} = \frac{2ie^2}{\pi^2} \int_{-1}^{1} \frac{d\eta}{2} \int_0^\infty du \sum_{k\ell} e^{-u(k_\eta+\mu^2-\frac{\omega^2}{4}(1-\eta^2))}$$

$$\left[\frac{1}{2}(T^{+-2}+T^{-+2})(-\mu^2-\frac{\omega^2}{4}(1-\eta^2)-\frac{1}{u})-2\sqrt{k\ell}T^{+-}T^{-+}\right]$$

$$P_{zz} = \frac{2ie^2}{\pi^2} \int_{-1}^{1} \frac{d\eta}{2} \int_0^\infty du \sum_{k\ell} e^{-u(k_\eta+\mu^2-\frac{\omega^2}{4}(1-\eta^2))} \tag{4.8}$$

$$\left[\frac{1}{2}(T^{++2}+T^{--\beta})(-\frac{\omega^2}{4}(1-\eta^2)-\mu^2)-2\sqrt{k\ell}T^{++}T^{--}\right]$$

where $k_\eta = k+\ell+\eta(k-\ell)$ and $\omega = qt$. The reason for putting the denominators into the exponential is to make the expression a power series in the discrete variables k and $\ell$.

$T^{+-}$, $T^{-+}$, and $T^{--}$ can be expressed in terms of $T^{++}$ and its derivatives, according to Equation 2.32, so it is only necessary to have one basic sum in order to do all the above sums. The required identity follows from Equation 10.12(20) of Erdelyi[16] (see Appendix 1):

$$\sum_{k\ell} |T^{++}|^2 e^{-u(k+\ell)-\eta u(k-\ell)} = \frac{e^u}{2\,\mathrm{sh}\,u} \exp\left[-z\,\frac{\mathrm{ch}\,u-\mathrm{ch}\,\eta u}{\mathrm{sh}\,u}\right] \tag{4.9}$$

where sh u and ch u are hyperbolic sines and cosines, $z = q_x^2/2$ and u and $\eta$ u are arbitrary. One can obtain sums with various integer powers of k and $\ell$ (as needed for $T^{+-}$, $T^{-+}$, and $T^{--}$) by taking appropriate derivatives with respect to u and $\eta$ u.

Applying this to Equation 4.8, and further simplifying by integrating terms of the form $\mu^2 f(u)\, e^{-\mu^2}$ du by parts with respect to u (to force all $\mu^2$ dependence into the exponent) we get the result

$$P_{yy} = \frac{ie^2}{2}\int_{-1}^{1}\frac{d\eta}{2}\int_0^\infty \frac{du}{shu}\exp\left[-\mu^2 u + u\frac{\omega^2}{4}(1-\eta^2) - \frac{q^2}{2}\frac{chu - ch\eta u}{shu}\right]\cdot$$

$$\left\{\frac{\omega^2}{2}\left(\frac{\eta chu\ sh\eta u}{shu} - ch\eta u\right) + \frac{q^2}{sh^2 u}(chu - ch\eta u)\right\}$$

$$P_{zz} = \frac{ie^2}{2}\int_{-1}^{1}\frac{d\eta}{2}\int_0^\infty \frac{du}{shu}\exp\left[-\mu^2 u + u\frac{\omega^2}{4} - \frac{q^2}{2}\frac{chu - ch\eta u}{shu}\right]\cdot \tag{4.10}$$

$$\left\{-\frac{\omega^2}{2}(1-\eta^2)chu + \frac{q^2}{2}\left(-\frac{\eta chu\ sh\eta u}{shu} + ch\eta u\right)\right\}$$

On doing the parts integration mentioned above, one gets infinite integrated parts, due to the quadratic divergence of the integral at short distances (small u). This can be repaired by using the normal gauge-invariant regularization techniques, as in Feynman[15]. This is done by integrating over a mass spectrum $G(m^2)$, which satisfies $\int dm^2\, G(m^2) = 0$, $\int dm^2\, m^2\, G(m^2) = 0$, and which approaches $\delta(m^2-\mu^2)$ for fixed $m^2$ as the cutoff parameter $\Lambda^2 \to \infty$. (A typical function would be $G(m^2) = \delta(m^2-\mu^2) - 2\,\delta(m^2-\Lambda^2-\mu^2) + \delta(m^2-2\Lambda^2-\mu^2)$.) The net effect of this is to zero out the integrated part.

Equation 4.10 is an exact expression for the first-order effect of vacuum polarization on photon propagation in a magnetic field. If we take the limit of small fields, ($\mu^2 \to \infty$, $q \to \infty$, $q/\mu$ constant) this expression agrees with the normal vacuum polarization calculation, where the vacuum current $J_{\mu\nu} = iP_{\mu\nu}$ (cf. reference 3).

This can be easily verified by working in the approximation $\mu^2$ large. The $e^{-\mu^2 u}$ term in the integral then requires that u be small. Expanding the various hyperbolic functions in this region gives for both amplitudes

$$P_{yy} = P_{zz} = \frac{ie^2}{\pi^2}\int_{-1}^{1}\frac{d\eta}{2}\int_0^\infty \frac{du}{u}\left\{\exp u\left[-\mu^2 + \frac{\omega^2-q^2}{4}(1-\eta^2)\right]\right\}\frac{1-\eta^2}{2}(q^2-\omega^2) \qquad (4.11)$$

When these integrals are done (with one subtraction near u = 0), the result is the empty-space electrodynamic result. The u integral is logarithmically divergent at small u, and must be renormalized in the conventional manner.

As shown above, the index of refraction for the high-field region is $n_{ee} = 1 + i/2\, P_{ee}(q)/q_t{}^2$ , where the photon momentum is assumed to be on the empty-space mass shell. At this point, the log-divergent parts of the integral disappear, giving a convergent result. For a photon with polarization e and momentum $q_\mu = (q,q,0,0)$, $z = q^2/2$, we have

$$n_{yy}-1 = \frac{e^2 z}{2\pi\pi^2}\int_{-1}^{1}\frac{d\eta}{2}\int_0^\infty \frac{du}{shu}\exp\left[-\mu^2 u + z\left(\frac{u}{2}(1-\eta^2) - \frac{chu-ch\eta u}{shu}\right)\right]$$

$$\left\{\frac{\eta chu\ sh\eta u}{shu} - ch\eta u + \frac{chu-ch\eta u}{sh^2 u}\right\}$$

$$n_{zz}-1 = \frac{e^2 z}{2\pi\pi^2}\int_{-1}^{1}\frac{d\eta}{2}\int_0^\infty du\,\frac{1}{shu}\exp\left[-\mu^2 u + z\left(\frac{u}{2}(1-\eta^2) - \frac{chu - ch\eta u}{shu}\right)\right] \qquad (4.12)$$

$$\left\{-(1-\eta^2)\ chu - \frac{\eta chu\ sh\eta u}{shu} + ch\eta u\right\}$$

If the photon energy $q_t \ll 2\mu$, then we can approximate

$$n_{yy}-1 = \frac{e^2}{\pi^2} 2z \int \frac{e^{-\mu^2 u}}{u^3 sh^3 u} \left[- chu\ sh^2 u + 2u^2 chu - ushu\right]$$

$$n_{zz}-1 = \frac{e^2}{\pi^2} \frac{2z}{3} \int \frac{e^{-\mu^2 u}}{u^2 sh^2 u} \left[-2u^2 shu\ chu - 3u + 3shu\ chu\right] \tag{4.13}$$

This result can be obtained somewhat more simply from the Euler-Heisenberg effective Lagrangian(17) which has also been derived by Schwinger(12). This effective Lagrangian is the (renormalized) effect of single electron loops in a strong field. In the low-energy limit, the field due to the photon is slowly varying in comparison to the distances over which the electron loops actually occur, so this effective Lagrangian can be treated as an additional term to the original Maxwell Lagrangian, and the above result can be derived by taking appropriate derivatives. This is done by Bialnicka-Birula and Bialnicki-Birula(5) and by Adler(7). The correction to the photon pole obtained in this way is the same as that calculated above.

Figure 4.4 is a plot of the index of refraction at 3 energies. As can be seen, in the elastic region the index does not depend a great deal on the energy. The plot shows that the index for y polarization does not depend strongly on $z/\mu^2$. For z polarization an approximate correction factor is:

$$\frac{1-n_{zz}(z)}{1-n_{zz}(0)} = 1 + \frac{.19}{\mu} \left(\frac{z}{\mu^2}\right)^{1.8} \tag{4.14}$$

If the external field is small, ($\mu^2 \gg 1$) we may expand this result in powers of the field strength, giving a first term

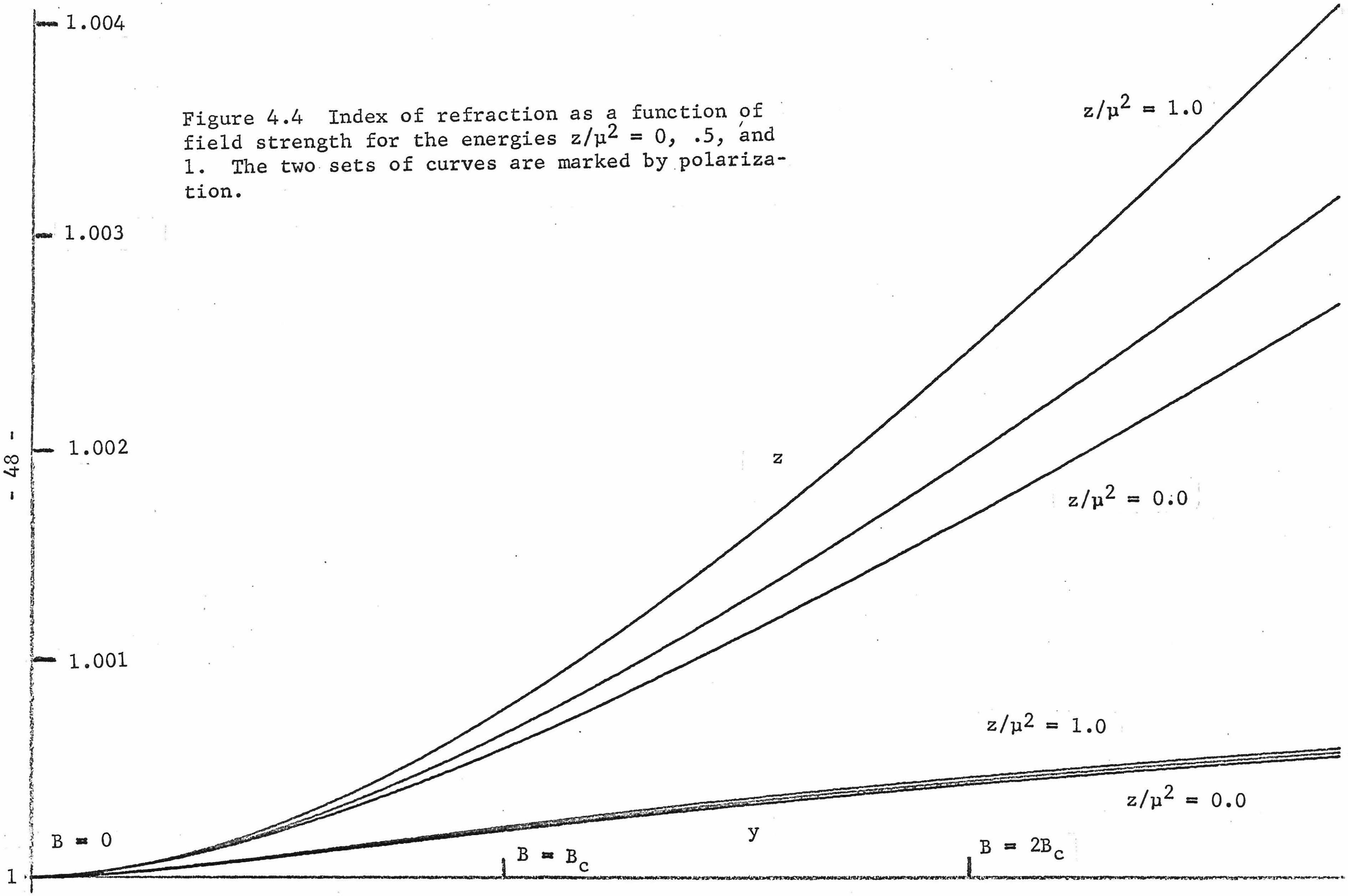



Figure 4.4 Index of refraction as a function of field strength for the energies $z/\mu^2 = 0$, .5, and 1. The two sets of curves are marked by polarization.

$$n_{yy}-1 = \frac{16B^2\alpha^2\omega^2}{45\mu^4}$$

$$n_{zz}-1 = \frac{28B^2\alpha^2\omega^2}{45\mu^4} \qquad (4.15)$$

where the magnetic field has been replaced in the equations, the substitution $e^2 = 2\pi\alpha$ has been made, and $\omega$ is the photon energy.

The results of Equations 4.10 through 4.13 apparently diverge exponentially if $q^2 > 4\mu^2$. This is because in deriving the identity used to sum over the discrete variables, the u integration contour was rotated away from the real direction, i.e., u as used above is the imaginary proper-time. To work in the inelastic region one should give the mass a small negative imaginary part, and integrate u from 0 to $+i\infty$. (In the derivation of identity 4.9, it was in fact necessary to use this contour; the identity was later put into the above form to make the identity explicitly real. See Appendix 1 for details.)

Adler[7] has calculated the index of refraction for arbitrary energies using Schwinger's methods, and has also calculated the low-energy approximation. Bialnicka-Birula and Bialnicki-Birula[5] have also calculated the low-energy limit of the index of refraction from the Euler-Heisenberg Lagrangian.

# 5. PHOTON SPLITTING

In the presence of a magnetic field, the process $\gamma \underset{B}{\rightarrow} \gamma\gamma$ is allowed: by this means a single photon can be split into two photons of lower energy. If we disregard index of refraction effects the two outgoing photons must have momenta parallel to that of the original photon. Photon splitting is represented to lowest order in the fine-structure constant by the Feynman diagram:

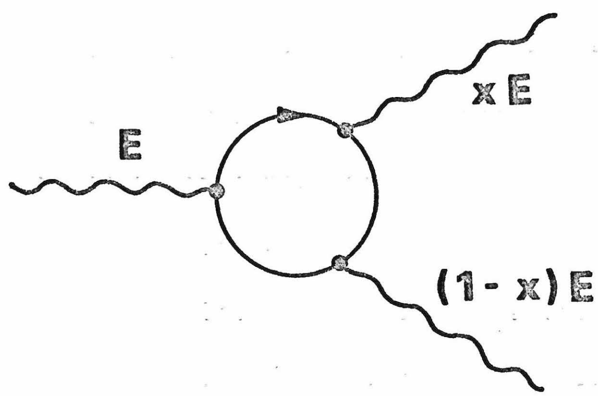



FIGURE 5.1

(where x is a number between 0 and 1) plus the diagram with outgoing photons exchanged.

I have made an attempt to calculate the photon splitting rate, but using the methods developed here, the algebra gets excessively complicated. One may put the discrete indices of the propagator into the exponent by assembling the denominators with two Feynman parameters, doing the time and z integrals and putting the remaining denominators into the exponent. If one then represents the vertices by an integral of Hermite polynomials (Eq. 2.29 with $q_y = 0$, corresponding to a momentum basis), then he can do the sums over intermediate states using the identity 1.11. We are left with the exact matrix elements in terms of three coupled Gaussian integrals. However, due to limitations of available time, I have not been able to finish the algebra to get a usable identity. (In the two photon case, this method can be used to calculate the identity Eq. 4.9.)

Adler[7] has calculated the exact result using the Schwinger method, which would be related to the above by a Fourier transform. The exact result given in the preprint of Adler's paper takes an entire page, and numerical calculations by him indicate that below pair-production threshold this result differs by at most 20% from the result calculated by taking appropriate derivatives of the Euler-Heisenberg Lagrangian. This low-frequency result, also derived by Bialnicka-Berula and Bialnicki-Bérula[5], is:

$$\text{Amplitude } (y \to yy) = -\frac{\alpha^{3/2}}{2\pi^2}\,\omega_o\omega_1\omega_2 \int\frac{du}{u}\,e^{-\mu^2 u}\,\frac{1}{u\,sh^4u}\left[-2u^3(2sh^2u+3) + 3u\,sh^2u + 3chu\,sh^3u\right]$$

$$\text{Amplitude } (y \to zz) = -\frac{\alpha^{3/2}}{2\pi^2}\,\omega_o\omega_1\omega_2 \int\frac{du}{u}\,e^{-\mu^2 u}\,\frac{1}{3u\,sh^3u}\left[2u^3shu+2u^2chu(sh^2u+3) + 3u\,shu - 9sh^2u\,chu\right] \tag{5.1}$$

where $\omega_o$, $\omega_1$ and $\omega_2$ are the three photon energies, and the polarizations refer to the electric vector. These amplitudes are plotted in Fig.5.2.

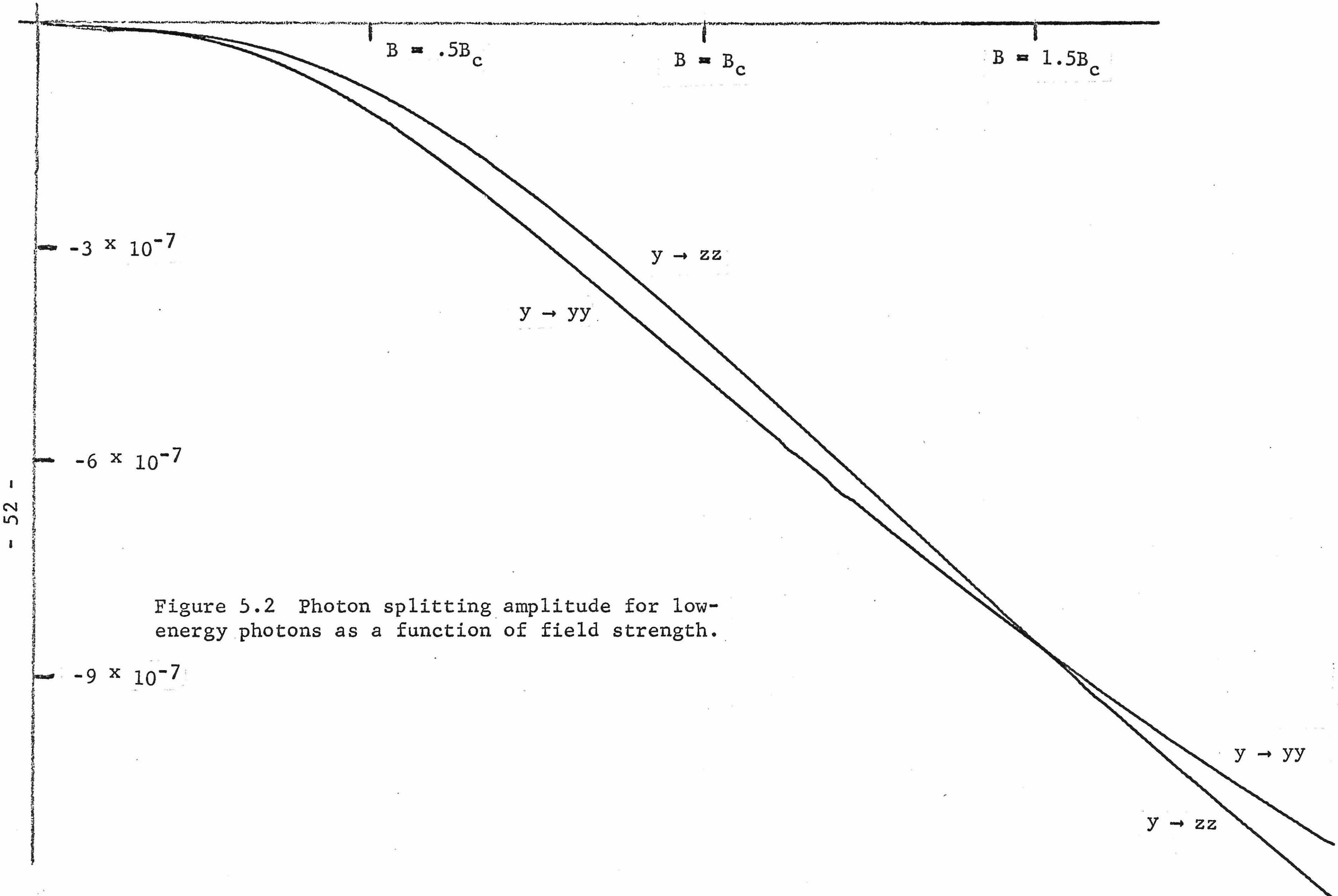



Figure 5.2 Photon splitting amplitude for low-energy photons as a function of field strength.

The second amplitude is also valid for any permutation of the polarizations. The amplitudes for any other polarization states are required to be 0 by CP invariance, as explained in the previous section. These two amplitudes have the low-field limits

$$\text{Amp}\ (y \rightarrow yy) = \frac{-16\ \alpha^{3/2}}{105\ \pi^2 \mu^8}\ \omega_o \omega_1 \omega_2$$

$$\text{Amp}\ (y \rightarrow zz) = \frac{-26\ \alpha^{3/2}}{315\ \pi^2 \mu^8}\ \omega_o \omega_1 \omega_2 \qquad (5.2)$$

and one can verify explicitly the cancellation of the terms of first order in B (the four-vertex "box" Feynman diagram). The above formula gives the behavior of the hexagon diagrams at arbitrary energies, since there is no inner product which can give a measure of the energy.

Due to index of refraction effects, there are additional kinematic constraints which disallow some processes. Adler gives an extensive analysis of the effects which might be expected near a neutron star, and a reader with a deeper interest in the subject is referred to his paper.

## 6. THE HIGH FIELD LIMIT

In this section I examine some of the phenomena which can occur in very strong fields, corresponding, in the units $eB = 1$, to $\mu^2 \ll 1$. Under these conditions, the first excited state lies much higher than the ground state, so that, for example, an electron in the first excited state will rapidly degrade emitting a photon. Except for a small region in the direction ($\pm$) of the magnetic field, this photon will normally be absorbed, creating an electron-positron pair. However, as the mass goes to zero, the polar phase space does also; and thus excited states will usually result in the emission of 2 electrons and 1 positron. The discussion of this section will relate to particles in the stable ground state.

The first interesting problem in a very strong field is the energy of the ground state. In this state, the energy due to the electron's magnetic dipole moment is negative, and for a Dirac electron this energy is - 1/2, exactly cancelling the + 1/2 ground-state energy of the harmonic motion. However, the electron has a slightly positive anomalous magnetic moment, which brings the ground state energy below the free space value. Some authors have speculated that the energy would become negative at sufficiently high fields(18). However, the magnetic field will alter the character of the self-interaction terms, since the electron must propagate through the magnetic field rather than empty space. Intuitively, the electron is confined by the field, so that at very high field strengths, the self-energy approaches the classical value more closely. It is thus necessary to calculate the change in the electron self-energy due to the magnetic field to all orders in the field strength. This is easily done using the formalism developed in Section 2.

The diagram we wish to calculate is

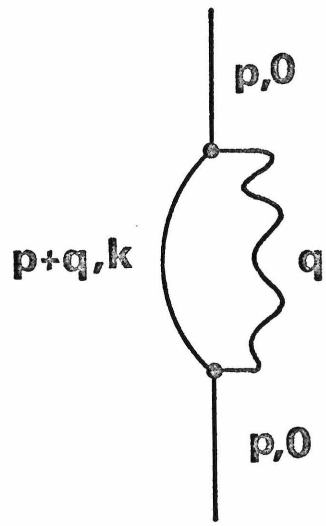



FIGURE 6.1

where the initial and final electrons are in the ground (k=0) state, and we integrate over all intermediate photon four-momenta. We thus have

$$\delta\mu = \int \frac{d^4q}{\pi^4} \frac{e^2}{q^2} \sum_k \frac{1}{(p+q)^2 - 2k - \mu^2} \Big[ \Sigma^- \gamma_\nu (T(0,k,q)\Sigma^- + T(0,(k-1),q)\Sigma^+) \; (p+a+\mu)(\Sigma^- T(k,0,-q) + \Sigma^+ T(k-1,0,-q))\gamma_\nu \Sigma^- \Big] \tag{6.1}$$

where we have dropped some terms from the vertices by using the fact that the electron spin is down.

Doing the spin calculation and using the vertex function 2.28, the factor in brackets above becomes

$$[\,] = \Sigma^{-}\left[2\mu \frac{z^k}{k!} + (-2(p+a) + 2\mu)\frac{z^{k-1}}{(k-1)!}\right]e^{-z}\,\Sigma^{-}\,, \tag{6.2}$$

where $\quad z = q_{\perp}{}^2/2$ .

Uniting the denominators with a Feynman parameter, calculating the $(q_t, q_z)$ integrals by Equation 4.7, putting the denominator into an exponent with a proper time integral, and then calculating the sum and the $(q_x, q_y)$ integrals, the self-energy correction reduces to

$$\delta\mu = \frac{2\mu\alpha}{\pi}\int_0^\infty ds \int_0^1 dx\, \frac{e^{-\mu^2x^2s}(1+xe^{-2sx})}{2s(1-x)+(1-e^{-2sx})} \tag{6.3}$$

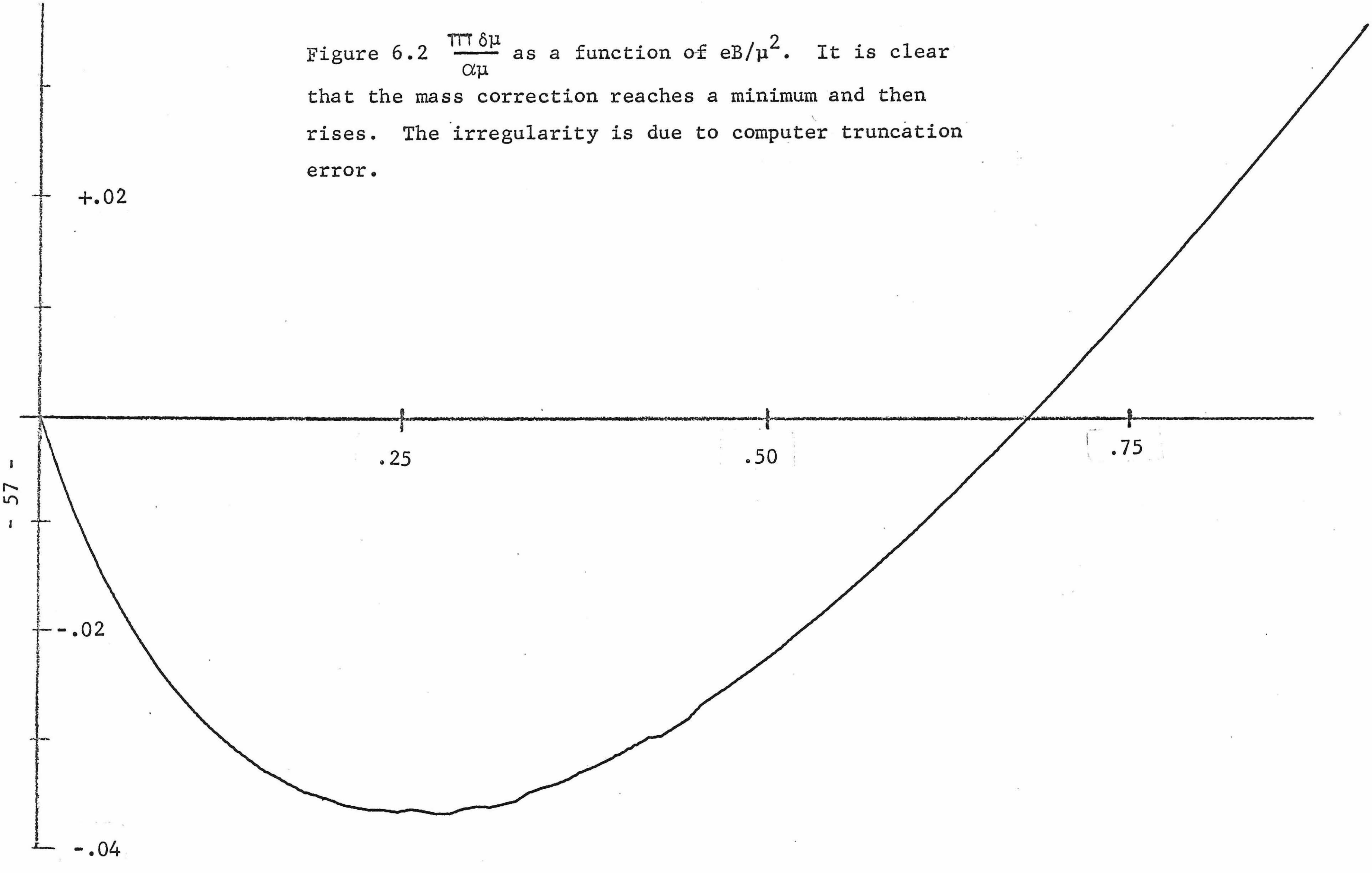
As expected, this integral diverges logarithmically as s approaches 0, requiring us to subtract $e^{-\mu^2x^2s}\,\frac{(1+x)}{2s}$ from the integrand. For small fields, ($\mu^2$ large) we get $\delta\mu \approx -\alpha/2\pi\mu$ , which is the same as the value calculated from the free-space anomalous magnetic moment.

B. Jancovici(19) has shown that for $\mu^2$ small (B large), the integrals in Equation 6.3 are proportional to $(\ln\mu)^2$, and reports the rising asymptotic value

$$\delta\mu \rightarrow \frac{\alpha\mu}{2\pi}\left[(\ln(\frac{2}{\mu^2}) - \gamma - \frac{3}{2})^2 + A\right] , \tag{6.4}$$

where $\gamma = .577216...$ is Euler's constant, and $-6 < A < 7$. I have evaluated this integral numerically, with the results plotted in Fig. 6.2. The electron's energy reaches a minimum of $E = \mu - \frac{\alpha\mu}{\pi}(.0396)$ at a field strength of $eB/\mu^2 = .255$.

Figure 6.2 $\frac{\pi\,\delta\mu}{\alpha\mu}$ as a function of $eB/\mu^2$. It is clear that the mass correction reaches a minimum and then rises. The irregularity is due to computer truncation error.



Another interesting phenomenon in very high fields is the existence of pair states where the two particles have a high probability to be at the same point, but are rigorously forbidden from annihilating. This is due to the infinite degeneracy of the electron and positron states and the conservation of the degenerate quantum number.

Using the linear momentum eigenstates, suppose we have an electron-positron pair in the ground state, with $p_z = 0$, $p_x = a$, $k = 0$; $p_z' = 0$, $p_x' = a$, $k' = 0$, where the primed variables refer to the positron. (The positron momenta here are the physical momenta, without the conventional change in sign.) In terms of position variables, the wave function is

$$\psi = \sqrt{\frac{2}{\pi}}\, \exp\left\{i\left[a(x+x') - \frac{xy}{2} + \frac{x'y'}{2}\right] - \frac{(y-a)^2}{2} - \frac{(y'+a)^2}{2}\right\}. \qquad (6.5)$$

It is clear that the total x-momentum of this state is 2a. If $a > \mu$, then the state cannot annihilate, since any combination of photons with total momentum greater than $2\mu$ need an energy greater than $2\mu$, the energy of the pair state. If the field is strong, $\mu^2 \ll 1$, the wave functions of the electron and positron may be only slightly displaced from each other in the y-direction. The probability distribution is

$$P = |\psi|^2 = \frac{2}{\pi} \exp(-(y-a)^2 - (y'+a)^2) \qquad (6.6)$$

If $1 \gg a > \mu$, the state will be forbidden from annihilating; yet the particles have strongly overlapping probability distributions, as shown in Fig. 6.2 for $a = .1$.

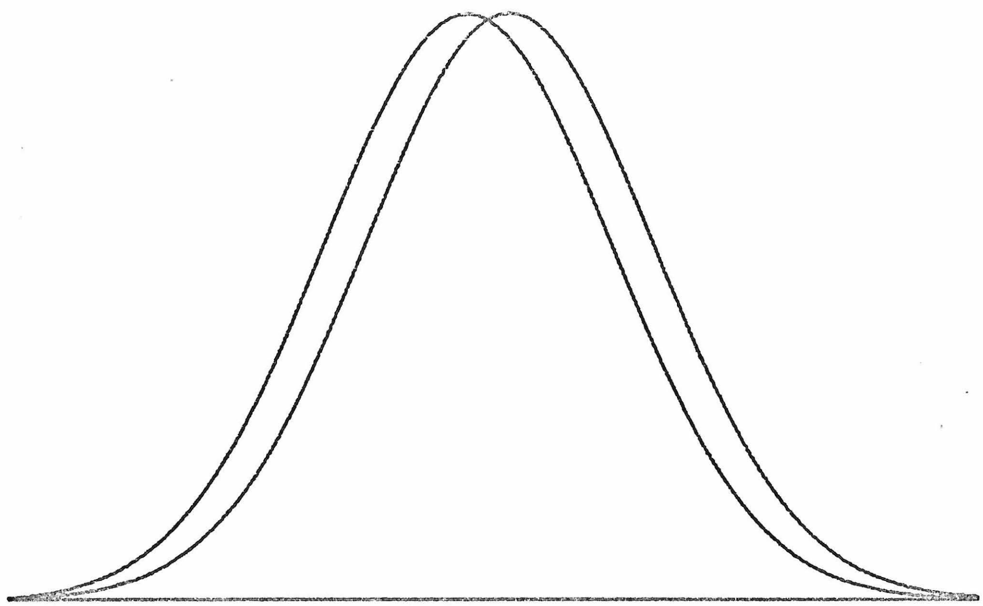


FIGURE 6.3

It is clear that we may select the degenerate momentum variable in any direction, so that there are non-annihilating states with momentum 2a in any given direction. To quantize an electron, using as the index of degenerate states the momentum in a direction at an angle $\varphi$ from the x-direction, we take as a basis the states (cf. Equation 1.8, 2.22)

$$\psi_a^- = \left(\frac{2}{\pi}\right)^{1/4} \exp \frac{1}{4}\left[-(x^2+y^2)+ (e^{i\varphi}(x-iy) - 2ia)^2 + 2a^2\right] \qquad (6.7)$$

The basis set for a positron can be taken to be $\psi_a^+ = (\psi_{-a}^-)^*$. An electron and positron quantized in the same direction $\varphi$, each with momentum a, then have the combined wave function

$$\psi_{a,\varphi}(x,y,x',y') = \left(\frac{2}{\pi}\right)^{1/2} \exp \frac{1}{4}\Big[-(x^2+y^2+x'^2+y'^2) + 4a^2 + (e^{i\varphi}(x-iy)-2ia)^2 + (e^{-i\varphi}(x'+iy')-2ia)^2\Big] \qquad (6.8)$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \exp \frac{1}{4}(-(x_+x_- + x_+'x_-') + 4a^2 + (e^{i\varphi}x_- - 2ia)^2 + (e^{i\varphi}x_+' - 2ia)^2)$$

where $x_+ = x+iy$, $x_- = x-iy$, etc.

We may obtain non-annihilating states in the angular momentum representation by superposing states quantized in various directions, getting

$$\psi_{a,L} = \frac{\sqrt{2}}{\pi^{3/2}} \int_0^{\pi} d\varphi \exp \frac{1}{4}\Big[-(x_+x_- + x_+'x_-') + 4a^2 + (e^{i\varphi}x_- - 2ia)^2$$

$$+ (e^{-i\varphi}x_+' - 2ia)^2\Big] e^{iL\varphi}$$

$$= \sqrt{\frac{2}{\pi}} \exp \frac{1}{4} (-(x_+x_- + x_+'x_-') -4a^2) \tag{6.9}$$

$$\sum_{j=-\infty}^{\infty} I_j(\frac{x_+'x_-}{2}) J_{2j+L}(2a(x_+'x_-)^{1/2})(-i)^{2j+L}(\frac{x_+'}{x_-})^{L/2}$$

where I is a modified Bessel function. Intuitively speaking, this state is a state of total momentum a and angular momentum L.

Unlike the individual momenta, which are required by translational invariance to have no effect on the energy, the sum of the electron and positron momenta in a non-annihilating state is clearly physically significant, since it controls a physical process. In fact, since the distance between the electron and positron in the perpendicular coordinate depends upon the total momentum 2a, the Coulomb energy of the pair should also depend on the total momentum.

Unlike the degenerate momenta of the individual particles (the ρ operators of section 2, defined in Eq.2.16), the x and y components of the degenerate momentum of the pair commute with each other and thus are simultaneously valid quantum numbers:

$$\left[(\rho_x + \rho_x'), (\rho_y + \rho_y')\right] = +i - i = 0$$

The pair thus has a well-defined momentum. This is a direct result of the overall neutrality of the state and the translational invariance of the system. A neutral particle cannot require a gauge transformation when it is moved, and thus the invariance of the physical system in the x, y plane requires that it have a well-defined momentum. The calculation of the dependence of the energy on the total momentum does not lead to any simple results.

## APPENDIX 1: DERIVATION OF INDEX OF REFRACTION IDENTITY

In this Appendix, I give the derivation of the identity 4.9:

$$\sum_{k\ell} |T^{++}|^2 \, e^{-u(k+\ell) - \eta u(k-\ell)} = \frac{e^u}{2\,\mathrm{sh}u} \exp\left[-z \, \frac{\mathrm{ch}u - \mathrm{ch}\eta u}{\mathrm{sh}u}\right]$$

Using the form

$$T^{++} = e^{-z/2} \sqrt{\frac{\ell!}{k!}} \, z^{(k-\ell)/2} \, L_\ell^{k-\ell}(z)$$

(which assumes that the photon momentum is in the x-direction), we have

$$\sum_{k,\ell=0}^{\infty} |T^{++}|^2 \, e^{-uk_+ - vk_-} = e^{-z} \sum_{k\ell} \frac{\ell!}{k!} \, z^{k_-} \left[L_\ell^{k_-}(z)\right]^2 e^{-2\ell u} \, e^{-(u+v)k_-} \qquad \text{(A1.1)}$$

where $k_+ = k + \ell$; $k_- = k - \ell$, and $v = \eta u$.

Erdelyi[16] gives the relation (Eq. 10.12(20))

$$\sum_{\ell=0}^{\infty} \frac{\ell!}{(\ell+\alpha)!} \left[L_\ell^{\alpha}(z)\right]^2 = (1-z)^{-1} \exp\left[-z \, \frac{2x}{1-z}\right] x^{-\alpha} \, z^{-\alpha/2} I_\alpha\left[\frac{2x\sqrt{z}}{1-z}\right],$$

$$|z| < 1 . \qquad \text{(A1.2)}$$

where $I_\alpha$ is a modified Bessel function. Substituting $\alpha \to k_-$, $z \to \exp(-2u)$, $x \to z$, we have

$$\sum_{\ell=0}^{\infty} \frac{\ell!}{(\ell+k_-)!} \left[L_\ell^{k_-}(z)\right]^2 e^{-2\ell u}$$

$$= \frac{e^u}{2\,\mathrm{sh}u} \exp\left[\frac{-ze^{-u}}{\mathrm{sh}u}\right] z^{-k_-} \, e^{k_- u} \, I_{k_-}\left(\frac{z}{\mathrm{sh}u}\right) \qquad \text{(A1.3)}$$

This gives

$$\sum_{k,\ell=0}^{\infty} |T^{++}|^2 e^{-uk_+ - vk_-}$$

$$= e^{-z} \sum_{k_-=-\infty}^{\infty} \frac{e^u}{2shu} \exp\left[\frac{-ze^{-u}}{shu}\right] I_{k_-}\left[\frac{z}{shu}\right] e^{-vk_-} \quad . \qquad \text{(A1.4)}$$

We now take u and v to be pure imaginary, u = iu', v = iv', which causes the u integral to converge for any value of $q^2$, given that $\mu^2$ has a small negative imaginary part. Bessel's integral representation for the $I_\alpha$ then becomes

$$e^{-i(u'+v')k_-} I_{k_-}\left(\frac{z}{i\sin u'}\right) = e^{-iv'k_-} i^{-k_-} J_{k_-}\left[\frac{z}{shu'}\right]$$

$$= \int_0^{\pi} \frac{d\theta}{\pi} e^{-i(u'+v')k_- - i\pi/4\, k_- + i(z/shu')\sin\theta - ik_-\theta} \quad .$$

Summing $k_-$ from $-\infty$ to $\infty$ now gives

$$\sum_{k_-=-\infty}^{\infty} e^{-v'k_-} I_{k_-}\left(\frac{z}{i\sin u'}\right) = \sum_{m=-\infty}^{\infty} \int_0^{\pi} \frac{d\theta}{\pi} \delta(\theta + u' + v' + \frac{\pi}{4} + m\pi) \cdot$$

$$\exp\left[i \frac{z}{shu'} \sin\theta\right], \qquad \text{(A1.5)}$$

where m is an integer which drops out when the $\theta$ integral is done.

We thus have the result

$$\sum_{k\ell=0}^{\infty} |T^{++}|^2 e^{-iu'k_+ - iv'k_-} = \frac{e^{iu'}}{2i\sin u'} \exp\left[-z \frac{\cos u' - \cos v'}{i\sin u'}\right] \qquad \text{(A1.6)}$$

If we are working below threshold, $q^2 < 4\mu^2$, all the u' integrals will converge if we rotate the contour back to its earlier position, iu' = u, iv' = v, so that all functions appearing are real:

$$\sum_{k\ell} |T^{++}|^2 \, e^{-u(k+\ell)-\eta u(k-\ell)} = \frac{e^u}{2shu} \exp\left[-z \, \frac{chu - ch\eta u}{shu}\right] \tag{A1.7}$$

I have slightly violated mathematical rigor in using the above relations A1.4 and A1.5 for $k_-$ negative. The derivation can be made rigorous by using $k_-$ slightly away from any integer, $k_- = k_-' + \varepsilon$, where $k_-$ is an integer and $\varepsilon$ is very small, and then taking the limit $\varepsilon \to 0$. Alternatively, one can handle the regions $k \leq \ell$ and $k > \ell$ separately, bringing them together only in Eq. A1.5. Either way leads to the above result.

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