Spin in conformal field theory

Thesis by
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In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

Caltech

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2018
(Defended May 30, 2018)

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To my family.

ACKNOWLEDGEMENTS

I feel grateful and indebted to many people who shaped my life as a graduate student during my years at Caltech. It appears next to impossible to express on these few pages my gratitude accumulated over this long time. I therefore have to ask the reader to multiply everything that follows by a large dimensionless factor.

First and foremost, I wish to thank my advisor Hirosi Ooguri for taking me on as his student at the time when I had no idea what doing research was like, and guiding me to where I am now—by always being ready to help, providing invaluable advice about physics, or life, and motivating me. During my graduate studies, I was fortunate enough to collaborate with many people, but this is only thanks to his help in navigating also the less rigorous aspects of academic life. It remains a mystery to me how he manages to be such a fantastic scientist and advisor, find time to take care of all the things he is responsible for, and pay an extraordinary amount of attention to details in all his endeavors.

Thanks to David Simmons-Duffin, who essentially co-advised me in the later years of my graduate studies and from whom I also learned a largely unbounded amount of physics, mathematics, and how to do research. He taught me that even the most (on the first sight) boring things, if understood well enough, can lead to beautiful physics. I would also like to use this opportunity to apologize for his time I spent making physics-related jokes of varying quality, although I cannot attest that I regret it.

I want to thank other members of theory group with whom I interacted on various occasions: Sergei Gukov, Anton Kapustin, Mark Wise. I learned a lot from them during classes, seminars, discussions, and TA meetings. Special thanks to Jason Alicea and Anton Kapustin for serving, together with Hirosi and David, on my candindacy and thesis committees.

This thesis wouldn’t be the same without the people with whom I had the luck to collaborate with: Tolya Dymarsky, Scott Collier, Gabriel Francisco Cuomo, Denis Karateev, Hyungrok Kim, Filip Kos, Ying-Hsuan Lin, David Poland, and Xi Yin. It was a pleasure to work with them, and everyone taught me something new.

Thanks to fellow students Mikhail Evtikhiev, Matthew Heydeman, Denis Karateev, Murat Koloğlu, Lev Spodyneiko, Alex Turzillo, Ke Ye, Minyoung You, and Stephan Zheng for being good friends and always being available for an intelligent (or not) discussion about anything. Special gratitude goes to Mikhail Evtikhiev for always giving valuable comments on my drafts, of all degrees of readability, and to Denis Karateev for exciting collaboration. I am also grateful to my officemates Grant Remmen and Tzu-Chen Huang for interesting discussions. Among postdocs at Caltech I would like to thank Mykola Dedushenko, Abhijit Gadde, and Ying-Hsuan Lin. Thanks to everyone with whom I had interacted over these years, and my most sincere apologies to those whom I forgot to mention.

Finally, a very special gratitude goes to my parents, who raised me and taught me how to learn and how to care. Throughout my life, they supported all my endeavors, showed me how to distinguish important from impressive, how to set goals and how to achieve them. Thanks to my grandfather, who along with my father taught me to analyze, build, and finish things. Thanks to my brother Valera, for his support and for not making fun of me when we played volleyball. This thesis is dedicated to my family.

ABSTRACT

Quantum field theory (QFT) is a powerful theoretical framework for studying a wide variety of physical phenomena, ranging from high energy scattering of elementary particles to condensed matter physics. The behavior of QFTs can differ dramatically between different energy or length scales. Renormalization group flows describe how the behavior of a QFT changes with the energy scale, and a typical flow starts and ends at fixed points. Such fixed points can often be described by non-trivial scale invariant QFTs, which in many cases also enjoy an enchanced – conformal – symmetry. Conformal quantum field theories (CFTs) are thus the simplest examples of QFTs, living at the endpoints of renormalization group flows. Any complete understanding of general RG flows (and thus general QFTs) must then necessarily include the understanding of these basic fixed points.

While two-dimensional conformal field theory is by now a classical textbook subject, only in the last decade has there been a significant advance in our understanding of general higher-dimensional CFTs. The work of Rattazzi, Rychkov, Tonni, and Vichi has revived the old subject of conformal bootstrap by applying numerical methods of linear programming to the so-called bootstrap equations. Since then a lot of progress has been made on both numerical and analytical frontiers. However, perhaps the majority of the work up to date concerns itself mainly with correlation functions of scalar local operators, which are the simplest objects in a conformal field theory. While in part this is simply because these objects provide a natural starting point, another important factor is the complexity of the description of non-scalar operators in higher dimensions.

In this dissertation we attempt to fill this gap by generalizing the existing methods to operators of general spin. This turns out to be a fruitful approach since in many cases the generalized point of view reveals a beautiful mathematical structure which allows us to obtain new results or find a more conceptual explanation of the existing ones. And, of course, simply having the technology to work with new types of objects allows us to perform calculations which were not possible before.

We begin in Chapter 2 by describing the kinematic structure of correlation functions of operators with spin. We reduce classification and construction of conformally-invariant tensor structures to simple representation-theoretic questions, generalizing and simplifying pre-existing approaches in a way that is useful for both numerical

and analytical analysis. In Chapters 2 and 3 we provide concrete tools for working with kinematics of 3d and 4d CFTs, and in the latter case we describe a `Mathematica` package which greatly simplifies the calculations.

In Chapter 4 we turn to the problem of computation of conformal blocks, which are the basic building blocks for four-point correlation functions. These functions are parametrized by spin representation of four "external" and one "intermediate" operator. It has been known for some time how to relate the conformal blocks with different external representations (but the same intermediate one) by means of conformally-invariant differential operators. We show that the basic objects in this approach are in fact conformally-covariant (as opposed to conformally-invariant) differential operators and give their complete classification. This point of view allows us to observe a multitude of new properties of these operators and solve the problem of changing the intermediate representation of a conformal block. This gives a concrete algorithm for computation of any conformal block in terms of the simplest one (with four external scalars).

However, this algorithm requires a non-trivial amount of symbolic calculation while for numerical purposes it is desirable to reduce this amount to a minimum. To this end, in Chapter 5 we generalize the exceptionally simple recursion relations for coefficients in a certain series expansion of scalar conformal blocks. These recursion relations follow from Casimir differential equation, which we rephrase in terms of representation-theoretic data, thus allowing a straightforward generalization. Our new recursion relations pave a way to a completely numerical algorithm for computing general conformal blocks. As a byproduct, we find that the general conformal blocks are naturally expanded in terms of $\mathrm{SO}(d)$ matrix elements in Gelfand-Tsetlin basis, which replace the Gegenbauer polynomials found in the scalar case.

Moving to the more analytical side, in Chapter 6 we consider the problem of inversion formulas, which give the scaling dimensions and three-point coefficients of primary operators in terms of a four-point function in which they are exchanged. Such inversion formulas in Euclidean signature have been known for a long time both for scalar operators and for operators with spin. Recently an intrinsically Lorentzian inversion formula was derived from these by Caron-Huot in the case of a four-point function of scalar operators. This new formula helps to systematize analytic results in large-spin perturbation theory and also shows that the three-point coefficients and scaling dimensions of local operators can be analytically continued in spin. We find a remarkably simple generalization of this formula to operators with spin. For this, we introduce a new class of conformally-invariant integral transforms, known as Knapp-Stein intertwining operators in mathematical literature, and use them to give a simple derivation of the Lorentzian inversion formula. Remarkably, we find that a version of this formula exists at the level of operators, providing an analytic continuation of physical operators in spin. The analytically-continued operators are non-local and we argue that they are localized on a null line. We discuss the relevance of these null-ray operators for Regge physics and prove a novel positivity condition for the leading twist light-ray operators in CFTs with light scalars.

In the rest of the dissertation we present some concrete computations in conformal bootstrap. First in Chapter 7 we discuss a simplified version of the bootstrap equations in a collinear configuration, in the limit of large external scaling dimensions and for a four-point function of scalar primaries. We show that a subset of the bootstrap equations can be solved analytically in this limit and imply a symmetry property for the coarse-grained spectral density of the operator product expansion (OPE). We also find another analytic bound, valid for finite scaling dimensions, which marginally strengthens previous bounds on OPE convergence and has an advantage of being a strict inequality instead of an asymptotic one.

Finally, in Chapter 8 we present a direct application of our analysis of conformal kinematics in 3d by performing numerical bootstrap study for the four-point function of the stress-energy tensor. We numerically reproduce the celebrated Hofman-Maldacena bounds on the coefficients of stress-energy tensor three-point function and find new universal upper bounds on the scaling dimensions of the lightest singlet operators in various spin sectors, valid for general 3d CFTs. For example, it follows from our analysis that any 3d conformal field theory must have light scalar operators and we conjecture that the 3d Ising model maximizes the scaling dimension of the lightest parity-odd scalar. Under reasonable assumptions, we put strong constraints on the stress-energy three-point function in this model.

PUBLISHED CONTENT AND CONTRIBUTIONS

Chapters 2-8 of this thesis are adapted from the papers [1–7] listed below. These works are the result of a close collaboration and represent an equal contribution of the authors.

[1] P. Kravchuk and D. Simmons-Duffin, *Counting Conformal Correlators*, *JHEP* **02** (2018) 096, [1612.08987].

[2] G. F. Cuomo, D. Karateev and P. Kravchuk, *General Bootstrap Equations in 4D CFTs*, *JHEP* **01** (2018) 130, [1705.05401].

[3] D. Karateev, P. Kravchuk and D. Simmons-Duffin, *Weight Shifting Operators and Conformal Blocks*, *JHEP* **02** (2018) 081, [1706.07813].

[4] P. Kravchuk, *Casimir recursion relations for general conformal blocks*, *JHEP* **02** (2018) 011, [1709.05347].

[5] P. Kravchuk and D. Simmons-Duffin, *Light-ray operators in conformal field theory*, 1805.00098.

[6] H. Kim, P. Kravchuk and H. Ooguri, *Reflections on Conformal Spectra*, *JHEP* **04** (2016) 184, [1510.08772].

[7] A. Dymarsky, F. Kos, P. Kravchuk, D. Poland and D. Simmons-Duffin, *The 3d Stress-Tensor Bootstrap*, *JHEP* **02** (2018) 164, [1708.05718].

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Chapter 1

INTRODUCTION

Conformal field theories (CFTs) play an important role in modern physics. In general, physical systems which exhibit some form locality, Poincaré (or Euclidean) invariance, and scale invariance can be reasonably expected to also enjoy conformal symmetry.

A large class of examples of this kind of behavior is provided by low-energy (or long-distance) limit of quantum field theories. To be more precise, if a quantum field theory (QFT) has no gap in its energy spectrum above the vacuum state then it can have a non-trivial low-energy scale-invariant fixed point. If the original QFT is furthermore local and unitary, most often it is the case that its fixed point is also conformally-invariant, thus being described by a CFT. Among the simplest ones are the Wilson-Fischer [9] and Banks-Zaks [10] fixed points.

Another class of examples is provided by long-distance behavior of second-order phase transitions in statistical physics systems, such as the critical point at the end of liquid-vapor transition line in ordinary water, order-disorder transitions in various types of magnets, and superfluid transition in ^{4}He. In these cases the CFT is most naturally understood in Euclidean signature and describes statistical correlations in equilibrium. Quantum criticality, for example in thin-film superconductors, on the other hand, leads to CFTs which are naturally Lorentzian and describe dynamics in real time.

In asymptotically safe quantum field theories the high-energy (UV) behavior is also often described by a CFT, and the original theory can be understood as a relevant deformation of the UV CFT [11].

Finally, certain (or even all) conformal filed theories are believed to be equivalent to theories of quantum gravity in Anti-de Sitter (AdS) space via the AdS/CFT correspondence [12–15]. Namely, a conformal field theory on the d-dimensional conformal boundary of $(d+1)$-dimensional asymptotically AdS space is equivalent to a UV-complete theory of quantum gravity inside the AdS space. Since conformal field theory is perfectly mathematically well-defined, AdS/CFT correspondence provides a rigorous handle on non-perturbative effects in quantum gravity.

This ubiquity of appearances of conformal field theories in modern physics makes them extremely interesting objects to study. In the rest of this chapter we first give a brief introduction into a mathematical description of CFTs and conformal bootstrap, and then overview the main results of the following chapters.

1.1 Formal conformal field theory

First and foremost, a conformal field theory is a quantum field theory. In this dissertation we mostly keep in mind those conformal field theories which satisfy the usual Wightman axioms [16], but of course our results on kinematics, which depend only on the properties of the conformal group, are valid more generally. In this section we review some of these axioms and specialize to the case of conformal symmetry. For a change [17–19], our starting point will be in Lorentzian signature.

We consider a quantum field theory on $\mathbb{R}^{1,d-1}$, with a positive-norm Hilbert space $\mathcal{H}$ of states defined on a spacial slice. Poincaré-invariance means that $\mathcal{H}$ is a unitary representation of the universal cover of Poincaré group. This representation can be described by anti-hermitian generators P_μ and $M_{\mu\nu}$ subject to commutation relations[1]

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\lambda] = \eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu, \\
&[M_{\mu\nu}, M_{\sigma\rho}] = \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\sigma\mu} - \eta_{\mu\rho} M_{\sigma\nu}.
\end{aligned} \tag{1.1}
$$

Physical momentum operators are given by $\mathcal{P}_\mu = iP_\mu$. In particular, the Hamiltonian is $H_{\text{Poincaré}} = iP^0$. We assume that the QFT satisfies energy positivity, which means that

$$
H_{\text{Poincaré}} \geq 0. \tag{1.2}
$$

Together with Poincaré invariance this implies that the spectrum of $\mathcal{P}^\mu$ is contained in forward null cone. We also assume a unique vacuum state $|\Omega\rangle$ which is invariant under all symmetries.

A QFT further possesses a set of local operators, denoted by $O(x)$, which transform naturally under the Poincare group as specified by finite-dimensional representations of the Lorentz group[2] $\mathrm{Spin}(1, d-1)$. The local operators are assumed to be operator-valued tempered distributions (in particular, we can take their Fourier transforms) with a sufficiently large dense domain of definition. Furthermore, the span of states

[1] We use mostly plus convention for Lorentz metric.

[2] To simplify the notation, we often leave the Lorentz indices implicit.

of the form

$$\int d^d x_1 \cdots d^d x_n f(x_1, \ldots, x_n) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) |\Omega\rangle \tag{1.3}$$

is dense in $\mathcal{H}$.[3] The axiom of micro-causality requires that for spacelike-separated points x and y, $(x-y)^2 > 0$,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0. \tag{1.4}$$

A theorem of Osterwalder-Schrader [20, 21] says that the Wightman functions of such a QFT, defined as the vacuum expectation values

$$\langle \Omega | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | \Omega \rangle, \tag{1.5}$$

can be analytically continued to Euclidean signature to yield correlation functions

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle, \tag{1.6}$$

satisfying Euclidean analogues of the above axioms and vice versa. The Euclidean correlators are reflection-positive, permutation-symmetric and, of course, covariant under Euclidean isometries.[4] Due to this theorem there is no real difference between studying Euclidean and Lorentizian QFTs.

We would like to study conformally-invariant QFTs, by which we mean QFTs whose Euclidean correlation functions are covariant under finite conformal transformations. To be more precise, the local operators split into primaries and descendants (which are spacetime derivatives of primary operators), and the conformal group acts homogeneously on primary operators (see chapter 2). Recall that for $d > 2$ the connected Euclidean conformal group is $\mathrm{SO}(1, d+1)$, and for $d = 2$ we will restrict to the global conformal group $\mathrm{SO}(1, 3)$.

Invariance of Euclidean correlation functions implies existence of new anti-hermitian symmetry generators D and K_μ which correspond to dilatations and special conformal transformations and satisfy the commutation relations

$$\begin{aligned}
&[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [D, M_{\mu\nu}] = 0, \\
&[K_\mu, K_\nu] = 0, \quad [K_\mu, P_\nu] = 2\eta_{\mu\nu} D - 2M_{\mu\nu}, \\
&[M_{\mu\nu}, K_\lambda] = \eta_{\nu\lambda} K_\mu - \eta_{\mu\lambda} K_\nu.
\end{aligned} \tag{1.7}$$

[3] More generally we can imagine states which can be created from the vacuum only by a non-local operator. In this thesis we mostly study correlation function of local operators, so we will not take this subtlety into account.

[4] We are omitting some details; for precise statements see [20, 21].

In Lorentzian signature this implies conformal invariance of correlation functions on the Lorentzian cylinder $\mathbb{R} \times S^{d-1}$ [22] (besides the obvious invariance under infinitesimal conformal transformations in Minkowski space $\mathbb{R}^{1,d-1}$).

Conformal symmetry implies that the asymptotic operator product expansion (OPE)

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \simeq \sum_{\mathcal{O}} c_{\mathcal{O}}(x)\mathcal{O}(0) \tag{1.8}$$

valid for small x actually converges [23] on the vacuum in the form

$$\mathcal{O}_1(x)\mathcal{O}_2(0)|\Omega\rangle = \sum_{\mathcal{O}} \int d^d x' f_{\mathcal{O}}(x,x')\mathcal{O}(x')|\Omega\rangle. \tag{1.9}$$

Together with completeness of the states (1.3) this implies that the full Hilbert space is densely spanned by single-operator states

$$\int d^d x f(x)\mathcal{O}(x)|\Omega\rangle. \tag{1.10}$$

It suffices to use primary operators above, and conformal symmetry implies orthogonality of such states corresponding to different primary operators. This is known as operator-state correspondence.

In what follows $\mathcal{O}(x)$ denotes primary operators, unless stated otherwise. As noted above, the two-point functions are diagonal in the sense

$$\langle\Omega|\mathcal{O}^\dagger(x)\mathcal{O}'(y)|\Omega\rangle \propto \delta_{\mathcal{O}\mathcal{O}'}, \tag{1.11}$$

and thus the form of the contribution of a primary $\mathcal{O}$ to the OPE (1.9) can be computed from the three-point function[5]

$$\langle\Omega|\mathcal{O}^\dagger(y)\mathcal{O}_1(x)\mathcal{O}_2(0)|\Omega\rangle = \int d^d x' f_{\mathcal{O}}(x,x')\langle\Omega|\mathcal{O}^\dagger(y)\mathcal{O}(x')|\Omega\rangle. \tag{1.12}$$

Since, as we discuss in chapter 2, the two- and three-point functions are fixed by conformal symmetry up to a finite[6] number of three-point coefficients (also known as OPE coefficients), the function $f_{\mathcal{O}}(x,x')$ can be determined from this equality in terms of OPE coefficients using only conformal symmetry. This implies that the knowledge of the quantum numbers of primary operators and of the discrete

[5] Specifically, by going to momentum space, the convolution in the right hand side becomes multiplication by the Fourier transform of the two-point function, and Fourier transform of $f_{\mathcal{O}}$ is trivially extracted. (Up to a natural ambiguity which arises due to the fact that $\mathcal{O}(x')|\Omega\rangle$ in (1.9) contains only positive-frequency modes in x'.)

[6] For each three-point function.

set of their OPE coefficients allows one to compute any Wightman function (or a Euclidean correlator) by a repeated application of (1.9). For this reason the set of quantum numbers of primary operators and OPE coefficients is called "the CFT data."

The knowledge of the CFT data thus completely defines the conformal field theory, at least as far as local correlation functions or the vacuum superselection sector is concerned. The ultimate goal thus would be to compute this data from the basic self-consistency conditions which we now discuss.

1.2 The conformal bootstrap

Since the operators at spacelike separations commute, it is possible to compute a given Wightman function using the OPE in several different ways. For example, consider a four-point function

$$\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle. \tag{1.13}$$

We can compute this four-point function by first taking $\mathcal{O}_1 \times \mathcal{O}_2$ OPE on the left vacuum and then $\mathcal{O}_3 \times \mathcal{O}_4$ OPE on the right vacuum. A simple way to describe this is to insert a complete set of states in the middle, and it is convenient to use the momentum eigenstates

$$|\mathcal{O}(p_1)\rangle \propto \int d^d x e^{ipx}\mathcal{O}(x)|\Omega\rangle, \tag{1.14}$$

normalized as[7]

$$\langle\mathcal{O}(p)|\mathcal{O}(q)\rangle = (2\pi)^d\delta^d(p-q). \tag{1.15}$$

We then have

$$\begin{aligned}&\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle\\ &=\sum_{\mathcal{O}}\int\frac{d^d p}{(2\pi)^d}\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\mathcal{O}(p)\rangle\langle\mathcal{O}(p)|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle.\end{aligned} \tag{1.16}$$

Since the three-point functions are kinematically determined up to a finite number of OPE coefficients, the above sum can be rewritten as

$$\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle = \sum_{\mathcal{O}} f_{12\mathcal{O}}f_{\mathcal{O}^\dagger 34}G_{1234,\mathcal{O}}(x_1,x_2,x_3,x_4), \tag{1.17}$$

[7] This formula is correct for scalar $\mathcal{O}$, for operators with spin see discussion in [24].

where G is a kinematically-determined function and f are the OPE coefficients (for simplicity we assume that there is a single OPE coefficient for each three-point function). Specifically, we have

$$G_{1234,\mathcal{O}}(x_1,x_2,x_3,x_4) = \int \frac{d^d p}{(2\pi)^d}\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\mathcal{O}(p)\rangle\langle\mathcal{O}(p)|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle, \tag{1.18}$$

where in the right hand side we use some standard three-point functions instead of physical ones. The function G is usually called the conformal block and the expansion (1.17) is known as the conformal block expansion. This expansion can be shown to converge exponentially fast for Euclidean configurations of x_i [25].

Now let us assume that all x_i are spacelike-separated. In this case, using microcausality, we can arbitrarily rearrange the operators in the Wightman function (1.13) and repeat the same procedure. For example, we can write

$$\begin{aligned}&\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle = \langle\Omega|\mathcal{O}_3(x_3)\mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\mathcal{O}_4(x_4)|\Omega\rangle\\ &= \sum_{\mathcal{O}} f_{32\mathcal{O}}f_{\mathcal{O}^\dagger 14}G_{3214,\mathcal{O}}(x_3,x_2,x_1,x_4).\end{aligned} \tag{1.19}$$

This leads to a non-trivial consistency condition

$$\sum_{\mathcal{O}} f_{12\mathcal{O}}f_{\mathcal{O}^\dagger 34}G_{1234,\mathcal{O}}(x_1,x_2,x_3,x_4) = \sum_{\mathcal{O}} f_{32\mathcal{O}}f_{\mathcal{O}^\dagger 14}G_{3214,\mathcal{O}}(x_3,x_2,x_1,x_4). \tag{1.20}$$

This equation is known as four-point crossing equation, or sometimes as the "bootstrap equation." It expresses a consequence of associativity of operator product expansion, and in fact one can show that if all possible four-point crossing equations are satisfied, then the operator product expansion is associative. In other words, no new constraints come from higher-point functions.

The "bootstrap" philosophy [26–28] is then to study CFTs by solving the full set of crossing equations (1.20). In 2-dimensional conformal field theory, where conformal algebra enhances to infinite-dimensional Virasoro algebra, the notion of a primary field is stronger and the sum in (1.20) can in rational theories be replaced by a finite sum over Virasoro primaries. Similarly, the number of crossing equations for Virasoro primaries becomes finite and an analytic solution is relatively simple [29]. However, little progress has been made for irrational theories in $d = 2$ or for general CFTs in $d > 2$ until the work [30] which showed how highly non-trivial information can be extracted even from a single crossing equation. For a comprehensive review we refer the reader to [18, 31]. Here we only review the subject as much as is useful for motivating the questions addressed in this thesis.

1.3 Numerical conformal bootstrap

For simplicity, let us consider the crossing equation (1.20) when all operators are identical real scalars ϕ. In this case it reduces to

$$\sum_{\mathcal{O}} |f_{\mathcal{O}}|^2 G_{\mathcal{O}}(x_1, x_2, x_3, x_4) = \sum_{\mathcal{O}} |f_{\mathcal{O}}|^2 G_{\mathcal{O}}(x_3, x_2, x_1, x_4). \tag{1.21}$$

Since we are considering a Wightman function of scalars, the conformal blocks G are scalar functions. From conformal invariance it follows that

$$G_{\mathcal{O}}(x_1, x_2, x_3, x_4) = \frac{1}{x_{12}^{2\Delta_\phi}} \frac{1}{x_{34}^{2\Delta_\phi}} G_{\mathcal{O}}(u, v), \tag{1.22}$$

where Δ_ϕ is the scaling dimension of ϕ while u and v are the conformally-invariant cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \tag{1.23}$$

The equation (1.21) then becomes

$$\sum_{\mathcal{O}} |f_{\mathcal{O}}|^2 F_{\mathcal{O}}(u, v) = 0, \tag{1.24}$$

$$F_{\mathcal{O}}(u, v) = u^{-2\Delta_\phi} G_{\mathcal{O}}(u, v) - v^{-2\Delta_\phi} G_{\mathcal{O}}(v, u). \tag{1.25}$$

In a way, the key idea of [30] is to lower our expectations. Instead of trying to find all possible solutions of this equation, let us try to prove that it doesn't have solutions. Of course, this shouldn't be possible since it is easy to construct a theory[8] with any value of Δ_ϕ allowed by unitarity. But we can try to impose some restrictions on what kind of operators $\mathcal{O}$ are allowed to appear in $\phi \times \phi$ OPE and show that under these assumptions there is no solution, thus proving that these assumptions are inconsistent. For example, we can try to prove that there must be a non-identity scalar in $\phi \times \phi$ OPE by assuming that there are no scalars and showing that then the equation has no solution.

How can we show that there is no solution? One way to do this is to find a linear functional α that can act on functions of u and v such that $\alpha[F_{\mathcal{O}}]$ is non-negative for

[8] Generalized free theory of scalar field.

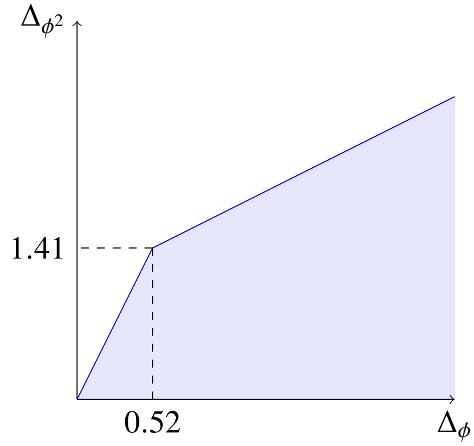


Figure 1.1: Schematic form of the allowed region (shaded) for pairs of Δ_ϕ and the dimension Δ_{ϕ^2} of the lightest scalar in $\phi \times \phi$ OPE in $d = 3$. The boundary is given by the plot of $\Delta^{\min}_{\phi^2}(\Delta_\phi)$. The kink happens close to the dimensions $\Delta_\sigma \approx 0.52$ and $\Delta_e \approx 1.41$ of the spin and energy fields in 3d Ising CFT. The current best bounds on these dimensions (obtained by bootstrap) are $\Delta_s = 0.5181489(10)$ and $\Delta_\epsilon = 1.412625(10)$ [8].

all $\mathcal{O}$ allowed by unitarity and positive for $\mathcal{O} = 1$ the unit operator. If we find such a functional, then we can prove that (1.24) has no solutions by applying α to it,

$$\sum_{\mathcal{O}} |f_{\mathcal{O}}|^2 \alpha[F_{\mathcal{O}}] = 0. \tag{1.26}$$

Indeed, since this is a sum of non-negative terms which is equal to zero, all terms have to be zero. But it is easy to check that $f_1 \neq 0$ and by assumption $\alpha[F_1] > 0$, so this is not possible.

As to be expected, it is impossible to find an α non-negative on all unitarity $\mathcal{O}$ (since by the above argument that would disprove existence of unitary CFTs with scalar operators), but a non-trivial result is that we *can* find an α which is non-negative on all unitary $\mathcal{O}$ except non-identity scalars [30]. This disproves existence of solutions to (1.24) without non-identity scalar operators and thus proves that non-trivial scalars must appear in $\phi \times \phi$ OPE in any unitary CFT.

Before explaining how one can find such linear functionals in practice, let us comment on how unreasonably powerful this general approach turns out to be. One can try to get more refined information by trying to find α as above, but also non-negative on scalars of scaling dimension above some Δ_{ϕ^2} (here ϕ^2 is just a notation and not a real operator). In other words,

$$\begin{aligned}&\alpha[F_1] > 0, \quad \alpha[F_\mathcal{O}] \geq 0 \quad \text{all non-scalar } \mathcal{O} \text{ allowed by unitarity,}\\ &\alpha[F_\mathcal{O}], \quad \text{all scalar } \mathcal{O} \text{ with } \Delta_\mathcal{O} > \Delta_{\phi^2}.\end{aligned} \tag{1.27}$$

If such an α exits it then follows that there must exist a scalar in $\phi \times \phi$ OPE with dimension below $\Delta^{\min}_{\phi^2}$. We can ask what is the minimal value of $\Delta^{\min}_{\phi^2}$ for which such an α can be found. Of course, there is a possibility that $\Delta^{\min}_{\phi^2} = \infty$, but in practice it turns out that $\Delta^{\min}_{\phi^2}$ is finite [30]. One can then plot the dependence of $\Delta^{\min}_{\phi^2}$ with respect to Δ_ϕ in $2 \leq d < 4$ and find a curve which is smooth everywhere except for a kink at Δ_ϕ extremely close to scaling dimension of spin field of d-dimensional Ising CFT and Δ_{ϕ^2} extremely close to scaling dimension of energy density field [32, 33] (figure 1.1). It is then a natural conjecture that Ising CFT saturates this bound precisely at the kink. If this conjecture is correct (for which we now have overwhelming evidence), this allows us to determine the scaling dimensions Δ_ϕ and Δ_{ϕ^2} using the above methods. Development of this idea has led to the most precise determinations of critical exponents of 3d Ising CFT [8, 34, 35]. What's more, using the fact that Ising CFT saturates the bound, it turns out to be possible to determine the entire low-lying spectrum of operators in $\phi \times \phi$ OPE and the corresponding $|f_\mathcal{O}|^2$ from the single (or a few) crossing equation (1.24) [31, 34, 36]. Other theories can be identified in a similar manner, such as critical O(N) models [8, 37, 38], Gross-Neveu-Yukawa models [39, 40], and many others. Furthermore, even without singling out a concrete theory, numerical bootstrap still yields strong universal bounds on CFT data [7, 41].

The search for α is most efficiently done numerically on a computer. For this, one first writes $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$ and looks for α in the form

$$\alpha[F] = \sum_{n,m} \alpha_{n,m} \partial_z^n \partial_{\bar{z}}^m F \Big|_{z=\bar{z}=\frac{1}{2}}, \tag{1.28}$$

where $\alpha_{n,m}$ are real coefficients. In practice one truncates the search space by $n + m \leq \Lambda$ for some large Λ. To check for positivity of $\alpha[F_\mathcal{O}]$ one uses the fact that only traceless-symmetric tensors of even spin can appear in $\phi \times \phi$ OPE due to conformal selection rules, and thus we can write $\alpha[F_{\Delta,J}]$ where J is even and non-negative and $\Delta \geq J + d - 2$ as required by unitarity [24, 42].[9] It turns out that large-J and large-Δ behavior of $\alpha[F_{\Delta,J}]$ is such that it typically suffices to check positivity

[9]For $J = 0$ we have $\Delta \geq \frac{d-2}{2}$.

for a finite number of spins J and a bounded range of Δ. In first papers [30, 43–46] the approach was to discretize Δ so that we get the inequalities, schematically,

$$\alpha[F_{\Delta_i,J}] \geq 0, \quad J = 0 \dots J_{max}, \quad i = 1 \dots N, \tag{1.29}$$

where $\Delta_i = J + d - 2 + \frac{i}{N}(\Delta_{\max} - J - d + 2)$. One can also normalize $\alpha[F_{0,0}] = 1$. The search for coefficients $\alpha_{n,m}$ then becomes a finite-dimensional linear program which can be solved numerically on a computer for any given value of Δ_ϕ.

A more modern way [35–37, 47] to ensure positivity of $\alpha[F_{\Delta,J}]$ is based on the fact that the conformal block $G_{\Delta,J}$ has a meromorphic representation in Δ [36, 37, 48, 49],

$$|\rho|^{-\Delta} G_{\Delta,J} = h_J^{(\infty)} + \sum_k \frac{1}{\Delta - \Delta_{J,k}} h_J^{(k)}. \tag{1.30}$$

Here ρ and h are functions of z and $\bar{z}$ but not Δ. The first important feature of this representation is that it can be truncated to keep only a finite number of poles in Δ, in a quickly convergent manner. The second is that the poles $\Delta_{J,k}$ are all below the unitarity bound $\Delta > J + d - 2$. This implies that using this approximation we can write for $\alpha[F_{\Delta,J}]$ for a given J

$$\alpha[F_{\Delta,J}] = Q_J(\Delta) \sum_{n,m} \alpha_{n,m} P_J^{n,m}(\Delta), \tag{1.31}$$

where $Q_J(\Delta)$ is some explicitly positive prefactor and $P_J^{n,m}(\Delta)$ are polynomials in Δ. We thus only need to make sure that

$$\sum_{n,m} \alpha_{n,m} P_J^{n,m}(\Delta) \geq 0 \tag{1.32}$$

for Δ in some range, which depends on our assumptions and which for simplicity we take to be $\Delta > \Delta_0 \geq J + d - 2$. It is then a theorem that the positivity holds iff we have a representation

$$\sum_{n,m} \alpha_{n,m} P^{n,m}(\Delta) = \sum_i r_i^2(\Delta) + (\Delta - \Delta_0) \sum_i p_i^2(\Delta) \tag{1.33}$$

for some polynomials r_i and p_i. Using this, one can phrase positivity as a finite-dimensional *semidefinite* problem which can be efficiently solved on a computer [35]. The advantage here is that this approach does not require discretization or a cutoff for Δ, which is nice conceptually and also simplifies and speeds up the calculations.

1.4 Overview of the results

1.4.1 Chapters 2 and 3

Part of the goal of this thesis is to develop techniques which allow to extend the methods of numerical bootstrap above to more general crossing equations (1.20). Specifically, the most general Wightman function, restoring the $\mathrm{Spin}(1, d-1)$ indices, has the form

$$\langle\Omega|O_1^{\alpha_1}(x_1)O_2^{\alpha_2}(x_2)O_3^{\alpha_3}(x_3)O_4^{\alpha_4}(x_4)|\Omega\rangle. \tag{1.34}$$

Correspondingly, its conformal block decomposition has the form

$$= \sum_{O}\sum_{a,b} f^a_{12O} f^b_{O^\dagger 34} (G^{\alpha_1\alpha_2\alpha_3\alpha_4}_{1234,O})_{(ab)}(x_1, x_2, x_3, x_4), \tag{1.35}$$

where a and b label the independent conformally-invariant tensor structures allowed for three-point functions $\langle O_1 O_2 O\rangle$ and $\langle O^\dagger O_3 O_4\rangle$ respectively.

Thus the very first step is to undertand the constraints of conformal invariance on three-point functions and on four-point functions (equivalently, on the index structure of the conformal block in (1.35), i.e., find the analog of (1.22)). This is the subject of chapters 2 and 3.

In the literature these questions have been most often addressed using embedding space (or essentially equivalent) methods [39, 50–56], which rewrite the correlation functions in terms of objects on which the conformal group acts linearly. This makes it easy to write out a few conformally-invariant building blocks and combine them in all possible ways to write out some tensor structures for physical correlators. A problem with this approach is that it often overcounts the tensor structures (i.e., produces an overcomplete basis), especially in the more physically relevant low dimensions. For example, for a four-point function of stress-energy tensors in 3d, which we discuss in chapter 8, these methods produce at least a hundred too many structures.

In chapter 2 we take a different approach,[10] which is based on the idea of "gauge-fixing" conformal symmetry. Specifically, we use the fact that if a correlator is known for some standard configuration of operator insertions, conformal symmetry then determines its values for some other configurations. For 1-, 2-, and 3-point functions just one standard configuration is enough to completely determine the correlator. For n-point functions with $n > 3$ there exist non-trivial conformal

[10] A similar but less general and systematic analysis have been performed in [23, 51].

moduli of n points, and we need a continuous family of standard configurations parametrized by several conformally-invariant cross-ratios. Restricting attention to standard configurations solves part of the constraints of conformal symmetry. However, even the values of the correlator in standard configurations have to satisfy certain invariance conditions.

We classify these invariance conditions and find counting rules for tensor structures of the most general n-point functions. For example, for $n > 2$ the tensor structures for an n-point function of distinct primaries in $\mathrm{Spin}(1, d-1)$ representations ρ_i are in one-to-one correspondence with the elements of the invariant subspace

$$(\rho_1 \otimes \cdots \otimes \rho_n)^{\mathrm{Spin}(d+2-m)}, \qquad m = \min(n, d+2). \tag{1.36}$$

Moreover, our analysis actually shows how one can not just count but also construct these tensor structures. We explain this in detail in the case of $d = 3$ (which will be used extensively in chapter 8). In general dimensions we work out the constraints of permutation symmetries, conservation conditions,[11] and establish a one-to-one correspondence of counting of CFT tensor structures with counting of tensor structures for massive scattering amplitudes in one dimension higher. We also study how analyticity of Euclidean correlators is related to analyticity of the functions of conformal-cross ratios u, v which multiply the conformally-invariant tensor structures in generalizations of (1.22), a question important for numerical bootstrap techniques.

In chapter 3 we review some results in the literature related to kinematics of 4d CFTs and complete them using the analysis of chapter 2. Furthermore, we implement the resulting techniques in a `Mathematica` package with a view towards applications in numerical bootstrap.

1.4.2 Chapter 4

Another important problem in applying numerical conformal bootstrap to operators with spin is to compute the general conformal blocks, especially in the form (1.30). There exist many approaches to this problem. The most direct one is to use Zamolodchikov-type recursion relations [36, 37, 48, 49], but it is problematic because these recursion relations are not known in general. Another approach is to solve conformal Casimir equations, satisfied by the conformal blocks, either analyt-

[11] For counting only.

ically [57, 58] or in power series [59, 60]. Analytic solution is difficult in general,[12] but in even dimensions can be obtained by combining methods described below and results of [57]. Power-series solutions are discussed in chapter 5.

Yet another approach is to use conformally-invariant differential operators to relate more complicated conformal blocks to simpler ones [39, 61, 62]. Specifically, one considers conformally-invariant differential operators $\mathscr{D}_{12}$ and $\mathscr{D}_{34}$ which act on coordinates and spin indices only of operators 1 and 2 or 3 and 4. It turns out that one can construct $\mathscr{D}$ which act on functions which transform according to one set of $\mathrm{Spin}(1, d-1)$ irreps, but produce functions which transform according to a new set. In other words, they change the spin of external operators. One can show that applying these differential operators to a conformal block

$$\mathscr{D}_{12,\alpha_1'\alpha_2'}^{\alpha_1\alpha_2}\mathscr{D}_{34,\alpha_3'\alpha_4'}^{\alpha_3\alpha_4}(G_{1234,\mathcal{O}}^{\alpha_1'\alpha_2'\alpha_3'\alpha_4'})_{(ab)}(x_1,x_2,x_3,x_4), \tag{1.37}$$

one obtains another conformal block which corresponds to a new set of $\mathrm{Spin}(1, d-1)$ irreps of external operators. To see this, one uses the representation (1.18) for the conformal block and studies the action of these operators on three-point structures $\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\mathcal{O}(p)\rangle$ and $\langle\mathcal{O}(p)|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle$, which simply produces new three-point structures of the same form, but with new representations.

This allows one to write any conformal block in terms of a simplest "seed" conformal block which exchanges $\mathcal{O}$ with given quantum numbers. For example, if $\mathcal{O}$ is a traceless-symmetric tensor, the seed block is the well-studied scalar conformal block [36, 37, 57, 63, 64]. However, since a scalar conformal block can only exchange traceless-symmetric tensors, for other types of $\mathcal{O}$ the seed blocks are more complicated and have to be computed in some other way.

In chapter 4 we greatly generalize these methods by observing that the conformally-invariant operators $\mathscr{D}_{12}$ can be written as contractions of conformally-*covariant* differential operators,

$$\mathscr{D}_{12} = \mathcal{D}_1^A\mathcal{D}_{2,A}, \tag{1.38}$$

where A is an index in a finite-dimensional irreducible representation (irrep) W of the conformal group, and operators $\mathcal{D}_i$ act on coordinates and indices of a single operator. We then show that these conformally-covariant operators $\mathcal{D}_i$ can be understood as computing the decomposition of tensor product of the Verma module

[12] Notably, it has been found in sufficient generality in 4d CFTs [58], although the solutions are perhaps too complicated to be convenient for numerical computations.

of $\mathcal{O}_i$ and the finite-dimensional irrep W into irreducible components, which allows us to give a complete classification of operators $\mathcal{D}_i$. This also clarifies how these operators should be constructed in general. Since these operators can change the representation of the primary operator they act on, in what follows we call them "weight-shifting operators."

This generalized point of view allows us to observe new properties of these operators and thus of the conformally-invariant operators. For example, applying a weight-shifting operator to a three-point function and contracting with a conformal Killing tensor, we find an object with transformations properties of a four-point function,

$$\langle(\mathcal{D}_1^A\mathcal{O}_1)(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle w_A(x_4). \tag{1.39}$$

Here w is a conformal Killing tensor, and as such satisfies a very constraining differential equation. In fact, one can show that the space of four-point functions which satisfy the same equation as (1.39) is finite dimensional. Moreover, we show that the set of four-point functions of the form (1.39), over all $\mathcal{D}_1$ and $\mathcal{O}_1$ such that the resulting four-point function has fixed quantum numbers, form a basis of this space. But then so do the objects

$$\langle\mathcal{O}_1(x_1)(\mathcal{D}_2^A\mathcal{O}_2)(x_2)\mathcal{O}_3(x_3)\rangle w_A(x_4), \tag{1.40}$$

since there was nothing special in operator $\mathcal{O}_1$. This implies that there must exist a linear relation between bases (1.39) and (1.40). These bases can be interpreted as conformal blocks, and this linear transformation can be interpreted as a finite-dimensional crossing transformation.

Using this transformation, we show how to compute expressions of the form

$$\mathscr{D}_{13,\alpha_1'\alpha_3'}^{\alpha_1\alpha_3}(G_{1234,\mathcal{O}}^{\alpha_1'\alpha_2\alpha_3'\alpha_4})_{(ab)}(x_1,x_2,x_3,x_4), \tag{1.41}$$

in terms of conformal blocks. This would be hard to do without our crossing transformation, since the operator $\mathscr{D}_{13}$ acts simultaneously on $\langle\Omega|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\mathcal{O}(p)\rangle$ and $\langle\mathcal{O}(p)|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|\Omega\rangle$ in (1.18). We show that this expression is equal to a linear combination of conformal blocks exchanging operators in the tensor product $W\otimes\mathcal{O}$, i.e., $\mathcal{D}_{13}$ changes the intermediate representation in a well-defined way. This observation allows us to reduce all seed blocks (and thus all blocks) to scalar conformal blocks, and even scalar $\mathcal{O}$.

We thus find a relatively simple algorithm which allows one to compute arbitrary conformal blocks. We explicitly reduce all seed blocks to the scalar case in 3d and 4d. Furthermore, our methods interact nicely with the other approaches to conformal blocks and explain some previously known formulas. For example, if one can find the scalar conformal block in a given dimension analytically (as is the case in all even dimensions), our techniques immediately yield analytic expressions for all conformal blocks. One can also use our techniques to derive Zamolodchikov-type recursion relations for general conformal blocks, as we illustrate in the case of fermionic blocks in 3d.

1.4.3 Chapter 5

In chapter 5 we study general conformal blocks from a more computational perspective. To facilitate numerical analysis of more general crossing equations, it is desirable to have a computer program which would be able to efficiently compute approximations of the form (1.30) for any required conformal block given the set of data which specifies it. This data is

1. scaling dimensions and $\mathrm{Spin}(1, d-1)$-irreps of external operators,
2. $\mathrm{Spin}(1, d-1)$-irrep of the intermediate operator,
3. a pair of three-point tensor structures to use on the left and on the right of (1.18),
4. technical information on the requested precision of the approximation.

While the methods described in chapter 4 do allow us to compute general conformal blocks, they still require a non-trivial amount of case-by-case analysis and symbolic calculation with differential operators. The goal of chapter 5 is to study the possibility of having a more numerically straightforward algorithm.

Our discussion is based on the following observation in the case of scalar blocks. Writing $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$, one can show [59] simply from scaling and $\mathrm{Spin}(d)$ invariance in Euclidean signature that the scalar conformal block in the right hand side of (1.22) can be written as (recall $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$)

$$G_{\mathcal{O}}(z, \bar{z}) = r^{\Delta_{\mathcal{O}}} \sum_{n=0}^{\infty} \sum_{j=j_{\mathcal{O}}-n}^{j_{\mathcal{O}}+n} \Lambda_{n,j} r^n C_j^{(\frac{d-2}{2})}(\cos\theta), \tag{1.42}$$

where $C_j^{(\nu)}$ are the Gegenbauer polynomials and $\Lambda_{n,j}$ are some yet undetermined coefficients (which are understood to be equal to 0 for $j < 0$), while $\Delta_{\mathcal{O}}$ and $j_{\mathcal{O}}$

are the scaling dimension and spin of $\mathcal{O}$. Using the Casimir differential equation satisfied by conformal blocks [57], one then shows that the coefficients $\Lambda_{n,j}$ satisfy a simple recursion relation

$$\Lambda_{n,j} = c^{+}_{n-1,j-1}\Lambda_{n-1,j-1} + c^{-}_{n-1,j+1}\Lambda_{n-1,j+1}. \tag{1.43}$$

for some known coefficients $c^{\pm}_{n,j}$. Starting from a normalization condition for $\Lambda_{0,j_{\mathcal{O}}}$ this allows one to efficiently compute the coefficients $\Lambda_{n,j}$. In section **??** we discuss how the series (1.42) can potentially be efficiently translated into approximations of the form (1.30).

Our approach is then to generalize the representation (1.42) and the recursion relation (1.43) to the case of arbitrary conformal blocks.[13] We first explain that a natural language to discuss the index structure of general conformal blocks is given by so-called Gelfand-Tsetlin (GT) bases for Spin(d) representations. The elements of a GT basis are classified by their transformation properties with respect to a chain of subgroups

$$\mathrm{Spin}(d) \supseteq \mathrm{Spin}(d-1) \supseteq \mathrm{Spin}(d-2) \supseteq \cdots \supseteq \mathrm{Spin}(2). \tag{1.44}$$

This sequence of subgroups is natural from the point of view of conformal correlation functions since $n \geq 3$ points in Euclidean $\mathbb{R}^d$ are left invariant by a Spin($d+2-n$) subgroup of the conformal group. In particular, we show that in general the Gegenbauer polynomials in (1.42) should be replaced by a matrix element of a particular rotation in a GT basis. We explain how these matrix elements can be efficiently computed using the known facts about representation theory in GT bases and provide lots of examples.

We then explain how the scalar recursion relation (1.43) can be derived from purely representation-theoretic manipulations, bypassing the Casimir differential equation (but still using the quadratic Casimir of the conformal group). This allows an almost straightforward generalization of (1.43) to general conformal blocks. We find that when the external operators have non-trivial spin, the appropriate generalization of the coefficients $c^{\pm}_{n,j}$ is expressed in terms of $6j$-symbols of Spin($d-1$). This makes numerical implementation of the generalized recursion relations straightforward in

[13] Steps in this direction were also taken in [60], albeit on a case-by-case basis and using more ad-hoc techniques. (A possible advantage of that work is that they write their recursion relations for a faster-converging series expansion of [59], although these recursion relations are much more complicated. It appears to us that it is perhaps easier to solve the recursion relations in our form and then convert the resulting series to the form of [59, 60].)

3d, 4d, and in some other cases which we discuss. We test our recursion relations in a number of examples, finding a perfect agreement with the previously known results. To demonstrate the power of our method, we explicitly compute the coefficients $c^{\pm}_{n,j}$ for general blocks in 3d and for fermion blocks in $2n$ dimensions.

1.4.4 Chapter 6

In chapter 6 we turn our attention to a rather different problem. A natural question one can ask when attempting an analytic solution of (1.20) is whether given a four-point function of primary operators one can recover the CFT data which it contains, i.e., the scaling dimensions and products $f_{12\mathcal{O}}f_{\mathcal{O}^\dagger 34}$ of OPE coefficients for all the intermediate operators $\mathcal{O}$. We will call any formula which accomplishes this an inversion formula. The reason this might be useful is that then one can try to plug the t-channel expansion (1.19) into an inversion formula for the s-channel expansion (1.17) and try to directly constrain the CFT data.

The most straightforward way to invert the s-channel expansion is to expand the four-point function in the OPE limit $1 \to 2$ and read off the contributing conformal blocks. This is possible since the OPE limit expansion is organized according to scaling dimension of intermediate operators. Another way to invert the expansion comes from harmonic analysis on the Euclidean conformal group $\mathrm{Spin}(1, d+1)$ [65]. One first defines a function $c(\Delta, j)$ as a conformally-invariant Euclidean integral

$$c(\Delta, j) = \int d^d x_1 \cdots d^d x_4 \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_4(x_4)\rangle \widetilde{F}_{\Delta,j}(x_1, x_2, x_3, x_4). \tag{1.45}$$

Here $\widetilde{F}_{\Delta,j}(x_1, x_2, x_3, x_4)$ is the conformal partial wave (CPW), a close cousin of conformal block G. Unlike G, $\widetilde{F}$ is single-valued in Euclidean space and can be defined by

$$\widetilde{F}_{\Delta,j}(x_1, x_2, x_3, x_4) = \alpha \widetilde{G}_{\Delta,j}(x_1, x_2, x_3, x_4) + \beta \widetilde{G}_{d-\Delta,j}(x_1, x_2, x_3, x_4) \tag{1.46}$$

for some known constants α and β, where $\widetilde{G}$ is the conformal block with external dimensions Δ_i replaced by their shadow dimensions $d - \Delta_i$. One then shows that the function $c(\Delta, j)$ has poles in complex Δ-plane at scaling dimensions of physical operators appearing in $\mathcal{O}_1 \times \mathcal{O}_2$ OPE with residues proportional to $f_{12\mathcal{O}}f_{\mathcal{O}^\dagger 34}$.

Unfortunately, both these inversions methods turn out to be not very helpful for analytic analysis of crossing equation, since they both probe the s-channel OPE limit of the four-point function, and in this limit any finite number of terms of t-channel OPE is useless. However, recently a Lorentzian inversion formula for scalar four-point functions was derived in [66, 67] which partially solves this problem. This formula computes the same function $c(\Delta, j)$ and has the form

$$c(\Delta, j) = \kappa_{\Delta,j} \int d^d x_1 \cdots d^d x_4 \langle O|[\phi_3, \phi_1][\phi_2, \phi_4]|\Omega\rangle \widetilde{G}_{j+d-1,\Delta-d+1}(x_1, x_2, x_3, x_4) + (1 \leftrightarrow 2), \tag{1.47}$$

where the integral is now over Minkowski space with some restrictions on causal relationship between the points and $\kappa_{\Delta,j}$ is some known coefficient given by a product of Γ-functions. The advantage of this formula is that the poles in $c(\Delta, j)$ come from the integral probing a lightcone limit of operators 1 and 2. The part of this limit which is important for large j is also the lightcone limit in t-channel, and the behavior of the four-point function in that limit can be approximated by a finite number of conformal blocks in t-channel, corresponding to the smallest values of "twist" $\Delta - j$. This allows to systematize and put on firm ground the analytical approach to solving the crossing equation known as large-spin perturbation theory [68, 69].

Another interesting feature of (1.47) is that it is manifestly analytic in spin j.[14] This implies that the CFT data computed by (1.47)—the scaling dimensions of local operators and the products of OPE coefficients—can be analytically continued in spin.[15] This implies that the local operators of different spins organize into families connected by this analytic continuation.

In chapter 6 we generalize the Lorentzian inversion formula (1.47) in two important directions: to operators of arbitrary spin and to operator level. In other words, we show that one can define the operators

$$\mathbb{O}_{\Delta,j}(x, z) = \int d^d x_1 d^d x_2 K_{\Delta,j}(x_1, x_2, x, z) O_1(x_1) O_2(x_2), \tag{1.48}$$

where z is a null polarization vector, $K_{\Delta,j}$ is a kernel well-defined for complex j, and as usual we suppress the Lorentz indices of local operators. The matrix elements of $\mathbb{O}_{\Delta,j}$ are then computed by an appropriate generalization of the scalar formula (1.47) and for integer j the residues of $\mathbb{O}_{\Delta,j}$ in Δ are related to the local operators of the theory. More generally, we argue that for any complex j the poles of $\mathbb{O}_{\Delta,j}$ come from the region of integration where O_1 and O_2 are confined to the light ray defined

[14] The conformal partial wave $\widetilde{F}_{\Delta,j}$ can also be analytically continued in spin, but it is not single-valued in the region of integration in (1.45) for non-integer j.

[15] This is true for the lowest dimension operators of every spin, but for higher-dimensional operators one can imagine that $c(\Delta, j)$ has cuts instead of poles at non-integral j. This question requires further investigation.

by z and thus the residues of these poles are non-local light-ray operators. For example, in the theory of generalized free scalar fields we show that residues of $\mathbb{O}_{\Delta,j}$ are proportional to

$$\frac{1}{\Gamma(-j)}\int ds du\, \phi_1((s+u)z)\, \phi_2((s-u)z)u^{-j-1} \tag{1.49}$$

when $\mathbb{O}_{\Delta,j}$ is inserted at past null infinity. For non-negative integer j this localizes[16] to $u=0$ and becomes a null integral of a local operator.

The key to these generalizations is to notice that the conformal block appearing in (1.47) has scaling dimension $j+d-1$ and spin $\Delta-d+1$, i.e., the roles of spin and scaling dimension are exchanged. From mathematical point of view, this is a particular affine Weyl reflection of a weight of the conformal group. It turns out that this Weyl reflection is not related to harmonic analysis on Euclidean conformal group $\mathrm{Spin}(1,d+1)$ but instead to harmonic analysis on Lorentzian conformal group $\widetilde{\mathrm{SO}}(2,d)$. It is an element of the so-called restricted Weyl group which is isomorphic to order-8 dihedral group. In harmonic analysis the restricted affine Weyl reflections are implemented by so-called Knapp-Stein intertwining operators [70, 71]. One can translate these intertwining operators to the CFT language, where they become conformally-invariant integral transforms which one can apply to primary operators. By studying properties of these integral transforms we simplify the derivation [66, 67] of (1.47) to the point where the generalizations discussed above become straightforward. We furthermore manage to rewrite the generalization of (1.47) in terms of natural objects so that the analogue of the non-trivial coefficient $\kappa_{\Delta,j}$ gets replaced by $(2\pi i)^{-1}$, thus giving an elegant formulation of the general result.

We also obtain other results which naturally follow from the above discussion. First, we give a generalization of some of the formulas used in conformal Regge theory in terms of the new integral transforms, arguing that the light-ray operators discussed above dominate, in an appropriate sense, the Regge limit of a time-ordered four-point function. Finally, we prove a novel continuous-spin version of (a higher-spin version of) averaged null-energy condition [72, 73] for CFTs which contain a sufficiently light scalar operator.

1.4.5 Chapter 7

In chapter 7 we study some truncations of the scalar crossing equation (1.21) and related equations. The goal here is to clarify some general questions about convergence rate and dominant contributions to the Euclidean OPE expansion.

[16] There is an appropriate $i\epsilon$-prescription which we have omitted for simplicity.

The first simplification that we make is to consider (1.21) only for $\sqrt{u}+\sqrt{v}=1$, which is the same as $z=\bar{z}=x$. We furthermore replace conformal blocks by their large scaling dimension limit. In the case of identical external scalars this leads to the following representation for the four-point function $G(x)$ [17,18]

$$G(x)=\int_0^\infty d\Delta\,\rho^\Delta x^{-2\Delta_\phi}g(\Delta)d\Delta, \tag{1.50}$$

where $\rho=\frac{4x}{(1+\sqrt{1-x})^2}$ and for convenience we have replaced a discrete sum by an integral over an "OPE density." We then study the derivatives of the crossing equation at $x=\frac{1}{2}$, which gives

$$\partial_x^{2k+1}G(x)\Big|_{x=\frac{1}{2}}=0, \qquad k=0,1,2,\ldots. \tag{1.51}$$

For large Δ and Δ_ϕ, derivatives with $k\ll\sqrt{\Delta}$ simplify as

$$\partial_x^{2k+1}\rho^\Delta x^{-2\Delta_\phi}\simeq\left(\frac{\partial_x\rho^\Delta x^{-2\Delta_\phi}}{\rho^\Delta x^{-2\Delta_\phi}}\right)^{2k+1}\rho^\Delta x^{-2\Delta_\phi}. \tag{1.52}$$

Using this simplification we show, for example, that for $x\geq\frac{1}{2}$ the four-point function (1.50) is dominated by states with

$$\Delta<\Delta_x\equiv\frac{2\Delta_\phi}{\sqrt{1-x}}, \tag{1.53}$$

and furthermore bound the contribution of states above this threshold as

$$\frac{1}{G(x)}\int_\Delta^\infty d\Delta\,\rho^\Delta x^{-2\Delta_\phi}g(\Delta)d\Delta\leq\frac{2}{1+T_{2k+1}\left(\frac{\Delta-\Delta_x/2}{\Delta_x/2}\right)}, \qquad (\Delta\geq\Delta_x) \tag{1.54}$$

where T is the Chebyshev polynomial, and $k\ll\sqrt{\Delta_\phi}$. We interpret this in terms of an approximate reflection symmetry which the crossing equations imply for the integrand of (1.50): it must be approximately reflection-symmetric around $\Delta_x/2$.

We repeat the same kind of analysis for several other crossing equations: for the modular invariance equation of 2d partition function, for "scaling block" version of four-point function, and for the four-point function in large dimension limit. We also derive a version of Cardy formula for these equations, which in the partition function case is equivalent to that of [74].

[17] We use notation $G(x)$ for the four-point function, which conflicts with our previous notation for the conformal block to match the notation of chapter 7. We hope this does not cause confusion.

[18] There is another technical approximation which goes into this. It is explained in chapter 7.

Finally, we consider the scaling block version of four-point function for finite scaling dimensions. This version is obtained from (1.50) by replacing $\rho \to x$ and can be interpreted as an exact truncation of the crossing equations. For it we derive an analytic upper bound on

$$\frac{1}{G(\frac{1}{2})}\int_{\Delta}^{\infty} d\Delta\, x^{\Delta-2\Delta_\phi} g^{(s)}(\Delta) d\Delta, \tag{1.55}$$

where (s) superscript distinguishes the OPE density in this case from the OPE density in (1.50). Our bound shows that this quantity decays exponentially at large Δ, improving the bounds of [25] in two aspects. First, it is asymptotically stronger by a factor of $\Delta^{-\frac{1}{2}}$ and second, it is valid for finite Δ (i.e., it is not asymptotic as in [25]).

1.4.6 Chapter 8

In chapter 8 we perform numerical bootstrap analysis of a particularly important four-point function—that of the stress-energy tensor. Importance of this four-point function comes from its universality, since stress-energy tensor is present in any local conformal field theory. We work in 3d and assume conservation of space parity (although some results are valid also in parity-violating theories).

Analysis of this four-point function is complicated by the fact that the stress-energy tensor is a conserved operator, i.e., we have

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.56}$$

as an operator equation. This leads to differential equations on its four-point function. Specifically, the Euclidean correlator can be written in the form

$$\langle TTTT\rangle = \sum_{I=1}^{97} \mathbb{Q}^I(x_1, x_2, x_3, x_4) g_I(z, \bar{z}), \tag{1.57}$$

where the conformally-invariant tensor structures $\mathbb{Q}_I$ carry all the spin indices and are constructed in chapter 2. Conservation equation (1.56) then leads to a system of first-order differential equations

$$\sum_{I=1}^{97}(A_{JI}\partial_z + \bar{A}_{JI}\partial_{\bar{z}} + C_{JI}) g_I(z, \bar{z}) = 0, \qquad J = 1, \ldots, 188. \tag{1.58}$$

There exist relations between these equations, which can be analyzed using methods of [75] and chapter 2. The analysis shows that these equations determine all 97 functions g_I in terms of 5 arbitrary functions and a set of boundary conditions. We carefully examine these equations and determine a complete and independent set of Taylor coefficients of functions g_I near $z = \bar{z} = \frac{1}{2}$. Since the conservation equations are crossing symmetric, it suffices to impose crossing symmetry only on this set.

Another challenge is the computation of the conformal blocks. For this we use the methods described in chapter 4, although since all bosonic representations in 3d are traceless-symmetric tensors the old results of [61] already suffice. Due to a large number of tensor structures it turns out to be crucial to adapt these methods to work with the construction of tensor structures in chapter 2.

After setting up the numerics, we first study lower bounds on the central charge C_T, defined as the coefficient of two-point function of stress-energy tensor

$$\langle TT \rangle \propto C_T, \tag{1.59}$$

as the function of the coefficients n_F and n_B of the three-point function

$$\langle TTT \rangle = n_B \langle TTT \rangle_B + n_F \langle TTT \rangle_F. \tag{1.60}$$

Here $\langle TTT \rangle_B$ and $\langle TTT \rangle_F$ are the three-point functions in the theories of a single free real scalar and a single free Majorana fermion respectively.[19] We find that the lower bound is of order 1 for non-negative n_B and n_F, but diverges if any of the parameters is less than 0. In this way we recover the celebrated Hofman-Maldacena bounds [76]

$$n_B, n_F \geq 0. \tag{1.61}$$

This represents a nice complement to the recent proof using analytic bootstrap methods [73, 77].

We also study the lower bound on C_T under additional assumptions about the spectrum. In particular, by imposing a lower bound on the scaling dimension of the lightest parity-odd scalar we find both upper and lower bounds on C_T, which force $C_T \sim 1$ and imply a small n_F/n_B ratio. We expect that 3d Ising CFT is consistent with the assumption imposed on the light spectrum, which allows us to estimate n_F and n_B in this theory by comparing our bounds with the known value of C_T. We find $0.01 \lesssim n_F \lesssim 0.02$.

We also study bounds on C_T assuming dimension gaps in other sectors, finding universal upper bounds on dimensions of lightest operators in these sectors. See section 8.4 for a complete summary.

[19] Due to the Ward identity $C_T = n_F + n_B$ our parameter is actually $\tan\theta = n_F/n_B$.

Chapter 2

COUNTING CONFORMAL CORRELATORS

This chapter is essentially identical to:

P. Kravchuk and D. Simmons-Duffin, *Counting Conformal Correlators*, *JHEP* **02** (2018) 096, [1612.08987].

2.1 Introduction

To apply conformal bootstrap techniques [26, 27, 30] to operators with spin, one must first understand the space of conformally-invariant tensor structures. This problem has been addressed previously for various types of operators in various dimensions [39, 50–56, 78, 79]. However, no completely general construction or classification of tensor structures currently exists in the literature.

The approaches [39, 53–56, 79] follow the strategy of defining basic conformally-invariant building blocks, and then multiplying them in all possible ways. While this strategy makes it easy to build conformally-invariant structures, it is not always convenient for bootstrap applications. This is because the building blocks satisfy nontrivial algebraic relations, which give rise to redundancies between structures built from them. As an example, of 201 possible parity-even combinations of the building blocks of [53] for the four-point function of identical spin-2 operators, only 97 are linearly independent in 3 dimensions. It is possible in principle to find relations between the 201 structures, and then choose a "standard" basis of 97 independent structures. However, this task is technically complicated and one may wonder if this step can be omitted completely.

In this chapter we discuss a different approach, which extends the formalism of [51, 78] to n-point functions. Based on the simple idea of "gauge-fixing" the conformal symmetry, our approach makes it possible to avoid the problem of algebraic relations completely in many cases. Furthermore, it applies uniformly to any operators in arbitrary representations of $SO(d)$, being essentially equivalent to invariant theory of orthogonal groups.

The basic idea is simple. Consider a three-point function $\langle O_1^{a_1}(x_1) O_2^{a_2}(x_2) O_3^{a_3}(x_3)\rangle$, where the operators O_i transform in representations ρ_i of the rotation group $SO(d)$, and a_i are indices for those representations. Using conformal transformations, we can place the operators in a standard configuration, say $\langle O_1^{a_1}(0) O_2^{a_2}(e) O_3^{a_3}(\infty)\rangle$, where e is a unit vector. The correlator must then be invariant under the "little group" for this configuration, which is the group $SO(d-1)$ of rotations that preserve the line through $0, e, \infty$. Such invariants are given by

$$\left(\mathrm{Res}^{SO(d)}_{SO(d-1)} \bigotimes_{i=1}^{3} \rho_i\right)^{SO(d-1)}, \tag{2.1}$$

where Res^G_H denotes restriction from a representation of G to a representation of $H \subseteq G$, and $(\rho)^H$ represents the H-invariant subspace of ρ (i.e., the singlet subrepresentations).

We generalize this argument in several directions: to arbitrary n-point functions, to incorporate permutation symmetries between identical operators, and most nontrivially to deal with conserved operators like currents J^μ and the stress-tensor $T^{\mu\nu}$. For three-point functions involving conserved operators, the conservation conditions become linear relations between tensor structures. However, for general n-point functions, conservation constraints become differential equations which are quite complicated to analyze [75]. The conclusion of [75] is that such correlators can be parametrized by a smaller number of functions of the conformal invariants of n points. For example, a parity-even four-point function of stress-tensors in 3d is parameterized by 5 scalar functions of conformal cross-ratios. We find a simple group-theoretic rule for counting these functions.

Besides simplicity, there are several motivations for characterizing the space of tensor structures in representation-theoretic language. Firstly, it is an obvious first step towards finding a general representation-theoretic formula for conformal blocks in $d > 2$ dimensions. Many examples of conformal blocks (not to mention superconformal blocks) have been computed using a variety of techniques [36, 37, 49, 54, 56–61, 63, 64, 80–82], but no one technique has yet proved completely general and efficient. Secondly, similar language might be helpful in classifying superconformally-invariant tensor structures, about which much less is known.

Importantly for numerical applications, our approach allows us to construct the tensor structures explicitly. We work out the tensor structures of non-conserved operators in 3d as an example.

It is well known [53, 56, 76] that the number of conformally-invariant tensor structures for a correlator in d-dimensions is equal to the number of Lorentz and gauge invariant tensor structures for a flat space scattering amplitude in $d+1$-dimensions. We demonstrate this relation by interpreting our group-theoretic counting rules in the S-matrix context.

2.2 Conformal correlators of long multiplets

In this section we describe in detail the construction and counting of tensor structures for correlators of long conformal mulptiplets (local operators not constrained by differential equations).

2.2.1 Conformal invariance

Consider a Euclidean CFT$_d$ on $\mathbb{R}^d$.[1] A conformally-invariant correlation function of n primary operators $\mathcal{O}_i^{a_i}(x_i)$ in representations ρ_i of $SO(d)$ can be expressed as

$$\left\langle \mathcal{O}_1^{a_1}(x_1)\dots\mathcal{O}_n^{a_n}(x_n)\right\rangle = \sum_{I=1}^{N}\mathbb{Q}_I^{a_1\dots a_n}(x_i)g^I(\mathbf{u}), \tag{2.2}$$

where g^I are scalar functions of the conformal invariants $\mathbf{u}$ of n points, and the possible tensor structures $\mathbb{Q}_I^{a_1\dots a_n}$ are constrained by conformal invariance. When some of the operators $\mathcal{O}_i$ are identical, these structures are further constrained by symmetry with respect to permutations. When one or more of the operators is a conserved current, the correlator also satisfies nontrivial differential equations.

Let $SO_0(d+1,1)$ be the identity component of the conformal group. Conformal transformations $U\in SO_0(d+1,1)$ act on primary operators as

$$U\mathcal{O}^a(x)U^{-1} = \Omega(x')^{\Delta}\rho^a{}_b(R(x')^{-1})\mathcal{O}^b(x'), \tag{2.3}$$

where

$$\Omega(x')R^\mu{}_\nu(x') = \frac{\partial x'^\mu}{\partial x^\nu}, \tag{2.4}$$

with $\Omega(x)>0$ and $R(x)\in SO(d)$. This leads to the following transformation of the correlator

$$\left\langle \mathcal{O}_1^{a_1}(x_1)\dots\mathcal{O}_n^{a_n}(x_n)\right\rangle = \left[\prod_{i=1}^{n}\Omega(x_i')^{\Delta_i}\rho_i^{a_i}{}_{b_i}(R(x_i')^{-1})\right]\left\langle \mathcal{O}_1^{b_1}(x_1')\dots\mathcal{O}_n^{b_n}(x_n')\right\rangle. \tag{2.5}$$

[1] Actually, we work on the conformal compactification S^d of $\mathbb{R}^d$, which means we can place operators at infinity. We will sometimes use the non-standard definition $\mathcal{O}(\infty)\equiv\lim_{L\to\infty}L^{2\Delta_\mathcal{O}}\mathcal{O}(Le)$, with e a fixed unit vector. The advantage of this definition is that we don't apply an inversion to $\mathcal{O}$, so $\mathcal{O}$ is treated more symmetrically with other operators in the correlator. The disadvantage is that the definition depends on e, so it breaks some rotational symmetries. However, in most of our computations these symmetries will already be broken by other operators in the correlator.

When some of the operators are fermionic, a small clarification is required. By construction, $R(x)$ is an element of $SO(d)$. However, it is the double cover $Spin(d)$ of $SO(d)$ that acts on a fermionic representation. One therefore must lift $R(x) \in SO(d)$ to some $\mathcal{R}(x) \in Spin(d)$. A natural point of view is to assign $\mathcal{R}(x)$ to an element r of the double cover $Spin(d+1,1)$ of the conformal group $SO_0(d+1,1)$: first we assign $\mathcal{R}(x) \equiv \text{id}$ to the identity of $Spin(d+1,1)$ and then define $\mathcal{R}$ on the rest of $Spin(d+1,1)$ by continuity. This is consistent because $Spin(d+1,1)$ is simply-connected. The invariance of correlation functions under the center of $Spin(d+1,1)$ is then simply the selection rule that the correlation function has to contain an even number of fermions.

To faciliate group-theoretic arguments, we write

$$g^{a_1\dots a_n}(x_1,\dots,x_n) = \left\langle \mathcal{O}_1^{a_1}(x_1)\dots\mathcal{O}_n^{a_n}(x_n)\right\rangle, \tag{2.6}$$

and define the action of the conformal group on g as follows. Let $r \in Spin(d+1,1)$ be a conformal transformation. It uniquely defines elements

$$\mathcal{R}_r(x') \in Spin(d), \quad \Omega_r(x') > 0, \tag{2.7}$$

as described above. We define the action of r on g by

$$(rg)^{a_1\dots a_n}(x_i,\dots,x_n) = \prod_{i=1}^n \Omega(x_i)^{-\Delta_i}\rho_i^{a_i}{}_{b_i}(\mathcal{R}_r(x_i))g^{b_1\dots b_n}(r^{-1}x_1,\dots,r^{-1}x_n). \tag{2.8}$$

With this definition, conformal invariane of the correlator is simply the statement that

$$rg = g. \tag{2.9}$$

We will often parametrize operators by polarizations, $\mathcal{O}(s,x) = s_a\mathcal{O}^a(x)$. In this case g becomes a function of s_i as well as x_i, and the above action becomes

$$(rg)(s_i,x_i) = \prod_{i=1}^n \Omega_r(x_i)^{-\Delta_i} g(\mathcal{R}_r(x_i)^{-1}s_i, r^{-1}x_i), \tag{2.10}$$

where for simplicity of notation we implicitly assume that s_i transforms in the dual representation $\rho_i^\vee$.

In a parity-preserving theory the above analysis should be extended to include reflections in $O(d)$. When fermions are present, one must specify a double cover $Pin(d)$ of $O(d)$ which will act on the spinor representations. In the following discussion this choice will be encapsulated in the representation theory of $Pin(d)$, and we therefore simply assume that a choice has been made which consistently defines an action of the disconnected conformal group on the correlators. In the following we will often refer to $SO(\cdot)$ or $O(\cdot)$ groups when we really mean their double covers if fermionic operators are involved. We hope that this will not cause confusion.

2.2.2 Conformal frame

Consider a four-point function of scalars,

$$g(x_1, x_2, x_3, x_4) = \langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle\,. \tag{2.11}$$

It is well-known that $g(x_i)$ only depends on two variables, the cross-ratios u and v,

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \tag{2.12}$$

where $x_{ij} = x_i - x_j$. The usual way to see this is to "fix" the conformal symmetry: choose a 2d half-plane α, a vector $e \in \partial\alpha$, and use conformal symmetry to set $x_1 = 0$, $x_3 = e$, and $x_4 = \infty$. The remaining symmetry is just the $SO(d-1)$ of rotations that fix e. Using these, we can put x_2 in α. Let us call the set of such configurations (when x_1, x_3 and x_4 are fixed and $x_2 \in \alpha$) a *conformal frame*.

Since any configuration can be mapped by a conformal transformation to a conformal frame configuration, it's clear that the full correlator g is uniquely fixed by its restriction g_0 to conformal frame configurations. These are parametrized by two coordinates for the point x_2 in α, which we can choose to be u and v.

With the coordinates x_i brought to a conformal frame configuration y_i, g_0 must still be invariant under the "little group." More precisely, let $St(y) \subset SO_0(d+1,1)$ be the group of conformal transformations that stabilize the y_i. Conformal invariance requires that for any $h \in St(y)$,

$$g_0(y_i) = (hg_0)(y_i). \tag{2.13}$$

For scalars this is automatic, since $St(y)$ is always a rotation group, and scalars are invariant under rotations. (For y in the interior of conformal frame, $St(y)$ is the $SO(d-2)$ of rotations orthogonal to α, and for y on the boundary $St(y)$ is the $SO(d-1)$ that fixes e.) Assuming that (2.13) holds, we can consistently define the full correlator g starting from g_0 by writing

$$g(x_i) = (r_x g_0)(x_i), \tag{2.14}$$

where r_x is any conformal transformation such that $y_i = r_x^{-1} x_i$ is in the conformal frame. The definition (2.14) doesn't depend on the choice of r_x for the usual reason: any other r'_x satisfies $r'_x = r_x h$ for some $h \in St(y)$, and this gives rise to the same $g(x_i)$ because of (2.13).

This approach clearly generalizes to n-point functions of operators in arbitrary $SO(d)$ representations — the only new ingredient is that the invariance (2.13) under the stabilizer subgroup $St(y)$ is now a non-trivial constraint. Quite generally, the configuration space of n points on the sphere splits into orbits under the action of the connected conformal group; we define the conformal frame to be a submanifold of the configuration space which intersects each orbit at precisely one point. Then all of the above works verbatim.

This is perhaps most striking for four-point functions in 3 dimensions. In this case, the stabilizer subgroup is generically the trivial $SO(3-2) = SO(1)$! So spinning four-point functions in 3d are almost no different from scalar ones. We return to this point in section 2.4.3.

Note that the above discussion showed that $St(y)$-invariance of g_0 is sufficient for g to be well-defined, but not necessarily smooth. If we require g to be smooth, we must impose more refined conditions for g_0 on the boundaries of the conformal frame. We discuss this point in appendix A.1. As we discuss in section 2.4.4, these conditions are important for formulating the bootstrap equations.

2.2.3 n-point functions

Consider the general case of $n \geq 3$ points. For convenience, we define $m = \min(n, d+2)$. To specify a conformal frame, we choose a flag of half-subspaces[2] α_i, $i = 2, \ldots m-2$, such that

$$\begin{aligned}
&\dim \alpha_i = i, \\
&\partial \alpha_i = \overline{\alpha}_{i-1}, \ \ i > 2, \\
&\partial \alpha_2 = \mathbb{R} e,
\end{aligned} \tag{2.15}$$

and $\overline{\alpha}_i$ is the linear subspace spanned by α_i. We first put operators $1, 2, 3$ at $0, e, \infty$, as before. We then use the remaining $SO(d-i+3)$ to bring the i-th operator to lie in α_{i-2}, for $i = 4, \ldots, m$. If $n > m$, we have already used all the conformal symmetry

[2] If $m = d+2$, then α_d should be the full linear subspace instead of a half-space. This is because when we fix the position of the last operator, we can only use $SO(d+3-m)$, which is trivial in this case.

to fix the positions of the first m operators, and the remaining $n-m$ operators can be anywhere.

After this is done, a generic conformal frame configuration has stabilizer subgroup $SO(d+2-m)$. It follows that the conformally-invariant tensor structures are given by

$$\left(\mathrm{Res}^{SO(d)}_{SO(d+2-m)} \bigotimes_{i=1}^{n} \rho_i\right)^{SO(d+2-m)}. \tag{2.16}$$

Again, Res^G_H denotes the restriction of a representation of G to a representation of $H \subseteq G$,[3] and ρ_i are the $SO(d)$ representations of the $\mathcal{O}_i$, and $(\rho)^H$ denotes the H-singlets in ρ.

This counting rule is consistent with the result of [56]. For simplicity, consider three-point functions. In [56], they show that the number of three-point structures for general tensor operators is the same as the number of traceless-symmetric tensors (TSTs) of $SO(d)$ in

$$\bigotimes_{i=1}^{3} \rho_i. \tag{2.17}$$

This is equivalent to (2.16) because the only $SO(d)$ representations that give singlets after restriction to $SO(d-1)$ are TSTs, and each TST gives exactly one singlet.

We can also count the dimension of the conformal moduli space $\overline{\mathcal{M}}_n = \mathcal{M}_n/SO(d+1,1)$ of n points, where $\mathcal{M}_n$ is the configuration space of n points on the sphere. By counting the unconstrained coordinates of the operators in conformal frame we get,

$$\dim \overline{\mathcal{M}}_n = \sum_{i=2}^{m-2} \dim \alpha_i + d(n-m) = \frac{m(m-3)}{2} + d(n-m). \tag{2.18}$$

This is of course also equal to

$$\dim \overline{\mathcal{M}}_n = \dim \mathcal{M}_n - \dim SO(d+1,1) + \dim SO(d+2-m). \tag{2.19}$$

Examples. Let us work out some simple examples of (2.16) in 3d. Let $\boldsymbol{\ell}$ denote the spin-ℓ representation of $SO(d)$, and (s) denote the charge-s representation of $SO(2) = U(1)$. For the trivial representation of the trivial group, we write $\bullet$.

[3] Because Res^G_H is a functor, we can restrict the representations before taking their tensor products. This sometimes simplifies calculations.

Consider an n-point function of non-identical vectors in 3d. When $n = 3$, the structures are given by $SO(2)$-singlets in

$$\begin{aligned}\left(\mathrm{Res}^{SO(3)}_{SO(2)}\mathbf{1}\right)^{\otimes 3} &= \left((1)\oplus(0)\oplus(-1)\right)^{\otimes 3} \\ &= (3)\oplus 3(2)\oplus 6(1)\oplus 7(0)\oplus 6(-1)\oplus 3(-2)\oplus(-3). \end{aligned} \tag{2.20}$$

In particular, there are 7 structures.

Let us emphasize that, despite the title of this chapter, (2.16) actually gives the *s*pace of structures, not just the number. For example, consider a three-point function of vectors $J_i(s_i, x_i) = s_i^\mu J_{i\mu}(x_i)$, where s_i^μ are polarization vectors. Restricting to the conformal frame configuration $\langle J_1(s_1, 0) J_2(s_2, e_1) J_3(s_3, \infty)\rangle$, we can write seven invariants under the $SO(2)$ of rotations in the 2-3 plane:

$$\begin{aligned} s_1^1 s_2^1 s_3^1, \quad s_1^1\, \delta_{ab} s_2^a s_3^b, \quad s_2^1\, \delta_{ab} s_3^a s_1^b, \quad s_3^1\, \delta_{ab} s_1^a s_2^b, \\ s_1^1\, \epsilon_{ab} s_2^a s_3^b, \quad s_2^1\, \epsilon_{ab} s_3^a s_1^b, \quad s_3^1\, \epsilon_{ab} s_1^a s_2^b, \end{aligned} \tag{2.21}$$

where δ_{ab} and ϵ_{ab} are the two-dimensional metric and epsilon symbol.

The correlator is then given by (2.14). Alternatively, we can map the structures (2.21) to the embedding-space structures of [53] using the dictionary[4]

$$\begin{aligned} s_i^1 &\mapsto V_i, \\ \delta_{ab} s_i^a s_j^b &\mapsto H_{ij} + V_i V_j, \\ \epsilon_{ab} s_i^a s_j^b &\mapsto 2\epsilon_{ij}. \end{aligned} \tag{2.22}$$

The resulting expressions will automatically be free of redundancies.

When $n \geq 4$, the stabilizer $SO(5-m)$ is trivial, and

$$\left(\mathrm{Res}^{SO(3)}_{1}\mathbf{1}\right)^{\otimes n} = (3\bullet)^{\otimes n} = 3^n \bullet, \tag{2.23}$$

so we have 3^n structures. In embedding space structures for $n \geq 5$, this corresponds to the fact that there are 3 linearly-independent V structures for each operator, and all H structures are redundant. For $n = 4$, we have two V structures per point and the H structures are replaced by $\epsilon(Z_i, P_1, P_2, P_3, P_4)$ in the notation of [53].

[4] Here, we use the nonstandard definition of an operator at infinity described in footnote 1.

2.2.4 Parity

If one wishes to distinguish parity-even and parity-odd structures, one has to note that the stabilizer group is actually $O(d+2-m)$ (for $n \geq 3$). There are two cases now, $n < d+2$ and $n \geq d+2$.

In the former case, $n < d+2$, the stabilizer subgroup contains a parity transformation. Therefore, parity of the correlator can be naturally defined on the conformal frame — parity-even structures are scalars under $O(d+2-m)$ and parity-odd structures are pseudo-scalars. Another way to state this is that reflection fixes the conformal frame and thus all the conformal invariants $\mathbf{u}$ of n points are parity even, and parity is a property of the tensor structure.

In the latter case, $n \geq d+2$, the stabilizer subgroup is trivial. Looking at the construction of the conformal frame, we see that parity actually acts within the conformal frame.[5] This means that there exist parity-odd conformal invariants $\mathbf{u}$ of n points, and it is actually quite easy to construct one. In the embedding-space formalism of [53] it can be written as

$$\frac{\epsilon(P_1 \cdots P_{d+2})}{\sqrt{P_{12}P_{23}\cdots P_{d+1,d+2}P_{d+2,1}}}. \tag{2.24}$$

Note that the condition $n \geq d+2$ enters this construction naturally. Using this invariant, all the tensor structures can be chosen to be parity-even. Parity of the correlator is then the property of the coefficient functions g^I.[6]

Examples. Let us apply the above discussion to n-point functions of parity-even vectors in 3d. We denote the parity-even/odd spin-ℓ representations of $O(3)$ by $\boldsymbol{\ell}^\pm$. The spin-ℓ representations of $O(2)$ are denoted $\boldsymbol{\ell}$ and the scalars/pseudoscalars are denoted $\mathbf{0}^\pm$.[7,8] Finally, the parity-even/odd representations of $O(1)$ are denoted $\bullet^\pm$.

[5] This is consistent with our definition of conformal frame, since that definition used only the connected component of the conformal group.

[6] If in the definition of conformal frame we used the full conformal group, then parity would not act on the conformal frame, but it also would not be a part of the stabilizer. Rather, r_x would contain the parity transformation for some x_i, and in that case the parity of the correlator would be supplied as extra information in the definition (2.14).

[7] Though we sometimes use the same notation for representations of different groups (for example scalars/pseudoscalars of $O(2)$ and $O(3)$), we hope that the relevant group will be clear from context.

[8] Note that spin-ℓ representations of $O(2)$ do not come in distinct parity-even and parity-odd versions. This is because $\epsilon_{\mu\nu}$ gives an isomorphism between the parity-even vector and the parity-odd vector in 2d. For spin-ℓ representations, we can act with $\epsilon_{\mu\nu}$ on one of the vector indices to get a parity-changing isomorphism. The only exception is the scalar representation, which comes in two versions $\mathbf{0}^\pm$, differing by a sign under reflections. Because of the ϵ isomorphism, we have $\mathbf{0}^\pm \otimes \boldsymbol{\ell} = \boldsymbol{\ell}$.

For three-point functions, we have

$$\left(\mathrm{Res}^{O(3)}_{O(2)}\mathbf{1}^+\right)^{\otimes 3} = \left(\mathbf{1}\oplus\mathbf{0}^+\right)^{\otimes 3} = \mathbf{3}\oplus 3\,\mathbf{2}\oplus 6\,\mathbf{1}\oplus 4\,\mathbf{0}^+\oplus 3\,\mathbf{0}^-, \tag{2.25}$$

so 4 of the 7 structures are parity-even and 3 are parity-odd, which is consistent with the explicit expressions (2.21). For four-point functions, we have

$$\left(\mathrm{Res}^{O(3)}_{O(1)}\mathbf{1}^+\right)^{\otimes 4} = \left(2\,\bullet^+\oplus\bullet^-\right)^{\otimes 4} = 41\,\bullet^+\oplus 40\,\bullet^-, \tag{2.26}$$

so 41 of the 81 structures are parity-even, and 40 are parity-odd. For $n \geq 5$, parity-odd cross-ratios exist and all structures can be chosen to be parity even. This is easily seen to be in accordance with the discussion after (2.23).

2.2.5 Permutation symmetry

In this section we consider the constraints of permutation symmetries from the point of view of the conformal frame. Derivations of some technical results of this section are collected in appendix A.2.

Correlators involving identical operators are (anti-)symmetric under permutations of those operators.[9] We can define the action of permutations on the correlator g by

$$(\pi g)^{a_1\ldots a_n}(x_1,\ldots,x_n) = \pm g^{a_{\pi(1)}\ldots a_{\pi(n)}}(x_{\pi(1)},\ldots,x_{\pi(n)}), \tag{2.27}$$

with a $-$ sign for an odd permutation of fermions. In terms of polarizations,

$$(\pi g)(s_i,x_i) = \pm g(s_{\pi(i)},x_{\pi(i)}). \tag{2.28}$$

Invariance under a permutation π is simply the statement that

$$\pi g = g. \tag{2.29}$$

Of course, in order to impose this consistently with conformal invariance, the quantum numbers of the exchanged operators should be equal.

Applying a permutation π to a conformal-frame configuration $p = \{x_i\}$ yields a new configuration πp which is generically not in the conformal frame. To compare the value of the correlator at πp with the value at p, one must find a conformal transformation that brings πp back to the conformal frame. More precisely, choose for every π a conformal transformation r_π such that the configuration $x'_i = r_\pi^{-1}x_{\pi(i)}$

[9] In principle it might be interesting to consider also permutations which exchange non-identical operators, in order to switch between conformal frames differing only by the ordering of operators.

belongs to conformal frame (in general r_π can depend on x_i). Then invariance (2.9) and (2.29) of the correlator requires

$$r_\pi \pi g = g. \tag{2.30}$$

By construction both the left hand side and right hand side depend only the values of g on the conformal frame and thus this requirement can be phrased in terms of g_0.

Depending on whether $x'_i = x_i$, this either restricts the number of tensor structures allowed for g_0 by constraining its value at a single point of the conformal frame, or simply relates values of g_0 at different points in the conformal frame. An example of the latter case is the crossing-symmetry equation for four-point functions. In the former case we say that the permutation is "kinematic". The permutations which satisfy $x'_i = x_i$ (and thus preserve the cross-ratios $\mathbf{u}$) form a subgroup $S_n^{\text{kin}} \subseteq S_n$.

For $n \leq 3$ the conformal frame consists of a single point, so permutations simply give linear relations between tensor structures and we have $S_n^{\text{kin}} = S_n$. For four-point functions, S_4^{kin} is the group of permutations that preserve u and v. This is $S_4^{\text{kin}} = \mathbb{Z}_2^2 = \{e, (12)(34), (13)(24), (14)(23)\}$ in cycle notation. For higher-point functions, S_n^{kin} is trivial because no nontrivial permutation preserves all the cross-ratios.

Let us be more explicit and assume that the correlator is invariant under a subgroup $\Pi \subseteq S_n$. In terms of polarizations we have for any $\pi \in \Pi$, using (2.8) and (2.27),

$$(r_\pi \pi g)(s_i, x_i) = (\pi g)(\mathcal{R}_{r_\pi}(x_i)^{-1} s_i, r_\pi^{-1} x_i) \prod_{i=1}^n \Omega_{r_\pi}^{-\Delta_i}(x_i) = g(s'_i, x'_i) \prod_{i=1}^n \Omega_{r_\pi}^{-\Delta_i}(x_i), \tag{2.31}$$

where

$$s'_i = \mathcal{R}_{r_\pi}(x_{\pi(i)})^{-1} s_{\pi(i)}, \tag{2.32}$$

and the scaling factor with Ω's is trivial if the scaling dimensions are invariant under π, which we assume. Suppose that the permutation is kinematic, $\pi \in \Pi^{\text{kin}}$, then the invariance condition becomes

$$g_0(s_i, x_i) = g_0(s'_i, x_i), \tag{2.33}$$

and basically constrains the value of $g_0(\cdot, x_i) \in \bigotimes_i \rho_i$. Therefore, we see that there is an action of Π^{kin} on $\bigotimes_i \rho_i$ which both permutes and twists the tensor factors. The tensor structures should be invariants of this action.

Since only S_3^{kin} and S_4^{kin} are non-trivial, it is easy to consider the permutations on a case by case basis. We do this in appendix A.2. In particular we describe there all r_π and the induced $\mathcal{R}_{r_\pi}$, which are required for practical calculations with tensor structures. For example, we use these results in our account of 3d tensor structures in section 2.4.

In the remainder of this section we derive group-theoretic rules for counting the permutation-symmetric tensor structures.

2.2.5.1 Three-point structures

In the case of three-point structures with non-trivial permutation symmetry we can have either $\Pi^{\text{kin}} = S_2$ or $\Pi^{\text{kin}} = S_3$.

Let us start with $\Pi^{\text{kin}} = S_2$, where we have two identical operators $\mathcal{O}_1 = \mathcal{O}_2$. Instead of going to the usual conformal frame, it is convenient to choose the configuration $\langle \mathcal{O}_1(-e)\mathcal{O}_3(0)\mathcal{O}_1(e)\rangle$, where e is a unit vector. This gives a function $\tilde{g}(s_i, e)$. By analogy with the usual conformal frame, it is sufficient to ensure that $\tilde{g}(s_i, e)$ is covariant under $SO(d)$ rotations (where we allow e to rotate as well as the s_i).

Before taking permutation symmetry into account, the tensor structures are in one-to-one correspondence with traceless symmetric tensors in $\rho_1 \otimes \rho_2 \otimes \rho_3$. (As we explained in section 2.2.3, this is equivalent to the space of singlets in 2.16.) Each such tensor of spin ℓ can be contracted with $e_{\mu_1} \ldots e_{\mu_\ell}$ to give the corresponding $\tilde{g}$. Now, permutation symmetry demands

$$\tilde{g}(s_1, s_2, s_3, e) = \pm\tilde{g}(s_2, s_1, s_3, -e) = \pm(-1)^\ell \tilde{g}(s_2, s_1, s_3, e), \tag{2.34}$$

where the $\pm$ sign is determined by the statistics of the operators $\mathcal{O}_1 = \mathcal{O}_2$, and the last equality is valid if $\tilde{g}$ comes from a spin-ℓ traceless-symmetric tensor in $\rho_1 \otimes \rho_2 \otimes \rho_3$. We find

Proposition 1 (S_2)**.** S_2-symmetric tensor structures are in one-to-one correspondence with even-spin traceless symmetric tensors in $\widehat{\mathrm{S}}^2\rho_1 \otimes \rho_3$ plus odd-spin traceless-symmetric tensors in $\widehat{\wedge}^2\rho_1 \otimes \rho_3$. Here, $\widehat{\mathrm{S}}^2$ denotes the symmetric square for bosonic arguments and exterior square for fermionic arguments, and $\widehat{\wedge}^2$ is defined analogously.

Now consider the case of S_3 symmetry with 3 identical operators. The full symmetry group is generated by permutations (12) and (123). We have already discussed

(12). We can generate the cyclic permutation (123) by exponentiating the action of $(P^\mu + K^\mu)e_\mu$. This moves the operators along the line spanned by e but does not rotate their polarizations, giving the condition

$$\tilde{g}(s_1, s_2, s_3, e) = \tilde{g}(s_3, s_1, s_2, e). \tag{2.35}$$

Together, (2.34) and (2.35) give the trivial representation of S_3 when ℓ is even and the sign representation when ℓ is odd. This leads to

Proposition 2 (S_3)**.** *S_3-symmetric tensor structures are in one-to-one correspondence with even-spin traceless symmetric tensors in $\mathrm{S}^3\rho_1$ plus odd-spin traceless-symmetric tensors in $\wedge^3\rho_1$.*[10]

In both propositions 1 and 2, the parity of the structure is determined by the intrinsic parity of the traceless symmetric representations.

2.2.5.2 Four-point structures

Let us now count four-point structures. Recall that in the absence of permutation symmetries, the space of tensor structures is

$$\left(\mathrm{Res}^{O(d)}_{O(d-2)} \bigotimes_{i=1}^{4} \rho_i\right)^{O(d-2)}. \tag{2.36}$$

The most natural generalization to symmetric correlators would be to symmetrize the tensor product by the kinematic symmetries of the correlator, including factors of (-1) for odd permutations of fermions. It turns out that this is almost correct, except that one does not need the (-1)'s. This is due to the fact that the conformal transformation that compares the permuted and unpermuted correlator also gives a (-1) for an exchange of fermions. The general statement is

Proposition 3 ($\mathbb{Z}_2$ and $\mathbb{Z}_2^2$)**.** *The space of tensor structures for four-point functions with permutation symmetry Π^{kin} is*

$$\left(\mathrm{Res}^{O(d)}_{O(d-2)} \left(\bigotimes_{i=1}^{4} \rho_i\right)^{\Pi^{\mathrm{kin}}}\right)^{O(d-2)}, \tag{2.37}$$

[10]The distinction between $\hat{S}$ and S has disappeared because all three operators are necessarily bosonic.

where Π^{kin} acts by a simple permutation on the tensor factors, regardless of the fermion/boson nature of the operators, and the parentheses mean taking the invariant subspace.[11]

We prove proposition 3 in appendix A.2.2.2. There are two non-trivial options for Π^{kin}: $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$. In the former case we simply need to compute the symmetric square of a representation. Indeed, without loss of generality assume that the non-trivial permutation is $(13)(24)$, and so $\rho_1 = \rho_3$ and $\rho_2 = \rho_4$. It is easy to see that

$$\left(\bigotimes_{i=1}^{4} \rho_i\right)^{\mathbb{Z}_2} = \mathrm{S}^2(\rho_1 \otimes \rho_2). \tag{2.38}$$

The latter case is a bit more involved. First, all the representations have to be identical, $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho$. The relevant formula is then, as we show in appendix A.3,

$$\left(\bigotimes_{i=1}^{4} \rho_i\right)^{\mathbb{Z}_2^2} = \rho^4 \ominus 3\left(\wedge^2\rho \otimes \mathrm{S}^2\rho\right), \tag{2.39}$$

where $\ominus$ represents the formal difference[12] in the character ring.

Examples. As examples, consider n-point correlators of identical parity-even vectors in 3d. For $n = 3$, we have the following identities among $O(3)$ representations:

$$\begin{aligned} \mathrm{S}^3\mathbf{1}^+ &= \mathbf{3}^+, \\ \wedge^3\mathbf{1}^+ &= \mathbf{0}^-. \end{aligned} \tag{2.40}$$

By proposition 2, it follows that there are no nontrivial three-point structures. For $n = 4$, using proposition 3 with $\Pi^{\text{kin}} = \mathbb{Z}_2^2$ and equation (2.39), we have

$$(2\,\bullet^+ \oplus \bullet^-)^4 \ominus 3\left(\wedge^2(2\,\bullet^+ \oplus \bullet^-) \otimes \mathrm{S}^2(2\,\bullet^+ \oplus \bullet^-)\right) = 17\,\bullet^+ \oplus 10\,\bullet^-, \tag{2.41}$$

so there are 17 parity-even structures and 10 parity-odd structures in a four-point function of identical vectors. Finally, for $n \geq 5$, kinematic permutations are absent, so there are 3^n structures (which can be taken to be parity-even).

[11] One can also project to singlets of Π^{kin} after applying $\mathrm{Res}^{O(d)}_{O(d-2)}$.

[12] One can think about representations in terms of characters. Since characters are functions, there is no problem with taking differences. Alternatively, one can think of a reducible representation as a formal sum of irreducible representations with non-negative coefficients. Then, taking a difference of representations is equivalent to taking differences of these coefficients. Some coefficients may end up being negative, in which case the result is called a "virtual" representation. The representation (2.39) is guaranteed not to be virtual.

Consider an example with two identical Majorana fermions and two identical scalars, all parity-even. We have the following $O(2,1)$ identity

$$S^2\frac{\mathbf{1}}{\mathbf{2}} = \mathbf{1}^+. \tag{2.42}$$

Using proposition 3 with $\Pi^{\text{kin}} = \mathbb{Z}_2$ and equation (2.38), we find the space of four-point structures

$$2\,\bullet^+ \oplus \bullet^-, \tag{2.43}$$

so there are 2 parity-even structures and 1 parity-odd structure. This agrees with [81]. Note that it was essential not to include (-1) for a permutation of fermions in proposition 3.

2.2.6 Summary: tensor structures of long multiplets

The discussion above can be summarized as the following theorem.

Theorem 1. *The conformal correlator involving $n \geq 3$ operators in representations ρ_i can be written as*

$$\left\langle O_1^{a_1}(x_1)\dots O_n^{a_n}(x_n)\right\rangle = \sum_I \mathbb{Q}_I^{a_1\dots a_n} g^I(\mathbf{u}), \tag{2.44}$$

where $\mathbf{u}$ is a set of coordinates on the conformal moduli space $\overline{\mathcal{M}}_n$ of n points $x_1\dots x_n$,

$$\dim\overline{\mathcal{M}}_n = \frac{m(m-3)}{2} + d(n-m), \quad m = \min(n, d+2), \tag{2.45}$$

and the conformally-invariant tensor structures $\mathbb{Q}_I$ are in one-to-one correspondence with scalars (for parity-even structures) and pseudo-scalars (for parity-odd structures) in the representation of $O(d+2-m)$ given by

$$\mathrm{Res}^{O(d)}_{O(d+2-m)} \bigotimes_{i=1}^n \rho_i. \tag{2.46}$$

If parity is not conserved, one simply replaces $O(\cdot)$ groups with $SO(\cdot)$ groups above. If $n \geq d+2$, then one can form parity-odd cross-ratios, and parity of the correlator is rather a property of the functions g^I rather that the structures $\mathbb{Q}_I$, which can all be chosen to be parity-even.

When $n = 3$ or $n = 4$ the correlator (2.44) can have a group Π^{kin} of permutation symmetries which leave $\mathbf{u}$ invariant, and thus impose constraints on the structures $\mathbb{Q}_I$. The spaces of structures in these cases are described in propositions 1, 2, and 3.

2.3 Conservation conditions

We now consider correlation functions of operators that satisfy conservation conditions. We are mainly interested in the number of "functional degrees of freedom" in such correlators — i.e., the number of functions of cross-ratios needed to completely specify the correlator [75]. For simplicity, we mostly restrict our attention to traceless symmetric tensor conserved currents, of which spin-1 currents and the stress tensor are prime examples. We describe the modifications required for more general operators at the end of this section.

Correlation functions involving conserved currents are constrained by differential equations such as

$$\frac{\partial}{\partial x_1^{\mu_1}}\langle J^{\mu_1\ldots\mu_\ell}(x_1)\ldots\rangle = \frac{\partial}{\partial x_1^{\mu_1}}\sum_{I=1}^{N}\mathbb{Q}_I^{\mu_1\ldots\mu_\ell\cdots}(x_i)g^I(\mathbf{u}) = \text{ contact terms.} \tag{2.47}$$

When $n \geq 4$, these are differential constraints on the functions $g_I(\mathbf{u})$. In general, the full set of conservation equations is not independent and this makes it not immediately clear how many degrees of freedom there actually are. The purpose of this section is to classify the relations between these equations and motivate a group-theoretic rule for the number of degrees of freedom of such correlators for $n \geq 4$.

Our rule will also classify "generic" three-point functions—i.e., three-point correlators where at least one operator has generic dimension Δ. When the dimensions of operators are non-generic, extra three-point structures can appear. The simplest example occurs for a three-point function of a conserved current and two scalars, $\langle J_\mu\phi_1\phi_2\rangle$. Generically, no structure exists for such a correlator, but a special structure becomes possible when the scalars have equal dimensions $\Delta_1 = \Delta_2$. These special structures are related to the contact terms on the right-hand side of (2.47). For higher-point correlators, non-generic structures have a fixed x_i dependence, so they do not contribute to the number of functional degrees of freedom.

Our strategy is to understand the relations between equations (2.47). In general, if we have a system of equations

$$D_1\mathbf{g} = 0, \tag{2.48}$$

where $\mathbf{g}$ is a vector of N_0 unknown functions and D_1 is a $N_1\times N_0$ matrix with differential operator coefficients, we say that there are relations between the equations (2.48) if there is an $N_2 \times N_1$ matrix D_2 such that

$$D_2D_1 = 0. \tag{2.49}$$

Note that here $D_2 D_1 \mathbf{g} = 0$ independently of (2.48). There is a sense in which D_2 can be complete. Namely, we say that D_2 is a compatibility[13] operator for D_1 iff any other $\tilde{D}_2$ satisfying $\tilde{D}_2 D_1 = 0$ can be expressed as $\tilde{D}_2 = Q D_2$ for some matrix differential operator Q. It can happen that there are further relations between the relations D_2, i.e., an $N_3 \times N_2$ matrix D_3 such that

$$D_3 D_2 = 0, \text{ etc.} \tag{2.50}$$

If at some point this sequence of compatibility operators terminates — i.e., for $i > i_0$ we have $N_i = 0$ — then we can compute a version of the Euler characteristic

$$N = \sum_{i=0}^{\infty} (-1)^i N_i. \tag{2.51}$$

We expect that N is the true number of functional degrees of freedom parametrizing a solution to (2.48). Note that by the number of functional degrees of freedom we mean the functional parameters which depend on the same number of variables as the original equation.

Consider first the simplest case of conservation of a spin-ℓ traceless-symmetric current,

$$\frac{\partial}{\partial x^{\mu_1}} J^{\mu_1 \cdots \mu_\ell}(x) = 0, \tag{2.52}$$

which can be phrased as setting to zero a spin-$(\ell - 1)$ operator

$$V^{\mu_1 \cdots \mu_{\ell-1}}(x) = \frac{\partial}{\partial x^{\mu}} J^{\mu \mu_1 \cdots \mu_{\ell-1}}(x). \tag{2.53}$$

If the current J has scaling dimension $\Delta_J = d + \ell - 2$, then the conservation equation is conformally-covariant, meaning simply that V transforms as a primary operator. Note that V is still conserved, but $\partial V = 0$ does not constitute a relation between the conservation equations in the above sense — it only holds if the original equation is satisfied. In fact, there is no differential operator which annihilates the left hand side of (2.52).

Since V is a primary, inserting it into a correlator we find

$$\langle V^{\mu_1 \cdots \mu_{\ell-1}} \ldots \rangle = \sum_{I=1, J=1}^{N_1, N} \tilde{\mathbb{Q}}_I^{\mu_1 \cdots \mu_{\ell-1} \cdots} (D_1)^I{}_J g^J(\mathbf{u}) = 0, \tag{2.54}$$

[13] This name comes from considering the equation $D_1 \mathbf{g} = \mathbf{f}$. The function $\mathbf{f}$ is compatible with this equation only if $D_2 \mathbf{f} = 0$. Systems of equations for which a non-trivial D_2 exists are known as overdetermined systems.

where the structures $\tilde{\mathbb{Q}}_I$ are the conformally invariant structures suitable for the correlator on the left. Note that the structures $\mathbb{Q}$ are in one-to-one correspondence with singlets in

$$[\ell \otimes \rho_2 \otimes \ldots] = [\ell] \otimes [\rho_2] \otimes \ldots, \tag{2.55}$$

where we use $[\,\cdot\,]$ to denote the restriction to $SO(d+2-m)$. On the other hand, the structures $\tilde{\mathbb{Q}}$ are given by the singlets in

$$[(\ell-1) \otimes \rho_2 \otimes \ldots] = [\ell-1] \otimes [\rho_2] \otimes \ldots. \tag{2.56}$$

If there is only one current in the correlator, then there are no relations between the equations and the number of degrees of freedom is given by the number of singlets in

$$\Big([\ell] \otimes [\rho_2] \otimes \ldots\Big) \ominus \Big([\ell-1] \otimes [\rho_2] \otimes \ldots\Big) = \Big([\ell] \ominus [\ell-1]\Big) \otimes [\rho_2] \otimes \ldots. \tag{2.57}$$

Here the $\ominus$ is the formal difference[14] in the character ring of $SO(d+2-m)$. The idea now is to note

$$\mathrm{Res}^{SO(d)}_{SO(d-1)}\ell \ominus \mathrm{Res}^{SO(d)}_{SO(d-1)}(\ell-1) = \ell', \tag{2.58}$$

where ℓ' is the spin-ℓ traceless symmetric representation of $SO(d-1)$.[15] Therefore, we see that the number of degrees of freedom is given by the singlets in

$$[\ell'] \otimes [\rho_2] \otimes \ldots \tag{2.59}$$

One may wonder if this rule holds more generally — i.e., whether one can compute the number of degrees of freedom in any correlator involving conserved operators by simply replacing the $SO(d)$ representations of these operators with their "effective" $SO(d-1)$ representations in Theorem 1. This is indeed so[16], and in section 2.3.1 we show in examples how this rule works in the situations when we have several conserved operators or when there are permutation symmetries.

In the example considered above the primary V obtained from J did not have any null states of its own, so it was easy to count the number of degrees of freedom in the correlator (2.54). For operators J satisfying more general conformally-invariant differential equations it may turn out that V itself has a null descendant V', and thus

[14] See footnote 12.

[15] Note that $SO(d-1)$ is the little group for massless particles in $d+1$ dimensions. We will make use of this fact in section 2.5.

[16] As we note in the beginning of this section, for three point functions this is only true for sufficiently generic scaling dimensions of the operators.

satisfies a conformally-invariant differential equation expressed as $V' = 0$. Now V' can turn out to have null descendants V'', and so on. A simple class of examples when this happens are the differential forms from the de Rham complex. Repeating the above analysis, we see that the effective $SO(d-1)$ representation we should use in this situation is

$$[\rho] \ominus [v] \oplus [v'] \ominus [v''] \oplus \ldots, \tag{2.60}$$

where ρ is the $SO(d)$ representation of J and v is the $SO(d)$ representation of V and so on.

We expect that quite generally this alternating sum gives an actual representation of $SO(d-1)$. Indeed, we have $V = \mathcal{D}J$ for some conformally invariant differential operator $\mathcal{D}$. Because of translation invariance $\mathcal{D}$ has constant coefficients, and thus the equation

$$\mathcal{D}J = 0 \tag{2.61}$$

is in momentum space a simple linear equation for the amplitude J. In particular, for each fixed momentum p, the space of solutions is a finite-dimensional representation of $SO(d-1)$ which leaves p invariant. It is easy to convince oneself that this is the representation which (2.60) is computing.

In applications to unitary conformal field theories we are only interested in operators J with the scaling dimension saturating some unitarity bound — these are the only operators which are unitary and have null descendants at the same time. A detailed classification of such operators can be found in section 5 of [83] (see also [42, 84]), here we only give a short summary. Among these operators, some can be classified as free and the rest, which we will call the unitary conserved currents, satisfy first-order differential equations. In 3d and 4d all unitary conserved currents are generalizations of $(d-1)$-forms and they do not have the analogue of V'. In 5d and 6d there appear unitary conserved currents which generalize $(d-2)$-forms, and they have V' but not V''. Given the classification in [83], it is an easy exercise to find the effective $SO(d-1)$ representation for arbitrary unitary conserved currents in $d \leq 6$.

2.3.1 Multiple conserved operators and permutation symmetries

Let us see how the rule (2.59) behaves when there are several conserved currents in the correlator. Consider for example the case of two currents J_1 and J_2. We then have the equations

$$\langle V_1 J_2 \ldots \rangle = 0, \tag{2.62}$$

$$\langle J_1 V_2 \ldots \rangle = 0. \tag{2.63}$$

But there is a relation between these equations. Taking the remaining divergences in both equations we arrive in both cases at

$$\langle V_1 V_2 \ldots \rangle = 0, \tag{2.64}$$

and by taking the difference we obtain 0 regardless of whether $V_i = 0$ or not. This thus leads to a number of relations. This number is equal to the number of tensor structures in $\langle V_1 V_2 \ldots \rangle$. Therefore, we need to add it to the number of degrees of freedom,

$$\begin{aligned}&\Big([\ell_1] \otimes [\ell_2]\Big) \ominus \Big([\ell_1] \otimes [\ell_2 - 1]\Big) \ominus \Big([\ell_1 - 1] \otimes [\ell_2]\Big) \oplus \Big([\ell_1 - 1] \otimes [\ell_2 - 1]\Big) \\ &= [\ell_1'] \otimes [\ell_2'].\end{aligned} \tag{2.65}$$

It is easy to see that this generalizes to any number of conserved operators.

Consider now the case when the operators J_1 and J_2 are identical, $\ell_1 = \ell_2 = \ell$ and there is a kinematic permutation expressing this. Assume that $n = 4$ and the other operators are scalars for simplicity. In this case the equations (2.62) and (2.63) are equivalent, since the tensor structures for $\langle J_1 J_2 \ldots \rangle$ are chosen to be symmetric. Then we can use just one equation, say (2.62). However, it is still subject to relations. In particular, if we take an extra divergence to get to the equation (2.64), we will find that it is symmetric in permutation of V's, and thus antisymmetrizing the V's we get 0. Since it is a non-trivial operation which we applied to (2.62), it constitutes a relation among equations (2.62). Therefore we need to look for scalars in

$$\mathrm{S}^2[\ell] \ominus \Big([\ell] \otimes [\ell - 1]\Big) \oplus \wedge^2[\ell - 1]. \tag{2.66}$$

Incidentally, the following relation holds in the character ring,

$$\mathrm{S}^2\,(\chi_1 - \chi_2) = \mathrm{S}^2\chi_1 - \chi_1\chi_2 + \wedge^2\chi_2. \tag{2.67}$$

It can be easily derived from the character formulas (A.32) and (A.33). We therefore see that the prescription works even when there is a permutation symmetry,

$$\mathrm{S}^2[\ell'] = \mathrm{S}^2[\ell] \ominus \Big([\ell] \otimes [\ell - 1]\Big) \oplus \wedge^2[\ell - 1]. \tag{2.68}$$

The techniques above also allow us to keep track of parity by simply replacing SO groups with O groups.

Examples (Conserved four-point functions in 3d and 4d)**.** As examples, let us compute the number of functional degrees of freedom in a four-point function of identical, conserved, parity-even, spin-ℓ currents in 3d and 4d. Applying proposition 3, equation (2.39), and the discussion above, we must find the number of $O(d-2)$ scalars $\bullet^+$ and pseudoscalars $\bullet^-$ in

$$\rho = [\ell']^4 \ominus 3\left(\wedge^2[\ell'] \otimes \mathrm{S}^2[\ell']\right). \tag{2.69}$$

In 3d, $[\ell']$ is the restriction of the spin-ℓ traceless symmetric tensor of $O(2)$ to $O(1)$, which is simply $[\ell'] = \bullet^+ \oplus \bullet^-$. Plugging in we easily find

$$\rho_{3\mathrm{d}} = 5\bullet^+ \oplus 2\bullet^-, \tag{2.70}$$

so there are 5 parity even and 2 parity odd degrees of freedom. Note that the answer is independent of ℓ. As we will see in section 2.5, this is related to the fact that massless particles in 4d always have two degrees of freedom, regardless of helicity.

In 4d, it is convenient to use characters of $O(2)$. $O(2)$ is a semidirect product

$$U(1) \rtimes \mathbb{Z}_2 = \{(x,s) : x \in U(1), s = \pm 1\}, \tag{2.71}$$

with the multiplication rule

$$(x_1, s_1)(x_2, s_2) = (x_1 x_2^{s_1}, s_1 s_2). \tag{2.72}$$

The spin-j representation $\mathbf{j}$ has character

$$\chi_{\boldsymbol{\ell}}(x,s) = \frac{1+s}{2}(x^j + x^{-j}), \tag{2.73}$$

while the scalars $\bullet^+$ and pseudoscalars $\bullet^-$ have characters 1 and s, respectively. $[\ell']$ is the restriction of the parity-even spin-ℓ representation of $O(3)$ to $O(2)$, namely

$$[\ell'] = \boldsymbol{\ell} \oplus (\boldsymbol{\ell}-\mathbf{1}) \oplus \cdots \oplus \mathbf{1} \oplus \bullet^+, \tag{2.74}$$

which has character

$$\chi_{[\ell']}(x,s) = \frac{1+s}{2}\frac{x^{\ell+\frac{1}{2}} - x^{-\ell-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} + \frac{1-s}{2}. \tag{2.75}$$

Plugging (2.75) into equation (A.36) for the character of a $\mathbb{Z}_2^2$-invariant tensor product, we find

$$\begin{aligned}
&\chi_{\rho_{4\mathrm{d}}}(x,s) \\
&= \frac{1+s}{2}\left(\frac{1}{4}\left(\frac{x^{\ell+\frac{1}{2}} - x^{-\ell-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}\right)^4 + \frac{3}{4}\left(\frac{x^{2\ell+1} - x^{-2\ell-1}}{x - x^{-1}}\right)^2\right) + \frac{1-s}{2}(3\ell^2 + 3\ell + 1) \\
&= \frac{(4\ell+3)(\ell+2)(\ell+1)}{6} + \frac{(4\ell+1)\ell(\ell-1)}{6}s + \ldots,
\end{aligned} \tag{2.76}$$

where "…" represents sums of spin-j characters (2.73). The constant term in (2.76) is the number of parity-even structures and the coefficient of s is the number of parity-odd structures. Plugging in $\ell = 1, 2$, we obtain $7 + 0s$ and $22 + 3s$, respectively, in agreement with [75].

2.4 Correlation functions in 3d

In this section we consider in detail correlation functions in three dimensions, in order to exemplify how our formalism gives the tensor structures rather than just their number, and how this can be applied in practice.

2.4.1 Conventions for $SO(2,1)$

In this section we will be working in Lorentzian signature in order to allow Majorana spinors. Our conventions for spinors will be those of [39]. In this subsection we describe the basic notation.

The primary operators in 2+1 dimensions transform in representations of $Spin(2,1) \simeq Sp(2,\mathbb{R})$. The smallest such representation is the two-component Majorana spinor $\mathbf{\frac{1}{2}}$, the fundamental of $Sp(2,\mathbb{R})$

$$\psi^\alpha. \tag{2.77}$$

This representation is equivalent to its dual

$$\psi_\alpha, \tag{2.78}$$

due to the invariant symplectic form of $Sp(2,\mathbb{R})$

$$\Omega^{\alpha\beta} = \Omega_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \psi_\alpha = \Omega_{\alpha\beta}\psi^\beta. \tag{2.79}$$

We have $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathrm{S}^2\mathbf{\frac{1}{2}} \oplus \wedge^2\mathbf{\frac{1}{2}} = \mathbf{1} \oplus \mathbf{0}$. The equivalence between $S^2\mathbf{\frac{1}{2}}$ and the vector representation of $Spin(2,1)$ is established by the gamma matrices $(\gamma^\mu)^\alpha{}_\beta$,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.80}$$

More precisely, we have

$$v^\mu = \Omega_{\alpha\sigma}(\gamma^\mu)^\sigma{}_\beta v^{(\alpha\beta)}. \tag{2.81}$$

Generally, all finite-dimensional representations of $Spin(2,1)$ are the symmetric powers of the Majorana representation, $\boldsymbol{\ell} = \mathrm{S}^{2\ell}\mathbf{\frac{1}{2}}$. We therefore represent an arbitrary real operator $\mathcal{O}$ of spin ℓ as

$$\mathcal{O}^{(\alpha_1\ldots\alpha_{2\ell})}(x), \tag{2.82}$$

and we will use index-free notation by introducing a polarization spinor s,

$$O(s,x) = s_{\alpha_1}\dots s_{\alpha_{2\ell}} O^{(\alpha_1\dots\alpha_{2\ell})}(x). \tag{2.83}$$

We need to make a choice of $Pin(2,1)$ group to consider parity. Reflection $x^1 \to -x^1$ is generated by

$$\psi \to \pm\gamma^1\psi, \tag{2.84}$$

and reflection $x^2 \to -x^2$ is generated by

$$\psi \to \pm\gamma^2\psi, \tag{2.85}$$

as can be checked by considering the induced action on the vector representation. The sign ambiguity reflects the fact that it is a double cover $Pin(2,1)$ of $O(2,1)$ which acts on spinors, so there are twice as many “reflections” as in $O(2,1)$.

2.4.2 Three-point structures

We choose the standard positions for the three operators by picking

$$x_1 = (0,0,0), \tag{2.86}$$

$$x_2 = (0,0,1), \tag{2.87}$$

$$x_3 = (0,0,L), \tag{2.88}$$

and considering the correlator

$$g_0(s_1,s_2,s_3) = \lim_{L\to+\infty} L^{2\Delta_3}\langle O_1(s_1,x_1)O_2(s_2,x_2)O_3(s_3,x_3)\rangle. \tag{2.89}$$

The connected component of the stabilizer subgroup in this case consists of boosts $s_i \to e^{-i\lambda\overline{K}_1}s_i$ with

$$\overline{K}_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}. \tag{2.90}$$

Writing

$$(s_i)_\alpha = \begin{pmatrix} \xi_i \\ \overline{\xi}_i,\end{pmatrix} \tag{2.91}$$

we see that ξ_i has charge $+1/2$ under these boosts, and $\overline{\xi}_i$ has charge $-1/2$.

According to the general rule, the three-point functions are in one-to-one correspondence with stabilizer-invariant functions $g_0(s_i)$. Clearly, one can choose a basis for such functions consisting of monomials

$$[q_1q_2q_3] = \prod_{i=1}^{3} \xi_i^{\ell_i+q_i}\overline{\xi}_i^{\ell_i-q_i}, \tag{2.92}$$

with $q_i \in \{-\ell_i, \ldots, \ell_i\}$ subject to

$$\sum_{i=1}^{3} q_i = 0. \tag{2.93}$$

If parity is conserved, then stabilizer subgroup also contains parity transformation $s_i \to \gamma_1 s_i$. This simply exchanges ξ_i and $\overline{\xi}_i$. Therefore, structures of definite parity are given by

$$[q_1 q_2 q_3]^{\pm} \equiv \prod_{i=1}^{3} \xi_i^{\ell_i+q_i} \overline{\xi}_i^{\ell_i-q_i} \pm \prod_{i=1}^{3} \xi_i^{\ell_i-q_i} \overline{\xi}_i^{\ell_i+q_i}, \tag{2.94}$$

and now sets q_i and $-q_i$ are identified.

Permutations. Consider the permutations, starting with the transposition (12). According to the general rule, we need to apply a transformation which brings the operators back to the conformal frame position after the permutation. We are interested in the $Spin(3)$ elements

$$\mathcal{R}_{r_\pi}^{-1}(x_i) \tag{2.95}$$

induced at the insertions of the operators. These are computed in the appendix A.2 with the result that for all transpositions there are $e^{\pm i\pi/2}$ at all insertions, inducing $s_i \mapsto \pm\gamma^0 s_{\pi(i)}$, under which $\xi_i \mapsto \pm\overline{\xi}_{\pi(i)}$ and $\overline{\xi}_i \mapsto \mp\xi_{\pi(i)}$. Taking into account the precise signs, we find the action of the permutations

$$(12) : [q_1 q_2 q_3]^{\pm} \mapsto \pm(-1)^{\ell_1+\ell_2-\ell_3} [q_2 q_1 q_3]^{\pm}, \tag{2.96}$$

$$(13) : [q_1 q_2 q_3]^{\pm} \mapsto \pm(-1)^{\ell_1+\ell_2+\ell_3} [q_3 q_2 q_1]^{\pm}, \tag{2.97}$$

$$(23) : [q_1 q_2 q_3]^{\pm} \mapsto \pm(-1)^{-\ell_1+\ell_2+\ell_3} [q_1 q_3 q_2]^{\pm}. \tag{2.98}$$

If the permutations are symmetries of the correlator, the signs in front of ℓ_i above can all be chosen to be +, since, e.g., for permutation (12) ℓ_3 has to be integral for the full correlator to be bosonic. Under these permutations the tensor structure has to be symmetric or anti-symmetric depending on whether the exchanged operators are bosons or fermions. Redefining the permutations as

$$(12)' : [q_1 q_2 q_3]^{\pm} \mapsto \pm(-1)^{\ell_3} [q_2 q_1 q_3]^{\pm}, \tag{2.99}$$

$$(13)' : [q_1 q_2 q_3]^{\pm} \mapsto \pm(-1)^{\ell_2} [q_3 q_2 q_1]^{\pm}, \tag{2.100}$$

$$(23)' : [q_1 q_2 q_3]^{\pm} \mapsto \pm(-1)^{\ell_1} [q_1 q_3 q_2]^{\pm}, \tag{2.101}$$

we now have the requirement that the tensor structure is symmetric regardless of the nature of the operators.

Counting. Let us now count the number of structures, assuming all the operators to be different. By counting all possible combinations of q_i one easily recovers the result of [53] for the number of 3-point structures,

$$N_{3d}(\ell_1,\ell_2,\ell_3) = (2\ell_1+1)(2\ell_2+1) - p(p+1), \tag{2.102}$$

where $p = \max(\ell_1+\ell_2-\ell_3, 0)$ and $\ell_1 \le \ell_2 \le \ell_3$. Unless all three operators are bosons, $q_i \equiv 0$ is not a solution, and thus there is an equal number of parity-even and parity-odd structures. In case all three operators are bosons, $q_i \equiv 0$ gives a valid parity-even structure. In this case the number of parity-even structures is larger than the number of parity-odd structures by 1. We then have for the number of definite-parity structures

$$N^{\pm}_{3d}(\ell_1,\ell_2,\ell_3) = \frac{N_{3d}(\ell_1,\ell_2,\ell_3)\pm\kappa}{2}, \tag{2.103}$$

where $\kappa = 1$ when all the operators are bosonic, and $\kappa = 0$ otherwise.

In the case when there are identical operators, there are two options. The first option is that there are two identical operators, say $\ell_1 = \ell_2$. The second is that all three operators are identical. In the first case one can show

$$N^{\pm}_{3d}(\ell_1\leftrightarrow\ell_2,\ell_3) = \frac{N^{\pm}_{3d}(\ell_1,\ell_1,\ell_3)}{2} + \frac{(-1)^{\ell_3}}{2}\left[\ell_1 + \tfrac{1\pm\kappa}{2} \pm \min(\lfloor \ell_1+\tfrac{1}{2}\rfloor, \lfloor \tfrac{\ell_3+1-\kappa}{2}\rfloor)\right], \tag{2.104}$$

and in the second case

$$N^{\pm}_{3d}(\ell) = \frac{1}{6}\left[N^{\pm}_{3d}(\ell,\ell,\ell) + (-1)^{\ell}\left(3\ell+\tfrac{3}{2}\pm 3\lfloor\tfrac{\ell}{2}\rfloor \pm \tfrac{3}{2}\right) + 1 \pm 1\right]. \tag{2.105}$$

These formulas can be obtained either from propositions 1, 2 and character formulas of appendix A.3 or from the above description of permutations by computing the character of S_2 or S_3 on the space of tensor structures $[q_1q_2q_3]^{\pm}$.

2.4.3 Four-point structures

For four operators, we choose the following conformal frame

$$x_1 = (0,0,0), \tag{2.106}$$

$$x_2 = (t,x,0), \tag{2.107}$$

$$x_3 = (0,1,0), \tag{2.108}$$

$$x_4 = (0,L,0), \tag{2.109}$$

and consider the correlator

$$g_0(s_i,t,x) = \lim_{L\to+\infty} L^{2\Delta_4}\langle O_1(s_1,x_1)O_2(s_2,x_2)O_3(s_3,x_3)O_3(s_4,x_4)\rangle\,. \tag{2.110}$$

We will mostly use the parameters

$$z = x - t, \quad \bar{z} = x + t, \tag{2.111}$$

such that under the continuation to Euclidean time $t_E = it$, we will get the usual holomorphic and anti-holomorphic coordinates.

Note that the stabilizer subgroup is just the $O(1)$ of reflections $x^2 \to -x^2$. Therefore, any function of s_i with appropriate homogeneous degrees will give us a valid 4-point structure. More precisely, we can write

$$g_0(s_i,z,\bar{z}) = \sum_{q_i}[q_1q_2q_3q_4]g_{[q_1q_2q_3q_4]}(z,\bar{z}), \tag{2.112}$$

where

$$[q_1q_2q_3q_4] = \prod_{i=1}^{4}\xi_i^{\ell_i+q_i}\bar{\xi}_i^{\ell_i-q_i} \tag{2.113}$$

with $\xi, \bar{\xi}$ as in (2.91) and $q_i \in \{-\ell_i\ldots\ell_i\}$.

The action of spatial parity is, according to (2.85), $s_i \mapsto \gamma_2 s_i$ or $\xi_i \mapsto \xi_i, \bar{\xi}_i \mapsto -\bar{\xi}_i$. Therefore,

$$[q_1q_2q_3q_4] \mapsto (-1)^{\sum_i \ell_i - q_i}[q_1q_2q_3q_4]. \tag{2.114}$$

We see that the structures we have chosen already have definite parity.

Permutations and crossing symmetry. Consider now how the four-point functions transform under the permutations. Since we are working in Lorentzian signature now, we need to perform an analytic continuation of the phases in appendix A.2. Doing this, we obtain the following formulas for the nontrivial permutations,

$$(12)(34)\ :\ [q_1q_2q_3q_4] \mapsto n((z-1)^{q_1+q_4-q_2-q_3})[q_2q_1q_4q_3], \tag{2.115}$$

$$(13)(24)\ :\ [q_1q_2q_3q_4] \mapsto n(z^{q_3+q_4-q_1-q_2}(1-z)^{q_1+q_4-q_2-q_3})[q_3q_4q_1q_2], \tag{2.116}$$

$$(14)(23)\ :\ [q_1q_2q_3q_4] \mapsto n((-z)^{q_3+q_4-q_1-q_2})[q_4q_3q_2q_1]. \tag{2.117}$$

Here $n(x) = x/\sqrt{x\bar{x}}$, where $\bar{x}$ is x with z and $\bar{z}$ exchanged. The possible (-1)'s from permutations of fermions are already taken into account. Note that if a structure is fixed by a permutation, the phase factor is automatically 1. This is due to the hidden triviality of these phases mentioned in the appendix A.2. This means that any structure can be symmetrized to give a non-zero result,

$$\langle q_1 q_2 q_3 q_4 \rangle_z = \frac{1}{n_{q_1 q_2 q_3 q_4}} \sum_{\pi \in \Pi^{\text{kin}}} \pi[q_1 q_2 q_3 q_4] \neq 0, \tag{2.118}$$

where $n_{q_1 q_2 q_3 q_4}$ is the number of elements in Π^{kin} stabilizing $[q_1 q_2 q_3 q_4]$. With this notation a Π^{kin}-symmetric four-point function can be rewritten as

$$g_0(s_i, z, \bar{z}) = \sum_{q_i/\Pi^{\text{kin}}} \langle q_1 q_2 q_3 q_4 \rangle_z \, g_{[q_1 q_2 q_3 q_4]}(z, \bar{z}), \tag{2.119}$$

where the sum is over some set of representatives of orbits of Π^{kin} action on the set of all tensor structures (possibly of definite parity).

For four-point functions it is convenient to also consider the action of the permutation (13), which is often used to write down a bootstrap equation for a four-point function containing identical operators. From the results of appendix A.2, it acts as

$$(13)\ :\ [q_1 q_2 q_3 q_4] \mapsto (-1)^{q_1+q_2-q_3-q_4} [q_3 q_2 q_1 q_4], \tag{2.120}$$

and this already accounts for the (-1) sign coming from a possible permutation of fermions. For the symmetrized structures the action is, including the change $z \to 1-z$

$$\langle q_1 q_2 q_3 q_4 \rangle_z \mapsto (-1)^{q_1+q_2-q_3-q_4} \langle q_3 q_2 q_1 q_4 \rangle_z \,. \tag{2.121}$$

The crossing equation for the full four-point function, in the case when the operators 1 and 3 are identical, is

$$\begin{aligned} &\sum_{q_i/\Pi^{\text{kin}}} \langle q_1 q_2 q_3 q_4 \rangle_z \, g_{[q_1 q_2 q_3 q_4]}(z, \bar{z}) \\ &= \sum_{q_i/\Pi^{\text{kin}}} \langle q_3 q_2 q_1 q_4 \rangle_z \, (-1)^{q_1+q_2-q_3-q_4} g_{[q_1 q_2 q_3 q_4]}(1-z, 1-\bar{z}). \end{aligned} \tag{2.122}$$

Note that the crossing permutation (13) maps orbits of Π^{kin} into orbits, so this basis essentially diagonalizes the crossing equation.

Counting. It is easy to count the number of four-point structures. Clearly, the total number of structures is

$$N_{3d}(\ell_1, \ell_2, \ell_3, \ell_4) = \prod_{i=1}^{4} (2\ell_i + 1), \tag{2.123}$$

and as discussed in section 2.2.3, this result is valid for all higher-point functions,

$$N_{3d}(\ell_1 \dots \ell_n) = \prod_{i=1}^{n}(2\ell_i + 1), \quad n \geq 4. \tag{2.124}$$

One can see from (2.114) that if there is at least one half-integer spin, then the number of parity even structures is equal to the number of parity odd structures (for such a spin $\ell_i - q_i$ is even exactly as often as it is odd). Performing an explicit computation in the case when all spins are integral, we arrive at the direct analog of (2.103)

$$N^{\pm}_{3d}(\ell_1, \ell_2, \ell_3, \ell_4) = \frac{N_{3d}(\ell_1, \ell_2, \ell_3, \ell_4) \pm \kappa}{2}, \tag{2.125}$$

where $\kappa = 1$ when all spins are integral and $\kappa = 0$ otherwise.

If there are non-trivial kinematic permutations, these are $\Pi^{\text{kin}} = \mathbb{Z}_2$ or $\Pi^{\text{kin}} = \mathbb{Z}_2^2$. In each case we can either use proposition 3 and (A.36) or count the number of orbits of Π^{kin} action on $[q_1 q_2 q_3 q_4]$ structures, which can be done using Burnside's lemma. The result in $\mathbb{Z}_2$ case is

$$N^{+}_{3d}(\ell_1 \leftrightarrow \ell_2, \ell_3 \leftrightarrow \ell_4) = \frac{1}{2}\left[N^{+}_{3d}(\ell_1, \ell_1, \ell_3, \ell_3) + (2\ell_1 + 1)(2\ell_3 + 1)\right], \tag{2.126}$$

$$N^{-}_{3d}(\ell_1 \leftrightarrow \ell_2, \ell_3 \leftrightarrow \ell_4) = \frac{1}{2}N^{-}_{3d}(\ell_1, \ell_1, \ell_3, \ell_3). \tag{2.127}$$

The result in $\mathbb{Z}_2^2$ case is

$$N^{+}_{3d}(\ell) = \frac{1}{4}\left[N^{+}_{3d}(\ell, \ell, \ell, \ell) + 3(2\ell + 1)^2\right], \tag{2.128}$$

$$N^{-}_{3d}(\ell) = \frac{1}{4}N^{-}_{3d}(\ell, \ell, \ell, \ell). \tag{2.129}$$

2.4.4 Example: 4 Majorana fermions

As an example, let us consider in detail the case of four identical Majorana fermions. This is a relatively simple yet non-trivial case for which we can compare to [39].

Let us start by analyzing the generic three-point functions for operators which appear in the OPE expansion. First, consider the three point function of two distinct Majorana fermions and a spin-ℓ_3 operator. Using (2.94) and (2.93), we find the following structures,

$$[\tfrac{1}{2}, \tfrac{1}{2}, -1]^{\pm}, \quad [\tfrac{1}{2}, -\tfrac{1}{2}, 0]^{\pm}. \tag{2.130}$$

For $\ell_3 = 0$ we can only have $q_3 = 0$, and thus only 1 parity-even and 1 parity-odd structures remain. If the fermions are identical, then we need only the structures symmetric under the exchange $(12)'$ given by (2.99). This leaves for even ℓ_3

$$[\tfrac{1}{2},\tfrac{1}{2},-1]^+,\quad [\tfrac{1}{2},-\tfrac{1}{2},0]^\pm, \tag{2.131}$$

and for odd ℓ_3

$$[\tfrac{1}{2},\tfrac{1}{2},-1]^-. \tag{2.132}$$

This is in complete agreement with [39].

Let us now turn to four-point functions. First, using (2.123), we immediately find that there are $2^4 = 16$ tensor structures. According to (2.125), 8 of them are parity-even and 8 are parity-odd. Using (2.114) we can write down the parity-even structures, denoting $q = +\frac{1}{2}$ with $\uparrow$ and $q = -\frac{1}{2}$ with $\downarrow$,

$$\begin{gathered}
[\uparrow\uparrow\uparrow\uparrow], [\downarrow\downarrow\downarrow\downarrow],\\
[\uparrow\uparrow\downarrow\downarrow], [\downarrow\downarrow\uparrow\uparrow],\\
[\uparrow\downarrow\uparrow\downarrow], [\downarrow\uparrow\downarrow\uparrow],\\
[\uparrow\downarrow\downarrow\uparrow], [\downarrow\uparrow\uparrow\downarrow].
\end{gathered} \tag{2.133}$$

Assuming that the fermions are identical, we simply perform the $\mathbb{Z}_2^2$ symmetrization (2.118) of these structures, obtaining $5 = (8+3\cdot 2^2)/4$ (c.f. (2.128)) independent parity-even structures,

$$\langle\uparrow\uparrow\uparrow\uparrow\rangle, \langle\uparrow\uparrow\downarrow\downarrow\rangle, \langle\uparrow\downarrow\uparrow\downarrow\rangle, \langle\downarrow\uparrow\uparrow\downarrow\rangle, \langle\downarrow\downarrow\downarrow\downarrow\rangle\,. \tag{2.134}$$

We can also easily form crossing-symmetric and anti-symmetric structures using (2.122),

$$\text{symmetric:}\ \langle\uparrow\uparrow\uparrow\uparrow\rangle, \langle\uparrow\downarrow\uparrow\downarrow\rangle, \langle\downarrow\downarrow\downarrow\downarrow\rangle, \langle\uparrow\uparrow\downarrow\downarrow\rangle + \langle\downarrow\uparrow\uparrow\downarrow\rangle, \tag{2.135}$$

$$\text{anti-symmetric:}\ \langle\uparrow\uparrow\downarrow\downarrow\rangle - \langle\downarrow\uparrow\uparrow\downarrow\rangle\,. \tag{2.136}$$

We thus have 4 crossing-even structures and 1 crossing-odd structure, which lead to 4 crossing-even equations and 1 crossing-odd equation.[17] This again coincides with the results of [39].

We can very explicitly write down the standard basis of crossing equations,

$$\partial^n\overline{\partial}^m g_{[\uparrow\uparrow\uparrow\uparrow]} = \partial^n\overline{\partial}^m g_{[\downarrow\downarrow\downarrow\downarrow]} = \partial^n\overline{\partial}^m g_{[\uparrow\downarrow\uparrow\downarrow]} = \partial^n\overline{\partial}^m (g_{[\uparrow\uparrow\downarrow\downarrow]} + g_{[\downarrow\uparrow\uparrow\downarrow]}) = 0,\quad n+m \text{ odd}, \tag{2.137}$$

$$\partial^n\overline{\partial}^m (g_{[\uparrow\uparrow\downarrow\downarrow]} - g_{[\downarrow\uparrow\uparrow\downarrow]}) = 0,\quad n+m \text{ even}, \tag{2.138}$$

[17] Note however that "crossing parity" is not a real invariant and can be modified by a structure redefinition.

where all functions are evaluated at $z = \overline{z} = 1/2$. However, there is an important subtlety. When we expand the four-point function in conformal blocks, we will find that the result is smooth (as a function of x_i). As we discuss in appendix A.1, not any choice of $g_{[q_1 q_2 q_3 q_4]}(z, \overline{z})$ leads to a smooth correlator, and a finite number of boundary conditions need to be imposed on derivatives of $g_{[q_1 q_2 q_3 q_4]}(z, \overline{z})$ at $z = \overline{z}$. This effectively gives relations between equations (2.137) and (2.138). These are easy to classify, and we work out the present example in appendix A.1.

Note that [39] used 4-point tensor structures constructed using embedding-space building blocks. They did not have to perform the aforementioned analysis of the boundary conditions. However, there was a different problem which required a similar analysis — since their coefficient functions, unlike those in the present work, do not represent physical values of the correlator but rather have to be multiplied by their tensor structures first, it is not guaranteed that they do not have singularities. In fact, it was found in [39] that their coefficient functions for conformal blocks diverge as $(z - \overline{z})^{-5}$ near $z = \overline{z}$. The solution was to multiply these functions by $(z - \overline{z})^5$ at the cost of introducing relations between the Taylor series coefficients, which are similar to ours. What is different is that in our case we have a simple classification of these relations, whereas in [39] they were handled in a brute-force way by numerically finding linearly independent vectors of crossing equations.

2.5 Scattering amplitudes

In this section we establish the equivalence of the counting of conformal correlators in CFT_d with counting of scattering amplitudes[18] in flat space QFT_{d+1}, generalizing results of [53, 56, 76] to arbitrary spin representations. The basic idea is quite simple — the conformal frame approach can be applied to scattering amplitudes in QFT_{d+1}, and it yields equivalent group-theoretic formulas.

Let us formulate the counting problem for amplitudes in the simplest case of traceless-symmetric spin ℓ particles (we will generalize to other representations later in this section). We can describe the scattering amplitude $\mathcal{A}(p_i, \zeta_i)$ as a Lorentz-invariant function of the momenta p_i, $p_i^2 = -m_i^2$, $\sum_i p_i = 0$, and traceless symmetric polarizations $\zeta_i^{\mu_1 \dots \mu_{\ell_i}}$. For all particles the polarizations satisfy the transversality condition $(p_i)_{\mu_1} \zeta_i^{\mu_1 \dots \mu_{\ell_i}} = 0$. For massless particles we in addition get

[18] The spaces of scattering amplitudes of spinning particles have been considered, for example, in [85–91]. We thank Massimo Taronna for pointing out these references to us.

the gauge equivalence

$$\zeta^{\mu_1\cdots\mu_{\ell_i}} \sim \zeta^{\mu_1\cdots\mu_{\ell_i}} + p^{(\mu_1}\lambda^{\mu_2\cdots\mu_\ell)}, \tag{2.139}$$

where λ is the parameter of the gauge transformation which is itself transverse. The scattering amplitude $\mathcal{A}(p_i, \zeta_i)$ should be invariant under this transformation. That is, $\mathcal{A}$ should be a function of the gauge equivalence classes of ζ_i.

A general solution to the above requirements has the form

$$\mathcal{A}(p_i, \zeta_i) = \sum_{I=1}^{N} \mathbb{T}_I(p_i, \zeta_i) f^I(s, t, \ldots), \tag{2.140}$$

where $\mathbb{T}_I$ are the tensor structures encoding the non-trivial dependence on the polarizations and momenta, and $s, t, \ldots$ are the kinematic invariants of n particles, i.e., the Mandelstam variables. Our goal in this section is to find the number N of tensor structures and prove that it is equal to the number of tensor structures in a certain conformal correlator.

2.5.1 Little group formulation

Note that for a fixed p, the solutions ζ to the transversality constraint $p_{\mu_1}\zeta^{\mu_1\cdots\mu_\ell} = 0$, as well as the gauge equivalence classes of such solutions are transformed into each other by the little group $L(p)$ which is the subgroup of the Lorentz group leaving p invariant. The little group in QFT$_{d+1}$ is $SO(d)$ in the massive case and $SO(d-1)$ in the massless case (formally it is $ISO(d-1)$, but for particles with a finite number of internal degrees of freedom the translations of ISO act trivially). In the case considered above ζ_i live in traceless symmetric representations of the respective little groups.

In order to have a general treatment, we will adopt this little group point of view on the particle polarizations. Instead of specifying a polarization ζ, we specify an element ε of some representation of $L(k)$, where k is a standard[19] momentum with $k^2 = p^2$. Accordingly, for each momentum p we specify a standard Lorentz transformation[20] $R(p)$ such that $R(p)k = p$. Now instead of $\mathcal{A}(p_i, \zeta_i)$, we have a function of the little group polarizations ε_i which we denote $S(p_i, \varepsilon_i)$.

[19]For concreteness, for massive particles of mass m we can choose $k = (m, 0, 0, \ldots)$ and for massless particles $k = (1, 1, 0, 0, \ldots)$ with signature $(-, +, +, \ldots)$.

[20]In general we need to allow $R(p)$ to belong to the disconnected components of the Lorentz group, since in general we may want to have momenta in the past lightcone (or treat in and out particles separately). Alternatively, we may consider the complexification of the whole setup, as anyway is required for the treatment of 3-point on-shell amplitudes. Either way, for simplicity of the discussion we ignore these subtleties.

To see the correspondence between the two descriptions, for example in the case of massless traceless symmetric particle, we can put ε into correspondence with a polarization $\zeta_k(\varepsilon)$ with transversality and gauge invariance defined by the momentum k. This then specifies $\zeta_p(\varepsilon) = R(p)\zeta_k(\varepsilon)$, which now satisfies transversality and gauge invariance defined by p. We can now set

$$S(p_i, \varepsilon_i) = \mathcal{A}(p_i, \zeta_{p_i}(\varepsilon_i)). \tag{2.141}$$

This establishes the isomorphism between the descriptions $S(p_i, \varepsilon_i)$ and $\mathcal{A}(p_i, \zeta_i)$. It also makes it easy to see how the Lorentz invariance is stated for $S(p_i, \varepsilon_i)$ — since for each Lorentz transformation Λ we have

$$\mathcal{A}(\Lambda p_i, \Lambda\zeta_i) = \mathcal{A}(p_i, \zeta_i), \tag{2.142}$$

then in terms of $S(p_i, \varepsilon_i)$ we should have

$$S(\Lambda p_i, R(\Lambda p_i)^{-1}\Lambda R(p_i)\varepsilon_i) = S(p_i, \varepsilon_i). \tag{2.143}$$

This formula makes sense because

$$R(\Lambda p_i)^{-1}\Lambda R(p_i)k_i = k_i \tag{2.144}$$

and thus

$$R(\Lambda p_i)^{-1}\Lambda R(p_i) \in L(k_i), \tag{2.145}$$

which can act on ε_i. This condition appears more complicated than (2.142), but the advantage is that this is the only condition we require of the amplitude (in contrast to requiring the gauge invariance and imposing the transversality constraints for $\mathcal{A}(p_i, \zeta_i)$). This makes it extremely easy to classify tensor structures for the amplitudes, as we now show.

2.5.2 Conformal frame for amplitudes

We now simply repeat the analysis of section 2.2.2 for the amplitudes. The Lorentz group acts on the configuration space of the momenta p_i, and splits this space into orbits. We chose a "scattering frame" — a submanifold of the momenta configuration space which intersects each orbit at precisely one point. It is easy to show that the dimension of scattering frame is the same as the dimension of the conformal frame at the same n (number of operators or particles) and d.

A scattering amplitude is now completely specified by its values on the scattering frame. These values, as in section 2.2.2, have to be invariant under the subgroup of Lorentz group which fixes the scattering frame.

It is easy to see what this subgroup is. First, n generic momenta, due to the conservation condition $\sum_i p_i = 0$, span an $(m-1)$-dimensional linear space $\mathcal{P}$, where $m = \min(d+2, n)$. The subgroup which fixes $\mathcal{P}$ depends only on the rank of the restriction of the Lorentz metric onto $\mathcal{P}$, which coincides with the rank of Gram matrix G of any $n-1$ momenta in $\mathcal{P}$. The determinant $\det G$ is an algebraic function of the particle masses m_i and the kinematic invariants $s, t, u, \ldots$.

For $n \geq 4$ we have non-trivial kinematic invariants, and thus for a generic set of these invariants $\det G \neq 0$ and the metric on $\mathcal{P}$ is full rank. This implies that $\mathcal{P}$ is stabilized by a subgroup $SO(d+1-(m-1)) = SO(d+2-m)$.

For $n = 3$[21] we have no non-trivial kinematic invariants, and $\det G$ is determined solely by the masses. For a generic set of masses, $\det G \neq 0$, and we again get $SO(d+2-m)$. This case corresponds to the generic three-point functions as discussed in section 2.3. For simplicity, we only consider this generic case.

Now, we need to understand how the stabilizing subgroup $St = SO(d+2-m)$ acts on the little group polarizations. Assume that Λ fixes all the p_i. In this case, we have

$$\varepsilon_i \to R(p_i)^{-1}\Lambda R(p_i)\varepsilon. \tag{2.146}$$

We can say, alternatively, that St is naturally a subgroup of each $L(p_i)$, which in turn are put in an isomorphism with $L(k_i)$ by

$$L(k_i) = R(p_i)^{-1} L(p) R(p_i). \tag{2.147}$$

This defines a restriction of representations of $L(k_i)$ to representations of $St = SO(d+2-m)$. Assume that the particles transform in representations ρ_i of $L(k_i)$. We then immediately find that the space of tensor structures for scattering amplitudes is

$$\left(\bigotimes_{i=1}^{n} \mathrm{Res}^{L(k_i)}_{SO(d+2-m)}\rho_i\right)^{SO(d+2-m)}. \tag{2.148}$$

Its dimension is equal to the number of tensor structures in a conformal correlator if the $SO(d)$ representations of the non-conserved local operators in CFT$_d$ are

[21]For $n = 3$ we need to consider complexified kinematics in order to have an on-shell amplitude.

identified[22] with the representations of the massive little group $SO(d)$ in QFT_{d+1}, and the effective $SO(d-1)$ representations of local operators (as described in section 2.3) are identified with the representations of the massless $SO(d-1)$ little group. It is in principle straightforward to extend this result to include parity and permutations symmetries. For example, it is not hard to check that kinematic permutation groups match in CFT_d and QFT_{d+1}.

Acknowledgements

We thank Chris Beem, Clay Córdova, Tolya Dymarsky, Mikhail Evtikhiev, Enrico Herrmann, Denis Karateev, Hirosi Ooguri and Chia-Hsien Shen for discussions. DSD is supported by a William D. Loughlin Membership at the Institute for Advanced Study, and Simons Foundation grant 488657 (Simons Collaboration on the Non-perturbative Bootstrap). This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Numbers DE-SC0009988 (DSD) and DE-SC0011632 (PK).

[22] This rule for massive case was also mentioned in [53, 56].

Chapter 3

GENERAL BOOTSTRAP EQUATIONS IN 4D CFTS

This chapter is essentially identical to (with the omission of section 5 of the original paper):

G. F. Cuomo, D. Karateev and P. Kravchuk, *General Bootstrap Equations in 4D CFTs*, *JHEP* **01** (2018) 130, [1705.05401].

3.1 Introduction

In recent years a lot of progress has been made in understanding Conformal Field Theories (CFTs) in $d \geq 3$ dimensions using the conformal bootstrap approach [26–28, 92, 93] (see [18, 19] for recent introduction). In this chapter we focus solely on $d = 4$. The 4D conformal bootstrap allows to study fixed points of 4D quantum field theories relevant for describing elementary particles and fundamental interactions. It promises to address the QCD conformal window [10] and may be useful for constraining the composite Higgs models; see [94] for discussion.

In the conformal bootstrap approach CFTs are described by the local CFT data, which consists of scaling dimensions and Lorentz representations of local primary operators together with structure constants of the operator product expansion (OPE). The observables of the theory are correlation functions which are computed by maximally exploiting the conformal symmetry and the operator product expansion. Remarkably, the CFT data is heavily constrained by the associativity of the OPE, which manifests itself in the form of consistency equations called the crossing or the bootstrap equations.

The bootstrap equations constitute an infinite system of coupled non-linear equations for the CFT data. In a seminal work [30] it was shown how constraints on a finite subset of the OPE data can be extracted numerically from these equations. In 4D the approach of [30] was further developed in [35, 43–47, 94–101]. In 3D a major advance came with the numerical identification of the 3D Ising [32, 34] and the $O(N)$ models [8, 36–38]. An analytic approach to the bootstrap equations was proposed in [68, 69] and further developed in [31, 102–110]. Other approaches include [101, 111–114].

Most of these studies, however, focus on correlation functions of scalar operators, and thus only have access to the scaling dimensions of traceless symmetric operators and their OPE coefficients with a pair of scalars. In order to derive constraints on the most general elements of the CFT data, one has to consider more general correlation functions. To the best of our knowledge, the only published numerical studies of a 4-point function of non-scalar operators in non-supersymmetric theories up to date were done in 3D for a 4-point function of Majorana fermions [39, 40] and for a 4-point function of conserved abelian currents [115].

One reason for the lack of results on 4-point functions of spinning operators is that such correlators are rather hard to deal with. In order to set up the crossing equations for a spinning 4-point function, first, one needs to find a basis of its tensor structures and second, to compute all the relevant conformal blocks. The difficulty of this task increases with the dimension d due to an increasing complexity of the d-dimesnional Lorentz group. For instance, the representations of the 4D Lorentz group are already much richer than the ones in 3D.

The problem of constructing tensor structures has a long history [39, 50–53, 56, 78, 79, 116, 117]. In 4D all the 3-point tensor structures were obtained in [54] and classified in [55] using the covariant embedding formalism approach. Unfortunately, in this approach 4- and higher-point tensor structures are hard to analyze due to a growing number of non-linear relations between the basic building blocks. This problem is alleviated in the conformal frame approach [1, 23, 51]. In [1] a complete classification of general conformally invariant tensor structures was obtained in a non-covariant form.

The problem of computing the conformal blocks for scalar 4-point functinons was solved by a variety of methods in [32, 37, 54, 59, 118–121]. Spinning conformal blocks were considered in [39, 49, 54, 56, 60, 61, 81, 82, 122, 123]. Remarkably, in [61] it was found that the Lorentz representations of external operators can be changed by means of differential operators. In 3D, this relates all bosonic conformal blocks to conformal blocks with external scalars. These results were extended to 3D fermions in [39, 81] completing in principle the program of computing general conformal blocks in 3D.

Results of [61] concerning traceless symmetric operators apply also to 4D, but are not sufficient even for the analysis of an OPE of traceless symmetric operators since such an OPE also contains non-traceless symmetric operators. The first expression for a 4D spinning conformal block was obtained in [122] for the case of 2 scalars and

2 vectors. A systematic study of conformal blocks in 4D with operators in arbitrary representations was done in [62], where the results of [61] were extended to reduce a general conformal block to a set of simpler conformal blocks called the seed blocks. In the consequent work [58] all the seed conformal blocks were computed.

The goal The results of [1, 55, 58, 62] are in principle sufficient for formulating the bootstrap equations for arbitrary correlators in 4D. Nevertheless, due to a large amount of scattered non-trivial and missing ingredients there is still a high barrier for performing 4D bootstrap computations. The goal of this chapter is to describe all the ingredients needed for setting up the 4D bootstrap equations in a coherent manner using consistent conventions and to implement all these ingredients into a `Mathematica` package.

In particular, we first unify the results of [55, 58, 62] with some extra developments and corrections. We then use the conformal frame approach [1] to solve the problem of constructing a complete basis of 4-point tensor structures in 4D in an extremely simple way. We provide a precise connection between the embedding and the conformal frame approaches making possible an easy transition between two formalisms at any time.

We implement the formalism in a `Mathematica` package which allows one to work with 2-, 3- and 4-point functions and to construct arbitrary spin crossing equations in 4D CFTs. The package can be downloaded from

https://gitlab.com/bootstrapcollaboration/CFTs4D.

Once it is installed one gets an access to a (hopefully) comprehensive documentation and examples. We also refer to the relevant functions from the package throughout the chapter as `[function]`.

Structure of the chapter In the main body of the chapter we describe the basic concepts applicable to the most generic correlators with no additional symmetries or conservation conditions. We comment on how these extra complications can be taken into account, and delegate a more detailed treatment to the appendices.

In section 3.2 we outline the path to the explicit crossing equations for operators of general spin, abstracting from a specific implementation. In section 3.3 we describe the implementation of the ideas from section 3.2 in the embedding formalism. In section 3.4 we give an alternative implementation in the conformal frame formalism. We conclude in section 3.5.

Appendices B.1 and B.2 summarize our conventions in 4D Minkowski space and 6D embedding space, as well as cover the action of $\mathcal{P}$- and $\mathcal{T}$-symmetries. appendix B.2 also contains details of the embedding formalism. In appendix B.3 we give details on normalization conventions for 2-point functions and seed conformal blocks. Appendices B.4 and B.5 contain details on explicitly covariant tensor structures. In appendix B.6 we describe all 3 Casimir generators of the four-dimensional conformal group. Appendices B.7 and B.8 cover conservation conditions and permutation symmetries.

3.2 Outline of the framework

The local operators in 4D CFT are labeled by $(\ell, \bar{\ell})$ representation of the Lorentz group $SO(1,3)$ and the scaling dimension Δ.[1] In a CFT one can distinguish a special class of *primary* operators, the operators which transform homogeneously under conformal transformations [92]. In a unitary CFT any local operator is either a primary or a derivative of a primary, in which case it is called a descendant operator. A primary operator in representation $(\ell, \bar{\ell})$ can be written as[2]

$$\mathcal{O}^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{\ell}}}_{\alpha_1\ldots\alpha_\ell}(x), \tag{3.1}$$

symmetric in spinor indices α_i and $\dot{\beta}_j$. Because of the symmetry in these indices, we can equivalently represent $\mathcal{O}$ by a homogeneous polynomial in auxiliary spinors s^α and $\bar{s}_{\dot{\beta}}$ of degrees ℓ and $\bar{\ell}$ correspondingly

$$\mathcal{O}(x, s, \bar{s}) = s^{\alpha_1} \cdots s^{\alpha_\ell} \bar{s}_{\dot{\beta}_1} \cdots \bar{s}_{\dot{\beta}_{\bar{\ell}}} \mathcal{O}^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{\ell}}}_{\alpha_1\ldots\alpha_\ell}(x). \tag{3.2}$$

We often call the auxiliary spinors s and $\bar{s}$ the spinor polarizations. The indices can be restored at any time by using

$$\mathcal{O}^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{\ell}}}_{\alpha_1\ldots\alpha_\ell}(x) = \frac{1}{\ell!\,\bar{\ell}!} \prod_{i=1}^{\ell} \prod_{j=1}^{\bar{\ell}} \frac{\partial}{\partial s^{\alpha_i}} \frac{\partial}{\partial \bar{s}_{\dot{\beta}_j}}\, \mathcal{O}(x, s, \bar{s}). \tag{3.3}$$

In principle the auxiliary spinors s and $\bar{s}$ are independent quantities; however without loss of generality we can assume them to be complex conjugates of each other,

[1] In this chapter we consider only the consequences of the conformal symmetry. In particular, we do not consider global (internal) symmetries because they commute with conformal trasformations and thus can be straightforwardly included. We also do not discuss supersymmetry.

[2] Our conventions relevant for 3+1 dimensional Minkowski spacetime are summarized in appendix B.1.

$s_\alpha = (\bar{s}_{\dot{\alpha}})^*$. This has the advantage that if $\mathcal{O}$ with $\ell = \bar{\ell}$ is a Hermitian operator, e.g., for $\ell = \bar{\ell} = 1$,

$$\mathcal{O}_{\alpha\dot{\beta}}(x) = \left(\mathcal{O}_{\beta\dot{\alpha}}(x)\right)^\dagger, \tag{3.4}$$

then so is $\mathcal{O}(x, s, \bar{s})$,

$$\mathcal{O}(x, s, \bar{s}) = (\mathcal{O}(x, s, \bar{s}))^\dagger\,. \tag{3.5}$$

More generally for non-Hermitian operators we define

$$\overline{\mathcal{O}}(x, s, \bar{s}) \equiv (\mathcal{O}(x, s, \bar{s}))^\dagger\,; \tag{3.6}$$

see (B.8) for the index-full version.

Conformal field theories possess an operator product expansion (OPE) with a finite radius of convergence [23, 25, 78, 124]

$$\mathcal{O}_1(x_1, s_1, \bar{s}_1)\mathcal{O}_2(x_2, s_2, \bar{s}_2) = \sum_{\mathcal{O}}\sum_a \lambda^a_{\langle\mathcal{O}_1\mathcal{O}_2\overline{\mathcal{O}}\rangle}\mathcal{B}_a(\partial_{x_2}, \partial_s, \partial_{\bar{s}}, \ldots)\mathcal{O}(x_2, s, \bar{s}), \tag{3.7}$$

where $\mathcal{B}_a$ are differential operators in the indicated variables (depending also on $x_1 - x_2, s_j, \bar{s}_j$, where $j = 1, 2$), which are fixed by the requirement of conformal invariance of the expansion. Here λ's are the OPE coefficients which are not constrained by the conformal symmetry. In general there can be several independent OPE coefficients for a given triple of primary operators, in which case we label them by an index a.

The OPE provides a way of reducing any n-point function to 2-point functions, which have canonical form in a suitable basis of primary operators. Therefore, the set of scaling dimensions and Lorentz representations of local operators, together with the OPE coefficients, completely determines all correlation functions of local operators in conformally flat $\mathbb{R}^{1,3}$. For this reason we call this set of data the CFT data in what follows.[3] The goal of the bootstrap approach is to constrain the CFT data by using the associativity of the OPE. In practice this is done by using the associativity inside of a 4-point correlation function, resulting in the crossing equations which can be analyzed numerically and/or analytically. In the remainder of this section we describe in detail the path which leads towards these equations.

[3] Besides the correlation functions of local operators one can consider extended operators, such as conformal defects, as well as the correlation functions on various non-trivial manifolds. In order to be able to compute these quantities one has to in general extend the notion of the CFT data.

3.2.1 Correlation functions of local operators

We are interested in studying n-point correlation functions

$$f_n(\mathbf{p}_1 \dots \mathbf{p}_n) \equiv \langle 0 | \mathcal{O}_{\Delta_1}^{(\ell_1,\bar{\ell}_1)}(\mathbf{p}_1) \; \dots \; \mathcal{O}_{\Delta_n}^{(\ell_n,\bar{\ell}_n)}(\mathbf{p}_n) | 0 \rangle, \tag{3.8}$$

where for convenience we defined a combined notation for dependence of operators on coordinates and auxiliary spinors

$$\mathbf{p}_i \equiv (x_i, s_i, \bar{s}_i). \tag{3.9}$$

We have labeled the primary operators with their spins and scaling dimensions. In general these labels do not specify the operator uniquely (for example in the presence of global symmetries); we ignore this subtlety for the sake of notational simplicity. For our purposes it will be sufficient to assume that all operators are space-like separated (this includes all Euclidean configurations obtained by Wick rotation), and thus the ordering of the operators will be irrelevant up to signs coming from permutations of fermionic operators.

The conformal invariance of the system puts strong constraints on the form of (3.8). By inserting an identity operator $\mathbf{1} = UU^\dagger$, where U is the unitary operator implementing a generic conformal transformation, inside this correlator and demanding the vacuum to be invariant $U|0\rangle = 0$, one arrives at the constraint

$$\langle 0 | \left(U^\dagger \mathcal{O}_{\Delta_1}^{(\ell_1,\bar{\ell}_1)} U \right) \dots \left(U^\dagger \mathcal{O}_{\Delta_n}^{(\ell_n,\bar{\ell}_n)} U \right) | 0 \rangle = \langle 0 | \mathcal{O}_{\Delta_1}^{(\ell_1,\bar{\ell}_1)} \dots \mathcal{O}_{\Delta_n}^{(\ell_n,\bar{\ell}_n)} | 0 \rangle. \tag{3.10}$$

The algebra of infinitesimal conformal transformations, as well as their action on the primary operators are summarized in our conventions in appendix B.1.

The general solution to the above constraint has the following form,

$$f_n(x_i, s_i, \bar{s}_i) = \sum_{I=1}^{N_n} g_n^I(\mathbf{u}) \; \mathbb{T}_n^I(x_i, s_i, \bar{s}_i), \tag{3.11}$$

where $\mathbb{T}_n^I$ are the conformally-invariant tensor structures which are fixed by the conformal symmetry up to a $\mathbf{u}$-dependent change of basis, and $\mathbf{u}$ are cross-ratios which are the scalar conformally-invariant combinations of the coordinates x_i. The structures $\mathbb{T}_n^I$ and their number N_n depend non-trivially on the $SO(1,3)$ representations of $\mathcal{O}_i$, but rather simply on Δ_i, so we can write

$$\mathbb{T}_n^I(x_i, s_i, \bar{s}_i) = \mathcal{K}_n(x_i) \hat{\mathbb{T}}_n^I(x_i, s_i, \bar{s}_i), \tag{3.12}$$

where all Δ_i-dependence is in the "kinematic" factor $\mathcal{K}_n$[4] and all the the Δ_i enter $\mathcal{K}_n$ through the quantity

$$\kappa \equiv \Delta + \frac{\ell + \bar{\ell}}{2}. \tag{3.13}$$

Note that $\mathbb{T}$ and $\hat{\mathbb{T}}$ are homogeneous polynomials in the auxiliary spinors, schematically,

$$\mathbb{T}_n^I, \hat{\mathbb{T}}_n^I \sim \prod_{i=1}^{n} s_i^{\ell_i} \bar{s}_i^{\bar{\ell}_i}. \tag{3.14}$$

In the rest of this subsection we give an overview of the structure of n-point correlation functions for various n, emphasizing the features specific to 4D.

2-point functions A 2-point function can be non-zero only if it involves two operators in complex-conjugate representations, $(\ell_1, \bar{\ell}_1) = (\bar{\ell}_2, \ell_2)$, and with equal scaling dimensions, $\Delta_1 = \Delta_2$. In fact, it is always possible to choose a basis for the primary operators so that the only non-zero 2-point functions are between Hermitian-conjugate pairs of operators. We always assume such a choice.

The general 2-point function `[n2CorrelationFunction]` then has an extremely simple form given by

$$\langle \overline{\mathcal{O}}_{\Delta}^{(\bar{\ell},\ell)}(\mathbf{p}_1) \mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}(\mathbf{p}_2) \rangle = c_{\langle \overline{\mathcal{O}} \mathcal{O} \rangle} \underbrace{x_{12}^{-2\kappa_1}}_{=\mathcal{K}_2} \underbrace{\left[\hat{\mathbb{I}}^{12}\right]^{\ell} \left[\hat{\mathbb{I}}^{21}\right]^{\bar{\ell}}}_{=\hat{\mathbb{T}}_2}, \tag{3.15}$$

where $c_{\langle \overline{\mathcal{O}} \mathcal{O} \rangle}$ is a constant. There is a single tensor structure $\hat{\mathbb{T}}_2$, and the building blocks $\hat{\mathbb{I}}^{ij}$ are defined in appendix B.4. Changing the normalization of $\mathcal{O}$ one can rescale the coefficient $c_{\langle \overline{\mathcal{O}} \mathcal{O} \rangle}$ by a positive factor. The phase is fixed by the requirement of unitarity, see appendix B.3. We can make the following choice

$$c_{\langle \overline{\mathcal{O}} \mathcal{O} \rangle} = i^{\ell - \bar{\ell}}, \quad c_{\langle \mathcal{O} \overline{\mathcal{O}} \rangle} = (-)^{\ell - \bar{\ell}} c_{\langle \overline{\mathcal{O}} \mathcal{O} \rangle} = i^{\bar{\ell} - \ell}, \tag{3.16}$$

where the factor $(-)^{\ell - \bar{\ell}}$ appears due to the spin statistics theorem.

3-point functions A generic form of a 3-point function `[n3ListStructures, n3ListStructuresAlternativeTS]` is given by[5]

$$\langle \mathcal{O}_{\Delta_1}^{(\ell_1,\bar{\ell}_1)}(\mathbf{p}_1) \mathcal{O}_{\Delta_2}^{(\ell_2,\bar{\ell}_2)}(\mathbf{p}_2) \mathcal{O}_{\Delta_3}^{(\ell_3,\bar{\ell}_3)}(\mathbf{p}_3) \rangle = \mathcal{K}_3 \sum_{a=1}^{N_3} \lambda^a_{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle} \hat{\mathbb{T}}_3^a, \tag{3.17}$$

[4]This does not uniquely fix the factorization, and we will make a choice based on convenience later.

[5]For notational convenience we use lowercase index a instead of capital index I to label the 3-point tensor structures.

where the kinematic factor `[n3KinematicFactor]` is given by

$$\mathcal{K}_3 = \prod_{i<j} |x_{ij}|^{-\kappa_i-\kappa_j+\kappa_k}. \tag{3.18}$$

The necessary and sufficient condition for the 3-point tensor structures $\hat{\mathbb{T}}_3^a$ to exist is that the 3-point function contains an even number of fermions and the following inequalities hold,

$$|\ell_i - \bar{\ell}_i| \leq \ell_j + \bar{\ell}_j + \ell_k + \bar{\ell}_k, \quad \text{for all distinct } i, j, k. \tag{3.19}$$

A general discussion on how to construct a basis of tensor structures $\hat{\mathbb{T}}_3^a$ is given in section 3.3. For convenience we summarize this construction for 3-point functions in appendix B.5.

The fact that the OPE coefficients enter 3-point functions follows simply from using the OPE (3.7) and the form of (3.15) in the left hand side of (3.17). It is also clear that one can always choose the bases for $\mathcal{B}_a$ and $\hat{\mathbb{T}}_3^a$ to be compatible.

There is a number of relations the OPE coefficients $\lambda^a_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}$ have to satisfy. The simplest one comes from applying complex conjugation to both sides of (3.17). On the left hand side one has

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle^* = \langle \overline{\mathcal{O}}_3\overline{\mathcal{O}}_2\overline{\mathcal{O}}_1\rangle. \tag{3.20}$$

Using the properties of tensor structures under conjugation summarized in appendix B.4, one obtains a relation of the form

$$\left(\lambda^a_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}\right)^* = C^{ab}\,\lambda^b_{\langle \overline{\mathcal{O}}_3\overline{\mathcal{O}}_2\overline{\mathcal{O}}_1\rangle}, \tag{3.21}$$

where the matrix C^{ab} is often diagonal with ± 1 entries. Other constraints arise from the possible $\mathcal{P}$- and $\mathcal{T}$-symmetries (see appendix B.1), conservation equations (see appendix B.7), and permutation symmetries (see appendix B.8). Importantly all these conditions give linear equations for λ's, which can be solved in terms of an independent set of real quantities $\hat{\lambda}$ as

$$\lambda^a_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle} = \sum_{\hat{a}=1}^{\hat{N}_3} P^{a\hat{a}}_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}\hat{\lambda}^{\hat{a}}_{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle}, \quad \hat{N}_3 < N_3. \tag{3.22}$$

It will be important for the calculation of conformal blocks that we can actually construct all the tensor structures $\mathbb{T}_3^a$ in (3.17) by considering a simpler 3-point function with two out of three operators having canonical spins $(\ell'_1, \bar{\ell}'_1)$ and $(\ell'_2, \bar{\ell}'_2)$, chosen in a way such that the 3-point function has a single tensor structure

$$\langle \mathcal{O}^{(\ell'_1,\bar{\ell}'_1)}_{\Delta'_1} \mathcal{O}^{(\ell'_2,\bar{\ell}'_2)}_{\Delta'_2} \mathcal{O}^{(\ell_3,\bar{\ell}_3)}_{\Delta_3} \rangle = \lambda\, \mathbb{T}_{seed}. \tag{3.23}$$

A simple choice is to set as many spin labels to zero as possible, for example

$$\ell'_1 = \bar{\ell}'_1 = \ell'_2 = 0, \quad \bar{\ell}'_2 = |\ell_3 - \bar{\ell}_3|. \tag{3.24}$$

As we review in section 3.3.2 one can then construct a set of differential operators $\mathbb{D}^a$ acting on the coordinates and polarization spinors of the first two operators such that

$$\mathbb{T}^a_3 = \mathbb{D}^a\, \mathbb{T}_{seed}. \tag{3.25}$$

We will call the canonical tensor structure $\mathbb{T}_{seed}$ a seed tensor structure in what follows. Our choice of seed structures is described in appendix B.3. When the third field is traceless symmetric, one has obviously $\bar{\ell}'_2 = 0$, thus relating a pair of generic operators to a pair of scalars [61].

4-point functions and beyond In the case $n = 4$ one has

$$\langle \mathcal{O}^{(\ell_1,\bar{\ell}_1)}_{\Delta_1}(\mathbf{p}_1) \mathcal{O}^{(\ell_2,\bar{\ell}_2)}_{\Delta_2}(\mathbf{p}_2) \mathcal{O}^{(\ell_3,\bar{\ell}_3)}_{\Delta_3}(\mathbf{p}_3) \mathcal{O}^{(\ell_3,\bar{\ell}_4)}_{\Delta_4}(\mathbf{p}_4) \rangle = \sum_{I=1}^{N_4} g^I_4(u,v)\, \mathbb{T}^I_4, \tag{3.26}$$

where $g^I_4(u,v)$ are not fixed by conformal symmetry and are functions of the 2 conformally invariant cross-ratios [formCrossRatios]

$$u = \frac{x^2_{12} x^2_{34}}{x^2_{13} x^2_{24}}, \quad v = \frac{x^2_{14} x^2_{23}}{x^2_{13} x^2_{24}}. \tag{3.27}$$

In most of the applications it will be more convenient to use another set of variables $(z, \bar{z})$ [changeVariables] defined as

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \tag{3.28}$$

We classify and construct all the 4-point tensor structures $\mathbb{T}_4$ [n4ListStructures, n4ListStructuresEF] in section 3.4. Following the literature we choose the kinematic factor [n4KinematicFactor] of the form[6]

$$\mathcal{K}_4 = \left(\frac{x_{24}}{x_{14}}\right)^{\kappa_1-\kappa_2} \left(\frac{x_{14}}{x_{13}}\right)^{\kappa_3-\kappa_4} \times \frac{1}{x_{12}^{\kappa_1+\kappa_2} x_{34}^{\kappa_3+\kappa_4}}. \tag{3.29}$$

[6] In section 3.4 we never separate the kinematic factor which has an extremely simple form $(z\bar{z})^{-\frac{\kappa_1+\kappa_2}{2}}$ in the conformal frame.

The case of $n \geq 5$ point functions is similar to the $n = 4$ case with a difference that the number of conformally invariant cross-ratios is $4n - 15$. We briefly discuss the classification of tensor structures for higher-point functions in section 3.4.

In general 4- and higher-point functions are subject to the same sort of conditions as 3-point functions. Reality conditions and implications of $\mathcal{P}$- and $\mathcal{T}$-symmetries are not conceptually different from the 3-point case. However, implications of permutation symmetries and conservation equations are more involved than those for 3-point functions, see [75], due to the existence of non-trivial conformal cross-ratios (3.27). See also appendices B.8 and B.7 for details.

3.2.2 Decomposition in conformal partial waves

Since the OPE data determines all the correlation functions, the functions $g_4^I(u,v)$ entering (3.26) can also be computed. To compute $g_4^I(u,v)$ we use the s-channel OPE, namely the OPE in pairs O_1O_2 and O_3O_4. One way to do this is to insert a complete orthonormal set of states in the correlator

$$f_4 \underset{s-OPE}{=} \langle \overbrace{O_1O_2}\overbrace{O_3O_4}\rangle = \sum_{|\Psi\rangle}\langle O_1O_2|\Psi\rangle\langle\Psi|O_3O_4\rangle. \tag{3.30}$$

By virtue of the operator-state correspondence, see for example [18, 19], the states $|\Psi\rangle$ are in one-to-one correspondence with the local primary operators O and their descendants $\partial^n O$. This allows us to express the inner products above in terms of the 3-point functions $\langle O_1O_2O\rangle$ and $\langle\overline{O}O_3O_4\rangle$ with the primary operator O and its conjugate $\overline{O}$, resulting in the following s-channel conformal partial wave decomposition

$$\langle \overbrace{O_1O_2}\overbrace{O_3O_4}\rangle = \sum_{O}\sum_{a,b}\lambda^a_{\langle O_1O_2O\rangle}W^{ab}_{\langle O_1O_2O\rangle\langle\overline{O}O_3O_4\rangle}\lambda^b_{\langle\overline{O}O_3O_4\rangle}. \tag{3.31}$$

The objects W^{ab} are called the conformal partial waves (CPWs).[7] The summation in (3.31) is over all primary operators O which appear in both 3-point functions $\langle O_1O_2O\rangle$ and $\langle\overline{O}O_3O_4\rangle$ and we can write explicitly

$$\sum_{O} = \sum_{|\ell-\bar{\ell}|=0}^{\infty}\ \sum_{\ell=0}^{\infty}\ \sum_{\Delta,i}, \tag{3.32}$$

where i labels the possible degeneracy of operators at fixed spin and scaling dimensions (coming, for example, from a global symmetry). Note that according to

[7] In this chapter "conformal partial waves" are what we usually call "conformal blocks" and "conformal blocks" mean what we would normally call "components of conformal block". We hope that this doesn't cause confusion.

properties of 3-point functions (3.19), there is a natural upper cut-off in the first summation

$$\sum_{|\ell-\bar{\ell}|=0}^{\infty} = \sum_{|\ell-\bar{\ell}|=0}^{|\ell-\bar{\ell}|_{max}}, \tag{3.33}$$

where

$$|\ell-\bar{\ell}|_{max} = \min(\ell_1+\bar{\ell}_1+\ell_2+\bar{\ell}_2,\ \ell_3+\bar{\ell}_3+\ell_4+\bar{\ell}_4). \tag{3.34}$$

Furthermore, if the operator O is bosonic then $|\ell-\bar{\ell}|$ assumes only even values; if the operator O is fermionic $|\ell-\bar{\ell}|$ assumes only odd values. The CPWs can be further rewritten in terms of conformal blocks (CB) and tensor structures as

$$W^{ab}_{\langle O_1O_2O\rangle\langle\bar{O}O_3O_4\rangle} = \sum_{I=1}^{N_4} G^{I,ab}_{\langle O_1O_2O\rangle\langle\bar{O}O_3O_4\rangle}(u,v)\ \mathbb{T}^I_4, \tag{3.35}$$

inducing the conformal block expansion for g_4^I

$$g_4^I(u,v) \underset{s-OPE}{=} \sum_{O}\sum_{a,b} \lambda^a_{\langle O_1O_2O\rangle} G^{I,ab}_{\langle O_1O_2O\rangle\langle\bar{O}O_3O_4\rangle}(u,v)\lambda^b_{\langle\bar{O}O_3O_4\rangle}. \tag{3.36}$$

Computation of conformal partial waves The computation of CPWs is rather difficult. Luckily there is a way of reducing them to simpler objects called the seed CPWs by means of differential operators [61, 62].

For example, the s-channel CPW appearing due to the exchange of a generic operator

$$O^{(\ell,\bar{\ell})}_{\Delta}, \quad p \equiv |\ell-\bar{\ell}| \tag{3.37}$$

by using (3.25) can be written as

$$W^{ab}_{\langle O_1O_2O\rangle\langle\bar{O}O_3O_4\rangle} = \mathbb{D}^a_{\langle O_1O_2O\rangle}\mathbb{D}^b_{\langle\bar{O}O_3O_4\rangle} W^{seed}_{\langle\mathcal{F}_1^{(0,0)}\mathcal{F}_2^{(p,0)}O\rangle\langle\bar{O}\mathcal{F}_3^{(0,0)}\mathcal{F}_4^{(0,p)}\rangle}, \tag{3.38}$$

where $\mathcal{F}_i$ are the operators with the same 4D scaling dimensions Δ_i as O_i, see section 3.3.2. The seed CPWs are defined as the s-channel contribution of (3.37) to the seed 4-point function

$$\langle\mathcal{F}_1^{(0,0)}\,\mathcal{F}_2^{(p,0)}\,\mathcal{F}_3^{(0,0)}\,\mathcal{F}_4^{(0,p)}\rangle. \tag{3.39}$$

An important property of the seed 4-point function (3.39) is that it has only $p+1$ tensor structures. We will distinguish two dual types of seed CPWs, following the convention of [58],

$$W^{(p)}_{seed} \equiv W^{seed}_{\langle\mathcal{F}_1^{(0,0)}\mathcal{F}_2^{(p,0)}O\rangle\langle\bar{O}\mathcal{F}_3^{(0,0)}\mathcal{F}_4^{(0,p)}\rangle}, \quad if\ \ell-\bar{\ell}\leq 0, \tag{3.40}$$

$$W^{(p)}_{dual\ seed} \equiv W^{seed}_{\langle\mathcal{F}_1^{(0,0)}\mathcal{F}_2^{(p,0)}O\rangle\langle\bar{O}\mathcal{F}_3^{(0,0)}\mathcal{F}_4^{(0,p)}\rangle}, \quad if\ \ell-\bar{\ell}\geq 0. \tag{3.41}$$

The case $W^{(0)}_{seed} = W^{(0)}_{dual\ seed}$ reproduces the classical scalar conformal block found by Dolan and Osborn [118, 119]. The seed CPWs `[seedCPW]` can be written in terms of a set of seed conformal blocks $H^{(p)}_e(z,\bar{z})$ and $\overline{H}^{(p)}_e(z,\bar{z})$ as[8]

$$W^{(p)}_{seed} = \mathcal{K}_4 \sum_{e=0}^{p} (-2)^{p-e}\, H^{(p)}_e(z,\bar{z}) \left[\hat{\mathbb{I}}^{42}\right]^e \left[\hat{\mathbb{I}}^{42}_{31}\right]^{p-e}, \tag{3.42}$$

$$W^{(p)}_{dual\ seed} = \mathcal{K}_4 \sum_{e=0}^{p} (-2)^{p-e}\, \overline{H}^{(p)}_e(z,\bar{z}) \left[\hat{\mathbb{I}}^{42}\right]^e \left[\hat{\mathbb{I}}^{42}_{31}\right]^{p-e}, \tag{3.43}$$

where the tensor structures are defined in appendix B.4.

The seed conformal blocks $H^{(p)}_e(z,\bar{z})$ and $\overline{H}^{(p)}_e(z,\bar{z})$ were found[9] `[plugSeedBlocks, plugDualSeedBlocks]` analytically in (5.36) and (5.37) in [58] up to an overall normalization factors, denoted there by $c^p_{0,-p}$ and $\bar{c}^p_{0,-p}$. Given the choice of seed 3-point tensor structures (B.96)-(B.99) and normalization of 2-point functions (3.16), we can fix these factors as

$$c^p_{0,-p} = (-1)^\ell\, i^p \quad \text{and} \quad \bar{c}^p_{0,-p} = 2^{-p}\,(-1)^\ell\, i^p; \tag{3.44}$$

see appendix B.3 for details. Other relevant functions are `[plugCoefficients, plugKFunctions, reduceKFunctionDerivatives, plugPolynomialsPQ]`.

The Casimir equation A very important property of the CPWs is that they satisfy the conformal Casimir eigenvalue equations [119, 120][10] which have the form

$$\left(\mathfrak{C}_n - E_n\right) W^{ab}_{\langle O_1 O_2 O\rangle\langle \overline{O} O_3 O_4\rangle} = 0, \tag{3.45}$$

where $n = 2, 3, 4$ and $\mathfrak{C}_2$, $\mathfrak{C}_3$ and $\mathfrak{C}_4$ are the quadratic, cubic and quartic Casimir differential operators respectively `[opCasimirnEF, opCasimir24D]`. They are defined in appendix B.6 together with their eigenvalues `[casimirEigenvaluen]`, where the conformal generators $\mathfrak{L}_{MN}$ given in appendix B.2 are taken to act on 2 different points

$$\mathfrak{L}_{MN} = \mathfrak{L}_{i\,MN} + \mathfrak{L}_{j\,MN}, \tag{3.46}$$

with $(ij) = (12)$ or $(ij) = (34)$ corresponding to the s-channel CPWs[11].

[8] The factors $(-2)^{p-e}$ are introduced here to match the original work [58].

[9] Notice slight change of notation $H_{here}(z,\bar{z}) \equiv G_{there}(z,\bar{z})$. This change is needed to distinguish $H_{here}(z,\bar{z}) = G_{here}(u(z,\bar{z}), v(z,\bar{z}))$.

[10] DK thanks Hugh Osborn for useful discussion on this topic.

[11] Notice that the eigenvalue of $\mathfrak{C}_3$ taken at $(ij) = (34)$ will differ by a minus sign from the eigenvalue of $\mathfrak{C}_3$ taken at $(ij) = (12)$.

The $n = 2$ Casimir equation was used in [58] for constructing the seed CPWs. Given that the seed CPWs are already known, in practice the Casimir equations can be used to validate the more general CPWs computed using the prescription above.

Conserved and identical operators, $\mathcal{P}-$ and $\mathcal{T}-$symmetries As noted in section 3.2.1, in general there are various constraints imposed on 3- and 4-point functions, such as reality conditions, permutation symmetries, conservation, and $\mathcal{P}-$ and $\mathcal{T}-$ symmetries. Recall that the most general CPW decomposition is given by (3.36),

$$g_4^I(u,v) \underset{s-\bar{O}PE}{=} \sum_{O}\sum_{a,b} \lambda^a_{\langle O_1 O_2 O\rangle} G^{I,ab}_{\langle O_1 O_2 O\rangle\langle \bar{O} O_3 O_4\rangle}(u,v)\lambda^b_{\langle \bar{O} O_3 O_4\rangle}. \tag{3.47}$$

According to the discussion around (3.22), the general solution to these constraints relevant for this expansion is

$$\lambda^a_{\langle O_1 O_2 O\rangle} = \sum_{\hat{a}} P^{a\,\hat{a}}_{\langle O_1 O_2 O\rangle}\hat{\lambda}^{\hat{a}}_{\langle O_1 O_2 O\rangle} \quad \text{and} \quad \lambda^b_{\langle \bar{O} O_3 O_4\rangle} = \sum_{\hat{b}} P^{b\,\hat{b}}_{\langle \bar{O} O_3 O_4\rangle}\hat{\lambda}^{\hat{b}}_{\langle \bar{O} O_3 O_4\rangle}. \tag{3.48}$$

Besides that, if the pair of operators O_1 and O_2 is the same as the pair of operatirs O_3 and O_4, there has to exist relations of the form

$$\lambda^b_{\langle \bar{O} O_3 O_4\rangle} = \sum_{b} N^{b\,c}_{\langle \bar{O} O_3 O_4\rangle}\lambda^c_{\langle O_1 O_2 \bar{O}\rangle}. \tag{3.49}$$

Once the relations (3.48) and (3.49) are inserted in the general expression (3.47), the resulting 4-point function will satisfy all the required constraints which preserve the s-channel.[12] In particular, the "reduced" CPWs corresponding to the coefficients $\hat{\lambda}$ will also satisfy these constraints automatically. Note that by construction the reduced CPWs are just the linear combinations of the generic CPWs.

3.2.3 The bootstrap equations

The conformal bootstrap equations are the equations which must be satisfied by the consistent CFT data. They arise as follows. The s-channel OPE (3.30) is not the only option to compute 4-point functions, there are in fact two other possibilities.

[12] Possible constraints which do not preserve s-channel are permutations of the form (13), etc. Such permutations, if present, are equivalent to the crossing equations discussed below.

One can use the t-channel OPE expansion

$$f_4 \underset{t-OPE}{=} \langle O_1(\mathbf{p}_1)O_2(\mathbf{p}_2)O_3(\mathbf{p}_3)O_4(\mathbf{p}_4)\rangle =$$
$$\pm\langle O_3(\mathbf{p}_1)O_2(\mathbf{p}_2)O_1(\mathbf{p}_3)O_4(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3} = \pm\langle O_1(\mathbf{p}_1)O_4(\mathbf{p}_2)O_3(\mathbf{p}_3)O_2(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_2\leftrightarrow\mathbf{p}_4} \tag{3.50}$$

or the u-channel OPE expansion

$$f_4 \underset{u-OPE}{=} \langle O_1(\mathbf{p}_1)O_2(\mathbf{p}_2)O_3(\mathbf{p}_3)O_4(\mathbf{p}_4)\rangle =$$
$$\pm\langle O_4(\mathbf{p}_1)O_2(\mathbf{p}_2)O_3(\mathbf{p}_3)O_1(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_4} = \pm\langle O_1(\mathbf{p}_1)O_3(\mathbf{p}_2)O_2(\mathbf{p}_3)O_4(\mathbf{p}_4)\rangle\Big|_{\mathbf{p}_2\leftrightarrow\mathbf{p}_3}. \tag{3.51}$$

In the above relations we permuted operators in the second and third equalities to get back the s-channel configuration. Minus signs are inserted for odd permutation of fermion operators.

In a consistent CFT the function f_4 is unique and does not depend on the channel used to computation it, leading to the requirement that the expressions (3.30), (3.50) and (3.51) must be equal. These equalities are the bootstrap equations. To be concrete we write the s-t consistency equation using (3.31) and (3.50)

$$f_4 \underset{s-OPE}{=} \sum_{O} \lambda^a_{\langle O_1 O_2 O\rangle} W^{ab}_{\langle O_1 O_2 O\rangle\langle \overline{O} O_3 O_4\rangle} \lambda^b_{\langle \overline{O} O_3 O_4\rangle}, \tag{3.52}$$

$$f_4 \underset{t-OPE}{=} \pm \sum_{O} \lambda^a_{\langle O_3 O_2 O\rangle} W^{ab}_{\langle O_3 O_2 O\rangle\langle \overline{O} O_1 O_4\rangle} \lambda^b_{\langle \overline{O} O_1 O_4\rangle}\Big|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3}. \tag{3.53}$$

In this example the tensor structures $\hat{\mathbb{T}}^I_n$ transform under permutation of points $\mathbf{p}_i \leftrightarrow \mathbf{p}_j$ as

$$\hat{\mathbb{T}}^I_{\langle O_3 O_2 O_1 O_4\rangle}\Big|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3} = M^{IJ}_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3}\, \hat{\mathbb{T}}^J_{\langle O_1 O_2 O_3 O_4\rangle}, \tag{3.54}$$

since they form a basis. Further decomposing these expressions using the basis of tensor structures one can compute the unknown $g^I_4(z,\overline{z})$

$$g^I_4(z,\overline{z}) \underset{s-OPE}{=} \sum_{O}\sum_{a,b} \lambda^a_{\langle O_1 O_2 O\rangle} G^{I,ab}_{\langle O_1 O_2 O\rangle\langle \overline{O} O_3 O_4\rangle}(z,\overline{z}) \lambda^b_{\langle \overline{O} O_3 O_4\rangle}, \tag{3.55}$$

$$g^I_4(z,\overline{z}) \underset{t-OPE}{=} \pm M^{IJ}_{\mathbf{p}_1\leftrightarrow\mathbf{p}_3} \sum_{O}\sum_{a,b} \lambda^a_{\langle O_3 O_2 O\rangle} G^{J,ab}_{\langle O_3 O_2 O\rangle\langle \overline{O} O_1 O_4\rangle}(1-z,1-\overline{z}) \lambda^b_{\langle \overline{O} O_1 O_4\rangle}. \tag{3.56}$$

Equating (3.55) and (3.56) we get N_4 independent equations. In a presence of additional constraints discussed in appendices B.1, B.7 and B.8, not all the N_4 equations are independent, and one should chose only those equations which correspond to the independent degrees of freedom. In the conventional numerical approach to conformal bootstrap, when Taylor expanding the crossing equations around $z = \bar{z} = 1/2$, one should also be careful to understand which Taylor coefficients are truly independent. Among other things, this depends on the analyticity properties of tensor structures $\mathbb{T}_4$, see appendix A of [1] for a discussion.

3.3 Embedding formalism

This section is meant to be a summary and a review of the embedding formalism (EF) [52–54, 125] approach to 4D correlators. The discussion is based on the works [55, 62] with some developments and corrections.

The key observation is that the 4D conformal group is isomorphic to $SO(4,2)$, the linear Lorentz group in 6D. It is then convenient to *embed* the 4D space into the 6D space where the group acts linearly, lifting the 4D operators to 6D operators. In particular, the linearity of the action of the conformal group in 6D allows one to easily build conformally invariant objects. However, non-trivial relations between these exist, posing problems for constructing the basis of tensor structures already in the case of 4-point functions. This motivates the introduction of a different formalism described in section 3.4.

The details of the 6D EF, its connection to the usual 4D formalism, and the relevant conventions are reviewed in appendix B.2. In this section we discuss only the construction of n-point tensor structures and the spinning differential operators. Our presentation focuses on the EF as a practical realization of the framework discussed in section 3.2.[13]

Embedding Let us first review the very basics of the EF. We label the points in the 6D space by $X^M = \{X^\mu,\, X^+,\, X^-\}$, with the metric given by

$$X^2 = X^\mu X_\mu + X^+ X^-. \tag{3.57}$$

The 4D space is then identified with the $X^+ = 1$ section of the lightcone $X^2 = 0$, and the coordinates on this section are chosen to be $x^\mu = X^\mu$.

[13]Note that most of the results discussed in section 3.2, like the explicit construction of 2- and 3-point tensor strucutures [53–55] and the existence of the spinning differential operators [61, 62] were originally obtained within the EF.

A generic 4D operator $O^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x)$ in spin-$(\ell,\bar\ell)$ representation can be uplifted according to (B.66) to a 6D operator $O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X)$ defined on the lightcone $X^2=0$ and totally symmetric in its both sets of indices. We can define an index-free operator $O(X,S,\overline{S})$ using the 6D polarizations S_a and $\overline{S}^b$ by

$$O(X,S,\overline{S}) \equiv O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X) S_{a_1}\ldots S_{a_\ell}\overline{S}^{b_1}\ldots\overline{S}^{b_{\bar\ell}}. \tag{3.58}$$

The 6D operators are homogeneous in X and the 6D polarizations,

$$O(X,S,\overline{S}) \sim X^{-\kappa} S^{\ell}\,\overline{S}^{\bar\ell}, \quad \kappa=\Delta+\frac{\ell+\bar\ell}{2}. \tag{3.59}$$

It is sometimes useful to assign the 4D scaling dimensions to the basic 6D objects as

$$\Delta[X]=-1 \quad \text{and} \quad \Delta[S]=\Delta[\overline{S}]=-\frac{1}{2}. \tag{3.60}$$

According to (B.69) there is a lot of freedom in choosing the lift $O(X,S,\overline{S})$. We can express this freedom by saying that the operators differing by gauge terms proportional to $S\mathbf{X}$, $\overline{S}\mathbf{X}$ or $\overline{S}S$ are equivalent. Note that $O(X,S,\overline{S})$ is a priori defined only on the lightcone $X^2=0$, but it is convenient to extend it arbitrarily to all values of X. This gives an additional redundancy that the operators differing by terms proportional to X^2 are equivalent.

The 4D field can be recovered via a projection operation defined in appendix B.2,

$$\mathcal{O}(x,s,\bar{s}) = O(X,S,\overline{S})\Big|_{proj}, \tag{3.61}$$

which essentially substitutes X, S, $\overline{S}$ with some expressions depending on x, s, $\bar{s}$ only. All the gauge terms proportional to $S\overline{\mathbf{X}}$, $\overline{S}\mathbf{X}$, $\overline{S}S$ or X^2 vanish under this operation.

Sometimes it is convenient to work with index-full form $O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X)$ and to fix part of the gauge freedom by requiring it to be traceless. We can restore the traceless form from the index-free expression $O(X,S,\overline{S})$ by

$$O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X) = \frac{2}{\ell!\,\bar\ell!\,(2+\ell+\bar\ell)!}\left(\prod_{i=1}^{\ell}\partial^{a_i}\right)\left(\prod_{j=1}^{\bar\ell}\partial_{b_j}\right)O(X,S,\overline{S}), \tag{3.62}$$

where[14]

$$\partial^a \equiv \left(S\cdot\frac{\partial}{\partial S}+\overline{S}\cdot\frac{\partial}{\partial \overline{S}}+3\right)\frac{\partial}{\partial S_a}-\overline{S}^a\left(\frac{\partial}{\partial S\cdot\partial\overline{S}}\right), \tag{3.63}$$

$$\partial_b \equiv \left(S\cdot\frac{\partial}{\partial S}+\overline{S}\cdot\frac{\partial}{\partial \overline{S}}+3\right)\frac{\partial}{\partial \overline{S}^b}-S_b\left(\frac{\partial}{\partial S\cdot\partial\overline{S}}\right). \tag{3.64}$$

Correlation functions A correlation function of 6D operators on the light cone must be SO(4,2) invariant and obey the homogeneity property (3.59). Consequently, it has the following generic form

$$\langle O_{\Delta_1}^{(\ell_1,\overline{\ell}_1)}(\mathbf{P}_1)\dots O_{\Delta_n}^{(\ell_n,\overline{\ell}_n)}(\mathbf{P}_n)\rangle = \sum_{I=1}^{N_n} g_I(\mathbf{U})T^I(X,S,\overline{S}), \tag{3.65}$$

where $T^I(X,S,\overline{S})$ are the 6D homogeneous $SU(2,2)$ invariant tensor structures and $g_I(\mathbf{U})$ are functions of 6D cross-ratios, i.e. homogeneous with degree zero SO(4,2) invariant functions of coordinates on the projective light cone. We also defined a short-hand notation

$$\mathbf{P} \equiv (X,S,\overline{S}). \tag{3.66}$$

Tensor structures split in a scaling-dependent and in a spin-dependent parts as

$$T^I(X,S,\overline{S}) = K_n\hat{T}^I(X,S,\overline{S}),\quad T^I,\hat{T}^I_n \sim \prod_{i=1}^{n} S_i^{\ell_i}\overline{S}_i^{\overline{\ell}_i}. \tag{3.67}$$

The object K_n is the 6D kinematic factor and $\hat{T}^I$ are the $SO(4,2)$ invariants of degree zero in each coordinate. The main invariant building block is the scalar product[15]

$$X_{ij} \equiv -2\,(X_i\cdot X_j). \tag{3.68}$$

The 6D kinematic factors `[n3KinematicFactor, n4KinematicFactor]` are given by

$$K_2 \equiv X_{12}^{-\frac{\kappa_1}{2}},\quad K_3 \equiv \prod_{i<j} X_{ij}^{-\frac{\kappa_i+\kappa_j-\kappa_k}{2}}, \tag{3.69}$$

[14]These operators are constructed to map terms proportional to $\overline{S}S$ to other terms proportional to $\overline{S}S$. In the equivalence class of uplifts, given an operator $O(X,S,\overline{S})$ one can find another operator $O'(X,S,\overline{S}) = O(X,S,\overline{S})+(\overline{S}S)(\dots)_O$ which differs from O by terms proportional to $\overline{S}S$ and encodes a traceless operator $O^{a_1\dots a_\ell}_{b_1\dots b_{\overline{\ell}}}(X)$. Since after taking the maximal number of derivatives the $\overline{S}S$ terms can only map to zero, we can safely replace O by O'. The action on $O'(X,S,\overline{S})$ is proportional to the action of $\frac{\partial}{\partial S_a}$ and $\frac{\partial}{\partial \overline{S}^a}$ and thus provides an inverse operation to (3.58).

[15]Notice a difference in the definition of X_{ij} compared to [55, 58, 62]: $X_{ij}^{\text{here}} = -2X_{ij}^{\text{there}}$.

and

$$K_4 \equiv \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\kappa_1-\kappa_2}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\kappa_3-\kappa_4}{2}} \times \frac{1}{X_{12}^{\frac{\kappa_1+\kappa_2}{2}} X_{34}^{\frac{\kappa_3+\kappa_4}{2}}}. \tag{3.70}$$

We also define the 6D cross-ratios by taking products of X_{ij} factors. For $n = 4$ only two cross ratios can be formed

$$U \equiv \frac{X_{12}^2 X_{34}^2}{X_{13}^2 X_{24}^2}, \quad V \equiv \frac{X_{14}^2 X_{23}^2}{X_{13}^2 X_{24}^2}. \tag{3.71}$$

With these definitions, under projection we recover the usual 4D expressions:

$$X_{ij}\Big|_{proj} = x_{ij}^2, \quad K_n\Big|_{proj} = \mathcal{K}_n, \quad U\Big|_{proj} = u, \quad V\Big|_{proj} = v. \tag{3.72}$$

Finally, given a correlator in the embedding space one can recover the 4D correlator

$$\langle O_{\Delta_1}^{(\ell_1,\bar\ell_1)}(\mathbf{p}_1)\dots O_{\Delta_n}^{(\ell_n,\bar\ell_n)}(\mathbf{p}_n)\rangle = \langle O_{\Delta_1}^{(\ell_1,\bar\ell_1)}(\mathbf{P}_1)\dots O_{\Delta_n}^{(\ell_n,\bar\ell_n)}(\mathbf{P}_n)\rangle\Big|_{proj}, \tag{3.73}$$

with the projections of the 6D invariants entering the 6D correlator given in the formula (3.72) and appendix B.4.

3.3.1 Construction of tensor structures

Let us discuss the construction of tensor structures $\hat{T}_n^I(X, S, \overline{S})$. In index-free notation, this is equivalent to finding all $SU(2,2)$ invariant homogeneous polynomials in S, $\overline{S}$. All $SU(2,2)$ invariants are built fully contracting the indices of the following objects:

$$\delta_b^a,\ \epsilon_{abcd},\ \overline{\epsilon}^{abcd},\ \mathbf{X}_{i\,ab},\ \overline{\mathbf{X}}_j^{ab},\ S_{k\,a},\ \overline{S}_l^a. \tag{3.74}$$

With the exception of taking traces over the coordinates $\mathrm{tr}[\mathbf{X}_i\overline{\mathbf{X}}_j \dots \mathbf{X}_k\overline{\mathbf{X}}_l]$, [16] all other tensor structures are built out of simpler invariants of degree two or four in S and $\overline{S}$.

List of non-normalized invariants By taking into account eq. (B.50) and the relations (B.68) and (B.72), it is possible to identify a set of invariants with the properties discussed above. These can be conveniently divided in five classes. The number of possible invariants increases with the number of points n. Below we provide a complete list of them for $n \leq 5$ and indicate their transformation property under the 4D parity. In what follows the indices $i, j, k, l, \dots$ are assumed to label different points.

[16] All such traces can be reduced to the scalar product $X_{ij} = -\mathrm{Tr}[\mathbf{X}_i\overline{\mathbf{X}}_j]/2$.

Class I constructed from $\overline{S}_i$ and S_j belonging to two different operators.

$$\begin{array}{llll} n \geq 2: & I^{ij} & \equiv & (\overline{S}_i S_j) & \xrightarrow{\mathcal{P}} & -I^{ji}, \\ n \geq 4: & I^{ij}_{kl} & \equiv & (\overline{S}_i \mathbf{X}_k \overline{\mathbf{X}}_l S_j) & \xrightarrow{\mathcal{P}} & -I^{ji}_{lk}, \\ n \geq 6: & \dots & & \dots & & \dots \end{array} \tag{3.75}$$

Class II constructed from $\overline{S}_i$ and S_i belonging to the same operator.

$$\begin{array}{llll} n \geq 3: & J^{i}_{jk} & \equiv & (\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_i) & \xrightarrow{\mathcal{P}} & -J^{i}_{kj} = J^{i}_{jk}, \\ n \geq 5: & J^{i}_{jklm} & \equiv & (\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k \mathbf{X}_l \overline{\mathbf{X}}_m S_i) & \xrightarrow{\mathcal{P}} & -J^{i}_{mlkj}, \\ n \geq 7: & \dots & & \dots & & \dots \end{array} \tag{3.76}$$

Class III constructed from S_i and S_j belonging to two different operators.

$$\begin{array}{llllllll} n \geq 3: & K^{ij}_{k} & \equiv & (S_i \overline{\mathbf{X}}_k S_j) & \xleftrightarrow{\mathcal{P}} & \overline{K}^{ij}_{k} & \equiv & (\overline{S}_i \mathbf{X}_k \overline{S}_j), \\ n \geq 5: & K^{ij}_{klm} & \equiv & (S_i \overline{\mathbf{X}}_k \mathbf{X}_l \overline{\mathbf{X}}_m S_j) & \xleftrightarrow{\mathcal{P}} & \overline{K}^{ij}_{klm} & \equiv & (\overline{S}_i \mathbf{X}_k \overline{\mathbf{X}}_l \mathbf{X}_m \overline{S}_j), \\ n \geq 7: & \dots & & \dots & & \dots & & \dots \end{array} \tag{3.77}$$

Class IV constructed from S_i and S_i belonging to the same operator.

$$\begin{array}{llllllll} n \geq 4: & L^{i}_{jkl} & \equiv & (S_i \overline{\mathbf{X}}_j \mathbf{X}_k \overline{\mathbf{X}}_l S_i) & \xleftrightarrow{\mathcal{P}} & \overline{L}^{i}_{jkl} & \equiv & (\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k \mathbf{X}_l \overline{S}_i), \\ n \geq 6: & \dots & & \dots & & \dots & & \dots \end{array} \tag{3.78}$$

Class V constructed from four S or four $\overline{S}$ belonging to different operators.

$$n \geq 4: \quad M^{ijkl} \equiv \epsilon(S_i S_j S_k S_l) \xleftrightarrow{\mathcal{P}} \overline{M}^{ijkl} \equiv \epsilon(\overline{S}_i \overline{S}_j \overline{S}_k \overline{S}_l). \tag{3.79}$$

Basic linear relations Simple properties `[applyEFProperties]` arise due to the relation (B.50). For instance

$$J^{i}_{jk} = -J^{i}_{kj}, \quad K^{ij}_{k} = -K^{ji}_{k}, \quad \overline{K}^{ij}_{k} = -\overline{K}^{ji}_{k} \tag{3.80}$$

for $n \geq 3$. Consequently not all these invariants are independent and it is convenient to work only with a subset of them, for instance $J^{i}_{j<k}$, $K^{i<j}_{k}$, $\overline{K}^{i<j}_{k}$. For $n \geq 4$ other properties must be taken into account:

$$I^{ij}_{kl} + I^{ij}_{lk} = -X_{kl} I^{ij}, \quad L^{i}_{jkl} = L^{i}_{[jkl]}, \quad M^{ijkl} = M^{[ijkl]}, \quad \overline{M}^{ijkl} = \overline{M}^{[ijkl]}. \tag{3.81}$$

These can be used in analogous manner to work only with a subset of invariants, for instance $I^{i<j}_{k<l}$, $I^{i>j}_{k>l}$, $L^{i}_{j<k<l}$, M^{1234} and $\overline{M}^{1234}$. Another important linear relation is

$$J^{i}_{[jk} X_{l]m} = 0, \tag{3.82}$$

where m is allowed to be equal to i.

Non-linear relations Unfortunately, even after taking into account all the linear relations above, many non-linear relations between products of invariant are present, see equations (B.122) - (B.125) for $n \geq 3$ relations `[applyJacobiRelations]` and appendix A in [62] for some $n \geq 4$ relations.[17] We expect that they all arise from (B.73).[18] As an example consider the following set of relations

$$M^{ijkl} = -2\, X_{ij}^{-1} \Big(K_i^{jk}\, K_j^{il} - K_i^{jl}\, K_j^{ik} \Big), \tag{3.83}$$

$$\overline{M}^{ijkl} = -2\, X_{ij}^{-1} \Big(\overline{K}_i^{jk}\, \overline{K}_j^{il} - \overline{K}_i^{jl}\, \overline{K}_j^{ik} \Big). \tag{3.84}$$

They show that M^{ijkl} and $\overline{M}^{ijkl}$ can be rewritten in terms of other invariants; hence class V objects are never used. All the relations obtained by fully contracting (3.74) with (B.73) in all possible ways, involve at most products of two invariants in class $I - IV$. In fact, we will see in section 3.4.2 that all non-linear relations have a quadratic nature. However, these quadratic relations can be combined together to form relations involving products of three or more invariants.[19] See appendix B.5 for an example of such phenomena in the $n = 3$ case.

Normalization of invariants The $\hat{T}_n^I(X, S, \overline{S})$ are required to be of degree zero in all coordinates. It is then convenient to introduce the following normalization factors

$$N_{ij} \equiv X_{ij}^{-1}, \quad N_k^{ij} \equiv \sqrt{\frac{X_{ij}}{X_{ik} X_{kj}}}, \quad N_{ijk} \equiv \frac{1}{\sqrt{X_{ij} X_{jk} X_{ki}}}. \tag{3.85}$$

Using these factors `[normalizeInvariants, denormalizeInvariants]` it is possible to define normalized type I and type II tensor structures

$$\hat{I}^{ij} \equiv I^{ij}, \quad \hat{I}_{kl}^{ij} \equiv N_{kl} I_{kl}^{ij}, \quad \hat{J}_{jk}^{i} \equiv N_{jk} J_{jk}^{i}, \quad \hat{J}_{jklm}^{i} \equiv N_{jk} N_{lm} J_{jklm}^{i}, \tag{3.86}$$

and normalized type III and type IV tensor structures

$$\hat{K}_k^{ij} \equiv N_k^{ij} K_k^{ij}, \quad \hat{K}_{klm}^{ij} \equiv N_{klm} K_{klm}^{ij}, \quad \hat{L}_{jkl}^{i} \equiv N_{jkl} L_{jkl}^{i}, \tag{3.87}$$

[17] Mind the difference in notation, see footnote 20 for details.

[18] In principle the Schouten identities might also contribute, see the footnote at page 26 of [54]; we found however that the Schouten identities, when contracted, give relations equivalent to (B.73) for $n \leq 4$.

[19] In other words, we have a graded ring of invariants and an ideal I of relations between them. The goal is to find a basis of independent invariants of a given degree modulo I. In principle, I is generated by a quadratic basis, but it is not trivial to reduce invariants modulo this basis. One would like to find a better basis, e.g., a Gröbner basis, which then will contain higher-order relations.

with the analogous expressions for parity conjugated invariants $\overline{\hat{K}}{}^{ij}_{k}$, $\overline{\hat{K}}{}^{ij}_{klm}$ and $\overline{\hat{L}}{}^{i}_{jkl}$. In appendix B.4 we provide an explicit 4D form of these invariants after projection. Notice the slight change of notation from previous works[20].

Basis of tensor structures Given an n-point function, one can construct a set of tensor structures [n3ListStructures, n3ListStructuresAlternativeTS] [n4ListStructuresEF] by taking products of basic invariants as

$$\hat{T}_n^I = \prod_{i,j,\ldots} \underbrace{\left[\hat{I}^{ij}\right]^{\#}}_{n\geq 2} \underbrace{\left[\hat{J}^{i}_{jk}\right]^{\#}\left[\hat{K}^{jk}_{i}\right]^{\#}\left[\overline{\hat{K}}{}^{jk}_{i}\right]^{\#}}_{n\geq 3} \underbrace{\left[\hat{I}^{ij}_{kl}\right]^{\#}\left[\hat{L}^{i}_{jkl}\right]^{\#}\left[\overline{\hat{L}}{}^{i}_{jkl}\right]^{\#}}_{n\geq 4} \underbrace{\left[\hat{J}^{i}_{jklm}\right]^{\#}\left[\hat{K}^{jk}_{ilm}\right]^{\#}\left[\overline{\hat{K}}{}^{jk}_{ilm}\right]^{\#}}_{n\geq 5} \cdots \tag{3.88}$$

The subscripts stress that for a given number of points n not all the invariants are defined. The non-negative exponents # are determined by requiring $\hat{T}_n^I$ to be of degree $(\ell_i, \bar{\ell}_i)$ in $(S_i, \overline{S}_i)$. Generally, not all tensor structures obtained in this way are independent, due to the properties and relations discussed above. The number of relations to take into account increase rapidly with n. For $n \leq 3$ the problem of constructing a basis of independent tensor structures has been succesfully solved in [54, 55]; we review the construction for $n = 3$ in appendix B.5. However the increasing number of relations makes this approach inefficient to study general correlators for $n \geq 4$, mainly because many relations which are cubic or higher order in invariants can be written. In section 3.4 an alternative method of identifying all the independent structures is provided. Using this method we will also prove in section 3.4.2 that any n-point function tensor structure is constructed out of $n \leq 5$ invariants, namely the invariants involving five or less points in the formula (3.88).

3.3.2 Spinning differential operators

Let us now discuss the EF realization of the *spinning differential operators* used in (3.25) which allow to relate 3-point tensor structures of correlators with different spins[21]

$$\langle O^{(\ell_i,\bar{\ell}_i)}_{\Delta_{O_i}} O^{(\ell_j,\bar{\ell}_j)}_{\Delta_{O_j}} O^{(\ell,\bar{\ell})}_{\Delta_O} \rangle \sim \mathbf{D}_{ij}\, \langle O^{(\ell'_i,\bar{\ell}'_i)}_{\Delta'_{O_i}} O^{(\ell'_j,\bar{\ell}'_j)}_{\Delta'_{O_j}} O^{(\ell,\bar{\ell})}_{\Delta_O} \rangle. \tag{3.89}$$

[20]The correspondence with the notation of [55, 58, 62] is as follows: $\hat{I}^{ij} \sim I_{ij}$, $-2\,\hat{I}^{ij}_{kl} \sim \hat{J}_{ij,\,kl}$, $-2\,\hat{J}^{i}_{jk} \sim J_{i,\,jk}$, $\sqrt{-2}\,\hat{K}^{ij}_{k} \sim K_{k,\,ij}$, $\sqrt{-2}\,\overline{\hat{K}}{}^{ij}_{k} \sim \overline{K}_{k,\,ij}$, $\sqrt{-8}\,\hat{L}^{i}_{jkl} \sim K_{i,jkl}$, $\sqrt{-8}\,\overline{\hat{L}}{}^{i}_{jkl} \sim \overline{K}_{i,jkl}$, where the expressions in the l.h.s. represent our notation and the expressions in the r.h.s. represent their notation.

[21]This relation is of course purely kinematic, it holds only at the level of tensor structures and does not hold at the level of the full correlator.

The operators[22] $\mathbf{D}_{ij}$ are written as a product of basic differential operators which were found in [62]

$$\mathbf{D}_{ij} = \Big\{ \prod_{i,j=1,2} \nabla_{ij}^{\#} I_{ij}^{\#} \overline{d}_{ij}^{\#} d_{ij}^{\#} D_{ij}^{\#} \widetilde{D}_{ij}^{\#} \Big\}. \tag{3.90}$$

The exponents are determined by matching the spins on both sides of (3.89). The basic spinning differential operators are constructed to be insensitive to pure gauge modifications and different extensions of fields outside of the light cone as stressed in (B.74). The action of these operators in 4D can be deduced by using the projection rules given in (B.76).

We provide here the list of basic differential operators[23] entering (3.90) arranging them in two sets according to the value of $\Delta\ell = |\ell_i + \ell_j - \overline{\ell}_i - \overline{\ell}_j| = 0, 2$. For $\Delta\ell = 0$ we have

$$\begin{aligned}
D_{ij} &\equiv \frac{1}{2}\overline{S}_i \Sigma^M \overline{\Sigma}^N S_i \Big(X_{jM} \frac{\partial}{\partial X_i^N} - X_{jN} \frac{\partial}{\partial X_i^M} \Big) && \sim \overline{S}_i S_i, \\
\widetilde{D}_{ij} &\equiv \overline{S}_i \mathbf{X}_j \overline{\Sigma}^N S_i \frac{\partial}{\partial X_j^N} + 2 I^{ij}\, S_{ia} \frac{\partial}{\partial S_{ja}} - 2 I^{ji}\, \overline{S}_i^a \frac{\partial}{\partial \overline{S}_j^a} && \sim \overline{S}_i S_i, \\
I^{ij} &\equiv \overline{S}_i S_j && \sim \overline{S}_i S_j, \\
\nabla_{ij} &\equiv \big[\mathbf{X}_i \overline{\mathbf{X}}_j \big]_a^b \frac{\partial^2}{\partial S_{i\,a}\, \partial \overline{S}_j^b} && \sim S_i^{-1} \overline{S}_j^{-1}.
\end{aligned} \tag{3.91}$$

For $\Delta\ell = 2$ we have

$$\begin{aligned}
d_{ij} &\equiv S_j \overline{X}_i \frac{\partial}{\partial \overline{S}_i} && \sim \overline{S}_i^{-1} S_j \\
\overline{d}_{ij} &\equiv \overline{S}_j X_i \frac{\partial}{\partial S_i} && \sim S_i^{-1} \overline{S}_j .
\end{aligned} \tag{3.92}$$

Note that for any differential operator $\mathbf{D}_{ij}$ we necessarily have $\Delta\ell$ even, since it has to preserve the total Fermi/Bose statistics of the pair of local operators.

The basic spinning differential operators described above carry the 4D scaling dimension according to (3.60), thus it is convenient to introduce an operator Ξ which formally shifts the 4D dimensions of external operators in a way that effectively makes the 4D scaling dimensions of $\mathbf{D}_{ij}$ vanish. The action of Ξ on basic spinning differential operators is defined as

$$\Xi[D_{ij}] f_n = (D_{ij} f_n)\Big|_{\Delta_j \to \Delta_j + 1}, \quad \Xi[\widetilde{D}_{ij}] f_n = (\widetilde{D}_{ij} f_n)\Big|_{\Delta_i \to \Delta_i + 1} \tag{3.93}$$

[22] We distinguish the operators $\mathbf{D}$ here and the operators $\mathbb{D}$ described in section 3.2.1 because acting on the seed tensor structures they generate different bases. The basis spanned by $\mathbf{D}$ is often called the differential basis.

[23] Notice a change in the normalization of the basic spinning differential operators compared to [62].

and

$$\Xi[op]f_n = (op\, f_n)\Big|_{\Delta_i \to \Delta_i + 1/2}\Big|_{\Delta_j \to \Delta_j + 1/2}, \tag{3.94}$$

where op denotes any of the remaining spinning differential operators.[24] These formal shifts of course make sense only if the scaling dimensions appear as variables in f_n. The use of the dimension-shifting operator Ξ allows to keep the same scaling dimensions in the seed CPWs and the CPW related by (3.38).

The relevant functions in the package are `[opDEF, opDtEF, opdEF, opdbEF, opIEF, opNEF]` and Ξ.

3.4 Conformal frame

For sufficiently complicated correlation functions one finds a lot of degeneracies in the embedding space construction of tensor structures. There exists an alternative construction [1, 51] which provides better control under degeneracies. More precisely, it reduces the problem of constructing tensor structures to the well studied problem of finding invariant tensors of orthogonal groups of small rank.

Our aim is to describe the correlation function $f_n(x, s, \overline{s})$ whose generic form is given in (3.11). The conformal symmetry relates the values of $f_n(x, s, \overline{s})$ at different values of x. There is a classical argument, usually applied to 4-point correlation functions, saying that it is sufficient to know only the value $f_n(x_{CF}, s, \overline{s})$ for some standard choices of x_{CF} such that all the other values of x can be obtained from some x_{CF} by a conformal transformation. This conformal transformation then allows one to compute $f_n(x, s, \overline{s})$ from $f_n(x_{CF}, s, \overline{s})$. The standard configurations x_{CF} are chosen in such a way that there are no conformal transformations relating two different standard configurations, so that the values $f_n(x_{CF}, s, \overline{s})$ can be specified independently. Following [1], we call the set of standard configurations x_{CF} the conformal frame (CF).

The usefulness of this construction lies in the fact that the values $f_n(x_{CF}, s, \overline{s})$ have to satisfy only a few constraints. In particular, these values have to be invariant only under the conformal transformations which do not change x_{CF} [1]. Such conformal transformations form a group which we call the "little group". The little group is $SO(d + 2 - n)$ for n-point functions in d dimensions.[25] For example, for 4-point functions in 4D it is $SO(2) \simeq U(1)$. One can already see a considerable

[24]The shift in the last formula can alternatively be implemented with multiplication by a factor $X_{ij}^{-1/2}$.

[25]For $n \geq 3$ and generic x. The little group is trivial for $n \geq d + 2$.

simplification offered by this construction for 4-point functions in 4D, since the invariants of $SO(2)$ are extremely easy to classify.

We use the following choice for the conformal frame configurations x_{CF} for $n \geq 3$,

$$x_1^\mu = (0, 0, 0, 0), \tag{3.95}$$
$$x_2^\mu = ((\bar{z} - z)/2, 0, 0, (z + \bar{z})/2), \tag{3.96}$$
$$x_3^\mu = (0, 0, 0, 1), \tag{3.97}$$
$$x_4^\mu = (0, 0, 0, L), \tag{3.98}$$
$$x_5^\mu = (x_5^0, x_5^1, 0, x_5^3), \tag{3.99}$$

where if $n = 3$ we can set $z = \bar{z} = 1/2$ and if we have more than 5 operators, the unspecified positions $x_{\geq 6}$ are completely unconstrained.

Here L is a fixed number, and we always take the limit $L \to +\infty$ to place the corresponding operator "at inifinity". In this limit one should use the rescaled operator O_4

$$O_4 \to O_4 \, L^{2\Delta_4} \tag{3.100}$$

inside all correlators to get a finite and non-zero result.

The variables z, $\bar{z}$, x_5^0, x_5^1, x_5^3, and the 4-vectors x_6, x_7, ... are the coordinates on the conformal frame and thus are essentially the conformal cross-ratios. Note that we have 2 conformal cross-ratios for 4 points, and $4\,n - 15$ for n points with $n \geq 5$. Notice also that for 4-point functions the analytic continuation with $z = \bar{z}^*$ corresponds to Euclidean kinematics. It is easy to check that there are no conformal generators which take the conformal frame configuration (3.95) - (3.99) to another nearby conformal frame configuration.

3.4.1 Construction of tensor structures

3.4.1.1 Three-point functions

As shown in appendix B.5, an independent basis for general 3-point tensor structures is relatively easy to construct in EF, and there is no direct need for the conformal frame construction. Nonetheless, in this section we employ the CF to construct 3-point tensor structures in order to illustrate how the formalism works in a familiar case.[26]

[26]The CF construction of 3-point functions is not implemented in the package.

The little group algebra $\mathfrak{so}(1,2)$ which fixes the points x_1, x_2, x_3 is defined by the following generators

$$M^{01}, \quad M^{02}, \quad M^{12}; \tag{3.101}$$

see appendix B.1 for details. According to our conventions, the corresponding generators acting on polarizations s_α are

$$\mathcal{S}^{01} = -\frac{1}{2}\sigma^1, \quad \mathcal{S}^{02} = -\frac{1}{2}\sigma^2, \quad \mathcal{S}^{12} = \frac{i}{2}\sigma^3, \tag{3.102}$$

and the generators acting on $\overline{s}^{\dot\alpha}$ are

$$\overline{\mathcal{S}}^{01} = \frac{1}{2}\sigma^1, \quad \overline{\mathcal{S}}^{02} = \frac{1}{2}\sigma^2, \quad \overline{\mathcal{S}}^{12} = \frac{i}{2}\sigma^3. \tag{3.103}$$

It is easy to see that if we introduce $t_\alpha \equiv s_\alpha$ and $\tilde{t}_\alpha \equiv \sigma^3_{\alpha\dot\beta}\overline{s}^{\dot\beta}$, then t and $\tilde{t}$ transform in the same representation of $\mathfrak{so}(1,2)$.

General 3-point structures are put in one-to-one correspondence with the $\mathfrak{so}(1,2) \simeq \mathfrak{su}(2)$ conformal frame invariants built out of t_i and $\tilde{t}_i$, $i = 1,2,3$. This gives an explicit implementation of the rule [1, 23, 51] which states that 3-point structures correspond to the invariants of $SO(d-1) = SO(3)$ group

$$\Big((\ell_1,\overline{\ell}_1)\otimes(\ell_2,\overline{\ell}_2)\otimes(\ell_3,\overline{\ell}_3)\Big)^{SO(3)} = \Big(\ell_1\otimes\overline{\ell}_1\otimes\ell_2\otimes\overline{\ell}_2\otimes\ell_3\otimes\overline{\ell}_3\Big)^{SO(3)}. \tag{3.104}$$

Using this rule, we can immediately build independent bases of 3-point structures, for example by first computing the tensor product decompositions

$$\ell_i\otimes\overline{\ell}_i = \bigoplus_{j_i=|\ell_i-\overline{\ell}_i|}^{\ell_i+\overline{\ell}_i} j_i, \qquad (j_i+\ell_i+\overline{\ell}_i \text{ even}) \tag{3.105}$$

and then for every set of j_i constructing the unique singlet in $j_1\otimes j_2\otimes j_3$ when it exists.

A more direct way, which does not however automatically avoid degeneracies, is to use the basic building blocks for $SO(3)$ invariants, which are the contractions of the form $t_i^\alpha t_{j\,\alpha}$, $t_i^\alpha \tilde{t}_{j\,\alpha}$ and $\tilde{t}_i^\alpha \tilde{t}_{j\,\alpha}$. It is then straightforward to establish the correspondence with the embedding formalism invariants

$$I^{ij} \propto \tilde{t}_i t_j, \quad J^i_{jk} \propto \tilde{t}_i t_i, \quad K^{ij}_k \propto t_i t_j, \quad \overline{K}^{ij}_k \propto \tilde{t}_i\tilde{t}_j, \tag{3.106}$$

where it is understood that i, j, k are all distinct. Up to the coefficients, this dictionary is fixed completely by matching the degrees of s and $\overline{s}$ on each side.

Correspondingly, as in the embedding space formalism, we have relations between these building blocks, which now come from the Schouten identity[27]

$$(AB)C_\alpha + (BC)A_\alpha + (CA)B_\alpha = 0. \tag{3.107}$$

For example we can take $A = t_i$, $B = t_k$, $C = \tilde{t}_j$ and contract (3.107) with $\tilde{t}_k$ to find

$$(t_i t_k)(\tilde{t}_j \tilde{t}_k) + (t_k \tilde{t}_j)(t_i \tilde{t}_k) + (\tilde{t}_j t_i)(t_k \tilde{t}_k) = 0, \tag{3.108}$$

which corresponds via the dictionary (3.106) to an identity of the form

$$\# K_j^{ik} \overline{K}_i^{jk} + \# I^{jk} I^{ki} + \# I^{ji} J_{ij}^k = 0. \tag{3.109}$$

This gives precisely the structure of the relation (B.122). We thus effectively reproduce the EF construction.

Finally, let us briefly comment on the action of $\mathcal{P}$ in the 3-point conformal frame. The parity transformation of operators (B.26) induces the following transformation of polarizations

$$s_\alpha \to i\bar{s}^{\dot\alpha}, \quad \bar{s}^{\dot\beta} \to i s_\beta \quad \Longrightarrow \quad t \to i\sigma^3 \tilde{t}, \quad \tilde{t} \to i\sigma^3 t. \tag{3.110}$$

The full parity transformation does not however preserve the conformal frame since it reflects all three spatial axes and thus moves the points x_2 and x_3. We can reproduce the correct parity action in the conformal frame by supplementing the full parity transformation with $i\pi$ boost in the 03 plane given by $e^{-i\pi S^{03}} = i\sigma_3$ on t and by $\sigma^3 e^{-i\pi S^{03}} \sigma^3 = -i\sigma_3$ on $\tilde{t}$. This leads to

$$t \to \tilde{t}, \quad \tilde{t} \to -t. \tag{3.111}$$

Note that according to (3.111) the transformations properties of (3.106) under parity match precisely the ones found in (3.75) - (3.77).

3.4.1.2 Four-point functions

In the $n = 4$ case the little group algebra $\mathfrak{so}(2) \simeq \mathfrak{u}(1)$ which fixes the points x_1, x_2, x_3, x_4 is given by the generator

$$M^{12}. \tag{3.112}$$

[27] Which itself follows from contracting $\epsilon^{\beta\gamma}$ with the identity $A_{[\alpha} B_\beta C_{\gamma]} = 0$ valid for two-component spinors.

Note that the algebra $\mathfrak{so}(2)$ is a subalgebra of the 3-point little group algebra $\mathfrak{so}(1,2)$ discussed above. According to (3.102), its action on both t and $\tilde{t}$ is given by

$$\mathcal{S}^{12} = \frac{i}{2}\sigma^3. \tag{3.113}$$

This generator acts diagonally on t and $\tilde{t}$, so that we can decompose

$$s_\alpha \equiv \begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad \bar{s}_{\dot\beta} \equiv \begin{pmatrix}\bar\xi\\ \bar\eta\end{pmatrix} \quad \Longrightarrow \quad t \equiv s_\alpha = \begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad \tilde{t} \equiv \sigma^3_{\alpha\dot\beta}\bar{s}^{\dot\beta} = \begin{pmatrix}\bar\eta\\ \bar\xi\end{pmatrix}. \tag{3.114}$$

Note that our convention $\bar{s}_{\dot\alpha} = (s_\alpha)^*$ implies that $\bar\xi = \xi^*$ and $\bar\eta = \eta^*$. Appropriately defining the $\mathfrak{u}(1)$ charge Q we can say that

$$Q[\xi] = Q[\bar\eta] = +1 \quad \textbf{and} \quad Q[\eta] = Q[\bar\xi] = -1. \tag{3.115}$$

Tensor structures of 4-point functions are just the products of $\xi, \bar\xi, \eta, \bar\eta$ of total charge $Q = 0$. These are given by [CF4pt,n4ListStructures]

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4\\ \bar q_1 & \bar q_2 & \bar q_3 & \bar q_4\end{bmatrix} \equiv \prod_{i=1}^4 \xi_i^{-q_i+\ell_i/2}\eta_i^{q_i+\ell_i/2}\bar\xi_i^{-\bar q_1+\bar\ell_i/2}\bar\eta_i^{\bar q_i+\bar\ell_i/2},$$
$$q_i \in \{-\ell_i/2,\dots,\ell_i/2\},\ \bar q_i \in \{-\bar\ell_i/2,\dots,\bar\ell_i/2\}, \tag{3.116}$$

subject to

$$\sum_{i=1}^4 (q_i - \bar q_i) = 0. \tag{3.117}$$

It is clear from the construction that these 4-point structures are all independent, i.e. there are no relations between them. It is in contrast with the embedding space formalism, where there are a lot of relations between various 4 point building blocks.

As a simple example, consider a 4-point function of a $(1,0)$ fermion at position 1, a $(0,1)$ fermion at position 2 and two scalars at position 3 and 4. The allowed 4-point tensor structures are then

$$\begin{bmatrix} +\frac12 & 0 & 0 & 0\\ 0 & +\frac12 & 0 & 0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac12 & 0 & 0 & 0\\ 0 & -\frac12 & 0 & 0\end{bmatrix}. \tag{3.118}$$

To compute the action of space parity, we need to supplement the full spatial parity (3.110) with a π rotation in, say, the 13 plane in order to make sure that parity preserves the 4-point conformal frame (3.95) - (3.98). In this case the combined transformation is simply a reflection in the 2'nd coordinate direction. It is easy to compute that this gives the action

$$\xi \to -i\overline{\xi}, \quad \overline{\xi} \to i\xi, \quad \eta \to -i\overline{\eta}, \quad \overline{\eta} \to i\eta. \tag{3.119}$$

Note that this does not commute with the action of $\mathfrak{u}(1)$ since the choice of the 13 plane was arbitrary – we could have also chosen the 23 plane, and $\mathfrak{u}(1)$ rotates between these two choices. It is only important that this reflection reverses the charges of $\mathfrak{u}(1)$ and thus maps invariants into invariants.

From (3.119) we find that the parity acts as

$$\mathcal{P}\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix} = i^{-\sum_i \ell_i - \overline{\ell}_i} \begin{bmatrix} \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}. \tag{3.120}$$

From the definition (3.116) we also immediately find the complex conjugation rule

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix}^* = \begin{bmatrix} \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}. \tag{3.121}$$

According to (B.36), by combining these two transformations we find the action of time reversal

$$\mathcal{T}\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix} = i^{\sum_i \ell_i - \overline{\ell}_i} \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix}. \tag{3.122}$$

3.4.1.3 Five-point functions and higher

In the $n \geq 5$ case there are no conformal generators which fix the conformal frame. It means that all $\xi, \overline{\xi}, \eta, \overline{\eta}$ are invariant by themselves.[28] This allows us to construct the n-point tensor structures

$$\begin{bmatrix} q_1 & q_2 & \dots & q_n \\ \overline{q}_1 & \overline{q}_2 & \dots & \overline{q}_n \end{bmatrix} \equiv \prod_{i=1}^{n} \xi_i^{-q_i+\ell_i/2} \eta_i^{q_i+\ell_i/2} \overline{\xi}_i^{-\overline{q}_1+\overline{\ell}_i/2} \overline{\eta}_i^{\overline{q}_i+\overline{\ell}_i/2}, \tag{3.123}$$

with the only restriction

$$q_i \in \{-\ell_i/2, \dots \ell_i/2\}, \quad \overline{q}_i \in \{-\overline{\ell}_i/2, \dots \overline{\ell}_i/2\}. \tag{3.124}$$

[28] More precisely, there is still the $\mathbb{Z}_2$ kernel of the projection $Spin(1,3) \to SO(1,3)$, which gives the selection rule that the full correlator should be bosonic (in this sense $\xi, \overline{\xi}, \eta, \overline{\eta}$ are not individually invariant).

3.4.2 Relation with the EF

In practical applications, 3- and 4-point functions are the most important objects. It is possible to treat 3-point functions in the CF or the EF. Since the latter is explicitly covariant, it is often more convenient. On the other hand, 4-point functions are treated most easily in the conformal frame approach. This creates a somewhat unfortunate situation when we have two formalisms for closely related objects. To remedy this, let us discuss how to go back and forth between the EF and the CF.

Embedding formalism to conformal frame It is relatively straightforward to find the map `[toConformalFrame]` from the embedding formalism tensor structures to the conformal frame ones. First one needs to project the 6D elements to the 4D ones and then to substitute the appropriate values of coordinates according to the choice of the conformal frame.

For 6D coordinates according to (B.65) and the definition of the conformal frame (3.95) - (3.98) one has

$$\begin{aligned} X_1 &= (0,0,0,0,1,0),\\ X_2 &= ((\bar{z}-z)/2, 0, 0, (z+\bar{z})/2, 1, -z\bar{z}),\\ X_3 &= (0,0,0,1,1,-1),\\ X_4 &= (0,0,0,L,1,-L^2), \end{aligned} \tag{3.125}$$

and for the 6D polarizations according to (B.71) one has

$$(S_i)_a = \begin{pmatrix} (s_i)_\alpha \\ -x_i^\mu \overline{\sigma}_\mu^{\dot\alpha\beta}(s_i)_\beta \end{pmatrix}, \quad (\overline{S}_i)^a = \begin{pmatrix} (\bar{s}_i)_{\dot\beta}\overline{\sigma}_\mu^{\dot\beta\alpha} x_i^\mu \\ (\bar{s}_i)_{\dot\alpha} \end{pmatrix}. \tag{3.126}$$

In the last expression it is understood that all the coordinates x belong to the conformal frame x_{CF} (3.95) - (3.98).

The final step is to perform the rescaling (3.100) and to take the limit $L \to +\infty$. There is a very neat way to do it by recalling that 6D operators O according to (3.59) are homogeneous in 6D coordinates and 6D polarizations, thus

$$O(S_4, \overline{S}_4, X_4)L^{2\Delta_4} = O(S_4, \overline{S}_4, X_4)L^{2\kappa_O-\ell_4-\bar\ell_4} = O(S_4/L, \overline{S}_4/L, X_4/L^2). \tag{3.127}$$

It is then clear that the final step is equivalent to the following substitution of the 6D coordinates at the 4th position

$$X_4 \to \lim_{L\to+\infty} X_4/L^2 = (0,0,0,0,0,-1) \tag{3.128}$$

and for the 6D polarizations

$$(S_4)_a \to \lim_{L\to+\infty}(S_4)_a/L = \begin{pmatrix} 0 \\ -\overline{\sigma}_3^{\dot{\alpha}\beta}(s_4)_\beta \end{pmatrix}, \quad (\overline{S}_4)^a \to \lim_{L\to+\infty}(\overline{S}_4)^a/L = \begin{pmatrix} (\overline{s}_4)_{\dot{\beta}}\overline{\sigma}_3^{\dot{\beta}\alpha} \\ 0 \end{pmatrix}. \tag{3.129}$$

Conformal frame to embedding formalism As discussed in section 3.4.1.2, 4-point tensor structures are given by products of $\xi_i, \overline{\xi}_i, \eta_i, \overline{\eta}_i$ with vanishing total $U(1)$ charge. It is easy to convince oneself that any such product can be represented (not uniquely) by a product of $U(1)$-invariant bilinears

$$\overline{\xi}_i\xi_j, \quad \overline{\eta}_i\eta_j, \quad \xi_i\eta_j, \quad \overline{\xi}_i\overline{\eta}_j, \tag{3.130}$$

where $i, j = 1\ldots4$. For $n \geq 5$-point a general tensor structure is still represented by a product of bilinears, see footnote 28, but since there is no $U(1)$-invariance condition, the following set of bilinears should also be taken into account

$$\xi_i\xi_j, \quad \eta_i\eta_j, \quad \overline{\xi}_i\overline{\xi}_j, \quad \overline{\eta}_i\overline{\eta}_j, \quad \overline{\eta}_i\xi_j, \quad \overline{\xi}_i\eta_j, \tag{3.131}$$

where $i, j = 1\ldots n$.

These bilinears themselves are tensor structures with low spin. Noticing that the EF invariants are also naturally bilinears in polarizations we can write a corresponding set of EF invariants with the same spin signatures. Translating these invariants to conformal frame via the procedure described above `[toConformalFrame]`, one can then invert the result and express the bilinears (3.130) and (3.131) in terms of covariant expressions. We could call this procedure covariantization `[toEmbeddingFormalism]`. The basis of EF structures is over-complete so the inversion procedure is ambiguous and one is free to choose one out of many options.

Since there is a finite number of bilinears (3.130) and (3.131) there will be a finite number of covariant tensor structures they can be expressed in terms of after the covariantization procedure. It is then very easy to see that one needs only the class of $n = 4$ tensor structures to cover all the bilinears (3.130) and the class of $n = 5$ tensor structures to cover all the bilinears (3.131).

The ambiguity of the inversion procedure mentioned above is related to the linear relations between EF structures. Non-linear relations between EF structures arise due to the tautologies such as

$$(\overline{\xi}_i\xi_j)(\overline{\eta}_k\eta_l) = (\overline{\xi}_i\overline{\eta}_k)(\xi_j\eta_l). \tag{3.132}$$

This observation in principle allows to classify all relations between $n \geq 4$ EF invariants.

Example. By going to the conformal frame we get

$$\hat{J}^1_{23} = \frac{z}{z-1}\,\overline{\xi}_1\xi_1 - \frac{\overline{z}}{\overline{z}-1}\,\overline{\eta}_1\eta_1, \quad \hat{J}^1_{24} = -z\,\overline{\xi}_1\xi_1 + \overline{z}\,\overline{\eta}_1\eta_1, \quad \hat{J}^1_{34} = -\overline{\xi}_1\xi_1 + \overline{\eta}_1\eta_1. \tag{3.133}$$

Inverting these relation one gets

$$\overline{\xi}_1\xi_1 = -\frac{z-1}{z\,(z-\overline{z})}\left((\overline{z}-1)\,\hat{J}^1_{23} + \hat{J}^1_{24}\right), \quad \overline{\eta}_1\eta_1 = -\frac{\overline{z}-1}{\overline{z}\,(z-\overline{z})}\left((z-1)\,\hat{J}^1_{23} + \hat{J}^1_{24}\right). \tag{3.134}$$

We see right away that the invariants J^1_{23}, J^1_{24} and J^1_{34} must be dependent. One can easily get a relation between them by plugging (3.134) to the third expression (3.133). The obtained relation will match perfectly the linear relation (3.82).

Note that there is a factor $1/(z-\overline{z})$ in (3.134), which suggests that the structure $\overline{\xi}_1\xi_1$ blows up at $z=\overline{z}$. This is not the case simply by the definition of ξ and $\overline{\xi}$; instead, it is the combination of structures on the right hand side which develops a zero giving a finite value at $z=\overline{z}$. However, this value will depend on the way the limit is taken. This is related to the enhancement of the little group from $U(1)=SO(2)$ to $SO(1,2)$ at $z=\overline{z}$. At $z=\overline{z}$ it is no longer true that $\overline{\xi}_1\xi_1$ is a little group invariant. This enhancement implies certain boundary conditions for the functions which multiply the conformal frame invariants. See appendix A of [1] for a detailed discussion of this point.

3.4.3 Differentiation in the conformal frame

Now we would like to understand how to implement the action of the embedding formalism differential operators such as (3.91) and (3.92) directly in the conformal frame. We need to make two steps. First, to understand the form of these differential operators in 4D space. This is done by using the projection of 6D differential operators to 4D given in appendix B.2. Second, to understand how to act with 4D differential operators directly in the conformal frame. We focus on this step in the remainder of this section. For simplicity, we restrict the discussion to the most important case of four points.

A correlation function in the conformal frame is obtained by restricting its coordinates x to the conformal frame configurations x_{CF}. The action of the derivatives $\partial/\partial s$ and $\partial/\partial\overline{s}$ in polarizations on this correlation function is straightforward, since nothing happens to polarizations during this restriction. The only non-trivial part is the coordinate derivatives $\partial/\partial x_i$: in the conformal frame a correlator only depends on the variables z and $\bar{z}$ which describe two degrees of freedom of the second operator and it is not immediately obvious how to take say the $\partial/\partial x_1$ derivatives.

The resolution is to recall that 4-point functions according to (3.10) are invariant under generic conformal transformation spanned by 15 conformal generators L_{MN}. By using (B.57) one can see that it is equivalent to 15 differential equations

$$(\mathfrak{L}_{1\,MN} + \mathfrak{L}_{2\,MN} + \mathfrak{L}_{3\,MN} + \mathfrak{L}_{4\,MN})\, f_4(x_i, s_i, \bar{s}_i) = 0. \tag{3.135}$$

The differential operators $\mathfrak{L}_{i\,MN}$ defined in (B.58) together with (B.76) and (B.77) are given by linear combinations of derivatives $\partial/\partial x_i$, $\partial/\partial s_i$ and $\partial/\partial\bar{s}_i$. Out of 15 differential equations (3.135) one equation (for L_{12}) expresses the little group invariance under rotations in the 12 plane and thus when restricted to the 4-point conformal frame (3.95) - (3.98) does not contain derivatives $\partial/\partial x_i$. The remaining 14 equations allow to express the 14 unknown derivatives $\partial/\partial x_i^\mu$ restricted to the conformal frame in terms of $\partial/\partial x_2^0$, $\partial/\partial x_2^3$, $\partial/\partial s_i$ and $\partial/\partial\bar{s}_i$. Higher-order derivatives can be obtained in a similar way by differentiating (3.135).

Computation of general derivatives can be cumbersome, but in practice it is easily automated with `Mathematica`. We provide a conformal frame implementation of the differential operators (3.91) - (3.92) `[opD4D, opDt4D, opd4D, opdb4D, opI4D, opN4D]` as well as of the quadratic Casimir operator `[opCasimir24D]` acting on 4-point functions. As a simple example (although it does not require differentiation in x), we display here the action of ∇_{12} on a generic conformal frame structure

$$\begin{aligned}
&\nabla_{12}\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2 & \bar{q}_3 & \bar{q}_4 \end{bmatrix} g(z,\bar{z}) \\
&= -\frac{(\ell_1+2q_1)(\bar{\ell}_2+2\bar{q}_2)}{4}\begin{bmatrix} q_1-\frac{1}{2} & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2-\frac{1}{2} & \bar{q}_3 & \bar{q}_4 \end{bmatrix} z g(z,\bar{z}) \\
&\quad + \frac{(\ell_1-2q_1)(\bar{\ell}_2-2\bar{q}_2)}{4}\begin{bmatrix} q_1+\frac{1}{2} & q_2 & q_3 & q_4 \\ \bar{q}_1 & \bar{q}_2+\frac{1}{2} & \bar{q}_3 & \bar{q}_4 \end{bmatrix} \bar{z} g(z,\bar{z}).
\end{aligned} \tag{3.136}$$

Other operators, e.g. (3.91), give rise to more complicated expressions which however can still be efficiently applied to the seed CPWs.

3.5 Conclusions

In this chapter we have described a framework for performing computations in 4D CFTs by unifying two different approaches, the covariant embedding formalism and the non-covariant conformal frame formalism. This framework allows to work with general 2-, 3-, and 4-point functions and thus to construct the 4D bootstrap equations for the operators in arbitrary spin representation, ready for further numerical or analytical analysis.

In the embedding formalism we have explained the recipe for constructing tensor structures of n-point functions in the 6D embedding space. We have also summarized the so called spinning differential operators relating generic CPWs to the seed CPWs. The conformally covariant expressions in 4D are easily obtained from the 6D expressions by using the so called projection operation. For the objects like kinematic factors and 2-, 3-, and 4-point tensor structures we have performed the projection operation explicitly.

The construction of a *basis* of tensor structures in the embedding formalism requires however the knowledge of a complete set of non-linear relations between products of the basic conformal invariants. Starting from $n = 4$ it is rather difficult to find such a set of relations and thus the embedding formalism turns out to be practically inefficient for $n \geq 4$. This problem is solved using the conformal frame approach.

In the conformal frame we have provided a complete basis for ($n \geq 3$)-point tensor structures in a remarkably simple form. For instance in the $n = 4$ case the tensor structures are simply monomials in polarization spinors with vanishing total charge under the $U(1)$ little group. In the $n < 4$ cases the little group is larger and constructing its singlets becomes harder whereas the embedding formalism is easily manageable. Since the embedding formalism is also explicitly covariant it becomes preferable for working with 2- and 3-point functions.

With practical applications in mind, we have found the action of various differential operators on 4-point functions in the conformal frame formalism. We have also shown how to apply permutations in the conformal frame. These results allow one to work with the 4-point functions (and, consequently, the crossing equations) entirely within the conformal frame formalism.

We have established a connection between the tensor structures constructed in the embedding and the conformal frame formalisms. The embedding formalism to conformal frame transition is straightforward and amounts to performing the 4D projection of the 6D structures and setting all the coordinates to the conformal frame. The conformal frame to the embedding formalism transition is slightly more complicated since it is not uniquely defined due to redundancies among the allowed 6D structures. After “translating” all the basic 6D structures to the conformal frame one inverts these relations by choosing only the independent 6D structures.

Finally, we have implemented our framework as a Mathematica package freely available at https://gitlab.com/bootstrapcollaboration/CFTs4D. It can perform any manipulations with 2-, 3- and 4-point functions in both formalism switching between them when needed. A detailed documentation is incorporated in the package with many explicit examples.

In the appendices we made our best effort to establish consistent conventions; we have provided a proper normalization of 2-point functions and the seed conformal blocks and summarized all the Casimir differential operators available in 4D. We have also given some extra details on permutation symmetries and conserved operators.

It is our hope that this work will aid the development of conformal bootstrap methods in 4D and will facilitate their application to spinning correlation functions, such as 4-point functions involving fermionic operators, global symmetry currents, and stress-energy tensors.

Acknowledgments

We thank Alejandra Castro, Tolya Dymarsky, Emtinan Elkhidir, Gabriele Ferretti, Diego Hofman, Hugh Osborn, João Penedones, Riccardo Rattazzi, Fernando Rejón-Barrera, Slava Rychkov, Volker Schomerus, David Simmons-Duffin, Marco Serone and Alessandro Vichi for useful discussions. We particularly thank Marco Serone for his valuable comments on the draft and Emtinan Elkhidir for collaboration on the initial stages of this work. DK and PK are grateful to the organizers of Boostrap 2016 workshop and the Galileo Galilei Institute for Theoretical Physics where the main ideas of this project were born. PK would like to thank the Institute for Advanced Study, where part of this work was completed, for hospitality. This work is supported in part by the DOE grant DE-SC0011632 (PK).

Chapter 4

WEIGHT-SHIFTING OPERATORS AND CONFORMAL BLOCKS

This chapter is essentially identical to:

D. Karateev, P. Kravchuk and D. Simmons-Duffin, *Weight Shifting Operators and Conformal Blocks*, *JHEP* **02** (2018) 081, [`1706.07813`].

4.1 Introduction

Concrete results in conformal representation theory have played a crucial role in the recent resurgence of the conformal bootstrap [6, 30–32, 34–39, 41, 43–47, 94–101, 110, 114, 122, 126–158]. Compact expressions for conformal blocks with external scalars [57, 63] were crucial for the development of modern numerical bootstrap techniques [30]. Subsequently, techniques for computing blocks of operators with spin [2, 39, 53, 54, 58, 60–62, 81, 82] have led to universal numerical bounds on wide classes of CFTs [39, 41, 159], in addition to analytical results like proofs of the conformal collider bounds [77, 160–162] and the average null energy condition [73], and new results on the Regge limit in CFTs [163–165]. In parallel developments, harmonic analysis on the conformal group [65] has played an important role in several recent works [66, 166–170], including the large-N solution of the SYK model [171–174]. Relationships between Witten diagrams and conformal blocks have also received recent attention [175–180].

More sophisticated analyses will require new results for operators with spin. Several efficient techniques for dealing with spinning operators have been developed over the last decade, including index-free/embedding-space methods [39, 53, 54, 61, 79, 82], the shadow formalism in the embedding space [54], "differential bases" for three-point functions [39, 61], and recursion relations [41, 49, 81]. While these methods are superior to naive approaches, they still aren't enough to solve some difficult problems. For example, the shadow formalism lets one write integral expressions for general blocks, but the integrals are difficult to evaluate in practice in all but the simplest cases. The differential basis approach lets one compute spinning blocks in terms of simpler "seed blocks," but doesn't explain how to compute the seed blocks.[1]

In this work, we introduce new tools that dramatically simplify computations in conformal representation theory, particularly involving operators with spin. The first key idea is to consider a (fictitious) operator $w(x)$ that transforms in a *finite-dimensional* representation W of the conformal group. By studying the OPE of this highly degenerate operator with a non-degenerate operator $\mathcal{O}(x)$, we find (in section 4.2) a large class of conformally-covariant differential operators $\mathcal{D}_A^v$ that can be used for computations. Here, $A = 1, \ldots, \dim W$ is an index for W, and v is a weight vector of W (i.e., a common eigenvector of the Cartan subalgebra).[2]

The action of $\mathcal{D}_A^v$ on $\mathcal{O}(x)$ shifts the weights of $\mathcal{O}$ by the weights of v, in addition to introducing a free A index. For this reason, we call $\mathcal{D}_A^v$ a *weight-shifting operator*. For example, weight-shifting operators can increase or decrease the spin of $\mathcal{O}$.[3] Weight-shifting operators can be written explicitly using the embedding space formalism [27, 39, 52–54, 82, 92, 125, 186–188], e.g. (4.45) in general spacetime dimensions, (4.72) in 3d, and (4.79) in 4d. However, our construction applies independently of the embedding space formalism, and in fact works for generalized Verma modules of any Lie (super-)algebra.[4]

A second key observation is that weight-shifting operators obey a type of crossing equation,

$$\mathcal{D}_{A,x_1}^{v}\langle\mathcal{O}_1'(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle^{(a)} = \sum_{\mathcal{O}_2',v',b}\{\cdots\}\,\mathcal{D}_{A,x_2}^{v'}\langle\mathcal{O}_1(x_1)\mathcal{O}_2'(x_2)\mathcal{O}_3(x_3)\rangle^{(b)}, \tag{4.1}$$

which we derive in section 4.3. Here, a and b label conformally-invariant three-point structures that can appear in a correlator of the given operators. The coefficients $\{\cdots\}$ are examples of $6j$ symbols (or Racah-Wigner coefficients) for the conformal group (which in this case are computable with simple algebra). Equation (4.1) lets

[1] A recursion relation for seed blocks in 3d was guessed in [81] by solving the Casimir equation order-by-order in an OPE expansion. Expressions for seed blocks in 4d were derived in [58] by solving the Casimir equation using a suitable ansatz.

[2] Some examples of such operators appear in the conformal tractor calculus, which originally deals with the case of tensor W [181, 182]. The theory of local twistors [183–185] deals with the case of spinor W. The primary interest of these theories is in curved conformal manifolds. Part of our results can be viewed as a classification of differential operators involving tractor or local twistor bundles in the conformally flat setting. It is an interesting question whether our results generalize to the curved setting.

[3] When $\mathcal{D}_A^v$ lowers the spin of $\mathcal{O}$, its missing spin degrees of freedom are (roughly speaking) transferred to the index A for W.

[4] Our construction is based on the "translation functor" of Zuckerman and Jantzen [189, 190].

us move a covariant differential operator acting on x_1 to an operator acting on x_2. As we will see, this provides enough flexibility to perform a variety of computations involving weight-shifting operators. We also introduce a diagrammatic language that makes these computations easy to understand.

As an application, in section 4.4 we focus on computing conformal blocks and understanding some of their properties. In section 4.4.3, we derive an expression for a general conformal block involving operators (both external and internal) in arbitrary representations of $\mathrm{SO}(d)$ in terms of derivatives of blocks with external scalars.[5] This generalizes the beautiful result of [61] for conformal blocks of symmetric traceless tensors (STTs). Our weight-shifting operators also explain where the differential operators of [61] come from (as we discuss in section 4.3.5). Our formula can be simplified in special cases. For example, in section 4.4.4 we give new expressions for so-called "seed blocks" in 3d and 4d CFTs in terms of derivatives of scalar blocks.

Our techniques also give a new way to understand many identities and recursion relations satisfied by conformal blocks. In section 4.4.5, we rederive and explain diagrammatically several identities relating scalar conformal blocks with different dimensions and spins.[6] In section 4.4.6, we discuss how to use derivative-based expressions for blocks to find recursion relations of the type introduced by Zamolodchikov [48, 193] and used in numerical bootstrap computations [36, 37, 41, 143, 149, 194].

In section 4.5, we comment on some additional applications beyond computing conformal blocks. Weight-shifting operators are helpful for studying inner products between conformal blocks that appear in inversion formulae [66, 168–170]. By integrating weight-shifting operators by parts, one can reduce inversion formulae for spinning operators to inversion formulae for scalars. In particular, one can express $6j$ symbols for arbitrary generalized Verma modules of the conformal group in terms of $6j$ symbols for four scalar (and two STT) representations. We pursue this idea in more detail in [195].

A related idea is "spinning-down" a crossing equation: by applying spin-lowering operators to both sides of a crossing equation, we can express it in terms of a crossing

[5] The rough idea is that weight-shifting operators allow us to exchange a tensor product $W \otimes V_{\Delta,\ell}$, where W is finite-dimensional, and $V_{\Delta,\ell}$ is the generalized Verma module of a symmetric traceless tensor (STT) operator. This tensor product then contains many new types of generalized Verma modules that can include operators in non-STT representations of $\mathrm{SO}(d)$.

[6] These identities can also be understood using techniques from integrability [123, 191, 192].

equations for scalar operators. Spinning-down may be useful in the numerical bootstrap—it could perhaps obviate the need to explicitly compute spinning blocks.

Finally, in section 4.6, we discuss further applications and future directions. We give several details and examples in the appendices.

4.2 Weight-shifting operators

4.2.1 Finite-dimensional conformal representations

Let W be a finite-dimensional irreducible representation of $\mathrm{SO}(d+1,1)$. We can think of W in two different ways. Firstly, W is a vector space with basis e^A ($A = 1, \ldots, \dim W$), in which the action of the conformal group is given by

$$g \cdot e^A = D_B{}^A(g) e^B, \tag{4.2}$$

where $D_B{}^A(g)$ are representation matrices.

Secondly, W is the conformal representation of a (very) degenerate primary operator $w^a(x)$. Under the subgroup $\mathrm{SO}(1,1)\times\mathrm{SO}(d) \subset \mathrm{SO}(d+1,1)$ generated by dilatations and and rotations, W decomposes into a direct sum[7]

$$W \to \bigoplus_{i=-j}^{j} (W_i)_i, \qquad j \in \tfrac{1}{2}\mathbb{N}. \tag{4.3}$$

Here, $(\rho)_\Delta$ denotes a representation of $\mathrm{SO}(1,1) \times \mathrm{SO}(d)$ with dimension Δ and $\mathrm{SO}(d)$ representation ρ. The dimensions in the decomposition (4.3) are integer-spaced and must be invariant under the Weyl reflection $\Delta \to -\Delta$, which implies that they are integers or half-integers.[8]

The lowest-dimension summand in (4.3) is spanned by the multiplet $w^a(0)$ which has scaling dimension $-j$ and carries an index a for the $\mathrm{SO}(d)$ representation W_{-j} (which is always irreducible). Because it has the lowest dimension in W, it is annihilated by K_μ and thus is a primary. The position-dependent operator $w^a(x) = e^{x\cdot P} w^a(0)$ is a polynomial in x of degree $2j$ because the representation W contains only $2j+1$ levels of descendants. In other words, almost all descendants of $w^a(x)$ are null and this is reflected in the fact that $w^a(x)$ satisfies a particular generalization of the conformal Killing equation that admits only polynomial solutions.

We can relate these two pictures by expanding $w^a(x)$ in our basis

$$w^a(x) = w^a_A(x) e^A. \tag{4.4}$$

[7] j is equal to the sum of all Dynkin labels of W, with spinor labels counted with multiplicity $\frac{1}{2}$, which is the same as the length of the first row of the $\mathrm{SO}(d+1,1)$ Young diagram for W.

[8] In general this Weyl reflection also acts non-trivially on the $\mathrm{SO}(d)$ representations.

The coefficients in this expansion $w^a_A(x)$ are conformal Killing (spin-)tensors. As an example, consider the adjoint representation $\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}$ of the conformal group. Under $\mathrm{SO}(1,1)\times\mathrm{SO}(d)$, it decomposes as (here and throughout, "$\bullet$" denotes the trivial representation)

$$\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array} \to (\square)_{-1} \oplus (\bullet \oplus \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array})_0 \oplus (\square)_1. \tag{4.5}$$

The operator $w^\mu(x)$ is thus a vector with dimension -1. A basis for $W = \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}$ is given by $e^A \in \{K^\mu, D, M^{\mu\nu}, P^\mu\}$, and the coefficients $w^\mu_A(x)$ in this basis are the usual conformal Killing vectors on $\mathbb{R}^d$,

$$w^\mu(x) = K^\mu - 2x^\mu D + (x_\rho\delta^\mu_\nu - x_\nu\delta^\mu_\rho)M^{\nu\rho} + (2x^\mu x_\nu - x^2\delta^\mu_\nu)P^\nu. \tag{4.6}$$

In this case the differential equation satisfied by $w^\mu(x)$ is the usual conformal Killing equation,

$$\partial^\mu w^\nu(x) + \partial^\nu w^\mu(x) - \text{trace} = 0. \tag{4.7}$$

4.2.2 Tensor products with finite-dimensional representations

Consider a primary operator $\mathcal{O}$ with $\mathrm{SO}(1,1)\times\mathrm{SO}(d)$ representation $(\rho)_\Delta$ for generic Δ. The conformal multiplet of $\mathcal{O}$ is a generalized Verma module which we denote $V_{\Delta,\rho}$.[9] Under a conformal transformation $x' = g(x)$, $\mathcal{O}$ transforms in the usual way[10]

$$\begin{aligned} g\cdot\mathcal{O}^a(x) &= \Omega(x')^\Delta \rho^a{}_b(R(x')^{-1})\mathcal{O}^b(x'), \\ \Omega(x')R^\mu{}_\nu(x') &= \frac{\partial x'^\mu}{\partial x^\nu}, \end{aligned} \tag{4.8}$$

where $R^\mu{}_\nu \in \mathrm{SO}(d)$ and $\rho^a{}_b(R^{-1})$ is the action of R^{-1} in the representation ρ.

We would like to understand the decomposition of the tensor product

$$W \otimes V_{\Delta,\rho}, \tag{4.9}$$

when W is finite-dimensional. This is equivalent to finding primary operators built out of $w^a(x)$ and $\mathcal{O}^b(x)$. Formally, we must take an OPE between $w^a(x)$ and $\mathcal{O}^b(x)$,

[9] Recall that a generalized Verma module (also called a parabolic Verma module) is roughly-speaking obtained by starting with a finite-dimensional representation of a subgroup (in this case $\mathrm{SO}(1,1)\times\mathrm{SO}(d)$) and acting with arbitrary products of lowering operators (in this case the momentum generators P_μ). See, e.g. [196]. This is the usual construction of long multiplets in conformal field theory.

[10] When we think of $\mathcal{O}^a(x)$ as an operator on a Hilbert space, then $g\cdot\mathcal{O}^a(x)$ means $U_g\mathcal{O}^a(x)U_g^{-1}$, where U_g is the unitary operator implementing g. Equation (4.8) should thus be understood as defining the action of g on the *value* $\mathcal{O}(x)$ rather than the function $\mathcal{O}$.

treating them as operators in decoupled theories.[11] The simplest primary in the OPE is

$$w^a(0) \otimes \mathcal{O}^b(0), \tag{4.10}$$

which is primary because it vanishes under the action of the special conformal generator $1 \otimes K_\mu + K_\mu \otimes 1$. This particular state is not generally in an irreducible representation of $\mathrm{SO}(d)$. Decomposing it further, we obtain primary states in irreducible representations $\lambda \in W_{-j} \otimes \rho$ of $\mathrm{SO}(d)$ and with scaling dimensions $\Delta - j$.

To find the other primaries in the OPE, we can use the following trick. Define $M = W \otimes V_{\Delta,\rho}$ and consider the factor space $M' = M/(\oplus_\mu P_\mu M)$, i.e., treat all total derivatives in M' as zero. Then any two states in M differing by a descendant will be equal in M'. As we show in appendix C.2, for generic Δ the tensor product M decomposes into a direct sum of simple generalized Verma modules, and in this case it is easy to see that the non-zero states in M' are in one-to-one correspondence with the primary states in M.

We can easily find a basis for M': given any expression of the form $\partial \cdots \partial w^a(0) \otimes \partial \cdots \partial \mathcal{O}(0)$, we can "integrate by parts" and move all the derivatives to act on w. Thus a basis for M' is given by the non-trivial states of the form[12]

$$\partial_{\mu_1} \cdots \partial_{\mu_m} w^a(0) \otimes \mathcal{O}^b(0). \tag{4.11}$$

Note that because w has a finite number of non-zero descendants, M' is finite-dimensional.

To find the primaries in M corresponding to this basis, we need to add total derivatives with the same scaling dimension to the above basis elements. This leads to the following ansatz with some undetermined coefficients c_k,

$$c_1 \partial_{\mu_1} \cdots \partial_{\mu_m} w^a(0) \otimes \mathcal{O}^b(0) + c_2 \partial_{\mu_1} \cdots \partial_{\mu_{m-1}} w^a(0) \otimes \partial_{\mu_m} \mathcal{O}^b(0) + \ldots . \tag{4.12}$$

After projecting onto an irreducible $\mathrm{SO}(d)$ representation $\lambda \in W_{-j+m} \otimes \rho$, we obtain an ansatz for a primary in representation $(\lambda)_{\Delta-j+m}$. We can fix the coefficients c_k

[11] We are not assuming that $w^a(x)$ is an operator in a physical theory — it is simply a mathematical object that serves as a useful tool for understanding consequences of conformal symmetry.

[12] If $\mathcal{O}^b(0)$ had null descendants (for example, if it itself were the primary of a finite-dimensional representation), it would be possible that some of these states are total derivatives and thus vanish in M'. Since we assume that Δ is generic, this does not happen.

up to an overall normalization by requiring that the state (4.12) is annihilated by $1 \otimes K_\mu + K_\mu \otimes 1$. In this way, we find a primary operator of scaling dimension $\Delta + i$ for each of the irreducible components in $W_i \otimes \rho$ and every $i = -j, \dots, j$.

It is not hard to confirm that these primaries account for all the states in $W \otimes V_{\Delta,\rho}$ by checking that the $\mathrm{SO}(1,1) \times \mathrm{SO}(d)$ characters agree. We thus conclude

$$W \otimes V_{\Delta,\rho} = \bigoplus_{i=-j}^{j} \bigoplus_{\lambda \in W_i \otimes \rho} V_{\Delta+i,\lambda}, \qquad (\text{generic } \Delta). \tag{4.13}$$

As a simple example, consider the case where $W = \square$ is the vector representation of $\mathrm{SO}(d+1,1)$ and ρ is the trivial representation of $\mathrm{SO}(d)$. We have the decomposition

$$\square \to (\bullet)_{-1} \oplus (\square)_0 \oplus (\bullet)_{+1}, \tag{4.14}$$

so the primary state of W is the scalar $w(0)$ of scaling dimension -1. We thus find

$$\square \otimes V_{\Delta,\bullet} = V_{\Delta-1,\bullet} \oplus V_{\Delta,\square} \oplus V_{\Delta+1,\bullet}. \tag{4.15}$$

According to the above discussion, we have the following ansatz for the primaries in this decomposition

$$\begin{aligned}
V_{\Delta-1,\bullet}: &\quad \phi_-(0) = w(0) \otimes \mathcal{O}(0), \\
V_{\Delta,\square}: &\quad V_\mu(0) = t_1 \partial_\mu w(0) \otimes \mathcal{O}(0) + t_2 w(0) \otimes \partial_\mu \mathcal{O}(0), \\
V_{\Delta+1,\bullet}: &\quad \phi_+(0) = b_1 \partial^2 w(0) \otimes \mathcal{O}(0) + b_2 \partial_\mu w(0) \otimes \partial^\mu \mathcal{O}(0) + b_3 w(0) \otimes \partial^2 \mathcal{O}(0).
\end{aligned} \tag{4.16}$$

Recalling that ∂_μ is the same as the action of P_μ and using the conformal algebra in appendix C.1, we find

$$\begin{aligned}
(1 \otimes K_\mu + K_\mu \otimes 1) \cdot \phi_-(0) &= 0, \\
(1 \otimes K_\mu + K_\mu \otimes 1) \cdot V_\nu(0) &= 2\delta_{\mu\nu}(\Delta t_2 - t_1) w(0) \otimes \mathcal{O}(0), \\
(1 \otimes K_\mu + K_\mu \otimes 1) \cdot \phi_+(0) &= 2(\Delta b_2 - d b_1) \partial_\mu w(0) \otimes \mathcal{O}(0) \\
&\quad + 2\left(b_3\,(2\Delta - d + 2) - b_2\right) w(0) \otimes \partial_\mu \mathcal{O}(0).
\end{aligned} \tag{4.17}$$

It follows that these states are primary if

$$\begin{aligned}
t_1 &= \Delta t_2, \\
b_1 &= \frac{\Delta b_3}{d}(2\Delta - d + 2), \quad b_2 = b_3(2\Delta - d + 2).
\end{aligned} \tag{4.18}$$

We must assume that Δ is generic because e.g. for $\Delta = 1$, V_μ becomes a primary descendant of ϕ_-, $V_\mu = \partial_\mu \phi_-$. In this special case, there are not sufficiently many primaries to account for all states of dimension Δ. In particular there is no combination of descendants which gives $\partial_\mu w(0) \otimes \mathcal{O}(0)$, and consequently $\square \otimes V_{1,\bullet}$ does not decompose into generalized Verma modules of primary operators. These subtleties will not be important in this work, and we will always assume Δ to be generic.

4.2.3 Covariant differential operators from tensor products

Consider now the primary state (4.12), and let us write it in the form

$$\mathcal{O}'^c(x) = e^A \otimes (\mathcal{D}_A)^c{}_b \mathcal{O}^b(x), \tag{4.19}$$

where the differential operators $\mathcal{D}_A$ are defined by[13]

$$\begin{aligned}(\mathcal{D}_A)^c{}_b \mathcal{O}^b(x) \equiv \pi^c_{ab\mu_1\cdots\mu_m} \Big(& c_1 \partial^{\mu_1} \cdots \partial^{\mu_m} w^a_A(x) \mathcal{O}^b(x) \\ & + c_2 \partial^{\mu_1} \cdots \partial^{\mu_{m-1}} w^a_A(x) \partial^{\mu_m} \mathcal{O}^b(x) + \dots \Big).\end{aligned} \tag{4.20}$$

Again, the c_i are chosen so that $\mathcal{O}'^c(0)$ is a primary transforming in the representation $(\lambda)_{\Delta'}$. Here, $\pi^c_{ab\mu_1\cdots\mu_m}$ is a projector onto the $\mathrm{SO}(d)$ representation $\lambda \in W_{-j+m} \otimes \rho$. By construction, $\mathcal{O}'$ transforms under a conformal transformation as

$$g \cdot \mathcal{O}'^c(x) = \Omega(x')^{\Delta'} \lambda^c{}_d(R^{-1}(x')) \mathcal{O}'^d(x'). \tag{4.21}$$

On the other hand, we also have

$$\begin{aligned} g \cdot \mathcal{O}'^c(x) &= g \cdot e^A \otimes g \cdot (\mathcal{D}_A \mathcal{O})^c(x) \\ &= D_B{}^A(g) e^B \otimes g \cdot (\mathcal{D}_A \mathcal{O})^c(x). \end{aligned} \tag{4.22}$$

It follows that

$$g \cdot (\mathcal{D}_A \mathcal{O})^c(x) = \Omega(x')^{\Delta'} \lambda^c{}_d(R^{-1}(x')) D_A{}^B(g^{-1}) (\mathcal{D}_B \mathcal{O})^d(x'). \tag{4.23}$$

In other words, $\mathcal{D}_A$ takes a primary operator that transforms in $(\rho)_\Delta$ to a primary operator that transforms in $(\lambda)_{\Delta'}$, up to the additional action of the finite-dimensional matrix $D_B{}^A(g^{-1})$. We summarize this situation by writing

$$\mathcal{D}_A : [\Delta, \rho] \to [\Delta', \lambda]. \tag{4.24}$$

[13] Note that $\mathcal{D}_A$ depends explicitly on x. This is because P_μ acts non-trivially on W and thus these operators are translation-covariant rather than translation-invariant.

Here, for all practical purposes $[\Delta, \rho]$ is just a convenient notation. We give it a precise meaning in appendix C.2.

Notice that $\mathcal{D}_A \mathcal{O}$ has a lowered index for W, so it transforms in the same way as the basis elements of the dual representation W^*. For this reason, we will say that $\mathcal{D}_A$ is *associated* with W^*. Similarly, exchanging W and W^*, $\mathcal{D}^A$ is associated with W. This convention will be useful when we discuss the action of differential operators on tensor structures in section 4.3.1.

This general construction shows that there exists a huge variety of conformally covariant differential operators, corresponding to tensor products with different finite-dimensional representations. In fact, as explained in appendix C.2, all conformally-covariant differential operators acting on generic Verma modules arise in this way. For reference, let us summarize this result in the following

Theorem 2. *The conformally-covariant operators $\mathcal{D}^A : [\Delta, \rho] \to [\Delta - i, \lambda]$ associated with W are (for generic Δ) in one-to-one correspondence with the irreducible components in the tensor product decomposition*

$$W^* \otimes V_{\Delta,\rho} = \bigoplus_{i=-j}^{j} \bigoplus_{\lambda \in (W_i)^* \otimes \rho} V_{\Delta - i, \lambda}. \tag{4.25}$$

When the Dynkin indices of ρ are sufficiently large, Brauer's formula (also known as Klimyk's rule) [197, 198] implies that the tensor products simplify, giving

$$W^* \otimes V_{\Delta,\rho} = \bigoplus_{(\delta,\pi) \in \Pi(W^*)} V_{\Delta+\delta, \rho+\pi}. \tag{4.26}$$

Here, $\Pi(W^*)$ denotes the weights of W^* (with multiplicity). A consequence of (4.26) is that for generic Δ, ρ, the number of differential operators acting on $[\Delta, \rho]$ and transforming in W is equal to $\dim(W^*)$. Further, each operator is labeled by a weight vector of W^* (i.e., an element of W^* which is an eigenvector of the Cartan subalgebra) and shifts (Δ, ρ) by that weight. For this reason, we call the $\mathcal{D}^A$ *weight-shifting operators*.

One of the most important weight-shifting operators comes from the adjoint representation of the conformal group, $W = {\tiny\yng(1,1)}$. The tensor product ${\tiny\yng(1,1)} \otimes V_{\Delta,\rho}$ always contains $V_{\Delta,\rho}$ itself as a factor. The corresponding $\mathcal{D}^A : [\Delta, \rho] \to [\Delta, \rho]$ are the usual differential operators generating the action of the conformal algebra (see e.g. [19]),

$$\mathcal{D}^A = w^A \cdot \partial + \frac{\Delta}{d}(\partial \cdot w^A) - \frac{1}{2}(\partial^\mu w^{A\nu}) \mathcal{S}_{\mu\nu}, \tag{4.27}$$

where $w^{A\mu}$ are conformal Killing vectors (4.6), and $\mathcal{S}_{\mu\nu}$ are the generators of $\mathrm{SO}(d)$ rotations in the representation ρ.

4.2.4 Algebra of weight-shifting operators

What is the algebra of weight-shifting operators?[14] Before answering this question, let us rephrase our construction in a slightly different language. Recall from (4.19) and (4.20) that we identify primaries in $W \otimes V_{\Delta,\rho}$ of the form

$$\mathcal{O}'^{c}(0) = e^{A} \otimes (\mathcal{D}_A)^{c}{}_{b}\mathcal{O}^{b}(0). \tag{4.28}$$

Note that $\mathcal{O}'^{c}(0) \in W \otimes V_{\Delta,\rho}$ but it transforms in the same way as the primary of $V_{\Delta',\lambda}$. This means that (4.28) gives a homomorphism

$$\Phi : V_{\Delta',\lambda} \to W \otimes V_{\Delta,\rho}, \tag{4.29}$$

defined by mapping the primary of $V_{\Delta',\lambda}$ to the right hand side of (4.28). The action of Φ on descendants follows by acting with $P_\mu \otimes 1 + 1 \otimes P_\mu$ on (4.28).

Composition of differential operators is equivalent to composition of the corresponding homomorphisms in the opposite order. Specifically, suppose

$$\begin{aligned}\Phi_1 &: V_{\Delta',\rho'} \to W_1 \otimes V_{\Delta,\rho},\\ \Phi_2 &: V_{\Delta'',\rho''} \to W_2 \otimes V_{\Delta',\rho'}.\end{aligned} \tag{4.30}$$

Then

$$\begin{aligned}(1 \otimes \Phi_1) \circ \Phi_2 &: V_{\Delta'',\rho''} \to W_2 \otimes W_1 \otimes V_{\Delta,\rho}\\ (1 \otimes \Phi_1)(\Phi_2(\mathcal{O}''(x))) &= e_2^{B} \otimes e_1^{A} \otimes \mathcal{D}_{2B}\mathcal{D}_{1A}\mathcal{O}(x).\end{aligned} \tag{4.31}$$

Thus, to find the algebra of weight-shifting operators, we must express the right-hand side of (4.31) in terms of homomorphisms associated to the irreducible factors of $W_2 \otimes W_1$.

As we will see in the next section, the embedding formalism lets us define weight-shifting operators that make sense even when ρ is a generic (i.e., not necessarily dominant) weight. For example, the spin ℓ of a symmetric traceless tensor operator can be written as $Z \cdot \frac{\partial}{\partial Z}$, where Z is a polarization vector. The operator $Z \cdot \frac{\partial}{\partial Z}$ is then well-defined when acting on functions of non-integer homogeneity in Z.

[14] The results of this section are not used in the rest of this work. The reader should feel free to skip this section on first reading.

The correct way to understand differential operators with generic weights is to consider homomorphisms between Verma modules as opposed to *generalized* Verma modules. Consider the triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+, \tag{4.32}$$

where $\mathfrak{h}$ is the Cartan subalgebra, and $\mathfrak{g}_\pm$ are generated by positive/negative roots of $\mathfrak{g}$. Let M_λ be the Verma module of $\mathfrak{g}$ with highest-weight λ, and denote the corresponding highest-weight vector by x_λ.[15]

Let W be a finite-dimensional representation of $\mathfrak{g}$. For each weight-vector[16] $w \in W$, we can construct a $\mathfrak{g}$-homomorphism

$$\Phi_\lambda^w : M_\lambda \to W \otimes M_\mu, \qquad \mu = \lambda - \mathrm{wt}\, w, \tag{4.33}$$

such that

$$\Phi_\lambda^w(x_\lambda) = w \otimes x_\mu + \dots. \tag{4.34}$$

Here, "…" is a sum of terms of the form

$$e_{\alpha_1} \cdots e_{\alpha_k} w \otimes e_{-\alpha_{k+1}} \cdots e_{-\alpha_m} x_\mu, \tag{4.35}$$

where $e_{\pm\alpha} \in \mathfrak{g}_\pm$ are raising/lowering operators. Their coefficients are fixed by demanding that $\Phi_\lambda^w(x_\lambda)$ is $\mathfrak{g}_+$-primary, i.e., that it is killed by $1 \otimes e_\alpha + e_\alpha \otimes 1$ for all positive roots α. Finally, the action of Φ_λ^w on $\mathfrak{g}_-$-descendants of x_λ is fixed by $\mathfrak{g}$-invariance. The construction of Φ_λ^w is completely analogous to the construction of Φ in (4.29) above. The vector (4.34) is the analog of the primary state (4.12).

Weight-shifting operators in the embedding space are in one-to-one correspondence with the homomorphisms Φ_λ^w. In particular, they are labeled by weight-vectors of W. This is consistent with our argument based on Brauer's formula in the previous section.

The homomorphisms (4.34) have been studied in [199]. Given two finite-dimensional representations V, W with weight-vectors $v \in V$, $w \in W$, they satisfy the algebra

$$(1 \otimes \Phi_{\lambda - \mathrm{wt}\, v}^w) \circ \Phi_\lambda^v = \Phi_\lambda^{J(\lambda)(v \otimes w)}, \tag{4.36}$$

[15] When $\lambda = (\Delta, \rho)$ with ρ a dominant weight of $\mathfrak{so}(d)$, then M_λ is reducible and contains the generalized Verma module $V_{\Delta,\rho}$ as a subfactor.

[16] Not to be confused with the conformal Killing tensors w^A from the previous section.

where

$$J(\lambda) \in \mathrm{Aut}(V \otimes W) \tag{4.37}$$

is an invertible operator called the *fusion operator*. The fusion operator thus completely encodes the algebra of weight-shifting operators. It satisfies a number of interesting properties, and is closely related to solutions of the Yang-Baxter equations and integrability [199]. Most importantly for our discussion, the Arnaudon-Buffenoir-Ragoucy-Roche equation gives an explicit expression for $J(\lambda)$ in terms of generators of $\mathfrak{g}$ [200]. In principle, this answers the question posed at the beginning of this section. In practice, we will not need such a general answer in this work. We leave further exploration of the fusion operator and its applications to the future.

Another point of view on the algebra of weight-shifting operators is given by a special kind of $6j$ symbols, as we explain in appendix C.4.

4.2.5 Weight-shifting operators in the embedding space

Our construction of weight-shifting operators is extremely general, but it is inconvenient for computations because it is cumbersome to find the primary states $\mathcal{O}'$. For practical computations, we can use the embedding formalism [27, 39, 52–54, 82, 92, 125, 186–188], where the conformal group acts linearly. The tradeoff is that coordinates in the embedding space satisfy constraints and gauge redundancies, and we must take care to find differential operators respecting these conditions. The above construction tells us precisely when this should be possible.

The formalism described in [53] makes it easy to study operators in tensor representations of $\mathrm{SO}(d)$. Symmetric traceless tensors (STTs) of $\mathrm{SO}(d)$ are particularly simple. We will describe this case first in order to make contact with the examples above. However, our primary interest is in general representations, and for these it will be useful to use specialized formalisms for different spacetime dimensions.

4.2.5.1 General dimensions

In the embedding formalism, the conformal compactification of $\mathbb{R}^d$ is realized as the projective null cone in $\mathbb{R}^{d+1,1}$. We take the metric on $\mathbb{R}^{d+1,1}$ to be

$$X^2 = \eta_{mn}X^m X^n = -X^+X^- + \sum_{\mu=1}^{d} X_\mu X^\mu. \tag{4.38}$$

A primary scalar $\mathcal{O}(x)$ lifts to a function on the null cone $\mathcal{O}(X)$ with homogeneity

$$\mathcal{O}(\lambda X) = \lambda^{-\Delta_{\mathcal{O}}}\mathcal{O}(X). \tag{4.39}$$

It is convenient to arbitrarily extend $\mathcal{O}(X)$ outside the null cone, introducing the gauge redundancy

$$\mathcal{O}(X) \sim \mathcal{O}(X) + X^2\Lambda(X). \tag{4.40}$$

A tensor operator $\mathcal{O}^{\mu_1\cdots\mu_\ell}(X)$ lifts to a tensor $\mathcal{O}^{m_1\cdots m_\ell}(X)$ in the embedding space, subject to gauge redundancies and transverseness

$$\mathcal{O}^{m_1\cdots m_\ell}(X) \sim \mathcal{O}^{m_1\cdots m_\ell}(X) + X^{m_i}\Lambda^{m_1\cdots\widehat{m}_i\cdots m_\ell}(X), \tag{4.41}$$

$$X_{m_i}\mathcal{O}^{m_1\cdots m_i\cdots m_\ell}(X) = 0, \tag{4.42}$$

in addition to the homogeneity condition (4.39). For symmetric tensors, it is useful to introduce a polarization vector Z^m and define

$$\mathcal{O}(X,Z) \equiv \mathcal{O}^{m_1\cdots m_\ell}(X)Z_{m_1}\cdots Z_{m_\ell}. \tag{4.43}$$

Because of (4.41), we must take $Z\cdot X = 0$, and because of (4.42), we must identify $Z \sim Z + \lambda X$. Finally, when $\mathcal{O}^{m_1\cdots m_\ell}$ is traceless, we can impose $Z^2 = 0$.

We can summarize these constraints as follows. Let I be the ideal generated by $\{X^2, X\cdot Z, Z^2\}$, and let R be the ring of functions of (X,Z) invariant under $Z \to Z + \lambda X$. Symmetric tensor operators are elements of $R/(R\cap I)$ which are homogeneous in both X and Z. For a differential operator in X, Z to be well-defined on this space, it must take $R \to R$ and also preserve the ideal $R\cap I$.

The construction in section 4.2.3 tells us when such operators should exist. For example, consider the case where $W = \square$ is the vector representation of $\mathrm{SO}(d+1,1)$ and $\mathcal{O}(X,Z)$ has spin ℓ and dimension Δ. Given the decomposition (4.14), we should be able to find differential operators with a vector index in the embedding space, taking[17]

$$\begin{aligned}
\mathcal{D}_m^{-0} &: [\Delta,\ell] \to [\Delta-1,\ell],\\
\mathcal{D}_m^{0-} &: [\Delta,\ell] \to [\Delta,\ell-1],\\
\mathcal{D}_m^{0+} &: [\Delta,\ell] \to [\Delta,\ell+1],\\
\mathcal{D}_m^{+0} &: [\Delta,\ell] \to [\Delta+1,\ell].
\end{aligned} \tag{4.44}$$

[17] There will also exist differential operators producing other representations in the tensor product of the vector and spin-ℓ representations of $\mathrm{SO}(d)$ (generically there is also the hook Young diagram). According to (4.26), when acting on general (non-STT) representations generically there are $d+2$ operators corresponding to the vector representation. However, to describe these we would need a formalism with more polarization vectors as in [82].

Our strategy for finding them is to start with a suitable ansatz and fix the coefficients by requiring that $\mathcal{D}_m$ preserve R and $R \cap I$. (We give more details in appendix C.3.) We find

$$\begin{aligned}
\mathcal{D}_m^{-0} &= X_m, \\
\mathcal{D}_m^{0-} &= \left((\Delta - d + 2 - \ell)\delta_m^n + X_m \frac{\partial}{\partial X_n}\right)\left((d - 4 + 2\ell)\frac{\partial}{\partial Z^n} - Z_n \frac{\partial^2}{\partial Z^2}\right), \\
\mathcal{D}_m^{0+} &= (\ell + \Delta) Z_m + X_m Z \cdot \frac{\partial}{\partial X}, \\
\mathcal{D}_m^{+0} &= c_1 \frac{\partial}{\partial X^m} + c_2 X_m \frac{\partial^2}{\partial X^2} + c_3 Z_m \frac{\partial^2}{\partial Z \cdot \partial X} + c_4 Z \cdot \frac{\partial}{\partial X}\frac{\partial}{\partial Z^m} \\
&\quad + c_5 X_m Z \cdot \frac{\partial}{\partial X}\frac{\partial^2}{\partial Z \cdot \partial X} + c_6 Z_m Z \cdot \frac{\partial}{\partial X}\frac{\partial^2}{\partial Z^2} + c_7 X_m \left(Z \cdot \frac{\partial}{\partial X}\right)^2 \frac{\partial^2}{\partial Z^2},
\end{aligned} \tag{4.45}$$

where the coefficients c_i are given in appendix C.3. For now, we simply quote

$$\frac{c_1}{c_2} = -2\left(\frac{d}{2} - 1 - \Delta\right). \tag{4.46}$$

Thus, when acting on scalar operators $\mathcal{O}(X)$, $\mathcal{D}_m^{+0}$ is proportional to the familiar Todorov operator [201]

$$\mathcal{D}_m^{+0} \propto \left(\frac{d}{2} + X \cdot \frac{\partial}{\partial X}\right)\frac{\partial}{\partial X^m} - \frac{1}{2} X_m \frac{\partial^2}{\partial X^2} + O\left(\frac{\partial}{\partial Z}\right). \tag{4.47}$$

This simplified version of $\mathcal{D}_m^{+0}$ (together with $\mathcal{D}_m^{-0}$) appears in tractor calculus, where it is known as Thomas operator [181, 182].

The overall normalization of our differential operators is a convention. It is useful to choose conventions where the coefficients c_i are polynomials in Δ, ℓ of the smallest possible degree. If we like, factors of Δ, ℓ can then be replaced with

$$\Delta = -X \cdot \frac{\partial}{\partial X}, \qquad \ell = Z \cdot \frac{\partial}{\partial Z}, \tag{4.48}$$

so that $\mathcal{D}$ can be expressed without reference to the operator it acts on.

Note that when acting on scalar $\mathcal{O}$ there a unique non-vanishing operator of the lowest scaling dimension, $\mathcal{D}_m^{-0}$. According to theorem 2, this is true in general. From the discussion in section 4.2.2 it follows that this operator should correspond to multiplication by the conformal Killing tensor $w_A^a(x)$ as in (4.10). This gives a general way of finding $w_A^a(x)$ from embedding space formalism.

For example, one can check that the primary operator $w(x)$ corresponding to the vector representation of the conformal group is given by

$$w(x) = w_m(x)e^m = e^m \mathcal{D}_m^{-0} = X^m e_m = e_+ + x^\mu e_\mu + x^2 e_-, \tag{4.49}$$

where e_+, e_- and e_μ form the light cone coordinate basis of the vector representation. It solves the equation

$$\partial_\mu \partial_\nu w(x) - \text{trace} = 0. \tag{4.50}$$

Let us now revisit the example from section 4.2.2. Let $\mathcal{O}(x)$ be a scalar primary of dimension Δ, as in section 4.2.2. We then compute[18]

$$\begin{aligned}
&e^m \otimes \mathcal{D}_m^{-0}\mathcal{O}(x) = w(x) \otimes \mathcal{O}(x),\\
&e^m \otimes \mathcal{D}_m^{0+}\mathcal{O}(x) = z^\mu \left(\Delta \partial_\mu w(x) \otimes \mathcal{O}(x) + w(x) \otimes \partial_\mu \mathcal{O}(x)\right),\\
&e^m \otimes \mathcal{D}_m^{+0}\mathcal{O}(x) = \frac{c_1 \Delta}{d} \partial^2 w(x) \otimes \mathcal{O}(x) + c_1 \partial_\mu w(x) \otimes \partial^\mu \mathcal{O}(x) + c_2 w(x) \otimes \partial^2 \mathcal{O}(x),
\end{aligned} \tag{4.51}$$

where c_i are as in (4.45). It is easy to see that this is consistent with (4.16) and (4.18). Naturally, $e^m \otimes \mathcal{D}_m^{0-}\mathcal{O}(x)$ vanishes when $\mathcal{O}(x)$ is a scalar.

4.2.5.2 1 dimension

To find the most general conformally-covariant differential operators, it is useful to employ a formalism specialized to the given spacetime dimension. The simplest case is 1-dimension, where the conformal group is $Spin(2,1)$.[19] The Lorentz group is $Spin(1) = \mathbb{Z}_2$ (see below) and the primary operators are labeled by a scaling dimension Δ and a spin $s = \pm$. We will denote the corresponding Verma modules by $V_{\Delta,s}$. Because the global 2-dimensional conformal group is a product of 1-dimensional groups, the results of this section can also be applied in 2-dimensions.

Note that the simply-connected conformal group is $Spin(2,1) \simeq SL(2,\mathbb{R})$. It acts by Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; x \to \frac{ax+b}{cx+d}, \qquad ad - bc = 1. \tag{4.52}$$

[18] Recall that on the Poincare section we have $X = (1, x^2, x^\mu)$ and $Z = (0, 2(x\cdot z), z^\mu) = z^\mu \partial_\mu X$ where the coordinates are ordered as (X^+, X^-, X^μ).

[19] We use the conventions of [39] for 2+1 dimensions.

The subgroup which fixes the origin is given by $b = 0$. We can exclude special conformal transformations by setting $c = 0$. The remaining subgroup is a product of dilatations $\mathbb{R}_+$ parametrized by $|a|$ and the Lorentz group $\mathbb{Z}_2$ parametrized by the sign of a. This is why we say that $Spin(1) = \mathbb{Z}_2$.[20]

The vector representation of $Spin(2,1)$ is equivalent to the symmetric square of the spinor represenation, and in the embedding formalism we can define

$$X_{(\alpha\beta)} = \gamma^m_{(\alpha\beta)} X_m, \qquad \gamma^m_{(\alpha\beta)} = \Omega_{\alpha\alpha'} (\gamma^m)^{\alpha'}{}_\beta. \tag{4.53}$$

In this notation the constraint $X^2 = 0$ can be solved as

$$X_{(\alpha\beta)} = \chi_\alpha \chi_\beta, \tag{4.54}$$

where χ_α is a real spinor in the fundamental representation of $SL(2, \mathbb{R})$. Note that χ is odd under the center of $SL(2, \mathbb{R})$. This parametrization has the advantage that now the embedding-space operators can be taken to depend on χ_α,

$$O(\lambda\chi) = \lambda^{-2\Delta_O} O(\chi), \quad \lambda > 0. \tag{4.55}$$

Notice that both χ and $-\chi$ correspond to the same X. The correct transformation property of $O(\chi)$ under this transformation comes from the $\mathbb{Z}_2$-spin,

$$O(-\chi) = sO(\chi). \tag{4.56}$$

This property will be important for the construction of tensor structures in section 4.3.4.1.

The embedding formalism in terms of χ is useful because the conformal group still acts linearly, but now there is no analogue of the ideal I which needs to be preserved by the embedding space differential operators. We have the following relation between χ and X derivatives,

$$\frac{\partial}{\partial \chi^\alpha} = (\gamma^m)^\beta{}_\alpha \chi_\beta \frac{\partial}{\partial X^m}. \tag{4.57}$$

Using this relation in an arbitrary differential operator written in terms of χ will automatically produce the terms necessary to preserve the ideal I in X-space. For example, we can recover the 1-dimensional version of the operator $\mathcal{D}_m^{+0}$ (c.f. (4.47)),

$$(\gamma^m)^{(\alpha\beta)} \frac{\partial}{\partial \chi^\alpha} \frac{\partial}{\partial \chi^\beta} \propto \left(X \cdot \frac{\partial}{\partial X} - \frac{1}{2} \right) \frac{\partial}{\partial X_m} - \frac{1}{2} X^m \frac{\partial^2}{\partial X^2}. \tag{4.58}$$

[20] The fields which have spin $s = +$ are the usual scalars on the circle. The fields which have $s = -$ are anti-periodic fermions.

A general embedding space differential operator is an arbitrary combination of χ_α and $\partial_\alpha = \frac{\partial}{\partial \chi^\alpha}$. The combinations irreducible under $Spin(2,1)$ are

$$\mathcal{D}^{j,i}_{\alpha_1\ldots\alpha_j} = \chi_{(\alpha_1}\cdots\chi_{\alpha_{j-i}}\partial_{\alpha_{j-i+1}}\cdots\partial_{\alpha_{2j})}, \qquad i=-j,\ldots j. \tag{4.59}$$

Of course, we can also add combinations of $\chi_\alpha\partial^\alpha$, but these simply act as scalars due to (4.55), so we can ignore this possibility. By construction, this differential operator transforms in the spin-j representation of $Spin(2,1)$, changes the scaling dimension by i, and exchanges bosons with fermions if j is half-integer,

$$\mathcal{D}^{j,i}:[\Delta,s]\to[\Delta+i,(-1)^{2j}s]. \tag{4.60}$$

It is easy to find the group-theoretic interpretation for $\mathcal{D}^{j,i}$. Indeed, the spin-j representation decomposes as

$$j\to\bigoplus_{i=-j}^{j}((-1)^{2j})_i, \tag{4.61}$$

which means that for a generic Δ we have the tensor product decomposition

$$j\otimes V_{\Delta,s}=\bigoplus_{i=-j}^{j}V_{\Delta+i,(-1)^{2j}s}. \tag{4.62}$$

Thus, we find explicitly the expected one-to-one correspondence between the differential operators $\mathcal{D}^{j,i}$ and the terms in this tensor product. We also see explicitly that the differential operators are labeled by the weights of the spin-j representation, in accordance with (4.26).

Let us see what our operators look like in x-coordinate space. It is easy to check that the usual Poincare section $X^+=1$ corresponds to $\chi^1=x$, $\chi^2=1$.[21] We can therefore write the embedding space operator in terms of the x-space operator as (multiplying also by sign χ^2 for $s=-$)

$$\mathcal{O}(\chi)=\frac{1}{|\chi^2|^{2\Delta}}\mathcal{O}\left(\frac{\chi^1}{\chi^2}\right). \tag{4.63}$$

We therefore see that χ_1 and χ_2 derivatives act as

$$\partial_1=\frac{\partial}{\partial\chi^1}=\frac{\partial}{\partial x}, \tag{4.64}$$

$$\partial_2=\frac{\partial}{\partial\chi^2}=-x\frac{\partial}{\partial x}-2\Delta. \tag{4.65}$$

These formulas are valid for higher order derivatives if we follow the convention that Δ in the last formula is increased by $\frac{1}{2}$ by every ∂_α.

[21] And also to minus these values, since there is a redundancy $\chi\sim-\chi$.

4.2.5.3 3 dimensions

In 3-dimensions, we use the formalism and conventions of [39].[22] The conformal group is $SO(3,2)$, which has $Sp(4,\mathbb{R})$ as a double cover. The most general Lorentz representation is the 2ℓ-th symmetric power of the spinor representation of $SO(2,1)$, where $\ell \in \frac{1}{2}\mathbb{N}$. An operator $O^{\alpha_1\cdots\alpha_{2\ell}}(x)$ lifts to an embedding space operator $O^{a_1\cdots a_{2\ell}}(X)$ with 2ℓ indices for the fundamental of $Sp(4,\mathbb{R})$, satisfying the homogeneity property

$$O^{a_1\cdots a_{2\ell}}(\lambda X) = \lambda^{-\Delta_O-\ell} O^{a_1\cdots a_{2\ell}}(X). \tag{4.66}$$

It is useful to introduce a polarization spinor S_a, and define

$$O(X,S) \equiv S_{a_1}\cdots S_{a_{2\ell}} O^{a_1\cdots a_{2\ell}}(X). \tag{4.67}$$

The polarization spinors are constrained to satisfy

$$S_a X^a{}_b = 0, \quad \text{where} \quad X^a{}_b \equiv X^m(\Gamma_m)^a{}_b, \tag{4.68}$$

where $(\Gamma_m)^a{}_b$ are generators of the Clifford algebra of $SO(3,2)$. For convenience, we also introduce the notation

$$X_{ab} = \Omega_{ac}X^c{}_b, \qquad X^{ab} = X^a{}_c\Omega^{cb}, \tag{4.69}$$

where $\Omega_{ac} = \Omega^{ac}$ is the symplectic form for $Sp(4,\mathbb{R})$.

Arbitrary finite-dimensional representations of $SO(3,2)$ can be obtained from tensors of the spinor representation $\mathcal{S}$. Thus, all the weight-shifting operators in 3d can be obtained from products of weight-shifting operators for $\mathcal{S}$. Under $SO(3,2) \to SO(1,1)\times SO(2,1)$, we have the decomposition

$$\mathcal{S} \to (\mathcal{S})_{-\frac{1}{2}} \oplus (\mathcal{S})_{\frac{1}{2}}. \tag{4.70}$$

Thus, we should be able to find differential operators with a fundamental index for $Sp(4,\mathbb{R})$ that take

$$\mathcal{D}_a^{\pm\pm} : [\Delta,\ell] \to \left[\Delta \pm \tfrac{1}{2}, \ell \pm \tfrac{1}{2}\right]. \tag{4.71}$$

[22] In particular, we use Lorentzian signature in this section.

Note that again the differential operators are labeled by weights of $\mathcal{S}$, consistently with (4.26). They are given by

$$\begin{aligned}
\mathcal{D}_a^{-+} &= S_a \\
\mathcal{D}_a^{--} &= X_{ab}\frac{\partial}{\partial S_b} \\
\mathcal{D}_a^{++} &= 2(\Delta-1)(\partial_X)_{ab}\Omega^{bc}S_c + S_a\left(S_b\Omega^{bc}(\partial_X)_{cd}\frac{\partial}{\partial S_d}\right) \\
\mathcal{D}_a^{+-} &= 4(\Delta-1)(1+\ell-\Delta)\Omega_{ab}\frac{\partial}{\partial S_b} - 2(1+\ell-\Delta)X_{ab}(\partial_X)^{bc}\Omega_{cd}\frac{\partial}{\partial S_d} \\
&\quad - S_a\left(\frac{\partial}{\partial S_c}X_{cd}(\partial_X)^{de}\Omega_{ef}\frac{\partial}{\partial S_f}\right).
\end{aligned} \tag{4.72}$$

We have determined the coefficients by demanding that these operators preserve the ideal generated by X^2 and $S_a X^a{}_b$. The differential operators (4.72) are analogous to χ_α and $\frac{\partial}{\partial\chi^\alpha}$ in the 1-dimensional case. By taking products of them, we can build weight-shifting operators in arbitrary representations of $\mathrm{SO}(3,2)$, analogous to the 1d operators (4.59). See also appendix C.4.

4.2.5.4 4 dimensions

In 4d, we can use the embedding space formalism of [2, 54, 55, 58, 62, 202]. Our conventions are those of [2]. A general Lorentz representation is now labeled by two weights $(\ell,\bar\ell)$, where $\ell,\bar\ell\in\mathbb{Z}_{\geq 0}$. (Spin-$\ell$ symmetric traceless tensor representations correspond to the case $\ell=\bar\ell$.) An operator $\mathcal{O}^{\alpha_1\cdots\alpha_\ell\dot\alpha_1\cdots\dot\alpha_{\bar\ell}}(x)$ lifts to an embedding space operator

$$\mathcal{O}(X,S,\bar S) = S_{a_1}\cdots S_{a_\ell}\bar S^{b_1}\cdots\bar S^{b_{\bar\ell}}\mathcal{O}^{a_1\cdots a_\ell}_{b_1\cdots b_{\bar\ell}}(X), \tag{4.73}$$

where we have introduced polarization spinors $S_a, \bar S^a$ transforming as left- and right-handed spinors of $\mathrm{SO}(4,2)$, or equivalently fundamentals and anti-fundamentals of $\mathrm{SU}(2,2)$. The polarization spinors satisfy

$$S_a\overline{\mathbf{X}}^{ab} = 0, \qquad \bar S^a\mathbf{X}_{ab} = 0, \qquad \bar S^a S_a = 0, \tag{4.74}$$

where[23]

$$\mathbf{X}_{ab} \equiv \Sigma^m_{ab}X_m, \qquad \overline{\mathbf{X}}^{ab} \equiv \bar\Sigma^{ab}_m X^m. \tag{4.75}$$

[23] Our conventions for the conformal algebra and embedding space in 4d are those of [2].

Let us also introduce the shorthand notation

$$\partial_{\overline{S},a} \equiv \frac{\partial}{\partial \overline{S}^a}, \qquad \partial_S^a \equiv \frac{\partial}{\partial S_a},$$
$$\partial_{ab} \equiv \Sigma^m_{ab}\frac{\partial}{\partial X^m}, \qquad \overline{\partial}^{ab} \equiv \overline{\Sigma}^{m\,ab}\frac{\partial}{\partial X^m}. \tag{4.76}$$

General representations of the conformal group SO(4, 2) can be obtained by tensoring with the left and right-handed spinors. Thus, our algebra of differential operators is generated by those associated with the spinor representations. To label these operators, it is convenient to use (4.26). Let us denote the weights so that the highest weight of the Verma module for $\mathcal{O}$ is $(2\Delta, \ell, \overline{\ell})$. Then the representations $\mathcal{S}$ and $\overline{\mathcal{S}}$ consist of the following weights,

$$\Pi(\mathcal{S}) = \{(-,+,0),(-,-,0),(+,0,+),(+,0,-)\}, \tag{4.77}$$
$$\Pi(\overline{\mathcal{S}}) = \{(-,0,+),(-,0,-),(+,+,0),(+,-,0)\}. \tag{4.78}$$

Note that basis vectors for $\mathcal{S}$ are e^a (so that we can contract them with S_a) and for $\overline{\mathcal{S}}$ the basis vectors are e_a.

According to (4.26), the operators $\mathcal{D}^a$ associated with $\mathcal{S}$ are then labeled by the weights (4.78) of $\mathcal{S}^* = \overline{\mathcal{S}}$, and the operators $\overline{\mathcal{D}}_a$ associated with $\overline{\mathcal{S}}$ are labeled by the weights (4.77) of $\overline{\mathcal{S}}^* = \mathcal{S}$. These operators have the following explicit expressions,

$$\begin{aligned}
\mathcal{D}^a_{-0+} &\equiv \overline{S}^a,\\
\mathcal{D}^a_{-0-} &\equiv \overline{\mathbf{X}}^{ab}\partial_{\overline{S},b},\\
\mathcal{D}^a_{++0} &\equiv \overline{a}\overline{\partial}^{ab}S_b + \overline{S}^a(S\overline{\partial}\partial_{\overline{S}}),\\
\mathcal{D}^a_{+-0} &\equiv \overline{b}c\partial^a_S + \overline{b}\overline{S}^a(\partial_S\partial_{\overline{S}}) + c\mathbf{X}_{bc}\overline{\partial}^{ab}\partial^c_S - \overline{S}^a(\mathbf{X}_{bc}\overline{\partial}^{bd}\partial^c_S\partial_{\overline{S},d}),\\
\overline{\mathcal{D}}_a^{-+0} &\equiv S_a,\\
\overline{\mathcal{D}}_a^{--0} &\equiv \mathbf{X}_{ab}\partial^b_S,\\
\overline{\mathcal{D}}_a^{+0+} &\equiv a\partial_{ab}\overline{S}^b + S_a(\overline{S}\partial\partial_S),\\
\overline{\mathcal{D}}_a^{+0-} &\equiv bc\partial_{\overline{S},a} + bS_a(\partial_{\overline{S}}\partial_S) + c\overline{\mathbf{X}}^{bc}\partial_{ab}\partial_{\overline{S},c} - S_a(\overline{\mathbf{X}}^{bc}\partial_{bd}\partial_{\overline{S},c}\partial^d_S),
\end{aligned} \tag{4.79}$$

where

$$a = 1 - \Delta + \tfrac{\ell}{2} - \tfrac{\overline{\ell}}{2}, \qquad \overline{a} = 1 - \Delta - \tfrac{\ell}{2} + \tfrac{\overline{\ell}}{2},$$
$$b = 2(\overline{\ell}+1), \qquad \overline{b} = 2(\ell+1),$$
$$c = -2 + \Delta - \tfrac{\ell+\overline{\ell}}{2}. \tag{4.80}$$

The coefficients above come from requiring that the operators preserve the ideal generated by the relations (4.74), together with $X^2 = 0$. We have added these operators to the `CFTs4D` Mathematica package described in [2].

4.3 Crossing for differential operators

The results of section 4.2 give us a large variety of conformally-covariant differential operators. In the present section we consider their action on conformally-invariant[24] correlation functions of local operators. The result of such an action is a conformally-covariant n-point function, which can also be interpreted as a conformally-invariant $(n+1)$-point function that includes the degenerate field $w^a(x)$. We will first describe the structure of such correlation functions and then establish a convenient graphical notation for the action of the differential operators. This will help us elucidate a rich structure of such actions at the end of this section.

4.3.1 Conformally-covariant tensor structures

Consider an n-point correlation function with an additional formal insertion of an element e^A of the finite-dimensional representation W of the conformal group $\mathrm{SO}(d+1,1)$,

$$\langle \mathcal{O}_1^{a_1}(x_1)\cdots\mathcal{O}_n^{a_n}(x_n)\rangle^A \equiv \langle \mathcal{O}_1^{a_1}(x_1)\cdots\mathcal{O}_n^{a_n}(x_n)e^A\rangle. \tag{4.81}$$

Note that this is a purely formal construct, i.e. this expression is simply a shorthand for a function of n points which carries indices a_i, A, and has transformation properties identical to those satisfied by a correlation function under the assumption that

$$U_g e^A U_g^{-1} = g\cdot e^A, \tag{4.82}$$

and $g\cdot e^A$ is defined by (4.2).

As discussed in section 4.2.1, we can also view (4.81) as a $(n+1)$-point conformally-invariant correlation function with the primary $w^b(y)$ of W,

$$\langle \mathcal{O}_1^{a_1}(x_1)\cdots\mathcal{O}_n^{a_n}(x_n)w^b(y)\rangle \equiv \langle \mathcal{O}_1^{a_1}(x_1)\cdots\mathcal{O}_n^{a_n}(x_n)e^A\rangle w_A^b(y), \tag{4.83}$$

subject to the conformal Killing differential equation satisfied by $w^b(y)$. This interpretation will be useful to us later on. In this section we stick with (4.81).

[24]We are making a distinction between conformally-covariant and conformally-invariant objects. For us, the former carry finite-dimensional $\mathrm{SO}(d+1,1)$ labels, whereas the latter do not.

Similarly to the usual conformally-invariant correlation functions, we have an expansion in tensor structures,

$$\langle \mathcal{O}_1^{a_1}(x_1)\cdots\mathcal{O}_n^{a_n}(x_n)e^A\rangle = \mathbb{T}_I^{a_1\dots a_n,A}(x_i)g^I(\mathbf{u}), \tag{4.84}$$

which now carry the $\mathrm{SO}(d+1,1)$ index A. Here $\mathbf{u}$ are the conformal cross-ratios of points x_i. The structures $\mathbb{T}^{a_1\dots a_n,A}$ can be constructed using embedding space methods, since there one explicitly works with objects which transform in fundamental representations of $\mathrm{SO}(d+1,1)$. In this subsection we are going to classify such tensor structures by extending the conformal frame approach of [1, 23].

The basic idea is to maximally use conformal symmetry to bring as many x_i as possible to some standard positions x_i'. The resulting configuration x_i' will be invariant under the subgroup $\mathcal{G}_n \subset \mathrm{SO}(d+1,1)$ of the conformal group that stabilizes n points. In particular

$$\mathcal{G}_n = \begin{cases} \mathrm{SO}(1,1)\times\mathrm{SO}(d) & n=2, \\ \mathrm{SO}(d+2-m) & n\geq 3, \end{cases} \tag{4.85}$$

where $m = \min(n, d+2)$. The tensor $\mathbb{T}(x_i')$ transforms as an element in[25]

$$W\otimes\bigotimes_{k=1}^{n}(\rho_k)_{\Delta_k}, \tag{4.86}$$

and by construction is invariant under $\mathcal{G}_n$. It is easy to check [1] that this is the only restriction for the tensor $\mathbb{T}^{a_1\dots a_n,A}(x_i')$ and the conformally-covariant tensor structures are then in one-to-one correspondence with the invariants of $\mathcal{G}_n$,[26]

$$\left(W\otimes\bigotimes_{k=1}^{n}(\rho_k)_{\Delta_k}\right)^{\mathcal{G}_n}. \tag{4.87}$$

In practice we always use the decomposition (4.3) in this formula and identify the tensor structures with

$$\bigoplus_{i=-j}^{j}\left((W_i)_i\otimes\bigotimes_{k=1}^{n}(\rho_k)_{\Delta_k}\right)^{\mathcal{G}_n}. \tag{4.88}$$

[25] In writing a tensor product of representations of different groups, we assume that each representation is restricted to the largest common subgroup. In (4.86), we implicitly restrict W to $\mathrm{SO}(1,1)\times\mathrm{SO}(d)\subset\mathrm{SO}(d+1,1)$.

[26] The notation $(\rho)^H$ denotes the H-invariant subspace of ρ, where ρ is a representation of G and $H\subseteq G$.

4.3.2 Tensor structures and diagrams

Let us work through some examples of covariant n-point functions and the counting rule (4.88). At the same time, we will introduce a useful diagrammatic language for describing tensor structures and differential operators.

4.3.2.1 Invariant two-point functions

Let us denote a conformally-invariant two-point structure by

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \quad = \quad \mathcal{O}_1 \longleftarrow\!\bullet\!\longrightarrow \mathcal{O}_2 \,. \tag{4.89}$$

It is well-known that there is at most one such structure, but let us re-derive this fact in the language of section 4.3.1, where it corresponds to the case $n = 2$ and $W = \bullet$.

Given x_1 and x_2, we can apply a conformal transformation to set $x_1 = 0$ and $x_2 = \infty$. Then the group $\mathcal{G}_2 = \mathrm{SO}(1,1) \times \mathrm{SO}(d)$ which fixes the two points consists of dilatations and rotations around 0. Sending the second operator to infinity has the effect that $\mathcal{O}_2$ effectively changes the sign of its scaling dimension, and transforms in the reflected representation[27] ρ_2^P under $\mathrm{SO}(d)$. Thus, two-point structures correspond to the $\mathcal{G}_2$-invariants in

$$(\rho_1)_{\Delta_1} \otimes (\rho_2^P)_{-\Delta_2}. \tag{4.90}$$

There is at most one such invariant, which exists iff $\rho_1 = (\rho_2^P)^*$ and $\Delta_1 = \Delta_2$. The dual-reflected representation, which we denote by $(\rho_2^P)^* \equiv \rho_2^\dagger$ is the same as the complex conjugate representation in Lorentzian signature.

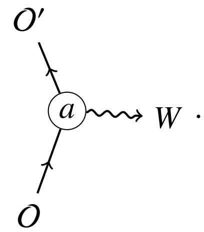
4.3.2.2 Differential operators

A differential operator $\mathcal{D}^A : \mathcal{O} \to \mathcal{O}'$ takes a conformally-invariant structure for $\mathcal{O}$ to a conformally-covariant structure for $\mathcal{O}'$, or equivalently an invariant structure for $\mathcal{O}'$ and W:

$$\mathcal{D}^A \langle \mathcal{O} \cdots \rangle \sim \langle e^A \mathcal{O}' \cdots \rangle. \tag{4.91}$$

[27] Given a representation ρ with generators $\rho_{\mu\nu}$ the reflected representation is defined as $\rho^P_{\mu\nu} = P_\mu^{\mu'} P_\nu^{\nu'} \rho_{\mu'\nu'}$, where P is a spatial reflection matrix. Formally, conjugating by reflection is an outer automorphism of $\mathrm{SO}(d)$, and hence permutes the representations of $\mathrm{SO}(d)$.

We denote such a differential operator by

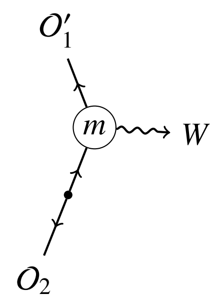
$$\mathcal{D}^{(a)A} \quad = \quad$$ 

(4.92)

The label a runs over the possible operators classified by theorem 2. We use a wavy line to indicate a finite-dimensional representation.

4.3.2.3 Covariant two-point functions

Consider acting with a differential operator $\mathcal{D}^{(m)A} : [\Delta_1, \rho_1] \to [\Delta_1', \lambda_1]$ on an invariant two-point function. In diagrammatic language, this is denoted by connecting an outgoing arrow from the two-point function with an incoming arrow for the differential operator,

$$(\mathcal{D}^{(m)A})^c{}_a \langle \mathcal{O}_1^a(x_1) \mathcal{O}_2^b(x_2) \rangle \quad = \quad$$ 

(4.93)

The result can be interpreted as a covariant two-point structure for $\mathcal{O}_1'$, $\mathcal{O}_2$, and W. Such structures are counted by $\mathrm{SO}(1,1) \times \mathrm{SO}(d)$-invariants in

$$\bigoplus_{i=-j/2}^{j/2} (W_i)_i \otimes (\lambda_1)_{\Delta_1'} \otimes (\rho_2^P)_{-\Delta_2}. \tag{4.94}$$

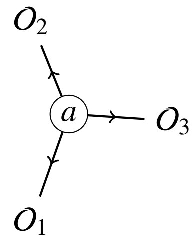
Invariants exist whenever $\Delta_1' = \Delta_2 - i = \Delta_1 - i$ and $\lambda_1 \in (W_i)^* \otimes (\rho_2^P)^* = (W_i)^* \otimes \rho_1$.[28] Note that these are exactly the conditions for the existence of $\mathcal{D}^A$ in theorem 2. Thus, the number of non-vanishing diagrams (4.93) is precisely equal to the number of tensor structures for $\langle \mathcal{O}_2 \mathcal{O}_1' e^A \rangle$. In other words, all covariant two-point structures can be obtained by acting with differential operators on an invariant two-point structure.

[28] We have assumed that $\Delta_1 = \Delta_2$ and $\rho_1 = (\rho_2^P)^*$ so that $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle$ is nonvanishing.

4.3.2.4 Invariant three-point functions

We denote conformally-invariant three-point structures by

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle^{(a)} \quad = \qquad\qquad\qquad . \tag{4.95}$$



The label a runs over possible tensor structures, which are classified by $\mathcal{G}_3 = \mathrm{SO}(d-1)$ singlets

$$(\rho_1 \otimes \rho_2 \otimes \rho_3)^{\mathrm{SO}(d-1)}. \tag{4.96}$$

A physical three-point function is a sum over tensor structures with different OPE coefficients λ_m,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \sum_{m=1}^{N_3} \lambda_m \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle^{(m)}, \tag{4.97}$$

where $N_3 = \dim(\rho_1 \otimes \rho_2 \otimes \rho_3)^{\mathrm{SO}(d-1)}$. When there is a unique three-point structure ($N_3 = 1$), we often omit the index m.[29]

4.3.2.5 Covariant three-point functions

Consider now acting on an invariant three-point structure with a differential operator. Let us begin with a three-point structure $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3' \rangle^{(a)}$, and suppose that $\mathcal{O}_3'$ transforms in the representation $[\Delta_3 + i, \lambda]$. The label a runs over singlets in

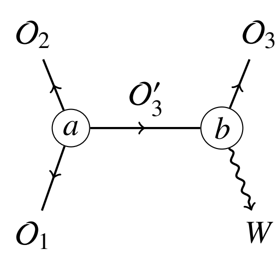
$$(\rho_1 \otimes \rho_2 \otimes \lambda)^{\mathrm{SO}(d-1)}. \tag{4.98}$$

By theorem 2, we have a differential operator $\mathcal{D}^{(b)A} : [\Delta_3 + i, \lambda] \to [\Delta_3, \rho_3]$ whenever

$$\rho_3 \in (W_i)^* \otimes \lambda \quad \Leftrightarrow \quad \lambda \in W_i \otimes \rho_3. \tag{4.99}$$

By acting with $\mathcal{D}^{(b)A}$ on $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3' \rangle^{(a)}$, we can form a covariant three-point structure for $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 e^A \rangle$,

$$(\mathcal{D}^{(b)A})^{a_3}{}_c \langle \mathcal{O}_1^{a_1}(x_1) \mathcal{O}_2^{a_2}(x_2) \mathcal{O}_3'^{c}(x_3) \rangle^{(a)} \quad = \qquad\qquad\qquad . \tag{4.100}$$



[29] Since we never work with physical three-point functions (4.97), there is no danger of confusion.

Let us count the number of diagrams (4.100) by summing over the allowed $\mathcal{O}_3'$, a and b. Taking into account the selection rule (4.99), we have

$$\begin{aligned}&\sum_{i=-j}^{j}\sum_{\lambda\in W_i\otimes\rho_3}\dim\left(\rho_1\otimes\rho_2\otimes\lambda\right)^{\mathrm{SO}(d-1)}\\&=\dim\left(\bigoplus_{i=-j}^{j}\bigoplus_{\lambda\in W_i\otimes\rho_3}\rho_1\otimes\rho_2\otimes\lambda\right)^{\mathrm{SO}(d-1)}=\dim\left(\bigoplus_{i=-j}^{j}W_i\otimes\rho_1\otimes\rho_2\otimes\rho_3\right)^{\mathrm{SO}(d-1)}.\end{aligned}\tag{4.101}$$

According to (4.88), this is precisely the total number of covariant three-point structures for $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3e^A\rangle$. In other words, generically, every conformally-covariant three-point structure can be obtained by acting with differential operators on conformally-invariant three-point structures.

Note that according to the discussion in section 4.2.1 we can interpret the conformally-covariant three-point functions as conformally-invariant four-point functions involving a degenerate primary $w^a(x)$. Analogously, we can interpret (4.100) as conformal blocks for these four-point functions. We have just proven a highly degenerate case of the folklore theorem which states that that the number of such conformal blocks is equal the dimension of the space of degenerate four-point functions.[30] Importantly, in our case this number is finite. This brings us to a very powerful observation.

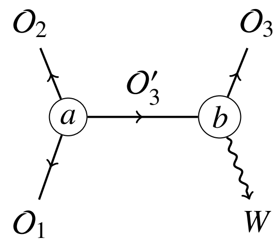
4.3.3 Crossing and $6j$ symbols

The diagrams (4.100) give a basis for the finite-dimensional space of covariant three-point structures $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3e^A\rangle$. However, this is not the only interesting basis. The distinguishing feature of (4.100) is that it selects a particular operator $\mathcal{O}_3'$ appearing in the $\mathcal{O}_1\times\mathcal{O}_2$ OPE. In other words, it diagonalizes the action of the Casimir $(L_1+L_2)^2$ acting simultaneously on $\mathcal{O}_1,\mathcal{O}_2$ (equivalently $\mathcal{O}_3, w$). However, we may wish to select an operator in a different channel, e.g. $\mathcal{O}_1'\in\mathcal{O}_2\times\mathcal{O}_3$. This would correspond to starting with a three-point structure $\langle\mathcal{O}_1'\mathcal{O}_2\mathcal{O}_3\rangle^{(m)}$ and acting with a differential operator $\mathcal{D}^{(n)A}:\mathcal{O}_1'\to\mathcal{O}_1$.

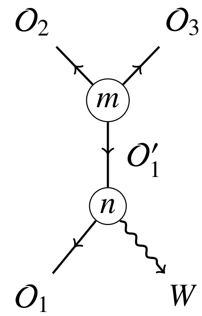
These two bases are related by a linear transformation, which gives a type of crossing

[30] In the non-degenerate case we have the number of families of conformal blocks and the number of "functional degrees of freedom."

equation for differential operators,



$$= \sum_{\mathcal{O}_1',m,n} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & W & \mathcal{O}_3' \end{Bmatrix}^{ab}_{mn}$$



. (4.102)

In equations, (4.102) reads

$$\begin{aligned} &\mathcal{D}_{x_3}^{(b)A}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3'(x_3)\rangle^{(a)} \\ &= \sum_{\mathcal{O}_1',m,n} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & W & \mathcal{O}_3' \end{Bmatrix}^{ab}_{mn} \mathcal{D}_{x_1}^{(n)A}\langle \mathcal{O}_1'(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle^{(m)}. \end{aligned} \tag{4.103}$$

Note that the sum over $\mathcal{O}_1'$ is finite with $\mathcal{O}_1'$ taking values in the tensor product $\mathcal{O}_1 \otimes W$. The coefficients in this transformation are called Racah coefficients, or $6j$ symbols.[31,32] The $6j$ symbols for operator representations (generalized Verma modules) of the conformal group have seen some recent interest for their role in the crossing equations for CFT four-point functions [168–170]. Here, we have a degenerate form of these objects, where one of the representations appearing is finite-dimensional. These degenerate $6j$ symbols enter in a degenerate crossing equation (4.102) where the objects on both sides live in a finite dimensional space. One can ask what happens if we consider $6j$ symbols with more finite-dimensional representations. As we show in appendix C.4, such $6j$ symbols are related to the algebra of conformally-covariant differential operators.

A useful analogy for understanding (4.102) is to consider a four-point function containing at least one degenerate Virasoro primary in a 2d CFT. The shortening condition on the degenerate primary implies that its four-point function lives in a finite-dimensional space spanned by a finite number of conformal blocks. The crossing transformation for these blocks is a finite-dimensional matrix. Similarly in (4.102), the left-hand side can be interpreted as the conformal block for $\mathcal{O}_3'$ exchange

[31] Technically, Racah coefficients and $6j$ symbols are sometimes defined to differ by various normalization factors. We will not distinguish between them and use both terms to refer to the coefficients in (4.102).

[32] $6j$ symbols depend only on a set of representations and three-point structures. However, for brevity, we often label them with operators $\mathcal{O}_i$ transforming in those representations, as in (4.102).

in a four-point function $\langle O_1 O_2 O_3 w\rangle$. Because w satisfies a highly-constraining differential equation, the crossing transformation for this block is a finite-dimensional matrix.

4.3.4 Examples

Because the space of covariant three-point structures is finite dimensional (its dimension is given by (4.101)), it is straightforward to find the degenerate $6j$ symbols by direct computation: we apply differential operators on both sides and invert a finite-dimensional matrix. Let us work through some examples.

4.3.4.1 $6j$ symbols in 1 dimension

3-point functions Before computing the $6j$ symbols, we need to choose a basis of three-point structures. The three-point functions in 1-dimension are not completely trivial, and it is important to get them right in order to have well-defined $6j$ symbols.

According to the discussion of section 4.2.5.2, there are two types of fields with different "spins" $s = \pm$. The fields with $s = +$ are the usual scalars. The simplest three-point function for the scalars is

$$\langle \Phi_1^+(\chi_1)\Phi_2^+(\chi_2)\Phi_3^+(\chi_3)\rangle^{(+)} = \frac{1}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2}}. \tag{4.104}$$

Here we have added the label $(+)$ to indicate that this is a parity-even three-point structure. We need this because there in fact exists a parity-odd three-point structure,

$$\langle \Phi_1^+(\chi_1)\Phi_2^+(\chi_2)\Phi_3^+(\chi_3)\rangle^{(-)} = \frac{(\chi_1\chi_2)(\chi_2\chi_3)(\chi_3\chi_1)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3+1}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1}}. \tag{4.105}$$

This is related to the fact that unless we allow reflections, all conformal transformations preserve the cyclic ordering of three points on the circle S^1. One can see that this structure is parity-odd from the parity transformation $\chi \to \gamma^2\chi$.

We will compute the $6j$ symbols for differential operators in the fundamental representation which, according to (4.60), change the spin s. Therefore, we will also need the parity even and parity odd structures for the three point function with two

$s = -$ operators,

$$\langle \Phi_1^-(\chi_1)\Phi_2^+(\chi_2)\Phi_3^-(\chi_3)\rangle^{(-)} = \frac{(\chi_3\chi_1)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1}}, \tag{4.106}$$

$$\langle \Phi_1^-(\chi_1)\Phi_2^+(\chi_2)\Phi_3^-(\chi_3)\rangle^{(+)} = \frac{(\chi_1\chi_2)(\chi_2\chi_3)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3+1}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2}}. \tag{4.107}$$

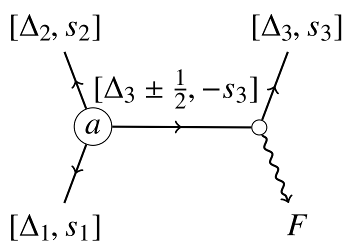
The difference between $s = +$ and $s = -$ tensor structures is in their transformation properties under (4.56).

$6j$ **symbols** As noted above, we will specialize to $W = F$ being the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$, which has weights $\Delta = \pm\frac{1}{2}$. The corresponding differential operators are

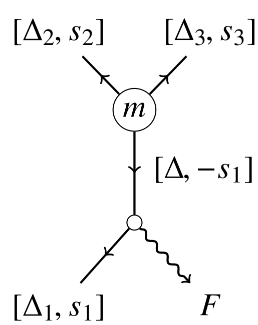
$$\mathcal{D}_\alpha^+ = \partial_\alpha, \qquad \mathcal{D}_\alpha^- = \chi_\alpha. \tag{4.108}$$

It will be convenient to contract each differential with a polarization spinor χ_4, giving $\chi_4^\alpha \mathcal{D}_\alpha^\pm$. This spinor may be interpreted as the coordinate of the fourth operator in representation $[-\frac{1}{2}, -]$. The operator $\chi_4\mathcal{D}^+$ is even under space parity, while the operator $\chi_4\mathcal{D}^-$ is odd under space parity.

The definition of $6j$ symbols in this case is

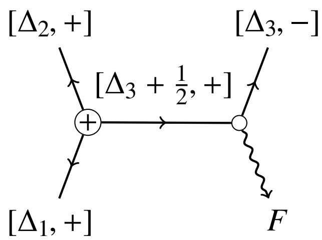


$$= \sum_{\substack{\Delta=\Delta_1\pm\frac{1}{2}\\ m}} \left\{\begin{matrix} [\Delta_1, s_1] & [\Delta_2, s_2] & [\Delta, -s_1] \\ [\Delta_3, s_3] & F & [\Delta_3 \pm \frac{1}{2}, -s_3] \end{matrix}\right\}_{m\bullet}^{a\bullet} \tag{4.109}$$

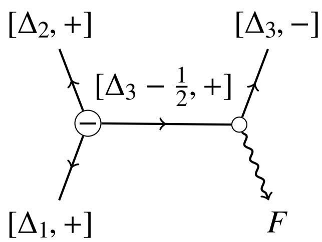


We don't need to label the vertices for differential operators, since there is always a unique choice of differential operator for the given dimensions. For example, on the left-hand side, when the internal line has dimension $\Delta_3 \pm \frac{1}{2}$, the F-differential operator must be $\mathcal{D}^{\mp}$. The notation "$\bullet$" on the $6j$ symbols means there is a unique corresponding structure or differential operator.

It is now straightforward to compute the objects above. Let us take for example $s_1 = s_2 = +$, $s_3 = -$ and specialize to the case when both sides of (4.109) are parity-odd. For the left-hand side we then have,

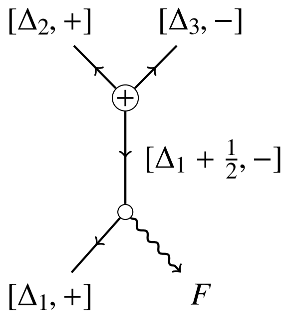


$$= \frac{(\chi_4\chi_3)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3-1/2}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1/2}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1/2}},$$

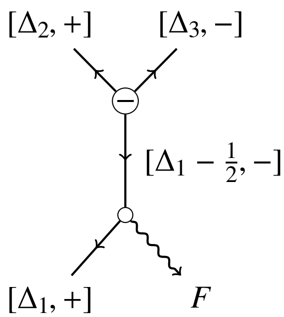


$$\begin{aligned} &= \frac{-(\Delta_1+\Delta_3-\Delta_2-1/2)(\chi_4\chi_1)(\chi_1\chi_2)(\chi_2\chi_3)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3+3/2}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1/2}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1/2}} \\ &\quad + \frac{(\Delta_2+\Delta_3-\Delta_1-1/2)(\chi_4\chi_2)(\chi_1\chi_2)(\chi_3\chi_1)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3+3/2}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1/2}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1/2}}. \end{aligned} \tag{4.110}$$

For the right-hand side,



$$= \frac{(\chi_4\chi_1)(\chi_1\chi_2)(\chi_2\chi_3)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3+3/2}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1/2}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1/2}}$$



$$\begin{aligned} &= \frac{(\Delta_1+\Delta_3-\Delta_2-1/2)(\chi_4\chi_3)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3-1/2}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1/2}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1/2}} \\ &\quad + \frac{-(\Delta_1+\Delta_2-\Delta_3-1/2)(\chi_4\chi_2)(\chi_1\chi_2)(\chi_3\chi_1)}{|\chi_1\chi_2|^{\Delta_1+\Delta_2-\Delta_3+3/2}|\chi_2\chi_3|^{\Delta_2+\Delta_3-\Delta_1+1/2}|\chi_3\chi_1|^{\Delta_3+\Delta_1-\Delta_2+1/2}}. \end{aligned} \tag{4.111}$$

After using the Schouten identity

$$(\chi_4\chi_1)(\chi_2\chi_3) + (\chi_4\chi_2)(\chi_3\chi_1) + (\chi_4\chi_3)(\chi_1\chi_2) = 0, \tag{4.112}$$

we can solve for the $6j$ symbols

$$\begin{Bmatrix} [\Delta_1,+] & [\Delta_2,+] & [\Delta_1+\frac{1}{2},-] \\ [\Delta_3,-] & F & [\Delta_3+\frac{1}{2},+] \end{Bmatrix}^{+\bullet}_{+\bullet} = -\frac{\Delta_1+\Delta_2-\Delta_3-1/2}{2\Delta_1-1}, \tag{4.113}$$

$$\begin{Bmatrix} [\Delta_1,+] & [\Delta_2,+] & [\Delta_1-\frac{1}{2},-] \\ [\Delta_3,-] & F & [\Delta_3+\frac{1}{2},+] \end{Bmatrix}^{+\bullet}_{-\bullet} = \frac{1}{2\Delta_1-1}, \tag{4.114}$$

$$\begin{Bmatrix} [\Delta_1,+] & [\Delta_2,+] & [\Delta_1+\frac{1}{2},-] \\ [\Delta_3,-] & F & [\Delta_3-\frac{1}{2},+] \end{Bmatrix}^{-\bullet}_{+\bullet} = -\frac{(\Delta_1+\Delta_3-\Delta_2-1/2)\,(\Delta_1+\Delta_2+\Delta_3-3/2)}{2\Delta_1-1}, \tag{4.115}$$

$$\begin{Bmatrix} [\Delta_1,+] & [\Delta_2,+] & [\Delta_1-\frac{1}{2},-] \\ [\Delta_3,-] & F & [\Delta_3-\frac{1}{2},+] \end{Bmatrix}^{-\bullet}_{-\bullet} = -\frac{\Delta_2+\Delta_3-\Delta_1-1/2}{2\Delta_1-1}. \tag{4.116}$$

4.3.4.2 $6j$ symbols in 3 dimensions

3-point functions It is also possible to find the general $6j$ symbols for the spinor representation $\mathcal{S}$ of the 3d conformal group. To do that, it is convenient to use the conformal frame basis of three-point structures from [1].[33] To construct this basis, one contracts the 3d primary operators with polarization spinors s_α,

$$\mathcal{O}(s,x) = s_{\alpha_1}\cdots s_{\alpha_{2\ell}}\mathcal{O}^{\alpha_1\ldots\alpha_{2\ell}}(x). \tag{4.117}$$

The three point-functions are then evaluated in the configuration

$$f_3(s_1,s_2,s_3) = \langle \mathcal{O}_1(s_1,0)\mathcal{O}_2(s_2,e)\mathcal{O}_3(s_3,\infty)\rangle, \tag{4.118}$$

where $e = (0,0,1)$ and $\mathcal{O}(s_3,\infty) = \lim_{L\to\infty} L^{2\Delta_3}\mathcal{O}(s_3,Le)$. The polynomial f_3 should be invariant under boosts in the 0-1 plane. A basis for such polynomials is given by the monomials

$$[q_1 q_2 q_3] = \prod_{i=1}^{3} \xi_i^{\ell_i+q_i}\overline{\xi}_i^{\ell-q_i}, \tag{4.119}$$

where $s_i = (\xi_i,\overline{\xi}_i)$ and $q_i = -\ell_i\ldots\ell_i$, subject to the constraint $\sum_i q_i = 0$.

It will also be convenient to think about the covariant three-point functions as four-point functions with the degenerate spinor primary $w^\alpha(x)$ of dimension $-\frac{1}{2}$. We construct an analogous basis for four-point tensor structures by evaluating

$$\langle \mathcal{O}_1(s_1,0)s_\alpha w^\alpha(ze)\mathcal{O}_2(s_2,e)\mathcal{O}_3(s_3,\infty)\rangle, \tag{4.120}$$

leading to a monomial basis $[q_1,q,q_2,q_3]$, where $q = \pm\frac{1}{2}$.[34] The configuration (4.120) is still invariant under boosts in the 0-1 plane, so we again have the condition $q+\sum q_i = 0$. We have introduced only one cross-ratio z because $w^\alpha(x)$ is a degenerate field. In fact, the general solution to its Killing equation is given by

$$w(x) = w_0 + x^\mu\gamma_\mu w_1, \tag{4.121}$$

and thus it is sufficient to know its values for $x = ze$ to determine it completely. Note also that this equation implies that a general four-point function of such form is linear in z.

[33] Our conventions in this section are those of [1, 39, 81].

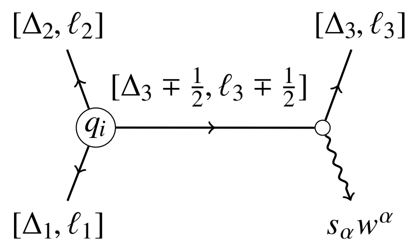
[34] Notice that we used a configuration different from the one used for four-point functions in [1].

To obtain these degenerate four-point functions, we think about the three-point functions as four-point functions with an identity operator at coordinate x and act with the operators

$$D_i^{\pm\pm} = \Omega^{ab}\mathcal{D}_{a,x}^{-+}\mathcal{D}_{b,x_i}^{\pm\pm}\Sigma_i^{\mp\mp}, \tag{4.122}$$

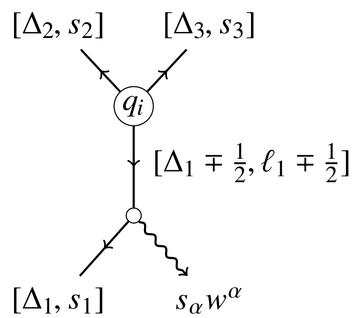
where $x = ze$, and $\Sigma_i^{\mp\mp}$ formally shifts the scaling dimension and spin of the operator i, so that $D_i^{\pm\pm}$ doesn't change the dimensions and spins.[35] In this notation we have[36]

$D_3^{\pm\pm}[q_1q_2q_3] \equiv$



,

$D_1^{\pm\pm}[q_1q_2q_3] \equiv$



. (4.123)

Our goal is therefore to find the transformation between the bases $D_3^{\pm\pm}[q_1q_2q_3]$ and $D_1^{\pm\pm}[q_1q_2q_3]$

$6j$ symbols It is obvious that since the operators $D_i^{\pm\pm}$ contain a finite number of derivatives in the polarization spinors, they take a three-point structure $[q_1q_2q_3]$ to four-point structures $[q_1', q, q_2', q_3']$ for (4.120) with q_i' differing from q_i by only finite shifts. We can say that $D_i^{\pm\pm}$ are local in q-space. It turns out that the inverse operation, which expresses an arbitrary four-point function (4.120) in terms of $D_i^{\pm\pm}[q_1q_2q_3]$, is also local in q-space. In this language the $6j$ symbols essentially give the composition of the inverse to $D_1^{\pm\pm}$ with $D_3^{\pm\pm}$ and are thus also local in q-space. This allows us to write down a general expression for these $6j$ symbols.

The number of shifts in q for which the $6j$ symbols are generically non-zero is however rather large. We therefore take an indirect approach in this section, describing

[35] In other words, the components of D_a^{-+} are essentially the conformal Killing spinors $s_\alpha w_a^\alpha(x)$.

[36] As in the 1d case, we omit the labels for the differential operators in the diagrams (4.123) because the differential operator is always fixed by the given representations.

how the $6j$ symbols can be straightforwardly generated from relatively simple expressions. Our strategy will be to write the action of $D_1^{\pm\pm}$ and $D_3^{\pm\pm}$ on $[q_1q_2q_3]$ in a form from which both the direct action and the inverse can be easily obtained. One can then simply substitute the inverse of $D_1^{\pm\pm}$ into the expressions for $D_3^{\pm\pm}[q_1q_2q_3]$ to generate the general $6j$ symbols.

First, we evaluate the expressions for $D_3^{\pm\pm}[q_1q_2q_3]$ and $D_1^{\pm\pm}[q_1q_2q_3]$. This can be done relatively easily in a computer algebra system. The result can be expressed in terms of the four-point tensor structures $[q_1, q, q_2, q_3]$, for instance,

$$D_1^{--}[q_1q_2q_3] = z(\ell_1 + q_1 + \tfrac{1}{2})[q_1 - \tfrac{1}{2}, +\tfrac{1}{2}, q_2, q_3] - z(\ell_1 - q_1 + \tfrac{1}{2})[q_1 + \tfrac{1}{2}, -\tfrac{1}{2}, q_2, q_3]. \tag{4.124}$$

We will now describe these actions in a compact form. We first define

$$\mathcal{A}_1^{\pm}[q_1q_2q_3] = \left(-D_1^{--} \mp (\ell_1 \mp q_1 + \tfrac{1}{2})D_1^{-+}\right)[q_1q_2q_3], \tag{4.125}$$

These operators satisfy

$$\mathcal{A}_1^{\pm}[q_1q_2q_3] = \mp z(2\ell_1 + 1)[q_1 \mp \tfrac{1}{2}, \pm\tfrac{1}{2}, q_2, q_3]. \tag{4.126}$$

Note that this solves the inversion problem for the linear terms $z[q_1, q, q_2, q_3]$ and is also sufficient to find the action $D_1^{-\pm}[q_1q_2q_3]$. We then define the analogous operators

$$\mathcal{B}_1^{\pm}[q_1q_2q_3] = \left(-D_1^{+-} \mp (\ell_1 \mp q_1 + \tfrac{1}{2})D_2^{++}\right)[q_1q_2q_3] + C_1^{\pm}[q_1q_2q_3], \tag{4.127}$$

where the correction term $C_1^{\pm}$ is a linear combination of $\mathcal{A}_1^{\pm}$ given below. The operators $\mathcal{B}_1^{\pm}$ act on $[q_1q_2q_3]$ as follows,

$$\begin{aligned}\left((\Delta_1 \pm q_1 - \tfrac{3}{2})\mathcal{B}_1^{\pm} + (\ell_1 \mp q_1 + \tfrac{1}{2})\mathcal{B}_1^{\mp}\right)[q_1q_2q_3] &= \\ = 4(2\ell_1 + 1)(\Delta_1 - \tfrac{3}{2})(\ell_1 + \Delta_1 - 1)(\ell_1 - \Delta_1 + 2)&[q_1 \mp \tfrac{1}{2}, \pm\tfrac{1}{2}, q_2, q_3].\end{aligned} \tag{4.128}$$

This solves the inversion problem for the constant terms $[q_1, q, q_2, q_3]$ and is also sufficient to write down the action of $\mathcal{B}_1^{\pm}$ and thus also of $D_1^{+\pm}$.

We can describe the action of $D_3^{\pm\pm}$ and its inverse in a similar fashion. In particular, we define

$$\mathcal{A}_3^{\pm}[q_1q_2q_3] = \left(-D_3^{--} \mp (\ell_3 \mp q_3 + \tfrac{1}{2})D_3^{-+}\right)[q_1q_2q_3], \tag{4.129}$$

$$\mathcal{B}_3^{\pm}[q_1q_2q_3] = \left(-D_3^{+-} \mp (\ell_3 \mp q_3 + \tfrac{1}{2})D_3^{++}\right)[q_1q_2q_3] - C_3^{\pm}[q_1q_2q_3]. \tag{4.130}$$

The correction term $C_3^{\pm}$ is defined below. For these operators we have the analogue of (4.126)

$$\mathcal{A}_3^{\pm}[q_1q_2q_3] = \pm(2\ell_3+1)[q_1, \pm\tfrac{1}{2}, q_2, q_3 \mp \tfrac{1}{2}], \tag{4.131}$$

and the analogue of (4.128),

$$\begin{aligned}\mathcal{B}_3^{\pm}[q_1q_2q_3] =& -4z(2\ell_3+1)(\Delta_3-\tfrac{3}{2})(\Delta_3 \mp q_3 - \tfrac{3}{2})[q_1, \pm\tfrac{1}{2}, q_2, q_3 \mp \tfrac{1}{2}]\\ &+4z(2\ell_3+1)(\Delta_3-\tfrac{3}{2})(\ell_3 \mp q_3 + \tfrac{1}{2})[q_1, \mp\tfrac{1}{2}, q_2, q_3 \pm \tfrac{1}{2}].\end{aligned} \tag{4.132}$$

We can use these expressions to find the action of $D_3^{\pm\pm}$ and then substitute the expressions (4.126) and (4.128) for the four-point functions $z[q_1, q, q_2, q_3]$ and $[q_1, q, q_2, q_3]$ in terms of $D_1^{\pm\pm}$ to find the $6j$ symbols. As a simple example, we find for $\ell_i = 0$,

$$\begin{aligned}D_3^{--}[000] =& -\frac{\Delta_1+\Delta_2-\Delta_3-2}{2(2\Delta_1-3)}D_1^{--}\left([-\tfrac{1}{2},0,\tfrac{1}{2}]+[\tfrac{1}{2},0,+\tfrac{1}{2}]\right)\\ &+\frac{1}{8(\Delta_1-\frac{3}{2})(\Delta_1-2)}D_1^{+-}\left([-\tfrac{1}{2},0,+\tfrac{1}{2}]-[+\tfrac{1}{2},0,-\tfrac{1}{2}]\right),\end{aligned} \tag{4.133}$$

from where we can read off the for example the following $6j$ symbol,

$$\begin{Bmatrix}[\Delta_1,0] & [\Delta_2,0] & [\Delta_1+\frac{1}{2},\frac{1}{2}]\\ [\Delta_3,0] & \mathcal{S} & [\Delta_3+\frac{1}{2},\frac{1}{2}]\end{Bmatrix}^{[000](--)}_{[-\frac{1}{2},0,+\frac{1}{2}](--)} = -\frac{\Delta_1+\Delta_2-\Delta_3-2}{2(2\Delta_1-3)}. \tag{4.134}$$

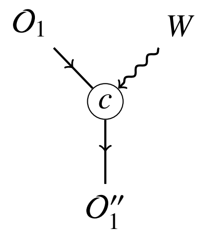
The correction term $C_1^{\pm}$ is given by

$$\begin{aligned}C_1^{\pm}[q_1q_2q_3] =& (\ell_1+q_1\mp\tfrac{1}{2})(\ell_3-q_3)\mathcal{A}_1^{\pm}[q_1-1,q_2,q_3+1]\\ &-(\ell_1-q_1\pm\tfrac{1}{2})(\ell_3+q_3)\mathcal{A}_1^{\pm}[q_1+1,q_2,q_3-1]\\ &-(\ell_1+q_1\mp\tfrac{1}{2})(\ell_2-q_2)\mathcal{A}_1^{\pm}[q_1-1,q_2+1,q_3]\\ &+(\ell_1-q_1\pm\tfrac{1}{2})(\ell_2+q_2)\mathcal{A}_1^{\pm}[q_1+1,q_2-1,q_3]\\ &\mp 2(\ell_3\mp q_3)(\Delta_1-2)\mathcal{A}_1^{\mp}[q_1\mp 1,q_2,q_3\pm 1]\\ &\pm 2(\ell_2\mp q_2)(\Delta_1-2)\mathcal{A}_1^{\mp}[q_1\mp 1,q_2\pm 1,q_3]\\ &\pm 2(\Delta_1\mp q_1-\tfrac{3}{2})(\Delta_1+\Delta_2-\Delta_3-\tfrac{3}{2})\mathcal{A}_1^{\pm}[q_1,q_2,q_3]\\ &\pm 2(\ell_1\mp q_1+\tfrac{1}{2})(\Delta_1-\tfrac{3}{2})\mathcal{A}_1^{\mp}[q_1,q_2,q_3].\end{aligned} \tag{4.135}$$

The correction term $C_3^{\pm}$ is obtained from the above expression by replacing $1 \leftrightarrow 3$ in the coefficients, replacing $\mathcal{A}_1$ by $\mathcal{A}_3$, and exchanging the shifts applied to q_1 and q_3 in the three-point structures. Note that $C_3^{\pm}$ enters (4.130) with a minus sign.

4.3.5 Differential bases from $6j$ symbols

The crossing equation (4.102) will be our key computational tool in this work. Using it, we can perform a variety of calculations with differential operators. As a brief example, consider contracting both sides of (4.102) with a differential operator $\mathcal{D}_A^{(c)} : \mathcal{O}_1 \to \mathcal{O}_1''$, which we denote



$$. \qquad (4.136)$$

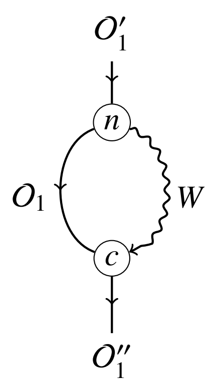
Here, the incoming arrow for W indicates that this operator is associated to the dual representation W^*. Let us connect the incoming W line in (4.136) with the outgoing W line in (4.102), i.e., contract the A indices. In equations, we find

$$\begin{aligned}
&\mathcal{D}^{(c)}_{A,x_1}\mathcal{D}^{(b)A}_{x_3}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3'(x_3)\rangle^{(a)} \\
&= \sum_{\mathcal{O}',m,n}\begin{Bmatrix}\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & W & \mathcal{O}_3'\end{Bmatrix}^{ab}_{mn}\mathcal{D}^{(c)}_{A,x_1}\mathcal{D}^{(n)A}_{x_1}\langle\mathcal{O}_1'(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle^{(m)},
\end{aligned} \qquad (4.137)$$

where we have given the differential operators subscripts x_i to indicate which leg they act on.

The composition of differential operators $\mathcal{D}^{(c)}_{A,x_1}\mathcal{D}^{(n)A}_{x_1}$ on a single leg corresponds to a bubble diagram

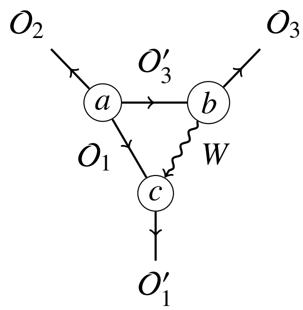
$$\mathcal{D}_A^{(c)}\mathcal{D}^{(n)A} \; = $$



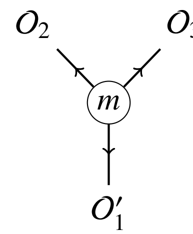
$$= \begin{pmatrix} \mathcal{O}_1' \\ \mathcal{O}_1 \; W \end{pmatrix}^{cn}\delta_{\mathcal{O}_1'\mathcal{O}_1''}. \qquad (4.138)$$

This vanishes unless the representations for $\mathcal{O}_1'$ and $\mathcal{O}_1''$ are the same, in which case it is proportional to the identity (at least for generic scaling dimensions Δ_1', Δ'). The reason is that (4.138) represents a homomorphism between generalized Verma modules, which are irreducible when the scaling dimensions are generic. The constant of proportionality, given by the symbol in parentheses on the right-hand side of (4.138), is actually related to another type of $6j$ symbol, as we explain in appendix C.4. For now, we take (4.138) as a definition of these symbols.

Using (4.138) with $\mathcal{O}_1' = \mathcal{O}_1''$, we can simplify the right-hand side of (4.137) to obtain



$$= \sum_{m,n} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1' \\ \mathcal{O}_3 & W & \mathcal{O}_3' \end{Bmatrix}^{ab}_{mn} \begin{pmatrix} \mathcal{O}_1' \\ \mathcal{O}_1 \; W \end{pmatrix}^{cn}$$

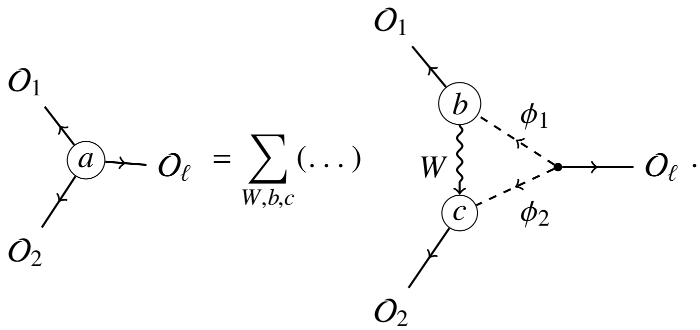


. (4.139)

The left-hand side of (4.139) is a conformally-invariant differential operator $\mathcal{D}^{(c)}_{A,x_1}\mathcal{D}^{(b)A}_{x_3}$ acting on a three-point structure at two different points. The right-hand side is a sum of structures where the representations at those points have been modified. The existence of such invariant two-point differential operators was a key observation of [61]. Here, we see that they factorize into a product of covariant differential operators, each acting on a single point. Indeed, it is easy to verify that all "basic" differential operators in [61] are of this form, with W being either the vector or the adjoint representations of the conformal group. Furthermore, from the discussion in section 4.2.4 and appendix C.4 it follows that arbitrary compositions of the basic differential operators of [61] are also of the form (4.139) with more complicated representations W. In this sense, (4.139) gives a more fundamental point of view on such operators.

The main purpose of the differential operators in [61] was to raise the spins of the operators they act on. Here, we see that it is also possible to *lower* spins, an idea that we discuss briefly in section 4.5.

Another observation of [61] is that (4.139) can sometimes be inverted to express a basis of tensor structures in terms of differential operators acting on simpler structures. For example, when one of the operators $\mathcal{O}_\ell$ is a traceless-symmetric tensor, one can write three-point structures involving $\mathcal{O}_\ell$ in terms of derivatives of three-pt structures involving scalars. In our notation, this reads



(4.140)

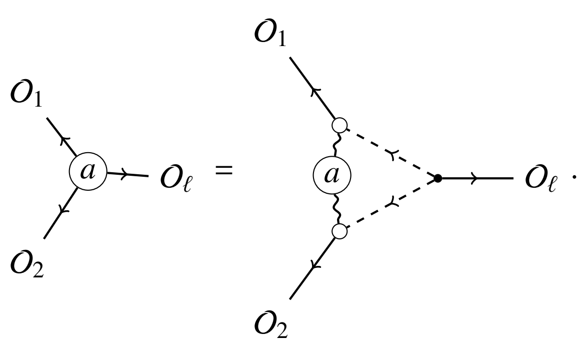
Here, the dashed lines denote scalar operators ϕ_1, ϕ_2. Note that the labels b, c determine the dimensions of ϕ_1, ϕ_2 in terms of $\Delta_{\mathcal{O}_1}, \Delta_{\mathcal{O}_2}$, respectively. Thus, the right-hand side will involve derivatives of scalar structures with dimensions shifted by half-/integers from those of $\mathcal{O}_1, \mathcal{O}_2$. In equations, we write

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_\ell \rangle^{(a)} = \mathscr{D}^{(a)\mathcal{O}_1\mathcal{O}_2}_{\phi_1\phi_2} \langle \phi_1 \phi_2 \mathcal{O}_\ell \rangle, \tag{4.141}$$

where $\mathscr{D}$ is a combination of derivatives $\partial_{x_1}, \partial_{x_2}$ and formal operators $\Sigma_{i,j} : \Delta_i \to \Delta_i + j$ that shift the dimensions Δ_1, Δ_2. We have suppressed $\mathrm{SO}(d)$ indices in (4.141) for simplicity.

The coefficients $(\ldots)$ expressing $\mathscr{D}^{(a)\mathcal{O}_1\mathcal{O}_2}_{\phi_1\phi_2}$ in terms of products of weight-shifting operators $\mathcal{D}^{(b)A}\mathcal{D}^{(c)}_A$ are determined by inverting (4.139). In writing (4.140), there are infinitely many possible choices of representation W and labels b, c. Generically, we expect that it should always be possible to choose enough W, b, c's to solve (4.140). This was shown explicitly in [61] when $\mathcal{O}_1, \mathcal{O}_2$ are traceless-symmetric tensors.[37]

For simplicity, we will sometimes write (4.140) as



(4.142)

[37]It would be interesting to characterize the minimal set of W's needed to build all possible structures.

4.4 Conformal blocks

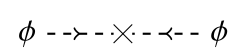
4.4.1 Gluing three-point functions

A general conformal block can be expressed as the integral of a product of three-point functions. For simplicity, consider the case where the external and internal operators are scalars. Given three-point functions $\langle\phi_1(x_1)\phi_2(x_3)\phi(x)\rangle$ and $\langle\phi(y)\phi_3(x_3)\phi_4(x_4)\rangle$, the following object is a solution to the conformal Casimir equation with the correct transformation properties to be a conformal block,

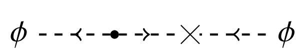
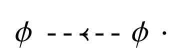
$$\frac{1}{\mathcal{N}_\Delta}\int d^d x\, d^d y \langle\phi_1(x_1)\phi_2(x_3)\phi(x)\rangle \frac{1}{(x-y)^{2(d-\Delta)}}\langle\phi(y)\phi_3(x_3)\phi_4(x_4)\rangle, \quad (4.143)$$

where $\Delta = \Delta_\phi$. This can be understood, for example, by writing the integral in a manifestly conformally-invariant way [54].[38],[39]

Let us denote the operation which glues two ϕ-correlators by[40]

$$|\phi\rangle \bowtie \langle\phi| \equiv \frac{1}{\mathcal{N}_\Delta}\int d^d x\, d^d y |\phi(x)\rangle \frac{1}{(x-y)^{2(d-\Delta)}}\langle\phi(y)| \quad = $$
 (4.144)

We should choose the normalization $\mathcal{N}_\Delta$ by demanding that

 =  (4.145)

That is, we demand that the shadow integral acting on a two-point function $\langle\phi\phi\rangle$ gives the identity transformation. In the case of scalars, this fixes the normalization factor to be [54, 63, 65]

$$\mathcal{N}_\Delta = \frac{\pi^d\Gamma(\Delta-\frac{d}{2})\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)\Gamma(d-\Delta)}. \quad (4.146)$$

---

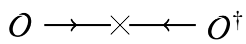
[38] In Euclidean signature, we take the range of integration of x, y to be all of $\mathbb{R}^d$. In this case (4.143) produces a solution to the conformal Casimir equation with the wrong boundary conditions to be a conformal block. However, the conformal block can be extracted by taking a suitable linear combination of analytic continuations of the integral [54]. One can alternatively isolate the conformal block by performing the integral in Lorentzian signature over a domain defined by the lightcones of the four points x_1, x_2, x_3, x_4 [203]. Calculations involving differential operators are insensitive to these issues because the differential operators always transform trivially under monodromy. Thus, our methods allow us to study spinning versions of any of the solutions to the Casimir equation.

[39] We expect that (4.143) only converges when Δ lies on the principal series $\Delta \in \frac{d}{2} + i\mathbb{R}$. We obtain a general conformal block by analytically continuing in Δ.

[40] Instead of thinking of the gluing operation (4.144) in terms of shadow integrals, we can alternatively think of it as simply a sum over normalized descendants of ϕ. The only properties of the gluing procedure that we use in this work are that it is bilinear, conformally-invariant, and satisfies the normalization condition (4.145).

For spinning operators, $\mathcal{O}$ glues to its dual-reflected representation $\mathcal{O}^\dagger$ — i.e., the representation with which $\mathcal{O}$ has a nonzero two-point function,

$$|\mathcal{O}_{\Delta,\rho}\rangle \bowtie \langle \mathcal{O}^\dagger_{\Delta,\rho^\dagger}| \equiv$$



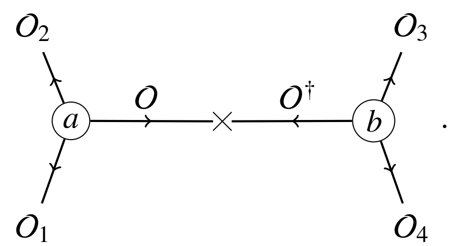
$$\equiv \frac{1}{\mathcal{N}_{\Delta,\rho}} \int d^d x\, d^d y |\mathcal{O}_{\Delta,\alpha}(x)\rangle \frac{t^{\alpha\overline{\alpha}}(x-y)}{(x-y)^{2(d-\Delta)}} \langle \mathcal{O}^\dagger_{\Delta,\overline{\alpha}}(y)|. \quad (4.147)$$

Here, $t^{\alpha\overline{\alpha}}(x-y)$ is the tensor structure appearing in the two point function of the shadow operators $\langle \widetilde{\mathcal{O}}\widetilde{\mathcal{O}}^\dagger\rangle$. We will not need the explicit expression, but simply the normalization condition

(4.148)

A general conformal block is given by

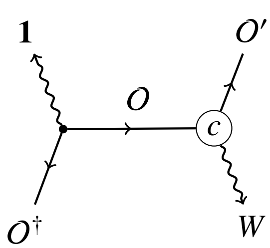
$$W^{ab} \equiv \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle^{(a)} \bowtie {}^{(b)}\langle \mathcal{O}^\dagger \mathcal{O}_3\mathcal{O}_4\rangle =$$



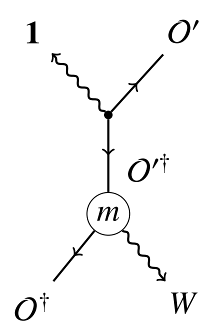
(4.149)

To perform computations with differential operators and shadow integrals, we must understand how to move differential operators from one side of a shadow integration to another — i.e., how to integrate by parts. This can be done purely diagrammatically, just from the definition (4.148).

First, consider a two-point function. Moving a differential operator past a two-point vertex is a special case of the definition of a $6j$ symbol,



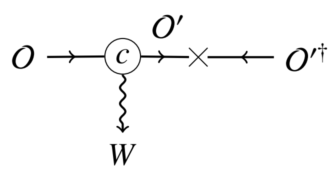
$$= \sum_m \left\{ \begin{matrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}'^\dagger \\ \mathcal{O}' & W & \mathcal{O} \end{matrix} \right\}^{\bullet c}_{\bullet m}$$



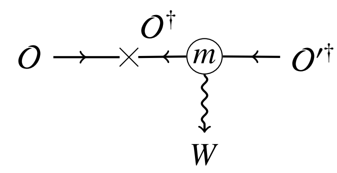
. (4.150)

A three-point vertex where one of the legs is the unit operator $\mathbf{1}$ is simply a two-point vertex. We could of course omit the unit operator from the above diagram, but we have temporarily included it to emphasize that (4.150) is a special case of (4.102).[41] Again, the notation "•" means there is a unique corresponding structure or differential operator.

Now, let us add shadow integrals onto both $\mathcal{O}$ and $\mathcal{O}'$ in the above diagram. Using (4.148), we find



$$= \sum_m \begin{Bmatrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}'^\dagger \\ \mathcal{O}' & W & \mathcal{O} \end{Bmatrix}^{\bullet c}_{\bullet m}$$



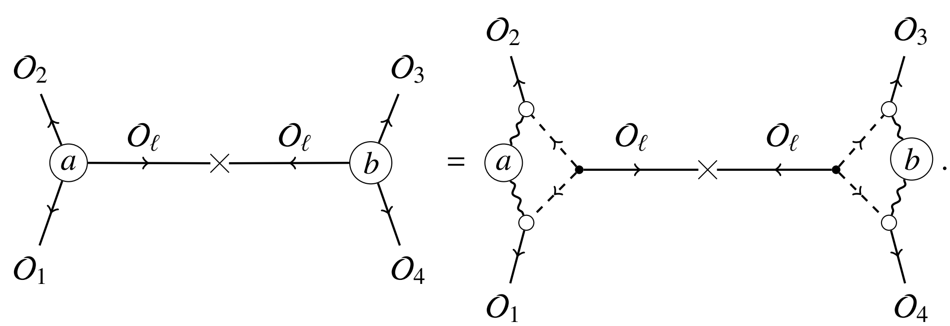
(4.151)

Equation (4.151) essentially implements two integrations by parts in the double integral (4.144), allowing us to move a differential operator from one side of a shadow integral to another. In symbolic notation it has the form

$$|\mathcal{D}^{(c)A}\mathcal{O}\rangle \bowtie \langle \mathcal{O}'^\dagger| = \sum_m \begin{Bmatrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}'^\dagger \\ \mathcal{O}' & W & \mathcal{O} \end{Bmatrix}^{\bullet c}_{\bullet m} |\mathcal{O}\rangle \bowtie \langle \mathcal{D}^{(m)A}\mathcal{O}'^\dagger|. \tag{4.152}$$

4.4.2 Spinning conformal blocks review

The expression (4.149) for a general block can be combined with the "differential basis" trick (4.140) to express certain conformal blocks as derivatives of scalar blocks [61]. Suppose the exchanged operator $\mathcal{O} = \mathcal{O}_\ell$ is a traceless-symmetric tensor of spin ℓ. Applying (4.140) twice, we find



(4.153)

Note that the right-hand side is a differential operator acting on conformal blocks with external scalars. In equations (4.153) reads

$$G^{(a,b)\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}_{\Delta,\ell}(x_i) = \mathscr{D}^{(a)\mathcal{O}_1\mathcal{O}_2}_{\phi_1\phi_2}\mathscr{D}^{(b)\mathcal{O}_3\mathcal{O}_4}_{\phi_3\phi_4} G^{\phi_1\phi_2\phi_3\phi_4}_{\Delta,\ell}(x_i). \tag{4.154}$$

[41] To be precise, we have established (4.102) only for non-degenerate operators $\mathcal{O}_i$. However, as explained in section 4.3.2.3, the objects on either side of (4.150) span the space of covariant two-point functions, which provides the missing ingredient.

The objects in (4.154) and (4.141) carry SO(d) indices which we have suppressed for simplicity.

Note that symmetric traceless tensors (STTs) are the only representations that can appear in an OPE of two scalars. Because $\mathscr{D}^{(a)\mathcal{O}_i\mathcal{O}_j}_{\phi_i\phi_j}$ can't change the representation of the exchanged operator, the expression (4.154) only works for conformal blocks with an exchanged STT. This is sufficient to compute all bosonic blocks in 3d, since all bosonic (irreducible) 3d Lorentz representations are STTs. However, in general there exist blocks which cannot be computed using (4.154).

To compute more general blocks, an approach advocated in [54, 61] is to identify the simplest set of blocks with general exchanged representations — so-called "seed" blocks — compute them using some other method and apply the trick (4.154) to those.[42] However, our new techniques will make it simple to modify (4.153) and (4.154) to compute any type of conformal block (including seed blocks).

4.4.3 Expression for general conformal blocks

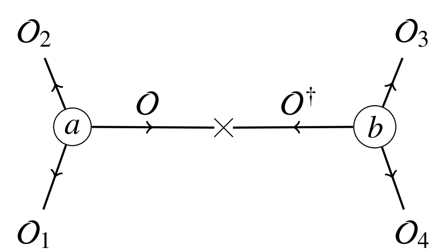
The basic idea is to allow the differential operators acting on the left and right to be conformally-covariant, instead of simply invariant,

$$G_{\mathcal{O}}^{(a,b)\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}(x_i) = \mathscr{D}^{(a)A}_{\text{left}}\mathscr{D}^{(b)}_{\text{right}\,A}G^{\phi_1\phi_2\phi_3\phi_4}_{\Delta,\ell}(x_i), \tag{4.155}$$

where A is an index for some finite-dimensional representation W of SO$(d+1,1)$. The exchanged operator then lives in the tensor product $W \otimes V_{\Delta,\ell}$, which can contain primaries with more general Lorentz representations. We must be careful to choose $\mathscr{D}^{(a)A}_{\text{left}}$ and $\mathscr{D}^{(b)}_{\text{right}\,A}$ so that precisely one irreducible subrepresentation of $W \otimes V_{\Delta,\ell}$ contributes. However, this can be done easily and systematically using the techniques we have developed.

Let us begin with the object we would like to compute: a conformal block for the exchange of an operator $\mathcal{O}$ transforming in $V_{\Delta,\rho}$,

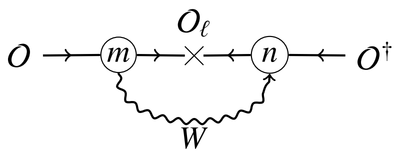
$$G_{\mathcal{O}}^{(a,b)\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}(x_i) =$$



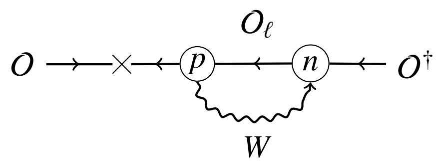
. (4.156)

[42] Seed blocks for 4d theories were classified in [62] and computed in [58] using the Casimir equation. In 3d, there are two types of seed blocks: external scalars with exchange of spin $\ell \in \mathbb{Z}$, and external fermion+scalars with exchange of spin-$\ell \in \mathbb{Z} + \frac{1}{2}$. A recursion relation for the latter type of 3d seed block was computed in [81].

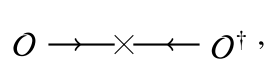
Let W be a finite-dimensional representation of the conformal group such that $W^* \otimes V_{\Delta,\rho}$ contains a spin-ℓ STT representation $\mathcal{O}_\ell$. We can introduce a bubble of W and $\mathcal{O}_\ell$ in the middle of the diagram, so that the shadow integral itself involves a spin-ℓ representation. Note that



$$= \sum_p \begin{Bmatrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}_\ell \\ \mathcal{O}_\ell & W & \mathcal{O} \end{Bmatrix}^{\bullet m}_{\bullet p}$$



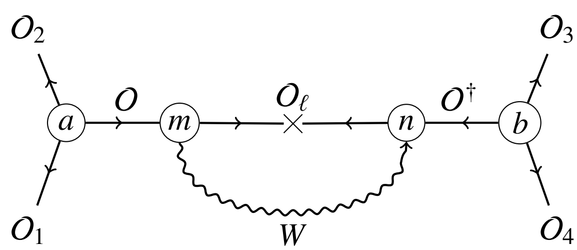
$$= \sum_p \begin{Bmatrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}_\ell \\ \mathcal{O}_\ell & W & \mathcal{O} \end{Bmatrix}^{\bullet m}_{\bullet p} \begin{pmatrix} \mathcal{O}^\dagger \\ \mathcal{O}_\ell W \end{pmatrix}^{pn}$$



$$, \tag{4.157}$$

where we have used (4.151) to move the differential operator $\mathcal{D}^{(m)A}$ from one side of the shadow integral to the other, and (4.138) to simplify a product of differential operators $\mathcal{D}^{(p)A}\mathcal{D}^{(n)}_A$ on a single leg. Thus, we have

$$G^{(a,b)\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}_{\mathcal{O}}(x_i) = \frac{1}{M_{mn}}$$



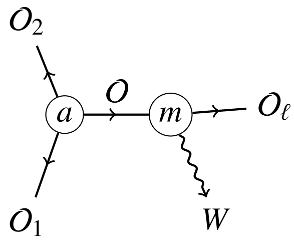
$$, \tag{4.158}$$

where

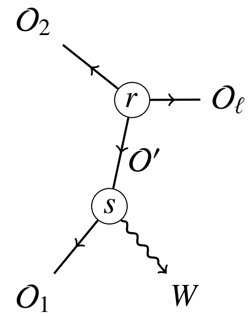
$$M_{mn} \equiv \sum_p \begin{Bmatrix} \mathcal{O}^\dagger & \mathbf{1} & \mathcal{O}_\ell \\ \mathcal{O}_\ell & W & \mathcal{O} \end{Bmatrix}^{\bullet m}_{\bullet p} \begin{pmatrix} \mathcal{O}^\dagger \\ \mathcal{O}_\ell W \end{pmatrix}^{pn}. \tag{4.159}$$

We do not sum over m, n in (4.158) — rather we can choose any m, n such that M_{mn} is nonzero.

Now we use crossing to move the W vertices to the external legs. Let us focus on the left-hand side of the diagram (4.158),

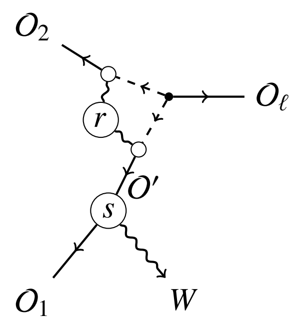


$$= \sum_{\mathcal{O}',r,s} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}' \\ \mathcal{O}_\ell & W & \mathcal{O} \end{Bmatrix}^{am}_{rs} \tag{4.160}$$



Now $\mathcal{O}_2$ and $\mathcal{O}'$ participate in a three-point vertex with an STT operator $\mathcal{O}_\ell$, so we can use (4.140) to obtain

$$= \sum_{\mathcal{O}',r,s} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}' \\ \mathcal{O}_\ell & W & \mathcal{O} \end{Bmatrix}^{am}_{rs} \tag{4.161}$$



.

Thus, we find

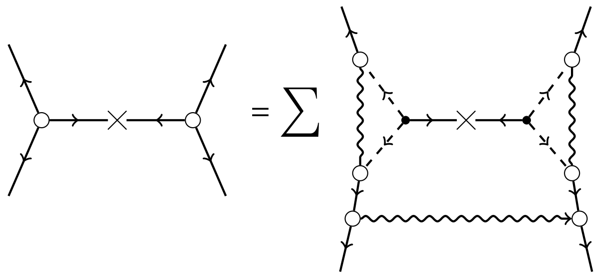
$$\begin{aligned} \mathscr{D}^{(a)A}_{\text{left}} \langle \phi_1 \phi_2 \mathcal{O}_\ell \rangle &= \frac{1}{\sqrt{M_{mn}}} \mathcal{D}^{(m)A}_x \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}(x) \rangle^{(a)}, \\ \text{where} \quad \mathscr{D}^{(a)A}_{\text{left}} &\equiv \frac{1}{\sqrt{M_{mn}}} \sum_{\mathcal{O}',r,s} \begin{Bmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}' \\ \mathcal{O}_\ell & W & \mathcal{O} \end{Bmatrix}^{am}_{rs} \mathcal{D}^{(s)A}_{x_1} \mathscr{D}^{(r)\mathcal{O}'\mathcal{O}_2}_{\phi_1\phi_2}, \end{aligned} \tag{4.162}$$

where the x subscript indicates that $\mathcal{D}^{(m)A}_x$ acts on the operator $\mathcal{O}(x)$. Similarly,

$$\begin{aligned} \mathscr{D}^{(b)}_{\text{right}\,A} \langle \phi_4 \phi_3 \mathcal{O}_\ell \rangle &= \frac{1}{\sqrt{M_{mn}}} \mathcal{D}^{(n)}_{A,x} \langle \mathcal{O}_4 \mathcal{O}_3 \mathcal{O}^\dagger(x) \rangle^{(b)} \\ \mathscr{D}^{(b)}_{\text{right}\,A} &\equiv \frac{1}{\sqrt{M_{mn}}} \sum_{\mathcal{O}',t,u} \begin{Bmatrix} \mathcal{O}_4 & \mathcal{O}_3 & \mathcal{O}' \\ \mathcal{O}_\ell & W^* & \mathcal{O}^\dagger \end{Bmatrix}^{bn}_{tu} \mathcal{D}^{(u)}_{x_4,A} \mathscr{D}^{(t)\mathcal{O}'\mathcal{O}_3}_{\phi_4\phi_3}. \end{aligned} \tag{4.163}$$

Together with (4.159), this gives (4.155).

Schematically, applying $\mathscr{D}^{(a)A}_{\text{left}} \mathscr{D}^{(b)}_{\text{right}\,A}$ to a scalar block results in a graph with the topology



(4.164)

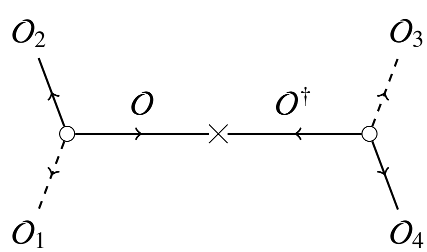
The inner object is a conformal block for external scalars (dashed lines). Weight-shifting operators dress it in a way such that (a component of) the tensor $W \otimes \mathcal{O}_\ell$ propagates from left to right.

The above calculation has the advantage of being extremely general. However, it requires us to make non-canonical choices of W and the differential operators m, n. Different choices for these objects will result in naively different, but equivalent expressions for our conformal block in terms of derivatives of scalar blocks. In some cases, to obtain the simplest possible expression, we may want to proceed slightly differently.

4.4.4 Expression for seed blocks

Let us consider for example the problem of computing the seed blocks. For simplicity of discussion, we will restrict to the case of even d. The case of odd d can be analyzed similarly[43] (for example, we construct the 3d seed block in section 4.4.4.1).

As mentioned above, seed blocks are the simplest conformal blocks that exchange a primary $\mathcal{O}$ in a given SO(d) representation. In particular, we can always choose the external operators in a way such that there exists a single three-point structure on either side of the block, for example

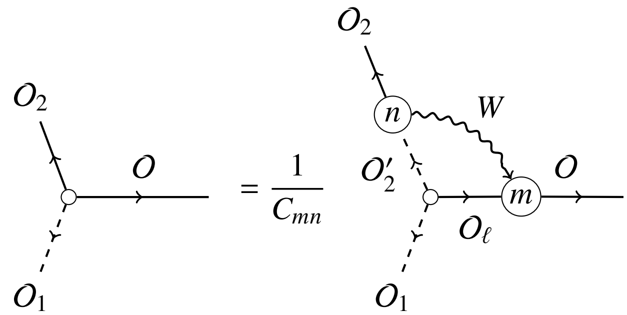


, (4.165)

[43] The complication in the case of odd d is that when $\mathcal{O}$ is a fermion, we cannot choose the external operators so that there is a single tensor structure on each side of the seed block. Instead, the minimum is two. This is related to the fact that the irreducible fermionic representations of SO($d-1$) are necessarily chiral when d is odd.

where $\mathcal{O}_1$ and $\mathcal{O}_3$ are scalars, while $\mathcal{O}_2$ and $\mathcal{O}_4$ transform in representations which are obtained from that of $\mathcal{O}$ by, for example, removing the first row of the SO(d) Young diagram.

To express this seed block in terms of scalar blocks, let us first focus on the left three-point structure. We can write



, (4.166)

where due to the uniqueness of the tensor structures, we are free to choose n, m and W as long as $\mathcal{O}_2'$ is a scalar and $\mathcal{O}_\ell$ is a STT. In what follows, we will perform manipulations with the operator labeled by m, but we will leave n untouched. For this reason, it is convenient to choose W and n so that n is a 0-th order differential operator. According to theorem 2, this means that the primary of W^* should transform in the same representation as $\mathcal{O}_2$, i.e., $(W^*)_{-j} = (W_j)^* = \rho_2$, where ρ_i is the SO(d) representation of $\mathcal{O}_i$.[44] On the other hand, the condition for existence of the structure on the left is

$$(\rho \otimes \rho_2)^{\mathrm{SO}(d-1)} \neq 0, \tag{4.167}$$

where ρ is the representation of $\mathcal{O}$. This is equivalent to saying that there is a STT in the tensor product $\rho \otimes \rho_2 = \rho \otimes (W_j)^*$. In turn, this leads to

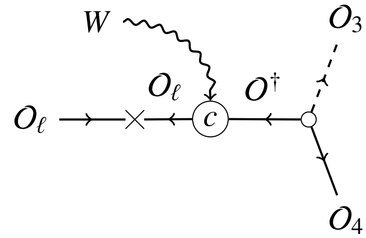
$$\rho \in \mathrm{STT} \otimes W_j. \tag{4.168}$$

According to theorem 2, this implies that we can use an order-$(2j+1)$ differential operator associated to W^* in place of m.

[44]Such a W^* always exists. In fact, there are infinitely many choices differing by the value of j, and the W^* with minimal j is obtained by prepending a 0 to the list of Dynkin labels of ρ_2 (in the natural ordering where the vector label is the first and the spinor labels are the last).

We can now use (4.151) to move m to the right three-point structure to find the piece

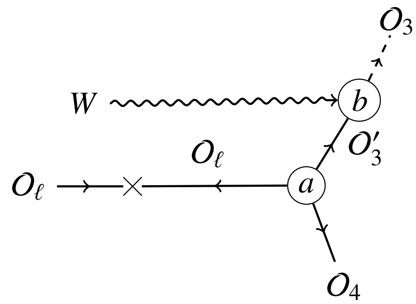
$$\sum_c \begin{Bmatrix} \mathcal{O}_\ell & \mathbf{1} & \mathcal{O}^\dagger \\ \mathcal{O} & W^* & \mathcal{O}_\ell \end{Bmatrix}_{\bullet c}^{\bullet m}$$



$$, \tag{4.169}$$

to which we can apply a crossing transformation to find

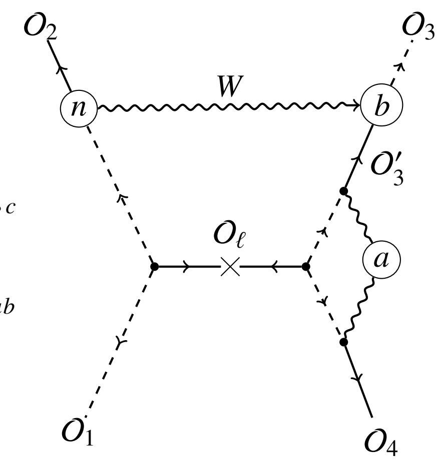
$$= \sum_c \sum_{\mathcal{O}_3', a, b} \begin{Bmatrix} \mathcal{O}_\ell & \mathbf{1} & \mathcal{O}^\dagger \\ \mathcal{O} & W^* & \mathcal{O}_\ell \end{Bmatrix}_{\bullet c}^{\bullet m} \begin{Bmatrix} \mathcal{O}_3 & \mathcal{O}_4 & \mathcal{O}_3' \\ \mathcal{O}_\ell & W^* & \mathcal{O}^\dagger \end{Bmatrix}_{ab}^{\bullet c}$$



$$. \tag{4.170}$$

We now use (4.140) to write the full seed block as

$$\frac{1}{C_{mn}} \sum_c \sum_{\mathcal{O}_3', a, b} \begin{Bmatrix} \mathcal{O}_\ell & \mathbf{1} & \mathcal{O}^\dagger \\ \mathcal{O} & W^* & \mathcal{O}_\ell \end{Bmatrix}_{\bullet c}^{\bullet m} \begin{Bmatrix} \mathcal{O}_3 & \mathcal{O}_4 & \mathcal{O}_3' \\ \mathcal{O}_\ell & W^* & \mathcal{O}^\dagger \end{Bmatrix}_{ab}^{\bullet c}$$



$$. \tag{4.171}$$

The advantage of this over the more general (4.164) is that we have been able to choose the differential operator n to be of zeroth order, and we also avoided acting with differential operators on one of the legs. This reduces the order of the full differential operator acting on the scalar conformal block relative to the general expression. Let us now consider some examples.

4.4.4.1 Example: seed block in 3d

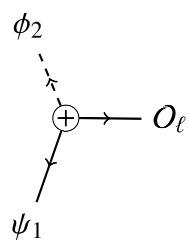
Our first example is the fermion seed block in 3 dimensions. The SO(3) representations are labeled by a single (half-)integer ℓ. If ℓ is integral, then the representation is bosonic, and operators O_ℓ can be exchanged in a four-point function of scalars. If ℓ is half-integral, then the representation is fermionic and O_ℓ can be exchanged in a scalar-fermion four-point function[45]

$$\langle \psi_{\Delta_1}(s_1, x_1)\phi_{\Delta_2}(x_2)\phi_{\Delta_3}(x_3)\psi_{\Delta_4}(s_4, x_4)\rangle. \tag{4.172}$$

It is therefore possible to express any conformal block in terms of a scalar or fermion-scalar block. The latter were computed in [81] by a Zamolodchikov type recursion relation. In this section we will show how the fermion-scalar block can be expressed as a third-order differential operator acting on a scalar conformal block, thus reducing all conformal blocks in 3d to derivatives of scalar blocks.

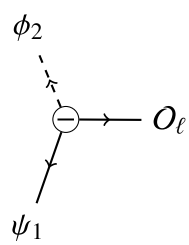
For ease of comparison, we will follow the conventions of [81]. Let us review basic properties of (4.172). On each side of conformal block there exist 2 three-point structures, which can be defined using the 5d embedding formalism as

$$\langle \psi_{\Delta_1}\phi_{\Delta_2}O_{\Delta,\ell}\rangle^{(+)} =$$



$$= \frac{\langle S_1 S_0\rangle\langle S_0 X_1 X_2 S_0\rangle^{\ell-\frac{1}{2}}}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+\ell-\frac{1}{2}}{2}} X_{20}^{\frac{\Delta_2+\Delta-\Delta_1+\ell-\frac{1}{2}}{2}} X_{01}^{\frac{\Delta_1+\Delta-\Delta_2+\ell+\frac{1}{2}}{2}}},$$

$$\langle \psi_{\Delta_1}\phi_{\Delta_2}O_{\Delta,\ell}\rangle^{(-)} =$$



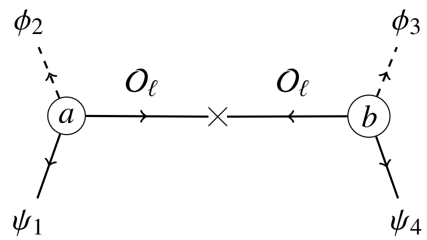
$$= \frac{\langle S_1 X_2 S_0\rangle\langle S_0 X_1 X_2 S_0\rangle^{\ell-\frac{1}{2}}}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+\ell+\frac{1}{2}}{2}} X_{20}^{\frac{\Delta_2+\Delta-\Delta_1+\ell+\frac{1}{2}}{2}} X_{01}^{\frac{\Delta_1+\Delta-\Delta_2+\ell-\frac{1}{2}}{2}}}, \tag{4.173}$$

and analogously for the right three-point function ($1 \to 4$, $2 \to 3$). Here the index 0 refers to the intermediate operator O_ℓ of dimension Δ, and we labeled the three-point structures by their P-parity. Accordingly, there exist 4 conformal blocks, which can

[45] Since our analysis is purely kinematical, we will label operators by their scaling dimensions and spins.

be expanded in a basis of four-point tensor structures,

$$G^{ab}_{\text{seed}}(s_1, s_4, x_i) =$$

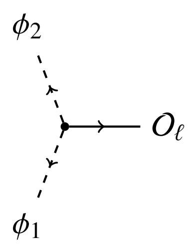


$$= \sum_{I=1}^{4} g_I^{ab}(z, \bar{z})\mathbb{T}_4^I(s_1, s_4, x_i). \tag{4.174}$$

As indicated, there exist 4 four-point tensor structures $\mathbb{T}_4^I$. Out of them, two structures are parity-even and participate in conformal blocks G^{++}, G^{--}, and two are parity-odd and participate in G^{+-} and G^{-+}. We give their exact form in appendix C.5.

We now compute the seed blocks using the algorithm[46] from section 4.4.4, and we will use the spinor representation $W = \mathcal{S}$ of the 3d conformal group to translate traceless-symmetric representations into fermionic representations. The first step is to write the left three-point structures in the form (4.166). Let us define the scalar three-point structures as

$$\langle \phi_{\Delta_1}\phi_{\Delta_2}O_{\Delta,\ell}\rangle =$$



$$= \frac{\langle S_0 X_1 X_2 S_0\rangle^\ell}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+\ell}{2}} X_{20}^{\frac{\Delta_2+\Delta-\Delta_1+\ell}{2}} X_{01}^{\frac{\Delta_1+\Delta-\Delta_2+\ell}{2}}}. \tag{4.175}$$

In (4.166) we will use the zeroth order operator $\mathcal{D}_a^{-+}$ in place of n. For m we can take any differential operator of the appropriate parity. A simple choice is to use $\mathcal{D}_a^{-+}$ for the parity even structure, and $\mathcal{D}_a^{--}$ for the parity-odd structure. We then have

$$\langle \psi_{\Delta_1}\phi_{\Delta_2}O_{\Delta,\ell}\rangle^{(\pm)} = \frac{1}{C_\pm}\langle \mathcal{D}_1^{-+}\mathcal{D}_0^{-\pm}\rangle\langle \phi_{\Delta_1+\frac{1}{2}}\phi_{\Delta_2}O_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}\rangle. \tag{4.176}$$

It is easy to find by a direct computation that

$$C_+ = 1, \qquad C_- = 2\ell + 1. \tag{4.177}$$

Note that $O_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}$ is the operator which is going to be exchanged in the scalar block. If we chose different operators for m (i.e., $\mathcal{D}^{+\mp}$) in (4.166), then we would relate the seed block to different scalar blocks (in particular, it doesn't make sense to mix these choices).

[46] Because we want to follow the conventions of [81], some minor modifications to the algorithm are required, such as reordering of the operators.

Crossing of 2-point functions The next step is to learn how to push the operators $\mathcal{D}_a^{-\pm}$ through the shadow integral. For that we need to fix the normalization of two-point functions, which we choose to be

$$\langle \mathcal{O}_{\Delta,\ell}(S_1,X_1)\mathcal{O}_{\Delta,\ell}(S_2,X_2)\rangle = i^{2\ell}\frac{\langle S_1S_2\rangle^{2\ell}}{X_{12}^{\Delta+\ell}}. \tag{4.178}$$

The definition of $6j$ symbols (4.150) is in our case

$$\begin{aligned}&\mathcal{D}_{2,a}^{-\pm}\langle \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}(S_1,X_1)\mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}(S_2,X_2)\rangle =\\ &= \begin{Bmatrix} \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}} & \mathbf{1} & \mathcal{O}_{\Delta,\ell} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}} \end{Bmatrix}^{\bullet(-\pm)}_{\bullet(+\mp)} \mathcal{D}_{1,a}^{+\mp}\langle \mathcal{O}_{\Delta,\ell}(S_1,X_1)\mathcal{O}_{\Delta,\ell}(S_2,X_2)\rangle.\end{aligned} \tag{4.179}$$

We can explicitly compute

$$\begin{Bmatrix} \mathcal{O}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}} & \mathbf{1} & \mathcal{O}_{\Delta,\ell} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}} \end{Bmatrix}^{\bullet(-+)}_{\bullet(+-)} = \frac{i}{8\ell(\Delta-1)(\Delta-\ell-1)}, \tag{4.180}$$

$$\begin{Bmatrix} \mathcal{O}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}} & \mathbf{1} & \mathcal{O}_{\Delta,\ell} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}} \end{Bmatrix}^{\bullet(--)}_{\bullet(++)} = \frac{i(2\ell+1)}{4(\Delta-1)(\Delta+\ell)}, \tag{4.181}$$

and use these coefficients in (4.151) to arrive at (4.169). At this point, we have expressed the seed block in the form

$$\begin{aligned}&G^{\pm b}_{\text{seed}}(s_1,s_4,x_i) =\\ &= \frac{1}{C_\pm}\begin{Bmatrix} \mathcal{O}_{\ell\mp\frac{1}{2}} & \mathbf{1} & \mathcal{O}_{\ell} \\ \mathcal{O}_{\ell} & \mathcal{S} & \mathcal{O}_{\ell\mp\frac{1}{2}} \end{Bmatrix}^{\bullet(-\pm)}_{\bullet(+\mp)} \Omega^{cd}\mathcal{D}_{1,c}^{-+}\langle\phi_{\Delta_1+\frac{1}{2}}\phi_{\Delta_2}\mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}\rangle \bowtie \mathcal{D}_{0,d}^{+\mp}\langle \mathcal{O}_{\Delta,\ell}\phi_{\Delta_3}\psi_{\Delta_4}\rangle^{(b)},\end{aligned} \tag{4.182}$$

where $\bowtie$ stands for shadow integral.

Crossing of three-point functions Now we are going to perform the crossing transformation on the right three-point function to write it as

$$\mathcal{D}^{+\mp}_{0,d}\langle \mathcal{O}_{\Delta,\ell}\phi_{\Delta_3}\psi_{\Delta_4}\rangle^{(b)} = \sum_{b'} \begin{Bmatrix} \phi_{\Delta_3} & \psi_{\Delta_4} & \psi_{\Delta_3+\frac{1}{2}} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}} \end{Bmatrix}^{b(+\mp)}_{b'(--)} \mathcal{D}^{--}_{3,d}\langle \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}\psi_{\Delta_3+\frac{1}{2}}\psi_{\Delta_4}\rangle^{(b')}$$
$$+ \begin{Bmatrix} \phi_{\Delta_3} & \psi_{\Delta_4} & \psi_{\Delta_3-\frac{1}{2}} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}} \end{Bmatrix}^{b(+\mp)}_{b'(+-)} \mathcal{D}^{+-}_{3,d}\langle \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}\psi_{\Delta_3-\frac{1}{2}}\psi_{\Delta_4}\rangle^{(b')}. \tag{4.183}$$

To proceed, we need to choose a basis of tensor structures for three-point functions of the type $\langle \mathcal{O}_{\Delta+\frac{1}{2},\ell\mp\frac{1}{2}}\psi_{\Delta_3\pm\frac{1}{2}}\psi_{\Delta_4}\rangle$. We define

$$t_1 = \frac{\langle S_1 S_2\rangle\langle S_3 X_1 X_2 S_3\rangle^{\ell}}{X_{12}^{\ell}} + \frac{\langle S_1 S_3\rangle\langle S_2 S_3\rangle\langle S_3 X_1 X_2 S_3\rangle^{\ell-1}}{X_{12}^{\ell-1}}, \tag{4.184}$$

$$t_2 = \frac{\langle S_1 S_2\rangle\langle S_3 X_1 X_2 S_3\rangle^{\ell}}{X_{12}^{\ell}} + 2\frac{\langle S_1 S_3\rangle\langle S_2 S_3\rangle\langle S_3 X_1 X_2 S_3\rangle^{\ell-1}}{X_{12}^{\ell-1}}, \tag{4.185}$$

$$t_3 = \frac{\langle S_3 X_1 X_2 S_3\rangle^{\ell-1}}{X_{12}^{\ell+\frac{1}{2}} X_{23}^{-\frac{1}{2}} X_{31}^{-\frac{1}{2}}} X_{23}\langle S_1 S_3\rangle\langle S_2 X_1 S_3\rangle, \tag{4.186}$$

$$t_4 = \frac{\langle S_3 X_1 X_2 S_3\rangle^{\ell-1}}{X_{12}^{\ell+\frac{1}{2}} X_{23}^{-\frac{1}{2}} X_{31}^{-\frac{1}{2}}} X_{13}\langle S_2 S_3\rangle\langle S_1 X_2 S_3\rangle, \tag{4.187}$$

where the first two structures are parity-even and the second two are parity-odd.[47] In terms of these structures we set

$$\langle \psi_1\psi_2\mathcal{O}_\ell\rangle^{(b)} = \frac{t_b}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3-\ell+1}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+\ell}{2}} X_{31}^{\frac{\Delta_3+\Delta_1-\Delta_3+\ell}{2}}}. \tag{4.188}$$

[47] We choose this peculiar basis only for the purposes of presentation, because in it $6j$ symbols have the simplest form. In practice we used the basis (4.119), in which we know the general $6j$ symbols for the spinor representation.

We can now compute the $6j$ symbols in (4.183). For example, the only non-vanishing symbols for $b = +$ and $\mathcal{D}^{++}$ on the left of (4.183) are

$$\begin{Bmatrix} \phi_{\Delta_3} & \psi_{\Delta_4} & \psi_{\Delta_3+\frac{1}{2}} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}} \end{Bmatrix}^{+(++)}_{1(--)} = (-1)^{\ell+\frac{1}{2}} \frac{(\Delta-\frac{3}{2})(\Delta+\ell+\Delta_3-\Delta_4-\frac{1}{2})(\Delta+\ell+\Delta_3+\Delta_4-\frac{3}{2})}{\Delta_3-\frac{3}{2}}, \tag{4.189}$$

$$\begin{Bmatrix} \phi_{\Delta_3} & \psi_{\Delta_4} & \psi_{\Delta_3+\frac{1}{2}} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}} \end{Bmatrix}^{+(++)}_{2(--)}] = (-1)^{\ell+\frac{1}{2}} \frac{(\Delta+\ell+\Delta_3-\Delta_4-\frac{1}{2})((\Delta-1)(\Delta+\ell+\Delta_3+\Delta_4-\frac{5}{2})-\frac{1}{2})}{\Delta_3-\frac{3}{2}}, \tag{4.190}$$

$$\begin{Bmatrix} \phi_{\Delta_3} & \psi_{\Delta_4} & \psi_{\Delta_3-\frac{1}{2}} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}} \end{Bmatrix}^{+(++)}_{3(+-)} = (-1)^{\ell+\frac{1}{2}} \frac{(\Delta+\ell+\Delta_3-\Delta_4-\frac{1}{2})}{4(\Delta_3-\frac{3}{2})(\Delta_3-2)}, \tag{4.191}$$

$$\begin{Bmatrix} \phi_{\Delta_3} & \psi_{\Delta_4} & \psi_{\Delta_3-\frac{1}{2}} \\ \mathcal{O}_{\Delta,\ell} & \mathcal{S} & \mathcal{O}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}} \end{Bmatrix}^{+(++)}_{4(+-)} = (-1)^{\ell+\frac{1}{2}} \frac{(\Delta-1)(\Delta+\ell-\Delta_3+\Delta_4+\frac{1}{2})}{2(\Delta_3-\frac{3}{2})(\Delta_3-2)}. \tag{4.192}$$

The other symbols vanish due to space parity. The are 12 more non-vanishing $6j$ symbols for other choices of b and of the operator on the left, which we won't list here since they represent only an intermediate step in our calculation.

Differential basis The final step is to express the three-point structures

$$\langle \mathcal{O}_{\Delta+\frac{1}{2},\ell\pm\frac{1}{2}} \psi_{\Delta_3\pm\frac{1}{2}} \psi_{\Delta_4} \rangle^{(b)} \tag{4.193}$$

in terms of derivatives acting on scalar thee-point structures. This is standard, and this particular case was solved in [39], so we do not explain it in detail. We only note that the operators which create the parity-even structures t_1 and t_2 should be parity even,

$$t_1,\ t_2 \sim \langle \mathcal{D}_3^{++}\mathcal{D}_4^{++}\rangle,\ \langle \mathcal{D}_3^{-+}\mathcal{D}_4^{-+}\rangle, \tag{4.194}$$

while operators which create parity-odd structures have to be parity-odd,

$$t_3,\ t_4 \sim \langle \mathcal{D}_3^{-+}\mathcal{D}_4^{++}\rangle,\ \langle \mathcal{D}_3^{++}\mathcal{D}_4^{-+}\rangle. \tag{4.195}$$

The recursion relation Assembling everything together, we arrive at the following expressions for the seed blocks in terms of third-order differential operators acting on scalar blocks,

$$\begin{aligned}
G^{++}_{\text{seed}}, G^{--}_{\text{seed}} =& v_1 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{--}_3 \rangle \langle \mathcal{D}^{-+}_3 \mathcal{D}^{++}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3+1} \phi_{\Delta_4-\frac{1}{2}} \rangle \\
&+ v_2 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{--}_3 \rangle \langle \mathcal{D}^{++}_3 \mathcal{D}^{-+}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3} \phi_{\Delta_4+\frac{1}{2}} \rangle \\
&+ v_3 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{+-}_3 \rangle \langle \mathcal{D}^{++}_3 \mathcal{D}^{++}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3-1} \phi_{\Delta_4-\frac{1}{2}} \rangle \\
&+ v_4 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{+-}_3 \rangle \langle \mathcal{D}^{-+}_3 \mathcal{D}^{-+}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3} \phi_{\Delta_4+\frac{1}{2}} \rangle,
\end{aligned} \tag{4.196}$$

$$\begin{aligned}
G^{+-}_{\text{seed}}, G^{-+}_{\text{seed}} =& v_1 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{--}_3 \rangle \langle \mathcal{D}^{++}_3 \mathcal{D}^{++}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3} \phi_{\Delta_4-\frac{1}{2}} \rangle \\
&+ v_2 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{--}_3 \rangle \langle \mathcal{D}^{-+}_3 \mathcal{D}^{-+}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3+1} \phi_{\Delta_4+\frac{1}{2}} \rangle \\
&+ v_3 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{+-}_3 \rangle \langle \mathcal{D}^{-+}_3 \mathcal{D}^{++}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3} \phi_{\Delta_4-\frac{1}{2}} \rangle \\
&+ v_4 \langle \mathcal{D}^{-+}_1 \mathcal{D}^{+-}_3 \rangle \langle \mathcal{D}^{++}_3 \mathcal{D}^{-+}_4 \rangle \langle \phi_{\Delta_1+\frac{1}{2}} \phi_{\Delta_2} \phi_{\Delta_3-1} \phi_{\Delta_4+\frac{1}{2}} \rangle.
\end{aligned} \tag{4.197}$$

The coefficients v_i are different for each of the blocks, and we give the explicit expressions in appendix C.5. The scalar blocks in the above expressions for $G^{+\pm}_{\text{seed}}$ correspond to exchange of $[\Delta + \frac{1}{2}, \ell - \frac{1}{2}]$, while for $G^{-\pm}_{\text{seed}}$ the exchanged primary is $[\Delta + \frac{1}{2}, \ell + \frac{1}{2}]$.

Decomposition into components Note that the scalar conformal blocks have the form

$$\langle \phi_{\Delta_1} \phi_{\Delta_2} \phi_{\Delta_3} \phi_{\Delta_4} \rangle = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{14}^2}{x_{24}^2} \right)^{\alpha} \left(\frac{x_{14}^2}{x_{13}^2} \right)^{\beta} G^{\alpha,\beta}_{\Delta,\ell}(z, \bar{z}), \tag{4.198}$$

where $\alpha = -\frac{1}{2}\Delta_{12}$, $\beta = \frac{1}{2}\Delta_{34}$, and depend essentially only on α and β and not the individual dimensions Δ_i. We then see that e.g. for G^{++}_{seed} we only need the scalar blocks $G^{\alpha-\frac{1}{4},\beta-\frac{1}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}$ and $G^{\alpha-\frac{1}{4},\beta+\frac{3}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}$. There exists a second-order differential operator (see [64] and section 4.4.5) which relates these two blocks,

$$G^{\alpha-\frac{1}{4},\beta+\frac{3}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}(z, \bar{z}) \sim (\partial_z \partial_{\bar{z}} + \ldots) G^{\alpha-\frac{1}{4},\beta-\frac{1}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}(z, \bar{z}). \tag{4.199}$$

In (4.196) only a first order operator acts on $G^{\alpha-\frac{1}{4},\beta+\frac{3}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}$, and thus we can use (4.199) to reduce (4.196) to another third-order operator acting on the single scalar block.

In particular, we can write

$$G^{++}_{\text{seed}} = g_1^{++}(z,\bar{z})\frac{[-\frac{1}{2},0,0,-\frac{1}{2}]+[\frac{1}{2},0,0,\frac{1}{2}]}{2} + g_2^{++}(z,\bar{z})\frac{[-\frac{1}{2},0,0,\frac{1}{2}]+[\frac{1}{2},0,0,-\frac{1}{2}]}{2}, \tag{4.200}$$

where the tensor structures are defined in appendix C.5 and

$$g_k^{++}(z,\bar{z}) = \frac{i(-1)^{\ell-\frac{1}{2}}}{\ell(\Delta-\ell-1)(\Delta-1)}(z\bar{z})^{-\frac{\Delta_1+\Delta_2+\frac{1}{2}}{2}}\mathfrak{D}_k^{++}G^{\alpha-\frac{1}{4},\beta-\frac{1}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}(z,\bar{z}). \tag{4.201}$$

The differential operators $\mathfrak{D}_k^{++}$ are given by[48]

$$\begin{aligned}
\mathfrak{D}_1^{++}(z,\bar{z}) =& \bar{z}\partial_{\bar{z}}D_z - z\partial_z D_{\bar{z}} - (z\partial_z - \bar{z}\partial_{\bar{z}})\frac{z\bar{z}}{2(z-\bar{z})}\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big) \\
&+ \frac{(\Delta-\ell)(\Delta-\ell-3)}{4}(z\partial_z - \bar{z}\partial_{\bar{z}}) + \frac{\Delta-\ell-3}{2}(D_z - D_{\bar{z}}),
\end{aligned} \tag{4.202}$$

$$\begin{aligned}
\mathfrak{D}_2^{++}(z,\bar{z}) =& \nabla_z D_{\bar{z}} + \nabla_{\bar{z}} D_z + (\nabla_z + \nabla_{\bar{z}})\frac{z\bar{z}}{2(z-\bar{z})}\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big) \\
&- \frac{(\Delta-\ell)(\Delta-\ell-3)}{4}(\nabla_z + \nabla_{\bar{z}}) + \frac{(2\ell+1)(\Delta-\ell-3)(\Delta-\frac{3}{2})}{4},
\end{aligned} \tag{4.203}$$

where

$$D_z = z^2(1-z)\partial_z^2 - (\alpha'+\beta'+1)z^2\partial_z - \alpha'\beta' z, \quad \alpha' = \alpha - \tfrac{1}{4}, \beta' = \beta - \tfrac{1}{4}, \tag{4.204}$$

$$\nabla_z = z\partial_z + \frac{z}{z-\bar{z}}, \tag{4.205}$$

and $D_{\bar{z}}$, $\nabla_{\bar{z}}$ are defined by exchanging z and $\bar{z}$.

The same reduction to a single block happens for G^{--}_{seed}. For G^{+-}_{seed} and G^{-+}_{seed} the situation is a little trickier since there is a second order differential operator acting on the "wrong" scalar block. However, it turns out that its second-order piece is in fact coming precisely from the dimension shifting operator, and we again can reduce to a third-order differential operator acting on a single scalar block. Explicit expressions for these blocks can be written in a compact form given in appendix C.5 together with an explanation of the normalization conventions.

[48] In simplifying these expressions for the differential operators we made use of the quadratic Casimir equation satisfied by the scalar conformal blocks.

4.4.4.2 Example: seed blocks in 4d

In 4-dimensions the operators in a generic spin representation are labeled by 2 non-negative integers[49] ℓ and $\bar{\ell}$

$$\mathcal{O}_{\Delta,\rho} = \mathcal{O}_{\Delta}^{(\ell,\bar{\ell})}. \tag{4.206}$$

It is convenient to distinguish different classes of representations by a parameter p defined as

$$p \equiv |\ell - \bar{\ell}|. \tag{4.207}$$

Operators with $p = 0$ are the symmetric traceless tensors. Using (4.96) one can easily check that any given four-point function can exchange operators with only a finite number of different values of p. This implies that contrary to the 3-dimensional case, in 4-dimensions we need infinitely many seed conformal blocks, parametrized by p.

A calculation of the general 4-dimensional seed conformal blocks was first performed in [58], where the explicit expressions for $p \leq 8$ were found. In this section we perform an alternative computation of the seed blocks by using our new machinery and the strategy outlined in section 4.4.4. Our approach is to express the p seed blocks in terms of the $p-1$ seed blocks. Knowing such a relation allows one to apply it recursively p times to get an expression of the p seed block in terms of the derivatives of the scalar $p = 0$ Dolan-Osborn block [57, 63]. Since the latter is known in terms of ${}_2F_1$ hypergeometric functions, this also gives hypergeometric expressions for the seed blocks, equivalent to those in [58].[50]

Let us note that the explicit hypergeometric expressions of [58] are quite complex already for $p = 2$. In numerical conformal bootstrap one usually requires simple rational approximations to conformal blocks [35, 36, 47], which are hard to construct from these expressions. On the other hand, our differential recurrence relation is rather simple, and we thus hope that it will find applications in the numerical bootstrap.

As in section 4.2.5.4, it will be convenient to use the 6d embedding formalism described in [2, 54, 55, 58, 62]. In what follows we use the conventions of [2], and all the computations are performed using the Mathematica package described

[49] Notice a difference in conventions relative to the 3-dimensional case where ℓ can be half-integer for fermionic operators.

[50] With normalization conventions derived in [2]. We performed the check for $p \leq 4$.

therein. To avoid repetition, the notation and conventions from [2] will be used in this section without explanation.[51]

A simple choice for the seed four-point function where the operator $O_\Delta^{(\ell,\bar{\ell})}$ with a given p can be exchanged in the s-channel is[52]

$$\langle \mathcal{F}_1^{(0,0)} \mathcal{F}_2^{(p,0)} \mathcal{F}_3^{(0,0)} \mathcal{F}_4^{(0,p)} \rangle. \tag{4.208}$$

The conformal block associated to the exchange of $O_\Delta^{(\ell,\bar{\ell})}$ in the seed 4-point function is

$$W_{\ell,\bar{\ell}}^{(p)} \equiv \langle F_{\Delta_1}^{(0,0)} F_{\Delta_2}^{(p,0)} O_\Delta^{(\ell,\bar{\ell})} \rangle \bowtie \langle \overline{O}_\Delta^{(\bar{\ell},\ell)} F_{\Delta_3}^{(0,0)} F_{\Delta_4}^{(0,p)} \rangle. \tag{4.209}$$

We distinguish 2 cases depending on the sign of $\ell - \bar{\ell}$. Using the convention of [58] we define the “seed"[53] and “dual seed" conformal blocks as

$$W_{seed}^{(p)} \equiv W_{\ell,\bar{\ell}}^{(p)}, \quad \ell \leq \bar{\ell}, \tag{4.210}$$
$$\overline{W}_{dual\ seed}^{(p)} \equiv W_{\ell,\bar{\ell}}^{(p)}, \quad \ell \geq \bar{\ell}. \tag{4.211}$$

The seed and the dual seed conformal blocks can be further decomposed into components as

$$W_{seed}^{(p)} = \mathcal{K}_4 \sum_{e=0}^{p} (-2)^{p-e} H_e^{(p)}(z,\bar{z}) \left[\hat{\mathbb{I}}^{42}\right]^e \left[\hat{\mathbb{I}}^{42}_{31}\right]^{p-e}, \tag{4.212}$$
$$\overline{W}_{dual\ seed}^{(p)} = \mathcal{K}_4 \sum_{e=0}^{p} (-2)^{p-e} \overline{H}_e^{(p)}(z,\bar{z}) \left[\hat{\mathbb{I}}^{42}\right]^e \left[\hat{\mathbb{I}}^{42}_{31}\right]^{p-e}. \tag{4.213}$$

The parameter $e = 0, \ldots, p$ labels the possible 4-point tensor structures. In this section we focus solely on the seed blocks $H_e^{(p)}(z,\bar{z})$. The case of the dual blocks $\overline{H}_e^{(p)}(z,\bar{z})$ is completely analogous and will be addressed in appendix C.6.

The calculation essentially follows the algorithm in section 4.4.4, the main difference being that we go from exchange of $(\ell, \ell+p)$ to $(\ell, \ell+p-1)$ instead of going directly to an STT exchange. The calculation is also largely analogous to the 3-dimensional

[51] The only difference is that we avoid using the terminology of [2, 58, 62] in which “conformal partial waves” refer to what we normally mean by conformal blocks, while “conformal blocks” refer to the coordinates in a basis of four-point tensor structures. When there is a danger of misinterpretation, we call the latter simply the components of conformal blocks. We do so to avoid the possible confusion with conformal partial waves from harmonic analysis.

[52] The seed 4-point functions are chosen so that there is a unique conformal block for the exchange of $O_\Delta^{(\ell,\bar{\ell})}$. There is an ambiguity in choosing the seed 4-point function, and here we use the convention of [58].

[53] In this paper we sometimes use “primal seed” to distinguish from the dual seeds.

calculation in section 4.4.4.1. For convenience, we start the algorithm from the right three-point structure instead of going from the left.

We first rewrite the right three-point function entering (4.209) as

$$\langle \overline{\mathcal{O}}_{\Delta}^{(\ell+p,\ell)} F_{\Delta_3}^{(0,0)} F_{\Delta_4}^{(0,p)} \rangle = (\overline{\mathcal{D}}_0^{-+0} \cdot \mathcal{D}_{4,-0+})\, \langle \overline{\mathcal{O}}_{\Delta+1/2}^{(\ell+p-1,\ell)} F_{\Delta_3}^{(0,0)} F_{\Delta_4+1/2}^{(0,p-1)} \rangle. \tag{4.214}$$

The subscript 0 indicates that $\overline{\mathcal{D}}_0^{-+0}$ acts on the internal operator $\overline{\mathcal{O}}$. We would like to move it across $\bowtie$ (integrate by parts) using the rule (4.152).

Crossing of 2-point functions The definition of the $6j$ symbol entering (4.152) in the present case is

$$\begin{aligned} &\overline{\mathcal{D}}_{2\,a}^{+0-} \langle \overline{\mathcal{O}}_{\Delta}^{(\ell+p,\ell)}(X_1, S_1, \overline{S}_1) \mathcal{O}_{\Delta}^{(\ell,\ell+p)}(X_2, S_2, \overline{S}_2) \rangle \\ &= \mathcal{A}\, \overline{\mathcal{D}}_{1\,a}^{-+0} \langle \overline{\mathcal{O}}_{\Delta+1/2}^{(\ell+p-1,\ell)}(X_1, S_1, \overline{S}_1) \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)}(X_2, S_2, \overline{S}_2) \rangle, \end{aligned} \tag{4.215}$$

where

$$\mathcal{A} \equiv \begin{Bmatrix} \overline{\mathcal{O}}_{\Delta}^{(\ell+p,\ell)} & \mathbf{1} & \overline{\mathcal{O}}_{\Delta+1/2}^{(\ell+p-1,\ell)} \\ \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} & \mathcal{S} & \mathcal{O}_{\Delta}^{(\ell,\ell+p)} \end{Bmatrix}^{\bullet(+0-)}_{\bullet(-+0)} = 2i(\ell+p)(\Delta - \tfrac{p}{2} - 1)(\Delta - \ell - \tfrac{p}{2} - 2). \tag{4.216}$$

Applying (4.152) and (4.214) to (4.209) we arrive at

$$W_{seed}^{(p)} = \mathcal{A}^{-1}\, (\overline{\mathcal{D}}_0^{+0-} \cdot \mathcal{D}_{4,-0+}) \langle F_{\Delta_1}^{(0,0)} F_{\Delta_2}^{(p,0)} \mathcal{O}_{\Delta}^{(\ell,\ell+p)} \rangle \bowtie \langle \overline{\mathcal{O}}_{\Delta+1/2}^{(\ell+p-1,\ell)} F_{\Delta_3}^{(0,0)} F_{\Delta_4}^{(0,p-1)} \rangle, \tag{4.217}$$

where $\overline{\mathcal{D}}_0^{+0-}$ now acts on the left three-point function.

Crossing of 3-point functions We now use the crossing equation for the 3-point function

$$\begin{aligned} \overline{\mathcal{D}}_{0\,a}^{+0-} \langle F_{\Delta_1}^{(0,0)} F_{\Delta_2}^{(p,0)} \mathcal{O}_{\Delta}^{(\ell,\ell+p)} \rangle = &\sum_{n=1}^{2} \mathcal{B}^{(n)} \overline{\mathcal{D}}_{1\,a}^{--0} \langle F_{\Delta_1+1/2}^{(1,0)} F_{\Delta_2}^{(p,0)} \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} \rangle^{(n)} + \\ &\sum_{n=1}^{2} C^{(n)} \overline{\mathcal{D}}_{1\,a}^{+0-} \langle F_{\Delta_1-1/2}^{(0,1)} F_{\Delta_2}^{(p,0)} \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} \rangle^{(n)}, \end{aligned} \tag{4.218}$$

where $\mathcal{B}^{(n)}$ and $C^{(n)}$ denote the $6j$ symbols

$$\begin{aligned} \mathcal{B}^{(n)} &\equiv \begin{Bmatrix} F_{\Delta_1}^{(0,0)} & F_{\Delta_2}^{(p,0)} & F_{\Delta_1+1/2}^{(1,0)} \\ \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} & \mathcal{S} & \mathcal{O}_{\Delta}^{(\ell,\ell+p)} \end{Bmatrix}^{\bullet(+0-)}_{(n)(--0)}, \\ C^{(n)} &\equiv \begin{Bmatrix} F_{\Delta_1}^{(0,0)} & F_{\Delta_2}^{(p,0)} & F_{\Delta_1-1/2}^{(0,1)} \\ \mathcal{O}_{\Delta+1/2}^{(\ell,\ell+p-1)} & \mathcal{S} & \mathcal{O}_{\Delta}^{(\ell,\ell+p)} \end{Bmatrix}^{\bullet(+0-)}_{(n)(+0-)}. \end{aligned} \tag{4.219}$$

The 3-point functions in the right-hand side of (4.218) have the following form

$$\langle F^{(1,0)}_{\Delta_1+1/2} F^{(p,0)}_{\Delta_2} \mathcal{O}^{(\ell,\ell+p-1)}_{\Delta+1/2} \rangle^{(i)} = \mathcal{K}_3 [\hat{I}^{32}]^{p-1} [\hat{J}^3_{12}]^{\ell-1} \begin{pmatrix} \hat{I}^{32} \hat{K}^{13}_2 \\ \hat{I}^{31} \hat{K}^{23}_1 \end{pmatrix},$$
$$\langle F^{(0,1)}_{\Delta_1-1/2} F^{(p,0)}_{\Delta_2} \mathcal{O}^{(\ell,\ell+p-1)}_{\Delta+1/2} \rangle^{(i)} = \mathcal{K}'_3 [\hat{I}^{32}]^{p-1} [\hat{J}^3_{12}]^{\ell-1} \begin{pmatrix} \hat{I}^{13} \hat{I}^{32} \\ \hat{I}^{12} \hat{J}^3_{12} \end{pmatrix}. \tag{4.220}$$

Again, we can find the $6j$ symbols $\mathcal{B}^{(n)}$ and $\mathcal{C}^{(n)}$ by an explicit calculation,

$$\begin{aligned}
\mathcal{B}^{(1)} &= \mathcal{B}^{(2)} - \frac{\ell(\Delta_1+\Delta_2+\Delta-\ell-p-6)}{4(\Delta_1-2)} \times \\
&\quad \left(4(\ell+p+1)(\Delta_1-\Delta_2+\ell+\frac{p}{2}+1) + (\Delta_1-\Delta_2+\Delta+\ell)(2\Delta-4\ell-3p-6)\right), \\
\mathcal{B}^{(2)} &= -\frac{p(\Delta_1-\Delta_2+\Delta+\ell)(2\Delta-2\ell-p-4)(\Delta_1+\Delta_2+\Delta-\ell-p-6)}{4(\Delta_1-2)}, \\
\mathcal{C}^{(1)} &= -\frac{\ell(2\Delta+p-2)(\Delta_1-\Delta_2-\Delta+\ell+p+2)}{4(\Delta_1-3)(\Delta_1-2)}, \\
\mathcal{C}^{(2)} &= \frac{p(-2\Delta+2\ell+p+4)(\Delta_1-\Delta_2-\Delta+\ell+p+2)}{4(\Delta_1-3)(\Delta_1-2)}.
\end{aligned} \tag{4.221}$$

Differential basis The last step is to relate the 3-point functions entering (4.218) to the seed 3-point functions $\langle F^{(0,0)}_{\Delta'_1} F^{(p-1,0)}_{\Delta'_2} \mathcal{O}^{(\ell,\ell+p-1)}_{\Delta+1/2} \rangle$ with shifted dimensions by using the differential basis trick. This is standard [61, 62], so we simply note that we use the following differential operators

$$\begin{aligned}
\langle F^{(1,0)}_{\Delta_1+1/2} F^{(p,0)}_{\Delta_2} \mathcal{O}^{(\ell,\ell+p-1)}_{\Delta+1/2} \rangle^{(n)} &\sim (\overline{\mathcal{D}}_1^{-+0} \cdot \mathcal{D}_{2,++0}), (\mathcal{D}_{1,++0} \cdot \overline{\mathcal{D}}_2^{-+0}), \\
\langle F^{(0,1)}_{\Delta_1-1/2} F^{(p,0)}_{\Delta_2} \mathcal{O}^{(\ell,\ell+p-1)}_{\Delta+1/2} \rangle^{(n)} &\sim (\overline{\mathcal{D}}_1^{+0+} \cdot \mathcal{D}_{2,++0}), (\mathcal{D}_{1,-0+} \cdot \overline{\mathcal{D}}_2^{-+0}).
\end{aligned} \tag{4.222}$$

The recursion relation Combining the expressions (4.217), (4.218), and the differential basis (4.222) we find the following recursion relation

$$\begin{aligned}
W^{(p)}_{\Delta,\ell;\, \Delta_1,\Delta_2,\Delta_3,\Delta_4} &= \\
&\mathcal{A}^{-1} \Big(v_1 (\overline{\mathcal{D}}_1^{--0} \cdot \mathcal{D}_{4,-0+})(\overline{\mathcal{D}}_1^{-+0} \cdot \mathcal{D}_{2,++0})\ W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\, \Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\quad + v_2 (\overline{\mathcal{D}}_1^{--0} \cdot \mathcal{D}_{4,-0+})(\mathcal{D}_{1,++0} \cdot \overline{\mathcal{D}}_2^{-+0})\ W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\, \Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\quad + v_3 (\overline{\mathcal{D}}_1^{+0-} \cdot \mathcal{D}_{4,-0+})(\overline{\mathcal{D}}_1^{+0+} \cdot \mathcal{D}_{2,++0})\ W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\, \Delta_1-1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \\
&\quad + v_4 (\overline{\mathcal{D}}_1^{+0-} \cdot \mathcal{D}_{4,-0+})(\mathcal{D}_{1,-0+} \cdot \overline{\mathcal{D}}_2^{-+0})\ W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\, \Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} \Big),
\end{aligned} \tag{4.223}$$

where the coefficients v_i are given explicitly by

$$
\begin{aligned}
v_1 &= \frac{(\Delta+\Delta_1-\Delta_2+\ell)(-\Delta-\Delta_1+\Delta_2+\ell+2)(\Delta+\Delta_1+\Delta_2-\ell-p-6)}{4(\Delta_1-2)(2\Delta_2+p-4)},\\
v_2 &= \frac{(-\Delta+\Delta_1-\Delta_2+\ell+p+2)(\Delta+\Delta_1-\Delta_2-\ell-2p-2)(\Delta+\Delta_1+\Delta_2-\ell-p-6)}{8(\Delta_1-2)(\Delta_1-1)},\\
v_3 &= \frac{-\Delta+\Delta_1-\Delta_2+\ell+p+2}{4(\Delta_1-3)(\Delta_1-2)^2(2\Delta_2+p-4)},\\
v_4 &= -\frac{(-\Delta+\Delta_1-\Delta_2+\ell+p+2)(-\Delta+\Delta_1+\Delta_2+\ell+2p-2)(\Delta+\Delta_1+\Delta_2-\ell-p-6)}{8(\Delta_1-3)(\Delta_1-2)}.
\end{aligned}
\tag{4.224}
$$

Decomposition into components By using (4.212) one can write the recursion relation (4.223) at the level of components of the seed conformal blocks $H_e^{(p)}(z,\bar{z})$.

First let us notice that according to [58] the components $H_e^{(p)}(z,\bar{z})$ of the seed blocks depend on the external scaling dimensions Δ_i only via the quantities

$$
a_e^p \equiv a^{(p)}, \quad b_e^p \equiv b^{(p)}+p-e, \quad c_e^p \equiv p-e, \tag{4.225}
$$

where

$$
a^{(p)} \equiv -\frac{\Delta_1-\Delta_2-p/2}{2}, \quad b^{(p)} \equiv +\frac{\Delta_3-\Delta_4-p/2}{2}. \tag{4.226}
$$

Let us now analyze the expression (4.223). Almost all the conformal blocks entering the right hand side of (4.223) correspond to the same parameters $a^{(p)}$ and $b^{(p)}$ (the difference in p is compensated by a difference in Δ_i). The only exception is the conformal block

$$
W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\;\Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}, \tag{4.227}
$$

which contains $a^{(p)}-1$ and $b^{(p)}$. Just as in the case of 3-dimensions in section 4.4.4.1, we can use a dimension shifting operator to simplify the structure of the recursion relation (4.223). The only difference is that we need to shift the external dimensions of a general seed block. This can be done by generalizing the construction of dimension-shifting operator outlined in section 4.4.5. We find

$$
W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\;\Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}} = \mathcal{E}^{-1}(\mathcal{D}_{1,+-0}\cdot\overline{\mathcal{D}}_2^{--0})(\mathcal{D}_{1,++0}\cdot\overline{\mathcal{D}}_2^{-+0})W^{(p-1)}_{\Delta+\frac{1}{2},\ell;\;\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}, \tag{4.228}
$$

where

$$
\mathcal{E} \equiv -(p+1)(\Delta_1-1)(\Delta_1-2)(\Delta+\Delta_1-\Delta_2+\ell)(\Delta+\Delta_1-\Delta_2-\ell-2). \tag{4.229}
$$

Note that this is in fact completely analogous to the differential basis trick, except that instead of changing the external spins, we change the external dimensions.

Plugging the relation (4.228) in (4.223), stripping off the kinematic factor and decomposing this relation into components according to (4.212) one obtains a recursion relation for the seed blocks of the form

$$H_e^{(p)}(z,\bar{z}) = -\frac{\mathcal{A}^{-1}}{z-\bar{z}}\left(D_0\, H_e^{(p-1)}(z,\bar{z}) - 2D_1\, H_{e-1}^{(p-1)}(z,\bar{z}) + 4c_{e-2}^{p-1} z\bar{z} D_2\, H_{e-2}^{(p-1)}(z,\bar{z})\right), \tag{4.230}$$

where the conformal block in the l.h.s depends on $[\Delta, \ell;\ \Delta_1, \Delta_2, \Delta_3, \Delta_4]$ while the conformal blocks in the r.h.s. depend on $[\Delta + \frac{1}{2}, \ell;\ \Delta_1, \Delta_2 + \frac{1}{2}, \Delta_3, \Delta_4 + \frac{1}{2}]$. The differential operators D_i are given by

$$\begin{aligned} D_0 \equiv& \nabla_{\bar{z}}[b_e^{p-1}] D_z^{(p-1,e)} - \nabla_z[b_e^{p-1}] D_{\bar{z}}^{(p-1,e)} \\ &+ k\left(D_z^{(p-1,e)} - D_{\bar{z}}^{(p-1,e)}\right) - (c_e^{p-1}+1) L[b_e^{p-1}] B\left[-\frac{k(k-2)}{1+c_e^{p-1}}\right], \end{aligned} \tag{4.231}$$

$$\begin{aligned} D_1 \equiv& z\nabla_{\bar{z}}[b_{e-1}^{p-1} + c_{e-1}^{p-1}] D_z^{(p-1,e-1)} - \bar{z}\nabla_z[b_{e-1}^{p-1} + c_{e-1}^{p-1}] D_{\bar{z}}^{(p-1,e-1)} \\ &+ k\left(z D_z^{(p-1,e-1)} - \bar{z} D_{\bar{z}}^{(p-1,e-1)}\right) \\ &+ (2c_{e-1}^{p-1}+1) z\bar{z} L[b_{e-1}^{p-1}](z-\bar{z})^{-1} L[a] - (k-2)(k-c_{e-1}^{p-1}-1)(z-\bar{z}) B[k], \end{aligned} \tag{4.232}$$

$$D_2 \equiv D_z^{(p-1,e-2)} - D_{\bar{z}}^{(p-1,e-2)} - L[a] B\left[k - c_{e-2}^{p-1} - 1\right], \tag{4.233}$$

where the coefficient k is

$$k \equiv \frac{4-\Delta+\ell}{2} + \frac{3p}{4}. \tag{4.234}$$

The elementary differential operators[54] used here are

$$D_x^{(a,b;c)} \equiv x^2(1-x)\partial_x^2 - \left((a+b+1)x^2 - cx\right)\partial_x - abx, \tag{4.235}$$

$$\nabla_x[\mu] \equiv -x(1-x)\partial_x + \mu\, x, \tag{4.236}$$

$$L[\mu] \equiv \nabla_z[\mu] - \nabla_{\bar{z}}[\mu], \tag{4.237}$$

$$B[\mu] \equiv \frac{z\bar{z}}{z-\bar{z}}\left((1-z)\partial_x - (1-\bar{z})\partial_{\bar{z}}\right) + \mu, \tag{4.238}$$

and we also use the following short-hand notation

$$D_x^{(p,e)} \equiv D_x^{(a_e^p, b_e^p; c_e^p)}. \tag{4.239}$$

[54]Exactly the same differential operators (except for $\nabla_x[\mu]$) enter the quadratic Casimir equation for the seed blocks [58]. Note that here the definition of L differs by a factor of $z-\bar{z}$.

4.4.5 Dimension-shifting and spin-shifting

Using our techniques, we can explain some of the identities for scalar conformal blocks which were derived by Dolan and Osborn in [64]. For the ease of comparison, in this section we use the notation of [64], which we now briefly recall. The scalar conformal block is defined as

$$\begin{aligned} &\langle \phi_{\Delta_1}(x_1)\phi_{\Delta_2}(x_2)|\mathcal{O}_{\Delta,\ell}|\phi_{\Delta_3}(x_3)\phi_{\Delta_4}(x_4)\rangle \\ &= \frac{1}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}\left(\frac{x_{24}}{x_{14}}\right)^{-2a}\left(\frac{x_{14}}{x_{13}}\right)^{2b} F_{\lambda_1\lambda_2}(a,b,x,\overline{x}), \end{aligned} \tag{4.240}$$

where x and $\overline{x}$ are the standard Dolan-Osborn coordinates denoted by z and $\overline{z}$ in the rest of this paper,

$$x\overline{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad (1-x)(1-\overline{x}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}, \tag{4.241}$$

and

$$a = -\frac{1}{2}\Delta_{12}, \qquad b = \frac{1}{2}\Delta_{34}, \tag{4.242}$$

while the parameters λ_i are defined as

$$\lambda_1 = \frac{1}{2}(\Delta+\ell), \qquad \lambda_2 = \frac{1}{2}(\Delta-\ell). \tag{4.243}$$

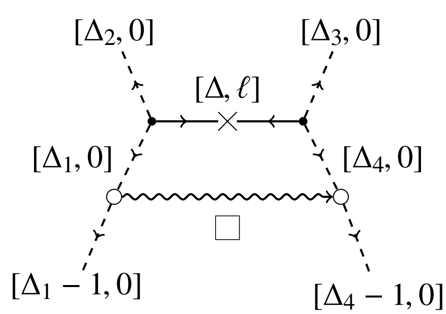
Operators $\mathcal{H}_k$ Let us consider acting on (4.240) with the following contraction of the vector operators (4.45),

$$-2\mathcal{D}_1^{-0}\cdot\mathcal{D}_4^{-0} = -2X_1\cdot X_4 = x_{14}^2. \tag{4.244}$$

The resulting four-point function will have scaling dimensions at positions 1 and 4 shifted by -1. Accordingly, we can remove the prefactor for the new set of scaling dimensions to find the resulting action of this operator on $F_{\lambda_1\lambda_2}$,

$$(x\overline{x})^{-\frac{1}{2}}F_{\lambda_1\lambda_2}(a,b,x,\overline{x}). \tag{4.245}$$

This operation is equivalent to the following diagram,



, (4.246)

and thus according to our general analysis can be expanded using the finite-dimensional crossing (4.102) in terms of scalar conformal blocks with shifted external dimensions and the internal representations appearing in

$$\square \otimes [\Delta, \ell] = [\Delta - 1, \ell] \oplus [\Delta, \ell + 1] \oplus [\Delta, \ell - 1] \oplus [\Delta + 1, \ell] \oplus \ldots, \tag{4.247}$$

where ... represents non-STT representations which do not appear in a four-point function of scalars. In the notation of [64], this corresponds to an equality of the form

$$\begin{aligned}(x\bar{x})^{-\frac{1}{2}} F_{\lambda_1\lambda_2}(a,b) = {}& r\, F_{\lambda_1-\frac{1}{2}\,\lambda_2-\frac{1}{2}}(a+\tfrac{1}{2}, b+\tfrac{1}{2}) + s\, F_{\lambda_1+\frac{1}{2}\,\lambda_2-\frac{1}{2}}(a+\tfrac{1}{2}, b+\tfrac{1}{2}) \\ & + t\, F_{\lambda_1-\frac{1}{2}\,\lambda_2+\frac{1}{2}}(a+\tfrac{1}{2}, b+\tfrac{1}{2}) + u\, F_{\lambda_1+\frac{1}{2}\,\lambda_2+\frac{1}{2}}(a+\tfrac{1}{2}, b+\tfrac{1}{2}),\end{aligned} \tag{4.248}$$

where the coefficients r, s, t, u are some combinations of the $6j$ symbols (4.102). This is precisely the equation (4.18) in [64]. Dolan and Osborn also introduce k-th order differential operators $\mathcal{H}_k$ for $k = 1, 2, 3$, which act on $F_{\lambda_1\lambda_2}$ in the same way but with different sets of coefficients r_k, s_k, t_k, u_k. In particular, they all increase a and b by $\frac{1}{2}$. In our formalism we can also find 3 other operators with such a property,

$$\begin{aligned}\mathcal{D}_{13} &= \mathcal{D}_1^{-0} \cdot \mathcal{D}_3^{+0},\\ \mathcal{D}_{24} &= \mathcal{D}_2^{+0} \cdot \mathcal{D}_4^{-0},\\ \mathcal{D}_{23} &= \mathcal{D}_2^{+0} \cdot \mathcal{D}_3^{+0},\end{aligned} \tag{4.249}$$

all of which also exchange the vector representation in a way similar to (4.246), and thus act in the same way as $\mathcal{H}_k$. In fact, one can express $\mathcal{H}_k$ in terms of these operators, and we provide explicit expressions in appendix C.7.

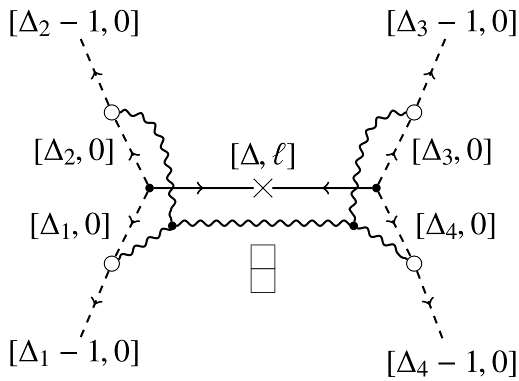
Operators $\mathcal{F}_k$ Another class of operators introduced in [64] can be interpreted as exchanges of the adjoint representation of conformal group. The simplest of such exchanges is given by

$$\mathcal{F}_0 = 8\mathcal{D}_{1,[m}^{-0}\mathcal{D}_{2,n]}^{-0}\mathcal{D}_3^{-0,[m}\mathcal{D}_4^{-0,n]}, \tag{4.250}$$

whose action on the functions $F_{\lambda_1\lambda_2}$ is equivalent to

$$\mathcal{F}_0 = \frac{1}{x} + \frac{1}{\bar{x}} - 1, \tag{4.251}$$

which is precisely how $\mathcal{F}_0$ is defined in [64]. The action of this operator on a conformal block corresponds to the following diagram,



$$, \tag{4.252}$$

where the individual differential operators have indices in the vector representation and are then joined into the adjoint representation $\begin{smallmatrix}\square\\\square\end{smallmatrix} \in \square \otimes \square$. Therefore, it decomposes into scalar blocks appearing in the tensor product

$$\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes [\Delta, \ell] = [\Delta-1, \ell+1] \oplus [\Delta-1, \ell-1] \oplus [\Delta+1, \ell+1] \oplus [\Delta+1, \ell-1] \oplus [\Delta, \ell] \oplus \ldots, \tag{4.253}$$

where "..." represents non-STT representations which do not appear in scalar conformal blocks. Thus there exists an identity of the form

$$\mathcal{F}_0 F_{\lambda_1\,\lambda_2} = r_0 F_{\lambda_1\,\lambda_2-1} + s_0 F_{\lambda_1-1\,\lambda_2} + t_0 F_{\lambda_1+1\,\lambda_2} + u_0 F_{\lambda_1\,\lambda_2+1} + w_0 F_{\lambda_1\,\lambda_2}, \tag{4.254}$$

with coefficients r_0, s_0, t_0, u_0, w_0 being some combinations of the $6j$ symbols (4.102). This is precisely (4.28) of [64]. The operators $\mathcal{F}_k$ with $k = 1, 2, 3$ can be constructed analogously.

Operator $\mathcal{D}^{(\varepsilon)}$ Finally, let us consider the identity (4.50) of [64], which is[55]

$$(x\bar{x})^{\varepsilon-b+1}\mathcal{D}^{(\varepsilon)}(x\bar{x})^{b-\varepsilon}F_{\lambda_1\,\lambda_2}(a, b, x, \bar{x}) = (\lambda_1+b)(\lambda_2+b-\epsilon)F_{\lambda_1\,\lambda_2}(a, b+1, x, \bar{x}). \tag{4.255}$$

We see that the left hand side of this expression gives a differential operator which shifts b by 1. In our formalism, it is extremely easy to construct this operator, namely

$$(x\bar{x})^{\varepsilon-b+1}\mathcal{D}^{(\varepsilon)}(x\bar{x})^{b-\varepsilon} = \frac{\mathcal{D}_3^{+0}\cdot\mathcal{D}_4^{-0}}{(\Delta_3-1)(d-2-\Delta_3)}. \tag{4.256}$$

From the definition it is clear that it simply shifts b by 1. The coefficient in the right hand side of (4.255) can be easily expressed in terms of $6j$ symbols (4.102).

[55]Note that there is a typo in the second part of (4.43) in [64]. The correct definition is $\mathcal{D}^{(\varepsilon)} = (x\bar{x})^{-\frac{1}{2}}\mathcal{H}_2$.

4.4.6 Recursion relations for conformal blocks

In sections 4.4.3 and 4.4.4 we have managed to express an arbitrary conformal block in terms of derivatives of scalar blocks, schematically

$$G_{\Delta,\rho} = \sum_k c_k(\Delta) D_k G^{\text{scalar}}_{\Delta+\delta_k,\ell_k}, \tag{4.257}$$

where $[\Delta, \rho]$ is the representation of the exchanged operator, D_k are some Δ-independent differential operators, and $c_k(\Delta)$ are rational functions. All ingredients in this formula implicitly depend on the dimensions and representations of the external operators, as well as on ρ. In practice we often have a generic spin parameter ℓ in ρ, and we can keep it generic in this formula as we did in the examples in sections 4.4.4.1 and 4.4.4.2. The spins ℓ_k are then finite shifts of ℓ, $\ell_k = \ell + \delta\ell_k$.

Explicit examples of such expressions are given in (4.196), (4.197) and (4.223). They readily allow us to compute the spinning conformal blocks numerically. But they also allow us to analytically infer properties of the spinning blocks from the known properties of the scalar blocks.

For example, a general method for numerical computation of conformal blocks is based on Zamolodchikov recursion relations [48, 193]. The basic idea is that for certain values Δ_i of the scaling dimension Δ the generalized Verma module for the representation $[\Delta, \rho]$ has null descendants $[\Delta'_i, \rho'_i]$, which lead to poles in the conformal block for $[\Delta, \rho]$ with the residue being proportional to the conformal block for $[\Delta'_i, \rho'_i]$,

$$G_{\Delta,\rho} \sim \frac{R_i}{\Delta - \Delta_i} G_{\Delta'_i,\rho'_i}, \tag{4.258}$$

where R_i are certain coefficients, which in the case of spinning blocks generically are matrices rotating the left and right three-point structures in G. For fixed ρ there are in general several infinite families of poles Δ_i. If we know the asymptotic behavior of the conformal blocks for $\Delta \to \infty$,

$$G_{\Delta,\rho} \sim r^{\Delta} h_{\infty,\rho}, \tag{4.259}$$

where r is the radial coordinate of [25, 59] and $h_{\infty,\rho}$ is some relatively easily computable function, then we can write the conformal block as a sum over residues [37, 49]. The resulting approximation is perfectly suited for numerical applications based on semidefinite methods [36, 37, 41].

To accomplish this program, one needs to understand the pole positions Δ_i, the representations of null states $[\Delta'_i, \rho'_i]$, and the residue matrices R_i. This data has been determined for general scalar blocks [36, 37] as well as some examples of spinning blocks [41, 49, 81]. Although the classification of the poles Δ_i and the null states $[\Delta_i', \rho_i']$ is known [49, 196, 204], the computation of the residue matrices R_i may not be an easy task.

Our expression (4.257) is perfectly suited for this problem. Indeed, from it the pole structure of $G_{\Delta,\rho}$ is completely apparent. In particular, the poles in $G_{\Delta,\rho}$ are given by the poles of the scalar blocks in the right hand side, and a *finite*[56] number of poles of the coefficients $c_k(\Delta)$. The residues of the poles are easy to compute. Indeed, any residue is given by a sum of differential operators D_k acting on some scalar blocks. Using the techniques of section 4.4.3, it is easy to express the action of D_k on a general scalar block as a sum over conformal blocks which can appear for the given external operators,[57]

$$D_k G^{\text{scalar}}_{\Delta,\ell} \sim \sum_{\Delta',\rho'} G_{\Delta',\rho'}. \tag{4.260}$$

In other words, our techniques allow us to translate the known recursion relations for scalar blocks into recursion relations for general conformal blocks. This approach has already been used in [41] for the exchange of traceless-symmetric representations. The new ingredient here is that we can now derive the recursion relation for general internal representations. For example, using the equations (4.196) and (4.197), we re-derived the recursion relation of [81] for the scalar-fermion seed blocks exchanging a fermionic representation.

In [49] the residues of the conformal blocks were computed explicitly by considering the action of the differential operators $\mathcal{D}_i$ corresponding to the null states on the three-point functions, and the behavior of the norm of the null state near the pole. We expect that the conformally-covariant differential operators can be useful also in this approach. For example, the null state differential operators $\mathcal{D}_i$ can be obtained by the translation functor from a set of basic operators [205]. In our language this means that one can write the operators $\mathcal{D}_i$ as

$$\mathcal{D}_i \propto \mathcal{D}_A \mathcal{D}_i' \mathcal{D}^A, \tag{4.261}$$

where $\mathcal{D}_i'$ are some simpler differential operators (for instance, many null states can be obtained from $\mathcal{D}' = d$ the exterior derivative acting on differential forms.). The action of $\mathcal{D}_i$ on a three-point function can then be computed by applying a crossing transformation to move $\mathcal{D}^A$ on a different leg and then acting with $\mathcal{D}_i'$.

[56] For a fixed ℓ.

[57] In particular, substituting these expressions in (4.257), we get a tautology.

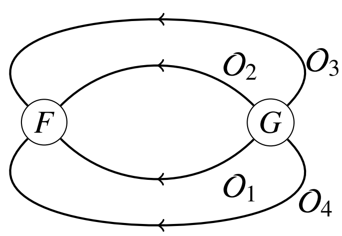
4.5 Further applications

4.5.1 Inversion formulae and "spinning-down" a four-point function

Orthogonality relations between conformal blocks are useful tools for analyzing crossing symmetry. By exploiting orthogonality, we can derive inversion formulae that express OPE data in terms of an integral of a conformal block against a four-point function [65, 201]. Applying an s-channel inversion formula to a t-channel conformal block expansion, we can study crossing directly in terms of CFT data.[58] The coefficients relating t-channel blocks and s-channel blocks are sometimes called "crossing kernels." Inversion formulae and crossing kernels for scalar operators have been discussed recently in [66, 168–170]. Here, we briefly describe how our techniques are perfectly suited for studying inversion formulae and crossing kernels for spinning operators. We will omit details, and simply highlight how weight-shifting operators can be used in these computations. We leave detailed discussion and examples for later work [195].

Our starting point is a conformally-invariant pairing between a four-point function of operators $\mathcal{O}_i$ in representations $[\Delta_i, \rho_i]$ and a four-point function of shadow operators $\widetilde{\mathcal{O}}_i$ in representations $[d-\Delta_i, \rho_i^*]$. This can be written

$$\langle F, G\rangle = \frac{1}{\mathrm{Vol}(\mathrm{SO}(d+1,1))}\int\left(\prod_{i=1}^{4} d^d x_i\right) F_{a_1a_2a_3a_4}(x_i)G^{a_1a_2a_3a_4}(x_i)$$

$=$ 

. (4.262)

In our diagrammatic language, an incoming line for $\mathcal{O}$ is equivalent to an outgoing line for $\widetilde{\mathcal{O}}$, and connecting lines means contracting indices and integrating over Euclidean space. To get a finite result for $\langle F, G\rangle$, we must divide by the volume of the conformal group acting on all four points x_i. In practice, this means gauge-fixing and inserting the appropriate Faddeev-Popov determinant.

Consider first the case of scalar operators $\mathcal{O}_i$. An orthogonal basis with respect to the pairing $\langle\cdot,\cdot\rangle$ is given by linear combinations of blocks that are single-valued in

[58] Note that the integral in an inversion formula in general does not commute with the sum over conformal blocks in the t-channel, so this analysis must be done carefully.

Euclidean space,

$$F_{\Delta,\ell} = \frac{1}{2}\left(G_{\Delta,\ell} + S_{\Delta,\ell}G_{d-\Delta,\ell}\right), \tag{4.263}$$

where $\Delta = \frac{d}{2} + i\nu$ is restricted to the principal series.[59] The constant $S_{\Delta,\ell}$ depends on Δ, ℓ and the external dimensions Δ_i, and will not be important for the current discussion. We call the $F_{\Delta,\ell}$ "Euclidean partial waves." Orthogonality follows from the fact that the Casimir operator is self-adjoint with respect to $\langle\cdot,\cdot\rangle$, together with the fact that $F_{\Delta,\ell}$ is single-valued so there are no boundary contributions from integrating by parts. See [66] for more details.

A four-point function of scalars has a Euclidean partial wave decomposition of the form

$$g(x_i) = 1 + \sum_\ell \oint_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} c(\Delta,\ell) F_{\Delta,\ell}(x_i) + \text{discrete series}. \tag{4.264}$$

The decomposition (4.264) is not the usual conformal block decomposition, but it is closely related. When $g(x_i)$ is a four-point function in a unitary CFT, we expect that $c(\Delta,\ell)$ has (shadow-symmetric) simple poles in Δ on the real axis

$$c(\Delta,\ell) \sim \sum_i -c_{\Delta_i,\ell}\left(\frac{1}{\Delta-\Delta_i} + \frac{S^{-1}_{\Delta,\ell}}{d-\Delta-\Delta_i}\right). \tag{4.265}$$

We can then deform the Δ-contour in (4.264) to the right for $G_{\Delta,\ell}$ and to the left for $G_{d-\Delta,\ell}$ to obtain

$$g(x_i) = 1 + \sum_{\Delta_i,\ell} c_{\Delta_i,\ell} G_{\Delta_i,\ell}(x_i). \tag{4.266}$$

Thus, positions of poles in $c(\Delta,\ell)$ encode the spectrum of the theory, and the residues encode products of OPE coefficients.[60]

For spinning operators, the Euclidean partial waves $F^{(a,b)}_{\Delta,\rho}$ and their coefficients $c_{(a,b)}(\Delta,\rho)$ are additionally labeled by a pair of three-point structures (a,b). An inversion formula for the coefficients is given by[61]

$$M^{(c,d)(a,b)}(\Delta,\rho)c_{(a,b)}(\Delta,\rho) = \langle \widetilde{F}^{(c,d)}_{\Delta,\rho}, g\rangle, \tag{4.267}$$

[59] We must also include the so-called "discrete series" in non-even dimensions [65].

[60] When deforming the Δ-contour, one must take into account poles in the blocks themselves, which interact in an intricate way [65, 66, 166].

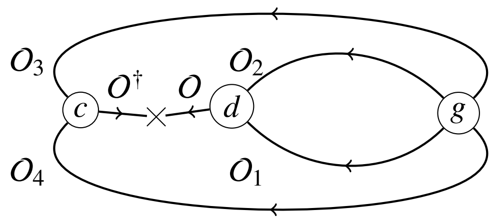
[61] We sum over raised and lowered pairs of three-point structures (a,b).

where, roughly speaking,[62]

$$\langle \widetilde{F}^{(c,d)}_{\Delta',\rho'}, F^{(a,b)}_{\Delta,\rho} \rangle \sim M^{(c,d)(a,b)}(\Delta,\rho)\delta_{\rho\rho'}\delta(\Delta-\Delta'). \tag{4.268}$$

Pictorially,

$$\langle \widetilde{F}^{(c,d)}_{\Delta,\rho}, g \rangle = $$



. (4.269)

One of our main observations is that spinning conformal blocks can be written as derivatives of scalar blocks. Schematically, we have

$$\begin{aligned} \widetilde{F}^{\text{spin}}_{\Delta,\rho} &= \mathfrak{D} F^{\text{scalar}}_{\Delta,\ell}, \\ \mathfrak{D} &= \sum_t d_t(\Delta,\rho) t_{ABCD} \mathcal{D}_1^{(a)A} \mathcal{D}_2^{(b)B} \mathcal{D}_3^{(c)C} \mathcal{D}_4^{(d)D}. \end{aligned} \tag{4.270}$$

The operators $\mathcal{D}_i^{(a_i)A_i}$ are spin-raising operators transforming in W_i, acting on the point x_i. Here, t runs over invariant tensors in $(W_1 \otimes W_2 \otimes W_3 \otimes W_4)^*$.

To compute the pairing (4.269), it is useful to integrate $\mathfrak{D}$ by parts,

$$\langle \widetilde{F}^{\text{spin}}_{\Delta,\rho}, g \rangle = \langle F^{\text{scalar}}_{\Delta,\ell}, \mathfrak{D}^* g \rangle, \tag{4.271}$$

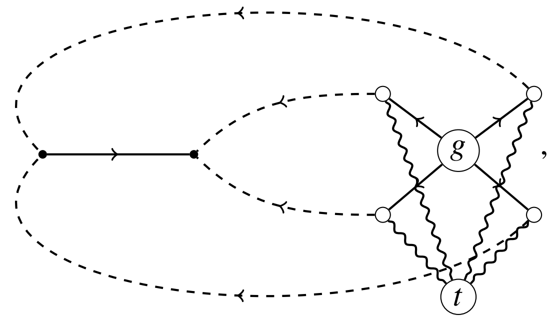
where $\mathfrak{D}^*$ is the adjoint of $\mathfrak{D}$ under the pairing $\langle \cdot, \cdot \rangle$, given by replacing each $\mathcal{D}_i^{(a)}$ with its adjoint $(\mathcal{D}_i^{(a)})^*$ (since we can integrate by parts individually on each leg). The adjoints $(\mathcal{D}_i^{(a_i)})^*$ are spin-*lowering* differential operators, and the right-hand side of (4.271) is a pairing between *scalar* four-point functions. We can thus proceed to study it in the same way as we study four-point functions of scalars. For example, one can derive spinning versions of the CFT Froissart-Gribov formula [66] using these techniques.[63] We call this trick "spinning-down" a four-point function.

[62] We are neglecting an additional term proportional to $\delta(\Delta + \Delta' - d)$ that is unimportant for the current discussion.

[63] One of the consequences of the Froissart-Gribov formula is that CFT data can be analytically continued in spin. When non-STTs can appear as internal operators, analytic continuation in spin can be understood by expressing $V_{\Delta,\rho}$ as a subrepresentation of $V_{\Delta',\ell} \otimes W$ for some fixed W, and then analytically continuing in ℓ. This is equivalent to analytically continuing in the length of the first row of the Young diagram for ρ.

In pictures, the right-hand side of (4.271) is

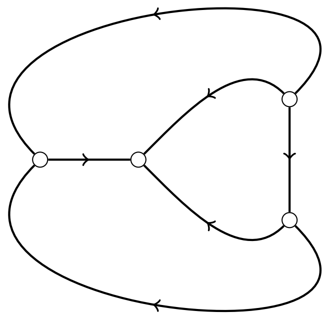
$$\langle \widetilde{F}^{\text{spin}}_{\Delta,\rho}, g\rangle \sim \sum_t \quad (4.272)$$



where the dashed lines represent scalars.

4.5.2 $6j$ symbols for infinite-dimensional representations

If we plug in a t-channel partial wave for g, then we can simplify (4.272) further by using crossing to move the differential operators to the internal leg:



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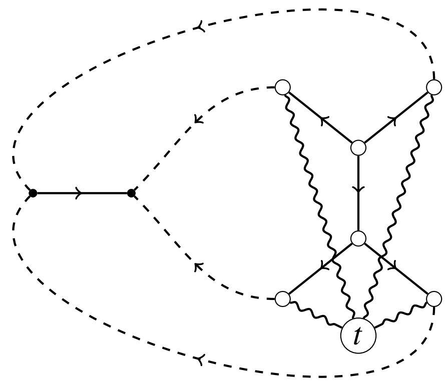
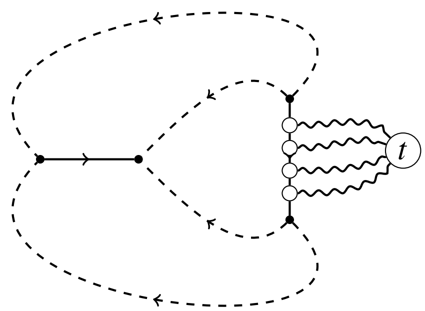
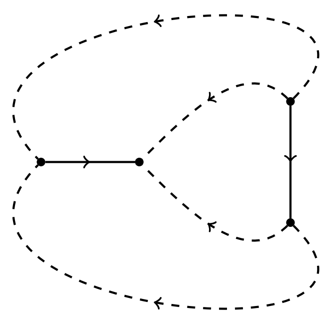


$$= \sum \left\{ \cdots \right\}^4$$



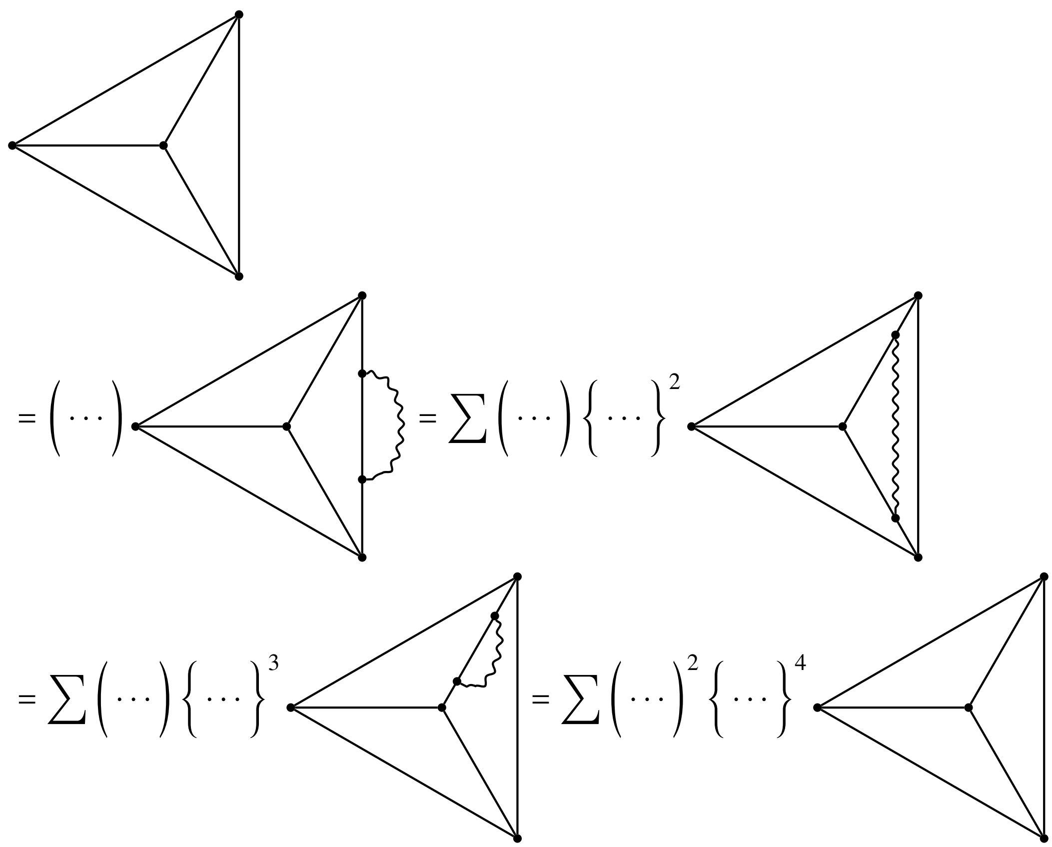
$$= \sum \left\{ \cdots \right\}^4 \left(\cdots \right)$$



(4.273)

The symbol $\{\cdots\}^4$ represents a product of four $6j$ symbols of the type in (4.102), and the factor $(\cdots)$ is the result of taking a conformally-invariant product of differential operators on the right internal leg. For simplicity, we have omitted labels and shown only the topology of the various diagrams. Dashed lines represent scalar operators, and solid lines represent operators with spin.

Equation (4.273) expresses an inner product of general spinning blocks in terms of inner products of scalar blocks. Such inner products are examples of $6j$ symbols for the conformal group, where all the representations are infinite-dimensional principal series representations. The corresponding graphs have the topology of a tetrahedron. The equality (4.273) is an example of a general set of relations between infinite-dimensional $6j$ symbols that we can derive as follows. We start with a tetrahedron graph and introduce a bubble with a finite-dimensional representation W on one of the lines. We can then move the vertices of the bubble to a different internal line and collapse it.



(4.274)

The above is essentially the pentagon identity for a mixture of finite-dimensional (degenerate) and infinite-dimensional representations. Because the crossing kernel for degenerate four-point functions is so simple, the pentagon identity becomes a useful tool for computing infinite-dimensional crossing kernels. The $6j$ symbol for six scalar representations of the conformal group was computed in [206] in terms of a four-fold Mellin-Barnes integral. That result, along with relations of the type illustrated in (4.274) in principle allows one to compute an arbitrary $6j$ symbol.

4.6 Discussion

In this work, we introduced new mathematical tools for computations in conformal representation theory. These include the construction of weight-shifting operators summarized in theorem 2, the observation that they satisfy the crossing equation (4.102), and our discussion of how weight-shifting operators interact with conformally-invariant projectors (4.151). For concrete computations, we introduced the embedding space operators (4.45), (4.72), and (4.79). We explored in detail how these tools can be applied to compute conformal blocks. We also discussed some applications to harmonic analysis and inversion formulae. We plan to expand on the latter in future work [195].

However, many directions remain unexplored. One natural question is how weight-shifting operators interact with short multiplets of the conformal group. For simplicity, we specialized to simple generalized Verma modules (long multiplets) in this paper. However, we expect new phenomena in the presence of shortening conditions. Some questions include: How is the tensor product decomposition 4.13 modified for short multiplets? How are shortening conditions reflected in the zeros and poles of $6j$ symbols? Is the spinning-down procedure of section 4.5.1 useful when external operators are in short multiplets?

Our construction of weight-shifting operators and their crossing equations is very general. As noted in the introduction, it also applies to generalized Verma modules of any Lie (super-)algebra.[64] In particular, supersymmetric weight-shifting operators should be useful for computing and studying superconformal blocks and tensor structures. It will be interesting to construct such operators and explore their applications. The question of how weight-shifting operators interact with shortening conditions becomes even more interesting in the superconformal case, since there are a wide variety of interesting short superconformal multiplets (see e.g. [83]).

As discussed in section 4.2.4, the algebra of weight-shifting operators is governed by the fusion matrix $J(\lambda)$, which is closely related to solutions to the Yang-Baxter equation and integrability [199]. Does this structure have an interesting role to play in conformal field theory? Is it related to the "superintegrability" of conformal blocks discussed in [123, 191, 192]?

It may also be interesting to explore the role of weight-shifting operators in holo-

[64] In the language of [168], it works in a GFT for any group G.

graphic calculations.[65] We expect that they should help in the computation of Witten diagrams for operators with spin. Natural questions include: What is the flat-space limit of weight-shifting operators? Are they useful for amplitudes calculations (for example are they related to the differential operators introduced in [208])? Weight-shifting operators may also be helpful for exploring spinning amplitudes in the conformal basis of [209].

Acknowledgements

We are grateful to Clay Córdova, Tolya Dymarsky, Abhijit Gadde, Mikhail Isachenkov, Eric Perlmutter, Fernando Rejon-Barrera, Douglas Stanford and Emilio Trevisani for discussions. DK and PK would like to thank the organizers of the Bootstrap 2017 workshop, where part of this work was completed. DSD is supported by DOE grant DE-SC0009988, a William D. Loughlin Membership at the Institute for Advanced Study, and Simons Foundation grant 488657 (Simons Collaboration on the Nonperturbative Bootstrap). PK is supported DOE grant DE-SC0011632.

[65] After this draft was completed, [207] appeared, which describes the special case of weight-shifting operators that shift the mass of a scalar field in AdS.

Chapter 5

CASIMIR RECURSION RELATIONS FOR GENERAL CONFORMAL BLOCKS

This chapter is essentially identical to:

P. Kravchuk, *Casimir recursion relations for general conformal blocks*, *JHEP* **02** (2018) 011, [1709.05347].

5.1 Introduction

Numerical conformal bootstrap is a very general and powerful approach to quantum conformal filed theories (CFTs), based on the idea of analyzing the crossing symmetry [26–28] of correlation functions in unitary CFTs by methods of semidefinite programming [30, 35–37, 47]. In recent years, this approach has proven to be extremely useful in extracting non-perturbative information about concrete CFTs, such as the critical exponents and structure constants of 3d Ising CFT, $O(N)$ and Gross-Neveu models [8, 31, 32, 34, 36–40], as well as a host of other results [6, 41, 43–46, 94–101, 110, 114, 122, 126–158]. Crossing symmetry of the four-point functions of such fundamental operators as spin-1 conserved currents or the energy-momentum tensor has also been instrumental in deriving universal constraints valid for general CFTs [7, 41].

The practical implementation of numerical conformal bootstrap relies heavily on two technical requirements: the knowledge of conformal blocks and the ability to efficiently solve the semidefinite programs. An efficient semidefinite solver `SDPB`, designed specifically for bootstrap applications, was introduced in [35]. This solver is able to solve the most general semidefinite programs which typically arise in conformal bootstrap, thus eliminating the technical obstructions related to semidefinite programming. The situation with conformal blocks is different. The simplest conformal blocks—those with external scalar operators—are very well studied by now and there exist simple and efficient techniques for their computation [37, 57, 59, 63, 64, 80]. Some of these techniques, such as Zamolodchikov-like recursion relations, iterative/-analytic solutions of conformal Casimir equations or shadow integrals have been extended to conformal blocks of operators with spins [41, 49, 54, 58, 60, 81, 82, 122]. Another approach to spinning conformal blocks is to relate them to simpler conformal blocks by means of differential operators [39, 61, 62]; recently it was shown that the most general conformal blocks can be reduced in this way to scalar blocks [3]. While these methods do allow us to calculate any given non-supersymmetric conformal block, all of them currently require a nontrivial amount of case-specific analysis.

In order to facilitate the conformal bootstrap studies with spinning operators it is therefore desirable to have a simple and general algorithm for numerical computation of conformal blocks which can be implemented on a computer, ideally avoiding the need for symbolic algebra. The first step in this direction was undertaken in [1], where a general classification and construction of conformally-invariant tensor structures was given. In this paper, we take another step towards this goal by formulating a general Casimir recursion relation for the z-coordinate series expansion of general spinning conformal blocks in any number of dimensions. For a conformal block exchanging a primary operator $\mathcal{O}$, the recursion relation takes the form

$$(C(\Delta_{p+1}, \widetilde{\mathbf{m}}_d) - C(\mathcal{O}))\, \Lambda^{ba}_{p+1,\widetilde{\mathbf{m}}_d} = \sum_{\mathbf{m}_d \in \square \otimes \widetilde{\mathbf{m}}_d} (\overline{\gamma}_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d} \Lambda_{p,\mathbf{m}_d} \gamma_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d})^{ba}, \qquad (5.1)$$

where the matrices $\Lambda_{p,\mathbf{m}_d}$ encode the contribution of descendants at level p and in $Spin(d)$ representation $\mathbf{m}_d$ in z-coordinates, $\Delta_p = \Delta_{\mathcal{O}} + p$, C give the conformal Casimir eigenvalues, while γ and $\overline{\gamma}$ are some matrices. Similar recursion relations have been recently considered in [60]. Our improvement over these results is in that the structure of our recursion relation is much simpler (in particular, it is one-step, i.e., it relates levels p and $p+1$, similarly to the scalar recursion relation in [59]) and we are able to remain completely general and write the coefficients γ and $\overline{\gamma}$ in terms $6j$ symbols (or Racah coefficients) of $Spin(d-1)$. Thus, in our form, the Casimir recursion relations can be immediately translated into a computer algorithm in all cases when the $6j$ can be computed algorithmically. This includes the general conformal blocks in 3 and 4 dimensions as well as seed blocks in general dimensions. Importantly, since we solve all representation-theoretic questions in terms of Clebsch-Gordan coefficients and $6j$ symbols, our analysis is applicable to all spin representations without any caveats, i.e., it applies equally well to spinor representations and is free from the redundancies which plague the less abstract approaches in low dimensions.[1]

[1] Assuming, of course, that Clebsch-Gordan coefficients are known.

This paper consists of three main parts. The first part is section 5.2 in which we review the basics of the representation theory of $Spin(d)$ and give a brief summary of the required facts from the theory of Gelfand-Tsetlin (GT) bases. The advantage of GT bases is that they allows us to work very explicitly with completely general representations in arbitrary d, at the same time being perfectly compatible with the conformal frame construction of [1]. Moreover, many explicit formulas for matrix elements and Clebsch-Gordan coefficients are available in these bases. These facts make them our main computational tool in this paper.

In section 5.3 we use these tools to study the contribution of a general $\mathbb{R}\times Spin(d)$ (dilatations×rotations) multiplet to a given four-point function. In section 5.3.1 we express the answer in terms of an explicit basis of three- and four-point functions (constructed using the Clebsch-Gordan coefficients of $Spin(d-1)$). The functions P which replace the Gegenbauer polynomials (which appear in scalar correlation functions) are some particular matrix elements of $e^{\theta M_{12}}$ in a GT basis. In sections 5.3.2-5.3.5 we consider the $\mathbb{R}\times Spin(d)$ contributions in some simple special cases. In section 5.3.6 we prove the folklore theorem which states that the number of four-point tensor structures is equal to the number of classes of conformal blocks. In section 5.3.7 we study the properties of P-functions and explain how they can be efficiently computed in practice by organizing them in so-called "matroms" [210] and deriving a recursion relation for these matroms. We also discuss the simplifications in the low-dimensional cases of $d = 3$ and $d = 4$. In appendix D.4 we relate the functions P to irreducible projectors studied recently in [82] in the case of tensor representations.

In section 5.4 we study the Casimir recursion relations for general conformal blocks. We start by rederiving the scalar result of [59] in section 5.4.1 using an abstract group-theoretic approach. In section 5.4.2 we extend this approach to general representations and derive the formulas (5.280) and (5.281) for γ and $\overline{\gamma}$ in terms of $6j$ symbols of $Spin(d-1)$. In sections 5.4.3-5.4.4 we discuss how these $6j$ symbols simplify in the case $d = 3$ and for the seed blocks in general d.[2] For more specific examples we explicitly work out the recursion relations for scalar-fermion seed blocks in $d = 3$ and $d = 2n$ and compare them to the known results. In section 5.4.5 we briefly discuss the problems associated with a practical solution of the Casimir recursion relation and suggest some possible workarounds.

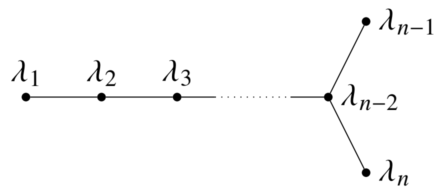
[2]We do not discuss the case of general blocks in $d = 4$, where these $6j$ symbols are also known, only to keep the size of the paper reasonable – the application of the general formula is completely mechanical.

We conclude in section 5.5. The appendices D.1 and D.2 contain some explicit formulas and details on our conventions. The appendix D.3 elaborates on comparison to known results. In appendix D.4 we explain the relation between GT and Cartesian bases for tensor representations.

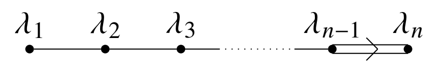
5.2 Representation theory of $Spin(d)$

We will be studying conformal blocks for the most general representations of $Spin(d)$, which requires a certain amount of mathematical machinery. In this section we review the relevant representation theory and establish important notation.

We will be working exclusively in the Euclidean signature (the results can be easily translated to Lorentz signature by Wick rotation). This means that we work with the compact real form of $Spin(d)$, which double covers $SO(d)$. As is well known, the basic properties of these groups depend on the parity of d. If $d = 2n$, then the Lie algebra of $Spin(d)$ is the simple[3] rank-n Lie algebra D_n with Dynkin diagram shown in Fig. 5.1a. If $d = 2n + 1$ then the relevant algebra is the simple rank-n Lie algebra B_n with Dynkin diagram shown in Fig. 5.1b.



(a) D_n algebra



(b) B_n algebra

Figure 5.1: Dynkin diagrams of $\mathfrak{so}(d)$ algebras.

It is standard to specify the irreducible representations[4] by non-negative integral Dynkin labels λ_i associated to the nodes in the Dynkin diagram. The representations in which only one λ_i is non-zero and equal to 1 are called the fundamental representations. The fundamental representation associated with λ_1 (i.e., the one with labels $\lambda_i = \delta_{i1}$) is the fundamental vector representation $\mathbb{R}^d$.[5] More generally, the fundamental representations associated with λ_i with $i < d/2 - 1$ are the exterior powers of the vector representation, $\wedge^i\mathbb{R}^d$. The nodes $\lambda_{n-1} = a$, $\lambda_n = c$ in D_n case

[3] Semi-simple for $d = 4$: $D_2 = A_1 \oplus A_1$ is equivalent to two copies of $\mathfrak{su}_2$ algebra.

[4] We are interested in representations over $\mathbb{C}$, since the physical Hilbert space is complex. However, we often treat the representations which are real (in the sense of being representable by real matrices) as being over $\mathbb{R}$.

[5] Unless $d \leq 4$ when λ_1 corresponds to one of the spinor representations.

correspond to the two chiral spinor representations. Similarly, the node $\lambda_n = b$ corresponds to the unique spinor representation in B_n case. A general representation can be obtained by tensoring the above “fundamental” representations together and taking the irreducible component with the highest weight (i.e., by imposing the maximal symmetry and tracelessness conditions on the resulting tensors).

For us it will be more convenient to label the representations by generalized Young diagrams, constructed as follows. To a given set of Dynkin labels of $Spin(d)$ we associate a vector of numbers $\mathbf{m}_d$ with components, for $d = 2n$,

$$m_{d,1} = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-2} + \frac{a+c}{2}, \tag{5.2}$$

$$m_{d,2} = \lambda_2 + \lambda_3 + \ldots + \lambda_{n-2} + \frac{a+c}{2}, \tag{5.3}$$

$$\vdots$$

$$m_{d,n-2} = \lambda_{n-2} + \frac{a+c}{2}, \tag{5.4}$$

$$m_{d,n-1} = \frac{a+c}{2}, \tag{5.5}$$

$$m_{d,n} = \frac{a-c}{2}, \tag{5.6}$$

and for $d = 2n+1$,

$$m_{d,1} = \lambda_1 + \lambda_2 + \ldots + \lambda_{n-1} + \frac{b}{2}, \tag{5.7}$$

$$m_{d,2} = \lambda_2 + \lambda_3 + \ldots + \lambda_{n-1} + \frac{b}{2}, \tag{5.8}$$

$$\vdots$$

$$m_{d,n-1} = \lambda_{n-1} + \frac{b}{2}, \tag{5.9}$$

$$m_{d,n} = \frac{b}{2}. \tag{5.10}$$

This gives all possible sequences satisfying

$$\begin{aligned} & m_{d,1} \geq m_{d,2} \geq \ldots m_{d,n-1} \geq |m_{d,n}|, && \text{for } d = 2n, \\ & m_{d,1} \geq m_{d,2} \geq \ldots m_{d,n} \geq 0, && \text{for } d = 2n+1, \end{aligned} \tag{5.11}$$

and consisting either entirely of intergers (bosonic representations) or entirely of half-integers (fermionic representations). The dimensions of these irreducible representations are given in appendix D.2.

When $\mathbf{m}_d$ is bosonic, we can think of $|m_{d,k}|$ as giving the length of k-th row in a Young diagram, with the caveat that for $d = 2n$ the diagrams of height n can correspond to self-dual tensors ($m_{d,n} > 0$) or anti-self-dual tensors ($m_{d,n} < 0$). Because of that, we will often represent the vectors $\mathbf{m}_d$ by Young diagrams, for example,

$$(5, 0, 0, \ldots) = \ydiagram{5}, \tag{5.12}$$

$$(5, 3, 1, 0, \ldots) = \ydiagram{5,3,1}, \tag{5.13}$$

$$(0, 0, \ldots) = \bullet. \tag{5.14}$$

Note that we denote the empty diagram corresponding to the trivial representation by $\bullet$. We will also sometimes use the notation

$$\mathbf{j} \equiv \square\square\cdots\square \quad (j \text{ boxes}), \tag{5.15}$$

$$(\mathbf{j}, \ydiagram{2,1}) \equiv \begin{array}{l}\square\square\square\square\cdots\square\\ \square\square\\ \square\end{array} \quad (j \text{ boxes in 1st row}). \tag{5.16}$$

Note, however, that we do *not* restrict our analysis to bosonic representations only.

For future convenience, we define

$$|\mathbf{m}_d| = \sum_{k=1}^{n} |m_{d,k}|, \tag{5.17}$$

which gives the number of boxes when $\mathbf{m}_d$ can be represented by a Young diagram.

Examples For example, consider $d = 2$. Strictly speaking, this case does not fall under the above discussion, since $Spin(2)$ is not semi-simple. However, the vectors $\mathbf{m}_2$ can still be used to label the representations, and this will be important to us in the following. The vectors $\mathbf{m}_2$ are one-dimensional, with a single (half-)integral entry $m = m_{2,1}$. The corresponding representation is the one-dimensional representation which associates to rotation $e^{\phi M_{12}}$ the phase factor $e^{-im\phi}$.[6] This is 4π-periodic for half-integral m, corresponding to the need to consider the double-cover $Spin(2)$ instead of $SO(2)$.

Now consider $d = 3$ corresponding to B_1 case. In this case the vector $\mathbf{m}_3$ consists of a single component equal to $b/2$, where b is the unique Dynkin label. In other words $\mathbf{m}_3 = (j)$, where j is the usual spin of $Spin(3)$.

The case $d = 4$ corresponds to D_2. We have two Dynkin labels, which we will denote by $l_L = a/2, l_R = c/2$. For example, the vector representation is given by

[6] We choose the minus sign for future convenience.

$(l_L, l_R) = (\frac{1}{2}, \frac{1}{2})$, while the Dirac spinors are $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The vector $\mathbf{m}_4$ is two dimensional with the components,

$$\mathbf{m}_4 = (l_L + l_R, l_L - l_R). \tag{5.18}$$

We see that for traceless-symmetric representations with $l_L = l_R$ we recover the one row Young diagram, while for example for the representations $(1, 0)$ or $(0, 1)$ we recover the diagram $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ with self- or anti-self-duality condition.

5.2.1 Dimensional reduction

Labeling the representations by the vectors $\mathbf{m}_d$ is convenient for describing the rule for dimensional reduction from $Spin(d)$ to $Spin(d-1)$. More precisely, an irreducible representation $\mathbf{m}_d$ decomposes into a direct sum of irreducible representations $\mathbf{m}_{d-1}$ of $Spin(d-1)$, which we can write as

$$\mathbf{m}_d = \bigoplus_{\mathbf{m}_{d-1} \in \mathbf{m}_d} N^{\mathbf{m}_d}_{\mathbf{m}_{d-1}} \, \mathbf{m}_{d-1}, \tag{5.19}$$

where $N^{\mathbf{m}_d}_{\mathbf{m}_{d-1}}$ denote the multiplicity with which $\mathbf{m}_{d-1}$ appears in the irreducible decomposition of $\mathbf{m}_d$. It turns out that all multiplicities are equal to one,

$$N^{\mathbf{m}_d}_{\mathbf{m}_{d-1}} = 1, \quad \forall \mathbf{m}_{d-1} \in \mathbf{m}_d. \tag{5.20}$$

We say that dimensional reduction is multiplicity-free. The representations $\mathbf{m}_{d-1} \in \mathbf{m}_d$ are described by the following rule [210]:

From $Spin(2n+1)$ to $Spin(2n)$: For an irreducible representation $\mathbf{m}_d$ of $Spin(d)$, $d = 2n + 1$, and an irreducible representation $\mathbf{m}_{d-1}$ of $Spin(d-1)$ the relation $\mathbf{m}_{d-1} \in \mathbf{m}_d$ holds iff both representations are of the same statistics (fermionic or bosonic) and satisfy

$$m_{d,1} \geq m_{d-1,1} \geq m_{d,2} \geq m_{d-1,2} \geq \ldots \geq m_{d,n} \geq |m_{d-1,n}| \geq 0. \tag{5.21}$$

From $Spin(2n)$ to $Spin(2n-1)$: For an irreducible representation $\mathbf{m}_d$ of $Spin(d)$, $d = 2n$, and an irreducible representation $\mathbf{m}_{d-1}$ of $Spin(d-1)$ the relation $\mathbf{m}_{d-1} \in \mathbf{m}_d$ holds iff both representations are of the same statistics (fermionic or bosonic) and satisfy

$$m_{d,1} \geq m_{d-1,1} \geq m_{d,2} \geq m_{d-1,2} \geq \ldots \geq m_{d-1,n-1} \geq |m_{d,n}| \geq 0. \tag{5.22}$$

Examples Consider first the reduction from $Spin(4)$ to $Spin(3)$. The constraint is

$$m_{4,1} \geq m_{3,1} \geq |m_{4,2}|, \tag{5.23}$$

which in terms of j, l_L, l_R reads

$$l_L + l_R \geq j \geq |l_L - l_R|. \tag{5.24}$$

Together with the constraint that the Fermi/Bose statistics is preserved, we find that

$$j = |l_L - l_R|, |l_L - l_R| + 1, \ldots, l_L + l_R. \tag{5.25}$$

This is the same as saying that $j \in l_L \otimes l_R$, where l_L and l_R are interpreted as $Spin(3)$ spins, which coincides with the familiar reduction rule.

Consider now the reduction from $Spin(3)$ to $Spin(2)$. For a given $\mathbf{m}_3 = (j)$ we have the following constraint on $\mathbf{m}_2 = (m)$,

$$j \geq |m| \geq 0, \tag{5.26}$$

and m should be (half-)integral simultaneously with j. In other words, $m = -j, -j + 1, \ldots j$. It is no accident that the relation between j and m is the same as in the basis elements $|j, m\rangle$, because the $Spin(2)$ irreps are one-dimensional. This in fact is a very powerful observation which generalizes to higher dimensions, as we now discuss.

5.2.2 Gelfand-Tsetlin basis

The fact that the dimensional reduction is multiplicity-free allows one to define a convenient basis for the irreducible representations of $Spin(d)$. To construct it, one first fixes a sequence of subgroups

$$Spin(d) \supset Spin(d-1) \supset Spin(d-2) \supset \ldots \supset Spin(2). \tag{5.27}$$

In practice, we pick an orthonormal basis $e_1, \ldots e_d$ in $\mathbb{R}^d$, and the $Spin(d-k)$ subgroup in the above sequence is defined as the one preserving the basis elements $e_1, \ldots, e_k$. Then, given a representation $\mathbf{m}_d$, we can consider an irreducible component $\mathbf{m}_{d-1} \in \mathbf{m}_d$ with respect to $Spin(d-1)$. Since the dimensional reduction is multiplicity-free, by specifying the numbers $\mathbf{m}_{d-1}$ we uniquely select an $Spin(d-1)$-irreducible subspace inside the representation space $V_{\mathbf{m}_d}$ of the representation $\mathbf{m}_d$. We can then continue to build a sequence

$$\mathbf{m}_d \ni \mathbf{m}_{d-1} \ni \mathbf{m}_{d-2} \ni \ldots \ni \mathbf{m}_2, \tag{5.28}$$

which uniquely selects a $Spin(2)$-irreducible subspace inside $V_{\mathbf{m}_d}$. Since $Spin(2)$ is abelian, all such subspaces are one-dimensional. Therefore, if we in addition make a choice of phases, the above sequence specifies a unit vector in $V_{\mathbf{m}_d}$.

Let us now denote a sequence of $\mathbf{m}_k$, $k = d, d-1, \ldots, 2$ by $\mathfrak{M}_d$. Call a sequence $\mathfrak{M}_d$ admissible if (5.28) is satisfied. The above construction associates to each admissible sequence a vector $|\mathfrak{M}_d\rangle$ in $V_{\mathbf{m}_d}$. It is an easy exercise to show that the set of $|\mathfrak{M}_d\rangle$ over all admissible sequences (with $\mathbf{m}_d$ fixed) forms an orthonormal basis in $V_{\mathbf{m}_d}$. This is the Gelfand-Tsetlin (GT) basis [211], and the sequences $\mathfrak{M}_d$ are known as Gelfand-Tsetlin patterns.

Analogously to the well-known formulas for the matrix elements of $Spin(3)$ generators between the $|j, m\rangle$ states, Gelfand and Tsetlin have derived formulas for the matrix elements of $Spin(d)$ generators in Gelfand-Tsetlin basis for arbitrary representations [210–213]. We provide these formulas for reference in section 5.2.3 and appendix D.2. Availability of such general formulas is one of the reasons why Gelfand-Tsetlin bases are useful. For our purposes the more important reason is that these bases play nicely with the inclusions (5.27), which appear naturally in construction of conformally invariant tensor structures [1].

Choice of phases Before proceeding further, let us make a general comment about the choice of phases for vectors $|\mathfrak{M}_d\rangle$. This choice is not going to be important in the discussion that follows – it only influences the explicit expressions for $Spin(d)$ matrix elements, Clebsch-Gordan coefficients, etc. Therefore, we should only worry about it when we compute these quantities, and we can make a choice which is the most convenient for our purposes. For example the formulas given in appendix D.2 correspond to some particular choice of phases. We have made this choice so that it is compatible with the explicit constructions in the the examples below, unless explicitly stated otherwise.

Notation As we mentioned above, for us the utility of GT bases comes from their compatibility with the nested sequence (5.27), which plays an important role in classification of conformally-invariant tensor structures [1]. Unfortunately, this means that we will have to dive into the structure of the sequences $\mathfrak{M}_d$ quite often. Because of that, it is important to establish a well-defined notation.

Firstly, we will always explicitly write the space dimension d to which a weight $\mathbf{m}_d$ corresponds as a subscript. Secondly, the GT patterns in representation with highest weight $\mathbf{m}_d$ will be denoted by the capital Fraktur letter $\mathfrak{M}_d$. Distinct patterns in the same $\mathbf{m}_d$ will be distinguished by primes, i.e., $\mathfrak{M}'_d$. The subscript on the pattern indicates the dimension d corresponding to the first weight in the pattern. This weight is kept fixed and equal to $\mathbf{m}_d$ when we write summation as

$$\sum_{\mathfrak{M}_d} . \tag{5.29}$$

In all summations it is assumed implicitly that only admissible sequences are included.

Furthermore, $\mathbf{m}_k$ for $k \leq d'$ is always used to denote the components of the GT pattern $\mathfrak{M}_{d'}$. In particular, this means that the pattern $\mathfrak{M}_{d-1}$ is the tail[7] of the pattern $\mathfrak{M}_d$ and we have, for example,

$$\sum_{\mathfrak{M}_d} \equiv \sum_{\mathbf{m}_{d-1}} \sum_{\mathfrak{M}_{d-1}} . \tag{5.30}$$

We also occasionally write $\mathfrak{M}_d = \mathbf{m}_d \, \mathfrak{M}_{d-1}$, etc, arranging the right hand side either vertically or horizontally, whichever way leads to more compact expressions. We also sometimes write out the GT patterns explicitly as

$$\mathfrak{M}_d \equiv \mathbf{m}_d, \mathbf{m}_{d-1}, \ldots, \mathbf{m}_2. \tag{5.31}$$

If we have $\mathbf{m}_k = \bullet$, then necessarily $\mathbf{m}_i = \bullet$ for $i \leq k$. We therefore often write the patterns out only to the first trivial representation, replacing the rest by dots. For example,

$$\mathfrak{M}_d = \ydiagram{5,2}, \ \ydiagram{3}, \bullet, \ldots \tag{5.32}$$

has $\mathbf{m}_k = \bullet$ for all $k \leq d-2$.

Different representations and patterns are distinguished either by different letters (i.e., $\mathbf{u}_d$ and $\mathfrak{U}_d$ vs $\mathbf{m}_d$ and $\mathfrak{M}_d$), accents other than primes (i.e., $\widetilde{\mathbf{m}}_d$ and $\widetilde{\mathfrak{M}}_d$ vs $\mathbf{m}_d$ and $\mathfrak{M}_d$), or upper indices (i.e., $\mathbf{m}^1_d$ and $\mathfrak{M}^1_d$ vs $\mathbf{m}_d$ and $\mathfrak{M}_d$). To reiterate, the lower index only "addresses" inside one pattern.

Our final comment concerns the use of GT patterns as indices. We will assume that the upper GT indices, such as

$$O^{\mathfrak{M}_d}, \tag{5.33}$$

[7] This is slightly in tension with our convention on primes. We will understand that $\mathfrak{M}'_{d-1}$ is the tail of $\mathfrak{M}'_d$, i.e., $\mathbf{m}'_{d-1}$ is not necessarily the same as $\mathbf{m}_{d-1}$ (which would be the case if we gave the priority to the prime notation rule and understood $\mathfrak{M}'_{d-1}$ as another pattern in $\mathbf{m}_{d-1}$). Note also that by this convention $\mathbf{m}'_{d-1} \in \mathbf{m}_d$, etc.

behave as ket states $|\mathfrak{M}_d\rangle$, while the lower indices behave as the dual bra states $\langle\mathfrak{M}_d|$, i.e.,

$$[M_{\mu\nu}, O^{\mathfrak{M}_d}(0)] = \sum_{\mathfrak{M}'_d} \langle\mathfrak{M}'_d|M_{\mu\nu}|\mathfrak{M}_d\rangle O^{\mathfrak{M}'_d}(0), \tag{5.34}$$

$$[O_{\mathfrak{M}_d}(0), M_{\mu\nu}] = \sum_{\mathfrak{M}'_d} \langle\mathfrak{M}_d|M_{\mu\nu}|\mathfrak{M}'_d\rangle O_{\mathfrak{M}'_d}(0). \tag{5.35}$$

5.2.2.1 Bilinear parings

The most basic invariants of $Spin(d)$ are the bilinear parings, such as the paring between a representation and its dual, or the invariant inner product in real representations. A bilinear pairing between irreducible representations $\mathbf{m}_d$ and $\mathbf{u}_d$ is a singlet in the tensor product

$$\mathbf{m}_d \otimes \mathbf{u}_d. \tag{5.36}$$

Schur's lemma implies that there is at most one such singlet, which exists iff $\mathbf{m}_d = \overline{\mathbf{u}_d}$, i.e., when the representations are mutually dual (equivalently, complex conjugate). The duality acts on the $Spin(d)$ irreps as follows. For odd d all irreps are self-dual, $\mathbf{m}_{2n+1} = \overline{\mathbf{m}_{2n+1}}$, as well as for d divisible by 4, $\mathbf{m}_{4k} = \overline{\mathbf{m}_{4k}}$. For $d = 4k + 2$ the duality acts non-trivially by exchanging the spinor nodes on D_{2k+1} Dynkin diagram, resulting in

$$m_{4k+2,i} = \overline{m_{4k+2,i}}, \quad i < n = 2k + 1, \tag{5.37}$$

$$m_{4k+2,2k+1} = -\overline{m_{4k+2,2k+1}}. \tag{5.38}$$

It is quite easy to write down the formula for the singlet in $\mathbf{m}_d \otimes \overline{\mathbf{m}_d}$ in GT basis. Indeed, it has to be singlet under all groups in 5.27 and thus the above discussion implies that it must be of the form

$$\sum_{\mathfrak{M}_d} \zeta_{\mathfrak{M}_d} |\mathfrak{M}_d\rangle \otimes |\overline{\mathfrak{M}_d}\rangle, \tag{5.39}$$

where $\overline{\mathfrak{M}_d}$ is obtained from GT pattern $\mathfrak{M}_d$ by replacing all representations with their duals, and the coefficients $\zeta_{\mathfrak{M}_d}$ are yet to be determined. Let us define

$$(-1)^{\mathbf{m}_{2n+1}} = 1, \tag{5.40}$$

$$(-1)^{\mathbf{m}_{4k}} = 1, \tag{5.41}$$

$$(-1)^{\mathbf{m}_{4k+2}} = (-1)^{m_{4k+2,2k+1}}, \tag{5.42}$$

$$(-1)^{\mathfrak{M}_d} = \prod_{k=2}^{d} (-1)^{\mathbf{m}_k}. \tag{5.43}$$

With the choice of phases as in appendix D.2, the coefficients $\zeta_{\mathfrak{M}_d}$ are proportional to $(-1)^{\mathfrak{M}_d}$.[8] In what follows, we will use the notation

$$\langle \mathfrak{M}_d, \overline{\mathfrak{M}'_d}|0\rangle \equiv \zeta_{\mathfrak{M}_d}\delta_{\mathfrak{M}_d,\mathfrak{M}'_d}, \tag{5.44}$$

so that the singlet (5.39) can be written as

$$\sum_{\mathfrak{M}_d,\mathfrak{M}'_d} \langle \mathfrak{M}_d, \overline{\mathfrak{M}'_d}|0\rangle\ |\mathfrak{M}_d\rangle \otimes |\overline{\mathfrak{M}'_d}\rangle. \tag{5.45}$$

Note that this is a special case of Clebsch-Gordan coefficients, which suggests the normalization condition

$$\sum_{\mathfrak{U}_d,\mathfrak{M}_d} \langle 0|\mathfrak{M}_d, \mathfrak{U}_d\rangle\langle \mathfrak{M}_d, \mathfrak{U}_d|0\rangle \equiv \sum_{\mathfrak{U}_d,\mathfrak{M}_d} (\langle \mathfrak{M}_d, \mathfrak{U}_d|0\rangle)^*\langle \mathfrak{M}_d, \mathfrak{U}_d|0\rangle = 1. \tag{5.46}$$

It corresponds to the requirement that (5.45) has unit norm. This implies

$$\langle \mathfrak{M}_d, \overline{\mathfrak{M}'_d}|0\rangle \equiv \frac{(-1)^{\mathfrak{M}_d}}{\sqrt{\dim \mathbf{m}_d}}\delta_{\mathfrak{M}_d,\mathfrak{M}'_d}. \tag{5.47}$$

Whenever $\mathbf{m}_d = \overline{\mathbf{m}_d}$ these coefficients have a definite symmetry under permutation of the two tensor factors. For bosonic representations they are always symmetric, while for fermionic they are symmetric if $d = 0, 1, 7 \mod 8$ and anti-symmetric for $d = 3, 4, 5 \mod 8$, as can be easily verified by using the explicit formula above.[9] Fermionic representations are never self-dual for $d = 2, 6 \mod 8$.

5.2.2.2 Vector representation

To gain some familiarity with GT bases, it is perhaps a good idea to start with the vector representation of $Spin(d)$. The vector representation is also going to play an extremely important role in section 5.4.

First of all, for $d > 3$, under dimensional reduction the d-dimensional vector representation splits into two irreducible components – a scalar and a $(d-1)$-dimensional vector. For $d = 3$ we obtain three representations, the $+1, \bullet, -1$ representations of $Spin(2)$. This means that the GT basis for vector representation consists of the

[8] We have not proven this statement, but we have checked it on a large sample of representations in various dimensions.

[9] If these coefficients are symmetric, then the self-dual $\mathbf{m}_d$ is real and otherwise it is pseudo-real (quaternionic). This statement is specific to Euclidean signature (in Lorentzian dual and complex conjugate representations are not the same), but the symmetry properties are signature-independent.

following elements,

$$|\square,\bullet,\bullet,\ldots,\bullet,\bullet\rangle, \tag{5.48}$$

$$|\square,\square,\bullet,\ldots,\bullet,\bullet\rangle, \tag{5.49}$$

$$|\square,\square,\square,\ldots,\bullet,\bullet\rangle, \tag{5.50}$$

$$\vdots$$

$$|\square,\square,\square,\ldots,\square,\bullet\rangle, \tag{5.51}$$

$$|\square,\square,\square,\ldots,\square,+1\rangle, \tag{5.52}$$

$$|\square,\square,\square,\ldots,\square,-1\rangle. \tag{5.53}$$

Given that each sequence contains the $d-1$ irreps (5.28), it is easy to see that the above gives exactly d basis vectors.

Let us consider the element (5.48). By definition, it lives in the trivial representation of $Spin(d-k)$ for $k \geq 1$ and thus has to be proportional to e_1. Similarly, (5.49) is invariant for $k \geq 2$ and thus has to be a linear combination of e_1 and e_2. Since it also has to be orthogonal to (5.48), it can only be proportional to e_2. Repeating this argument, and making a choice of phases, we find

$$|\square,\bullet,\bullet,\ldots,\bullet,\bullet\rangle = (-1)^d e_1, \tag{5.54}$$

$$|\square,\square,\bullet,\ldots,\bullet,\bullet\rangle = (-1)^{d-1} e_2, \tag{5.55}$$

$$|\square,\square,\square,\ldots,\bullet,\bullet\rangle = (-1)^{d-2} e_3, \tag{5.56}$$

$$\vdots$$

$$|\square,\square,\square,\ldots,\square,\bullet\rangle = (-1)^3 e_{d-2}, \tag{5.57}$$

$$|\square,\square,\square,\ldots,\square,+1\rangle = (-1)^2 \frac{e_{d-1}+ie_d}{\sqrt{2}}, \tag{5.58}$$

$$|\square,\square,\square,\ldots,\square,-1\rangle = (-1)^1 \frac{e_{d-1}-ie_d}{\sqrt{2}}. \tag{5.59}$$

In the above expressions the phases are chosen to be consistent with the formulas for the matrix elements in appendix D.2 and the interpretation that M_{ij} "rotates from i to j",

$$M_{ij} e_i = e_j. \tag{5.60}$$

Note that according to our conventions for $Spin(2)$ representations described earlier, we have

$$M_{d-1,d}|\square,\square,\square,\ldots,\square,\pm1\rangle = \mp i|\square,\square,\square,\ldots,\square,\pm1\rangle. \tag{5.61}$$

This approach generalizes to other representations. In appendix D.4 we consider the relation between GT and Cartesian bases in tensor representations of $Spin(d)$.

Let us now look at the inner product between vectors. Note that $\mathbf{m}_{4k+2,2k+1}$ is only non-zero in the GT patterns (5.58) and (5.59) and for $k = 0$. Thus $(-1)^{\mathfrak{M}_d}$ is -1 for these two patterns and 1 otherwise. Finally, these two patterns are mutually dual, while all other patterns are self-dual, so that according to (5.45) and (5.47) we get the following pairing, up to normalization,

$$\begin{aligned}&|\,\square, \bullet, \ldots\rangle \otimes |\,\square, \bullet, \ldots\rangle + |\,\square, \square, \bullet, \ldots\rangle \otimes |\,\square, \square, \bullet, \ldots\rangle + \ldots \\ &- |\,\square, \ldots, \square, +1\rangle \otimes |\,\square, \ldots, \square, -1\rangle - |\,\square, \ldots, \square, -1\rangle \otimes |\,\square, \ldots, \square, +1\rangle.\end{aligned} \tag{5.62}$$

From (5.54)-(5.59) we see that this is equal to

$$\sum_{i=1}^{d} e_i \otimes e_i, \tag{5.63}$$

which is the usual pairing between vectors.

5.2.2.3 General representations in 3 dimensions

We now consider the case of general representations in $d = 3$ ($n = 1$). As before, the representations $\mathbf{m}_3$ are labeled by a (half-)integer $j \equiv m_{3,1} \geq 0$, which is the usual spin, and the representations $\mathbf{m}_2$ are labeled by a (half-)integer $m \equiv m_{2,1}$. The representations $\mathbf{m}_2 \in \mathbf{m}_3$ are given by $m = -j, -j+1, \ldots, j$. The GT basis vectors are then

$$|\mathfrak{M}_3\rangle \equiv |\mathbf{m}_3, \mathbf{m}_2\rangle \equiv |j, m\rangle. \tag{5.64}$$

We can choose conventions such that this coincides with the basis of $Spin(3)$ representations familiar from the theory of angular momenta. Indeed, let us first define the anti-Hermitian generators

$$I_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\lambda} M_{\nu\lambda}, \tag{5.65}$$

which are then subject to the commutation relation (see appendix D.1),

$$[I_\mu, I_\nu] = \epsilon_{\mu\nu\lambda} I_\lambda. \tag{5.66}$$

Their Hermitian analogues $J_\mu = iI_\mu$ satisfy the familiar $Spin(3)$ commutation relations

$$[J_\mu, J_\nu] = i\epsilon_{\mu\nu\lambda} J_\lambda. \tag{5.67}$$

If we now define

$$\hat{1} \equiv 2, \quad \hat{2} \equiv 3, \quad \hat{3} \equiv 1, \tag{5.68}$$

then the operators $J_{\hat{\mu}}$ satisfy the same commutation relations. By definition, we have

$$J_{\hat{3}}|j,m\rangle = iI_1|j,m\rangle = iM_{23}|j,m\rangle = i(-im)|j,m\rangle = m|j,m\rangle. \tag{5.69}$$

We have performed the index relabeling (5.68) precisely so that $|j,m\rangle$ are eigenstates of $J_{\hat{3}}$, making contact with standard angular momentum conventions. In particular, the standard [214] formulas for action of $J_{\hat{\mu}}$ coincide with $d = 3$ case of formulas in appendix D.2.

5.2.2.4 General representations in 4 dimensions

In $d = 4$ ($n = 2$), we have

$$\mathbf{m}_4 = (\ell_1, \ell_2) = (l_L + l_R, l_L - l_R), \tag{5.70}$$

$$\mathbf{m}_3 = j = |l_L - l_R|, |l_L - l_R| + 1, \ldots, l_L + l_R, \quad \Leftrightarrow \quad j \in l_L \otimes l_R, \tag{5.71}$$

$$\mathbf{m}_2 = m = -j, -j+1, \ldots, j, \tag{5.72}$$

and thus we can write

$$|\mathfrak{M}_4\rangle \equiv |l_L, l_R; j, m\rangle. \tag{5.73}$$

It will be convenient to connect this to the basis which arises from the exceptional isomorphism $Spin(4) \simeq SU(2) \times SU(2)$. To define this latter basis, we write

$$Q_\mu \equiv M_{1\mu}, \quad I_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda}M_{\nu\lambda}, \quad \mu,\nu,\lambda \in \{2,3,4\}, \tag{5.74}$$

where $\epsilon_{234} = 1$. Then the Hermitian operators

$$J^L_\mu \equiv iI^L_\mu \equiv \frac{i}{2}(I_\mu + Q_\mu), \quad J^R_\mu \equiv iI^R_\mu \equiv \frac{i}{2}(I_\mu - Q_\mu) \tag{5.75}$$

obey the commutation relations

$$[J^L_\mu, J^L_\nu] = i\epsilon_{\mu\nu\lambda}J^L_\lambda, \tag{5.76}$$

$$[J^R_\mu, J^R_\nu] = i\epsilon_{\mu\nu\lambda}J^R_\lambda, \tag{5.77}$$

$$[J^L_\mu, J^R_\nu] = 0. \tag{5.78}$$

We can then define, similarly to 3 dimensions,

$$\hat{1} \equiv 3, \quad \hat{2} \equiv 4, \quad \hat{3} \equiv 2, \tag{5.79}$$

and construct the conventional basis states for the algebras $J^L_{\hat{\mu}}, J^R_{\hat{\mu}}$,

$$|l_L, m_L; l_R, m_R\rangle \tag{5.80}$$

subject to the usual condition

$$J^L_{\hat{3}}|l_L, m_L; l_R, m_R\rangle = m_L|l_L, m_L; l_R, m_R\rangle, \tag{5.81}$$

$$J^R_{\hat{3}}|l_L, m_L; l_R, m_R\rangle = m_R|l_L, m_L; l_R, m_R\rangle. \tag{5.82}$$

Let us now relate the bases (5.73) and (5.80). First, note that the generators $J_{\hat{\mu}} \equiv iI_{\hat{\mu}}$ of the $Spin(3)$ which preserves the first axis are given by

$$J_{\hat{\mu}} = J^L_{\hat{\mu}} + J^R_{\hat{\mu}}, \tag{5.83}$$

and thus under this $Spin(3)$ the state (5.80) transforms as a tensor product state in $l_L \otimes l_R$. We can therefore simply set

$$|l_L, l_R; j, m\rangle \equiv \sum_{m_L+m_R=m} \langle l_L, m_L; l_R, m_R|j, m\rangle |l_L, m_L; l_R, m_R\rangle, \tag{5.84}$$

where

$$\langle l_L, m_L; l_R, m_R|j, m\rangle \tag{5.85}$$

are the Clebsch-Gordan coefficients of $Spin(3)$. It is easy to check that this definition is consistent with the definition of GT basis. Note that (5.84) essentially fixes our choice of phases through the phases of $Spin(3)$ CG coefficients. The resulting phase conventions are consistent with appendix D.2 if one uses CG coefficients $\langle j_1, m_1; j_2; m_2|j, m\rangle$ which differ from [214] by a factor of $i^{j-j_1-j_2}$.[10]

For future reference, let us give the expression for $M_{12} = Q_2$. We have

$$M_{12} = Q_2 = -iJ^L_2 + iJ^R_2 = -iJ^L_{\hat{3}} + iJ^R_{\hat{3}}. \tag{5.86}$$

[10]These CG coefficients will still differ from the vector CG coefficients of D.2 by a factor of $-i$ when $j = j_1$ and $j_2 = 1$, but the matrix elements in 4d will be consistent.

5.2.3 Clebsch-Gordan coefficients and matrix elements

In the next sections we will find that a lot of calculations (for example, three-point tensor structures and Casimir recursion relations) involve manipulations with Clebsch-Gordan coefficients (CG coefficients). In this section we therefore discuss the structure of these coefficients in GT bases.

CG coefficients essentially establish an equivalence between a tensor product and its decomposition into irreducible representations,

$$V_{\mathbf{m}_d^1} \otimes V_{\mathbf{m}_d^2} \simeq \bigoplus_{\mathbf{m}_d \in \mathbf{m}_d^1 \otimes \mathbf{m}_d^2} V_{\mathbf{m}_d}. \tag{5.87}$$

More specifically, we have the relation between basis vectors

$$|\mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle = \sum_{\mathbf{m}_d \in \mathbf{m}_d^1 \otimes \mathbf{m}_d^2} \sum_{\mathfrak{M}_d} \langle \mathfrak{M}_d | \mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle \; |\mathfrak{M}_d\rangle, \tag{5.88}$$

where $\langle \mathfrak{M}_d | \mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle$ are the CG coefficients. This equation has to be modified somewhat if there are multiplicities in the tensor product,

$$|\mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle = \sum_{(\mathbf{m}_d, t) \in \mathbf{m}_d^1 \otimes \mathbf{m}_d^2} \sum_{\mathfrak{M}_d} \langle \mathfrak{M}_d, t | \mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle \; |\mathfrak{M}_d, t\rangle. \tag{5.89}$$

Here t counts the possible degeneracy. Inverse transformation is given by

$$|\mathfrak{M}_d, t\rangle = \sum_{\mathfrak{M}_d^1 \mathfrak{M}_d^2} \langle \mathfrak{M}_d^1 \mathfrak{M}_d^2 | \mathfrak{M}_d, t\rangle \; |\mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle, \tag{5.90}$$

where $\langle \mathfrak{M}_d^1 \mathfrak{M}_d^2 | \mathfrak{M}_d, t\rangle = \langle \mathfrak{M}_d, t | \mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle^*$. Note that there is an ambiguity in the definition of CG coefficients. Indeed, the decomposition

$$|\mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle = \sum_{(\mathbf{m}_d, t) \in \mathbf{m}_d^1 \otimes \mathbf{m}_d^2} \sum_{\mathfrak{M}_d, t'} U_{tt'} \langle \mathfrak{M}_d, t' | \mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle \; |\mathfrak{M}_d, t\rangle, \tag{5.91}$$

where U is a unitary matrix, is also perfectly fine from the point of view of $Spin(d)$ invariance. One thus has to fix this freedom for every choice of $\mathbf{m}_d^1$ and $\mathbf{m}_d^2$. We will not try to fix the general conventions here, and work on a case-by-case basis in the examples.

GT bases exhibit a set of relations between the CG coefficients of the nested groups (5.27). Indeed, let us write the GT patters in CG coefficients (5.89) in the form $\mathfrak{M}_d = \mathbf{m}_d \, \mathfrak{M}_{d-1}$,

$$\langle \mathfrak{M}_d, t | \mathfrak{M}_d^1 \mathfrak{M}_d^2\rangle \equiv \langle \mathbf{m}_d \, \mathfrak{M}_{d-1}, t | \mathbf{m}_d^1 \, \mathfrak{M}_{d-1}^1; \mathbf{m}_d^2 \, \mathfrak{M}_{d-1}^2\rangle. \tag{5.92}$$

Thinking about $Spin(d-1)$-invariance, we see that must necessarily have

$$\langle \mathbf{m}_d\, \mathfrak{M}_{d-1}\, t|\mathbf{m}_d^1, \mathfrak{M}_{d-1}^2; \mathbf{m}_d^2\, \mathfrak{M}_{d-1}^2\rangle = \sum_{t'} \left(\begin{array}{c|cc} \mathbf{m}_d & \mathbf{m}_d^1 & \mathbf{m}_d^2 \\ \mathbf{m}_{d-1} & \mathbf{m}_{d-1}^1 & \mathbf{m}_{d-1}^2 \end{array}\right)_{tt'} \langle \mathfrak{M}_{d-1}, t'|\mathfrak{M}_{d-1}^1 \mathfrak{M}_{d-1}^2\rangle \tag{5.93}$$

where the constants

$$\left(\begin{array}{c|cc} \mathbf{m}_d & \mathbf{m}_d^1 & \mathbf{m}_d^2 \\ \mathbf{m}_{d-1} & \mathbf{m}_{d-1}^1 & \mathbf{m}_{d-1}^2 \end{array}\right)_{tt'} \tag{5.94}$$

are the so-called $Spin(d) : Spin(d-1)$ isoscalar factors,[11] while $\langle \mathfrak{M}_{d-1}, t'|\mathfrak{M}_{d-1}^1 \mathfrak{M}_{d-1}^2\rangle$ are the CG coefficients of $Spin(d-1)$. This can be iterated, and since the CG coefficients of $Spin(2)$ are extremely simple,

$$\langle m|m^1 m^2\rangle = \delta_{m,m^1+m^2}, \tag{5.95}$$

it follows that the knowledge of CG coefficients of $Spin$ groups is equivalent to the knowledge of the isoscalar factors.

For example, the $Spin(3) : Spin(2)$ isoscalar factors are essentially the $Spin(3)$ CG coefficients, due to the aforementioned triviality of $Spin(2)$ CG coefficients. One can show that the $Spin(4) : Spin(3)$ isoscalar factors are essentially equivalent to $Spin(3)$ $9j$ symbols [215].

For our applications we in principle need the most general CG coefficients of $Spin(d-1)$ groups – simply the knowledge of all possible conformally-invariant three-point tensor structures already implies the knowledge of all possible $Spin(d-1)$ CG coefficients (see section 5.3.1). We are not aware of a general formula for $Spin(d-1)$ CG coefficients valid for general d.[12] For the most physically relevant cases $d = 4, 3$ one can use the well-known CG coefficients of $Spin(3) \simeq SU(2)$ or the trivial CG coefficients of $Spin(2) \simeq U(1)$. Due to the exceptional isomorphism $Spin(4) \simeq SU(2)\times SU(2)$, we also know the general CG coefficients of $Spin(d-1)$ for $d = 5$. Let us note that the case $d \geq 6$ is qualitatively different since tensor products in $Spin(5)$ and larger groups are not multiplicity-free. Luckily, for each particular choice of a four-point function there is only a finite number of relevant three-point tensor structures and thus also of $Spin(d-1)$ CG coefficients. For any given tensor product, the problem of finding CG coefficients is a finite-dimensional

[11] Also known as reduced CG, reduced Wigner coefficients, or reduction factors.
[12] See [216, 217] for partial progress in this direction.

linear algebra problem and can in principle be solved on a computer, although phase conventions and resolution of multiplicities need to be carefully addressed. See [218] for an approach to $Spin(5)$ CG coefficients.

For the applications to Casimir recursion relations, we will need a special infinite class of CG coefficients of $Spin(d)$ – the CG coefficients involving a vector representation. The good news are that these CG coefficients are known for general d in closed form.

$Spin(d)$ **matrix elements and Clebsch-Gordan coefficients with vector representation** It turns out that Clebsch-Gordan coefficients for vector representation are closely related to the matrix elements of $Spin(d)$ generators. Indeed, let us consider the matrix elements of $M_{1\mu}$,

$$M_{1\mu}|\mathfrak{M}_d\rangle = \sum_{\mathfrak{M}'_d} \langle \mathfrak{M}'_d|M_{1\mu}|\mathfrak{M}_d\rangle\, |\mathfrak{M}'_d\rangle, \qquad \mathbf{m}'_d = \mathbf{m}_d. \tag{5.96}$$

The piece $M_{1\mu}|\mathfrak{M}_d\rangle$ transforms under $Spin(d-1)$ in the representation $\square \otimes \mathbf{m}_{d-1}$. The vectors on the right, on the other hand, transform in irreducible representations of $Spin(d-1)$. For fixed $\mathbf{m}_d, \mathbf{m}_{d-1}$ this therefore has precisely the form required of a CG decomposition, so that we have

$$\langle \mathfrak{M}'_d|M_{1\mu}|\mathfrak{M}_d\rangle = \left(\begin{matrix} \mathbf{m}_d \\ \mathbf{m}'_{d-1} \end{matrix} \middle| M^{\square} \middle| \begin{matrix} \mathbf{m}_d \\ \mathbf{m}_{d-1} \end{matrix} \right) \langle \mathfrak{M}'_{d-1}|\mathfrak{M}_{d-1}, \mu\rangle \tag{5.97}$$

for some constants

$$\left(\begin{matrix} \mathbf{m}'_d \\ \mathbf{m}'_{d-1} \end{matrix} \middle| M^{\square} \middle| \begin{matrix} \mathbf{m}_d \\ \mathbf{m}_{d-1} \end{matrix} \right) \tag{5.98}$$

known as reduced matrix elements. This is essentially a version of Wigner-Eckart theorem. Note that the tensor product with vector representation is always multiplicity free and thus we don't need any extra labels. This follows from Brauer's formula [197] and the fact that all weights in the vector representation have multiplicity 1. The $\square$ label for M is supposed to indicate that we are looking at $M_{1\mu}$, which is a vector under $Spin(d-1)$.

Let us consider an example by setting $\mu = 2$, which is equivalent to $\mu = [\square, \bullet, \ldots]$

in terms of GT patterns. We then find

$$
\begin{aligned}
\langle \mathfrak{M}'_d | M_{12} | \mathfrak{M}_d \rangle =& (-1)^{d-1} \left(\begin{matrix} \mathbf{m}_d \\ \mathbf{m}'_{d-1} \end{matrix} \middle| M\square \middle| \begin{matrix} \mathbf{m}_d \\ \mathbf{m}_{d-1} \end{matrix} \right) \langle \mathfrak{M}'_{d-1} | \mathfrak{M}_{d-1}; \square, \bullet, \ldots \rangle \\
=& (-1)^{d-1} \left(\begin{matrix} \mathbf{m}_d \\ \mathbf{m}'_{d-1} \end{matrix} \middle| M\square \middle| \begin{matrix} \mathbf{m}_d \\ \mathbf{m}_{d-1} \end{matrix} \right) \left(\begin{array}{c|cc} \mathbf{m}'_{d-1} & \mathbf{m}_{d-1} & \square \\ \mathbf{m}'_{d-2} & \mathbf{m}_{d-2} & \bullet \end{array} \right) \langle \mathfrak{M}'_{d-2} | \mathfrak{M}_{d-2}; \bullet, \ldots \rangle \\
=& (-1)^{d-1} \left(\begin{matrix} \mathbf{m}_d \\ \mathbf{m}'_{d-1} \end{matrix} \middle| M\square \middle| \begin{matrix} \mathbf{m}_d \\ \mathbf{m}_{d-1} \end{matrix} \right) \left(\begin{array}{c|cc} \mathbf{m}'_{d-1} & \mathbf{m}_{d-1} & \square \\ \mathbf{m}'_{d-2} & \mathbf{m}_{d-2} & \bullet \end{array} \right) \delta_{\mathfrak{M}_{d-2}, \mathfrak{M}'_{d-2}}.
\end{aligned}
\tag{5.99}
$$

Here we used the definition of the isoscalar factor (5.93) and the triviality of CG coefficients when one of the factors is the trivial representation. We also made use of the relation (5.55). Note that this implies the constraint $\mathbf{m}'_{d-1} \in \square \otimes \mathbf{m}_{d-1}$. Due to the structure of the nested sequence (5.27) the matrix elements of $M_{k,k+1}$ for all $1 \leq k \leq d-1$ follow from the matrix elements of M_{12} for $Spin(d-k+1)$. It is an easy exercise to show that $M_{k,k+1}$ generate the whole Lie algebra of $Spin(d)$.

We therefore find that the reduced matrix elements (5.98) and the simplest vector isoscalar factors

$$
\left(\begin{array}{cc|c} \mathbf{m}_d & \square & \mathbf{m}'_d \\ \mathbf{m}_{d-1} & \bullet & \mathbf{m}_{d-1} \end{array} \right)
\tag{5.100}
$$

allow the computation of the most general $Spin(d)$ matrix elements. There exist relatively simple closed-form expressions for these quantities [212, 213], which we provide in appendix D.2 for the ease of reference.[13]

These quantities in fact also completely determine the vector CG coefficients. Indeed, given the isoscalar factor (5.100), it only remains to find the second isoscalar factor[14]

$$
\left(\begin{array}{cc|c} \mathbf{m}_{d-1} & \square & \mathbf{m}'_{d-1} \\ \mathbf{m}_{d-2} & \square & \mathbf{m}'_{d-2} \end{array} \right).
\tag{5.101}
$$

It can be easily computed by considering the expression

$$
\langle \mathfrak{M}_d; \square, \bullet, \ldots | M_{12} | \mathfrak{M}'_d \rangle
\tag{5.102}
$$

[13] Note that our phase conventions differ from those in [212, 213].

[14] For $d = 3$ we can have (± 1) instead of lower $\square$ in (5.101). The corresponding isoscalar factors can be obtained completely analogously. See appendix D.2.3.

and evaluating it via isoscalar factors and reduced matrix elements in two different ways (acting with M on the left and on the right). Action on the left produces, among other terms, the term

$$\langle \mathfrak{M}_d; \square, \square, \bullet, \ldots | \mathfrak{M}'_d \rangle, \tag{5.103}$$

which is proportional to the sought for isoscalar factor. See appendix D.2.2 for details.

5.3 Structure of spinning correlation functions and conformal blocks

In this section we apply the formalism of GT bases to study the general structure of radially-quantized correlators or conformal blocks. At this stage, no distinction is made between correlation functions and individual conformal blocks, so we use these two terms interchangeably.

5.3.1 Contribution of a $\mathbb{R} \times Spin(d)$-multiplet

Consider a 4-point correlation function, radially quantized so that the points 1 and 2 lie inside the unit sphere, whereas the points 3 and 4 lie outside (or on) the unit sphere. One can then insert a complete basis of states on the unit sphere, organized in representations of $\mathbb{R} \times Spin(d)$ (dilatations $\times$ rotations), and ask what is the contribution of a single representation. This question was answered in [59] for four-point functions with external scalar operators, exchanging traceless-symmetric tensors on the unit sphere (the only representations allowed in this this case). The case of four-point functions of tensor operators was addressed in [60]. Unfortunately, as mentioned in the introduction, the approach of [60] requires a non-trivial amount of case-by-case analysis and the knowledge of irreducible projectors. The goal of this section is to give a more general alternative treatment.

For concreteness, we will work in the radial kinematics of [59].[15] Namely, we chose an orthonormal basis in $\mathbb{R}^d$, labeling the axes by integers from 1 to d, and we introduce a complex coordinate w in plane 1-2 as

$$w = x_1 + i x_2. \tag{5.104}$$

We then place all four operators in this plane, setting their coordinates to

$$w_1 = -\rho, \quad w_2 = \rho \quad w_3 = 1, \quad w_4 = -1, \tag{5.105}$$

[15] The same approach also works in other kinematics. For examples, we will switch to Dolan-Osborn [57, 63] kinematics in section 5.4. The analysis in that case is only slightly different due to the presence of an operator at infinity.

for some $\rho \in \mathbb{C}$. Any non-coincident configuration of four points can be brought to a configuration of the above form by a conformal transformation, with ρ being related to the familiar cross-ratios u and v. We assume $|\rho| < 1$.

We also fix the sequence of groups (5.27), defining $Spin(d-k)$ to be the subgroup of $Spin(d)$ which fixes the first k axes. This defines for us Gelfand-Tsetlin bases for the representations of $Spin(d)$. We will accordingly denote the primary operators by

$$\mathcal{O}_i^{\mathfrak{M}_d^i}(w_i), \tag{5.106}$$

where the sequences $\mathfrak{M}_d^i$ label the Gelfand-Tsetlin basis vectors as in section 5.2.2, and we use the upper index i to label the operators in order to avoid confusion with the dimension label, $\mathfrak{M}_d^i = \mathbf{m}_d^i, \mathbf{m}_{d-1}^i, \ldots, \mathbf{m}_2^i$.

We are interested in the radially-quantized four-point function

$$\langle 0|\mathcal{O}_4^{\mathfrak{M}_d^4}(-1)\mathcal{O}_3^{\mathfrak{M}_d^3}(1)\mathcal{O}_2^{\mathfrak{M}_d^2}(\rho)\mathcal{O}_1^{\mathfrak{M}_d^1}(-\rho)|0\rangle. \tag{5.107}$$

It turns out that it is more convenient to work with

$$\langle 0|\mathcal{O}_4^{\mathfrak{M}_d^4}(-1)\mathcal{O}_3^{\mathfrak{M}_d^3}(1)r^D e^{\theta M_{12}}\mathcal{O}_2^{\mathfrak{M}_d^2}(1)\mathcal{O}_1^{\mathfrak{M}_d^1}(-1)|0\rangle, \tag{5.108}$$

where $\rho = re^{i\theta}$, D is the dilatation operator and $M_{\mu\nu}$ is the anti-hermitian rotation generator in the plane μ-ν.[16] The relation between (5.107) and (5.108) is given by

$$\begin{aligned}\langle 0|\mathcal{O}_4^{\mathfrak{M}_d^4}(-1)\mathcal{O}_3^{\mathfrak{M}_d^3}(1)r^D e^{\theta M_{12}}\mathcal{O}_2^{\mathfrak{M}_d^2}(1)\mathcal{O}_1^{\mathfrak{M}_d^1}(-1)|0\rangle &= \\ = r^{\Delta_1+\Delta_2}\sum_{\mathfrak{M}_d'^1,\mathfrak{M}_d'^2} R^{\mathfrak{M}_d^1}_{\mathfrak{M}_d'^1}(\theta)R^{\mathfrak{M}_d^2}_{\mathfrak{M}_d'^2}(\theta)\langle 0|\mathcal{O}_4^{\mathfrak{M}_d^4}(-1)&\mathcal{O}_3^{\mathfrak{M}_d^3}(1)\mathcal{O}_2^{\mathfrak{M}_d'^2}(\rho)\mathcal{O}_1^{\mathfrak{M}_d'^1}(-\rho)|0\rangle,\end{aligned} \tag{5.109}$$

where R are the matrix elements of the rotations in the plane 1-2 in Gelfand-Tsetlin basis,

$$R^{\mathfrak{M}_d^i}_{\mathfrak{M}_d'^i}(\theta) = \langle \mathfrak{M}_d'^i|e^{\theta M_{12}}|\mathfrak{M}_d^i\rangle. \tag{5.110}$$

Recall that according to our conventions the primed patterns belong to the same representations as unprimed ones. Clearly, the two forms can be used interchangeably. The reader may recognize the factor $r^{-\Delta_1-\Delta_2}$, which appears in many formulas

[16] See appendix D.1 for our conventions on conformal algebra. Our definition of $M_{\mu\nu}$ differs by a sign from e.g. [19].

for scalar four-point functions, and is often stripped off as in here by multiplying by $r^{+\Delta_1+\Delta_2}$. The matrices R play a similar role for the spinning degrees of freedom.[17]

Consider now a contribution from a $\mathbb{R}\times Spin(d)$ multiplet with scaling dimension Δ and in representation $\mathbf{m}_d$ of $Spin(d)$,

$$\sum_{\mathfrak{M}_d}\langle 0|O_4^{\mathfrak{M}_d^4}(-1)O_3^{\mathfrak{M}_d^3}(1)|\Delta,\mathfrak{M}_d\rangle\langle\Delta,\mathfrak{M}_d|r^D e^{\theta M_{12}}O_2^{\mathfrak{M}_d^2}(1)O_1^{\mathfrak{M}_d^1}(-1)|0\rangle =$$
$$= \sum_{\mathfrak{M}_d,\mathfrak{M}'_d} r^{\Delta}\langle 0|O_4^{\mathfrak{M}_d^4}(-1)O_3^{\mathfrak{M}_d^3}(1)|\Delta,\mathfrak{M}_d\rangle\langle\mathfrak{M}_d|e^{\theta M_{12}}|\mathfrak{M}'_d\rangle\langle\Delta,\mathfrak{M}'_d|O_2^{\mathfrak{M}_d^2}(1)O_1^{\mathfrak{M}_d^1}(-1)|0\rangle. \tag{5.111}$$

Here $\mathbf{m}'_d = \mathbf{m}_d$. This expression consists of three main ingredients: the two three-point functions

$$\langle 0|O_4^{\mathfrak{M}_d^4}(-1)O_3^{\mathfrak{M}_d^3}(1)|\Delta,\mathfrak{M}_d\rangle \quad\text{and}\quad \langle\Delta,\mathfrak{M}'_d|O_2^{\mathfrak{M}_d^2}(1)O_1^{\mathfrak{M}_d^1}(-1)|0\rangle, \tag{5.112}$$

and the matrix elements

$$\langle\mathfrak{M}_d|e^{\theta M_{12}}|\mathfrak{M}'_d\rangle. \tag{5.113}$$

In order to proceed further, we need to understand the structure of these objects.

5.3.1.1 Three-point functions

The three-point functions (5.112) are some tensors in the Gelfand-Tsetlin indices, whose values are constrained by the requirement of conformal invariance. To be precise, for three-point functions involving $\mathbb{R}\times Spin(d)$ multiplets, the only intrinsic restrictions come from $\mathbb{R}\times Spin(d)$ invariance.[18] Of these, only the $Spin(d-1)$ subgroup which fixes the first axis imposes the restriction directly on (5.112), while the other generators in $\mathbb{R}\times Spin(d)$ can be used to determine the values of these three-point functions for different positions of O_i (we have essentially done this above). Even in the case when the $\mathbb{R}\times Spin(d)$ multiplet in question is a conformal primary, $Spin(d-1)$-invariance is the only restriction on the tensors (5.112) [1].

In particular, the allowed tensor structures for, e.g.,

$$\langle\Delta,\mathfrak{M}'_d|O_2^{\mathfrak{M}_d^2}(1)O_1^{\mathfrak{M}_d^1}(-1)|0\rangle \tag{5.114}$$

---

[17] Importantly, the action of R here is only on the labels of the external operators. Because it commutes with the stabilizer group $Spin(d-2)$ of four points, it can be though of as a change of the basis of four-point tensor structures. We study the matrix elements such as R further in sections 5.3.1.2 and 5.3.7.

[18] The extrinsic restrictions, relating the contribution of the descendant multiplets to the primary, are discussed in section 5.4.

are in one-to-one correspondence with the $Spin(d-1)$ invariant subspace

$$\left(\overline{\mathbf{m}_d}\otimes\mathbf{m}_d^1\otimes\mathbf{m}_d^2\right)^{Spin(d-1)},\tag{5.115}$$

where the bar indicates taking the dual[19] representation. Because dimensional reduction is multiplicity-free, such singlets are in one-to-one correspondence with singlets in

$$\overline{\mathbf{m}'_{d-1}}\otimes\mathbf{m}_{d-1}^1\otimes\mathbf{m}_{d-1}^2\tag{5.116}$$

over all $\mathbf{m}'_{d-1}\in\mathbf{m}_d$, $\mathbf{m}_{d-1}^i\in\mathbf{m}_d^i$. Such a singlet exists whenever $\mathbf{m}'_{d-1}$ appears in $\mathbf{m}_{d-1}^1\otimes\mathbf{m}_{d-2}^2$, in which case we write

$$(\mathbf{m}'_{d-1},t')\in\mathbf{m}_{d-1}^1\otimes\mathbf{m}_{d-1}^2,\tag{5.117}$$

where the extra label t' is needed if $\mathbf{m}'_{d-1}$ appears in the tensor product with multiplicity.[20] If (5.117) holds, we can build an invariant using $Spin(d-1)$ Clebsch-Gordan coefficients. More explicitly, we have

$$\langle\Delta,\mathfrak{M}'_d|O_2^{\mathfrak{M}_d^2}(1)O_1^{\mathfrak{M}_d^1}(-1)|0\rangle=\sum_{t'}\lambda^{\mathbf{m}_{d-1}^1,\mathbf{m}_{d-1}^2}_{\mathbf{m}'_{d-1},t'}\langle\mathfrak{M}'_{d-1},t'|\mathfrak{M}_{d-1}^1,\mathfrak{M}_{d-1}^2\rangle,\tag{5.118}$$

where λ's are the three-point coefficients unconstrained by symmetry, and we recall that $\mathfrak{M}_{d-1}$ is defined as

$$\mathfrak{M}_d=\mathbf{m}_d,\mathbf{m}_{d-1},\ldots,\mathbf{m}_2\Longrightarrow\mathfrak{M}_{d-1}\equiv\mathbf{m}_{d-1},\mathbf{m}_{d-2},\ldots,\mathbf{m}_2.\tag{5.119}$$

It is understood that if $\mathbf{m}'_{d-1}\notin\mathbf{m}_{d-1}^1\otimes\mathbf{m}_{d-1}^2$, then the Clebsch-Gordan coefficient vanishes and the corresponding λ is undefined.

Analogously, for the second three-point function we have[21]

$$\langle 0|O_4^{\mathfrak{M}_d^4}(-1)O_3^{\mathfrak{M}_d^3}(1)|\Delta,\mathfrak{M}_d\rangle=\sum_{t}\overline{\lambda}^{\mathbf{m}_{d-1}^3,\mathbf{m}_{d-1}^4}_{\mathbf{m}_{d-1},t}\langle 0|\mathfrak{M}_{d-1}^3,\mathfrak{M}_{d-1}^4,\mathfrak{M}_{d-1},t\rangle,\tag{5.120}$$

where we now have a $3j$ symbol instead of Clebsch-Gordan coefficients (the distinction is of course rather formal).

Note that (5.118) and (5.120) give a somewhat unusual way of writing the three-point function, since the spin indices of the operators directly select which three-point

[19] Equivalently complex-conjugate, since all representations of compact $Spin(d)$ are unitary.

[20] If $d\leq 5$, then tensor products in $Spin(d-1)$ are multiplicity-free and the sum over t' can be dropped.

[21] The coefficients $\overline{\lambda}$ are in general not complex conjugates of λ.

coefficients λ appear in the right hand side. A perhaps more intuitive equivalent form of (5.118) is

$$\sum_{\widetilde{\mathbf{m}}^i_{d-1}} \sum_{\widetilde{\mathbf{m}}'_{d-1},t'} \lambda^{\widetilde{\mathbf{m}}^1_{d-1},\widetilde{\mathbf{m}}^2_{d-1}}_{\widetilde{\mathbf{m}}'_{d-1},t'} \left\{\delta_{\mathbf{m}^1_{d-1},\widetilde{\mathbf{m}}^1_{d-1}} \delta_{\mathbf{m}^2_{d-1},\widetilde{\mathbf{m}}^2_{d-1}} \delta_{\mathbf{m}'_{d-1},\widetilde{\mathbf{m}}'_{d-1}} \langle \mathfrak{M}'_{d-1}, t' | \mathfrak{M}^1_{d-1}, \mathfrak{M}^2_{d-1}\rangle \right\}, \tag{5.121}$$

where the object in the curly braces is the three point tensor structure, and it is made explicit that the three-point coefficients are labeled by two $Spin(d-1)$ representations $\widetilde{\mathbf{m}}^1_{d-1}$ and $\widetilde{\mathbf{m}}^2_{d-1}$ and a pair $(\widetilde{\mathbf{m}}'_{d-1}, t') \in \widetilde{\mathbf{m}}^1_{d-1} \otimes \widetilde{\mathbf{m}}^2_{d-1}$. We will sometimes use a shorthand notation to denote such composite labels. Namely, for the right three point function we use the label

$$a = (\widetilde{\mathbf{m}}^1_{d-1}, \widetilde{\mathbf{m}}^2_{d-1}, \widetilde{\mathbf{m}}'_{d-1}, t'), \qquad (\widetilde{\mathbf{m}}'_{d-1}, t') \in \widetilde{\mathbf{m}}^1_{d-1} \otimes \widetilde{\mathbf{m}}^2_{d-1}. \tag{5.122}$$

Similarly, for the left three-point function we use

$$b = (\widetilde{\mathbf{m}}^3_{d-1}, \widetilde{\mathbf{m}}^4_{d-1}, \widetilde{\mathbf{m}}_{d-1}, t), \qquad (\overline{\widetilde{\mathbf{m}}_{d-1}, t}) \in \widetilde{\mathbf{m}}^3_{d-1} \otimes \widetilde{\mathbf{m}}^4_{d-1}. \tag{5.123}$$

It is instructive to consider the case of 3 dimensions. In this case, we are considering the three-point functions

$$\langle \Delta, j', m' | \mathcal{O}_2^{j_2,m_2}(1) \mathcal{O}_1^{j_1,m_1}(-1) | 0 \rangle. \tag{5.124}$$

The $Spin(2)$ invariance basically tells us that the spin projection has to be conserved, $m' = m_1 + m_2$, and the Spin(2) Clebsch-Gordan coefficients are

$$\langle m' | m_1, m_2 \rangle = \delta_{m', m_1+m_2}. \tag{5.125}$$

We can therefore write

$$\langle \Delta, j', m' | \mathcal{O}_2^{j_2,m_2}(1) \mathcal{O}_1^{j_1,m_1}(-1) | 0 \rangle = \delta_{m',m_1+m_2} \lambda^{m_1,m_2}_{m'}. \tag{5.126}$$

Analogously, for the other three-point function we have

$$\langle 0 | \mathcal{O}_4^{j_4,m_4}(-1) \mathcal{O}_3^{j_3,m_3}(-1) | \Delta, j, m \rangle = \overline{\lambda}^{m_3,m_4}_{m} \delta_{0,m_3+m_4+m}. \tag{5.127}$$

We discuss the 3d case further in section 5.3.3.

In order to study the most general four-point functions, we need to know the most general three-point functions (5.118) and (5.120) and thus the most general $Spin(d-1)$ CG coefficients. Unfortunately, as discussed in section 5.2.3, to the best of our knowledge there is no general closed-form expression for such CG coefficients valid for general d available in the literature, but there are important special cases when such expressions are available.

Besides the cases considered in section 5.2.3, an important scenario is when, say, $\mathbf{m}_d^1 = \mathbf{m}_d^4 = \bullet$, in which case the required CG coefficients are trivial in any d. This happens, for example, in a certain choice of four-point functions for the so-called seed blocks. These are the simplest conformal blocks which exchange a given intermediate $Spin(d)$ representation $\mathbf{m}_d$. We discuss this case further in section 5.3.5.

5.3.1.2 Matrix elements

Consider now the matrix elements (5.113). An important feature is that the $Spin(d)$ element $e^{\theta M_{12}}$ commutes with the standard $Spin(d-2)$ subgroup which fixes the axes 1 and 2. On the other hand, the $Spin(d)$ representation $\mathbf{m}_d$ decomposes into irreducibles under $Spin(d-2)$, and by Schur's lemma this implies that $e^{\theta M_{12}}$ acts by identity times a constant inside of these irreducible components. More precisely, we have

$$\langle \mathfrak{M}_d | e^{\theta M_{12}} | \mathfrak{M}'_d \rangle = P^{\mathbf{m}_d, \mathbf{m}_{d-2}}_{\mathbf{m}_{d-1}, \mathbf{m}'_{d-1}}(\theta) \delta_{\mathfrak{M}_{d-2}, \mathfrak{M}'_{d-2}}. \tag{5.128}$$

One can arrive at the same conclusion by examining (5.99). The functions $P^{\mathbf{m}_d, \mathbf{m}_{d-2}}_{\mathbf{m}_{d-1}, \mathbf{m}'_{d-1}}(\theta)$ will play the role of Gegenbauer polynomials for the spinning conformal blocks. We will describe their structure, basic properties, and how to compute them in section 5.3.7. For now, note that they are labeled by an $Spin(d)$ representation $\mathbf{m}_d$, two $Spin(d-1)$ representations $\mathbf{m}_{d-1}, \mathbf{m}'_{d-1} \in \mathbf{m}_d$, and one $Spin(d-2)$ representation $\mathbf{m}_{d-2} \in \mathbf{m}_{d-1}, \mathbf{m}'_{d-1}$.

It is again useful to look at the case of three dimensions. Here, $Spin(d-2) = Spin(1)$ is trivial, and according to (5.128) we have (recall that $\mathbf{m}_3 \equiv j$ and $\mathbf{m}_2 \equiv m$)

$$P^j_{m,m'}(\theta) = \langle j, m | e^{\theta M_{12}} | j, m' \rangle = \langle j, m | e^{-i\theta J_{\hat{2}}} | j, m' \rangle = d^j_{m,m'}(-\theta), \tag{5.129}$$

where $d^j_{m,m'}(\theta)$ is the small Wigner d-matrix familiar from the representation theory of $Spin(3)$. For other examples see section 5.3.7 and appendix D.4.

5.3.1.3 Putting everything together

We can now combine (5.118), (5.120) and (5.128) to rewrite (5.111) in the following terrifying form,

$$\sum_{\mathfrak{M}_d}\langle 0|O_4^{\mathfrak{M}_d^4}(-1)O_3^{\mathfrak{M}_d^3}(1)|\Delta,\mathfrak{M}_d\rangle\langle\Delta,\mathfrak{M}_d|r^D e^{\theta M_{12}}O_2^{\mathfrak{M}_d^2}(1)O_1^{\mathfrak{M}_d^1}(-1)|0\rangle =$$

$$= \sum_{\widetilde{\mathbf{m}}^i_{d-1}}\sum_{\substack{\mathbf{m}_{d-1},t\\ \mathbf{m}'_{d-1},t'}}\sum_{\mathbf{m}_{d-2}}\lambda^{\widetilde{\mathbf{m}}^1_{d-1},\widetilde{\mathbf{m}}^2_{d-1}}_{\mathbf{m}'_{d-1},t'}\overline{\lambda}^{\widetilde{\mathbf{m}}^3_{d-1},\widetilde{\mathbf{m}}^4_{d-1}}_{\mathbf{m}_{d-1},t}r^\Delta P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\times$$

$$\times\left[\begin{matrix}\mathfrak{M}^3_d\\ \mathfrak{M}^4_d\end{matrix}\middle|\begin{matrix}\widetilde{\mathbf{m}}^3_{d-1}\\ \widetilde{\mathbf{m}}^4_{d-1}\end{matrix}\ \mathbf{m}_{d-1},t\middle|\mathbf{m}_{d-2}\middle|\mathbf{m}'_{d-1},t'\ \begin{matrix}\widetilde{\mathbf{m}}^1_{d-1}\\ \widetilde{\mathbf{m}}^2_{d-1}\end{matrix}\middle|\begin{matrix}\mathfrak{M}^1_d\\ \mathfrak{M}^2_d\end{matrix}\right], \quad (5.130)$$

where following selection rules on the summation variables hold,

$$\begin{aligned}\widetilde{\mathbf{m}}^i_{d-1} &\in \mathbf{m}^i_d,\\ (\mathbf{m}'_{d-1},t') &\in \widetilde{\mathbf{m}}^1_{d-1}\otimes\widetilde{\mathbf{m}}^2_{d-1},\\ (\overline{\mathbf{m}_{d-1}},t) &\in \widetilde{\mathbf{m}}^3_{d-1}\otimes\widetilde{\mathbf{m}}^4_{d-1},\\ \mathbf{m}_{d-2} &\in \mathbf{m}_{d-1},\mathbf{m}'_{d-1}\in\mathbf{m}_d.\end{aligned} \quad (5.131)$$

Using the shorthand notation (5.122) and (5.123) for the three-point tensor structures, we can rewrite (5.130) as

$$= \sum_{a,b}\sum_{\mathbf{m}_{d-2}}\lambda^a\overline{\lambda}^b r^\Delta P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\times\left[\begin{matrix}\mathfrak{M}^3_d\\ \mathfrak{M}^4_d\end{matrix}\middle|b\middle|\mathbf{m}_{d-2}\middle|a\middle|\begin{matrix}\mathfrak{M}^1_d\\ \mathfrak{M}^2_d\end{matrix}\right]. \quad (5.132)$$

We have also introduced a four-point tensor structure

$$\left[\begin{matrix}\mathfrak{M}^3_d\\ \mathfrak{M}^4_d\end{matrix}\middle|\begin{matrix}\widetilde{\mathbf{m}}^3_{d-1}\\ \widetilde{\mathbf{m}}^4_{d-1}\end{matrix}\ \mathbf{m}_{d-1},t\middle|\mathbf{m}_{d-2}\middle|\mathbf{m}'_{d-1},t'\ \begin{matrix}\widetilde{\mathbf{m}}^1_{d-1}\\ \widetilde{\mathbf{m}}^2_{d-1}\end{matrix}\middle|\begin{matrix}\mathfrak{M}^1_d\\ \mathfrak{M}^2_d\end{matrix}\right] \quad (5.133)$$

which we will define momentarily. Before doing that, let us comment briefly on the structure of (5.130) and (5.132).

There are two complications compared to the case of external scalar operators. First, there are many possible three-point tensor structures, and we have to sum over the contributions from different pairs of three-point structures. This is done in the first two sums in (5.130) or equivalently the first sum in (5.132). Indeed, according to the discussion around (5.118), the set $a = (\widetilde{\mathbf{m}}^1_{d-1},\widetilde{\mathbf{m}}^2_{d-1},\mathbf{m}'_{d-1},t)$ such that $\mathbf{m}'_{d-1},t$ selects an irreducible component in $\widetilde{\mathbf{m}}^1_{d-1}\otimes\widetilde{\mathbf{m}}^2_{d-1}$ uniquely determines a three-point tensor structure for the operators 1 and 2, and an analogous statement holds for

b and the operators 3 and 4. Second, there are many four-point structures, and a single pair of three-point structures can contribute to many four-point structures. This is the last sum in (5.130) and (5.132). As we discuss below, the role of $\mathbf{m}_{d-2}$ representation is to specify a way of gluing the two three-point structures into a four-point structure. Note that the three-point structures do not depend on $\mathbf{m}_{d-2}$, but the angular functions P and the four-point tensor structures do. We stress that the structures (5.133) form a basis of all four-point tensor structures, as we now explain.

The definition of (5.133) follows straightforwardly from the construction,

$$\left[\begin{matrix}\mathfrak{M}_d^3 \\ \mathfrak{M}_d^4\end{matrix}\middle|\begin{matrix}\widetilde{\mathbf{m}}_{d-1}^3 \\ \widetilde{\mathbf{m}}_{d-1}^4\end{matrix}\;\mathbf{m}_{d-1}, t\middle|\mathbf{m}_{d-2}\middle|\mathbf{m}'_{d-1}, t'\;\begin{matrix}\widetilde{\mathbf{m}}_{d-1}^1 \\ \widetilde{\mathbf{m}}_{d-1}^2\end{matrix}\middle|\begin{matrix}\mathfrak{M}_d^1 \\ \mathfrak{M}_d^2\end{matrix}\right] =$$
$$= \sum_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} \langle 0|\mathfrak{M}_{d-1}^3, \mathfrak{M}_{d-1}^4, \mathfrak{M}_{d-1}, t\rangle \delta_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} \langle \mathfrak{M}'_{d-1}, t'|\mathfrak{M}_{d-1}^1, \mathfrak{M}_{d-1}^2\rangle \times$$
$$\times\, \delta_{\mathbf{m}_{d-1}^1,\widetilde{\mathbf{m}}_{d-1}^1} \delta_{\mathbf{m}_{d-1}^2,\widetilde{\mathbf{m}}_{d-1}^2} \delta_{\mathbf{m}_{d-1}^3,\widetilde{\mathbf{m}}_{d-1}^3} \delta_{\mathbf{m}_{d-1}^4,\widetilde{\mathbf{m}}_{d-1}^4}. \tag{5.134}$$

Here $\mathbf{m}'_{d-2} = \mathbf{m}_{d-2}$. Note that for every choice of $\widetilde{\mathbf{m}}_{d-1}^i, \mathbf{m}_{d-1}, \mathbf{m}'_{d-1}, \mathbf{m}_{d-2}, t, t'$, this is a function of $\mathfrak{M}_d^i$, i.e., an element of

$$\mathbf{m}_d^1 \otimes \mathbf{m}_d^2 \otimes \mathbf{m}_d^3 \otimes \mathbf{m}_d^4. \tag{5.135}$$

Furthermore, it is clear from the definition that it is $Spin(d-2)$ invariant. This means that it is an element of

$$\left(\mathbf{m}_d^1 \otimes \mathbf{m}_d^2 \otimes \mathbf{m}_d^3 \otimes \mathbf{m}_d^4\right)^{Spin(d-2)}, \tag{5.136}$$

which is the space of four-point tensor structures [1, 123].

The set of structures (5.134) with the parameters restricted by (5.131) spans (5.136). Indeed, we have

$$\left(\mathbf{m}_d^1 \otimes \mathbf{m}_d^2 \otimes \mathbf{m}_d^3 \otimes \mathbf{m}_d^4\right)^{Spin(d-2)} = \bigoplus_{\substack{\mathbf{m}_{d-1}^{12} \in \mathbf{m}_d^1 \otimes \mathbf{m}_d^2 \\ \overline{\mathbf{m}_{d-1}^{34}} \in \mathbf{m}_d^3 \otimes \mathbf{m}_d^4}} (\mathbf{m}_{d-1}^{12} \otimes \overline{\mathbf{m}_{d-1}^{34}})^{Spin(d-2)}, \tag{5.137}$$

where the sum is taken with multiplicities. Because the dimensional reduction is multiplicity-free, we have that $Spin(d-2)$ singlets in $\mathbf{m}_{d-1}^{12} \otimes \overline{\mathbf{m}_{d-1}^{34}}$ are in one-to-one correspondence with $\mathbf{m}_{d-2}^{1234} \in \mathbf{m}_{d-1}^{12}, \mathbf{m}_{d-1}^{34}$.

This enumeration is implemented by (5.131) as follows. By specifying $\widetilde{\mathbf{m}}_{d-1}^1, \widetilde{\mathbf{m}}_{d-1}^2, \mathbf{m}'_{d-1}, t'$ we first select a general $Spin(d-1)$ representation $\mathbf{m}_{d-1}^{12} \simeq \mathbf{m}'_{d-1}$ in $\mathbf{m}_d^1 \otimes \mathbf{m}_d^2$. Similarly, $\widetilde{\mathbf{m}}_{d-1}^3, \widetilde{\mathbf{m}}_{d-1}^4, \overline{\mathbf{m}_{d-1}}, t$ select a general $Spin(d-1)$ irrep $\overline{\mathbf{m}_{d-1}^{34}} \simeq \overline{\mathbf{m}_{d-1}}$ in $\mathbf{m}_d^3 \otimes \mathbf{m}_d^4$. The "gluing" representation $\mathbf{m}_{d-2}^{1234}$ is then identified with $\mathbf{m}_{d-2}$.

5.3.2 Example: Scalar correlators

Let us see how we can recover the Genegenbauer expansion for scalar four-point functions. For scalars we have $\mathbf{m}_d^i = (0, \dots 0) = \bullet$, and the only Gelfand-Tsetlin patterns are $[\bullet] \equiv (\bullet, \dots \bullet)$. Similarly, $\widetilde{\mathbf{m}}_{d-1}^i = \bullet$. In (5.130) we only need to sum over $\mathbf{m}'_{d-1} \in \widetilde{\mathbf{m}}_{d-1}^1 \otimes \widetilde{\mathbf{m}}_{d-1}^2$, thus only $\mathbf{m}'_{d-1} = \bullet$ is allowed and there is no need in t' label. Similarly, $\mathbf{m}_{d-1} = \bullet$. The sum over $\mathbf{m}_{d-2}$ is restricted to $\mathbf{m}_{d-2} \in \mathbf{m}_{d-1}, \mathbf{m}'_{d-1}$, and thus we only have $\mathbf{m}_{d-2} = \bullet$. The unique component of the unique four-point structure is

$$\left[\begin{matrix}[\bullet] \\ [\bullet]\end{matrix}\middle|\begin{matrix}\bullet \\ \bullet\end{matrix}\ \bullet\middle|\bullet\middle|\bullet\ \begin{matrix}\bullet \\ \bullet\end{matrix}\middle|\begin{matrix}[\bullet] \\ [\bullet]\end{matrix}\right] = 1. \tag{5.138}$$

Equation (5.130) collapses then to

$$\begin{aligned}&\sum_{\mathfrak{M}_d}\langle 0|\mathcal{O}_4^{[\bullet]}(-1)\mathcal{O}_3^{[\bullet]}(1)|\Delta,\mathfrak{M}_d\rangle\langle\Delta,\mathfrak{M}_d|r^D e^{\theta M_{12}}\mathcal{O}_2^{[\bullet]}(1)\mathcal{O}_1^{[\bullet]}(-1)|0\rangle \\ &= \lambda_{\bullet}^{\bullet,\bullet}\overline{\lambda}_{\bullet}^{\bullet,\bullet} r^\Delta P_{\bullet,\bullet}^{\mathbf{m}_d,\bullet}(\theta).\end{aligned} \tag{5.139}$$

We need $\mathbf{m}_{d-1}, \mathbf{m}'_{d-1} \in \mathbf{m}_d$, and thus for scalars we get the condition $\mathbf{m}_d \ni \bullet$, which is only satisfied if $\mathbf{m}_d$ is traceless-symmetric, $\mathbf{m}_d = \mathbf{j} = (j, 0, \dots, 0)$. Finally, as we show in (5.202) later in this section, $P_{\bullet,\bullet}^{\mathbf{j},\bullet}(\theta)$ is proportional to a Gegenbauer polynomial. Taking (5.202) into account, we reproduce the result of [59]

$$\begin{aligned}&\sum_{\mathfrak{M}_d}\langle 0|\mathcal{O}_4^{[\bullet]}(-1)\mathcal{O}_3^{[\bullet]}(1)|\Delta,\mathfrak{M}_d\rangle\langle\Delta,\mathfrak{M}_d|r^D e^{\theta M_{12}}\mathcal{O}_2^{[\bullet]}(1)\mathcal{O}_1^{[\bullet]}(-1)|0\rangle \\ &= \lambda_{\bullet}^{\bullet,\bullet}\overline{\lambda}_{\bullet}^{\bullet,\bullet} r^\Delta \frac{C_j^{(\nu)}(\theta)}{C_j^{(\nu)}(1)}.\end{aligned} \tag{5.140}$$

5.3.3 Example: General 3d correlators

Consider now the case $d = 3$. Let us first write the four-point tensor structure (5.134). Since $d = 3$, the sums in (5.134) are trivial, as well as $\mathbf{m}_{d-2}$ is. Furthermore, $Spin(d-1) = Spin(2)$ tensor products are multiplicity-free, so the labels t and t' are also trivial. We then find, using (5.127) and (5.126),

$$\left[\begin{matrix}j_3, m_3 \\ j_4, m_4\end{matrix}\middle|\begin{matrix}\widetilde{m}_3 \\ \widetilde{m}_4\end{matrix}\ m\middle\|m'\ \begin{matrix}\widetilde{m}_1 \\ \widetilde{m}_2\end{matrix}\middle|\begin{matrix}j_1, m_1 \\ j_2, m_2\end{matrix}\right] = \delta_{m',m_1+m_2}\delta_{0,m+m_3+m_4}\delta_{m_1,\widetilde{m}_1}\delta_{m_2,\widetilde{m}_2}\delta_{m_3,\widetilde{m}_3}\delta_{m_4,\widetilde{m}_4}. \tag{5.141}$$

Since the tensor product of $Spin(2)$ representations $\widetilde{m}_1, \widetilde{m}_2$ contains only one representation, $\widetilde{m}_1 + \widetilde{m}_2$, we do not need to specify m' separately. The same holds for

m. We can thus simplify this tensor structure as

$$\left[\begin{matrix} j_3, m_3 \\ j_4, m_4 \end{matrix}\middle|\begin{matrix} \widetilde{m}_3\ \widetilde{m}_1 \\ \widetilde{m}_4\ \widetilde{m}_2 \end{matrix}\middle|\begin{matrix} j_1, m_1 \\ j_2, m_2 \end{matrix}\right] \equiv \delta_{m_1,\widetilde{m}_1}\delta_{m_2,\widetilde{m}_2}\delta_{m_3,\widetilde{m}_3}\delta_{m_4,\widetilde{m}_4}. \tag{5.142}$$

Before moving further, let us understand the meaning of this expression. It is a four-point tensor structure in the sense that by fixing $\widetilde{m}_i$ we have a tensor with indices m_i, i.e., an element of

$$j_1 \otimes j_2 \otimes j_3 \otimes j_4. \tag{5.143}$$

Note that these structures form a complete basis for such tensors, which is consistent with the fact that $Spin(d-2) = Spin(1)$ is trivial and so there is no invariance constraint on conformal frame four-point structures [1].[22]

As noted above, we can essentially drop $\mathbf{m}_{d-1}, \mathbf{m}'_{d-1}, \mathbf{m}_{d-2}, t, t'$ in (5.130). Using (5.129) and (5.142) we can rewrite (5.130) as

$$\sum_m \langle 0|O_4^{j_4,m_4}(-1)O_3^{j_3,m_3}(1)|\Delta, j, m\rangle\langle\Delta, j, m|r^D e^{\theta M_{12}} O_2^{j_2,m_2}(1)O_1^{j_1,m_1}(-1)|0\rangle =$$
$$= \sum_{\widetilde{m}^i} \lambda^{\widetilde{m}_1,\widetilde{m}_2}\overline{\lambda}^{\widetilde{m}_3,\widetilde{m}_4} r^\Delta d^j_{-\widetilde{m}_3-\widetilde{m}_4,\widetilde{m}_1+\widetilde{m}_2}(-\theta)\left[\begin{matrix} j_3, m_3 \\ j_4, m_4 \end{matrix}\middle|\begin{matrix} \widetilde{m}_3\ \widetilde{m}_1 \\ \widetilde{m}_4\ \widetilde{m}_2 \end{matrix}\middle|\begin{matrix} j_1, m_1 \\ j_2, m_2 \end{matrix}\right], \tag{5.144}$$

where summation is over

$$\widetilde{m}_i = -j_i, -j_i + 1, \ldots j_i, \tag{5.145}$$

and the last line of (5.131) also restricts

$$|\widetilde{m}_1 + \widetilde{m}_2|,\ |\widetilde{m}_3 + \widetilde{m}_4| \leq j \tag{5.146}$$

as well as that $\widetilde{m}_1 + \widetilde{m}_2$ and $\widetilde{m}_3 + \widetilde{m}_4$ are integral or half-integral simultaneously with j, so that small Wigner d-matrix is well-defined.

5.3.4 Example: General 4d correlators

We now consider the case of the general correlation functions in $d = 4$. The usefulness of this example comes from the fact that while being not very different from the most general case, it can still be formulated using only the familiar ingredients from representation theory of $Spin(d-1) = Spin(3) \simeq SU(2)$.

[22] One can be more pedantic by taking $Spin(1) = \mathbb{Z}_2$, in which case there is a constraint which simply says that $\widetilde{m}_1 + \widetilde{m}_2 + \widetilde{m}_3 + \widetilde{m}_4$ (equivalently, $j_1 + j_2 + j_3 + j_4$) must be an integer, i.e., the correlator should contain an even number of fermions.

First, we need to construct the three-point tensor structures. Consider for example the right tensor structure (5.118) parametrized by the data (5.122). We can write in 4d

$$a = (\widetilde{j}_1, \widetilde{j}_2, \widetilde{j}'), \tag{5.147}$$

where $\widetilde{j}_i \in l_L^i \otimes l_R^i$ and $\widetilde{j}' \in l_L \otimes l_R$ where (l_L, l_R) is the representation of the exchanged operator. The constraint in (5.122) then takes form $\widetilde{j}' \in \widetilde{j}_1 \otimes \widetilde{j}_2$. In particular, we do not need a multiplicity label because the tensor products in $Spin(3)$ are multiplicity-free. The three-point functions take the form

$$\begin{aligned}
&\langle \Delta, l_L, l_R; j', m' | O_2^{j_2, m_2}(1) O_1^{j_1, m_1}(-1) | 0 \rangle \\
&= \lambda_{(j_1, j_2, j')} \langle j', m' | j_1, m_1; j_2, m_2 \rangle \\
&= \sum_{a = (\widetilde{j}_1, \widetilde{j}_2, \widetilde{j}')} \lambda_{(\widetilde{j}_1, \widetilde{j}_2, \widetilde{j}')} \{ \delta_{j_1 \widetilde{j}_1} \delta_{j_2 \widetilde{j}_2} \delta_{j' \widetilde{j}'} \langle j', m' | j_1, m_1; j_2, m_2 \rangle \}.
\end{aligned} \tag{5.148}$$

Here, for notational simplicity, we have omitted the $\mathbf{m}_4^i$ part of the GT pattern for the primary operators O_i. The second line of this equation gives the more traditional form of the three-point functions as a sum over tensor structures labeled by a. Finally, $\langle j', m' | j_1, m_1; j_2, m_2 \rangle$ is the $SU(2)$ Clebsch-Gordan coefficient. Similarly, for the left three-point function we have

$$\begin{aligned}
&\langle 0 | O_4^{j_4, m_4}(-1) O_3^{j_3, m_3}(1) | \Delta, l_L, l_R; j, m \rangle \\
&= \overline{\lambda}_{(j_3, j_4, j)} \langle 0 | j_4, m_4; j_3, m_3; j, m \rangle \\
&= \sum_{b = (\widetilde{j}_3, \widetilde{j}_4, \widetilde{j})} \lambda_{(\widetilde{j}_3, \widetilde{j}_4, \widetilde{j})} \{ \delta_{j_3, \widetilde{j}_3} \delta_{j_4, \widetilde{j}_4} \delta_{j, \widetilde{j}} \langle 0 | j_4, m_4; j_3, m_3; j, m \rangle \},
\end{aligned} \tag{5.149}$$

and the constraint from (5.123) is simply $\widetilde{j} \in \widetilde{j}_3 \otimes \widetilde{j}_4$ since all $SU(2)$ irreps are self-conjugate. Here $\langle 0 | j_4, m_4; j_3, m_3; j, m \rangle$ is essentially the $SU(2)$ $3j$ symbol. Note that this parametrization of three-point structures is essentially the same as the one mentioned in [2].

The four-point tensor structures (5.134) can also be computed as

$$\begin{aligned}
&\left[\begin{matrix} j_3, m_3 \\ j_4, m_4 \end{matrix} \middle| \begin{matrix} \widetilde{j}_3 \\ \widetilde{j}_4 \end{matrix} \, j \middle| m \middle| j' \, \begin{matrix} \widetilde{j}_1 \\ \widetilde{j}_2 \end{matrix} \middle| \begin{matrix} j_1, m_1 \\ j_2, m_2 \end{matrix} \right] \\
&= \langle 0 | j_4, m_4; j_3, m_3; j, m \rangle \langle j', m | j_1, m_1; j_2, m_2 \rangle \delta_{j_1, \widetilde{j}_1} \delta_{j_2, \widetilde{j}_2} \delta_{j_3, \widetilde{j}_3} \delta_{j_4, \widetilde{j}_4}.
\end{aligned} \tag{5.150}$$

Recall that the labels m_i parametrize the representations of the $Spin(2)$ which rotates in the plane 3-4. This plane is orthogonal to the plane 1-2 in which we place our operators, and thus this $Spin(2)$ it the stabilizer group of the four points and, as usual, the four-point tensor structures have to be invariant under it. Using the constraints $m_4 + m_3 + m = 0$ and $m = m_1 + m_2$ coming from the CG coefficients, we find $m_4 + m_3 + m_2 + m_1 = 0$ which is precisely the required invariance condition. Of course, this comes as no surprise since it was guaranteed by construction. Note that this basis of four-point tensor structures is different from the one in [2], since it is not an eigenbasis for rotations in plane 1-2.

The final formula (5.130) takes the following form in $4d$,

$$\sum_{j,m}\langle 0|\mathcal{O}_4^{j_4,m_4}(-1)\mathcal{O}_3^{j_3,m_3}(1)|\Delta,l_L,l_R;j,m\rangle\langle\Delta,l_L,l_R;j,m|e^{\theta M_{12}}r^D\mathcal{O}_2^{j_2,m_2}(1)\mathcal{O}_1^{j_1,m_1}(-1)|0\rangle =$$

$$= \sum_{a,b}\sum_{m}\lambda^a\overline{\lambda}^b\begin{bmatrix} j_3,m_3 \\ j_4,m_4 \end{bmatrix}\!\!\left|b\right|m\left|a\right|\!\!\begin{bmatrix} j_1,m_1 \\ j_2,m_2\end{bmatrix} P^{l_L,l_R;m}_{\widetilde{j},\widetilde{j}'}(\theta), \quad (5.151)$$

where the four-point tensor structure and the three-point labels a, b are described above, while the P-function is given below in section 5.3.7 by equation (5.212). The range of summation over m is restricted to be $-\min(\widetilde{j},\widetilde{j}'), -\min(\widetilde{j},\widetilde{j}')+1, \ldots, \min(\widetilde{j},\widetilde{j}')$.

5.3.5 Example: Seed conformal blocks in general dimensions

Our last example concerns an especially simple case which occurs for every d. The simplification is based on the fact that the CG coefficients are trivial when one of the representations is trivial. Choosing two of the four operators operators to be scalars, we can ensure that the CG coefficients for both the right and the left three-point function simplify, with the correlator itself still being sufficiently general. If fact, as will be clear from the construction, the so-called seed blocks for arbitrary intermediate representations can be chosen to be of this form.

Let us choose the operators $\mathcal{O}_1$ and $\mathcal{O}_3$ to be scalars. Then the general result (5.130) simplifies as

$$\sum_{\mathfrak{M}_d}\langle 0|\mathcal{O}_4^{\mathfrak{M}_d^4}(-1)\mathcal{O}_3(1)|\Delta,\mathfrak{M}_d\rangle\langle\Delta,\mathfrak{M}_d|r^D e^{\theta M_{12}}\mathcal{O}_2^{\mathfrak{M}_d^2}(1)\mathcal{O}_1(-1)|0\rangle =$$

$$= \sum_{\widetilde{\mathbf{m}}^i_{d-1}}\sum_{\mathbf{m}_{d-2}}\lambda^{\bullet,\widetilde{\mathbf{m}}^2_{d-1}}_{\widetilde{\mathbf{m}}^2_{d-1}}\overline{\lambda}^{\bullet,\widetilde{\mathbf{m}}^4_{d-1}}_{\overline{\widetilde{\mathbf{m}}^4_{d-1}}}r^\Delta P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\widetilde{\mathbf{m}}^2_{d-1},\overline{\widetilde{\mathbf{m}}^4_{d-1}}}(\theta)\times$$

$$\times\left[\begin{matrix}\bullet \\ \mathfrak{M}^4_d\end{matrix}\middle|\begin{matrix}\bullet \\ \widetilde{\mathbf{m}}^4_{d-1}\end{matrix}\;\overline{\widetilde{\mathbf{m}}^4_{d-1}}\middle|\mathbf{m}_{d-2}\middle|\widetilde{\mathbf{m}}^2_{d-1}\;\begin{matrix}\bullet \\ \widetilde{\mathbf{m}}^2_{d-1}\end{matrix}\middle|\begin{matrix}\bullet \\ \mathfrak{M}^2_d\end{matrix}\right], \quad (5.152)$$

with the four point-structures given by the specialization of (5.134),

$$
\begin{aligned}
&\left[\begin{array}{c|c|c|c|c|c|c} \bullet & \bullet & & & & \bullet & \bullet \\ \mathfrak{M}_d^4 & \widetilde{\mathbf{m}}_{d-1}^4 & \overline{\widetilde{\mathbf{m}}_{d-1}^4} & \mathbf{m}_{d-2} & \widetilde{\mathbf{m}}_{d-1}^2 & \widetilde{\mathbf{m}}_{d-1}^2 & \mathfrak{M}_d^2 \end{array}\right] = \\
&= \sum_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} \langle 0|\mathfrak{M}_{d-1}^4, \mathfrak{M}_{d-1}\rangle \delta_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} \langle \mathfrak{M}'_{d-1}|\mathfrak{M}_{d-1}^2\rangle \delta_{\mathbf{m}_{d-1}^2,\widetilde{\mathbf{m}}_{d-1}^2} \delta_{\mathbf{m}_{d-1}^4,\widetilde{\mathbf{m}}_{d-1}^4} = \\
&= \sum_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} \frac{(-1)^{\mathfrak{M}_{d-1}^4}}{\sqrt{\dim \widetilde{\mathbf{m}}_{d-1}^4}} \delta_{\mathfrak{M}_{d-1}^4,\overline{\mathfrak{M}_{d-1}}} \delta_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} \delta_{\mathfrak{M}'_{d-1},\mathfrak{M}_{d-1}^2} \delta_{\mathbf{m}_{d-1}^2,\widetilde{\mathbf{m}}_{d-1}^2} \delta_{\mathbf{m}_{d-1}^4,\widetilde{\mathbf{m}}_{d-1}^4} = \\
&= \frac{(-1)^{\mathfrak{M}_{d-1}^4}}{\sqrt{\dim \widetilde{\mathbf{m}}_{d-1}^4}} \delta_{\mathbf{m}_{d-1}^2,\widetilde{\mathbf{m}}_{d-1}^2} \delta_{\mathbf{m}_{d-1}^4,\widetilde{\mathbf{m}}_{d-1}^4} \delta_{\mathbf{m}_{d-2}^4,\overline{\mathbf{m}_{d-2}}} \delta_{\mathbf{m}_{d-2}^2,\mathbf{m}_{d-2}} \delta_{\mathfrak{M}_{d-2}^4,\overline{\mathfrak{M}_{d-2}^2}},
\end{aligned}
\tag{5.153}
$$

where we made use of (5.47). The constraints (5.131) reduce in this case to

$$\widetilde{\mathbf{m}}_{d-1}^i \in \mathbf{m}_d^i, \quad i = 2, 4, \tag{5.154}$$

$$\mathbf{m}_{d-2} \in \widetilde{\mathbf{m}}_{d-1}^2 \in \mathbf{m}_d, \tag{5.155}$$

$$\mathbf{m}_{d-2} \in \overline{\widetilde{\mathbf{m}}_{d-1}^4} \in \mathbf{m}_d. \tag{5.156}$$

Note that for any $\mathbf{m}_d$ there exists a choice of $\mathbf{m}_d^i$ such that these constraints can be satisfied, and thus arbitrary intermediate representations can be exchanged in this simplified setup. In fact, for a given $\mathbf{m}_d$, in even d, we can always choose $\mathbf{m}_d^i$ so that there is a unique choice available for $\widetilde{\mathbf{m}}_{d-1}^i$ (and thus a unique three-point function on either side). For this, set, for example[23]

$$
\begin{aligned}
m_{d,k}^2 &= m_{d,k+1}, \quad 1 \leq k < n, \\
m_{d,n}^2 &= 0 \text{ or } \tfrac{1}{2}, \\
\mathbf{m}_d^4 &= \overline{\mathbf{m}_d^2},
\end{aligned}
\tag{5.157}
$$

where the choice in the second equality is determined by the statistics of $\mathbf{m}_d$. In odd d, this only reduces down to two choices for each of $\widetilde{\mathbf{m}}_{d-1}^i$ if the representations are fermionic (but still one choice for bosonic representations). This is because in the case of odd d the outer automorphism of $Spin(d-1)$ (given by reflection) necessarily acts non-trivially on fermionic representations of $Spin(d-1)$, but trivially on the representations of $Spin(d)$. Therefore, the number of three-point tensor structures

[23]This is choice is different from the one used in $d = 4$ in [2]. In fact, in even d it doesn't matter what we choose $m_{d,n}^2$ to be, and the choice in [2] corresponds to $m_{d,n}^2 = |m_{d,n}|$. Our choice (5.157) has the advantage that is also works in odd dimensions, see below. Also, note that there is some freedom in choosing $\mathbf{m}_d^4$ independently of $\mathbf{m}_d^2$.

involving fermionic representations is always even, and we simply cannot have less than 2 non-trivial structures.

If we think about the state $|\Delta, \mathfrak{M}_d\rangle$ as being a conformal primary, then the choices of external representations described above give us a valid choice for the so-called seed blocks for exchange of primary $\mathbf{m}_d$ – they lead to the minimum number of three-point tensor structures on both sides of the four-point function. The equations (5.152) and (5.153) then give the leading contribution to the OPE limit of such seed conformal blocks.

As a concrete example, consider the scalar-fermion blocks in even dimensions. Specifically, we take

$$\mathbf{m}_d^2 = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, +\tfrac{1}{2}), \tag{5.158}$$

$$\mathbf{m}_d^4 = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}). \tag{5.159}$$

This is slightly different from the prescription (5.157) unless $d = 4k + 2$, but it is more convenient to have a uniform choice of representations for all even d. Under dimensional reduction both $\mathbf{m}_d^2$ and $\mathbf{m}_d^4$ restrict to a single representation, and thus necessarily

$$\widetilde{\mathbf{m}}_{d-1}^2 = \widetilde{\mathbf{m}}_{d-1}^4 = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}). \tag{5.160}$$

These representations further restrict to a direct sum of $(\frac{1}{2}, \ldots, +\frac{1}{2})$ and $(\frac{1}{2}, \ldots, -\frac{1}{2})$ in $d - 2$ dimensions, so that there are two four-point tensor structures

$$t_\pm \equiv \left[\begin{array}{c} \bullet \\ \mathfrak{M}_d^4 \end{array} \middle| \begin{array}{c} \bullet \\ (\frac{1}{2}, \ldots, \frac{1}{2}) \end{array} (\tfrac{1}{2}, \ldots, \tfrac{1}{2}) \middle| (\tfrac{1}{2}, \ldots, \pm\tfrac{1}{2}) \middle| (\tfrac{1}{2}, \ldots, \tfrac{1}{2}) \begin{array}{c} \bullet \\ (\frac{1}{2}, \ldots, \frac{1}{2}) \end{array} \middle| \begin{array}{c} \bullet \\ \mathfrak{M}_d^2 \end{array} \right]. \tag{5.161}$$

Correspondingly, there are two types of $\mathbf{m}_d$ that can be exchanged, each with a single three-point tensor structure on either side,

$$\mathbf{m}_d^\pm = (j, \tfrac{1}{2}, \ldots, \pm\tfrac{1}{2}). \tag{5.162}$$

From (5.130) we find that the contribution of the representation $\mathbf{m}_d^\pm$ to the four-point function (5.108) is given by

$$\sum_\pm \Lambda_\pm r^\Delta \left(P^{(j,\frac{1}{2},\ldots,\pm\frac{1}{2}),(\frac{1}{2},\ldots,+\frac{1}{2})}_{(\frac{1}{2},\ldots,\frac{1}{2})(\frac{1}{2},\ldots,\frac{1}{2})}(\theta) t_+ + P^{(j,\frac{1}{2},\ldots,\pm\frac{1}{2}),(\frac{1}{2},\ldots,-\frac{1}{2})}_{(\frac{1}{2},\ldots,\frac{1}{2})(\frac{1}{2},\ldots,\frac{1}{2})}(\theta) t_- \right), \tag{5.163}$$

where

$$\Lambda_\pm = \lambda^{\bullet, \widetilde{\mathbf{m}}_{d-1}^2}_{\widetilde{\mathbf{m}}_{d-1}^2, \mathbf{m}_d^\pm} \overline{\lambda}^{\bullet, \widetilde{\mathbf{m}}_{d-1}^4}_{\widetilde{\mathbf{m}}_{d-1}^4, \mathbf{m}_d^\pm}. \tag{5.164}$$

Here $\mathbf{m}_d^{\pm}$ index of OPE coefficients labels the exchanged representation. We find explicit expressions for the above P-functions in section 5.3.7.3, with the result given in (5.196).

5.3.6 Example: Conformal block/Four-point tensor structure correspondence

As another simple application of the above formalism, let us discuss the folklore theorem which states that the number of classes of conformal blocks which contribute to a given four-point function is equal to the number of four-point tensor structures [55, 75]. We will consider the simplest case where the only relevant symmetry is the connected conformal group (i.e., no space parity or permutation symmetries for identical operators). In our formalism this theorem becomes essentially a tautology. Because of that, this section basically reiterates what was already said, with a slightly different focus.

First, let us explain what is meant by classes of conformal blocks. Each conformal block contributing to a four-point function is parametrized by the dimension Δ and the $Spin(d)$ representation $\mathbf{m}_d$ of the exchanged primary operator, as well as by a pair of three-point functions a and b. From the previous discussion, we can parametrize the three-point functions as follows,

$$\begin{aligned} a &= (\widetilde{\mathbf{m}}^1_{d-1}, \widetilde{\mathbf{m}}^2_{d-1}, \mathbf{m}'_{d-1}, t'), \\ b &= (\widetilde{\mathbf{m}}^3_{d-1}, \widetilde{\mathbf{m}}^4_{d-1}, \mathbf{m}_{d-1}, t), \end{aligned} \tag{5.165}$$

subject to (5.131). In particular, the constraint

$$\widetilde{\mathbf{m}}^i_{d-1} \in \mathbf{m}^i_d \tag{5.166}$$

gives us finitely many choices for $\widetilde{\mathbf{m}}^i_{d-1}$ for fixed $\mathbf{m}^i_d$ and the constraints

$$\begin{aligned} (\mathbf{m}'_{d-1}, t') &\in \widetilde{\mathbf{m}}^1_{d-1} \otimes \widetilde{\mathbf{m}}^2_{d-1}, \\ (\overline{\mathbf{m}_{d-1}}, t) &\in \widetilde{\mathbf{m}}^3_{d-1} \otimes \widetilde{\mathbf{m}}^4_{d-1}, \end{aligned} \tag{5.167}$$

thus give us finitely many choices of $(\mathbf{m}'_{d-1}, t')$ and $(\mathbf{m}_{d-1}, t)$. The intermediate representation is then constrained by $\mathbf{m}_{d-1}, \mathbf{m}'_{d-1} \in \mathbf{m}_d$. This leaves *infinitely* many choices of $\mathbf{m}_d$ for a given four-point function. However, the allowed $\mathbf{m}_d$ organize into natural families. Indeed, let us denote $\mathbf{m}_d = (j, \widetilde{\mathbf{m}}_{d-2})$, i.e., j is the length of the first row of the generalized Young diagram of j and $\widetilde{\mathbf{m}}_{d-2}$ encodes the remaining rows.[24] The following two statements are then equivalent,

$$\mathbf{m}_{d-1}, \mathbf{m}'_{d-1} \in \mathbf{m}_d = (j, \widetilde{\mathbf{m}}_{d-2}) \quad \Longleftrightarrow \quad \widetilde{\mathbf{m}}_{d-2} \in \mathbf{m}_{d-1}, \mathbf{m}'_{d-1} \text{ and } j \geq m_{d-1,1}, m'_{d-1,1}. \tag{5.168}$$

This leaves only a finite number of choices for $\widetilde{\mathbf{m}}_{d-2}$.

The infiniteness of the number of conformal blocks is therefore only due to the generic parameters Δ and j. If we consider any two conformal blocks differing by only these two parameters to belong to the same *class*, we obtain a finite set of classes parametrized by a pair of three-point structures (5.165) subject to (5.166)-(5.167) and a $\widetilde{\mathbf{m}}_{d-2}$ subject to

$$\widetilde{\mathbf{m}}_{d-2} \in \mathbf{m}_{d-1}, \mathbf{m}'_{d-1}. \tag{5.169}$$

The statement of the theorem is that the number of such classes is equal to the number of four-point tensor structures. Indeed, we already saw that the four-point tensor structures (5.134) are parameterized by exactly the same data.

For conformal blocks this statement is, strictly speaking, only a counting statement and thus it would be interesting to get a more physical understanding of this. Note however that the matroms $\mathbf{P}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}$, as discussed in section 5.3.7.3 link together, in some sense, the spaces of $\mathbb{R} \times Spin(d)$ blocks and four-point tensor structures.

5.3.7 *P*-functions

In this section we discuss general properties of the GT matrix elements P, as well as their explicit calculation in various situations. This section is rather technical and mostly independent from the sections to follow, and thus can be skipped on the first reading.

5.3.7.1 Basic properties

First, recall the definition (5.128)

$$\langle \mathfrak{M}_d | e^{\theta M_{12}} | \mathfrak{M}'_d \rangle = P^{\mathbf{m}_d, \mathbf{m}_{d-2}}_{\mathbf{m}_{d-1}, \mathbf{m}'_{d-1}}(\theta) \delta_{\mathfrak{M}_{d-2}, \mathfrak{M}'_{d-2}}. \tag{5.170}$$

There are a lot of properties of P which follow immediately from this definition as a matrix element. For example, the simplest property of P is given by substituting $\theta = 0$,

$$P^{\mathbf{m}_d, \mathbf{m}_{d-2}}_{\mathbf{m}_{d-1}, \mathbf{m}'_{d-1}}(0) = \delta_{\mathbf{m}_{d-1}, \mathbf{m}'_{d-1}}. \tag{5.171}$$

[24] Note that indeed $\widetilde{\mathbf{m}}_{d-2}$ is always a dominant weight for $Spin(d-2)$.

Furthermore, P is 2π-periodic for bosonic representations and 2π-antiperiodic for fermionic representations. More generally, we know from the standard representation theory arguments that the spectrum of iM_{12} consists of (half-)integers ranging from $-m_{d,1}$ to $m_{d,1}$, and thus all P-functions have the form

$$P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta) = \sum_{m=-m_{d,1}}^{m_{d,1}} c_m e^{im\theta}, \tag{5.172}$$

where c_m are coefficients which depend on the indices of P, some of which may vanish.

Reality properties can be obtained by applying Hermitian conjugation to the definition above and noting that $M_{\mu\nu}$ are anti-Hermitian, resulting in, for real θ,

$$\left(P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\right)^* = P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}'_{d-1},\mathbf{m}_{d-1}}(-\theta). \tag{5.173}$$

Note that we also have

$$\sum_{\mathfrak{M}'_d}\langle 0|\mathfrak{M}_d\overline{\mathfrak{M}'}_d\rangle\langle\overline{\mathfrak{M}'}_d|e^{\theta M_{12}}|\overline{\mathfrak{M}''}_d\rangle = \sum_{\mathfrak{M}'_d}\langle 0|\mathfrak{M}'_d\overline{\mathfrak{M}''}_d\rangle\langle\mathfrak{M}'_d|e^{-\theta M_{12}}|\mathfrak{M}_d\rangle \tag{5.174}$$

due to the invariance of $\langle 0|\mathfrak{M}_d\overline{\mathfrak{M}'}_d\rangle$. Contracting with $\langle\mathfrak{M}'''_d\overline{\mathfrak{M}''}_d|0\rangle$ on both sides we find

$$\dim\mathbf{m}_d\sum_{\mathfrak{M}'_d,\mathfrak{M}''_d}\langle 0|\mathfrak{M}_d\overline{\mathfrak{M}'}_d\rangle\langle\overline{\mathfrak{M}'}_d|e^{\theta M_{12}}|\overline{\mathfrak{M}''}_d\rangle\langle\mathfrak{M}'''_d\overline{\mathfrak{M}''}_d|0\rangle = \langle\mathfrak{M}'''_d|e^{-\theta M_{12}}|\mathfrak{M}_d\rangle. \tag{5.175}$$

This implies, in terms of P-functions,

$$(-1)^{\mathbf{m}_{d-1}-\mathbf{m}'_{d-1}}P^{\overline{\mathbf{m}_d},\overline{\mathbf{m}_{d-2}}}_{\overline{\mathbf{m}_{d-1}},\overline{\mathbf{m}'_{d-1}}}(\theta) = P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}'_{d-1},\mathbf{m}_{d-1}}(-\theta) = \left(P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\right)^*. \tag{5.176}$$

The group composition property for the matrix elements

$$\sum_{\mathfrak{M}'_d}\langle\mathfrak{M}_d|e^{\theta_1 M_{12}}|\mathfrak{M}'_d\rangle\langle\mathfrak{M}'_d|e^{\theta_2 M_{12}}|\mathfrak{M}''_d\rangle = \langle\mathfrak{M}_d|e^{(\theta_1+\theta_2)M_{12}}|\mathfrak{M}''_d\rangle \tag{5.177}$$

gives the sum rule

$$\sum_{\mathbf{m}'_{d-1}} P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta_1)P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}'_{d-1},\mathbf{m}''_{d-1}}(\theta_2) = P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}''_{d-1}}(\theta_1+\theta_2). \tag{5.178}$$

In particular, substituting $\theta_2 = -\theta_1$, $\mathbf{m}''_{d-1} = \mathbf{m}_{d-1}$, we find, for real θ,

$$\sum_{\mathbf{m}'_{d-1}}|P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta_1)|^2 = 1, \tag{5.179}$$

and thus

$$|P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)| \leq 1. \tag{5.180}$$

5.3.7.2 Orthogonality relations

The matrix elements of group representations obey Schur orthogonality relations which read as

$$\int_{Spin(d)} \langle \mathfrak{M}_d|R|\mathfrak{M}'_d\rangle (\langle \widetilde{\mathfrak{M}}_d|R|\widetilde{\mathfrak{M}}'_d\rangle)^* dR = \frac{1}{\dim \mathbf{m}_d}\delta_{\mathfrak{M}_d,\widetilde{\mathfrak{M}}_d}\delta_{\mathfrak{M}'_d,\widetilde{\mathfrak{M}}'_d}. \tag{5.181}$$

Here the δ-symbols also compare $\mathbf{m}_d$ with $\widetilde{\mathbf{m}}_d$. The group integral in the left hand side is understood to be over Haar measure normalized as

$$\int_{Spin(d)} dR = 1. \tag{5.182}$$

Let us set $\mathfrak{M}_{d-1} = \widetilde{\mathfrak{M}}_{d-1}$ and $\mathfrak{M}'_{d-1} = \widetilde{\mathfrak{M}}'_{d-1}$ in (5.181) and do $Spin(d-1)$ sums. Equation (5.181) then becomes

$$\begin{aligned}\sum_{\mathfrak{M}_{d-1},\mathfrak{M}'_{d-1}} \int_{Spin(d)} \langle \mathbf{m}_d\mathfrak{M}_{d-1}|R|\mathbf{m}_d\mathfrak{M}'_{d-1}\rangle\langle \widetilde{\mathbf{m}}_d\mathfrak{M}'_{d-1}|R^{-1}|\widetilde{\mathbf{m}}_d\mathfrak{M}_{d-1}\rangle dR = \\ = \frac{\dim \mathbf{m}_{d-1}\dim \mathbf{m}'_{d-1}}{\dim \mathbf{m}_d}\delta_{\mathbf{m}_d,\widetilde{\mathbf{m}}_d}.\end{aligned} \tag{5.183}$$

We then write R as $R = KAK'$, where $A = e^{\theta M_{12}}$ for some θ and $K, K' \in Spin(d-1)$.[25] In the left hand side K and K' cancel out due to $Spin(d-1)$ invariance of the contractions, resulting in

$$\begin{aligned}\sum_{\mathfrak{M}_{d-1},\mathfrak{M}'_{d-1}} \int_{Spin(d)} \langle \mathbf{m}_d\mathfrak{M}_{d-1}|e^{\theta(R)M_{12}}|\mathbf{m}_d\mathfrak{M}'_{d-1}\rangle\langle \widetilde{\mathbf{m}}_d\mathfrak{M}'_{d-1}|e^{-\theta(R)M_{12}}|\widetilde{\mathbf{m}}_d\mathfrak{M}_{d-1}\rangle dR = \\ = \sum_{\mathbf{m}_{d-2}} \dim \mathbf{m}_{d-2}\int_{Spin(d)} P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta(R))\left(P^{\widetilde{\mathbf{m}}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta(R))\right)^* dR.\end{aligned} \tag{5.184}$$

By using explicit coordinates on $Spin(d)$ one can show that, for $d > 2$

$$\int_{Spin(d)} f(\theta(R))dR = \frac{\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d-1}{2})}\int_0^\pi \sin^{d-2}\theta f(\theta)d\theta. \tag{5.185}$$

Putting everything together, we obtain the following orthogonality relation

$$\begin{aligned}\sum_{\mathbf{m}_{d-2}} \dim \mathbf{m}_{d-2}\int_0^\pi P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\left(P^{\widetilde{\mathbf{m}}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\right)^* \sin^{d-2}\theta d\theta = \\ = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}\frac{\dim \mathbf{m}_{d-1}\dim \mathbf{m}'_{d-1}}{\dim \mathbf{m}_d}\delta_{\mathbf{m}_d,\widetilde{\mathbf{m}}_d}.\end{aligned} \tag{5.186}$$

[25]This follows from a standard choice of coordinates on $Spin(d)$, which follows from $Spin(d)/Spin(d-1) = S^{d-1}$: an element on the sphere can be obtained from a fixed point by KA and K' comes from $Spin(d-1)$ equivalence class.

5.3.7.3 Computational techniques

In the remainder of this section we discuss how P-functions can be computed in practice, first in general and then in specific examples.

The conceptually simplest computational scheme follows immediately from the definition (5.128) as a matrix element of $e^{\theta M_{12}}$. Indeed, since we know the matrix elements of M_{12} (see section 5.2.3 and appendix D.2), we can find the matrix corresponding to M_{12} in any given representation and then exponentiate it by the standard methods. When doing this, one can reduce the amount of calculation by taking note of the structure of the right hand side of (5.128). Following this strategy, we simultaneously produce

$$P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta) \tag{5.187}$$

with fixed $\mathbf{m}_d$ and $\mathbf{m}_{d-2}$ for all choices of $\mathbf{m}_{d-1}$ and $\mathbf{m}'_{d-1}$.

This strategy is therefore somewhat of an overkill for our purposes, since in a four-point function the possible choices of representations $\mathbf{m}_{d-1}$ and $\mathbf{m}'_{d-1}$ are prescribed by the spins of external representations, while $\mathbf{m}_d$ and $\mathbf{m}_{d-2}$ take on all the values allowed by each pair of $\mathbf{m}_{d-1}$ and $\mathbf{m}'_{d-1}$.[26] Fortunately, there exist techniques which compute $P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)$ for fixed $\mathbf{m}_{d-1}$ and $\mathbf{m}'_{d-1}$.

Let us fix $\mathbf{m}_{d-1}$ and $\mathbf{m}'_{d-1}$. Furthermore, write $\mathbf{m}_d = (j, \widetilde{\mathbf{m}}_{d-2})$, i.e., define $j \equiv m_{d,1}$ and think of the rest of $\mathbf{m}_d$ as a $(d-2)$-dimensional weight $\widetilde{\mathbf{m}}_{d-2}$. Note that $\mathbf{m}_{d-1}, \mathbf{m}'_{d-1} \in \mathbf{m}_d$ requires $j \geq \max(m_{d-1,1}, m'_{d-1,1})$. Assuming that this holds, it is easy to check that the following two statements are equivalent,

$$\mathbf{m}_d \ni \mathbf{m}_{d-1}, \mathbf{m}'_{d-1} \Longleftrightarrow \widetilde{\mathbf{m}}_{d-2} \in \mathbf{m}_{d-1}, \mathbf{m}'_{d-1}. \tag{5.188}$$

In other words, $\widetilde{\mathbf{m}}_{d-2}$ satisfies the same requirements as $\mathbf{m}_{d-2}$. This means that we can arrange $P^{(j,\widetilde{\mathbf{m}}_{d-2}),\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)$ into a square matrix $\mathbf{P}^j_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)$ with rows and columns labeled by $\widetilde{\mathbf{m}}_{d-2}$ and $\mathbf{m}_{d-2}$ respectively. Such matrices are discussed, for example, in [210] (and references therein), where they are shown to satisfy certain second-order matrix differential equations, and methods for solving these equations are developed. Following the terminology of [210], we will refer to these matrices as “matroms.” Note that the size of the matrom is independent of j and is only determined by $\mathbf{m}_{d-1}$ and $\mathbf{m}'_{d-1}$. Furthermore, all (if any) components of a given matrom appear in a given four-point function.

[26] Also, the size of the matrix which one needs to exponentiate grows with the spin $m_{d,1}$, which makes this approach computationally more intensive.

Potentially, the results described in [210] may allow one to find analytic in j expressions for the matroms $\mathbf{P}^j_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}$ in terms of known special functions. Unfortunately, we were not able to devise a complete computational algorithm based on these results.[27] However, since in numerical applications one requires $\mathbf{P}^j_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}$ for all j up to a certain cutoff, it is convenient to use a recursion relation in j as described below. Expressions analytic in j can still be obtained in a number of cases, as we discuss in the next subsections.

The basic idea is to consider the product

$$\langle\mathfrak{M}_d|e^{\theta M_{12}}|\mathfrak{M}'_d\rangle\langle\square,\bullet,\ldots|e^{\theta M_{12}}|\square,\bullet,\ldots\rangle = \langle\mathfrak{M}_d|e^{\theta M_{12}}|\mathfrak{M}'_d\rangle\cos\theta. \tag{5.189}$$

The left hand side is a matrix element in $\mathbf{m}_d\otimes\square$ and thus can be decomposed as a sum of matrix elements in various irreducible representations,

$$\begin{aligned}&\langle\mathfrak{M}_d|e^{\theta M_{12}}|\mathfrak{M}'_d\rangle\langle\square,\bullet,\ldots|e^{\theta M_{12}}|\square,\bullet,\ldots\rangle =\\ &=\sum_{\widetilde{\mathbf{m}}_d\in\mathbf{m}_d\otimes\square}\langle\widetilde{\mathbf{m}}_d\mathfrak{M}_{d-1}|e^{\theta M_{12}}|\widetilde{\mathbf{m}}_d\mathfrak{M}'_{d-1}\rangle\left(\begin{array}{cc|c}\mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d\\ \mathbf{m}_{d-1} & \bullet & \mathbf{m}_{d-1}\end{array}\right)\left(\begin{array}{cc|c}\mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d\\ \mathbf{m}'_{d-1} & \bullet & \mathbf{m}'_{d-1}\end{array}\right)^*.\end{aligned} \tag{5.190}$$

One can easily see that in terms of matroms this leads to the following recursion relation,

$$\mathbf{A}^+_j\mathbf{P}^{j+1}+\mathbf{A}^-_j\mathbf{P}^{j-1}+\mathbf{B}_j\mathbf{P}^j=\cos\theta\,\mathbf{P}^j, \tag{5.191}$$

where $\mathbf{A}^\pm_j$, $\mathbf{B}^j$ are some matrices,[28] and we have suppressed the dependence of everything on $\mathbf{m}_{d-1},\mathbf{m}'_{d-1}$ for simplicity of notation. Starting from the smallest possible j (for which we can compute $\mathbf{P}^j$ by, say, exponentiation), one can use this relation to find $\mathbf{P}^j$ for higher j.

As an example, consider the matroms in $d=2n$ with $\mathbf{m}_{d-1}=\mathbf{m}'_{d-1}=(\frac{1}{2},\ldots,\frac{1}{2})$, which will be useful in the example of section 5.3.5. There are two representations in the dimensional reduction of $\mathbf{m}_{d-1}=\mathbf{m}'_{d-1}$, $\mathbf{m}_{d-2}=(\frac{1}{2},\ldots,\pm\frac{1}{2})$, i.e., the two fermionic representations in $d-2$ dimensions. We therefore have a 2×2 matrom

$$\mathbf{P}^j=\begin{pmatrix}P^{(j,\frac{1}{2},\ldots,+\frac{1}{2}),(\frac{1}{2},\ldots,+\frac{1}{2})}_{(\frac{1}{2},\ldots,\frac{1}{2}),(\frac{1}{2},\ldots,\frac{1}{2})}(\theta) & P^{(j,\frac{1}{2},\ldots,+\frac{1}{2}),(\frac{1}{2},\ldots,-\frac{1}{2})}_{(\frac{1}{2},\ldots,\frac{1}{2}),(\frac{1}{2},\ldots,\frac{1}{2})}(\theta)\\ P^{(j,\frac{1}{2},\ldots,-\frac{1}{2}),(\frac{1}{2},\ldots,+\frac{1}{2})}_{(\frac{1}{2},\ldots,\frac{1}{2}),(\frac{1}{2},\ldots,\frac{1}{2})}(\theta) & P^{(j,\frac{1}{2},\ldots,-\frac{1}{2}),(\frac{1}{2},\ldots,-\frac{1}{2})}_{(\frac{1}{2},\ldots,\frac{1}{2}),(\frac{1}{2},\ldots,\frac{1}{2})}(\theta)\end{pmatrix}. \tag{5.192}$$

[27] It is an interesting problem to complete the results described in [210] to find a general algorithm for constructing analytic expressions for generic matroms.

[28] The matrices $\mathbf{A}$ are, importantly, diagonal, which makes it easy to invert $\mathbf{A}^+_j$.

For example, one can easily check that for any d

$$\mathbf{P}^{\frac{1}{2}} = \begin{pmatrix} e^{-i\theta/2} & e^{+i\theta/2} \\ e^{+i\theta/2} & e^{-i\theta/2} \end{pmatrix}. \tag{5.193}$$

By using the explicit formulas for the isoscalar factors from appendix D.2.2, one can show that the recursion relation (5.191) reduces in this case to

$$\frac{j+2n-\frac{3}{2}}{j+n-\frac{1}{2}}\mathbf{P}^{j+1} + \frac{j-\frac{1}{2}}{j+n-\frac{3}{2}}\mathbf{P}^{j-1} + \frac{n-1}{(j+n-\frac{1}{2})(j+n-\frac{3}{2})}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{P}^{j} = 2\cos\theta\,\mathbf{P}^{j}, \tag{5.194}$$

where $n = d/2$. For instance, applying this relation twice, we find

$$\mathbf{P}^{2+\frac{1}{2}} = \frac{n-1}{2n-1}\begin{pmatrix} \frac{1}{2}\frac{n+1}{n-1}e^{-\frac{5}{2}i\theta} + e^{-\frac{1}{2}i\theta} + \frac{1}{2}e^{+\frac{3}{2}i\theta} & \frac{1}{2}\frac{n+1}{n-1}e^{+\frac{5}{2}i\theta} + e^{+\frac{1}{2}i\theta} + \frac{1}{2}e^{-\frac{3}{2}i\theta} \\ \frac{1}{2}\frac{n+1}{n-1}e^{+\frac{5}{2}i\theta} + e^{+\frac{1}{2}i\theta} + \frac{1}{2}e^{-\frac{3}{2}i\theta} & \frac{1}{2}\frac{n+1}{n-1}e^{-\frac{5}{2}i\theta} + e^{-\frac{1}{2}i\theta} + \frac{1}{2}e^{+\frac{3}{2}i\theta} \end{pmatrix}, \tag{5.195}$$

valid for any $d = 2n$. The general solution can be expressed in terms of Jacobi polynomials as[29]

$$\mathbf{P}^{j} = \frac{(j-\frac{1}{2})!}{(n-\frac{1}{2})_{j-\frac{1}{2}}}\left[\cos\tfrac{\theta}{2}P^{(n-\frac{3}{2},n-\frac{1}{2})}_{j-\frac{1}{2}}(\cos\theta)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \sin\tfrac{\theta}{2}P^{(n-\frac{1}{2},n-\frac{3}{2})}_{j-\frac{1}{2}}(\cos\theta)\begin{pmatrix} -i & i \\ i & -i \end{pmatrix}\right]. \tag{5.196}$$

5.3.7.4 Contribution of $\mathbb{R}\times Spin(d)$ multiplets in terms of matroms

Having introduced the matroms $\mathbf{P}$ in the previous subsection, it makes sense to reanalyse (5.132) in terms of them. For fixed a and b as in (5.122) and (5.123) denote by

$$\mathbf{T}^{ba} \tag{5.197}$$

the column vector built out of four-point tensor structures

$$\begin{bmatrix} \mathfrak{M}^3_d \\ \mathfrak{M}^4_d \end{bmatrix} b \bigg| \mathbf{m}_{d-2} \bigg| a \bigg| \begin{matrix} \mathfrak{M}^1_d \\ \mathfrak{M}^2_d \end{matrix} \bigg] \tag{5.198}$$

[29] To find this solution, we first diagonalized the recursion relation and then matched it to the recursion relation for Jacobi polynomials. The Jacobi polynomials entering this expression can in principle be expressed in terms of linear combinations of Gegenbauer polynomials.

with $\mathbf{m}_{d-2}$ running through all allowed values. Also, denote $\mathbf{P}^j_{ba} \equiv \mathbf{P}^j_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}$. Finally, let

$$\mathbf{\Lambda}^{ba}_j \tag{5.199}$$

be the row vector built out of

$$\overline{\lambda}^b_{\Delta,\mathbf{m}_d}\lambda^a_{\Delta,\mathbf{m}_d} \tag{5.200}$$

corresponding to all $\mathbf{m}_d = (j, \widetilde{\mathbf{m}}_{d-2})$ which can contribute to the given pair a, b according to (5.131), summed over degenerate multiplets. If we are considering the contribution of a single $\mathbb{R}\times Spin(d)$ multiplet, then this vector contains a single non-zero element, but at this point it is convenient to also allow several contributions. We then have

$$\begin{aligned}\sum_{\mathbf{m}_d, m_{d,1}=j}\sum_{a,b}\sum_{\mathbf{m}_{d-2}} \overline{\lambda}^b_{\Delta,\mathbf{m}_d}\lambda^a_{\Delta,\mathbf{m}_d} r^\Delta P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta) \times \begin{bmatrix}\mathfrak{M}^3_d \\ \mathfrak{M}^4_d\end{bmatrix}\!\!\Big|b\Big|\mathbf{m}_{d-2}\Big|a\Big|\!\!\begin{bmatrix}\mathfrak{M}^1_d \\ \mathfrak{M}^2_d\end{bmatrix} = \\ = r^\Delta \sum_{a,b} \mathbf{\Lambda}^{ba}_j \cdot \mathbf{P}^j_{ba}(\theta)\cdot \mathbf{T}^{ba}.\end{aligned} \tag{5.201}$$

As we discuss in section 5.3.6, in this equation $\mathbf{\Lambda}^{ba}_j$ correspond, roughly speaking, to the space of conformal blocks, while $\mathbf{T}^{ba}$ correspond to the space of four-point tensor structures. The matroms link these two spaces together, giving a realization of the folklore theorem [55, 75] (see section 5.3.6).

In the rest of this section we consider some more explicit examples. First, we recover the Gegenbauer polynomials relevant to the scalar correlation functions and then we consider the low-dimensional cases $d = 3$ and $d = 4$.

5.3.7.5 Scalar matrom

Let us consider the simplest P-function $P^{\mathbf{j},\bullet}_{\bullet,\bullet}(\theta)$, which is the only component of the simplest scalar matrom $\mathbf{P}^j_{\bullet,\bullet}(\theta)$. Analogously to the example considered above, we could write down the recursion relation (5.191) for this matrom and recognize that, together with the initial condition $P^{\bullet,\bullet}_{\bullet,\bullet}(\theta) \equiv 1$, it is solved by

$$P^{\mathbf{j},\bullet}_{\bullet,\bullet}(\theta) = \frac{C^{(\nu)}_j(\cos\theta)}{C^{(\nu)}_j(1)}, \tag{5.202}$$

where $\nu = (d-2)/2$. However, it is instructive to take another approach to arrive at this result. Consider the tensor given by

$$e^{\mu_1}_1 \cdots e^{\mu_j}_1 - \text{traces}. \tag{5.203}$$

Obviously, this tensor is an element of $\mathbf{j}$ of $Spin(d)$. On the other hand, it transforms under the trivial representation of $Spin(d-1)$. Therefore, we have

$$e_1^{\mu_1}\cdots e_1^{\mu_j} - \text{traces} \propto |\mathbf{j},\bullet,\ldots\rangle. \tag{5.204}$$

Acting with $e^{\theta M_{12}}$, we find that

$$e^{\theta M_{12}}|\mathbf{j},\bullet,\ldots\rangle \propto e_1^{\mu_1}(\theta)\cdots e_1^{\mu_j}(\theta) - \text{traces}, \tag{5.205}$$

where $e_1(\theta) = \cos\theta e_1 + \sin\theta e_2$. This implies

$$\begin{aligned} &P^{\mathbf{j};\bullet}_{\bullet;\bullet}(\theta) \\ &= \langle \mathbf{j},\bullet,\ldots|e^{\theta M_{12}}|\mathbf{j},\bullet,\ldots\rangle \propto (e_{1,\mu_1}\cdots e_{1,\mu_j} - \text{traces})(e_1^{\mu_1}(\theta)\cdots e_1^{\mu_j}(\theta) - \text{traces}). \end{aligned} \tag{5.206}$$

The right hand side of this equation is known to be proportional to the Gegenbauer polynomial $C_j^{(\nu)}(e_1\cdot e_1(\theta)) = C_j^{(\nu)}(\cos\theta)$. Combining this with the normalization condition $P^{\mathbf{j};\bullet}_{\bullet;\bullet}(0) = 1$, we recover (5.202).

This strategy generalizes to other tensor representations and also allows one to relate P-functions to the irreducible projectors studied recently in [82]. We discuss this further in appendix D.4.

5.3.7.6 3 dimensions

We now consider the case $d = 3$. As discussed in section 5.3.1, the 3-dimensional GT matrix elements $P^j_{m,m'}(\theta)$ are given by (5.129),[30]

$$P^j_{m,m'}(\theta) = \langle j,m|e^{\theta M_{12}}|j,m'\rangle = \langle j,m|e^{-i\theta J_{\hat{2}}}|j,m'\rangle = d^j_{m,m'}(-\theta). \tag{5.207}$$

Note that in $3d$ all matroms are 1×1 and coincide with the above functions. There is not much to add here, except for the explicit formula for the small Wigner d-matrix $d^j_{m,m'}(\theta)$,

$$\begin{aligned} &d^j_{m,m'}(\theta) \\ &= (-1)^{m-m'}\sqrt{\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}}\left(\sin\frac{\beta}{2}\right)^{m'-m}\left(\cos\frac{\beta}{2}\right)^{m'+m} P^{(m'-m,m'+m)}_{j-m'}(\cos\theta), \end{aligned} \tag{5.208}$$

where in this expression $P_n^{(a,b)}$ are the Jacobi polynomials.

[30] We use the convention consistent with Mathematica's `WignerD[{`j, m, m'`},`θ`]`.

5.3.7.7 4 dimensions

In 4d we have the following definition of GT matrix elements $P^{l_L,l_R;m}_{j,j'}(\theta)$,

$$P^{l_L,l_R;m}_{j,j'}(\theta) = \langle l_L, l_R; j, m|e^{\theta M_{12}}|l_L, l_R; j', m\rangle. \tag{5.209}$$

We can compute them by going to the $SU(2)\times SU(2)$ basis,

$$\begin{aligned} P^{l_L,l_R;m}_{j,j'}(\theta) &= \langle l_L, l_R; j, m|e^{\theta M_{12}}|l_L, l_R; j', m\rangle \\ &= \sum_{m_L+m_R=m}\ \sum_{m'_L+m'_R=m} \langle l_L, m_L; l_R, m_R|e^{\theta M_{12}}|l_L, m'_L; l_R, m'_R\rangle\times \\ &\qquad \times \langle j, m|l_L, m_L; l_R, m_R\rangle\langle l_L, m'_L; l_R, m'_R|j', m\rangle. \end{aligned} \tag{5.210}$$

Using (5.86), we find

$$\begin{aligned} \langle l_L, m_L; l_R, m_R|e^{\theta M_{12}}|l_L, m'_L; l_R, m'_R\rangle &= \langle l_L, m_L; l_R, m_R|e^{-i\theta J^L_{\hat{3}}+i\theta J^R_{\hat{3}}}|l_L, m'_L; l_R, m'_R\rangle \\ &= e^{-i(m_L-m_R)}\delta_{m_L m'_L}\delta_{m_R m'_R}, \end{aligned} \tag{5.211}$$

and thus

$$P^{l_L,l_R;m}_{j,j'}(\theta) = \sum_{k=-l_L-l_R}^{l_L+l_R} \left\langle j, m\middle| l_L, \frac{m+k}{2}; l_R, \frac{m-k}{2}\right\rangle\left\langle l_L, \frac{m+k}{2}; l_R, \frac{m-k}{2}\middle| j', m\right\rangle e^{-ik\theta}. \tag{5.212}$$

Note that in this formula the summation is over (half-)integral values of k for (half-)integral values of $\ell_1 = l_L + l_R$, and whenever the Clebsch-Gordan coefficient is undefined, we assume that it is equal to zero. Thus the range of summation is effectively restricted to

$$\{-2l_L - m, \ldots, 2l_L - m\} \cap \{-2l_R + m, \ldots, 2l_R + m\}. \tag{5.213}$$

For example, if $m = l_L + l_R$, then only $k = l_L - l_R$ enters the sum. (Also necessarily $j = j' = l_L + l_R$.)

5.4 Casimir equation

In this section we derive Casimir recursion relation for the series expansion of spinning conformal blocks. We first rederive the results of [59] for scalar conformal blocks in a more streamlined way and then extend these results to arbitrary spinning conformal blocks. As an example, we explicitly work out the recursion relations for general 3d conformal blocks and for general seed blocks in arbitrary d.

In this section we will work in coordinates different from those in section 5.3. In particular, we set

$$w_1 = 0, \quad w_2 = z, \quad w_3 = 1, \quad w_4 = +\infty. \tag{5.214}$$

We use the following definition of $\mathcal{O}_4(+\infty)$,

$$\mathcal{O}_4(+\infty) \equiv \lim_{L\to+\infty} L^{2\Delta_4}\mathcal{O}_4(Le_1). \tag{5.215}$$

Note that we do not act in any way on the spin indices of $\mathcal{O}_4$ when taking this limit.[31] The results of section 5.3 translate to this case without essential modification (except for changing the insertion point of the operators in all formulas).

We use (5.214) because the Casimir recursion relations take the simplest form in these coordinates, analogously to the case of scalar blocks [59]. The recursion relations in ρ-coordinates, unfortunately, take a much more complicated form [60, 80].

5.4.1 Review of scalar conformal blocks

Consider the scalar conformal block for exchange of a primary operator $\mathcal{O}$

$$G_{\mathcal{O}}(s,\phi) \equiv \langle 0|\phi_4(\infty)\phi_3(1)|\mathcal{O}|s^D e^{\theta M_{12}}\phi_2(1)\phi_1(0)|0\rangle, \tag{5.216}$$

where $z = se^{i\theta}$, we have used the convention (5.108) for writing the four-point functions,[32] and $|\mathcal{O}|$ is the projection operator on the conformal family of $\mathcal{O}$,

$$|\mathcal{O}| = \sum_{p\geq 0,\mathbf{m}_d,\mathfrak{M}_d,q} |\Delta_p,\mathfrak{M}_d,q\rangle\langle\Delta_p,\mathfrak{M}_d,q|, \tag{5.217}$$

where the sum is over an orthonormal basis of descendants of $\mathcal{O}$. Here $\Delta_p = \Delta_{\mathcal{O}} + p$ is the scaling dimension of a level-p descendant, $\mathbf{m}_d$ is the $Spin(d)$ representation of the descendant, and q labels the possible degeneracies which arise when there are several descendants in representation $\mathbf{m}_d$ at level p.

[31] When $\mathcal{O}_4$ is tensor, one often acts on its indices with reflection along e_1 when taking this limit. This is done because $\mathcal{O}_4(\infty)$ defined our way effectively transforms in the representation reflected to $\mathbf{m}_d^4$. When $\mathbf{m}_d^4$ is tensor, its reflection is equivalent to $\mathbf{m}_d^4$ and thus one may find it convenient to act on $\mathcal{O}_4$ with the map which furnishes this equivalence. More generally, the reflected representation can be different from $\mathbf{m}_d^4$ and thus there is no benefit in acting on spin indices of $\mathcal{O}_4$ within our general treatment.

[32] In the scalar case (5.108) differs from (5.107) only by the factor $s^{\Delta_1+\Delta_2}$.

The results of section 5.3 and in particular 5.3.2 tell us what is the most general contribution of a single term of (5.217) to (5.216). We therefore have

$$G_{\mathcal{O}}(s,\phi)=\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\sum_{q}\lambda^{\bullet,\bullet}_{\bullet,p,j,q}\overline{\lambda}^{\bullet,\bullet}_{\bullet,p,j,q}s^{\Delta_p}P^{\mathbf{j},\bullet}_{\bullet,\bullet}(\theta)=\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\Lambda_{p,j}s^{\Delta_{\mathcal{O}}+p}\frac{C_j^{(\nu)}(\cos\theta)}{C_j^{(\nu)}(1)}. \tag{5.218}$$

We have defined

$$\Lambda_{p,j}\equiv\sum_{q}\lambda^{\bullet,\bullet}_{\bullet,p,j,q}\overline{\lambda}^{\bullet,\bullet}_{\bullet,p,j,q}. \tag{5.219}$$

The range of j is in fact restricted by the spectrum of descendants at each level p so that $|j-j_{\mathcal{O}}|\leq p$, but we will ignore this by assuming that $\Lambda_{p,j}=0$ for p,j outside this range. While this expansion respects $\mathbb{R}\times Spin(d)$ symmetry, it doesn't tell us what the coefficients $\Lambda_{p,j}$ are.

These coefficients are constrained by consistency of expansion 5.217 with the full conformal symmetry. It was noticed in [57] that it suffices to ensure consistency with the action of the quadratic conformal Casimir operator. Usually this is condition is formulated in a form of differential equation [57, 64]. When applied to (5.218), this equation immediately yields a one-step recursion relation for the coefficients $\Lambda_{p,j}$ [59],

$$(C_{p,j}-C_{0,j_{\mathcal{O}}})\Lambda_{p,j}=\Gamma^{+}_{p-1,j-1}\Lambda_{p-1,j-1}+\Gamma^{-}_{p-1,j+1}\Lambda_{p-1,j+1}, \tag{5.220}$$

where coefficients $\Gamma^{\pm}_{p,j}$ are given by[33]

$$\begin{aligned}\Gamma^{+}_{p,j}&=\frac{(\Delta_p+j-\Delta_{12})(\Delta_p+j+\Delta_{34})(j+d-2)}{2j+d-2},\\ \Gamma^{-}_{p,j}&=\frac{(\Delta_p-j-d+2-\Delta_{12})(\Delta_p-j-d+2+\Delta_{34})j}{2j+d-2},\end{aligned} \tag{5.221}$$

with $\Delta_{ij}=\Delta_i-\Delta_j$, while the Casimir eigenvalues are given by

$$C_{p,j}=\Delta_p(\Delta_p-d)+j(j+d-2). \tag{5.222}$$

This result is remarkably simple, much simpler than the intermediate steps in the derivation of [59] would suggest. In fact, it is not a priori obvious from that derivation that the recursion relation should take such a simple form. For example,

[33] In [59] these coefficients are given with $\Delta_{12}=\Delta_{34}=0$, but it is trivial to generalize their argument.

when repeated in ρ-coordinates, essentially the same derivation leads to a much more complicated recursion relation. We are therefore motivated to look for a more conceptual derivation of (5.220), which manifests this simple structure.

Let us start from the definition of the conformal Casimir operator,

$$C = D(D-d) + C_{Spin(d)} - P \cdot K, \tag{5.223}$$

where $C_{Spin(d)}$ is the $Spin(d)$ quadratic Casimir defined as

$$C_{Spin(d)} = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu}. \tag{5.224}$$

The key property of C is that it commutes with all conformal generators and thus acts on all the descendants of $\mathcal{O}$ by the same eigenvalue as on $\mathcal{O}$. That eigenvalue can be computed by

$$C|\mathcal{O}\rangle = \left(D(D-d) + C_{Spin(d)} - P \cdot K\right)|\mathcal{O}\rangle = C(\mathcal{O})|\mathcal{O}\rangle, \tag{5.225}$$

$$C(\mathcal{O}) = \Delta_{\mathcal{O}}(\Delta_{\mathcal{O}} - d) + C_{Spin(d)}(\mathbf{m}_d^{\mathcal{O}}), \tag{5.226}$$

where we used $K_\mu|\mathcal{O}\rangle = 0$, and $C_{Spin(d)}(\mathbf{m}_d)$ is the $Spin(d)$ quadratic Casimir eigenvalue corresponding to the $Spin(d)$ representation $\mathbf{m}_d$. It is given by

$$C_{Spin(d)}(\mathbf{m}_d) = \sum_{k=1}^{\lfloor d/2 \rfloor} m_{d,k}(m_{d,k} + d - 2k). \tag{5.227}$$

For future convenience, let us define for any (not necessarily primary) $\mathbb{R} \times Spin(d)$ multiplet the number

$$C(\Delta, \mathbf{m}_d) \equiv \Delta(\Delta - d) + C_{Spin(d)}(\mathbf{m}_d). \tag{5.228}$$

It is the eigenvalue of the operator

$$\widetilde{C} \equiv C + P \cdot K = D(D-d) + C_{Spin(d)}. \tag{5.229}$$

Note that $P \cdot K = K^\dagger \cdot K \geq 0$ for Δ above unitarity bound and thus we always have in such cases

$$\widetilde{C} \geq C. \tag{5.230}$$

Since C takes the same eigenvalue on all states in a conformal multiplet, we have

$$|\mathcal{O}|C = |\mathcal{O}|C(\mathcal{O}). \tag{5.231}$$

This implies the following operator version of the Casimir equation,

$$\langle 0|\phi_4\phi_3|\mathcal{O}|C s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle = C(\mathcal{O})\langle 0|\phi_4\phi_3|\mathcal{O}|s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle. \tag{5.232}$$

For notational simplicity, we have omitted the positions of the operators, which are the same as in (5.216). The standard Casimir differential equation can be obtained by acting with C on the right in the left hand side of this equation and expressing this action in terms of derivatives in θ and s. We will take another approach, rewriting the left hand side instead as

$$\begin{aligned}
&\langle 0|\phi_4\phi_3|\mathcal{O}|C s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle \\
&= \langle 0|\phi_4\phi_3|\mathcal{O}|(\widetilde{C} - P^\mu K_\mu) s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle \\
&= \langle 0|\phi_4\phi_3|\mathcal{O}|\widetilde{C} s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle - \langle 0|\phi_4\phi_3|\mathcal{O}|P^\mu K_\mu s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle \\
&= \langle 0|\phi_4\phi_3|\mathcal{O}|\widetilde{C} s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle - \langle 0|\phi_4\phi_3 P^\mu|\mathcal{O}|K_\mu s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle,
\end{aligned} \tag{5.233}$$

where in the last line we have used the conformal invariance of the projector $|\mathcal{O}|$, i.e., that it commutes with all conformal generators. Rearranging, we find

$$\langle 0|\phi_4\phi_3|\mathcal{O}|(\widetilde{C} - C) s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle = \langle 0|\phi_4\phi_3 P^\mu|\mathcal{O}|K_\mu s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle. \tag{5.234}$$

We will now derive the recursion relation (5.220) by evaluating both sides of this equation with the help of (5.217).

5.4.1.1 Left hand side

To warm up, let us consider the left hand side of this equation first. Using (5.217), we find

$$\begin{aligned}
&\langle 0|\phi_4\phi_3|\mathcal{O}|(\widetilde{C} - C) s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle \\
&\quad = \sum_{p,\mathbf{m}_d,\mathfrak{M}_d,q} \langle 0|\phi_4\phi_3|\Delta_p,\mathfrak{M}_d,q\rangle\langle\Delta_p,\mathfrak{M}_d,q|(\widetilde{C} - C) s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle \\
&\quad = \sum_{p,\mathbf{m}_d,\mathfrak{M}_d,q} \Big(C(\Delta_p,\mathbf{m}_d) - C(\Delta_{\mathcal{O}},\mathbf{j}_{\mathcal{O}})\Big) \langle 0|\phi_4\phi_3|\Delta_p,\mathfrak{M}_d,q\rangle\langle\Delta_p,\mathfrak{M}_d,q|s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle \\
&\quad = \sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(C_{p,j} - C_{0,j_{\mathcal{O}}})\Lambda_{p,j} s^{\Delta_{\mathcal{O}}+p}\frac{C_j^{(\nu)}(\cos\phi)}{C_j^{(\nu)}(1)},
\end{aligned} \tag{5.235}$$

where the last line follows similarly to (5.218), and we also made use of the fact that we arranged the descendants into $\mathbb{R}\times Spin(d)$ multiplets. We can already see that we are on the right track – the coefficients in this expansion exactly reproduce the left hand side of the recursion relation (5.220).

5.4.1.2 Right hand side

Let us now analyze the less trivial right hand side of (5.234). We first look at the contribution of a single term of (5.217). For simplicity of notation, we will omit the degeneracy index q for now and restore it later. We thus consider

$$\sum_{\mathfrak{M}_d}\langle 0|\phi_4\phi_3 P^\mu|\Delta_p,\mathfrak{M}_d\rangle\langle\Delta_p,\mathfrak{M}_d|K_\mu s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle. \tag{5.236}$$

Left three-point structure We will first evaluate the left three-point function by commuting P on the left. We have (see appendix D.1 for our conventions on conformal algebra)

$$\langle 0|\phi_4(\infty)\phi_3(1)P_\mu|\Delta_p,\mathfrak{M}_d\rangle = -\langle 0|\phi_4(\infty)\partial_\mu\phi_3(1)|\Delta_p,\mathfrak{M}_d\rangle. \tag{5.237}$$

The crucial point is that the knowledge of $\langle 0|\phi_4(\infty)\phi_3(1)|\Delta_p,\mathfrak{M}_d\rangle$ and $\mathbb{R}\times Spin(d)$ invariance allow us to evaluate

$$\langle 0|\phi_4(\infty)\phi_3(x)|\Delta_p,\mathfrak{M}_d\rangle \tag{5.238}$$

for any $x\in\mathbb{R}^d$. In particular, we can compute the right hand side of (5.237). For example, note that

$$\begin{aligned}\langle 0|\phi_4(\infty)\partial_1\phi_3(1)|\Delta_p,\mathfrak{M}_d\rangle &= -\langle 0|\phi_4(\infty)\phi_3(1)(D+\Delta_3-\Delta_4)|\Delta_p,\mathfrak{M}_d\rangle\\ &= -(\Delta_p+\Delta_3-\Delta_4)\langle 0|\phi_4(\infty)\phi_3(1)|\Delta_p,\mathfrak{M}_d\rangle.\end{aligned} \tag{5.239}$$

Here the first equality follows from action of D on the left while the second equality follows from action on the right. The minus sign in front of Δ_4 is due to the fact that we placed O_4 at infinity. Analogously, for $\mu\neq 1$,

$$\langle 0|\phi_4(\infty)\partial_\mu\phi_3(1)|\Delta_p,\mathfrak{M}_d\rangle = -\langle 0|\phi_4(\infty)\phi_3(1)M_{1\mu}|\Delta_p,\mathfrak{M}_d\rangle. \tag{5.240}$$

Here we can act with $M_{1\mu}$ on the right by using the representation $\mathbf{m}_d$ for $M_{1\mu}$. As we discussed in section 5.2.3, such actions can be described by means of a reduced matrix element,

$$\langle\mathfrak{M}'_d|M^{1\,\mathfrak{U}_{d-1}}|\mathfrak{M}_d\rangle = \left(\begin{matrix}\mathbf{m}_d\\ \mathbf{m}'_{d-1}\end{matrix}\middle|M^{\square}\middle|\begin{matrix}\mathbf{m}_d\\ \mathbf{m}_{d-1}\end{matrix}\right)\langle\mathfrak{M}'_{d-1}|\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}\rangle. \tag{5.241}$$

We conclude

$$\begin{aligned}\langle 0|\phi_4(\infty)\phi_3(1)P^{\mathfrak{U}_d}|\Delta_p,\mathfrak{M}_d\rangle = \sum_{\mathfrak{M}'_d}\left(\begin{matrix}\mathbf{m}_d\\ \mathbf{m}'_{d-1}\end{matrix}\middle|M^{\square}\middle|\begin{matrix}\mathbf{m}_d\\ \mathbf{m}_{d-1}\end{matrix}\right)\langle\mathfrak{M}'_{d-1}|\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}\rangle\times\\ \times\langle 0|\phi_4(\infty)\phi_3(1)|\Delta_p,\mathfrak{M}'_d\rangle,\end{aligned} \tag{5.242}$$

where $\mathbf{u}_d = \mathbf{u}_{d-1} = \square$.

Note that the states $P^{\mathfrak{U}_d}|\Delta_p, \mathfrak{M}_d\rangle$ are just some other descendants of $\mathcal{O}$. It is convenient to decompose them into the irreducible representations of $Spin(d)$ by defining the states

$$|P, \Delta_p, \mathbf{m}_d; \widetilde{\mathfrak{M}}_d\rangle \equiv \sum_{\mathfrak{M}_d, \mathfrak{U}_d} \langle \mathfrak{M}_d \mathfrak{U}_d | \widetilde{\mathfrak{M}}_d\rangle \; P^{\mathfrak{U}_d}|\Delta_p, \mathfrak{M}_d\rangle, \tag{5.243}$$

where $\widetilde{\mathbf{m}}_d \in \square \otimes \mathbf{m}_d$ and $\langle \mathfrak{M}_d \mathfrak{U}_d | \widetilde{\mathfrak{M}}_d\rangle$ are the vector Clebsch-Gordan coefficients. We can decompose this sum according to $Spin(d-1)$ symmetry of the three-point functions as

$$\begin{aligned}
|P, \Delta_p, \mathbf{m}_d; \widetilde{\mathfrak{M}}_d\rangle = &\sum_{\mathfrak{M}_d} P^{\square, \bullet, \dots}|\Delta_p, \mathfrak{M}_d\rangle \langle \mathfrak{M}_d; \; \square, \bullet, \dots | \widetilde{\mathfrak{M}}_d\rangle \\
&+ \sum_{\substack{\mathfrak{M}_d, \mathfrak{U}_d \\ \mathbf{u}_{d-1} = \square}} P^{\mathfrak{U}_d}|\Delta_p, \mathfrak{M}_d\rangle \langle \mathfrak{M}_d \mathfrak{U}_d | \widetilde{\mathfrak{M}}_d\rangle, \\
= &P^{\square, \bullet, \dots}|\Delta_p, \mathbf{m}_d \, \widetilde{\mathfrak{M}}_{d-1}\rangle \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} & \bullet & \widetilde{\mathbf{m}}_{d-1} \end{array}\right) \\
&+ \sum_{\substack{\mathfrak{M}_d, \mathfrak{U}_d \\ \mathbf{u}_{d-1} = \square}} P^{\mathfrak{U}_d}|\Delta_p, \mathfrak{M}_d\rangle \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \mathbf{m}_{d-1} & \square & \widetilde{\mathbf{m}}_{d-1} \end{array}\right) \langle \mathfrak{M}_{d-1} \mathfrak{U}_{d-1} | \widetilde{\mathfrak{M}}_{d-1}\rangle.
\end{aligned} \tag{5.244}$$

Here we made use of (5.93) and of the triviality of CG coefficients involving the trivial representation. Using equations (5.54), (5.239) and (5.242) we then find

$$\langle 0|\phi_4(\infty)\phi_3(1)|P, \Delta_p, \mathbf{m}_d; \widetilde{\mathfrak{M}}_d\rangle = \begin{bmatrix} \widetilde{\mathbf{m}}_d \; \mathbf{m}_d \\ \widetilde{\mathbf{m}}_{d-1} \end{bmatrix}^{34}_p \langle 0|\phi_4(\infty)\phi_3(1)|\Delta_p, \mathbf{m}_d \, \widetilde{\mathfrak{M}}_{d-1}\rangle, \tag{5.245}$$

where

$$\begin{aligned}
\begin{bmatrix} \widetilde{\mathbf{m}}_d \; \mathbf{m}_d \\ \widetilde{\mathbf{m}}_{d-1} \end{bmatrix}^{34}_p = &(-1)^d (\Delta_p + \Delta_3 - \Delta_4) \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} & \bullet & \widetilde{\mathbf{m}}_{d-1} \end{array}\right) \\
&+ \sum_{\mathbf{m}_{d-1}} \left(\begin{array}{c|c|c} \mathbf{m}_d & & \mathbf{m}_d \\ \widetilde{\mathbf{m}}_{d-1} & M^{\square} & \mathbf{m}_{d-1} \end{array}\right) \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \mathbf{m}_{d-1} & \square & \widetilde{\mathbf{m}}_{d-1} \end{array}\right).
\end{aligned} \tag{5.246}$$

As we discuss in appendix D.2.4, the two terms in the last expression are in fact proportional to each other,

$$\sum_{\mathbf{m}_{d-1}} \left(\begin{array}{c|c|c} \mathbf{m}_d & & \mathbf{m}_d \\ \widetilde{\mathbf{m}}_{d-1} & M^{\square} & \mathbf{m}_{d-1} \end{array}\right) \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \mathbf{m}_{d-1} & \square & \widetilde{\mathbf{m}}_{d-1} \end{array}\right) = (-1)^{d-1} (\mathbf{m}_d \, \square | \widetilde{\mathbf{m}}_d) \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} & \bullet & \widetilde{\mathbf{m}}_{d-1} \end{array}\right), \tag{5.247}$$

where $(\mathbf{m}_d\,\square|\widetilde{\mathbf{m}}_d)$ is given by (D.36)-(D.38). This leads to

$$\begin{bmatrix} \widetilde{\mathbf{m}}_d & \mathbf{m}_d \\ \widetilde{\mathbf{m}}_{d-1} & \end{bmatrix}^{34}_p = (-1)^d\Big(\Delta_p + \Delta_{34} - (\mathbf{m}_d\,\square|\widetilde{\mathbf{m}}_d)\Big)\left(\begin{matrix} \mathbf{m}_d & \square \\ \widetilde{\mathbf{m}}_{d-1} & \bullet \end{matrix}\,\middle|\,\begin{matrix} \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} \end{matrix}\right). \tag{5.248}$$

Note that we have not yet actually specialized to the case of scalar operators, except in deriving (5.242).[34] Let us do this now.

We start by observing that we necessarily have $\widetilde{\mathbf{m}}_{d-1} = \bullet$ in order for both sides of (5.245) to be non-trivial – both sides are proportional to $Spin(d-1)$ CG coefficient $\langle\bullet,\ldots;\bullet,\ldots|\widetilde{\mathfrak{M}}_{d-1}\rangle$ which defines the three-point structures, see equation (5.120). The selection rule $\widetilde{\mathbf{m}}_d \in \mathbf{m}_d \otimes \square$, combined with the requirement that in the scalar case $\mathbf{m}_d = \mathbf{j}$ and $\widetilde{\mathbf{m}}_d$ are both traceless-symmetric, leaves only two options, $\widetilde{\mathbf{m}}_d = \mathbf{j}(\pm 1)$, in notation of appendix D.2. We therefore only need to compute

$$\begin{bmatrix} \mathbf{j}(\pm1) & \mathbf{j} \\ \bullet & \end{bmatrix}^{34}_p. \tag{5.249}$$

According to (5.248) we have

$$\begin{bmatrix} \mathbf{j}(\pm1) & \mathbf{j} \\ \bullet & \end{bmatrix}^{34}_p = (-1)^d\Big(\Delta_p + \Delta_{34} - (\mathbf{j}\,\square|\mathbf{j}(\pm1)\Big)\left(\begin{matrix} \mathbf{j} & \square \\ \bullet & \bullet \end{matrix}\,\middle|\,\begin{matrix} \mathbf{j}(\pm1) \\ \bullet \end{matrix}\right). \tag{5.250}$$

By using the explicit expressions from appendix D.2 we find

$$\begin{bmatrix} \mathbf{j}(-1) & \mathbf{j} \\ \bullet & \end{bmatrix}^{34}_p = (-1)^d(\Delta_p + \Delta_{34} - j - d + 2)\sqrt{\frac{j}{2j+d-2}}, \tag{5.251}$$

$$\begin{bmatrix} \mathbf{j}(+1) & \mathbf{j} \\ \bullet & \end{bmatrix}^{34}_p = (-1)^d(\Delta_p + \Delta_{34} + j)\sqrt{\frac{j+d-2}{2j+d-2}}. \tag{5.252}$$

One can already recognize here parts of the recursion coefficients $\Gamma^{\pm}_p$ in (5.221). In order to obtain the complete expressions, we need to consider the right three-point structure.

Right three-point structure We now consider the right part of (5.236),

$$\langle\Delta_p, \mathfrak{M}_d|K_\mu s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle = s^{\Delta_p+1}\langle\Delta_p, \mathfrak{M}_d|K_\mu e^{\theta M_{12}}\phi_2\phi_1|0\rangle. \tag{5.253}$$

Let us denote

$$\langle\Delta_p, \mathfrak{M}_d; K, \mathfrak{U}_d| \equiv \langle\Delta_p, \mathfrak{M}_d|K_{\mathfrak{U}_d}, \tag{5.254}$$

[34] For more general operators there will be extra contributions (which we discuss in section 5.4.2) to (5.242) and thus also to (5.245). The formula (5.248) for the universal contribution (5.245) will remain the same.

and write

$$\langle \Delta_p, \mathfrak{M}_d | K_{\mathfrak{U}_d} e^{\theta M_{12}} \phi_2 \phi_1 | 0 \rangle = \sum_{\mathfrak{M}'_d, \mathfrak{U}'_d} \langle \mathfrak{M}_d \mathfrak{U}_d | e^{\theta M_{12}} | \mathfrak{M}'_d \mathfrak{U}'_d \rangle \times$$
$$\times \langle \Delta_p, \mathfrak{M}'_d | K_{\mathfrak{U}'_d} \phi_2 \phi_1 | 0 \rangle. \tag{5.255}$$

We first compute $\langle \Delta_p, \mathfrak{M}'_d | K_{\mathfrak{U}'_d} \phi_2 \phi_1 | 0 \rangle$ in the same way as we computed the left three-point function. We can make a shortcut by noting

$$\langle \Delta_p, \mathfrak{M}'_d | K_{\mathfrak{U}'_d} \phi_2 \phi_1 | 0 \rangle = \left(\langle 0 | \phi_2 \phi_1 P^{\mathfrak{U}'_d} | \Delta, \mathfrak{M}'_d \rangle \right)^* \tag{5.256}$$

and reusing the results for the left three-point function. This gives us

$$\langle K, \Delta_p, \mathbf{m}_d ; \widetilde{\mathfrak{M}}'_d | \phi_2 \phi_1 | 0 \rangle = \left(\begin{bmatrix} \widetilde{\mathbf{m}}'_d & \mathbf{m}_d \\ \multicolumn{2}{c}{\widetilde{\mathbf{m}}'_{d-1}} \end{bmatrix}^{21}_p \right)^* \langle \Delta_p, \mathbf{m}_d \, \widetilde{\mathfrak{M}}'_{d-1} | \phi_2 \phi_1 | 0 \rangle, \tag{5.257}$$

where

$$\begin{bmatrix} \widetilde{\mathbf{m}}'_d & \mathbf{m}_d \\ \multicolumn{2}{c}{\widetilde{\mathbf{m}}'_{d-1}} \end{bmatrix}^{21}_p \tag{5.258}$$

is given by an analogue (5.248) with Δ_3, Δ_4 replaced by Δ_2, Δ_1, and we defined

$$\langle K, \Delta_p, \mathbf{m}_d ; \widetilde{\mathfrak{M}}'_d | = \sum_{\mathfrak{M}'_d \mathfrak{U}'_d} \langle \widetilde{\mathfrak{M}}'_d | \mathfrak{M}'_d \mathfrak{U}'_d \rangle \langle \Delta_p, \mathfrak{M}'_d | K_{\mathfrak{U}'_d}. \tag{5.259}$$

Finally, note that we can rewrite the $Spin(d)$ matrix element in (5.255) as

$$\langle \mathfrak{M}_d \mathfrak{U}_d | e^{\theta M_{12}} | \mathfrak{M}'_d \mathfrak{U}'_d \rangle = \sum_{\widetilde{\mathbf{m}}_d = \widetilde{\mathbf{m}}'_d} \sum_{\widetilde{\mathfrak{M}}_d, \widetilde{\mathfrak{M}}'_d} \langle \mathfrak{M}_d \mathfrak{U}_d | \widetilde{\mathfrak{M}}_d \rangle \langle \widetilde{\mathfrak{M}}_d | e^{\theta M_{12}} | \widetilde{\mathfrak{M}}'_d \rangle \langle \widetilde{\mathfrak{M}}'_d | \mathfrak{M}'_d \mathfrak{U}'_d \rangle, \tag{5.260}$$

where the summation is over $\widetilde{\mathbf{m}}_d \in \mathbf{m}_d \otimes \square$. Note that the CG coefficients here are the same as in (5.243) and (5.259), explaining the usefulness of these definitions.

Combining the results By combining equations (5.243), (5.245), (5.255), (5.257), (5.259) and (5.260) we can rewrite (5.236) as

$$\sum_{\mathfrak{M}_d} \langle 0 | \phi_4 \phi_3 P_\mu | \Delta_p, \mathfrak{M}_d \rangle \langle \Delta_p, \mathfrak{M}_d | K^\mu s^D e^{\theta M_{12}} \phi_2 \phi_1 | 0 \rangle =$$
$$= s^{\Delta_p + 1} \sum_{\widetilde{\mathbf{m}}_d \in \square \otimes \mathbf{m}_d} \sum_{\widetilde{\mathfrak{M}}_d, \widetilde{\mathfrak{M}}'_d} \begin{bmatrix} \widetilde{\mathbf{m}}_d & \mathbf{m}_d \\ \multicolumn{2}{c}{\widetilde{\mathbf{m}}_{d-1}} \end{bmatrix}^{34}_p \left(\begin{bmatrix} \widetilde{\mathbf{m}}_d & \mathbf{m}_d \\ \multicolumn{2}{c}{\widetilde{\mathbf{m}}'_{d-1}} \end{bmatrix}^{21}_p \right)^* \times$$
$$\times \langle 0 | \phi_4(\infty) \phi_3(1) | \Delta_p, \mathbf{m}_d \, \widetilde{\mathfrak{M}}_{d-1} \rangle \times$$
$$\times \langle \widetilde{\mathfrak{M}}_d | e^{\theta M_{12}} | \widetilde{\mathfrak{M}}'_d \rangle \times$$
$$\times \langle \Delta_p, \mathbf{m}_d \, \widetilde{\mathfrak{M}}'_d | \phi_2(1) \phi_1(0) | 0 \rangle. \tag{5.261}$$

Here $\widetilde{\mathbf{m}}'_d = \widetilde{\mathbf{m}}_d$. The right hand side of (5.261) now has the same form as the generic contribution (5.111), except that the state $(\Delta_p, \mathbf{m}_d)$ now contributes as a state with dimension Δ_{p+1} and spin $\widetilde{\mathbf{m}}_d \in \square \otimes \mathbf{m}_d$ with a relative coefficient determined by the representation-theoretic data through (5.248). In the scalar correlator case these contributions have the form determined by (5.139). It is trivial to account for possible degeneracies and arrive at the following result

$$\sum_{p,\mathbf{m}_d,\mathfrak{M}_d,q} \langle 0|\phi_4\phi_3 P_\mu|\Delta_p, \mathfrak{M}_d, q\rangle\langle\Delta_p, \mathfrak{M}_d, q|K^\mu s^D e^{\theta M_{12}}\phi_2\phi_1|0\rangle =$$
$$= \sum_{p=0}^{\infty}\sum_{j=0}^{\infty}(\Gamma^+_{p-1,j-1}\Lambda_{p-1,j-1} + \Gamma^-_{p-1,j+1}\Lambda_{p-1,j+1})s^{\Delta_O+p}\frac{C_j^{(\nu)}(\cos\theta)}{C_j^{(\nu)}(1)}, \tag{5.262}$$

where

$$\Gamma^\pm_{p,j} = \begin{bmatrix} \mathbf{j}(\pm 1) & \mathbf{j} \\ & \bullet \end{bmatrix}^{34}_p \left(\begin{bmatrix} \mathbf{j}(\pm 1) & \mathbf{j} \\ & \bullet \end{bmatrix}^{21}_p\right)^*. \tag{5.263}$$

Given the definition of (5.258) together with the formulas (5.251) and (5.252) we immediately recover the result (5.221) of [59]. By comparing (5.262) with (5.235) we also recover the required recursion relation (5.220).

This derivation may seem much more elaborate than that of [59]. However, it has several advantages. The first is that the recursion relation is determined not by some particular identities satisfied by Gegenbauer polynomials,[35] but instead by a simple set of representation-theoretic data – by the reduced matrix elements and isoscalar factors. The second is that it is completely general and only a few modifications are required to find the recursion relations for the most general conformal blocks, as we now discuss.

5.4.2 Spinning conformal blocks

5.4.2.1 Difference from the scalar case

Let us now consider the general case of spinning conformal blocks. Looking at the derivation of scalar recursion relation, one can see that the first essential deviation in the spinning case happens in (5.240), which needs to be replaced by (recall that

[35] Of course, given the representation-theoretic interpretation of Gegenbauer polynomials from (5.202), the identities satisfied by Gegenbauer polynomials can also be understood from representation-theoretic point of view.

$\mu \neq 1$ in this context)

$$\begin{aligned}\langle 0|O_4^{\mathfrak{M}_d^4}(\infty)\partial_\mu O_3^{\mathfrak{M}_d^3}(1)|\Delta_p,\mathfrak{M}_d\rangle = &-\langle 0|O_4^{\mathfrak{M}_d^4}(\infty)O_3^{\mathfrak{M}_d^3}(1)M_{1\mu}|\Delta_p,\mathfrak{M}_d\rangle\\ &-\sum_{\mathfrak{M}_d'^3}\langle\mathfrak{M}_d'^3|M_{1\mu}|\mathfrak{M}_d^3\rangle\langle 0|O_4^{\mathfrak{M}_d^4}(\infty)O_3^{\mathfrak{M}_d'^3}(1)|\Delta_p,\mathfrak{M}_d\rangle\\ &+\sum_{\mathfrak{M}_d'^4}\langle\mathfrak{M}_d'^4|M_{1\mu}|\mathfrak{M}_d^4\rangle\langle 0|O_4^{\mathfrak{M}_d'^4}(\infty)O_3^{\mathfrak{M}_d^3}(1)|\Delta_p,\mathfrak{M}_d\rangle.\end{aligned} \tag{5.264}$$

Analogously to (5.239), the relative sign for action on O_4 is required because we have placed that operator at infinity. This forces this operator to transform in the reflected representation, which is essentially defined by replacing the generators for $M_{1\mu}$ with $-M_{1\mu}$, hence the relative sign.[36] Note that this does not affect the $Spin(d-1)$ representations, and so the results of section 5.3 regarding three-point functions still hold.

To proceed, we need to put these new contributions into a form similar to (5.245). Let us focus on the contribution from O_3 which is proportional to

$$\sum_{\mathfrak{M}_d'^3}\langle\mathfrak{M}_d'^3|M^{1\,\mathfrak{U}_{d-1}}|\mathfrak{M}_d^3\rangle\langle 0|O_4^{\mathfrak{M}_d^4}(\infty)O_3^{\mathfrak{M}_d'^3}(1)|\Delta_p,\mathfrak{M}_d\rangle. \tag{5.265}$$

As is already familiar, we start by writing out the matrix element as

$$\langle\mathfrak{M}_d'^3|M^{1\,\mathfrak{U}_{d-1}}|\mathfrak{M}_d^3\rangle = \left(\begin{matrix}\mathbf{m}_d^3\\ \mathbf{m}_{d-1}'^3\end{matrix}\middle| M^{\square}\middle|\begin{matrix}\mathbf{m}_d^3\\ \mathbf{m}_{d-1}^3\end{matrix}\right)\langle\mathfrak{M}_{d-1}'^3|\mathfrak{M}_{d-1}^3\mathfrak{U}_{d-1}\rangle. \tag{5.266}$$

We then recall from (5.244) that in the end we would like to contract (5.265) with $\langle\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}|\widetilde{\mathfrak{M}}_{d-1}\rangle$. We are therefore led to consider the combination (we have temporarily omitted the summation over $\mathbf{m}_{d-1}'^3$ and $\mathbf{m}_{d-1}$)

$$\sum_{\mathfrak{U}_{d-1}\mathfrak{M}_{d-1}'^3\mathfrak{M}_{d-1}}\langle\mathfrak{M}_{d-1}'^3|\mathfrak{M}_{d-1}^3\mathfrak{U}_{d-1}\rangle\langle\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}|\widetilde{\mathfrak{M}}_{d-1}\rangle\langle 0|O_4^{\mathfrak{M}_d^4}(\infty)O_3^{\mathfrak{M}_d'^3}(1)|\Delta_p,\mathfrak{M}_d\rangle = \tag{5.267}$$

At this point, we should recall the structure of the three-point functions (5.120), leading to

$$= \sum_{\mathfrak{U}_{d-1}\mathfrak{M}_{d-1}'^3\mathfrak{M}_{d-1},t}\langle\mathfrak{M}_{d-1}'^3|\mathfrak{M}_{d-1}^3\mathfrak{U}_{d-1}\rangle\langle\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}|\widetilde{\mathfrak{M}}_{d-1}\rangle\langle 0|\mathfrak{M}_{d-1}^4\mathfrak{M}_{d-1}'^3\mathfrak{M}_{d-1},t\rangle\overline{\lambda}^{\mathbf{m}_{d-1}'^3,\mathbf{m}_{d-1}^4}_{\mathbf{m}_{d-1},t}. \tag{5.268}$$

[36] This is most easily understood by considering the radial quantization as the limit of NS quantization [18] with poles at the positions of O_1 and O_4 as O_4 is taken to $+\infty$.

By separating the sum over t, we find the objects

$$\sum_{\mathfrak{U}_{d-1}\mathfrak{M}'^{3}_{d-1}\mathfrak{M}_{d-1}} \langle \mathfrak{M}'^{3}_{d-1}|\mathfrak{M}^{3}_{d-1}\mathfrak{U}_{d-1}\rangle\langle \mathfrak{M}_{d-1}\mathfrak{U}_{d-1}|\widetilde{\mathfrak{M}}_{d-1}\rangle\langle 0|\mathfrak{M}^{4}_{d-1}\mathfrak{M}'^{3}_{d-1}\mathfrak{M}_{d-1},t\rangle = \tag{5.269}$$

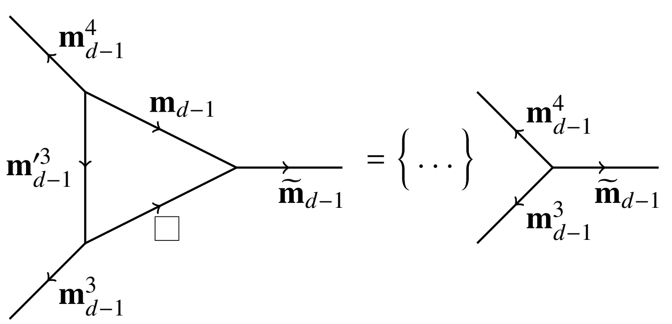
These objects have the same invariance properties as $3j$ symbols, and thus should be expressible in terms of them,

$$= \sum_{t'} \begin{Bmatrix} \mathbf{m}^{4}_{d-1} & \mathbf{m}^{3}_{d-1} & \widetilde{\mathbf{m}}_{d-1} \\ \square & \mathbf{m}_{d-1} & \mathbf{m}'^{3}_{d-1} \end{Bmatrix}^{(3)}_{tt'} \langle 0|\mathfrak{M}^{4}_{d-1}\mathfrak{M}^{3}_{d-1}\widetilde{\mathfrak{M}}_{d-1},t'\rangle. \tag{5.270}$$

The constants

$$\begin{Bmatrix} \mathbf{m}^{4}_{d-1} & \mathbf{m}^{3}_{d-1} & \widetilde{\mathbf{m}}_{d-1} \\ \square & \mathbf{m}_{d-1} & \mathbf{m}'^{3}_{d-1} \end{Bmatrix}^{(3)}_{tt'} \tag{5.271}$$

are known as $6j$-symbols or Racah coefficients of $Spin(d-1)$.[37,38] We added a label (3) to the notation for the $6j$ symbol to distinguish its definition from the definitions (5.282)-(5.285) for the operators $1,2,4$ which will appear later.[39] Note that we can represent this equality schematically as



(5.272)

Restoring the OPE coefficients and the summations over $t, \mathbf{m}_{d-1}$ and $\mathbf{m}'^{3}_{d-1}$, and

[37] Up to inessential normalization conventions. We will not make a distinction between the two terms.

[38] Interestingly, a different kind of $6j$ symbols recently played an important role in another approach to conformal blocks [3].

[39] Of course, there is only one type of $6j$ symbols for a given group, and this label is superficial. The $6j$ symbols with different labels can be obtained from the $6j$ symbols of the form (5.271) by certain permutations of columns and introduction of normalization factors. Such relations are, however, convention-dependent, and we therefore avoid using them and instead use the labels such as (3).

adding the isoscalar factor from (5.244) we find

$$\sum_{\mathfrak{M}'^3_d,\mathfrak{M}_d,\mathfrak{U}_{d-1}} \langle \mathfrak{M}'^3_d | M^{1\,\mathfrak{U}_{d-1}} | \mathfrak{M}^3_d \rangle \langle 0 | O_4^{\mathfrak{M}^4_d}(\infty) O_3^{\mathfrak{M}'^3_d}(1) | \Delta_p, \mathfrak{M}_d \rangle \langle \mathfrak{M}_{d-1} \mathfrak{U}_{d-1} | \widetilde{\mathfrak{M}}_{d-1} \rangle \times$$
$$\times \left(\begin{matrix} \mathbf{m}_d & \square \\ \mathbf{m}_{d-1} & \square \end{matrix} \middle| \begin{matrix} \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} \end{matrix} \right) =$$
$$= \langle 0 | O_4^{\mathfrak{M}^4_d}(\infty) O_3^{\mathfrak{M}^3_d}(1) | \Delta_p, \mathbf{m}_d\, \widetilde{\mathfrak{M}}_{d-1} \rangle', \tag{5.273}$$

where prime on the three-point function indicates that the OPE coefficients $\overline{\lambda}$ have been replaced with $\overline{\lambda}'$ defined as

$$(\overline{\lambda}')^{\mathbf{m}^3_{d-1},\mathbf{m}^4_{d-1}}_{\widetilde{\mathbf{m}}_{d-1},t'} =$$
$$\sum_{\mathbf{m}_{d-1},\mathbf{m}'^3_{d-1},t} \left(\begin{matrix} \mathbf{m}^3_d \\ \mathbf{m}'^3_{d-1} \end{matrix} \middle| M \square \middle| \begin{matrix} \mathbf{m}^3_d \\ \mathbf{m}^3_{d-1} \end{matrix} \right) \left(\begin{matrix} \mathbf{m}_d & \square \\ \mathbf{m}_{d-1} & \square \end{matrix} \middle| \begin{matrix} \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} \end{matrix} \right) \begin{Bmatrix} \mathbf{m}^4_{d-1} & \mathbf{m}^3_{d-1} & \widetilde{\mathbf{m}}_{d-1} \\ \square & \mathbf{m}_{d-1} & \mathbf{m}'^3_{d-1} \end{Bmatrix}_{tt'} \overline{\lambda}^{\mathbf{m}'^3_{d-1},\mathbf{m}^4_{d-1}}_{\mathbf{m}_{d-1},t}. \tag{5.274}$$

We can easily perform a similar analysis for the contribution of O_4 as well as for the operators O_1 and O_2 in the right three-point function. Note that the right hand side in (5.273) has essentially the same form as the universal contribution (5.245), and thus we can continue to derive the recursion relation in an exact analogy with the scalar case.

5.4.2.2 The general form of the recursion relation

It is now straightforward to finish the derivation of the Casimir recursion relation. The operator version of the Casimir equation is given by the spinning analogue of (5.234),

$$\langle 0 | O_4^{\mathfrak{M}^4_d} O_3^{\mathfrak{M}^3_d} | O | (\widetilde{C} - C) s^D e^{\theta M_{12}} O_2^{\mathfrak{M}^2_d} O_1^{\mathfrak{M}^1_d} | 0 \rangle = \langle 0 | O_4^{\mathfrak{M}^4_d} O_3^{\mathfrak{M}^3_d} P^\mu | O | K_\mu s^D e^{\theta M_{12}} O_2^{\mathfrak{M}^2_d} O_1^{\mathfrak{M}^1_d} | 0 \rangle. \tag{5.275}$$

Completely analogously to the scalar case, the contribution of a single term of (5.217) to the left hand side of this equation is given by (5.130) multiplied by the difference of $\widetilde{C}$ and C eigenvalues,

$$\sum_{\mathfrak{M}_d}\langle 0|O_4^{\mathfrak{M}_d^4}O_3^{\mathfrak{M}_d^3}|\Delta_p,\mathfrak{M}_d\rangle\langle\Delta_p,\mathfrak{M}_d|(\widetilde{C}-C)s^D e^{\theta M_{12}}O_2^{\mathfrak{M}_d^2}O_1^{\mathfrak{M}_d^1}|0\rangle =$$
$$=\sum_{\widetilde{\mathbf{m}}_{d-1}^i}\sum_{\substack{\mathbf{m}_{d-1},t\\ \mathbf{m}'_{d-1},t'}}\sum_{\mathbf{m}_{d-2}}(C(\Delta_p,\mathbf{m}_d)-C(O))\lambda^{\widetilde{\mathbf{m}}^1_{d-1},\widetilde{\mathbf{m}}^2_{d-1}}_{\mathbf{m}'_{d-1},t'}\overline{\lambda}^{\widetilde{\mathbf{m}}^3_{d-1},\widetilde{\mathbf{m}}^4_{d-1}}_{\mathbf{m}_{d-1},t}s^{\Delta_p}P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\times$$
$$\times\left[\begin{matrix}\mathfrak{M}_d^3\\ \mathfrak{M}_d^4\end{matrix}\middle|\begin{matrix}\widetilde{\mathbf{m}}^3_{d-1}\\ \widetilde{\mathbf{m}}^4_{d-1}\end{matrix}\;\mathbf{m}_{d-1},t\middle|\mathbf{m}_{d-2}\middle|\mathbf{m}'_{d-1},t'\;\begin{matrix}\widetilde{\mathbf{m}}^1_{d-1}\\ \widetilde{\mathbf{m}}^2_{d-1}\end{matrix}\middle|\begin{matrix}\mathfrak{M}_d^1\\ \mathfrak{M}_d^2\end{matrix}\right]. \tag{5.276}$$

Introducing the shorthand notation (5.122) and (5.123), restoring the dependence of λ on $p,\mathbf{m}_d$ and q, and summing over the possible degeneracies q we find

$$\sum_{\mathfrak{M}_d,q}\langle 0|O_4^{\mathfrak{M}_d^4}O_3^{\mathfrak{M}_d^3}|\Delta_p,\mathfrak{M}_d,q\rangle\langle\Delta_p,\mathfrak{M}_d,q|(\widetilde{C}-C)s^D e^{\theta M_{12}}O_2^{\mathfrak{M}_d^2}O_1^{\mathfrak{M}_d^1}|0\rangle =$$
$$=\sum_{a,b}\sum_{\mathbf{m}_{d-2}}(C(\Delta_p,\mathbf{m}_d)-C(O))\,\Lambda^{ba}_{p,\mathbf{m}_d}\,s^{\Delta_p}P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\left[\begin{matrix}\mathfrak{M}_d^3\\ \mathfrak{M}_d^4\end{matrix}\middle|b\middle|\mathbf{m}_{d-2}\middle|a\middle|\begin{matrix}\mathfrak{M}_d^1\\ \mathfrak{M}_d^2\end{matrix}\right], \tag{5.277}$$

where the OPE matrix Λ is defined as

$$\Lambda^{ba}_{p,\mathbf{m}_d}\equiv\sum_q\lambda^a_{p,\mathbf{m}_d,q}\overline{\lambda}^b_{p,\mathbf{m}_d,q}. \tag{5.278}$$

Following the discussion of scalar recursion relations in section 5.4.1 and the modifications mentioned in the beginning of this section, we can find

$$\sum_{\mathfrak{M}_d,\mathfrak{U}_d}\langle 0|O_4^{\mathfrak{M}_d^4}O_3^{\mathfrak{M}_d^3}P^{\mathfrak{U}_d}|\Delta_p,\mathfrak{M}_d\rangle\langle\Delta_p,\mathfrak{M}_d|K_{\mathfrak{U}_d}s^D e^{\theta M_{12}}O_2^{\mathfrak{M}_d^2}O_1^{\mathfrak{M}_d^1}|0\rangle =$$
$$=\sum_{\widetilde{\mathbf{m}}_d\in\square\otimes\mathbf{m}_d}\sum_{a,b}\sum_{\mathbf{m}_{d-2}}(\overline{\gamma}_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d}\overline{\lambda})^b(\lambda\gamma_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d})^a s^{\Delta_p+1}P^{\widetilde{\mathbf{m}}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\left[\begin{matrix}\mathfrak{M}_d^3\\ \mathfrak{M}_d^4\end{matrix}\middle|b\middle|\mathbf{m}_{d-2}\middle|a\middle|\begin{matrix}\mathfrak{M}_d^1\\ \mathfrak{M}_d^2\end{matrix}\right]. \tag{5.279}$$

Here the matrix γ is defined as

$$(\lambda\gamma_{p,\mathbf{m}_d\widetilde{\mathbf{m}}_d})^{\mathbf{m}^1_{d-1}\mathbf{m}^2_{d-1}}_{\widetilde{\mathbf{m}}'_{d-1},t''}=(-1)^d\Big(\Delta_p-\Delta_{12}-(\mathbf{m}_d\,\square|\widetilde{\mathbf{m}}_d)^*\Big)\left(\begin{matrix}\widetilde{\mathbf{m}}_d\\ \widetilde{\mathbf{m}}'_{d-1}\end{matrix}\middle|\begin{matrix}\mathbf{m}_d & \square\\ \widetilde{\mathbf{m}}'_{d-1} & \bullet\end{matrix}\right)\lambda^{\mathbf{m}^1_{d-1}\mathbf{m}^2_{d-1}}_{\widetilde{\mathbf{m}}'_{d-1},t''}$$
$$+\sum_{\mathbf{m}'^2_{d-1},t',\mathbf{m}'_{d-1}}\left(\begin{matrix}\mathbf{m}^2_d\\ \mathbf{m}^2_{d-1}\end{matrix}\middle|M\square\middle|\begin{matrix}\mathbf{m}^2_d\\ \mathbf{m}'^2_{d-1}\end{matrix}\right)^*\left(\begin{matrix}\widetilde{\mathbf{m}}_d\\ \widetilde{\mathbf{m}}'_{d-1}\end{matrix}\middle|\begin{matrix}\mathbf{m}_d & \square\\ \mathbf{m}'_{d-1} & \square\end{matrix}\right)\left\{\begin{matrix}\widetilde{\mathbf{m}}'_{d-1} & \mathbf{m}^2_{d-1} & \mathbf{m}^1_{d-1}\\ \mathbf{m}'^2_{d-1} & \mathbf{m}'_{d-1} & \square\end{matrix}\right\}^{(2)}_{t't''}\lambda^{\mathbf{m}^1_{d-1},\mathbf{m}'^2_{d-1}}_{\mathbf{m}'_{d-1},t'}$$
$$-\sum_{\mathbf{m}'^1_{d-1},t',\mathbf{m}'_{d-1}}\left(\begin{matrix}\mathbf{m}^1_d\\ \mathbf{m}^1_{d-1}\end{matrix}\middle|M\square\middle|\begin{matrix}\mathbf{m}^1_d\\ \mathbf{m}'^1_{d-1}\end{matrix}\right)^*\left(\begin{matrix}\widetilde{\mathbf{m}}_d\\ \widetilde{\mathbf{m}}'_{d-1}\end{matrix}\middle|\begin{matrix}\mathbf{m}_d & \square\\ \mathbf{m}'_{d-1} & \square\end{matrix}\right)\left\{\begin{matrix}\widetilde{\mathbf{m}}'_{d-1} & \mathbf{m}^2_{d-1} & \mathbf{m}^1_{d-1}\\ \mathbf{m}'^1_{d-1} & \square & \mathbf{m}'_{d-1}\end{matrix}\right\}^{(1)}_{t't''}\lambda^{\mathbf{m}'^1_{d-1},\mathbf{m}^2_{d-1}}_{\mathbf{m}'_{d-1},t'}, \tag{5.280}$$

while the matrix $\overline{\gamma}$ is defined as

$$
\begin{aligned}
(\overline{\gamma}_{p,\mathbf{m}_d\widetilde{\mathbf{m}}_d}\overline{\lambda})^{\mathbf{m}^3_{d-1}\mathbf{m}^4_{d-1}}_{\widetilde{\mathbf{m}}_{d-1},t'} &= (-1)^d\Big(\Delta_p+\Delta_{34}-(\mathbf{m}_d\,\square|\widetilde{\mathbf{m}}_d)\Big)\left(\begin{array}{cc|c}\mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d\\ \widetilde{\mathbf{m}}_{d-1} & \bullet & \widetilde{\mathbf{m}}_{d-1}\end{array}\right)\overline{\lambda}^{\mathbf{m}^3_{d-1}\mathbf{m}^4_{d-1}}_{\widetilde{\mathbf{m}}_{d-1},t'}\\
&+\sum_{\mathbf{m}'^3_{d-1},t,\mathbf{m}_{d-1}}\left(\begin{array}{c|c|c}\mathbf{m}^3_d & & \mathbf{m}^3_d\\ \mathbf{m}'^3_{d-1} & M\square & \mathbf{m}^3_{d-1}\end{array}\right)\left(\begin{array}{cc|c}\mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d\\ \mathbf{m}_{d-1} & \square & \widetilde{\mathbf{m}}_{d-1}\end{array}\right)\begin{Bmatrix}\mathbf{m}^4_{d-1} & \mathbf{m}^3_{d-1} & \widetilde{\mathbf{m}}_{d-1}\\ \square & \mathbf{m}_{d-1} & \mathbf{m}'^3_{d-1}\end{Bmatrix}^{(3)}_{tt'}\overline{\lambda}^{\mathbf{m}'^3_{d-1},\mathbf{m}^4_{d-1}}_{\mathbf{m}_{d-1},t}\\
&-\sum_{\mathbf{m}'^4_{d-1},t,\mathbf{m}_{d-1}}\left(\begin{array}{c|c|c}\mathbf{m}^4_d & & \mathbf{m}^4_d\\ \mathbf{m}'^4_{d-1} & M\square & \mathbf{m}^4_{d-1}\end{array}\right)\left(\begin{array}{cc|c}\mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d\\ \mathbf{m}_{d-1} & \square & \widetilde{\mathbf{m}}_{d-1}\end{array}\right)\begin{Bmatrix}\mathbf{m}^4_{d-1} & \mathbf{m}^3_{d-1} & \widetilde{\mathbf{m}}_{d-1}\\ \mathbf{m}_{d-1} & \square & \mathbf{m}'^4_{d-1}\end{Bmatrix}^{(4)}_{tt'}\overline{\lambda}^{\mathbf{m}^3_{d-1},\mathbf{m}'^4_{d-1}}_{\mathbf{m}_{d-1},t}.
\end{aligned}
\tag{5.281}
$$

The $6j$ symbols are defined as solutions to the following equations:

$$
\sum_{\mathfrak{M}'^1_{d-1},\mathfrak{U}'_{d-1},\mathfrak{M}'_{d-1}}\langle\mathfrak{M}'_{d-1},t'|\mathfrak{M}^2_{d-1}\mathfrak{M}'^1_{d-1}\rangle\langle\mathfrak{M}'^1_{d-1}\mathfrak{U}'_{d-1}|\mathfrak{M}^1_{d-1}\rangle\langle\widetilde{\mathfrak{M}}'_{d-1}|\mathfrak{M}'_{d-1}\mathfrak{U}'_{d-1}\rangle =
$$

$$
=\sum_{t''}\begin{Bmatrix}\widetilde{\mathbf{m}}'_{d-1} & \mathbf{m}^2_{d-1} & \mathbf{m}^1_{d-1}\\ \mathbf{m}'^1_{d-1} & \square & \mathbf{m}'_{d-1}\end{Bmatrix}^{(1)}_{t't''}\langle\widetilde{\mathfrak{M}}'_{d-1},t''|\mathfrak{M}^2_{d-1}\mathfrak{M}^1_{d-1}\rangle, \tag{5.282}
$$

$$
\sum_{\mathfrak{M}'^2_{d-1},\mathfrak{U}'_{d-1},\mathfrak{M}'_{d-1}}\langle\mathfrak{M}'_{d-1},t'|\mathfrak{M}'^2_{d-1}\mathfrak{M}^1_{d-1}\rangle\langle\mathfrak{M}'^2_{d-1}\mathfrak{U}'_{d-1}|\mathfrak{M}^2_{d-1}\rangle\langle\widetilde{\mathfrak{M}}'_{d-1}|\mathfrak{M}'_{d-1}\mathfrak{U}'_{d-1}\rangle =
$$

$$
=\sum_{t''}\begin{Bmatrix}\widetilde{\mathbf{m}}'_{d-1} & \mathbf{m}^2_{d-1} & \mathbf{m}^1_{d-1}\\ \mathbf{m}'^2_{d-1} & \mathbf{m}'_{d-1} & \square\end{Bmatrix}^{(2)}_{t't''}\langle\widetilde{\mathfrak{M}}'_{d-1},t''|\mathfrak{M}^2_{d-1}\mathfrak{M}^1_{d-1}\rangle, \tag{5.283}
$$

$$
\sum_{\mathfrak{M}'^3_{d-1},\mathfrak{U}_{d-1},\mathfrak{M}_{d-1}}\langle 0|\mathfrak{M}^4_{d-1}\mathfrak{M}'^3_{d-1}(\mathfrak{M}_{d-1},t)\rangle\langle\mathfrak{M}'^3_{d-1}|\mathfrak{M}^3_{d-1}\mathfrak{U}_{d-1}\rangle\langle\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}|\widetilde{\mathfrak{M}}_{d-1}\rangle =
$$

$$
=\sum_{t'}\begin{Bmatrix}\mathbf{m}^4_{d-1} & \mathbf{m}^3_{d-1} & \widetilde{\mathbf{m}}_{d-1}\\ \square & \mathbf{m}_{d-1} & \mathbf{m}'^3_{d-1}\end{Bmatrix}^{(3)}_{tt'}\langle 0|\mathfrak{M}^4_{d-1}\mathfrak{M}^3_{d-1}(\widetilde{\mathfrak{M}}_{d-1},t')\rangle, \tag{5.284}
$$

$$
\sum_{\mathfrak{M}'^4_{d-1},\mathfrak{U}_{d-1},\mathfrak{M}_{d-1}}\langle 0|\mathfrak{M}'^4_{d-1}\mathfrak{M}^3_{d-1}(\mathfrak{M}_{d-1},t)\rangle\langle\mathfrak{M}'^4_{d-1}|\mathfrak{M}^4_{d-1}\mathfrak{U}_{d-1}\rangle\langle\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}|\widetilde{\mathfrak{M}}_{d-1}\rangle =
$$

$$
=\sum_{t'}\begin{Bmatrix}\mathbf{m}^4_{d-1} & \mathbf{m}^3_{d-1} & \widetilde{\mathbf{m}}_{d-1}\\ \mathbf{m}_{d-1} & \square & \mathbf{m}'^4_{d-1}\end{Bmatrix}^{(4)}_{tt'}\langle 0|\mathfrak{M}^4_{d-1}\mathfrak{M}^3_{d-1}(\widetilde{\mathfrak{M}}_{d-1},t')\rangle. \tag{5.285}
$$

Reintroducing the degeneracy index q in (5.279) we find

$$
\sum_{\mathfrak{M}_d,\mathfrak{U}_d,q}\langle 0|O_4^{\mathfrak{M}^4_d}O_3^{\mathfrak{M}^3_d}P^{\mathfrak{U}_d}|\Delta_p,\mathfrak{M}_d\rangle\langle\Delta_p,\mathfrak{M}_d|K_{\mathfrak{U}_d}s^D e^{\theta M_{12}}O_2^{\mathfrak{M}^2_d}O_1^{\mathfrak{M}^1_d}|0\rangle =
$$

$$
=\sum_{\widetilde{\mathbf{m}}_d\in\square\otimes\mathbf{m}_d}\sum_{a,b}\sum_{\mathbf{m}_{d-2}}(\overline{\gamma}_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d}\Lambda_{p,\mathbf{m}_d}\gamma_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d})^{ba}s^{\Delta_p+1}P^{\widetilde{\mathbf{m}}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)\left[\begin{array}{c|c|c|c|c}\mathfrak{M}^3_d & & & & \mathfrak{M}^1_d\\ & b & \mathbf{m}_{d-2} & a & \\ \mathfrak{M}^4_d & & & & \mathfrak{M}^2_d\end{array}\right].
\tag{5.286}
$$

Comparing (5.277) and (5.286) we arrive at the following recursion relation

$$(C(\Delta_{p+1}, \widetilde{\mathbf{m}}_d) - C(\mathcal{O}))\, \Lambda^{ba}_{p+1,\widetilde{\mathbf{m}}_d} = \sum_{\mathbf{m}_d \in \square \otimes \widetilde{\mathbf{m}}_d} (\overline{\gamma}_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d} \Lambda_{p,\mathbf{m}_d} \gamma_{p,\mathbf{m}_d,\widetilde{\mathbf{m}}_d})^{ba}. \quad (5.287)$$

Equation (5.287) represents the main result of this paper. It gives a recursion relation for the power series coefficients Λ of a completely general conformal block. This relation has the same structure as the scalar recursion relation (5.220) and can be solved starting from $p = 0$ in a straightforward way. The main difficulty lies in evaluation of the coefficient matrices γ and $\overline{\gamma}$, so let us discuss this in some more detail.

Suppose that we have chosen a concrete four-point function for which we wish to evaluate the conformal blocks, i.e., we made a choice of $\mathbf{m}^i_d$. If we look at, say, (5.281), we see that all the sums are finite and the number of terms is independent of $m_{d,1}$ or $\widetilde{m}_{d,1}$, which are the only weights that can be arbitrarily large for the given four-point function. Moreover, each term contributes to a single element of the matrix $\overline{\gamma}$. Furthermore, we see that $\mathbf{m}_d$ and $\widetilde{\mathbf{m}}_d$ only enter into the simple quantities (isoscalar factors for vector representation and reduced matrix elements) for which closed-form expressions are known (see appendix D.2). Similar remarks apply to (5.280). This means that if we compute for the given four-point function a finite number of $6j$ symbols (5.282)-(5.285), we can then express the matrices γ and $\overline{\gamma}$ as closed-form analytic expressions in $\mathbf{m}_d$ and $\widetilde{\mathbf{m}}_d$, thus obtaining a closed-form analytic expression for the recursion relation (5.287). If we know all the CG coefficients in (5.282)-(5.285), then the calculation of a finite number of $6j$ symbols is a simple linear algebra problem, so we can assume their knowledge to be equivalent to the knowledge of CG coefficients.

As discussed in section 5.2.3, in several important cases the CG coefficients are known analytically (and so are $6j$ symbols). In these cases we can write closed-form expressions for $\overline{\gamma}$ and γ. In the rest of this section we consider two such situations: general blocks in $d = 3$ and seed blocks for general d.

5.4.3 Example: General conformal blocks in 3 dimensions

As discussed above, the only non-trivial ingredients in the recursion relation (5.287) are the $6j$ symbols entering the expressions (5.280) and (5.281). In $d = 3$ these symbols simplify dramatically. However, before computing them, we need to understand a small subtlety which arises in $d = 3$.

In the derivation of the recursion relation, we have encountered isoscalar factors such as

$$\left(\begin{array}{cc|c} \mathbf{m}_d & \square & \mathbf{m}'_d \\ \mathbf{m}_{d-1} & \square & \mathbf{m}'_{d-1} \end{array}\right). \tag{5.288}$$

In $d = 3$ this presents a problem since we should instead use the isoscalar factors

$$\left(\begin{array}{cc|c} \mathbf{m}_3 & \square & \mathbf{m}'_3 \\ \mathbf{m}_2 & \pm 1 & \mathbf{m}'_2 \end{array}\right), \tag{5.289}$$

because the vector representation is reducible in 2d. One can still use the formulas of appendix D.2 to compute the value of (5.288), but we need to interpret it in terms of (5.289). Such an interpretation, together with a analogous discussion for reduced matrix elements is given in D.2.3. Using these, one can check that (5.246) still holds in $d = 3$ and we can still simplify it using the sum rule from appendix D.2.4. The formulas of section 5.4.2 can also be seen to remain valid if we interpret the sum over $\mathfrak{U}_2$ in (5.282)-(5.285) as a sum over $\mathbf{u}_2 = (+1)$ and $\mathbf{u}_2 = (-1)$.

Consider, for example, the equation (5.282) for the $6j$ symbol related to O_1,

$$\sum_{\mathfrak{M}'^1_{d-1},\mathfrak{U}'_{d-1},\mathfrak{M}'_{d-1}} \langle \mathfrak{M}'_{d-1}, t' | \mathfrak{M}^2_{d-1} \mathfrak{M}'^1_{d-1} \rangle \langle \mathfrak{M}'^1_{d-1} \mathfrak{U}'_{d-1} | \mathfrak{M}^1_{d-1} \rangle \langle \widetilde{\mathfrak{M}}'_{d-1} | \mathfrak{M}'_{d-1} \mathfrak{U}'_{d-1} \rangle =$$

$$= \sum_{t''} \begin{Bmatrix} \widetilde{\mathbf{m}}'_{d-1} & \mathbf{m}^2_{d-1} & \mathbf{m}^1_{d-1} \\ \mathbf{m}'^1_{d-1} & \square & \mathbf{m}'_{d-1} \end{Bmatrix}^{(1)}_{t't''} \langle \widetilde{\mathfrak{M}}'_{d-1}, t'' | \mathfrak{M}^2_{d-1} \mathfrak{M}^1_{d-1} \rangle. \tag{5.290}$$

$$\tag{5.291}$$

In $d = 3$, taking into account the subtlety discussed above, this equation simplifies to

$$\sum_{u'=\pm 1} \delta_{m', m_2 + m'_1} \delta_{m'_1 + u', m_1} \delta_{\widetilde{m}', m' + u'} = \begin{Bmatrix} \widetilde{m}' & m_2 & m_1 \\ m'_1 & \square & m' \end{Bmatrix}^{(1)} \delta_{\widetilde{m}', m_2 + m_1}. \tag{5.292}$$

It is solved by

$$\begin{Bmatrix} \widetilde{m}' & m_2 & m_1 \\ m'_1 & \square & m' \end{Bmatrix}^{(1)} = \begin{cases} 1, & m_1 - m'_1 = \widetilde{m}' - m' = \pm 1 \\ 0, & \text{otherwise} \end{cases}. \tag{5.293}$$

Similarly, we find

$$\begin{Bmatrix} \widetilde{m}' & m_2 & m_1 \\ m_2' & m' & \square \end{Bmatrix}^{(2)} = \begin{cases} 1, & m_2 - m_2' = \widetilde{m}' - m' = \pm 1 \\ 0, & \text{otherwise} \end{cases}, \tag{5.294}$$

$$\begin{Bmatrix} m_4 & m_3 & \widetilde{m} \\ \square & m & m_3' \end{Bmatrix}^{(3)} = \begin{cases} 1, & m_3' - m_3 = \widetilde{m} - m = \pm 1 \\ 0, & \text{otherwise} \end{cases}, \tag{5.295}$$

$$\begin{Bmatrix} m_4 & m_3 & \widetilde{m} \\ m & \square & m_4' \end{Bmatrix}^{(4)} = \begin{cases} 1, & m_4' - m_4 = \widetilde{m} - m = \pm 1 \\ 0, & \text{otherwise} \end{cases}. \tag{5.296}$$

Recall that the right OPE coefficients in 3d are parametrized as $\lambda^{m_1 m_2}$. We then have, according to (5.280) for $\widetilde{j} = j \pm 1$,

$$\begin{aligned}
&(\lambda\gamma_{p,j,j\pm1})^{m_1 m_2} = \\
&-(\Delta_p - \Delta_{12} \pm j - \delta_{\pm,-})\sqrt{\frac{(j - m_1 - m_2 + \delta_{\pm,+})(j + m_1 + m_2 + \delta_{\pm,+})}{(2j+1)(j+\delta_{\pm,+})}}\lambda^{m_1 m_2} \\
&-\sum_{u=\pm1} \pm\sqrt{\frac{(j_2 + um_2)(j_2 - um_2 + 1)(j \pm um_1 \pm um_2)(j \pm um_1 \pm um_2 + 1)}{4(j+\delta_{\pm,+})(2j+1)}}\lambda^{m_1(m_2-u)} \\
&+\sum_{u=\pm1} \pm\sqrt{\frac{(j_1 + um_1)(j_1 - um_1 + 1)(j \pm um_1 \pm um_2)(j \pm um_1 \pm um_2 + 1)}{4(j+\delta_{\pm,+})(2j+1)}}\lambda^{(m_1-u)m_2},
\end{aligned} \tag{5.297}$$

and for $\widetilde{j} = j$

$$\begin{aligned}
&(\lambda\gamma_{p,j,j})^{m_1 m_2} = \\
&-(\Delta_p - \Delta_{12} - 1)\frac{m_1 + m_2}{\sqrt{j(j+1)}}\lambda^{m_1 m_2} \\
&+\sum_{u=\pm1} u\sqrt{\frac{(j_2 + um_2)(j_2 - um_2 + 1)(j + um_1 + um_2)(j - um_1 - um_2 + 1)}{4j(j+1)}}\lambda^{m_1(m_2-u)} \\
&-\sum_{u=\pm1} u\sqrt{\frac{(j_1 + um_1)(j_1 - um_1 + 1)(j + um_1 + um_2)(j - um_1 - um_2 + 1)}{4j(j+1)}}\lambda^{(m_1-u)m_2}.
\end{aligned} \tag{5.298}$$

Similarly, from (5.281) we find

$$
\begin{aligned}
&(\overline{\gamma}_{p,j,j\pm 1}\overline{\lambda})^{m_3m_4} = \\
&-(\Delta_p+\Delta_{34}\pm j-\delta_{\pm,-})\sqrt{\frac{(j-m_3-m_4+\delta_{\pm,+})(j+m_3+m_4+\delta_{\pm,+})}{(2j+1)(j+\delta_{\pm,+})}}\overline{\lambda}^{m_3m_4} \\
&-\sum_{u=\pm 1}\pm\sqrt{\frac{(j_3-um_3)(j_3+um_3+1)(j\mp um_3\mp um_4)(j\mp um_3\mp um_4+1)}{4(j+\delta_{\pm,+})(2j+1)}}\overline{\lambda}^{(m_3+u)m_4} \\
&+\sum_{u=\pm 1}\pm\sqrt{\frac{(j_4-um_4)(j_4+um_4+1)(j\mp um_3\mp um_4)(j\mp um_3\mp um_4+1)}{4(j+\delta_{\pm,+})(2j+1)}}\overline{\lambda}^{m_3(m_4+u)},
\end{aligned}
\tag{5.299}
$$

$$
\begin{aligned}
&(\overline{\gamma}_{p,j,j}\overline{\lambda})^{m_3m_4} = \\
&\quad(\Delta_p+\Delta_{34}-1)\frac{m_3+m_4}{\sqrt{j(j+1)}}\overline{\lambda}^{m_3m_4} \\
&+\sum_{u=\pm 1}u\sqrt{\frac{(j_3-um_3)(j_3+um_3+1)(j-um_3-um_3)(j+um_3+um_4+1)}{4j(j+1)}}\overline{\lambda}^{(m_3+u)m_4} \\
&-\sum_{u=\pm 1}u\sqrt{\frac{(j_4-um_4)(j_4+um_4+1)(j-um_3-um_4)(j+um_3+um_4+1)}{4j(j+1)}}\overline{\lambda}^{m_3(m_4+u)}.
\end{aligned}
\tag{5.300}
$$

5.4.3.1 Scalar-fermion block in 3 dimensions

As a concrete example, consider the scalar-fermion blocks in 3d [3, 81]. In this case we have $j_1 = j_4 = \frac{1}{2}$ and $j_2 = j_3 = 0$. Matrices Λ then have the indices

$$
\Lambda^{m_4,m_1}_{p,j}, \qquad m_1, m_4 = \pm\tfrac{1}{2}. \tag{5.301}
$$

In terms of these coefficients the conformal block takes the form, according to (5.144),

$$
\langle 0|\psi_4^{m_4}\phi_3|\mathcal{O}|s^D e^{\theta M_{12}}\phi_2\psi_1^{m_1}|0\rangle = \sum_{\widetilde{m}_1,\widetilde{m}_4}\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\Lambda^{\widetilde{m}_4,\widetilde{m}_1}_{p,j}\, d^j_{-\widetilde{m}_4,\widetilde{m}_1}(-\theta)\delta_{m_4,\widetilde{m}_4}\delta_{m_1,\widetilde{m}_1}. \tag{5.302}
$$

The intermediate representations are $\mathbf{m}_3 = (j)$ with half-integral $j \geq \frac{1}{2}$. The Casimir eigenvalue is given by

$$
C_{p,j} = \Delta_p(\Delta_p-3)+j(j+1). \tag{5.303}
$$

Using equations (5.297)-(5.300) we find

$$(\lambda\gamma_{p,j,j+1})^{\pm\frac{1}{2}} = -(\Delta_p - \Delta_{12} + j)\sqrt{\frac{j+\frac{3}{2}}{2(j+1)}}\lambda^{\pm\frac{1}{2}} + \tfrac{1}{2}\sqrt{\frac{j+\frac{3}{2}}{2(j+1)}}\lambda^{\mp\frac{1}{2}}, \tag{5.304}$$

$$(\lambda\gamma_{p,j,j-1})^{\pm\frac{1}{2}} = -(\Delta_p - \Delta_{12} - j - 1)\sqrt{\frac{j-\frac{1}{2}}{2j}}\lambda^{\pm\frac{1}{2}} - \tfrac{1}{2}\sqrt{\frac{j-\frac{1}{2}}{2j}}\lambda^{\mp\frac{1}{2}}, \tag{5.305}$$

$$(\lambda\gamma_{p,j,j})^{\pm\frac{1}{2}} = \mp(\Delta_p - \Delta_{12} - 1)\tfrac{1}{2}\frac{1}{\sqrt{j(j+1)}}\lambda^{\pm\frac{1}{2}} \mp \tfrac{1}{2}\frac{(j+\frac{1}{2})}{\sqrt{j(j+1)}}\lambda^{\mp\frac{1}{2}}, \tag{5.306}$$

$$(\overline{\gamma}_{p,j,j+1}\overline{\lambda})^{\pm\frac{1}{2}} = -(\Delta_p + \Delta_{34} + j)\sqrt{\frac{j+\frac{3}{2}}{2(j+1)}}\overline{\lambda}^{\pm\frac{1}{2}} + \tfrac{1}{2}\sqrt{\frac{j+\frac{3}{2}}{2(j+1)}}\overline{\lambda}^{\mp\frac{1}{2}}, \tag{5.307}$$

$$(\overline{\gamma}_{p,j,j-1}\overline{\lambda})^{\pm\frac{1}{2}} = -(\Delta_p + \Delta_{34} - j - 1)\sqrt{\frac{j-\frac{1}{2}}{2j}}\lambda^{\pm\frac{1}{2}} - \tfrac{1}{2}\sqrt{\frac{j-\frac{1}{2}}{2j}}\overline{\lambda}^{\mp\frac{1}{2}}, \tag{5.308}$$

$$(\overline{\gamma}_{p,j,j}\overline{\lambda})^{\pm\frac{1}{2}} = \pm(\Delta_p + \Delta_{34} - 1)\tfrac{1}{2}\frac{1}{\sqrt{j(j+1)}}\lambda^{\pm\frac{1}{2}} \pm \tfrac{1}{2}\frac{(j+\frac{1}{2})}{\sqrt{j(j+1)}}\overline{\lambda}^{\mp\frac{1}{2}}. \tag{5.309}$$

Using this in (5.287) we immediately obtain the recursion relation for coefficients (5.301). For example, we have

$$\begin{aligned}(C_{p,j} - C_{0,j_O})\Lambda_{p,j}^{+\frac{1}{2},+\frac{1}{2}} &= (\Delta_{p-1} - \Delta_{12} + j - 1)(\Delta_{p-1} + \Delta_{34} + j - 1)\frac{j+\frac{1}{2}}{2j}\Lambda_{p-1,j-1}^{+\frac{1}{2},+\frac{1}{2}} \\ &\quad - (\Delta_{p-1} - \Delta_{12} + j - 1)\tfrac{1}{2}\frac{j+\frac{1}{2}}{2j}\Lambda_{p-1,j-1}^{-\frac{1}{2},+\frac{1}{2}} \\ &\quad - \tfrac{1}{2}(\Delta_{p-1} + \Delta_{34} + j - 1)\frac{j+\frac{1}{2}}{2j}\Lambda_{p-1,j-1}^{+\frac{1}{2},-\frac{1}{2}} \\ &\quad + \frac{1}{4}\frac{j+\frac{1}{2}}{2j}\Lambda_{p-1,j-1}^{-\frac{1}{2},-\frac{1}{2}} + \ldots,\end{aligned} \tag{5.310}$$

where "..." represent contributions from $\Lambda_{p-1,j}$ and $\Lambda_{p-1,j+1}$. We compare the conformal block generated by this recursion relation with the known results [3, 81] in appendix D.3, finding a perfect agreement.

5.4.4 Example: Seed conformal blocks in general dimensions

We have already considered the seed blocks in section 5.3.5. Here, as in previous subsections, we start by computing the $6j$ symbols (5.282)-(5.285). Since in the seed block case the operators $\mathcal{O}_1$ and $\mathcal{O}_3$ are scalars, we do not need the $6j$ symbols for them.

For O_2 the equation for the $6j$ symbol specializes to

$$\sum_{\mathfrak{M}'^2_{d-1},\mathfrak{U}'_{d-1},\mathfrak{M}'_{d-1}} \langle \mathfrak{M}'_{d-1}|\mathfrak{M}'^2_{d-1}\rangle\langle \mathfrak{M}'^2_{d-1}\mathfrak{U}'_{d-1}|\mathfrak{M}^2_{d-1}\rangle\langle \widetilde{\mathfrak{M}}'_{d-1}|\mathfrak{M}'_{d-1}\mathfrak{U}'_{d-1}\rangle =$$

$$= \begin{Bmatrix} \mathbf{m}^2_{d-1} & \mathbf{m}^2_{d-1} & \bullet \\ \mathbf{m}'^2_{d-1} & \mathbf{m}'^2_{d-1} & \square \end{Bmatrix}^{(2)} \langle \widetilde{\mathfrak{M}}'_{d-1}|\mathfrak{M}^2_{d-1}\rangle, \tag{5.311}$$

and we can simplify the left-hand side to

$$\langle \widetilde{\mathfrak{M}}'_{d-1}|\mathfrak{M}^2_{d-1}\rangle \tag{5.312}$$

which implies that simply

$$\begin{Bmatrix} \mathbf{m}^2_{d-1} & \mathbf{m}^2_{d-1} & \bullet \\ \mathbf{m}'^2_{d-1} & \mathbf{m}'^2_{d-1} & \square \end{Bmatrix}^{(2)} = 1, \tag{5.313}$$

whenever the selection rules are satisfied. Similarly, equation (5.285) specializes to

$$\sum_{\mathfrak{M}'^4_{d-1},\mathfrak{U}_{d-1},\mathfrak{M}_{d-1}} \langle 0|\mathfrak{M}'^4_{d-1}\mathfrak{M}_{d-1}\rangle\langle \mathfrak{M}'^4_{d-1}|\mathfrak{M}^4_{d-1}\mathfrak{U}_{d-1}\rangle\langle \mathfrak{M}_{d-1}\mathfrak{U}_{d-1}|\widetilde{\mathfrak{M}}_{d-1}\rangle =$$

$$= \begin{Bmatrix} \overline{\mathbf{m}^4_{d-1}} & \bullet & \overline{\mathbf{m}^4_{d-1}} \\ \mathbf{m}'^4_{d-1} & \square & \mathbf{m}'^4_{d-1} \end{Bmatrix}^{(4)} \langle 0|\mathfrak{M}^4_{d-1}\widetilde{\mathfrak{M}}_{d-1}\rangle, \tag{5.314}$$

and the left hand side can be reduced to

$$\pm\langle 0|\mathfrak{M}^4_{d-1}\widetilde{\mathfrak{M}}_{d-1}\rangle, \tag{5.315}$$

where the sign is equal to[40] $(-1)^{\mathbf{m}^4_{d-1}-\mathbf{m}'^4_{d-1}}$ unless $\mathbf{m}^4_{d-1} = \mathbf{m}'^4_{d-1}$ and $d = 4k$ in which case it is equal to -1. To see this, one can use the identity

$$\sum_{\mathfrak{M}'^4_{d-1}} \langle 0|\mathfrak{M}'^4_{d-1}\mathfrak{M}_{d-1}\rangle\langle \mathfrak{M}'^4_{d-1}|\mathfrak{M}^4_{d-1}\mathfrak{U}_{d-1}\rangle = \pm \sum_{\mathfrak{M}'_{d-1}} \langle 0|\mathfrak{M}^4_{d-1}\mathfrak{M}'_{d-1}\rangle\langle \mathfrak{M}'_{d-1}|\mathfrak{M}_{d-1}\mathfrak{U}_{d-1}\rangle, \tag{5.316}$$

where the sign is as above. Up to normalization, it has to be true because both sides have the same $Spin(d-1)$ invariance properties. Up to a phase, the normalization can be determined by fully contracting each side with its Hermitian conjugate. The sign can then be found by setting $\mathfrak{U}_{d-1} = (\square, \bullet, \ldots)$ and examining the phase on both sides using (5.47) and the formulas in section D.2.2.

[40] Here, as before, $(-1)^{\mathbf{m}_d}$ is defined as 1 unless $d = 4k+2$ in which case it is equal to $(-1)^{m_{4k+2,2k+1}}$.

This implies that

$$\left\{\begin{matrix} \mathbf{m}^4_{d-1} & \bullet & \overline{\mathbf{m}^4_{d-1}} \\ \overline{\mathbf{m}'^4_{d-1}} & \square & \mathbf{m}'^4_{d-1} \end{matrix}\right\}^{(4)} = \begin{cases} -1 & \mathbf{m}^4_{d-1} = \mathbf{m}'^4_{d-1} \text{ and } d = 4k \\ (-1)^{\mathbf{m}^4_{d-1} - \mathbf{m}'^4_{d-1}} & \text{otherwise} \end{cases}. \tag{5.317}$$

It is now straightforward to substitute these $6j$ symbols into the expressions (5.280) and (5.281) for the matrices γ and $\overline{\gamma}$ to obtain closed-form analytic expressions for them. The final general expression is not particularly illuminating, so we do not write it out explicitly. Instead, let us again consider a specific example, the scalar-fermion seed blocks in $d = 2n$ dimensions.

5.4.4.1 Scalar-fermion blocks in $d = 2n$ dimensions

We have considered the structure of these blocks in section 5.3.5. The OPE matrices Λ are 1×1 and there are two types of exchanged representations, $\mathbf{j}^{\pm} \equiv (j, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$. Thus, we can label the OPE matrices as

$$\Lambda_{p,j,\pm}. \tag{5.318}$$

We can arrange them into a vector as in section 5.3.7.4,

$$\mathbf{\Lambda}_{p,j} = \begin{pmatrix} \Lambda_{p,j,+} \\ \Lambda_{p,j,-} \end{pmatrix}. \tag{5.319}$$

We furthermore have

$$\square \otimes \mathbf{j}^{\pm} = (\mathbf{j}+1)^{\pm} \oplus (\mathbf{j}-1)^{\pm} \oplus \mathbf{j}^{\mp}. \tag{5.320}$$

Equation (5.280) reduces to

$$\begin{aligned}
\lambda\gamma_{p,\mathbf{j}^{\pm},(\mathbf{j}+1)^{\pm}} &= (\Delta_p - \Delta_{12} + j)\sqrt{\frac{1}{2}\frac{j+2n-\frac{3}{2}}{j+n-\frac{1}{2}}}\lambda \\
&\quad - \left(\frac{i}{2}\sqrt{2n-1}\right)\left(\frac{\pm i}{\sqrt{2n-1}}\sqrt{\frac{1}{2}\frac{j+2n-\frac{3}{2}}{j+n-\frac{1}{2}}}\right)\lambda \\
&= (\Delta_p - \Delta_{12} + j \pm \tfrac{1}{2})\sqrt{\frac{1}{2}\frac{j+2n-\frac{3}{2}}{j+n-\frac{1}{2}}}\lambda, \qquad (5.321)\\
\lambda\gamma_{p,\mathbf{j}^{\pm},(\mathbf{j}-1)^{\pm}} &= (\Delta_p - \Delta_{12} - j - d + 2)\sqrt{\frac{1}{2}\frac{j-\frac{1}{2}}{j+n-\frac{3}{2}}}\lambda \\
&\quad - \left(\frac{i}{2}\sqrt{2n-1}\right)\left(\frac{\mp i}{\sqrt{2n-1}}\sqrt{\frac{1}{2}\frac{j-\frac{1}{2}}{j+n-\frac{3}{2}}}\right)\lambda \\
&= (\Delta_p - \Delta_{12} - j - d + 2 \mp \tfrac{1}{2})\sqrt{\frac{1}{2}\frac{j-\frac{1}{2}}{j+n-\frac{3}{2}}}\lambda, \qquad (5.322)\\
\lambda\gamma_{p,\mathbf{j}^{\pm},\mathbf{j}^{\mp}} &= (\Delta_p - \Delta_{12} - n + \tfrac{1}{2})\sqrt{\frac{2n-2}{(2j+2n-3)(2j+2n-1)}}\lambda \\
&\quad - \left(\frac{i}{2}\sqrt{2n-1}\right)\left(\frac{\mp 2i(j+n-1)}{\sqrt{2n-1}}\sqrt{\frac{2n-2}{(2j+2n-3)(2j+2n-1)}}\right)\lambda \\
&= (\Delta_p - \Delta_{12} - n + \tfrac{1}{2} \mp (j+n-1))\sqrt{\frac{1}{2}\frac{n-1}{(j+n-\frac{3}{2})(j+n-\frac{1}{2})}}\lambda. \qquad (5.323)
\end{aligned}$$

Similarly, we find from (5.281)

$$\overline{\gamma}_{p,\mathbf{j}^{\pm},(\mathbf{j}+1)^{\pm}}\lambda = (\Delta_p + \Delta_{34} + j \pm (-1)^{n-1}\tfrac{1}{2})\sqrt{\frac{1}{2}\frac{j+2n-\frac{3}{2}}{j+n-\frac{1}{2}}}\lambda, \qquad (5.324)$$

$$\overline{\gamma}_{p,\mathbf{j}^{\pm},(\mathbf{j}-1)^{\pm}}\lambda = (\Delta_p + \Delta_{34} - j - d + 2 \mp (-1)^{n-1}\tfrac{1}{2})\sqrt{\frac{1}{2}\frac{j-\frac{1}{2}}{j+n-\frac{3}{2}}}\lambda, \qquad (5.325)$$

$$\overline{\gamma}_{p,\mathbf{j}^{\pm},\mathbf{j}^{\mp}}\lambda = (\Delta_p + \Delta_{34} - n + \tfrac{1}{2} \mp (-1)^{n-1}(j+n-1))\sqrt{\frac{1}{2}\frac{n-1}{(j+n-\frac{3}{2})(j+n-\frac{1}{2})}}\lambda. \qquad (5.326)$$

Finally, the Casimir eigenvalue is given, according to (5.227) and (5.228),

$$C_{p,j} = \Delta_p(\Delta_p - 2n) + j(j+2n-2) + \frac{(2n-2)(2n-3)}{8}. \qquad (5.327)$$

The recursion relation (5.287) can then be put into the form

$$(C_{p,j} - C_{0,j_O})\mathbf{\Lambda}_{p,j} = \Gamma^{+}_{p-1,j-1}\mathbf{\Lambda}_{p-1,j-1} + \Gamma^{-}_{p-1,j-1}\mathbf{\Lambda}_{p-1,j+1} + \Gamma^{0}_{p-1,j-1}\mathbf{\Lambda}_{p-1,j}, \quad (5.328)$$

where

$$\begin{aligned}\Gamma^{+}_{p,j} = {} & (\Delta_p - \Delta_{12} + j + \tfrac{1}{2})(\Delta_p + \Delta_{34} + j + (-1)^{n-1}\tfrac{1}{2})\frac{j+2n-\frac{3}{2}}{2j+2n-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \\ & + (\Delta_p - \Delta_{12} + j - \tfrac{1}{2})(\Delta_p + \Delta_{34} + j - (-1)^{n-1}\tfrac{1}{2})\frac{j+2n-\frac{3}{2}}{2j+2n-1}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\end{aligned} \quad (5.329)$$

$$\begin{aligned}\Gamma^{-}_{p,j} = {} & (\Delta_p + \Delta_{12} - j - 2n + 2 - \tfrac{1}{2})(\Delta_p + \Delta_{34} - j - 2n + 2 - (-1)^{n-1}\tfrac{1}{2})\frac{j-\frac{1}{2}}{2j+2n-3}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \\ & + (\Delta_p + \Delta_{12} - j - 2n + 2 + \tfrac{1}{2})(\Delta_p + \Delta_{34} - j - 2n + 2 + (-1)^{n-1}\tfrac{1}{2})\frac{j-\frac{1}{2}}{2j+2n-3}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\end{aligned} \quad (5.330)$$

$$\begin{aligned}\Gamma^{0}_{p,j} = {} & (\Delta_p - \Delta_{12} - j - 2n + \tfrac{3}{2})(\Delta_p + \Delta_{34} - n + \tfrac{1}{2} - (-1)^{n-1}(j+n-1))\times \\ & \times \frac{2n-2}{(2j+2n-3)(2j+2n-1)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \\ & + (\Delta_p - \Delta_{12} + j - \tfrac{1}{2})(\Delta_p + \Delta_{34} - n + \tfrac{1}{2} + (-1)^{n-1}(j+n-1))\times \\ & \times \frac{2n-2}{(2j+2n-3)(2j+2n-1)}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.\end{aligned} \quad (5.331)$$

The full conformal block can then be expanded by using a generalization of (5.201),

$$\langle 0|\overline{\psi}_4\psi_3|O|s^{\Delta}e^{\theta M_{12}}\psi_2\phi_1|0\rangle = \sum_{p=0}^{\infty}\sum_{j=0}^{\infty} s^{\Delta+p}\mathbf{\Lambda}_{p,j}\cdot\mathbf{P}^{j}(\theta)\cdot\mathbf{T}, \quad (5.332)$$

where $\mathbf{T} = (t_+, t_-)$ and the matrom $\mathbf{P}^j$ is given by (5.196). In appendix D.3 we compare the conformal blocks obtained from this recursion relation with the known expressions in 2d ($n = 1$) and 4d ($n = 2$), finding a perfect agreement.

5.4.5 An efficient implementation?

We have derived the Casimir recursion relation for general conformal blocks. Our derivation relies on the knowledge of a number of $6j$-symbols of $Spin(d-1)$. As we have discussed, there are important cases, such as general blocks in 3d and 4d or seed blocks in general dimensions, where these symbols are readily available. In other cases, they can be computed as soon as the relevant Clebsch-Gordan coefficients are known. These Clebsch-Gordan coefficients are needed anyway for the three-point functions (and can be derived from them), so it is reasonable to assume that the $6j$ symbols are computable in all cases of interest.

If the relevant $6j$ symbols are known, then our results provide a closed-form expression for the recursion relation (5.287). This is a quite general result, so it is interesting to discuss the possibility of employing it for an efficient computation of spinning conformal blocks. Assume that we have fixed numerical values for scaling dimensions and spins of the external operators and the spin of the intermediate primary and would like to compute the conformal block and its derivatives as a function of the intermediate dimension Δ_O. The simplest approach is to naively iterate the recursion relation and find the coefficients of the power series expansion in z-coordinates.

This approach has several obvious disadvantages. Firstly, the z-coordinate expansion converges much slower than the ρ-coordinate expansion [59]. Secondly, the coefficients of the expansion are going to be some complicated rational functions of Δ_O, manipulations with which are costly. Moreover, the difference of Casimir eigenvalues in (5.287),

$$C(\Delta_O + n, \widetilde{\mathbf{m}}_d) - C(O) = 2n\Delta_O + n^2 - nd + C(\widetilde{\mathbf{m}}_d) - C(\mathbf{m}_d^O), \tag{5.333}$$

produces a lot of apparent poles at various rational values of Δ_O. We however know that the conformal blocks can only have poles at (half-)integral values of Δ_O [49]. This implies that there must be a lot of cancellations, which make the direct analytic even less optimal. Let us discuss some possible solutions to these problems.

The first problem can be in principle avoided by converting the z-coordinate expansion into a ρ-coordinate expansion. It is possible because we have the relation $z = 4\rho + O(\rho^2)$, so if we know the expansion of $f(z)$ to order z^N, we can compute expansion of $f(z(\rho))$ to the same order ρ^N. If the coefficients in expansion of $f(z)$ are numbers, and we aim to evaluate $f(\frac{1}{2})$, then this conversion can be done efficiently by defining z_N^k to be equal to the ρ-series of z^k, truncated at order ρ^N and with ρ set to $\rho = 3 - 2\sqrt{2}$ (the value corresponding to $z = \frac{1}{2}$). Then the number $f(1/2)$ can be computed by simply replacing z^k in its z-expansion by the numbers z_N^k. These numbers can be precomputed once for any given N.

However, as we noted above, in our case the coefficients of z-expansion are complicated rational functions and thus this conversion would have to be performed using symbolic algebra. To solve this problem, it is convenient to recall that for any conformal block $G(\Delta_O)$ (for simplicity of notation we keep the dependence only on Δ_O explicit) the function $H(\Delta_O) = |\rho|^{-\Delta_O} G(\Delta_O)$ is a meromorphic function of Δ_O with either single or double[41] poles and a finite limit at infinity[42] [37, 48, 49]. In odd dimensions this function only has single poles, so let us consider this case for simplicity.[43] We then can write

$$H(\Delta_O) = H(\infty) + \sum_i \frac{R_i}{\Delta_O - \Delta_i}, \tag{5.334}$$

where Δ_i are the locations of the poles and R_i are some coefficients.[44] The function $H(\infty)$ can be computed in closed form for a general conformal block by a suitable choice of the basis of four-point structures. Expansion (5.334) is often used to derive rational approximations to conformal blocks, required for numerical analysis using SDPB [35, 37]. For this, note that different terms in this expansion are suppressed by powers ρ^{n_i} for some positive n_i. Thus, one can keep only the finite number of terms with $n_i \leq M$ for some sufficiently large M. Since the derivatives of G are determined by derivatives of H, it is sufficient to compute the derivatives of R_i and $H(\infty)$ numerically in order to obtain the rational approximations required for numerical bootstrap applications.

Our recursion relation can be used to determine R_i and their derivatives numerically. Indeed, on each step of the recursion relation we explicitly divide by a linear function of Δ_O (5.333). Thus, we know exactly when we produce poles and we can compute their residues and how they change on each step of the recursion. If we select a subset of Δ_i, we only need to track the derivatives of the residues at these poles, which are simply numbers. We can avoid dealing with the apparent poles at rational Δ_O by tracking only the Δ_i allowed by representation theory [49]. This is similar in spirit to multiplication of polynomials in Fourier space (as in FFT polynomial multiplication), except we are working with rational functions. This approach should

[41] We are not aware of a direct proof that at most second-order poles appear in even d (see e.g. [49, 196] for a discussion). However, since the scalar blocks have at most second-order poles, the results of [3] imply that there are at most finitely many higher-order poles in any given conformal block. Also, standard arguments from complex analysis show that at most double poles can appear from collision of two single poles, which can possibly be used to show that at least the blocks which can be analytically continued in dimension d have at most second-order poles.

[42] At least for Δ_O-independent choice of three-point functions.

[43] The same approach should work in even dimensions, with minor modifications.

[44] R_i are known to be proportional to other conformal blocks. We do not use this fact here.

allow us to efficiently compute the numerical z-series of derivatives of R_i. We can then use the aforementioned procedure to resum it into ρ-series at $z = \frac{1}{2}$.

Note that in this scheme it is most convenient to take the derivatives in z-coordinate. These derivatives do not necessarily have the fastest rate of convergence among other simple choices.[45] A related problem is that it is not obvious what is the best basis of four-point tensor structures in terms of convergence.[46] The approach based on (5.334) somewhat solves this ambiguity – it is a well-defined procedure to keep a finite number of poles in (5.334), and we can then compute R_i to an order N higher than M, eliminating the possible discrepancies between various choices. Indeed, if we keep the number of poles that we track fixed, then the complexity of computing each new order grows only because the range of allowed values for $m_{d,1}$ expands.

In order for the above program to succeed, we need to be able to efficiently compute derivatives of these P-functions. It appears that this problem is largely solved by the recursion relation (5.191) which can be easily implemented numerically for any choice of representations given the availability of closed-form formulas for vector isoscalar factors. We still need an initial condition for the recursion relation. As we discussed previously, it can be obtained by direct exponentiation of M_{12}. However, in numerical applications we do not even need this. We only need a first few derivatives of P-functions at $\theta = 0$, which are given by matrix elements of powers of M_{12}, making the computation even easier.

5.5 Conclusions

The two major results of this paper are

1. The general form (5.130) of a $\mathbb{R} \times Spin(d)$-multiplet contribution to a general four-point function of operators with spins.

2. The Casimir recursion relation (5.287) (and the formulas (5.280) and (5.281)) for the relevant coefficients) for the amplitudes $\Lambda_{p,\mathbf{m}_d}$ of these contributions to a general spinning conformal block.

The first result is expressed in terms of certain special functions P (5.128), which we have studied in detail in section 5.3.7. We have described the basic properties of

[45] Choice of the coordinate matters: the derivative $df(z)/df(z)$ converges much faster than the derivative of $df(z)/dz$.

[46] The choice of basis matters as well, because the bases can differ by z-dependent factors: even if $f(z)$ converges quickly, $f(z)/(1-z)^{100}$ may converge much slower.

these functions (including orthogonality relations) as well as a practical approach to their calculation. In appendix D.4 we have furthermore related these functions to the irreducible projectors of [82].[47] We have studied how (5.130) simplifies in some special cases, namely for $d = 3, 4$ and for seed blocks in general d. We have also proven the folklore theorem which states that the number of four-point tensor structures is the same as the number of classes of conformal blocks.

Our second result paves a way to an algorithmic computation of general conformal blocks. The expressions (5.287), (5.280) and (5.281) give a closed-form recursion relation for the coefficients of the z-coordinate expansion of a general conformal block, if the relevant $6j$ symbols of $Spin(d-1)$ are known. There is a finite number of such $6j$ symbols for any given conformal block, and they can be straightforwardly computed if the corresponding Clebsch-Gordan coefficients are known. The required CG coefficients are indeed known in many important cases. In particular, we have explicitly worked out the case of general conformal blocks in 3 dimensions and the seed blocks in general dimensions. To illustrate the recursion relation in explicit examples, we have studied the scalar-fermion seed blocks in $d = 3$ and $d = 2n$, comparing to the known results when possible. Finally, in section 5.4.5 we have briefly discussed a strategy for an efficient numerical implementation of the recursion relation (5.287).

Many extensions of these results are possible. For example, the scalar-fermion seed blocks can also be straightforwardly obtained for $d = 2n + 1$, we have omitted this case only to keep the size of the paper reasonable. For the same reason we have not written down the explicit formulas for the case of general blocks in $d = 4$, even though these can be obtained (in terms of $SU(2)$ $6j$-symbols) mechanically from the general expressions. Extension to $d = 5$ is also possible, due to $Spin(5-1) \simeq SU(2) \times SU(2)$. An interesting problem is to develop a numerical algorithm for computation of general $Spin(d-1)$ CG coefficients and $6j$ symbols. Combined with the recursion relation (5.287) this would constitute the first completely general algorithm for computation of conformal blocks.[48] It is also interesting to implement this recursion relation efficiently, perhaps along the lines of section 5.4.5. Finally, there is always the question whether these results can be

[47]We believe that this is not the most optimal way for computation of explicit examples of functions P, and one instead should use the methods described in 5.3.7. Nevertheless, this relation does provide expressions which may be useful in analytical applications.

[48]Here by an "algorithm" we mean an actual complete algorithm which can be straightforwardly translated into a computer program. Techniques (not algorithms) for computing completely general spinning conformal blocks are already known [3, 49, 54, 60].

extended to superconformal case. We hope to address some of these questions in future work.

Acknowledgments

I would like to thank Denis Karateev, João Penedones, Fernando Rejón-Barerra, Slava Rychkov, David Simmons-Duffin, Emilio Trevisani, and the participants of Simons Bootstrap Collaboration workshop on numerical bootstrap for valuable discussions. Special thanks to David Simmons-Duffin for comments on the draft. I am grateful to the authors of [82] for making their `Mathematica` code publicly available. I also thank the Institute for Advanced Study, where part of this work was completed, for their hospitality. This work was supported by DOE grant DE-SC0011632.

Chapter 6

LIGHT-RAY OPERATORS IN CONFORMAL FIELD THEORY

This chapter is essentially identical to:

[5] P. Kravchuk and D. Simmons-Duffin, *Light-ray operators in conformal field theory*, 1805.00098.

6.1 Introduction

Singularities of Euclidean correlators in conformal field theory (CFT) are described by the operator product expansion (OPE). However, in Lorentzian signature there exist singularities that cannot be described in a simple way using the OPE. One of the most important is the Regge limit of a time-ordered four-point function (figure 6.1) [166, 188, 219–222].[1] The Regge limit is the CFT version of a high-energy scattering process: operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_3(x_3)$ create excitations that move along nearly lightlike trajectories, interact, and then are measured by operators $\mathcal{O}_2(x_2)$ and $\mathcal{O}_4(x_4)$. In holographic theories, the Regge limit is dual to high-energy forward scattering in the bulk [224].

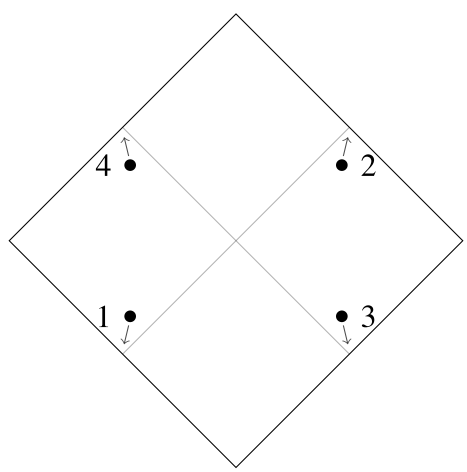


Figure 6.1: The Regge limit of a four-point function: the points $x_1, \ldots, x_4$ approach null infinity, with the pairs x_1, x_2 and x_3, x_4 becoming nearly lightlike separated.

In Lorentzian signature, the OPE $\mathcal{O}_i \times \mathcal{O}_j$ converges if the product $\mathcal{O}_i\mathcal{O}_j$ acts on the

[1]In perturbation theory, Lorentzian singularities correspond to Landau diagrams [223]. It is possible that this is also true nonperturbatively.

vacuum (either past or future) [78]. That is, we have an equality of states

$$\mathcal{O}_i\mathcal{O}_j|\Omega\rangle = \sum_k f_{ijk}\mathcal{O}_k|\Omega\rangle, \tag{6.1}$$

where k runs over local operators of the theory (we suppress position dependence, for brevity). Thus, in figure 6.1 the OPEs $\mathcal{O}_1\times\mathcal{O}_3$ and $\mathcal{O}_1\times\mathcal{O}_4$ converge because they act on the past vacuum, and the OPEs $\mathcal{O}_2\times\mathcal{O}_3$ and $\mathcal{O}_2\times\mathcal{O}_4$ converge because they act on the future vacuum. (Here we use the fact that spacelike-separated operators commute to rearrange the operators in the time-ordered correlator to apply (6.1).) However, each of these OPEs is converging very slowly in the Regge limit. They can be used to prove results like analyticity and boundedness in the Regge limit [160, 225], but they are less useful for computations (unless one has good control over the theory). Meanwhile, the OPEs $\mathcal{O}_1\times\mathcal{O}_2$ and $\mathcal{O}_3\times\mathcal{O}_4$ are invalid in the Regge regime.

The problem of describing four-point functions in the Regge regime was partially solved in [166, 167, 224]. The behavior of the correlator is controlled by the analytic continuation of data in the $\mathcal{O}_1\times\mathcal{O}_2$ and $\mathcal{O}_3\times\mathcal{O}_4$ OPEs to non-integer spin. For example, in a planar theory, the Regge correlator behaves (very) schematically as

$$\frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\rangle\langle\mathcal{O}_3\mathcal{O}_4\rangle} \sim 1 - f_{12\mathcal{O}}(J_0)f_{34\mathcal{O}}(J_0)e^{t(J_0-1)} + \dots. \tag{6.2}$$

Here, $f_{12\mathcal{O}}(J)$ and $f_{34\mathcal{O}}(J)$ are OPE coefficients that have been analytically continued in the spin J of $\mathcal{O}$. The parameter t measures the boost of $\mathcal{O}_1,\mathcal{O}_2$ relative to $\mathcal{O}_3,\mathcal{O}_4$. $J_0\in\mathbb{R}$ is the Regge/Pomeron intercept, and is determined by the analytic continuation of the dimension $\Delta_\mathcal{O}$ to non-integer J.[2] The "..." in (6.2) represent higher-order corrections in $1/N^2$ and also terms that grow slower than $e^{t(J_0-1)}$ in the Regge limit $t\to\infty$.

A missing link in this story was provided recently by Caron-Huot, who proved that OPE coefficients and dimensions have a natural analytic continuation in spin in any CFT [66]. The analytic continuation of OPE data in a scalar four-point function $\langle\phi_1\phi_2\phi_3\phi_4\rangle$ can be computed by a "Lorentzian inversion formula," given by the integral of a double-commutator $\langle[\phi_4,\phi_1][\phi_2,\phi_3]\rangle$ times a conformal block $G_{J+d-1,\Delta-d+1}$ with unusual quantum numbers. Specifically, Δ and J are replaced with

$$(\Delta, J)\to(J+d-1,\Delta-d+1) \tag{6.3}$$

[2] In $d=2$, the Regge regime is the same as the chaos regime. In $d\geq 3$, it is related to chaos in hyperbolic space. See [174, 226] for discussions. Note that J_0-1 plays the role of a Lyapunov exponent, and it is constrained by the chaos bound to be less than 1 [225, 227].

relative to a conventional conformal block. Caron-Huot's Lorentzian inversion formula has many other useful applications, for example to large-spin perturbation theory and the lightcone bootstrap [31, 68, 69, 102–104, 106, 162, 228, 229], and to the SYK model [172, 173, 230, 231].[3]

However, Caron-Huot's result raises some obvious questions:

- Can operators themselves (not just their OPE data) be analytically continued in spin?
- What is the space of continuous spin operators in a given CFT?
- Do continuous-spin operators have a Hilbert space interpretation (similar to how integer-spin operators correspond to CFT states on S^{d-1})?
- What is the meaning of the funny block in the Lorentzian inversion formula, and how do we generalize it?

Answering these questions is important for making sense of the Regge limit, and more generally for understanding how to write a convergent OPE in non-vacuum states.

It is easy to describe continuous-spin operators mathematically. Consider first a primary operator $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$ with integer spin J. Let us introduce a null polarization vector z_μ and contract it with the indices of $\mathcal{O}$ to form a function of (x, z):

$$\mathcal{O}(x, z) \equiv \mathcal{O}^{\mu_1\cdots\mu_J}(x) z_{\mu_1} \cdots z_{\mu_J}, \qquad (z^2 = 0). \tag{6.4}$$

The tensor $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$ can be recovered from the function $\mathcal{O}(x, z)$ by stripping off the z's and subtracting traces. Thus, $\mathcal{O}(x, z)$ is a valid alternative description of a traceless symmetric tensor. Note that $\mathcal{O}(x, z)$ is a homogeneous polynomial of degree J in z. The generalization to a continuous spin operator $\mathbb{O}$ is now straightforward: we simply drop the requirement that $\mathbb{O}(x, z)$ be polynomial in z and allow it to have non-integer homogeneity,

$$\mathbb{O}(x, \lambda z) = \lambda^J \mathbb{O}(x, z), \qquad \lambda > 0, \; J \in \mathbb{C}. \tag{6.5}$$

Continuous-spin operators are necessarily nonlocal. This follows from Mack's classification of positive-energy representations of the Lorentzian conformal group

[3] In the 1-dimensional SYK model, the analog of analytic continuation in spin is analytic continuation in the weight of discrete states in the conformal partial wave expansion [67, 172].

$\widetilde{\mathrm{SO}}(d,2)$ [232], which only includes nonnegative integer spin representations.[4] CFT states have positive energy, so by the state-operator correspondence, local operators must have nonnegative integer spin, and conversely continuous-spin operators must be nonlocal. Mack's classification also shows that continuous-spin operators must annihilate the vacuum:

$$\mathbb{O}(x,z)|\Omega\rangle = 0 \qquad (J \notin \mathbb{Z}_{\geq 0}), \tag{6.6}$$

otherwise $\mathbb{O}(x,z)|\Omega\rangle$ would transform in a nontrivial continuous-spin representation, which would include a state with negative energy.

If continuous-spin operators annihilate the vacuum, how can we analytically continue the local operators of a CFT, which certainly do not annihilate the vacuum? The answer is that we must first turn local operators into something nonlocal that annihilates the vacuum, and then analytically continue that. The correct object turns out to be the integral of a local operator along a null line,

$$\int_{-\infty}^{\infty} d\alpha\, \mathcal{O}(\alpha z, z) = \int_{-\infty}^{\infty} d\alpha\, \mathcal{O}^{\mu_1\cdots\mu_J}(\alpha z) z_{\mu_1}\cdots z_{\mu_J}. \tag{6.7}$$

This can be written more covariantly by performing a conformal transformation to bring the beginning of the null line to a generic point x:[5]

$$\mathbf{L}[\mathcal{O}](x,z) \equiv \int_{-\infty}^{\infty} d\alpha (-\alpha)^{-\Delta-J} \mathcal{O}\left(x - \frac{z}{\alpha}, z\right). \tag{6.8}$$

This defines an integral transform $\mathbf{L}$ that we call the "light transform." The expression (6.7) corresponds to $\mathbf{L}[\mathcal{O}](-\infty z, z)$, where $x = -\infty z$ is a point at past null infinity.

After reviewing some representation theory in sections 6.2.1 and 6.2.2, we show in section 6.2.3 that if $\mathcal{O}_{\Delta,J}$ has dimension Δ and spin J, then $\mathbf{L}[\mathcal{O}_{\Delta,J}](x,z)$ transforms like a primary operator with dimension $1-J$ and spin $1-\Delta$:

$$\mathbf{L} : (\Delta, J) \to (1-J, 1-\Delta). \tag{6.9}$$

In particular, $\mathbf{L}[\mathcal{O}_{\Delta,J}]$ can have non-integer spin. The average null energy operator $\mathcal{E} = \mathbf{L}[T]$ (the light transform of the stress tensor) is a special case, having dimension

[4] For non traceless-symmetric tensor operators, we define spin as the length of the first row of the Young diagram for their $\mathrm{SO}(d)$ representation. For fermionic representations spin is a half-integer and for simplicity of language we include this case into the notion of "integer spin" operators.

[5] As $\alpha \to 0^-$, the point $x - z/\alpha$ diverges to future null infinity, and the integration contour should be understood as extending into the next Poincare patch on the Lorentzian cylinder. We give more detail in section 6.2.3.2.

-1 and spin $1-d$. We will see that $\mathbf{L}$ is part of a dihedral group (D_8) of intrinsically Lorentzian integral transforms that generalize the Euclidean shadow transform [54, 233]. These Lorentzian transforms implement affine Weyl reflections that preserve the Casimirs of the conformal group. For example, the quadratic Casimir eigenvalue is given by

$$C_2(\Delta, J) = \Delta(\Delta - d) + J(J + d - 2), \tag{6.10}$$

and this is indeed invariant under (6.9). The transformation (6.3) appearing in Caron-Huot's formula is another affine Weyl reflection. The Lorentzian transforms do not give precisely a representation of D_8, but instead satisfy an interesting "anomalous" algebra that we derive in section 6.2.7. Mack's classification implies that $\mathbf{L}[\mathcal{O}_{\Delta,J}]$ must annihilate the vacuum whenever $\mathcal{O}_{\Delta,J}$ is a local operator. This is also easy to see directly by deforming the α contour into the complex plane, as we show in section 6.2.4.

We claim that the operators $\mathbf{L}[\mathcal{O}_{\Delta,J}]$ can be analytically continued in J, and their continuations are light-ray operators.[6] As an example, consider Mean Field Theory (a.k.a. Generalized Free Fields) in $d = 2$ with a scalar primary ϕ. This theory contains "double-trace" operators

$$[\phi\phi]_J(u, v) \equiv :\phi(u, v)\partial_v^J \phi(u, v): + \partial_v(\dots) \tag{6.11}$$

with dimension $2\Delta_\phi + J$ and even spin J. Here, $: :$ denotes normal ordering and we have written out the definition up to total derivatives (which are required to ensure that this is a primary operator). We are using lightcone coordinates $u = x-t$, $v = x+t$, and for simplicity focusing on operators with ∂_v derivatives only. The corresponding analytically-continued light-ray operators are

$$\begin{aligned}&\mathbb{O}_J(0, -\infty)\\ &= \frac{i\Gamma(J+1)}{2^J} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} \frac{ds}{2\pi} \left(\frac{1}{(s + i\epsilon)^{J+1}} + \frac{1}{(-s + i\epsilon)^{J+1}} \right) :\phi(0, v + s)\phi(0, v - s): .\end{aligned} \tag{6.12}$$

When J is an even integer, we have

$$\frac{i\Gamma(J+1)}{2\pi} \left(\frac{1}{(s + i\epsilon)^{J+1}} - \frac{1}{(s - i\epsilon)^{J+1}} \right) = \frac{\partial^J \delta(s)}{\partial s^J} \qquad (J \in 2\mathbb{Z}_{\geq 0}). \tag{6.13}$$

[6] Note that $\mathbf{L}[\mathcal{O}_{\Delta,J}](x, z)$ has dimension $1-J$ and spin $1-\Delta$. Thus, analytic continuation in J is really analytic continuation in the *dimension* of $\mathbf{L}[\mathcal{O}_{\Delta,J}]$ away from negative integer values. We will continue to refer to it as analytic continuation in spin, since J labels the spin of local operators.

Thus, when J is an even integer, $\mathbb{O}_J$ becomes

$$\begin{aligned}\mathbb{O}_J(0,-\infty) &= 2^{-J}\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} ds\, \frac{\partial^J \delta(s)}{\partial s^J} :\phi(0, v+s)\phi(0,v-s): \\ &= \int_{-\infty}^{\infty} dv :\phi\partial_v^J\phi:(0,v) = \mathbf{L}[[\phi\phi]_J](0,-\infty) \qquad (J \in 2\mathbb{Z}_{\geq 0}). \end{aligned} \quad (6.14)$$

By contrast, when J is not an even integer, $\mathbb{O}_J$ is a legitimately nonlocal light-ray operator whose correlators are analytic continuations of the correlators of $\mathbf{L}[[\phi\phi]_J]$. In particular, three-point functions $\langle \mathcal{O}_1\mathcal{O}_2\mathbb{O}_J\rangle$ give an analytic continuation of the three-point coefficients of $\langle \mathcal{O}_1\mathcal{O}_2[\phi\phi]_J\rangle$.

Similar light-ray operators have a long history in the gauge-theory literature [234, 235] (see [236–239] for recent discussions). There, one often considers a bilocal integral of operators inserted along a null Wilson line. Such operators were discussed in [76], where they were argued to control OPEs of the average null energy operator $\mathcal{E}$. In perturbation theory, it is reasonable to imagine constructing more operators like (6.12). However, it is less clear how to define them in a nonperturbative context where normal ordering is not well-defined, and there can be complicated singularities when two operators become lightlike-separated. It is also not clear what a null Wilson line means in an abstract CFT.

Our tool for constructing analogs of $\mathbb{O}_J$ in general CFTs will be harmonic analysis [65]. Given primary operators $\mathcal{O}_1, \mathcal{O}_2$, we find in section 6.3 an integration kernel $K_{\Delta,J}(x_1,x_2,x,z)$ such that

$$\mathbb{O}_{\Delta,J}(x,z) = \int d^dx_1 d^dx_2 K_{\Delta,J}(x_1,x_2,x,z)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \quad (6.15)$$

transforms like a primary with dimension $1-J$ and spin $1-\Delta$ (when inserted in a time-ordered correlator). The object $\mathbb{O}_{\Delta,J}$ is meromorphic in Δ and J and has poles of the form

$$\mathbb{O}_{\Delta,J}(x,z) \sim \frac{1}{\Delta-\Delta_i(J)}\mathbb{O}_{i,J}(x,z). \quad (6.16)$$

We conjecture based on examples that poles must come from the region where x_1, x_2 are close to the light ray $x + \mathbb{R}_{\geq 0}z$ (we have not established this rigorously in a general CFT). The residues of the poles can thus be interpreted as light-ray operators $\mathbb{O}_{i,J}(x,z)$ that make sense in arbitrary correlators. Furthermore, when J is an integer, the residues are light-transforms of local operators $\mathbf{L}[\mathcal{O}]$. Thus the $\mathbb{O}_{i,J}$ give a analytic continuations of $\mathbf{L}[\mathcal{O}]$ for all $\mathcal{O} \in \mathcal{O}_1 \times \mathcal{O}_2$.

In section 6.4, we show that $\langle \mathcal{O}_3\mathcal{O}_4\mathbb{O}_{\Delta,J}\rangle$ can be computed via the integral of a double-commutator $\langle[\mathcal{O}_4,\mathcal{O}_1][\mathcal{O}_2,\mathcal{O}_3]\rangle$ over a Lorentzian region of spacetime. This leads to a simple proof of Caron-Huot's Lorentzian inversion formula. The contour manipulation from [67] is crucial for this computation. However, the light-ray perspective makes our proof simpler than the one in [67]. In particular, it makes it clearer why the unusual conformal block $G_{J+d-1,\Delta-d+1}$ appears. The reason is that the quantum numbers $(J+d-1,\Delta-d+1)$ are dual to those of the light-transform $(1-J,1-\Delta)$ in the sense that the product

$$d^d x\, d^d z\, \delta(z^2)\, \mathcal{O}_{1-J,1-\Delta}(x,z)\mathcal{O}_{J+d-1,\Delta-d+1}(x,z) \tag{6.17}$$

has dimension zero and spin zero. Our perspective also leads to a natural generalization of Caron-Huot's formula to the case of arbitrary operator representations, which we describe in section 6.4.2. Subsequently in section 6.5, we generalize conformal Regge theory to arbitrary operator representations as well, along the way showing that light-ray operators describe part of the Regge limit of four-point functions as conjectured in [222].

As mentioned above, the average null energy operator $\mathcal{E}=\mathbf{L}[T]$ is an example of a light-ray operator. The average null energy condition (ANEC) states that $\mathcal{E}$ is positive-semidefinite, i.e., its expectation value in any state is nonnegative. Some implications of the ANEC in CFTs are discussed in [76, 240, 241]. The ANEC was recently proven in [72] using techniques from information theory and in [73] using causality. By expressing $\mathcal{E}$ as the residue of an integral of a pair of real operators $\phi(x_1)\phi(x_2)$, we find a new proof of the ANEC in section 6.6.[7] Furthermore, $\mathcal{E}$ is part of a family of light-ray operators $\mathcal{E}_J$ labeled by continuous spin J, and our construction of light-ray operators applies to this entire family. This lets us derive a novel generalization of the ANEC to continuous spin. More precisely, we show that

$$\langle\Psi|\mathcal{E}_J|\Psi\rangle \geq 0, \qquad (J\in\mathbb{R}_{\geq J_{\min}}), \tag{6.18}$$

where $\mathcal{E}_J$ is the family of light-ray operators whose values at even integer J are given by

$$\mathcal{E}_J=\mathbf{L}[\mathcal{O}_{\Delta_{\min}(J),J}] \qquad (J\in 2\mathbb{Z},\ J\geq 2), \tag{6.19}$$

where $\mathcal{O}_{\Delta_{\min}(J),J}$ is the operator with spin J of minimal dimension. Here, $J_{\min}\leq 1$ is the smallest value of J for which the Lorentzian inversion formula holds [66].

[7] Our proof requires the dimension Δ_ϕ to be sufficiently low, though we expect it should be possible to relax this restriction.

We conclude in section 6.7 with discussion and numerous questions for the future. The appendices contain useful mathematical background, further technical details, and some computations needed in the main text. In particular, appendix E.1 includes a general discussion of continuous-spin tensor structures and their analyticity properties, appendix E.3 contains a lightning review of harmonic analysis for the Euclidean conformal group, and appendix E.8 gives details on conformal blocks with continuous spin.

Notation

In this work, we use the convention that correlators in the state $|\Omega\rangle$ represent physical correlators in a CFT. For example,

$$\langle\Omega|\mathcal{O}_1\cdots\mathcal{O}_n|\Omega\rangle \tag{6.20}$$

is a physical Wightman function, and

$$\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle_\Omega \equiv \langle\Omega|T\{\mathcal{O}_1\cdots\mathcal{O}_n\}|\Omega\rangle \tag{6.21}$$

is a physical time-ordered correlator.

Often, we discuss two- and three-point structures that are fixed by conformal invariance up to a constant. These structures do not represent physical correlators — they are simply known functions of spacetime points. We write them as correlators in the ficticious state $|0\rangle$. For example, if ϕ_i are scalar primaries with dimensions Δ_i, then

$$\begin{aligned}&\langle 0|\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)|0\rangle\\ &= \frac{1}{(x_{12}^2+i\epsilon t_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{23}^2+i\epsilon t_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(x_{13}^2+i\epsilon t_{13})^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}}\end{aligned} \tag{6.22}$$

denotes the unique conformally-invariant three-point structure for scalars with dimensions Δ_i, with the $i\epsilon$-prescription appropriate for the given Wightman ordering. Similarly,

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{1}{(x_{12}^2+i\epsilon)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{23}^2+i\epsilon)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(x_{13}^2+i\epsilon)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}} \tag{6.23}$$

denotes the unique conformally-invariant structure with the $i\epsilon$-prescription for a time-ordered correlator. In particular, (6.22) and (6.23) do not include OPE coefficients.

6.2 The light transform

This section is devoted to mathematical background and results that will be needed for constructing and studying light-ray operators. We first review some basic facts about the Lorentzian conformal group and its representation theory, with an emphasis on continuous spin operators. We then introduce a set of intrinsically Lorentzian integral transforms, which generalize the well-known Euclidean shadow transform, and study their properties. One of these transforms is the "light transform" mentioned in the introduction. It will play a key role in the sections that follow.

6.2.1 Review: Lorentzian cylinder

Similarly to Euclidean space $\mathbb{R}^d$, Minkowski space $\mathcal{M}_d = \mathbb{R}^{d-1,1}$ is not invariant under finite conformal transformations. In Euclidean space, this problem is easily solved by studying CFTs on S^d, the conformal compactification of $\mathbb{R}^d$. In Lorentzian signature, the problem is more subtle.

The simplest extension of Minkowski space $\mathcal{M}_d = \mathbb{R}^{d-1,1}$ that is invariant under the Lorentzian conformal group $\mathrm{SO}(d,2)$ is its conformal compactification $\mathcal{M}_d^c$. The space $\mathcal{M}_d^c$ can be easily described by the embedding space construction [27, 52, 92, 125, 186–188]: it is the projectivization of the null cone in $\mathbb{R}^{d,2}$ on which $\mathrm{SO}(d,2)$ acts by its vector representation. If we choose coordinates on $\mathbb{R}^{d,2}$ to be $X^{-1}, X^0, \ldots X^d$ with the metric

$$X^2 = -(X^{-1})^2 - (X^0)^2 + (X^1)^2 + \ldots + (X^d)^2, \tag{6.24}$$

then the null cone is defined by

$$(X^{-1})^2 + (X^0)^2 = (X^1)^2 + \ldots + (X^d)^2. \tag{6.25}$$

If we mod out by positive rescalings (i.e., by $\mathbb{R}_+$), we can set both sides of this equation to 1, identifying the space of solutions with $S^1 \times S^{d-1}$, where the S^1 is timelike. To get $\mathcal{M}_d^c$, we mod out by $\mathbb{R}$ rescalings,[8] obtaining $\mathcal{M}_d^c = S^1 \times S^{d-1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ identifies antipodal points in both S^1 and S^{d-1}. Minkowski space $\mathcal{M}_d \subset \mathcal{M}_d^c$ can be obtained by introducing lightcone coordinates in $\mathbb{R}^{d,2}$,

$$X^{\pm} = X^{-1} \pm X^d, \tag{6.26}$$

[8] In the Euclidean embedding space construction based on $\mathbb{R}^{d+1,1}$ we usually just take the future null cone instead of considering negative rescalings, but in $\mathbb{R}^{d,2}$ the null cone is connected and this is not possible.

and considering points with $X^+ \neq 0$. Using $\mathbb{R}$ rescalings we can set $X^+ = 1$ for such points, and the null cone equation becomes

$$X^- = -(X^0)^2 + (X^1)^2 + \ldots + (X^{d-1})^2. \tag{6.27}$$

If we set $x^\mu = X^\mu$ for $\mu = 0, \ldots d-1$, this gives the standard embedding of $\mathbb{R}^{d-1,1}$,

$$(X^+, X^-, X^\mu) = (1, x^2, x^\mu). \tag{6.28}$$

One can check that the action of $\mathrm{SO}(d,2)$ on X induces the usual conformal group action on x^μ. The points that lie in $\mathcal{M}_d^c \backslash \mathcal{M}_d$ have $X^+ = 0$ and thus $X^\mu X_\mu = 0$ with arbitrary X^-. They correspond to space-time infinity[9] ($X^\mu = 0$) and null infinity ($X^\mu \neq 0$).

By construction, $\mathcal{M}_d^c$ has an action of $\mathrm{SO}(d,2)$ and is thus a natural candidate for the space on which a conformally-invariant QFT can live. However, it is unsuitable for this purpose due to the existence of closed timelike curves that are evident from its description as $S^1 \times S^{d-1}/\mathbb{Z}_2$ with timelike S^1. This problem can be fixed by instead considering the universal cover $\widetilde{\mathcal{M}}_d = \mathbb{R} \times S^{d-1}$,[10] which is simply the Lorentzian cylinder. It was shown in [22] that Wightman functions of a CFT on $\mathbb{R}^{d-1,1}$ can be analytically continued to $\widetilde{\mathcal{M}}_d$. Indeed, one can first Wick-rotate the CFT to $\mathbb{R}^d$, map it conformally to the Euclidean cylinder $\mathbb{R} \times S^{d-1}$, and then Wick-rotate to $\widetilde{\mathcal{M}}_d$ (of course the actual proof in [22] is more involved).

To describe coordinates on $\widetilde{\mathcal{M}}_d$, it is convenient to first consider the null cone in $\mathbb{R}^{d,2}$ mod $\mathbb{R}_+$. It is equivalent to $S^1 \times S^{d-1}$ defined by

$$(X^{-1})^2 + (X^0)^2 = (X^1)^2 + \ldots + (X^d)^2 = 1, \tag{6.29}$$

and we can use the parametrization

$$\begin{aligned} X^{-1} &= \cos\tau, \\ X^0 &= \sin\tau, \\ X^i &= e^i, \quad i = 1 \ldots d, \end{aligned} \tag{6.30}$$

where $\vec{e}$ is a unit vector in $\mathbb{R}^d$. Here τ is the coordinate on S^1 with identification $\tau \sim \tau + 2\pi$, and taking the universal cover is equivalent to removing this identification.

[9] In $\mathcal{M}_d^c$ the infinite future, the infinite past and the spatial infinity of Minkowski space are identified. The past neighborhood of the future infinity, the future neighborhood of the past infinity and the spacelike neighborhood of the spatial infinity together form a complete neighbourhood of the space-time infinity in $\mathcal{M}_d^c$.

[10] For $d = 2$ this is not the universal cover.

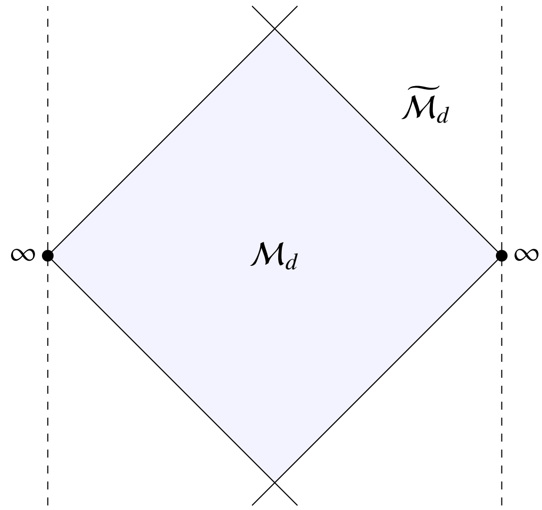


Figure 6.2: Poincare patch $\mathcal{M}_d$ (blue, shaded) inside the Lorentzian cylinder $\widetilde{\mathcal{M}}_d$ in the case of 2 dimensions. The spacelike infinity of $\mathcal{M}_d$ is marked by ∞. The dashed lines should be identified.

The coordinates $(\tau, \vec{e})$ with $\tau \in \mathbb{R}$ then cover $\widetilde{\mathcal{M}}_d$ completely. Minkowski space $\mathcal{M}_d$ can be conformally identified with a particular region in $\widetilde{\mathcal{M}}_d$ by using the embedding (6.28). This gives

$$x^0 = \frac{\sin\tau}{\cos\tau + e^d},$$
$$x^i = \frac{e^i}{\cos\tau + e^d}, \quad i = 1, \ldots d-1, \tag{6.31}$$

in the region where $\cos\tau + e^d > 0$ and $-\pi < \tau < \pi$. This region consists of points spacelike separated from $\tau = 0$, $\vec{e} = (0, \ldots, 0, -1)$, which is the spatial infinity of $\mathcal{M}_d$ (see figure 6.2). We will refer to this particular region as the (first) Poincare patch. Note that the null cone in $\mathbb{R}^{d,2}$ modulo $\mathbb{R}_+$ contains two Poincare patches – one with $X^+ > 0$ and one with $X^+ < 0$. The relation between Wightman functions on $\mathcal{M}_d$ and $\widetilde{\mathcal{M}}_d$ (in their natural metrics) for operators reads as[11]

$$\langle\Omega|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|\Omega\rangle_{\mathcal{M}_d} = \prod_{i=1}^{n}(\cos\tau_i + e_i^d)^{\Delta_i}\langle\Omega|\mathcal{O}_1(\tau_1, \vec{e}_1)\cdots\mathcal{O}_n(\tau_n, \vec{e}_n)|\Omega\rangle_{\widetilde{\mathcal{M}}_d}. \tag{6.32}$$

Let us discuss the action of the conformal group on $\widetilde{\mathcal{M}}_d$. First of all, because we have taken the universal cover of $\mathcal{M}_d^c$, it is no longer true that $\mathrm{SO}(d,2)$ acts on $\widetilde{\mathcal{M}}_d$.

[11]When applied to operators with spin, this identity does not produce a nice function on $\widetilde{\mathcal{M}}_d$, because in typical bases of spin indices on Minkowski space translations in τ act by matrices which have singularities. Therefore, in order to have nice functions on $\widetilde{\mathcal{M}}_d$ one has to perform a redefinition of spin indices [22].

Instead, the universal covering group $\widetilde{\mathrm{SO}}(d,2)$ acts on $\widetilde{\mathcal{M}}_d$. Indeed, the rotation generator $M_{-1,0}$ generates shifts in τ and in $\mathrm{SO}(d,2)$ we have $e^{2\pi M_{-1,0}} = 1$, whereas this is definitely not true on $\widetilde{\mathcal{M}}_d$ because $\tau \nsim \tau + 2\pi$. In the universal cover $\widetilde{\mathrm{SO}}(d,2)$, this direction gets decompactified so that the action becomes consistent.

6.2.1.1 Symmetry between different Poincare patches

There exists an important symmetry $\mathcal{T}$ of $\widetilde{\mathcal{M}}_d$ that commutes with the action of $\widetilde{\mathrm{SO}}(d,2)$. Namely, if we take a point with coordinates $p = (\tau, \vec{e})$ and send light rays in all future directions, they will all converge at the point $\mathcal{T}p \equiv (\tau + \pi, -\vec{e})$. The points p and $\mathcal{T}p$ in $\widetilde{\mathcal{M}}_d$ correspond to the same point in $\mathcal{M}_d^c$ and thus $\mathcal{T}$ commutes with infinitesimal conformal generators and therefore also with the full $\widetilde{\mathrm{SO}}(d,2)$.

When d is even, $\mathcal{T}$ lies in the center of $\widetilde{\mathrm{SO}}(d,2)$ and we can take

$$\mathcal{T} = e^{\pi M_{-1,0}} e^{\pi M_{1,2} + \pi M_{3,4} + \ldots + \pi M_{d-1,d}}. \tag{6.33}$$

For odd d only $\mathcal{T}^2$ lies in $\widetilde{\mathrm{SO}}(d,2)$. But if the theory preserves parity, i.e., we have an operator P that maps $x^1 \to -x^1$ in the first Poincare patch, then we can take

$$\mathcal{T} = e^{\pi M_{0,-1} + \pi M_{23} + \ldots + \pi M_{d-1,d}} P. \tag{6.34}$$

If the theory doesn't preserve parity, $\mathcal{T}$ can still be defined as an operation on correlation functions in the sense specified below.

If $\mathcal{T}$ exists as a unitary operator on the Hilbert space (d even or parity-preserving theory in odd d), then we can consider its action on local operators. For scalars we clearly have

$$\mathcal{T}\phi(x)\mathcal{T}^{-1} = \phi(\mathcal{T}x), \tag{6.35}$$

up to intrinsic parity in odd d. To understand the action of $\mathcal{T}$ on operators with spin, it is convenient to work in the embedding space, where we have for tensor operators

$$\mathcal{T}\mathcal{O}(X, Z_1, Z_2, \ldots Z_n)\mathcal{T}^{-1} = \mathcal{O}(-X, -Z_1, -Z_2, \ldots, -Z_n). \tag{6.36}$$

Here the point $-X$ is interpreted as the point in the Poincare patch which is in immediate future of the first Poincare patch, and Z_i are null polarizations corresponding to the various rows of the Young diagram of $\mathcal{O}$. Again, in odd dimensions we might need to add a factor of intrinsic parity.

Note that the above action on tensor operators can be defined regardless of the dimension d or whether or not the theory preserves parity. We will thus define $\mathcal{T}$ as an operator which can act on functions on $\widetilde{\mathcal{M}}_d$ according to

$$(\mathcal{T} \cdot O)(X, Z_1, Z_2, \dots Z_n) \equiv O(-X, -Z_1, -Z_2, \dots, -Z_n), \tag{6.37}$$

where again $-X$ is interpreted as corresponding to $\mathcal{T}x$. As discussed above, in even dimensions this always comes from a unitary symmetry of the theory defined by (6.33), but in odd dimensions it may not be a symmetry (even if the theory preserves parity). In such cases we can still use $\mathcal{T}$ thus defined to study conformally-invariant objects, similarly to how we can separate tensor structures into parity-odd and parity-even regardless of whether the theory preserves parity. To have a uniform discussion, we will use this definition of $\mathcal{T}$ action in what follows.

Finally, let us note that in even dimensions for tensor operators

$$\begin{aligned}\mathcal{T}O(x)|\Omega\rangle &= e^{i\pi(\Delta+N)}O(x)|\Omega\rangle,\\ \langle\Omega|O(x)\mathcal{T} &= e^{i\pi(\Delta+N)}\langle\Omega|O(x),\end{aligned} \tag{6.38}$$

where N is the total number of boxes in the $\mathrm{SO}(d-1,1)$ Young diagram of O. This follows from the fact that the representation generated by O acting on the vacuum is irreducible. One can check the eigenvalue by considering this identity inside a Wightman two-point function. The same relation holds in parity-even structures in odd dimensions (in particular, in two-point functions) and with a minus sign in parity-odd structures.

6.2.1.2 Causal structure

The action of $\widetilde{\mathrm{SO}}(d,2)$ on $\widetilde{\mathcal{M}}_d$ preserves the causal structure of the Lorentzian cylinder [22]. This property will allow us to define conformally-invariant integration regions. We usually label points in $\widetilde{\mathcal{M}}_d$ by natural numbers and we write $1 < 2$ when point 1 is inside the past lightcone of 2 and $1 \approx 2$ when 1 is spacelike from 2. Furthermore, we write $1^{\pm}$ for $\mathcal{T}^{\pm 1}1$ (more generally, $1^{\pm k}$ for $\mathcal{T}^{\pm k}1$). That is, 1^+ is the point in the "next" Poincare patch with the same Minkowski coordinates as 1. Similarly, 1^- is the point in the "previous" Poincare patch with the same Minkowski coordinates as 1. Some causal relationships between points can be written in different ways, e.g., $1 \approx 2$ if and only if $2^- < 1 < 2^+$ (figure 6.3).

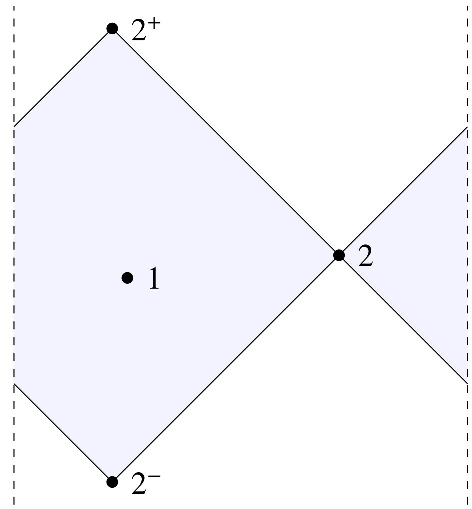


Figure 6.3: 1 is spacelike from 2 ($1 \approx 2$) if and only if 1 is in the future of 2^- and the past of 2^+ ($2^- < 1 < 2^+$). The figure shows the Lorentzian cylinder in 2-dimensions. The dashed lines should be identified.

6.2.2 Review: Representation theory of the conformal group

We will also need some facts from unitary representation theory of the conformal groups $\mathrm{SO}(d+1,1)$ and $\mathrm{SO}(d,2)$. These groups are non-compact and their unitary representations are infinite-dimensional. We will mostly be interested in a particular class of unitary representations known as principal series representations, and also their non-unitary analytic continuations.

Unitary principal series representations of $\mathrm{SO}(d+1,1)$ are the easiest to describe. In this case, a principal series representation $\mathcal{E}_{\Delta,\rho}$ is labeled by a pair (Δ, ρ), where Δ is a scaling dimension of the form $\Delta = \frac{d}{2} + is$ with $s \in \mathbb{R}$ and an ρ is an irreducible $\mathrm{SO}(d)$ representation. The elements of $\mathcal{E}_{\Delta,\rho}$ are functions on $\mathbb{R}^d$ (more precisely, on the conformal sphere S^d) that transform under $\mathrm{SO}(d+1,1)$ as primary operators with scaling dimension Δ and $\mathrm{SO}(d)$ representation ρ. The inner product between two functions $f^a(x)$ and $g^a(x)$ (where a is an index for ρ) is defined by

$$(f,g) \equiv \int d^d x (f^a(x))^* g^a(x). \tag{6.39}$$

This is positive-definite by construction. It is conformally-invariant because while g transforms with scaling dimension $\Delta = \frac{d}{2} + is$ in ρ of $\mathrm{SO}(d)$, f^* transforms with scaling dimension $\Delta^* = \frac{d}{2} - is$ in ρ^* of $\mathrm{SO}(d)$, and thus the integrand is a scalar of scaling dimension $\Delta + \Delta^* = d$, as required for conformal invariance. The representations $\mathcal{E}_{\Delta,\rho}$ are important because the representations of primary operators that appear in CFTs are their analytic continuations to real Δ.[12] Also, $\mathcal{E}_{\Delta,\rho}$ appear in partial wave analysis of Euclidean correlators [65].

The pair (Δ, ρ) can be thought of as a weight of the algebra $\mathfrak{so}_{\mathbb{C}}(d+2)$ if we define $-\Delta$ to be the length of the first row of a Young diagram, and use the Young diagram of ρ for the remaining rows. Through this identification, the unitary representations of $\mathrm{SO}(d+2)$ have non-positive (half-)integer Δ. For $\mathrm{SO}(d+1,1)$, we instead have continuous Δ because the corresponding Cartan generator $D \propto M_{-1,d+1}$ of $\mathrm{SO}(d+1,1)$ is noncompact (i.e., it must be multiplied by i in order to relate the Lie algebra $\mathfrak{so}(d+1,1)$ to the compact form $\mathfrak{so}(d+2)$).

In $\mathrm{SO}(d,2)$ there are two noncompact Cartan generators (D and M_{01}), and both of their weights become continuous. Thus, the unitary principal series representations $\mathcal{P}_{\Delta,J,\lambda}$ for $\mathrm{SO}(d,2)$ are parametrized by a triplet (Δ, J, λ), where $\Delta \in \frac{d}{2} + i\mathbb{R}$, $J \in -\frac{d-2}{2} + i\mathbb{R}$ and λ is an irrep of $\mathrm{SO}(d-2)$. Here the pair (J, λ) can be thought of as a weight of $\mathrm{SO}(d)$, where J is the component corresponding to the length of the first row of a Young diagram. In this sense we have a continuous-spin generalization of $\mathrm{SO}(d)$ irreps.

To make sense of functions with continuous spin, we follow the logic described in the introduction. Let us first review the case of integer spin, and take λ to be trivial for simplicity. The elements of integer spin representations are tensors that are traceless and symmetric in their indices

$$f^{\mu_1\cdots\mu_J}(x). \tag{6.40}$$

We can always contract f with a null polarization vector z^μ to obtain a homogeneous polynomial of degree J in z,

$$f(x,z) \equiv f^{\mu_1\cdots\mu_J}(x) z_{\mu_1}\cdots z_{\mu_J}. \tag{6.41}$$

The tensor $f^{\mu_1\cdots\mu_J}(x)$ can be recovered from $f(x,z)$ via

$$f_{\mu_1\cdots\mu_J}(x) = \frac{1}{J!(\frac{d-2}{2})_J} D_{\mu_1}\cdots D_{\mu_J} f(x,z), \tag{6.42}$$

where

$$D_\mu = \left(\frac{d-2}{2} + z\cdot\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z^\mu} - \frac{1}{2} z_\mu \frac{\partial^2}{\partial z^2} \tag{6.43}$$

[12] It will not be important to give a precise meaning to this "analytic continuation"; in most of the discussion we only use $\mathcal{E}_{\Delta,\rho}$ as a guide for writing conformally-invariant formulas. The same remark concerns representations of $\widetilde{\mathrm{SO}}(d,2)$ below.

is the Thomas/Todorov operator [181, 182, 201]. Thus, the two ways (6.40) and (6.41) of representing f are equivalent.

The generalization to continuous spin is now as stated in the introduction: we can consider functions $f(x,z)$ that are homogeneous of degree J in z, where J is no longer an integer and $f(x,z)$ is no longer a polynomial in z. More precisely, the elements of $\mathcal{P}_{\Delta,J}$ are functions $f(x,z)$ with $x \in \mathcal{M}_d^c$ and $z \in \mathbb{R}_+^{d-1,1}$ a future-pointing null vector that are constrained to satisfy

$$f(x, \alpha z) = \alpha^J f(x,z), \quad \alpha > 0. \tag{6.44}$$

The object $f(x,z)$ transforms under conformal transformations in the same way as functions of the form (6.41) would. The operation of recovering the underlying tensor (6.42) only makes sense when J is a nonnegative integer.[13]

To describe representations $\mathcal{P}_{\Delta,J,\lambda}$ with non-trivial λ, we can make use of an analogy between the space of polarization vectors z and the embedding space. The embedding space lets us lift functions on $\mathbb{R}^d$ with indices for an $\mathrm{SO}(d)$ representation to functions on the null cone in $d+2$ dimensions with indices for an $\mathrm{SO}(d+1,1)$ representation. In the present case, λ is a representation of $\mathrm{SO}(d-2)$, so we can lift it to a representation of $\mathrm{SO}(d-1,1)$ defined on the null cone $z^2 = 0$ in a similar way. For example, if λ is a rank-k tensor representation of $\mathrm{SO}(d-2)$, then we consider functions

$$f^{a_1 \dots a_k}(x,z), \tag{6.45}$$

with a_i being $\mathrm{SO}(d-1,1)$-indices, where f obeys gauge redundancies and transverseness constraints [53]

$$f^{a_1\dots a_k}(x,z) \sim f^{a_1\dots a_k}(x,z) + z^{a_i} h^{a_1\dots a_{i-1}a_{i+1}\dots a_k}(x,z), \tag{6.46}$$

$$z_{a_i} f^{a_1\dots a_k}(x,z) = 0. \tag{6.47}$$

Additionally, f should be homogeneous (6.44) and satisfy the same tracelessness and symmetry conditions in a_i as λ-tensors of $\mathrm{SO}(d-2)$.[14] Other types of representations

[13] Also, $f(x,z)$ should satisfy a differential equation in z. This differential equation is conformally invariant and is essentially a generalization of the $(d-2)$-dimensional conformal Killing equation, similarly to the equations discussed in [3]. Such equations only exist for nonnegative integer J and express the fact that $f(x,z)$ is actually polynomial in z.

[14] To make more direct contact with integer spin, instead of (6.46) one can use

$$D_{a_i} f^{a_1\dots a_k}(x,z) = 0, \tag{6.48}$$

where D is the Todorov operator acting on z. In this case, for integer spin tensors the function $f^{a_1\dots a_k}(x,z)$ is given simply by contracting z_μ with the first-row indices of the tensor.

can be described by adapting other embedding space formalisms. In most of this chapter we focus on trivial λ for simplicity.

We can define an inner product for Lorentzian principal series representations by

$$(f,g) \equiv \int d^d x D^{d-2} z f^*(x,z) g(x,z), \tag{6.49}$$

$$D^{d-2} z \equiv \frac{d^d z \theta(z^0)\delta(z^2)}{\mathrm{vol}\,\mathbb{R}_+}. \tag{6.50}$$

Here the integral over z replaces the index contraction that we would use for integer J. The measure for z is manifestly Lorentz-invariant and supported on the null cone. Together with the measure, the integrand is invariant under rescaling of z. Thus, we obtain a finite result by dividing by the volume of the group of positive rescalings, $\mathrm{vol}\,\mathbb{R}_+$. The z-integral is exactly the kind of integral considered in [54] in the context of the embedding space formalism. Here, we have adapted it to describe $\mathrm{SO}(d-1,1)$-invariant integration on the null cone $z^2 = 0$.

In section 6.2.3 we will use analytic continuations of $\mathcal{P}_{\Delta,J,\lambda}$ to find interesting relations for primary operators in Lorentzian CFTs. But before we can do this, we should note that these representations are constructed on $\mathcal{M}_d^c$, which is unsatisfactory from the physical point of view. We can construct similar representations of $\widetilde{\mathrm{SO}}(d,2)$ consisting of functions on $\widetilde{\mathcal{M}}_d$, which we call $\widetilde{\mathcal{P}}_{\Delta,J,\lambda}$. These representations behave very similarly to $\mathcal{P}_{\Delta,J,\lambda}$ but there is an important distinction. While the representations $\mathcal{P}_{\Delta,J,\lambda}$ are generically irreducible, their analogues $\widetilde{\mathcal{P}}_{\Delta,J,\lambda}$ are not. Indeed, the action of $\mathcal{T}$ on $\widetilde{\mathcal{M}}_d$ commutes with the action of $\widetilde{\mathrm{SO}}(d,2)$ and thus $\widetilde{\mathcal{P}}_{\Delta,J,\lambda}$ decompose into a direct integral of irreducible subrepresentations in which $\mathcal{T}$ acts by a constant phase.

6.2.3 Weyl reflections and integral transforms

Given the principal series representations described in section 6.2.2, we can ask whether there exist equivalences between them. Equivalent representations must have the same eigenvalues of the Casimir operators,[15] and these eigenvalues are polynomials in the weights (Δ,ρ) (for $\mathrm{SO}(d+1,1)$) and (Δ,J,λ) (for $\mathrm{SO}(d,2)$). For example, the quadratic and quartic Casimir eigenvalues for $\mathcal{P}_{\Delta,J}$ (with trivial λ) are

$$\begin{aligned} C_2(\mathcal{P}_{\Delta,J}) &= \Delta(\Delta-d) + J(J+d-2), \\ C_4(\mathcal{P}_{\Delta,J}) &= (\Delta-1)(d-\Delta-1)J(2-d-J). \end{aligned} \tag{6.51}$$

[15] Here we mean all Casimir operators, not just the quadratic Casimir.

The "restricted Weyl group" W' is a finite group that acts on these weights, doesn't mix discrete and continuous labels, and leaves the Casimir eigenvalues invariant. Conversely, if two principal series weights have the same Casimirs, they can be related by an element of W'.

For example, in the case of $\mathrm{SO}(d+1,1)$, the restricted Weyl group is $W' = \mathbb{Z}_2$. Its non-trivial element $\mathrm{S}_E \in W'$ acts by

$$\mathrm{S}_E(\Delta, \rho) = (d - \Delta, \rho^R), \tag{6.52}$$

where ρ^R is the reflection of ρ. Other transformations exist that leave all Casimir eigenvalues invariant, but S_E is the only one that does not mix the integral weights of ρ with the continuous weight Δ.

In the case of $\mathrm{SO}(d,2)$, there are two continuous parameters that can mix, and thus the restricted Weyl group W' is larger. It is isomorphic to a dihedral group of order 8, $W' = \mathrm{D}_8$.[16] This group has a faithful representation on $\mathbb{R}^2$ where it acts as symmetries of the square. Its action on $\Delta = \frac{d}{2} + is$ and $J = -\frac{d-2}{2} + iq$ can be described by taking s and q to be Cartesian coordinates in this $\mathbb{R}^2$. It is easy to see that this action preserves the eigenvalues (6.51). Altogether, the elements of W' are given in table 6.1.[17]

As mentioned above, the representations defined by weights in an orbit of W' have equal Casimir eigenvalues, which means that potentially they can be equivalent. This indeed turns out to be true [70, 71]. Equivalence of representations means that there exist intertwining maps between $\mathcal{E}_{(\Delta,\rho)}$ and $\mathcal{E}_{w(\Delta,\rho)}$, as well as between $\mathcal{P}_{(\Delta,J,\lambda)}$ and $\mathcal{P}_{w(\Delta,J,\lambda)}$ for all $w \in W'$.

The intertwining map between $\mathrm{SO}(d+1,1)$ representations $\mathcal{E}_{\Delta,\rho}$ and $\mathcal{E}_{d-\Delta,\rho^R}$ is well-known [54, 65, 233]: it is given by the so-called shadow transform

$$\widetilde{\mathcal{O}}^a(x) = \mathbf{S}_E[\mathcal{O}]^a(x') \equiv \int d^d x' \langle \widetilde{\mathcal{O}}^a(x) \widetilde{\mathcal{O}}^\dagger_b(x') \rangle \mathcal{O}^b(x'). \tag{6.53}$$

Here $\widetilde{\mathcal{O}} \in \mathcal{E}_{d-\Delta,\rho^R}$, $\mathcal{O} \in \mathcal{E}_{\Delta,\rho}$, we use dagger to denote taking the dual reflected representation of $\mathrm{SO}(d)$, and $\langle \widetilde{\mathcal{O}}^a(x) \widetilde{\mathcal{O}}^\dagger_b(x') \rangle$ is a standard choice of two-point

[16] This also turns out to be the Weyl group of BC_2 root system, which was recently studied in the context of conformal blocks in [191, 242]. It would be interesting to better understand the connection of the present discussion with that work.

[17] To check that the action on λ is as in the table, one can consider the 4d case. The eigenvalues of all 3 Casimirs of $\widetilde{\mathrm{SO}}(2,4)$ are written out, for example, in appendix F of [2] with $\ell = J + \lambda$, $\bar{\ell} = J - \lambda$ and $\lambda^R = -\lambda$. More generally, by solving the system of polynomial equations expressing invariance of these explicit Casimir eigenvalues, one can check that W' is indeed isomorphic to D_8.

w		order	Δ'	J'	λ'
1		1	Δ	J	λ
S_Δ	$= \mathrm{LS}_J\mathrm{L}$	2	$d-\Delta$	J	λ^R
S_J		2	Δ	$2-d-J$	λ^R
S	$= (\mathrm{S}_J\mathrm{L})^2$	2	$d-\Delta$	$2-d-J$	λ
L		2	$1-J$	$1-\Delta$	λ
F	$= \mathrm{S}_J\mathrm{LS}_J$	2	$J+d-1$	$\Delta-d+1$	λ
R	$= \mathrm{S}_J\mathrm{L}$	4	$1-J$	$\Delta-d+1$	λ^R
$\overline{\mathrm{R}}$	$= \mathrm{LS}_J$	4	$J+d-1$	$1-\Delta$	λ^R

Table 6.1: The elements of the restricted Weyl group $W' = \mathrm{D}_8$ of $\mathrm{SO}(d,2)$. Each element w takes the weights (Δ, J, λ) to (Δ', J', λ'). The order 2 elements other than S are the four reflection symmetries of the rectangle, while S is the rotation by π. The center of the group is $Z\mathrm{D}_8 = \{1, \mathrm{S}\}$. Finally, the element R is a $\pi/2$ rotation. The group is generated by L and S_J, with the relations $\mathrm{L}^2 = \mathrm{S}_J^2 = (\mathrm{LS}_J)^4 = 1$.

function for the operators in their respective representations. The integration region is the full $\mathbb{R}^d$ (more precisely, the conformal sphere S^d).

According to our discussion above, in Lorentzian signature there should exist 6 new integral transforms, corresponding to the other non-trivial elements of W'. There in fact exists a general formula for these transforms, valid for any element of W' [70, 71].[18] However, it is most naturally written using a different construction of $\mathcal{P}_{\Delta,J,\lambda}$, and the conversion to the form appropriate for our purposes is cumbersome.[19] Thus instead of deriving these transforms from the general result we will simply give the final expressions and check that they are indeed conformally-invariant. Furthermore, we will lift these transforms to representations $\widetilde{\mathcal{P}}_{\Delta,J,\lambda}$ of $\widetilde{\mathrm{SO}}(d,2)$.

Although the Lorentzian transforms we define are only necessarily isomorphisms when acting on principal series representations $\mathcal{P}_{\Delta,J,\lambda}$, it is still interesting to consider the analytic continuation of their action on other representations, like those associated to physical CFT operators. For example the action of L will be well-defined on physical local operators. The result of this action will generically be a primary operator with non-integer spin. One can then ask how such operators make sense in a CFT and what properties do they have. In this and the following sections we will be able to answer these question by studying the examples provided by integral transforms. In appendix E.1 we study the same questions on more general

[18] In the mathematical literature, these transforms are known as Knapp-Stein intertwining operators.

[19] See [65] for an example of this conversion in the case of the shadow transform (6.53).

grounds (by using unitarity, positivity of energy, and conformal symmetry) and reach similar conclusions.

6.2.3.1 Transforms for $\mathrm{S}_\Delta, \mathrm{S}_J, \mathrm{S}$

Let us start with the Lorentzian analogue of (6.53). The idea is to essentially keep the form (6.53) while generalizing to continuous spin,

$$\mathbf{S}_\Delta[\mathcal{O}](x,z) \equiv i \int_{x'\approx x} d^d x' \frac{1}{(x-x')^{2(d-\Delta)}} \mathcal{O}(x', I(x-x')z), \tag{6.54}$$

$$I^\mu_\nu(x) = \delta^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2}. \tag{6.55}$$

The integrand is conformally-invariant because $I(x-x')$ performs a conformally-invariant translation of a vector at x to a vector at x'. The factor of i is to match a Wick-rotated version of the Euclidean shadow transform, although we still have $\mathbf{S}_E = (-2)^J \mathbf{S}_\Delta$ after Wick rotation because of our convention for two-point functions (E.24).

We must specify a conformally-invariant integration region for x'. The essentially unique choice is to integrate over the region spacelike separated from x. If x is at spatial infinity of $\mathcal{M}_d$, then this region is the full Poincare patch $\mathcal{M}_d \subset \widetilde{\mathcal{M}}_d$, and for integer J the integral is simply the Wick rotation of the Euclidean shadow integral (6.53). If, however, x is inside the first Poincare patch, then the integral extends beyond the first Poincare patch on the Lorentzian cylinder $\widetilde{\mathcal{M}}_d$. All other conformally-invariant regions defined by x are translations of the spacelike region by powers of $\mathcal{T}$ or unions thereof. The two-point function in these regions differs from the two-point function in the spacelike region only by a constant phase, and thus the most general choice of $\mathbf{S}_\Delta$ differs from the above by multiplication by a function of $\mathcal{T}$.[20] The possibility of multiplying by a function of $\mathcal{T}$ is present for all the transforms we consider and we just make the simplest choice. The choice (6.54) is natural because of its relation to (6.53).

For S_J, the integral transform is

$$\mathbf{S}_J[\mathcal{O}](x,z) \equiv \int D^{d-2}z' (-2\, z\cdot z')^{2-d-J} \mathcal{O}(x,z'), \tag{6.56}$$

where the measure $D^{d-2}z$ is defined in (6.50). We call this the "spin shadow transform." Note that this is essentially the same as the shadow transform in the embedding space [54], with X replaced by z and d replaced by $d-2$.

[20] In particular, there is no ambiguity in representations $\mathcal{P}_{\Delta,J,\lambda}$ of $\mathrm{SO}(d,2)$.

The transform for S, which we call the "full shadow transform," is simply the composition of the commuting transforms for S_Δ and S_J,

$$\begin{aligned}\mathbf{S}[O](x,z) \equiv (\mathbf{S}_J\mathbf{S}_\Delta)[O](x,z) &= i\int_{x'\approx x} d^dx' D^{d-2}z' \frac{(-2\,z\cdot z')^{2-d-J}}{(x-x')^{2(d-\Delta)}} O(x', I(x-x')z') \\ &= (\mathbf{S}_\Delta\mathbf{S}_J)[O](x,z) = i\int_{x'\approx x} d^dx' D^{d-2}z' \frac{(-2\,z\cdot I(x-x')z')^{2-d-J}}{(x-x')^{2(d-\Delta)}} O(x',z').\end{aligned} \tag{6.57}$$

These two forms of $\mathbf{S}$ are equivalent because $I(x-x')^2 = 1$, for spacelike $x-x'$ $I(x-x')$ is an element of the orthochronous Lorentz group $\mathrm{O}^+(d-1,1)$, and the measure of the z-integration is invariant under $\mathrm{O}^+(d-1,1)$.

The second line of (6.57) can also be written as

$$\mathbf{S}[O](x,z) = i\int_{x'\approx x} d^dx' D^{d-2}z' \langle O^{\mathrm{S}}(x,z) O^{\mathrm{S}}(x',z')\rangle O(x',z'), \tag{6.58}$$

where O^{S} denotes the representation with dimension $d-\Delta$ and spin $2-d-J$. Here, we are using the following convention for a two-point structure

$$\langle O(x_1,z_1) O(x_2,z_2)\rangle = \frac{(-2z_1\cdot I(x_{12})\cdot z_2)^J}{x_{12}^{2\Delta}}, \tag{6.59}$$

which differs by a factor of $(-2)^J$ from some more traditional conventions. Our conventions for two- and three-point structures are summarized in appendix E.1.3

6.2.3.2 Transform for L

The integral transform corresponding to L is

$$\mathbf{L}[O](x,z) = \int_{-\infty}^{+\infty} d\alpha\, (-\alpha)^{-\Delta-J} O\left(x - \frac{z}{\alpha}, z\right). \tag{6.60}$$

Because it involves integration along a null direction, we call $\mathbf{L}$ the "light transform." Although most of the transforms in this section are only well-defined on nonphysical representations like Lorentzian principal series representations, the light transform is significant because it can be applied to physical operators as well. Note that it converges near $\alpha = \pm\infty$ only for $\Delta + J > 1$.[21] In unitary theories it can therefore be applied to all non-scalar operators and to scalars with dimension $\Delta > 1$ (which includes all non-trivial scalars in $d \geq 4$).

[21] For Lorentzian principal series $\mathrm{Re}(\Delta + J) = 1$ but for non-zero $\mathrm{Im}(\Delta + J)$ the integral still makes sense.

Before discussing conformal invariance, let us describe the contour of integration in more detail. The integral starts at $\alpha = -\infty$, in which case the argument of $\mathcal{O}$ is simply x. It then increases to $\alpha = -0$, and in the process $\mathcal{O}$ moves along z to future null infinity in $\mathcal{M}_d$. As α crosses 0, the integration contour leaves the first Poincare patch $\mathcal{M}_d$ and enters the second Poincare patch $\mathcal{T}\mathcal{M}_d \subset \widetilde{\mathcal{M}}_d$. Finally, at $\alpha = +\infty$ it ends at $\mathcal{T}x \in \mathcal{T}\mathcal{M}_d$. In other words, the integration contour is a null geodesic in $\widetilde{\mathcal{M}}_d$ from x to $\mathcal{T}x$ with direction defined by z (figure 6.4). This is obviously a conformally-invariant contour.

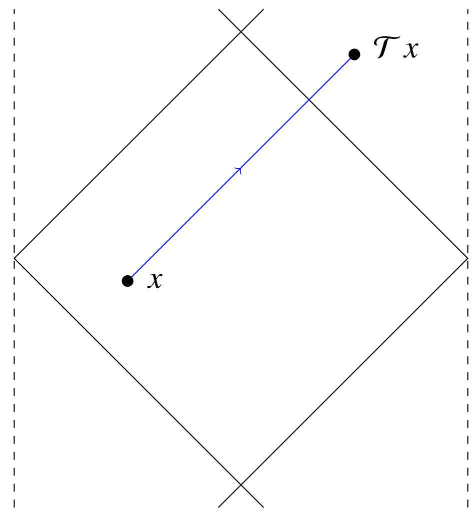


Figure 6.4: The contour prescription for the light-transform. The contour starts at $x \in \mathcal{M}_d$ and moves along the z direction to the point $x^+ = \mathcal{T}x$ in the next Poincare patch $\mathcal{T}\mathcal{M}_d$.

It turns out that no phase prescription is necessary to define $(-\alpha)^{-\Delta-J}$ for $\alpha > 0$, because the naive singularity at $\alpha = 0$ is cancelled in correlators of $\mathcal{O}$. To see this, note that (6.60) is equivalent to the following integral in the embedding formalism of [53],

$$\begin{aligned}\mathbf{L}[\mathcal{O}](X,Z) &= \int_{-\infty}^{+\infty} d\alpha\, (-\alpha)^{-\Delta-J}\mathcal{O}\left(X - \frac{Z}{\alpha}, Z\right)\\ &= \int_{-\infty}^{+\infty} d\alpha\, \mathcal{O}(Z-\alpha X, X),\end{aligned} \tag{6.61}$$

where in the second equality we used the homogeneity properties of $\mathcal{O}(X,Z)$ in the region $\alpha < 0$, together with gauge invariance $\mathcal{O}(X, Z+\beta X) = \mathcal{O}(X,Z)$. In (6.61) it is clear that the point $\alpha = 0$ is not special (see also appendix E.2.1 for yet another explanation).

The embedding space integral (6.61) makes conformal invariance of the light-transform manifest: it is $\mathrm{SO}(d, 2)$ invariant, and gauge invariance

$$\mathbf{L}[\mathcal{O}](X, Z + \beta X) = \mathbf{L}[\mathcal{O}](X, Z) \tag{6.62}$$

can be proved by shifting α by β in the integral. It is also clear from homogeneity in X and Z that the dimension and spin of $\mathbf{L}[\mathcal{O}](X, Z)$ are $1 - J$ and $1 - \Delta$, respectively. (Note that the parameter α carries homogeneity 1 in Z and -1 in X.) Finally, (6.61) confirms the prescription that the integral goes between x and $\mathcal{T} x$. Indeed, according to the discussion in section 6.2.1 the embedding space covers two Poincare patches and $\mathcal{T} X$ is simply $-X$. The integral in (6.61) starts at the argument $Z + \infty X$ which is the same as X modulo $\mathbb{R}_+$ and ends at $Z - \infty X$ which is $-X = \mathcal{T} X$ modulo $\mathbb{R}_+$.

Let us describe another way of writing $\mathbf{L}$ that will be useful. Equation (6.60) expresses $\mathbf{L}$ in a conformal frame where x is in the interior of a Poincare patch. In this case, the integration contour extends from one patch into the next. However, if we place x at past null infinity, the integration contour fits entirely within a single Poincare patch. Specifically, in the integral (6.61), let us set[22]

$$\begin{aligned} Z &= (1, y^2, y), \\ X &= (0, -2y \cdot z, -z) \end{aligned} \tag{6.63}$$

to obtain

$$\mathbf{L}[\mathcal{O}](x, z) = \int_{-\infty}^{\infty} d\alpha \, \mathcal{O}(y + \alpha z, z). \tag{6.64}$$

Here, $x = y - \infty z$. Equation (6.64) is simply the integral of $\mathcal{O}$ along a null ray from past null infinity to future null infinity, contracted with a tangent vector to the ray. As an example, the "average null energy" operator is given by

$$\mathcal{E} = \int_{-\infty}^{\infty} d\alpha T_{\mu\nu}(\alpha z) z^\mu z^\nu = \mathbf{L}[T](-\infty z, z), \tag{6.65}$$

where $T_{\mu\nu}$ is the stress tensor. It follows from our discussion that $\mathcal{E}$ transforms like a primary with dimension -1 and spin $1 - d$, centered at $-\infty z$.

[22] This choice reverses the role of X, Z relative to the usual Poincare section gauge fixing. However, it still satisfies the required conditions $X^2 = Z^2 = X \cdot Z = 0$. To obtain these expressions, consider the usual Poicare coordinates for a point shifted by $-Lz$ for large L,

$$\begin{aligned} X &= (1, (x - Lz)^2, x - Lz) \simeq L \times (0, -2x \cdot z, -z), \\ Z &= (0, 2z \cdot x, z) = L^{-1} \times \left((1, x^2, x) - X\right), \end{aligned}$$

from where the new gauge-fixing follows.

6.2.3.3 Transforms for F, R, $\overline{\mathrm{R}}$

The transforms for the remaining elements $\mathrm{F}, \mathrm{R}, \overline{\mathrm{R}} \in \mathrm{D}_8$ are compositions

$$\begin{aligned}
\mathbf{F} &\equiv \mathbf{S}_J \mathbf{L} \mathbf{S}_J, \\
\mathbf{R} &\equiv \mathbf{S}_J \mathbf{L}, \\
\overline{\mathbf{R}} &\equiv \mathbf{L} \mathbf{S}_J.
\end{aligned} \tag{6.66}$$

For example,

$$\begin{aligned}
\mathbf{F}[\mathcal{O}](x,z) \equiv &\int d^d\zeta D^{d-2}z' \delta(\zeta^2)\theta(\zeta^0)(-2\,\zeta\cdot z')^{-J-d+2}(-2\,\zeta\cdot z)^{\Delta-d+1}\mathcal{O}(x+\zeta,z') \\
&+ \int d^d\zeta D^{d-2}z' \delta(\zeta^2)\theta(\zeta^0)(-2\,\zeta\cdot z')^{-J-d+2}(-2\,\zeta\cdot z)^{\Delta-d+1}(\mathcal{T}\mathcal{O})(x-\zeta,z').
\end{aligned} \tag{6.67}$$

Note that here the second term involves an integral over the second Poincare patch $\mathcal{T}\mathcal{M}_d$. Similarly to the light transform, here we integrate over all future-directed null geodesics from x to $\mathcal{T}x$. Because we integrate over all null directions, we call $\mathbf{F}$ the "floodlight transform."

Similarly, we have

$$\begin{aligned}
\mathbf{R}[\mathcal{O}](x,z) = &\int d^d\zeta \delta(\zeta^2)\theta(\zeta^0)(-2z\cdot\zeta)^{1-d+\Delta}\mathcal{O}(x+\zeta,\zeta) \\
&+ \int d^d\zeta \delta(\zeta^2)\theta(\zeta^0)(-2z\cdot\zeta)^{1-d+\Delta}(\mathcal{T}\mathcal{O})(x-\zeta,\zeta),
\end{aligned} \tag{6.68}$$

$$\overline{\mathbf{R}}[\mathcal{O}](x,z) = \int d\alpha D^{d-2}z'(-\alpha)^{-\Delta-2+d+J}(-2\,z\cdot z')^{2-d-J}\mathcal{O}\left(x-\frac{z}{\alpha},z'\right). \tag{6.69}$$

As an example, $\mathbf{R}[T] = \mathbf{S}_J[\mathbf{L}[T]]$ is given by integrating the average null energy operator $\mathcal{E} = \mathbf{L}[T]$ over null directions. This is equivalent to integrating the stress tensor over a complete null surface, which produces a conformal charge. We can understand this more formally as follows. Note that the dimension and spin of $\mathbf{R}[T]$ are given by

$$\mathrm{R}(d,2) = (-1,1). \tag{6.70}$$

These are exactly the weights of the adjoint representation of the conformal group. Conservation of $T^{\mu\nu}$ ensures that $\mathbf{R}[T]$ transforms irreducibly, so that it transforms precisely in the adjoint representation. In other words, conservation equation for T

becomes the conformal Killing equation for $\mathbf{R}[T]$. It can thus be written as a linear combination of conformal Killing vectors (CKVs):[23]

$$\begin{aligned}\mathbf{R}[T](x,z) &= Q^A w_A^\mu(x) z_\mu \\ &= K\cdot z - 2(x\cdot z)D + (x_\rho z_\nu - x_\nu z_\rho)M^{\nu\rho} + 2(x\cdot z)(x\cdot P) - x^2(z\cdot P).\end{aligned} \tag{6.71}$$

Here, A is an index for the adjoint representation of the conformal group, $w_A^\mu(x)$ are CKVs, and the Q^A are the associated charges. On the second line, we've given the charges their usual names. We can see from (6.71) that inserting $\mathbf{R}[T]$ at spatial infinity $x = \infty$ gives the momentum charge. This is a familiar fact from "conformal collider physics" [76]. Similarly, when J is a conserved spin-1 current, $\mathbf{R}[J]$ has dimension-0 and spin-0, which are the correct quantum numbers for a conserved charge.

6.2.4 Some properties of the light transform

As noted above, the light transform of the stress-energy tensor is the average null energy operator $\mathbf{L}[T] = \mathcal{E}$. The average null energy condition (ANEC) states that $\mathcal{E}$ is non-negative,

$$\langle\Psi|\mathcal{E}|\Psi\rangle \geq 0. \tag{6.72}$$

Non-negative operators with vanishing vacuum expectation value $\langle\Omega|\mathcal{E}|\Omega\rangle = 0$ must necessarily annihilate the vacuum $|\Omega\rangle$ [243].[24,25] Indeed, using the Cauchy-Schwarz inequality for the inner product defined by $\mathcal{E}$, we find

$$|\langle\Psi|\mathcal{E}|\Omega\rangle|^2 \leq \langle\Psi|\mathcal{E}|\Psi\rangle\langle\Omega|\mathcal{E}|\Omega\rangle = 0 \tag{6.73}$$

for any state $|\Psi\rangle$. Thus $\mathcal{E}|\Omega\rangle = 0$.

In fact, we know that $\mathbf{L}[\mathcal{O}]|\Omega\rangle = 0$ for any local primary operator $\mathcal{O}$ — not just the stress tensor. Indeed, if $\mathcal{O}$ has scaling dimension Δ, then $\mathbf{L}[\mathcal{O}]$ has spin $1-\Delta$, which in a unitary theory is a non-negative integer only if $\Delta = 0$ or $\Delta = 1$. However, in these cases $J = 0$ and the light transform diverges. For all other scaling dimensions $\mathbf{L}[\mathcal{O}]$ is a continuous-spin operator and thus must annihilate the vacuum. This

[23] See [3] for more discussion of writing finite-dimensional representations of the conformal group in terms of fields on spacetime.

[24] We thank Clay Córdova for discussion on this point.

[25] Intuitively, the vacuum must contain the same amount of positive-$\mathcal{E}$ states and negative-$\mathcal{E}$ states in order for $\langle\Omega|\mathcal{E}|\Omega\rangle$ to vanish. Since there are no negative-$\mathcal{E}$ states, the vacuum only contains vanishing-$\mathcal{E}$ states and is thus annihilated by $\mathcal{E}$.

makes it possible for other null positivity conditions (like those proved in [73] and section 6.6) to hold as well. In the rest of this subsection we check explicitly that $\mathbf{L}[\mathcal{O}]|\Omega\rangle = 0$ for all $\Delta + J > 1$ and make some general comments about properties of $\mathbf{L}$.

Lemma 1. *The light transform of a local primary operator, when exists (i.e $\Delta + J > 1$), annihilates the vacuum,* [26]

$$\mathbf{L}[\mathcal{O}]|\Omega\rangle = 0. \tag{6.74}$$

Proof. We will show that for any local operators V_i,

$$\langle\Omega|V_n(x_n)\cdots V_1(x_1)\mathbf{L}[\mathcal{O}](y,z)|\Omega\rangle = 0, \tag{6.75}$$

which implies the result. Let us work in a Poincare patch where y is at past null infinity and for simplicity assume that the x_i fit in this patch; other configurations can be obtained by analytic continuation. Using a Lorentz transformation we can set $z = (1, 1, 0, \dots, 0)$ and parameterize the light transform contour as $x_0 = \left(\frac{v-u}{2}, \frac{v+u}{2}, 0, 0, \dots\right)$ for $v \in (-\infty, \infty)$. We are then computing

$$\begin{aligned}
&\int_{-\infty}^{\infty} dv \langle\Omega|V_n(x_n)\cdots V_1(x_1)\mathcal{O}(x_0,z)|\Omega\rangle = \\
&= \lim_{\epsilon\to+0}\int_{-\infty}^{\infty} dv \langle\Omega|V_n(x_n - in\epsilon\hat{e}_0)\cdots V_1(x_1 - i\epsilon\hat{e}_0)\mathcal{O}(x_0,z)|\Omega\rangle,
\end{aligned} \tag{6.76}$$

where $\hat{e}_0$ is the future-pointing unit vector in the time direction. The above $i\epsilon$ prescription arranges the operators so that they are time-ordered in Euclidean time, and this is precisely how the Wightman function should be defined as a distribution. Let us now write

$$x_k - ik\epsilon\hat{e}_0 = y_k + i\zeta_k, \quad k = 0, 1, \dots n, \tag{6.77}$$

where both y_k and ζ_k are real vectors. Positivity of energy implies that Wightman functions are analytic if ζ_k is in the absolute future of ζ_{k+1} for all k [16]:[27]

$$\zeta_0 > \zeta_1 > \cdots > \zeta_n. \tag{6.78}$$

This condition clearly holds when the x_k are real. If we then give an arbitrary positive imaginary part to v while keeping u and other components of x_0 fixed,

[26] For general spin representations J must be replaced by the sum of all Dynkin labels with spinor labels taken with weight $\frac{1}{2}$.

[27] For example, it is easy to check that under this condition $(y_{ik} + i\zeta_{ik})^2 \neq 0$ for all y_{ik}, and thus there are no obvious null cone singularities. More generally, see appendix E.1.

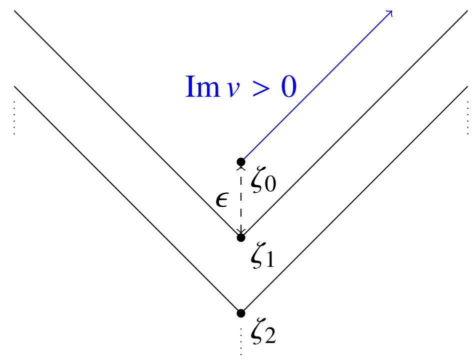


Figure 6.5: Relationships between the imaginary parts ζ_k. A deformation of v in the positive imaginary direction is shown in blue.

$\zeta_0 = \mathrm{Im}(v)z$ will remain in the future of $\zeta_1 = -\epsilon\hat{e}_0$ (see figure 6.5). Therefore, the integrand is an analytic function of v in the upper half plane. If we can close the v contour in the upper half plane, that would imply the required result.

According to the discussion around (6.60), conformal invariance implies that the integral (6.60) is regular as $\alpha \to -0$, which in turn implies that the integrand of (6.76) decays as $|v|^{-\Delta-J}$ for real v. We will now show that this is also true for complex v in the upper half-plane, so we can close the contour as long as $\Delta + J > 1$.

To compute the rate of decay in v, we can use the OPE for the operators V_i, which converges acting on the left vacuum.[28] The leading contribution at large v will be from $\mathcal{O}$ in this OPE, leading to a two-point function of $\mathcal{O}$. Because v is moving in the direction of its polarization z, the decay of this two-point function is governed not by Δ but by $\Delta + J$. Indeed, we need to consider the two-point function

$$\langle \mathcal{O}(0, z')\mathcal{O}(u, v; z)\rangle. \tag{6.79}$$

The problem is then essentially two-dimensional: the statement that v is along z means that $\mathcal{O}$ has definite left and right-moving weights of the 2d conformal subgroup. Invariance under the 2d conformal subgroup then selects the component of z' with the same weights, so the two-point function is proportional to

$$\langle \mathcal{O}(0, z')\mathcal{O}(u, v; z)\rangle \propto \frac{(z'^1 - z'^0)^J}{u^{\Delta-J} v^{\Delta+J}}. \tag{6.80}$$

[28]For this argument it is important that $i\epsilon$-prescriptions and positive imaginary part of v smear the operators so that we are working with normalizable states. An argument from the Euclidean OPE is that the $i\epsilon$ shifts separate the operators on the Euclidean cylinder, and Lorentzian times do not affect convergence of the OPE. The operators in the right hand side of the OPE can be placed anywhere in Euclidean future of $\mathcal{O}$. Alternatively to (but not logically independently from) the OPE argument, we could have just started with $\langle\Omega|\mathcal{O}\mathbf{L}[\mathcal{O}]|\Omega\rangle$ in the first place, since states of the form $\int d^d x f(x)\langle\Omega|\mathcal{O}(x)$ are dense in the space of states which can have a non-zero overlap with $\mathbf{L}[\mathcal{O}]|\Omega\rangle$.

Let us see this explicitly in the case of traceless-symmetric tensor $\mathcal{O}$,

$$\langle \mathcal{O}(0,z')\mathcal{O}(u,v;z)\rangle \propto \frac{(z'_\mu I^{\mu\nu}(x_0)z_\nu)^J}{(uv)^\Delta}, \tag{6.81}$$

where we have $x_0 = \frac{1}{2}vz + \frac{1}{2}uz^\perp$. Here $z^\perp = (-1,1,0,\ldots)$ is the basis vector for the u coordinate and we have $(z\cdot z^\perp) = 2$. The numerator is then

$$z'_\mu I^{\mu\nu}(x_0)z_\nu = (z'\cdot z) - \frac{2u}{uv}\left(\tfrac{1}{2}(z'\cdot z)v + \tfrac{1}{2}(z'\cdot z^\perp)u\right) = (z'^1 - z'^0)\frac{u}{v}. \tag{6.82}$$

This indeed leads to the expected form (6.80).

In summary, we can close the v contour in the upper half plane to give zero whenever $\Delta + J > 1$. □

Recall that the condition $\Delta + J > 1$ is true for all non-scalar operators in unitary CFTs, and for all non-identity scalar operators in $d \geq 4$ dimensions.

As as simple corollary of lemma 1, light transforms of local operators not acting on the vacuum can be expressed in terms of commutators. For example,

$$\langle\Omega|\mathcal{O}_1\mathbf{L}[\mathcal{O}_3]\mathcal{O}_2|\Omega\rangle = \langle\Omega|[\mathcal{O}_1,\mathbf{L}[\mathcal{O}_3]]\mathcal{O}_2|\Omega\rangle = \langle\Omega|\mathcal{O}_1[\mathbf{L}[\mathcal{O}_3],\mathcal{O}_2]|\Omega\rangle. \tag{6.83}$$

Note that these commutators vanish at spacelike separations, so the integral in the light transforms only receives contributions from timelike separations. More explicitly, we can understand the commutators (6.83) as follows. In the integral

$$\int_{-\infty}^{\infty} d\alpha(-\alpha)^{-\Delta-J}\langle\Omega|\mathcal{O}_1\mathcal{O}_3(x-z/\alpha,z)\mathcal{O}_2|\Omega\rangle, \tag{6.84}$$

there is one singularity in the lower half-plane where 3 becomes lightlike from 1 and another in the upper half-plane where 3 becomes lightlike from 2 (figure 6.6). If we deform the contour to wrap around the first singularity ($3\sim 1$), we obtain the commutator $[\mathcal{O}_1,\mathcal{O}_3]$; if we deform the contour around the second singularity ($3\sim 2$), we obtain $[\mathcal{O}_3,\mathcal{O}_2]$.

Lemma 1 has the following simple consequence for time-ordered correlators:

Lemma 2. *Let $\mathcal{O}$ be a local primary operator with $\Delta + J > 1$. In a time-ordered correlator*

$$\langle V_1\ldots V_n\mathbf{L}[\mathcal{O}]\rangle_\Omega, \tag{6.85}$$

if the integration contour of $\mathbf{L}[\mathcal{O}]$ crosses only past or only future null cones, the transform is zero. Note that on the Lorentzian cylinder, generically, the contour crosses the null cone of each V_i exactly once.

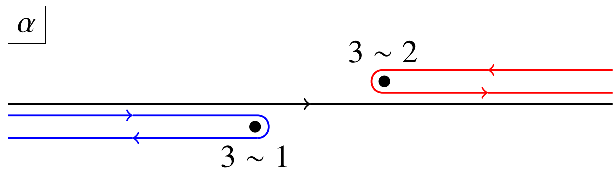


Figure 6.6: Contour prescriptions for the α integral in the light transform of a three-point function (6.83). The black contour corresponds to $\langle\Omega|O_1\mathbf{L}[O_3]O_2|\Omega\rangle$, the blue contour corresponds to $\langle\Omega|[O_1,\mathbf{L}[O_3]]O_2|\Omega\rangle$, and the red contour corresponds to $\langle\Omega|O_1[\mathbf{L}[O_3],O_2]|\Omega\rangle$.

Note that here the notation (6.85) means that $\mathbf{L}$ is applied to a physical time-ordered correlation function, as opposed to time-ordering acting on the continuous spin operator $\mathbf{L}[O]$. (Since continuous spin operators are necessarily non-local, it is unclear how to define the latter time-ordering in a Lorentz-invariant way, see appendix E.1.) We also use the subscript Ω to stress that we mean a physical correlation function, as opposed to a conformally-invariant tensor structure.

Finally, let us note that if we use the usual Wightman $i\epsilon$-prescription,[29] the light transform of a Wightman function is an analytic function of its arguments, including the polarizations. This follows simply from the fact that it is an integral of an analytic function. This is consistent with our statements concerning analyticity of Wightman functions of continuous-spin operators in appendix E.1.

6.2.5 Light transform of a Wightman function

As a concrete example, and because it will play an important role later, let us compute the light-transform of the Wightman function

$$\langle 0|\phi_1(x_1)O(x_3,z)\phi_2(x_2)|0\rangle = \frac{\left(2z\cdot x_{23}\,x_{13}^2 - 2z\cdot x_{13}\,x_{23}^2\right)^J}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}\,x_{13}^{\Delta_1+\Delta-\Delta_2+J}\,x_{23}^{\Delta_2+\Delta-\Delta_1+J}}, \tag{6.86}$$

where ϕ_i are scalar operators with dimensions Δ_i, and O has dimension Δ and spin J. (Our three-point structure normalization differs by a factor of 2^J from some more conventional normalizations. Our conventions are summarized in appendix E.1.3.) In the above expression, the Wightman $i\epsilon$ prescription is implicit. As discussed at the end of the introduction, we use the convention that expectation values in the state $|\Omega\rangle$ denote physical correlation functions, whereas the expectation values in the state $|0\rangle$ denote two- or three-point tensor structures fixed by conformal invariance.

[29] In other words, add small Euclidean times to the operators to make the expectation value time-ordered in Euclidean time.

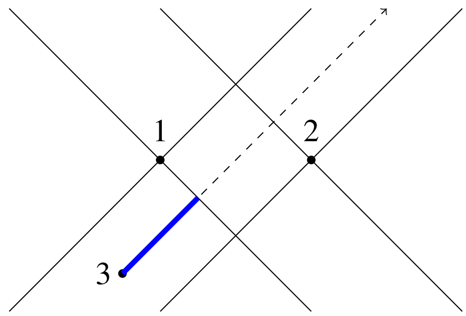


Figure 6.7: Causal relationships between points in the light transform (6.87). The original integration contour is the union of the solid blue line and the dashed line. The solid blue line shows the region where the commutator $[\phi_1, \mathcal{O}]$ is non-zero.

The same comment applies to time-ordered correlation functions $\langle\cdots\rangle_\Omega$ and $\langle\cdots\rangle$ respectively.

Because the light-transform of a local operator annihilates the vacuum (lemma 1), it is equivalent to the commutators

$$\langle 0|\phi_1 \mathbf{L}[\mathcal{O}]\phi_2|0\rangle = \langle 0|\phi_1[\mathbf{L}[\mathcal{O}], \phi_2]|0\rangle = \langle 0|[\phi_1, \mathbf{L}[\mathcal{O}]]\phi_2|0\rangle. \tag{6.87}$$

Specifically, let us compute the third expression above,

$$\begin{aligned}&\langle 0|\left[\phi_1(x_1), \mathbf{L}[\mathcal{O}](x_3, z)\right]\phi_2(x_2)|0\rangle \\ &= \int_{-\infty}^{+\infty} d\alpha(-\alpha)^{-\Delta-J}\langle 0|\left[\phi_1(x_1), \mathcal{O}\left(x_3 - \frac{z}{\alpha}, z\right)\right]\phi_2(x_2)|0\rangle.\end{aligned} \tag{6.88}$$

Since the light transform of a Wightman function is analytic (see section 6.2.4 and appendix E.1), we can compute it for any choice of causal relationships, and obtain the answer for other configurations by analytic continuation. We will work with the configuration in figure 6.7. All points lie in a single Poincare patch. The points 1 and 2 are spacelike separated, and the integration contour starts at $3 < 1$ and ends at $3^+ > 2$. The commutator $[\phi_1, \mathcal{O}]$ vanishes at spacelike separation, so the upper limit of the integral (6.88) gets restricted to the value of α when 3 crosses the past null cone of 1.

In our configuration, we have

$$(z \cdot x_{13}) < 0, \tag{6.89}$$

$$-2\frac{(z\cdot x_{13})}{x_{13}^2} < -2\frac{(z\cdot x_{23})}{x_{23}^2}. \tag{6.90}$$

The first inequality follows because z and x_{13} are future-pointing and x_{13} is not null. The second inequality expresses the fact that the null cone of 1 is crossed before the null cone of 2.

Taking into account that $x_{13}^2 = e^{i\pi}|x_{13}^2|$ for the ordering $\phi_1\mathcal{O}$ and $x_{13}^2 = e^{-i\pi}|x_{13}^2|$ for the ordering $\mathcal{O}\phi_1$, and restricting the range of integration to the past lightcone of 1, we find

$$\begin{aligned}&\langle 0|[\phi_1(x_1), \mathbf{L}[\mathcal{O}]](x_3,z)\phi_2(x_2)|0\rangle =\\ &= -2i\sin\pi\frac{\Delta_1+\Delta-\Delta_2+J}{2}\int_{-\infty}^{-\frac{2(z\cdot x_{13})}{x_{13}^2}} d\alpha(-\alpha)^{-\Delta-J}\frac{\left(2z\cdot x_{23}\,x_{13}^2-2z\cdot x_{13}\,x_{23}^2\right)^J}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}|x_{13'}|^{\Delta_1+\Delta-\Delta_2+J}x_{23'}^{\Delta_2+\Delta-\Delta_1+J}},\end{aligned} \tag{6.91}$$

where $x_3' = x_3 - z/\alpha$. Note that the factor $(\ldots)^J$ in the numerator is independent of α because z is null. We thus need to compute

$$\begin{aligned}&\int_{-\infty}^{-\frac{2(z\cdot x_{13})}{x_{13}^2}} d\alpha(-\alpha)^{-\Delta-J}\frac{1}{|x_{13'}|^{\Delta_1+\Delta-\Delta_2+J}x_{23'}^{\Delta_2+\Delta-\Delta_1+J}}\\ &= \int_{\frac{2(z\cdot x_{13})}{x_{13}^2}}^{+\infty} d\alpha\frac{1}{|\alpha x_{13}^2-2(z\cdot x_{13})|^{\frac{\Delta_1+\Delta-\Delta_2+J}{2}}(\alpha x_{23}^2-2(z\cdot x_{23}))^{\frac{\Delta_2+\Delta-\Delta_1+J}{2}}}\\ &= \frac{\Gamma(\Delta+J-1)\Gamma\left(1-\frac{\Delta+\Delta_1-\Delta_2+J}{2}\right)}{\Gamma\left(\frac{\Delta-\Delta_1+\Delta_2+J}{2}\right)}\frac{1}{|x_{13}|^{\Delta_1+\Delta-\Delta_2+J}x_{23}^{\Delta_2+\Delta-\Delta_1+J}}\left(\frac{2(z\cdot x_{13})}{x_{13}^2}-\frac{2(z\cdot x_{23})}{x_{23}^2}\right)^{1-\Delta-J}.\end{aligned} \tag{6.92}$$

By (6.89), α has constant sign, which allows us to go to the second line. Because of (6.90), the function of z which enters $(\ldots)^{1-\Delta-J}$ is positive, so the result is well-defined.

Putting everything together, we find

$$\begin{aligned}&\langle 0|\phi_1(x_1)\,\mathbf{L}[\mathcal{O}](x_3,z)\,\phi_2(x_2)|0\rangle\\ &= L(\phi_1\phi_2[\mathcal{O}])\frac{\left(2z\cdot x_{23}\,x_{13}^2-2z\cdot x_{13}\,x_{23}^2\right)^{1-\Delta}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-(1-J)+(1-\Delta)}{2}}(-x_{13}^2)^{\frac{\Delta_1+(1-J)-\Delta_2+(1-\Delta)}{2}}(x_{23}^2)^{\frac{\Delta_2+(1-J)-\Delta_1+(1-\Delta)}{2}}},\end{aligned} \tag{6.93}$$

where

$$L(\phi_1\phi_2[\mathcal{O}]) \equiv -2\pi i\frac{\Gamma(\Delta+J-1)}{\Gamma(\frac{\Delta+\Delta_1-\Delta_2+J}{2})\Gamma(\frac{\Delta-\Delta_1+\Delta_2+J}{2})}. \tag{6.94}$$

The result (6.93) indeed takes the form of a conformally-invariant correlation function of ϕ_1 and ϕ_2 with an operator of dimension $1-J$ and spin $1-\Delta$. Note how continuous spin structures arise in a natural way from the light transform. Note also that (6.93) is pure negative-imaginary in the configuration of figure 6.7, where all quantities in the denominator are real. This is related to Rindler positivity as we discuss in section 6.6.1.

Although we did the computation in a specific configuration, we have expressed the result in terms of an analytic function of the positions. Because the result should be analytic, the resulting expression (6.93) is valid for any configuration. The $i\epsilon$-prescription in (6.93) is the same as for the original Wightman function. In particular, if we move x_3 back into a configuration where all the points are spacelike separated, we obtain a phase

$$e^{i\pi\frac{\Delta_1+(1-J)-\Delta_2+(1-\Delta)}{2}} \tag{6.95}$$

coming from $-x_{13}^2$ becoming negative. This phase will play a role in section 6.2.7.

6.2.6 Light transform of a time-ordered correlator

Finally, let us discuss the light-transform of a time-ordered correlator $\langle \mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}_3]\rangle$. By lemma (2), this is nonzero only if $2^- < 3 < 1$ (as in figure 6.7) or $1^- < 3 < 2$. In the first nonzero configuration $2^- < 3 < 1$, the time-ordered correlator is equivalent to the Wightman function $\langle 0|\mathcal{O}_1\mathcal{O}_3\mathcal{O}_2|0\rangle$ along the entire integration contour of the light transform. The other nonzero configuration differs by $1 \leftrightarrow 2$. Thus, we have

$$\langle \mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}_3]\rangle = \langle 0|\mathcal{O}_1\mathbf{L}[\mathcal{O}_3]\mathcal{O}_2|0\rangle\theta(2^- < 3 < 1) + \langle 0|\mathcal{O}_2\mathbf{L}[\mathcal{O}_3]\mathcal{O}_1|0\rangle\theta(1^- < 3 < 2). \tag{6.96}$$

Note that here the standard Wightman functions $\langle 0|\mathcal{O}_1\mathcal{O}_3\mathcal{O}_2|0\rangle$ and $\langle 0|\mathcal{O}_2\mathcal{O}_3\mathcal{O}_1|0\rangle$ (on which the light transforms act) are related to each other by analytic continuation and not by merely by relabeling the operators in the standard tensor structures $\langle 0|\ldots|0\rangle$.

For example, consider the three-point structure (6.86), now assumed to have $i\epsilon$ prescriptions appropriate for a time-ordered correlator. From (6.96) and our computation for the Wightman function (6.93), the light-transform is

$$
\begin{aligned}
&\langle \phi_1\phi_2 \mathbf{L}[O](x_3,z)\rangle = L(\phi_1\phi_2[O]) \\
&\times\left[\frac{\left(2z\cdot x_{23}\,x_{13}^2 - 2z\cdot x_{13}\,x_{23}^2\right)^{1-\Delta}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-(1-J)+(1-\Delta)}{2}}(-x_{13}^2)^{\frac{\Delta_1+(1-J)-\Delta_2+(1-\Delta)}{2}}(x_{23}^2)^{\frac{\Delta_2+(1-J)-\Delta_1+(1-\Delta)}{2}}}\theta(2^- < 3 < 1)\right. \\
&\quad\left.+\frac{(-1)^J\left(2z\cdot x_{13}\,x_{23}^2 - 2z\cdot x_{23}\,x_{13}^2\right)^{1-\Delta}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-(1-J)+(1-\Delta)}{2}}(x_{13}^2)^{\frac{\Delta_1+(1-J)-\Delta_2+(1-\Delta)}{2}}(-x_{23}^2)^{\frac{\Delta_2+(1-J)-\Delta_1+(1-\Delta)}{2}}}\theta(1^- < 3 < 2).\right]
\end{aligned}
\tag{6.97}
$$

The factor of $(-1)^J$ in the second term comes from the fact that the original structure $\langle\phi_1\phi_2 O\rangle$ picks up $(-1)^J$ when we swap $1 \leftrightarrow 2$.[30]

6.2.7 Algebra of integral transforms

The $\mathbf{L}$-transformation in (6.93) has the curious property that $\mathbf{L}^2$ is a nontrivial function of $\Delta_1, \Delta_2, \Delta$ and J, even though it originates from a Weyl reflection $(\Delta, J) \leftrightarrow (1-J, 1-\Delta)$ that squares to 1. Specifically, its square acting on a three-point Wightman function is given by

$$
\langle 0|\phi_1(x_1)\,\mathbf{L}^2[O](x_3,z)\,\phi_2(x_2)|0\rangle = \alpha_{\Delta_1,\Delta_2,\Delta,J}\langle 0|\phi_1(x_1)O(x_3,z)\phi_2(x_2)|0\rangle, \tag{6.98}
$$

where

$$
\begin{aligned}
\alpha_{\Delta_1,\Delta_2,\Delta,J} &= e^{i\pi\frac{\Delta_1+\Delta-\Delta_2+J}{2}}L(\phi_1\phi_2[O^{\mathrm{L}}])\times e^{i\pi\frac{\Delta_1+(1-J)-\Delta_2+(1-\Delta)}{2}}L(\phi_1\phi_2[O]) \\
&= \frac{\pi}{(\Delta+J-1)\sin\pi(\Delta+J)}(e^{i\pi(\Delta_1-\Delta_2)}-e^{i\pi(\Delta+J)})(e^{i\pi(\Delta_1-\Delta_2)}-e^{-i\pi(\Delta+J)}).
\end{aligned}
\tag{6.99}
$$

The phases in the first line of (6.99) are from (6.95).

Note that the square of the light transform *does* give back a three-point function of the same functional form as the original. However, the coefficient $\alpha_{\Delta_1,\Delta_2,\Delta,J}$ depends on Δ_1, Δ_2 in a non-trivial way that cannot be removed by redefining $\mathbf{L}$ by some function of Δ, J alone. This is in contrast to the Euclidean shadow transform, which squares to a coefficient $\mathcal{N}(\Delta, J)$ that is independent of the correlation function it acts on (appendix E.3.2).

[30] As we explain in appendix E.1, time-ordered correlators with continuous spin do not make sense, so we must assume J is an integer in this computation. This means that the factor $(-1)^J$ is unambiguous. The light transform $\langle\phi_1\phi_2\mathbf{L}[O]\rangle$ still gives a sensible continuous-spin structure because the result (6.97) is no longer a time-ordered correlator, e.g. it has θ-functions.

This "anomaly" in the group relation $L^2 = 1$ occurs for the following reason. The group-theoretic origin of $\mathbf{L}$ only guarantees that it squares to a multiple of the identity when acting on principal series representations $\mathcal{P}_{\Delta,J}$ defined on the conformal compactification of Minkowski space $\mathcal{M}_d^c$. However, here we are applying it to the space $\widetilde{\mathcal{P}}_{\Delta,J}$ defined on the universal cover $\widetilde{\mathcal{M}}_d$. The squared transformation $\mathbf{L}^2$ still commutes with $\widetilde{\mathrm{SO}}(d,2)$, so it becomes a non-trivial automorphism of the representation $\widetilde{\mathcal{P}}_{\Delta,J}$.

By Schur's lemma, nontrivial automorphisms can only occur in reducible representations. Indeed, as discussed in section 6.2.2, $\widetilde{\mathcal{P}}_{\Delta,J}$ is reducible and its irreducible components are the eigenspaces of $\mathcal{T}$. Within these irreducible components $\mathbf{L}^2$ must act by a constant, and thus we should have

$$\mathbf{L}^2 = f_{\mathrm{L}}(\Delta, J, \mathcal{T}). \tag{6.100}$$

Furthermore, note that $\mathbf{L}^2[\mathcal{O}](x,z)$ only depends on the values of $\mathcal{O}$ between x and $\mathcal{T}^2 x$. This means that $f_{\mathrm{L}}(\Delta, J, \mathcal{T})$ must be at most a quadratic polynomial in $\mathcal{T}$. Finally, because $\mathbf{L}^2[\mathcal{O}]$ vanishes when acting on the past or future vacuum, $f_{\mathrm{L}}(\Delta, J, \mathcal{T})$ should have roots at the eigenvalues of $\mathcal{T}$ in $\mathcal{O}|\Omega\rangle$ and $\langle\Omega|\mathcal{O}$ inside a correlation function,[31] which are $e^{\pm i\pi(\Delta+J)}$. In fact, as we show explicitly in appendix E.2.1,

$$\mathbf{L}^2 = f_{\mathrm{L}}(\Delta, J, \mathcal{T}) = \frac{\pi}{(\Delta + J - 1)\sin\pi(\Delta + J)}(\mathcal{T} - e^{i\pi(\Delta+J)})(\mathcal{T} - e^{-i\pi(\Delta+J)}). \tag{6.101}$$

This immediately implies (6.99) because $e^{i\pi(\Delta_1-\Delta_2)}$ is the eigenvalue of t acting on $\mathcal{O}$ in the Wightman function $\langle 0|\phi_1(x_1)\mathcal{O}(x_3,z)\phi_2(x_2)|0\rangle$. To see this, write the action of $\mathcal{T}$ on $\mathcal{O}$ as

$$\langle 0|\phi_1(x_1)\mathcal{T}\mathcal{O}(x_3,z)\mathcal{T}^{-1}\phi_2(x_2)|0\rangle \tag{6.102}$$

and use (6.38).

In fact, we can also turn this reasoning around and use the relatively simple computation (6.99) to fix the polynomial $f_{\mathrm{L}}(\Delta, J, \mathcal{T})$ in general. This will be helpful in appendix E.7 where we will need the statement that for general Lorentz irreps ρ the ratio

$$\frac{f_L(\Delta, \rho, \mathcal{T})}{(\mathcal{T} - \gamma)(\mathcal{T} - \gamma^{-1})}, \tag{6.103}$$

[31] Here we need the adjoint action as $\mathcal{O} \to \mathcal{T}\mathcal{O}\mathcal{T}^{-1}$, c.f. equation (6.38).

where γ is the eigenvalue in (6.38) corresponding to (Δ, ρ), is independent of $\mathcal{T}$.

More generally, this reasoning implies that relations between restricted Weyl reflections $w \in \mathrm{D}_8$ also hold for the corresponding integral transforms, but only up to multiplication by polynomials in $\mathcal{T}$ with coefficients depending on Δ and J. In the remainder of this section we derive these modified relations between integral transforms.

First of all, some relations hold by construction given the definitions in section 6.2.3,

$$\begin{aligned}
\mathbf{S} &= \mathbf{S}_J\mathbf{S}_\Delta = \mathbf{S}_\Delta\mathbf{S}_J, \\
\mathbf{F} &= \mathbf{S}_J\mathbf{L}\mathbf{S}_J, \\
\mathbf{R} &= \mathbf{S}_J\mathbf{L}, \\
\overline{\mathbf{R}} &= \mathbf{L}\mathbf{S}_J.
\end{aligned} \tag{6.104}$$

Furthermore, we already know that (for simplicity, we consider only $\widetilde{\mathcal{P}}_{\Delta,J,\lambda}$ with trivial λ)

$$\mathbf{L}^2 = f_{\mathrm{L}}(\Delta, J, \mathcal{T}), \tag{6.105}$$

$$\mathbf{S}_J^2 = f_J(J), \tag{6.106}$$

where we have suppressed the dependence on t. Here f_{L} is a quadratic polynomial in t defined in (6.101), while $f_J(J)$ depends only on J and is equal to the square of Euclidean shadow transform in $d-2$ dimensions:

$$f_J(J) = \frac{\pi^{d-1}}{(J+\frac{d-2}{2})\sin\pi(J+\frac{d}{2})}\frac{1}{\Gamma(-J)\Gamma(J+d-2)}. \tag{6.107}$$

That is, $f_J(J) = \mathcal{N}(-J, 0)$ in $d-2$ dimensions, where $\mathcal{N}(\Delta, J)$ in d dimensions is given in (E.53). These equations allow us to compute

$$\mathbf{R}\overline{\mathbf{R}} = f_{\mathrm{L}}(\Delta, 2-d-J, \mathcal{T})f_J(J), \tag{6.108}$$

$$\overline{\mathbf{R}}\mathbf{R} = f_{\mathrm{L}}(\Delta, J, \mathcal{T})f_J(1-\Delta). \tag{6.109}$$

As we show in appendix E.2.2, there is another relation,

$$\mathbf{S}_\Delta = i\mathcal{T}^{-1}\mathbf{L}\mathbf{S}_J\mathbf{L}. \tag{6.110}$$

Together with $\mathbf{S} = \mathbf{S}_J\mathbf{S}_\Delta = \mathbf{S}_\Delta\mathbf{S}_J$ this implies

$$\mathbf{S} = i\mathcal{T}^{-1}\mathbf{R}^2 = i\mathcal{T}^{-1}\overline{\mathbf{R}}^2, \tag{6.111}$$

and thus we find

$$\begin{aligned}\mathbf{S}^2 &= -\mathcal{T}^{-2}\mathbf{R}^2\overline{\mathbf{R}}^2\\ &= -\mathcal{T}^{-2} f_{\mathrm{L}}(\Delta, 2-d-J, \mathcal{T}) f_J(J) f_{\mathrm{L}}(J+d-1, 1-d+\Delta, \mathcal{T}) f_J(1-\Delta).\end{aligned} \tag{6.112}$$

Due to $\mathbf{S}^2 = -\mathcal{T}^{-2}(\mathbf{S}_J\mathbf{L})^4 = -\mathcal{T}^{-2}(\mathbf{L}\mathbf{S}_J)^4$, we also have

$$(\mathbf{L}\mathbf{S}_J)^4 = (\mathbf{S}_J\mathbf{L})^4 = f_{\mathrm{L}}(\Delta, 2-d-J, \mathcal{T}) f_J(J) f_{\mathrm{L}}(J+d-1, 1-d+\Delta, \mathcal{T}) f_J(1-\Delta). \tag{6.113}$$

At this point it is obvious that f_J and f_{L} completely determine the relations between all integral transforms, since D_8 is generated by L and S_J modulo $\mathrm{L}^2 = \mathrm{S}_J^2 = (\mathrm{S}_J\mathrm{L})^4 = 1$ and we have already found the generalization of these relations to the integral transforms $\mathbf{L}$ and $\mathbf{S}_J$ in (6.105), (6.106), and (6.113).

A convenient way to summarize these results is by using normalized versions of $\mathbf{L}$ and $\mathbf{S}_J$. Specifically, we define

$$\hat{\mathbf{L}} \equiv \mathbf{L}\frac{1}{\Gamma(\Delta+J-1)(\mathcal{T} - e^{i\pi(\Delta+J)})}, \tag{6.114}$$

$$\hat{\mathbf{S}}_J \equiv \mathbf{S}_J\frac{\Gamma(-J)}{\pi^{\frac{d-2}{2}}\Gamma(J+\frac{d-2}{2})}, \tag{6.115}$$

where Δ and J in there right hand side should be understood as operators reading off the dimension and spin of the functions they act upon. One can then check the following relations

$$\hat{\mathbf{L}}^2 = 1, \qquad \hat{\mathbf{S}}_J^2 = 1, \qquad (\hat{\mathbf{L}}\hat{\mathbf{S}}_J)^4 = (\hat{\mathbf{S}}_J\hat{\mathbf{L}})^4 = 1. \tag{6.116}$$

These normalized transforms therefore generate the dihedral group D_8 without any extra coefficients. Note that $\hat{\mathbf{L}}$ is very non-local because it has $\mathcal{T}$ in the denominator. In particular, by doing a Taylor expansion in $\mathcal{T}$ we see that it involves a sum over an infinite number of different Poincare patches. Thus, even though $\hat{\mathbf{L}}$ satisfies a simpler algebra, we mostly prefer to work with $\mathbf{L}$.

6.3 Light-ray operators

In this section, we explain how to fuse a pair of local operators $\mathcal{O}_1, \mathcal{O}_2$ into a light-ray operator $\mathbb{O}_{i,J}$ which gives an analytic continuation in spin J of the light-transform of local operators in the $\mathcal{O}_1 \times \mathcal{O}_2$ OPE. This amounts to defining correlation functions

$$\langle\Omega|V_1 \ldots V_k \mathbb{O}_{i,J} V_{k+1} \ldots V_n|\Omega\rangle \tag{6.117}$$

in terms of those of $\mathcal{O}_1$ and $\mathcal{O}_2$,

$$\langle\Omega|V_1 \ldots V_k \mathcal{O}_1 \mathcal{O}_2 V_{k+1} \ldots V_n|\Omega\rangle. \tag{6.118}$$

When J is an integer, $\mathbb{O}_{i,J}$ is related to a local operator in the $\mathcal{O}_1\mathcal{O}_2$ OPE, and these correlation functions are linked by Euclidean harmonic analysis [65]. Our strategy will be to start with this relation, rephrase it in Lorentzian signature, and then analytically continue in J. By the operator-state correspondence, it suffices to consider just two insertions V_i, and for simplicity we will also restrict to scalars $\mathcal{O}_1 = \phi_1$ and $\mathcal{O}_2 = \phi_2$. (The generalization to arbitrary spin of $\mathcal{O}_1, \mathcal{O}_2$ will be straightforward.)

6.3.1 Euclidean partial waves

Consider a Euclidean correlation function $\langle\phi_1\phi_2 V_3 V_4\rangle_\Omega$, where the V_3 and V_4 are local operators of any spin (not necessarily primary) and ϕ_1, ϕ_2 are local primary scalars. By the Plancherel theorem for SO$(d+1,1)$ (due to Harish-Chandra [244]), such a correlation function can be expanded in partial waves $P_{\Delta,J}$ that diagonalize the action of the conformal Casimirs acting simultaneously on points 1 and 2 [65],[32]

$$\langle V_3 V_4 \phi_1\phi_2\rangle_\Omega = \sum_{J=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i}\mu(\Delta, J) \int d^d x P^{\mu_1\cdots\mu_J}_{\Delta,J}(x_3, x_4, x)\langle\widetilde{\mathcal{O}}^\dagger_{\mu_1\cdots\mu_J}(x)\phi_1\phi_2\rangle. \tag{6.119}$$

Here, $\mathcal{O}$ has spin J and dimension $\Delta \in \frac{d}{2} + i\mathbb{R}^+$ on the principal series. The factor $\mu(\Delta, J)$ is the Plancherel measure (E.53), which we have inserted in order to simplify later expressions. For traceless-symmetric $\mathcal{O}$ there is no difference between representations $\widetilde{\mathcal{O}}^\dagger$ and $\widetilde{\mathcal{O}}$, but we will keep the daggers in what follows with the view towards the more general case.

Let us make two technical comments about the applicability of this formula. It follows directly from $L^2(G)$ harmonic analysis on SO$(d+1,1)$ if $\Delta_1 - \Delta_2$ is pure imaginary (possibly 0) and $\langle V_3 V_4\phi_1\phi_2\rangle_\Omega$ is square-integrable in the sense that

$$\int d^d x_1 d^d x_2\, x_{12}^{-2d+4\,\mathrm{Re}\,\Delta_1}\langle V_3V_4\phi_1\phi_2\rangle_\Omega(\langle V_3V_4\phi_1\phi_2\rangle_\Omega)^* < \infty. \tag{6.120}$$

[32]For general spin operators we should also include contributions from a discrete series of partial waves.

This is precisely the situation when the conformal Casimir operators acting on points 1 and 2 are self-adjoint and we can perform their spectral analysis.[33] Neither of these conditions is satisfied by a typical correlator in a physically-relevant CFT. Lifting the restriction of square integrability is conceptually easy and is similar to the usual Fourier transform: non-square integrable correlation functions can be interpreted as distributions (of some kind) and their partial waves also become distributions.[34]

Relaxing the restriction $\Delta_1-\Delta_2 \in i\mathbb{R}$, on the other hand, seems to be hard to do from first principles, since the Casimir operators are not self-adjoint anymore. We will thus not attempt to do this here and instead adopt the following pedestrian approach: we will imagine multiplying correlation functions by products of scalar two-point functions $x_{ij}^{\kappa\delta_{ij}}$ with $\kappa = 1$ so that the scaling dimensions of external operators will formally become principal series (this will of course modify the conformal block decomposition of these functions).[35] We perform harmonic analysis for these modified functions and then remove the auxiliary two-point functions by sending $\kappa \to 0$. For this to make sense we have to assume that the final expressions can be analytically continued to $\kappa = 0$.

With these comments in mind, we may proceed with (6.119). Using the bubble integral (E.52), we find that $P_{\Delta,J}$ is given by

$$\begin{aligned}
&P^{\mu_1\cdots\mu_J}_{\Delta,J}(x_3,x_4,x)\\
&=\left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E^{-1}\int d^dx_1d^dx_2\langle V_3V_4\phi_1\phi_2\rangle_\Omega\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O^{\mu_1\cdots\mu_J}(x)\rangle,
\end{aligned} \tag{6.121}$$

where

$$\left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E = \frac{2^{2J}\hat{C}_J(1)}{2^d\mathrm{vol}(\mathrm{SO}(d-1))}. \tag{6.122}$$

is the three-point pairing defined in appendix E.3.1. In anticipation of performing the light-transform, let us contract spin indices of O with a null polarization vector

[33] The reason why it is important to have $\Delta_1-\Delta_2 \in i\mathbb{R}$ is that the adjoint of a Casimir operator acts on functions with conjugate shadow scaling dimensions $\widetilde{\Delta}_i^*$. This is a different space of functions than the one $\langle V_3V_4\phi_1\phi_2\rangle_\Omega$ lives in unless $\widetilde{\Delta}_i^* = \Delta_i$, which is the case when $\Delta_i \in \frac{d}{2}+i\mathbb{R}$ are principal series representations. It furthermore turns out that only $\Delta_1-\Delta_2$ is important for the argument, since $\Delta_1+\Delta_2$ can be changed by multiplying $\langle V_3V_4\phi_1\phi_2\rangle_\Omega$ by a two-point function x_{12}^δ for some δ, and such two-point functions cancel out in equations.

[34] The distributional contribution to the partial wave can be analyzed by subtracting a finite number of contributions of low dimensional operators to make the function better behaved. This analysis was essentially performed in [66] and in generic cases amounts to a deformation of Δ-contour in (6.119).

[35] Note that such two-point functions have the right Wightman analyticity properties, and thus do not spoil the analyticity of physical correlators which we use in the arguments below.

z^μ to give

$$P_{\Delta,J}(x_3,x_4,x,z) = \left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle, \langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E^{-1} \int d^dx_1 d^dx_2 \langle V_3V_4\phi_1\phi_2\rangle_\Omega \langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O(x,z)\rangle, \tag{6.123}$$

where $O(x,z) = O^{\mu_1\cdots\mu_J}(x)z_{\mu_1}\cdots z_{\mu_J}$.

Physical correlation functions $\langle V_3V_4O_*\rangle_\Omega$ of operators O_* in the $\phi_1\times\phi_2$ OPE are residues of the partial waves,

$$f_{12*}\langle V_3V_4O_*(x,z)\rangle_\Omega = -\operatorname*{Res}_{\Delta=\Delta_*} \mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{O}^\dagger])P_{\Delta,J}(x_3,x_4,x,z)\Big|_{J=J_*}. \tag{6.124}$$

Here, $S_E(\phi_1\phi_2[\widetilde{O}^\dagger])$ is the shadow transform coefficient (E.55), and f_{12*} is the OPE coefficient of $O_*\in\phi_1\times\phi_2$. Equation (6.124) is a simple generalization of the standard result for primary four-point functions. We derive it in appendix E.3.3.

6.3.2 Wick-rotation to Lorentzian signature

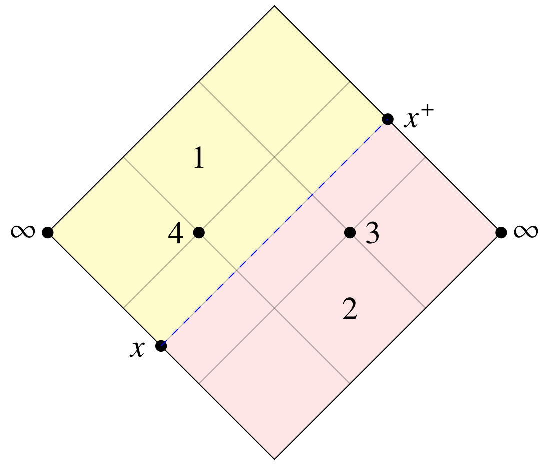
To obtain the promised analytic continuation of $\mathbf{L}[O]$, we need to first go to Lorentzian signature, and then apply the light transform.

We thus Wick-rotate all the operators ϕ_1,ϕ_2,V_3,V_4,O to Lorentzian signature by setting

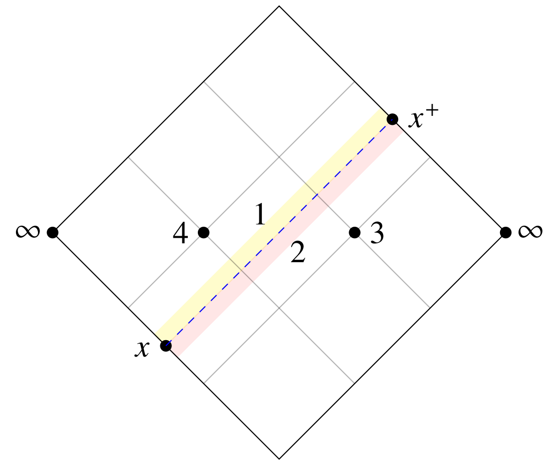
$$\tau = (i+\epsilon)t, \tag{6.125}$$

where τ and t are Euclidean and Lorentzian time, respectively. In more detail, we simultaneously rotate the time coordinates of each of the operators ϕ_1,ϕ_2,V_3,V_4,O. For the operators V_3,V_4,O, this means we analytically continue in the coordinates x_3,x_4,x. The operators ϕ_1,ϕ_2 are being integrated over in (6.123), and we rotate their respective integration contours simultaneously with the analytic continuation of x_3,x_4,x. Simultaneous Wick-rotation turns Euclidean correlators into time-ordered Lorentzian correlators. The result is a double-integral of time-ordered correlators over Minkowski space

$$\begin{aligned}&P_{\Delta,J}(x_3,x_4,x,z)\\ &= -\left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle, \langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E^{-1} \int_{\infty\approx 1,2} d^dx_1 d^dx_2 \langle V_3V_4\phi_1\phi_2\rangle_\Omega \langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O(x,z)\rangle.\end{aligned} \tag{6.126}$$



(a) Integration region in (6.128)



(b) Region S in (6.128) which contributes to the residue.

Figure 6.8: The configuration of points within the Poincare patch of ∞. Point 4 is in the future of x and 3 is in the past of x^+, while x is null separated and in the past of ∞. The shaded yellow (red) region is the region of integration for 1 (2) after taking the light transform, in the first term in equations (6.127) and (6.128). The dashed null line is spanned by z. Note that in (b), for $d > 2$ the region S extends in and out of the picture, while the dashed null line doesn't.

Here, we have chosen a generic point x_∞ on the Lorentzian cylinder $\widetilde{\mathcal{M}}_d$ and written Minkowski space as the Poincare patch that is spacelike from this point.[36,37] All the points $1, 2, 3, 4, x$ are constrained to lie within this patch. The minus sign in (6.126) comes from two Wick rotations in the measure $d\tau_1 d\tau_2 = -dt_1 dt_2$.

6.3.3 The light transform and analytic continuation in spin

Let us now move $\mathcal{O}(x, z)$ to past null infinity and perform the light transform. We choose 3, 4 such that $3^- < x < 4$, so that the left-hand side is nonzero, see figure 6.8a. Since $\mathcal{O}$ is on the Euclidean principal series, the condition $\mathrm{Re}(\Delta + J) > 1$ is satisfied and we can plug in (6.96) to find

$$
\begin{aligned}
&\mathbf{L}[P_{\Delta,J}](x_3, x_4, x, z) \\
&= -\left(\langle \phi_1\phi_2\widetilde{\mathcal{O}}^\dagger\rangle, \langle \widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger\mathcal{O}\rangle\right)_E^{-1} \int_{\substack{2^- < x < 1 \\ \infty \approx 1,2}} d^dx_1 d^dx_2 \langle V_3 V_4 \phi_1\phi_2\rangle_\Omega \langle 0|\widetilde{\phi}_1^\dagger \mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2^\dagger|0\rangle \\
&\qquad + (1 \leftrightarrow 2). \qquad (6.127)
\end{aligned}
$$

[36] In particular the result must be independent of which point we choose for x_∞. The spurious dependence of formulas on x_∞ will go away soon.

[37] Note that we do not place $\mathcal{O}(x, z)$ at infinity before performing the Wick rotation, in contrast to [67]. The reason is that in our case the region of integration for 1, 2 is independent of the position of $\mathcal{O}$ so it is easier to analytically continue in the position of $\mathcal{O}$.

See the discussion below (6.96) for the precise meaning of the $(1 \leftrightarrow 2)$ term.

Let us now define

$$
\begin{aligned}
&\mathbb{O}_{\Delta,J}(x,z) \\
&\equiv \frac{\mu(\Delta,J) S_E(\phi_1\phi_2[\widetilde{O}^\dagger])}{\left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle, \langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E} \int_{\substack{2^- < x < 1 \\ \infty \approx 1,2}} d^d x_1 d^d x_2 \langle 0|\widetilde{\phi}_1^\dagger \mathbf{L}[O](x,z)\widetilde{\phi}_2^\dagger|0\rangle \phi_1\phi_2 + (1 \leftrightarrow 2).
\end{aligned}
\tag{6.128}
$$

It is implicit here that x is null separated from ∞. This expression makes sense (at least formally) for continuous J. The euclidean three-point structure $\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle$ that we started with is single-valued only for integer J. However, due to the particular Wightman ordering the structures in (6.128) are well-defined for any J, as discussed in appendix E.1. In order to continue to non-integer J, we must also choose an analytic continuation of the prefactors in (6.128), which we discuss in more detail below. One consequence is that we have two different analytic continuations: one from even values of J that we denote $\mathbb{O}^+_{\Delta,J}$, and one from odd values of J that we denote $\mathbb{O}^-_{\Delta,J}$.

For integer J, (6.127) and (6.124) imply that the residues $\mathbb{O}^\pm_{i,J}$, defined by

$$
\mathbb{O}^\pm_{\Delta,J}(x,z) \sim \frac{1}{\Delta - \Delta_i^\pm(J)} \mathbb{O}^\pm_{i,J}(x,z),
\tag{6.129}
$$

have the same three-point functions as light-transforms of local operators in the $\phi_1 \times \phi_2$ OPE. (We include a $\pm$ subscript on $\Delta_i^\pm(J)$ because the positions of poles in the (Δ, J) plane are in general different for the even/odd cases.) To be precise, when J is an integer, the residue of a *time-ordered* correlator, where time-ordering acts on ϕ_1 and ϕ_2 inside the definition of $\mathbb{O}^\pm_{\Delta,J}$,

$$
\langle V_3 V_4 \mathbb{O}^\pm_{\Delta,J}(x,z)\rangle_\Omega,
\tag{6.130}
$$

agrees with

$$
f_{12O}\langle V_3 V_4 \mathbf{L}[O_{i,J}]\rangle_\Omega.
\tag{6.131}
$$

for a local operator $O_{i,J}$, where $\pm$ is determined by $(-1)^J = \pm 1$.

We now claim that, for any J, the residue in (6.130) comes from a region S where ϕ_1 and ϕ_2 are simultaneously almost null-separated from x and from each other; see figure 6.8b. Indeed, we always expect singularities in correlators when points are null-separated. In integrated correlators, such singularities can be removed by $i\epsilon$-prescriptions. However, lightlike singularities in the region S are not removed because they coincide with boundaries in the integration regions for x_1, x_2. In a time-ordered correlator, we can also have singularities at coincident points. However, we expect singularities related to the $\phi_1 \times \phi_2$ OPE to come from 1 being lightlike to 2 and not from other coincident limits.

Let us focus on the first term of (6.128). For this term, it is guaranteed that $1 \geq 3$, $2 \leq 4$, and $1 \geq 2$. In the region S we furthermore have $1 \leq 4$ and $2 \geq 3$, i.e., we have the ordering $4 \geq 1 \geq 2 \geq 3$, and the contribution of the first term of (6.128) to the time-ordered correlator (6.130) agrees with its contribution to the Wightman function

$$\langle\Omega|V_4\mathbb{O}^{\pm}_{\Delta,J}V_3|\Omega\rangle. \tag{6.132}$$

The same obviously holds for the second term, and, moreover, (6.131) agrees with the Wightman function

$$f_{12\mathcal{O}}\langle\Omega|V_4\mathbf{L}[\mathcal{O}_{i,J}]V_3|\Omega\rangle. \tag{6.133}$$

Since any state in CFT can be approximated by local operators V_i acting on the vacuum in an arbitrarily small region, this implies that we can interpret (6.128) and (6.129) as operator equations. Furthermore, by construction, for non-negative integer J we must have, as an operator equation,

$$\mathbb{O}^{\pm}_{i,J} = f_{12\mathcal{O}}\mathbf{L}[\mathcal{O}_{i,J}] \qquad (J \in \mathbb{Z}_{\geq 0},\ (-1)^J = \pm 1) \tag{6.134}$$

for some local operator $\mathcal{O}_{i,J}$.

For non-integer J the definition (6.128) with (6.129) provides an analytic continuation in J of $\mathbf{L}[\mathcal{O}_{i,J}]$. As we will show in section 6.4, it is precisely the matrix elements of $\mathbb{O}^{\pm}_{\Delta,J}$ and $\mathbb{O}^{\pm}_{i,J}$ which are computed by Caron-Huot's Lorentzian inversion formula. As discussed above, the residues $\mathbb{O}^{\pm}_{i,J}$ should only depend on the region of the integral where ϕ_1 and ϕ_2 are almost null-separated. In fact, it is natural to expect that the residue is further localized onto the null line defined by z. Thus we refer to them as light-ray operators. In the next subsection we show this explicitly in the case of mean field theory (MFT).

In our argument for the existence of light-ray operators, it is not necessary that $\mathbb{O}^{\pm}_{\Delta,J}$ be a meromorphic function with simple poles. We expect that any non-analyticity in

$\mathbb{O}^{\pm}_{\Delta,J}$ in the (Δ, J) plane should come from the region where ϕ_1 and ϕ_2 are lightlike-separated. Thus, for example, it should be possible to define light-ray operators by taking discontinuities across branch cuts of $\mathbb{O}^{\pm}_{\Delta,J}$ (if they exist). Determining the analyticity structure of $\mathbb{O}^{\pm}_{\Delta,J}$ in the (Δ, J) plane is an important problem for the future.

As mentioned above, to analytically continue $\mathbb{O}^{\pm}_{\Delta,J}$ in spin, we must choose an analytic continuation in J of the prefactors

$$\begin{aligned}&\frac{\mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{O}^\dagger])}{\left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E}\\&=(-1)^J\frac{\Gamma(J+\frac{d}{2})\Gamma(d+J-\Delta)\Gamma(\Delta-1)}{2\pi^d\Gamma(J+1)\Gamma(\Delta-\frac{d}{2})\Gamma(\Delta+J-1)}\frac{\Gamma(\frac{\Delta+J+\Delta_1-\Delta_2}{2})\Gamma(\frac{\Delta+J-\Delta_1+\Delta_2}{2})}{\Gamma(\frac{d-\Delta+J+\Delta_1-\Delta_2}{2})\Gamma(\frac{d-\Delta+J-\Delta_1+\Delta_2}{2})}.\end{aligned} \tag{6.135}$$

Additionally, the term in (6.128) with $(1 \leftrightarrow 2)$ has a prefactor differing by $(-1)^J$. Because of the $(-1)^J$'s, we must make two separate analytic continuations from even and odd J, leading to $\mathbb{O}^{\pm}_{\Delta,J}$. In general, we expect the spectrum of light-ray operators to be different in the odd and even cases. For example, in MFT with a real scalar ϕ, the analytic continuation of even-J two-ϕ operators is nontrivial, but there are no odd-J two-ϕ operators.

The analytic continuation of the remaining Γ-function factors in (6.135) is determined by requiring that they be meromorphic and polynomially bounded at infinity in the right half-plane. This is important for the Sommerfeld-Watson resummation discussed in section 6.5.2. The expression (6.135) satisfies these conditions, so provides a good analytic continuation. When ϕ_1, ϕ_2 are not scalars, then we can relate the prefactor to a rational function of J times (6.135) using weight-shifting operators [3, 195], and this provides a good analytic continuation in that case as well.

Although we have assumed scalar ϕ_1, ϕ_2 in this section for simplicity, the generalization to arbitrary representations O_1, O_2 is straightforward. We discuss some aspects of the general case in section 6.4.2.

6.3.4 Light-ray operators in Mean Field Theory

In this section we explicitly show that $\mathbb{O}^{\pm}_{i,J}$ are light-ray operators in Mean Field Theory (MFT). For simplicity, we assume that the scalar operators in (6.128) are distinct fundamental MFT scalars. More generally, we can imagine that they belong to two decoupled CFTs.

The kernel in (6.128) is obtained from (6.93) by sending x_3 to past null infinity according to the rule

$$\mathcal{O}(-z\infty, z) = \lim_{L\to+\infty} L^{\Delta+J}\mathcal{O}(-Lz, z), \tag{6.136}$$

i.e.

$$\begin{aligned}&\langle 0|\widetilde{\phi}_1^\dagger \mathbf{L}[\mathcal{O}]\widetilde{\phi}_2^\dagger|0\rangle = \\ &= L(\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger|\mathcal{O})\frac{2^{J-1}\left(z\cdot x_2\, x_1^2 - z\cdot x_1\, x_2^2\right)^{1-\Delta}}{(x_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2+J-\Delta}{2}}(-z\cdot x_1)^{\frac{\widetilde{\Delta}_1-\widetilde{\Delta}_2+2-\Delta-J}{2}}(z\cdot x_2)^{\frac{\widetilde{\Delta}_2-\widetilde{\Delta}_1+2-\Delta-J}{2}}}.\end{aligned} \tag{6.137}$$

The expression (6.93) was written for $1 > 3, 3 \approx 2, 1 \approx 2$. With these conditions, the ratio above is positive. In the integral we need to relax $1 \approx 2$, which is done by adding $i\epsilon$ to x_2^0 and $-i\epsilon$ to x_1^0, according to the Wightman ordering above. We now introduce lightcone coordinates by writing

$$x_i = \tfrac{1}{2}zv_i + \tfrac{1}{2}z'u_i + \mathbf{x}_i \tag{6.138}$$

with $z'^2 = 0$, $z'\cdot z = 2$ and $\mathbf{x}_i\cdot z = \mathbf{x}_i\cdot z' = 0$. Since this requires z' to be past-directed, the $i\epsilon$-prescription is equivalent to adding a positive imaginary part to u_1 and v_2 and negative to u_2 and v_1. We then find for the integral in the first term of (6.128)

$$\frac{1}{4}\int du_1du_2dv_1dv_2d^{d-2}\mathbf{x}_1d^{d-2}\mathbf{x}_2\frac{2^{J-1}\left(u_1u_2v_{12} + u_2\mathbf{x}_1^2 - u_1\mathbf{x}_2^2\right)^{1-\Delta}\phi_1(x_1)\phi_2(x_2)}{(u_{12}v_{12}+\mathbf{x}_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2+J-\Delta}{2}}(-u_1)^{\frac{\widetilde{\Delta}_1-\widetilde{\Delta}_2+2-\Delta-J}{2}}u_2^{\frac{\widetilde{\Delta}_2-\widetilde{\Delta}_1+2-\Delta-J}{2}}}. \tag{6.139}$$

We have temporarily suppressed the light transform coefficient $L(\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger[\mathcal{O}])$.

The integration region has $u_1 < 0$ and $u_2 > 0$. Let us assume for now that $v_2 > v_1$ and make the change of variables

$$\begin{aligned}u_1 &= -r\alpha,\\ u_2 &= r(1-\alpha),\\ \mathbf{x}_i &= (rv_{21})^{\frac{1}{2}}\mathbf{w}_i.\end{aligned} \tag{6.140}$$

The integral becomes

$$\frac{1}{4}\int_0^1 d\alpha \int dv_1 dv_2 d^{d-2}\mathbf{w}_1 d^{d-2}\mathbf{w}_2 \frac{2^{J-1} v_{21}^{-1-\frac{\Delta-\Delta_1-\Delta_2+J}{2}} \left(\alpha(1-\alpha)+(1-\alpha)\mathbf{w}_1^2+\alpha\mathbf{w}_2^2\right)^{1-\Delta}}{(1+\mathbf{w}_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2+J-\Delta}{2}} \alpha^{\frac{\widetilde{\Delta}_1-\widetilde{\Delta}_2+2-\Delta-J}{2}} (1-\alpha)^{\frac{\widetilde{\Delta}_2-\widetilde{\Delta}_1+2-\Delta-J}{2}}} \times$$
$$\times \int_0^\infty \frac{dr}{r} r^{-\frac{\Delta-\Delta_1-\Delta_2-J}{2}} \phi_1(-r\alpha, v_1, (rv_{21})^{\frac{1}{2}}\mathbf{w}_1)\phi_2(r(1-\alpha), v_2, (rv_{21})^{\frac{1}{2}}\mathbf{w}_2). \quad (6.141)$$

In the second line, we have isolated the integral

$$\int_0^\infty \frac{dr}{r} r^{-\frac{\Delta-\Delta_1-\Delta_2-J}{2}} \phi_1(-r\alpha, v_1, (rv_{21})^{\frac{1}{2}}\mathbf{w}_1)\phi_2(r(1-\alpha), v_2, (rv_{21})^{\frac{1}{2}}\mathbf{w}_2). \quad (6.142)$$

The region $r \sim 0$ corresponds to ϕ_1 and ϕ_2 being localized near the light ray defined by z.

Now imagine expanding the product of field operators in a power series in r. This is possible since we have assumed that ϕ_1 and ϕ_2 do not interact and thus there is no lightcone singularity between them.[38] We find terms of the form

$$r^{n+m+\frac{1}{2}(a+b)}(-\alpha)^n(1-\alpha)^m v_{21}^{\frac{1}{2}(a+b)} \mathbf{w}_1^a \mathbf{w}_2^b. \quad (6.143)$$

Only terms with even values of $a+b$ contribute, since the $\mathbf{w}_i$ integral is invariant under $\mathbf{w}_i \to -\mathbf{w}_i$. Therefore, $N = n+m+\frac{1}{2}(a+b) \geq 0$ is an integer and the integral over r takes the form

$$\int_0^\infty \frac{dr}{r} r^{-\frac{\Delta-\Delta_1-\Delta_2-J-2N}{2}} \sim -\frac{2}{\Delta-\Delta_1-\Delta_2-J-2N}. \quad (6.144)$$

The pole comes from the region of small r. We can see this by imposing an upper cutoff on r: the residue will be independent of it. (In particular, we can make the cutoff depend on α and $\mathbf{w}_i$ thereby cutting out arbitrary regions around the null ray and the residue won't change.) The pole is at

$$\Delta = \Delta_1 + \Delta_2 + J + 2N, \quad (6.145)$$

which for integer J are precisely the locations of double-trace operators $[\phi_1\phi_2]_{N,J}$. For every N, the residue of (6.142) only depends on a finite number of derivatives of ϕ_i on the null ray, and thus is localized on it, as promised in the introduction.

[38] If we consider $\phi_1 = \phi_2 = \phi$, then in MFT we have $\phi(x_1)\phi(x_2) =: \phi(x_1)\phi(x_2) : +\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle$. The singular term is positive-energy in x_2 and negative-energy in x_1. But in (6.128) we are integrating against $\langle 0|\widetilde{\phi}_1\mathbf{L}[\mathcal{O}]\widetilde{\phi}_2|0\rangle$, which has the same energy conditions on x_1 and x_2. Since the integrals pick out the term with vanishing total energy in both x_1 and x_2, the singular piece does not contribute to (6.128).

For simplicity, let us focus on the leading twist trajectory with $N = 0$. The residue of (6.142) is then

$$-2\phi_1(0, v_1, 0)\phi_2(0, v_2, 0) \tag{6.146}$$

and the residue of the integral (6.141) becomes

$$\begin{aligned}
&\frac{-1}{2}\int_0^1 d\alpha \int d^{d-2}\mathbf{w}_1 d^{d-2}\mathbf{w}_2 \frac{2^{J-1}\left(\alpha(1-\alpha)+(1-\alpha)\mathbf{w}_1^2+\alpha\mathbf{w}_2^2\right)^{1-\Delta_1-\Delta_2-J}}{(1+\mathbf{w}_{12}^2)^{d-\Delta_1-\Delta_2}\alpha^{-\Delta_1+1-J}(1-\alpha)^{-\Delta_2+1-J}}\times \\
&\times \int dv_1 dv_2 (v_{21}+i\epsilon)^{-1-J}\phi_1(0,v_1,0)\phi_2(0,v_2,0).
\end{aligned} \tag{6.147}$$

The first line is an overall coefficient which we compute in appendix E.4 and here simply denote by $\mathcal{R}(\Delta_1, \Delta_2, J)$. In the second line, we have restored the $i\epsilon$ prescription for v_i, which allows us to relax the assumption $v_2 > v_1$. (The factor $(v_{21} + i\epsilon)^{-1-J}$ is understood to be positive for positive v_{21} and real J.)

Combining everything together, we conclude that the leading twist operators $\mathbb{O}_{0,J}$ are given by

$$\begin{aligned}
&\mathbb{O}_{0,J}(-z\infty, z) \\
&= i\frac{(-1)^J}{4\pi}\int dsdt\left((t+i\epsilon)^{-1-J}+(-1)^J(-t+i\epsilon)^{-1-J}\right)\phi_1(0,s-t,0)\phi_2(0,s+t,0),
\end{aligned} \tag{6.148}$$

where we have included the contribution of the second term in (6.128), performed the change of variables $v_1 = s - t$, $v_2 = s + t$, and used the identity

$$L(\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger[O])\mathcal{R}(\Delta_1,\Delta_2,J)\frac{\mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{O}^\dagger])}{\left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E} = i\frac{(-1)^J 2^{J-2}}{\pi}. \tag{6.149}$$

The analytic continuations from even and odd J are

$$\begin{aligned}
\mathbb{O}^+_{0,J}(-z\infty, z) &= +\frac{i}{4\pi}\int dsdt\left((t+i\epsilon)^{-1-J}+(-t+i\epsilon)^{-1-J}\right)\phi_1(0,s-t,0)\phi_2(0,s+t,0), \\
\mathbb{O}^-_{0,J}(-z\infty, z) &= -\frac{i}{4\pi}\int dsdt\left((t+i\epsilon)^{-1-J}-(-t+i\epsilon)^{-1-J}\right)\phi_1(0,s-t,0)\phi_2(0,s+t,0).
\end{aligned} \tag{6.150}$$

These are exactly the null-ray operators advertised in the introduction. We can check that they are indeed primary by lifting their definitions to the embedding space, where they are variants of

$$\sim \int_{-\infty}^{+\infty} d\alpha d\beta\, \phi_1(Z-\alpha X)\phi_2(Z-\beta X)(\alpha-\beta)^{-J-1}. \tag{6.151}$$

We discuss conformal invariance of this embedding-space integral in the next subsection.

For integer J both kernels for the t-integral are equal to

$$\begin{aligned}&(t+i\epsilon)^{-1-J}+(-1)^J(-t+i\epsilon)^{-1-J}=\\&=\frac{(-1)^J}{\Gamma(J+1)}\frac{\partial^J}{\partial t^J}\left((t+i\epsilon)^{-1}-(t-i\epsilon)^{-1}\right)=-2\pi i\frac{(-1)^J}{\Gamma(J+1)}\frac{\partial^J}{\partial t^J}\delta(t).\end{aligned}\tag{6.152}$$

Thus, for integer J we find

$$\mathbb{O}_{0,J}(-z\infty,z)=\frac{(-1)^J}{\Gamma(J+1)}\int\frac{ds}{2}\,\phi_1(0,s,0)(\overleftrightarrow{\partial}_s)^J\phi_2(0,s,0)=\mathbf{L}[[\phi_1\phi_2]_{0,J}](-z\infty,z).\tag{6.153}$$

Since total derivatives vanish in the integral over s, it follows that for integer spin $\mathbb{O}_{0,J}$ is given by the light transform of a primary double-twist operator of the form

$$[\phi_1\phi_2]_{0,J}(x,z)\equiv\frac{(-1)^J}{\Gamma(J+1)}\phi_1(x)(z\cdot\partial)^J\phi_2(x)+(z\cdot\partial)(\ldots).\tag{6.154}$$

Let us check that these operators are correctly normalized. It was found in [245] that the full expression for the primary $[\phi_1\phi_2]_{0,J}$ is

$$\begin{aligned}&[\phi_1\phi_2]_{0,J}(x,z)\\&=c_J\sum_{k=0}^{J}\frac{(-1)^k}{k!(J-k)!\Gamma(\Delta_1+k)\Gamma(\Delta_2+J-k)}(z\cdot\partial)^k\phi_1(x)(z\cdot\partial)^{J-k}\phi_2(x)\end{aligned}\tag{6.155}$$

and in our case c_J is given by

$$c_J=\frac{(-1)^J}{\Gamma(J+1)}\left(\sum_{k=0}^{J}\frac{1}{k!(J-k)!\Gamma(\Delta_1+k)\Gamma(\Delta_2+J-k)}\right)^{-1}.\tag{6.156}$$

If we write now

$$\langle\phi_1\phi_2[\phi_1\phi_2]_{0,J}\rangle_\Omega=f_{12J}\langle\phi_1\phi_2\mathcal{O}_J\rangle,\tag{6.157}$$

and

$$\langle[\phi_1\phi_2]_{0,J}[\phi_1\phi_2]_{0,J}\rangle_\Omega=C_J\langle\mathcal{O}_J\mathcal{O}_J\rangle,\tag{6.158}$$

where in the right hand side we use the standard structures defined in appendix E.1.3, then our normalization conventions are such that $C_J/f_{12J} = 1$.[39] It is a straightforward exercise to show using (6.155) that

$$\frac{C_J}{f_{12J}} = (-1)^J \Gamma(J+1) c_J \sum_{k=0}^{J} \frac{1}{k!(J-k)!\Gamma(\Delta_1+k)\Gamma(\Delta_2+J-k)} = 1. \tag{6.159}$$

In doing the calculation it is convenient to use the same null polarization vector for both operators in (6.158).

6.3.4.1 Subleading families and multi-twist operators

Although we will not compute the residue of $\mathbb{O}_{\Delta,J}$ for $N > 0$, let us comment on the form of the light-ray operators that we expect to obtain, as well as on some further interesting generalizations. For simplicity, in this section we ignore $i\epsilon$-prescriptions, the difference between even and odd J, and normalization factors. As mentioned above, the leading double-twist operators are essentially the primaries

$$\mathbb{O}_{0,J}(X,Z) \equiv \int d\alpha\, d\beta\, \phi_1(Z-\alpha X)\phi_2(Z-\beta X)(\alpha-\beta)^{-J-1}. \tag{6.160}$$

The fact that $\mathbb{O}$ is a primary follows from conformal invariance of the integral on the right-hand side. According to the usual rules of the embedding space formalism [53], conformal invariance is equivalent to

1. homogeneity in X and Z with degrees $-\Delta_{\mathbb{O}}$ and $J_{\mathbb{O}}$, and
2. invariance under $Z \to Z + \lambda X$.

The former requirement is fulfilled due to homogeneity of the measure $d\alpha\, d\beta$, the "wavefunction" $(\alpha-\beta)^{-J-1}$, and the original primaries ϕ_i, which leads to

$$\begin{aligned} \Delta_{\mathbb{O}} &= 1 - J, \\ J_{\mathbb{O}} &= 1 - \Delta_1 - \Delta_2 - J. \end{aligned} \tag{6.161}$$

The latter requirement is due to translational invariance of the measure $d\alpha\, d\beta$ and the wavefunction $(\alpha-\beta)^{-J-1}$.

[39] To be more precise, if O is an operator in $\phi_1 \times \phi_2$ OPE, we are computing $[\phi_1\phi_2]_J = f_{12O} O / C_O$, which is independent of the normalization of O. Using $[\phi_1\phi_2]_J$ instead of O then yields the claimed normalization condition.

This leads to two simple observations. The first is that since the only requirement on ϕ_i is that of being a primary, we can dress them with weight-shifting operators [3]. For example, let D_m be the Thomas/Todorov differential operator which increases the scaling dimension of a primary by 1 and carries a vector embedding space index m. Then we can define

$$\begin{aligned}&\mathbb{O}_{N,J}(X,Z)\\&=\int d\alpha d\beta (D_{m_1}\cdots D_{m_N}\phi_1)(Z-\alpha X)(D^{m_1}\cdots D^{m_N}\phi_2)(Z-\beta X)(\alpha-\beta)^{-J-1}.\end{aligned} \tag{6.162}$$

By construction, we now have

$$\begin{aligned}\Delta_{\mathbb{O}} &= 1-J,\\ J_{\mathbb{O}} &= 1-\Delta_1-\Delta_2-J-2N.\end{aligned} \tag{6.163}$$

With appropriate $i\epsilon$-prescriptions for α- and β-contours, for integer J these operators reduce to light transforms of the local family $[\phi_1\phi_2]_{N,J}$. It is clear how (at least in principle) this construction generalizes to non-scalar ϕ_i.

The second observation is that this construction straightforwardly generalizes to multi-twist operators. In particular, define

$$\mathbb{O}_\psi(X,Z)=\int d\alpha_1\cdots d\alpha_n\phi_1(Z-\alpha_1 X)\cdots\phi_n(Z-\alpha_n X)\psi(\alpha_1,\ldots,\alpha_n), \tag{6.164}$$

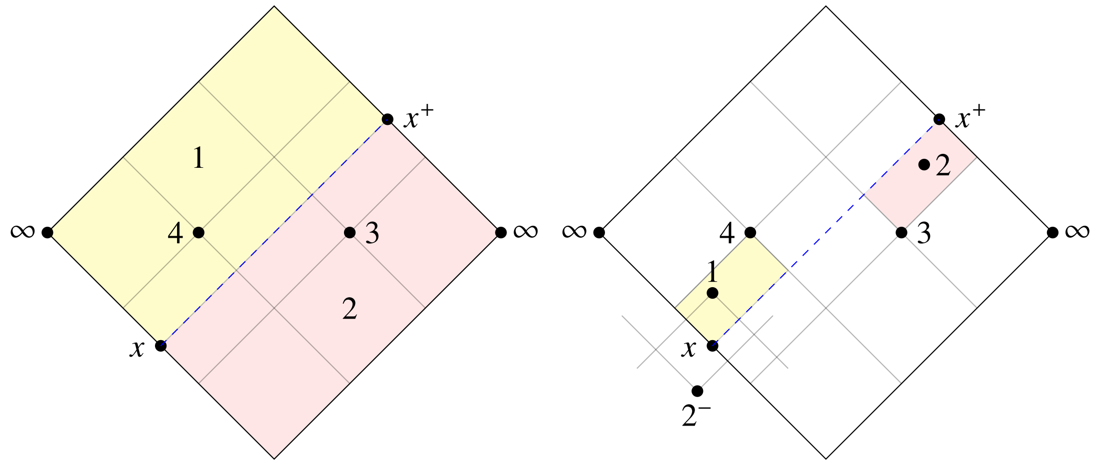
where ψ is a wavefunction which is translationally-invariant and homogeneous,

$$\begin{aligned}\psi(\alpha_1+\beta,\ldots,\alpha_n+\beta)&=\psi(\alpha_1,\ldots,\alpha_n),\\ \psi(\lambda\alpha_1,\ldots,\lambda\alpha_n)&=\lambda^{-J-1}\psi(\alpha_1,\ldots,\alpha_n).\end{aligned} \tag{6.165}$$

We can easily check that $\mathbb{O}_\psi$ is a primary with scaling dimension and spin given by

$$\begin{aligned}\Delta_{\mathbb{O}} &= 1-J,\\ J_{\mathbb{O}} &= 1-J+\sum_{i=1}^{n}\Delta_n.\end{aligned} \tag{6.166}$$

Subleading families can be obtained as above, by dressing with weight-shifting operators. The generalization to non-scalar ϕ_i is also clear.



(a) After taking the light transform but before reducing to a double commutator.

(b) After reducing to a double commutator.

Figure 6.9: The configuration of points within the Poincare patch of x_∞ at various stages of the derivation. The blue dashed line shows the support of light transform of $\mathcal{O}(x, z)$. The yellow (red) shaded region shows the allowed region for 1 (2). In the right-hand figure, we indicate that x is constrained to satisfy $2^- < x < 1$. Note that after reducing to a double-commutator, the yellow and red regions are independent of x_∞ (as long as x is lightlike from x_∞).

6.4 Lorentzian inversion formulae

In this section we show that matrix elements of $\mathbb{O}_{\Delta,J}$ are computed by a Lorentzian inversion formula of the type discussed by Caron-Huot [66]. Our derivation will borrow some key steps from [67]. However the light transform will simplify the derivation to the point where its generalization to external spinning operators is obvious. In particular, after using the light transform in the appropriate way, it will be immediately clear why the conformal block $G_{J+d-1,\Delta-d+1}$ and its generalizations appear. For simplicity, we will present most of the derivation with scalar operators and generalize to spinning operators at the end.

6.4.1 Inversion for the scalar-scalar OPE

6.4.1.1 The double commutator

Our starting point is the light-transformed expression (6.127). Let us concentrate on the first term in (6.127). Because of the restrictions $3^- < x < 4$ and $2^- < x < 1$, the lightcone of x splits Minkowski space into two regions, with $2, 3$ in the lower region and $1, 4$ in the upper, see figure 6.9a. Thus, we can write the integrand as

$$\langle\Omega|T\{V_4\phi_1\}T\{\phi_2 V_3\}|\Omega\rangle\langle 0|\widetilde{\phi}_1^\dagger \mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2^\dagger|0\rangle. \tag{6.167}$$

Recall that in our notation, expectation values in the state $|\Omega\rangle$ denote physical correlation functions, whereas expectation values in the state $|0\rangle$ denote two- or three-point structures that are fixed by conformal invariance. (For instance, three-point structures $\langle 0|\cdots|0\rangle$ don't include OPE coefficients.)

We can now use the reasoning in lemma 1 to obtain a double commutator.[40] Consider a modified integrand where ϕ_1 acts on the future vacuum,

$$\langle\Omega|\phi_1 V_4 T\{\phi_2 V_3\}|\Omega\rangle\langle 0|\widetilde{\phi}_1^\dagger \mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2^\dagger|0\rangle. \tag{6.168}$$

Imagine integrating ϕ_1 over a lightlike line in the direction of z, with coordinate v_1 along the line. Because ϕ_1 acts on the future vacuum, the correlator is analytic in the lower half v_1-plane. Furthermore, at large v_1, the product of correlators goes like

$$\frac{1}{v_1^{\Delta_1}} \times \frac{1}{v_1^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2+\Delta+J-2}{2}}}. \tag{6.169}$$

Here, the first factor comes from the estimate (6.80) of $\langle\Omega|\phi_1\cdots|\Omega\rangle$ using the OPE and the second factor comes from direct computation using the three-point function (6.93). Thus, we can deform the v_1 contour in the lower half-plane to give zero whenever

$$\mathrm{Re}(2(d-2)+\Delta_1-\Delta_2+\Delta+J)>0. \tag{6.170}$$

This condition is certainly true for $\Delta\in\frac{d}{2}+i\infty$ and $J\geq 0$, assuming (for now) that $\mathrm{Re}(\Delta_2-\Delta_1)=0$ (see section 6.3.1).

Consequently, the x_1 integral vanishes if we replace (6.167) with (6.168), so we can freely replace

$$T\{V_4\phi_1\} \;\to\; T\{V_4\phi_1\}-\phi_1 V_4 \;=\; [V_4,\phi_1]\theta(1<4). \tag{6.171}$$

By similar reasoning, we can replace

$$T\{\phi_2 V_3\}\to[\phi_2,V_3]\theta(3<2). \tag{6.172}$$

[40] This argument is the same as the contour manipulation in [67].

Overall, we find a double commutator in the integrand, together with some extra restrictions on the region of integration

$$\int_{\substack{x<1<4\\3<2<x^+}} d^d x_1 d^d x_2 \langle\Omega|[V_4,\phi_1][\phi_2,V_3]|\Omega\rangle\langle 0|\widetilde{\phi}_1^\dagger \mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2^\dagger|0\rangle + (1\leftrightarrow 2). \tag{6.173}$$

Note that the spurious dependence on the point at infinity x_∞ has disappeared because the commutators are only nonzero if $x<1<4$ and $3<2<x^+$, and these restrictions imply that $1,2$ lie in the same Poincare patch as $3,4,x$.

In terms of $\mathbb{O}_{\Delta,J}$ we have

$$\begin{aligned}&\langle V_4 \mathbb{O}_{\Delta,J}(x,z) V_3\rangle_\Omega = \\ &= \frac{\mu(\Delta,J) S_E(\phi_1\phi_2[\widetilde{\mathcal{O}}^\dagger])}{\left(\langle\phi_1\phi_2\widetilde{\mathcal{O}}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger\mathcal{O}\rangle\right)_E}\int_{\substack{x<1<4\\3<2<x^+}} d^d x_1 d^d x_2 \langle\Omega|[V_4,\phi_1][\phi_2,V_3]|\Omega\rangle\langle 0|\widetilde{\phi}_1^\dagger \mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2^\dagger|0\rangle \\ &\quad + (1\leftrightarrow 2).\end{aligned} \tag{6.174}$$

This gives a Lorentzian inversion formula analogous to the Euclidean inversion formula (6.121). It is different from Caron-Huot's formula [66] in that it is not formulated in terms of cross-ratio integrals and it is valid for non-primary or non-scalar V_i. The form of the inversion formula above will be useful in section 6.6 where we discuss the average null energy condition and its generalizations. Note also that the generalization to operators $\mathcal{O}_1$ and $\mathcal{O}_2$ with nonzero spin is straightforward. In the rest of this subsection we show how to reduce (6.174) to a cross-ratio integral in the form of [66].

6.4.1.2 Inversion for a four-point function of primaries

To obtain an integral over cross-ratios, let us specialize to the case where $V_3=\phi_3$ and $V_4=\phi_4$ are primary scalars. The partial wave $P_{\Delta,J}$ in this case is fixed by conformal invariance up to a coefficient:

$$\mu(\Delta,J) S_E(\phi_1\phi_2[\widetilde{\mathcal{O}}^\dagger]) P_{\Delta,J}(x_3,x_4,x,z) = C(\Delta,J)\langle\phi_3\phi_4\mathcal{O}(x,z)\rangle. \tag{6.175}$$

OPE data is encoded in the resiudes of $C(\Delta,J)$ by (6.124),

$$f_{12\mathcal{O}_*} f_{34\mathcal{O}_*} = -\operatorname*{Res}_{\Delta=\Delta_*} C(\Delta,J_*). \tag{6.176}$$

The matrix element $\langle\phi_4\mathbb{O}_{\Delta,J}(x,z)\phi_3\rangle_\Omega$ is the light-transform of (6.175), so (6.174) becomes

$$
\begin{aligned}
&C(\Delta,J)\langle 0|\phi_4\mathbf{L}[\mathcal{O}](x,z)\phi_3|0\rangle\\
&=-\frac{\mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{\mathcal{O}}^\dagger])}{\left(\langle\phi_1\phi_2\widetilde{\mathcal{O}}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger\mathcal{O}\rangle\right)}\int_{\substack{x<1<4\\3<2<x^+}}d^dx_1d^dx_2\langle\Omega|[\phi_4,\phi_1][\phi_2,\phi_3]|\Omega\rangle\langle 0|\widetilde{\phi}_1^\dagger\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2^\dagger|0\rangle\\
&\quad+(1\leftrightarrow 2).
\end{aligned}
\tag{6.177}
$$

For reasons that will become clear in a moment, let us replace $x_4\to x_4^+$ (equivalently act with $\mathcal{T}_4$ on both sides). This converts the condition $3^-<x<4$ into $3^-<x<4^+$. At the same time, let us make the change of variables $x_2\to x_2^+$ in the integral. We obtain

$$
\begin{aligned}
&C(\Delta,J)\langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}](x,z)\phi_3|0\rangle\\
&=-\frac{\mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{\mathcal{O}}])}{\left(\langle\phi_1\phi_2\widetilde{\mathcal{O}}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger\mathcal{O}\rangle\right)_E}\times\\
&\times\int_{3^-<2<x<1<4^+}d^dx_1d^dx_2\,\langle\Omega|[\phi_{4^+},\phi_1][\phi_{2^+},\phi_3]|\Omega\rangle\langle 0|\widetilde{\phi}_1^\dagger\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_{2^+}^\dagger|0\rangle\\
&\quad+(1\leftrightarrow 2).
\end{aligned}
\tag{6.178}
$$

Explicitly, the structure on the left-hand side is (under the additional constraint $3>4$)

$$
\begin{aligned}
&\langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}](x_0,z)\phi_3|0\rangle\\
&=L(\phi_3\phi_4[\mathcal{O}])\frac{(-1)^J\left(2z\cdot x_{40}\,x_{30}^2-2z\cdot x_{30}\,x_{40}^2\right)^{1-\Delta}}{(-x_{43}^2)^{\frac{\Delta_4+\Delta_3+J-\Delta}{2}}(x_{40}^2)^{\frac{\Delta_4-\Delta_3+2-\Delta-J}{2}}(x_{30}^2)^{\frac{\Delta_3-\Delta_4+2-\Delta-J}{2}}},
\end{aligned}
\tag{6.179}
$$

where $L(\phi_3\phi_4[\mathcal{O}])$ is given by (6.94). This expression comes from making the replacements $1,2,3\to 3,4^+,0$ in the second line of (6.97) and using $x_{i4^+}^2=-x_{i4}^2$ and $z\cdot x_{4^+0}=-z\cdot x_{40}$.[41] Similarly, the structure in the right hand side is

$$
\begin{aligned}
&\langle 0|\widetilde{\phi}_1^\dagger\,\mathbf{L}[\mathcal{O}](x_0,z)\,\widetilde{\phi}_{2^+}^\dagger|0\rangle\\
&=L(\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger[\mathcal{O}])\frac{\left(2z\cdot x_{10}\,x_{20}^2-2z\cdot x_{20}\,x_{10}^2\right)^{1-\Delta}}{(-x_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2+J-\Delta}{2}}(-x_{10}^2)^{\frac{\widetilde{\Delta}_1-\widetilde{\Delta}_2+2-\Delta-J}{2}}(-x_{20}^2)^{\frac{\widetilde{\Delta}_2-\widetilde{\Delta}_1+2-\Delta-J}{2}}}>0,
\end{aligned}
\tag{6.180}
$$

[41] These relations follow from the embedding space representation of these quantities as inner products with X_4. An alternative way to obtain this result is to use $\langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}]\phi_3|0\rangle=\langle 0|\mathcal{T}\phi_4\mathcal{T}^{-1}\mathbf{L}[\mathcal{O}]\phi_3|0\rangle=e^{-i\pi\Delta_4}\langle 0|\phi_4\mathbf{L}[\mathcal{O}]\phi_3|0\rangle$ and then (6.93) with replacements $1\to 4,2\to 3,3\to 0$, analytically continued. The factor $(-1)^J$ comes from the fact that the standard structure (E.25) depends on formal ordering of operators and we need $\langle\phi_3\phi_4\mathcal{O}\rangle$ by convention.

which follows from (6.93) by using the same rules.

We would now like to express the coefficient $C(\Delta, J)$ as an integral of the double-commutator $\langle\Omega|[\phi_{4^+}, \phi_1][\phi_{2^+}, \phi_3]|\Omega\rangle$ against a conformal block. Both sides of the above equation transform like conformal three-point functions. We can pick out the coefficient $C(\Delta, J)$ by taking a conformally-invariant pairing of both sides with a three-point structure that is "dual" to the one on the left-hand side.

In other words, in order to isolate $C(\Delta, J)$, we should find a structure T such that

$$\Big(T, \langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}](x,z)\phi_3|0\rangle\Big)_L = 1, \tag{6.181}$$

with the pairing $(\cdot,\cdot)_L$ defined in equation (E.79) as

$$\begin{aligned}&\Big(\langle\mathcal{O}_3\mathcal{O}_4\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}_3^\dagger\widetilde{\mathcal{O}}_4^\dagger\mathcal{O}^{\mathrm{S}\dagger}\rangle\Big)_L\\ &\equiv \int_{\substack{4<3\\ x\approx 3,4}} \frac{d^dx_3d^dx_4d^dxD^{d-2}z}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle\mathcal{O}_3(x_3)\mathcal{O}_4(x_2)\mathcal{O}(x,z)\rangle\langle\widetilde{\mathcal{O}}_3^\dagger(x_3)\widetilde{\mathcal{O}}_4^\dagger(x_4)\mathcal{O}^{\mathrm{S}\dagger}(x,z)\rangle.\end{aligned} \tag{6.182}$$

(Note the causal restrictions in the integral.) It will be convenient to write (6.181) using the shorthand notation

$$T = \langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}](x,z)\phi_3|0\rangle^{-1}. \tag{6.183}$$

For the pairing (6.181) to be well-defined, $\langle 0|\phi_{4+}\mathbf{L}[\mathcal{O}]\phi_3|0\rangle^{-1}$ must transform like a three-point function with representations $\langle\widetilde{\phi}_4^\dagger\mathcal{O}^{\mathrm{F}\dagger}\widetilde{\phi}_3^\dagger\rangle$, where $\mathcal{O}^{\mathrm{F}}$ has dimension and spin

$$\begin{aligned}\Delta_{\mathcal{O}^{\mathrm{F}}} &= J+d-1,\\ J_{\mathcal{O}^{\mathrm{F}}} &= \Delta-d+1.\end{aligned} \tag{6.184}$$

The quantum numbers of $\mathcal{O}^{\mathrm{F}}$ are precisely those appearing in Caron-Huot's block. We will see shortly that this is not a coincidence. Explicitly, the dual structure $\langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}]\phi_3|0\rangle^{-1}$ is given by (again for $3>4$)

$$\begin{aligned}&\langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}](x_0,z)\phi_3|0\rangle^{-1}\\ &\quad= \frac{2^{2d-2}\mathrm{vol}(\mathrm{SO}(d-2))}{L(\phi_3\phi_4[\mathcal{O}])}\frac{(-1)^J\left(2z\cdot x_{40}\,x_{30}^2-2z\cdot x_{30}\,x_{40}^2\right)^{\Delta-d+1}}{(-x_{43}^2)^{\frac{\widetilde{\Delta}_4+\widetilde{\Delta}_3-J+\Delta-2d+2}{2}}(x_{40}^2)^{\frac{\widetilde{\Delta}_4-J-\widetilde{\Delta}_3-\Delta+2}{2}}(x_{30}^2)^{\frac{\widetilde{\Delta}_3-J-\widetilde{\Delta}_4-\Delta+2}{2}}}.\end{aligned} \tag{6.185}$$

This follows easily from the alternative characterization of the paring (6.182) given in appendix E.5.

Finally, pairing both sides of (6.178) with $\langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}]\phi_3|0\rangle^{-1}$, we obtain

$$C(\Delta,J)=\int_{\substack{1>2\\3>4}}\frac{d^dx_1\cdots d^dx_4}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle\Omega|[\phi_{4^+},\phi_1][\phi_{2^+},\phi_3]|\Omega\rangle H_{\Delta,J}(x_i)+(1\leftrightarrow 2),\tag{6.186}$$

where

$$\begin{aligned}&H_{\Delta,J}(x_i)\\&=-\frac{\mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{\mathcal{O}}])}{\left(\langle\phi_1\phi_2\widetilde{\mathcal{O}}^\dagger\rangle,\langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger\mathcal{O}\rangle\right)_E}\int_{2<x<1}d^dxD^{d-2}z\langle 0|\widetilde{\phi}_1^\dagger\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_{2^+}^\dagger|0\rangle\langle 0|\phi_{4^+}\mathbf{L}[\mathcal{O}](x,z)\phi_3|0\rangle^{-1}.\end{aligned}\tag{6.187}$$

In the integral for $C(\Delta,J)$, all the pairs of points x_i are spacelike separated except for $1>2$ and $3>4$. The causal relations in (6.186) and (6.187) come from the causal relations in (6.178) and (6.182) which are, together,

$$4^-<3^-<2<x<1<4^+<3^+.\tag{6.188}$$

Recalling that $a\approx b$ is equivalent to $a^-<b<a^+$ (figure 6.3), we easily find that the above relations are the same as

$$\begin{aligned}&1>x>2,\quad 3>4,\\&1\approx 3,\quad 1\approx 4,\quad 2\approx 3,\quad 2\approx 4.\end{aligned}\tag{6.189}$$

Now the benefit of performing the light-transform becomes clear. The integral (6.187) over the diamond $2<x<1$ precisely takes the form of a well-known Lorentzian integral for a conformal block. Note that the integral (6.187) is conformally-invariant and is an eigenfunction of the conformal Casimir operators acting on points $1,2$ (equivalently $3,4$) by construction. Importantly, the integral over x stays away from the region near $3,4$, see figure 6.10. Thus, we can determine its behavior in the OPE limit $3\to 4$ by simply taking the limit inside the integrand. (This limit corresponds to the Regge limit of the physical operators at $1,2^+,3,4^+$.) Any eigenfunction of the conformal Casimirs is fixed by its OPE limit, so this determines the full function. Thus, it's clear that $H_{\Delta,J}$ is proportional to a conformal block, with external operators $\widetilde{\phi}_1^\dagger,\dots,\widetilde{\phi}_4^\dagger$, and an exchanged operator with the quantum numbers of $\mathcal{O}^{\mathrm{F}\dagger}$.

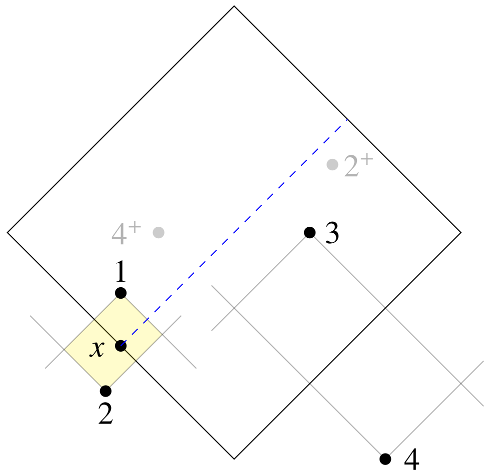


Figure 6.10: After (temporarily) relabeling the points $2^- \to 2$ and $4^- \to 4$, we have a configuration where $1 > x > 2$ and $3 > 4$, with all other pairs of points spacelike separated. This is the same configuration as in figure E.2 of appendix E.8.2, where we compute the Lorentzian integral for a conformal block. The integration region for x is shaded yellow. Importantly, it stays away from 3 and 4, so the $3 \to 4$ limit can be computed inside the integrand.

We perform this analysis in detail in appendix E.8.2. Using the result (E.179), we find

$$H_{\Delta,J}(x_i) = \frac{q_{\Delta,J}}{(-x_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2}{2}}(-x_{34}^2)^{\frac{\widetilde{\Delta}_3+\widetilde{\Delta}_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{\widetilde{\Delta}_2-\widetilde{\Delta}_1}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\widetilde{\Delta}_3-\widetilde{\Delta}_4}{2}} G^{\widetilde{\Delta}_i}_{J+d-1,\Delta-d+1}(\chi,\overline{\chi}), \tag{6.190}$$

where

$$\begin{aligned} q_{\Delta,J} &= -(-1)^J \frac{2^{2d-2}\mathrm{vol}(\mathrm{SO}(d-2))}{\left(\langle\phi_1\phi_2\widetilde{O}^\dagger\rangle, \langle\widetilde{\phi}_1^\dagger\widetilde{\phi}_2^\dagger O\rangle\right)_E} \frac{\mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{O}])L(\widetilde{\phi}_1\widetilde{\phi}_2[O])}{L(\phi_3\phi_4[O])} b^{\widetilde{\Delta}_1,\widetilde{\Delta}_2}_{J+d-1,\Delta-d+1} \\ &= -2^{2d}\mathrm{vol}(\mathrm{SO}(d-2)) \frac{\Gamma(\frac{\Delta+J+\Delta_1-\Delta_2}{2})\Gamma(\frac{\Delta+J-\Delta_1+\Delta_2}{2})\Gamma(\frac{\Delta+J+\Delta_3-\Delta_4}{2})\Gamma(\frac{\Delta+J-\Delta_3+\Delta_4}{2})}{16\pi^2\Gamma(\Delta+J)\Gamma(\Delta+J-1)}. \end{aligned} \tag{6.191}$$

(The quantity $b^{\Delta_1,\Delta_2}_{\Delta,J}$ is defined in (E.178) and the conformal block G is defined in appendix E.8.1.) Factors other than $b^{\Delta_1,\Delta_2}_{\Delta,J}$ come from (6.187) and the structures (6.180) and (6.185). In the proof of the Lorentzian inversion formula in [67], performed without using the light transform, one obtains an expression for $H_{\Delta,J}$ as an integral over a region totally spacelike from $1, 2^+, 3, 4^+$, which is harder to understand.

6.4.1.3 Writing in terms of cross-ratios

Finally, let us replace $2^+ \to 2$ and $4^+ \to 4$ so that the physical operators are again at the points $1, 2, 3, 4$. The inversion formula reads

$$C(\Delta, J) = \int_{\substack{4>1 \\ 2>3}} \frac{d^d x_1 \cdots d^d x_4}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} \langle \Omega | [\phi_4, \phi_1][\phi_2, \phi_3] | \Omega \rangle (\mathcal{T}_2^{-1} \mathcal{T}_4^{-1} H_{\Delta,J}(x_i)) + (1 \leftrightarrow 2). \tag{6.192}$$

Here, $\mathcal{T}_i^{-1}$ denotes a shift $x_i \to x_i^-$ or, more generally, application of the $\mathcal{T}^{-1}$ to the operator at i-th position. In the integrand, we can isolate quantities that depend only on cross-ratios, times a universal dimensionful factor $|x_{12}|^{-2d}|x_{34}|^{-2d}$,

$$\begin{aligned} &\langle \Omega | [\phi_4, \phi_1][\phi_2, \phi_3] | \Omega \rangle (\mathcal{T}_2^{-1} \mathcal{T}_4^{-1} H_{\Delta,J}(x_i)) \\ &= \frac{1}{|x_{12}|^{2d}|x_{34}|^{2d}} \frac{\langle \Omega | [\phi_4, \phi_1][\phi_2, \phi_3] | \Omega \rangle}{T^{\Delta_i}(x_i)} G^{\widetilde{\Delta}_i}_{J+d-1, \Delta-d+1}(\chi, \overline{\chi}), \end{aligned} \tag{6.193}$$

where

$$T^{\Delta_i}(x_i) \equiv \frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{|x_{14}|}{|x_{24}|}\right)^{\Delta_2-\Delta_1} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_3-\Delta_4}. \tag{6.194}$$

Since we now have a fixed causal ordering of the points, we do not have to worry about an $i\epsilon$ prescription in these expressions and we can simply take absolute values of spacetime intervals.

We can gauge-fix (6.192) to obtain an integral over cross-ratios alone. As explained in [67],[42] the measure becomes

$$\int \frac{d^d x_1 \cdots d^d x_4}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} \frac{1}{|x_{12}|^{2d}|x_{34}|^{2d}} \to \frac{1}{2^{2d}\mathrm{vol}(\mathrm{SO}(d-2))} \int_0^1 \int_0^1 \frac{d\chi d\overline{\chi}}{\chi^2 \overline{\chi}^2} \left| \frac{\overline{\chi} - \chi}{\chi \overline{\chi}} \right|^{d-2}. \tag{6.195}$$

Putting everything together, we find

$$\begin{aligned} &C(\Delta, J) \\ &= \frac{q_{\Delta,J}}{2^{2d}\mathrm{vol}(\mathrm{SO}(d-2))} \left[\int_0^1 \int_0^1 \frac{d\chi d\overline{\chi}}{\chi^2 \overline{\chi}^2} \left| \frac{\overline{\chi} - \chi}{\chi \overline{\chi}} \right|^{d-2} \frac{\langle \Omega | [\phi_4, \phi_1][\phi_2, \phi_3] | \Omega \rangle}{T^{\Delta_i}(x_i)} G^{\widetilde{\Delta}_i}_{J+d-1, \Delta-d+1}(\chi, \overline{\chi}) \right. \\ &\qquad \left. + (-1)^J \int_{-\infty}^0 \int_{-\infty}^0 \frac{d\chi d\overline{\chi}}{\chi^2 \overline{\chi}^2} \left| \frac{\overline{\chi} - \chi}{\chi \overline{\chi}} \right|^{d-2} \frac{\langle \Omega | [\phi_4, \phi_2][\phi_1, \phi_3] | \Omega \rangle}{T^{\Delta_i}(x_i)} \hat{G}^{\widetilde{\Delta}_i}_{J+d-1, \Delta-d+1}(\chi, \overline{\chi}) \right]. \end{aligned} \tag{6.196}$$

[42] We use a definition of the measure on $\widetilde{\mathrm{SO}}(d,2)$ which differs from the one [67] by a factor of 2^d.

Here, $\hat{G}_{\Delta,J}(\chi,\overline{\chi})$ denotes the solution to the Casimir equation that behaves as $(-\chi)^{\frac{\Delta-J}{2}}(-\overline{\chi})^{\frac{\Delta+J}{2}}$ for negative cross-ratios satisfying $|\chi| \ll |\overline{\chi}| \ll 1$. This precisely coincides with Caron-Huot's Lorentzian inversion formula.

6.4.1.4 A natural formula for the Lorentzian block

To make it easy to generalize the above result to arbitrary representations, let us write it in a more transparent way. First we need to introduce more flexible notation for a conformal block. Let

$$\frac{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\langle \mathcal{O}_3\mathcal{O}_4\mathcal{O}\rangle}{\langle \mathcal{O}\mathcal{O}\rangle} \tag{6.197}$$

denote the conformal block formed by gluing the three-point structures in the numerator using the two-point structure in the denominator. We describe the gluing procedure in more detail in appendix E.8.1. In particular, the gluing procedure is well-defined (for a restricted causal configuration) even if $\mathcal{O}$ is a continuous-spin operator. Using this notation, the coefficient function $C(\Delta, J)$ is defined by

$$\langle \phi_1\phi_2\phi_3\phi_4\rangle_\Omega = \sum_{J=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}C(\Delta,J)\frac{\langle \phi_1\phi_2\mathcal{O}\rangle\langle\phi_3\phi_4\mathcal{O}\rangle}{\langle\mathcal{O}\mathcal{O}\rangle}, \tag{6.198}$$

where $\mathcal{O}$ has dimension Δ and spin J.

Using the same notation, we claim that the function $H_{\Delta,J}(x_i)$ in (6.192) is given by

$$H_{\Delta,J}(x_i) = -\frac{1}{2\pi i}\frac{(\mathcal{T}_2\langle\phi_1\phi_2\mathbf{L}[\mathcal{O}]\rangle)^{-1}(\mathcal{T}_4\langle\phi_3\phi_4\mathbf{L}[\mathcal{O}]\rangle)^{-1}}{\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle^{-1}}, \qquad (1>2, 3>4). \tag{6.199}$$

In the numerator, $(\mathcal{T}_2\langle\phi_1\phi_2\mathbf{L}[\mathcal{O}]\rangle)^{-1}$ is the dual structure to $\mathcal{T}_2\langle\phi_1\phi_2\mathbf{L}[\mathcal{O}]\rangle$ via the three-point pairing (E.79). It is given by (6.185), with the replacement $3,4\to 1,2$. Note that while we have written the structures in the numerators in terms of light transforms of time ordered products, they can alternatively be written in terms of Wightman functions for the kinematics we are considering, since

$$\begin{aligned}
\mathcal{T}_2\langle\phi_1\phi_2\mathbf{L}[\mathcal{O}]\rangle &= \mathcal{T}_2\langle 0|\phi_2\mathbf{L}[\mathcal{O}]\phi_1|0\rangle \qquad (\text{when } 1>2, \quad 1,2\approx 0),\\
\mathcal{T}_4\langle\phi_3\phi_4\mathbf{L}[\mathcal{O}]\rangle &= \mathcal{T}_4\langle 0|\phi_4\mathbf{L}[\mathcal{O}]\phi_3|0\rangle \qquad (\text{when } 3>4, \quad 3,4\approx 0).
\end{aligned} \tag{6.200}$$

The structure $\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle^{-1}$ in the denominator is dual to the double light-transform of the time-ordered two-point function $\langle\mathcal{O}\mathcal{O}\rangle$ via the conformally-invariant two-point pairing,

$$\left(\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle^{-1}, \langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle\right)_L = 1. \tag{6.201}$$

Here the pairing $(\cdot,\cdot)_L$ for two-point functions is defined in (E.72). In order for the pairing in (6.201) to be conformally-invariant, $\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle^{-1}$ must transform like a two-point function of $\mathcal{O}^{\mathrm{F}}$.

We have already computed the three-point structures in the numerator, so to verify (6.199), we need to compute $\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle$. Here, it is important to treat two-point structures as distributions. By lemma 2, $\langle \mathcal{O}(x_1,z_1)\mathbf{L}[\mathcal{O}](x_2,z_2)\rangle$ vanishes if $x_2 > x_1$ or $x_2 < x_1$ — i.e., it vanishes almost everywhere. However, it is nonzero if x_1 is precisely lightlike from x_2. Specifically, $\langle \mathcal{O}(x_1,z_1)\mathbf{L}[\mathcal{O}](x_2,z_2)\rangle$ is a distribution localized where x_2 is on the past lightcone of x_1.[43] In fact, it is proportional to the integral kernel for the "floodlight transform" $\mathbf{F}$.

Let us now actually compute $\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle$. It is useful to think of this structure as an integral kernel K, defined by

$$(Kf)(x,z) \equiv \int d^d x' D^{d-2}z' \, \langle \mathbf{L}[\mathcal{O}](x,z)\mathbf{L}[\mathcal{O}](x',z')\rangle f(x',z'). \tag{6.202}$$

In (6.202), we can integrate one of the $\mathbf{L}$-transforms by parts, giving

$$(Kf)(x,z) = \int d^d x' D^{d-2}z' \, \langle \mathbf{L}[\mathcal{O}](x,z)\mathcal{O}(x',z')\rangle (\mathcal{T}^{-1}\mathbf{L}[f])(x',z'). \tag{6.203}$$

To simplify (6.203) further, we can express the time-ordered two-point function $\langle \mathcal{O}\mathcal{O}\rangle$ in terms of integral transforms and use the algebra derived in section 6.2.7. When x, x' are spacelike, $\langle \mathcal{O}(x,z)\mathcal{O}(x',z')\rangle$ is precisely the kernel for $\mathbf{S}$. However, $\mathbf{S}$ is supported only in the region $x \approx x'$, whereas the time-ordered two-point function has support everywhere. More precisely, keeping track of the phases as we move x, x' into different Poincare patches, we have

$$\begin{aligned}
\langle \mathcal{O}(x,z)\mathcal{O}(x',z')\rangle &= \frac{(-2z\cdot z'(x-x')^2 + 4z\cdot(x-x')z'\cdot(x-x'))^J}{((x-x')^2 + i\epsilon)^{\Delta+J}} \\
&= \mathbf{S}\left(1 + \sum_{n=1}^{\infty} e^{-in\pi(\Delta+J)}\mathcal{T}^n + \sum_{n=1}^{\infty} e^{-in\pi(\Delta+J)}\mathcal{T}^{-n}\right) \\
&= \mathbf{S}\frac{-2i\mathcal{T}\sin\pi(\Delta+J)}{(\mathcal{T}-e^{i\pi(\Delta+J)})(\mathcal{T}-e^{-i\pi(\Delta+J)})}.
\end{aligned} \tag{6.204}$$

[43] Note that this is different from treating two-point functions as physical Wightman functions, so there is no contradiction with previous discussion.

Plugging this into (6.203), we find

$$
\begin{aligned}
K &= \mathbf{L}\mathbf{S}\frac{-2i\mathcal{T}\sin\pi(\Delta+J)}{(\mathcal{T}-e^{i\pi(\Delta+J)})(\mathcal{T}-e^{-i\pi(\Delta+J)})}\mathcal{T}^{-1}\mathbf{L} \\
&= \mathbf{S}\frac{-2i\sin\pi(\Delta+J)}{(\mathcal{T}-e^{i\pi(\Delta+J)})(\mathcal{T}-e^{-i\pi(\Delta+J)})}\mathbf{L}^2 \\
&= \frac{-2\pi i}{\Delta+J-1}\mathbf{S},
\end{aligned}
\tag{6.205}
$$

where in the second line we used that $\mathbf{L}, \mathbf{S}, \mathcal{T}$ commute with each other, together with the formula $\mathbf{L}^2 = f_L(J+d-1, \Delta-d+1, \mathcal{T})$, where f_L is given in equation (6.101). The arguments of f_L come from the fact that K acts on a representation with dimension $J+d-1$ and spin $\Delta-d+1$.

The kernel of $\mathbf{S}$ in the last line is the two-point function of an operator with spin $1-\Delta$ and dimension $1-J$. Thus, using our two-point pairing (E.72), we find

$$
\langle \mathbf{L}[O]\mathbf{L}[O]\rangle^{-1} = -\frac{\Delta+J-1}{2\pi i}2^{2d-2}\mathrm{vol}(\mathrm{SO}(d-2))\langle O^{\mathrm{F}}O^{\mathrm{F}}\rangle,
\tag{6.206}
$$

where $\langle O^{\mathrm{F}}O^{\mathrm{F}}\rangle$ is the standard two-point structure (E.24) for an operator with dimension $J+d-1$ and spin $\Delta-d+1$. Combining this with the three-point structures in the numerator, and comparing with the result (6.190) for $H_{\Delta,J}(x_i)$, we verify (6.199).

Note that (6.199) is independent of a choice of normalization of the integral transform $\mathbf{L}$. In fact, it depends only on the three-point structures $\langle\phi_1\phi_2 O\rangle, \langle\phi_3\phi_4 O\rangle$, the two-point structure $\langle OO\rangle$, and the existence of a conformally-invariant map between representations $\mathcal{P}_{\Delta,J,\lambda}$ and $\mathcal{P}_{1-J,1-\Delta,\lambda}$ (which $\mathbf{L}$ implements). The formula would still be true if we chose different normalization conventions for two and three-point functions, because this would change the definition of $C(\Delta, J)$ in a compatible way, via (6.198). Because it is essentially independent of conventions, we call (6.199) a “natural” formula.

6.4.2 Generalization to arbitrary representations

6.4.2.1 The light transform of a partial wave

The derivation in the previous section is straightforward to generalize to the case of arbitrary conformal representations $\phi_i \to O_i$. In this case, three-point functions admit multiple conformally-invariant structures $\langle O_1 O_2 O\rangle^{(a)}$, so partial waves $P_{O,(a)}$

carry an additional structure label.[44] They are defined by

$$\langle V_3 V_4 O_1 O_2\rangle_\Omega = \sum_{\rho,a}\int_{\frac{d}{2}}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}\mu(\Delta,J)\int d^d x P_{O,(a)}(x_3,x_4,x)\langle\widetilde{O}^\dagger(x)O_1O_2\rangle^{(a)}. \tag{6.207}$$

(Here, we implicitly contract the SO(d) indices of $P_{O,(a)}$ and the operator $\widetilde{O}^\dagger$.)

The logic leading to the double-commutator integral (6.173) is essentially unchanged. We find

$$\begin{aligned}
&\mathbf{L}[P_{O,(a)}](x_3,x_4,x,z)\\
&= -(\langle O_1O_2\widetilde{O}^\dagger\rangle^{(a)},\langle\widetilde{O}_1^\dagger\widetilde{O}_2^\dagger O\rangle^{(b)})_E^{-1}\times\\
&\times\int_{\substack{x<1<4\\3<2<x^+}} d^dx_1 d^dx_2\langle\Omega|[V_4,O_1][O_2,V_3]|\Omega\rangle\langle 0|\widetilde{O}_1^\dagger\mathbf{L}[O](x,z)\widetilde{O}_2^\dagger|0\rangle^{(b)}\\
&\quad+(1\leftrightarrow 2),
\end{aligned} \tag{6.208}$$

where $(\langle O_1O_2\widetilde{O}^\dagger\rangle^{(a)},\langle\widetilde{O}_1^\dagger\widetilde{O}_2^\dagger O\rangle^{(b)})_E^{-1}$ is the inverse of the three-point pairing (E.50) defined by

$$(\langle O_1O_2\widetilde{O}^\dagger\rangle^{(a)},\langle\widetilde{O}_1^\dagger\widetilde{O}_2^\dagger O\rangle^{(b)})_E^{-1}(\langle O_1O_2\widetilde{O}^\dagger\rangle^{(c)},\langle\widetilde{O}_1^\dagger\widetilde{O}_2^\dagger O\rangle^{(b)})_E = \delta_a^c. \tag{6.209}$$

6.4.2.2 The generalized Lorentzian inversion formula

To generalize the remaining steps leading to the Lorentzian inversion formula, we seemingly need to understand of all the factors entering the expression for $H_{\Delta,J}(x_i)$ (6.190). However, this is unnecessary because the generalization is obvious from the natural formula (6.199).

The coefficient function $C_{ab}(\Delta,\rho)$ we would like to compute is defined by

$$\langle O_1\cdots O_4\rangle_\Omega = \sum_{\rho,a,b}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}C_{ab}(\Delta,\rho)\frac{\langle O_1O_2O^\dagger\rangle^{(a)}\langle O_3O_4O\rangle^{(b)}}{\langle OO^\dagger\rangle}, \tag{6.210}$$

where O has dimension Δ and SO(d)-representation ρ. Here, we sum over principal series representations $\mathcal{E}_{\Delta,\rho}$, as well as three-point structures a,b. The obvious

[44] The possible structures in a three-point function of spinning operators are classified in [1].

generalization of (6.186) and (6.199) is

$$\begin{aligned}
&C_{ab}(\Delta,\rho)\\
&= -\frac{1}{2\pi i}\int_{\substack{4>1\\2>3}} \frac{d^dx_1\cdots d^dx_4}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle\Omega|[\mathcal{O}_4,\mathcal{O}_1][\mathcal{O}_2,\mathcal{O}_3]|\Omega\rangle\\
&\qquad\qquad\times \mathcal{T}_2^{-1}\mathcal{T}_4^{-1}\frac{\left(\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)}\right)^{-1}\left(\mathcal{T}_4\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle^{(b)}\right)^{-1}}{\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}^\dagger]\rangle^{-1}}\\
&\quad + (1\leftrightarrow 2).
\end{aligned}\tag{6.211}$$

The dual structures in the numerator are defined by

$$\begin{aligned}
\left(\left(\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)}\right)^{-1},\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(c)}\right)_L &= \delta_a^c,\\
\left(\left(\mathcal{T}_4\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle^{(b)}\right)^{-1},\mathcal{T}_4\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle^{(d)}\right)_L &= \delta_b^d,
\end{aligned}\tag{6.212}$$

where $(\cdot,\cdot)_L$ is the three-point pairing defined in (E.79). The two-point structure in the denominator is the dual of $\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}^\dagger]\rangle$ via the two-point pairing (E.72).

Note that the structure $\left(\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)}\right)^{-1}$ transforms like a three-point function of representations $\langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_2^\dagger\mathcal{O}^{\dagger\mathrm{F}}\rangle$ and similarly for the operators 3 and 4. In (6.211), we are implicitly contracting Lorentz indices of $\mathcal{O}_i$ with their dual indices in these structures.

6.4.2.3 Proof using weight-shifting operators

Equation (6.211) follows if we prove the generalization of the expression (6.199) for H, with H defined using the appropriate generalization of (6.187). Specifically, the definition of H becomes

$$\begin{aligned}
&H_{\Delta,\rho,(ab)}(x_i)\\
&= -\mu(\Delta,\rho^\dagger)S_E(\mathcal{O}_1\mathcal{O}_2[\widetilde{\mathcal{O}}^\dagger])^c{}_a(\langle\mathcal{O}_1\mathcal{O}_2\widetilde{\mathcal{O}}^\dagger\rangle^{(c)},\langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_2^\dagger\mathcal{O}\rangle^{(d)})^{-1}\times\\
&\quad\times\int_{2<x<1} d^dxD^{d-2}z\langle 0|\widetilde{\mathcal{O}}_1^\dagger\mathbf{L}[\mathcal{O}](x,z)\widetilde{\mathcal{O}}_{2^+}^\dagger|0\rangle^{(d)}(\langle 0|\mathcal{O}_{4^+}\mathbf{L}[\mathcal{O}](x,z)\mathcal{O}_3|0\rangle^{(b)})^{-1}.
\end{aligned}\tag{6.213}$$

We want to prove that

$$H_{\Delta,\rho,(ab)}(x_i) = -\frac{1}{2\pi i}\frac{\left(\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)}\right)^{-1}\left(\mathcal{T}_4\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle^{(b)}\right)^{-1}}{\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}^\dagger]\rangle^{-1}}.\tag{6.214}$$

Our proof will proceed in two steps. Here we are going to show that if for a given ρ (6.214) is valid for some "seed" choice of SO(d) irreps of external operators, it is then valid for all choices of external irreps. In appendix E.7 using methods of [3] we show that validity of (6.214) for traceless-symmetric ρ implies its validity for seed blocks for all ρ. Together these statements imply (6.214) in full generality.

Generalizing the external representations It is convenient to consider the structure defined by

$$T_a \equiv \mu(\Delta,\rho^\dagger)S_E(\mathcal{O}_1\mathcal{O}_2[\widetilde{\mathcal{O}}^\dagger])^c{}_a(\langle\mathcal{O}_1\mathcal{O}_2\widetilde{\mathcal{O}}^\dagger\rangle^{(c)},\langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_2^\dagger\mathcal{O}\rangle^{(d)})_E^{-1}\langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_{2^+}^\dagger\mathcal{O}\rangle^{(d)}. \tag{6.215}$$

We can check that

$$T_a = (\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E(\langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle^{(a)})_E^{-1}, \tag{6.216}$$

where all pairings and inverses are Euclidean. Indeed, we can compute the Euclidean paring

$$\begin{aligned}(T_d,\langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle^{(a)})_E =&S_E(\mathcal{O}_1\mathcal{O}_2[\mathcal{O}^\dagger])^a{}_b(T_a,\langle\mathcal{O}_1\mathcal{O}_2\widetilde{\mathcal{O}}^\dagger\rangle^{(b)})_E\\ =&\mu(\Delta,\rho^\dagger)S_E(\mathcal{O}_1\mathcal{O}_2[\mathcal{O}^\dagger])^a{}_bS_E(\mathcal{O}_1\mathcal{O}_2[\widetilde{\mathcal{O}}^\dagger])^b{}_d\\ =&\mu(\Delta,\rho^\dagger)\mathcal{N}(\Delta,\rho^\dagger)\delta^a_d = (\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E\delta^a_d.\end{aligned} \tag{6.217}$$

Here we used the relation (E.56) between the Plancherel measure and the square of the Euclidean shadow transform. Importance of the structures T_a comes from the fact that it is the light transform of their Wick rotation which enters (6.213).

We now choose some other SO(d) irreps ρ_1' and ρ_2' for operators $\mathcal{O}_1'$ and $\mathcal{O}_2'$ such that there is a unique tensor structure[45]

$$\langle\mathcal{O}_1'\mathcal{O}_2'\widetilde{\mathcal{O}}^\dagger\rangle. \tag{6.218}$$

We then can write

$$T_a = (\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E\mathcal{T}_2^{-1}\mathcal{D}_{12,a}\mathcal{T}_2(\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}^\dagger]\rangle)_E^{-1}, \tag{6.219}$$

where $\mathcal{D}_{12,a}$ are contractions of weight-shifting operators acting on points 1 and 2 [3, 61].[46] We can use this to write

$$H_{\Delta,\rho,(ab)}(x_i) = \mathcal{D}_{12,a}H'_{\Delta,\rho,(b)}(x_i), \tag{6.220}$$

[45] In odd dimensions and for fermionic ρ the number of tensor structures is always even, and so it is not possible to make this choice. However, there we can make a choice such that there is only one parity-even structure, which will be good enough.

[46] Note that $\mathcal{T}_2^{-1}\mathcal{D}_{12,d}\mathcal{T}_2$ are differential operators which can be interpreted in Euclidean signature. In particular, if $\mathcal{D}_{12,d} = D_{1,A}D_2^A$ for A transforming in an irreducible representation W of the conformal group then $\mathcal{T}_2^{-1}\mathcal{D}_{12,d}\mathcal{T}_2$ is proportional to $\mathcal{D}_{12,d}$ with coefficient equal to the eigenvalue of $\mathcal{T}$ in W.

where H' is given by (6.213) with $\mathcal{O}_1'$ and $\mathcal{O}_2'$ instead of $\mathcal{O}_1$ and $\mathcal{O}_2$, and using the unique tensor structure on the left of H'.

On the other hand, we can write

$$\begin{aligned}\delta_d^a &= \frac{1}{(\langle\mathcal{O}^\dagger\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}(T_d, \langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle^{(a)})_E \\ &= (\mathcal{T}_2^{-1}\mathcal{D}_{12,d}\mathcal{T}_2(\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}^\dagger]\rangle)^{-1}, \langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle^{(a)})_E \\ &= ((\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}^\dagger]\rangle)^{-1}, (\mathcal{T}_2^{-1}\mathcal{D}_{12,d}\mathcal{T}_2)^*\langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle^{(a)})_E, \end{aligned} \tag{6.221}$$

where we integrated the differential operators $\mathcal{T}_2^{-1}\mathcal{D}_{12,d}\mathcal{T}_2$ by parts inside the Euclidean pairing. This produces new operators $\mathcal{D}_{12,d}^*$, which are again contractions of weight-shifting operators.[47] We thus conclude that

$$(\mathcal{T}_2^{-1}\mathcal{D}_{12,d}\mathcal{T}_2)^*\langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle^{(a)} = \delta_d^a\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}^\dagger]\rangle. \tag{6.222}$$

Canceling $\mathbf{S}_E$ on both sides (it is invertible on generic tensor structures) we find

$$(\mathcal{T}_2^{-1}\mathcal{D}_{12,d}\mathcal{T}_2)^*\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}^\dagger\rangle^{(a)} = \delta_d^a\langle\mathcal{O}_1'\mathcal{O}_2'\mathcal{O}^\dagger\rangle. \tag{6.223}$$

We now want to show that

$$\mathcal{D}_{12,a}(\mathcal{T}_2\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1} = (\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)})_L^{-1}, \tag{6.224}$$

where the inverse structure is understood with respect to Lorentzian pairing. This follows by doing the above calculation in reverse and in Lorentzian signature. First, we apply $\mathbf{L}$ to both sides of (6.223) and use $\mathcal{T}^* = \mathcal{T}^{-1}$,

$$\mathcal{T}_2^{-1}\mathcal{D}_{12,d}^*\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)} = \delta_d^a\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{L}[\mathcal{O}^\dagger]\rangle, \tag{6.225}$$

Then, we apply $\mathcal{T}_2$ to both sides and take Lorentzian contraction with $(\mathcal{T}_2\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1}$

$$((\mathcal{T}_2\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1}, \mathcal{D}_{12,d}^*\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)})_L = \delta_d^a, \tag{6.226}$$

and finally integrate by parts,

$$(\mathcal{D}_{12,d}(\mathcal{T}_2\langle\mathcal{O}_1'\mathcal{O}_2'\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1}, \mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle^{(a)})_L = \delta_d^a. \tag{6.227}$$

This is equivalent to (6.224) The crucial point here is that integration by parts leads to the same operation on the weight-shifting operators both in Euclidean

[47] For details see appendix E.6 and [3, 195].

and Lorentzian signature (on integer-spin operators). A way to summarize this calculation is by saying that

$$(\mathcal{T}_2\langle \mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1} \quad \text{and} \quad \mathcal{T}_2(\langle \mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle)_E^{-1} \tag{6.228}$$

have the same transformation properties under weight-shifting operators acting on 1 and 2.

This implies that if (6.214) is true for $\mathcal{O}_1'$ and $\mathcal{O}_2'$, it is also true for $\mathcal{O}_1$ and $\mathcal{O}_2$, since we can simply apply $\mathcal{D}_{12,a}$ in both (6.213) and (6.214). Since exactly the same tensor structure appears for the operators $\mathcal{O}_3, \mathcal{O}_4$ in (6.213) and (6.214), an analogous (even simpler) argument works for this tensor structure as well. In conclusion, if (6.214) holds for a seed conformal block, it holds for all conformal blocks with the same ρ.

6.5 Conformal Regge theory

6.5.1 Review: Regge kinematics

Consider a time-ordered four-point function of scalar operators $\langle\phi_1\cdots\phi_4\rangle$. Its conformal block expansion in the $12 \to 34$ channel takes the form

$$\begin{aligned}\langle\phi_1(x_1)\cdots\phi_4(x_4)\rangle &= \sum_{\Delta,J} p_{\Delta,J} G^{\Delta_i}_{\Delta,J}(x_i)\\ &= \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{\Delta_2-\Delta_1}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}}\sum_{\Delta,J} p_{\Delta,J} G^{\Delta_i}_{\Delta,J}(\chi,\overline{\chi}),\end{aligned} \tag{6.229}$$

where $p_{\Delta,J}$ are products of OPE coefficients. This expansion is convergent whenever $\chi, \overline{\chi} \in \mathbb{C}\backslash[1,\infty)$ [25]. However, it fails to converge in the Regge limit.[48]

To reach the Regge regime, which was originally described for CFT correlators in [166], let us place the operators in a 2d Lorentzian plane with lightcone coordinates

$$\begin{aligned} x_1 &= (-\rho, -\overline{\rho}),\\ x_2 &= (\rho, \overline{\rho}),\\ x_3 &= (1,1),\\ x_4 &= (-1,-1). \end{aligned} \tag{6.230}$$

The usual cross-ratios are given by

$$\chi = \frac{4\rho}{(1+\rho)^2}, \qquad \overline{\chi} = \frac{4\overline{\rho}}{(1+\overline{\rho})^2}. \tag{6.231}$$

[48] The other OPE channels $14 \to 23$ and $13 \to 24$ are still convergent, though they are approaching the boundaries of their regimes of validity, as discussed in the introduction.

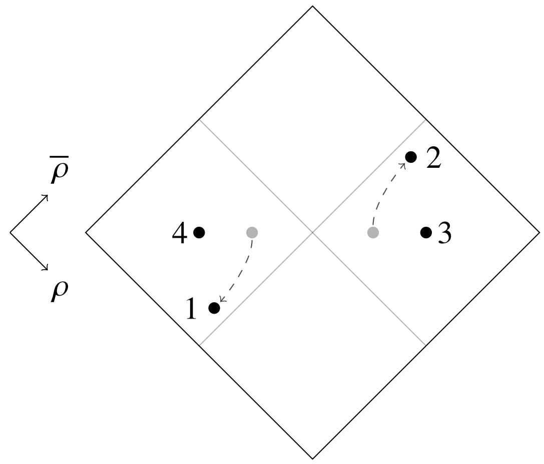


Figure 6.11: The Regge limit in the configuration (6.230). We boost points 1 and 2 while keeping points 3 and 4 fixed. This configuration is related by an overall boost to the one in figure 6.1.

It is also useful to introduce polar coordinates

$$\rho = re^{i\theta} = rw, \qquad \overline{\rho} = re^{-i\theta} = rw^{-1}. \tag{6.232}$$

In Euclidean signature, r and θ are real. By contrast in Lorentzian signature, r is real, θ becomes pure-imaginary (it is conjugate to a boost), and $\rho, \overline{\rho}$ become independent real variables. To reach the Regge regime, we apply a large boost to operators 1 and 2, while keeping 3 and 4 fixed (figure 6.11). More precisely, we take

$$\theta = it + \epsilon, \qquad (t \to \infty), \tag{6.233}$$

so that

$$\rho = re^{-t+i\epsilon}, \quad \overline{\rho} = re^{t-i\epsilon}, \qquad (t \to \infty). \tag{6.234}$$

Here, we use the correct $i\epsilon$ prescription to compute a time-ordered Lorentzian correlator when $t > 0$. With this prescription, the cross-ratios behave as follows. As t-increases, χ moves toward zero. Meanwhile, $\overline{\chi}$ initially increases, then goes counterclockwise around 1, and finally decreases back to zero (figure 6.12).

The only difference between the Regge and $1 \to 2$ OPE limits from the perspective of the cross-ratios $\chi, \overline{\chi}$ is the continuation of $\overline{\chi}$ around 1. In both cases, we take $\chi, \overline{\chi} \to 0$. This is because the Regge limit resembles an OPE limit between points in different Poincare patches. This observation was made in [236]. Specifically, the configuration in figure 6.11 is related by a boost to the one in figure 6.13. The

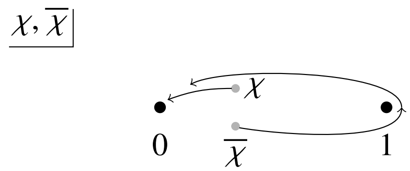


Figure 6.12: The paths of the cross ratios $\chi, \overline{\chi}$ when moving from the Euclidean regime to the Regge regime. In the Euclidean regime, $\chi, \overline{\chi}$ are complex conjugates (gray points). As we boost x_1, x_2, the cross ratio χ decreases towards zero, while $\overline{\chi}$ moves counterclockwise around 1 before decreasing towards zero. For sufficiently large t, $\overline{\chi}$ follows the same path as χ, but we have separated the paths to clarify the figure.

Regge limit can thus be described as $1 \to 2^-$ and $3 \to 4^-$. The cross-ratios $\chi, \overline{\chi}$ are unchanged when we apply $\mathcal{T}$ to any of the points, which is why they still go to zero in this limit.

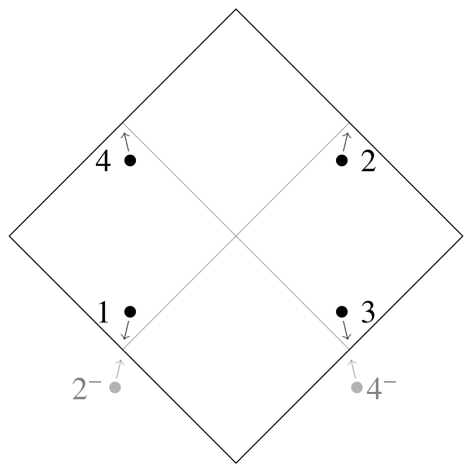


Figure 6.13: Another description of the Regge limit is $x_1 \to x_2^-$ and $x_3 \to x_4^-$. The points x_2^-, x_4^- are shown in gray. The cross-ratios $\chi, \overline{\chi}$ associated with the points $1, 2, 3$ and 4 are the same as those associated with $1, 2^-, 3$ and 4^-.

To understand what happens to the conformal block expansion (6.229) in the Regge regime, we must compute the monodromy of $G^{\Delta_i}_{\Delta,J}(\chi, \overline{\chi})$ from taking $\overline{\chi}$ counterclockwise around 1. This was described in [66]. Firstly, we have the decomposition

$$G^{\Delta_i}_{\Delta,J}(\chi, \overline{\chi}) = g^{\text{pure}}_{\Delta,J}(\chi, \overline{\chi}) + \frac{\Gamma(J+d-2)\Gamma(-J-\frac{d-2}{2})}{\Gamma(J+\frac{d-2}{2})\Gamma(-J)} g^{\text{pure}}_{\Delta,2-d-J}(\chi, \overline{\chi}), \tag{6.235}$$

where $g^{\text{pure}}_{\Delta,J}$ is the solution to the conformal Casimir equation defined by

$$g^{\text{pure}}_{\Delta,J}(\chi, \overline{\chi}) = \chi^{\frac{\Delta-J}{2}} \overline{\chi}^{\frac{\Delta+J}{2}} \times (1 + \text{integer powers of } \chi/\overline{\chi}, \overline{\chi}) \qquad (\chi \ll \overline{\chi} \ll 1). \tag{6.236}$$

For small χ, $g^{\mathrm{pure}}_{\Delta,J}$ has a simple form in terms of a hypergeometric function [64],

$$g^{\mathrm{pure}}_{\Delta,J}(\chi,\overline{\chi}) = \chi^{\frac{\Delta-J}{2}} k_{\Delta+J}(\overline{\chi}) \times (1+O(\chi)) \qquad (\chi \ll 1), \tag{6.237}$$

$$k_{2\overline{h}}(\overline{\chi}) = \overline{\chi}^{\overline{h}} {}_2F_1\left(\overline{h}-\frac{\Delta_{12}}{2}, \overline{h}+\frac{\Delta_{34}}{2}, 2\overline{h}, \overline{\chi}\right), \tag{6.238}$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$. The monodromy of $g^{\mathrm{pure}}_{\Delta,J}$ as $\overline{\chi}$ goes around 1 can then be determined from (6.237) using elementary hypergeometric function identities, keeping χ small so that the approximation (6.237) remains valid.

Let us defer discussing the precise form of the monodromy until section 6.5.3, and focus on one important feature. Note that $k_{2\overline{h}}(\overline{\chi})$ is a conformal block for $\mathrm{SL}(2,\mathbb{R})$. In particular, it is a solution to the conformal Casimir equation (a second-order differential equation) with eigenvalue $\overline{h}(\overline{h}-1)$. Under monodromy, it will mix with the other solution, which differs by $\overline{h} \to 1-\overline{h}$. In terms of Δ, J, this becomes

$$(\Delta, J) \to (1-J, 1-\Delta), \tag{6.239}$$

i.e., it is the affine Weyl reflection associated to the light transform. After monodromy, in the limit $\chi, \overline{\chi} \to 0$ each block contains a term

$$\chi^{\frac{\Delta-J}{2}} \overline{\chi}^{\frac{1-\Delta+1-J}{2}} \sim e^{(J-1)t} \qquad (t \gg 1). \tag{6.240}$$

In other words, the monodromy of each block grows as $e^{(J-1)t}$ in the Regge limit. Because the sum (6.229) includes arbitrarily large J, the OPE expansion formally diverges as $t \to \infty$.

In what follows, it will be important to understand the large-J limit of conformal blocks in slightly more detail. We compute this in appendix E.8.3. The result is ($|J| \gg 1$)

$$g^{\mathrm{pure}}_{\Delta,J}(\chi,\overline{\chi}) \sim \frac{4^{\Delta} f_{1-\Delta}(\frac{1}{2}(r+\frac{1}{r}))w^{-J}}{(1-w^2)^{\frac{d-2}{2}}(r^2+\frac{1}{r^2}-w^2-\frac{1}{w^2})^{\frac{1}{2}}} \left(\frac{(1-\frac{r}{w})(1-rw)}{(1+\frac{r}{w})(1+rw)}\right)^{\frac{\Delta_{12}-\Delta_{34}}{2}}, \tag{6.241}$$

where $w = e^{i\theta}$ and $f_{1-\Delta}(x)$ is given in (E.185). For us, the most important feature of (6.241) is that its J-dependence is w^{-J}. Note that the small-w limit of (6.241) is consistent with the claim that $g^{\mathrm{pure}}_{\Delta,J}$ grows as $w^{1-J} = e^{(J-1)t}$ in the limit $t \to \infty$.

6.5.2 Review: Sommerfeld-Watson resummation

Taking the monodromy of $\overline{\chi}$ around 1 requires leaving the region $|\overline{\rho}| < 1$ where the sum over Δ in the conformal block expansion converges. The conformal partial wave expansion gives a way to avoid this problem: we replace a sum of the form $\sum_\Delta |\rho\overline{\rho}|^{\Delta/2}$ with an integral over $\Delta \in \frac{d}{2} + i\mathbb{R}$. This integral is better-behaved when $|\overline{\rho}| > 1$.

In the Regge limit we still have the problem that each individual block grows like $e^{(J-1)t}$. This can be dealt with in a similar way: by replacing the sum over J with an integral in the imaginary direction. This trick is called the Sommerfeld-Watson transform.

Let us begin with the conformal partial wave expansion

$$\begin{aligned} \langle\phi_1(x_1)\cdots\phi_4(x_4)\rangle &= \sum_{J=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}C(\Delta,J)F^{\Delta_i}_{\Delta,J}(x_i),\\ F^{\Delta_i}_{\Delta,J}(x_i) &\equiv \frac{1}{2}\left(G^{\Delta_i}_{\Delta,J}(x_i) + \frac{S_E(\phi_1\phi_2[O])}{S_E(\phi_3\phi_4[\widetilde{O}])}G^{\Delta_i}_{d-\Delta,J}(x_i)\right). \end{aligned} \tag{6.242}$$

For integer J, the coefficient function $C(\Delta, J)$ can be written

$$C(\Delta,J) = C^t(\Delta,J) + (-1)^J C^u(\Delta,J), \qquad (J\in\mathbb{Z}), \tag{6.243}$$

where C^t comes from the first term in the Lorentzian inversion formula (6.196), and C^u comes from the second term with $1 \leftrightarrow 2$. (The superscripts t and u stand for "t-channel" and "u-channel.") Each of the functions $C^{t,u}(\Delta, J)$ has a natural analytic continuation in J that is bounded in the right half-plane. This follows from (6.196), since the conformal block $G^{\widetilde{\Delta}_i}_{J+d-1,\Delta-d+1}(\chi,\overline{\chi})$ is well-behaved in the square $\chi,\overline{\chi}\in[0,1]$ when J is in the right half-plane.

Let us split the partial wave $F^{\Delta_i}_{\Delta,J}$ into two pieces

$$F^{\Delta_i}_{\Delta,J}(x_i) = \mathcal{F}_{\Delta,J}(x_i) + \mathcal{H}_{\Delta,J}(x_i), \tag{6.244}$$

where $\mathcal{F}_{\Delta,J}$ behaves like w^{-J} at large J,

$$\begin{aligned} &\mathcal{F}_{\Delta,J}(x_i) \equiv \\ &\frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{\Delta_2-\Delta_1}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}}\frac{1}{2}\left(g^{\text{pure}}_{\Delta,J}(\chi,\overline{\chi}) + \frac{S_E(\phi_1\phi_2[O])}{S_E(\phi_3\phi_4[\widetilde{O}])}g^{\text{pure}}_{d-\Delta,J}(\chi,\overline{\chi})\right), \end{aligned} \tag{6.245}$$

and $\mathcal{H}_{\Delta,J}(x_i)$ represents the remaining terms, which behave like w^{J+d-2} at large J. We must treat the two terms in (6.244) differently in the Sommerfeld-Watson transform. Let us focus on the first term. The sum over integer spins can be written as a contour integral

$$\sum_{J=0}^{\infty} C(\Delta, J)\mathcal{F}_{\Delta,J}(x_i) = -\oint_{\Gamma} dJ \frac{C^t(\Delta, J) + e^{-i\pi J} C^u(\Delta, J)}{1 - e^{-2\pi i J}} \mathcal{F}_{\Delta,J}(x_i)$$
$$(\mathrm{Re}(\theta) \in (0, \pi), \mathrm{Im}(\theta) = 0), \qquad (6.246)$$

where the contour Γ encircles all the nonnegative integers clockwise. Here, we have carefully chosen the analytic continuation of $C(\Delta, J)$ so that the integrand is bounded at large J in the right half-plane whenever θ satisfies the given conditions. For this, we use the fact that $\mathcal{F}_{\Delta,J}(x_i)$ behaves as w^{-J} at large J. Because the other term in (6.244) behaves as w^{J+d-2} at large J, we must replace $e^{-i\pi J} \to e^{i\pi J}$ to get an integral for that term that is valid in the same range of θ.

The contour integral (6.246) is more suitable than a naïve sum over spins for continuing to the Regge regime. Recall that the issue with a sum over J was that a conformal block with spin J grows as $e^{(J-1)t}$ in the Regge limit. Because the integrand in (6.246) is well-behaved at large J, we can deform the contour Γ to a region where $\mathrm{Re}(J) < 1$, so that its contributions die as $t \to \infty$.[49] In doing so, we may pick up new poles in $C^{u,t}(\Delta, J)$ with real part $\mathrm{Re}(J) > 1$. The rightmost such pole will dominate the correlator in the Regge limit. Denote the deformed contour, including these new poles, by Γ' (figure 6.14).

After deforming the contour, we now have a representation of the correlator that is valid in the strip

$$\mathrm{Re}(\theta) \in (0, \pi), \quad \mathrm{Im}(\theta) > 0, \qquad (6.247)$$

which includes the angle $\theta = it + \epsilon$ required for a time-ordered Lorentzian correlator. Thus, we can continue to the Regge regime. The continuation of $\mathcal{H}_{\Delta,J}(x_i)$ does not give a growing contribution in the Regge limit, so let us ignore it for the moment. We find that the four-point function behaves as

$$\langle \phi_1(x_1) \cdots \phi_4(x_4) \rangle \sim -\oint_{\Gamma'} dJ \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{C^t(\Delta, J) + e^{-i\pi J} C^u(\Delta, J)}{1 - e^{-2\pi i J}} \mathcal{F}_{\Delta,J}(x_i)^{\circlearrowleft}, \qquad (6.248)$$

[49] A natural choice is the Lorentzian principal series $\mathrm{Re}(J) = -\frac{d-2}{2}$.

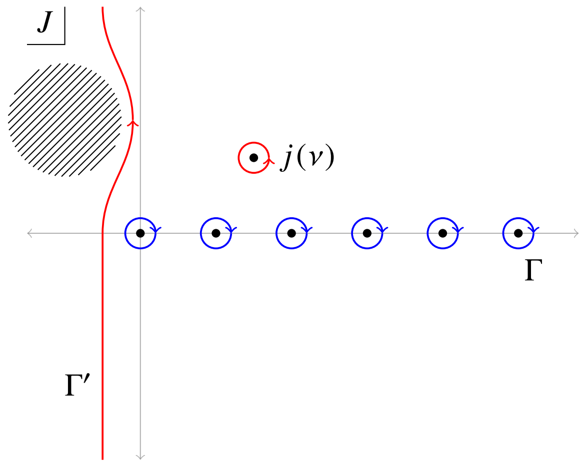


Figure 6.14: Integration contours in the J plane. The contour Γ (blue) encircles all the integers clockwise. The deformed contour Γ' runs parallel to the imaginary axis, asymptotically approaching $\mathrm{Re}(J) = -\frac{d-2}{2}$ at large imaginary J. In deforming the contour, we must ensure that Γ' avoids non-analyticities, like a pole at non-integer J, branch cuts, or other singularities. Here, we show a single non-integer pole at $J = j(\nu)$ and possible non-analyticities in the shaded region. However, this is only an example—we don not know the structure of the J-plane in general.

where $\mathcal{F}_{\Delta,J}(x_i)^{\circlearrowleft}$ denotes the continuation to Regge kinematics, including the monodromy of $\overline{\chi}$ around 1 and phases arising from the prefactor in (6.245).[50]

In planar large-N theories, the rightmost feature of Γ' is conjectured to be an isolated pole $J = j(\nu)$ where $\Delta = \frac{d}{2} + i\nu$. Assuming this is the case, we obtain

$$\begin{aligned}&\langle\phi_1(x_1)\cdots\phi_4(x_4)\rangle\\ &\sim -2\pi i\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\operatorname*{Res}_{J=j(\nu)}\frac{C^t(\frac{d}{2}+i\nu,J)+e^{-i\pi J}C^u(\frac{d}{2}+i\nu,J)}{1-e^{-2\pi i J}}\mathcal{F}_{\frac{d}{2}+i\nu,J}(x_i)^{\circlearrowleft}.\end{aligned} \tag{6.249}$$

6.5.3 Relation to light-ray operators

The appearance of the affine Weyl transform (6.239) is suggestive that Regge kinematics should be related to the light transform and light-ray operators. To see how, let us finally compute $\mathcal{F}_{\Delta,J}(x_i)^{\circlearrowleft}$ using (6.237). We find

$$\begin{aligned}\mathcal{F}_{\Delta,J}(x_i)^{\circlearrowleft} &= -\frac{2i\pi^3\Gamma(\Delta+J)\Gamma(\Delta+J-1)}{\Gamma(\frac{\Delta+J+\Delta_{12}}{2})\Gamma(\frac{\Delta+J-\Delta_{12}}{2})\Gamma(\frac{\Delta+J+\Delta_{34}}{2})\Gamma(\frac{\Delta+J-\Delta_{34}}{2})}T^{\Delta_i}(x_i)G_{1-J,1-\Delta}(\chi,\overline{\chi})\\ &\quad+\dots,\end{aligned} \tag{6.250}$$

[50]Representing the correlator as an integral over both Δ and J is natural from the point of view of Lorentzian harmonic analysis, where principal series representations are labeled by continuous $\Delta = \frac{d}{2} + is$ and $J = -\frac{d-2}{2} + it$. However, it is not immediately obvious how the representation (6.248) is related to the Plancherel theorem for $\widetilde{\mathrm{SO}}(d,2)$. We leave this question for future work.

where $T^{\Delta_i}(x_i)$ is the product of $|x_{ij}|$'s given in (6.194). Here, we have explicitly written the term that is growing in the Regge limit. The "..." represent other solutions of the Casimir equations that do not grow in the Regge limit, coming from both $\mathcal{F}_{\Delta,J}$ and $\mathcal{H}_{\Delta,J}$. The above expression is valid in the configuration $4 > 1, 2 > 3$, with other points spacelike-separated.

Comparing with (6.190) and (6.199), we immediately recognize

$$\mathcal{F}_{\Delta,J}(x_i)^{\circlearrowleft} = \pi^2 \mathcal{T}_2^{-1}\mathcal{T}_4^{-1} \frac{(\mathcal{T}_2\langle\phi_1\phi_2\mathbf{L}[O^\dagger]\rangle)(\mathcal{T}_4\langle\phi_3\phi_4\mathbf{L}[O]\rangle)}{\langle\mathbf{L}[O]\mathbf{L}[O^\dagger]\rangle} + \dots, \tag{6.251}$$

where we use the notation for a conformal block introduced in section 6.4.1.4. Equation (6.251) is the main observation of this section. In the case where Regge kinematics is dominated by an isolated pole (6.249), the residue $\mathrm{Res}_{J=j(\nu)}$ means that coefficients in the integrand can be interpreted as products of OPE coefficients for light-ray operators. This is because a nontrivial residue comes from the neighborhood of the light ray.[51] Plugging (6.251) into (6.249), we find a sum/integral of conformal blocks for these light-ray operators.

In the gauge-theory literature, the object that controls the Regge limit of a planar amplitude is called the "Pomeron" [246, 247]. Here, we see that for planar CFT correlation functions, the Pomeron is a light-ray operator: it is proportional to the rightmost residue in J of $\mathbb{O}_{\Delta,J}$, for $\Delta \in \frac{d}{2} + i\mathbb{R}$.

The observation (6.251) also lets us immediately generalize conformal Regge theory to arbitrary operator representations. In the Regge limit, we have

$$\begin{aligned}
&\langle O_1(x_1)\cdots O_4(x_4)\rangle \\
&\sim -\pi^2 \sum_{\lambda,a,b} \oint_{\Gamma'} dJ \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{C_{ab}(\Delta,J,\lambda)}{1-e^{-2\pi i J}} \\
&\qquad\qquad \times \mathcal{T}_2^{-1}\mathcal{T}_4^{-1} \frac{(\mathcal{T}_2\langle O_1O_2\mathbf{L}[O^\dagger]\rangle^{(a)})(\mathcal{T}_4\langle O_3O_4\mathbf{L}[O]\rangle^{(b)})}{\langle\mathbf{L}[O]\mathbf{L}[O^\dagger]\rangle}.
\end{aligned} \tag{6.252}$$

Here, $C_{ab}(\Delta,J,\lambda)$ is the unique analytic continuation of $C_{ab}(\Delta,\rho)$ such that $\frac{C_{ab}(\Delta,J,\lambda)}{1-e^{-2\pi i J}}e^{-i\theta J}$ is bounded for large J in the right-half plane and $\theta \in (0,\pi)$. The weight J is the length of the first row of the Young diagram of ρ, and λ represents the remaining weights of ρ, as discussed in section 6.2.2. The indices a, b run over three-point structures.

[51] The same is true if the Regge limit is dominated by a cut instead of a pole, though now we have a doubly-continuous family of light-ray operators, parameterized by ν and J along the cut.

As before, it is straightforward to argue that (6.252) is the only possibility consistent with the scalar case and with weight-shifting operators. It would be interesting to verify it more directly, and in general to characterize all monodromies of blocks in terms of the integral transforms in section 6.2.3. Note that (6.252) displays a beautiful duality with the generalized Lorentzian inversion formula (6.211).

We can try to interpret (6.251) as a contribution to the non-vacuum OPE of $\phi_1\phi_2$ in the following way. We construct light-ray operators as an integral of the form (6.15), which together with conformal symmetry implies that we should be able to write, schematically,

$$\phi_1\phi_2 = \int d\nu\, \mathcal{B}_{\nu,j(\nu)}[\mathbb{O}_{0,j(\nu)}] + \text{other contributions}. \tag{6.253}$$

Here $\mathcal{B}$ is a kind of OPE kernel which is fixed by conformal symmetry, and the equation should be interpreted in an operator sense. The representation (6.251) suggests that (6.253) is a good version of the OPE in non-vacuum states, with the first term giving the only possibly-growing contribution in the Regge limit.

The "other contributions" can perhaps be understood by studying the terms that we ignored above, coming form $\mathcal{H}_{\Delta,J}$ and part of $\mathcal{F}^{\circlearrowleft}_{\Delta,J}$. We expect that they can be understood more systematically using harmonic analysis on the Lorentzian conformal group $\widetilde{\mathrm{SO}}(d,2)$. (We hope to address this in future work.) In a finite-N CFT, the correlator saturates in the Regge limit — i.e., it eventually stops growing. Thus, the details of these terms will presumably be important for determining the actual behavior of the correlator in the Regge limit.[52]

6.6 Positivity and the ANEC

The average null energy condition (ANEC) states that $\mathcal{E} = \mathbf{L}[T]$ is a positive-semidefinite operator. The ANEC was proven in [72] using information theory and in [73] using causality. The causality-based proof [73] proceeds by isolating the contribution of $\mathcal{E}$ in a correlation function and using Rindler positivity to show that the contribution is positive. Isolating $\mathcal{E}$ requires using the OPE outside its naïve regime of validity. However, the authors of [73] give an argument that one can still trust the leading term in the OPE in an asymptotic expansion in the lightcone limit.

From our work in section 6.3, we now have an alternative construction of $\mathcal{E}$ as a special case of a light-ray operator. Using this construction, we can avoid asymptotic expansions and any technical issues associated with using the OPE outside its regime

[52] We thank Sasha Zhiboedov for discussions on this point.

of validity. Beyond technical convenience, our approach gives extra flexibility. The authors of [73] also prove a higher-spin version of the ANEC:

$$\mathcal{E}_J \equiv \mathbf{L}[X_J] \geq 0, \qquad (J = 2, 4, \dots), \tag{6.254}$$

where X_J is the lowest-dimension operator with spin J.[53,54] Our construction lets us generalize this statement to

$$\mathcal{E}_J \geq 0, \qquad (J \in \mathbb{R}_{\geq J_{\text{min}}}), \tag{6.255}$$

where $J_{\text{min}} \leq 1$ is the smallest value of J for which the Lorentzian inversion formula holds [66]. Here, $\mathcal{E}_J(x, z)$ denotes the light-ray operator with dimension and spin $(1 - J, 1 - \Delta)$, where Δ, J are real and Δ is minimal. This result follows by writing a sum rule for *all* light-ray operators, and simply observing that it is positive by Rindler positivity when (Δ, J) satisfy the above conditions. When J is an integer, (6.255) reduces to (6.254). However, when J is not an integer, (6.255) is a new condition.

A possible connection between Lorentzian inversion formulae and the ANEC was first suggested by Caron-Huot using a toy dispersion relation [66]. In this section, we are simply making the connection more precise.

6.6.1 Rindler positivity

Rindler positivity is a key ingredient in the causality-based proof of the ANEC [73], so let us review it. Given $x = (t, y, \vec{x}) \in \mathbb{R}^{d+1,1}$, define the Rindler reflection

$$\overline{x} = \overline{(t, y, \vec{x})} = (-t^*, -y^*, \vec{x}). \tag{6.256}$$

Rindler conjugation maps an operator in the right Rindler wedge to an operator in the left Rindler wedge. For traceless-symmetric tensors, it is defined by

$$\overline{O(x, z)} = O^\dagger(\overline{x}, \overline{z}). \tag{6.257}$$

More generally, Rindler conjugation is given by $\overline{O} = JOJ$, where $J = U(R(y, \pi))\mathrm{CPT}$, with $R(y, \pi)$ a rotation by π in the $y\tau$ plane, where $\tau = it$. Note that Rindler conjugation does not change the order of operators

$$\overline{O_1 O_2} = \overline{O}_1 \overline{O}_2. \tag{6.258}$$

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[53] More precisely, X_J can be the lowest-dimension operator with spin J in any OPE of the form $O^\dagger \times O$.

[54] The proof of the higher-spin ANEC in [73] relies on some assumptions about subleading terms when the OPE is used as an asymptotic expansion outside of its regime of convergence. We thank Tom Hartman for discussion on this point.

The statement of Rindler positivity is that

$$\langle\Omega|\overline{O}_1\cdots\overline{O}_n O_1\cdots O_n|\Omega\rangle \geq 0, \tag{6.259}$$

where O_i are restricted to the right Rindler wedge

$$\mathcal{W}_R = \{(u,v,\vec{x}) \ : \ uv > 0, \ \arg v \in (-\tfrac{\pi}{2},\tfrac{\pi}{2}), \ \vec{x}\in\mathbb{R}^{d-2}\}. \tag{6.260}$$

(Here, we use lightcone coordinates $u = y - t, v = y + t$.)

To establish (6.259) for general causal configurations of the O_i, [248] appeals to Tomita-Takesaki theory. However, this is not necessary as argued in [73]. We can summarize their argument as follows. Because the operators $O_1\cdots O_n$ act on the vacuum, we can perform the OPE to replace

$$O_1\cdots O_n|\Omega\rangle = \sum_{O} C(x_i,x,\partial_x)O(x)|\Omega\rangle, \tag{6.261}$$

where $C(x_i,x,\partial_x)$ is a differential operator. We are free to choose x to be any point in $\mathcal{W}_R$ (we cannot choose x to be timelike from the $\overline{x}_i$). Truncating the sum, we approximate the right hand side by a local operator. The expectation value (6.259) then becomes a Rindler-reflection symmetric two-point function. Positivity of this two-point function is a consequence of reflection-positivity, since the two points are spacelike-separated.

6.6.2 The continuous-spin ANEC

Following [73], we will prove

$$i\langle\Omega|\overline{V}\mathcal{E}'_J V|\Omega\rangle \geq 0, \tag{6.262}$$

where V is any local operator located at a point $x_V = (0,\delta,\mathbf{0}) \in \mathcal{W}_R$ in the right Rindler wedge. Here, $\mathcal{E}'_J$ is a continuous-spin light-ray operator of spin-J with lowest twist, oriented along the null direction $z = (1,1,\vec{0})$. As argued in [73], it follows that $\mathcal{E}'_J$ satisfies the positivity condition

$$e^{i\frac{\pi}{2}J}\langle\Omega|(R\cdot V)^\dagger(t=-i\delta)\,\mathcal{E}'_J\,(R\cdot V)(t=i\delta)|\Omega\rangle \geq 0, \tag{6.263}$$

where R rotates by $\frac{\pi}{2}$ in the Euclidean $y\tau$-plane, with $\tau = it$, and $R\cdot V$ represents the action of R on V at the origin. States of the form $(R\cdot V)(t=i\delta)|\Omega\rangle \in \mathcal{H}$ are dense in $\mathcal{H}$, by the state-operator correspondence. Thus,

$$\mathcal{E}_J \equiv e^{i\frac{\pi}{2}J}\mathcal{E}'_J \tag{6.264}$$

is a positive operator.

Let ϕ be a real scalar primary. We will produce $\mathcal{E}'_J$ by smearing two ϕ insertions. For simplicity, we will not attempt to divide by OPE coefficients in the $\phi\times\phi$ OPE. Thus, when J is an integer, we will actually have $\mathcal{E}_J = f_{\phi\phi X_J}\mathbf{L}[X_J]$, where X_J is the lowest-twist operator of spin-J in the $\phi\times\phi$ OPE and $f_{\phi\phi X_J}$ is an OPE coefficient. In particular $\mathcal{E}_2$ in this section differs from the usual ANEC operator by a factor of $f_{\phi\phi T}$.

From (6.174), we have

$$\begin{aligned} i\langle \overline{V}\mathbb{O}^+_{\Delta,J}(-\infty z,z)V\rangle &= \int_{\substack{-\infty z<x_1<\overline{x}_V\\ x_V<x_2<\infty z}} d^dx_1 d^dx_2 \langle\Omega|[\overline{V},\phi(x_1)][\phi(x_2),V]|\Omega\rangle K_{\Delta,J}(x_1,x_2),\\ K_{\Delta,J}(x_1,x_2) &= \frac{2i\mu(\Delta,J)S_E(\phi\phi[\widetilde{O}])}{(\langle\phi\phi\widetilde{O}\rangle,\langle\widetilde{\phi}\widetilde{\phi}O\rangle)_E}\langle 0|\widetilde{\phi}(x_1)\mathbf{L}[O](-\infty z,z)\widetilde{\phi}(x_2)|0\rangle. \end{aligned} \tag{6.265}$$

We have included a factor of 2 from the term $1\leftrightarrow 2$ in (6.174), and we should interpret the prefactors in $K_{\Delta,J}$ as being analytically continued from even J. The matrix elements of $\mathcal{E}_J$ are defined by

$$i\langle\Omega|\overline{V}\mathcal{E}'_J V|\Omega\rangle = \operatorname*{Res}_{\Delta=\Delta_*} i\langle\overline{V}\mathbb{O}^+_{\Delta,J}(-\infty z,z)V\rangle, \tag{6.266}$$

where Δ_* is the location of the pole in $\mathbb{O}^+_{\Delta,J}$ with minimal real Δ. The expression (6.265) is guaranteed to be convergent for $\Delta\in\frac{d}{2}+i\mathbb{R}$ on the principal series. In particular it converges at $\Delta=\frac{d}{2}$. Our strategy will be to show that $i\langle\overline{V}\mathbb{O}^+_{\Delta,J}(x,z)V\rangle$ is strictly negative as we move rightward along the real axis starting from $\Delta=\frac{d}{2}$ (figure 6.15). It follows that the first pole we encounter must have positive residue.[55]

The kernel $K_{\Delta,J}$ is given by

$$K_{\Delta,J}(x_1,x_2) = \frac{2^J i\mu(\Delta,J)S_E(\phi\phi[\widetilde{O}])L(\widetilde{\phi}\widetilde{\phi}[O])}{(\langle\phi\phi\widetilde{O}\rangle,\langle\widetilde{\phi}\widetilde{\phi}O\rangle)_E}\frac{(z\cdot x_2 x_1^2 - z\cdot x_1 x_2^2)^{1-\Delta}}{x_{12}^{2\widetilde{\Delta}_\phi-\Delta+J}(-z\cdot x_1)^{\frac{2-J-\Delta}{2}}(z\cdot x_2)^{\frac{2-J-\Delta}{2}}}. \tag{6.267}$$

We would like to show that $K_{\Delta,J}(x_1,x_2)$ is a positive-definite kernel when integrated against Rindler-symmetric configurations of x_1,x_2. Note that this is a stronger condition than $K_{\Delta,J}(\overline{x},x)\geq 0$ point-wise.

[55] Requiring negativity for all Δ between $\frac{d}{2}$ and the first pole is stronger than necessary. It should be possible to improve our proof by establishing negativity only for Δ sufficiently close to the first pole.

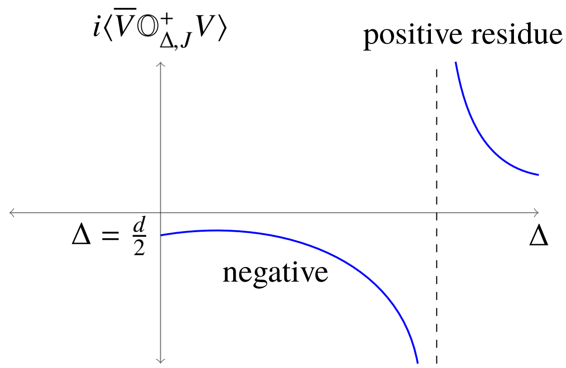


Figure 6.15: We show that $i\langle \overline{V}\mathbb{O}^+_{\Delta,J}V\rangle$ is negative for Δ between $\frac{d}{2}$ (the principal series) and the first pole. It follows that the first pole has positive residue.

Consider first an inversion $x \mapsto x' = \frac{x}{x^2}$ that places $\mathcal{E}_J$ at null infinity. In this conformal frame, the three-point structure $\langle 0|\widetilde{\phi}\mathbf{L}[O]\widetilde{\phi}|0\rangle$ becomes translationally invariant. Thus our kernel should be a translationally-invariant function of x_1', x_2', times some scale-factors that depend independently on x_1, x_2. Indeed, it is easy to check

$$
\begin{aligned}
&\frac{\left(z\cdot x_2\, x_1^2 - z\cdot x_1\, x_2^2\right)^{1-\Delta}}{x_{12}^{2\widetilde{\Delta}_\phi-\Delta+J}(-z\cdot x_1)^{\frac{2-J-\Delta}{2}}(z\cdot x_2)^{\frac{2-J-\Delta}{2}}} \\
&\qquad = x_1'^{2\widetilde{\Delta}_\phi} x_2'^{2\widetilde{\Delta}_\phi}\left(-z\cdot x_1'\right)^{\frac{J+\Delta-2}{2}}\left(z\cdot x_2'\right)^{\frac{J+\Delta-2}{2}}\frac{\left(z\cdot(x_2'-x_1')\right)^{1-\Delta}}{(x_2'-x_1')^{2\widetilde{\Delta}_\phi-\Delta+J}}.
\end{aligned}
\tag{6.268}
$$

Because our kernel originates from the light-transform of a three-point structure, it inherits Rindler positivity properties. These are made clear by going to a kind of complexified Fourier-space in the inverted coordinates x_i'. Define lightcone coordinates $x^- = u = y-t$ and $x^+ = v = y+t$. One can prove the following identity which is valid in the right Rindler wedge $u, v > 0$:

$$
\begin{aligned}
\frac{u^{1-\Delta}}{(uv+\vec{x}^2)^{\frac{2\widetilde{\Delta}_\phi-\Delta+J}{2}}} &= \frac{2^{2-2\widetilde{\Delta}_\phi-J}}{\pi^{\frac{d-2}{2}}\Gamma(\frac{2\widetilde{\Delta}_\phi-\Delta+J}{2})\Gamma(\frac{2\widetilde{\Delta}_\phi+J+\Delta-d}{2})} \\
&\quad\times\int_{k>0} d^dk\,(-k^2)^{\frac{2\widetilde{\Delta}_\phi+\Delta+J-d-2}{2}}(-k^-)^{1-\Delta} f_k(x) \\
f_k(x) &\equiv e^{-\frac{1}{2}k^+u+\frac{1}{2}k^-v+i\vec{k}\cdot\vec{x}}.
\end{aligned}
\tag{6.269}
$$

Here, the notation $k>0$ indicates that k is restricted to the interior of the forward null cone. This ensures that k^+u is positive and k^-v is negative, so that the integral is convergent. The complexified plane wave $f_k(x)$ is designed to satisfy

$$f_k(x)^* = f_k(-\overline{x}). \tag{6.270}$$

Putting everything together, we find

$$K_{\Delta,J}(x_1,x_2) = \mathcal{K}_{\Delta,J}\int_{k>0} d^dk\,(-k^2)^{\frac{2\widetilde{\Delta}_\phi+\Delta+J-d-2}{2}}(-k^-)^{1-\Delta}\psi_k(x_2)(\psi_k(\overline{x}_1))^*, \tag{6.271}$$

where

$$\psi_k(x) \equiv \frac{1}{x^{2\widetilde{\Delta}_\phi}}\left(\frac{u}{x^2}\right)^{\frac{J+\Delta-2}{2}}\exp\left(\frac{-\frac{1}{2}k^+u+\frac{1}{2}k^-v+i\vec{k}\cdot\vec{x}}{x^2}\right), \tag{6.272}$$

$$\mathcal{K}_{\Delta,J} = \frac{2^{1-d-\Delta+J+2\Delta_\phi}\Gamma(J+\frac{d}{2})\Gamma(\frac{J+d+1-\Delta}{2})\Gamma(\Delta-1)}{\pi^{\frac{3(d-1)}{2}}\Gamma(J+1)\Gamma(\frac{d+J-\Delta}{2})\Gamma(\Delta-\frac{d}{2})\Gamma(\frac{J+\Delta+d-2\Delta_\phi}{2})\Gamma(\frac{J-\Delta+2d-2\Delta_\phi}{2})}. \tag{6.273}$$

Consequently, we can write

$$i\langle\overline{V}\mathbb{O}^+_{\Delta,J}(-\infty z,z)V\rangle = -\mathcal{K}_{\Delta,J}\int_{k>0} d^dk\,(-k^2)^{\frac{2\widetilde{\Delta}_\phi+\Delta+J-d-2}{2}}(-k^-)^{1-\Delta}\langle\overline{\Theta_k}\Theta_k\rangle,$$
$$\Theta_k = \int_{x_V<x<\infty z} d^dx\,\psi_k(x)[\phi(x),V]. \tag{6.274}$$

The coefficient $\mathcal{K}_{\Delta,J}$ is positive whenever

$$d > \Delta - J > 2(d-\Delta_\phi). \tag{6.275}$$

This is also the condition for $K_{\Delta,J}(x_1,x_2)$ to be integrable without an $i\epsilon$ prescription. When these conditions hold, the minus sign in (6.274) ensures that the first nontrivial residue in Δ is positive. This proves the ANEC and its continuous spin generalization in this case.

Let us understand the condition $\Delta - J > 2(d-\Delta_\phi)$ in more detail. When this inequality fails, two things happen. Firstly, the factor

$$\Gamma\left(\frac{J-\Delta+2d-2\Delta_\phi}{2}\right) \tag{6.276}$$

in $\mathcal{K}_{\Delta,J}$ may no longer be positive. Secondly, the kernel $K_{\Delta,J}(x_1,x_2)$ develops a naively non-integrable singularity along the lightcone. To make sense of this singularity, one must take into account the appropriate $i\epsilon$ prescription for x_1,x_2. This turns $K_{\Delta,J}(x_1,x_2)$ into a non-sign-definite distribution, and then we cannot conclude anything about the sign of (6.274). To get the strongest result, we should pick ϕ to be the lowest-dimension scalar in the theory. The spin-2 ANEC then follows if $\Delta_\phi \leq \frac{d+2}{2}$. Large-spin perturbation theory [31, 68, 69, 77, 102–104, 106, 162, 228, 229] and Nachtmann's theorem [68, 160, 249, 250] imply that the minimum twist $\Delta - J$ at each spin J is always less than $2\Delta_\phi$. Thus, we can ensure $\Delta - J > 2(d - \Delta_\phi)$ if $\Delta_\phi \leq \frac{d}{2}$. This condition is also sufficient to ensure $d > \Delta - J$. Thus, the continuous-spin ANEC follows if $\Delta_\phi \leq \frac{d}{2}$.

6.6.3 Example: Mean Field Theory

The continuous spin version of ANEC is easy to check in MFT. (This is essentially the same calculation as in [73, 251].) We have already computed the leading twist operators $\mathcal{E}'_J = \mathbb{O}^+_{0,J}$ in section 6.3.4. In this section we need the straightforward generalization of (6.150) to the case of identical operators,

$$\mathcal{E}'_J = \mathbb{O}^+_{0,J} = \frac{i}{2\pi}\int ds dt (t+i\epsilon)^{-1-J} :\phi\left(\frac{s+t}{2}z\right)\phi\left(\frac{s-t}{2}z\right):, \tag{6.277}$$

with a future-directed null z. We can explicitly compute these operators in terms of creation-annihilation operators using

$$\phi(x) = N_{\Delta_\phi}^{-\frac{1}{2}}\int_{p>0}\frac{d^dp}{(2\pi)^d}|p|^{\Delta_\phi-\frac{d}{2}}\left(a^\dagger(p)e^{-ipx} + a(p)e^{ipx}\right), \tag{6.278}$$

where Δ_ϕ is the scaling dimension of ϕ and

$$N_\Delta = \frac{2^{2\Delta-1}\pi^{\frac{d-2}{2}}}{(2\pi)^d}\Gamma(\Delta)\Gamma(\Delta - \tfrac{d-2}{2}) > 0. \tag{6.279}$$

The creation-annihilation operators satisfy the commutation relation

$$[a(p), a^\dagger(q)] = (2\pi)^d\delta^d(p-q). \tag{6.280}$$

Plugging (6.278) into (6.277), we find

$$\begin{aligned}\mathcal{E}'_J = \frac{iN_{\Delta_\phi}^{-\frac{1}{2}}}{2\pi}\int_{p,q>0}\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\int ds dt (t+i\epsilon)^{-1-J}\Big[&a^\dagger(p)a^\dagger(q)e^{-\frac{i}{2}(p+q)\cdot zs-\frac{i}{2}(p-q)\cdot zt}\\ &+ a(p)a(q)e^{\frac{i}{2}(p+q)\cdot zs+\frac{i}{2}(p-q)\cdot zt}\\ &+ a^\dagger(p)a(q)e^{-\frac{i}{2}(p-q)\cdot zs-\frac{i}{2}(p+q)\cdot zt}\\ &+a^\dagger(q)a(p)e^{\frac{i}{2}(p-q)\cdot zs+\frac{i}{2}(p+q)\cdot zt}\Big].\end{aligned} \tag{6.281}$$

The first two terms under the integral vanish because s-integration restricts $(p+q)\cdot z = 0$, which is impossible since both p and q are in the forward null cone. This is consistent with the requirement that $\mathbb{O}^+_{0,J}$ should annihilate both past and future vacua. Since $(p+q)\cdot z < 0$ we can close the t-contour in the upper half-plane for the third term (for $J > 0$) and thus it also vanishes. We are left with the last term, where we can close the t-contour in the lower half-plane. Specifically, we get for s and t integrals

$$\int dsdt(t+i\epsilon)^{-1-J}e^{\frac{i}{2}(p-q)\cdot zs+\frac{i}{2}(p+q)\cdot zt} = \frac{2\pi^2\delta((p-q)\cdot z)e^{-\frac{i\pi}{2}(J+1)}}{\Gamma(J+1)}\left(\frac{-(p+q)\cdot z}{2}\right)^J. \tag{6.282}$$

Combining with the rest of the expression we find, using the lightcone coordinates $p = zp_v/2 - z'p_u/2 + \mathbf{p}$ with $z\cdot z' = 2$,

$$\mathcal{E}'_J = \frac{\pi e^{-\frac{i\pi}{2}J}N_{\Delta_\phi}^{-\frac{1}{2}}}{\Gamma(J+1)}\int_0^\infty dp_u p_u^J A^\dagger(p_u)A(p_u), \tag{6.283}$$

where

$$A(p_u) \equiv \int_{|\mathbf{p}|<p_u p_v} \frac{dp_v d^{d-2}\mathbf{p}}{(2\pi)^d} a(p_u, p_v, \mathbf{p}). \tag{6.284}$$

For $\mathcal{E}_J = e^{\frac{i\pi}{2}J}\mathcal{E}'_J$ we then obtain

$$\mathcal{E}_J = \frac{\pi N_{\Delta_\phi}^{-\frac{1}{2}}}{\Gamma(J+1)}\int_0^\infty dp_u p_u^J A^\dagger(p_u)A(p_u) \geq 0, \tag{6.285}$$

which is manifestly non-negative.

6.6.4 Relaxing the conditions on Δ_ϕ

The conditions (6.275) are stronger than necessary because we have not assumed anything about the quantity that $K_{\Delta,J}(x_1,x_2)$ is integrated against. We can somewhat relax them as follows. Note that poles in $i\langle\overline{V}\mathbb{O}_{\Delta,J}(-\infty z, z)V\rangle$ come from the region where x_1, x_2 are near the lightray $\mathbb{R}z$. In this region, we expect the correlator $\langle\Omega|[\overline{V},\phi(x_1)][\phi(x_2),V]|\Omega\rangle$ to depend most strongly on the positions v_1, v_2 of the operators along the light-ray and simple invariants built out of the relative position $x_1 - x_2$, since $V, \overline{V}$ are far from the light ray.

To be more precise, consider the integral over x_1, x_2 in the coordinates of section 6.3.4,

$$\frac{2^J i\mu(\Delta,J) S_E(\phi\phi[\widetilde{O}]) L(\widetilde{\phi}\widetilde{\phi}[O])}{(\langle\phi\phi\widetilde{O}\rangle, \langle\widetilde{\phi}\widetilde{\phi}O\rangle)_E}$$
$$\times \frac{1}{4}\int \frac{dr}{r} dv_1 dv_2 d\alpha d^{d-2}\mathbf{w}_1 d^{d-2}\mathbf{w}_2 \frac{2^{J-1} v_{21}^{-1-\frac{\Delta-\Delta_1-\Delta_2+J}{2}} \left(\alpha(1-\alpha)+(1-\alpha)\mathbf{w}_1^2+\alpha\mathbf{w}_2^2\right)^{1-\Delta}}{(1+\mathbf{w}_{12}^2)^{\frac{\widetilde{\Delta}_1+\widetilde{\Delta}_2+J-\Delta}{2}} \alpha^{\frac{\widetilde{\Delta}_1-\widetilde{\Delta}_2+2-\Delta-J}{2}} (1-\alpha)^{\frac{\widetilde{\Delta}_2-\widetilde{\Delta}_1+2-\Delta-J}{2}}}$$
$$\times r^{-\frac{\Delta-\Delta_1-\Delta_2-J}{2}} \phi(-r\alpha, v_1, (rv_{21})^{\frac{1}{2}}\mathbf{w}_1)\phi(r(1-\alpha), v_2, (rv_{21})^{\frac{1}{2}}\mathbf{w}_2). \tag{6.286}$$

The most important quantities built from x_{12} are

$$v_{21}, \qquad x_{12}^2 = rv_{21}(1+\mathbf{w}_-^2). \tag{6.287}$$

Let us make the approximation that, to leading order in r, the correlator $\langle[\overline{V},\phi][\phi,V]\rangle$ depends only on v_1, v_2 and x_{12}^2. That is, let us replace

$$\begin{aligned}&\phi(-r\alpha, v_1, (rv_{21})^{\frac{1}{2}}\mathbf{w}_1)\phi(r(1-\alpha), v_2, (rv_{21})^{\frac{1}{2}}\mathbf{w}_2)\\ &\sim \phi\left(-\frac{r}{2}(1+\mathbf{w}_-^2), v_1, 0\right)\phi\left(\frac{r}{2}(1+\mathbf{w}_-^2), v_2, 0\right).\end{aligned} \tag{6.288}$$

This approximation would be valid, for example, if we could perform the OPE $\phi(x_1)\times\phi(x_2)$, since the leading terms in the OPE depend only on v_{21} and x_{12}^2. However, our assumption is weaker than assuming that we can perform the OPE.

After rescaling $r \to r/(1+\mathbf{w}_-^2)$, we can now perform the integrals over α and $\mathbf{w}_\pm$, following the methods in appendix E.4. The result is

$$\begin{aligned}&i\langle\overline{V}\mathbb{O}^+_{\Delta,J}(-\infty z, z)V\rangle\\ &\sim \frac{2^{d+J-4}}{\pi}\int\frac{dr}{r}dv_1 dv_2\, r^{\frac{2\Delta_\phi-\Delta+J}{2}} v_{21}^{\frac{2\Delta_\phi-\Delta-J-2}{2}} \langle\Omega|[\overline{V},\phi(-\tfrac{r}{2},v_1,0)][\phi(\tfrac{r}{2},v_2,0),V]|\Omega\rangle\\ &= -\frac{2^{d+J-4}}{\pi\Gamma\left(\frac{\Delta+J+2-2\Delta_\phi}{2}\right)}\int\frac{dr}{r}r^{\frac{2\Delta_\phi-\Delta+J}{2}}\int_0^\infty dk\, k^{\frac{\Delta+J-2\Delta_\phi}{2}}\langle\Omega|\overline{\Theta_k(r)}\Theta_k(r)|\Omega\rangle,\end{aligned} \tag{6.289}$$

where

$$\Theta_k(r) \equiv \int_0^\infty dv\, e^{-kv}[\phi(\tfrac{r}{2},v,0),V]. \tag{6.290}$$

The integrand in (6.289) should be correct to leading order at small r, which means the leading residue of $i\langle\overline{V}\mathbb{O}_{\Delta,J}(-\infty z, z)V\rangle$ should be correct. This residue is manifestly positive whenever

$$\Delta_\phi < \frac{\Delta + J + 2}{2}. \tag{6.291}$$

For example, this proves the continuous spin ANEC for all $J \geq 2$ if the lowest-dimension scalar in the theory has dimension $\Delta_\phi \leq \frac{d+4}{2}$.

6.7 Discussion

We have argued that every CFT contains light-ray operators that provide an analytic continuation in spin of the light-transforms of local operators. This gives a physical interpretation of Caron-Huot's Lorentzian inversion formula [66]. Our construction involves smearing two primary operators $\mathcal{O}_1, \mathcal{O}_2$ against a kernel to produce an object $\mathbb{O}_{\Delta,J}$, and then taking residues in Δ to localize the operators along a null ray. We have not shown rigorously that the integral localizes to a null ray (as opposed to a lightcone). However, we expect this is true based on the example of MFT and the fact that it's true for integer J. More generally, we expect that any singularity in the (Δ, J)-plane should lead to a light-ray operator. (For instance, one could take the discontinuity across a branch cut instead of a residue.) It would be nice to understand better the structure of the (Δ, J)-plane in general CFTs. We know that for nonnegative integer J, the object $\mathbb{O}_{\Delta,J}$ has simple poles in Δ at the locations of local operator dimensions. However, we do not know how it behaves for general complex J.[56] We also have not addressed the question of whether different operators $\mathcal{O}_1, \mathcal{O}_2$ produce different light-ray operators. We expect that in a nonperturbative theory, the same set of light-ray operators should appear in every product $\mathcal{O}_i\mathcal{O}_j$, if allowed by symmetry. It would be nice to show this rigorously.

Light-ray operators have the advantage over local operators that they fit into a more rigid structure, due to analyticity in spin. However, unlike local operators, they are not included in the Hilbert space of the CFT on S^{d-1} because they annihilate the vacuum. One way to realize them as states is to double the Hilbert space (with time running forwards in one copy and backwards in the other). The $\mathbb{O}_{i,J}$ then become states in the doubled Hilbert space.[57] A general message is that the doubled Hilbert space contains interesting structure that is not visible in a single copy, and it would be interesting to explore this idea further.

[56] In planar $\mathcal{N} = 4$ SYM, beautiful pictures of the (Δ, J)-plane have been constructed using integrability [252–255].

[57] $\mathbb{O}_{i,J}$ itself is a somewhat violent state. However, we can regularize it by acting on the thermofield double state with some temperature β. We thank Alexei Kitaev for this suggestion.

We have seen that light-ray operators enter the Regge limit of CFT four-point functions. It would be nice to understand the actual spectrum and OPE coefficients of continuous-spin light-ray operators in important physical theories (e.g. the 3d Ising model, $\mathcal{N} = 4$ SYM, and more), in order to determine what the Regge limit actually looks like in those theories.[58] Such operators have been explored in weakly-coupled gauge theories (see e.g. [234–239]), and it would be interesting to study other perturbative examples. For example, can one write a continuous-spin generalization of the Hamiltonian of the Wilson-Fisher theory [256]?

Another important question is the extent to which light-ray operators form a complete basis for describing the Regge regime. Indeed, in our discussion in section 6.5, we ignored certain non-growing contributions in the Regge limit. It would be interesting to include them and give them operator interpretations. Perhaps lightcone operators or other types of nonlocal operators play a role. This question is also interesting in 1 dimension, where the analog of the Regge regime is the so-called "chaos regime" of a four-point function.

In any spacetime dimension, we can ask: is there a complete basis of nonlocal operators transforming as primaries in Lorentzian signature? Identifying a complete basis could help in developing a generalization of the OPE that is valid in non-vacuum states. (The usual OPE still works as an asymptotic expansion in non-vacuum states, but we would like to find a convergent expansion.) Such a generalization would be a powerful tool for studying Lorentzian physics.

Relatedly, it would be interesting to study OPEs of light-ray operators with each other, especially the ANEC operator $\mathcal{E} = \mathbf{L}[T]$.[59] In "conformal collider physics" [76] one considers ANEC operators starting at the same point $\mathcal{E}(x, z_1)\mathcal{E}(x, z_2)$ (usually taken to be spatial infinity $x = \infty$, so that the light-rays lie along future null infinity), and it is natural to study the limit where their polarization vectors coincide $z_1 \to z_2$. This question was explored in [76], where it was argued that the leading term in the $\mathcal{E} \times \mathcal{E}$ OPE in $\mathcal{N} = 4$ SYM is a particular spin-3 light-ray operator that can be described in bulk string theory using the Pomeron vertex operator of [224]. It would be nice to determine a systematic expansion for this limit in a general CFT. Such an expansion could be useful for computing energy correlators and studying jet substructure in CFTs. Light-ray operators could also be useful for understanding

[58] Besides planar $\mathcal{N} = 4$ SYM, another CFT where the Regge limit of a four-point function has been computed is the 2d (supersymmetric) SYK model [174].

[59] We thank Sasha Zhiboedov for discussion on this point.

aspects of deep inelastic scattering and PDFs.[60]

In this work, inspired by Caron-Huot's beautiful result [66], we have been led to an unusual hybrid of Euclidean and Lorentzian harmonic analysis, i.e., harmonic analysis with respect to the groups $\mathrm{SO}(d+1,1)$ and $\widetilde{\mathrm{SO}}(d,2)$. However, many of the resulting formulae suggest that it might be fruitful to start with $\widetilde{\mathrm{SO}}(d,2)$ from the beginning. For example, after applying the Sommerfeld-Watson trick, Regge correlators are written as an integral over Δ and J, which is suggestive of an expansion in Lorentzian principal series representations (this observation was also made recently in [257]). It will be important to develop this area further and explore its implications for many of the above questions.[61]

The intrinsically Lorentzian integral transforms introduced in section 6.2.3 have been a key computational tool in this work. These transforms have a natural group-theoretic origin as Knapp-Stein intertwining operators for $\mathrm{SO}(d,2)$, but they can also be applied to representations of $\widetilde{\mathrm{SO}}(d,2)$. In this work, we have focused primarily on the light-transform, but the remaining transforms may also have interesting applications. For example, it would be interesting to compute the full monodromy matrix for spinning conformal blocks in terms of intertwining operators, generalizing (6.251). Steps in this direction have already been taken in [242].

One concrete result of this work is a generalization of Caron-Huot's Lorentzian inversion formula to four-point correlators of operators in arbitrary Lorentz representations. Caron-Huot's original formula has already proven useful in a variety of contexts [258–264],[62] and we hope that our generalization will be similarly useful. For example, one might try to determine all four-point functions in theories with weakly-broken higher spin symmetry, generalizing the results of [262]. It would also be interesting to study inversion formulae in the context of stress-tensor four-point functions, perhaps making contact with the sum rules in [240, 267].

An important application of Lorentzian inversion formulae is to the lightcone bootstrap and large-spin perturbation theory [31, 68, 69, 77, 102–104, 106, 162, 228, 229]. Lorentzian inversion formulae make it particularly simple to study OPE coefficients and anomalous dimensions of "double-twist operators" [68, 69] and averaged OPE data for "multi-twist" operators (see e.g. [263, 264]). An important problem

[60] We thank Juan Maldacena for this suggestion.

[61] We thank Abhijit Gadde for emphasizing this idea.

[62] See also [265, 266] for applications of Lorentzian inversion formulae to quantities other than vacuum four-point functions. It would be interesting to understand whether light-ray operators offer a useful perspective on these works.

for the future is to disentangle individual multi-twist trajectories. It is likely that this will require studying crossing symmetry for higher-point functions. We hope that light-ray operators will offer a useful perspective on this problem.

Another result of this work is a new proof of the average null energy condition (ANEC), obtained by combining the causality-based proof of [73] with the idea of an inversion formula. Our proof has some technical advantages over [73]. For example, it does not use the OPE outside its regime of validity, and it also allows one to move away from the asymptotic lightcone limit. However, it also has disadvantages. In particular, our proof requires the CFT to contain a sufficiently low-dimension operator, and this condition is absent in [73]. It would be interesting to understand whether this condition can be relaxed further while still using an inversion formula. Another technical point that is worth clarifying is the role/necessity of Rindler positivity, as opposed to the more easily-established "wedge reflection positivity" [248] or the traditional positivity of norms.

The ANEC has a growing list of interesting applications in conformal field theory [76, 240, 241, 268–270]. However its higher-spin generalizations [73] have been less well-explored. We have additionally proven that the ANEC holds for continuous spin — i.e., on the entire leading Regge trajectory. It would be interesting to understand the implications of this result, for example in a holographic context. (See [271] for recent work on shockwave operators, which are holographically dual to light-ray operators.) It would also be interesting to understand the information-theoretic role of continuous-spin operators. How do they behave under modular flow? Can they appear in OPEs of entangling twist defects? The ANEC can be improved to the quantum null energy condition (QNEC) [272, 273], which was recently proven in [274] together with a higher integer spin generalization. Is there a continuous-spin version of the QNEC?

Acknowledgements

We thank Clay Córdova, Thomas Dumitrescu, Abhijit Gadde, Luca Iliesiu, Daniel Jafferis, Alexei Kitaev, Murat Koloğlu, Raghu Mahajan, Eric Perlmutter, Matt Strassler, and Aron Wall for helpful discussions. We thank Simon Caron-Huot, Tom Hartman, Denis Karateev, Juan Maldacena, Douglas Stanford, and Sasha Zhiboedov for discussions and comments on the draft. DSD is supported by Simons Foundation grant 488657 (Simons Collaboration on the Nonperturbative Bootstrap). This work was supported by DOE grant DE-SC0011632 and the Walter Burke In-

stitute for Theoretical Physics.

Chapter 7

REFLECTIONS ON CONFORMAL SPECTRA

This chapter is essentially identical to:

H. Kim, P. Kravchuk and H. Ooguri, *Reflections on Conformal Spectra*, *JHEP* **04** (2016) 184, [1510.08772].

7.1 Introduction

Modular invariance and crossing symmetry relate ultraviolet and infrared properties of conformal field theory and impose strong constraints on its energy spectrum and operator product expansion (OPE). In two dimensions, the partition function,

$$Z(\tau) = \mathrm{tr}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \tag{7.1}$$

is invariant under the modular transformation, $\tau \to -1/\tau$, where $q = e^{2\pi i \tau}$ and τ is the torus modulus. In any number of dimensions, a four-point function on the sphere,

$$G(x) = \langle 0|\phi(\infty)\phi(1)\phi(x)\phi(0)|0\rangle, \tag{7.2}$$

is invariant under the crossing transformation, $x \to 1 - x$, where x is the Dolan-Osborn coordinate [118]. The use of modular invariance was initiated in [275]. The conformal bootstrap program to exploit crossing symmetry was pioneered in [26, 27], was developed further in two dimensions starting with [29], and is currently undergoing a renaissance in higher dimensions starting with [30].

The quintessential application of modular invariance is the Cardy formula [275], which describes the spectral density for a large scaling dimension Δ with a fixed value of the central charge c. In [25], crossing symmetry was used to estimate the spectral density weighted by the OPE coefficients, for large Δ with a fixed value of the scaling dimension Δ_0 of the external operator ϕ in (7.2).

In this paper, we will study the different limits:

$$\Delta, c \to \infty, \quad \text{with } \Delta/c: \ \text{fixed}, \tag{7.3}$$

for the partition function,

$$\Delta, \Delta_0 \to \infty, \quad \text{with } \Delta/\Delta_0: \ \text{fixed}, \tag{7.4}$$

and

$$\Delta, \Delta_0, d \to \infty, \quad \text{with } \Delta/d,\ \Delta_0/d: \ \text{fixed}, \tag{7.5}$$

for the four-point function. Here d is the spacetime dimension.

The limit (7.3) for the partition function was considered in [74], where it was shown that the Cardy formula holds for $\Delta > c/6$ under a certain condition on light spectrum, strengthening the result of [275], which held only in the limit $\Delta \gg c$. In this paper, we will describe an approximate symmetry of spectral decomposition of the partition function, which can be used to motivate this result. Moreover, this symmetry suggests some bounds for the spectral density, which we derive by independent techniques. We employ a similar approach to study the limit (7.4) of the four-point function to derive properties of the spectral density weighted by the OPE coefficients as a function of Δ. This approach proves to be universal and we apply it also to the case of large spacetime dimension.

7.2 Results

7.2.1 Partition function

To study the partition function in two dimensions, we will use the following simplified expression:

$$Z(\tau) = \int_0^\infty q^{\Delta - \frac{c}{12}} n(\Delta) d\Delta, \tag{7.6}$$

where $n(\Delta)$ is the density of conformal primary states with scaling dimension Δ. This formula ignores contributions from Virasoro descendants, which will turn out to be subleading in $1/c$ in what follows. Another interpretation is that $n(\Delta)$ is the density of all states, not just the primaries, in which case the above formula is valid literally. The spins of primary states are not visible when q is real and τ is pure imaginary, which we will assume throughout the paper.

Our basic observation is that modular invariance $Z(\tau) = Z(-1/\tau)$ implies the following approximate reflection symmetry in the space of scaling dimension Δ:

$$\overline{\omega}_\tau(\Delta) \simeq \overline{\omega}_{-1/\tau}\left((1+|\tau|^2)\frac{c}{12} - |\tau|^2\Delta\right) \times |\tau|^2, \tag{7.7}$$

where $\overline{\omega}_\tau(\Delta)$ is defined by

$$\omega_\tau(\Delta) = \frac{1}{Z(\tau)} q^{\Delta - \frac{c}{12}} n(\Delta), \tag{7.8}$$

$$\overline{\omega}_\tau(\Delta) = K_c(\Delta) * \omega_\tau(\Delta), \tag{7.9}$$

and $*$ denotes convolution, $(f * g)(x) = \int f(x-y)g(y)dy$. Here the kernel K_c smears the integrand of (7.6) over the interval of size ε, $\sqrt{c} \ll \varepsilon \ll c$. Note however that K_c decays rather slowly outside of this interval – see section III.A.1. With this definition, $\overline{\omega}_\tau$ measures the significance of Δ in the partition function averaged over the small interval of the size ε to smooth out the sum of delta-functions in $n(\Delta)$. Since Δ is bounded below by 0 in any unitary theory, $\overline{\omega}_\tau(\Delta)$ approximately vanishes for $\Delta \lesssim -\varepsilon$. The reflection symmetry (7.7) maps this to

$$\overline{\omega}_\tau(\Delta \gtrsim \Delta_\tau) \simeq 0, \tag{7.10}$$

where the edge Δ_τ is given by

$$\Delta_\tau = \left(1 + \frac{1}{|\tau|^2}\right)\frac{c}{12}. \tag{7.11}$$

We can estimate how fast the integrand of (7.6) decays above this threshold $\Delta > \Delta_\tau$, $|\tau| < 1$, as

$$\int_\Delta^\infty \omega_\tau(\Delta')d\Delta' \leq \frac{2}{1 + T_{2k_0+1}\left(\frac{\Delta - \Delta_\tau/2}{\Delta_\tau/2}\right)}, \tag{7.12}$$

where $T_{2k_0+1}(x)$ is the degree $(2k_0+1)$ Chebyshev polynomial of the first kind and k_0 is chosen so that $k_0 \ll \sqrt{c}$. In the limit of $c \to \infty$, the half decay width of the right hand side is $\sim c/k_0 \gg \sqrt{c}$.

Of course, from Cardy formula one expects exponential rather than polynomial decay, but this formula shows the specific threshold value Δ_τ, beyond which there can be no dominant contribution to $Z(\tau)$. From the discussion in [74] it follows that there exist theories which essentially saturate this bound, i.e., for which the integral (7.6) is dominated by states at Δ_τ.

This happens in theories satisfying the sparse light spectrum condition, defined in [74] as

$$\begin{gathered} n(\Delta') \lesssim e^{2\pi\Delta}, \\ \text{for } 0 \leq \Delta < c/12, \end{gathered} \tag{7.13}$$

where the inequality should be understood in an averaged sense. The essence of this condition is that the partition function for the low temperature phase $|\tau| > 1$ is dominated by the vacuum state (in particular, the maximum of $\omega_\tau(\Delta)$ is at $\Delta = 0$). In this case, the reflection symmetry shows that the maximum of $\omega_\tau(\Delta)$ jumps to the edge Δ_τ in the high temperature phase $|\tau| < 1$, and gives a prediction on the value of this maximum. With τ changing in the high-temperature phase the maximum at Δ_τ scans through the region $\Delta > c/6$, allowing one to obtain information on $n(\Delta)$ in this region. Rigorous microscopic estimates were made in [74], and the resulting Cardy-like formula is

$$\overline{n}(\Delta) \simeq \exp\left(2\pi\sqrt{\frac{c}{3}\left(\Delta - \frac{c}{12}\right)} + O(c^\alpha)\right), \tag{7.14}$$

for $\Delta > c/6$ and the average density of states,

$$\overline{n}(\Delta) = \frac{1}{\varepsilon'}\int_{\Delta}^{\Delta+\varepsilon'} n(\Delta')d\Delta', \tag{7.15}$$

with $\varepsilon' \sim c^\alpha$, $1/2 < \alpha < 1$.

7.2.2 Four-point function

In this section we consider the four-point function of identical scalar operators of scaling dimension Δ_0. We insert the four operators on one two-dimensional plane, which we identify with the complex plane of variable x. We insert three scalars at $0, 1, \infty$ and the fourth scalar at the Dolan-Osborn coordinate x. This four-point function (7.2) can be expressed as a sum of the spectral density weighted by the OPE coefficients and the conformal block $\mathcal{F}_{\Delta,\ell}(x)$ for the scaling dimension Δ and the spin ℓ, see e.g., [25]. Here and throughout the paper, we assume that the coordinate x is real and $0 < x < 1$.

As a by-product of our work, we find an expression for $\mathcal{F}_{\Delta,\ell}(x)$ for general ℓ in the scaling limit (7.4), when external operators are identical scalars. In Appendix A, we will solve the fourth order differential equation derived in [80] for the conformal block to show, for $x < 1$,

$$\begin{aligned}\mathcal{F}_{\Delta,\ell}(x) \simeq \rho^\Delta \left(1 - \frac{\rho^2}{16}\right)^{-\frac{d}{2}+\kappa(\Delta,\ell,\rho)} \\ \times \left(1 + O(1/\Delta)\right),\end{aligned} \tag{7.16}$$

where ρ is the radial coordinate,

$$\rho = \frac{4x}{(1+\sqrt{1-x})^2}, \tag{7.17}$$

introduced in [25] and discussed further in [59]. Note that this approximation breaks down when $x \to 1$. This should be kept in mind when interpreting the formulas below. In general the results of this section apply to the limit $\Delta_0 \to \infty$ with x kept fixed. In this limit, spin dependence of the conformal block is only through the exponent $\kappa(\Delta, \ell, \rho)$, which behaves as

$$\begin{aligned} \kappa(\Delta, \ell, \rho) &\to 0, \quad (\Delta - \ell \sim \Delta), \\ &\to \frac{1}{2}, \quad (\Delta = \ell + d - 2 : \text{unitarity bound}). \end{aligned} \tag{7.18}$$

Here in the first case ℓ can be on the order of Δ, but has to stay away from the unitarity bound. Between the two cases κ can acquire ρ dependence. However, the results in the two regimes suggest that the factor $(1 - \rho^2/16)^{-d/2+\kappa(\Delta,\ell,\rho)}$ in the conformal block (7.16) is altogether negligible in the large Δ analysis in this paper, just as Virasoro descendants are negligible in the partition function as in (7.6). Thus, we can express the four-point function in the scaling limit (7.4) as

$$G(x) = \int_0^\infty \rho^\Delta x^{-2\Delta_0} g(\Delta) d\Delta, \tag{7.19}$$

where $g(\Delta)$ is the spectral density weighted by the square of the OPE coefficients, which is non-negative when ϕ's are identical. One can of course keep this subleading factor in what follows without affecting the conclusions. Note that though we made no assumptions on the spins of the intermediate states, the spectral decomposition of $G(x)$ is blind to them for real x and large scaling dimensions.

One can also view (7.19) as an exact expansion, in which we have discarded the structure of conformal multiplets and treat primary and descendant operators on equal footing. This is the radial coordinates expansion of [25, 59]. Below we also consider another kind of "descendant" expansion, which corresponds to a different choice of coordinates.

Since the spectral decomposition of the four-point function (7.19) is similar to that of the partition function (7.6) in these limits, crossing symmetry $G(x) = G(1 - x)$ implies a similar reflection symmetry in Δ. Let us introduce the "branching ratio" of $\phi(x) \times \phi(0)$ turning into operators of dimension Δ,

$$\gamma_x(\Delta) = \frac{1}{G(x)} \rho^\Delta x^{-2\Delta_0} g(\Delta), \tag{7.20}$$

$$\overline{\gamma}_x(\Delta) = K_{\Delta_0}(\Delta) * \gamma_x(\Delta), \tag{7.21}$$

with K_{Δ_0} averaging over intervals of the size $\sqrt{\Delta_0} \ll \varepsilon \ll \Delta_0$. In terms of this quantity, the approximate reflection symmetry is expressed as

$$\overline{\gamma}_x(\Delta) \simeq \overline{\gamma}_{1-x}\left(\frac{1}{\sqrt{x}}\left(2\Delta_0 - \sqrt{1-x}\Delta\right)\right)\sqrt{\frac{1-x}{x}}. \tag{7.22}$$

The reflection of $\overline{\gamma}_x(\Delta \lesssim -\varepsilon) = 0$ is then

$$\overline{\gamma}_x(\Delta \gtrsim \Delta_x) \simeq 0, \tag{7.23}$$

where the edge Δ_x is given by

$$\Delta_x = \frac{2}{\sqrt{1-x}}\Delta_0. \tag{7.24}$$

As in the case of the partition function (7.12), we can estimate how fast $\gamma_x(\Delta)$ decays above the threshold $\Delta > \Delta_x$, $x > 1/2$ as

$$\int_\Delta^\infty \gamma_x(\Delta')d\Delta' \leq \frac{2}{1 + T_{2k_0+1}\left(\frac{\Delta-\Delta_x/2}{\Delta_x/2}\right)}, \tag{7.25}$$

with $k_0 \ll \sqrt{\Delta_0}$. Note that the half-decay width is $\sim \Delta_0/k_0 \gg \sqrt{\Delta_0}$. This can be compared to the conformal block expansion of the correlation function of the generalized free field,

$$G(x) = \frac{1}{x^{2\Delta_0}} + \frac{1}{(1-x)^{2\Delta_0}} + 1, \tag{7.26}$$

which can be shown, as long as x is away from 1, to have a saddle point at $\Delta = \Delta_x$ of width $\sim \sqrt{\Delta_0}$. In this order-of-magnitude sense the bound (7.25) is almost saturated.

We can also perform the “descendant” expansion in the standard coordinates described in the beginning of this section (see e.g., [25]), again treating primary and descendant operators on equal footing,

$$G(x) = \int_0^\infty x^{\Delta-2\Delta_0} g^{(s)}(\Delta)d\Delta, \tag{7.27}$$

where we added the superscript $^{(s)}$ to $g(\Delta)$ to note the fact that we are expanding $G(x)$ in what we will henceforth call “scaling blocks”. We use a similar notation for branching ratios $\gamma_x^{(s)}$, $\overline{\gamma}_x^{(s)}$. All of the above results also hold in this case, with the modification that now

$$\Delta_x = \frac{2}{1-x}, \tag{7.28}$$

the reflection relation is

$$\overline{\gamma}_x^{(s)}(\Delta) \simeq \overline{\gamma}_{1-x}^{(s)}\left(\frac{2}{x}\Delta_0 - \frac{1-x}{x}\Delta\right)\frac{1-x}{x}. \tag{7.29}$$

7.2.2.1 Finite-Δ_0 bounds

So far, our statements have been in the limit (7.4) of large Δ and Δ_0. In the case of the scaling block decomposition of four-point function, we can derive inequalities which are valid at finite Δ and Δ_0. For example, for $2 < 4\Delta_0 < \Delta$,

$$\int_{\Delta}^{\infty} \gamma_{1/2}^{(s)}(\Delta')d\Delta' \leq \frac{1}{1+\frac{\Gamma(\Delta-2\Delta_0+1)\Gamma(2\Delta_0)}{\Gamma\left(\frac{\Delta+3}{2}\right)\Gamma\left(\frac{\Delta-1}{2}\right)}}, \tag{7.30}$$

where

$$\gamma_x^{(s)}(\Delta) = \frac{1}{G(x)} x^{\Delta-2\Delta_0} g^{(s)}(\Delta). \tag{7.31}$$

Note that this bound also implies a bound on individual delta-function contributions to $g^{(s)}$, since they are all positive. If we keep Δ_0 finite and take $\Delta \to \infty$, this inequality becomes

$$\int_{\Delta}^{\infty} \gamma_{1/2}^{(s)}(\Delta')d\Delta' \leq \sqrt{2\pi}\frac{\Delta^{2\Delta_0-\frac{1}{2}}}{2^{\Delta}\Gamma(2\Delta_0)}. \tag{7.32}$$

In this limit, this inequality is stronger than the asymptotic bound of [25],

$$\int_{\Delta}^{\infty} \gamma_{1/2}^{(s)}(\Delta')d\Delta' \lesssim \frac{2^{-2\Delta_0}}{G(1/2)}\frac{\Delta^{2\Delta_0}}{2^{\Delta}\Gamma(2\Delta_0+1)}. \tag{7.33}$$

However, the Cardy-like asymptotic of [25],

$$\int_{0}^{\Delta} g^{(s)}(\Delta')d\Delta' \sim \frac{\Delta^{2\Delta_0}}{\Gamma(2\Delta_0+1)}, \tag{7.34}$$

suggests by differentiation that one can expect the stronger convergence rate of

$$\int_{\Delta}^{\infty} \gamma_x^{(s)}(\Delta')d\Delta' \propto \Delta^{2\Delta_0-1}2^{-\Delta}. \tag{7.35}$$

While (7.30) is weaker than this expectation, it has the advantage that it is rigorous and holds for finite Δ and Δ_0.

In fact, the method we use for proving this bound is quite general and can be used for construction of finite Δ and Δ_0 analytic bounds for (7.19) as well. We have checked that these bounds are asymptotically at least as strong as those of [25], still having the advantage of being valid for finite values of Δ. Given the improvement of (7.30) over (7.33), one might expect that an improvement is possible for (7.19) as well. We hope to return to this question in future.

So far, we did not assume that the four-point function is dominated by a saddle point. If we make this assumption, our results have simple explanation. Let the location of the saddle point in the expansion of $G(x)$ be $\Delta(x)$, which has to obey the reflection relation imposed by crossing symmetry,

$$\frac{\Delta(x) - 2\Delta_0}{x} = -\frac{\Delta(1-x) - 2\Delta_0}{1-x}. \tag{7.36}$$

This is most easy to see if we note that $\Delta(x) - 2\Delta_0 = \frac{\partial \log G(x)}{\partial \log x}$ and apply the crossing relation $G(x) = G(1-x)$. In unitary theory $\Delta(x) \geq 0$, which implies, by the above relation,

$$\Delta(x) \leq \Delta_x = \frac{2\Delta_0}{1-x}. \tag{7.37}$$

7.2.2.2 Cardy formula

An analogue of the sparse light spectrum condition (7.13) for the four point function can be introduced, namely,

$$g^{(s)}(\Delta') \lesssim 2^{\Delta},$$
$$\text{for } 0 < \Delta < 2\Delta. \tag{7.38}$$

Again, this should be understood in some averaged sense, such that this condition would imply that the four-point function for $|x| < 1/2$ is dominated by the vacuum state. Then, by the reflection symmetry, the maximum of $\gamma_x(\Delta)$ jumps to the edge Δ_τ for $|x| > 1/2$. This, exactly as in the case of the partition function, can be translated into a statement on $g^{(s)}(\Delta)$, which reads, for $\Delta > \Delta_{1/2} = 4\Delta_0$,

$$\overline{g}_s(\Delta) = \exp\left[-\Delta \log\left(1 - \frac{2\Delta_0}{\Delta}\right) + \right.$$
$$\left. +2\Delta_0 \log\left(\frac{\Delta}{2\Delta_0} - 1\right) + O(\Delta_0^{\alpha})\right], \tag{7.39}$$

where $1/2 < \alpha < 1$, and $\overline{g}^{(s)}$ is $g^{(s)}$ integrated on the scale $\delta \sim \Delta_0^{\alpha}$.

7.2.3 Four-point function in large spacetime dimension

So far we have only discussed the limits where the operators considered were heavy compared to any other scales we had, and in particular far away from the unitarity bounds. Some interesting phenomena happen near unitarity bounds, such as that a scalar field has to become free as its scaling dimension is pushed toward the bound. In this section we consider a limit in which we take not only the scaling dimension of the external scalars, but also the number of spacetime dimensions d to be large. In fact, when the number of spacetime dimensions is taken to be large, the unitarity bounds force all the operators to become heavy. We are then able to apply the same methods as before, but now to all operators in the theory.

Recall that the unitarity bounds are

$$\Delta \geq \frac{d-2}{2} \sim \frac{d}{2} \qquad \text{for non-identity scalars,} \tag{7.40}$$

$$\Delta \geq \ell + d - 2 \sim \ell + d \qquad \text{for operators with spin,} \tag{7.41}$$

and thus the natural limit is the double scaling $\Delta_0 \sim d \to \infty$. In this limit we can see the gap between the identity and the lightest allowed scalars and the difference between the lightest scalars and the lightest spin operator. Appearance of these features means that now we have to distinguish several classes of operators.

It turns out that for us there is no difference between spin and scalar operators, since on the real line $x = \overline{x}$ the conformal blocks in a large number of spacetime dimension do not depend on spin (for details on conformal blocks see Appendix F.2). However, the gap above the identity is important and the identity operator has to be treated separately.

As mentioned before, we can apply almost the same methods as we used in other limits. A new feature is that the duality relation is now non-linear and is not as pleasant to manipulate as in the above discussions. However, it carries more information, since we are now able to take our external scalars close to the unitarity bound.

Let us introduce the duality relation. We state it in the following form,

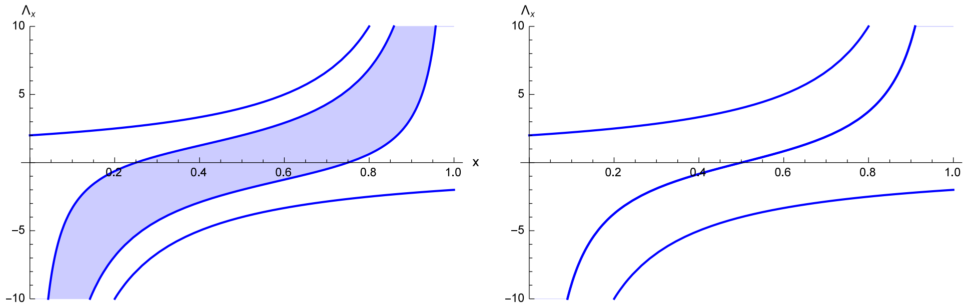
$$\Lambda_x(\Delta) = -\Lambda(\Delta')_{1-x}. \tag{7.42}$$

This has to be understood as an implicit relation between the symmetry-related scaling dimensions Δ in the conformal block expansion at x and Δ' at $1-x$. Here Λ_x is given by

$$\Lambda_x(\Delta) = \frac{1}{\Delta_0}\frac{\partial \log x^{-2\Delta_0} F_\Delta(x)}{\partial x}, \tag{7.43}$$

and F_Δ is the spin-independent conformal block. The explicit form of Λ_x is cumbersome, but is straightforwardly obtained from (F.36) for $\Delta > 0$. For the identity operator $F_0(x) = 1$, and so we get $\Lambda_x = -2/x$. One can easily obtain the range of Λ_x corresponding to the unitary range $\Delta \in \{0\} \cup [d/2, +\infty)$. It is given by

$$\Lambda_x \in \left\{-\frac{2}{x}\right\} \cup \left[\frac{1}{2\delta_0 x(1-x)} - \frac{2}{x}, +\infty\right), \tag{7.44}$$



(a) $\delta_0 = 1$ (b) $\delta_0 = 1/2$

Figure 7.1: Allowed range for Λ_x as a function of x for $\delta_0 = 1$ and $\delta_0 = 1/2$.

where $\delta_0 = \Delta_0/d$. Now, let us apply the duality relation to this range – in this way we will obtain the allowed range for the saddle point in the conformal block decomposition. The result is, in terms of Λ,

$$\Lambda_x \in \left\{-\frac{2}{x}\right\} \cup \left\{\frac{2}{1-x}\right\} \cup \left[\frac{1}{2\delta_0 x(1-x)} - \frac{2}{x}, -\frac{1}{2\delta_0 x(1-x)} + \frac{2}{1-x}\right]. \tag{7.45}$$

This range is plotted in Fig. 7.1. The case of $\delta_0 = 1$ is generic and is shown in Fig. 7.1a. As the external scalar gets heavier, δ_0 gets larger and the range fills the region between the curves corresponding to the identity operator and its dual image.

An interesting thing happens as δ_0 approaches the unitarity bound $1/2$, Fig 7.1b. The allowed range for Λ_x shrinks into three points. This is the manifestation of the fact that a scalar at the unitarity bound has to be free. Let us remind the reader of the reasoning. The unitarity bound $\Delta \geq (d-2)/2$ expresses non-negativity of the norm of a descendant of ϕ, which thus becomes null at the unitarity bound. This implies that ϕ satisfies the free field equation of motion $\Delta\phi = 0$ as an operator equation, and all the correlation functions of ϕ are harmonic away from singularities. Then one can take for example the four point function of ϕ and subtract the free field four point function. The result G_4' is still harmonic and the OPE limits imply that it has singularities weaker than those of free field, $1/|x|^{d-2}$. But $1/|x|^{d-2}$ is the weakest singularity a harmonic function can have. Thus, G_4' is harmonic everywhere, tends to zero at infinity, and is therefore 0. So the four point function of ϕ is that of the free field, which in turn implies that the $\phi\phi$ OPE is also free.

Note that the above argument explicitly imposes the equation of motion of ϕ on the four point function. It is not a priori obvious that the crossing equation for this four point function alone should also imply that ϕ must be free at the unitarity bound. However, it seems to be the case as the numerical results suggest (e.g., [30] in four spacetime dimensions). From our perspective, it is true as long as one excludes the middle curve in Fig. 7.1b. If this is done, then duality at $x = 1/2$ tells us that there are to equally important saddle points, and for other values of x one of them dominates, just as in the previous discussion. The resulting behavior is characteristic of the free field, to the accuracy of our approximation.

Section III is devoted to derivations. Some of technical details are discussed in appendices, including the derivation of (7.16).

7.3 Derivations

7.3.1 Modular Invariance

7.3.1.1 Reflection Symmetry

Here we discuss the derivation of the reflection symmetry (7.7). We do not try to make the derivation very detailed or completely rigorous, since we only use (7.7) as a heuristic device, and our other derivations are independent of it.

Parametrizing τ in the partition function as,

$$\tau = ie^{x-\frac{1}{2}}, \tag{7.46}$$

the modular transformation $\tau \to -1/\tau$ becomes the reflection $x \to 1-x$. Therefore,

$$\frac{\partial^{2k+1}}{\partial x^{2k+1}} Z(\tau(x))\Big|_{x=1/2} = 0, \quad k = 0, 1, 2, \cdots, \tag{7.47}$$

and this can be expressed the integral constraints on $\omega_\tau(\Delta)$ as,

$$\int_0^\infty \left[\Delta - \frac{c}{12}\right]^{(2k+1)} \omega_{\tau=i}(\Delta) d\Delta = 0, \tag{7.48}$$

where $\omega_\tau(\Delta)$ is defined by (7.9) and the bracket symbol $[\Delta - c/12]^{(2k+1)}$ is defined by,

$$\begin{aligned} [y]^{(N)} &\equiv e^{2\pi y e^x} \left(-\frac{1}{2\pi}\frac{\partial}{\partial x}\right)^N e^{-2\pi y e^x}\Big|_{x=0}, \\ &= y^N \left(1 + \frac{N(N-1)}{2y} + \cdots\right). \end{aligned} \tag{7.49}$$

When $N \ll \sqrt{|y|}$, we can approximate $[y]^{(N)}$ by the monomial y^N. Note that if we use the full Virasoro character instead of $q^{\Delta - c/12}$, this approximation is still valid. It is in this sense in which we said previously that Virasoro descendants are subleading. Therefore,

$$\int_0^\infty \left(\Delta - \frac{c}{12}\right)^{2k+1} \omega_i(\Delta) d\Delta \simeq 0, \tag{7.50}$$

for $k \ll \sqrt{\Delta}, \sqrt{\Delta_0}$, assuming that the region near $\Delta = c/12$ does not make a major contribution to the integral, which is consistent with results we will find. This suggests that $\omega_i(\Delta)$ is approximately symmetric under reflection at $\Delta = c/12$:

$$\omega_i(\Delta) \simeq \omega_i\left(\frac{c}{6} - \Delta\right). \tag{7.51}$$

If the dominant contribution came from $c/12$, approximate symmetry like this would be self-evident.

Of course, one cannot expect a literal equality like this – in the end, we only have a finite number of equations (7.50). To formulate a more precise statement, let us look at the case of general τ. For $\tau \neq i$, we have for any $k \geq 0$,

$$\frac{\partial^k}{\partial x^k} Z(\tau(x)) = (-1)^k \frac{\partial^k}{\partial x^k} Z(\tau(1-x)), \tag{7.52}$$

which, with similar approximations, translates into

$$\begin{aligned}\int_0^\infty \left[2\pi|\tau|\left(\Delta - \frac{c}{12}\right)\right]^k \omega_\tau(\Delta) d\Delta &= \\ &= \int_0^\infty \left[\frac{2\pi}{|\tau|}\left(\frac{c}{12} - \Delta\right)\right]^k \omega_{-1/\tau}(\Delta) d\Delta,\end{aligned} \tag{7.53}$$

for $k \ll \sqrt{c}$. This is now an equality between some polynomial moments of ω_τ and $\omega_{-1/\tau}$, which after some linear changes of arguments and densities ω can be translated into

$$\int_{-\pi|\tau|c/6}^\infty \lambda^k \omega'_\tau(\lambda) d\lambda = \int_{-\infty}^{\pi c/6|\tau|} \lambda^k \omega'_{-1/\tau}(-\lambda) d\lambda, \tag{7.54}$$

where λ is a rescaled version of Δ, and ω' is the rescaled and renormalized version of ω. We will see below that with k bounded above by $\sqrt{c}$, the integrals can be restricted to finite intervals of size $\sim c$, up to $1/c$ errors. Then one has an equality of polynomial moments of two functions on finite interval. In other words, their convolutions with any polynomial kernel coincide, provided the degree of the polynomial is bounded by $\sqrt{c}$. One can then try to pick a delta-like kernel $K'_c(\lambda)$, for example,

$$K'_c(\lambda) = \left(\frac{l^2 - \lambda^2}{l^2}\right)^{k/2}, \tag{7.55}$$

where l is twice the size of the interval to which we restrict the integrals in (7.53). Then, restoring the original variables, we have the required claim (7.7). Note that this particular delta-like kernel would average over regions of size $\gg c^{3/4}$. One can do better, for details see [276].

7.3.1.2 Bound on Tail

As discussed in Section 7.2.1, the reflection symmetry (7.7) suggests that $\omega_\tau(\Delta)$ approximately vanishes for $\Delta > \Delta_\tau$. To understand how good the statement is, we should estimate an upper bound on $\omega_\tau(\Delta)$ when Δ goes above the threshold Δ_τ. At $\tau = i$, the conditions on $\omega_i(\Delta)$ are,

$$\begin{aligned} &\int_0^\infty \omega_i(\Delta) d\Delta = 1, \\ &\int_0^\infty \left[\Delta - \frac{c}{12}\right]^{(2k-1)} \omega_i(\Delta) d\Delta = 0, \end{aligned} \tag{7.56}$$

and,

$$\omega_i(\Delta) \geq 0. \tag{7.57}$$

What we want to do is to estimate an upper bound on $\omega_i(\Delta)$ at a particular value $\widehat{\Delta}$ by maximizing the value of $\omega_i(\Delta)$ under these conditions. This is a typical linear optimization problem.

Generally speaking, the maximum value (optimal value for the primal problem) of $\vec{c} \cdot \vec{x}$ subject to

$$A\vec{x} = \vec{b}, \quad \text{and} \quad \vec{x} \geq 0, \tag{7.58}$$

is equal to the minimum value (optimal value for the dual problem) of $\vec{b} \cdot \vec{y}$ subject to

$$A^T \vec{y} \geq \vec{c}. \tag{7.59}$$

This is a statement of the strong duality theorem of linear programming [277], which is valid for finite-dimensional vector spaces. In our case, $\vec{x}$ is an infinite dimensional vector whose entries are values of $\omega_i(\Delta)$ at different values of Δ, A is a set of integral transforms mapping $\omega_i(\Delta)$ to the left-hand side of (7.56), and $\vec{b} = (1, 0, 0, \cdots)$ as in its right-hand side. Although we still expect the strong duality to hold in our case, we really need only the weak duality, which says that the optimal value for the dual problem (in fact, any feasible value) puts an upper bound on the optimal value of the primal problem. This weaker duality is straightforward to see. Indeed, let x be a solution to (7.58), then for any y a solution to (7.59) we have

$$\vec{b} \cdot \vec{y} = \vec{x} \cdot A^T \vec{y} \geq \vec{x} \cdot \vec{c}. \tag{7.60}$$

Before discussing what the dual problem is in our case, we first note that maximizing $\omega_i(\widehat{\Delta})$ does not make much sense, since ω_i appears only inside the integrals in the constraint equations, and thus its value at a point is irrelevant unless ω_i has a delta-function singularity at $\widehat{\Delta}$. Therefore, it only makes sense to maximize the coefficient of delta-singularity in ω_i at $\widehat{\Delta}$.

It is an easy exercise to check that in our case the dual minimization problem then is to minimize y_0, subject to

$$P_0(\Delta) \geq 0, \quad \forall \Delta \geq 0, \tag{7.61}$$

$$P_0(\widehat{\Delta}) \geq 1, \tag{7.62}$$

where

$$P_0(\Delta) = y_0 + \sum_{k=1}^{\infty} \left[\Delta - \frac{c}{12}\right]^{(2k-1)} y_k. \tag{7.63}$$

Setting $\Delta = c/12$ we get $y_0 \geq 0$, and thus if $P_0(\widehat{\Delta}) > 1$, we can always decrease y_0 by dividing $\vec{y}$ by $P_0(\widehat{\Delta})$. Thus we may assume $P_0(\widehat{\Delta}) = 1$.

For convenience, we consider $\vec{\lambda} = \vec{y}/y_0$, and then the minimum value of y_0 is equal to the minimal value of $1/P(\widehat{\Delta})$, where

$$P(\Delta) = 1 + \sum_{k=1}^{\infty} \left[\Delta - \frac{c}{12}\right]^{(2k-1)} \lambda_k, \tag{7.64}$$

with λ_k's being variables, subject to $P(\Delta) \geq 0$ for all Δ. This is the form of the dual problem most suitable for our purposes. For a different perspective on this problem see [278].

We can find a weaker bound on $\omega_\tau(\Delta)$ by utilizing the conditions (7.56) for a restricted set of k's, such as $k = 0, 1, 2, \ldots, k_0$ for some $k_0 \ll c$. Let us first consider the case of $\tau = i$ again. For $k \ll c$, we can approximate $[\Delta - c/12]^{(k)}$ by the monomial $(\Delta - c/12)^k$. Our task is then to minimize $1/P_{k_0}(\widehat{\Delta})$, where

$$P_{k_0}(\Delta) = 1 + \sum_{k=1}^{k_0} \left(\Delta - \frac{c}{12}\right)^{2k-1} \lambda_k, \tag{7.65}$$

under the condition $P_{k_0}(\Delta) \geq 0$ for $\Delta \geq 0$. This is the same problem as maximizing the degree $(2k_0 - 1)$ odd polynomial,

$$Q_{k_0}(\Delta) = \sum_{k=1}^{k_0} \left(\Delta - \frac{c}{12}\right)^{2k-1} \lambda_k, \tag{7.66}$$

under the condition, $Q_{k_0}(\Delta) \geq -1$ for $\Delta \geq 0$. Since $Q_{k_0}(\Delta)$ is odd under the reflection $\Delta \to c/6 - \Delta$, within the reflection symmetric interval $0 \leq \Delta \leq c/6$, $Q_{k_0}(\Delta) \geq -1$ also implies $Q_{k_0}(\Delta) \leq 1$. Namely,

$$|Q_{k_0}(\Delta)| \leq 1, \quad \text{for } 0 \leq \Delta \leq c/6. \tag{7.67}$$

Under the condition $\widehat{\Delta} > c/6$, the maximum of $Q_{k_0}(\widehat{\Delta})$ is achieved by the degree $(2k_0 - 1)$ Chebyshev polynomial of the first kind $T_{2k_0-1}(x)$ with $x = \frac{\Delta - c/12}{c/12}$ [279]. Notably, the polynomial is independent of $\widehat{\Delta}$.

We were so far optimizing the coefficient of delta function in $\omega_i(\widehat{\Delta})$. However, it turns out that the bound we found is also a bound for the integral $\int_{\widehat{\Delta}}^{\infty} \omega_i(\Delta) d\Delta$. Indeed, optimizing this integral would replace (7.62) with $P_0(\Delta) \geq 1$ for all $\Delta \geq \widehat{\Delta}$. It is easy to check that P_0 corresponding to the Chebyshev polynomial solution satisfies this stronger constraint as well. This in fact can be generalized to many cases of the form $\int_{\widehat{\Delta}}^{\infty} f(\Delta)\omega_i(\Delta) d\Delta$. Therefore,

$$\int_{\Delta}^{\infty} \omega_i(\Delta') d\Delta' \leq \frac{1}{1 + T_{2k_0+1}\left(\frac{\Delta - c/12}{c/12}\right)}, \tag{7.68}$$

for $\Delta > c/6$. Similarly, for a general value of $|\tau| < 1$, the tail at the threshold Δ_τ can be bounded as,

$$\int_{\Delta}^{\infty} \omega_\tau(\Delta') d\Delta' \leq \frac{2}{1 + T_{2k_0-1}\left(\frac{\Delta - \Delta_\tau/2}{\Delta_\tau/2}\right)}, \tag{7.69}$$

for $\Delta > \Delta_\tau$. To see this, recall the condition (7.54), which for odd powers of λ can rewritten as

$$\int_{-a}^{\infty} \lambda^{2k-1} \omega''_\tau(\lambda) d\lambda = 0, \tag{7.70}$$

and $a = \max\{\pi|\tau|c/6, \pi c/6|\tau|\}$ and $\omega''_\tau(\lambda) = \frac{1}{2}[w'_\tau(\lambda) + w'_{-1/\tau}(\lambda)]$. Here it is understood that $\omega_\tau(\Delta) = 0$ for $\Delta < 0$. It is also easy to see the normalization

$$\int_{-a}^{\infty} \omega''_\tau(\lambda) d\lambda = 1, \tag{7.71}$$

and thus the problem is reduced to $\tau = i$ case. It then follows

$$\int_{\widehat{\lambda}}^{\infty} \omega''_\tau(\lambda) d\lambda \leq \frac{1}{1 + T_{2k_0-1}(\widehat{\lambda}/a)}, \tag{7.72}$$

for $\widehat{\lambda} > a$ which then easily implies the claim.

Note that in the inequality (7.69) in the denominator is the polynomial which has the largest value for $\Delta > \Delta_\tau$, subject to the requirement of taking values in $[0, 2]$ for $0 \leq \Delta \leq \Delta_\tau$. In this way, it wins over any polynomial such as (7.55), especially if one takes l to be asymptotically larger than Δ_τ and the degree of K_c smaller than that of the Chebyshev polynomial. More precisely, K_c can be used as f in the aforementioned generalized bound on $\int f(\Delta)\omega_\tau(\Delta)d\Delta$. This justifies truncating the integrals in (7.54).

7.3.2 Crossing Symmetry

Unlike the case of the partition function in two dimensions, where contributions from Virasoro descendants are subleading in $1/c$, conformal descendants play an important role in the large Δ asymptotics in the four-point function (unless one makes a careful choice of the configuration of the four points [59]). For example, the large Δ conformal block behaves as ρ^Δ as we saw in (7.16) whereas the contribution of each local operator is x^Δ, and their difference is not negligible in the large Δ limit. On the other hand, it is easier to derive various bounds on the spectral decomposition of the four-point function if we use x^Δ. Thus, we will start with the warm-up exercise with the expansion,

$$G(x) = \int_0^\infty x^{\Delta - 2\Delta_0} g^{(s)}(\Delta) d\Delta, \tag{7.73}$$

where we treat all the local operators (including conformal descendants) independently.

7.3.2.1 Reflection and Bounds

Crossing symmetry $G(x) = G(1-x)$ means $G(x)$ is symmetric under reflection at $x = 1/2$, and therefore,,

$$\frac{\partial^{2k+1}}{\partial x^{2k+1}} G(x)\Big|_{x=1/2} = 0, \quad k = 0, 1, 2, \cdots, \tag{7.74}$$

which is equivalent to,

$$\int_0^\infty [\Delta - 2\Delta_0]^{(2k+1)} \gamma_{1/2}^{(s)}(\Delta) d\Delta = 0. \tag{7.75}$$

The bracket symbol $[\Delta - 2\Delta_0]^{(k)}$ in this subsection is different from the previous one and is the falling Pochhammer symbol,

$$\begin{aligned} [y]^{(N)} &\equiv x^{N-y} \frac{\partial^N}{\partial x^N} x^y \Big|_{x=1/2}, \\ &= y(y-1)(y-2)\cdots(y-N+1). \end{aligned} \tag{7.76}$$

When $N \ll \sqrt{|y|}$, we can approximate $[y]^{(N)} \sim y^N$. We can then repeat the analysis for the partition function and find that $\gamma^{(s)}_{1/2}(\Delta)$ is approximately reflection symmetric,

$$\overline{\gamma}^{(s)}_{1/2}(\Delta) \simeq \overline{\gamma}^{(s)}_{1/2}(4\Delta_0 - \Delta). \tag{7.77}$$

In general,

$$\overline{\gamma}^{(s)}_{x}(\Delta) \simeq \overline{\gamma}^{(s)}_{1-x}\left(\frac{2}{x}\Delta_0 - \frac{1-x}{x}\Delta\right)\frac{1-x}{x}. \tag{7.78}$$

In particular, $\gamma^{(s)}_x(\Delta < 0) = 0$ means,

$$\overline{\gamma}^{(s)}_{x}(\Delta > \Delta_x) \simeq 0, \tag{7.79}$$

where

$$\Delta_x = \frac{2}{1-x}\Delta_0. \tag{7.80}$$

In this limit, we can also solve the linear optimization problem to find,

$$\int_\Delta^\infty \gamma^{(s)}_{1/2}(\Delta')d\Delta' \leq \frac{1}{1 + T_{2k_0+1}\left(\frac{\Delta - 2\Delta_0}{2\Delta_0}\right)}, \tag{7.81}$$

for $\Delta > 4\Delta_0$. For general x, we can bound $\gamma_x(\Delta)$ for $\Delta > \Delta_x$ by,

$$\int_\Delta^\infty \gamma^{(s)}_{x}(\Delta')d\Delta' \leq \frac{2}{1 + T_{2k_0+1}\left(\frac{\Delta - \Delta_x/2}{\Delta_x/2}\right)}. \tag{7.82}$$

The bound is stronger for $x = 1/2$ since $\gamma^{(s)}_{1/2}(\Delta)$ is invariant under the reflection as in (7.77), while the reflection symmetry for $x \neq 1/2$ relates $\gamma^{(s)}_x$ to $\gamma^{(s)}_{1-x}$ as in (7.78). The latter bound can be improved in a neighborhood of $x = 1/2$.

For $\gamma^{(s)}_x(\Delta)$, we can also derive bounds at finite values of Δ and Δ_0, without approximating $[y]^{(N)}$ by y^N because of the simple structure (7.76) of the bracket symbol. As we explained in the case of partition function, the problem is to maximize $P(\Delta)$ given by

$$P(\Delta) = 1 + \sum_{k=0}^{\infty} \left[\Delta - 2\Delta_0\right]^{(2k+1)} \lambda_k, \tag{7.83}$$

at a particular value of Δ while maintaining $P(\Delta) \geq 0$ for all values of Δ.

However, as we noted before, any $P(\Delta)$ satisfying the constraints will lead to an upper bound on the optimal value of the primal problem. We can use

$$P(\Delta) = 1 - \frac{[\Delta - 2\Delta_0]^{(2k+1)}}{[-2\Delta_0]^{(2k+1)}}, \tag{7.84}$$

as an ansatz for such a $P(\Delta)$. To check that $P(\Delta) \geq 0$, we note that $[-2\Delta_0]^{(2k+1)} < 0$ and $[\Delta - 2\Delta_0]^{(2k+1)} > 0$ for $\Delta - 2\Delta_0 > 2k$, and it is easy to show that

$$\left| \frac{[\Delta - 2\Delta_0]^{(2k+1)}}{[-2\Delta_0]^{(2k+1)}} \right| \leq 1, \tag{7.85}$$

for $\Delta - 2\Delta_0 \leq 2k$, provided $\Delta_0 \geq 1/2$ (this condition can be weakened). Maximizing this ansatz $P(\Delta)$ at a particular value of Δ by using k as a variable gives the bound (after a natural interpolation of the right hand side, which happens not to invalidate the bound),

$$\int_{\Delta}^{\infty} \gamma_{1/2}^{(s)}(\Delta')d\Delta' \leq \frac{1}{1 + \frac{\Gamma(\Delta - 2\Delta_0 + 1)\Gamma(2\Delta_0)}{\Gamma\left(\frac{\Delta+3}{2}\right)\Gamma\left(\frac{\Delta-1}{2}\right)}}. \tag{7.86}$$

The above analysis of the limit $\Delta_0 \to \infty$ is easily carried over to the case of conformal blocks. One just has to note that

$$\begin{aligned} \frac{\partial^n}{\partial x^n}\rho^{\Delta}x^{-2\Delta_0} &\simeq \left(\frac{\partial \log \rho^{\Delta}x^{-2\Delta_0}}{\partial x}\right)^n \rho^{\Delta}x^{-2\Delta_0} \\ &= \left(\frac{\Delta}{x\sqrt{1-x}} - \frac{2\Delta_0}{x}\right)^n \rho^{\Delta}x^{-2\Delta_0}, \end{aligned} \tag{7.87}$$

to see that a polynomial approximation can be made again. It is then straightforward to derive the corresponding formulas for the conformal block case.

7.3.2.2 Cardy formula

Derivation of the Cardy-like formula (7.39) for the OPE coefficients is essentially equivalent to the partition function case in [74]. We outline the main steps here.

First, the analogue of light sparse spectrum condition is interpreted using crossing symmetry as, for $x > 1/2$,

$$\log G(x) = -2\Delta_0 \log(1-x) + O(1). \tag{7.88}$$

Then, one divides the spectrum into light and heavy parts, $L = [0, 2\Delta_0 + \epsilon)$ and $H = [2\Delta_0 + \epsilon, +\infty)$. Here ϵ is some fixed positive number, which can be taken exponentially small in $\sqrt{\Delta_0}$. A scaling dimension $\widehat{\Delta}$ is then picked inside the heavy spectrum and the latter is further split into three parts,

$$H_1 = [2\Delta_0 + \epsilon, \widehat{\Delta} - \delta), \quad H_3 = (\widehat{\Delta} + \delta, +\infty), \tag{7.89}$$

$$H_2 = [\widehat{\Delta} - \delta, \widehat{\Delta} + \delta]. \tag{7.90}$$

Here δ is some averaging scale which will turn out to be restricted by $\delta \sim \Delta_0^\alpha$, $\alpha \in (1/2, 1)$.

The idea is now to show that if $\widehat{\Delta} = 2\Delta_0/(1-x)$, then $G(x)$ is essentially due to contributions from H_2, $G \simeq G[H_2]$. To that end, one first bounds $G[H_2] \leq G$, as well as

$$G[H_2] = \int_{H_2} x^{\Delta - 2\Delta_0} g^{(s)}(\Delta) d\Delta \geq x^{\widehat{\Delta}-2\Delta_0+\delta} \overline{g}_\delta^{(s)}(\widehat{\Delta}), \tag{7.91}$$

where

$$\overline{g}_\delta^{(s)}(\widehat{\Delta}) = \int_{H_2} g^{(s)}(\Delta) d\Delta. \tag{7.92}$$

This leads to an inequality for $\overline{g}^{(s)}(\widehat{\Delta})$, which, upon picking an optimal value of x, reads for $\widehat{\Delta} > 4\Delta_0$ as

$$\begin{aligned}\log \overline{g}_\delta^{(s)}(\widehat{\Delta}) \leq -\widehat{\Delta} \log\left(1 - \frac{2\Delta_0}{\widehat{\Delta}}\right) + \\ + 2\Delta_0 \log\left(\frac{\widehat{\Delta}}{2\Delta_0} - 1\right) - \delta \log\left(1 - \frac{2\Delta_0}{\widehat{\Delta}}\right).\end{aligned} \tag{7.93}$$

One also gets a different inequality for $2\Delta_0 \leq \widehat{\Delta} \leq 4\Delta_0$. Then one replaces δ in these inequalities with a new δ' and takes the latter to be sufficiently small while keeping the δ in H_i fixed. This allows one to bound the contribution from H_1 and H_3 up to $\log \Delta_0$ error terms. The contribution from L is also bounded [74]. It then follows that given $\delta \sim \Delta_0^\alpha$, $\alpha \in (1/2, 1)$ H_2 dominates the 4-point function, and the inequality (7.93) turns into the equality (7.39).

7.4 Discussion

In the present paper we studied implications of modular invariance and crossing symmetry in certain scaling limits. We have found that all these cases share certain general features, in particular

1. A truncated set of crossing equations limits to a problem about polynomial moments of the branching ratios. This leads to an approximate duality relation for the branching ratios at crossing symmetric points.

2. The duality relation motivates tail bounds for the integrals of the branching ratios. These bounds are threshold bounds in the sense that they constrain the set of dominant scaling dimensions.

3. "Sparseness" of the light spectrum implies universality of the couplings of heavy spectrum. Such theories almost saturate the tail bounds. We discussed this only in two cases, but it is clear that this is a general feature.

These facts have a natural explanation if one assumes that a single saddle point dominates the expansions. Indeed, in this case the location of saddle point can be determined easily by taking appropriate log-derivative of the four-point or partition function. The crossing relation then imposes an equation on this location in a straightforward way. Note, however, that at no point we made such an assumption. In fact, one can assume that several competing saddle points may exist at some points, and in this case our duality relation maps their positions to the crossing symmetric expansion. This happens for example for generalized free field, which at $x = 1/2$ exhibits two saddle points – one at $\Delta = 0$ and one at $\Delta = 4\Delta_0$ (in scaling blocks). These two saddles are correctly related by the duality relation.

Besides this general features, we have also found features specific for some of the cases, in particular

1. For scaling block expansion of four point function we were able to use an ansatz incorporating infinitely many derivatives to produce an exponentially decaying tail bound. This bound is a strict inequality valid without taking any limit whatsoever.

2. For the large spacetime dimension limit of the conformal block expansion, we were able to see a manifestation of unitarity bound for external scalars without the use of the free scalar equation of motion.

Most of our results used some kind of a limit, and thus are not applicable to the bootstrap of light operators. However, one may hope that some qualitative features also carry over to the case of light operators, and thus may provide useful intuition. Let us discuss possible implications for numerical analysis.

In some cases, the four-point amplitude $G(x)$ is dominated by operators near the saddle point $\Delta(x)$. This observation may have applications to numerical bootstrap methods. which often employ derivatives of the crossing relation at $x = 1/2$. This mostly probes operators near the saddle point $\Delta(1/2)$. To learn about the other parts of the spectrum, apart from taking more and more derivatives at $x = 1/2$, one may consider the crossing relation at different values of x. In the case of scaling blocks, it is natural to expect that $O(1)$ changes in $1/(1-x)$ result in $O(\Delta_0)$ changes in $\Delta(x)$ and that the width of saddle point is on the order of $\sqrt{\Delta_0}$. Therefore, in order to have the spectrum up to $\Delta = \Lambda$ evenly covered, one may use the bootstrap equation at $O(\Lambda\sqrt{\Delta_0})$ points x so that $1/(1-x)$ is distributed evenly with spacing of the order of $O(1/\sqrt{\Delta_0})$.

Another observation is that gaps in OPE spectrum can render the parts of spectrum symmetric to them difficult to study. An example is the generalized free field four point function, which is

$$G(x) = \frac{1}{x^{2\Delta_0}} + \frac{1}{(1-x)^{2\Delta_0}} + 1. \tag{7.94}$$

It can be easily seen to be dominated by the vacuum term $x^{-2\Delta_0}$ for $x < 1/2$. As discussed above, this forces a discontinuity in $\Delta(x)$ at $x = 1/2$, with $\Delta(1/2+0) = 4\Delta_0$. In this theory there are no operators in the interval $(0, 2\Delta_0)$, but there are operators in $[2\Delta_0, 4\Delta_0)$, which by the approximate symmetry never dominate the four-point function.

Acknowledgments

We thank S. El-Showk, N. Hunter-Jones, C. Keller, Z. Komargodski, Y. Nakayama, S. Rychkov, D. Simmons-Duffin, B. Stoica, and W. Yan for discussion. We also thank S. Rychkov and D. Simmons-Duffin for their comments on the draft of this paper. This work is supported in part by U.S. DOE grant DE-SC0011632, by the WPI Initiative of MEXT, by JSPS KAKENHI Grant Numbers C-26400240 and 15H05895, by the Simons Investigator Award, and by the Walter Burke Institute for Theoretical Physics and the Moore Center for Theoretical Cosmology and Physics at Caltech. We thank the hospitality of the Institute for Advanced Study, where HO is Director's Visiting Professor. HO also thanks the Aspen Center for Physics and the Simons Center for Geometry and Physics, where parts of this work were done.

Chapter 8

THE 3D STRESS-TENSOR BOOTSTRAP

This chapter is essentially identical to:

A. Dymarsky, F. Kos, P. Kravchuk, D. Poland and D. Simmons-Duffin, *The 3d Stress-Tensor Bootstrap*, *JHEP* **02** (2018) 164, [1708.05718].

8.1 Introduction

The conformal bootstrap [26–28, 30] (see [18, 19, 280] for reviews) uses basic consistency conditions to bound the space of conformal field theories. By making fewer assumptions about the theories being studied, one can derive more universal bounds.[1] The original bounds [30, 43, 44, 46, 47, 95, 96, 126] apply to theories with scalar operators of various dimensions. Bounds from fermionic correlators [39, 40, 81] apply to theories with fermions, and the recent bounds in [41] apply to any 3d CFT with a continuous global symmetry.

Perhaps the minimal possible assumption about a CFT is the existence of a stress tensor. Indeed, a stress tensor (i.e. a conserved spin-2 operator whose integrals are the conformal charges) is necessarily present in any local CFT.[2] In this work, we study the constraints of conformal symmetry and unitarity on a four-point function of stress tensors in 3d CFTs. For simplicity, we also assume a parity symmetry, so our bounds apply universally to any unitary parity-preserving local 3d CFT. This birds-eye view of local CFTs with spacetime symmetry $O(3,2)$ is similar in spirit to the views of superconformal theories achieved in [133, 140, 151, 281].

An advantage of a numerical approach is that we can make contact with analytic results, but we also have the flexibility to perform more sophisticated studies that are currently not analytically tractable. For instance, we numerically recover the conformal collider bounds [73, 76, 77, 287], but we can additionally study how these bounds are modified under various assumptions about the spectrum of the CFT. As

[1]By contrast, one can study a specific theory by inputting characteristic features that distinguish the theory in question. In this sense, the conformal bootstrap was successfully applied to extract precise properties of the 3d Ising model [8, 31, 32, 34, 36, 111]. Families of critical $O(N)$ models [8, 37, 38, 45, 137], Gross-Neveu-Yukawa models [39, 40], and various supersymmetric theories [133, 136, 138–144, 150–152, 281, 282] have also been studied in this way.

[2]Examples of theories without a stress tensor include boundary/defect theories [127, 129, 283] and nonlocal theories like the Long-Range Ising model [284–286].

we discuss below, we also find a host of new universal bounds constraining, e.g., the spectrum of low-dimension scalar operators.

The bootstrap equations are consistency conditions on the conformal block decomposition of 4-point functions. Written in terms of CFT data, they are quadratic constraints on OPE coefficients. Self-consistency or "feasibility" of these constraints can be efficiently analyzed using semidefinite programming [19, 35, 36, 47]. Formulating the bootstrap constraints for stress tensors in a way suitable for semidefinite programming involves several steps, which we briefly describe below. First is the task of writing 3- and 4-point functions of stress tensors in an explicitly conformally-invariant way. We do this using a combination of the embedding formalism of [53] and the conformal frame formalism of [1]. The second step is to get rid of the degeneracies associated with permutation symmetry and conservation. This is done by identifying a minimal set of linearly-independent crossing equations, slightly refining the approach of [75]. These steps are explained in detail in section 8.2. Finally, the third step is the calculation of conformal blocks which is done in section 8.3 by translating the approach of [61] to the conformal frame formalism. In this way we obtain a set of bootstrap equations suitable for numerical analysis.

In the rest of the paper we analyze the bootstrap constraints supplemented by various additional assumptions about the spectrum. In section 8.4.2 we numerically reproduce, in full generality, the conformal collider bounds on the "central charges" of unitary theories [76, 287], previously discussed in the context of the analytic bootstrap in [77, 162]. Our main result here is a lower bound on the central charge C_T as a function of the independent parameter in the stress-tensor three-point function, characterized by the angle θ defined in (8.80). In section 8.4.3 we study constraints on the spectrum of the lightest parity-even and parity-odd scalars in general unitary 3d CFTs. Some of the results are shown in figure 8.8. In particular, we find that any unitary CFT must necessarily have both light parity-even and light parity-odd singlet scalars in its spectrum. This is similar to a recent finding that unitary 3d CFTs with global symmetries must have low-dimension scalars in the OPE of two conserved currents [41].

Quite generally, we find that when the gaps in the spectrum of scalar operators are sufficiently large to exclude large N theories (by excluding some double-trace operators), the allowed region for OPE coefficients C_T and θ is compact—in particular, there exists an upper bound on the central charge. This suggests that theories with large C_T must necessarily have double-trace operators in $T \times T$ OPE. Furthermore, this may potentially point to the existence of new strongly-coupled theories residing inside these compact regions. We observe the same phenomenon when imposing a gap on the dimension of the second lightest spin-2 operator in section 8.4.4.

In section 8.4.5 we discuss theories with a gap Δ_4 in the spectrum of spin-4 parity-even operators. In full consistency with the Nachtmann theorem, we observe that when Δ_4 approaches 6, the lower bound on C_T grows indefinitely for all θ, in accord with the expectation that the corresponding theory is dual to weakly coupled gravity in AdS_4. Finally, section 8.4.6 is devoted to studies of the 3d Ising model. Under the assumption of no relevant parity-odd scalars, and by imposing the known values of the central charge and the dimensions of certain light operators, we obtain a window $0.01 < \theta < 0.05$. Under stronger but still plausible assumptions we obtain a tighter bound $0.010 < \theta < 0.019$. We also find an upper bound on the parity-odd scalar gap $\Delta_{\text{odd}} < 11.2$. We conclude with a discussion in section 8.5.

8.2 Conformal structures

8.2.1 3-point structures

To set up the bootstrap equations for the 4-point function $\langle TTTT\rangle$ in 3d CFTs preserving parity, we first need to understand the possible 3-point functions $\langle TT\mathcal{O}\rangle$ between the stress tensor $T^{\mu\nu}$ and various operators $\mathcal{O}$ in the CFT. The purpose of this section is to classify such 3-point functions, and thus the operators which can be exchanged in the OPE decomposition of $\langle TTTT\rangle$.

First of all, only bosonic operators $\mathcal{O}$ can appear in $T \times T$ OPE, and so without loss of generality we can assume that $\mathcal{O}$ is a traceless symmetric tensor primary of spin ℓ. Furthermore, since T is a singlet under all global symmetries, $\mathcal{O}$ must be a singlet as well. However, $\mathcal{O}$ may be even or odd under space parity.

The 3-point functions $\langle TT\mathcal{O}\rangle$ should be conformally-invariant, symmetric with respect to permutation of the two T insertions, and satisfy the conservation equation for the stress tensor,

$$\partial_\mu T^{\mu\nu} = 0 + \text{contact terms}. \tag{8.1}$$

Such 3-point functions have the form

$$\langle TT\mathcal{O}\rangle = \sum_{a=1}^{N_{TT\mathcal{O}}} \lambda^{(a)}_{TT\mathcal{O}} \langle TT\mathcal{O}\rangle_{(a)}, \tag{8.2}$$

where $\langle TTO\rangle_{(a)}$ are 3-point tensor structures which form a basis of solutions to the above constraints, and $\lambda^{(a)}_{TTO}$ are OPE coefficients. We can always choose a basis such that $\lambda^{(a)}_{TTO}$ are real.

The 3-point tensor structures $\langle TTO\rangle_{(a)}$ can be classified using e.g. the conformal frame formalism of [1]. We will also need to perform manipulations with explicit expressions, which we can obtain by constructing the tensor structures using the 5d embedding space formalism of [53, 61].

In this latter formalism, the parity-even 3-point tensor structures are constructed from basic invariants denoted by H_{ij} and V_i, where i and j index the operators in the 3-point function. The structure H_{ij} increases the spin by one unit for operators i and j, while V_i does so only for the operator i. For example, a general 3-point structure for $\langle TT\phi\rangle$ with a scalar ϕ of dimension Δ is given by[3]

$$\langle TT\phi\rangle = \frac{\alpha H_{12}^2 + \beta H_{12}V_1V_2 + \gamma V_1^2V_2^2}{(-2X_1\cdot X_2)^{\frac{10-\Delta}{2}}(-2X_2\cdot X_3)^{\frac{\Delta}{2}}(-2X_3\cdot X_1)^{\frac{\Delta}{2}}}, \tag{8.3}$$

where the constants α, β, γ are subject to linear constraints coming from conservation of T and permutation symmetry, while X_i are the embedding space coordinates of the operators [53]. For sufficiently large ℓ there are 14 different combinations of H_{ij} and V_i which give the correct spins for the three operators in $\langle TTO\rangle$. Not all of them are independent, since there exist non-linear relations between the invariants H and V, which were classified in [53]. In our case there is a single redundant structure

$$H_{12}H_{23}H_{31}V_3^{\ell-2}, \tag{8.4}$$

which can be expressed in terms of other structures.

Using the results of [53], it is straightforward to impose permutation and conservation constraints on these tensor structures. An analogous construction works for parity-odd tensor structures [53]. We will not need the explicit expressions for the tensor structures in this "algebraic" basis, but rather in the so called differential basis, which we describe in section 8.3.[4] The explicit expressions in the differential basis are provided in appendix G.1.

Here, let us summarize the counting of 3-point tensor structures. Let O_ℓ denote a primary operator of spin ℓ and a scaling dimension Δ strictly above the unitarity

[3] We assume that the stress tensors are at positions 1 and 2, while the intermediate operator is at position 3.

[4] We will still use input from the algebraic basis to perform calculations in the differential basis.

bound. This restriction is important since the number of solutions to conservation equations can increase at special values of Δ.[5] In fact, this is what happens for $\Delta = 3$ and $\ell = 2$, i.e. when $\mathcal{O}_{\ell=2} = T$ is the stress tensor itself. With these conventions, the counting of 3-point tensor structures is given by the table:

$\mathcal{O}$	$N_{TT\mathcal{O}}$
$\mathcal{O}_0$	$1^+ + 1^-$
$\mathcal{O}_2$	$1^+ + 1^-$
T	$2^+ + 1^-$
$\mathcal{O}_{2n}, n \geq 2$	$2^+ + 1^-$
$\mathcal{O}_{2n+1}, n \geq 2$	1^-

where we have separated parity-even and parity-odd tensor structures (indicated by the $\pm$ superscripts). For $\mathcal{O} = T$, the tensor structures are invariant under permutations of all three operators. Note that the parity-odd tensor structure for $\langle TTT \rangle$ does not appear in a parity-preserving theory, since T is necessarily parity-even, as can be seen from the Ward identity discussed below.

8.2.1.1 Ward identities

As mentioned above, the 3-point function $\langle TTT \rangle$ has two allowed parity-even tensor structures, which can be realized in the theories of a free real scalar and a free Majorana fermion,

$$\langle TTT \rangle = n_B \langle TTT \rangle_B + n_F \langle TTT \rangle_F. \tag{8.5}$$

There exists a non-trivial Ward identity for this correlator. Indeed, one can construct the dilatation current $J_D^\mu = x_\nu T^{\mu\nu}$ from one of the three stress-tensor operators, and integrate it over a surface surrounding another stress-tensor operator put at $x = 0$ to obtain, schematically,

$$\int x \langle TTT \rangle dS = \Delta_T \langle TT \rangle. \tag{8.6}$$

This Ward identity implies a linear relation between the coefficients n_B, n_F and the 2-point function $\langle TT \rangle$. The latter can be parametrized as

$$\langle TT \rangle = C_T \langle TT \rangle_B, \tag{8.7}$$

[5]Note that the conservations constraints are linear with coefficients dependent on Δ. The rank of a parameter-dependent linear system is always constant at generic values of the parameters and can only decrease at special values.

where $\langle TT\rangle_B$ is the 2-point function $\langle TT\rangle$ in the theory of a free real scalar and C_T is the “central charge." The Ward identity then must be of the form

$$C_B n_B + C_F n_F = C_T. \tag{8.8}$$

The constants C_B, C_F are simply the central charges of the free real scalar and free Majorana fermion respectively, where our normalization for C_T implies $C_B = C_F = 1$. However, in the sections below we will often write results in terms of the ratio C_T/C_B so that they also hold for other normalizations of C_T.

8.2.2 4-point structures

The 4-point function $\langle TTTT\rangle$ should satisfy the following properties, which interact with each other in nontrivial ways:

- conformal invariance,
- permutation symmetry,
- conservation,
- regularity (analyticity).

We will address each property in turn, culminating in a minimal set of crossing symmetry equations suitable for applying numerical bootstrap techniques.

It is useful to use index-free notation to encode different tensor structures. Let us write

$$T(w,x) = w_\mu w_\nu T^{\mu\nu}(x), \tag{8.9}$$

where w_μ is an auxiliary polarization vector. Because $T^{\mu\nu}$ is traceless, we can take w_μ to be null, $w^2 = 0$. We can recover $T^{\mu\nu}$ as

$$T^{\mu\nu}(x) = D_w^\mu D_w^\nu T(w,x), \tag{8.10}$$

where D_w^μ is the Todorov operator [201]

$$D_w^\mu = \left(\frac{d-2}{2} + w\cdot\frac{\partial}{\partial w}\right)\frac{\partial}{\partial w_\mu} - \frac{1}{2}w^\mu\frac{\partial^2}{\partial w\cdot\partial w}, \tag{8.11}$$

with $d = 3$ the spacetime dimension. Note that the Todorov operator preserves the ideal generated by w^2,

$$D_w^\mu(w^2 f(w)) = w^2(\dots), \tag{8.12}$$

so it is well-defined even though w is constrained to be null.

8.2.2.1 Conformal invariance

To study the above properties, it is useful to fix a conformal frame and use representation theory of stabilizer groups to classify tensor structures, following [1]. This approach makes it easy to deal with degeneracies between tensor structures in low spacetime dimensions, and will also help us understand regularity conditions on the $z = \bar{z}$ line. We work in Euclidean signature throughout.

Using conformal transformations we can place the four operators in the 1-2 plane in the following configuration:

$$g(z, \bar{z}, w_i) = \langle T(w_1, 0)T(w_2, z)T(w_3, 1)T(w_4, \infty)\rangle. \tag{8.13}$$

We have $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$, with the direction perpendicular to the plane being x^3. For brevity, we have written only the holomorphic coordinate of each operator.

We define the operator at infinity in a non-standard way, where we do not act with an inversion on the polarization vector,

$$T(w, \infty) \equiv \lim_{L\to\infty} L^{2\Delta_T} T(w, L), \quad \Delta_T = 3\ . \tag{8.14}$$

The virtue of this convention is that the polarization vectors are treated more symmetrically, so it will be easier to understand the action of permutations.

We will consider parity-preserving theories, so the group of spacetime symmetries is $O(4, 1)$. The points $0, z, 1, \infty$ are stabilized by an $O(1) = \mathbb{Z}_2$ subgroup of $O(4, 1)$ consisting of reflections in the x^3 direction (perpendicular to the plane). The 4-point function $g(z, \bar{z}, w_i)$ must be invariant under this stabilizer subgroup or “little-group." Little-group invariance then guarantees that $g(z, \bar{z}, w_i)$ can be extended to an $O(4, 1)$-invariant function for arbitrary configurations of the $T(w_i, x_i)$.

Let $\boldsymbol{\ell}^\pm$ denote the parity-even/odd spin-ℓ representation of $O(3)$, and let $\bullet^\pm$ denote the even and odd representations of $O(1)$. Each operator $T(w, x)$ transforms in the representation $\mathbf{2}^+$ of $O(3)$. Little-group invariants are $O(1)$ singlets in

$$\left(\mathrm{Res}^{O(3)}_{O(1)}\mathbf{2}^+\right)^{\otimes 4} = \left(3\bullet^+ \oplus 2\bullet^-\right)^{\otimes 4} = 313\bullet^+ \oplus 312\bullet^-, \tag{8.15}$$

where $\mathrm{Res}^G_H\rho$ denotes the restriction of a representation ρ of G to a representation of $H \subseteq G$. In particular, there are 313 parity-even tensor structures (and 312 parity-odd tensor structures).

These structures are easy to enumerate. Define components of the polarization vectors

$$\begin{aligned} \omega &= w^z = w^1 + iw^2 \\ \overline{\omega} &= w^{\overline{z}} = w^1 - iw^2 \\ \omega^0 &= w^3. \end{aligned} \tag{8.16}$$

For each "helicity" $h \in \{-2, -1, 0, 1, 2\}$, we can construct a unique monomial $[h]$ with degree 2 and charge h under rotation in the z-plane,

$$[-2] = \overline{\omega}^2, \quad [-1] = \overline{\omega}\omega^0, \quad [0] = \omega\overline{\omega}, \quad [1] = \omega\omega^0, \quad [2] = \omega^2. \tag{8.17}$$

(Using the fact that $w_\mu w^\mu = (\omega^0)^2 + \omega\overline{\omega} = 0$, we can ensure that the degree in ω^0 is at most one.) Let $[h_1 h_2 h_3 h_4]$ denote a product of the corresponding monomials for each polarization vector w_i^μ.[6] It is easy to verify that there are 313 structures $[h_1 h_2 h_3 h_4]$ which are even under parity $\omega^0 \to -\omega^0$, i.e. such that $\sum_i h_i \equiv 0 \bmod 2$. The 4-point function is a linear combination of these structures, with coefficients that are functions of z and $\overline{z}$,

$$g(z, \overline{z}, w_i) = \sum_{\sum_i h_i \text{ even}} [h_1 h_2 h_3 h_4] g_{[h_1 h_2 h_3 h_4]}(z, \overline{z}). \tag{8.18}$$

Using rotations around the x_1 axis, we can relate the point $(z, \overline{z})$ to its reflection in the imaginary direction $(\overline{z}, z)$. Invariance of the full correlator under this transformation implies

$$g_{[h_1 h_2 h_3 h_4]}(z, \overline{z}) = g_{[-h_1, -h_2, -h_3, -h_4]}(\overline{z}, z). \tag{8.19}$$

Meanwhile, reality[7] of g implies

$$g_{[h_1 h_2 h_3 h_4]}(z, \overline{z}) = \overline{g}_{[-h_1, -h_2, -h_3, -h_4]}(\overline{z}, z), \tag{8.20}$$

where we used the notation $\overline{f}(\overline{z}, z) \equiv (f(z, \overline{z}))^*$, from which it follows that

$$g_{[h_1 h_2 h_3 h_4]}(z, \overline{z}) = \overline{g}_{[h_1 h_2 h_3 h_4]}(z, \overline{z}). \tag{8.21}$$

In other words, the functions $g_{[h_1 h_2 h_3 h_4]}(z, \overline{z})$ must have real coefficients in a Taylor series expansion in powers of z and $\overline{z}$.

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[6] This definition differs from the one based on spinor polarizations in [1] by a numerical factor.

[7] Reality of $\langle TTTT \rangle$ follows from a combination of space parity and Euclidean Hermitian conjugation.

8.2.2.2 Permutation invariance

The 4-point function $\langle T(w_1, x_1) \cdots T(w_4, x_4)\rangle$ must be invariant under permutations of the four operators. Permutations that change the cross-ratios $z, \bar{z}$ lead to non-trivial crossing equations that we explore later. However, permutations that leave $z, \bar{z}$ invariant, which we call "kinematic permutations," give constraints on tensor structures alone [1, 75]. In our case, the group of kinematic permutations is (in cycle notation)

$$\Pi^{\text{kin}} = \{\text{id}, (12)(34), (13)(24), (14)(23)\} = \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{8.22}$$

As shown in [1], Π^{kin}-invariant tensor structures are in one-to-one correspondence with

$$\left(\bigotimes_{i=1}^{4} \text{Res}^{O(3)}_{O(1)} \mathbf{2}^+\right)^{\Pi^{\text{kin}}}, \tag{8.23}$$

where Π^{kin} acts on tensor factors in the natural way, and $(\rho)^G$ denotes the G-invariant subspace of ρ. These can be counted using

$$(\rho^{\otimes 4})^{\mathbb{Z}_2\times\mathbb{Z}_2} = \rho^4 \ominus 3(\wedge^2\rho \otimes \mathrm{S}^2\rho), \tag{8.24}$$

where $\ominus$ represents the formal difference in the character ring. Plugging in $\rho = 3\,\bullet^+ \oplus 2\,\bullet^-$ to (8.24), we find

$$((3\,\bullet^+ \oplus 2\,\bullet^-)^{\otimes 4})^{\mathbb{Z}_2\times\mathbb{Z}_2} = 97\,\bullet^+ \oplus 78\,\bullet^-, \tag{8.25}$$

so there are 97 permutation-invariant parity-even structures.

	r_1	r_2	r_3	r_4
id	1	1	1	1
(12)(34)	$-(1-z)$	$-(1-\bar{z})$	$-(1-\bar{z})$	$-(1-z)$
(13)(24)	$\bar{z}(1-z)$	$\bar{z}(1-\bar{z})$	$z(1-\bar{z})$	$z(1-z)$
(14)(23)	$-\bar{z}$	$-\bar{z}$	$-z$	$-z$

Table 8.1: Permutation phases for a 4-point function of identical operators, computed in [1].

To write the structures explicitly, we must be more specific about the action of permutations on polarization vectors. A permutation $\pi \in \Pi^{\text{kin}}$ acts on a monomial $[h_i]$ as

$$\pi : [h_i] \mapsto \qquad n(r_i(\pi))^{h_i}[h_{\pi(i)}], \tag{8.26}$$

where $n(x) = \sqrt{x/\bar{x}}$ is a phase and the $r_i(\pi)$ are given in the table 8.1. Permutation-invariant structures are given by symmetrizing with respect to this action:

$$\begin{aligned}\langle h_1 h_2 h_3 h_4\rangle_z \equiv &\frac{1}{m_{h_1h_2h_3h_4}}\Big([h_1h_2h_3h_4] \\ &+ n(1-z)^{-h_1+h_2+h_3-h_4}[h_2h_1h_4h_3] \\ &+ n(z)^{h_1+h_2-h_3-h_4}[h_4h_3h_2h_1] \\ &+ n(z)^{h_1+h_2-h_3-h_4}n(1-z)^{-h_1+h_2+h_3-h_4}[h_3h_4h_1h_2]\Big),\end{aligned} \tag{8.27}$$

where $m_{h_1h_2h_3h_4}$ is the number of elements Π^{kin} which stabilize $[h_1h_2h_3h_4]$. We have also added an index z to the symmetric tensor structures to indicate that they depend on z and $\bar{z}$. Here, it's clear that independent Π^{kin}-invariant structures are in one-to-one correspondence with orbits of $\mathbb{Z}_2 \times \mathbb{Z}_2$ when acting on quadruples $[h_1h_2h_3h_4]$. Making a choice of representative for each of the 97 parity-even orbits, we can write

$$g(z,\bar{z},w_i) = \sum_{\substack{h_i/\mathbb{Z}_2^2 \\ \sum_i h_i \text{ even}}} \langle h_1h_2h_3h_4\rangle_z\, g_{[h_1h_2h_3h_4]}(z,\bar{z}). \tag{8.28}$$

Note that the functions $g_{[h_1h_2h_3h_4]}(z,\bar{z})$ are the same as those appearing in (8.18).

8.2.2.3 Conservation

Imposing conservation of $T^{\mu\nu}(x)$ gives nontrivial differential equations relating the functions $g_{[h_1h_2h_3h_4]}(z,\bar{z})$. These equations can be solved up to some undetermined functions of $z,\bar{z}$ that we call "functional degrees of freedom." Conversely, after imposing conservation, the functional degrees of freedom fix the entire correlator (modulo boundary terms that we discuss below). Thus, an independent set of crossing-symmetry equations should make reference to functional degrees of freedom alone.

In [75], it was shown that there are 5 functional degrees of freedom in a 4-point function of stress tensors in 3d. We can obtain the number 5 with a simple group-theoretic rule from [1]. To account for conservation, we simply replace

$$\operatorname{Res}^{O(3)}_{O(1)}\mathbf{2}^+ \to \operatorname{Res}^{O(2)}_{O(1)}\mathbf{2} = \bullet^+ \oplus \bullet^- \tag{8.29}$$

in (8.23). Here, $O(2)$ can be interpreted as the little group of a massless particle in 4 dimensions, and $\mathbf{2}$ on the right-hand side of the arrow represents the spin-2 representation of $O(2)$. Plugging $\rho = \bullet^+ \oplus \bullet^-$ into (8.24), we find $5\,\bullet^+ \oplus 2\,\bullet^-$, so there are indeed 5 parity-even functional degrees of freedom.

Let us see more explicitly how these 5 degrees of freedom come about. Because the permutation group Π^{kin} acts freely on the four points, it suffices to impose conservation at one of the points, say x_2. The conservation equation is

$$D_{w_2} \cdot \frac{\partial}{\partial x_2} \langle T(w_2, x_2) \cdots \rangle = 0, \tag{8.30}$$

where D_w is the Todorov operator (8.11). Restricting to the conformal frame configuration (8.13), this gives[8]

$$\left(\left(\frac{3}{2} - \omega \partial_\omega \right) \partial_{\overline{\omega}} \partial_z + \left(\frac{3}{2} - \overline{\omega} \partial_{\overline{\omega}} \right) \partial_\omega \partial_{\overline{z}} + \frac{i D_w^3 \mathcal{L}_{23}}{z - \overline{z}} \right) g(z, \overline{z}, w_i) = 0, \tag{8.31}$$

where

$$\mathcal{L}_{23} = i \sum_k \left(\omega_k^0 \left(\partial_{\omega_k} - \partial_{\overline{\omega}_k} \right) + \frac{1}{2} (\omega_k - \overline{\omega}_k) \partial_{\omega_k^0} \right) \tag{8.32}$$

is the generator of rotations in the 2-3 plane acting on polarization vectors. In (8.31), $\omega, \overline{\omega}, \omega^0$ refer to $\omega_2, \overline{\omega}_2, \omega_2^0$, respectively. The last term in the conservation equation is naively singular at $z = \overline{z}$. However, the singularity will be cancelled by zeros in the action of $\mathcal{L}_{23}$. These complications stem from the fact that $z = \overline{z}$ is a locus of enhanced symmetry, where the little group becomes $O(2)$ instead of $O(1)$. We will study these issues in more detail below.

Following [75], we can solve (8.31) by thinking of one of the directions in the z-$\overline{z}$ plane as "time" t and the other as "space" ξ and integrating away from a constant time slice. The conservation equation then has the structure

$$(A \partial_t + B \partial_\xi + C) g = 0, \tag{8.33}$$

where A, B, C are linear operators on the space of tensor structures. The number of functional degrees of freedom is the dimension of the kernel of A.

In our case, it is convenient to choose z as the time direction, with $\overline{z}$ as the space direction. The operator A is then $A_z = \left(\frac{3}{2} - \omega_2 \partial_{\omega_2} \right) \partial_{\overline{\omega}_2}$, which vanishes on any structure that is independent of $\overline{\omega}_2$. This restricts the helicity h_2 to be either 1 or

[8] The Todorov operator in the first two terms simplifies because of our choice of tensor structures (8.17), which is at most linear in ω^0.

2. Because permutations Π^{kin} act freely, all helicities must be either 1 or 2, so the kernel of A_z is spanned by the five structures

$$\langle 2222\rangle_z, \quad \langle 1111\rangle_z, \quad \langle 1212\rangle_z, \quad \langle 1122\rangle_z, \quad \langle 2112\rangle_z. \tag{8.34}$$

When integrating the conservation equation, we can set the coefficients of these structures to anything we like. In practice, it will be useful to use a slightly different basis of functional degrees of freedom. Let

$$\langle h_1 h_2 h_3 h_4\rangle_z^{\pm} = \frac{1}{2}\left(\langle h_1 h_2 h_3 h_4\rangle_z \pm \langle -h_1, -h_2, -h_3, -h_4\rangle_z\right), \tag{8.35}$$

and define the corresponding coefficient functions

$$g^{\pm}_{[h_1 h_2 h_3 h_4]}(z,\bar{z}) = g_{[h_1 h_2 h_3 h_4]}(z,\bar{z}) \pm g^{\pm}_{[-h_1,-h_2,-h_3,-h_4]}(z,\bar{z}). \tag{8.36}$$

Equation (8.19) implies

$$g^{\pm}_{[h_1 h_2 h_3 h_4]}(\bar{z},z) = \pm g^{\pm}_{[h_1 h_2 h_3 h_4]}(z,\bar{z}). \tag{8.37}$$

We will take the functions $g^{+}_{[h_1 h_2 h_3 h_4]}(z,\bar{z})$ as our functional degrees of freedom. Fixing these functions is sufficient to remove ambiguities when integrating the conservation equation in the z-direction. By working in a Taylor expansion in $z, \bar{z}$, it is easy to argue that fixing $g^{+}_{[h_1 h_2 h_3 h_4]}(z,\bar{z})$ removes ambiguities when integrating in any direction. In particular, later we will integrate the conservation equation in the $x_2 = \operatorname{Im} z$ direction.

As explained in [75], in order to consistently integrate (8.33) away from a spatial slice, the initial data might need to satisfy additional constraints. Suppose N is a matrix such that $NA = 0$. Acting with N on (8.33), we obtain

$$(NB\partial_\xi + NC)g = 0. \tag{8.38}$$

This constraint turns out to be first class, meaning that we only need to impose it on the initial data. Our initial slice will be the line $z = \bar{z}$. Because this is a locus of enhanced symmetry, we must take care while analyzing the conservation equation around it.

8.2.2.4 Regularity and boundary conditions

For numerical bootstrap applications, we would like to write the crossing equations in a Taylor series expansion around the point $z = \bar{z} = \frac{1}{2}$. The line $z = \bar{z}$ corresponds to the four points x_i becoming collinear, which means the stabilizer group is enhanced from $O(1) \to O(2)$. Since the tensor structures have to be invariant under the stabilizer group, we can see that there are boundary conditions at $z = \bar{z}$ which the functions $g_{[h_1h_2h_3h_4]}$ have to satisfy in a well-defined correlator. As we will now show, smoothness of the correlator places further constraints on the Taylor expansion of $g_{[h_1h_2h_3h_4]}$ around this locus.

Consider the 4-point function after fixing x_1, x_3, x_4, but before rotating x_2 into the 1-2 plane,

$$g(x_2, w_i) = \langle T(w_1, 0)T(w_2, x_2)T(w_3, e)T(w_4, \infty)\rangle. \tag{8.39}$$

Here, $e = (1, 0, 0)$ is a unit vector in the 1-direction. We want the correlator to be smooth in x_2. In particular, it should have a Taylor expansion in the directions orthogonal to e,

$$g(x_2, w_i) = \sum_{n=0,\ell=0}^{\infty} g_n^{\mu_1\cdots\mu_\ell}(w_i, x) y_{\mu_1} \cdots y_{\mu_\ell} y^{2n}, \tag{8.40}$$

where $y_\mu = (x_2)_\mu - e_\mu(x_2 \cdot e)$ is the projection of x_2 onto the directions orthogonal to e, and $x = e \cdot x_2$. The coefficient functions $g_n^{\mu_1\cdots\mu_\ell}(w_i, x)$ are symmetric tensors of the stabilizer group $O(2)$, built out of polarization vectors. Let us count them. Let $\mathbf{0}^\pm$ denote the parity-even/odd scalar of $O(2)$, and let $\boldsymbol{\ell}$ denote the spin-ℓ representation of $O(2)$. Each operator transforms in the representation

$$\rho = \mathrm{Res}^{O(3)}_{O(2)} \mathbf{2}^+ = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}^+. \tag{8.41}$$

Although $\mathbb{Z}_2 \times \mathbb{Z}_2$ permutations act in a way that depends on x and y_μ, the leading-order in y action is simply the obvious permutation of polarization vectors, because the phases $n(r_i(\pi))$ are trivial on the line $z = \bar{z}$.[9] Thus, for the sake of counting new permutation-invariant tensor structures at each order in y_μ, we can use (8.24), which gives

$$(\rho^{\otimes 4})^{\mathbb{Z}_2\times\mathbb{Z}_2} = 22\, \mathbf{0}^+ \oplus 3\, \mathbf{0}^- \oplus \ldots. \tag{8.42}$$

Equation (8.40) implies that a polarization structure transforming in $\boldsymbol{\ell}$ of $O(2)$ can appear starting at order ℓ in the y-expansion. From (8.42) we see that at zeroth order in y, there are 22 parity-even permutation-invariant structures that can appear (out

[9] In fact, as shown in [1], we can define polarization vectors $\tilde{w}_i = w_i + O(y)$, which permute with trivial phases to all orders in y. We can then use these polarization vectors in (8.40).

of 97 total).[10] In order for the 4-point function to be well-defined at $z = \bar{z}$, only the coefficients of these 22 structures can be nonzero.

It turns out that thanks to the conservation equation, this is the only condition that we have to worry about. In general, since (8.42) gives $O(2)$ spins up to **8**, in the absence of the conservation equation we would have to write similar conditions for the first 8 orders in $\operatorname{Im} z$. However, as the derivation above shows, these constraints follow from $O(2)$ invariance. In particular, the conservation equation is compatible with (8.40) in the sense that it produces a recursion relation for the coefficients g_n. Therefore, as long as the zeroth order constraints are satisfied, higher orders follow automatically.[11] We have explicitly verified this by working order-by order in a Taylor expansion in $\operatorname{Im} z$.

Thus, our initial conditions include 22 undetermined functions of a single variable $\operatorname{Re} z$. We can take 5 of these to be the restrictions of our two-variable degrees of freedom to the $z = \bar{z}$ line, $g^{+}_{[h_1 h_2 h_3 h_4]}(\operatorname{Re} z, \operatorname{Re} z)$ where the h_i are given in (8.34). Even though the structures $\langle h_1 h_2 h_3 h_4 \rangle^{+}_{z}$ do not lie in the 22-dimensional subspace of $O(2)$ singlets, we can choose the coefficients of other structures to cancel the non-$O(2)$-invariant parts. The projection of the 5 bulk structures onto the $O(2)$-invariant subspace at $\operatorname{Im} z = 0$ is five-dimensional. Thus, there are exactly $22 - 5 = 17$ remaining one-variable degrees of freedom.

Finally, the constraints (8.38) give 8 independent first-order equations that these univariate functions must satisfy. Thus, in addition to 5 two-variable degrees of freedom, we have 9 one-variable degrees of freedom and 8 integration constants. We are free to choose these however we like, as long as the projection of the corresponding structures to the $O(2)$-invariant subspace is 22-dimensional.

8.2.2.5 Summary and crossing equations

Altogether, we choose the following functions as our undetermined degrees of freedom.

[10] Incidentally, 22 is also the number of functional degrees of freedom in a 4-point function of stress tensors in 4d. This is because the stabilizer group of a generic configuration of 4-points in 4d is $O(2)$, while the little group for massless particles in 5d is $O(3)$. Thus, the representation theory computation is the same as the one here (see [1, 75]).

[11] One should make sure that the choice of independent two-variable degrees of freedom does not contradict the regularity constraints. Or, equivalently, that these degrees of freedom are indeed independent from the point of view of the recursion relation for (8.40). We have checked that it is true for our choice of two-variable degrees of freedom.

- Two-variable degrees of freedom:

$$\begin{aligned}
&g^+_{[2222]}(z,\bar{z}), \quad g^+_{[1111]}(z,\bar{z}), \quad g^+_{[1212]}(z,\bar{z}),\\
&g^+_{[1122]}(z,\bar{z}), \quad g^+_{[2112]}(z,\bar{z}).
\end{aligned} \tag{8.43}$$

- One-variable degrees of freedom:

$$\begin{aligned}
&g^+_{[0000]}(z), \quad g^+_{[0101]}(z), \quad g^+_{[0202]}(z),\\
&g^+_{[0112]}(z), \quad g^+_{[1012]}(z),\\
&g^+_{[0011]}(z), \quad g^+_{[1001]}(z),\\
&g^+_{[0,0,-1,1]}(z), \quad g^+_{[-1,0,0,1]}(z).
\end{aligned} \tag{8.44}$$

- Integration constants:

$$\begin{aligned}
&g^+_{[0022]}(1/2), \quad g^+_{[2002]}(1/2),\\
&g^+_{[0,1,-1,2]}(1/2), \quad g^+_{[-1,1,0,2]}(1/2),\\
&g^+_{[0,-1,1,2]}(1/2), \quad g^+_{[1,-1,0,2]}(1/2),\\
&g^+_{[1,-1,-1,1]}(1/2), \quad g^+_{[-1,-1,1,1]}(1/2).
\end{aligned} \tag{8.45}$$

The statement of crossing symmetry is simply

$$g^+_{[h_1h_2h_3h_4]}(z,\bar{z}) = g^+_{[h_3h_2h_1h_4]}(1-z, 1-\bar{z}). \tag{8.46}$$

We have chosen the set of helicities in our independent degrees of freedom (8.43), (8.44), and (8.45) to be invariant under $h_1 \leftrightarrow h_3$. Thus, crossing symmetry becomes a constraint on these degrees of freedom alone.

As usual, we Taylor-expand the crossing equations around $z = \bar{z}$ to obtain the following system, parametrized by $n \le \bar{n}$, $n + \bar{n} \le \Lambda$.

- Two-variable equations:

$$\begin{aligned}
&\partial_z^n \partial_{\bar{z}}^{\bar{n}} g^+_{[2222]}(1/2, 1/2) = 0, \qquad (n+\bar{n} \text{ odd}),\\
&\partial_z^n \partial_{\bar{z}}^{\bar{n}} g^+_{[1111]}(1/2, 1/2) = 0, \qquad (n+\bar{n} \text{ odd}),\\
&\partial_z^n \partial_{\bar{z}}^{\bar{n}} g^+_{[1212]}(1/2, 1/2) = 0, \qquad (n+\bar{n} \text{ odd}),\\
&\partial_z^n \partial_{\bar{z}}^{\bar{n}} g^+_{[1122]}(1/2, 1/2) = (-)^{n+\bar{n}} \partial_z^n \partial_{\bar{z}}^{\bar{n}} g^+_{[2112]}(1/2, 1/2).
\end{aligned} \tag{8.47}$$

- One-variable equations

$$\begin{aligned}
\partial_z^n g^+_{[0000]}(1/2) &= 0, \qquad (n \text{ odd}),\\
\partial_z^n g^+_{[0101]}(1/2) &= 0, \qquad (n \text{ odd}),\\
\partial_z^n g^+_{[0202]}(1/2) &= 0, \qquad (n \text{ odd}),\\
\partial_z^n g^+_{[0112]}(1/2) &= (-)^n \partial_z^n g^+_{[1102]}(1/2),\\
\partial_z^n g^+_{[0011]}(1/2) &= (-)^n \partial_z^n g^+_{[1001]}(1/2),\\
\partial_z^n g^+_{[0,0,-1,1]}(1/2) &= (-)^n \partial_z^n g^+_{[-1,0,0,1]}(1/2).
\end{aligned} \tag{8.48}$$

- Integration constants

$$\begin{aligned}
g^+_{[0022]}(1/2) &= g^+_{[2002]}(1/2),\\
g^+_{[0,1,-1,2]}(1/2) &= g^+_{[-1,1,0,2]}(1/2),\\
g^+_{[0,-1,1,2]}(1/2) &= g^+_{[1,-1,0,2]}(1/2),\\
g^+_{[1,-1,-1,1]}(1/2) &= g^+_{[-1,-1,1,1]}(1/2).
\end{aligned} \tag{8.49}$$

Note that the analysis of the conservation constraints was necessary to make sure that the crossing equations we write are independent. We have explicitly verified that this indeed is the case by Taylor expanding to some finite order Λ and checking that, modulo the conservation equation, the full set of crossing equations is indeed equivalent to (8.47)-(8.49) and that there are no linear dependencies among the equations (8.47)-(8.49).

8.3 Conformal blocks

We compute the conformal blocks for $\langle TTTT\rangle$ using the approach of [61]. In this approach, the conformal blocks for external operators with large spins are obtained by acting with differential operators on simpler conformal blocks, known as seed blocks, exchanging the same intermediate representation. Since in our case we only need the conformal blocks for the exchange of traceless symmetric operators, we can take the scalar blocks as our seeds. This is exactly the case studied in [61].

Consider the contribution of a single primary state $|\mathcal{O}^\alpha\rangle$ and its descendants $P^{\{A\}}|\mathcal{O}^\alpha\rangle$ to the 4-point function,

$$\sum_{\{A\},\{B\}} \langle T(w_4,x_4)T(w_3,x_3)P^{\{B\}}|\mathcal{O}^\beta\rangle Q_{\beta\{B\},\alpha\{A\}}\langle \mathcal{O}^\alpha|K^{\{A\}}T(w_2,x_2)T(w_1,x_1)\rangle. \tag{8.50}$$

Here α and β are indices in the $SO(3)$ irrep of O, $\{A\}$ and $\{B\}$ are multi-indices such that

$$P^{\{A\}} = P^{A_1} \cdots P^{A_n}, \tag{8.51}$$

and $Q_{\alpha\{A\},\beta\{B\}}$ is the matrix inverse to $\langle O^\beta | K^{\{B\}} P^{\{A\}} | O^\alpha \rangle$. The inner products in (8.50) are derivatives of the 3-point functions

$$\langle O^\beta | T(w_2, x_2) T(w_1, x_1) \rangle = \lambda^{(a)}_{TTO} \langle O^\beta | T(w_2, x_2) T(w_1, x_1) \rangle_{(a)}, \tag{8.52}$$

$$\langle T(w_4, x_4) T(w_3, x_3) | O^\alpha \rangle = \left(\lambda^{(a)}_{TTO}\right)^* {}_{(a)}\langle T(w_4, x_4) T(w_3, x_3) | O^\alpha \rangle, \tag{8.53}$$

where λ are the OPE coefficients and the objects multiplying them are the tensor structures. We choose our tensor structures so that the OPE coefficients λ_{TTO} are real. The sum over contributions (8.50) can be then written as

$$\langle T(w_4, x_4) T(w_3, x_3) T(w_2, x_2) T(w_1, x_1) \rangle = \sum_O \lambda^{(a)}_{TTO} \lambda^{(b)}_{TTO} G_{O,ab}(w_i, x_i), \tag{8.54}$$

where we defined the conformal block

$$G_{O,ab}(w_i, x_i) \equiv \sum_{\{A\},\{B\}} {}_{(b)}\langle T(w_4, x_4) T(w_3, x_3) P^{\{B\}} | O^\beta \rangle Q_{\beta\{B\},\alpha\{A\}} \langle O^\alpha | K^{\{A\}} T(w_2, x_2) T(w_1, x_1) \rangle_{(a)}. \tag{8.55}$$

Note that if O is parity-even then both a and b should correspond to parity-even structures, and if O is parity-odd then both a and b should correspond to parity-odd structures. The corresponding conformal blocks will have different properties in what follows, and we hence refer to these cases as even-even and odd-odd respectively.

The main observation in [61] was that one can find conformally-invariant differential operators $\mathcal{D}^{(a)}_{ij}(w_i, w_j)$ acting on a pair of points such that[12]

$$\begin{aligned} \langle O^\alpha | T(w_2, x_2) T(w_1, x_1) \rangle_{(a)} &= \mathcal{D}^{(a)}_{12}(w_1, w_2) \langle O^\alpha | \phi_2(x_2) \phi_1(x_1) \rangle, \\ {}_{(b)}\langle T(w_4, x_4) T(w_3, x_3) | O^\beta \rangle &= \mathcal{D}^{(b)}_{34}(w_3, w_4) \langle \phi_4(x_4) \phi_3(x_3) | O^\beta \rangle. \end{aligned} \tag{8.56}$$

Here in the right-hand side the operators act on some standard scalar 3-point functions,[13] which we choose to be, in the formalism of [53],

$$\langle \phi_1 \phi_2 O_3 \rangle \equiv \frac{V_3^{\ell_3}}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3-\ell_3}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+\ell_3}{2}} X_{31}^{\frac{\Delta_3+\Delta_1-\Delta_2+\ell_3}{2}}}, \quad X_{ij} = -2 X_i \cdot X_j. \tag{8.57}$$

[12] The existence of the $\mathcal{D}^{(a)}_{ij}$ can be understood in terms of "weight-shifting operators" [3].

[13] Of course, this relation is purely kinematical (i.e., between tensor structures), and the operators ϕ_i do not actually exist in the physical theory.

Conformal invariance of these differential operators means that the same relations (8.56) hold even if we insert $P^{\{B\}}$ or $K^{\{A\}}$ in these 3-point functions. We thus find

$$G_{a,b}(w_i, x_i) = \mathcal{D}_{12}^{(a)}(w_1, w_2)\mathcal{D}_{34}^{(b)}(w_3, w_4)G_{\text{scalar}}(x_i), \tag{8.58}$$

where the scalar block is given by

$$G_{\text{scalar}}(w_i, x_i) = \sum_{\{A\},\{B\}} \langle \phi_4(x_4)\phi_3(x_3)P^{\{B\}}|O^\beta\rangle Q_{\beta\{B\},\alpha\{A\}}\langle O^\alpha|K^{\{A\}}\phi_2(x_2)\phi_1(x_1)\rangle. \tag{8.59}$$

This relation can also be seen directly from the OPE as discussed in [61]. The problem of calculating conformal blocks then reduces to three subproblems:

1. Construction of the conformally-invariant differential operators $\mathcal{D}_{ij}^{(a)}$ which satisfy (8.56).
2. Computation of the scalar conformal blocks G_{scalar}.
3. Performing the differentiation in the right-hand side of (8.58).

8.3.1 Differential basis

Construction of the differential operators $\mathcal{D}_{ij}^{(a)}$ has been discussed in [61]. Let us first consider the operators $\mathcal{D}_{12}^{(a)}$ and restrict ourselves to parity-even structures. They are constructed as products of the basic operators

$$D_{11},\ D_{12},\ D_{21},\ D_{22},\ H_{12}, \tag{8.60}$$

where the first order operators D_{ij} increase spin at position i by 1 while decreasing the scaling dimension at position j by 1. The operator H_{12} is just multiplication by the structure H_{12} and it increases the spin and the scaling dimension by 1 at both positions. These operators do not commute, but their algebra closes, so that one can consider the following general ansatz,

$$\mathcal{D}_{12}^{(a)} = \sum_{n_{ij},m_k} c^{(a)}_{n_{12},n_{23},n_{13},m_1,m_2} H_{12}^{n_{12}} D_{12}^{n_{13}} D_{21}^{n_{23}} D_{11}^{m_1} D_{22}^{m_2} \Sigma_1^{n_{12}+n_{23}+m_1} \Sigma_2^{n_{12}+n_{13}+m_2}, \tag{8.61}$$

where the parameters in the sum are constrained so that the resulting operator increases spin by 2 at both points. Here Σ_i is a formal operator which increases the scaling dimension at position i by 1. This is needed because various terms in the sum change the scaling dimensions by different amounts. Accordingly, (8.58) should actually contain several types of scalar blocks differing by the scaling dimensions of the external operators. We will return to this issue when we discuss the calculation of these scalar blocks.

One can check that the differential basis ansatz (8.61) contains 14 different operators. This is the same as the number of *algebraic* (not yet conserved or symmetric) tensor structures for $\langle TT\mathcal{O}_\ell\rangle$ one can build out of H_{ij} and V_i for $\ell \geq 4$. We can therefore find a change of basis between the algebraic and differential bases.

We can then easily formulate the conservation and the permutation symmetry constraints for $\langle TT\mathcal{O}_\ell\rangle$ in the algebraic basis and then translate these constraints to the differential basis. This results in a system of linear equations for the coefficients c,

$$\sum_{n_{ij},m_k} M^{\alpha}_{n_{ij},m_k}(\Delta)c^{(a)}_{n_{ij},m_k} = 0. \tag{8.62}$$

The coefficients in this equation are rational functions of the dimension Δ of the exchanged primary $\mathcal{O}$, and thus the solutions are rational functions of Δ as well. Consistently with the discussion in section 8.2.1, we find that there exist 2 solutions for even $\ell \geq 4$. To simplify the numerical evaluation of (8.58), we choose a basis of the solutions $c^{(a)}_{n_{ij},m_k}$ which is polynomial in Δ of the lowest possible degree. These degrees are 6 and 4 for the two solutions.

In the above discussion we have glossed over a slight subtlety that in the algebraic basis in 3d, there is one tensor structure (8.4) which is redundant and can be expressed in terms of other structures, so the number of independent structures is actually 13. There is also a corresponding relation in the differential basis. If we were to ignore this relation, we would find more solutions to the conservation constraints. Taking it into account, we can use it to simplify the form of the solutions $c^{(a)}_{n_{ij},m_k}$.

A similar procedure works for $\ell \leq 4$, the only difference being that there appear new relations in the differential basis (while the algebraic basis simply becomes smaller). These relations are easily controlled by the transformation matrix which expresses the differential basis structures in terms of the algebraic ones. We then use these relations to find the simplest form of the non-redundant solutions of (8.62).

The parity-odd structures can be treated in a similar way, except that we generally find more redundancies than in the parity-even case. We describe the construction of parity-odd differential basis in appendix G.1, together with the explicit expressions for the coefficients $c^{(a)}_{n_{ij},m_k}$. In both the parity-even and the parity-odd cases the operators $\mathcal{D}_{34}^{(a)}$ can be obtained by applying a simple permutation to the operators $\mathcal{D}_{12}^{(a)}$.

8.3.2 Computing the scalar blocks

Since (8.61) involves the formal dimension-shifting operators $\Sigma_{1,2}$, there are several scalar conformal blocks entering (8.58), which differ by the dimensions Δ_i of the external scalars.

Let us analyze the dimensions of the scalar at positions 1 and 2. The exponents in (8.61) are constrained by the spins of the stress tensors

$$n_{12} + n_{13} + m_1 = n_{12} + n_{23} + m_2 = 2. \tag{8.63}$$

On the other hand, the dimensions of the scalar operators in each term are given by

$$\Delta_1 = \Delta_T + n_{12} + n_{23} + m_1, \tag{8.64}$$

$$\Delta_2 = \Delta_T + n_{12} + n_{13} + m_2. \tag{8.65}$$

It follows that the sum

$$\Delta_1 + \Delta_2 = 2\Delta_T + 4 = 10 \tag{8.66}$$

is the same for all the terms. On the other hand, the difference is

$$\Delta_{12} = \Delta_1 - \Delta_2 = n_{23} - n_{13} + m_1 - m_2 = 2(m_1 - m_2), \tag{8.67}$$

and one can see that it takes all even values $-4 \leq \Delta_{12} \leq 4$. The same is true for Δ_{34}.

The analysis for parity-odd operators is similar, with the result that $\Delta_1 + \Delta_2 = 9$, while Δ_{12} assumes all odd values $-3 \leq \Delta_{12} \leq 3$. The same is true for Δ_{34}.

Note that the scalar blocks essentially depend only on the differences Δ_{12} and Δ_{34}. Furthermore, there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of permutations of the external operators which preserves the OPE s-channel and the cross-ratios,[14] and thus acts in a simple way on the conformal blocks. The elements of this group change the scaling dimensions of the scalar blocks according to

$$(12)(34): \quad \Delta_{12} \to -\Delta_{12},\ \Delta_{34} \to -\Delta_{34}, \tag{8.68}$$

$$(13)(24): \quad \Delta_{12} \leftrightarrow \Delta_{34}, \tag{8.69}$$

$$(14)(23): \quad \Delta_{12} \leftrightarrow -\Delta_{34}. \tag{8.70}$$

[14] Of course, we can also use the permutations which change the cross-ratios, but in practice it is easier to have all scalar blocks with the same arguments.

We thus only need to compute the scalar blocks with Δ_{12} and Δ_{34} in a fundamental domain for these transformations, and then all the other blocks can be easily inferred. It is easy to check that a fundamental domain is given by

$$\Delta_{12} \geq |\Delta_{34}|. \tag{8.71}$$

The resulting fundamental set of the parameters Δ_{12}, Δ_{34} for the scalar blocks is

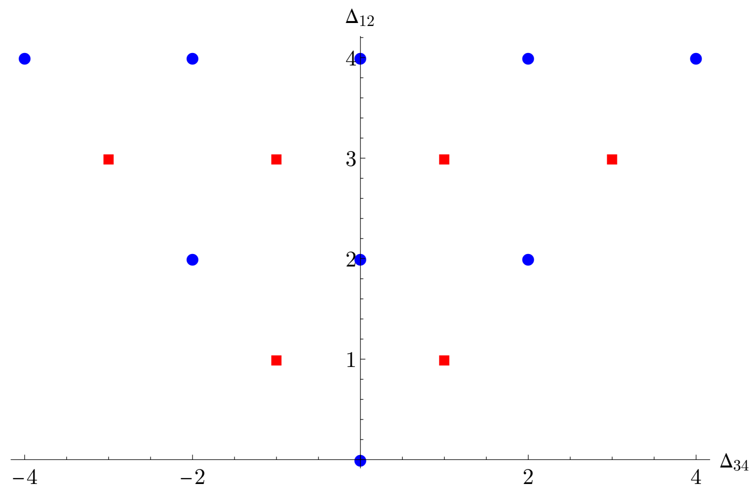

Figure 8.1: Parameters of scalar conformal blocks for the even-even (blue dots) and odd-odd (red squares) cases.

shown in figure 8.1. There are 9 scalar blocks required for the computation of even-even $\langle TTTT\rangle$ blocks, and 6 scalar blocks required for the computation of odd-odd $\langle TTTT\rangle$ blocks.[15] In practice we compute them efficiently using the pole expansion of [36, 49] evaluated on the diagonal $z = \bar{z}$ combined with the recursion relation implied by the Casimir equation to evaluate scalar block derivatives away from the diagonal.

8.3.3 Applying the differential operators

To finish the calculation of the stress-tensor conformal blocks, it is necessary to apply the differential operators $\mathcal{D}_{ij}^{(a)}$ to the scalar blocks. The embedding-space definition of these operators, given in [61], seems inadequate for this purpose because the embedding-space 4-point tensor structures in 3d contain many degeneracies. Therefore, it is convenient to reformulate these operators directly in the conformal frame basis constructed in section 8.2.2.1.

[15] Note that by using the dimension-shifting differential operators [3, 120] we can reduce this set to just one scalar conformal block for each parity.

The first step is to convert the embedding-space expression for the differential operators to explicit expressions in 3 dimensions. For this purpose, we consider an explicit uplift of 3 dimensional primary operators to embedding space operators,

$$\mathcal{O}(Z,X) = \frac{1}{(X^+)^\Delta}\mathcal{O}\left(Z^\mu - Z^+\frac{X^\mu}{X^+}, \frac{X^\mu}{X^+}\right), \tag{8.72}$$

where on the right-hand side we have the 3d operator $\mathcal{O}(w,x)$. Applying embedding-space differential operators to this expression, we reproduce on the right-hand side the corresponding differential operators in 3 dimensions. Choosing a different uplift will yield the same result due to the consistency conditions imposed on the embedding space differential operators.

With the 3-dimensional expressions at hand, we can understand the action of the differential operators in the conformal frame. In the conformal frame, some of the operators are placed at fixed positions. In order to apply derivatives in these constrained directions, we simply solve the equations

$$\sum_{k=1}^{4} L_{k\,AB}\langle TTTT\rangle = 0 \tag{8.73}$$

for these derivatives. Here L_k are the conformal generators acting on point k. For example, consider the equation corresponding to $L_{AB} = D$ the dilatation operator,

$$\sum_{k=1}^{4}(x_k\cdot\frac{\partial}{\partial x_k} + \Delta_T)\langle TTTT\rangle = 0. \tag{8.74}$$

Here $\Delta_T = 3$ is the scaling dimension of T. We give expressions for the other generators in appendix G.2. Evaluating this equation in the conformal frame[16] (8.13) we find

$$(z\partial_z + \bar{z}\partial_{\bar{z}} + \frac{\partial}{\partial x_3^1} + 6)g(z,\bar{z},w_i) = 0. \tag{8.75}$$

Here $\frac{\partial}{\partial x_3^1}g(z,\bar{z},w_i)$ should be understood as $\frac{\partial}{\partial x_3^1}\langle TTTT\rangle$ evaluated in conformal frame. This allows us to conclude

$$\frac{\partial}{\partial x_3^1}g(z,\bar{z},w_i) = -(z\partial_z + \bar{z}\partial_{\bar{z}} + 6)g(z,\bar{z},w_i). \tag{8.76}$$

By using (8.73) with L_{AB} equal to translations, special conformal transformations, and rotations we find $3+3+3=9$ more equations which allow us to solve for the

[16]And taking into account that we should replace $x_4\cdot\frac{\partial}{\partial x_4}$ by $-2\Delta_T$ since we put operator 4 at infinity. This has to do with the fact that the correlator decays as $x_4^{-2\Delta_T}$.

remaining 9 derivatives – all derivatives in x_1 and x_4, 2 unknown derivatives in x_3 and 1 unknown derivative in x_2.[17] Note that the equations for special conformal and rotation generators will involve derivatives in w_i in addition to z and $\bar{z}$ (see appendix G.2). In practice we solve these equations in `Mathematica`. We do not write out the solution explicitly since it is rather complicated. Note that if we need higher-order derivatives, we can differentiate (8.73) and proceed analogously.

As a result, taking into account also (8.18), we can write for any 3d differential operator D

$$D\left([h_1 h_2 h_3 h_4] g_{[h_1 h_2 h_3 h_4]}(z,\bar{z})\right) = \sum_{h'_i} [h'_1 h'_2 h'_3 h'_4] D^{[h'_1 h'_2 h'_3 h'_4]}_{[h_1 h_2 h_3 h_4]} g_{[h_1 h_2 h_3 h_4]}(z,\bar{z}), \tag{8.77}$$

where $D^{[h'_i]}_{[h_i]}$ are differential operators in z and $\bar{z}$. In this equation, we can keep the spins ℓ_i and the parameters h_i as variables, in which case h'_i differ from h_i by finite shifts. Using in place of D the basic differential operators (8.60) and their parity-odd analogs, we obtain their counterparts in the conformal frame.

This allows us to efficiently compute the more complicated compositions (8.61) directly in conformal frame without encountering any redundancies in tensor structures in intermediate steps. In the end, we find expressions for the $\langle TTTT \rangle$ blocks of the form

$$\left(G_{\Delta,\ell,ab}\right)_{[h_1 h_2 h_3 h_4]}(z,\bar{z}) = \sum_{i=1}^{N_{\text{scalar}}} \sum_{m,n} a^{i,mn,ab}_{[h_1 h_2 h_3 h_4]}(\Delta,\ell,z,\bar{z}) \partial_z^m \partial_{\bar{z}}^n G^{\Delta^{(i)}_{12},\Delta^{(i)}_{34}}_{\Delta,\ell}(z,\bar{z}), \tag{8.78}$$

where a are some rational functions of $z, \bar{z}, \ell$, and polynomial in Δ,[18] while $\Delta^{(i)}_{12}$ and $\Delta^{(i)}_{34}$ are the parameters of the scalar conformal blocks from the fundamental region (8.71). The derivative order is $m+n \leq 8$ for even-even blocks and $m+n \leq 10$ for odd-odd blocks; N_{scalar} is 9 and 6 respectively.

The functions a contain powers of $(z - \bar{z})$ in their denominators, but these get canceled when one takes into account that the scalar blocks are symmetric under $z \leftrightarrow \bar{z}$. For example, if we rewrite the above expression in coordinates $z + \bar{z}$ and $(z-\bar{z})^2$, then the functions a manifestly have only the OPE singularities. This is to be expected, since the functions entering the decomposition (8.18) must have the same

[17] We have just found 1 derivative in x_3 from $L_{AB} = D$ and the two derivatives in x_2 are simply the z and $\bar{z}$ derivatives.

[18] Because of our polynomial choice of the solutions $c^{(a)}_{n_{ij},m_k}$ to (8.62).

singularities as the physical correlator. Therefore, we can take further derivatives directly in this expression, and then evaluate it at $z = \bar{z} = 1/2$ to find the derivatives of $\langle TTTT \rangle$ blocks in terms of linear combinations of the derivatives of scalar blocks with coefficients polynomial in Δ. Substituting rational approximations for the derivatives of the scalar blocks then immediately yields rational approximations for $\langle TTTT \rangle$ blocks suitable for use in SDPB [35].

8.4 Numerical bounds

In this section we discuss how to use the crossing equations and conformal blocks derived in the previous sections to compute numerical bounds on the OPE coefficients and scaling dimensions appearing in the $T \times T$ OPE. Further details of our numerical implementation are given in appendix G.3.

8.4.1 Initial comments: C_T and θ

To begin, let us return to the conformal block decomposition of the stress-tensor 4-point function in a general 3d CFT,

$$\langle TTTT \rangle = \lambda_{TT\mathbf{1}}^2 G_{\mathbf{1}} + \frac{1}{C_T} \lambda_{TTT}^{(a)} \lambda_{TTT}^{(b)} G_{T,ab} + \sum_{\mathcal{O}} \lambda_{TT\mathcal{O}}^{(a)} \lambda_{TT\mathcal{O}}^{(b)} G_{\mathcal{O},ab}, \tag{8.79}$$

where we have explicitly separated the contribution of the identity operator and the stress tensor itself. We have also assumed that the CFT in question possesses a unique stress tensor. The factor $\frac{1}{C_T}$ comes from the fact that C_T enters the 2-point function of the canonically-normalized stress tensor T.

The OPE coefficient $\lambda_{TT\mathbf{1}}$ of the identity operator is just the coefficient in the 2-point function $\langle TT \rangle$, and thus is essentially the central charge C_T. At the same time, the OPE coefficients for the stress tensor itself are given by $\lambda_{TTT}^{(1)} = n_B$ and $\lambda_{TTT}^{(2)} = n_F$. Due to the Ward identity constraint (8.8), these three coefficients are not independent. It is therefore convenient to introduce the following parametrization,[19]

$$n_B = C_T \frac{\cos\theta}{\sin\theta + \cos\theta}, \tag{8.80}$$
$$n_F = C_T \frac{\sin\theta}{\sin\theta + \cos\theta}. \tag{8.81}$$

Note that $\theta = \tan^{-1}(n_F/n_B)$ is π-periodic, so we can assume that $\theta \in (-\pi/4, 3\pi/4)$, where the denominators are positive. We also renormalize the 4-point function

[19] Another, perhaps more natural, parametrization would be $n_B = C_T \cos^2\theta'$, $n_F = C_T \sin^2\theta'$. However this parametrization doesn't allow us to numerically test negative values of n_B and n_F so we adopt the one in the text in order to probe the conformal collider bounds.

$\langle TTTT\rangle$ so that C_T appears only in one of the terms,

$$C_T^{-2}\langle TTTT\rangle = G_{\mathbf{1}} + \frac{1}{C_T}\Theta^{ab}G_{T,ab} + \sum_{O}\hat{\lambda}^{(a)}_{TTO}\hat{\lambda}^{(b)}_{TTO}G_{O,ab}$$
$$= G_{\mathbf{1}} + \frac{1}{C_T}\Theta^{ab}G_{T,ab} + \sum_{\Delta,\rho}M^{ab}_{\Delta,\rho}G_{\Delta,\rho,ab}, \tag{8.82}$$

where $\hat{\lambda}^{(a)}_{TTO} = C_T^{-1}\lambda^{(a)}_{TTO}$ and the positive-semidefinite matrix Θ^{ab} is given by

$$\Theta = \frac{1}{(\sin\theta+\cos\theta)^2}\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}. \tag{8.83}$$

We have also defined the positive-semidefinite OPE matrix $M^{ab}_{\Delta,\rho}$ to be the sum of $\hat{\lambda}^{(a)}_{TTO}\hat{\lambda}^{(b)}_{TTO}$ over the operators O with scaling dimension Δ and in the $O(3)$ representation ρ. Of course, the operators appearing in the $T\times T$ OPE are singlets of global symmetries and we generically do not expect there to be any degeneracies. Therefore, we expect that all matrices $M_{\Delta,\rho}$ have rank 1. However, without additional assumptions the operators are allowed to have arbitrarily close scaling dimensions, which is numerically indistinguishable from a degeneracy in the spectrum. In other words, even if we had a way of constraining all $M_{\Delta,\rho}$ to have rank 1, numerically this would make no difference unless we also input assumptions about gaps between operators. The stress-tensor four-point function written in the form (8.82) is suitable for numerical analysis using the standard methods which we review in appendix G.3. Here, let us make some initial comments about our assumptions and on the kind of bounds we can expect to find.

Note that $C_T^{-1}\Theta$ is essentially a special case of the OPE matrices $M_{\Delta,\rho}$. We only consider the theories with a unique spin-$\mathbf{2}^+$ conserved operator, and this is reflected in the fact that we explicitly assume Θ to have rank 1 by writing (8.83). Unlike in the case of generic $M_{\Delta,\rho}$, this constraint matters. Indeed, parity-even spin-2 operators *strictly above* the unitarity bound only have a single OPE coefficient and thus are clearly distinguishable from T even if their scaling dimension is arbitrarily close to 3. It is therefore more appropriate to think about T as an isolated operator.[20]

It is important to note that although this assumption on the form of Θ is non-trivial, it does not necessarily imply that this CFT has a unique conserved spin-$\mathbf{2}^+$ operator. Indeed, consider a decoupled system of any number $N\geq 2$ of CFTs, all of which

[20]Although not completely appropriate — there is still a direction in the 3-dimensional space of symmetric matrices Θ which can be "altered" by spin-$\mathbf{2}^+$ operators with $\Delta = 3+\epsilon$. This direction, however, coincides with (8.83) only if $\theta\to -\pi/4+\pi k$.

satisfy (8.83) with the same value of θ. If the stress tensors in these theories are T_i, then the stress tensor of the full system is

$$T = \sum_{i=1}^{N} T_i. \tag{8.84}$$

We also have $C_T = \sum_i C_{T_i}$. It is easy to check that $\langle TTTT \rangle$ in this system satisfies (8.82) and (8.83), even though each T_i is a distinct conserved spin-$\mathbf{2}^+$ operator.

This also shows that for any value of θ which is allowed by the crossing symmetry of (8.82) the central charge C_T is unbounded from above – we can simply take N copies of the same CFT for arbitrarily large N. In the limit $N \to \infty$, the corresponding four-point function approaches that of the mean field theory (MFT). The stress-tensor 4-point function in MFT is dual to the 4-point scattering of free spin-2 massless particles in AdS_4 and is given by Wick's theorem,

$$\langle TTTT \rangle = \langle TT \rangle \langle TT \rangle + \langle TT \rangle \langle TT \rangle + \langle TT \rangle \langle TT \rangle. \tag{8.85}$$

In this theory C_T is formally infinite. In other words, it gives a unitary solution to crossing symmetry for which the second term in (8.82) vanishes. In particular, its existence shows that any value of θ is formally allowed unless one excludes $C_T = \infty$.

From the above discussion it follows that we cannot put upper bounds on C_T or constrain θ without extra assumptions which go beyond unitarity, parity invariance, crossing symmetry and existence of a unique stress tensor. Importantly, this is not a technical obstruction of the associated semidefinite problem. As we noted, T is effectively an isolated operator and thus there is no a-priori problem with such bounds. The problem is more physical in nature and ultimately due to existence of the MFT. We will repeatedly see that as soon as MFT is excluded by additional assumptions, these bounds become possible.

8.4.2 General theories

Given that MFT has infinite central charge, we can hope to exclude some values of θ by assuming that C_T is finite. One way this can be possible is if there exists a θ-dependent lower bound on C_T which diverges for some values of θ. Of course, numerically we might not reproduce the divergence but instead see a finite bound which grows as we improve our numerical approximation (i.e. increase the derivative order Λ).

This is indeed what happens. In figure 8.2 we show a series of lower bounds on C_T as a function of θ for derivative orders $\Lambda = 3, \ldots, 19$, with no assumptions beyond unitarity, crossing symmetry, parity conservation, and the existence of a unique stress tensor. The behavior of the bound differs dramatically depending on whether $\theta \in [0, \pi/2]$ or not. For $\theta \in [0, \pi/2]$, the bound appears to converge to a finite value. Strikingly, for $\theta < 0$ or $\theta > \pi/2$ the bound diverges with growing Λ.

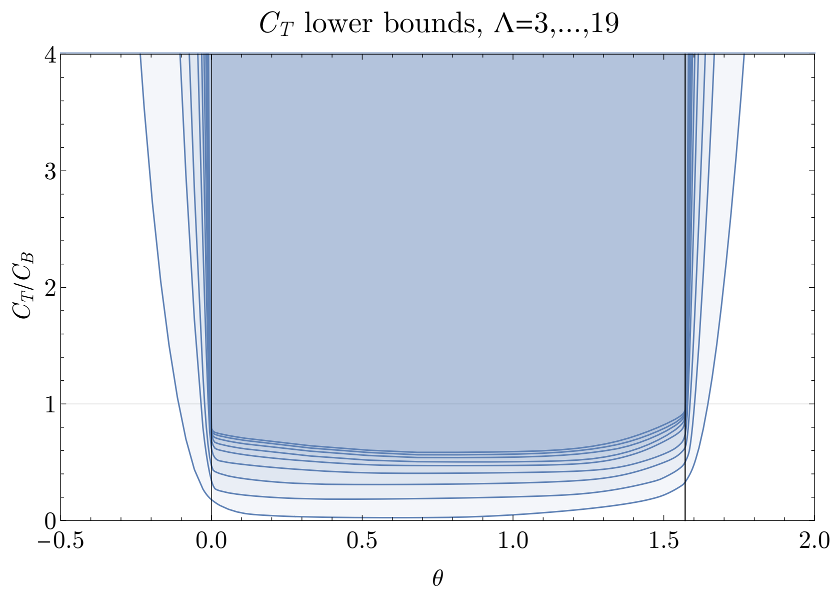

Figure 8.2: A series of lower bounds on C_T as a function of θ, valid in any unitary parity-preserving 3d CFT. The shaded region is allowed.

These numerical results strongly suggest that for unitary parity-preserving theories with finite C_T, θ necessarily lies in the interval $[0, \pi/2]$. Note that $\theta \in [0, \pi/2]$ corresponds to $n_B, n_F \geq 0$, which is equivalent to the conformal collider bounds [76, 287]. We have thus essentially recovered the stress-tensor conformal collider bounds using the numerical bootstrap.[21] Note that the recent analytical proof [77] of the conformal collider bounds uses the lightcone limit of the crossing equation. The analysis of [31] suggests that numerical bootstrap techniques at high derivative order can probe the lightcone limit of the crossing equation (despite the fact that the numerical bootstrap usually involves expanding the crossing equation around a Euclidean point). Thus, it is perhaps unsurprising that we make contact with analytical results at large Λ.

When the conformal collider bounds are saturated ($n_F = 0$ or $n_B = 0$), the theory is expected to be free [288]. Our lower bounds at $\theta = 0, \pi/2$ are consistent with the

[21] Similar conformal collider bounds for OPE coefficients of conserved currents were recovered numerically in [41].

existence of the free boson theory ($\theta = 0$) and the free fermion theory ($\theta = \pi/2$), though they are not yet saturated by those theories. However, the bounds continue to change as we increase the derivative order Λ. It is possible that at sufficiently large Λ, our lower bound will become C_B at each endpoint. We do not currently have enough data to perform a reliable extrapolation to $\Lambda = \infty$ (as in, e.g. [140]).

8.4.3 Scalar gaps

8.4.3.1 Parity-even scalar gaps

Let us now explore how the bounds on C_T and θ change when we impose further restrictions on the CFT data. It is natural to ask: what is the allowed space of (θ, C_T) in theories with no relevant parity-even scalars in the $T \times T$ OPE — i.e. CFTs in which no tuning would be required if all global symmetries (including parity) were preserved microscopically. Denoting the dimension of the lowest-dimension parity-even scalar by Δ_{even}, we show a bound on theories with $\Delta_{\text{even}} \geq 3$ in figure 8.3. The free fermion at $\theta = \pi$ is allowed (the lowest-dimension parity-even singlet in the free-fermion theory is $\psi^2 \partial_\mu \psi^\alpha \partial^\mu \psi_\alpha$, which has $\Delta = 6$), whereas the free boson is of course excluded. The lower bound on C_T falls quickly as θ varies between 0 and π, dipping below C_B only for a small range $\theta \in [1.3, \pi]$.

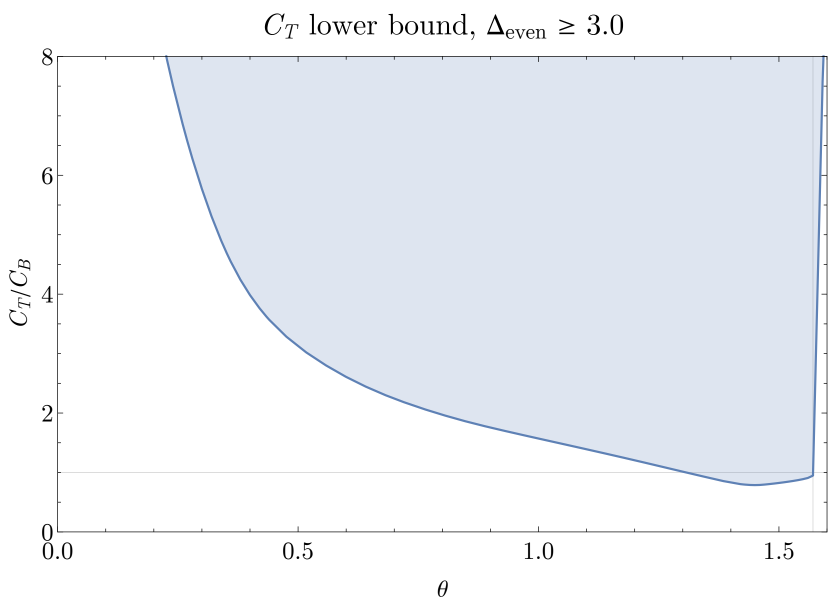

Figure 8.3: A lower bound on C_T as a function of θ in 3d CFTs with no relevant parity-even scalars.

As we increase the imposed gap in the parity-even scalar sector, $\Delta_{\text{even}} \geq \Delta_{\text{even}}^{\text{min}}$, the lower bounds on C_T get stronger, while still remaining consistent with the existence of the free fermion up to $\Delta_{\text{even}}^{\text{min}} = 6$. We illustrate these bounds in figure 8.4. Note that it is not possible to place upper bounds on C_T when $\Delta_{\text{even}}^{\text{min}} < 6$, because of the existence of MFT, which has $\Delta_{\text{even}} = 6$ (associated with $\mathcal{O}_{\text{even}} = T_{\mu\nu}T^{\mu\nu}$) and infinite C_T. However, when $\Delta_{\text{even}}^{\text{min}} > 6$, upper bounds become possible, and indeed C_T and θ become confined to a small island in the vicinity of the free fermion point. For example, when $\Delta_{\text{even}}^{\text{min}} = 6.8$, we find $\theta \in [1.54, 1.57]$ and $C_T/C_B \in [1.2, 2.6]$. It is interesting to ask whether any CFT realizes these values. For even larger values of $\Delta_{\text{even}}^{\text{min}}$, the allowed region disappears.

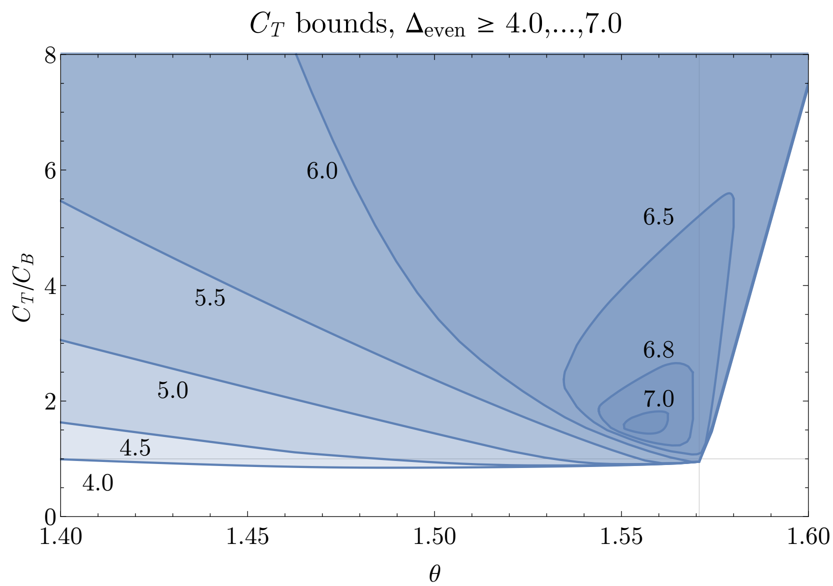

Figure 8.4: Bounds on (θ, C_T) with varying gaps in the parity-even scalar sector. When $\Delta_{\text{even}}^{\text{min}} = 4.0, \ldots, 6.0$, we have a series of lower bounds on C_T as a function of θ. When $\Delta_{\text{even}}^{\text{min}} > 6.0$, we have closed islands which eventually shrink to zero size.

8.4.3.2 Parity-odd scalar gaps

Next we study the effect of a gap in the parity-odd scalar operators. In figure 8.5, we show a series of bounds on C_T as a function of θ, for various gaps in the parity-odd scalar sector, $\Delta_{\text{odd}} \geq \Delta_{\text{odd}}^{\text{min}}$. The bounds are roughly a mirror image of those in the previous subsection. For $\Delta_{\text{odd}}^{\text{min}} = 2, \ldots, 7$, we find a series of increasingly strong bounds pushing the allowed region towards smaller θ. When $\Delta_{\text{odd}}^{\text{min}} > 7$, our assumption excludes MFT (which has $\mathcal{O}_{\text{odd}} = \epsilon_{\mu\nu\rho}T^{\mu\sigma}\partial^{\nu}T^{\rho}{}_{\sigma}$, of dimension 7), and it becomes possible to find both upper and lower bounds on C_T. Indeed, we find a series of islands (figure 8.6), which finally exclude the free-boson theory when

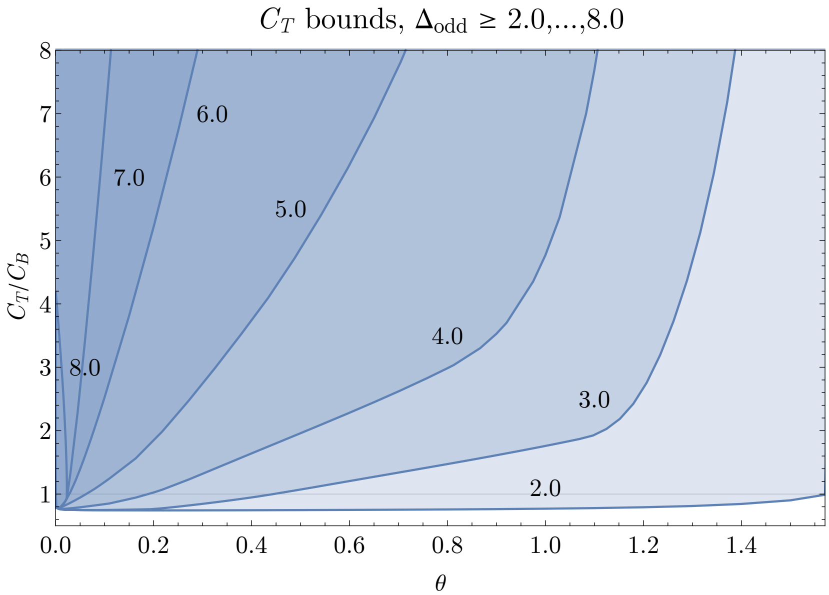

Figure 8.5: Bounds on (θ, C_T) with varying gaps in the parity-odd scalar sector. When the value of the gap $\Delta_{\rm odd}^{\rm min} > 7$, it becomes possible to find both upper and lower bounds on C_T as.

$\Delta_{\rm odd} \gtrsim 11$.[22] A common corner point of these islands is very close to the C_T value of the 3d Ising CFT. We return to this point in section 8.4.6, where we will see that further imposing known gaps in the 3d Ising CFT slightly reduces this apparent upper bound on $\theta_{\rm Ising}$.

Finally, note that these bounds imply that any CFT with a large parity-odd gap must have a stress-tensor 3-point function close to the bosonic one, with $\theta < .023$.

8.4.3.3 Scalar gaps in both sectors

In figure 8.7, we show a bound constraining the space of “dead-end" CFTs, i.e. theories with no parity-preserving or parity-breaking relevant deformations. Strictly speaking, our bound only assumes the absence of relevant scalar deformations that are singlet under other global symmetries (so they are allowed to appear in the $T \times T$ OPE). We see from this plot that such theories must have $C_T \gtrsim 2$. In addition, for a given C_T, θ is constrained to lie towards the middle of the range $[0, \pi/2]$.

For each of the parity-even and parity-odd sectors, we have seen that there exists a maximal gap beyond which no CFT can exist (figures 8.4 and 8.6). In figure 8.8,

[22] The lightest parity-odd $\mathbb{Z}_2$-even scalar in the theory of a single free boson is the dimension-11 scalar $\epsilon^{\mu\nu\rho}\phi(\partial_\alpha\partial_{\beta_1}\partial_{\beta_2}\partial_\mu\phi)(\partial^\alpha\partial_\nu\phi)(\partial^{\beta_1}\partial^{\beta_2}\partial_\rho\phi)$ + desc.

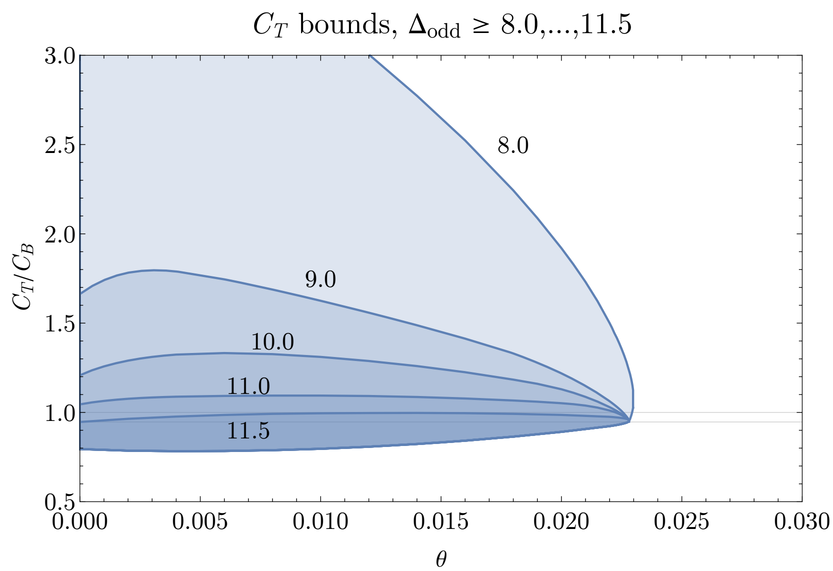

Figure 8.6: Closed regions for (θ, C_T), given various large gaps in the parity-odd scalar sector. The lower horizontal line shows the value of C_T in the 3d Ising CFT.

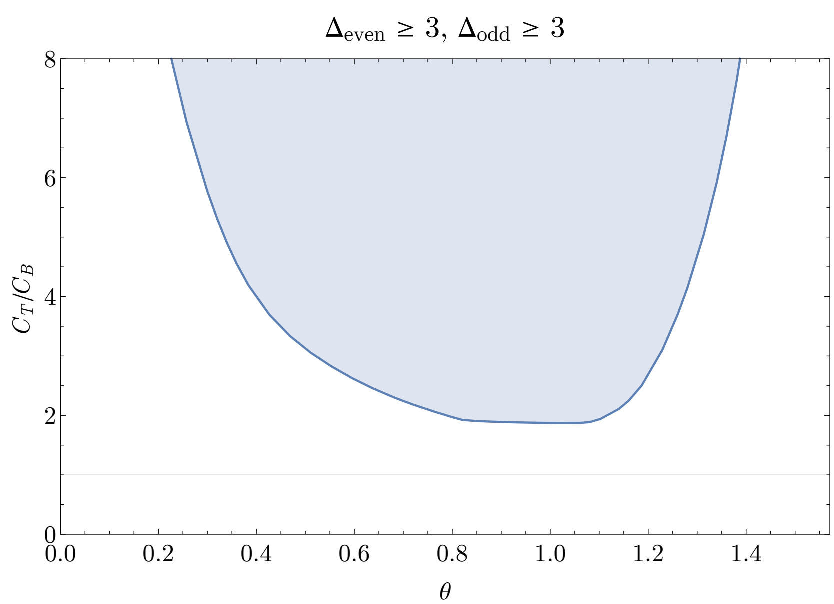

Figure 8.7: Lower bound on C_T as a function of θ assuming no relevant scalar operators.

we show the full space of allowed gaps in the both sectors. Along the axes, this plot reproduces the gaps at which the islands disappear in figures 8.4 and 8.6. The full bound shows several interesting features that approximately coincide with known theories. Notable points include MFT at $(\Delta_{\text{even}}, \Delta_{\text{odd}}) = (6, 7)$, the free Majorana fermion at $(6,2)$, the free real scalar at $(1,11)$,[23] and the $N = \infty$ limit of the $O(N)$ models at $(2,7)$. We also see the maximal possible gaps $\Delta_{\text{even}} \leq 7.0$ and $\Delta_{\text{odd}} \leq 11.78$.

The known scaling dimension $\Delta_\epsilon = 1.412625(10)$ [8] of the energy operator ϵ in the 3d Ising CFT is shown in figure 8.8 by a vertical line. We see that while most features seem to be related to free theories, there appears to be a sharp transition in the upper part of the allowed region, very close to the Ising line. We return to this point in section 8.4.6.

There is also a feature near $(\Delta_{\text{even}}, \Delta_{\text{odd}}) = (7,1)$, which does not seem to correspond to a known theory. Such a theory, if exists, is constrained by the bound in figure 8.4 to have $C_T/C_B \sim 2$ and a value of θ very close to but lower than the free fermion value, $1.55 < \theta < 1.563$. Since this putative theory requires a very light parity-odd operator $\mathcal{O}_{\text{odd}}$, such a large parity-even gap should be excluded by the bootstrap constraints for 4-point functions of $\mathcal{O}_{\text{odd}}$ unless the $\mathcal{O}_{\text{odd}} \times \mathcal{O}_{\text{odd}}$ OPE contains an additional parity-even scalar not present in the $T \times T$ OPE. We leave it as an open question whether this can occur and if this region has any physical significance.

Note that every point which is allowed in this plot must be allowed together with a rectangular region to its lower left. Because of this, a large part of the allowed region is due to existence of MFT. It is therefore interesting to study analogous bounds under assumptions which would exclude the MFT. We leave this question for future work.

8.4.4 Spin-2 gaps

Next we turn to imposing gaps in the spin-2 spectrum. First we ask how the gap until the second parity-even spin-2 operator T' of dimension Δ_2 affects the lower bounds on C_T. This is shown for gaps $\Delta_2 \geq 3, \ldots, 6$ in figure 8.9. We can see that such gaps have a minimal effect on the lower bound. The gap $\Delta_2 = 6$ is special because this dimension occurs for the operator $T'_{\mu\nu} = T_{\mu\sigma}T^{\sigma}_{\nu}$ in a number of different CFTs, including free theories, $O(N)$ models at large N, and MFT. Thus it is not surprising that the full range of θ is still allowed at this gap and that the bound is not very strong.

However, we expect that if the $\Delta_2^{\min}$ is raised above 6, then we may be able to start excluding MFT and large N theories by obtaining an upper bound on C_T.

[23] Note that the fundamental field in a free scalar theory is charged under a $\mathbb{Z}_2$ symmetry and thus does not appear in the $T \times T$ OPE.

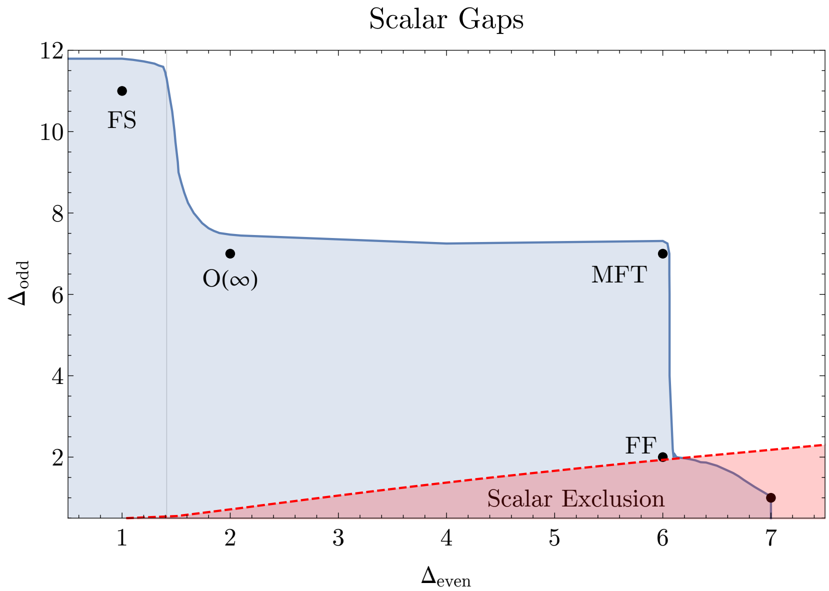

Figure 8.8: Bound on the allowed gaps in parity-even and parity-odd scalar sectors (imposed simultaneously). The blue shaded region is allowed by the $\langle TTTT\rangle$ bootstrap. The vertical grey line indicates the scaling dimension of ϵ in the Ising model. The red region is excluded from the scalar bootstrap for 4-point functions $\langle \mathcal{O}_{\text{odd}}\mathcal{O}_{\text{odd}}\mathcal{O}_{\text{odd}}\mathcal{O}_{\text{odd}}\rangle$ assuming $\mathcal{O}_{\text{even}}$ appears in both the $\mathcal{O}_{\text{odd}} \times \mathcal{O}_{\text{odd}}$ and $T \times T$ OPEs.

This is because the “double-trace" operator $T_{\mu\sigma}T^{\sigma}_{\nu}$ in large C_T theories will have a dimension $\Delta_2 = 6 + \mathcal{O}(1/C_T)$, so imposing a gap above 6 will exclude some set of these theories. This is realized in figures 8.10 and 8.11, where for gaps slightly above 6 the upper bound is fairly weak, but as it is raised further it becomes very strong and for gaps near 8.5 the closed region shrinks to a small island around $C_T/C_B \sim 1$ and $.4 \lesssim \theta \lesssim .9$. It is interesting to ask if there is a unitary CFT with such a large spin-2 gap and $\theta \approx \pi/4$ which lives inside of this allowed region.

8.4.5 Spin-4 gaps

In this section we move on to considering the constraints resulting from imposing a bound on the dimension of lightest spin-four operator Δ_4. Consistency of crossing with the OPE in Minkowski space when two operators are light-like separated imposes a number of non-trivial constraints on the spectrum of “intermediate” operators. In particular the “Nachtmann theorem” stipulates that the leading twist, defined as the twist of the lightest primary of spin ℓ appearing in the OPE $\mathcal{O} \times \mathcal{O}$,

$$\tau_\ell = \Delta_\ell - \ell\,, \tag{8.86}$$

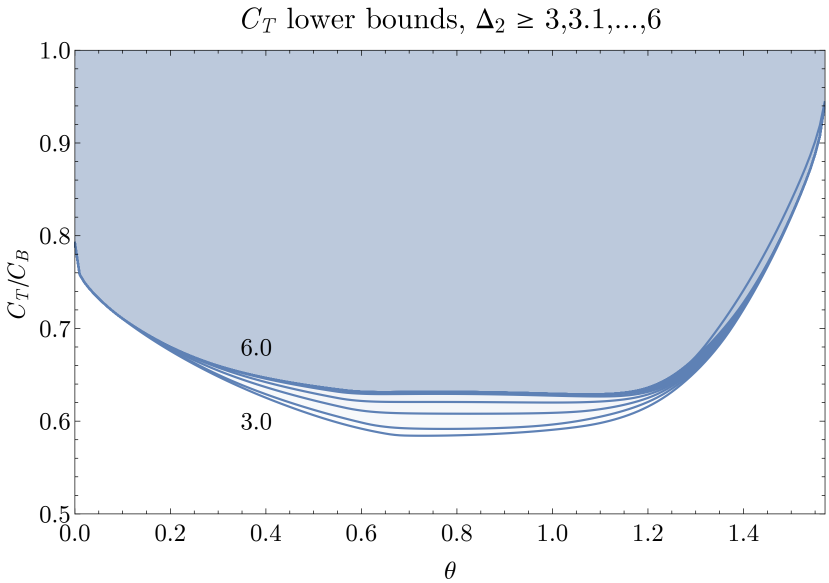

Figure 8.9: Lower bounds on C_T as a function of θ in 3d CFTs for different gaps between the stress tensor and the second parity-even spin-2 operator.

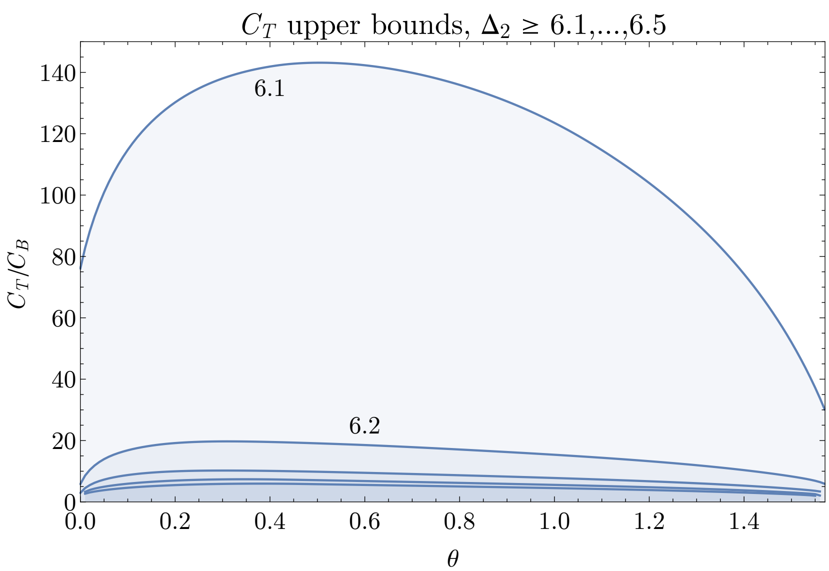

Figure 8.10: Upper bounds on C_T as a function of θ in 3d CFTs for different gaps between the stress tensor and the second parity-even spin-2 operator.

is a monotonically non-decreasing convex function of ℓ which asymptotes to $2\tau_O$ [68, 69, 162, 249, 250]. So far this has been rigorously established for scalar O and even ℓ, although the result is expected to hold more generally, for primary O of any spin. Applying this to the stress tensor one finds that the dimension of the lightest

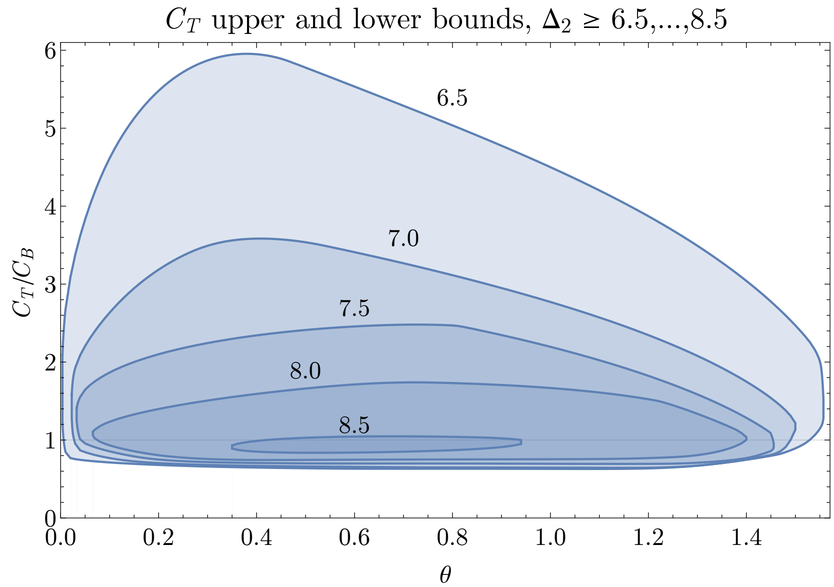

Figure 8.11: Upper and lower bounds on C_T as a function of θ in 3d CFTs for different gaps between the stress tensor and the second parity-even spin-2 operator.

operator of spin ℓ should not exceed $\ell + 2$. For the leading spin-4 operator this implies inconsistency of unitary theories with $\Delta_4 > 6$. Moreover, when $\Delta_4 = 6$, the lightest operators of spin $\ell > 4$ must have dimensions exactly equal to $\ell + 2$. The corresponding theory is a MFT dual to pure gravity in AdS_4 with Newton's constant taken to zero. The operators in question are double-trace operators, schematically $T\partial^{\ell-4}T$, where we omit indices for simplicity.

When Δ_4 approaches 6 from below, by convexity all higher spin operators must approach $\ell + 2$. This is exactly the behavior expected for a theory dual to weakly coupled gravity in AdS_4. The double-trace anomalous dimensions $\Delta_\ell - \ell - 2$ are due to graviton exchange in the bulk, which is proportional to Newton's constant $G_N \sim 1/C_T$. This picture suggests that imposing a gap $\Delta_4 > 6 - \epsilon$ should result in a numerical bound on the central charge $C_T \geq C_T^*$, with C_T^* going to infinity as $C_T^* \sim 1/\epsilon$.

Such behavior was observed previously in the context of the $\mathcal{N} = 8$ numerical supersymmetric bootstrap in $3d$ [133]. There the lower bound on C_T was studied as a function of the dimensions of spin-0 and spin-2 long multiplets, Δ_0^* and Δ_2^* respectively. When the dimensions approached the values associated with $N \to \infty$ ABJM theory, the exclusion region for C_T grew accordingly, with the lower bound on C_T scaling as $1/(2 - \Delta_2^*)$. Another related result is in the context of the numerical bootstrap of four conserved currents [41]. In this case imposing $\Delta_4 = 6$ resulted in the lower bound on C_T growing indefinitely as the numerical precision (the derivative order Λ) increased.

The numerical results of imposing a gap on Δ_4 are shown in figure 8.12, with some projections at smaller values of Δ_4 shown in figure 8.13. For each value of Δ_4 and $0 \le \theta \le \pi/2$ we find a minimal allowed value of C_T. This value is quite sensitive to θ, generally reaching maximal values for $\theta \to 0, \pi/2$ and remaining relatively small around $\theta \approx \pi/4$. At the same time when Δ_4 approaches 6 the bound rapidly grows for all value of θ, and seems to diverge (numerically we see bounds of $O(600-700)$) as $\Delta_4 \to 6$, consistent with the Nachtmann theorem. Our bounds do not seem to show sufficient convergence to read off the expected $1/\epsilon$ scaling, but it will be interesting to study this divergent behavior more closely in future work.

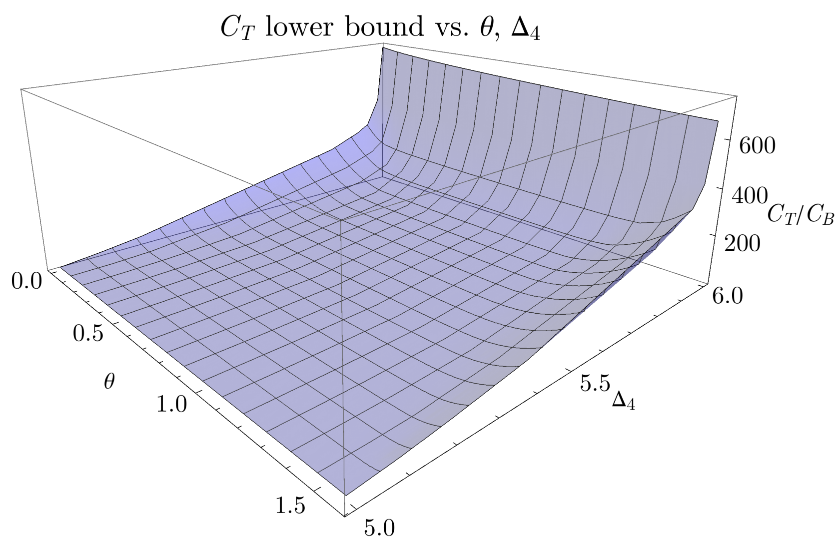

Figure 8.12: Lower bounds on C_T as a function of θ and the spin-4 gap Δ_4.

8.4.6 Ising-like spectrum

Next we focus our attention on what can be learned about the 3d Ising model from the $\langle TTTT\rangle$ bootstrap. In earlier numerical bootstrap work [34], a precise determination of the central charge $C_T^{\text{Ising}}/C_B = 0.946534(11)$ was found. As far as we are aware, no determinations of the $\langle TTT\rangle$ 3-point function in the 3d Ising model have been made previously.

The Ising model has a $\mathbb{Z}_2$ global symmetry, but only $\mathbb{Z}_2$-even operators appear in the $T \times T$ OPE. Such operators can be either even or odd under spacetime parity. The

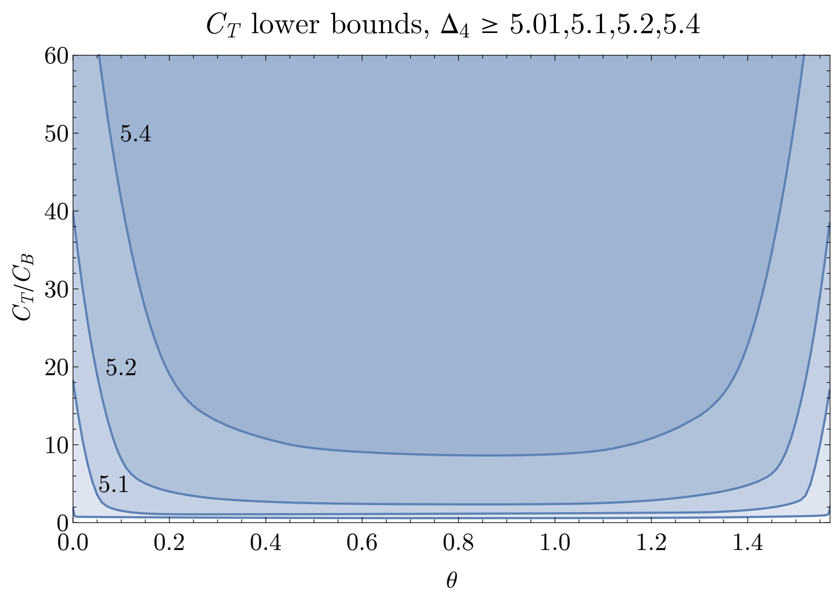

Figure 8.13: Lower bounds on C_T as a function of θ for spin-4 gaps $\Delta_4 \geq 5.01, 5.1, 5.2, 5.4$.

scaling dimensions of the leading parity-even operators in the 3d Ising spectrum have been computed to high precision using numerical bootstrap methods (see table 2 of [31] for a summary). However, as far as we are aware very little is known about the parity-odd spectrum.

In figure 8.14 we show the result of inputting the approximate known scaling dimensions for the leading parity-even scalars $\{\epsilon, \epsilon'\}$, the second spin-2 operator T', and the leading spin-4 operator. The horizontal lines show the 3d Ising value of C_T as well as the free scalar value. Regions very close to $\theta = 0$ and $\theta = \pi/2$ are excluded (primarily due to the spin-4 gap) but otherwise this data does not place a very strong constraint.

On the other hand, we find that imposing a parity-odd gap places a very strong constraint on the allowed region. In figure 8.15 we show the effect of inputting the expectation (e.g., from the ϵ-expansion) that the leading parity-odd scalar is irrelevant,[24] in addition to inputting the leading parity-even scalar dimensions. Only a tiny window at small θ is compatible with the 3d Ising value of C_T. We show a zoom of this region in figure 8.16, where it can be seen that these assumptions imply $.01 < \theta < .05$.

[24]It would be nice to directly confirm this by identifying a system in the Ising universality class with parity (or time-reversal) symmetry breaking at the microscopic level. We thank Slava Rychkov for discussions on this issue.

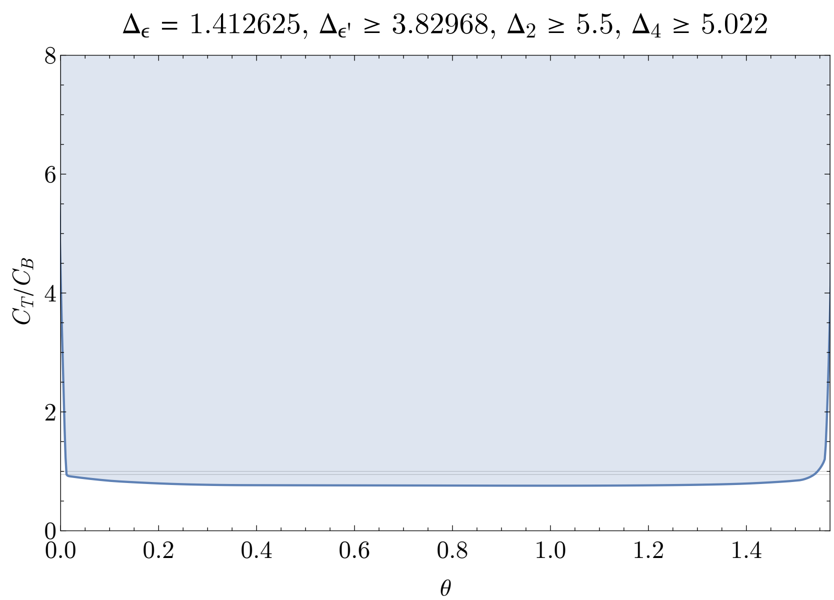

Figure 8.14: Lower bound on C_T as a function of θ assuming known low-lying gaps in the parity-even spectrum in the 3d Ising CFT.

In fact, it is likely that the parity-odd scalar gap in the 3d Ising model is significantly larger than 3. E.g., it may be close to the free scalar value $\Delta_{\text{odd}} = 11$. This large gap is also plausible given figure 8.8, where it can be seen that a sharp transition in the allowed region occurs near the Ising value of Δ_{even}. In light of this plot, if the gap is maximal we see that it may be as large as $\Delta_{\text{odd}} \lesssim 11.2$.

Previously in figure 8.6 we saw that a parity-odd gap close to this value on its own imposes a robust restriction $\theta < .023$, with an allowed region compatible with C_T^{Ising}. In figure 8.17 we show the result on the allowed region of additionally imposing the known values of Δ_ϵ and $\Delta_{\epsilon'}$, combined with the sequence of assumptions $\Delta_{\text{odd}} \geq 9, 10, 11, 11.1, 11.2$. These assumptions lead to closed islands and if the gap is close to being saturated allow us to make the tighter determination $.01 < \theta < .018 - .019$, with the precise upper bound depending on the gap.

8.5 Discussion

In this work we used the numerical conformal bootstrap to study the space of unitary parity-preserving CFTs in three dimensions. Assuming the existence of a unique stress tensor (conserved spin-2 current) and imposing crossing symmetry of its four-point correlation function, we found a number of universal bounds on CFT data. One striking discovery is the necessity of both light parity-even ($\Delta_{\text{even}} \leq 7$) and

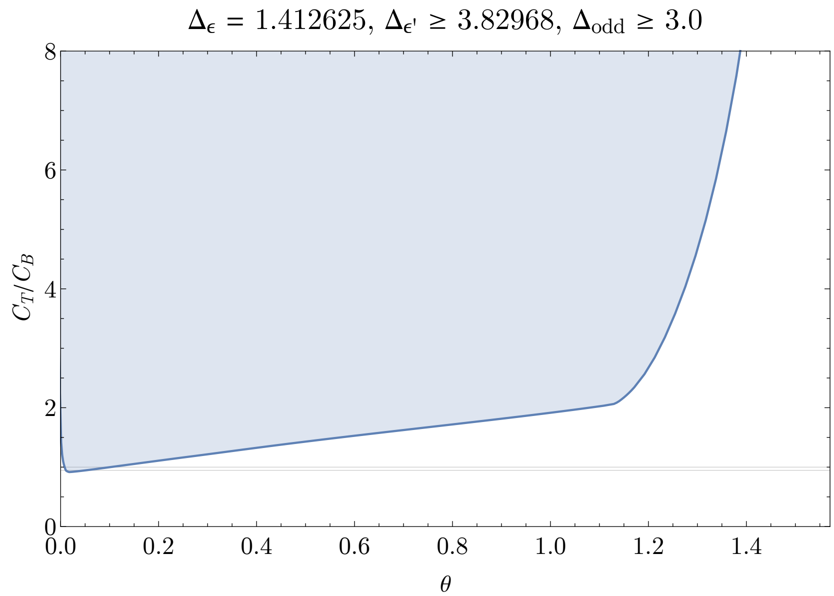

Figure 8.15: Lower bound on C_T as a function of θ assuming known low-lying gaps in the parity-even scalar spectrum in the 3d Ising CFT, combined with the assumption that the leading parity-odd scalar is irrelevant.

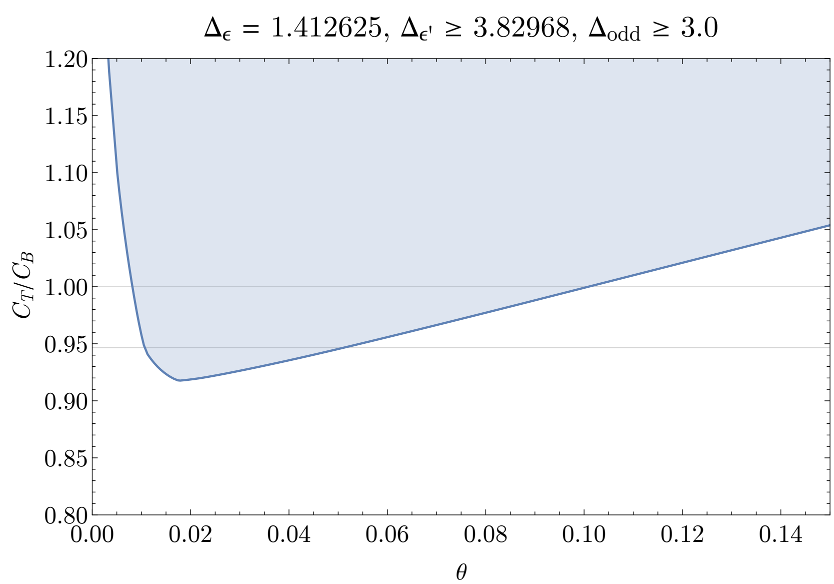

Figure 8.16: Lower bound on C_T as a function of θ assuming known low-lying gaps in the parity-even scalar spectrum in the 3d Ising CFT, combined with the assumption that the leading parity-odd scalar is irrelevant.

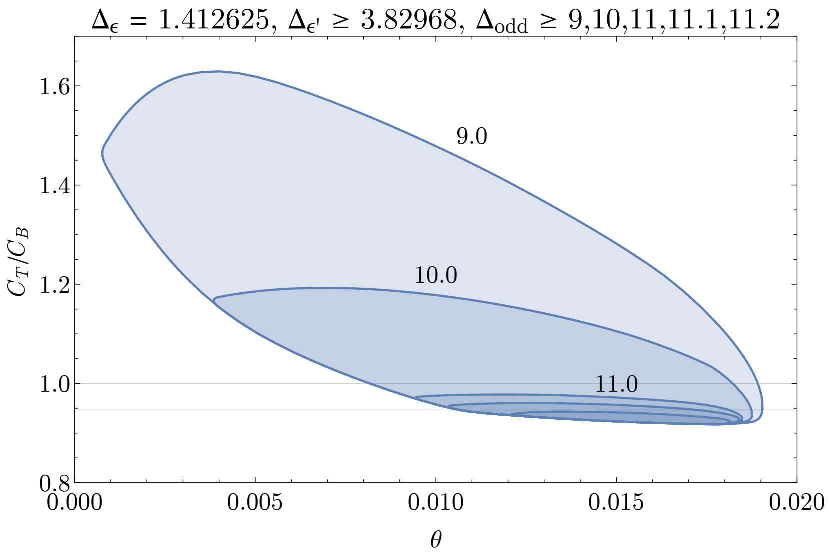

Figure 8.17: Lower and upper bounds on (θ, C_T) assuming known low-lying gaps in the parity-even scalar spectrum in the 3d Ising CFT, combined with various larger gaps in the parity-odd spectrum. A gap $\Delta_{\text{odd}} = 11.1$ is compatible with C_T^{Ising} (shown as the lower horizontal line) but a gap $\Delta_{\text{odd}} = 11.2$ is not.

parity-odd ($\Delta_{\text{odd}} \leq 11.78$) scalars in the spectrum of *any* consistent local unitary CFT, see figure 8.8. Among other universal results are those limiting the value of the central charge C_T modulo additional assumptions. For example, in hypothetical "dead-end" CFTs without any relevant scalars C_T is constrained to be larger than roughly twice the central charge of a free 3d scalar or Majorana fermion. These, and other similar findings presented in this paper are of a new kind, in the sense that they cannot be derived (as far as we know) using any theoretical tools other than the numerical bootstrap.

There is another class of discoveries presented in this paper which further support and extend previously established theoretical results. Our numerical results reproduce the "conformal collider" bounds, see figure 8.2. Imposing scalar or spin-2 gaps above the values they take in holographic theories further allows us to place upper bounds on C_T. Similarly, imposing a gap on the dimension of the lightest spin-4 operator discussed in section 8.4.5, $\Delta_4 \geq 6 - \epsilon$, $\epsilon \to 0$, forces the CFT in question to have an apparently diverging central charge and a spectrum likely dual to weakly coupled gravity in AdS_4, in full consistency with the Nachtmann theorem [68, 69, 162, 249, 250]. Reproducing these results is a strong consistency check on our numerical setup.

Many exclusion plots in this work exhibit characteristic features potentially signaling the existence of an underlying theory saturating the corresponding bounds. The scalar exclusion plot in figure 8.8 has a kink that tentatively corresponds to the 3d Ising model, in addition to reassuring corners that coincide with other known free or mean-field solutions. This gives hope to extend our results to further elucidate precise properties of particular theories. The first few steps in this direction for the 3d Ising model were already undertaken in section 8.4.6, where known dimensions of light scalar operators[25] were used to obtain a strong bound $0.01 < \theta < 0.05$ on the OPE coefficient controlling the 3pt function of stress tensors (8.80). By assuming larger gaps in the parity-odd scalar sector this window can be reduced down to $0.010 < \theta < 0.019$. We also find closed islands in Figs. 8.4 and 8.11 which may indicate new nontrivial solutions to the bootstrap equations and could be interesting to study further.

Our work paves the way for many future investigations. Below we briefly describe only some of the possible directions, which we find particularly interesting and important. A substantial extension of this work would be to combine stress tensors with other operators, such as scalars, fermions, or global symmetry currents, using a larger mixed correlator bootstrap. In this way one should be able to isolate e.g. theories with global $O(N)$ symmetry and obtain a host of new constraints pertaining to such theories. One can also extend our work to CFTs with varying amounts of supersymmetry, requiring additional computation of the necessary superconformal blocks. From the technical point of view these generalizations are relatively straightforward and only require combining previously developed ingredients.

Yet another natural generalization is to extend the analysis of this paper to parity-breaking theories. This direction is interesting in part because it would help us gain a better understanding of the large family of Chern-Simons-matter theories in three dimensions, recently understood to be interconnected by a large web of RG flows and dualities (e.g. [289–291]). From the technical point of view such an extension would require the straightforward task of generalizing the analysis of sections 8.2 and 8.3 to additional parity-breaking structures.

Finally, the numerical analysis performed in this paper, and the theoretical developments which it required, constitute significant progress in the development of the conformal bootstrap in $d = 3$ dimensions. It would be very interesting to generalize the current analysis to higher dimensions, first to $d = 4$. The needed conformal

[25] Assuming that the lightest parity-odd scalar is irrelevant.

blocks in four dimensions were recently calculated implicitly in a number of works [2, 3, 55, 58, 62, 82]. Accordingly, the bootstrap for the stress tensor and other operators with spin in four dimensions is now accessible in principle, although it still represents a substantial technical challenge. We hope to address this problem in the future. This research program can also be potentially extended to arbitrary d yielding universal constraints on CFTs in $d = 5, 6$ and beyond. We hope this study will eventually yield new non-trivial results contributing to our understanding of interacting CFTs, or their absence, in $d > 6$.

Acknowledgements

We are grateful to Clay Córdova, Daliang Li, David Meltzer, João Penedones, Eric Perlmutter, Slava Rychkov, Marco Serone, Emilio Trevisani, Alessandro Vichi, and Alexander Zhiboedov for discussions. We also thank Revant Nayar for collaboration in the initial stages of this work. Many thanks to the organizers and participants of the bootstrap collaboration workshops at Yale, Princeton, and ICTP São Paulo where part of this work was completed. AD is supported by NSF grant PHY-1720374. DSD is supported by DOE grant DE-SC0009988, a William D. Loughlin Membership at the Institute for Advanced Study, and Simons Foundation grant 488657 (Simons Collaboration on the Nonperturbative Bootstrap). PK is supported by DOE grant DE-SC0011632. DP is supported by NSF grant PHY-1350180 and Simons Foundation grant 488651. The computations in this paper were run on the Omega and Grace computing clusters supported by the facilities and staff of the Yale University Faculty of Arts and Sciences High Performance Computing Center, on the Hyperion computing cluster supported by the School of Natural Sciences Computing Staff at the Institute for Advanced Study and on the computing clusters of the National Energy Research Scientific Computing Center, a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

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Appendix A

APPENDICES TO CHAPTER 2

A.1 Smoothness conditions on correlators

The analysis of section 2.2 did not take into account smoothness of g. In order for g to be continuous, it is sufficient for g_0 to be continuous and to satisfy the stabilizer invariance condition (2.13). Note that with the choice of conformal frame discussed in section 2.2.3 the stabilizer subgroup is the same $SO(d-m+2)$ for generic y, but it enhances an the boundaries of conformal frame, essentially giving a boundary condition for the otherwise $SO(d+m-2)$-invariant g_0. We will now see that this boundary condition needs to be refined further if we want g to be smooth.

For simplicity, let us consider only the most important case of 4-point functions. It is easy to convince oneself that g as given by (2.14) will be smooth for y in the interior of conformal frame as soon as g_0 is smooth there. What is non-trivial is the smoothness on the boundary of conformal frame. Let us start with a smooth g and see what kind of g_0 it leads to.

We split the reduction to conformal frame into two steps. First, we fix the coordinates x_1, x_3, x_4 as in section 2.2.2. This leads to a function $g_1(x_2)$ which is to be invariant under $SO(d-1)$. Note that its smoothness is equivalent to smoothness of g. We can expand g_1 in Taylor series along the directions orthogonal to e,

$$g_1(s_i, x_2) = \sum_{n=0}^{N} g_1^{\mu_1\ldots\mu_n}(s_i, e\cdot x_2) z_{\mu_1}\ldots z_{\mu_n} + o(z^N), \tag{A.1}$$

where z is the $(d-1)$-dimensional projection of x_2 onto the subspace orthogonal to e. From the invariance equation (2.10) we read off the condition that for every $e\cdot x_2$ the value $g_1^{\mu_1\ldots\mu_n}(\,\cdot\,, e\cdot x_2)$ is a singlet in

$$\hat{\mathbf{n}} \otimes \mathrm{Res}^{O(d)}_{O(d-1)} \bigotimes_{i=1}^{4} \rho_i, \tag{A.2}$$

where $\hat{\mathbf{n}}$ is the reducible symmetric tensor representation of $O(d-1)$.[1] The symmetric tensor decomposes into symmetric traceless tensors as

$$\hat{\mathbf{n}} = \mathbf{n} + (\mathbf{n}-\mathbf{2}) + \ldots + (\mathbf{n} \mod 2). \tag{A.3}$$

[1] We also easily take into account the kinematic permutation symmetries by using in (A.1) the trivialized polarizations $\tilde{s}_i$ constructed in appendix A.2.2.2.

Now when we finally restrict to the conformal frame by taking z inside the half-plane α, which we will assume to be along 1st and 2nd coordinate axes, with e being along the 1st axis, we find

$$g_0(s_i, x_2^1, x_2^2) = \sum_{n=0}^{N} g_1^{2\ldots 2}(s_i, x_2^1)(x_2^2)^n + o\left((x_2^2)^N\right). \tag{A.4}$$

Note that theorem 1 tells us to look for $O(d-1)$ symmetric traceless tensors[2] in

$$\mathrm{Res}^{O(d)}_{O(d-1)} \bigotimes_{i=1}^{4} \rho_i. \tag{A.5}$$

Equation (A.2) therefore tells us at which orders in Taylor series (A.4) which traceless symmetric tensors of (A.5) can contribute. For example, the spin-3 symmetric traceless tensor representation **3**, if appears in (A.5), defines a tensor structure whose coefficient function can contribute to (A.4) at orders $(x_2^2)^3, (x_2^2)^5, (x_2^2)^7, \ldots$ but not $(x_2^2)^1$ or $(x_2^2)^{2n}$.

In other words, (A.2) restricts the expansion of the coefficient functions of our structures by specifying their parity under $x_2^2 \to -x_2^2$ and the rate at which they go to zero on the boundary of conformal frame. Note that the x_2^2 parity of the coefficient function can also be extracted from how the corresponding structure behaves under a π rotation in the plane, say, 2-3, which is more convenient in practice than (A.2).

As the most basic example, consider the scalar four-point function. In this case, the Taylor coefficients $g_1^{\mu_1\ldots\mu_n}$ are singlets in

$$\hat{\mathbf{n}} \otimes \bullet = \hat{\mathbf{n}}, \tag{A.6}$$

and thus only exist for even n, according to (A.3). This tells us that scalar correlation functions restrict to g_0 with even expansion in x_2^2 and this is why we can parametrize them by u and v (which are also even).

A.1.1 Example: 4 Majorana fermions

Consider now the example of section 2.4.4. There are two aspects of the smoothness analysis which are important for actual numerical analysis. For convenience, we use the t and x coordinates of section 2.4.3 below.

The first is that some of the coefficient functions are restricted to be even or odd in $t \sim z - \bar{z}$. This is easy to handle by hand, since as noted above, this is determined

[2] This is equivalent to taking singlets in further restriction to $O(d-2)$.

by the behavior of the structure under π rotation in the plane 0-2. Via analytic continuation this rotation is equivalent to exchange of $\uparrow$ and $\downarrow$. Therefore, we can consider structures

$$\begin{aligned}
\langle\uparrow\uparrow\uparrow\uparrow\rangle^{\pm} &= \langle\uparrow\uparrow\uparrow\uparrow\rangle \pm \langle\downarrow\downarrow\downarrow\downarrow\rangle, \\
\langle\uparrow\uparrow\downarrow\downarrow\rangle^{+} &= \langle\uparrow\uparrow\downarrow\downarrow\rangle + \langle\downarrow\downarrow\uparrow\uparrow\rangle, \\
\langle\uparrow\downarrow\uparrow\downarrow\rangle^{+} &= \langle\uparrow\downarrow\uparrow\downarrow\rangle + \langle\downarrow\uparrow\downarrow\uparrow\rangle, \\
\langle\downarrow\uparrow\uparrow\downarrow\rangle^{+} &= \langle\uparrow\downarrow\downarrow\uparrow\rangle + \langle\downarrow\uparrow\uparrow\downarrow\rangle,
\end{aligned} \tag{A.7}$$

each of which have definite parity under $t \to -t$. Note that we didn't form the difference in the last three structures since the terms on the right side in each line lie in the same orbit of $\mathbb{Z}_2^2$.

The second is that some of the coefficient functions should vanish faster than is required by their t-parity. We compute[3], using (2.39)

$$\mathrm{Res}^{O(3)}_{O(2)} \left(\mathbf{\tfrac{1}{2}}^{\otimes 4}\right)^{\mathbb{Z}_2^2} = \mathbf{2} \oplus \mathbf{1} \oplus 3\ \bullet^{+}\ . \tag{A.8}$$

According to (A.2), this means that from 5 coefficient functions of parity-even structures, 4 are even in t, of which 3 start with t^0 and 1 starts with t^2, and one is odd in t and starts with t^1. We see that there is one t-even coefficient function which should vanish as t^2, which is faster than required by its t-parity. This means that there is a linear relation between t^0 coefficients of the coefficient functions $g_{\langle\uparrow\uparrow\uparrow\uparrow\rangle^+}, g_{\langle\uparrow\uparrow\downarrow\downarrow\rangle^+}, g_{\langle\uparrow\downarrow\uparrow\downarrow\rangle^+}, g_{\langle\downarrow\uparrow\uparrow\downarrow\rangle^+}$, i.e.

$$\alpha_1 g_{\langle\uparrow\uparrow\uparrow\uparrow\rangle^+}(0,x) + \alpha_2 g_{\langle\uparrow\uparrow\downarrow\downarrow\rangle^+}(0,x) + \alpha_3 g_{\langle\uparrow\downarrow\uparrow\downarrow\rangle^+}(0,x) + \alpha_4 g_{\langle\downarrow\uparrow\uparrow\downarrow\rangle^+}(0,x) = 0, \tag{A.9}$$

where the first argument is $t = 0$. One can check that $\alpha_1 \neq 0$, and we can then use this equation to find $g_{\langle\uparrow\uparrow\uparrow\uparrow\rangle^+}(0,x)$.

More generally, to find such relations, it is convenient to consider the quadratic Casimir operator for the $SO(d-1)$ subgroup. Since $SO(d-2) \subset SO(d-1)$, it commutes with $SO(d-2)$ generators and thus maps $SO(d-2)$-invariants to $SO(d-2)$-invariants. This means that it is a linear operator on the space of four-point tensor structures, and it detects the $SO(d-1)$ representations to which these structures belong. Since only traceless-symmetric representations can appear, the quadratic Casimir eigenvalues completely characterize them. The recipe is

[3] In general one may need to be a little more careful with the permutation phases than we have been in this simple example.

then to organize the four-point tensor structures according to eigenvalues of this Casimir, demand the coefficients of the structures with $SO(d-1)$ Casimir eigenvalue $k(k+d-3)$ vanish as t^k (and are even or odd in t, depending on the parity of k).

Summarizing the discussion in section 2.4.4 and in this appendix, one can use the following independent system of crossing equations,

$$\begin{aligned}
\partial_t^{2n}\partial_x^{2m+1} g_{\langle\uparrow\uparrow\uparrow\uparrow\rangle^+} &= 0, \quad n \geq 1, m \geq 0,\\
\partial_t^{2n}\partial_x^{2m+1} g_{\langle\uparrow\downarrow\uparrow\downarrow\rangle^+} &= 0, \quad n \geq 0, m \geq 0,\\
\partial_t^{2n}\partial_x^{2m+1}\left(g_{\langle\uparrow\uparrow\downarrow\downarrow\rangle^+} + g_{\langle\downarrow\uparrow\uparrow\downarrow\rangle^+}\right) &= 0, \quad n \geq 0, m \geq 0,\\
\partial_t^{2n}\partial_x^{2m}\left(g_{\langle\uparrow\uparrow\downarrow\downarrow\rangle^+} - g_{\langle\downarrow\uparrow\uparrow\downarrow\rangle^+}\right) &= 0, \quad n \geq 0, m \geq 0,\\
\partial_t^{2n+1}\partial_x^{2m} g_{\langle\uparrow\uparrow\uparrow\uparrow\rangle^-} &= 0, \quad n \geq 0, m \geq 0,
\end{aligned} \tag{A.10}$$

where everything is evaluated at $t = 0,\ x = 1/2$.

A.2 More on permutations

A.2.1 Kinematic permutations

In this section we prove that $\{S_n^{\text{kin}}\}_{n=1}^{\infty} = \{0, S_2, S_3, \mathbb{Z}_2^2, 0, 0, \ldots\}$, where 0 stands for the trivial group. The first three cases are, as noted in the main text, trivial, since the conformal moduli space $\overline{\mathcal{M}}_n$ of $n = 1, 2, 3$ points consists of one point, and thus $S_n^{\text{kin}} = S_n$.

Now suppose $n \geq 4$. Consider the set U of all conformal cross-ratios of the form

$$u_{ij,kl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}, \quad x_{ij}^2 = (x_i - x_j)^2, \tag{A.11}$$

with i, j, k, l all different. Permutations of points x_i act on this set by permutations, and permutations from S_n^{kin} should leave these cross-ratios invariant. Since for a generic configuration there are no two exactly equal cross-ratios (even though there are relations between them), this means that the permutations induced on U should be trivial.

Suppose a permutation maps $i \to j$, $i \neq j$. Then by looking at $u_{ij,kl}$ (with i, j, k, l all different) we see that necessarily $j \to i$, otherwise this cross-ratio will change. But then also $k \leftrightarrow l$. Since we were free to choose k, l, this leads to a contradiction unless $n = 4$ and only one choice of k, l is possible. This establishes that $S_n^{\text{kin}} = 0$ for $n > 4$. For $n = 4$ it means that the allowed permutations are products of 2-cycles and an explicit check shows that all possible products are allowed, giving

$$S_4^{\text{kin}} = \{e, (12)(34), (13)(24), (14)(23)\} = \mathbb{Z}_2^2. \tag{A.12}$$

A.2.2 Conformal transformations for permutations

We now analyze explicitly the conformal transformations r_π induced by permutations. We only do so for three and four-point functions, since these are the only cases when there are interesting kinematic permutations.

For both three- and four-point functions we choose the r_π to preserve the plane α in which all the operators lie (for three points we choose some such plane). Such conformal transformations restrict on α to the fractional linear transformations, and we can describe them by a mapping

$$x \mapsto x' = \frac{ax+b}{cx+d}, \tag{A.13}$$

where we identified α with $\mathbb{C}$. Note that we can choose these transformation to give trivial rotations in the planes orthogonal to α. We therefore only need to compute $Spin(2)$ elements induced by r_π inside the plane, and the problem is entirely two-dimensional.

The group of fractional linear transformations is double covered by $SL(2,\mathbb{C})$. Thus $r_\pi \in SL(2,\mathbb{C})$. The correspondence is

$$r_\pi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \Rightarrow r_\pi x = \frac{ax+b}{cx+d}, \quad ad - bc = 1. \tag{A.14}$$

This is 2 to 1 because r_π and $-r_\pi$ give the same transformation. Recall that the basic condition for r_π is that

$$r_\pi x'_i = x_{\pi(i)}, \tag{A.15}$$

for x'_i in the conformal frame. In the case of kinematic permutations we have $x'_i = x_i$. Thus we have the following equation for r_π,

$$\frac{ax_i+b}{cx_i+d} = x_{\pi(i)}. \tag{A.16}$$

This has two solutions differing by a sign. Since the correlator is bosonic in total, we are free to choose either of them.

The $SO(2)$ element $R_{r_\pi}(x_i)$ is given by (upon identification of $SO(2)$ with the unit circle in complex plane)

$$R_{r_\pi}(x_i) = n\left(\frac{dx'}{dx}\right)\bigg|_{x=x_{\pi^{-1}(i)}}, \tag{A.17}$$

where $n(x) = x/|x|$. The implementation of the lifting from $SO(2)$ to $Spin(2)$ discussed in section 2.2.1 is straightforward in two dimensions. Note that for $ad - bc = 1$,

$$\frac{d}{dz}\left(\frac{ax+b}{cx+d}\right) = \frac{1}{(cz+d)^2}. \tag{A.18}$$

This is invariant under $r_\pi \to -r_\pi$ and the phase gives an element of $SO(2)$ as above. Lifting to an element of $Spin(2)$ is essentially equivalent to choosing a square root of this expression, with the most natural choice being

$$\sqrt{\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right)} = \frac{1}{cx+d}. \tag{A.19}$$

This is not invariant over $r_\pi \to -r_\pi$, which means that this is only a map from the double cover $SL(2,\mathbb{C})$ of the conformal group to $Spin(2)$, but not from the conformal group $PSL(2,\mathbb{C}) = SO(3,1)$ itself. This is in accord with the discussion in section 2.2.1. Therefore, we find that

$$\mathcal{R}_{r_\pi}(x_i) = n(cx+d)^{-1}|_{x=x_{\pi^{-1}(i)}}. \tag{A.20}$$

In the following table we summarize the locations of the operators in the conformal frame we choose, by specifying the complex coordinates

	x_1	x_2	x_3	x_4
3-point	0	1	∞	-
4-point	0	z	1	∞

As discussed before, the operator at infinity is inserted by putting it at L and then taking the limit $L \to \infty$ along the real axis. This is done in order to avoid using inversion when defining the operator at infinity. A safe way of determining the phases is working with finite L and then taking the limit.

In the following we compute the transformations r_π and

$$\mathcal{R}_{r_\pi}(x_i)^{-1} = n(h_i(\pi)). \tag{A.21}$$

Note that the $SO(2)$ rotation angle is given by the phase of $n(h_i(\pi))^2$. We write the permutations in cycle notation. For example, $\pi = (134)(25)$ is the permutation $\pi(1) = 3, \pi(3) = 4, \pi(4) = 1, \pi(2) = 5, \pi(5) = 2$.

A.2.2.1 3-point functions

For three-point functions we have the following parameters a, b, c, d for the transformations and the induced h_i:

	h_1	h_2	h_3	a	b	c	d
id	1	1	1	1	0	0	1
(12)	i	i	$-i$	-1	1	0	1
(13)	i	i	i	0	1	1	0
(23)	$-i$	i	i	1	0	1	-1
(123)	-1	1	1	0	1	-1	1
(132)	1	1	-1	1	-1	1	0

A.2.2.2 4-point functions

For four-point functions we have

	h_1	h_2	h_3	h_4
id	1	1	1	1
(12)(34)	$i\sqrt{1-z}$	$i\sqrt{1-\bar{z}}$	$-i\sqrt{1-\bar{z}}$	$-i\sqrt{1-z}$
(13)(24)	$-\sqrt{\bar{z}(1-z)}$	$-\sqrt{\bar{z}(1-\bar{z})}$	$\sqrt{z(1-\bar{z})}$	$\sqrt{z(1-z)}$
(14)(23)	$i\sqrt{\bar{z}}$	$i\sqrt{\bar{z}}$	$i\sqrt{z}$	$i\sqrt{z}$

Note that these transformations have to be accompanied by a $-$ sign for an odd permutation of fermions. If we assume that we use the permutations to exchange identical operators then we can instead use the following table, but *without* the extra minus sign for the odd fermion permutation,

	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{h}_3$	$\tilde{h}_4$
id	1	1	1	1
(12)(34)	$i\sqrt{1-z}$	$-i\sqrt{1-\bar{z}}$	$i\sqrt{1-\bar{z}}$	$-i\sqrt{1-z}$
(13)(24)	$\sqrt{\bar{z}(1-z)}$	$\sqrt{\bar{z}(1-\bar{z})}$	$\sqrt{z(1-\bar{z})}$	$\sqrt{z(1-z)}$
(14)(23)	$i\sqrt{\bar{z}}$	$-i\sqrt{\bar{z}}$	$i\sqrt{z}$	$-i\sqrt{z}$

The trick now is that these $\tilde{h}_i(\pi)$ satisfy the group property

$$n(\tilde{h}_i(\pi\sigma)) = n(\tilde{h}_i(\pi))n(\tilde{h}_{\pi^{-1}(i)}(\sigma)), \tag{A.22}$$

which is an identity in $Spin(2)$, while it is only trivial that it holds in $SO(2)$.

This fact together with the fact that the action of $\mathbb{Z}_2^2$ is free actually implies that these phases can be trivialized in the following way. Suppose for concreteness that the full symmetry is the $\mathbb{Z}_2^2$, the argument for subgroups is similar. Thus, assume that all polarizations s_i transform in the same representation ρ and denote by $\rho(h)$ the action of $n(h) \in Spin(2)$. First, define the new polarizations

$$\begin{aligned}
\tilde{s}_1 &= s_1, \\
\tilde{s}_2 &= \rho(-i\sqrt{1-\bar{z}})s_2, \\
\tilde{s}_3 &= \rho(\sqrt{z(1-\bar{z})})s_3, \\
\tilde{s}_4 &= \rho(-i\sqrt{z})s_4.
\end{aligned} \tag{A.23}$$

Then recall that the action of the permutation, say, $(14)(23)$ is

$$\begin{aligned}
&s_1 \to \rho(-i\sqrt{z})s_4, \quad s_4 \to \rho(i\sqrt{\bar{z}})s_1, \\
&s_2 \to \rho(i\sqrt{z})s_3, \quad s_3 \to \rho(-i\sqrt{\bar{z}})s_2.
\end{aligned} \tag{A.24}$$

This induces the following action on the redefined polarizations,

$$\begin{aligned}
&\tilde{s}_1 \to \rho(-i\sqrt{z})s_4 = \tilde{s}_4, \\
&\tilde{s}_2 \to \rho(-i\sqrt{1-\bar{z}})\rho(i\sqrt{z})s_3 = \tilde{s}_3, \\
&\tilde{s}_3 \to \rho(\sqrt{z(1-\bar{z})})\rho(-i\sqrt{\bar{z}})s_2 = \tilde{s}_2, \\
&\tilde{s}_4 \to \rho(-i\sqrt{z})\rho(i\sqrt{\bar{z}})s_1 = \tilde{s}_1.
\end{aligned} \tag{A.25}$$

It is easy to check that the same holds for all other permutations. Since the redefinition commutes with the action of the stabilizing $O(d-2)$, we conclude that for the purposes of counting the structures we simply look at the tensor product $\bigotimes_{i=1}^4 \rho_i$ symmetrized by the kinematic symmetry group of the correlator *without* the fermionic $-$ sign, and then extract the $O(d-2)$ singlets.

For completeness we also consider the non-kinematic permutations. It is sufficient to consider (12) and (13) since these together with the kinematic permutations generate the full S_4. For these permutations $x'_i \neq x_i$, but rather

	x'_1	x'_2	x'_3	x'_4
(12)	0	$z/(z-1)$	1	∞
(13)	0	$1-z$	1	∞

We find the following permutation phases

	h_1	h_2	h_3	h_4
(12)	$\sqrt{1-\bar{z}}$	$\sqrt{1-\bar{z}}$	$\sqrt{1-\bar{z}}$	$\sqrt{1-z}$
(13)	i	i	i	$-i$

Again, we can define $\tilde{h}$ to automatically account for fermionic "–" sign,

	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{h}_3$	$\tilde{h}_4$
(12)	$\sqrt{1-\bar{z}}$	$-\sqrt{1-\bar{z}}$	$\sqrt{1-\bar{z}}$	$\sqrt{1-z}$
(13)	i	i	$-i$	$-i$

A.3 Character formula for symmetrized tensor products

Consider a tensor product

$$W = V^{\otimes n}, \tag{A.26}$$

and the subspace of it invariant under a subgroup $\Pi \subseteq S_n$ of permutations of tensor factors,

$$Z = \left[V^{\otimes n}\right]^{\Pi}. \tag{A.27}$$

More generally, we can allow Π to act by multiplication by permutations followed by a multiplication by a one-dimensional character χ_Π of Π. As an example, we can have $\Pi = S_n$ and $\chi_\Pi(\pi) \equiv 1$, in which case Z is the n-th symmetric tensor power, or $\chi_\Pi(\pi) = \operatorname{sign}\pi$, in which case Z is the n-th antisymmetric power of V. For simplicity, we will consider only these two choices of χ_Π, but leave Π completely general.

Assume that V is a representation of some group G, given by $\rho : G \to GL(V)$. Then W and Z are also representations of G, and out goal is to compute the character of G on Z, χ_Z.

Define the operator

$$P = \frac{1}{|\Pi|}\sum_{\pi\in\Pi} \pi \in GL(W), \tag{A.28}$$

where π acts as described above. Let $\rho_n = \rho^{\otimes n}$. Note that

$$P^2 = \frac{1}{|\Pi|^2}\sum_{\pi,\sigma\in\Pi} \pi\sigma = \frac{1}{|\Pi|^2}\sum_{\pi,\sigma'\in\Pi} \pi\pi^{-1}\sigma' = P, \tag{A.29}$$

where $\sigma' = \pi\sigma$. Since $P^2 = P$, P is a projection and W decomposes into a sum of eigenspaces of P, $W = W_0 \oplus W_1$, with the explicit decomposition being $w = (1-P)w + Pw$. It is easy to see that Pw is Π-invariant and if w is Π-invariant, then $Pw = w$. This shows $W_1 = Z$. Since ρ_n commutes with P, this decomposition is also a decomposition of W into representations of G. It then follows that

$$\chi_Z(g) = \operatorname{tr} P\rho_n(g), \tag{A.30}$$

as can be shown by choosing a basis diagonal for P. It is a simple exercise to show in some choice of basis that

$$\chi_Z(g) = \frac{1}{|\Pi|}\sum_{c\in C}\left[|c|\chi_\Pi(c)\prod_i \chi_\rho(g^{c_i})\right], \tag{A.31}$$

where C is the set of cycle types of permutations in Π, $|c|$ is the number of elements of cycle type c in Π, and c_i are the cycle lengths in the cycle type c. For example, the cycle type of the trivial permutation is $c = 1^n$, i.e. it is a product of n cycles of length 1, and $|c| = 1$. Therefore the contribution of the identity to the sum is always $\chi_\rho^n(g)$. Since we restricted χ_Π to come from a one-dimensional character of S_n, it takes the same value on all elements with the same cycle type, so that notation $\chi_\Pi(c)$ is well-defined.

The examples relevant in this paper are $\Pi = \mathbb{Z}_2 \subset S_2$, $\Pi = S_3$ and $\Pi = \mathbb{Z}_2^2 \subset S_4$. In the first case we have two cycle types, 1^2 and 2^1, each occuring once, and therefore we obtain for the trivial χ_Π

$$\chi_{\mathbb{Z}_2}(g) = \mathrm{S}^2\chi(g) = \frac{1}{2}\left[\chi^2(g) + \chi(g^2)\right], \tag{A.32}$$

the well-known formula for the symmetric square. For the exterior square one has, using $\chi_\Pi = \mathrm{sign}$,

$$\wedge^2\chi(g) = \frac{1}{2}\left[\chi^2(g) - \chi(g^2)\right]. \tag{A.33}$$

In the second case we have the symmetric and exterior cube relevant for proposition 2. In this case we have $\Pi = S_3$ and cycle types $1^3, 2^2 1^1, 3^1$ with multiplicities $1, 3, 2$. We find from (A.31),

$$\mathrm{S}^3\chi(g) = \frac{1}{6}\left[\chi(g)^3 + 3\chi(g^2)\chi(g) + 2\chi(g^3)\right], \tag{A.34}$$

$$\wedge^3\chi(g) = \frac{1}{6}\left[\chi(g)^3 - 3\chi(g^2)\chi(g) + 2\chi(g^3)\right]. \tag{A.35}$$

In the third case we have cycle types 1^4 and 2^2 with the latter occuring thrice, so that we find

$$\chi_{\mathbb{Z}_2^2}(g) = \frac{1}{4}\left[\chi^4(g) + 3\chi^2(g^2)\right]. \tag{A.36}$$

In practice this can be computed as

$$\rho^4 \ominus 3\left(\wedge^2\rho \otimes \mathrm{S}^2\rho\right), \tag{A.37}$$

which easily can be checked using the above formulas. The case $\chi_\Pi = \text{sign}$ is equivalent to $\chi_\Pi \equiv 1$.

Appendix B

APPENDICES TO CHAPTER 3

B.1 Details of the 4D formalism

We work in the signature $-+++$ and denote the diagonal 4D Minkowski metric by $h_{\mu\nu}$. We mostly follow the conventions of Wess and Bagger [292].

The representations of the connected Lorentz group in 4D are labeled by a pair of non-negative integers $(\ell, \bar{\ell})$. These representations can be constructed as the highest-weight irreducible components in a tensor product of the two basic spinor representations $(1,0)$ and $(0,1)$.

We denote the objects in the left-handed spinor representation $(1,0)$ as ψ_α, $\alpha = 1,2$, and the objects in its dual representation as ψ^α. The original and the dual representations are equivalent via the identification

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \tag{B.1}$$

where

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = +1. \tag{B.2}$$

Because of the equivalence between $(1,0)$ and its dual representation, we will not be careful to distinguish them in the text, the distinction in formulas will be clear from the location of indices.

The right-handed spinor representation $(0,1)$ is the complex conjugate of the left-handed spinor representation, and the objects transforming in $(0,1)$ representation will be denoted as $\chi_{\dot{\alpha}}$. Here the dot should not be considered as part of the index, but rather as an indication that this index transforms in $(0,1)$ and not in $(1,0)$ representation. For example, the definition of $(0,1)$ representation is essentially

$$\psi^\dagger_{\dot{\alpha}} = (\psi_\alpha)^\dagger. \tag{B.3}$$

The dual of $(0,1)$ is equivalent to $(0,1)$ via the conjugation of (B.1)

$$\chi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\chi^{\dot{\beta}}, \quad \chi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi_{\dot{\beta}}, \tag{B.4}$$

where $\epsilon_{\dot{\alpha}\dot{\beta}} \equiv \epsilon_{\alpha\beta}$, $\epsilon^{\dot{\alpha}\dot{\beta}} \equiv \epsilon^{\alpha\beta}$. We use the contraction conventions

$$\psi_1\psi_2 = \psi_1^\alpha\psi_{2\alpha}, \quad \chi_1\chi_2 = \chi_{1\dot{\alpha}}\chi_2^{\dot{\alpha}}. \tag{B.5}$$

The tensor product $(1,0)\otimes(0,1)=(1,1)$ is equivalent to the vector representation, and the equivalence is established by the 4D sigma matrices $\sigma^{\mu}_{\alpha\dot\beta}$ and $\overline{\sigma}^{\mu\dot\alpha\beta}$, which we define as

$$\sigma^0=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\quad \sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{B.6}$$

and $\overline{\sigma}^0=\sigma^0$, $\overline{\sigma}^1=-\sigma^1$, $\overline{\sigma}^2=-\sigma^2$, $\overline{\sigma}^3=-\sigma^3$. For a convenient summary of relations involving sigma-matrices see for example [293].[1]

For primary operators we adopt the convention to write them out with dotted indices upstairs and the undotted indices downstairs

$$O^{\dot\alpha_1\ldots\dot\alpha_{\overline{\ell}}}_{\beta_1\ldots\beta_\ell}.\tag{B.7}$$

In this notation the index-full version of (3.6) is

$$\overline{O}^{\dot\beta_1\ldots\dot\beta_\ell}_{\alpha_1\ldots\alpha_{\overline{\ell}}}\equiv(-1)^{\overline{\ell}-\ell}\;\epsilon_{\alpha_1\alpha'_1}\ldots\epsilon_{\alpha_{\overline{\ell}}\alpha'_{\overline{\ell}}}\epsilon^{\dot\beta_1\dot\beta'_1}\ldots\epsilon^{\dot\beta_\ell\dot\beta'_\ell}O^{\dagger\,\alpha'_1\ldots\alpha'_{\overline{\ell}}}_{\quad\dot\beta'_1\ldots\dot\beta'_\ell}.\tag{B.8}$$

Action of conformal generators We denote the conformal generators by P, K, D, M. We choose to work with anti-Hermitian generators (related to the Hermitian ones by a factor of i)

$$D^\dagger=-D,\quad P^\dagger=-P,\quad K^\dagger=-K,\quad M^\dagger=-M,\tag{B.9}$$

which allow us to avoid many factors of i in the formulas below (note that even though D is anti-Hermitian, its adjoint action has real eigenvalues). These generators satisfy the following algebra

$$[D,D]=0,\quad [D,P_\mu]=P_\mu,\quad [D,K_\mu]=-K_\mu,\tag{B.10}$$

$$[P_\mu,P_\nu]=0,\quad [K_\mu,K_\nu]=0,\quad [K_\mu,P_\nu]=2h_{\mu\nu}D-2M_{\mu\nu},\tag{B.11}$$

$$[M_{\mu\nu},D]=0,\quad [M_{\mu\nu},P_\rho]=h_{\nu\rho}P_\mu-h_{\mu\rho}P_\nu,\quad [M_{\mu\nu},K_\rho]=h_{\nu\rho}K_\mu-h_{\mu\rho}K_\nu,\tag{B.12}$$

$$[M_{\mu\nu},M_{\rho\sigma}]=h_{\nu\rho}M_{\mu\sigma}-h_{\mu\rho}M_{\nu\sigma}-h_{\nu\sigma}M_{\mu\rho}+h_{\mu\sigma}M_{\nu\rho}.\tag{B.13}$$

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[1] One should download and compile the version with mostly plus metric. Notice also a factor of i difference between their $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ and ours $\mathcal{S}^{\mu\nu}$ and $\overline{\mathcal{S}}^{\mu\nu}$.

The action of the conformal generators on primary fields is given by

$$[D, O(x,s,\bar{s})] = (x^\mu \partial_\mu + \Delta)\, O(x,s,\bar{s}), \tag{B.14}$$

$$[P_\mu, O(x,s,\bar{s})] = \partial_\mu\, O(x,s,\bar{s}), \tag{B.15}$$

$$[K_\mu, O(x,s,\bar{s})] = (2x_\mu x^\sigma - x^2\delta_\mu^\sigma)\partial_\sigma O(x,s,\bar{s}) + 2(\Delta\, x_\mu - x^\sigma \mathcal{M}_{\mu\sigma})O(x,s,\bar{s}), \tag{B.16}$$

$$[M_{\mu\nu}, O(x,s,\bar{s})] = (x_\nu\partial_\mu - x_\mu\partial_\nu)O(x,s,\bar{s}) + \mathcal{M}_{\mu\nu}O(x,s,\bar{s}), \tag{B.17}$$

where the spin generators are

$$\mathcal{M}_{\mu\nu}O(x,s,\bar{s}) = \left(-s^\alpha (\mathcal{S}_{\mu\nu})_\alpha{}^\beta \frac{\partial}{\partial s^\beta} - \bar{s}_{\dot{\alpha}} (\overline{\mathcal{S}}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \frac{\partial}{\partial \bar{s}_{\dot{\beta}}}\right) O(x,s,\bar{s}). \tag{B.18}$$

We have defined here the generators of the left- and right-handed spinor representations

$$(\mathcal{S}_{\mu\nu})_\alpha{}^\beta = -\frac{1}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)_\alpha{}^\beta, \tag{B.19}$$

$$(\overline{\mathcal{S}}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{1}{4}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{B.20}$$

which satisfy the same commutation relations as $M_{\mu\nu}$. Notice that as usual the differential operators in the right hand side of (B.14)-(B.17) have the commutation relations opposite to those of the Hilbert space operators in the left hand side. This is because if the Hilbert space operators A and B act on fields by differential operators $\mathfrak{A}$ and $\mathfrak{B}$, then their product AB acts by $\mathfrak{B}\mathfrak{A}$.

Action of space parity If a theory preserves parity, there exists a unitary operator $\mathcal{P}$ with the following commutation rule with Lorentz generators

$$\mathcal{P}M_{0i}\mathcal{P}^{-1} = -M_{0i}, \quad \mathcal{P}M_{ij}\mathcal{P}^{-1} = M_{ij}, \tag{B.21}$$

where $i,j = 1,2,3$. Applying this to (B.17) at $x = 0$, we see that

$$[M_{\mu\nu}, \mathcal{P}O_\alpha(0)\mathcal{P}^{-1}] = (\overline{\mathcal{S}}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\mathcal{P}O_\beta(0)\mathcal{P}^{-1}. \tag{B.22}$$

This implies that we can define an operator $\widetilde{O}$ as

$$\widetilde{O}^{\dot{\alpha}}(x) \equiv -i\mathcal{P}O_\alpha(\mathcal{P}x)\mathcal{P}^{-1} \tag{B.23}$$

which transform as a primary operator in the representation $(0,1)$. We also have $\mathcal{P}x^0 = x^0$, $\mathcal{P}x^k = -x^k$, $k = 1,2,3$. More generally, it is easy to check that we can consistently define

$$\widetilde{O}^{\dot{\alpha}_1\ldots\dot{\alpha}_\ell}_{\beta_1\ldots\beta_{\bar{\ell}}}(x) \equiv (-i)^{\ell+\bar{\ell}}\mathcal{P}O^{\dot{\beta}_1\ldots\dot{\beta}_{\bar{\ell}}}_{\alpha_1\ldots\alpha_\ell}(\mathcal{P}x)\mathcal{P}^{-1}. \tag{B.24}$$

The factor of i was introduced to reproduce the standard parity action on traceless symmetric operators in the $\widetilde{O} = O$ case.

The above definition provides the most generic action of parity on the operators O which can be slightly rewritten as

$$\mathcal{P} O_{\alpha_1 \ldots \alpha_\ell}^{\dot\beta_1 \ldots \dot\beta_{\bar\ell}}(x) \mathcal{P}^{-1} = i^{\ell+\bar\ell} \widetilde{O}_{\beta_1 \ldots \beta_{\bar\ell}}^{\dot\alpha_1 \ldots \dot\alpha_\ell}(\mathcal{P}x), \tag{B.25}$$

or equivalently in index-free notation

$$\mathcal{P} O(x, s, \bar s) \mathcal{P}^{-1} = \widetilde{O}(\mathcal{P}x, \mathcal{P}s, \mathcal{P}\bar s), \quad (\mathcal{P}\bar s)_{\dot\alpha} = i s^\alpha, \quad (\mathcal{P}s)^\alpha = i\bar s_{\dot\alpha}. \tag{B.26}$$

Notice that if O transforms in the $(\ell, \bar\ell)$ representation then the operator $\widetilde{O}$ transforms in $(\bar\ell, \ell)$ and may or may not be related to the operator $\overline{O}$ defined in (3.6) or to O itself if $\ell = \bar\ell$. This depends on a specific theory. What is important for us is that in a theory which preserves $\mathcal{P}$ there is a relation between correlators involving O_i and $\widetilde{O}_i$

$$\begin{aligned}\langle 0| O_1(x_1, s_1, \bar s_1) \cdots O_n(x_n, s_n, \bar s_n) |0\rangle = & \\ =& \langle 0| \mathcal{P} O_1(x_1, s_1, \bar s_1) \mathcal{P}^{-1} \cdots \mathcal{P} O_n(x_n, s_1, \bar s_1) \mathcal{P}^{-1} |0\rangle \\ =& \langle 0| \widetilde{O}_1(\mathcal{P}x_1, \mathcal{P}s_1, \mathcal{P}\bar s_1) \cdots \widetilde{O}_n(\mathcal{P}x_n, \mathcal{P}s_n, \mathcal{P}\bar s_n) |0\rangle. \end{aligned} \tag{B.27}$$

Written in terms of tensor structures this equality reads as

$$\sum_I \mathbb{T}_n^I g_n^I = \sum_I (\mathcal{P} \widetilde{\mathbb{T}}_n^I) \widetilde{g}_n^I, \tag{B.28}$$

where $\mathcal{P}\widetilde{\mathbb{T}}_n^I$ is given by $\widetilde{\mathbb{T}}_n^I$ with $x \to \mathcal{P}x$, $s \to \mathcal{P}s$, $\bar s \to \mathcal{P}\bar s$ and $\widetilde{\mathbb{T}}_n^I$ are the tensor structures appropriate to the correlators with the operators $\widetilde{O}_i$.[2] We provide the rules for the action of $\mathcal{P}$ on various tensor structures in equations (B.114), (B.115), and (3.120) `[applyPParity]`.

Action of time reversal If a theory has time reversal symmetry, there exists an anti-unitary operator $\mathcal{T}$ with the following commutation rule with Lorentz generators

$$\mathcal{T} M_{0i} \mathcal{T}^{-1} = -M_{0i}, \quad \mathcal{T} M_{ij} \mathcal{T}^{-1} = M_{ij}, \tag{B.29}$$

where $i, j = 1, 2, 3$. Applying it to (B.17) at $x = 0$, we see that

$$[M_{\mu\nu}, \mathcal{T} O_\alpha(0) \mathcal{T}^{-1}] = \left[(\overline{\mathcal{S}}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \right]^* \mathcal{T} O_\beta(0) \mathcal{T}^{-1}. \tag{B.30}$$

[2] If there are any parity-odd cross-ratios (i.e. $n \geq 6$) then $\widetilde{g}$ should have these with reversed signs.

This implies that $\mathcal{T}\mathcal{O}_\beta(0)\mathcal{T}^{-1}$ transforms as ψ^β and we can define the operator $\widehat{\mathcal{O}}$ as

$$\widehat{\mathcal{O}}_\alpha(x) \equiv -i\epsilon_{\alpha\beta}\mathcal{T}\mathcal{O}_\beta(\mathcal{T}x)\mathcal{T}^{-1}, \tag{B.31}$$

where $\mathcal{T}x^0 = -x^0$, $\mathcal{T}x^k = x^k$, $k = 1,2,3$. One can similarly define

$$\widehat{\mathcal{O}}^{\dot\alpha}(x) \equiv i\epsilon^{\dot\alpha\dot\beta}\mathcal{T}\mathcal{O}^{\dot\beta}(\mathcal{T}x)\mathcal{T}^{-1} \tag{B.32}$$

and extend the above definitions to arbitrary representations in an obvious way. For traceless symmetric operators in the $\widehat{\mathcal{O}} = \mathcal{O}$ case, this reproduces the standard time reversal action. In index-free notation we can write[3]

$$\mathcal{T}\mathcal{O}(x,s,\bar s)\mathcal{T}^{-1} = \widehat{\mathcal{O}}(\mathcal{T}x,\mathcal{T}s,\mathcal{T}\bar s), \quad (\mathcal{T}s)^\alpha = is^*_{\dot\alpha},\ (\mathcal{T}\bar s)_{\dot\alpha} = -i(\bar s^*)^\alpha. \tag{B.33}$$

Again, $\widehat{\mathcal{O}}$ may or may not be related to $\mathcal{O}$ depending on a theory. The only important point is that there is a relation between correlators with $\mathcal{O}_i$ and $\widehat{\mathcal{O}}_i$ in a theory preserving the time reversal symmetry

$$\begin{aligned}\langle 0|\mathcal{O}_1(x_1,s_1,\bar s_1)\cdots\mathcal{O}_n(x_n,s_n,\bar s_n)|0\rangle &= \\ &= \left[\langle 0|\mathcal{T}\mathcal{O}_1(x_1,s_1,\bar s_1)\mathcal{T}^{-1}\cdots\mathcal{T}\mathcal{O}_n(x_n,s_1,\bar s_1)\mathcal{T}^{-1}|0\rangle\right]^* \\ &= \left[\langle 0|\widehat{\mathcal{O}}_1(\mathcal{T}x_1,\mathcal{T}s_1,\mathcal{T}\bar s_1)\cdots\widehat{\mathcal{O}}_n(\mathcal{T}x_n,\mathcal{T}s_n,\mathcal{T}\bar s_n)|0\rangle\right]^*,\end{aligned} \tag{B.34}$$

where the conjugation happens because of the anti-unitarity of $\mathcal{T}$.[4] Written in terms of tensor structures this equality reads as

$$\sum_I \mathbb{T}_n^I g_n^I = \sum_I (\mathcal{T}\widehat{\mathbb{T}}_n^I)(\widehat{g}_n^I)^*, \tag{B.35}$$

where $\mathcal{T}\widehat{\mathbb{T}}_n^I$ is given by $(\widehat{\mathbb{T}}_n^I)^*$ with the replacements $x\to\mathcal{T}x$, $s\to\mathcal{T}s$, $\bar s\to\mathcal{T}\bar s$ made before the conjugation and $\widehat{\mathbb{T}}_n^I$ are the structures appropriate for the operators $\widehat{\mathcal{O}}_i$.

Computing $\mathcal{T}\widehat{\mathbb{T}}_n^I$ is easy, since we can construct $\mathcal{T}$ conjugation from $\mathcal{P}$ and the rotation $e^{i\pi M^{03}+\pi M^{12}}$. The latter rotation sends $s\to s$, $\bar s\to -\bar s$, which takes $\mathcal{T}s$ and $\mathcal{T}\bar s$ to $\mathcal{P}s$ and $\mathcal{P}\bar s$. The end result is

$$\mathcal{T}\widehat{\mathbb{T}}_n^I = \left(\mathcal{P}\widehat{\mathbb{T}}_n^I\right)^*. \tag{B.36}$$

We list the rules for the action of $\mathcal{T}$ on tensor structures in equations (B.116), (B.117), and (3.122) `[applyTParity]`.

[3]Note that $\mathcal{T}s$ and $\mathcal{T}\bar s$ are not complex conjugates of each other even if s and $\bar s$ are, so to avoid confusion here we do not assume that s and $\bar s$ are complex-conjugate. There is always a second complex conjugation (see below), so this is only intermediate.

[4]As an extreme example $\mathcal{T}i\mathcal{T}^{-1} = -i$, so we have $i = \langle 0|i|0\rangle = [\langle 0|\mathcal{T}i\mathcal{T}^{-1}|0\rangle]^* \neq \langle 0|\mathcal{T}i\mathcal{T}^{-1}|0\rangle$.

B.2 Details of the 6D formalism

In this appendix we describe our conventions for the 6D embedding space. We mostly follow [54, 55].

We work in the signature $\{- + + + + -\}$, and we denote the 6D metric by h_{MN}. We often use the lightcone coordinates

$$X^{\pm} \equiv X^4 \pm X^5, \tag{B.37}$$

and write the components of 6D vectors as

$$X^M = \{X^\mu, X^+, X^-\}. \tag{B.38}$$

The metric in lightcone coordinates has the components

$$h_{+-} = h_{-+} = \frac{1}{2}, \quad h^{+-} = h^{-+} = 2. \tag{B.39}$$

The 6D Lorentz group $Spin(2,4)$ is isomorphic to the $SU(2,2)$ group. The latter can be defined as the group of 4 by 4 matrices U which act on 4-component complex vectors V_a and preserve the sesquilinear form

$$\langle V, W\rangle = g^{\overline{a}b}(V_a)^* W_b, \quad \langle UV, UW\rangle = \langle V, W\rangle. \tag{B.40}$$

Here the metric tensor $g^{\overline{a}b}$ is a Hermitian matrix with eigenvalues $\{+1, +1, -1, -1\}$, which we choose to be

$$g^{\overline{a}b} \equiv g^{b\overline{a}} \equiv \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}_{ab}. \tag{B.41}$$

The bar over the index $\overline{a}$ indicates that this index transforms in a complex conjugate representation. In other words, we say that V_a transforms in the fundamental representation while

$$V^*_{\overline{a}} \equiv (V_a)^* \tag{B.42}$$

transforms in the complex conjugate of the fundamental representation (that is, by matrices U^*). The metric $g^{\overline{a}b}$ establishes an isomorphism between the complex conjugate representation and the dual representation

$$\overline{V}^a \equiv g^{a\overline{b}}\overline{V}_{\overline{b}}. \tag{B.43}$$

We say that $\overline{V}^a$ transforms in the anti-fundamental representation (that is, the anti-fundamental representation is the dual of the fundamental representation). The inverse isomorphism is established by the tensor

$$g_{\overline{a}b} \equiv g_{b\overline{a}} \equiv -g^{\overline{a}b}. \tag{B.44}$$

We have the relations

$$g_{a\overline{b}}g^{\overline{b}c} = g^{c\overline{b}}g_{\overline{b}a} = \delta^c_a, \quad (g^{a\overline{b}})^* = g^{\overline{a}b}. \tag{B.45}$$

The isomorphism between $Spin(2,4)$ and $SU(2,2)$ can be established by identifying the vector representation of $Spin(2,4)$ with the exterior square of the fundamental or anti-fundamental representations of $SU(2,2)$.[5] This equivalence is provided by the invariant tensors Σ^M_{ab} and $\overline{\Sigma}^{M\,ab}$ defined by

$$\Sigma^\mu_{ab} = \begin{pmatrix} 0 & -(\sigma^\mu\epsilon)_\alpha{}^{\dot\beta} \\ (\overline{\sigma}^\mu\epsilon)^{\dot\alpha}{}_\beta & 0 \end{pmatrix}, \quad \Sigma^+_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \quad \Sigma^-_{ab} = \begin{pmatrix} -2\,\epsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & 0 \end{pmatrix}, \tag{B.46}$$

and

$$\overline{\Sigma}^{\mu\,ab} = \begin{pmatrix} 0 & -(\epsilon\sigma^\mu)^\alpha{}_{\dot\beta} \\ (\epsilon\overline{\sigma}^\mu)_{\dot\alpha}{}^\beta & 0 \end{pmatrix}, \quad \overline{\Sigma}^{+\,ab} = \begin{pmatrix} -2\,\epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{\Sigma}^{-\,ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}. \tag{B.47}$$

These tensors have the following simple conjugation properties,

$$\left(\Sigma^M_{ab}\right)^* = g_{\overline{a}a'}g_{\overline{b}b'}\overline{\Sigma}^{M\,a'b'} \qquad \left(\overline{\Sigma}^{M\,ab}\right)^* = g^{\overline{a}a'}g^{\overline{b}b'}\Sigma^M_{a'b'}. \tag{B.48}$$

The above sigma-matrices satisfy many useful relations, for an incomplete list of them see appendix A in [55]. Using the sigma matrices we define the coordinate matrices

$$\mathbf{X}_{ab} \equiv X_M\Sigma^M_{ab} = -\mathbf{X}_{ba}, \qquad \overline{\mathbf{X}}^{ab} \equiv X_M\overline{\Sigma}^{M\,ab} = -\overline{\mathbf{X}}^{ba}, \tag{B.49}$$

which satisfy the algebra

$${}_a(\mathbf{X}_i\overline{\mathbf{X}}_j)^b + {}_a(\mathbf{X}_j\overline{\mathbf{X}}_i)^b = 2\,(X_i\cdot X_j)\delta^b_a. \tag{B.50}$$

We can now identify the $SU(2,2)$ generators corresponding to the standard 6D Lorentz generators

$$\Sigma^{MN} \equiv \frac{1}{4}\,(\Sigma^M\overline{\Sigma}^N - \Sigma^N\overline{\Sigma}^M), \quad \overline{\Sigma}^{MN} \equiv \frac{1}{4}\,(\overline{\Sigma}^M\Sigma^N - \overline{\Sigma}^N\Sigma^M), \tag{B.51}$$

[5] The fundamental and anti-fundamental representations themselves are the two spinor representations of $Spin(2,4)$.

satisfying the commutation relations

$$[\Sigma^{MN},\Sigma^{PQ}] = h^{NP}\Sigma^{MQ} - h^{MP}\Sigma^{NQ} - h^{NQ}\Sigma^{MP} + h^{MQ}\Sigma^{NP}, \tag{B.52}$$

$$[\overline{\Sigma}^{MN},\overline{\Sigma}^{PQ}] = h^{NP}\overline{\Sigma}^{MQ} - h^{MP}\overline{\Sigma}^{NQ} - h^{NQ}\overline{\Sigma}^{MP} + h^{MQ}\overline{\Sigma}^{NP}, \tag{B.53}$$

thus establishing the isomorphism $Spin(2,4) \simeq SU(2,2)$ at Lie algebra level.

By comparing the expressions for $\Sigma_{\mu\nu}$ and $\overline{\Sigma}^{\mu\nu}$ with $\mathcal{S}^{\mu\nu}$ and $\overline{\mathcal{S}}^{\mu\nu}$, we find that under the Lorentz $Spin(1,3)$ subgroup of $Spin(2,4)$ the fundamental and anti-fundamental representations of $SU(2,2)$ decompose as

$$V_a = \begin{pmatrix} V_\alpha \\ V^{\dot\alpha} \end{pmatrix}, \quad \overline{W}^a = \begin{pmatrix} \overline{W}^\alpha \\ \overline{W}_{\dot\alpha} \end{pmatrix}. \tag{B.54}$$

In other words, we write V_α or $V^{\dot\alpha}$ to refer to first two or second two components of V_a, and analogously for $\overline{W}^a$.

Conformal algebra in 6D notation We can identify explicitly the conformal generators with the 6D Lorentz algebra

$$M_{\mu\nu} = L_{\mu\nu}, \quad D = L_{45}, \quad P_\mu = L_{5\mu} - L_{4\mu}, \quad K_\mu = -L_{4\mu} - L_{5\mu}. \tag{B.55}$$

With these conventions, the generators L_{MN} satisfy the algebra

$$[L_{MN}, L_{PQ}] = h_{NP}L_{MQ} - h_{MP}L_{NQ} - h_{NQ}L_{MP} + h_{MQ}L_{NP}. \tag{B.56}$$

These generators act on the 6D primary operators as

$$[L_{MN}, O(X,S,\overline{S})] = \mathfrak{L}_{MN}O(X,S,\overline{S}), \tag{B.57}$$

where the differential 6D generator is defined as

$$\mathfrak{L}_{MN} \equiv -(X_M\partial_N - X_N\partial_M) - S\overline{\Sigma}_{MN}\partial_S - \overline{S}\Sigma_{MN}\partial_{\overline{S}}. \tag{B.58}$$

It is sometimes convenient to work with the conformal generators in $SU(2,2)$ notation

$$L_a{}^b \equiv \left[\Sigma^{MN}\right]_a{}^b L_{MN}, \quad L_{i\,MN} = -\frac{1}{2}L_a{}^b\left[\Sigma_{MN}\right]_b{}^a. \tag{B.59}$$

In this notation the conformal generators obey the commutation relations

$$\left[L_a{}^b, L_c{}^d\right] = 2\delta_c^b\, L_a{}^d - 2\delta_a^d\, L_c{}^b. \tag{B.60}$$

We also have the following action on the primary operators

$$[L_a{}^b, O(X, S, \overline{S})] = \mathfrak{L}_a{}^b\, O(X, S, \overline{S}), \tag{B.61}$$

where $\mathfrak{L}_a{}^c$ is the differential operator associated to the 6D generator $L_a{}^c$ in Hilbert space

$$\mathfrak{L}_a{}^b \equiv -\frac{1}{2}\left[\left(\mathbf{X}\overline{\Sigma}^M\right)_a{}^b \partial_M - \left(\Sigma^M\overline{\mathbf{X}}\right)_a{}^b \partial_M\right] + \frac{1}{2}\delta_a^b\left(S\cdot\partial_S - \overline{S}\cdot\partial_{\overline{S}}\right) - 2\left(S_a\partial_S^b - \overline{S}^b\partial_{\overline{S}\,a}\right). \tag{B.62}$$

Embedding formalism In the embedding formalism the flat 4D space is identified with a particular section of the 6D light cone $X^2 = 0$. Namely, we take the Poincare section $X^+ = 1$, which then implies

$$X^- = -X^\mu X_\mu. \tag{B.63}$$

The 4D coordinates x_μ are identified on this section as

$$x^\mu = X^\mu. \tag{B.64}$$

In particular, on the Poincare section we have

$$X^M\Big|_{\text{Poincare}} = \{x^\mu,\ 1,\ -x^2\}. \tag{B.65}$$

Consider an operator $O^{a_1\ldots a_l}_{b_1\ldots b_{\bar{l}}}(X)$, defined on the light cone $X^2 = 0$, symmetric in its two sets of indices. Following [54], it can be projected down to a 4D operator $O^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x)$ as

$$O^{\dot\beta_1\ldots\dot\beta_{\bar\ell}}_{\alpha_1\ldots\alpha_\ell}(x) = \mathbf{X}_{\alpha_1 a_1}\ldots\mathbf{X}_{\alpha_\ell a_\ell}\overline{\mathbf{X}}^{\dot\beta_1 b_1}\ldots\overline{\mathbf{X}}^{\dot\beta_{\bar\ell} b_{\bar\ell}}\, O^{a_1\ldots a_\ell}_{b_1\ldots b_{\bar\ell}}(X)\Big|_{\text{Poincare}}. \tag{B.66}$$

If the 6D operator satisfies the homogeneity property

$$O^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(\lambda X) = \lambda^{-\kappa_O} O^{a_1\ldots a_l}_{b_1\ldots b_{\bar l}}(X), \tag{B.67}$$

where κ_O is defined in (3.13), then the resulting 4D operator will transform as a primary operator of dimension Δ_O under conformal transformations. We call O a 6D uplift of O.

Notice that the 6D uplift O is not uniquely defined. Indeed as a consequence of the light cone condition in terms of the matrices in (B.50),

$$X^2 = 0 \quad\Longrightarrow\quad {}_a(\mathbf{X}\overline{\mathbf{X}})^b = 0 \quad\text{and}\quad {}^a(\overline{\mathbf{X}}\mathbf{X})_b = 0, \tag{B.68}$$

the 6D operator is defined up to terms which vanish in (B.66), leading to the following equivalence relation

$$O^{a_1\dots a_\ell}_{b_1\dots b_{\bar\ell}} \sim O^{a_1\dots a_\ell}_{b_1\dots b_{\bar\ell}} + \overline{\mathbf{X}}^{a_1\, c} A^{a_2\dots a_\ell}_{c\, b_1\dots b_{\bar\ell}} + \mathbf{X}_{b_1 c}\, B^{c\, a_1\dots a_\ell}_{b_2\dots b_{\bar\ell}} + \delta^{a_1}_{b_1}\, C^{a_2\dots a_\ell}_{b_2\dots b_{\bar\ell}}. \tag{B.69}$$

Furthermore, in order to simplify the treatment of derivatives in the embedding space, it is convenient to arbitrarily extend $O(X)$ away from the light cone $X^2 = 0$ and treat all the extensions as equivalent. This means that we can also add to $O(X)$ terms proportional to X^2. Following the terminology of [61], we refer to this possibility as a gauge freedom and the terms proportional to $\mathbf{X}_{ab}$,$\overline{\mathbf{X}}^{ab}$, δ^a_b or X^2 will be called pure gauge terms.

It is convenient to use the index-free notation (3.58). Contracting the 4D auxiliary spinors with (B.66), we find that

$$O(x, s, \bar{s}) = O(X, S, \overline{S})\Big|_{proj}\,, \tag{B.70}$$

where we introduced the formal operation $|_{proj}$ defined as

$$X^M\Big|_{proj} \equiv X^M\Big|_{\text{Poincare}}, \qquad S_a\Big|_{proj} \equiv s^\alpha \mathbf{X}_{\alpha a}\Big|_{\text{Poincare}}, \qquad \overline{S}^a\Big|_{proj} \equiv \bar{s}_{\dot\beta}\overline{\mathbf{X}}^{\dot\beta b}\Big|_{\text{Poincare}}. \tag{B.71}$$

As a consequence of the gauge freedom, the index-free 6D uplift $O(X, S, \overline{S})$ is defined up to pure gauge terms proportional to $S\overline{\mathbf{X}}$, $\overline{S}\mathbf{X}$, $\overline{S}S$ or X^2. Note that they all vanish under the operation of projection (B.70) due to (B.68)

$$\overline{\mathbf{X}}^{ab} S_b\Big|_{proj} = 0, \quad \overline{S}^b \mathbf{X}_{ba}\Big|_{proj} = 0, \quad \overline{S}^a S_a\Big|_{proj} = 0, \quad X^2\Big|_{proj} = 0, \tag{B.72}$$

We will always work modulo the gauge terms (B.72). In practice this is taken into account by treating (B.72) as explicit relations in the embedding formalism even before the projection. Note then that as a consequence of the relations (B.68), (B.72), the anti-symmetric properties (B.49) and the relations (A.7) in appendix A of [55], the following identities hold[6] which we call the 6D Jacobi identities

$$S_{[a}\mathbf{X}_{bc]} = 0, \quad \overline{S}^{[a}\overline{\mathbf{X}}^{bc]} = 0, \quad \mathbf{X}_{[ab}\mathbf{X}_{c]d} = 0, \quad \overline{\mathbf{X}}^{[ab}\overline{\mathbf{X}}^{c]d} = 0. \tag{B.73}$$

Differential operators In section 3.2 we commented upon the importance of some differential operators, such as the conservation operator (B.142), spinning

[6]We thank Emtinan Elkhidir for showing this simple derivation.

differential operators (3.91), (3.92) and the Casimir operators entering (3.45). To consistently define these operators in embedding space, we require their action to be insensitive to different extensions of fields outside the light cone and the other gauge terms in (B.72). This results in the requirement[7]

$$D\left(\frac{\partial}{\partial X^M}, \frac{\partial}{\partial S_a}, \frac{\partial}{\partial \overline{S}^a}, X, S, \overline{S}\right) \cdot O(X^2, S\overline{\mathbf{X}}, \overline{S}\mathbf{X}, \overline{S}S) = O(X^2, S\overline{\mathbf{X}}, \overline{S}\mathbf{X}, \overline{S}S). \quad \text{(B.74)}$$

To go from 6D differential operators to 4D differential operators, we need to find an explicit uplift of the 4D operators $\mathcal{O}(x, s, \overline{s})$ to the 6D operators $O(X, S, \overline{S})$. As noted above, there are infinitely many such uplifts differing by gauge terms, but all lead to the same result for 4D differential operators if the 6D operator satisfies (B.74). For example, we can choose the uplift

$$O(X, S, \overline{S}) = (X^+)^{-\kappa_O}\mathcal{O}(X^\mu/X^+, S_\alpha, \overline{S}_{\dot\alpha}). \quad \text{(B.75)}$$

In particular, X^-, $S^{\dot\beta}$, $\overline{S}^{\beta}$ derivatives of this uplift of $\mathcal{O}$ vanish. By applying 6D derivatives to this expression we automatically obtain the required 4D derivatives on the right hand side. For instance, we find for the first order derivatives after the 4D projection

$$\partial/\partial X^M\Big|_{proj} = \{\partial/\partial x^\mu, -\kappa_O - x^\nu\partial/\partial x^\nu, 0\}\,, \quad \text{(B.76)}$$

$$\partial/\partial S_a\Big|_{proj} = \{\partial/\partial s_\alpha, 0\}\,, \qquad \partial/\partial \overline{S}^a\Big|_{proj} = \{0, \partial/\partial \overline{s}_{\dot\alpha}\}\,. \quad \text{(B.77)}$$

Reality properties of the basic invariants Using the reality properties (B.48) of the sigma matrices, the projection rules (B.71) for S and $\overline{S}$, and the reality convention for 4D auxiliary polarizations $s_\alpha = (\overline{s}_{\dot\alpha})^*$, we can find the following reality properties for the basic objects hold

$$(\mathbf{X}_{ab})^* = \overline{\mathbf{X}}_{\overline{a}\overline{b}}, \quad \left(\overline{\mathbf{X}}^{ab}\right)^* = \mathbf{X}^{\overline{a}\overline{b}}, \quad (S_a)^* = i\overline{S}_{\overline{a}}, \quad (\overline{S}^a)^* = iS^{\overline{a}}. \quad \text{(B.78)}$$

Due to the relations such as $Y^a W_a = Y_{\overline{a}} W^{\overline{a}}$, we have an extremely simple conjugation rule for the expressions such as $\left(\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_l\right)$: replace $\mathbf{X} \leftrightarrow \overline{\mathbf{X}}$, $S \leftrightarrow \overline{S}$ and add a factor of i for each S and $\overline{S}$.

Action of space parity To analyze space parity, let us denote by P_N^M the 6x6 matrix which relfects the spacial components of X^μ. We also denote by $\hat{a}$ indices transforming in the representation reflected relative to the one of a.[8] Note that the

[7]In this equation O stands for the usual big-O notation and not the 6D operator.

[8]The reflected representation is the representation with the Lorentz generators $\mathcal{M}^{\text{refl}}_{MN}$ given by $\mathcal{M}^{\text{refl}}_{MN} = P_M^{M'} P_N^{N'} \mathcal{M}_{M'N'}$, where $\mathcal{M}$ are the original generators.

reflection of the fundamental representation is equivalent to anti-fundamental and vice versa and this equivalence should be implemented by some matrices $p^{\hat{a}b}$ and $p_{\hat{a}b}$. In terms of these matrices we then have

$$P_N^M \Sigma^N_{ab} = \overline{\Sigma}^N_{\hat{a}\hat{b}} = p_{\hat{a}a'} p_{\hat{b}b'} \overline{\Sigma}^{M\,a'b'}, \tag{B.79}$$

$$P_N^M \overline{\Sigma}^{N\,ab} = \Sigma^{N\,\hat{a}\hat{b}} = p^{\hat{a}a'} p^{\hat{b}b'} \Sigma^M_{a'b'}. \tag{B.80}$$

It is easy to check that these identities (as well as the equivalence between the representations) are achieved by choosing

$$p^{\hat{a}b} = p^{b\hat{a}} = -p_{\hat{a}b} = -p_{b\hat{a}} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}_{ab} . \tag{B.81}$$

From the above we deduce the action of parity on on $\mathbf{X}$ and $\overline{\mathbf{X}}$

$$\mathbf{X}_{ab} \mapsto \overline{\mathbf{X}}_{\hat{a}\hat{b}}, \quad \overline{\mathbf{X}}^{ab} \mapsto \mathbf{X}^{\hat{a}\hat{b}}. \tag{B.82}$$

We can also check, based on 4D projections of S and $\overline{S}$, that

$$S_a \mapsto -\overline{S}_{\hat{a}}, \quad \overline{S}^a \mapsto S^{\hat{a}}. \tag{B.83}$$

Due to the identities such as $Y^a W_a = Y_{\hat{a}} W^{\hat{a}}$, we have the following parity conjugation rule for the products like $\left(\overline{S}_i \mathbf{X}_j \overline{\mathbf{X}}_k S_l\right)$: replace $\mathbf{X} \leftrightarrow \overline{\mathbf{X}}$, $S \leftrightarrow \overline{S}$ and a factor of -1 for each S in the original expression.

Action of time reversal As discussed in appendix B.1, see equation (B.36), the time reversal transformation can be implemented by combining the space parity with complex conjugation. Using the above rule, $\mathcal{T}$ acts simply as a multiplication by $i^{\sum_i \ell_i - \bar{\ell}_i}$ on each structure.

B.3 Normalization of two-point functions and seed CPWs

In this appendix our goal is to fix the normalization constants of 2-point functions (3.16) and the seed CPWs (3.44).

The phase of 2-point functions is constrained by unitarity. A simple manifestation of the unitarity is the requirement that all the states in a theory have non-negative norms

$$\langle \Psi | \Psi \rangle \geq 0. \tag{B.84}$$

Our strategy is to define a state whose norm is related to 2-point functions (3.15) and use this relation to fix the phase (3.16). In particular, we set

$$|\mathcal{O}(s,\bar{s})\rangle \equiv \mathcal{O}(x_0,s,\bar{s})\,|0\rangle, \quad x_0^\mu \equiv \{i\epsilon,0,0,0\}, \tag{B.85}$$

where $\epsilon > 0$. Here we are working in the standard Lorentzian quantization where the states are defined on spacelike hyperplanes. The state $|\mathcal{O}(s,\bar{s})\rangle$ can then be interpreted as a NS-quantization state in a Euclidean CFT [18]. Note that we have

$$|\mathcal{O}(s,\bar{s})\rangle = e^{-\epsilon H}\mathcal{O}(0,s,\bar{s})|0\rangle. \tag{B.86}$$

Here $H = -iP_0$ is the Hamiltonian[9] of the theory, and thus its spectrum is bounded from below. Therefore, we need $\epsilon > 0$ in order for $|\mathcal{O}(s,\bar{s})\rangle$ to have a finite norm. To compute this norm, we first consider the conjugate state

$$\langle\mathcal{O}(s,\bar{s})| = \langle 0|(\mathcal{O}(x_0,s,\bar{s}))^\dagger = \langle 0|\overline{\mathcal{O}}(-x_0,s,\bar{s}), \tag{B.87}$$

where we used $x_0^* = -x_0$. Then the norm is given by

$$\langle\mathcal{O}(s,\bar{s})|\mathcal{O}(s,\bar{s})\rangle = \langle 0|\overline{\mathcal{O}}(-x_0,s,\bar{s})\mathcal{O}(x_0,s,\bar{s})|0\rangle. \tag{B.88}$$

By using (3.15) to further rewrite (B.88), with the invariants x_{12}^2, $\mathbb{I}^{21}$ and $\mathbb{I}^{12}$ taking the form

$$x_{12}^2 = 4\epsilon^2, \quad \mathbb{I}^{21} = 2i\epsilon\ s^\dagger s, \quad \mathbb{I}^{12} = -2i\epsilon\ s^\dagger s, \tag{B.89}$$

we find

$$\langle 0|\overline{\mathcal{O}}(-x_0,s,\bar{s})\mathcal{O}(x_0,s,\bar{s})|0\rangle = c_{\langle\overline{\mathcal{O}}\mathcal{O}\rangle}(2\epsilon)^{-2\Delta}(s^\dagger s)^{\ell+\bar{\ell}}i^{\bar{\ell}-\ell} \geq 0, \tag{B.90}$$

where $s^\dagger s = |s_1|^2 + |s_2|^2 \geq 0$. This equation fixes the phase of $c_{\langle\overline{\mathcal{O}}\mathcal{O}\rangle}$, and we can consistently set

$$c_{\langle\overline{\mathcal{O}}\mathcal{O}\rangle} = i^{\ell-\bar{\ell}}. \tag{B.91}$$

Normalization of seed CPWs One can find the leading OPE behavior of the seed and the dual seed conformal blocks by taking the limit $z,\bar{z} \to 0$, $z \sim \bar{z}$, of the solutions obtained in [58]. In particular, for the seed blocks we find

$$\lim_{z,\bar{z}\to 0} H_e^{(p)} = c_{0,-p}^p\,\frac{(-2)^{e-p}\,p!\,(p-e+1)_e}{e!\,(\ell+1)_p}\,(z\bar{z})^{\frac{\Delta+e-p/2}{2}}\,C_{\ell-p+e}^{(p+1)}\left(\frac{z+\bar{z}}{2\,(z\bar{z})^{1/2}}\right), \tag{B.92}$$

[9] Recall that in our conventions P is anti-Hermitian.

and for the dual seed blocks

$$\lim_{z,\bar{z}\to 0} \overline{H}_e^{(p)} = (-2)^p\, \bar{c}^{\,p}_{0,-p}\, \frac{(-2)^{e-p}\, p!\,(p-e+1)_e}{e!\,(\ell+1)_p}\, (z\bar{z})^{\frac{\Delta+e-p/2}{2}}\, C^{(p+1)}_{\ell-e}\left(\frac{z+\bar{z}}{2\,(z\bar{z})^{1/2}}\right), \tag{B.93}$$

where $C_j^{(\nu)}(x)$ are the Gegenbauer polynomials, which in the limit $0 < z \ll \bar{z} \ll 1$ read as

$$C_s^{(p+1)}\left(\frac{z+\bar{z}}{2\,(z\bar{z})^{1/2}}\right) \approx \frac{(p+1)_s}{s!}\, z^{-\frac{s}{2}}\, \bar{z}^{\frac{s}{2}}. \tag{B.94}$$

In the equations above $c^p_{0,-p}$ and $\bar{c}^{\,p}_{0,-p}$ are some overall normalization coefficients defined in [58]. The purpose of this paragraph is to find the values of these coefficients appropriate for our conventions for 2- and 3-point functions.

In order to fix these coefficients, it suffices to consider the leading term in the s-channel OPE in the seed 4-point functions. We have checked that the OPE exactly reproduces the form of (B.92) and (B.93) if one sets

$$c^p_{0,-p} = 2^p\, \bar{c}^{\,p}_{0,-p} = (-1)^\ell\, i^p. \tag{B.95}$$

Let us stress that this normalization factor is fixed by the convention (3.15) and (3.16) for the 2-point functions, and the definitions of the seed 3-point functions. The seed 3-point tensor structures are defined as

$$\langle \mathcal{F}_1^{(0,0)}(\mathbf{p}_1)\, \mathcal{F}_2^{(p,0)}(\mathbf{p}_2)\, \mathcal{O}_\Delta^{(\ell,\,\ell+p)}(\mathbf{p}_3)\rangle = [\hat{\mathbb{I}}^{32}]^p\, [\hat{\mathbb{J}}^3_{12}]^\ell \mathcal{K}_3, \tag{B.96}$$

$$\langle \overline{\mathcal{O}}_\Delta^{(\ell+p,\,\ell)}(\mathbf{p}_2)\, \mathcal{F}_3^{(0,0)}(\mathbf{p}_3)\, \mathcal{F}_4^{(0,p)}(\mathbf{p}_4)\,\rangle = [\hat{\mathbb{I}}^{42}]^p\, [\hat{\mathbb{J}}^2_{34}]^\ell \mathcal{K}_3, \tag{B.97}$$

and the dual seed 3-point functions are defined as

$$\langle \mathcal{F}_1^{(0,0)}(\mathbf{p}_1)\, \mathcal{F}_2^{(p,0)}(\mathbf{p}_2)\, \overline{\mathcal{O}}_\Delta^{(\ell+p,\,\ell)}(\mathbf{p}_3)\rangle = [\hat{\mathbb{K}}^{23}_1]^p\, [\hat{\mathbb{J}}^3_{12}]^\ell \mathcal{K}_3, \tag{B.98}$$

$$\langle \mathcal{O}_\Delta^{(\ell,\,\ell+p)}(\mathbf{p}_2)\, \mathcal{F}_3^{(0,0)}(\mathbf{p}_3)\, \mathcal{F}_4^{(0,p)}(\mathbf{p}_4)\,\rangle = [\hat{\overline{\mathbb{K}}}{}^{24}_3]^p\, [\hat{\mathbb{J}}^2_{34}]^\ell \mathcal{K}_3, \tag{B.99}$$

where in each equation $\mathcal{K}_3$ has to be replaced with the appropriate 3-point kinematic factor as defined in (3.18).

Equation (B.95) can be derived from these three-point functions and the corresponding leading OPE terms

$$\begin{aligned} &\mathcal{F}_1^{(0,0)}(0)\mathcal{F}_2^{(p,0)}(x_2,s_2) \\ &= \frac{(-i)^p}{\ell!(\ell+p)!}|x_2|^{\Delta-\Delta_1-\Delta_2-\ell}(s_2\partial_s)^p(x_2^\mu\partial_s\sigma_\mu\partial_{\bar{s}})^\ell\,\overline{\mathcal{O}}_\Delta^{(\ell+p,\ell)}(0,s,\bar{s}) + \ldots, \end{aligned} \tag{B.100}$$

$$\begin{aligned} &\mathcal{F}_1^{(0,0)}(0)\mathcal{F}_2^{(p,0)}(x_2,s_2) \\ &= \frac{i^p}{\ell!(\ell+p)!}|x_2|^{\Delta-\Delta_1-\Delta_2-\ell-p}(x_2^\mu s_2\sigma_\mu\partial_{\bar{s}})^p(x_2^\mu\partial_s\sigma_\mu\partial_{\bar{s}})^\ell\,\mathcal{O}_\Delta^{(\ell,\ell+p)}(0,s,\bar{s}) + \ldots, \end{aligned} \tag{B.101}$$

where we have defined

$$(\partial_s)^{\alpha} \equiv \frac{\partial}{\partial s_{\alpha}}, \quad (\partial_{\bar{s}})^{\dot{\alpha}} \equiv \frac{\partial}{\partial \bar{s}_{\dot{\alpha}}}. \tag{B.102}$$

The normalization coefficients in these OPEs can be computed by substituting the OPEs into (B.96) and (B.98) and using the two-point function (3.16). The normalization coefficients for the CPWs are then obtained by using these OPEs in the seed four-point function

$$\langle \mathcal{F}_1^{(0,0)}\, \mathcal{F}_2^{(p,0)}\, \mathcal{F}_3^{(0,0)}\, \mathcal{F}_4^{(0,p)} \rangle \tag{B.103}$$

and utilizing the 3-point function definitions (B.97) and (B.99). In practice, when comparing the normalization coefficients, we found it convenient to use the conformal frame (3.95) - (3.98) in the limit $0 < z \ll \bar{z} \ll 1$ and further set $\eta_2 = 0$ and $e = p$ for the seed CPWs or $\xi_2 = 0$ and $e = 0$ for the dual seed CPWs.

B.4 4D form of basic tensor invariants

Here we provide the form of basic tensor invariants in 4D for $n \leq 4$ point functions. They are obtained by applying the projection operation (B.71) to the basic 6D tensor invariants constructed in section 3.3.1

$$(\hat{\mathbb{I}}^{ij},\ \hat{\mathbb{I}}^{ij}_{kl},\ \hat{\mathbb{J}}^{k}_{ij},\ \hat{\mathbb{K}}^{ij}_{k},\ \hat{\overline{\mathbb{K}}}{}^{ij}_{k},\ \hat{\mathcal{L}}^{i}_{jkl},\ \hat{\overline{\mathcal{L}}}{}^{i}_{jkl}) \equiv (\hat{I}^{ij},\ \hat{I}^{ij}_{kl},\ J^{k}_{ij},\ K^{ij}_{k},\ \overline{K}^{ij}_{k},\ L^{i}_{jkl},\ \overline{L}^{i}_{jkl})\Big|_{proj}, \tag{B.104}$$

where

$$\hat{\mathbb{I}}^{ij} = x^{\mu}_{ij}\,(\bar{s}_i \bar{\sigma}_{\mu} s_j), \tag{B.105}$$

$$\hat{\mathbb{I}}^{ij}_{kl} = \frac{1}{2\,x^2_{kl}} \times \Big((x^2_{ik} x^{\mu}_{jl} - x^2_{il} x^{\mu}_{jk}) + (x^2_{jk} x^{\mu}_{il} - x^2_{jl} x^{\mu}_{ik}) - x^2_{ij} x^{\mu}_{kl} - x^2_{kl} x^{\mu}_{ij} - 2i\epsilon^{\mu\nu\rho\sigma} x_{ik\,\nu} x_{lj\,\rho} x_{lk\,\sigma} \Big) \times (\bar{s}_i \bar{\sigma}_{\mu} s_j), \tag{B.106}$$

$$\hat{\mathbb{J}}^{k}_{ij} = \frac{x^2_{ik} x^2_{jk}}{x^2_{ij}} \times \left(\frac{x^{\mu}_{ik}}{x^2_{ik}} - \frac{x^{\mu}_{jk}}{x^2_{jk}} \right) \times (\bar{s}_k \bar{\sigma}_{\mu} s_k), \tag{B.107}$$

$$\hat{\mathbb{K}}^{ij}_{k} = \frac{1}{2} \frac{|x_{ij}|}{|x_{ik}||x_{jk}|} \times \Big((x^2_{ik} + x^2_{jk} - x^2_{ij})(s_i s_j) - 4 x^{\mu}_{ik} x^{\nu}_{jk}\,(s_i \sigma_{\mu\nu} s_j) \Big), \tag{B.108}$$

$$\hat{\overline{\mathbb{K}}}{}^{ij}_{k} = \frac{1}{2} \frac{|x_{ij}|}{|x_{ik}||x_{jk}|} \times \Big((x^2_{ik} + x^2_{jk} - x^2_{ij})(\bar{s}_i \bar{s}_j) - 4 x^{\mu}_{ik} x^{\nu}_{jk}\,(\bar{s}_i \bar{\sigma}_{\mu\nu} \bar{s}_j) \Big), \tag{B.109}$$

$$\hat{\mathcal{L}}^{i}_{jkl} = \frac{2}{|x_{jk}||x_{kl}||x_{lj}|} \times \Big(x^2_{ij} x^{\mu}_{kl} x^{\nu}_{il} + x^2_{ik} x^{\mu}_{lj} x^{\nu}_{ij} + x^2_{il} x^{\mu}_{jk} x^{\nu}_{ik} \Big) \times \big(s_i \sigma_{\mu\nu} s_i \big), \tag{B.110}$$

$$\hat{\overline{\mathcal{L}}}{}^{i}_{jkl} = \frac{2}{|x_{jk}||x_{kl}||x_{lj}|} \times \Big(x^2_{ij} x^{\mu}_{kl} x^{\nu}_{il} + x^2_{ik} x^{\mu}_{lj} x^{\nu}_{ij} + x^2_{il} x^{\mu}_{jk} x^{\nu}_{ik} \Big) \times \big(\bar{s}_i \bar{\sigma}_{\mu\nu} \bar{s}_i \big). \tag{B.111}$$

We recall that $x_{ij}^{\mu} \equiv x_i^{\mu} - x_j^{\mu}$ and $\epsilon_{0123} = -1$ in our conventions. From these expressions it is possible to derive the conjugation properties of the invariants. They read as follows:

$$\left(\hat{\mathbb{I}}^{ij}\right)^* = -\hat{\mathbb{I}}^{ji}, \qquad \left(\hat{\mathbb{I}}^{ij}_{kl}\right)^* = -\hat{\mathbb{I}}^{ji}_{lk}, \qquad \left(\hat{\mathbb{J}}^{k}_{ij}\right)^* = \hat{\mathbb{J}}^{k}_{ij}, \tag{B.112}$$

$$\left(\hat{\mathbb{K}}^{ij}_{k}\right)^* = -\hat{\overline{\mathbb{K}}}{}^{ij}_{k}, \qquad \left(\hat{\mathcal{L}}^{i}_{jkl}\right)^* = -\hat{\overline{\mathcal{L}}}{}^{i}_{jkl}. \tag{B.113}$$

Their parity transformation can be deduced from (B.26)

$$\mathcal{P}\,\hat{\mathbb{I}}^{ij} = -\hat{\mathbb{I}}^{ji}, \qquad \mathcal{P}\,\hat{\mathbb{I}}^{ij}_{kl} = -\hat{\mathbb{I}}^{ji}_{lk}, \qquad \mathcal{P}\,\hat{\mathbb{J}}^{k}_{ij} = \hat{\mathbb{J}}^{k}_{ij}, \tag{B.114}$$

$$\mathcal{P}\,\hat{\mathbb{K}}^{ij}_{k} = \hat{\overline{\mathbb{K}}}{}^{ij}_{k}, \qquad \mathcal{P}\,\hat{\mathcal{L}}^{i}_{jkl} = \hat{\overline{\mathcal{L}}}{}^{i}_{jkl}. \tag{B.115}$$

Finally, according to (B.36) one gets transformations under time reversal

$$\mathcal{T}\,\hat{\mathbb{I}}^{ij} = \hat{\mathbb{I}}^{ij}, \qquad \mathcal{T}\,\hat{\mathbb{I}}^{ij}_{kl} = \hat{\mathbb{I}}^{ij}_{kl}, \qquad \mathcal{T}\,\hat{\mathbb{J}}^{k}_{ij} = \hat{\mathbb{J}}^{k}_{ij}, \tag{B.116}$$

$$\mathcal{T}\,\hat{\mathbb{K}}^{ij}_{k} = -\hat{\mathbb{K}}^{ij}_{k}, \qquad \mathcal{T}\,\hat{\mathcal{L}}^{i}_{jkl} = -\hat{\mathcal{L}}^{i}_{jkl}. \tag{B.117}$$

The same properties follow from the discussion of $\mathcal{P}$-, $\mathcal{T}$-symmetries, and conjugation in appendix B.2.

B.5 Covariant bases of three-point tensor structures

Let us review the construction `[n3ListStructures]` of 3-point function tensor structures [55]. According to the discussion below (3.88) one has

$$\hat{\mathbb{T}}_3^a = \left\{ \prod_{i\neq j} \left[\hat{\mathbb{I}}^{ij}\right]^{m_{ij}} \times \prod_{i,\,j<k} \left[\hat{\mathbb{J}}^{i}_{jk}\right]^{n_i} \left[\hat{\mathbb{K}}^{jk}_{i}\right]^{k_i} \left[\hat{\overline{\mathbb{K}}}{}^{jk}_{i}\right]^{\overline{k}_i} \right\}, \tag{B.118}$$

where the exponents satisfy the following system

$$\ell_i = \sum_{l\neq i} m_{li} + \sum_{l\neq i} k_l + n_i, \tag{B.119}$$

$$\overline{\ell}_i = \sum_{l\neq i} m_{il} + \sum_{l\neq i} \overline{k}_l + n_i. \tag{B.120}$$

Let us also define the quantity

$$\Delta\ell \equiv \sum_i (\ell_i - \overline{\ell}_i). \tag{B.121}$$

Due to relations among products of invariants, not all the structures obtained this way are independent and constraints on possible values of the exponents in (B.118) must be imposed. Theses relations come from the Jacobi identities (B.73) by contracting them with 6D polarizations and 6D coordinate matrices in all possible ways.

The first set of relations reads

$$\widehat{\mathbb{K}}^{ik}_{j}\widehat{\overline{\mathbb{K}}}^{jk}_{i} = -\,\widehat{\mathbb{I}}^{ki}\widehat{\mathbb{I}}^{jk} - \widehat{\mathbb{I}}^{ji}\widehat{\mathbb{J}}^{k}_{ij}\,, \tag{B.122}$$

$$\widehat{\mathbb{K}}^{ij}_{k}\widehat{\overline{\mathbb{K}}}^{ij}_{k} = \widehat{\mathbb{I}}^{ij}\widehat{\mathbb{I}}^{ji} - \widehat{\mathbb{J}}^{jk}_{i}\widehat{\mathbb{J}}^{ik}_{j}\,. \tag{B.123}$$

If $\Delta\ell \neq 0$ we use these relations to set $\overline{k}_i = 0$ or $k_i = 0$ for $\forall\, i$ in the expression (B.118); if $\Delta\ell = 0$ we set instead $k_i = \overline{k}_i = 0 \;\; \forall\, i$.

The second set of relations reads

$$\widehat{\mathbb{J}}^{j}_{ik}\widehat{\mathbb{K}}^{ik}_{j} = \widehat{\mathbb{I}}^{ji}\widehat{\mathbb{K}}^{jk}_{i} - \widehat{\mathbb{I}}^{jk}\widehat{\mathbb{K}}^{ij}_{k}\,, \tag{B.124}$$

$$\widehat{\mathbb{J}}^{j}_{ik}\widehat{\overline{\mathbb{K}}}^{ik}_{j} = \widehat{\mathbb{I}}^{ij}\widehat{\overline{\mathbb{K}}}^{kj}_{i} + \widehat{\mathbb{I}}^{kj}\widehat{\overline{\mathbb{K}}}^{ij}_{k}\,. \tag{B.125}$$

This allows to set either $n_i = 0$ or $k_i = 0$ if $\Delta\ell > 0$ and either $n_i = 0$ or $\overline{k}_i = 0$ if $\Delta\ell < 0$ in (B.118).

If $\Delta\ell = 0$ it might seem that the relations (B.124) and (B.125) do not play any role, since all K and $\overline{K}$ are removed by mean of (B.122) and (B.123). However it is not the case, by combining (B.124) and (B.125) with (B.122) and (B.123) one gets a third order relation

$$\widehat{\mathbb{J}}^{1}_{23}\widehat{\mathbb{J}}^{2}_{13}\widehat{\mathbb{J}}^{3}_{12} = \left(\widehat{\mathbb{I}}^{23}\widehat{\mathbb{I}}^{32}\widehat{\mathbb{J}}^{1}_{23} - \widehat{\mathbb{I}}^{13}\widehat{\mathbb{I}}^{31}\widehat{\mathbb{J}}^{2}_{13} + \widehat{\mathbb{I}}^{12}\widehat{\mathbb{I}}^{21}\widehat{\mathbb{J}}^{3}_{12}\right) - \left(\widehat{\mathbb{I}}^{21}\widehat{\mathbb{I}}^{13}\widehat{\mathbb{I}}^{32} - \widehat{\mathbb{I}}^{12}\widehat{\mathbb{I}}^{31}\widehat{\mathbb{I}}^{23}\right). \tag{B.126}$$

This allows to set in (B.118) either $n_1 = 0$ or $n_2 = 0$ or $n_3 = 0$ when $\Delta\ell = 0$[10]. It can be verified that no other independent relations exist.

In the case when all operators are trace-less symmetric, i.e. $\ell_i = \overline{\ell}_i$ for each field, it is convenient to work in terms of structures manifestly even or odd under parity. Following [62], the most general parity definite tensor structure reads as

$$\widehat{\mathbb{T}}^{a}_{3} = \left\{\left(\widehat{\mathbb{I}}^{21}\widehat{\mathbb{I}}^{13}\widehat{\mathbb{I}}^{32} + \widehat{\mathbb{I}}^{12}\widehat{\mathbb{I}}^{31}\widehat{\mathbb{I}}^{23}\right)^{p} \times \prod_{i,j}\left(\widehat{\mathbb{I}}^{ij}\widehat{\mathbb{I}}^{ji}\right)^{m_{ij}} \times \prod_{i,\,j<k}\left[\widehat{\mathbb{J}}^{i}_{jk}\right]^{n_i}\right\}, \tag{B.127}$$

where the structure is even if $p = 0$ and the structure is odd if $p = 1$. The form of this basis is structurally identical to the one found in [53]. This basis has extremely simple properties under complex conjugation, parity and time reversal

$$\left(\widehat{\mathbb{T}}^{a}_{3}\right)^{*} = (-1)^{p}\,\widehat{\mathbb{T}}^{a}_{3}, \quad \mathcal{P}\,\widehat{\mathbb{T}}^{a}_{3} = (-1)^{p}\,\widehat{\mathbb{T}}^{a}_{3}, \quad \mathcal{T}\,\widehat{\mathbb{T}}^{a}_{3} = \widehat{\mathbb{T}}^{a}_{3}. \tag{B.128}$$

This basis can be constructed using `[n3ListStructuresAlternativeTS]`.

[10] Notice that for $\Delta\ell \neq 0$ at least one n_i is always 0 and hence (B.126) does not give new constraints.

B.6 Casimir differential operators

The Lie algebra of the 4D conformal group is a real form of the simple rank-3 algebra $\mathfrak{so}(6)$. Therefore, it has three independent Casimir operators, which can be defined using the 6D Lorentz generators (B.57) as follows:

$$C_2 \equiv \frac{1}{2} L_{MN}\, L^{NM}, \tag{B.129}$$

$$C_3 \equiv \frac{1}{24i}\, \epsilon^{MNPQRS}\, L_{MN}\, L_{PQ}\, L_{RS}, \tag{B.130}$$

$$C_4 \equiv \frac{1}{2} L_{MN}\, L^{NP}\, L_{PQ}\, L^{QM}, \tag{B.131}$$

where $\epsilon^{012345} = \epsilon_{012345} = +1$.

To write out the Casimir eigenvalues for primary operators, it is convenient to introduce also the $SO(1,3)$ Casimir operators using the 4D Lorentz generator (B.17). There are two such Casimirs

$$c_2^+ \equiv -\frac{1}{2} L_{\mu\nu} L^{\mu\nu}, \quad c_2^- \equiv \frac{1}{4i} \epsilon^{\mu\nu\rho\sigma} L_{\mu\nu} L_{\rho\sigma}, \tag{B.132}$$

with the eigenvalues

$$e_2^+ = \frac{1}{2}\ell(\ell+2) + \frac{1}{2}\bar{\ell}(\bar{\ell}+2), \quad e_2^- = \frac{1}{2}\ell(\ell+2) - \frac{1}{2}\bar{\ell}(\bar{\ell}+2). \tag{B.133}$$

The conformal Casimir eigenvalues are then given by

$$E_2 \equiv \Delta(\Delta-4) + e_2^+, \tag{B.134}$$

$$E_3 \equiv \left(\Delta - 2\right) e_2^-, \tag{B.135}$$

$$E_4 \equiv \Delta^2(\Delta-4)^2 + 6\,\Delta(\Delta-4) + \left(e_2^+\right)^2 - \frac{1}{2}\left(e_2^-\right)^2. \tag{B.136}$$

Note that c_2^- is parity-odd and therefore e_2^- changes the sign under $\ell \leftrightarrow \bar{\ell}$. The same comment applies to C_3 and E_3.

It is convenient to write the Casimir Operators in the $SU(2,2)$ language by plugging (B.59) into the expression (B.129), (B.130), and (B.131)

$$C_2 = \frac{1}{4}\, \mathrm{tr}\, L^2, \tag{B.137}$$

$$C_3 = \frac{1}{12}\left(\mathrm{tr}\, L^3 - 16\, C_2\right), \tag{B.138}$$

$$C_4 = -\frac{1}{8}\left(\mathrm{tr}\, L^4 - 8\ \mathrm{tr}\, L^3 - 12\, C_2^2 + 16\, C_2\right). \tag{B.139}$$

Let us emphasize that the Casimir operators C_n are the Hilbert space operators. Their differential form $\mathfrak{C}_n$ can be obtained by replacing the Hilbert space operators L_{MN} and $L_a{}^c$ with their differential representations $\mathfrak{L}_{MN}$ and $\mathfrak{L}_a{}^c$ given in (B.58) and (B.62) together with reverting[11] the order of operators L_{MN} and $L_a{}^c$ in equations (B.129) - (B.131) and (B.137) - (B.139).

B.7 Conserved operators

By conserved operators we mean primary operators in short representations of the conformal group, i.e. those possessing null descendants and thus satisfying differential equations. In a unitary 4D CFT all local primary operators satisfy the unitarity bounds [24, 42][12]

$$\Delta \geq 1 + \frac{\ell + \bar{\ell}}{2}, \quad \ell = 0 \textbf{ or } \bar{\ell} = 0, \tag{B.140}$$

$$\Delta \geq 2 + \frac{\ell + \bar{\ell}}{2}, \quad \ell \neq 0 \textbf{ and } \bar{\ell} \neq 0, \tag{B.141}$$

and unitary null states can only appear when these bounds are saturated.

The operators of the type $\ell = 0$ **or** $\bar{\ell} = 0$ with $\Delta = 1 + (\ell + \bar{\ell})/2$ satisfy the free wave equation[13] $\partial^2 O_\Delta^{(\ell,\bar{\ell})} = 0$ [294], which immediately implies that such operators can only come from a free subsector of the CFT. The operators of the second type, $\ell\bar{\ell} \neq 0$, $\Delta = 2 + (\ell + \bar{\ell})/2$, are the conserved currents which satisfy the following operator equation[14]

$$\partial \cdot O_\Delta^{(\ell,\bar{\ell})}(x, s, \bar{s}) = 0, \qquad \partial \equiv (\epsilon\sigma^\mu)^\alpha_{\ \dot{\beta}}\, \partial_\mu \frac{\partial^2}{\partial s^\alpha\, \partial \bar{s}_{\dot{\beta}}}. \tag{B.142}$$

Of particular importance are the spin-1 currents J^μ in representation $(1, 1)$, the stress tensor $T^{\mu\nu}$ in representation $(2, 2)$ and the supercurrents J^μ_α and $J^\mu_{\dot{\alpha}}$ in representations $(2, 1)$ and $(1, 2)$. Note that an appearance of traceless symmetric higher-spin currents is known to imply an existence of a free subsector [295, 296].

The conservation condition results in the following Ward identity for n-point functions

$$\partial \cdot \langle \ldots O_\Delta^{(\ell,\bar{\ell})}(x, s, \bar{s}) \ldots \rangle = 0 + \text{contact terms}, \tag{B.143}$$

[11] See the discussion below (B.20).

[12] An operator with $\ell = \bar{\ell} = 0$ has an extra option $\Delta = 0$. This is the identity operator.

[13] This is not the conformally-invariant differential equation satisfied by these operators, but rather its consequence.

[14] The operator ∂ can be applied in the conformal frame `[opConservation4D]` or in the embedding formalism `[opConservationEF]`.

where the contact terms encode charges of operators under the symmetry generated by the conserved current $O_{\Delta}^{(\ell,\bar{\ell})}$. Note that since $\partial\cdot O_{\Delta}^{(\ell,\bar{\ell})}$ is itself a primary operator in representation $(\ell-1,\bar{\ell}-1)$, $\Delta = 3+(\ell+\bar{\ell})/2$, the left hand side of the above equation has the transformation properties of a correlation function of primary operators and thus can be expanded in a basis of appropriate tensor structures.

For 3-point functions, the Ward identities imply two kind of constraints. First, the validity of (B.143) at generic configurations of points x_i implies homogeneous linear relations between the OPE coefficients entering 3-point functions. Second, the validity of (B.143) at coincident points relates some of the OPE coefficients to the charges of the other two operators in a given 3-point function (this happens only if special relations between scaling dimensions of these operators are satisfied). The solution of these constraints is of the form (3.22), where some of $\hat{\lambda}$ can be related to the charges.

For 4-point functions the situation is more complicated, since (B.143) at non-coincident points leads to a system of first order differential equations for the functions $g_4^I(u,v)$ of the form

$$B^{AJ}(u,v,\partial_u,\partial_v)\; g_4^J(u,v) = 0, \tag{B.144}$$

where A runs through the number of tensor structures for the correlator in the left hand side of (B.143). The constraints implied by these equations were analysed in [75]. It turns out that one can solve these equations by aribtrarily specifying a smaller number N_4' of the functions $g_4^I(u,v)$ and a number of boundary conditions for the remaining $g_4^I(u,v)$.[15] It is generally important to take this into account when formulating an independent set of crossing symmetry equations. We refer the reader to [75] for details. In [75] the value N_4' was found for 4 identical conserved spin 1 and spin 2 operators. The same values N_4' were found later by other means in [62] and a general counting rule was proposed in [1].

Conservation operator in the embedding formalism The conservation condition (B.142) can be consistently reformulated in the embedding space [opConservationEF] as follows:

$$D\; O_{\Delta_O}^{(\ell,\bar{\ell})}\left(X,S,\bar{S}\right) = 0, \quad \Delta_O = 2+\frac{\ell+\bar{\ell}}{2} \tag{B.145}$$

[15] DK thanks Anatoly Dymarsky, João Penedones, and Alessandro Vichi for discussions on this issue.

and the differential operator originally found in [55] is given by[16]

$$D \equiv \frac{2}{\ell\,\bar{\ell}\,\left(2+\ell+\bar{\ell}\right)} \left(X_M \Sigma^{MN} \partial_N\right)_a^{\ b}\ \partial_b^{\ a}, \tag{B.146}$$

where we have defined

$$\partial_b^{\ a} \equiv \frac{1}{1+\ell+\bar{\ell}}\,\partial^a \partial_b =$$
$$\left(4 + S\cdot\frac{\partial}{\partial S} + \bar{S}\cdot\frac{\partial}{\partial \bar{S}}\right)\frac{\partial}{\partial S_a}\frac{\partial}{\partial \bar{S}^b} - S_b \frac{\partial}{\partial S_a}\frac{\partial^2}{\partial S\cdot\partial\bar{S}} - \bar{S}^a \frac{\partial}{\partial \bar{S}^b}\frac{\partial^2}{\partial S\cdot\partial\bar{S}}. \tag{B.147}$$

In this identity we dropped the terms which project to zero upon contraction with $\left(X_M \Sigma^{MN} \partial_N\right)_a^{\ b}$.

B.8 Permutations symmetries

When the points in (3.8) are space-like separated, the ordering of operators is not important up to signs coming from permutations of fermions. In particular, if some operator enters the expectation value more than once, say at points $\mathbf{p}_i$ and $\mathbf{p}_j$, the function f_n enjoys the permutation symmetry

$$f_n(\ldots,\mathbf{p}_i,\ldots,\mathbf{p}_j,\ldots) = [(ij)f_n](\ldots,\mathbf{p}_i,\ldots,\mathbf{p}_j,\ldots) \equiv \pm f_n(\ldots,\mathbf{p}_j,\ldots,\mathbf{p}_i,\ldots). \tag{B.148}$$

Here we used the cycle notation for permutations, for instance (123) denotes $1\to 2$, $2\to 3$, $3\to 1$. In general, there may be more identical operators in the right hand side of (3.8) in which case f_n is invariant under some subgroup of permutations $\Pi \subseteq S_n$.

The degrees of freedom in f_n are described by the functions g_n^I defined via (3.11)

$$f_n(x_i, s_i, \bar{s}_i) = \sum_{I=1}^{N_n} g_n^I(\mathbf{u})\ \mathbb{T}_n^I(x_i, s_i, \bar{s}_i). \tag{B.149}$$

One can then find the implications of the permutation symmetries directly for g_n^I. Note that since the exchanged operators are identical, a permutation $\pi \in \Pi$ acting on a tensor structure gives a tensor structure of the same kind, and thus we can expand it in the same basis

$$\pi \mathbb{T}_n^I = \sum_J \pi_I^J(\mathbf{u}) \mathbb{T}_n^J. \tag{B.150}$$

[16] We note that there is a mistake in the original paper [55] due to a wrong choice of the analogue of (3.62).

This means that in general the consequence of a permutation symmetry is

$$g_n^I(\mathbf{u}) = \sum_J \pi^I_J(\mathbf{u}) g_n^J(\pi\mathbf{u}). \tag{B.151}$$

At this point we should divide all the permutations into two classes. We call the permutations which preserve the cross-rations ($\pi\mathbf{u} = \mathbf{u}$) the kinematic permutations and all the other permutations will be referred to as non-kinematic. The group of kinematic permutations Π_n^{kin} is S_n for $n \leq 3$ since there are no non-trivial cross-ratios in these cases. We also have $\Pi_4^{\text{kin}} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ and Π_n^{kin} is trivial for $n \geq 5$.

This distinction is important because for kinematic permutations the constraint (B.151) becomes a simple local linear constraint,

$$g_n^I(\mathbf{u}) = \sum_J \pi^I_J(\mathbf{u}) g_n^J(\mathbf{u}), \tag{B.152}$$

which we can be solved as

$$g_n^I(\mathbf{u}) = \sum_A P^I_A(\mathbf{u}) \hat{g}_n^A(\mathbf{u}). \tag{B.153}$$

In the case of 3-point functions the solution (B.153) has a particularly simple form (3.22).

Applying permutation `[permutePoints]` and computing $\pi^I_J(\mathbf{u})$ is straightforward in the EF – we simply need to permute the coordinates X_i and the polarizations $S_i, \overline{S}_i$. It is somewhat trickier to figure out the permutations in the CF [1], and we describe the case $n = 4$ in the remainder of this section. We also comment on how to permute non-identical operators, which is required, for example, in order to exchange s- and t-channels.

Semi-covariant CF Structues First, we describe a slight generalization of the conformal frame, which is convenient for computing the action of permutations on the CF structures. Note that the 4-point tensor structures constructed in section 3.4.1.2 are covariant under the conformal transformations acting in z plane. Indeed, it is easy to see that the structures (3.116) transform with 2d spin $q_i + \overline{q}_i$ at each point. Taking into account the scaling dimensions of the operators, we see that we can assign the left- and right-moving weights

$$h_i = \frac{\Delta_i + q_i + \overline{q}_i}{2}, \quad \overline{h}_i = \frac{\Delta_i - q_i - \overline{q}_i}{2} \tag{B.154}$$

to each tensor structure. We can then easily write the value of the 4-point function represented on the conformal frame by

$$f_4(0, z, 1, \infty, s_i, \overline{s}_i) = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix} g_{\{q_i, \overline{q}_i\}}(z, \overline{z}) \tag{B.155}$$

in a generic configuration of the four points z_i in z-plane as `[cfEvaluateInPlane]`

$$f_4(z_1, z_2, z_3, z_4, s_i, \overline{s}_i) = \left[\begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \\ \hline z_1 & z_2 & z_3 & z_4 \end{array}\right] g_{\{q_i, \overline{q}_i\}}(z, \overline{z}), \tag{B.156}$$

where

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \quad \overline{z} = \frac{(\overline{z}_1 - \overline{z}_2)(\overline{z}_3 - \overline{z}_4)}{(\overline{z}_1 - \overline{z}_3)(\overline{z}_2 - \overline{z}_4)}, \tag{B.157}$$

and, defining $z_{ij} = z_i - z_j$,

$$\begin{aligned} \left[\begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \\ \hline z_1 & z_2 & z_3 & z_4 \end{array}\right] = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix} & \times (z_{31}^{-h_1-h_2-h_3+h_4} z_{41}^{-h_1+h_2+h_3-h_4} z_{42}^{-2h_2} z_{43}^{h_1+h_2-h_3-h_4}) \\ & \times (\overline{z}_{31}^{-\overline{h}_1-\overline{h}_2-\overline{h}_3+\overline{h}_4} \overline{z}_{41}^{-\overline{h}_1+\overline{h}_2+\overline{h}_3-\overline{h}_4} \overline{z}_{42}^{-2\overline{h}_2} \overline{z}_{43}^{\overline{h}_1+\overline{h}_2-\overline{h}_3-\overline{h}_4}). \end{aligned} \tag{B.158}$$

Note that the definition is chosen in such a way that the semi-covariant structure transforms with the required left and right weights and[17]

$$\left[\begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \\ \hline 0 & z & 1 & \infty \end{array}\right] = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ \overline{q}_1 & \overline{q}_2 & \overline{q}_3 & \overline{q}_4 \end{bmatrix}. \tag{B.159}$$

In general we might need to specify the branches of the fractional powers in (B.158). The kinematic factor in this equation can be split into products of

$$(z_{ij}\overline{z}_{ij})^{f(\Delta_k)} \quad \text{and} \quad \left(\frac{z_{ij}}{\overline{z}_{ij}}\right)^{\tilde{f}(q_k + \overline{q}_k)}. \tag{B.160}$$

In the region of the configuration space where all pairs of points are spacelike separated[18], we have $z_{ij}\overline{z}_{ij} > 0$, so there is no branching for the factors of the first

[17] Recall that the limit $z_4 = \infty$ is defined with an extra factor $|x_4|^{2\Delta_4}$ in order to obtain a non-zero result.

[18] In particular, in the whole Euclidean region.

kind. The exponent of the factors of the second kind is always half-integral, thus we only need to specify the branch of $\sqrt{\frac{z_{ij}}{\bar{z}_{ij}}}$ which can be chosen

$$\sqrt{\frac{z_{ij}}{\bar{z}_{ij}}} = \sqrt{\frac{z_{ij}^2}{z_{ij}\bar{z}_{ij}}} = \frac{z_{ij}}{\sqrt{z_{ij}\bar{z}_{ij}}}. \tag{B.161}$$

This is valid because it gives a smooth choice for the whole spacelike region and reduces the kinematic factor to 1 in the standard configuration $\{z_1, z_2, z_3, z_4\} = \{0, 1, z, \infty\}$.

The above discussion gives a version of the CF 4-point tensors structures which is defined for any configuration of the four points in the z-plane. This is sufficient for computing the action of arbitrary permutations on the tensor structures (3.116). Explicit formulas for permutations between identical operators can be found in [1]. General permutations are implemented in `CFTs4D` package in the function `[permutePoints]`.

Appendix C

APPENDICES TO CHAPTER 4

C.1 Conformal algebra

We use the following conventions for the conformal algebra,

$$[D, K_\mu] = -K_\mu, \quad [D, P_\mu] = P_\mu, \tag{C.1}$$

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu} D - 2M_{\mu\nu}, \tag{C.2}$$

$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu, \tag{C.3}$$

$$[M_{\mu\nu}, K_\rho] = \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu, \tag{C.4}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\rho\mu} - \delta_{\mu\sigma} M_{\rho\nu}, \tag{C.5}$$

and all other commutators vanish. In Lorentzian signature, all generators are anti-Hermitian. In Euclidean signature $D = D^\dagger$, $K = P^\dagger$ and M is anti-Hermitian. Notice how (C.2) expresses the conformal Killing equation for the adjoint representation by saying that the rank-2 symmetric traceless tensor does not appear among level-1 descendants of the primary K_μ.

C.2 Verma modules and differential operators

In the main text we have seen that for every irreducible component $V_{\Delta',\rho}$ in the tensor product $W \otimes V_{\Delta,\rho}$ there is a conformally-covariant differential operator $\mathcal{D}_A : [\Delta, \rho] \to [\Delta', \lambda]$ with a W^*-index A. Here we would like to state this relation more carefully and show that there is in fact a one-to-one correspondence.

Theorem 3. *For generic Δ the decomposition (4.13) holds. The irreducible components in the tensor product decomposition (4.13) are in one-to-one correspondence with the conformally-covariant differential operators $\mathcal{D}_A : [\Delta, \rho] \to [\Delta', \lambda]$ with an index A transforming in a finite-dimensional representation W of* $\mathrm{SO}(d+1, 1)$.

Proof. First we show that the tensor product decomposition (4.13) holds. The discussion in section 4.2.2 essentially shows that the characters on the both sides agree. This statement holds for all Δ. This however does not necessarily imply (4.13) as an isomorphism between the representations. So our first step is to construct the isomorphism (4.13).

We can define on $W \otimes V_{\Delta,\rho}$ a conformally-invariant inner product, induced from the inner products on W and $V_{\Delta,\rho}$. Suppose that there is a submodule $Y \subseteq W \otimes V_{\Delta,\rho}$. If the conformally-invariant inner product is non-degenerate[1] on Y, it follows that Y is in fact a direct summand,

$$M \equiv W \otimes V_{\Delta,\rho} = Y \oplus Y^{\perp}. \tag{C.6}$$

Starting from this observation, it is a standard argument to show that (4.13) holds. We reproduce it here for completeness. The states (4.10) are always primary because they have the smallest possible scaling dimension $\Delta - j$. We can decompose them into mutually orthogonal irreducibles of $\mathrm{SO}(d)$. Considering all the descendants of these states we form the submodule

$$Y_{-j} = \bigoplus_{\lambda \in W_{-j} \otimes \rho} V_{\Delta-j,\lambda}. \tag{C.7}$$

For generic Δ the generalized Verma modules in this sum are irreducible, and thus the inner product is non-degenerate (otherwise the null states form a submodule). By (C.6) we then have

$$M = Y_{-j} \oplus M_1, \quad M_1 \equiv Y_{-j}^{\perp}. \tag{C.8}$$

We can now look at the states of the smallest scaling dimension inside of M_0. These all are again primary, and we can consider the submodule Y_{-j+1} which they generate. Since we already know (4.13) as a character identity, we know that

$$Y_{-j+1} = \bigoplus_{\lambda \in W_{-j+1} \otimes \rho} V_{\Delta-j+1,\lambda}. \tag{C.9}$$

Again, from (C.6) we find

$$M_1 = Y_{-j+1} \oplus M_2. \tag{C.10}$$

We then continue recursively until we exhaust all states as controlled by (4.13) as a character identity. Collecting everything together, we arrive at (4.13) as a direct sum decomposition.

From the discussion in the main text it follows that the primaries which we identify in the tensor product $W \otimes V_{\Delta,\rho}$ give rise to conformally-covariant differential operators. At the same time, as observed in section 4.2.4, they give rise to homomorphisms (4.29). In fact, there is a one-to-one correspondence between these objects.

[1] Note that if the inner-product is non-degenerate but not positive-definite, there still can exist subspaces on which it is degenerate. Finite-dimensional representations of non-compact groups such as $\mathrm{SO}(d+1,1)$ or $\mathrm{SO}(d,2)$ necessarily have indefinite inner products.

Lemma 3. *For any fixed $\Delta, \Delta', \rho, \lambda$ the conformally-covariant differential operators $\mathcal{D}_A : [\Delta, \rho] \to [\Delta', \lambda]$ are in a one-to-one correspondence with the homomorphisms of the form* (4.29).

The map implied by this lemma is essentially constructed in section 4.2.3. Looking at it one can easily convince oneself that the lemma is almost a tautology. We give a formal proof later in this appendix.

Given lemma 3, to finish the proof of the theorem it only remains to show that generically the only homomorphisms of the form (4.29) are those which come from the embeddings of the direct summands in (4.13). This follows immediately from Schur's lemma and the fact that Verma modules are irreducible for generic scaling dimensions. □

Proof of lemma 3. For $W = \bullet$, lemma 3 is standard material in representation theory of generalized Verma modules [297], and we need to only slightly modify it by introducing the non-trivial W. Let us give an elementary review of the proof with the appropriate modifications.

First, we need to give the precise meaning to $[\Delta, \rho]$, which is in fact a vector bundle. The sections of $[\Delta, \rho]$ are the functions $f^a(x)$ on the conformal sphere S^d with index a in ρ which transform as[2]

$$(gf)^a(x) = \Omega(x)^{-\Delta} \rho^a{}_b(R(x)) f^b(g^{-1}x), \quad g \in \mathrm{SO}(d+1, 1). \tag{C.11}$$

We also associate a vector bundle $\mathcal{W}$ to W. The sections of $\mathcal{W}$ are the functions f_A which transform as

$$(gf)_A(x) = D_A{}^B(g) f_B(g^{-1}x). \tag{C.12}$$

The conformally-covariant differential operator $\mathcal{D}_A$ is then a differential operator between the vector bundles

$$\mathcal{D} : [\rho, \lambda] \to \mathcal{W} \otimes [\Delta', \lambda], \tag{C.13}$$

which commutes with the action of the conformal group. We will refer to this property as equivariance. The idea now is to note that if we know that $\mathcal{D}$ is equivariant, then it is completely specified by its action at zero, i.e. by the expression

$$(\mathcal{D}f)^a_A(0) = \text{derivatives of } f \text{ at } 0. \tag{C.14}$$

[2]The difference with (4.8) comes from the fact that here we are defining the action on functions rather than operators, and the appearance of g^{-1} in the argument of f on the right hand side is dictated by compatibility with the group multiplication $(gh)f = g(h(f))$.

Indeed, let t_x be the translation which takes 0 to x. Then we can compute $\mathcal{D}f$ at any x by writing

$$(\mathcal{D}f)^a_A(x) = (t_x\mathcal{D}t_{-x}f)^a_A(x) = D_A{}^B(t_x)(\mathcal{D}t_{-x}f)^a_B(0), \tag{C.15}$$

and using (C.14) for $t_{-x}f$. As usual, the only condition the expression (C.14) has to satisfy in order for this construction to be self-consistent is that it has to be equivariant with respect to the transformations which fix the origin – in our case with respect to dilatations, rotations and special conformal transformations, the algebra of which we will denote by $\mathfrak{p}$.[3]

Instead of studying this condition in detail, we can just map it to the similar problem for Verma modules. If $\mathcal{D}$ is of order k, the equation (C.14) can be understood as the map

$$\mathcal{D} : J^k_0[\Delta, \rho] \to W \otimes J^0_0[\Delta', \lambda], \tag{C.16}$$

where $J^k_0[\Delta, \rho]$ is the space of k-jets of sections of $[\Delta, \rho]$ at 0, i.e. the space of formal power series of sections of $[\Delta, \rho]$ around the origin, truncated to k-th order. One can extend the action of conformal algebra to these jets, and the problem of finding a $\mathfrak{p}$-equivariant map (C.14) is equivalent to finding $\mathfrak{p}$-equivariant maps (C.16). Using (C.15) we can extend such maps to $\mathfrak{so}(d+1,1)$-equivariant maps

$$\mathcal{D} : J^\infty_0[\Delta, \rho] \to W \otimes J^\infty_0[\Delta', \lambda], \tag{C.17}$$

between the formal power series. These are the same as Verma module homomorphisms because $V_{\Delta,\rho}$ consists of the states like $\partial_{\mu_1}\cdots\partial_{\mu_n}\mathcal{O}^a(0)$, which are naturally linear functionals on the formal power series $J^\infty_0[\Delta, \rho]$. In fact, one can show that as $\mathfrak{so}(d+1,1)$-representations,

$$V_{\Delta,\rho} \simeq \left(J^\infty_0[\Delta, \rho]\right)^*. \tag{C.18}$$

Thus by taking the dual of (C.17) we obtain a homomorhism

$$\mathcal{D}^* : W^* \otimes V_{\Delta',\lambda} \to V_{\Delta,\rho}. \tag{C.19}$$

As usual, we can replace W^* on the left with a W on the right: we can define $\mathcal{D}'(v) = e^A \otimes \mathcal{D}^*(e^*_A \otimes v)$, so that

$$\mathcal{D}' : V_{\Delta',\lambda} \to W \otimes V_{\Delta,\rho} \tag{C.20}$$

is a homomorphism of the form (4.29). All the steps that we took to get from the differential operator $\mathcal{D}_A$ to $\mathcal{D}'$ were invertible, so we get a one-to-one correspondence. □

[3]This is not to be confused with the subalgebra generated by translations. We use this notation to be consistent with the mathematics literature, where $\mathfrak{p}$ stands for "parabolic".

C.3 Weight-shifting operators for the vector representation

Let us give more detail about the computation of the weight-shifting operators for the vector representation (4.45). Recall that traceless symmetric tensor operators are homogeneous elements of $R/(R \cap I)$, where R is the ring of functions of $X, Z \in \mathbb{R}^{d+1,1}$ that are invariant under $Z \to Z + \lambda X$ (equivalently they are killed by $X \cdot \frac{\partial}{\partial Z}$), and I is the ideal generated by $\{X^2, X \cdot Z, Z^2\}$. For a differential operator $\mathcal{D}$ to be well-defined on $R/(R \cap I)$, it must satisfy

$$\mathcal{D}R \subseteq R, \tag{C.21}$$

$$\mathcal{D}(R \cap I) \subseteq R \cap I. \tag{C.22}$$

Because we are searching for homogeneous differential operators, it suffices to consider their action on homogeneous elements of R. It is not hard to convince oneself that a general homogeneous element of R can be written as a linear combination of functions of the form

$$f_{\Delta,\ell}(X,Z) \equiv (X \cdot Y)^{-\Delta-\ell}((Z \cdot P)(X \cdot Q) - (Z \cdot Q)(X \cdot P))^{\ell}, \tag{C.23}$$

for various Y, P, Q.

To find the weight-shifting operators $\mathcal{D}_m^{(a)}$, we start by enumerating conformally-covariant terms with the correct homogeneity in X and Z, modulo $X \cdot \frac{\partial}{\partial X}$, $Z \cdot \frac{\partial}{\partial Z}$, and $X \cdot \frac{\partial}{\partial Z}$ (which act as $-\Delta$, ℓ, and 0, respectively). There are a finite number of such terms, and this leads to the ansatz (4.45) with undetermined coefficients that are functions of Δ, ℓ.

To fix the coefficients, it suffices to check (C.21) and (C.22) for a sufficient number of functions. In particular, we impose (C.21) in the form

$$\left(X \cdot \frac{\partial}{\partial Z}\right) \mathcal{D}_m^{(a)} f_{\Delta,\ell}(X,Z) = 0, \tag{C.24}$$

and (C.22) in the form

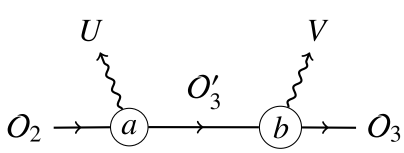
$$\begin{aligned}
\mathcal{D}_m^{(a)}((S \cdot Z)^2(X \cdot X) - 2(S \cdot X)(S \cdot Z)(X \cdot Z) + (S \cdot X)^2(Z \cdot Z))f_{\Delta+2,\ell-2}(X,Z) &\in R \cap I, \\
\mathcal{D}_m^{(a)}((X \cdot X)(Z \cdot Z) - (X \cdot Z)^2)f_{\Delta+2,\ell-2}(X,Z) &\in R \cap I, \\
\mathcal{D}_m^{(a)}((X \cdot X)(Z \cdot S) - (X \cdot S)(X \cdot Z))f_{\Delta+2,\ell-1}(X,Z) &\in R \cap I, \\
\mathcal{D}_m^{(a)}(X \cdot X)f_{\Delta+2,\ell}(X,Z) &\in R \cap I,
\end{aligned} \tag{C.25}$$

where $S, Y, P, Q \in \mathbb{R}^{d+1,1}$ are arbitrary vectors. (Because of (C.21), to check whether the left hand sides of (C.25) are in $R \cap I$, it suffices to check whether they are in I. That is, we set $X^2, X \cdot Z, Z^2$ to zero and check whether the result is zero.) These conditions are sufficient to fix the unknown coefficients. In particular, for the most complicated weight-shiting operator $\mathcal{D}_m^{+0}$, we find

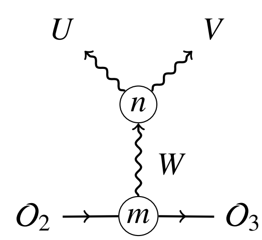
$$\begin{aligned}
c_1 &= \left(\frac{d}{2} - \Delta - 1\right)(\Delta + \ell - 1)(d - \Delta + \ell - 2) \\
c_2 &= -\frac{1}{2}(\Delta + \ell - 1)(d - \Delta + \ell - 2) \\
c_3 &= -\left(\frac{d}{2} - \Delta - 1\right)(\Delta + \ell - 2) \\
c_4 &= -\left(\frac{d}{2} - \Delta - 1\right)(d - \Delta + \ell - 2) \\
c_5 &= \frac{d}{2} + \ell - 2 \\
c_6 &= \frac{d}{2} - \Delta - 1 \\
c_7 &= -\frac{1}{2}.
\end{aligned} \tag{C.26}$$

C.4 $6j$ symbols and the algebra of operators

In this appendix we consider the crossing equation which is obtained by replacing $\mathcal{O}_1$ in (4.102) by a finite-dimensional representation,[4]



$$= \sum_{W,m,n} \begin{Bmatrix} U & \mathcal{O}_2 & W \\ \mathcal{O}_3 & V & \mathcal{O}_3' \end{Bmatrix}^{ab}_{mn}$$



.

(C.27)

Here, the sum is over $W \in U \otimes V$. Since restricting $\mathcal{O}_1$ to a finite-dimensional representation changes the counting of structures on both sides, we should check that the numbers still agree. Let us assume that $\Delta_3 = \Delta_2 - l$. According to theorem 2, the number of structures on the left is

$$\sum_{i+k=l} \dim(\rho_2^* \otimes U_i \otimes V_k \otimes \rho_3)^{\mathrm{SO}(d)}, \tag{C.28}$$

[4] We then find a third finite-dimensional representation arising from the tensor product of the first two.

while the number of structures on the right is

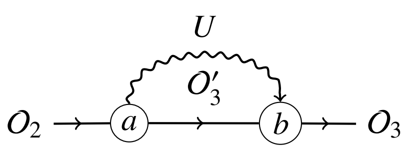
$$\sum_{W \in U \otimes V} \dim(\rho_2^* \otimes W_l \otimes \rho_3)^{\mathrm{SO}(d)} \times \dim(U \otimes V \otimes W^*)^{\mathrm{SO}(d+1,1)}. \tag{C.29}$$

These numbers are the same due to

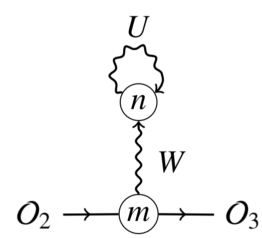
$$\bigoplus_{W \in U \otimes V} \dim(U \otimes V \otimes W^*)^{\mathrm{SO}(d+1,1)} \times W_l \simeq \bigoplus_{i+k=l} U_i \otimes V_k. \tag{C.30}$$

The crossing transformation (C.27) defines the algebra of weight-shifting operators for general ρ_2 and generic Δ_2. In section 4.2.4 we described the same algebra in the situation when both ρ_2 and Δ_2 are generic. As in section 4.2.4, (C.27) essentially expresses the associativity of the tensor product.

As a simple application, suppose that $U = V^*$ and let us contract U and V indices in (C.27) to form the bubble diagram,

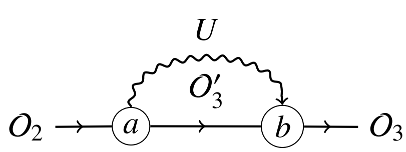


$$= \sum_{W,m,n} \begin{Bmatrix} U & \mathcal{O}_2 & W \\ \mathcal{O}_3 & U^* & \mathcal{O}_3' \end{Bmatrix}^{ab}_{mn}$$



. (C.31)

The tadpole on the right can be non-zero only if $W = \bullet$ is the representation of the identity operator $\mathbf{1}$. But then m exists only if $\Delta_2 = \Delta_3$ and $\rho_2 = \rho_3$. In this case there exists a unique structure for both n and m. We can erase the line for the trivial W, and the U-loop gives a factor of $\dim U$. We thus find

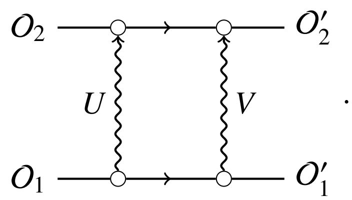


$$= (\dim U) \begin{Bmatrix} U & \mathcal{O}_2 & \mathbf{1} \\ \mathcal{O}_3 & U^* & \mathcal{O}_3' \end{Bmatrix}^{ab}_{\bullet\bullet} \; \mathcal{O}_2 \longrightarrow \mathcal{O}_3 \, . \tag{C.32}$$

We thus conclude

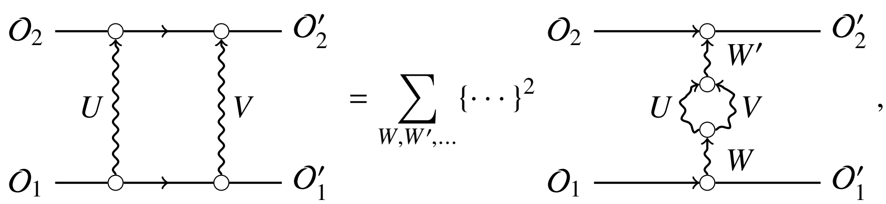
$$\begin{pmatrix} & \mathcal{O}_3 \\ \mathcal{O}_3' & U \end{pmatrix}^{ba} = (\dim U) \begin{Bmatrix} U & \mathcal{O}_3 & \mathbf{1} \\ \mathcal{O}_3 & U^* & \mathcal{O}_3' \end{Bmatrix}^{ab}_{\bullet\bullet}. \tag{C.33}$$

The algebra (C.27) also shows tells us how to compose the two-point operators of [61]. Indeed, suppose we have a composition of two two-point operators, ignoring the operator labels,



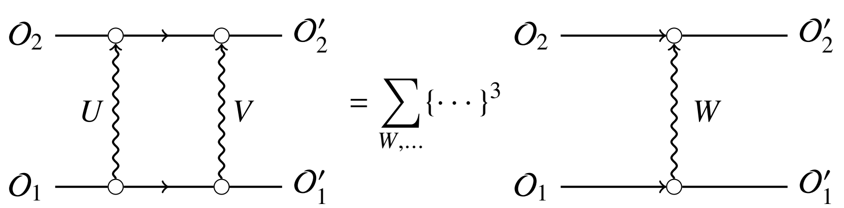
(C.34)

We can apply (C.27) at the top and at the bottom to find, schematically,



(C.35)

where $\{\cdots\}^2$ is a product of two $6j$ symbols. By Schur's lemma, the bubble diagram in the middle can be non-zero only if $W = W'$, in which case it is a scalar. This scalar can be determined from finite-dimensional $6j$ symbols. We thus arrive at



(C.36)

where $\{\cdots\}^3$ are some coefficients involving three $6j$ symbols, and the sum is over $W \in U \otimes V$.

C.5 Seed blocks in 3d

Basis of four-point tensor structures. For the four-point tensor structures we use the conformal frame structures

$$[q_1 q_2 q_3 q_4] \tag{C.37}$$

that we introduced in section 4.3.4.2. It is analogous to the basis used in [1], but we make a different choice of the conformal frame,

$$\begin{aligned}
x_1 &= (0,0,0), \\
x_2 &= (\frac{\bar{z} - z}{2}, 0, \frac{\bar{z} + z}{2}), \\
x_3 &= (0,0,1), \\
x_4 &= (0,0,+\infty).
\end{aligned} \tag{C.38}$$

The configuration used in section 4.3.4.2 corresponds then to $z = \bar{z}$.

In terms of these structures, for parity-even four-point functions (G^{++}_{seed} and G^{--}_{seed}) we use the basis

$$G = g_1(z,\bar{z})\frac{[-\frac{1}{2},0,0,-\frac{1}{2}]+[\frac{1}{2},0,0,\frac{1}{2}]}{2} + g_2(z,\bar{z})\frac{[\frac{1}{2},0,0,-\frac{1}{2}]+[-\frac{1}{2},0,0,\frac{1}{2}]}{2}, \tag{C.39}$$

and for parity-odd four-point functions (G^{+-}_{seed} and G^{-+}_{seed}) we use the basis

$$G = g_1(z,\bar{z})\frac{[-\frac{1}{2},0,0,-\frac{1}{2}]-[\frac{1}{2},0,0,\frac{1}{2}]}{2} + g_2(z,\bar{z})\frac{[\frac{1}{2},0,0,-\frac{1}{2}]-[-\frac{1}{2},0,0,\frac{1}{2}]}{2}. \tag{C.40}$$

We will now provide explicit expressions for $g_i^{\pm\pm}(z,\bar{z})$.

Explicit expressions for $g_i^{\pm\pm}(z,\bar{z})$. First we strip off some normalization factors,

$$\begin{aligned}
g_k^{++}(z,\bar{z}) &= \frac{i(-1)^{\ell-\frac{1}{2}}}{\ell(\Delta-\ell-1)(\Delta-1)}(z\bar{z})^{-\frac{\Delta_1+\Delta_2+\frac{1}{2}}{2}}\mathfrak{D}_i^{++}G^{\alpha-\frac{1}{4},\beta-\frac{1}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}(z,\bar{z}),\\
g_k^{--}(z,\bar{z}) &= \frac{i(-1)^{\ell-\frac{1}{2}}}{(\ell+\frac{1}{2})(\Delta+\ell)(\Delta-1)}(z\bar{z})^{-\frac{\Delta_1+\Delta_2+\frac{1}{2}}{2}}\mathfrak{D}_i^{--}G^{\alpha-\frac{1}{4},\beta-\frac{1}{4}}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}}(z,\bar{z}),\\
g_k^{+-}(z,\bar{z}) &= \frac{i(-1)^{\ell-\frac{1}{2}}}{\ell(\Delta-\ell-1)(\Delta-1)}(z\bar{z})^{-\frac{\Delta_1+\Delta_2+\frac{1}{2}}{2}}\mathfrak{D}_i^{+-}G^{\alpha-\frac{1}{4},\beta+\frac{1}{4}}_{\Delta+\frac{1}{2},\ell-\frac{1}{2}}(z,\bar{z}),\\
g_k^{-+}(z,\bar{z}) &= \frac{i(-1)^{\ell-\frac{1}{2}}}{(\ell+\frac{1}{2})(\Delta+\ell)(\Delta-1)}(z\bar{z})^{-\frac{\Delta_1+\Delta_2+\frac{1}{2}}{2}}\mathfrak{D}_i^{-+}G^{\alpha-\frac{1}{4},\beta+\frac{1}{4}}_{\Delta+\frac{1}{2},\ell+\frac{1}{2}}(z,\bar{z}).
\end{aligned} \tag{C.41}$$

Here, $\alpha = -(\Delta_1-\Delta_2)/2$ and $\beta = (\Delta_3-\Delta_4)/2$, where Δ_i are the dimensions of the external operators in (4.172). To write down the expressions for $\mathfrak{D}_i^{\pm\pm}$, we introduce the following operators,

$$\begin{aligned}
D_z &= z^2(1-z)\partial_z^2 - (\alpha'+\beta'+1)z^2\partial_z - \alpha'\beta' z,\\
d_z &= z\partial_z,\\
\nabla_z &= \frac{1}{z-\bar{z}}d_z(z-\bar{z}) = z\partial_z + \frac{z}{z-\bar{z}},\\
\widetilde{d}_z &= (1-z)d_z - \alpha' z,\\
\widetilde{\nabla}_z &= (1-z)\nabla_z - (\alpha'-1)z,
\end{aligned} \tag{C.42}$$

as well as their conjugates which are obtained by exchanging z and $\bar{z}$. The variables α' and β' in the above formulas are equal to the parameters of the scalar conformal blocks $G^{\alpha',\beta'}_{\Delta',\ell'}(z,\bar{z})$ on which the differential operators act in (C.41).

The differential operators $\mathfrak{D}_i^{\pm\pm}$ are given by[5]

$$\begin{aligned}\mathfrak{D}_1^{\pm\pm} =& d_{\bar{z}}D_z - d_z D_{\bar{z}} - (d_z - d_{\bar{z}})\frac{z\bar{z}}{2(z-\bar{z})}\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big)\\ &+ a^{\pm\pm}(d_z - d_{\bar{z}}) + b^{\pm\pm}(D_z - D_{\bar{z}}),\\ \mathfrak{D}_2^{\pm\pm} =& \nabla_z D_{\bar{z}} + \nabla_{\bar{z}} D_z + (\nabla_z + \nabla_{\bar{z}})\frac{z\bar{z}}{2(z-\bar{z})}\Big((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\Big)\\ &- a^{\pm\pm}(\nabla_z + \nabla_{\bar{z}}) + c^{\pm\pm},\end{aligned} \tag{C.43}$$

where coefficients $a^{\pm\pm}$, $b^{\pm\pm}$ and $c^{\pm\pm}$ are given below and, additionally, *in* $\mathfrak{D}^{+-}$ *and* $\mathfrak{D}^{-+}$ *the operators* d *and* ∇ *need to be replaced by* $\widetilde{d}$ *and* $\widetilde{\nabla}$ *respectively.* We have

$$\begin{aligned}a^{++} &= a^{+-} = \frac{(\Delta-\ell)(\Delta-\ell-3)}{4},\\ b^{++} &= b^{+-} = \frac{\Delta-\ell-3}{2},\\ c^{++} &= c^{+-} = \frac{(2\ell+1)(\Delta-\ell-3)(\Delta-\frac{3}{2})}{4},\end{aligned} \tag{C.44}$$

and the coefficients for parity-odd left structure are obtained by replacing $\ell \to -\ell-1$,

$$\begin{aligned}a^{--} &= a^{-+} = \frac{(\Delta+\ell+1)(\Delta+\ell-2)}{4},\\ b^{--} &= b^{-+} = \frac{\Delta+\ell-2}{2},\\ c^{--} &= c^{-+} = -\frac{(2\ell+1)(\Delta+\ell-2)(\Delta-\frac{3}{2})}{4}.\end{aligned} \tag{C.45}$$

Normalization conventions Our normalization conventions are fixed by our choice of two-point functions (4.178), the scalar-fermion three-point functions (4.173) and the scalar three-point functions (4.175). These conventions agree with [81]. In particular, if the scalar blocks are normalized as

$$G^{\alpha,\beta}_{\Delta,\ell}(z,\bar{z}) \sim \frac{(-1)^\ell (1)_\ell}{(1/2)_\ell}(z\bar{z})^{\Delta/2} P_\ell\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right), \quad z,\bar{z} \ll 1, \tag{C.46}$$

where P_ℓ are Legendre polynomials, then the resulting seed blocks $G^{\pm\pm}_{\text{seed}}$ are normalized as in [81] with their $c_O = 1$. To obtain the blocks at other values of c_O, one should divide our formulas by c_O.

[5] Note again that in order to simplify these expressions we made use of the quadratic Casimir equation satisfied by the scalar conformal blocks.

The coefficients v_i For G^{++} we have

$$
\begin{aligned}
v_1 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta-2\Delta_3-2\Delta_4+2\ell+9)\,(2\Delta+2\Delta_3-2\Delta_4+2\ell-1)}{64(\Delta-1)\,(2\Delta_3-3)\,(2\Delta_4-3)\,\ell(-\Delta+\ell+1)} \\
&\quad\times(-2\Delta-2\Delta_3+2\Delta_4+2\ell+1)\,, \\
v_2 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta-2\Delta_3-2\Delta_4+2\ell+9)\,(-2\Delta+2\Delta_3-2\Delta_4+2\ell+3)}{128(\Delta-1)\,(\Delta_3-1)\,(2\Delta_3-3)\,\ell(-\Delta+\ell+1)} \\
&\quad\times(-2\Delta-2\Delta_3+2\Delta_4+2\ell+3)\,, \\
v_3 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta+2\Delta_3-2\Delta_4+2\ell+3)}{64(\Delta-1)\,(\Delta_3-2)^{\,2}\,(2\Delta_3-3)\,(2\Delta_4-3)\,\ell(-\Delta+\ell+1)}, \\
v_4 &= \frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta-2\Delta_3-2\Delta_4+2\ell+9)\,(-2\Delta+2\Delta_3-2\Delta_4+2\ell+3)}{128(\Delta-1)\,(\Delta_3-2)\,(2\Delta_3-3)\,\ell(-\Delta+\ell+1)} \\
&\quad\times(-2\Delta+2\Delta_3+2\Delta_4+2\ell-3)\,.
\end{aligned}
\tag{C.47}
$$

For G^{--} we have

$$
\begin{aligned}
v_1 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(2\Delta+2\Delta_3-2\Delta_4+2\ell+1)\,(-2\Delta-2\Delta_3+2\Delta_4+2\ell+3)}{32(\Delta-1)\,(2\Delta_3-3)\,(2\Delta_4-3)\,(2\ell+1)(\Delta+\ell)} \\
&\quad\times(2\Delta+2\Delta_3+2\Delta_4+2\ell-7)\,, \\
v_2 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(2\Delta+2\Delta_3-2\Delta_4+2\ell-1)\,(2\Delta-2\Delta_3+2\Delta_4+2\ell-1)}{64(\Delta-1)\,(\Delta_3-1)\,(2\Delta_3-3)\,(2\ell+1)(\Delta+\ell)} \\
&\quad\times(2\Delta+2\Delta_3+2\Delta_4+2\ell-7)\,, \\
v_3 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(2\Delta-2\Delta_3+2\Delta_4+2\ell-1)}{32(\Delta-1)\,(\Delta_3-2)^{\,2}\,(2\Delta_3-3)\,(2\Delta_4-3)\,(2\ell+1)(\Delta+\ell)}, \\
v_4 &= \frac{i(-1)^{\ell-\frac{1}{2}}\,(2\Delta-2\Delta_3-2\Delta_4+2\ell+5)\,(2\Delta-2\Delta_3+2\Delta_4+2\ell-1)}{64(\Delta-1)\,(\Delta_3-2)\,(2\Delta_3-3)\,(2\ell+1)(\Delta+\ell)} \\
&\quad\times(2\Delta+2\Delta_3+2\Delta_4+2\ell-7)\,.
\end{aligned}
\tag{C.48}
$$

For G^{+-} we have

$$
\begin{aligned}
v_1 &= \frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta-2\Delta_3-2\Delta_4+2\ell+9)\,(-2\Delta-2\Delta_3+2\Delta_4+2\ell+3)}{64(\Delta-1)\,(\Delta_3-1)\,(2\Delta_3-3)\,(2\Delta_4-3)\,\ell(-\Delta+\ell+1)}, \\
v_2 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta-2\Delta_3-2\Delta_4+2\ell+7)\,(2\Delta-2\Delta_3-2\Delta_4+2\ell+3)}{128(\Delta-1)\,(2\Delta_3-3)\,\ell(-\Delta+\ell+1)}\times \\
&\qquad\times(-2\Delta-2\Delta_3+2\Delta_4+2\ell+3)\,(2\Delta+2\Delta_3+2\Delta_4+2\ell-7)\,, \\
v_3 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta+2\Delta_3-2\Delta_4+2\ell+3)\,(-2\Delta-2\Delta_3+2\Delta_4+2\ell+3)}{64(\Delta-1)\,(\Delta_3-2)\,(2\Delta_3-3)\,(2\Delta_4-3)\,\ell(-\Delta+\ell+1)}, \\
v_4 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta+2\Delta_3-2\Delta_4+2\ell+1)\,(2\Delta-2\Delta_3+2\Delta_4+2\ell-1)}{128(\Delta-1)\,(\Delta_3-2)^{\,2}\,(2\Delta_3-3)\,\ell(-\Delta+\ell+1)}.
\end{aligned}
\tag{C.49}
$$

For G^{-+} we have

$$
\begin{aligned}
v_1 &= -\frac{i(-1)^{\ell-\frac{1}{2}}\,(2\Delta+2\Delta_3-2\Delta_4+2\ell-1)\,(2\Delta+2\Delta_3+2\Delta_4+2\ell-7)}{32(\Delta-1)\,(\Delta_3-1)\,(2\Delta_3-3)\,(2\Delta_4-3)\,(2\ell+1)(\Delta+\ell)},\\
v_2 &= \frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta-2\Delta_3-2\Delta_4+2\ell+9)\,(2\Delta+2\Delta_3-2\Delta_4+2\ell-1)}{64(\Delta-1)\,(2\Delta_3-3)\,(2\ell+1)(\Delta+\ell)}\times\\
&\qquad\times(-2\Delta+2\Delta_3+2\Delta_4+2\ell-1)\,(2\Delta+2\Delta_3+2\Delta_4+2\ell-5)\,,\\
v_3 &= \frac{i(-1)^{\ell-\frac{1}{2}}\,(2\Delta+2\Delta_3-2\Delta_4+2\ell-1)\,(2\Delta-2\Delta_3+2\Delta_4+2\ell-1)}{32(\Delta-1)\,(\Delta_3-2)\,(2\Delta_3-3)\,(2\Delta_4-3)\,(2\ell+1)(\Delta+\ell)},\\
v_4 &= \frac{i(-1)^{\ell-\frac{1}{2}}\,(-2\Delta+2\Delta_3-2\Delta_4+2\ell+3)\,(2\Delta-2\Delta_3+2\Delta_4+2\ell+1)}{64(\Delta-1)\,(\Delta_3-2)^{\,2}\,(2\Delta_3-3)\,(2\ell+1)(\Delta+\ell)}.
\end{aligned}
\tag{C.50}
$$

C.6 Dual seed blocks in 4d

In this appendix we provide the final expression for the dual seed conformal blocks recursion relation omitting all the derivations. All the quantities below carry a bar to distinguish them from their analogous in the seed case.

By performing the calculation completely analogous to the one in section 4.4.4.2, we find that the dual seed conformal blocks obey the following recursion relation

$$
\begin{aligned}
&\overline{W}^{(p)}_{\Delta,\ell;\,\Delta_1,\Delta_2,\Delta_3,\Delta_4} =\\
&\quad\overline{\mathcal{A}}^{-1}\Bigg(\bar v_1\,(\overline{\mathcal{D}}_1^{--0}\cdot\mathcal{D}_{4,-0+})\,(\overline{\mathcal{D}}_1^{-+0}\cdot\mathcal{D}_{2,++0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1+1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}\\
&\qquad+\bar v_2\,(\overline{\mathcal{D}}_1^{--0}\cdot\mathcal{D}_{4,-0+})\,(\mathcal{D}_{1,++0}\cdot\overline{\mathcal{D}}_2^{-+0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}\\
&\qquad+\bar v_3\,(\overline{\mathcal{D}}_1^{+0-}\cdot\mathcal{D}_{4,-0+})\,(\overline{\mathcal{D}}_1^{+0+}\cdot\mathcal{D}_{2,++0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1-1,\Delta_2-\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}\\
&\qquad+\bar v_4\,(\overline{\mathcal{D}}_1^{+0-}\cdot\mathcal{D}_{4,-0+})\,(\mathcal{D}_{1,-0+}\cdot\overline{\mathcal{D}}_2^{-+0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1,\Delta_2+\frac{1}{2},\Delta_3,\Delta_4+\frac{1}{2}}\Bigg),
\end{aligned}
\tag{C.51}
$$

where the coefficients are[6]

$$
\overline{\mathcal{A}} = -\frac{i\,(\ell+p)(\Delta+\Delta_3-\Delta_4+\ell+p-2)}{2\Delta+2\ell+p-2}
\tag{C.52}
$$

[6]Here $\overline{\mathcal{A}}$ is not the $6j$ symbol analogous to $\mathcal{A}$, but simply an overall coefficient.

and

$$
\begin{aligned}
\overline{v}_1 &= \frac{(\Delta-\Delta_1-\Delta_2+\ell+p+2)(-\Delta-\Delta_1+\Delta_2+\ell+p+2)}{2(\Delta_1-2)(2\Delta+p-4)(2\Delta_2+p-4)},\\
\overline{v}_2 &= -\frac{(\Delta-\Delta_1-\Delta_2+\ell+p+2)(\Delta-\Delta_1+\Delta_2+\ell+2p-2)}{4(\Delta_1-2)(\Delta_1-1)(2\Delta+p-4)},\\
\overline{v}_3 &= -\frac{1}{2(\Delta_1-3)(\Delta_1-2)^2(2\Delta+p-4)(2\Delta_2+p-4)},\\
\overline{v}_4 &= -\frac{(\Delta-\Delta_1-\Delta_2+\ell+p+2)(\Delta+\Delta_1+\Delta_2+\ell+2p-6)}{4(\Delta_1-3)(\Delta_1-2)(2\Delta+p-4)}.
\end{aligned}
\tag{C.53}
$$

Analogously to the primal seed case, we replace one of the conformal blocks on the right hand side of (C.51) by using the dimension-shifting operator

$$
\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1+1,\,\Delta_2-\frac{1}{2},\,\Delta_3,\,\Delta_4+\frac{1}{2}} = \overline{\mathcal{E}}^{-1}\,(\mathcal{D}_{1,+-0}\cdot\overline{\mathcal{D}}_2^{\,--0})(\mathcal{D}_{1,++0}\cdot\overline{\mathcal{D}}_2^{\,-+0})\,\overline{W}^{(p-1)}_{\Delta-\frac{1}{2},\ell;\,\Delta_1,\,\Delta_2+\frac{1}{2},\,\Delta_3,\,\Delta_4+\frac{1}{2}},
\tag{C.54}
$$

where

$$
\overline{\mathcal{E}} \equiv (p+1)(\Delta_1-2)(\Delta_1-1)(\Delta+\Delta_1-\Delta_2+l+p-2)(-\Delta-\Delta_1+\Delta_2+l+p+2).
\tag{C.55}
$$

Decomposition into components Plugging the relation (C.54) in (C.51), stripping off the kinematic factor and decomposing this relation into four-point tensor structures according to (4.213) one obtains a recursion relation for the components of seed blocks of the form analogous to (4.230)

$$
\overline{H}^{(p)}_e(z,\overline{z}) = -\frac{\overline{\mathcal{A}}'^{-1}}{z-\overline{z}}\Big(D_0\,\overline{H}^{(p-1)}_e(z,\overline{z}) - 2D_1\,\overline{H}^{(p-1)}_{e-1}(z,\overline{z}) + 4c^{p-1}_{e-2}z\overline{z}D_2\,\overline{H}^{(p-1)}_{e-2}(z,\overline{z})\Big),
\tag{C.56}
$$

where the blocks in the l.h.s depend on $[\Delta,\ \ell;\ \Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4]$ and the blocks in the r.h.s. depend on $[\Delta-\frac{1}{2},\ \ell;\ \Delta_1,\ \Delta_2+\frac{1}{2},\ \Delta_3,\ \Delta_4+\frac{1}{2}]$.

The overall coefficient is

$$
\overline{\mathcal{A}}' \equiv -(\Delta+\frac{p}{2}-2)(\Delta+\Delta_1-\Delta_2+l+p-2)\,\overline{\mathcal{A}}.
\tag{C.57}
$$

The differential operators D_i are given by the expression (4.231)-(4.233) with the parameter k replaced by $\overline{k}$

$$
\overline{k} \equiv \frac{\Delta+\ell}{2}+\frac{3p}{4}.
\tag{C.58}
$$

C.7 Operators $\mathcal{H}_k$

First, let us define normalized versions of operators (4.249),

$$
\begin{aligned}
\hat{\mathcal{D}}_{13} &= \frac{\mathcal{D}_1^{-0} \cdot \mathcal{D}_3^{+0}}{(\Delta_3 - 1)(d - \Delta_3 - 2)}, \\
\hat{\mathcal{D}}_{24} &= \frac{\mathcal{D}_2^{+0} \cdot \mathcal{D}_4^{-0}}{(\Delta_2 - 1)(d - \Delta_2 - 2)}, \\
\hat{\mathcal{D}}_{23} &= \frac{\mathcal{D}_2^{+0} \cdot \mathcal{D}_3^{+0}}{(\Delta_3 - 1)(d - \Delta_3 - 2)(\Delta_2 - 1)(d - \Delta_2 - 2)}.
\end{aligned}
\tag{C.59}
$$

In terms of these, the operators $\mathcal{H}_k$ have the following expressions,[7]

$$
\begin{aligned}
\mathcal{H}_1 &= \frac{\hat{\mathcal{D}}_{13} - \hat{\mathcal{D}}_{24}}{\Delta_{12}^+ - \Delta_{34}^+} + \frac{1}{4}(\Delta_{12}^+ + \Delta_{34}^+ - 2\varepsilon)(x\bar{x})^{-\frac{1}{2}}, \\
\mathcal{H}_2 &= \frac{\hat{\mathcal{D}}_{13} + \hat{\mathcal{D}}_{24}}{2} + \frac{\Delta_{12}^+ + \Delta_{34}^+}{2}\mathcal{H}_1 - \frac{\Delta_{12}^+(\Delta_{12}^+ - 2\varepsilon) + \Delta_{34}^+(\Delta_{34}^+ - 2\varepsilon)}{8}(x\bar{x})^{-\frac{1}{2}}, \\
\mathcal{H}_3 &= \frac{2\hat{\mathcal{D}}_{23}}{\Delta_{12}^+ + \Delta_{34}^+ - 4\varepsilon - 2} - (\Delta_{12}^+ + \Delta_{34}^+ - 2(\varepsilon - 1))\mathcal{H}_2 + (2c_2 + \Delta_{12}^+\Delta_{34}^+)\mathcal{H}_1 + \kappa_0(x\bar{x})^{-\frac{1}{2}},
\end{aligned}
\tag{C.60}
$$

where we defined[8]

$$
\Delta_{ij}^+ = \Delta_i + \Delta_j, \quad \varepsilon = \frac{d-2}{2}, \tag{C.61}
$$

and the Casimir eigenvalues and the coefficient κ_0 are given by

$$
\begin{aligned}
c_2 &= \lambda_1(\lambda_1 - 1) + \lambda_2(\lambda_2 - 2\epsilon - 1), \\
c_4 &= (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2 + 2\varepsilon)(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 - 1 - 2\varepsilon), \\
\kappa_0 &= \frac{(\Delta_{12}^+ - 2\varepsilon)(\Delta_{34}^+ - 2\varepsilon)(\Delta_{12}^+\Delta_{34}^+ + 4c_2) - 4(c_4 - c_2(c_2 + 2\varepsilon))}{4(\Delta_{12}^+ + \Delta_{34}^+ - 4\varepsilon - 2)}.
\end{aligned}
\tag{C.62}
$$

Note that the identity for the operator $\mathcal{H}_3$ is valid up to quadratic and quartic Casimir equations (i.e. only when acting on scalar conformal blocks $F_{\lambda_1\lambda_2}(a, b)$).

[7] Here and below in this section we sometimes abuse the notation by using the same symbols for the embedding-space differential operators and their action on $F_{\lambda_1\lambda_2}$.

[8] Notice that in [64] ε is defined as here, whereas in the earlier work [57] the definition $\varepsilon = d - 2$ was used.

Appendix D

APPENDICES TO CHAPTER 5

D.1 Conformal algebra and its representation on local operators

Here we describe our conventions for the conformal algebra. The commutation relations are as follows,

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \tag{D.1}$$

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu} D + 2M_{\mu\nu}, \tag{D.2}$$

$$[M_{\mu\nu}, P_\rho] = \delta_{\mu\rho} P_\nu - \delta_{\nu\rho} P_\mu, \quad [M_{\mu\nu}, K_\rho] = \delta_{\mu\rho} K_\nu - \delta_{\nu\rho} K_\mu, \tag{D.3}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\mu\rho} M_{\nu\sigma} - \delta_{\nu\rho} M_{\mu\sigma} + \delta_{\mu\sigma} M_{\rho\nu} - \delta_{\nu\sigma} M_{\rho\mu}. \tag{D.4}$$

The generators obey the following conjugation properties,

$$D^\dagger = D, \quad P^\dagger = K, \quad M_{\mu\nu}^\dagger = -M_{\mu\nu}. \tag{D.5}$$

The generators act on primary operators as follows,

$$[D, \mathcal{O}(x)] = x \cdot \partial \mathcal{O}(x) + \Delta \mathcal{O}(x), \tag{D.6}$$

$$[P_\mu, \mathcal{O}(x)] = \partial_\mu \mathcal{O}(x), \tag{D.7}$$

$$[M_{\mu\nu}, \mathcal{O}(x)] = (x_\mu \partial_\nu - x_\nu \partial_\mu) \mathcal{O}(x) + S_{\mu\nu} \mathcal{O}(x), \tag{D.8}$$

$$[K_\mu, \mathcal{O}(x)] = (2x_\mu x^\sigma - x^2 \delta_\mu^\sigma) \partial_\sigma \mathcal{O}(x) + 2x^\sigma (\Delta \delta_{\mu\sigma} + S_{\mu\sigma}) \mathcal{O}(x). \tag{D.9}$$

Here $S_{\mu\nu}$ are the generators which act on the spin indices of $\mathcal{O}(x)$ and satisfy the commutation relations opposite to $M_{\mu\nu}$. Our convention for $M_{\mu\nu}$ differs by a minus sign from that of [19]. $M_{\mu\nu}$ in our case has the interpretation of rotating e_μ towards e_ν.

D.2 Reduced matrix elements and vector isoscalar factors

In order to write down the formulas for isoscalar factors and reduced matrix elements, we need to take some preliminary steps. First, let us consider the decomposition of the tensor product $\mathbf{m}_d \otimes \square$. Generically, we have in even dimensions, according to Brauer's formula,

$$\mathbf{m}_d \otimes \square \simeq \bigoplus_{i=1}^{n} \mathbf{m}_d(+i) \oplus \mathbf{m}_d(-i), \qquad d = 2n, \tag{D.10}$$

where $\mathbf{m}_d(\pm i)$ is the same as $\mathbf{m}_d$ but with the component $m_{d,i}$ shifted by ± 1. Similarly, in odd dimensions we have, generically,

$$\mathbf{m}_d \otimes \square \simeq \mathbf{m}_d \oplus \bigoplus_{i=1}^{n} \mathbf{m}_d(+i) \oplus \mathbf{m}_d(-i), \qquad d = 2n+1. \tag{D.11}$$

These formulas are valid for generic $\mathbf{m}_d$, i.e. those with all components non-zero and sufficiently large. For concrete representations, some of the direct summands may disappear if there are non-dominant weights in the right hand side. By applying Brauer's formula, we see that to find the final tensor product rule we just need to remove all non-dominant weights and, if $d = 2n+1$ and $m_{d,n} = 0$, also remove $\mathbf{m}_d$.[1]

We now define the following new parameters,

$$x_{2n+1,j} = m_{2n+1,j} + n - j, \tag{D.12}$$

$$x_{2n,j} = m_{2n,j} + n - j. \tag{D.13}$$

Note that regardless of the dimension, $m_{d,j}$ is a non-increasing function of j. Since we add to it a strictly decreasing function of j, we find that $x_{d,j}$ is a strictly decreasing function of j. In particular $x_{d,j} \neq x_{d,i}$ for $i \neq j$. Furthermore, $x_{d,j} > 0$ except possibly for $j = n$ when it can be zero (for $d = 2n+1$) or negative (for $d = 2n$). We can also easily check that $|x_{d,j}|$ is strictly decreasing and thus in fact $x_{d,j} \neq \pm x_{d,i}$ for $i \neq j$.

In terms of these parameters the dimensions of the representations $\mathbf{m}_d$ have the following expressions

$$\dim \mathbf{m}_{2n} = \prod_{1\leq i<j\leq n} \frac{(x_{2n,i}+x_{2n,j})(x_{2n,i}-x_{2n,j})}{(y_{2n,i}+y_{2n,j})(y_{2n,i}-y_{2n,j})}, \tag{D.14}$$

$$\dim \mathbf{m}_{2n+1} = \prod_{i=1}^{n} \frac{x_{2n+1,i}+\frac{1}{2}}{y_{2n+1,i}+\frac{1}{2}} \prod_{1\leq i<j\leq n} \frac{(x_{2n+1,i}+x_{2n+1,j}+1)(x_{2n+1,i}-x_{2n+1,j})}{(y_{2n+1,i}+y_{2n+1,j}+1)(y_{2n+1,i}-y_{2n+1,j})}, \tag{D.15}$$

where $y_{d,k} = n - k$ is $x_{d,k}$ for the trivial representation (so that $\dim \bullet = 1$).

[1]This can be seen by analyzing the situations in which $\mathbf{m}_d(\pm i)$ may fail to be dominant. It turns out that in most cases there is an affine Weyl reflection which stabilizes the non-dominant $\mathbf{m}_d(\pm i)$ and thus such weights simply have to be removed. The exception is the case $m_{2n+1,n} = 0$: $\mathbf{m}_{2n+1}(-n)$ can be turned into $\mathbf{m}_{2n+1}$ with one affine Weyl reflection, and thus cancels it.

D.2.1 Reduced matrix elements

We are now ready to write the formulas for the reduced matrix elements (5.98). We will give formulas for

$$\left(\begin{matrix}\mathbf{m}_d\\ \mathbf{m}_{d-1}\end{matrix}\middle| M_{12} \middle| \begin{matrix}\mathbf{m}_d\\ \mathbf{m}'_{d-1}\end{matrix}\right) \equiv (-1)^{d-1}\left(\begin{matrix}\mathbf{m}_d\\ \mathbf{m}_{d-1}\end{matrix}\middle| M^{\square} \middle| \begin{matrix}\mathbf{m}_d\\ \mathbf{m}'_{d-1}\end{matrix}\right), \tag{D.16}$$

which is more natural from the point of view of (5.99). We have in odd dimensions

$$\left(\begin{matrix}\mathbf{m}_{2n+1}\\ \mathbf{m}_{2n}\end{matrix}\middle| M_{12} \middle| \begin{matrix}\mathbf{m}_{2n+1}\\ \mathbf{m}_{2n}(\pm j)\end{matrix}\right) = \pm\sqrt{\frac{\prod_{k=1}^{n}(x_{2n+1,k} \mp x_{2n,j})(x_{2n+1,k} \pm x_{2n,j}+1)}{2\prod_{\substack{k=1\\k\neq j}}^{n}(x_{2n,k}-x_{2n,j})(x_{2n,k}+x_{2n,j})}}. \tag{D.17}$$

According to (D.10) this gives all possible reduced matrix elements in even dimensions. Note that according to the discussion above, the factors in the denominator are never zero (assuming that all weights are dominant).

In even dimensions we have

$$\left(\begin{matrix}\mathbf{m}_{2n}\\ \mathbf{m}_{2n-1}\end{matrix}\middle| M_{12} \middle| \begin{matrix}\mathbf{m}_{2n}\\ \mathbf{m}_{2n-1}\end{matrix}\right) = \frac{-i\prod_{k=1}^{n} x_{2n,k}}{\sqrt{\prod_{k=1}^{n-1} x_{2n-1,k}(x_{2n-1,k}+1)}}, \tag{D.18}$$

$$\left(\begin{matrix}\mathbf{m}_{2n}\\ \mathbf{m}_{2n-1}\end{matrix}\middle| M_{12} \middle| \begin{matrix}\mathbf{m}_{2n}\\ \mathbf{m}_{2n-1}(\pm i)\end{matrix}\right) =$$

$$= \pm\sqrt{-\frac{\prod_{k=1}^{n}(x_{2n,k}-x_{2n-1,i}-\delta_{\pm,+})(x_{2n,k}+x_{2n-1,i}+\delta_{\pm,+})}{(x_{2n-1,i}+\delta_{\pm,+})(2x_{2n-1,i}+1)\prod_{\substack{k=1\\k\neq i}}^{n-1}(x_{2n-1,k}-x_{2n-1,i})(x_{2n-1,k}+x_{2n-1,i}+1)}}, \tag{D.19}$$

where $\delta_{\pm,+}$ is equal to 1 for + sign and to 0 for − sign. According to (D.11), this account for all reduced matrix elements in even dimensions. The only potential zero in the denominator of (D.18) is from $x_{2n-1,n-1}$. However, if $x_{2n-1,n-1} = m_{2n-1,n-1} = 0$, then $\mathbf{m}_{2n-1}$ does not appear in $\mathbf{m}_{2n-1} \otimes \square$, and this reduced matrix element has to be set to 0. Similarly, the only potential zero in the denominator of (D.19) appears for (−) sign and $i = n-1$, when we have a factor of $x_{2n-1,n-1}$. Again, it is only a problem if $x_{2n-1,n-1} = m_{2n-1,n-1} = 0$, in which case $\mathbf{m}_{2n-1}(-n+1)$ does not appear in $\mathbf{m}_{2n-1} \otimes \square$ so we need to set this matrix element to 0.

D.2.2 Isoscalar factors

The isoscalar factors are given by formulas of a very similar form. In odd dimensions we have

$$\left(\begin{array}{cc|c} \mathbf{m}_{2n+1} & \square & \mathbf{m}_{2n+1} \\ \mathbf{m}_{2n} & \bullet & \mathbf{m}_{2n} \end{array}\right) = \frac{\prod_{k=1}^{n} x_{2n,k}}{\sqrt{\prod_{k=1}^{n} x_{2n+1,k}(x_{2n+1,k}+1)}}, \tag{D.20}$$

$$\left(\begin{array}{cc|c} \mathbf{m}_{2n+1} & \square & \mathbf{m}_{2n+1}(\pm i) \\ \mathbf{m}_{2n} & \bullet & \mathbf{m}_{2n} \end{array}\right) =$$
$$= \sqrt{\frac{\prod_{k=1}^{n}(x_{2n+1,i}-x_{2n,k}+\delta_{\pm,+})(x_{2n+1,i}+x_{2n,k}+\delta_{\pm,+})}{(x_{2n+1,i}+\delta_{\pm,+})(2x_{2n+1,i}+1)\prod_{\substack{k=1\\k\neq i}}^{n}(x_{2n+1,i}-x_{2n+1,k})(x_{2n+1,i}+x_{2n+1,k}+1)}}, \tag{D.21}$$

and the same comments as for the reduced matrix elements apply about the possible zeros in denominators. In even dimensions the isoscalar factors are given by

$$\left(\begin{array}{cc|c} \mathbf{m}_{2n} & \square & \mathbf{m}_{2n}(\pm i) \\ \mathbf{m}_{2n-1} & \bullet & \mathbf{m}_{2n-1} \end{array}\right) = \sqrt{\frac{\prod_{k=1}^{n-1}(x_{2n,i}\mp x_{2n-1,k})(x_{2n,i}\pm x_{2n-1,k}\pm 1)}{2\prod_{\substack{k=1\\k\neq i}}^{n}(x_{2n,i}-x_{2n,k})(x_{2n,i}+x_{2n,k})}}. \tag{D.22}$$

To derive the isoscalar factor for $(\square, \square)$ pattern in vector representation, we consider the following expression,

$$\langle \mathfrak{M}_d; \square, \bullet, \ldots | M_{12} | \mathfrak{M}'_d \rangle. \tag{D.23}$$

Acting with M_{12} on the left, we find

$$\begin{aligned}
&\langle \mathfrak{M}_d; \square, \square, \bullet, \ldots | \mathfrak{M}'_d \rangle + \sum_{\widetilde{\mathfrak{M}}_d} \langle \mathfrak{M}_d | M_{12} | \widetilde{\mathfrak{M}}_d \rangle \langle \widetilde{\mathfrak{M}}_d; \square, \bullet, \ldots | \mathfrak{M}'_d \rangle = \\
&= \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \mathbf{m}'_d \\ \mathbf{m}_{d-1} & \square & \mathbf{m}'_{d-1} \end{array}\right) \left(\begin{array}{cc|c} \mathbf{m}_{d-1} & \square & \mathbf{m}'_{d-1} \\ \mathbf{m}_{d-2} & \bullet & \mathbf{m}'_{d-2} \end{array}\right) \delta_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} - \\
&\quad - \sum_{\widetilde{\mathfrak{M}}_d} \left(\begin{array}{c|c|c} \mathbf{m}_d & & \mathbf{m}_d \\ \widetilde{\mathbf{m}}_{d-1} & M_{12} & \mathbf{m}_{d-1} \end{array}\right)^{*} \left(\begin{array}{cc|c} \mathbf{m}_{d-1} & \square & \widetilde{\mathbf{m}}_{d-1} \\ \mathbf{m}_{d-2} & \bullet & \widetilde{\mathbf{m}}_{d-2} \end{array}\right) \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \mathbf{m}'_d \\ \widetilde{\mathbf{m}}_{d-1} & \bullet & \mathbf{m}'_{d-1} \end{array}\right) \delta_{\mathfrak{M}_{d-2},\widetilde{\mathfrak{M}}_{d-2}} \delta_{\mathfrak{M}'_{d-1},\widetilde{\mathfrak{M}}_{d-1}} \\
&= \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \mathbf{m}'_d \\ \mathbf{m}_{d-1} & \square & \mathbf{m}'_{d-1} \end{array}\right) \left(\begin{array}{cc|c} \mathbf{m}_{d-1} & \square & \mathbf{m}'_{d-1} \\ \mathbf{m}_{d-2} & \bullet & \mathbf{m}'_{d-2} \end{array}\right) \delta_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}} - \\
&\quad - \left(\begin{array}{c|c|c} \mathbf{m}_d & & \mathbf{m}_d \\ \mathbf{m}'_{d-1} & M_{12} & \mathbf{m}_{d-1} \end{array}\right)^{*} \left(\begin{array}{cc|c} \mathbf{m}_{d-1} & \square & \mathbf{m}'_{d-1} \\ \mathbf{m}_{d-2} & \bullet & \mathbf{m}'_{d-2} \end{array}\right) \left(\begin{array}{cc|c} \mathbf{m}_d & \square & \mathbf{m}'_d \\ \mathbf{m}'_{d-1} & \bullet & \mathbf{m}'_{d-1} \end{array}\right) \delta_{\mathfrak{M}_{d-2},\mathfrak{M}'_{d-2}}.
\end{aligned} \tag{D.24}$$

Action on the right gives, on the other hand,

$$
\begin{aligned}
&\sum_{\widetilde{\mathfrak{M}}_d} \langle \mathfrak{M}_d; \square, \bullet, \ldots | \widetilde{\mathfrak{M}}_d \rangle \langle \widetilde{\mathfrak{M}}_d | M_{12} | \mathfrak{M}'_d \rangle = \\
&= -\sum_{\widetilde{\mathfrak{M}}_d} \left(\begin{matrix} \mathbf{m}_d & \square \\ \mathbf{m}_{d-1} & \bullet \end{matrix}\middle| \begin{matrix} \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} \end{matrix}\right) \left(\begin{matrix} \mathbf{m}'_d \\ \mathbf{m}'_{d-1} \end{matrix}\middle| M_{12} \middle| \begin{matrix} \mathbf{m}'_d \\ \widetilde{\mathbf{m}}_{d-1} \end{matrix}\right)^* \left(\begin{matrix} \widetilde{\mathbf{m}}_{d-1} & \square \\ \widetilde{\mathbf{m}}_{d-2} & \bullet \end{matrix}\middle| \begin{matrix} \mathbf{m}'_{d-1} \\ \mathbf{m}'_{d-2} \end{matrix}\right) \delta_{\mathfrak{M}_{d-1}, \widetilde{\mathfrak{M}}_{d-1}} \delta_{\mathfrak{M}'_{d-2}, \widetilde{\mathfrak{M}}_{d-2}} \\
&= -\left(\begin{matrix} \mathbf{m}_d & \square \\ \mathbf{m}_{d-1} & \bullet \end{matrix}\middle| \begin{matrix} \mathbf{m}'_d \\ \mathbf{m}_{d-1} \end{matrix}\right) \left(\begin{matrix} \mathbf{m}'_d \\ \mathbf{m}'_{d-1} \end{matrix}\middle| M_{12} \middle| \begin{matrix} \mathbf{m}'_d \\ \mathbf{m}_{d-1} \end{matrix}\right)^* \left(\begin{matrix} \mathbf{m}_{d-1} & \square \\ \mathbf{m}'_{d-2} & \bullet \end{matrix}\middle| \begin{matrix} \mathbf{m}'_{d-1} \\ \mathbf{m}'_{d-2} \end{matrix}\right) \delta_{\mathfrak{M}_{d-2}, \mathfrak{M}'_{d-2}}.
\end{aligned}
\tag{D.25}
$$

By comparing these expressions and choosing $\mathbf{m}'_{d-2}$ such that

$$
\left(\begin{matrix} \mathbf{m}_{d-1} & \square \\ \mathbf{m}'_{d-2} & \bullet \end{matrix}\middle| \begin{matrix} \mathbf{m}'_{d-1} \\ \mathbf{m}'_{d-2} \end{matrix}\right) \tag{D.26}
$$

is non-vanishing, we conclude

$$
\begin{aligned}
\left(\begin{matrix} \mathbf{m}_d & \square \\ \mathbf{m}_{d-1} & \square \end{matrix}\middle| \begin{matrix} \mathbf{m}'_d \\ \mathbf{m}'_{d-1} \end{matrix}\right) = &-\left(\begin{matrix} \mathbf{m}_d & \square \\ \mathbf{m}_{d-1} & \bullet \end{matrix}\middle| \begin{matrix} \mathbf{m}'_d \\ \mathbf{m}_{d-1} \end{matrix}\right) \left(\begin{matrix} \mathbf{m}'_d \\ \mathbf{m}'_{d-1} \end{matrix}\middle| M_{12} \middle| \begin{matrix} \mathbf{m}'_d \\ \mathbf{m}_{d-1} \end{matrix}\right)^* \\
&+ \left(\begin{matrix} \mathbf{m}_d & \square \\ \mathbf{m}'_{d-1} & \bullet \end{matrix}\middle| \begin{matrix} \mathbf{m}'_d \\ \mathbf{m}'_{d-1} \end{matrix}\right) \left(\begin{matrix} \mathbf{m}_d \\ \mathbf{m}'_{d-1} \end{matrix}\middle| M_{12} \middle| \begin{matrix} \mathbf{m}_d \\ \mathbf{m}_{d-1} \end{matrix}\right)^*.
\end{aligned}
\tag{D.27}
$$

D.2.3 Comments on $d = 3$

A few modifications to the above formulas are required in the case $d = 3$. This is because the $d - 1 = 2$ and vector representation in $d = 2$ is not irreducible.

The formulas for the reduced matrix elements of remain valid if they are used together with (D.22). Indeed, we can compute

$$
\langle j, m \pm 1 | M_{12} | j, m \rangle = \left(\begin{matrix} j \\ m \pm 1 \end{matrix}\middle| M_{12} \middle| \begin{matrix} j \\ m \end{matrix}\right) \left(\begin{matrix} m \pm 1 \\ \bullet \end{matrix}\middle| \begin{matrix} \square & m \\ \bullet & \bullet \end{matrix}\right) = \mp \frac{1}{2} \sqrt{(j \mp m)(j \pm m + 1)},
\tag{D.28}
$$

which coincides with the standard expression for M_{12} which follows from

$$
M_{12} = -i J_{\hat{2}} = -\frac{J_+}{2} + \frac{J_-}{2},
\tag{D.29}
$$

as discussed in section 5.2.2.3. Alternatively, the formula for the reduced matrix can be interpreted as

$$
\left(\begin{matrix} j \\ m \pm 1 \end{matrix}\middle| M_{12} \middle| \begin{matrix} j \\ m \end{matrix}\right) = \left(\begin{matrix} j \\ m \pm 1 \end{matrix}\middle| M^{1,+1} \middle| \begin{matrix} j \\ m \end{matrix}\right) - \left(\begin{matrix} j \\ m \pm 1 \end{matrix}\middle| M^{1,-1} \middle| \begin{matrix} j \\ m \end{matrix}\right),
\tag{D.30}
$$

where $M^{1,\pm} = -\frac{J_\pm}{\sqrt{2}}$ are defined according to (5.58) and (5.59) (treating the second index of M as a vector index). The matrix elements in the right hand side should be used with the CG coefficients of $Spin(2)$, $\langle m \pm 1| \pm 1, m\rangle = 1$.

The isoscalar factors can interpreted as

$$\left(\begin{array}{cc|c} j & \square & j' \\ m & \square & m' \end{array}\right) = \left(\begin{array}{cc|c} j & \square & j' \\ m & +1 & m' \end{array}\right) - \left(\begin{array}{cc|c} j & \square & j' \\ m & -1 & m' \end{array}\right). \tag{D.31}$$

The isoscalar factors in the right hand side are to be combined with the CG coefficients of $Spin(2)$, $\langle m \pm 1| \pm 1, m\rangle = 1$. This can be checked against the known formulas for $Spin(3)$ CG coefficients.[2]

D.2.4 A sum rule for reduced matrix elements and isoscalar factors

As discussed in the main text, the following identity holds,

$$\sum_{\mathbf{m}_{d-1}} \left(\begin{array}{c} \mathbf{m}_d \\ \widetilde{\mathbf{m}}_{d-1} \end{array}\middle| M^{\square} \middle| \begin{array}{c} \mathbf{m}_d \\ \mathbf{m}_{d-1} \end{array}\right)\left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \mathbf{m}_{d-1} & \square & \widetilde{\mathbf{m}}_{d-1} \end{array}\right) = (-1)^{d-1}(\mathbf{m}_d \square | \widetilde{\mathbf{m}}_d)\left(\begin{array}{cc|c} \mathbf{m}_d & \square & \widetilde{\mathbf{m}}_d \\ \widetilde{\mathbf{m}}_{d-1} & \bullet & \widetilde{\mathbf{m}}_{d-1} \end{array}\right). \tag{D.32}$$

We are not aware of a simple derivation of this fact or of the coefficients $(\mathbf{m}_d \square | \widetilde{\mathbf{m}}_d)$. We note, however, that this identity is required for existence of certain weight-shifting operators in vector representation. The coefficients $(\mathbf{m}_d \square | \widetilde{\mathbf{m}}_d)$ are given by the following formulas

$$(\mathbf{m}_{2n} \square | \mathbf{m}_{2n}(\pm i)) = n \mp x_{2n,i} - 1, \tag{D.33}$$

$$(\mathbf{m}_{2n+1} \square | \mathbf{m}_{2n+1}(\pm i)) = n \mp x_{2n,i} - \delta_{\pm +}, \tag{D.34}$$

$$(\mathbf{m}_{2n+1} \square | \mathbf{m}_{2n+1}) = n. \tag{D.35}$$

We found these formulas by considering a few low-dimensional cases and guessing the general result, which was then verified on a large set of representations in various dimensions. In terms of $m_{d,k}$ these coefficients can be rewritten as

$$(\mathbf{m}_d \square | \mathbf{m}_d(+i)) = -m_{d,i} + i - 1, \tag{D.36}$$

$$(\mathbf{m}_d \square | \mathbf{m}_d(-i)) = m_{d,i} + d - i - 1, \tag{D.37}$$

$$(\mathbf{m}_{2n+1} \square | \mathbf{m}_{2n+1}) = n. \tag{D.38}$$

[2] Recall that the m-independent phase of CG coefficients is convention-dependent. The formulas given here agree with the conventions of [214] (the conventions used in `Mathematica` as of version `11.0`) for $j' = j, j+1$ and differ by a sign for $j' = j - 1$.

D.3 Scalar-fermion blocks in various dimensions

D.3.1 Comparison in 2 dimensions

Interestingly, the formulas for scalar-fermion seed blocks in section 5.4.4 also work in the case $n = 1$, i.e. $d = 2$. We have the following identity,

$$s^D e^{\theta M_{12}} = (se^{i\theta})^{\frac{D-iM_{12}}{2}} (se^{-i\theta})^{\frac{D+iM_{12}}{2}}, \tag{D.39}$$

and so if we define

$$L_0 = \frac{D - iM_{12}}{2}, \quad L_{-1} = \frac{P_1 - iP_2}{2}, \quad L_{+1} = \frac{K_1 + iK_2}{2}, \tag{D.40}$$

$$\overline{L}_0 = \frac{D + iM_{12}}{2}, \quad \overline{L}_{-1} = \frac{P_1 + iP_2}{2}, \quad \overline{L}_{+1} = \frac{K_1 - iK_2}{2}, \tag{D.41}$$

we find that the conformal block in the form (5.108) is given by

$$\langle 0|O_4^{m_4} O_3^{m_3}|O|z^{L_0}\overline{z}^{\overline{L}_0} O_2^{m_2} O_1^{m_1}|0\rangle. \tag{D.42}$$

The algebras (D.40) and (D.41) satisfy the usual commutation relations

$$[L_m, L_n] = (m - n)L_{m+n}, \tag{D.43}$$

$$[\overline{L}_m, \overline{L}_n] = (m - n)\overline{L}_{m+n}. \tag{D.44}$$

The configuration considered in section 5.4.4 is $m_2 = -m_4 = \frac{1}{2}$ and $m_1 = m_3 = 0$. This corresponds to holomorphic and anti-holomorphic dimensions

$$h_1 = \tfrac{1}{2}\Delta_1, \quad h_2 = \tfrac{1}{2}\Delta_2 - \tfrac{1}{4}, \quad h_3 = \tfrac{1}{2}\Delta_3, \quad h_4 = \tfrac{1}{2}\Delta_4 + \tfrac{1}{4}, \tag{D.45}$$

$$\overline{h}_1 = \tfrac{1}{2}\Delta_1, \quad \overline{h}_2 = \tfrac{1}{2}\Delta_2 + \tfrac{1}{4}, \quad \overline{h}_3 = \tfrac{1}{2}\Delta_3, \quad \overline{h}_4 = \tfrac{1}{2}\Delta_4 - \tfrac{1}{4}, \tag{D.46}$$

while the intermediate representation $\mathbf{j}^{\pm}$ corresponds to $h_O = \frac{1}{2}\Delta_O \mp \frac{1}{2}j$ and $\overline{h}_O = \frac{1}{2}\Delta_O \pm \frac{1}{2}j$. The conformal block for exchange of $\mathbf{j}^{\pm}$ is equal to the usual expression

$$z^{h_O} {}_2F_1(h_O - h_{12}, h_O + h_{34}; 2h_O; z) \times \overline{z}^{\overline{h}_O} {}_2F_1(\overline{h}_O - \overline{h}_{12}, \overline{h}_O + \overline{h}_{34}; 2\overline{h}_O; \overline{z}). \tag{D.47}$$

It is straightforward to expand this expression in power series in s and check that it is consistent with the recursion relation (5.328).

D.3.2 Comparison in 3 dimensions

To perform the comparison with the known 3d results, we first need to relate the GT basis to the standard basis for 3d fermions. There is a unique fermionic representation in 3d, $\mathbf{m}_3 = (\frac{1}{2})$, with the allowed GT patterns

$$\mathfrak{M}_{3,\pm} = (\tfrac{1}{2}), (\pm\tfrac{1}{2}), \tag{D.48}$$

consistently with the representation being two-dimensional. For 3d spinors we use the conventions as in [1, 3, 81], and we will be comparing with the scalar-fermion blocks in the form of [3]. These papers use Lorentz signature and thus we need to perform Wick rotation by defining

$$M^{\mu\nu} = -i^{\delta_{\mu,0}+\delta_{\nu,0}} M_L^{\mu\nu}, \tag{D.49}$$

where $M_L^{\mu\nu}$ are the Lorentz generators from [3]. We also added a $(-)$ due to the difference in conventions for conformal algebra. Furthermore, we need to relabel the indices by defining

$$1_{\text{here}} = 2_{\text{there}}, \quad 2_{\text{here}} = 0_{\text{there}}, \quad 3_{\text{here}} = 1_{\text{there}}. \tag{D.50}$$

This is required because of the way the conformal frame is defined in [3]. Using the explicit expression for the Lorentz generators and the correspondence above, we can identify

$$O^1 = O^{\mathfrak{M}_{3,-}}, \quad O^2 = iO^{\mathfrak{M}_{3,+}}. \tag{D.51}$$

Contracting the structures (5.142) with polarization vectors s_α as in [3], we find

$$\left[\begin{array}{c|cc|c} 0,0 & 0 & +\frac{1}{2} & \frac{1}{2}, m_1 \\ \frac{1}{2}, m_4 & +\frac{1}{2} & 0 & 0,0 \end{array}\right] \to -\bar{\xi}_4\bar{\xi}_1 = -[-\tfrac{1}{2}, 0, 0, -\tfrac{1}{2}], \tag{D.52}$$

$$\left[\begin{array}{c|cc|c} 0,0 & 0 & -\frac{1}{2} & \frac{1}{2}, m_1 \\ \frac{1}{2}, m_4 & +\frac{1}{2} & 0 & 0,0 \end{array}\right] \to i\bar{\xi}_4\xi_1 = i[\tfrac{1}{2}, 0, 0, -\tfrac{1}{2}], \tag{D.53}$$

$$\left[\begin{array}{c|cc|c} 0,0 & 0 & +\frac{1}{2} & \frac{1}{2}, m_1 \\ \frac{1}{2}, m_4 & -\frac{1}{2} & 0 & 0,0 \end{array}\right] \to i\bar{\xi}_1\xi_4 = i[-\tfrac{1}{2}, 0, 0, \tfrac{1}{2}], \tag{D.54}$$

$$\left[\begin{array}{c|cc|c} 0,0 & 0 & -\frac{1}{2} & \frac{1}{2}, m_1 \\ \frac{1}{2}, m_4 & -\frac{1}{2} & 0 & 0,0 \end{array}\right] \to \xi_4\xi_1 = [\tfrac{1}{2}, 0, 0, \tfrac{1}{2}], \tag{D.55}$$

where the right hand side is in the notation of [3]. The results (for parity-even components) of [3] are given in the form

$$\begin{aligned} &\langle 0|\psi_4(\infty, s_4)\phi_3(1)|\mathcal{O}|\phi_2(z,\bar{z})\psi_1(0,s_1)|0\rangle \\ &= \tfrac{1}{2}g_1(z,\bar{z})[-\tfrac{1}{2},0,0,-\tfrac{1}{2}] + \tfrac{1}{2}g_2(z,\bar{z})[\tfrac{1}{2},0,0,-\tfrac{1}{2}]+ \\ &+ \tfrac{1}{2}g_2(z,\bar{z})[-\tfrac{1}{2},0,0,\tfrac{1}{2}] + \tfrac{1}{2}g_1(z,\bar{z})[\tfrac{1}{2},0,0,\tfrac{1}{2}]. \end{aligned} \tag{D.56}$$

This implies that

$$
\begin{aligned}
&s^{-\Delta_1-\Delta_2}\langle 0|\psi_4\phi_3|O|s^D e^{\theta M_{12}}\phi_2\psi_1|0\rangle \\
&= \tfrac{1}{2}\left(\cos\tfrac{\theta}{2}\, g_1(z,\bar{z}) + i\sin\tfrac{\theta}{2}\, g_2(z,\bar{z})\right)[-\tfrac{1}{2},0,0,-\tfrac{1}{2}]+ \\
&+ \tfrac{1}{2}\left(\cos\tfrac{\theta}{2}\, g_2(z,\bar{z}) + i\sin\tfrac{\theta}{2}\, g_1(z,\bar{z})\right)[\tfrac{1}{2},0,0,-\tfrac{1}{2}]+ \\
&+ \tfrac{1}{2}\left(\cos\tfrac{\theta}{2}\, g_2(z,\bar{z}) + i\sin\tfrac{\theta}{2}\, g_1(z,\bar{z})\right)[-\tfrac{1}{2},0,0,\tfrac{1}{2}]+ \\
&+ \tfrac{1}{2}\left(\cos\tfrac{\theta}{2}\, g_1(z,\bar{z}) + i\sin\tfrac{\theta}{2}\, g_2(z,\bar{z})\right)[\tfrac{1}{2},0,0,\tfrac{1}{2}].
\end{aligned}
\tag{D.57}
$$

Using this result, we can compute the expansion (5.302) in terms of functions g_1 and g_2. These functions are conveniently computed by acting with the differential operators of [3] on the scalar conformal block obtained from the recursion relation (5.220).[3] We have checked that the resulting expansion is consistent with the recursion relation which follows from (5.304)-(5.309) at the first few levels for various choices of j_O.

D.3.3 Comparison in 4 dimensions

To perform the comparison with the known 4d results, we first need to relate the GT basis to the standard basis for Weyl fermions. We are considering the two fermionic representations $\mathbf{m}_4^\pm = (\frac{1}{2}, \pm\frac{1}{2})$. The allowed GT patterns are

$$
\mathfrak{M}^\pm_{4,+} = (\tfrac{1}{2}, \pm\tfrac{1}{2}), (\tfrac{1}{2}), (+\tfrac{1}{2}), \tag{D.58}
$$

$$
\mathfrak{M}^\pm_{4,-} = (\tfrac{1}{2}, \pm\tfrac{1}{2}), (\tfrac{1}{2}), (-\tfrac{1}{2}), \tag{D.59}
$$

consistently with the representations being two-dimensional. For 4d Weyl spinors, we will use the conventions of [2]. We need to make a few adaptations from conventions there to the present conventions. First, we need to perform Wick rotation by defining

$$
M^{\mu\nu} = -i^{\delta_{\mu,0}+\delta_{\nu,0}} M_L^{\mu\nu}, \tag{D.60}
$$

where M_L are the Lorentz generators of [2]. We also added a $(-)$ due to the difference in conventions for conformal algebra. Furthermore, we need to relabel the indices by defining

$$
1_{\text{here}} = 3_{\text{there}}, \quad 2_{\text{here}} = 0_{\text{there}}, \quad 3_{\text{here}} = 1_{\text{there}}, \quad 4_{\text{here}} = 2_{\text{there}}. \tag{D.61}
$$

This is required because of the way the conformal frame is defined in [2].

[3] Alternatively, one can use Zamolodchikov-type recursion relations of [81].

Comparing the transformation properties of $|\mathfrak{M}^{\pm}_{4,\pm}\rangle$ and the operators $O^{\dot\alpha}$ and O_α, we find that we can set

$$O_1 = O^{\mathfrak{M}^+_{4,-}}, \quad O_2 = -iO^{\mathfrak{M}^+_{4,+}}, \tag{D.62}$$
$$O^{\dot 1} = O^{\mathfrak{M}^-_{4,-}}, \quad O^{\dot 2} = +iO^{\mathfrak{M}^-_{4,+}}. \tag{D.63}$$

According to (5.153) we find the following non-zero components of tensor structures (5.161)

$$\left[\begin{matrix}\bullet\\ \mathfrak{M}^-_{4,-}\end{matrix}\middle|\begin{matrix}\bullet\\ (\frac{1}{2})\end{matrix}\,(\tfrac{1}{2})\middle|(+\tfrac{1}{2})\middle|(\tfrac{1}{2})\,\begin{matrix}\bullet\\ (\frac{1}{2})\end{matrix}\middle|\begin{matrix}\bullet\\ \mathfrak{M}^+_{4,+}\end{matrix}\right] = \frac{-i}{\sqrt{2}}, \tag{D.64}$$
$$\left[\begin{matrix}\bullet\\ \mathfrak{M}^-_{4,+}\end{matrix}\middle|\begin{matrix}\bullet\\ (\frac{1}{2})\end{matrix}\,(\tfrac{1}{2})\middle|(-\tfrac{1}{2})\middle|(\tfrac{1}{2})\,\begin{matrix}\bullet\\ (\frac{1}{2})\end{matrix}\middle|\begin{matrix}\bullet\\ \mathfrak{M}^+_{4,-}\end{matrix}\right] = \frac{i}{\sqrt{2}}. \tag{D.65}$$

Contracting with polarization vectors as in [2], we find

$$t_+ = +\frac{\xi_2\overline{\xi}_4}{\sqrt{2}} = +\frac{1}{\sqrt{2}}\begin{bmatrix}0 & -\frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2}\end{bmatrix}, \tag{D.66}$$
$$t_- = -\frac{\eta_2\overline{\eta}_2}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\begin{bmatrix}0 & +\frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & +\frac{1}{2}\end{bmatrix}. \tag{D.67}$$

Using this correspondence, we can find that the primal conformal block has the form

$$\begin{aligned}\langle 0|\overline{\psi}_4\phi_3|O|s^D e^{\theta M_{12}}\psi_2\phi_1|0\rangle = &-\sqrt{2}\left(2\sqrt{z}H_1^0(z,\overline{z}) + \frac{1}{\sqrt{z}}H_1^1(z,\overline{z})\right)t_+\\ &-\sqrt{2}\left(2\sqrt{z}H_1^0(z,\overline{z}) + \frac{1}{\sqrt{z}}H_1^1(z,\overline{z})\right)t_-.\end{aligned} \tag{D.68}$$

In our terminology it corresponds to exchange of a primary in representation $(\ell+\frac{1}{2},\frac{1}{2})$ with ℓ as in [2]. Using explicit expressions for functions H [58] in normalization of [2], we can check that the leading term in $s = |z|$ coincides with (5.332) and (5.196) with

$$\mathbf{\Lambda}_{0,j_O} = \begin{pmatrix} -i\frac{(\ell+2)(-1)^\ell}{\sqrt{2}}\\ 0\end{pmatrix}. \tag{D.69}$$

We can then use the recursion relation (5.328) to compute higher order coefficients and plug them into the expansion (5.332). We can compute the same expansion by plugging the explicit expressions for functions H into (D.68) using `CFTs4D` package from [2]. We checked that both expansions coincide at the first few levels.

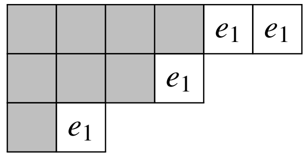


Figure D.1: The relationship between Young diagrams of $\mathbf{m}_d$ and $\mathbf{m}_{d-1}$. The boxes which belong to $\mathbf{m}_{d-1}$ are shaded.

D.4 Gelfand-Tsetlin bases for tensor representations

To gain some familiarity with GT bases in general dimensions, let us consider how it is related to the usual Cartesian bases for tensor representations. For simplicity of discussion, we avoid dealing with self-duality constraints. This restricts us to the representations $\mathbf{m}_d$ with $m_{d,k} = 0$ for $k \geq d/2$, i.e. to Young diagrams with less than $d/2$ rows. In particular, we will only consider the GT patters in which all representations are of this kind.[4]

Our goal is for a given GT pattern $\mathfrak{M}_d$ to find the explicit tensor $T^{\mu_1\ldots\mu_{|\mathbf{m}_d|}}_{\mathfrak{M}_d}$ which gives the corresponding basis element $|\mathfrak{M}_d\rangle$, up to a multiplicative factor. We do this recursively, by explicitly constructing the *dimensional induction map*

$$I^{\mathbf{m}_d}_{\mathbf{m}_{d-1}} : V_{\mathbf{m}_{d-1}} \to V_{\mathbf{m}_d}, \quad \mathbf{m}_{d-1} \in \mathbf{m}_d, \tag{D.70}$$

which is defined, up to normalization, by the requirement that it is $Spin(d-1)$-equivariant and non-trivial. By irreducibility of $\mathbf{m}_{d-1}$ it follows that I establishes an isomorphism between $V_{\mathbf{m}_{d-1}}$ and the subspace in $V_{\mathbf{m}_d}$ which transforms according to $\mathbf{m}_{d-1}$ under $Spin(d-1)$. Since dimensional reduction is multiplicity-free, this subspace is uniquely determined.

It then immediately follows from the definition of GT basis that the following relationship between GT basis vectors holds,

$$|\mathbf{m}_d, \mathbf{m}_{d-1}, \mathbf{m}_{d-2}, \ldots\rangle \propto I^{\mathbf{m}_d}_{\mathbf{m}_{d-1}} |\mathbf{m}_{d-1}, \mathbf{m}_{d-2}, \ldots\rangle. \tag{D.71}$$

In particular, if $\mathbf{m}_{d-k} = \bullet$ is the trivial representation, we find

$$|\mathbf{m}_d, \mathbf{m}_{d-1}, \mathbf{m}_{d-2}, \ldots\rangle \propto I^{\mathbf{m}_d}_{\mathbf{m}_{d-1}} I^{\mathbf{m}_{d-1}}_{\mathbf{m}_{d-2}} I^{\mathbf{m}_{d-2}}_{\mathbf{m}_{d-3}} \ldots I^{\mathbf{m}_{d-k+1}}_{\mathbf{m}_{d-k}} 1. \tag{D.72}$$

To construct $I^{\mathbf{m}_d}_{\mathbf{m}_{d-1}}$ explicitly, start with a general $U^{\mu_1\ldots\mu_{|\mathbf{m}_{d-1}|}} \in V_{\mathbf{m}_{d-1}}$. For convenience we assume that the indices of U run from 2 to d.[5] We first extend the

[4] The same general approach works even without these assumptions, and the details are not hard to recover.

[5] Recall that by our choice of $Spin(d-1) \subset Spin(d)$, $Spin(d-1)$ stabilizes e_1.

definition of U to allow its indices to assume the value 1 by setting $U^{\mu_1\ldots\mu_{|\mathbf{m}_{d-1}|}} = 0$ whenever at least one of $\mu_i = 1$. We then define

$$T'^{\mu_1\ldots\mu_{|\mathbf{m}_d|}} = U^{\mu_1\ldots\mu_{|\mathbf{m}_{d-1}|}} e_1^{\mu_{|\mathbf{m}_{d-1}|+1}} \cdots e_1^{\mu_{|\mathbf{m}_d|}} - \text{traces}. \tag{D.73}$$

A generic relationship between the Young diagrams $\mathbf{m}_d$ and $\mathbf{m}_{d-1}$ is shown in figure D.1. We can associate the indices of e_1 in (D.73) to the unshaded boxes in figure D.1 and apply to T' the Young symmetrizer $Y_{\mathbf{m}_d}$ corresponding to $\mathbf{m}_d$ to define

$$I^{\mathbf{m}_d}_{\mathbf{m}_{d-1}} U \equiv Y_{\mathbf{m}_d} T'. \tag{D.74}$$

Note that it is guaranteed by the dimensional reduction rules from section 5.2.1 that no two indices of e_1 land in the same column of $\mathbf{m}_d$.

As explained above, this map allows us to reconstruct Gelfand-Tsetlin basis vectors up to a phase. Let us look at some examples. First, consider the GT basis vector

$$|\square\square\cdots\square, \bullet, \ldots\rangle. \tag{D.75}$$

From (D.72) we find

$$|\square\square\cdots\square, \bullet, \ldots\rangle \propto I_{\bullet}^{\square\square\cdots\square} 1 = e_1^{\mu_1} \cdots e_1^{\mu_j} - \text{traces}. \tag{D.76}$$

This reproduces the result of section 5.3.7.5.

As a more complicated example, consider

$$|\square\square\cdots\square, \square, \bullet, \ldots\rangle. \tag{D.77}$$

From (D.72) we find

$$\begin{aligned} |\square\square\cdots\square, \square, \bullet, \ldots\rangle &\propto I^{\square\square\cdots\square}_{\square} |\square, \bullet, \ldots\rangle \\ &\propto I^{\square\square\cdots\square}_{\square} e_2^{\mu_1} = e_2^{(\mu_1} e_1^{\mu_2} \cdots e_1^{\mu_j)} - \text{traces}. \end{aligned} \tag{D.78}$$

Similarly, we can find that

$$|\square\square\cdots\square, \square, \square, \bullet, \ldots\rangle \propto e_3^{(\mu_1} e_1^{\mu_2} \cdots e_1^{\mu_j)} - \text{traces}, \tag{D.79}$$

and so on. In the case $j = 1$ this reproduces the results of section 5.2.2.2 for vector representation.

Consider now the simplest non-STT example,

$$\left|\begin{smallmatrix}\square\square\cdots\square \\ \square\end{smallmatrix}, \square, \bullet, \ldots\right\rangle. \tag{D.80}$$

Note that $\square$ is the simplest representation to which $\ydiagram{2,1}\cdots\ydiagram{1}$ can reduce. This differs from (D.78) only in the Young symmetrizer,

$$|\ydiagram{2,1}\cdots\ydiagram{1}, \square, \bullet, \ldots\rangle \propto Y_{\ydiagram{2,1}\cdots\ydiagram{1}}(e_2^\nu e_1^{\mu_1}\cdots e_1^{\mu_j} - \text{traces}) =$$
$$= \tfrac{1}{2}e_2^\nu e_1^{(\mu_1}e_1^{\mu_2}\cdots e_1^{\mu_j)} - \tfrac{1}{2}e_1^\nu e_2^{(\mu_1}e_1^{\mu_2}\cdots e_1^{\mu_j)} - \text{traces}. \quad \text{(D.81)}$$

Similarly,

$$|\ydiagram{2,1}\cdots\ydiagram{1}, \square, \square, \bullet, \ldots\rangle \propto \tfrac{1}{2}e_3^\nu e_1^{(\mu_1}e_1^{\mu_2}\cdots e_1^{\mu_j)} - \tfrac{1}{2}e_1^\nu e_3^{(\mu_1}e_1^{\mu_2}\cdots e_1^{\mu_j)} - \text{traces}, \quad \text{(D.82)}$$

and so on.

It is important that we perform trace subtraction and Young symmetrization in all steps of dimensional induction. Consider for example the state

$$|\ydiagram{3}, \ydiagram{2}, \bullet, \ldots\rangle. \quad \text{(D.83)}$$

We have first in $d-1$ dimensions

$$|\ydiagram{2}, \bullet, \ldots\rangle \propto e_2^{\mu_1}e_2^{\mu_2} - \frac{1}{d-1}\delta^{\mu_1\mu_2}, \quad \text{(D.84)}$$

and when we lift it to d dimensions, we have agreed to set the new entries of this tensor to 0, which in this case amounts to replacing

$$\delta^{\mu_1\mu_2} \to \widetilde{\delta}^{\mu_1\nu_1} \equiv \delta^{\mu_1\mu_2} - e_1^{\mu_1}e_1^{\mu_2}, \quad \text{(D.85)}$$

so that indeed $\widetilde{\delta}^{1\mu_2} = 0$. We thus have[6]

$$|\ydiagram{3}, \ydiagram{2}, \bullet, \ldots\rangle \propto Y_{\ydiagram{3}}\left(e_2^{\mu_1}e_2^{\mu_2} - \frac{1}{d-1}\left(\delta^{\mu_1\mu_2} - e_1^{\mu_1}e_1^{\mu_2}\right)\right)e_1^{\mu_3}. \quad \text{(D.86)}$$

Clearly, if we didn't take care with $\widetilde{\delta}$, or had postponed trace subtraction to d dimensions, we would have never obtained a term $e_1^{\mu_1}e_1^{\mu_2}e_1^{\mu_3}$. These choices would be wrong since for $\mu_1 = \mu_2 = \mu_3 = 1$ they would reduce to a non-zero constant and thus their dimensional reduction has a component along the trivial representation of $Spin(d-1)$. On the other hand, (D.86) is non-zero iff only one of μ_i is set to 1, in which case it reduces to $\ydiagram{2}$, as required.

[6] This object is automatically traceless in d dimensions so we don't have to subtract d-dimensional traces.

Similarly, care should be taken with compositions of Young symmetrizers between dimensions. Consider the state

$$|\,\ydiagram{2,1},\ \ydiagram{2},\ \ydiagram{1},\ \bullet, \ldots\rangle. \tag{D.87}$$

We have successively

$$|\,\ydiagram{1},\ \bullet, \ldots\rangle \propto e_3^{\mu_1}, \tag{D.88}$$

$$|\,\ydiagram{2},\ \ydiagram{1},\ \bullet, \ldots\rangle \propto e_2^{(\mu_1}e_3^{\mu_2)}, \tag{D.89}$$

$$|\,\ydiagram{2,1},\ \ydiagram{2},\ \ydiagram{1},\ \bullet, \ldots\rangle \propto \tfrac{1}{2}e_1^{\nu}e_2^{(\mu_1}e_3^{\mu_2)} - \frac{1}{4}\left(e_1^{\mu_1}e_2^{(\nu}e_3^{\mu_2)} + e_1^{\mu_2}e_2^{(\nu}e_3^{\mu_1)}\right). \tag{D.90}$$

Here we have applied Young symmetrizer both in (D.89) and (D.90). Had we only applied the d-dimensional symmetrizer, we would find

$$|\,\ydiagram{2,1},\ \ydiagram{2},\ \ydiagram{1},\ \bullet, \ldots\rangle \propto \tfrac{1}{2}e_1^{\nu}e_2^{(\mu_1}e_3^{\mu_2)} - \frac{1}{4}\left(e_1^{\mu_1}e_2^{\nu}e_3^{\mu_2} + e_1^{\mu_2}e_2^{\nu}e_3^{\mu_1}\right). \tag{D.91}$$

It is easy to see that (D.91) is wrong: setting $\mu_2 = 1$ we obtain $-\tfrac{1}{2}e_2^{\nu}e_3^{\mu_1}$, which is a tensor with no definite symmetry. On contrary, setting $\mu_2 = 1$ in (D.90), we find $-\tfrac{1}{2}e_2^{(\nu}e_3^{\mu_1)}$ which belongs to $\ydiagram{2}$ as required. We thus see that the symmetrizers from different dimensions interact non-trivially to ensure that the dimensional reductions are irreducible.

We have so far avoided the question of normalization of the tensors $T_{\mathfrak{M}_d}$. Up to a phase it is determined by the requirement that GT vectors have unit length. This is straightforward to implement on the tensor side. Sometimes we would like to know the normalization factor as a function of the length of the first row j – this is perhaps most easily implemented using the irreducible projectors as we explain below. The phases can be chosen based on convenience,[7] unless one wants to make contact with the GT formulas in appendix D.2. We have not attempted to find the general prescription which would match the phase conventions of these formulas.

D.4.1 P-functions

In this section we relate $P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta)$ in tensor representations to the irreducible projectors studied in [82].

[7]Of course, for every GT pattern this choice should be made once and for all in order to have consistent expressions.

We start by utilizing the tensor representation of GT basis vectors in the definition of P,

$$\begin{aligned} P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta) &\equiv \langle \mathbf{m}_d, \mathbf{m}_{d-1}, \mathbf{m}_{d-2}, \ldots | e^{\theta M_{12}} | \mathbf{m}_d, \mathbf{m}'_{d-1}, \mathbf{m}_{d-2}, \ldots \rangle \\ &= T^{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}_{\mathfrak{M}_d} (e^{\theta M_{12}})_{\mu_1 \ldots \mu_{|\mathbf{m}_d|}, \nu_1 \ldots \nu_{|\mathbf{m}_d|}} T^{\nu_1 \ldots \nu_{|\mathbf{m}_d|}}_{\mathfrak{M}'_d} \\ &= T^{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}_{\mathfrak{M}_d} T^{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}_{\mathfrak{M}'_d}(\theta), \end{aligned} \tag{D.92}$$

where $T_{\mathfrak{M}'_d}(\theta)$ is equal to $T_{\mathfrak{M}'_d}$ in which all occurrences of e_1 and e_2 have been replaced with

$$e_1(\theta) = e^{\theta M_{12}} e_1 = \cos\theta e_1 + \sin\theta e_2, \tag{D.93}$$

$$e_2(\theta) = e^{\theta M_{12}} e_2 = -\sin\theta e_1 + \cos\theta e_2. \tag{D.94}$$

Note that in the first line of (D.92) $\ldots$ represent the same sequence in both vectors, which can be chosen arbitrarily. For example, if $\mathbf{m}_{d-2}$ is STT, we can choose all representations in $\ldots$ to be trivial. We have also assumed that we had chosen the tensors $T_{\mathfrak{M}_d}$ to be real for all relevant $\mathfrak{M}_d$.[8]

We can further trivially rewrite the last line of (D.92) as

$$T^{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}_{\mathfrak{M}_d} T^{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}_{\mathfrak{M}'_d}(\theta) = T^{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}_{\mathfrak{M}_d}\, {}_{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}\pi_{\nu_1 \ldots \nu_{|\mathbf{m}_d|}} T^{\nu_1 \ldots \nu_{|\mathbf{m}_d|}}_{\mathfrak{M}'_d}(\theta) = T_{\mathfrak{M}_d} \cdot \pi \cdot T_{\mathfrak{M}'_d}, \tag{D.95}$$

where ${}_{\mu_1 \ldots \mu_{|\mathbf{m}_d|}}\pi_{\nu_1 \ldots \nu_{|\mathbf{m}_d|}}$ is the projector onto the irreducible representation $\mathbf{m}_d$. From our construction of tensors $T_{\mathfrak{M}_d}$ we know that we can write $T_{\mathfrak{M}_d}$ in terms of the basis vectors e_i and Kronecker deltas $\delta_{\mu_i \mu_j}$. We can thus write

$$T_{\mathfrak{M}_d} = T^{(e)}_{\mathfrak{M}_d} + \text{terms containing } \delta^{\mu_i \mu_j}, \tag{D.96}$$

$$T_{\mathfrak{M}'_d} = T^{(e)}_{\mathfrak{M}'_d} + \text{terms containing } \delta^{\mu_i \mu_j}. \tag{D.97}$$

We then conclude

$$P^{\mathbf{m}_d,\mathbf{m}_{d-2}}_{\mathbf{m}_{d-1},\mathbf{m}'_{d-1}}(\theta) = T^{(e)}_{\mathfrak{M}_d} \cdot \pi \cdot T^{(e)}_{\mathfrak{M}'_d}(\theta). \tag{D.98}$$

Note that it is easy to compute $T^{(e)}_{\mathfrak{M}_d}$ for generic $m_{d,1}$, because we do not need to explicitly remove traces in the last step of dimensional induction, while the number of indices in the preceding steps is independent from $m_{d,1}$.

[8] This might not be possible it the GT patterns do not satisfy the assumptions discussed in the beginning of this appendix. In that case one needs to add some complex conjugations in the formulas.

Furthermore, the right hand side of (D.98) contains the irreducible projector π contracted with a bunch of vectors (basis vectors e_i or $e_1(\theta), e_2(\theta)$) on both sides. These are precisely the contractions studied recently in [82]. Given their results, we then obtain a simple algorithm for computation of P-functions. It is best illustrated in examples.

Matrix element $P^{\mathbf{m}_d,\bullet}_{\bullet,\bullet}(\theta)$

We start with the simplest example,

$$P^{\mathbf{m}_d,\bullet}_{\bullet,\bullet}(\theta). \tag{D.99}$$

Since in this case $\mathbf{m}_{d-1} = \bullet$, we necessarily have $\mathbf{m}_d = \mathbf{j}$ is a traceless-symmetric tensor representation. Recall from (D.76) that

$$T^{\mu_1\ldots\mu_j}_{\mathbf{j},\bullet,\ldots} = N_j\left(e_1^{\mu_1}\ldots e_1^{\mu_j} - \text{traces}\right), \tag{D.100}$$

where we also introduced the normalization factor N_j. We thus conclude

$$T^{(e),\mu_1\ldots\mu_j}_{\mathbf{j},\bullet,\ldots} = N_j e_1^{\mu_1}\ldots e_1^{\mu_j}, \tag{D.101}$$

$$T^{(e),\mu_1\ldots\mu_j}_{\mathbf{j},\bullet,\ldots}(\theta) = N_j e_1^{\mu_1}(\theta)\ldots e_1^{\mu_j}(\theta). \tag{D.102}$$

The results of [82] are formulated in the following way. They define the function

$$\pi_j(z_1,\bar z_1) = z_1^{\mu_1}\ldots z_1^{\mu_j}{}_{\mu_1\ldots\mu_j}\pi_{\nu_1\ldots\nu_j}\bar z_1^{\mu_1}\ldots\bar z_1^{\mu_j}, \tag{D.103}$$

where π is the projector on traceless-symmetric spin-j representation. This function completely encodes the projector since the components can be recovered by taking repeated derivatives in z_1 and $\bar z_1$.[9] It is then can be shown that

$$\pi_j(z_1,\bar z_1) = \frac{j!}{2^j(\nu)_j}|z_1|^j|\bar z_1|^j C_j^{(\nu)}\left(\frac{z_1\cdot\bar z_1}{|z_1||\bar z_1|}\right), \tag{D.104}$$

where $\nu = \frac{d-2}{2}$. We then immediately find that

$$\begin{aligned}
P^{\mathbf{j},\bullet}_{\bullet,\bullet}(\theta) &= T^{(e)}_{\mathbf{j},\bullet,\ldots}\cdot\pi\cdot T^{(e)}_{\mathbf{j},\bullet,\ldots}(\theta)\\
&= N_j^2\pi(z_1,\bar z_1)\Big|_{z_1=e_1,\,\bar z_1=e_1(\theta)}\\
&= N_j^2\frac{j!}{2^j(\nu)_j}|e_1|^j|e_1(\theta)|^j C_j^{(\nu)}\left(\frac{e_1\cdot e_1(\theta)}{|e_1||e_1(\theta)|}\right)\\
&= \frac{N_j^2 j!}{2^j(\nu)_j}C_j^{(\nu)}(\cos\theta).
\end{aligned} \tag{D.105}$$

[9]Note that we do not require $z_1\cdot z_1 = 0$.

Note that the normalization condition for $|\mathbf{j}, \bullet, \ldots\rangle$ is equivalent to $P^{\mathbf{j},\bullet}_{\bullet,\bullet}(0) = 1$, and thus using

$$C_j^{(\nu)}(1) = \frac{(2\nu)_j}{j!}. \tag{D.106}$$

we find

$$1 = \frac{N_j^2 j!}{2^j (\nu)_j} C_j^{(\nu)}(1) = N_j^2 \frac{(2\nu)_j}{2^j (\nu)_j}, \tag{D.107}$$

from where we conclude that[10]

$$N_j = \sqrt{\frac{2^j (\nu)_j}{(2\nu)_j}}, \tag{D.108}$$

while

$$P^{\mathbf{j},\bullet}_{\bullet,\bullet}(\theta) = \frac{j!}{(2\nu)_j} C_j^{(\nu)}(\cos\theta). \tag{D.109}$$

Matrix element $P^{\mathbf{j},\bullet}_{\square,\square}(\theta)$

We now consider the matrix elements

$$P^{\mathbf{m}_d,\bullet}_{\square,\square}(\theta). \tag{D.110}$$

Note that now both $\mathbf{m}_{d-1}$ and $\mathbf{m}'_{d-1}$ are equal to $\square$ and thus $\mathbf{m}_d$ can be either a traceless-symmetric tensor $\mathbf{j}$ or a hook diagram $(\mathbf{j}, \square)$.

We start from the traceless-symmetric case and will return to the hook exchange later. From (D.78) we find

$$T^{(e),\mu_1,\ldots,\mu_j}_{\mathbf{j},\square,\bullet,\ldots} = N_{j,\square} e_2^{(\mu_1} e_1^{\mu_2} \cdots e_1^{\mu_j)}, \tag{D.111}$$

$$T^{(e),\mu_1,\ldots,\mu_j}_{\mathbf{j},\square,\bullet,\ldots}(\theta) = N_{j,\square} e_2^{(\mu_1}(\theta) e_1^{\mu_2}(\theta) \cdots e_1^{\mu_j)}(\theta). \tag{D.112}$$

We then find

$$\begin{aligned} P^{\mathbf{j},\bullet}_{\square,\square}(\theta) &= T^{(e)}_{\mathbf{j},\square,\bullet,\ldots} \cdot \pi \cdot T^{(e)}_{\mathbf{j},\square,\bullet,\ldots}(\theta) \\ &= \frac{1}{j^2} N^2_{j,\square} (e_2 \cdot \partial_{z_1})(e_2(\theta) \cdot \partial_{\bar{z}_1}) \pi(z_1, \bar{z}_1)\Big|_{z_1 = e_1,\, \bar{z}_1 = e_1(\theta)} \\ &= \frac{N^2_{j,\square} j!}{j^2 2^j (\nu)_j} \left(\cos\theta \partial C_j^{(\nu)}(\cos\theta) - \sin^2\theta \partial^2 C_j^{(\nu)}(\cos\theta) \right) \\ &= -\frac{N^2_{j,\square} j!}{j^2 2^j (\nu)_j} \partial_\theta^2 C_j^{(\nu)}(\cos\theta). \end{aligned} \tag{D.113}$$

[10] Here we essentially make a choice of phase for $|\mathbf{j}, \bullet, \ldots\rangle$.

Again, we have the normalization condition $P^{\mathbf{j},\bullet}_{\square,\square}(1) = 1$. To solve for $N_{j,\square}$, we need to know $\partial C^{(\nu)}_j(1)$, which can be computed using the identity

$$\partial_x C^{(\nu)}_j(x) = 2\nu\, C^{(\nu+1)}_{j-1}(x). \tag{D.114}$$

We thus find

$$\frac{N^2_{j,\square}\, j!}{j^2 2^j (\nu)_j} \frac{2\nu(2\nu+2)_{j-1}}{(j-1)!} = 1, \tag{D.115}$$

and therefore (adding a phase for future convenience)

$$N_{j,\square} = -\sqrt{\frac{2^j j(\nu)_j}{2\nu(2\nu+2)_{j-1}}}, \tag{D.116}$$

$$P^{\mathbf{j},\bullet}_{\square,\square}(\theta) = -\frac{(j-1)!}{2\nu(2\nu+2)_{j-1}} \partial^2_\theta C^{(\nu)}_j(\cos\theta). \tag{D.117}$$

Matrix elements $P^{\mathbf{m}_d,\bullet}_{\square,\bullet}(\theta)$ and $P^{\mathbf{m}_d,\bullet}_{\bullet,\square}(\theta)$

Having determined the normalization factors N_j and $N_{j,\square}$, we can now address the matrix elements

$$P^{\mathbf{m}_d,\bullet}_{\square,\bullet}(\theta), \qquad P^{\mathbf{m}_d,\bullet}_{\bullet,\square}(\theta), \tag{D.118}$$

which are not subject to a simple normalization condition at $\theta = 0$. In particular, their phases are convention-dependent. We have

$$\begin{aligned} P^{\mathbf{m}_d,\bullet}_{\square,\bullet}(\theta) &= T^{(e)}_{\mathbf{j},\square,\bullet,\ldots} \cdot \pi \cdot T^{(e)}_{\mathbf{j},\bullet,\ldots}(\theta) \\ &= N_j N_{j,\square} j^{-1} (e_2 \cdot \partial_{z_1}) \pi_j(z_1, \bar{z}_1)\Big|_{z_1=e_1, \bar{z}_1=e_1(\theta)} \\ &= -\frac{2\nu j!}{(2\nu)_j} \sqrt{\frac{2\nu+1}{j(2\nu+j)}} \sin\theta\, C^{(\nu+1)}_{j-1}(\cos\theta). \end{aligned} \tag{D.119}$$

An analogous calculation shows that

$$P^{\mathbf{m}_d,\bullet}_{\bullet,\square}(\theta) = \frac{2\nu j!}{(2\nu)_j} \sqrt{\frac{2\nu+1}{j(2\nu+j)}} \sin\theta\, C^{(\nu+1)}_{j-1}(\cos\theta) = \left(P^{\mathbf{m}_d,\bullet}_{\square,\bullet}(-\theta)\right)^*, \tag{D.120}$$

consistently with (5.173). One can check in explicit examples that these results coincide with the direct exponentiation of M_{12}, providing a non-trivial check of the formalism and normalization factors.

Matrix element $P^{(\mathbf{j},\square),\bullet}_{\square,\square}(\theta)$

Consider now the case of the hook exchange $\mathbf{m}_d = (\mathbf{j}, \square)$ in (D.110). We are now dealing with a new type of representations. Correspondingly, in [82] the following function is defined

$$\pi_{(j,1)}(z_1, z_2, \bar{z}_1, \bar{z}_2) = z_2^{\nu} z_1^{\mu_1} \cdots z_1^{\mu_j}{}_{\nu,\mu_1\ldots\mu_j}\pi_{\bar{\nu},\bar{\mu}_1\ldots\bar{\mu}_j}\bar{z}_2^{\bar{\nu}}\bar{z}_1^{\bar{\mu}_1}\cdots\bar{z}_1^{\bar{\mu}_j}. \tag{D.121}$$

The expression for the full projector is somewhat complicated, so we do not reproduce it here. In practice, we used the Mathematica code supplied with [82] to perform the calculations with these projectors.

From equation (D.81) we find

$$T^{(e),\nu,\mu_1,\ldots,\mu_j}_{(\mathbf{j},\square),\square,\bullet,\ldots} = N_{(\mathbf{j},\square),\square}\left[e_1^{\nu}e_2^{(\mu_1}e_1^{\mu_2}\cdots e_1^{\mu_j)} - e_2^{\nu}e_1^{(\mu_1}e_1^{\mu_2}\cdots e_1^{\mu_j)}\right], \tag{D.122}$$

$$\begin{aligned} &T^{(e),\nu,\mu_1,\ldots,\mu_j}_{(\mathbf{j},\square),\square,\bullet,\ldots}(\theta) \\ &= N_{(\mathbf{j},\square),\square}\left[e_1^{\nu}(\theta)e_2^{(\mu_1}(\theta)e_1^{\mu_2}(\theta)\cdots e_1^{\mu_j)}(\theta) - e_2^{\nu}(\theta)e_1^{(\mu_1}(\theta)e_1^{\mu_2}(\theta)\cdots e_1^{\mu_j)}(\theta)\right]. \end{aligned} \tag{D.123}$$

This implies

$$\begin{aligned} &N^{-2}_{(\mathbf{j},\square),\square}T^{(e)}_{(\mathbf{j},\square),\square,\bullet,\ldots}\cdot\pi\cdot T^{(e)}_{(\mathbf{j},\square),\square,\bullet,\ldots}(\theta) \\ &= j^{-2}(e_2\cdot\partial_{z_1})(e_2(\theta)\cdot\partial_{\bar{z}_1})\pi_{(j,1)}(e_1, e_1, e_1(\theta), e_1(\theta)) \\ &- j^{-1}(e_2\cdot\partial_{z_1})\pi_{(j,1)}(e_1, e_1, e_1(\theta), e_2(\theta)) \\ &- j^{-1}(e_2(\theta)\cdot\partial_{\bar{z}_1})\pi_{(j,1)}(e_1, e_2, e_1(\theta), e_1(\theta)) \\ &+ \pi_{(j,1)}(e_1, e_2, e_1(\theta), e_2(\theta)), \end{aligned} \tag{D.124}$$

where the values of the arguments of $\pi_{(j,1)}(z_1, z_2, \bar{z}_1, \bar{z}_2)$ should be substituted after taking the derivatives. Using the explicit form of the projector [82], and using the normalization condition $P^{(\mathbf{j},\square),\bullet}_{\square,\square}(0) = 1$, we find

$$N_{(\mathbf{j},\square),\square} = \sqrt{\frac{2^j j(d+j-3)(\nu)_j}{2\nu^2(j+1)^2(2\nu+2)_{j-1}}}, \tag{D.125}$$

$$P^{(\mathbf{j},\square),\bullet}_{\square,\square}(\theta) = \frac{(j-1)!}{(2\nu+2)_{j-1}}C^{(\nu+1)}_{j-1}(\cos\theta). \tag{D.126}$$

Appendix E

APPENDICES TO CHAPTER 6

E.1 Correlators and tensor structures with continuous spin

In this appendix we assume that there exists a continuous-spin operator $\mathbb{O}(x, z)$ and study its Wightman functions. Note that here we are concerned with physical correlators. In other parts of chapter 6 we discuss the existence of continuous-spin conformal invariants for fixed causal relations between the operator insertions, which is a very different problem – Wightman functions must be well defined for arbitrary causal relationships between points.

E.1.1 Analyticity properties of Wightman functions

Recall that Wightman functions of local operators are analytic in their arguments when the appropriate $i\epsilon$ prescription is introduced. More precisely, consider a Wightman function of local operators (suppressing polarization vectors for simplicity)

$$\langle\Omega|\mathcal{O}_n(x_n)\cdots\mathcal{O}_1(x_1)|\Omega\rangle, \tag{E.1}$$

and let us split each x_k into its real and imaginary parts,

$$x_k = y_k + i\zeta_k, \qquad y_k, \zeta_k \in \mathbb{R}^{d-1,1}. \tag{E.2}$$

The Wightman function (E.1) is analytic in the following region [16, 298] (see [160] for a nice review):[1]

$$\zeta_1 > \zeta_2 > \cdots > \zeta_n. \tag{E.3}$$

Here, the notation $p > q$ means that $p - q$ is timelike and future-pointing. We will refer to this analyticity property as *positive-energy analyticity*.

Positive-energy analyticity can be derived in the following way. We first represent the Wightman function (E.1) as a Fourier transform

$$\langle\Omega|\mathcal{O}_n(x_n)\cdots\mathcal{O}_1(x_1)|\Omega\rangle = \int \frac{d^d p_1}{(2\pi)^d}\cdots\frac{d^d p_n}{(2\pi)^d} e^{-ip_1x_1\ldots-ip_nx_n}\langle\Omega|\mathcal{O}_n(p_n)\cdots\mathcal{O}_1(p_1)|\Omega\rangle. \tag{E.4}$$

[1] In fact, these functions are analytic in an even larger region [16, 298], but we do not study consequences of this extended analyticity in this work.

The existence of the Fourier transform follows from the Wightman temperedness axiom. The Heisenberg equation implies

$$[H, \mathcal{O}_i(x_i)] = -i\frac{\partial}{\partial x_i^0}\mathcal{O}_i(x_i) \qquad \Longrightarrow \qquad [H, \mathcal{O}_i(p_i)] = p_i^0 \mathcal{O}_i(p_i), \tag{E.5}$$

and thus

$$H\mathcal{O}_i(p_i)\cdots\mathcal{O}_1(p_1)|\Omega\rangle = (p_1^0 + \ldots + p_i^0)\mathcal{O}_i(p_i)\cdots\mathcal{O}_1(p_1)|\Omega\rangle. \tag{E.6}$$

In physical theories, all states have positive energies. Furthermore, positivity should hold in any Lorentz frame. Thus, we conclude that whenever $\langle\Omega|\mathcal{O}_n(p_n)\cdots\mathcal{O}_1(p_1)|\Omega\rangle$ is nonvanishing,

$$p_1 + \ldots + p_i \geq 0 \qquad (i = 1, \ldots, n). \tag{E.7}$$

Here, the notation $p \geq 0$ means that p is timelike or null and future-pointing. Note that the real part of the exponential factor in (E.4) is given by

$$\begin{aligned}\exp\left(\zeta_1 \cdot p_1 + \ldots \zeta_n \cdot p_n\right) = \exp[&(p_n + \ldots + p_1)\cdot\zeta_n \\ &+ (p_{n-1} + \ldots + p_1)\cdot(\zeta_{n-1} - \zeta_n) \\ &+ (p_{n-2} + \ldots + p_1)\cdot(\zeta_{n-2} - \zeta_{n-1}) \\ &+ \ldots \\ &+ p_1 \cdot (\zeta_1 - \zeta_2)],\end{aligned} \tag{E.8}$$

where $\zeta_k = \mathrm{Im}(x_k)$. By translation-invariance, the first term in the exponential $(p_n + \ldots + p_1)\cdot\zeta_n$ can be replaced with zero. Suppose that the ζ_k satisfy (E.3). Due to (E.7), all other terms in the exponential are non-positive and serve to damp the integral (E.4). Thus, we can make sense of the Wightman function as an analytic function in this region.

The above discussion in no way depends on locality properties of $\mathcal{O}_i$. The only information about $\mathcal{O}_i$ that we needed was the Heisenberg equation (E.5). This is of course also satisfied by continuous-spin primary operators $\mathbb{O}(x, z)$, because it is simply part of the definition of being primary. This means that positive-energy analyticity also holds for Wightman functions involving continuous-spin operators. In the main text we construct examples of continuous-spin operators for which positive-energy analyticity can be checked explicitly.

This clarifies the properties of $\mathbb{O}(x, z)$ with respect to x. However, $\mathbb{O}(x, z)$ is also a non-trivial function of z, and it is interesting to study analyticity in z. For this, assume that we have already adopted the appropriate $i\epsilon$-prescription. By using Lorentz and translation symmetries we can assume that we have inserted $\mathbb{O}(x, z)$ at $x = i\epsilon\hat{e}_0 = (i\epsilon, 0, \ldots, 0)$ with $\epsilon > 0$. Then we have for $i, j = 1 \ldots d-1$

$$[M_{ij}, \mathbb{O}(i\epsilon\hat{e}_0, z)] = (z^j\partial_{z^i} - z^i\partial_{z^j})\mathbb{O}(i\epsilon\hat{e}_0, z), \tag{E.9}$$

and so we have an $\mathrm{Spin}(d-1) \subset \widetilde{\mathrm{SO}}(d,2)$ subgroup which stabilizes position of $\mathbb{O}$ and allows us to change z. In particular, together with the homogeneity property (6.44) it allows us to relate all future-directed null z to $z = \hat{e}_0 + \hat{e}_1 = (1, 1, 0, \ldots, 0)$. Let $U_z \in \mathrm{Spin}(d-1)$ that takes $\alpha_z(\hat{e}_0 + \hat{e}_1)$ with $\alpha_z > 0$ to z. Then for a Wightman function with a single continuous-spin operator we can write

$$\begin{aligned}\langle\Omega|\mathcal{O}_n(x_n)\cdots\mathcal{O}_k(x_k)\mathbb{O}(i\epsilon\hat{e}_0, z)\mathcal{O}_{k-1}(x_{k-1})\cdots\mathcal{O}_1(x_1)|\Omega\rangle =\\ = \alpha_z^J\langle\Omega|\mathcal{O}_n(x_n)\cdots\mathcal{O}_k(x_k)U_z\mathbb{O}(i\epsilon\hat{e}_0, \hat{e}_0 + \hat{e}_1)U_z^\dagger\mathcal{O}_{k-1}(x_{k-1})\cdots\mathcal{O}_1(x_1)|\Omega\rangle,\end{aligned} \tag{E.10}$$

and compute the right hand side by acting with U_z and $U_z^\dagger$ on the left and on the right. This action will act on the spin indices of local operators and also shift their positions. Change in the positions will, however, preserve the ordering of imaginary parts ζ_k (E.2), and thus the Wightman function will remain in the region of analyticity.[2] Since we can take U_z to depend on z analytically in a neighborhood of any given z, this implies that in the absence of other continuous-spin operators the left hand side of (E.10) should be analytic in z.

It would be interesting to understand the analyticity conditions in z in presence of other continuous spin operators. This might depend on some extra assumptions about the nature of such operators, but it is natural to expect them to still be analytic. At least this is the case for the integral transforms defined in section 6.2.3, since at fixed $i\epsilon$-prescription these involve integrals of analytic functions.

E.1.2 Two- and three-point functions

Let us now study examples of Wightman functions of continuous-spin operators from the point of view of positive-energy analyticity. This is especially interesting in CFTs because the analytic structure of two- and three-point functions is fixed by conformal symmetry, and this turns out to be in strong tension with positive-energy analyticity. For simplicity, we focus on correlation functions involving the minimal

[2] Note that in principle the stabilizer of $i\epsilon\hat{e}_0$ includes a full $\mathrm{Spin}(d) \in \widetilde{\mathrm{SO}}(d,2)$. However, some of the transformations in $\mathrm{Spin}(d)\backslash\mathrm{Spin}(d-1)$ will change ordering of ζ_k and thus move Wightman function out of the region of analyticity.

number of continuous-spin operators. We also restrict to traceless-symmetric tensor operators. However, the same statements hold for general representations because the part of the tensor structure responsible for the discrete spin labels λ is always positive-energy analytic.

A conformally-invariant two-point function of traceless-symmetric operators has the form

$$\langle \mathbb{O}(x_1, z_1)\mathbb{O}(x_2, z_2)\rangle \propto \frac{\Big(2(x_{12}\cdot z_1)(x_{12}\cdot z_2) - x_{12}^2(z_1\cdot z_2)\Big)^J}{x_{12}^{2(\Delta+J)}}. \tag{E.11}$$

It is easy to check that the denominator is positive-energy analytic for any choice of Wightman ordering, and we only need to study the numerator. For generic z_1 and z_2 we can write

$$x_{12} = \alpha z_1 + \beta z_2 + x_\perp, \tag{E.12}$$

where $x_\perp \cdot z_i = 0$. Note that $x_\perp$ is spacelike, because it is orthogonal to the timelike vector $z_1 + z_2$. (Recall that all polarization vectors are null and future-directed.) The numerator then takes the form

$$\Big(2(x_{12}\cdot z_1)(x_{12}\cdot z_2) - x_{12}^2(z_1\cdot z_2)\Big)^J = (-z_1\cdot z_2)^J x_\perp^{2J} > 0. \tag{E.13}$$

On the one hand, we see that this is positive and well-defined for all real x_i and z_i. On the other hand, we can show that it is only positive-energy analytic for integer $J \geq 0$. Indeed, selecting a Wightman ordering and adding appropriate imaginary parts as in (E.2), in any case we find that $\zeta_\perp$ is a spacelike vector (we can make it non-zero), because it is orthogonal to $z_1 + z_2$. This means that by choosing an appropriate y_{12} we can achieve

$$x_\perp^2 = y_\perp^2 - \zeta_\perp^2 + 2i(y_\perp\cdot\zeta_\perp) = 0, \tag{E.14}$$

and in particular wind $x_\perp^2$ around zero without leaving the region of positive-energy analyticity.[3,4] Thus (E.13) can not be analytic there unless J is a non-negative integer.

[3] To be specific, we can wind $x_\perp^2$ around 0 once with y_{12} returning to the original position, and thus for (E.13) to be single-valued, we need $J \in \mathbb{Z}$.

[4] This argument doesn't work in $d = 3$ because then $y_\perp$ and $\zeta_\perp$ are forced to lie in the same 1-dimensional subspace. In that case we are still free to change both $y_\perp$ and $\zeta_\perp$, and thus $x_\perp = y_\perp + i\zeta_\perp$, in a neighborhood of 0. This leads to a weaker requirement that $J \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. This has to do with the fact that for $d = 3$ the null-cone is not simply-connected and it makes sense to consider multi-valued functions of z. In fact, fermionic operators can be described by double-valued functions of z. (If we write $z_\mu = \chi_\alpha\chi_\beta\sigma_\mu^{\alpha\beta}$ for a real spinor χ, then we get polynomial functions of χ.) Our argument thus shows that only single- and double-valued functions of z are consistent with positive-energy analyticity. In higher dimensions we cannot describe fermionic representations by using a single null polarization and thus we do not get this subtlety.

This implies that the only way the Wightman two-point function of a generic continuous spin operator $\mathbb{O}$ can be positive-energy analytic is by being zero,[5]

$$\langle\Omega|\mathbb{O}(x_1, z_1)\mathbb{O}(x_2, z_2)|\Omega\rangle = 0. \tag{E.15}$$

In unitary theories vanishing of this two-point function implies

$$\mathbb{O}(x, z)|\Omega\rangle = 0. \tag{E.16}$$

This gives another derivation of the fact stated in the introduction: continuous-spin operators must annihilate the vacuum.

Let us now consider a three-point function with a single continuous-spin operator $\mathbb{O}$,

$$\langle O_1(x_1, z_1)O_2(x_2, z_2)\mathbb{O}(x_3, z_3)\rangle \propto f(x_i, z_i)\left(\frac{x_{13}\cdot z_3}{x_{13}^2} - \frac{x_{23}\cdot z_3}{x_{23}^2}\right)^{J_3-n_3}, \tag{E.17}$$

where $f(x_i, z_i)$ is the part of the tensor structure which is manifestly positive-energy analytic, and is a homogeneous polynomial in z_3 with degree $n_3 \geq 0$. The non-trivial part of the correlator can be written as

$$\left(\frac{x_{13}\cdot z_3}{x_{13}^2} - \frac{x_{23}\cdot z_3}{x_{23}^2}\right)^{J_3-n_3} = (v_{12,3}\cdot z_3)^{J_3-n_3}, \tag{E.18}$$

where

$$v_{12,3}^2 = \left(\frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}\right)^2 = \frac{x_{12}^2}{x_{13}^2 x_{23}^2}. \tag{E.19}$$

We see that $v_{12,3}$ can be both spacelike and timelike, depending on the causal relationship between the three points x_i. This immediately implies that, for example, when all x_{ij} are spacelike, the inner product $v_{12,3}\cdot z_3$ is not sign-definite and we need to invoke $i\epsilon$-prescriptions to define $(v_{12,3}\cdot z_3)^{J_3-n_3}$, even for purely Euclidean configurations. For the $i\epsilon$-prescriptions to make sense, the tensor structure must be positive-energy analytic. This means that in this situation, positive-energy analyticity is not only required for correlators to make physical sense, but also simply for the tensor structures to be single-valued.[6] To proceed, note that in the region of positive-energy analyticity $x_{ij}^2 \neq 0$ and furthermore the map

$$x \mapsto \frac{x}{x^2} \tag{E.20}$$

[5] We derived this for generic z_1 and z_2, but as discussed in the previous section, we expect the Wightman functions to be continuous in polarizations.

[6] This is in contrast to the two-point Wightman function case considered above, where (E.13) is single-valued without the $i\epsilon$-prescription.

preserves the set of $x = y + i\zeta$ with future-directed (past-directed) timelike ζ.[7] Since it is also its own inverse, this implies that by varying x_{13} and x_{23} within the region of positive-energy analyticity, we can reproduce any pair of values for $q_1 = \frac{x_{13}}{x_{13}^2}$ and $q_2 = \frac{x_{23}}{x_{23}^2}$ with imaginary parts satisfying the same constraints as those of x_{13} and x_{23} respectively. This means that in the region of positive-energy analyticity for the orderings

$$\langle 0|O_2 \mathbb{O} O_1|0\rangle \quad \text{and} \quad \langle 0|O_1 \mathbb{O} O_2|0\rangle, \tag{E.21}$$

the vector $v_{12,3} = q_1 - q_2$ has a timelike imaginary part restricted to be future-directed or past-directed respectively, while for the orderings

$$\langle 0|O_i O_j \mathbb{O}|0\rangle \quad \text{and} \quad \langle 0|\mathbb{O} O_i O_j|0\rangle \tag{E.22}$$

this imaginary part is not restricted at all. In the former case $v_{12,3} \cdot z_3$ has either negative or positive imaginary part, and thus the inner product cannot vanish or wind around zero, while in the latter case this inner product can vanish or wind around zero. We thus conclude that the Wightman functions (E.21) are positive-energy analytic for any value of J_3, while the Wightman functions (E.22) are positive-energy analytic only for integer $J_3 \geq n_3$.[8]

Again, recalling that the physical Wightman functions of continuous-spin operators must be positive-energy analytic, we are forced to conclude that Wightman functions (E.22) vanish,

$$\langle\Omega|O_1 O_2 \mathbb{O}|\Omega\rangle = \langle\Omega|\mathbb{O} O_1 O_2|\Omega\rangle = 0, \tag{E.23}$$

which of course consistent with the fact that $\mathbb{O}$ annihilates the vacuum. An interesting observation is that the distinction we made above between the Wightman orderings (E.21) and (E.22) conflicts with microcausality, because for spacelike-separated points all these Wightman functions would be equal.[9] This means that

[7] If $x^2 = (y + i\zeta)^2 = y^2 - \zeta^2 + 2iy \cdot \zeta = 0$ with timelike ζ, then $y \cdot \zeta = 0$, which implies that y is spacelike and thus $y^2 - \zeta^2 > 0$, leading to contradiction. Imaginary part of $\frac{x}{x^2}$ is, up to a positive factor, $\zeta(y^2 - \zeta^2) - 2y(y \cdot \zeta)$. For $y = 0$ this is timelike and has the same direction as ζ. For any y, this squares to $\zeta^2((y^2 - \zeta^2)^2 + 4(y \cdot z)^2) < 0$, and thus by continuity $\mathrm{Im}\frac{x}{x^2}$ remains timelike in the direction of ζ.

[8] Recall that $n_3 \leq J_3$ is the standard condition that we encounter when dealing with integer-spin tensor structures, it just means that $f(x_i, z_i)$ must be a polynomial in z_3 of degree at most J_3. The 3d subtlety we discussed in footnote 4 would be visible here as well, if we allowed f to be double-valued in z (and polynomial in χ), which would correspond to making the product $O_1 O_2$ fermionic, thus forcing J to be half-integer.

[9] Recall that as noted above, the region of spacelike separation is the problematic one, because there $v_{12,3}$ is spacelike and $v_{12,3} \cdot z_3$ is not sign-definite.

non-trivial continuous-spin operators *must* be non-local, as stated in the introduction, in the sense that they cannot satisfy microcausality.

A consequence of non-locality is that a physical correlator involving a continuous-spin operator is not well-defined without specifying an operator ordering even if all the distances are spacelike. This in particular means that time-ordered correlators are not quite well-defined in the presence of continuous-spin operators (i.e. how do we order $\mathbb{O}$ when it is spacelike from something?). This also makes it unclear how one would define Euclidean correlators for continuous spin (the usual Wick-rotation to Euclidean signature requires micro-causality). Another problem with attempting to describe continuous-spin operators in Euclidean signature is that under Euclidean rotation group $\mathrm{SO}(d)$ the orbit of a single null direction in $\mathbb{R}^{d-1,1}$ consists of all null directions in $\mathbb{C}^d$. Thus we would need to define $\mathbb{O}(x, z)$ for all complex null z, but above it was very important to have *future-directed real* z to establish positive-energy analyticity of at least some Wightman functions.

E.1.3 Conventions for two- and three-point tensor structures

When working with integer spin the simplest way to specify standard tensor structures is to give their expressions in Euclidean signature or, equivalently, in Lorentzian signature with all points are spacelike separated. With continuous spin, Euclidean signature is not an option, and as we saw above even for spacelike separations in Lorentzian signature care must be taken to define phases of three-point functions. In this section we briefly record our conventions for symmetric tensor operators.

We will choose the following convention for a two-point function in Lorentzian signature:

$$\begin{aligned}\langle O(x_1, z_1)O(x_2, z_2)\rangle &= \frac{(-2z_1 \cdot I(x_{12})z_2)^J}{x_{12}^{2\Delta}}\\ I^\mu{}_\nu(x) &= \delta^\mu{}_\nu - 2\frac{x^\mu x_\nu}{x^2}.\end{aligned} \tag{E.24}$$

The nonstandard numerator is so that the two-point function is positive when 1 and 2 are spacelike separated and $z_{1,2}$ are future-pointing null vectors. For local operators this completely defines standard Wightman two-point functions via $i\epsilon$ prescriptions. For continuous-spin operators physical Wightman functions vanish, but we still need two-point conformal invariants in some calculations (like the definition of the **S**-transform), and for these purposes it suffices to specify the two-point invariant for spacelike x_{12}.

Now consider a three-point function $\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}(x_3,z)\rangle$, where ϕ_1 and ϕ_2 are scalars and $\mathcal{O}$ has dimension Δ and spin J. We demand that the correlator (either Wightman or time-ordered) should be positive when $1,2,3$ are mutually spacelike and $z\cdot x_{23}\,x_{13}^2 - z\cdot x_{13}\,x_{23}^2 > 0$. Our precise convention is

$$\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}(x_3,z)\rangle = \frac{\left(2z\cdot x_{23}\,x_{13}^2 - 2z\cdot x_{13}\,x_{23}^2\right)^J}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}\,x_{13}^{\Delta_1+\Delta-\Delta_2+J}\,x_{23}^{\Delta_2+\Delta-\Delta_1+J}}. \tag{E.25}$$

This is unambiguous for local operators since at spacelike separations there is no difference between various Wightman orderings and time-ordering.[10] If J is continuous, we are necessarily talking about a Wightman function and we need to specify the ordering. Our choice is

$$\langle 0|\phi_1(x_1)\mathbb{O}(x_3,z)\phi_2(x_2)|0\rangle = \frac{\left(2z\cdot x_{23}\,x_{13}^2 - 2z\cdot x_{13}\,x_{23}^2\right)^J}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}\,x_{13}^{\Delta_1+\Delta-\Delta_2+J}\,x_{23}^{\Delta_2+\Delta-\Delta_1+J}}, \tag{E.26}$$

defined to be positive under the same conditions as (E.25).

The nontraditional factors of 2 in (E.24) and (E.25) are so that the associated conformal blocks have simple behavior in the limit of small cross-ratios

$$\frac{\langle\phi_1\phi_2\mathcal{O}\rangle\langle\mathcal{O}\phi_3\phi_4\rangle}{\langle\mathcal{O}\mathcal{O}\rangle} \sim \left(\prod x_{ij}^{\#}\right)\chi^{\frac{\Delta-J}{2}}\overline{\chi}^{\frac{\Delta+J}{2}} \qquad \chi \ll \overline{\chi} \ll 1. \tag{E.27}$$

They also simplify several formulae in the main text.

E.2 Relations between integral transforms

E.2.1 Square of light transform

In this appendix we explicitly compute the square of the light transform. In order to do this, we need to assume that the operator that the light transform acts upon belongs to the Lorentzian principal series

$$\Delta = \frac{d}{2} + is, \quad J = -\frac{d-2}{2} + iq, \tag{E.28}$$

so that $\Delta + J = 1 + i(s+q) = 1 + i\omega$ and $\Delta^L + J^L = 2 - \Delta - J = 1 - i(s+q) = 1 - i\omega$ and thus both the first and the second light transforms make sense if $w \neq 0$.

It will also be convenient to use the expression for the light transform in the coordinates $(\tau, \vec{e})$ on $\widetilde{\mathcal{M}}_d$. In these coordinates the polarization vector z can be described

[10] Note however that this notation for the standard structure is somewhat abusive. For physical correlators we of course have $\langle\phi_1\phi_2\mathcal{O}\rangle_\Omega = \langle\phi_2\phi_1\mathcal{O}\rangle_\Omega$, but the standard structure (E.25) gains a $(-1)^J$ under this permutation. This leads to several appearances of $(-1)^J$ in our formulas which are awkward to explain.

as $(z^0, \vec{z})$ where $\vec{z}$ is tangent to S^{d-1} at $\vec{e}$, i.e. $\vec{z}\cdot\vec{e} = 0$, and we have $(z^0)^2 = |\vec{z}|^2$. We then have

$$\begin{aligned}&\mathbf{L}[O](\tau, \vec{e}; z^0, \vec{z})\\ &= \int_0^\pi d\kappa\,(\sin\kappa)^{\Delta+J-2}(z^0)^{1-\Delta} O(\tau+\kappa, \cos\kappa\,\vec{e} + \sin\kappa\,\tfrac{\vec{z}}{z^0}; 1, \cos\kappa\,\tfrac{\vec{z}}{z^0} - \sin\kappa\,\vec{e}).\end{aligned} \tag{E.29}$$

Note that this form also makes it manifest that there is no singularity associated to $\alpha = 0$ in (6.60).

The square of light transform becomes

$$\begin{aligned}\mathbf{L}^2[O](\tau, \vec{e}; z^0, \vec{z}) &= \int_0^\pi\int_0^\pi d\kappa d\kappa'(z^0)^J(\sin\kappa')^{-\Delta-J}(\sin\kappa)^{\Delta+J-2}\\ &\quad\times O(\tau+\kappa+\kappa', \cos(\kappa+\kappa')\vec{e} + \sin(\kappa+\kappa')\tfrac{\vec{z}}{z^0}; 1, \cos(\kappa+\kappa')\tfrac{\vec{z}}{z^0} - \sin(\kappa+\kappa')\vec{e})\\ &= \int_0^{2\pi} d\kappa K(\kappa)(z^0)^J O(\tau+\kappa, \cos\kappa\,\vec{e} + \sin\kappa\,\tfrac{\vec{z}}{z^0}; 1, \cos\kappa\,\tfrac{\vec{z}}{z^0} - \sin\kappa\,\vec{e}),\end{aligned} \tag{E.30}$$

where

$$K(\kappa) = \int_{\max(-\kappa/2, \kappa/2-\pi)}^{\min(\kappa/2, \pi-\kappa/2)} d\eta(\sin\frac{\kappa}{2} - \eta)^{-1-i\omega}(\sin\frac{\kappa}{2} + \eta)^{-1+i\omega}. \tag{E.31}$$

To compute $K(\kappa)$, for $\kappa \neq 0, \pi, 2\pi$ we can use the substitution

$$e^\beta = \frac{\sin\left(\frac{\kappa}{2} + \eta\right)}{\sin\left(\frac{\kappa}{2} - \eta\right)}, \tag{E.32}$$

which turns the integral into

$$K(\kappa) = \frac{1}{\sin\kappa}\int_{-\infty}^{+\infty} d\beta e^{iw\beta} = 0, \quad (\omega \neq 0). \tag{E.33}$$

This means that $K(\kappa)$ is supported at $\kappa = 0, \pi, 2\pi$. Let us thus consider first the contribution near $\kappa = 0$. Near $\kappa = 0$ we can expand both sines and find, introducing a regulator ϵ,

$$\begin{aligned}K(\kappa) &= \int_{-\kappa/2}^{\kappa/2} d\eta\left(\frac{\kappa}{2} - \eta\right)^{-1-i\omega+\epsilon}\left(\frac{\kappa}{2} + \eta\right)^{-1+i\omega+\epsilon}\\ &= \kappa^{-1+2\epsilon}\int_{-1/2}^{1/2} d\eta\left(\frac{1}{2} - \eta\right)^{-1-i\omega+\epsilon}\left(\frac{1}{2} + \eta\right)^{-1+i\omega+\epsilon}\\ &= (2\epsilon)\kappa^{-1+2\epsilon}\frac{\Gamma(i\omega+\epsilon)\Gamma(-i\omega+\epsilon)}{(2\epsilon)\Gamma(2\epsilon)}. \qquad (\kappa \ll 1)\end{aligned} \tag{E.34}$$

For $\epsilon \to 0$, using

$$(2\epsilon)\kappa^{-1+2\epsilon} \to \delta(\kappa), \quad (\kappa > 0) \tag{E.35}$$

we find

$$K(\kappa) = \Gamma(-i\omega)\Gamma(i\omega)\delta(\kappa) = \frac{\pi}{(\Delta + J - 1)\sin\pi(\Delta + J)}\delta(\kappa), \quad (\kappa \ll 1). \tag{E.36}$$

The calculation near $\kappa = 2\pi$ is the same and thus we have

$$K(\kappa) = \frac{\pi}{(\Delta + J - 1)\sin\pi(\Delta + J)}(\delta(\kappa) + \delta(\kappa - 2\pi)) + \langle\text{contribution from } \pi\rangle \tag{E.37}$$

To find the contribution from $\kappa = \pi$, write $\kappa = \pi - r$ for small $0 < r \ll 1$.[11] We have now

$$\begin{aligned}
K(\kappa) &= \int_{-\frac{\pi}{2}+\frac{r}{2}}^{\frac{\pi}{2}-\frac{r}{2}} d\eta \left(\sin\frac{\pi}{2} - \frac{r}{2} - \eta\right)^{-1-i\omega} \left(\sin\frac{\pi}{2} - \frac{r}{2} + \eta\right)^{-1+i\omega} \\
&= \int_0^{\pi-r} d\eta \, (\sin r + \eta)^{-1-i\omega} \, (\sin\eta)^{-1+i\omega} \\
&\approx \int_0^{Nr} d\eta \, (r+\eta)^{-1-i\omega+\epsilon} \, \eta^{-1+i\omega+\epsilon} + \int_0^{Nr} d\eta \, (r+\eta)^{-1+i\omega+\epsilon} \, \eta^{-1-i\omega+\epsilon} \\
&= r^{-1+2\epsilon}\left[\int_0^{\infty} d\eta \, (1+\eta)^{-1+i\omega+\epsilon} \, \eta^{-1-i\omega+\epsilon} + \int_0^{\infty} d\eta \, (1+\eta)^{-1-i\omega+\epsilon} \, \eta^{-1+i\omega+\epsilon}\right] \\
&= r^{-1+2\epsilon}\frac{\pi\Gamma(1-2\epsilon)}{\Gamma(2-J-\Delta-\epsilon)\Gamma(J+\Delta-\epsilon)}\left(\csc(\pi(J+\Delta-\epsilon)) - \csc(\pi(J+\Delta+\epsilon))\right).
\end{aligned} \tag{E.38}$$

Here $0 \ll r \ll Nr \ll 1$ and the two terms come from the two sides of the integral. We can now compute for small $\Lambda > 0$

$$\lim_{\epsilon\to 0}\int_{\pi-\Lambda}^{\pi} K(\kappa)d\kappa = -\frac{\pi\cos\pi(\Delta + J)}{(\Delta + J - 1)\sin\pi(\Delta + J)}. \tag{E.39}$$

Recalling also that there is also a contribution from the negative values of r, we find the final result

$$K(\kappa) = \frac{\pi}{(\Delta + J - 1)\sin\pi(\Delta + J)}\left(\delta(\kappa) - 2\cos\pi(\Delta + J)\delta(\kappa - \pi) + \delta(\kappa - 2\pi)\right). \tag{E.40}$$

In terms of action on $\mathcal{O}$ this immediately implies

$$\begin{aligned}
\mathbf{L}^2 &= \frac{\pi}{(\Delta + J - 1)\sin\pi(\Delta + J)}\left(1 - 2\cos\pi(\Delta + J)\mathcal{T} + \mathcal{T}^2\right) \\
&= \frac{\pi}{(\Delta + J - 1)\sin\pi(\Delta + J)}\left(\mathcal{T} - e^{i\pi(\Delta+J)}\right)\left(\mathcal{T} - e^{-i\pi(\Delta+J)}\right).
\end{aligned} \tag{E.41}$$

[11] There is going a similar contribution from $r < 0$.

E.2.2 Relation between shadow transform and light transform

In this appendix we prove the relation (6.110). As in the preceding part of this appendix, we must assume that (6.110) acts on an operator in the Lorentzian principal series so that this action is well-defined. We have

$$\begin{aligned}&\mathbf{L}\mathbf{S}_J\mathbf{L}[\mathcal{O}](x,z)\\&=\int D^{d-2}z'd\alpha_1 d\alpha_2(-\alpha_1)^{-\Delta-J}(-\alpha_2)^{d-2+J-\Delta}(-2z\cdot z')^{1-d+\Delta}\mathcal{O}(x-z'/\alpha_1-z/\alpha_2,z')\end{aligned}\tag{E.42}$$

Let us write $x'=x-z'/\alpha_1-z/\alpha_2$. Then we have

$$I(x-x')z=z-2\frac{(z'/\alpha_1+z/\alpha_2)(z'/\alpha_1+z/\alpha_2)\cdot z}{(z'/\alpha_1+z/\alpha_2)^2}=-\frac{\alpha_2}{\alpha_1}z'.\tag{E.43}$$

Considering the integral in the region of large negative α_1 and α_2 we find

$$\begin{aligned}&\int D^{d-2}z'd\alpha_1 d\alpha_2(-\alpha_1)^{-\Delta-J}(-\alpha_2)^{d-2+J-\Delta}\left(-\alpha_1\alpha_2(x-x')^2\right)^{1-d+\Delta}\left(\frac{\alpha_1}{\alpha_2}\right)^J\mathcal{O}(x',-I(x-x')z)\\&=\int D^{d-2}z'd\alpha_1 d\alpha_2(-\alpha_1)^{1-d}(-\alpha_2)^{-1}\left(-(x-x')^2\right)^{1-d+\Delta}\mathcal{O}(x',-I(x-x')z)\end{aligned}\tag{E.44}$$

We would now like to replace the integral $\int D^{d-2}z'd\alpha_1 d\alpha_2$ by $\int d^dx'$. For this we write

$$1=\int d^dx'\delta^d(x-x'-z'/\alpha_1-z/\alpha_2)\tag{E.45}$$

and then compute

$$\begin{aligned}&\int D^{d-2}z'd\alpha_1 d\alpha_2(-\alpha_1)^{1-d}(-\alpha_2)^{-1}\delta^d(x-x'-z'/\alpha_1-z/\alpha_2)\\&=\int\frac{d^dz'd\alpha_1 d\alpha_2}{\mathrm{vol}\mathbb{R}}\theta(z'^0)\delta(z'^2)(-\alpha_1)(-\alpha_2)^{-1}\delta^d(-\alpha_1(x-x')+z'+\alpha_1 z/\alpha_2)\\&=\int\frac{d\alpha_1 d\alpha_2}{\mathrm{vol}\mathbb{R}}\delta(((x-x')-z/\alpha_2)^2)(-\alpha_1)^{-1}(-\alpha_2)^{-1}\\&=-(x-x')^{-2}.\end{aligned}\tag{E.46}$$

We thus conclude that (E.44) is equal to

$$\int d^dx'(-(x-x')^2)^{\Delta-d}\mathcal{O}(x',-I(x-x')z).\tag{E.47}$$

More precisely, it is the contribution to (E.42) from the region of large negative α_i. We recognize that it has precisely the form of $\mathcal{T}$-shifted Lorentzian shadow integral (6.54), i.e.

$$\mathbf{S}_\Delta=i\mathcal{T}^{-1}\mathbf{L}\mathbf{S}_J\mathbf{L}.\tag{E.48}$$

E.3 Harmonic analysis for the Euclidean conformal group

E.3.1 Pairings between three-point structures

The conformal representation of an operator $\mathcal{O}$ is labeled by a scaling dimension Δ and an SO(d) representation ρ. The representation $\widetilde{\mathcal{O}}^\dagger$ has dimension $d-\Delta$ and SO(d) representation ρ^* (the dual of ρ). Thus, there is a natural conformally-invariant pairing between n-point functions of $\mathcal{O}_i$'s and n-point functions of $\widetilde{\mathcal{O}}_i^\dagger$'s, given by multiplying and integrating over all points modulo the conformal group,

$$\left(\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle,\langle\widetilde{\mathcal{O}}_1^\dagger\cdots\widetilde{\mathcal{O}}_n^\dagger\rangle\right)_E=\int\frac{d^dx_1\cdots d^dx_n}{\mathrm{vol}(\mathrm{SO}(d+1,1))}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle\langle\widetilde{\mathcal{O}}_1^\dagger\cdots\widetilde{\mathcal{O}}_n^\dagger\rangle. \tag{E.49}$$

Here, we are implicitly contracting Lorentz indices between each pair $\mathcal{O}_i$ and $\widetilde{\mathcal{O}}_i^\dagger$. The "$E$" subscript stands for "Euclidean."

This pairing is particularly simple for three-point structures. In that case, we can use conformal transformations to set $x_1=0, x_2=e, x_3=\infty$ (with e a unit vector), and no integrations are necessary. The pairing becomes simply

$$\begin{aligned}&\left(\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle,\langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_2^\dagger\widetilde{\mathcal{O}}_3^\dagger\rangle\right)_E\\&=\frac{1}{2^d\mathrm{vol}(\mathrm{SO}(d-1))}\langle\mathcal{O}_1(0)\mathcal{O}_2(e)\mathcal{O}_3(\infty)\rangle\langle\widetilde{\mathcal{O}}_1^\dagger(0)\widetilde{\mathcal{O}}_2^\dagger(e)\widetilde{\mathcal{O}}_3^\dagger(\infty)\rangle.\end{aligned} \tag{E.50}$$

The factor 2^{-d} comes from the Fadeev-Popov determinant for the above gauge-fixing.[12] The factor vol(SO($d-1$)) is the volume of the stabilizer group of three points.

As an example, a scalar-scalar-spin-J correlator has a single tensor structure $\langle\phi_1\phi_2\mathcal{O}_{3,J}\rangle$ given in (E.25). The pairing in that case is

$$\begin{aligned}&\left(\langle\phi_1\phi_2\mathcal{O}_{3,J}\rangle,\langle\widetilde{\phi}_1\widetilde{\phi}_2\widetilde{\mathcal{O}}_{3,J}\rangle\right)_E\\&=\frac{2^{2J}}{2^d\mathrm{vol}(\mathrm{SO}(d-1))}(e^{\mu_1}\cdots e^{\mu_J}-\text{traces})(e_{\mu_1}\cdots e_{\mu_J}-\text{traces})\\&=\frac{2^{2J}\hat{C}_J(1)}{2^d\mathrm{vol}(\mathrm{SO}(d-1))},\end{aligned} \tag{E.51}$$

where $\hat{C}_J(x)$ is defined in (E.152).

[12] Note that [67] used a convention where vol(SO($d+1,1$)) was defined to include an extra factor of 2^{-d} to cancel the Fadeev-Popov determinant. Here, we prefer not to cancel this factor because it simplifies other formulae in this work.

E.3.2 Euclidean conformal integrals

Suppose $\mathcal{O}, \mathcal{O}'$ are principal series representations, with dimensions $\Delta = \frac{d}{2} + is$, $\Delta' = \frac{d}{2} + is'$ with $s, s' > 0$ and $\mathrm{SO}(d)$ representations ρ, ρ'. A "bubble" integral of two three-point functions is proportional to their three-point pairing,

$$\begin{aligned} \int d^d x_1 d^d x_2 \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}^a(x) \rangle \langle \widetilde{\mathcal{O}}_1^\dagger \widetilde{\mathcal{O}}_2^\dagger \widetilde{\mathcal{O}}_b'^\dagger(x') \rangle &= \frac{\left(\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O} \rangle, \langle \widetilde{\mathcal{O}}_1^\dagger \widetilde{\mathcal{O}}_2^\dagger \widetilde{\mathcal{O}}^\dagger \rangle \right)_E}{\mu(\Delta, \rho)} \delta_b^a \delta(x - x') \delta_{\mathcal{O}\mathcal{O}'}, \\ \delta_{\mathcal{O}\mathcal{O}'} &\equiv 2\pi \delta(s - s') \delta_{\rho\rho'}. \end{aligned} \tag{E.52}$$

The right-hand side contains a term $\delta_{\mathcal{O}\mathcal{O}'}$ restricting the representations $\mathcal{O}, \mathcal{O}'$ to be the same, since this is the only possibility allowed by conformal invariance.[13] Here, a, b are indices for the representations ρ, ρ^* of $\mathrm{SO}(d)$, respectively. We have suppressed the $\mathrm{SO}(d)$ indices of the other operators, for brevity.

The factor $\mu(\Delta, \rho)$ in the denominator is called the Plancherel measure. It is known in great generality [65] (see [195] for an elementary derivation). In this work, we will only need $\mu(\Delta, J)$ for symmetric traceless tensors:

$$\begin{aligned} \mu(\Delta, J) &= \frac{\dim \rho_J}{2^d \mathrm{vol}(\mathrm{SO}(d))} \frac{\Gamma(\Delta - 1)\Gamma(d - \Delta - 1)(\Delta + J - 1)(d - \Delta + J - 1)}{\pi^d \Gamma(\Delta - \frac{d}{2})\Gamma(\frac{d}{2} - \Delta)}, \\ \dim \rho_J &= \frac{\Gamma(J + d - 2)(2J + d - 2)}{\Gamma(J + 1)\Gamma(d - 1)}. \end{aligned} \tag{E.53}$$

Here $\dim \rho_J$ is the dimension of the spin-J representation of $\mathrm{SO}(d)$.

Another conformal integral we will need is the Euclidean shadow transform of a three-point function of two scalars and a symmetric traceless tensor

$$\begin{aligned} \langle \phi_1 \phi_2 \mathbf{S}_E[\mathcal{O}](y) \rangle &= \int d^d x \langle \widetilde{\mathcal{O}}(y) \widetilde{\mathcal{O}}^\dagger(x) \rangle \langle \phi_1 \phi_2 \mathcal{O}(x) \rangle \\ &= S_E(\phi_1 \phi_2 [\mathcal{O}]) \langle \phi_1 \phi_2 \widetilde{\mathcal{O}}(y) \rangle, \end{aligned} \tag{E.54}$$

where

$$S_E(\phi_1 \phi_2 [\mathcal{O}]) = (-2)^J \frac{\pi^{d/2} \Gamma(\Delta - \frac{d}{2}) \Gamma(\Delta + J - 1)}{\Gamma(\Delta - 1)\Gamma(d - \Delta + J)} \frac{\Gamma(\frac{d - \Delta + \Delta_1 - \Delta_2 + J}{2}) \Gamma(\frac{d - \Delta + \Delta_2 - \Delta_1 + J}{2})}{\Gamma(\frac{\Delta + \Delta_1 - \Delta_2 + J}{2}) \Gamma(\frac{\Delta + \Delta_2 - \Delta_1 + J}{2})}. \tag{E.55}$$

The factor of $(-2)^J$ relative to [67] is because we are using a different normalization convention for the two-point function (E.24).

[13] Eq. (E.52) is sometimes written including two terms — one with $\delta(s - s')$ and another with $\delta(s + s')$. Here we have only one term because we have restricted $s, s' > 0$. The other term can be obtained by performing the shadow transform on either $\mathcal{O}$ or $\widetilde{\mathcal{O}}'^\dagger$.

The square of the shadow transform is related to the Plancherel measure by [65] (see [195] for an elementary derivation)

$$\mathbf{S}_E^2 = \frac{1}{\mu(\Delta,\rho)} \frac{\langle \mathcal{O}(0)\mathcal{O}^\dagger(\infty)\rangle\langle \widetilde{\mathcal{O}}(\infty)\widetilde{\mathcal{O}}^\dagger(0)\rangle}{2^d \mathrm{vol}(\mathrm{SO}(d))} \equiv \mathcal{N}(\Delta,\rho), \tag{E.56}$$

where the indices in two-point functions are implicitly contracted. In the case of a spin-J representation, we have

$$\mathcal{N}(\Delta, J) = \frac{2^{2J} \dim \rho_J}{2^d \mu(\Delta, J)\mathrm{vol}(\mathrm{SO}(d))}, \tag{E.57}$$

Indeed, we can easily verify

$$S_E(\phi_1\phi_2[\mathcal{O}])S_E(\phi_1\phi_2[\widetilde{\mathcal{O}}]) = \mathcal{N}(\Delta, J). \tag{E.58}$$

E.3.3 Residues of Euclidean partial waves

In this section, we prove 6.124. The proof for primary four-point functions is standard (see e.g. [65, 67]). We now give a slightly more complicated argument that works for n-point functions. However, the key ingredients are identical to the standard argument.

Consider the integral in the completeness relation (6.119),

$$I = \int d^d x P_{\Delta,J}(x)\langle \widetilde{\mathcal{O}}(x)\phi_1\phi_2\rangle. \tag{E.59}$$

The partial wave $P_{\Delta,J}$ also depends on the coordinates $x_3, \dots, x_k$, but they don't play a role in the current discussion so we have suppressed them. We have also suppressed Lorentz indices. When we have a product of an operator and its shadow at coincident points, we will assume their Lorentz indices are contracted.

Note that I is an eigenvector of the Casimirs of the conformal group acting simultaneously on points 1 and 2. Thus, it is completely determined by its behavior in the OPE limit $x_1 \to x_2$. There are two contributions in this limit. The first comes from the regime where x is sufficiently far from x_1, x_2 that we can use the 1×2 OPE inside the integrand:

$$\langle \phi_1\phi_2\widetilde{\mathcal{O}}(x)\rangle = C_{12\widetilde{\mathcal{O}}}(x_1, x_2, x', \partial_{x'})\langle \widetilde{\mathcal{O}}(x')\widetilde{\mathcal{O}}(x)\rangle. \tag{E.60}$$

Here, $C_{12\widetilde{\mathcal{O}}}$ is a differential operator that encodes the sum over descendants in the $\phi_1 \times \phi_2$ OPE. The point x' can be chosen arbitrarily inside a sphere separating x_1, x_2 from all other points. We will abbreviate the right-hand side of (E.60) as

$C_{12\widetilde{O}}(x')\langle\widetilde{O}(x')\widetilde{O}(x)\rangle$. Inserting (E.60) and applying the shadow transform to the definition of $P_{\Delta,J}$ (6.121), we find

$$I \supset C_{12\widetilde{O}}(x') \int d^d x \langle\widetilde{O}(x')\widetilde{O}(x)\rangle P_{\Delta,J}(x) = S_E(\phi_1\phi_2[O])C_{12\widetilde{O}}(x)P_{\widetilde{\Delta},J}(x). \tag{E.61}$$

The second contribution to I comes from the regime where x is near both x_1, x_2 but far away from all other points. In this case, we can insert a shadow transform and then perform the OPE:

$$\begin{aligned} I &= S_E(\phi_1\phi_2[O])^{-1} \int d^d x d^d x' P_{\Delta,J}(x)\langle\widetilde{O}(x)\widetilde{O}(x')\rangle\langle O(x')\phi_1\phi_2\rangle \\ &\supset S_E(\phi_1\phi_2[O])^{-1} \int d^d x d^d x' P_{\Delta,J}(x)\langle\widetilde{O}(x)\widetilde{O}(x')\rangle C_{12O}(x'')\langle O(x'')O(x')\rangle \\ &= S_E(\phi_1\phi_2[O])^{-1}\mathcal{N}(\Delta,J)C_{12O}(x)P_{\Delta,J}(x) \\ &= S_E(\phi_1\phi_2[\widetilde{O}])C_{12O}(x)P_{\Delta,J}(x). \end{aligned} \tag{E.62}$$

Where we have used (E.58).

The two contributions (E.61) and (E.62) are already eigenvectors of the conformal Casimirs, so together they give the full answer for I. The two terms differ simply by the replacement $\Delta \leftrightarrow d-\Delta$. Thus, we can plug them into the completeness relation (6.119) and use $\Delta \leftrightarrow d-\Delta$ symmetry to extend the Δ integral along the entire imaginary axis,

$$\langle V_3\cdots V_k O_1 O_2\rangle_\Omega = \sum_{J=0}^{\infty}\int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}\mu(\Delta,J)S_E(\phi_1\phi_2[\widetilde{O}])C_{12O}P_{\Delta,J}(x). \tag{E.63}$$

Because C_{12O} dies exponentially at large positive Δ, we can now close the Δ contour to the right and pick up poles along the positive real axis. Comparing to the physical operator product expansion gives (6.124).

E.4 Computation of $\mathcal{R}(\Delta_1,\Delta_2,J)$

In this appendix we compute the coefficient $\mathcal{R}$ appearing in the first line of (6.147)

$$\begin{aligned} &\mathcal{R}(\Delta_1,\Delta_2,J) \\ &\equiv -2^{J-2}\int d\alpha d^{d-2}\mathbf{w}_1 d^{d-2}\mathbf{w}_2 \frac{2^{J-1}\left(\alpha(1-\alpha)+(1-\alpha)\mathbf{w}_1^2+\alpha\mathbf{w}_2^2\right)^{1-\Delta_1-\Delta_2-J}}{(1+\mathbf{w}_{12}^2)^{d-\Delta_1-\Delta_2}\alpha^{-\Delta_1+1-J}(1-\alpha)^{-\Delta_2+1-J}}. \end{aligned} \tag{E.64}$$

As the first step, we do the $\mathbf{w}_i$ integrals. We define $\mathbf{w}_- = \mathbf{w}_{12}$ and $\mathbf{w}_+ = \mathbf{w}_1 + \mathbf{w}_2$. The integral over dw_i becomes (without the -2^{J-2} and $\mathbf{w}$-independent factors)

$$2^{2(\Delta_1+\Delta_2+J)-d} \int d^{d-2}\mathbf{w}_+ d^{d-2}\mathbf{w}_- \frac{\left(4\alpha(1-\alpha) + \mathbf{w}_+^2 + \mathbf{w}_-^2 + 2(1-2\alpha)\mathbf{w}_+ \cdot \mathbf{w}_-\right)^{1-\Delta_1-\Delta_2-J}}{(1+\mathbf{w}_-^2)^{d-\Delta_1-\Delta_2}}. \tag{E.65}$$

Now we shift $\mathbf{w}_+ \to \mathbf{w}_+ - (1-2\alpha)\mathbf{w}_-$ to find

$$2^{2(\Delta_1+\Delta_2+J)-d} \int d^{d-2}\mathbf{w}_+ d^{d-2}\mathbf{w}_- \frac{\left(4\alpha(1-\alpha)(1+\mathbf{w}_-^2) + \mathbf{w}_+^2\right)^{1-\Delta_1-\Delta_2-J}}{(1+\mathbf{w}_-^2)^{d-\Delta_1-\Delta_2}}. \tag{E.66}$$

Rescaling $\mathbf{w}_+$ we find

$$\begin{aligned}\int d^{d-2}\mathbf{w}_+ d^{d-2}\mathbf{w}_- &\frac{(\alpha(1-\alpha))^{1-\Delta_1-\Delta_2-J+\frac{d-2}{2}} \left(1+\mathbf{w}_+^2\right)^{1-\Delta_1-\Delta_2-J}}{(1+\mathbf{w}_-^2)^{J+\frac{d}{2}}} = \\ = (\alpha(1-\alpha))^{1-\Delta_1-\Delta_2-J+\frac{d-2}{2}} \times \pi^{d-2} &\frac{\Gamma(J+1)\Gamma(-\frac{d}{2}+J+\Delta_1+\Delta_2)}{\Gamma(J+\frac{d}{2})\Gamma(J+\Delta_1+\Delta_2-1)}.\end{aligned} \tag{E.67}$$

The remaining α-integral becomes

$$\int d\alpha \alpha^{-\Delta_2+\frac{d-2}{2}}(1-\alpha)^{-\Delta_1+\frac{d-2}{2}} = \frac{\Gamma(\frac{d}{2}-\Delta_1)\Gamma(\frac{d}{2}-\Delta_2)}{\Gamma(d-\Delta_1-\Delta_2)}. \tag{E.68}$$

Combining everything together we find

$$\mathcal{R}(\Delta_1,\Delta_2,J) = -2^{J-2}\pi^{d-2}\frac{\Gamma(J+1)\Gamma(-\frac{d}{2}+J+\Delta_1+\Delta_2)}{\Gamma(J+\frac{d}{2})\Gamma(J+\Delta_1+\Delta_2-1)}\frac{\Gamma(\frac{d}{2}-\Delta_1)\Gamma(\frac{d}{2}-\Delta_2)}{\Gamma(d-\Delta_1-\Delta_2)}. \tag{E.69}$$

E.5 Parings of continuous-spin structures

In this section we describe the natural conformally-invariant pairing between continuous spin structures. Recall that the Euclidean pairings are constructed from the basic invariant integral

$$\int d^d x \mathcal{O}(x)\widetilde{\mathcal{O}}^\dagger(x), \tag{E.70}$$

where contraction of SO(d) indices is implicit. This integral is conformally-invariant because if $\mathcal{O}$ transforms in (Δ,ρ) then $\widetilde{\mathcal{O}}^\dagger$ transforms in $(d-\Delta,\rho^*)$, where ρ^* is the SO(d) irrep dual to ρ. We can therefore contract SO(d) indices and the dimensions in the integrand add up to 0 (taking into account the measure $d^d x$).

To pair continuous-spin structures in Lorentzian, we need to make use of the integral

$$\int d^d x D^{d-2} z \mathcal{O}(x,z)\mathcal{O}^{\mathrm{S}\dagger}(x,z) \tag{E.71}$$

If $\mathcal{O}$ transforms in (Δ, J, λ), then $\mathcal{O}^{\mathrm{S}\dagger}$ transforms in $(d-\Delta, 2-d-J, \lambda^*)$. The integrand has 0 homogeneity in x and z, and λ-indices can be contracted.[14]

E.5.1 Two-point functions

Let us start with two-point functions. As discussed in section E.1, two-point functions of continuous-spin operators do not make sense as Wightman functions, so in order to discuss them, we have to think about them simply as some conformal invariants defined at least for spacelike separated points.

That said, given a two-point structure for $\mathcal{O}$ in representation (Δ, J, λ) and a two-point function for $\mathcal{O}^{\mathrm{S}}$ in representation $\mathbf{S}[(\Delta, J, \lambda)] = (d-\Delta, 2-d-J, \lambda)$, we can define the two-point pairing by

$$\begin{aligned}&\frac{(\langle \mathcal{O}\mathcal{O}^\dagger\rangle, \langle \mathcal{O}^{\mathrm{S}}\mathcal{O}^{\mathrm{S}\dagger}\rangle)_L}{\mathrm{vol}(\mathrm{SO}(1,1))^2}\\ &\equiv \int_{x_1\approx x_2} \frac{d^d x_1 d^d x_2 D^{d-2}z_1 D^{d-2}z_2}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))}\langle \mathcal{O}^a(x_1,z_1)\mathcal{O}^{b\dagger}(x_2,z_2)\rangle\langle \mathcal{O}^{\mathrm{S}}_b(x_2,z_2)\mathcal{O}^{\mathrm{S}\dagger}_a(x_1,z_1)\rangle,\end{aligned} \tag{E.72}$$

where factor $\mathrm{vol}(\mathrm{SO}(1,1))^2$ is for future convenience[15] and the subscript "L" stands for "Lorentzian." On the right hand side, we divide by the volume of the conformal group since the integral is invariant under it. Formally, this means that we should compute the integral by gauge-fixing the action of conformal group and introducing an appropriate Faddeev-Popov determinant. To perform gauge-fixing, we can first put x_1 and x_2 into some standard configuration. A natural choice is to set $x_1 = 0$ and

[14] Given that $\mathcal{O}^S$ transforms in $(d-\Delta, 2-d-J, \lambda)$, it is a bit non-trivial to understand why $\mathcal{O}^{S\dagger}$ has λ^*. In odd dimensions λ and λ^* is the same irrep, so there is no question here. In even dimension $\dagger$ changes the sign of the last row of Young diagram of $(d-\Delta, 2-d-J, \lambda)$ in the same way as it does for all $\mathfrak{so}(d)$-weights. In other words, it flips the sign if $d = 4k$ and does nothing for $d = 4k+2$. However, this last row is also the last row of λ and λ is an $\mathrm{SO}(d-2)$-irrep. It then turns out that from the $\mathrm{SO}(d-2)$ point of view, this action is equivalent to taking the dual. Another way to see this is that $\dagger$ is complex conjugation for $\mathrm{SO}(d-1,1)$, and thus for $\mathrm{SO}(d-2)$, which can be thought of as a subgroup of $\mathrm{SO}(d-1,1)$. But since $\mathrm{SO}(d-2)$ is compact, for it complex conjugation is the same as taking the dual.

[15] Similarly to the Euclidean case [195], the right hand side can be alternatively computed in terms of Plancherel measure divided by $\mathrm{vol}(\mathrm{SO}(1,1))^2$. In Euclidean we get only one power of $\mathrm{vol}(\mathrm{SO}(1,1))$, which corresponds to the fact that there we have only one continuous parameter Δ, while in Lorentzian we have both Δ and J.

$x_2 = \infty$ (spacelike infinity).[16] This configuration is still invariant under dilatation and Lorentz transformations. Thus we have

$$\begin{aligned}&\frac{(\langle \mathcal{O}\mathcal{O}^\dagger\rangle, \langle \mathcal{O}^{\mathrm{S}}\mathcal{O}^{\mathrm{S}\dagger}\rangle)_L}{\mathrm{vol}(\mathrm{SO}(1,1))^2}\\ &= \int \frac{D^{d-2}z_1 D^{d-2}z_2}{2^d \mathrm{vol}(\mathrm{SO}(1,1)\times \mathrm{SO}(d-1,1))}\langle \mathcal{O}^a(0,z_1)\mathcal{O}^{b\dagger}(\infty,z_2)\rangle\langle \mathcal{O}^{\mathrm{S}}_b(\infty,z_2)\mathcal{O}^{\mathrm{S}\dagger}_a(0,z_1)\rangle,\end{aligned} \tag{E.73}$$

where 2^d comes from the Faddeev-Popov determinant.[17] If we define $z_2^R = (z_2^0, -z_2^1, z_2^2, \ldots, z_2^{d-1})$, so that Lorentz group transforms z_1 and z_2^R in the same way, the integral

$$\int \frac{D^{d-2}z_1 D^{d-2}z_2^R}{\mathrm{vol}(\mathrm{SO}(d-1,1))} \tag{E.74}$$

essentially becomes the $(d-2)$-dimensional Euclidean conformal two-point integral. It can also be computed by gauge-fixing, i.e. by setting $z_1^\mu = z_0^\mu \equiv (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$, which is the embedding-space representation of the origin of $\mathbb{R}^{d-2}$, $z_2^{R\,\mu} = z_\infty^\mu \equiv (\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0)$, which is the embedding-space representation of the infinity of $\mathbb{R}^{d-2}$. The stabilizer group of this configuration is $\mathrm{SO}(1,1)\times\mathrm{SO}(d-2)$, which consists of $(d-2)$-dimensional dilatations and rotations. We thus conclude

$$\begin{aligned}&(\langle \mathcal{O}\mathcal{O}^\dagger\rangle, \langle \mathcal{O}^{\mathrm{S}}\mathcal{O}^{\mathrm{S}\dagger}\rangle)_L\\ &= \frac{1}{2^d 2^{d-2}\mathrm{vol}(\mathrm{SO}(d-2))}\langle \mathcal{O}^a(0,z_0)\mathcal{O}^{b\dagger}(\infty,z_\infty^R)\rangle\langle \mathcal{O}^{\mathrm{S}}_b(\infty,z_\infty^R)\mathcal{O}^{\mathrm{S}\dagger}_a(0,z_0)\rangle,\end{aligned} \tag{E.75}$$

where we included another Faddeev-Popov determinant. Note that the right hand side is proportional to $\dim\lambda$.

We can summarize this result as follows. Note that the product

$$\langle \mathcal{O}^a(x_1,z_1)\mathcal{O}^{b\dagger}(x_2,z_2)\rangle\langle \mathcal{O}^{\mathrm{S}}_b(x_2,z_2)\mathcal{O}^{\mathrm{S}\dagger}_a(x_1,z_1)\rangle \tag{E.76}$$

transforms in representation $(\Delta, J, \lambda) = (d, 2-d, \bullet)$ at both x_1 and x_2. Thus we must have

$$\langle \mathcal{O}^a(x_1,z_1)\mathcal{O}^{b\dagger}(x_2,z_2)\rangle\langle \mathcal{O}^{\mathrm{S}}_b(x_2,z_2)\mathcal{O}^{\mathrm{S}\dagger}_a(x_1,z_1)\rangle = A\frac{(-2z_1\cdot I(x_{12})z_2)^{2-d}}{x_{12}^{2d}}. \tag{E.77}$$

For some constant A. Using (E.75), we find

$$(\langle \mathcal{O}\mathcal{O}^\dagger\rangle, \langle \mathcal{O}^{\mathrm{S}}\mathcal{O}^{\mathrm{S}\dagger}\rangle)_L = \frac{A}{2^{2d-2}\mathrm{vol}(\mathrm{SO}(d-2))}. \tag{E.78}$$

[16] We define $\mathcal{O}(\infty) = \lim_{L\to\infty} L^{2\Delta}\mathcal{O}(Le)$, where e is a conventional spacelike unit vector. We choose $e = (0,1,0,\ldots,0)$.

[17] A fixed power of 2 also goes into what we mean by $\mathrm{vol}(\mathrm{SO}(1,1))$.

E.5.2 Three-point pairings

We can analogously define a three-point pairing for continuous-spin structures,

$$\begin{aligned}&\Big(\langle O_1 O_2 O\rangle, \langle \widetilde{O}_1^\dagger \widetilde{O}_2^\dagger O^{S\dagger}\rangle\Big)_L \\ &\equiv \int_{\substack{2<1\\ x\approx 1,2}} \frac{d^d x_1 d^d x_2 d^d x D^{d-2} z}{\mathrm{vol}(\widetilde{\mathrm{SO}}(d,2))} \langle O_1(x_1) O_2(x_2) O(x,z)\rangle \langle \widetilde{O}_1^\dagger(x_1) \widetilde{O}_2^\dagger(x_2) O^{S\dagger}(x,z)\rangle.\end{aligned} \tag{E.79}$$

Here, finite-dimensional Lorentz indices are implicitly contracted. Note that due to the fixed causal relationships between the points the continuous-spin structures are single-valued without $i\epsilon$ prescriptions (see appendix E.1). As in the Euclidean case, Lorentzian three-point pairings are simple to compute because they don't involve any actual integrals over positions. We can use the conformal group to fix all three points to a standard configuration consistent with the given causal relationships, for example

$$x_4 = 0, \quad x_3 = e^0, \quad x = \infty, \tag{E.80}$$

where e^0 is a unit vector in the t direction. The Fadeev-Popov determinant associated with this choice is 2^{-d}. All that remains is an integral over the polarization vector z,

$$= \frac{1}{2^d \mathrm{vol}(\mathrm{SO}(d-1))} \int D^{d-2} z \, \langle O_1(e^0) O_2(0) O(\infty,z)\rangle \langle \widetilde{O}_1^\dagger(e^0) \widetilde{O}_2^\dagger(0) O^{S\dagger}(\infty,z)\rangle, \tag{E.81}$$

where $\mathrm{vol}(\mathrm{SO}(d-1))$ is the volume of the stabilizer group of the three points.[18] In practice, we can avoid doing the integral over z as well. This is because the product in the integrand must be proportional to a three-point function of two scalars with dimension d and a spinning operator with dimension d and spin $2-d$. The integral of the z-dependent part of this product is always

$$\frac{1}{2^d \mathrm{vol}(\mathrm{SO}(d-1))} \int D^{d-2} (-2z\cdot e^0)^{2-d} = \frac{1}{2^{2d-2}\mathrm{vol}(\mathrm{SO}(d-2))}. \tag{E.82}$$

Thus, we can write

$$\begin{aligned}&\Big(\langle O_1 O_2 O\rangle, \langle \widetilde{O}_1^\dagger \widetilde{O}_2^\dagger O^{S\dagger}\rangle\Big)_L \\ &= \frac{1}{2^{2d-2}\mathrm{vol}(\mathrm{SO}(d-2))} \frac{\langle O_1(e^0) O_2(0) O(\infty,z)\rangle \langle \widetilde{O}_1^\dagger(e^0) \widetilde{O}_2^\dagger(0) O^{S\dagger}(\infty,z)\rangle}{(-2z\cdot e^0)^{2-d}}.\end{aligned} \tag{E.83}$$

[18] Note that the stabilizer group depends on the causal relationships of the points. For example, three spacelike points have stabilizer group $\mathrm{SO}(d-2,1)$.

E.6 Integral transforms, weight-shifting operators and integration by parts

In this appendix we elaborate on the interplay between integral transforms, weight-shifting operators, and conformally-invariant pairings, following [195] and generalizing the discussion to Lorentzian signature. For simplicity of discussion, we ignore possible signs coming from odd permutations of fermions.

E.6.1 Euclidean signature

In Euclidean signature we have one integral transform, $\mathbf{S}_E$, and a conformally-invariant pairing

$$(\mathcal{O}, \widetilde{\mathcal{O}}^\dagger) \equiv \int d^d x \mathcal{O}(x) \widetilde{\mathcal{O}}^\dagger(x), \tag{E.84}$$

where the spin indices are implicitly contracted. With respect to this paring we can define a conjugation on weight-shifting operators and on the integral transform,

$$\begin{aligned}(\mathcal{D}\mathcal{O}, \widetilde{\mathcal{O}}^\dagger) &= (\mathcal{O}, \mathcal{D}^* \widetilde{\mathcal{O}}^\dagger),\\ (\mathbf{S}_E \mathcal{O}, \widetilde{\mathcal{O}}^\dagger) &= (\mathcal{O}, \mathbf{S}_E^* \widetilde{\mathcal{O}}^\dagger).\end{aligned} \tag{E.85}$$

We have $*^2 = 1$ and $\mathbf{S}_E^* = \mathbf{S}_E$.

Furthermore, we can define Weyl reflection on weight-shifting operators according to

$$\mathbf{S}_E \mathcal{D} = (\mathbf{S}_E[\mathcal{D}]) \mathbf{S}_E. \tag{E.86}$$

We then have

$$\mathbf{S}_E^2 \mathcal{D} = \mathbf{S}_E (\mathbf{S}_E[\mathcal{D}]) \mathbf{S}_E = (\mathbf{S}_E^2[\mathcal{D}]) \mathbf{S}_E^2, \tag{E.87}$$

and since $\mathbf{S}_E^2 = \mathcal{N}(\Delta, \rho)$, we have when acting on operators transforming in (Δ, ρ)

$$\mathbf{S}_E^2[\mathcal{D}] = \frac{\mathcal{N}(\Delta + \delta_\Delta, \rho + \delta_\rho)}{\mathcal{N}(\Delta, \rho)} \mathcal{D}, \tag{E.88}$$

where $(\delta_\Delta, \delta_\rho)$ is the weight by which $\mathcal{D}$ shifts. Conjugating (E.86) we find

$$\mathbf{S}_E (\mathbf{S}_E[\mathcal{D}])^* = \mathcal{D}^* \mathbf{S}_E, \tag{E.89}$$

and thus

$$\mathbf{S}_E[\mathcal{D}]^* = \mathbf{S}_E^{-1}[\mathcal{D}^*]. \tag{E.90}$$

We also note that the crossing equation for weight-shifting operators acting on a two-point function [3] can be written in terms of shadow transform and conjugation. Namely, we can interpret $\mathbf{S}_E\mathcal{D}^*$ as convolution with the kernel

$$\langle\widetilde{O}(\mathcal{D}\widetilde{O}^\dagger)\rangle, \tag{E.91}$$

while, on the other hand, it is equal to $\mathbf{S}_E[\mathcal{D}^*]\mathbf{S}$ which is convolution with (assume that $\mathcal{D}\widetilde{O}^\dagger$ transforms as $\widetilde{O}'^\dagger$)

$$\langle(\mathbf{S}_E[\mathcal{D}^*]\widetilde{O}')\widetilde{O}'^\dagger\rangle. \tag{E.92}$$

We thus find the crossing equation

$$\langle\widetilde{O}(\mathcal{D}\widetilde{O}^\dagger)\rangle = \langle(\mathbf{S}_E[\mathcal{D}^*]\widetilde{O}')\widetilde{O}'^\dagger\rangle. \tag{E.93}$$

E.6.2 Lorentzian signature

The above discussion has an analogue in Lorentzian signature. Now we have more integral transforms, so let us denote a generic one by $\mathbb{W}$. We also have a new pairing, given by

$$(O, O^{\mathrm{S}\dagger})_L = \int d^d x D^{d-2} z O(x,z) O^{\mathrm{S}\dagger}(x,z), \tag{E.94}$$

where the $\mathrm{SO}(d-2)$ indices are implicit and contracted. This pairing leads to a new conjugation operation on weight-shifting operators and on integral transforms,

$$\begin{aligned}(\mathcal{D}O, \widetilde{O}^\dagger)_L &= (O, \overline{\mathcal{D}}\widetilde{O}^\dagger)_L,\\ (\mathbb{W}O, \widetilde{O}^\dagger)_L &= (O, \overline{\mathbb{W}}\widetilde{O}^\dagger)_L.\end{aligned} \tag{E.95}$$

Note that in general the Lorentzian and Euclidean conjugations do not commute (see below). Analogously to the Euclidean case, we find

$$\overline{\mathbb{W}[\mathcal{D}]} = \mathbb{W}^{-1}[\overline{\mathcal{D}}]. \tag{E.96}$$

As in Euclidean signature, we can define the action of integral transforms on weight-shifting operators by

$$\mathbb{W}\mathcal{D} = (\mathbb{W}[\mathcal{D}])\mathbb{W}. \tag{E.97}$$

In principle $\mathbb{W}[\mathcal{D}]$ can be a differential operator with coefficients which depend on $\mathcal{T}$. However, when acting on a function, the left hand side of this expression depends only on the values of this function in a set which fits in one Poincare patch.

If $\mathbb{W}[\mathcal{D}]$ had non-trivial t dependence, the same would not hold for the right hand side. Therefore $\mathbb{W}[\mathcal{D}]$ has to be a local weight-shifting differential operator.

It is easy to check that if two integral transforms commute, then their actions on weight-shifting operators also commute. Similarly to Euclidean case, relations such as $\mathbf{L}^2 = f_L(\Delta, J, \mathcal{T})$ generalize to action on weight-shifting operators. Let us write down the square of an order two transform (any transform except $\mathbf{R}$ and $\overline{\mathbf{R}}$)

$$\mathbb{W}^2[\mathcal{D}] = f_W(\Delta, \rho, \mathcal{T})\mathcal{D} f_W^{-1}(\Delta, \rho, \mathcal{T}), \tag{E.98}$$

where Δ and ρ are understood as operators which read off the scaling dimension and representation of whatever they act on. Let us comment on this formula in the case of $\mathbf{S}_\Delta$. Modulo Wick rotation, we have the relation $\mathbf{S}_E = (-2)^J \mathbf{S}_\Delta$ for traceless-symmetric operators. It follows that (E.88) and (E.98) should be compatible. That is, we should have

$$\frac{\mathcal{N}(\Delta + \delta_\Delta, J + \delta_J)}{\mathcal{N}(\Delta, J)} = \frac{4^{J+\delta_J} f_\Delta(\Delta + \delta_\Delta, J + \delta_J, c\mathcal{T})}{4^J f_\Delta(\Delta, J, \mathcal{T})}, \tag{E.99}$$

where δ_Δ, δ_J are the weights by which $\mathcal{D}$ shifts, and c is defined by

$$\mathcal{T}\mathcal{D}\mathcal{T}^{-1} = c\mathcal{D}, \tag{E.100}$$

i.e., c is the eigenvalue of $\mathcal{T}$ in the finite-dimensional irrep of conformal group to which $\mathcal{D}$ is associated. For example, for vector representation $c = -1$. To check this relation, we can use the results of section 6.2.7 and in particular the relation (6.110) which implies (we consider traceless-symmetric case for simplicity)

$$f_\Delta(\Delta, J, \mathcal{T}) = -\mathcal{T}^{-2} f_L(\Delta, \rho, \mathcal{T}) f_J(1 - \Delta) f_L(1 - J, 1 - d + \Delta, \mathcal{T}). \tag{E.101}$$

It is then an easy exercise to verify that (E.99) holds for vector weight-shifting operators [3].

Another useful result is obtained by substituting $\mathcal{D} \to \mathbb{W}^{-1}[\mathcal{D}]$ into (E.98) to find

$$\mathbb{W}^{-1}[\mathcal{D}] = f_W^{-1}(\Delta, \rho, \mathcal{T})\mathbb{W}[\mathcal{D}] f_W(\Delta, \rho, \mathcal{T}). \tag{E.102}$$

For example,

$$\mathbf{L}^{-1}[\mathcal{D}] = \mathbf{L}[\mathcal{D}] \frac{f_L(\Delta, \rho, \mathcal{T})}{f_L(\Delta + \mathbf{L}[\delta_\Delta], \rho + \mathbf{L}[\delta_\rho], c\mathcal{T})}, \tag{E.103}$$

where we kept explicit dependence of f_L on t, $(\mathbf{L}[\delta_\Delta], \mathbf{L}[\delta_\rho])$ is the weight by which $\mathbf{L}[\mathcal{D}]$ shifts. It is easy to check that $\mathcal{T}$-dependence indeed cancels out for $\mathcal{D}$ in vector representation.

We can derive two-point crossing in terms of Lorentzian conjugation and $\mathbf{S}$ transform,

$$\langle \mathcal{O}^{\mathrm{S}}(\mathcal{D}\mathcal{O}^{\mathrm{S}\dagger})\rangle = \langle (\mathbf{S}[\overline{\mathcal{D}}]\mathcal{O}'^{\mathrm{S}})\mathcal{O}'^{\mathrm{S}\dagger}\rangle. \tag{E.104}$$

Comparing to the Euclidean form of two-point crossing leads to a useful relation

$$\mathbf{S}_E[\mathcal{D}^*] = \mathbf{S}[\overline{\mathcal{D}}]. \tag{E.105}$$

We will need a version of this relation with order of integral transforms and conjugations interchanged. First, (E.105) implies

$$(\mathbf{S}_E^{-1}[\mathcal{D}])^* = \overline{\mathbf{S}^{-1}[\mathcal{D}]}. \tag{E.106}$$

Then we use that $\mathbf{S}_E$ and $\mathbf{S}$ are proportional to their inverses. In particular, we find from (E.102)

$$\begin{aligned}
(f_E^{-1}(\Delta,\rho,\mathcal{T})\mathbf{S}_E[\mathcal{D}]f_E(\Delta,\rho,\mathcal{T}))^* &= \overline{(f_S^{-1}(\Delta,\rho,\mathcal{T})\mathbf{S}[\mathcal{D}]f_S(\Delta,\rho,\mathcal{T}))},\\
f_E(\Delta,\rho,\mathcal{T})(\mathbf{S}_E[\mathcal{D}])^* f_E^{-1}(\Delta,\rho,\mathcal{T}) &= f_S(\Delta,\rho,\mathcal{T})\overline{\mathbf{S}[\mathcal{D}]}f_S^{-1}(\Delta,\rho,\mathcal{T}),
\end{aligned} \tag{E.107}$$

where we temporarily interpret $\mathbf{S}_E$ as a Lorentzian transform defined by $(-2)^J\mathbf{S}_\Delta$. We can now use

$$f_S(\Delta,\rho,\mathcal{T}) = \mathbf{S}^2 = \mathbf{S}_\Delta^2\mathbf{S}_J^2 = 4^{-J}\mathbf{S}_E^2\mathbf{S}_J^2 = 4^{-J}f_E(\Delta,\rho,\mathcal{T})f_J(\rho) \tag{E.108}$$

to conclude

$$\overline{\mathbf{S}[\mathcal{D}]} = 4^J f_J^{-1}(\rho)(\mathbf{S}_\Delta[\mathcal{D}])^* 4^{-J} f_J(\rho). \tag{E.109}$$

E.7 Proof of (6.214) for seed blocks

In this appendix we prove (6.214) for seed blocks by starting from the scalar case. For simplicity we consider only bosonic representations. We assume that $\mathcal{O}_i$ are in $\mathrm{SO}(d)$ representations appropriate for the seed block for intermediate ρ which we are interested in. As discussed in section 4.4 of [3], we can assume that $\mathcal{O}_2$ and $\mathcal{O}_4$ are scalars in all seed blocks, so we don't have to change their representations. We start with the identity

$$(\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E(\langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle)_E^{-1} = (\langle\mathcal{O}'^\dagger\mathcal{O}'\rangle,\langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E\mathcal{D}_{1,A}\widetilde{\mathcal{D}}^A(\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}'^\dagger]\rangle)_E^{-1}, \tag{E.110}$$

where $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are some weight-shifting operators,[19] while $\mathcal{O}_1'$ and $\mathcal{O}'$ come from a seed block for which we already know that (6.214) holds. A possible proportionality

[19]Here tilde isn't related to shadow transform and $\widetilde{\mathcal{D}}$ acts on the third position. The representation of index A can be assumed to be vector.

coefficient can be absorbed into the definition of either the weight-shifting operators or the tensor structures. Consider pairing both sides with $\langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle$ to obtain

$$\frac{(\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle\mathcal{O}'^\dagger\mathcal{O}'\rangle,\langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E}=(\langle\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathcal{O}^\dagger]\rangle,\mathcal{D}_{1,A}\widetilde{\mathcal{D}}^A(\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}'^\dagger]\rangle)_E^{-1})_E. \quad \text{(E.111)}$$

Integrating by parts and using definitions of appendix E.6 we find

$$\frac{(\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle\mathcal{O}'^\dagger\mathcal{O}'\rangle,\langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E}=(\langle\mathcal{D}^*_{1,A}\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathbf{S}_E^{-1}[\widetilde{\mathcal{D}}^*]^A\mathcal{O}^\dagger]\rangle,(\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}'^\dagger]\rangle)_E^{-1})_E, \quad \text{(E.112)}$$

which allows us to conclude

$$\langle\mathcal{D}^*_{1,A}\mathcal{O}_1\mathcal{O}_2\mathbf{S}_E[\mathbf{S}_E^{-1}[\widetilde{\mathcal{D}}^*]^A\mathcal{O}^\dagger]\rangle=\frac{(\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle\mathcal{O}'^\dagger\mathcal{O}'\rangle,\langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E}\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{S}_E[\mathcal{O}'^\dagger]\rangle, \quad \text{(E.113)}$$

or, canceling $\mathbf{S}_E$ on both sides,

$$\langle\mathcal{D}^*_{1,A}\mathcal{O}_1\mathcal{O}_2(\mathbf{S}_E^{-1}[\widetilde{\mathcal{D}}^*]^A\mathcal{O}^\dagger)\rangle=\frac{(\langle\mathcal{O}^\dagger\mathcal{O}\rangle,\langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle\mathcal{O}'^\dagger\mathcal{O}'\rangle,\langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E}\langle\mathcal{O}_1\mathcal{O}_2'\mathcal{O}'^\dagger\rangle. \quad \text{(E.114)}$$

We will use this characterization of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ later in the proof.

For now, let us apply (E.110) to (6.213) and find that H is given by

$$\begin{aligned}&H_{\Delta,\rho}(x_i)\\&=-\mu(\Delta,\rho'^\dagger)(\mathcal{O}_1\mathcal{O}_2'\mathbf{S}_E[\widetilde{\mathcal{O}}'^\dagger])(\langle\mathcal{O}_1\mathcal{O}_2'\widetilde{\mathcal{O}}'^\dagger\rangle,\langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_2'^\dagger\mathcal{O}'\rangle)_E^{-1}\times\\&\quad\times\int_{2<x<1}d^dxD^{d-2}z\langle0|\mathcal{D}_{1,A}\widetilde{\mathcal{O}}_1^\dagger\mathbf{L}[\widetilde{\mathcal{D}}^A\mathcal{O}](x,z)\widetilde{\mathcal{O}}'^\dagger_{2^+}|0\rangle(\langle0|\mathcal{O}_{4^+}\mathbf{L}[\mathcal{O}](x,z)\mathcal{O}_3|0\rangle)_L^{-1}.\end{aligned} \quad \text{(E.115)}$$

We now use

$$\mathbf{L}[\widetilde{\mathcal{D}}^A\mathcal{O}]=\mathbf{L}[\widetilde{\mathcal{D}}]^A\mathbf{L}[\mathcal{O}], \quad \text{(E.116)}$$

and integrate $\mathbf{L}[\widetilde{\mathcal{D}}]$ by parts. This gives

$$\begin{aligned}&H_{\Delta,\rho}(x_i)\\&=-\mu(\Delta,\rho'^\dagger)(\mathcal{O}_1\mathcal{O}_2'\mathbf{S}_\Delta[\widetilde{\mathcal{O}}'^\dagger])(\langle\mathcal{O}_1\mathcal{O}_2'\widetilde{\mathcal{O}}'^\dagger\rangle,\langle\widetilde{\mathcal{O}}_1^\dagger\widetilde{\mathcal{O}}_2'^\dagger\mathcal{O}'\rangle)_E^{-1}\times\\&\quad\times\int_{2<x<1}d^dxD^{d-2}z\langle0|\mathcal{D}_{1,A}\widetilde{\mathcal{O}}_1^\dagger\mathbf{L}[\mathcal{O}](x,z)\widetilde{\mathcal{O}}'^\dagger_{2^+}|0\rangle\overline{\mathbf{L}[\widetilde{\mathcal{D}}]}^A(\langle0|\mathcal{O}_{4^+}\mathbf{L}[\mathcal{O}](x,z)\mathcal{O}_3|0\rangle)_L^{-1},\end{aligned} \quad \text{(E.117)}$$

where $\overline{\mathbf{L}[\widetilde{\mathcal{D}}]}$ acts on the middle position in the right three-point structure. We can further apply a crossing transformation on the right three-point structure as in [3] to make all differential operators act on the external operators only. We will not do this in detail, because we will anyway reverse this step in a moment. Let us denote the resulting differential operator acting on external operators by $\mathfrak{D}$.

The conclusion of the above calculation is schematically that

$$H_\rho = \mathfrak{D} H_{\rho'}, \tag{E.118}$$

where $H_{\rho'}$ is some conformal for which we know (6.214) to hold. We can thus apply $\mathfrak{D}$ to (6.214) written for $H_{\rho'}$. Since the right three-point structure in (6.214) and (6.213) is the same, we can unwind the steps in the derivation of $\mathfrak{D}$ which were performed solely on the right three-point structure to conclude

$$H_{\Delta,\rho}(x_i) = -\frac{1}{2\pi i}\frac{\mathcal{D}_{1,A}\left(\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{L}[\mathcal{O}'^\dagger]\rangle\right)_L^{-1}\overline{\mathbf{L}[\widetilde{\mathcal{D}}]}^A\,(\mathcal{T}_4\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle)_L^{-1}}{(\langle\mathbf{L}[\mathcal{O}']\mathbf{L}[\mathcal{O}']\rangle)_L^{-1}}. \tag{E.119}$$

We can use (E.167) to write this as

$$\begin{aligned} &H_{\Delta,\rho}(x_i) \\ &= -\frac{1}{2\pi i}\frac{(\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle)_L^{-1}}{(\langle\mathbf{L}[\mathcal{O}']\mathbf{L}[\mathcal{O}']\rangle)_L^{-1}}\frac{\mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]^A\mathcal{D}_{1,A}\left(\mathcal{T}_2\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{L}[\mathcal{O}'^\dagger]\rangle\right)_L^{-1}(\mathcal{T}_4\langle\mathcal{O}_4\mathcal{O}_3\mathbf{L}[\mathcal{O}]\rangle)_L^{-1}}{(\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle)_L^{-1}}. \end{aligned} \tag{E.120}$$

We now want to express

$$\mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]^A\mathcal{D}_{1,A}(\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{L}[\mathcal{O}'^\dagger]\rangle)_L^{-1} \tag{E.121}$$

in terms of

$$(\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1}. \tag{E.122}$$

To do this, let us consider the Lorentzian pairing

$$\begin{aligned} &\left(\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle, \mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]^A\mathcal{D}_{1,A}(\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{L}[\mathcal{O}'^\dagger]\rangle)_L^{-1}\right)_L \\ &= \left(\overline{\mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]}^A\mathcal{D}^*_{1,A}\langle\mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle, (\langle\mathcal{O}_1\mathcal{O}_2'\mathbf{L}[\mathcal{O}'^\dagger]\rangle)_L^{-1}\right)_L. \end{aligned} \tag{E.123}$$

We can use the results of appendix E.6 and 6.2.7 to write

$$\overline{\mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]} = \overline{\mathbf{L}[\mathbf{S}[\widetilde{\mathcal{D}}]]} = \mathbf{L}^{-1}[\overline{\mathbf{S}[\widetilde{\mathcal{D}}]}] = \frac{f_L(\mathbf{L}[\Delta],\mathbf{L}[\rho^\dagger],\mathcal{T})}{f_L(\mathbf{L}[\Delta]+\mathbf{L}[\delta_\Delta],\mathbf{L}[\rho^\dagger]+\mathbf{L}[\delta_\rho],c\mathcal{T})}\mathbf{L}[\overline{\mathbf{S}[\widetilde{\mathcal{D}}]}], \tag{E.124}$$

where $(\delta_\Delta, \delta_\rho)$ is the weight by which $\overline{\mathbf{S}[\widetilde{\mathcal{D}}]}$ shifts and c is defined by (E.100) for $\widetilde{\mathcal{D}}$. Since we consider only bosonic representations, $c = \pm 1$ ($c = -1$ for vector weight-shifting operators). We have $(\Delta + \delta_\Delta, \rho^\dagger + \delta_\rho) = (\Delta', \rho'^\dagger)$. We furthermore have

$$\mathbf{L}[\overline{\mathbf{S}[\widetilde{\mathcal{D}}]}]\mathbf{L}[\mathcal{O}^\dagger] = \mathbf{L}[\overline{\mathbf{S}[\widetilde{\mathcal{D}}]}\mathcal{O}^\dagger] = \frac{4^{-J} f_J(\rho^\dagger)}{4^{-J'} f_J(\rho'^\dagger)}\mathbf{L}[(\mathbf{S}_\Delta[\widetilde{\mathcal{D}}])^*\mathcal{O}^\dagger] \tag{E.125}$$

and thus

$$\begin{aligned}&\overline{\mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]}^A \mathcal{D}^*_{1,A}\langle \mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle \\ &= \frac{4^{-J} f_J(\rho^\dagger)}{4^{-J'} f_J(\rho'^\dagger)}\frac{f_L(\mathbf{L}[\Delta], \mathbf{L}[\rho^\dagger], \mathcal{T})}{f_L(\mathbf{L}[\Delta'], \mathbf{L}[\rho'^\dagger], c\mathcal{T})}\langle \mathcal{O}_1\mathcal{D}^*_{1,A}\mathcal{O}_2\mathbf{L}[(\mathbf{S}_E[\widetilde{\mathcal{D}}])^*\mathcal{O}^\dagger]\rangle.\end{aligned} \tag{E.126}$$

Now use $(\mathbf{S}_E[\widetilde{\mathcal{D}}])^* = \mathbf{S}_E^{-1}[\widetilde{\mathcal{D}}^*]$, apply $\mathbf{L}$ to both sides of (E.114) and conclude

$$\begin{aligned}&\overline{\mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]}^A \mathcal{D}^*_{1,A}\langle \mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle \\ &= \frac{4^{-J} f_J(\rho^\dagger)}{4^{-J'} f_J(\rho'^\dagger)}\frac{f_L(\mathbf{L}[\Delta], \mathbf{L}[\rho^\dagger], \mathcal{T})}{f_L(\mathbf{L}[\Delta'], \mathbf{L}[\rho'^\dagger], c\mathcal{T})}\frac{(\langle \mathcal{O}^\dagger\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle \mathcal{O}'^\dagger\mathcal{O}'\rangle, \langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E}\langle \mathcal{O}_1\mathcal{O}'_2\mathbf{L}[\mathcal{O}'^\dagger]\rangle.\end{aligned} \tag{E.127}$$

This implies that the pairing (E.123) is equal to

$$\frac{4^{-J} f_J(\rho^\dagger)}{4^{-J'} f_J(\rho'^\dagger)}\frac{f_L(\mathbf{L}[\Delta], \mathbf{L}[\rho^\dagger], \mathcal{T})}{f_L(\mathbf{L}[\Delta'], \mathbf{L}[\rho'^\dagger], c\mathcal{T})}\frac{(\langle \mathcal{O}^\dagger\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle \mathcal{O}'^\dagger\mathcal{O}'\rangle, \langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E} \tag{E.128}$$

and thus

$$\begin{aligned}&\mathbf{S}[\mathbf{L}[\widetilde{\mathcal{D}}]]^A \mathcal{D}_{1,A}\langle \mathcal{O}_1\mathcal{O}'_2\mathbf{L}[\mathcal{O}'^\dagger]\rangle^{-1} \\ &= \frac{4^{-J} f_J(\rho^\dagger)}{4^{-J'} f_J(\rho'^\dagger)}\frac{f_L(\mathbf{L}[\Delta], \mathbf{L}[\rho^\dagger], \mathcal{T})}{f_L(\mathbf{L}[\Delta'], \mathbf{L}[\rho'^\dagger], c\mathcal{T})}\frac{(\langle \mathcal{O}^\dagger\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle \mathcal{O}'^\dagger\mathcal{O}'\rangle, \langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E}(\langle \mathcal{O}_1\mathcal{O}_2\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1}.\end{aligned} \tag{E.129}$$

Collecting all the pieces, we find that (E.120) implies (6.214) for the seed H if

$$C = \frac{(\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle)_L^{-1}}{(\langle\mathbf{L}[\mathcal{O}']\mathbf{L}[\mathcal{O}']\rangle)_L^{-1}}\frac{4^{-J} f_J(\rho^\dagger)}{4^{-J'} f_J(\rho'^\dagger)}\frac{f_L(\mathbf{L}[\Delta], \mathbf{L}[\rho^\dagger], \mathcal{T})}{f_L(\mathbf{L}[\Delta'], \mathbf{L}[\rho'^\dagger], c\mathcal{T})}\frac{(\langle \mathcal{O}^\dagger\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle \mathcal{O}'^\dagger\mathcal{O}'\rangle, \langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E} = 1. \tag{E.130}$$

Proof that $C = 1$ First, we note that

$$\frac{(\langle \mathcal{O}^\dagger\mathcal{O}\rangle, \langle\widetilde{\mathcal{O}}^\dagger\widetilde{\mathcal{O}}\rangle)_E}{(\langle \mathcal{O}'^\dagger\mathcal{O}'\rangle, \langle\widetilde{\mathcal{O}}'^\dagger\widetilde{\mathcal{O}}'\rangle)_E} = \frac{4^J \dim\rho^\dagger}{4^{J'}\dim\rho'^\dagger}. \tag{E.131}$$

Furthermore, f_J is square of shadow transform in $d-2$ dimensions. Thus if we write $\rho^\dagger = (J, \lambda)$ then (similarly to appendix E.3)

$$f_J(\rho^\dagger) \propto \frac{\dim \lambda}{\mu(\rho^\dagger)}, \tag{E.132}$$

where μ is the Plancherel measure for $\mathrm{SO}(d-1,1)$. Furthermore, the ratio

$$\frac{\mu(\rho^\dagger)}{\dim \rho^\dagger} \tag{E.133}$$

is independent of ρ [65, 195]. This implies that

$$\frac{4^{-J} f_J(\rho^\dagger)}{4^{-J'} f_J(\rho'^\dagger)} \frac{(\langle \mathcal{O}^\dagger \mathcal{O}\rangle, \langle \widetilde{\mathcal{O}}^\dagger \widetilde{\mathcal{O}}\rangle)_E}{(\langle \mathcal{O}'^\dagger \mathcal{O}'\rangle, \langle \widetilde{\mathcal{O}}'^\dagger \widetilde{\mathcal{O}}'\rangle)_E} = \frac{\dim \lambda}{\dim \lambda'}. \tag{E.134}$$

Furthermore, we can write

$$\frac{\dim \lambda}{\dim \lambda'} = \frac{(\langle \mathcal{O}'\mathcal{O}'^\dagger\rangle)_L^{-1}}{(\langle \mathcal{O}\mathcal{O}^\dagger\rangle)_L^{-1}}, \tag{E.135}$$

which is due to

$$(\langle \mathcal{O}\mathcal{O}^\dagger\rangle, \langle \mathcal{O}^S \mathcal{O}^{S\dagger}\rangle)_L \propto \dim \lambda, \tag{E.136}$$

and similarly for primed quantities (see appendix E.5).

Now we need to recall the calculation of $\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}^\dagger]\rangle$. We have for the kernel which is represented by the time-ordered two-point function $\langle \mathcal{O}\mathcal{O}^\dagger\rangle$,

$$\langle \mathcal{O}\mathcal{O}^\dagger\rangle = \mathbf{S}(1 + \sum_{n=1}^{\infty} \gamma^{-n}(\mathcal{T}^n + \mathcal{T}^{-n})), \tag{E.137}$$

where γ is the eigenvalue of $\mathcal{T}$ corresponding to $\mathcal{O}$, see (6.38). The calculation in section 6.4.1.4 then yields, in the same sense as above,

$$\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}^\dagger]\rangle = \mathbf{S}(1 + \sum_{n=1}^{\infty} \gamma^{-n}(\mathcal{T}^n + \mathcal{T}^{-n}))\mathcal{T}^{-1} f_L(\mathbf{F}[\Delta], \mathbf{F}[\rho], \mathcal{T}). \tag{E.138}$$

Since $\mathbf{L}$ commutes with $\mathbf{S}$, we find that we can replace $f_L(\mathbf{F}[\Delta], \mathbf{F}[\rho], \mathcal{T})$ by $f_L(\mathbf{L}[\Delta], \mathbf{L}[\rho], \mathcal{T})$. This implies

$$\frac{(\langle \mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle)_L^{-1}}{(\langle \mathbf{L}[\mathcal{O}']\mathbf{L}[\mathcal{O}']\rangle)_L^{-1}} = \frac{(1 + \sum_{n=1}^{\infty} \gamma'^{-n}(\mathcal{T}^n + \mathcal{T}^{-n})) f_L(\mathbf{L}[\Delta'], \mathbf{L}[\rho'], \mathcal{T})}{(1 + \sum_{n=1}^{\infty} \gamma^{-n}(\mathcal{T}^n + \mathcal{T}^{-n})) f_L(\mathbf{L}[\Delta], \mathbf{L}[\rho], \mathcal{T})} \frac{(\langle \mathcal{O}\mathcal{O}^\dagger\rangle)_L^{-1}}{(\langle \mathcal{O}'\mathcal{O}'^\dagger\rangle)_L^{-1}}. \tag{E.139}$$

Recall that $\overline{\mathbf{S}[\widetilde{\mathcal{D}}]}$ takes $\mathcal{O}$ to $\mathcal{O}'$ and $c\widetilde{\mathcal{D}} = \mathcal{T}\widetilde{\mathcal{D}}\mathcal{T}^{-1}$, which implies $\gamma' = c\gamma = \pm g$. (Recall we consider only bosonic representations.) Thus we have

$$\begin{aligned}
&\frac{(1+\sum_{n=1}^{\infty}\gamma'^{-n}(\mathcal{T}^n+\mathcal{T}^{-n}))}{(1+\sum_{n=1}^{\infty}\gamma^{-n}(\mathcal{T}^n+\mathcal{T}^{-n}))} f_L(\mathbf{L}[\Delta'],\mathbf{L}[\rho'],\mathcal{T}) \\
&= \frac{(1+\sum_{n=1}^{\infty}\gamma'^{-n}(\mathcal{T}^n+\mathcal{T}^{-n}))}{(1+\sum_{n=1}^{\infty}\gamma'^{-n}((c\mathcal{T})^n+(c\mathcal{T})^{-n}))} f_L(\mathbf{L}[\Delta'],\mathbf{L}[\rho'],\mathcal{T}) \\
&= \frac{(c\mathcal{T}-\gamma)(c\mathcal{T}-\gamma^{-1})}{(\mathcal{T}-\gamma)(\mathcal{T}-\gamma^{-1})} f_L(\mathbf{L}[\Delta'],\mathbf{L}[\rho'],\mathcal{T}) \\
&= f_L(\mathbf{L}[\Delta'],\mathbf{L}[\rho'],c\mathcal{T}),
\end{aligned} \tag{E.140}$$

where we used the fact that (6.103) is $\mathcal{T}$-independent. We thus conclude that

$$\frac{(\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}^\dagger]\rangle)_L^{-1}}{(\langle\mathbf{L}[\mathcal{O}']\mathbf{L}[\mathcal{O}'^\dagger]\rangle)_L^{-1}} = \frac{f_L(\mathbf{L}[\Delta'],\mathbf{L}[\rho'],c\mathcal{T})}{f_L(\mathbf{L}[\Delta],\mathbf{L}[\rho],\mathcal{T})}\frac{(\langle\mathcal{O}\mathcal{O}^\dagger\rangle)_L^{-1}}{(\langle\mathcal{O}'\mathcal{O}'^\dagger\rangle)_L^{-1}}. \tag{E.141}$$

By combining this equation with (E.134) and (E.135) we see that indeed[20]

$$C = 1. \tag{E.142}$$

E.8 Conformal blocks with continuous spin

E.8.1 Gluing three-point structures

Consider two three-point structures $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle$ and $\langle\mathcal{O}\mathcal{O}_3\mathcal{O}_4\rangle$. We can glue them into a conformal block as follows. We find a linear operator $B_{12\mathcal{O}}(x_{12})$ such that in the OPE limit $1\to 2$, the first three-point structure becomes

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}^\dagger(x)\rangle \sim B_{12\mathcal{O}}(x_{12})\langle\mathcal{O}(x_2)\mathcal{O}^\dagger(x)\rangle, \qquad (|x_{12}| \ll |x_1-x|,|x_2-x|). \tag{E.143}$$

For example, when $\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}$ are all scalars, we have

$$B_{12\mathcal{O}}(x_{12}) = x_{12}^{\Delta_{\mathcal{O}}-\Delta_1-\Delta_2}. \tag{E.144}$$

($B_{12\mathcal{O}}$ can be extended to a differential operator such that (E.143) becomes an equality away from the $1\to 2$ limit, but this is not necessary for the current discussion.) Note that to define $B_{12\mathcal{O}}$ we must choose a normalization of the two-point structure $\langle\mathcal{O}\mathcal{O}\rangle$.

[20]Since we for simplicity restricted to bosonic representations, we haven't been very careful with distinguishing ρ and $\rho^\dagger$. (There is no difference except possibly for self-dual tensors.) It would be interesting to repeat our argument in a more careful manner, accounting for fermionic representations as well.

We define a conformal block $G_{\mathcal{O}}^{\mathcal{O}_i}(x_i)$ as the conformally-invariant solution to the conformal Casimir equation [57] whose OPE limit is

$$G_{\mathcal{O}}^{\mathcal{O}_i}(x_i) \sim B_{12\mathcal{O}}(x_{12})\langle \mathcal{O}(x_2)\mathcal{O}_3\mathcal{O}_4\rangle, \qquad (|x_{12}| \ll |x_{ij}|). \tag{E.145}$$

It is very useful to introduce the following notation for a conformal block, which makes manifest the choices of two- and three-point structures needed to define it

$$G_{\mathcal{O}}^{\mathcal{O}_i}(x_i) = \frac{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}^\dagger\rangle\langle \mathcal{O}\mathcal{O}_3\mathcal{O}_4\rangle}{\langle \mathcal{O}\mathcal{O}^\dagger\rangle}. \tag{E.146}$$

In our convention $\mathcal{O}$ appears in the OPE $\mathcal{O}_1 \times \mathcal{O}_2$ and $\mathcal{O}^\dagger$ in the OPE $\mathcal{O}_3 \times \mathcal{O}_4$.

E.8.1.1 Example: integer spin in Euclidean signature

As an example, let us review the case of external scalars $\phi_1, \ldots, \phi_4$ and an exchanged operator $\mathcal{O}$ with integer spin J,

$$G_{\Delta,J}^{\Delta_i}(x_i) = \frac{\langle \phi_1\phi_2\mathcal{O}\rangle\langle \phi_3\phi_4\mathcal{O}\rangle}{\langle \mathcal{O}\mathcal{O}\rangle}, \tag{E.147}$$

where $\langle \phi_1\phi_2\mathcal{O}\rangle$ and $\langle \phi_3\phi_4\mathcal{O}\rangle$ are the standard three-point structures (E.25) and $\langle \mathcal{O}\mathcal{O}\rangle$ is the standard two-point structure (E.24). We will assume that all points are in Euclidean signature.

In the OPE limit $1 \to 2$, we have

$$\begin{aligned}\langle \phi_1\phi_2\mathcal{O}(x_0, z)\rangle &\sim \frac{1}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}} \frac{(-2z \cdot I(x_{20}) \cdot x_{12})^J}{x_{20}^{2\Delta}} \\ &= \frac{1}{x_{12}^{\Delta_1+\Delta_2-\Delta+J}} x_{12}^{\mu_1} \cdots x_{12}^{\mu_J} \langle \mathcal{O}_{\mu_1\cdots\mu_J}(x_2)\mathcal{O}(x_0, z)\rangle.\end{aligned} \tag{E.148}$$

To compute the leading behavior of the block, it suffices to take the limit $3 \to 4$ in $\langle \phi_3\phi_4\mathcal{O}\rangle$,

$$\langle \phi_3\phi_4\mathcal{O}_{\mu_1\cdots\mu_J}(x_2)\rangle = \frac{1}{x_{34}^{\Delta_3+\Delta_4-\Delta+J}} \frac{(-2I(x_{42}) \cdot x_{34})_{\mu_1} \cdots (-2I(x_{42}) \cdot x_{34})_{\mu_J} - \text{traces}}{x_{42}^{2\Delta}}. \tag{E.149}$$

(This limit is identical to the first line of (E.148) after replacing $1, 2, 0 \to 3, 4, 2$ and stripping off the polarization vector z.) Thus the OPE limit of the resulting block is

$$\begin{aligned}G_{\Delta,J}^{\Delta_i}(x_i) &\sim \frac{x_{12}^{\mu_1} \cdots x_{12}^{\mu_J}}{x_{12}^{\Delta_1+\Delta_2-\Delta+J} x_{34}^{\Delta_3+\Delta_4-\Delta+J}} \frac{(-2I(x_{42}) \cdot x_{34})_{\mu_1} \cdots (-2I(x_{42}) \cdot x_{34})_{\mu_J} - \text{traces}}{x_{42}^{2\Delta}} \\ &= \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{12}^2 x_{34}^2}{x_{42}^4}\right)^{\Delta/2} 2^J \hat{C}_J\left(\frac{-x_{12} \cdot I(x_{42}) \cdot x_{34}}{|x_{12}||x_{34}|}\right).\end{aligned} \tag{E.150}$$

Here, we've used the identity

$$(m^{\mu_1}\cdots m^{\mu_J})(n_{\mu_1}\cdots n_{\mu_J}-\text{traces}) = |m|^J|n|^J\hat{C}_J\left(\frac{m\cdot n}{|m||n|}\right), \tag{E.151}$$

where

$$\hat{C}_J(\eta) = \frac{\Gamma(\frac{d-2}{2})\Gamma(J+d-2)}{2^J\Gamma(J+\frac{d-2}{2})\Gamma(d-2)}{}_2F_1\left(-J, J+d-2, \frac{d-1}{2}, \frac{1-\eta}{2}\right) \tag{E.152}$$

is proportional to a Gegenbauer polynomial (note in particular that for $\eta = 1$ the hypergeometric function reduces to 1). Factoring out some standard kinematical factors, we find

$$G^{\Delta_i}_{\Delta,J}(x_i) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{\Delta_2-\Delta_1}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} G^{\Delta_i}_{\Delta,J}(\chi,\overline{\chi}), \tag{E.153}$$

where $G^{\Delta_i}_{\Delta,J}(\chi,\overline{\chi})$ is a solution to the conformal Casimir equations normalized so that

$$G^{\Delta_i}_{\Delta,J}(\chi,\overline{\chi}) \sim (\chi\overline{\chi})^{\Delta/2}\left(\frac{\chi}{\overline{\chi}}\right)^{-J/2}, \qquad (\chi \ll \overline{\chi} \ll 1). \tag{E.154}$$

Here, $\chi, \overline{\chi}$ are conformal cross-ratios defined by $u = \chi\overline{\chi}$, $v = (1-\chi)(1-\overline{\chi})$. This is the standard conformal block in the normalization convention of [66, 67].

E.8.1.2 Example: continuous spin in Lorentzian signature

Our definition of a conformal block also works when $\mathcal{O}$ has continuous spin. However, now we must allow $B_{12\mathcal{O}}$ to be an integral operator in the polarization vector of $\mathcal{O}$. Let us again consider external scalars $\phi_1, \ldots, \phi_4$. For later applications, we work in a Lorentzian configuration where all four points $1, 2, 3, 4$ are in the same Minkowski patch, with the causal relationships $1 > 2$, $3 > 4$, and all other pairs spacelike-separated, see figure E.1.

We also modify the three-point structures by taking $x_{34}^2 \to -x_{34}^2$ and $x_{12}^2 \to -x_{12}^2$ so that they are positive when x_0 is spacelike from $1, 2$ and $3, 4$, since precisely these positive structures will appear later. Specifically, let

$$T^{\Delta_1,\Delta_2}_{\Delta,J}(x_1,x_2,x_0,z) = \frac{(2z\cdot x_{20}x_{10}^2 - 2z\cdot x_{10}x_{20}^2)^J}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta+J}{2}}(x_{10}^2)^{\frac{\Delta_1+\Delta-\Delta_2+J}{2}}(x_{20}^2)^{\frac{\Delta_2+\Delta-\Delta_1+J}{2}}}. \tag{E.155}$$

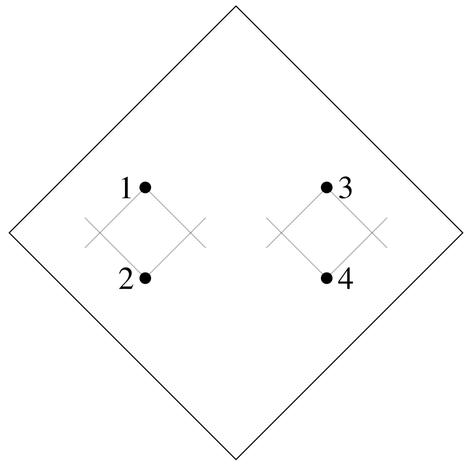


Figure E.1: A configuration of points where $1 > 2$ and $3 > 4$, with all other pairs of points spacelike-separated. The three-point structure (E.155) is positive in this configuration.

We will study the block

$$\frac{T^{\Delta_1,\Delta_2}_{\Delta,J} T^{\Delta_3,\Delta_4}_{\Delta,J}}{\langle \mathcal{O}\mathcal{O}\rangle}, \tag{E.156}$$

where $\langle \mathcal{O}\mathcal{O}\rangle$ is the two-point structure (E.24). To define a block, our structures only need to be defined when x_0 is spacelike from the other points, so we do not need to give an $i\epsilon$ prescription here.

In the OPE limit $1 \to 2$, we have

$$T^{\Delta_1,\Delta_2}_{\Delta,J}(x_1, x_2, x_0, z) \sim \frac{1}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta+J}{2}}} \frac{(-2z\cdot I(x_{20})\cdot x_{12})^J}{(x_{20}^2)^\Delta} \qquad (1\to 2). \tag{E.157}$$

The quantity on the right differs from the two-point structure $\langle \mathcal{O}(x_2, z')\mathcal{O}(x_0, z)\rangle$ by the replacement $z' \to x_{12}$. We can no longer strip off z' and contract indices with x_{12}. However, the replacement can be achieved via an integral transform:

$$\begin{aligned}
&T^{\Delta_1,\Delta_2}_{\Delta,J}(x_1, x_2, x_0, z) \sim B_{12\mathcal{O}}\langle \mathcal{O}(x_2, z')\mathcal{O}(x_0, z)\rangle \\
&B_{12\mathcal{O}} f(x', z') = \\
&= \frac{1}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta-J-d+2}{2}}} \frac{\Gamma(J+d-2)}{\pi^{\frac{d-2}{2}}\Gamma(J+\frac{d-2}{2})} \int D^{d-2}z'(-2x_{12}\cdot z')^{2-d-J} f(x', z').
\end{aligned} \tag{E.158}$$

Now let us apply $B_{12\mathcal{O}}$ to the three-point structure $T^{\Delta_3,\Delta_4}_{\Delta,J}(x_3, x_4, x_2, z)$, working in the limit $3 \to 4$ (since this is sufficient to determine the small cross-ratio dependence of the resulting block). In doing so, we need the identity

$$\int D^{d-2}z'\,(-2x_{12}\cdot z')^{2-d-J}(-2z'\cdot I(x_{42})\cdot x_{34})^J$$
$$= (-x_{12}^2)^{\frac{2-d-J}{2}}(-x_{34}^2)^{\frac{J}{2}}\frac{2^{2-d}\mathrm{vol}(S^{d-2})}{\hat{C}_J(1)}\hat{C}_J\left(\frac{-x_{12}\cdot I(x_{42})\cdot x_{34}}{(-x_{12}^2)^{1/2}(-x_{34}^2)^{1/2}}\right), \tag{E.159}$$

where $\hat{C}_J(\eta)$ is given in (E.152). (Here, it is important that we use the correct definition of $\hat{C}_J$ for non-integer J.) Using (E.159), we find that in the OPE limit

$$\frac{T^{\Delta_1,\Delta_2}_{\Delta,J}T^{\Delta_3,\Delta_4}_{\Delta,J}}{\langle OO\rangle} \sim \frac{1}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{12}^2x_{34}^2}{x_{42}^4}\right)^{\Delta/2}2^J\hat{C}_J\left(\frac{-x_{12}\cdot I(x_{42})\cdot x_{34}}{(-x_{12}^2)^{1/2}(-x_{34}^2)^{1/2}}\right), \tag{E.160}$$

so that

$$\frac{T^{\Delta_1,\Delta_2}_{\Delta,J}T^{\Delta_3,\Delta_4}_{\Delta,J}}{\langle OO\rangle} = \frac{1}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{\Delta_2-\Delta_1}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}}G^{\Delta_i}_{\Delta,J}(\chi,\overline{\chi}). \tag{E.161}$$

This is the same result we would have gotten by pretending J was an integer and performing the computation in the previous subsection. However, here we see that a conformal block with non-integer J is well-defined and completely specified by continuous-spin two- and three-point structures.

E.8.1.3 Rules for weight-shifting operators

Let us consider how the gluing rule described in E.8.1 interacts with weight-shifting operators changing the internal representation. Suppose we can write

$$\langle O_1O_2O^\dagger(x)\rangle = \langle O_1(\mathcal{D}_AO_2')(\widetilde{\mathcal{D}}^AO'^\dagger)\rangle \tag{E.162}$$

for a pair of weight-shifting operators $\mathcal{D}$ and $\widetilde{\mathcal{D}}$. By acting with the same weight-shifting operators on (E.143) for primed operators we find

$$\langle O_1O_2O^\dagger(x)\rangle \sim (\mathcal{D}_{2,A}B_{12O})(x_{12})\langle O(x_2)(\widetilde{\mathcal{D}}^AO'^\dagger)(x)\rangle. \tag{E.163}$$

Recall the crossing equation (E.104), which holds when the two-point structures are related to the kernel of $\mathbf{S}$-transform. Let us assume for now that this is the case. Then we find

$$\langle O_1O_2O^\dagger(x)\rangle \sim (\mathcal{D}_{2,A}B_{12O})(x_{12})\langle(\mathbf{S}[\overline{\widetilde{\mathcal{D}}}]^AO)(x_2)O'^\dagger(x)\rangle. \tag{E.164}$$

Substituting this into (E.145), we find

$$G_{\mathcal{O}}^{\mathcal{O}_i}(x_i) \sim (\mathcal{D}_{2,A}B_{12\mathcal{O}})(x_{12})\langle(\mathbf{S}[\overline{\widetilde{\mathcal{D}}}]^A\mathcal{O})(x_2)\mathcal{O}_3\mathcal{O}_4\rangle. \tag{E.165}$$

Using notation (E.146) we can summarize this as[21]

$$\frac{\langle\mathcal{O}_1(\mathcal{D}_A\mathcal{O}_2)(\widetilde{\mathcal{D}}^A\mathcal{O}'^\dagger)\rangle\langle\mathcal{O}\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}\mathcal{O}\rangle} = \frac{\langle\mathcal{O}_1(\mathcal{D}_A\mathcal{O}_2)\mathcal{O}'^\dagger\rangle\langle(\mathbf{S}[\overline{\widetilde{\mathcal{D}}}]^A\mathcal{O})\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}'\mathcal{O}'\rangle}. \tag{E.166}$$

This holds if the two-point functions for $\mathcal{O}$ and $\mathcal{O}'$ are standard in the sense of being related to $\mathbf{S}$-kernel. Generalization of this to arbitrary two-point functions is given by

$$\frac{\langle\mathcal{O}_1(\mathcal{D}_A\mathcal{O}_2)(\widetilde{\mathcal{D}}^A\mathcal{O}'^\dagger)\rangle\langle\mathcal{O}\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}\mathcal{O}\rangle} = \frac{\langle\mathcal{O}'\mathcal{O}'\rangle}{\langle\mathcal{O}\mathcal{O}\rangle}\frac{\langle\mathcal{O}_1(\mathcal{D}_A\mathcal{O}_2)\mathcal{O}'^\dagger\rangle\langle(\mathbf{S}[\overline{\widetilde{\mathcal{D}}}]^A\mathcal{O})\mathcal{O}_3\mathcal{O}_4\rangle}{\langle\mathcal{O}'\mathcal{O}'\rangle}, \tag{E.167}$$

where the ratio of two-point functions is a scalar defined as

$$\frac{\langle\mathcal{O}'\mathcal{O}'\rangle}{\langle\mathcal{O}\mathcal{O}\rangle} \equiv \frac{\langle\mathcal{O}'\mathcal{O}'\rangle}{\langle\mathcal{O}'\mathcal{O}'\rangle_0}\frac{\langle\mathcal{O}\mathcal{O}\rangle_0}{\langle\mathcal{O}\mathcal{O}\rangle}, \tag{E.168}$$

where the structures with subscript 0 are standard and related to $\mathbf{S}$-kernel. Note that we can reverse (E.167) by replacing $\widetilde{\mathcal{D}} \to \overline{\mathbf{S}^{-1}[\widetilde{\mathcal{D}}]}$. However, due to (E.96) we have $\overline{\mathbf{S}^{-1}[\widetilde{\mathcal{D}}]} = \mathbf{S}[\overline{\widetilde{\mathcal{D}}}]$ and so we get the same rule for moving the operator from right to left.

E.8.2 A Lorentzian integral for a conformal block

Conformal blocks in Euclidean signature can be computed via a "shadow representation," where one integrates a product of three-point functions over Euclidean space [54, 118, 233]. However, this integral produces a linear combination of a standard block $G_{\Delta,J}^{\Delta_i}$ and the so-called "shadow block" $G_{d-\Delta,J}^{\Delta_i}$. The shadow block comes from regions of the integral where the OPE is not valid inside the integrand.

By contrast, there is a simple integral representation for a block alone (without its shadow) in Lorentzian signature [203]. The reason is that in Lorentzian signature, we can integrate over a conformally-invariant region that stays away from two of the points, say $x_{3,4}$. Thus, the $x_3 \to x_4$ OPE limit can be taken inside the integrand and dictates the behavior of the result.

[21] The results of [3] concerning weight-shifting of the internal representation are recovered by further using crossing for the weight-shifting operator acting on the right three-point structure.

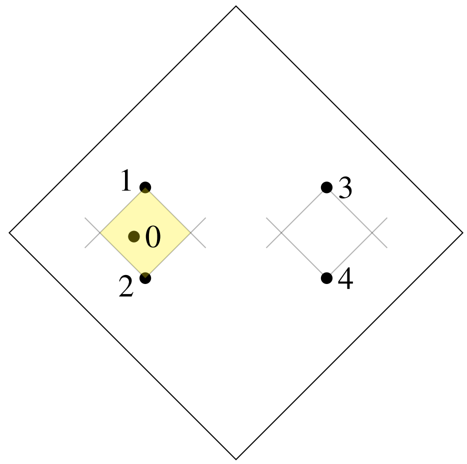


Figure E.2: In the Lorentzian integral for a conformal block, the point x_0 is integrated over the diamond $2 < 0 < 1$ (yellow). Because the integration region is far away from points 3, 4, the 3×4 OPE is valid inside the integral.

The Lorentzian integral for a conformal block plays an important role in section 6.4.1.2, so let us compute it. Consider the same configuration as in the previous subsection where $1, 2, 3, 4$ are in the same Poincare patch, with $1 > 2$ and $3 > 4$, and other pairs of points spacelike separated from each other (figure E.2). We can produce a conformal block in the $1, 2 \to 3, 4$ channel by performing a shadow-like integral over the causal diamond $2 < 0 < 1$,

$$\mathcal{G}_{\Delta,J} \equiv \int_{2<0<1} d^d x_0 D^{d-2} z \, |T^{\Delta_1,\Delta_2}_{d-\Delta,2-d-J}(x_1, x_2, x_0, z)| T^{\Delta_3,\Delta_4}_{\Delta,J}(x_3, x_4, x_0, z). \quad \text{(E.169)}$$

The notation $|T^{\Delta_1,\Delta_2}_{d-\Delta,2-d-J}|$ means that spacetime intervals x_{ij} should appear with absolute values $|x_{ij}|$, so that the integrand is positive in the configuration we are considering. (This notation is somewhat imprecise, since when $\Delta_1, \Delta_2, \Delta, J$ are complex, we do not mean one should take the absolute value of the whole expression.) When J is an integer, there is a similar integral expression for a Lorentzian block with $\int D^{d-2} z$ replaced by index contractions. However (E.169) also works for continuous spin.

The expression (E.169) is proportional to $G_{\Delta,J}$ because it is a conformally-invariant solution to the Casimir equation whose OPE limit agrees with the OPE limit of $T^{\Delta_3,\Delta_4}_{\Delta,J}$ (because the integration point stays away from $x_{3,4}$). The behavior of the integral in the limit $1 \to 2$ is not immediately obvious. However, conformal invariance requires that this limit must be the same as $3 \to 4$.

More precisely, in the OPE limit $3 \to 4$, we have

$$T^{\Delta_3,\Delta_4}_{\Delta,J}(x_3,x_4,x_0,z) \sim B_{34\mathcal{O}}\langle \mathcal{O}(x_4,z')\mathcal{O}(x_0,z)\rangle \qquad (3\to 4,\ \ 0\approx 3,4), \tag{E.170}$$

where $B_{34\mathcal{O}}$ is the linear operator defined in (E.158). Plugging this in, we find $(3 \to 4)$

$$\mathcal{G}_{\Delta,J} \sim B_{34\mathcal{O}}\int_{2<0<1} d^d x_0 D^{d-2}z\,|T^{\Delta_1,\Delta_2}_{d-\Delta,2-d-J}(x_1,x_2,x_0,z)|\langle \mathcal{O}(x_4,z')\mathcal{O}(x_0,z)\rangle. \tag{E.171}$$

The integral in the OPE limit now takes the form of an **S**-transform.

E.8.2.1 Shadow transform in the diamond

Let us evaluate the integral (E.171) by splitting it into two steps: first we apply $\mathbf{S}_\Delta$ and then subsequently $\mathbf{S}_J$. For notational convenience, define

$$\begin{aligned}\Delta_0 &\equiv d-\Delta\\ J_0 &\equiv 2-d-J.\end{aligned}\tag{E.172}$$

The $\mathbf{S}_\Delta$ transform is fixed by conformal invariance up to a coefficient $a^{\Delta_1,\Delta_2}_{\Delta_0,J_0}$,

$$\begin{aligned}&\mathbf{S}_{\Delta 0}[|T^{\Delta_1,\Delta_2}_{d-\Delta,2-d-J}(x_1,x_2,x_0,z)|\theta(2<0<1)]\\ &= \int_{2<0<1} d^d x_0 \frac{1}{x_{04}^{2(d-\Delta_0)}}|T^{\Delta_1,\Delta_2}_{d-\Delta,2-d-J}(x_1,x_2,x_0,I(x_{04})z)|\\ &= a^{\Delta_1,\Delta_2}_{\Delta_0,J_0}\frac{|2z\cdot x_{14}x_{24}^2-2z\cdot x_{24}x_{14}^2|^{J_0}}{|x_{12}|^{\Delta_1+\Delta_2-(d-\Delta_0)+J_0}|x_{14}|^{\Delta_1+(d-\Delta_0)-\Delta_2+J_0}|x_{24}|^{\Delta_2+(d-\Delta_0)-\Delta_1+J_0}}.\end{aligned}\tag{E.173}$$

Here, we are writing expressions valid in the kinematical configuration we are considering, namely $2<0<1$ and $4\approx 1,0,2$. To find the coefficient, we choose the following configuration in lightcone coordinates

$$\begin{aligned} x_0 &= (u,v,x_\perp),\\ x_1 &= (1,0,0),\\ x_2 &= (0,1,0),\\ x_4 &= (\infty,\infty,0),\\ w &= I(x_{04})z = (2,0,0),\end{aligned}\tag{E.174}$$

where the metric is $x^2 = uv + x_\perp^2$. Note that since 4 is sent to infinity, w is actually independent of x_0. Our integral becomes

$$\begin{aligned} a^{\Delta_1,\Delta_2}_{\Delta_0,J_0} &= \frac{1}{2^{J_0+1}}\int du\,dv\,dx_\perp \frac{|2w\cdot x_{10}x_{20}^2 - 2w\cdot x_{20}x_{10}^2|^{J_0}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_0+J_0}|x_{10}|^{\Delta_1+\Delta_0-\Delta_2+J_0}|x_{20}|^{\Delta_2+\Delta_0-\Delta_1+J_0}} \\ &= \frac{\mathrm{vol}(S^{d-3})}{2}\int du\,dv\,dr\,r^{d-3}\frac{(u(1-u)-r^2)^{J_0}}{(u(1-v)-r^2)^{\frac{\Delta_1-\Delta_2+\Delta_0+J_0}{2}}(v(1-u)-r^2)^{\frac{\Delta_2-\Delta_1+\Delta_0+J_0}{2}}}. \end{aligned} \tag{E.175}$$

It is now straightforward to perform the v integral over $v \in [\frac{r^2}{1-u}, \frac{u-r^2}{u}]$, followed by the r integral over $r \in [0, \sqrt{u(1-u)}]$, and finally the u integral over $u \in [0,1]$. The result is

$$\begin{aligned} &a^{\Delta_1,\Delta_2}_{\Delta_0,J_0} \\ &= \frac{\pi^{\frac{d-2}{2}}\Gamma(2-\Delta_0)\Gamma(\frac{2-J_0-\Delta_0+\Delta_1-\Delta_2}{2})\Gamma(\frac{d+J_0-\Delta_0+\Delta_1-\Delta_2}{2})\Gamma(\frac{2-J_0-\Delta_0-\Delta_1+\Delta_2}{2})\Gamma(\frac{d+J_0-\Delta_0-\Delta_1+\Delta_2}{2})}{2\Gamma(1+\frac{d}{2}-\Delta_0)\Gamma(2-J_0-\Delta_0)\Gamma(d+J_0-\Delta_0)}. \end{aligned} \tag{E.176}$$

Note that $a^{\Delta_1,\Delta_2}_{\Delta_0,J_0} = a^{\Delta_1,\Delta_2}_{\Delta_0,2-d-J_0}$, which is consistent with the requirement that $\mathbf{S}_\Delta$ commute with $\mathbf{S}_J$. We can additionally perform $\mathbf{S}_J$ using

$$\begin{aligned} &\int D^{d-2}z'(-2z\cdot z')^{2-d-J_0}(-2z'\cdot v)^{J_0} \\ &= \frac{\pi^{\frac{d-2}{2}}\Gamma(-J_0-\frac{d-2}{2})}{\Gamma(-J_0)}(-v^2)^{\frac{d-2}{2}+J_0}(-2z\cdot v)^{2-d-J_0}. \end{aligned} \tag{E.177}$$

Combining everything together, we find

$$\begin{aligned} \mathbf{S}_0[|T^{\Delta_1,\Delta_2}_{d-\Delta,2-d-J}(x_1,x_2,x_0,z)|\theta(2<0<1)] &= b^{\Delta_1,\Delta_2}_{\Delta,J}T^{\Delta_1,\Delta_2}_{\Delta,J}(x_1,x_2,x_4,z) \\ b^{\Delta_1,\Delta_2}_{\Delta,J} &\equiv \frac{\pi^{\frac{d-2}{2}}\Gamma(J+\frac{d-2}{2})}{\Gamma(J+d-2)}a^{\Delta_1,\Delta_2}_{d-\Delta,2-d-J}. \end{aligned} \tag{E.178}$$

Plugging this into (E.171) and using (E.161), we conclude

$$\mathcal{G}_{\Delta,J}(x_i) = \frac{b^{\Delta_1,\Delta_2}_{\Delta,J}}{(-x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(-x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{\Delta_2-\Delta_1}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}}G^{\Delta_i}_{\Delta,J}(\chi,\overline{\chi}). \tag{E.179}$$

E.8.3 Conformal blocks at large J

In this appendix, we compute the large-J behavior of a conformal block. Recall that we have the decomposition

$$G^{\Delta_i}_{\Delta,J}(\chi,\overline{\chi}) = g^{\mathrm{pure}}_{\Delta,J}(\chi,\overline{\chi}) + \frac{\Gamma(J+d-2)\Gamma(-J-\frac{d-2}{2})}{\Gamma(J+\frac{d-2}{2})\Gamma(-J)}g^{\mathrm{pure}}_{\Delta,2-d-J}(\chi,\overline{\chi}). \tag{E.180}$$

Thus it suffices to compute the large-J behavior of $g^{\text{pure}}_{\Delta,J}$.

The Casimir equation was solved in the large-Δ limit in [36, 37]. We can use this result together with an affine Weyl reflection to determine $g^{\text{pure}}_{\Delta,J}$ at large J. The solution from [36] is given by

$$\frac{r^\Delta f_J(\cos\theta)}{(1-r^2)^{\frac{d-2}{2}}(1+r^2+2r\cos\theta)^{\frac{1}{2}(1+\Delta_{12}-\Delta_{34})}(1+r^2-2r\cos\theta)^{\frac{1}{2}(1+\Delta_{34}-\Delta_{12})}} \quad (|\Delta|\gg 1), \tag{E.181}$$

where r and θ are defined by

$$\rho = re^{i\theta}, \quad \overline{\rho} = re^{-i\theta}, \qquad \chi = \frac{4\rho}{(1+\rho)^2}, \quad \overline{\chi} = \frac{4\overline{\rho}}{(1+\overline{\rho})^2}. \tag{E.182}$$

From studying the regime $r \ll 1$, we find that $f_J(\cos\theta)$ must obey the Gegenbauer differential equation.

Note that the conformal Casimir equation has the following symmetries:

$$\begin{aligned} (\Delta, J) &\leftrightarrow (1-J, 1-\Delta), \\ r &\leftrightarrow w = e^{i\theta}. \end{aligned} \tag{E.183}$$

The first is an affine Weyl reflection that preserves the Casimir eigenvalue. The second transformation is equivalent to $\overline{\rho} \leftrightarrow 1/\overline{\rho}$, which leaves $\overline{\chi}$ invariant, and therefore also leaves the Casimir equation invariant. Applying these transformations to (E.181), we find ($|J| \gg 1$)

$$\frac{w^{1-J} f_{1-\Delta}\left(\frac{1}{2}(r+\frac{1}{r})\right)}{(1-w^2)^{\frac{d-2}{2}}(1+w^2+w(r+1/r))^{\frac{1}{2}(1+\Delta_{12}-\Delta_{34})}(1+w^2-w(r+1/r))^{\frac{1}{2}(1+\Delta_{34}-\Delta_{12})}}. \tag{E.184}$$

Note in particular that we have replaced large-Δ with large-J. Demanding pure power behavior as $r \to 0$ requires us to choose the following solution to the Gegenbauer equation:

$$f_J(x) = (2x)^J {}_2F_1\left(\frac{-J}{2}, \frac{1-J}{2}, 2-J-\frac{d}{2}, \frac{1}{x^2}\right). \tag{E.185}$$

Finally, fixing the constant out front and rearranging terms, we find (6.241).

Appendix F

APPENDICES TO CHAPTER 7

F.1 Asymptotic form of conformal blocks on the diagonal $x = \overline{x}$

In [80] a fourth-order differential equation was derived for the conformal blocks in d dimensions on the diagonal $x = \overline{x}$. The derivation is based on combining the quadratic and quartic Casimir equations.

The equation has the following form

$$D_4 f(x) = 0, \tag{F.1}$$

where the differential operator D_4 is defined below and $f(x) = F_{\Delta,\ell}(x = \overline{x})$. This equation is equipped with the boundary condition

$$f(x) \sim x^{\Delta},\ x \to 0, \tag{F.2}$$

which also fixes our normalization.

Another way to phrase our normalization is to say that the conformal block for complex x is given by

$$F_{\Delta,\ell}(x,\overline{x}) \sim r^{\Delta} \frac{C_\ell^{(\epsilon)}(\cos\theta)}{C_\ell^{(\epsilon)}(1)}, \tag{F.3}$$

where $r = |x| \to 0$, $\theta = \arg x$, $\epsilon = d/2 - 1$ and $C_\ell^{(\epsilon)}$ is the Gegenbauer polynomial.

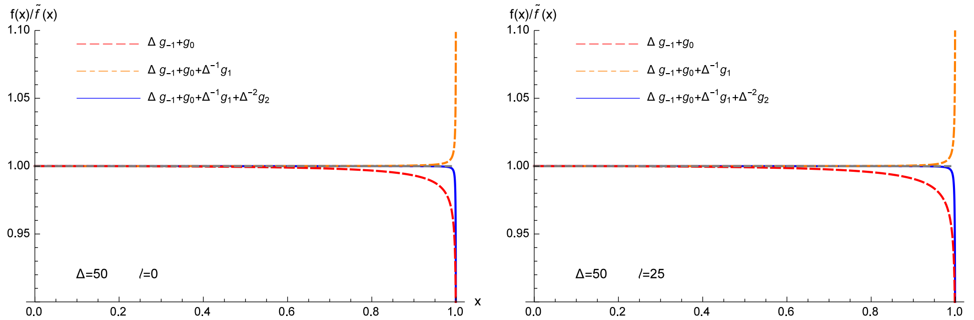
The operator D_4 is given by

$$\begin{aligned} D_4 =& (x-1)^3 x^4 \frac{d^4}{dx^4} + \sum_{r=2}^{3} (x-1)^{r-1} p_r(x) x^r \frac{d^r}{dx^r} + \\ &+ \sum_{r=0}^{1} p_r(x) x^r \frac{d^r}{dx^r}, \end{aligned} \tag{F.4}$$

where p_3, p_2, p_1, p_0 are known [80] polynomials in x of degrees 1, 2, 3, 3 respectively, whose coefficients depend on the differences between external operator scaling dimensions, which we set to 0 (then p_0 is degree 2), as well as on the spin and scaling dimension of the intermediate operator. The dependence of p_r on Δ and ℓ is through the quadratic and quartic Casimir invariants c_2 and c_4,

$$c_2 = \frac{1}{2}\left[\ell(\ell+2\epsilon) + \Delta(\Delta - 2 - 2\epsilon)\right], \tag{F.5}$$

$$c_4 = \ell(\ell+2\epsilon)(\Delta-1)(\Delta-1-2\epsilon). \tag{F.6}$$



(a) $\Delta = 50,\ \ell = 0$ (b) $\Delta = 50,\ \ell = 25$

Figure F.1: Plots of $f(x)/\tilde{f}(x)$ for different orders of approximation and values of angular momenta in four dimensions.

We will be considering the double-scaling limit with $\lambda = \ell/\Delta$ fixed and Δ large. We will do so in order to allow for large angular momenta. Our results will turn out to be applicable to small angular momenta as well by setting the ratio λ to be 0.

With this scaling assumed, $c_2 \propto \Delta^2$ and $c_4 \propto \Delta^4$. The polynomials have the leading behavior

$$p_0 \simeq c_4(x-1), \tag{F.7}$$

$$p_1 \simeq c_2(1-2\epsilon)x^2 + c_2(1+6\epsilon)x - 2c_2(1+2\epsilon), \tag{F.8}$$

$$p_2 \simeq 2c_2(x-1), \tag{F.9}$$

$$p_3 = O(1). \tag{F.10}$$

We would like to see whether there is a WKB-like solution of the form $f(x) \sim e^{\Delta g(x)}$, where $g(x) = O(1)$. It is easy to see that the leading power of Δ produced by action of (F.4) on such a solution will be Δ^4 since each derivative produces a power of Δ, and the polynomials p_r have scaling Δ^k with $k \le 4 - r$. We see that in the leading Δ^4 order only p_0 and p_2 appear. This results in the equation for g (for $x < 1$)

$$\left[\sqrt{1-x}xg'(x)\right]^4 - 2\frac{c_2}{\Delta^2}\left[\sqrt{1-x}xg'(x)\right]^2 + \frac{c_4}{\Delta^4} = 0. \tag{F.11}$$

Here we are only allowed to keep the leading terms in the Casimir invariants. We then find the following solutions,

$$\left[\sqrt{1-x}xg'(x)\right]^2 = 1 \qquad \left[\sqrt{1-x}xg'(x)\right]^2 = \frac{\ell^2}{\Delta^2}. \tag{F.12}$$

With our boundary condition we are interested in $\sqrt{1-x}xg'(x) = 1$ which produces

$$g(x) = \log\rho, \tag{F.13}$$

where

$$\rho = \frac{4x}{(1+\sqrt{1-x})^2}. \tag{F.14}$$

We thus find that $\log f(x) = \log \rho^\Delta + O(1)$ is a solution. We can perform the analysis more systematically by substituting $f(x) = e^{G(x)}$ in (F.1) and looking for g in the form

$$G(x) = \Delta g_{-1}(x) + g_0(x) + \frac{1}{\Delta} g_1(x) + \frac{1}{\Delta^2} g_2(x) + \dots \tag{F.15}$$

Then we will be able to solve the resulting equation order by order in Δ. We already found $g_{-1}(x) = \log \rho$. The next order gives

$$f(x) = \left(1 - \frac{\rho^2}{16}\right)^{-\epsilon-1} \rho^\Delta e^{O(\frac{1}{\Delta})}, \tag{F.16}$$

this not depending on whether we scale ℓ with Δ or not.

Order by order we have

$$g_{-1} = \log \rho, \tag{F.17}$$

$$g_0 = -(1+\epsilon) \log\left(1 - \frac{\rho^2}{16}\right), \tag{F.18}$$

$$g_1 = \frac{\rho^2}{16} \frac{1}{1-\frac{\rho^2}{16}} \frac{(1+\epsilon-\epsilon^2)\Delta^2 + \epsilon(\epsilon-1)\ell^2}{\Delta^2 - \ell^2}, \tag{F.19}$$

$$\dots \tag{F.20}$$

The higher order terms get more messy, but are not hard to compute in principle. We can see that g_2 contains a negative power of $\Delta^2 - \ell^2$, which scales as Δ^2 and is supposed to be canceling Δ^2 scaling in numerator. This means that applicability of our expansion is limited to the region where $\Delta^2 - \ell^2$ is not too small. Higher order terms have higher powers of $\Delta^2 - \ell^2$ in denominators. We also observe that that the subleading terms become singular in the limit $\rho \to 4$ corresponding to $x \to 1$. Therefore, the above approximation works as an asymptotic expansion when

1. $\frac{|\Delta-\ell|}{\Delta}$ is greater than some fixed positive number
2. $x \leq x_0$, where $x_0 < 1$ and is fixed.

We compare the proposed expansion with the exact conformal block in four dimensions in Fig. F.1. There $\tilde{f}$ is the approximate conformal block given by our expansion. We include various numbers of terms in the expansion, up to $\Delta^{-2}g_2$. We see that the approximation works almost equally well for scalar (Fig. F.1a) and large-spin (Fig. F.1b) operators. We also observe the promised singularity at $x = 1$. See Fig. F.2 for comparison at the unitarity bound.

We can get an understanding of how the conformal block behaves when $\ell \to \Delta$ independently of the above thanks to the decoupling of large numbers of descendants for leading twist operators [59]. The unitarity limits the maximal spin of an operator to be $\ell = \Delta - d + 2 = \Delta - 2\epsilon$. It is shown in [59] that for the maximal allowed spin the conformal block on the diagonal $x = \bar{x}$ can be expressed as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(\ell+\epsilon)_n(\ell+2\epsilon)_n}{n!(2\ell+2\epsilon)_n} x^{\Delta+n} = \\ &= x^{\Delta}{}_2F_1(\Delta-\epsilon, \Delta; 2\Delta-2\epsilon; x). \end{aligned} \tag{F.21}$$

We can then use the standard representation

$$B(b, c-d)_2F_1(a, b; c; x) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-xx)^{-a}dx \tag{F.22}$$

to compute the asymptotic expansion of the hypergeometric function by saddle-point method. This leads to

$$f(x) = \left(1 - \frac{\rho^2}{16}\right)^{-\epsilon-1/2} \rho^{\Delta}\left[1 + O\left(\frac{1}{\Delta}\right)\right], \tag{F.23}$$

valid at the unitarity bound $\ell = \Delta - 2\epsilon$.

We see that for most values of ℓ, the conformal block can be well approximated by (F.16). This approximation breaks down as we approach the unitarity bound $\ell = \Delta - 2\epsilon$ due to higher-order terms becoming large. However, at the exact unitarity bound the formula (F.23) is valid. The two formulas are compared in Fig. F.2.

The most important part of the conformal block for us is the factor ρ^{Δ}, which encapsulates the leading behavior in the limit of large Δ. We see that in all regimes this factor is present and is not modified.

F.2 Asymptotic form of conformal blocks on the diagonal $x = \bar{x}$ in the large-dimension limit

The previous derivation holds for fixed number of spacetime dimensions and large conformal dimensions of the intermediate state; it therefore captures the behavior

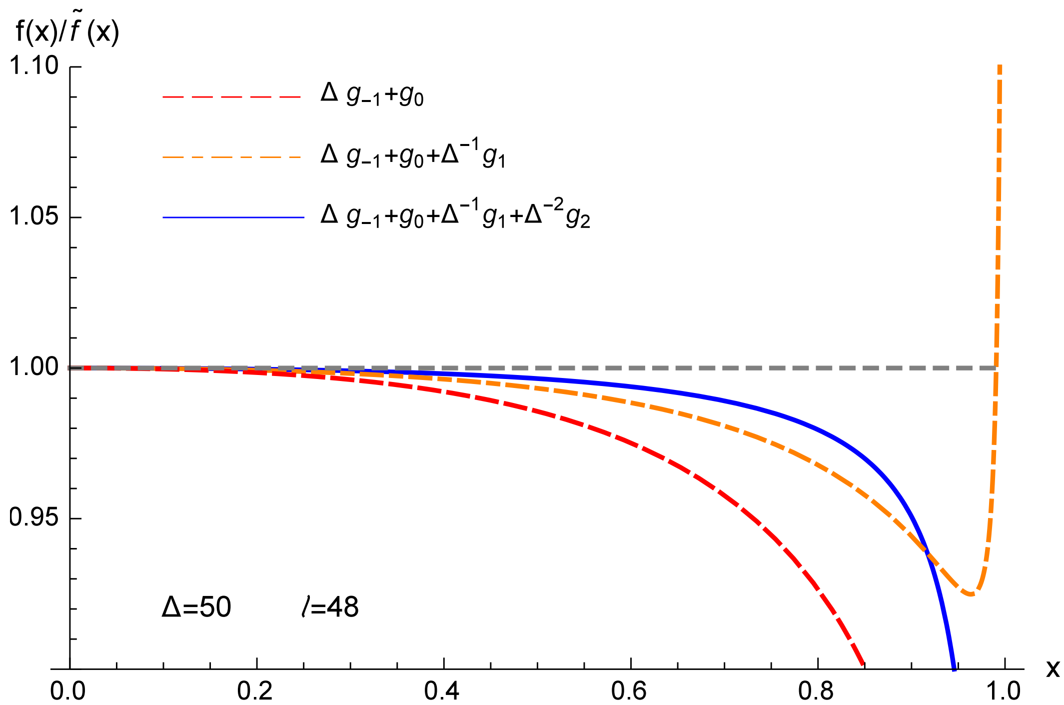


Figure F.2: Plot of $f(x)/\tilde{f}(x)$ for different orders of approximation and the value of angular momenta at the unitarity bound in four dimensions.

of states very far from the unitarity bound. If we additionally adjust the number of spacetime dimensions d, however, we can take analytic approximations that capture the behavior of states close to the unitarity bound. Such a limit was already described in [299], where the authors derive an expression for the conformal block in the scaling limit

$$d \to \infty, \qquad \Delta \to \infty, \tag{F.24}$$

$$\alpha = 2 - d/\Delta \ \text{ fixed.} \tag{F.25}$$

If one takes $d \to \infty$, then the unitarity bound means that Δ and ℓ must scale as well.

To express the conformal block in this limit, define

$$y_+ = \frac{x\bar{x}}{(1 + |1 - x|)^2} = \frac{x^2}{(2 - x)^2}, \tag{F.26}$$

$$y_- = \frac{x\bar{x}}{(1 - |1 - x|)^2} = 1, \tag{F.27}$$

where the second equality in each line holds on the real line $x = \bar{x}$. The conformal block then becomes, in normalization of [299]

$$F_{\Delta,\ell}(x) \approx \frac{2^{\Delta+\ell}}{\sqrt{y_- - y_+}} A_\Delta(y_+) A_{1-\ell}(y_-) = \tag{F.28}$$

$$= \mathcal{N}_\ell 2^\Delta \sqrt{\frac{y_-}{y_- - y_+}} A_\Delta(y_+) C_\ell^{\left(\frac{d-2}{2}\right)}\left(y_-^{-\frac{1}{2}}\right), \tag{F.29}$$

where

$$A_\beta(x) = x^{\beta/2} {}_2F_1\left(\frac{\beta-1}{2}, \frac{\beta}{2}, \beta - \frac{d-2}{2}; x\right), \tag{F.30}$$

$$\mathcal{N}_\ell = \frac{\Gamma(\ell+1)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\ell + \frac{d-2}{2}\right)}, \tag{F.31}$$

and $C_n^{(\lambda)}(x)$ are the Gegenbauer polynomials.

Notice that the spin dependence factorizes. In particular, when $y_- = 1$, spin-dependent factors carry no dependence on y_+. This immediately implies that in the normalization of this paper the block has no ℓ-dependence on real line $x = \bar{x}$. In fact, we have in our normalization

$$F_{\Delta,\ell}(x) \approx \frac{(4y_+)^{\Delta/2}}{\sqrt{1-y_+}} {}_2F_1\left(\frac{\Delta-1}{2}, \frac{\Delta}{2}, \Delta - \frac{d-2}{2}; y_+\right). \tag{F.32}$$

The saddle-point approximation for the hypergeometric function gives

$$\lim_{c\to\infty} \frac{\log {}_2F_1(c,c;\alpha c;y)}{c} = \log \frac{\alpha t_0}{1-yt_0}\left(\frac{\alpha(1-t_0)}{\alpha-1}\right)^{\alpha-1}, \tag{F.33}$$

where

$$t_0(\alpha, y) = \frac{\alpha - \sqrt{\alpha^2 + 4(1-\alpha)y}}{2(\alpha-1)y}. \tag{F.34}$$

Therefore, up to $O(1)$ factors we have

$$F_{\Delta,\ell}(x) \sim \left(\frac{4\alpha y_+ t_0}{1-y_+t_0}\left(\frac{\alpha(1-t_0)}{\alpha-1}\right)^{\alpha-1}\right)^{\Delta/2}. \tag{F.35}$$

A simple computation then gives

$$\frac{\partial \log F_{\Delta,\ell}}{\partial x} = \frac{d(2-x)}{4x(1-x)}\left(1 + \sqrt{1 + \frac{16(1-x)}{(2-x)^2}\delta(\delta-1)}\right), \tag{F.36}$$

where $\delta = \Delta/d$. This is obviously a non-decreasing function of Δ, which for $\delta \gg 1$ asymptotes to

$$d^{-1}\frac{\partial \log F_{\Delta,\ell}}{\partial x} \approx \frac{\delta - 1/2}{x\sqrt{1-x}} + \frac{2-x}{4x(1-x)}. \tag{F.37}$$

Note that for $\Delta_0 \gg d$ we expect $\Delta \gg d$ to be important and thus $\delta \gg 1$, so we regain from this expression the previously discussed case of large Δ_0.

Note that the image of $\beta \in [\frac{1}{2}, \infty)$ under $d^{-1}\partial \log F_{\Delta,\ell}/\partial x$ is

$$[\tfrac{1}{2}, \infty) \mapsto [(2x(1-x))^{-1}, \infty), \tag{F.38}$$

as used in the main text.

Appendix G

APPENDICES TO CHAPTER 8

G.1 Tensor structures

In this section we give the explicit expressions for the three-point tensor structures in the differential basis as required for the computation of conformal blocks in section 8.3.

G.1.1 Parity-even structures in differential basis

For a given spin ℓ, we define the basis of parity-even differential operators for $\langle TT\mathcal{O}_\ell\rangle$ as

$$\mathcal{D}_{n_{23},n_{13},n_{12}} = H_{12}^{n_{12}} D_{12}^{n_{13}} D_{21}^{n_{23}} D_{11}^{m_1} D_{22}^{m_2} \Sigma_1^{n_{12}+n_{23}+m_1} \Sigma_2^{n_{12}+n_{13}+m_2}, \tag{G.1}$$

where $m_1 = 2 - n_{12} - n_{13}$ and $m_2 = 2 - n_{12} - n_{23}$.

Structures for $\langle TT\mathcal{O}_0\rangle$ There exists a single parity-even tensor structure for $\langle TT\mathcal{O}_0\rangle$, given by the differential operator

$$\mathcal{D}_{\mathbf{0}^+}^{(1)} = -\mathcal{D}_{0,0,0} + (\Delta-5)(\Delta+2)\mathcal{D}_{0,0,1} - \frac{1}{8}(\Delta-5)(\Delta-3)\Delta(\Delta+2)\mathcal{D}_{0,0,2}. \tag{G.2}$$

Structures for $\langle TT\mathcal{O}_2\rangle$ There exists a single parity-even tensor structure for $\langle TT\mathcal{O}_2\rangle$, with $\Delta > 3$, given by the differential operator

$$\begin{aligned}\mathcal{D}_{\mathbf{2}^+}^{(1)} = &-8\left(7\Delta^2 - 13\Delta + 30\right)\mathcal{D}_{0,0,0} + 16(\Delta+2)(5\Delta-11)\mathcal{D}_{1,0,0} \\ &- 16(\Delta+2)(\Delta+4)\mathcal{D}_{2,0,0} + 16(\Delta+2)(5\Delta-11)\mathcal{D}_{0,1,0} \\ &- 32\Delta(2\Delta-5)\mathcal{D}_{1,1,0} - 16(\Delta+2)(\Delta+4)\mathcal{D}_{0,2,0} + 8\Delta\left(\Delta^2+29\Delta-78\right)\mathcal{D}_{0,0,1} \\ &- 8(\Delta-3)(\Delta+2)\left(\Delta^2-2\Delta-2\right)\mathcal{D}_{1,0,1} - 8(\Delta-2)(\Delta+2)\left(\Delta^2-3\Delta+8\right)\mathcal{D}_{0,1,1} \\ &+ 8(\Delta-2)^2(\Delta-1)\Delta\mathcal{D}_{1,1,1} + (\Delta-2)(\Delta-1)\Delta\left(\Delta^3-6\Delta^2-25\Delta+78\right)\mathcal{D}_{0,0,2}.\end{aligned} \tag{G.3}$$

$\langle TTT\rangle$ structures There exist two parity-even tensor structures for $\langle TTT\rangle$, one realized in the theory of a single free scalar field, and the other in the theory of single free Majorana fermion. They are given by the following differential operators

$$\begin{aligned}\mathcal{D}_T^{(B)} = &-\frac{9}{128\pi^3}\mathcal{D}_{0,0,0}+\frac{35}{256\pi^3}\mathcal{D}_{1,0,0}-\frac{245}{1024\pi^3}\mathcal{D}_{2,0,0}+\frac{35}{256\pi^3}\mathcal{D}_{0,1,0}-\frac{33}{512\pi^3}\mathcal{D}_{1,1,0}\\ &-\frac{245}{1024\pi^3}\mathcal{D}_{0,2,0}+\frac{153}{1024\pi^3}\mathcal{D}_{0,0,1}-\frac{35}{256\pi^3}\mathcal{D}_{1,0,1}-\frac{159}{1024\pi^3}\mathcal{D}_{0,1,1}-\frac{63}{1024\pi^3}\mathcal{D}_{1,1,1},\end{aligned} \tag{G.4}$$

$$\begin{aligned}\mathcal{D}_T^{(F)} = &-\frac{9}{64\pi^3}\mathcal{D}_{0,0,0}+\frac{5}{16\pi^3}\mathcal{D}_{1,0,0}-\frac{35}{64\pi^3}\mathcal{D}_{2,0,0}+\frac{5}{16\pi^3}\mathcal{D}_{0,1,0}-\frac{9}{64\pi^3}\mathcal{D}_{1,1,0}-\frac{35}{64\pi^3}\mathcal{D}_{0,2,0}\\ &+\frac{45}{128\pi^3}\mathcal{D}_{0,0,1}-\frac{5}{16\pi^3}\mathcal{D}_{1,0,1}-\frac{39}{128\pi^3}\mathcal{D}_{0,1,1}-\frac{9}{64\pi^3}\mathcal{D}_{1,1,1}.\end{aligned} \tag{G.5}$$

Structures for $\langle TT\mathcal{O}_\ell\rangle$ There exists two parity-even tensor structure for $\langle TT\mathcal{O}_\ell\rangle$ for even $\ell \geq 4$, given by the differential operators

$$\begin{aligned}
\mathcal{D}^{(1)}_{\ell^+} =&(\Delta^4-6\Delta^3+43\Delta^2-102\Delta+3\ell^4+6\ell^3-4\Delta^2\ell^2\\
&\quad+12\Delta\ell^2-35\ell^2-4\Delta^2\ell+12\Delta\ell-38\ell+184)\mathcal{D}_{0,0,0}\\
&-2(-\Delta+\ell+1)(\Delta+\ell)\left(-\Delta^2+3\Delta+\ell^2+\ell-14\right)\mathcal{D}_{1,0,0}\\
&+(-\Delta+\ell-1)(-\Delta+\ell+1)(\Delta+\ell)(\Delta+\ell+2)\mathcal{D}_{2,0,0}\\
&-2(-\Delta+\ell+1)(\Delta+\ell)\left(-\Delta^2+3\Delta+\ell^2+\ell-14\right)\mathcal{D}_{0,1,0}\\
&-4\left(-\Delta^4+6\Delta^3-13\Delta^2+12\Delta+\ell^4+2\ell^3-7\ell^2-8\ell+44\right)\mathcal{D}_{1,1,0}\\
&+2(-\Delta+\ell+1)(\Delta+\ell)\left(\Delta^2-3\Delta+\ell^2+\ell-10\right)\mathcal{D}_{2,1,0}\\
&+(-\Delta+\ell-1)(-\Delta+\ell+1)(\Delta+\ell)(\Delta+\ell+2)\mathcal{D}_{0,2,0}\\
&+2(-\Delta+\ell+1)(\Delta+\ell)\left(\Delta^2-3\Delta+\ell^2+\ell-10\right)\mathcal{D}_{1,2,0}\\
&+(\Delta^4-6\Delta^3-5\Delta^2+42\Delta+\ell^4+2\ell^3-\ell^2-2\ell+40)\mathcal{D}_{2,2,0}\\
&-2(\ell-1)(\ell+2)\left(12\Delta^2-36\Delta+\ell^4+2\ell^3-\Delta^2\ell^2+3\Delta\ell^2-13\ell^2\right.\\
&\quad\left.-\Delta^2\ell+3\Delta\ell-14\ell+72\right)\mathcal{D}_{0,0,1}\\
&-12\left(\ell^2+\ell-4\right)(-\Delta+\ell+1)(\Delta+\ell)\mathcal{D}_{1,0,1}\\
&-8\ell(\ell+1)(-\Delta+\ell+1)(\Delta+\ell)\mathcal{D}_{0,1,1}-8(\ell-1)\ell(\ell+1)(\ell+2)\mathcal{D}_{1,1,1}\\
&+\frac{1}{4}(\ell-1)\ell(\ell+1)(\ell+2)\left(-\Delta^4+6\Delta^3+5\Delta^2-42\Delta+\ell^4+2\ell^3-17\ell^2\right.\\
&\quad\left.-18\ell+104\right)\mathcal{D}_{0,0,2}, \qquad\qquad (\text{G.6})\\
\mathcal{D}^{(2)}_{\ell^+} =&(-\Delta^2+3\Delta-\ell^2-\ell+36)\mathcal{D}_{0,0,0}+2(-\Delta+\ell+1)(\Delta+\ell)\mathcal{D}_{1,0,0}\\
&+2(-\Delta+\ell+1)(\Delta+\ell)\mathcal{D}_{0,1,0}+4\left(\Delta^2-3\Delta+\ell^2+\ell-6\right)\mathcal{D}_{1,1,0}\\
&+(\Delta^4-6\Delta^3-5\Delta^2+42\Delta+\ell^4+2\ell^3-17\ell^2-18\ell+72)\mathcal{D}_{0,0,1}\\
&+2(-\Delta+\ell+1)(\Delta+\ell)\mathcal{D}_{1,0,1}\\
&+\frac{1}{8}\left(-\Delta^6+9\Delta^5-13\Delta^4-57\Delta^3+86\Delta^2+120\Delta-\ell^6-3\ell^5-\Delta^2\ell^4+3\Delta\ell^4\right.\\
&\quad+15\ell^4-2\Delta^2\ell^3+6\Delta\ell^3+35\ell^3-\Delta^4\ell^2+6\Delta^3\ell^2+6\Delta^2\ell^2-45\Delta\ell^2-54\ell^2\\
&\quad\left.-\Delta^4\ell+6\Delta^3\ell+7\Delta^2\ell-48\Delta\ell-72\ell\right)\mathcal{D}_{0,0,2}. \qquad\qquad (\text{G.7})
\end{aligned}$$

G.1.2 Parity-odd structures in differential basis

To construct the differential operators for parity-odd tensor structures, we use the differential operators derived in [61],

$$Q_1 = \epsilon\left(Z_1, Z_2, X_1, X_2, \frac{\partial}{\partial X_1}\right), \tag{G.8}$$

$$Q_2 = \epsilon\left(Z_1, Z_2, X_1, X_2, \frac{\partial}{\partial X_2}\right), \tag{G.9}$$

$$\widetilde{D}_1 = \epsilon\left(Z_1, X_1, \frac{\partial}{\partial X_1}, X_2, \frac{\partial}{\partial X_2}\right), \tag{G.10}$$

$$\widetilde{D}_2 = \epsilon\left(Z_2, X_2, \frac{\partial}{\partial X_2}, X_1, \frac{\partial}{\partial X_1}\right). \tag{G.11}$$

Note that the operators $\widetilde{D}_i$ satisfy all consistency conditions of [61] only when operators 1 and 2 have spin 0.[1]

Using these, we can define the operators

$$E_{13} = \widetilde{D}_1, \tag{G.12}$$

$$E_{23} = \widetilde{D}_2, \tag{G.13}$$

$$E_{12} = \frac{1}{2}\left(Q_1\Sigma_1^1 + Q_2\Sigma_2^1\right). \tag{G.14}$$

We define the basis of parity-odd differential operators for $\langle TT\mathcal{O}_\ell\rangle$ as

$$\mathcal{D}^-_{n_{23},n_{13},n_{12},1} = \mathcal{D}_{n_{23},n_{13},n_{12}}E_{23}, \tag{G.15}$$

$$\mathcal{D}^-_{n_{23},n_{13},n_{12},2} = \mathcal{D}_{n_{23},n_{13},n_{12}}E_{13}, \tag{G.16}$$

$$\mathcal{D}^-_{n_{23},n_{13},n_{12},3} = \mathcal{D}_{n_{23},n_{13},n_{12}}E_{12}. \tag{G.17}$$

Here $\mathcal{D}_{n_{23},n_{13},n_{12}}$ are the parity-even differential operators with m_1, m_2 defined depending on which E_{ij} it multiplies so that the total spins at points 1 and 2 agree.

Structures for $\langle TT\mathcal{O}_0\rangle$ There exists a unique parity-odd tensor structure for $\langle TT\mathcal{O}_0\rangle$, given by the differential operator

$$\tilde{\mathcal{D}}^{(1)}_{\mathbf{0}^-} = -4\mathcal{D}^-_{0,0,0,3} + (\Delta-4)(\Delta+1)\mathcal{D}^-_{0,0,1,3}. \tag{G.18}$$

There is a slight complication in this case, since the transition matrix between the differential and algebraic bases vanishes at $\Delta = 1$. Thus any differential basis

[1] In [61] these operators are defined with extra terms containing derivatives in polarizations. However, even with that definition $\widetilde{D}_1$ does not commute with $X_1 \cdot \frac{\partial}{\partial Z_1}$ and one needs to add extra terms to ensure full consistency for action on generic operators.

structure with polynomial coefficients vanishes for $\Delta = 1$, which is undesirable since we would like to have a non-zero conformal block for every $\Delta \geq 1/2$. We therefore in this case consider the non-polynomial solution given by

$$\mathcal{D}^{(1)}_{\mathbf{0}^-} = \frac{1}{\Delta - 1}\tilde{\mathcal{D}}^{(1)}_{\mathbf{0}^-}. \tag{G.19}$$

In practice, we work with $\tilde{\mathcal{D}}^{(1)}_{\mathbf{0}^-}$ and only in the end divide the numerator of the resulting rational approximation to the parity-odd scalar block by $(\Delta-1)^2$.[2] The construction guarantees that this division is possible.

Structures for $\langle TT\mathcal{O}_2\rangle$ There exists a unique parity-odd tensor structure for $\langle TT\mathcal{O}_2\rangle$, given by the differential operator

$$\begin{aligned}\mathcal{D}^{(1)}_{\mathbf{2}^-} = &-4\mathcal{D}^-_{0,1,0,1} - 2(\Delta-2)(\Delta+3)\mathcal{D}^-_{0,1,0,3} + (\Delta^4 - 6\Delta^3 - 13\Delta^2 + 66\Delta + 144)\mathcal{D}^-_{0,0,1,3} \\ &+ 2(\Delta-6)(\Delta+2)\mathcal{D}^-_{0,1,1,1} - (4)\mathcal{D}^-_{1,0,0,2} - 2(\Delta-2)(\Delta+3)\mathcal{D}^-_{1,0,0,3} + 8(\Delta+6)\mathcal{D}^-_{1,1,0,3} \\ &+ 2(\Delta-6)(\Delta+2)\mathcal{D}^-_{1,0,1,2}.\end{aligned} \tag{G.20}$$

Structures for $\langle TT\mathcal{O}_\ell\rangle$ for even ℓ There exists a unique parity-odd tensor structure for $\langle TT\mathcal{O}_\ell\rangle$ for even $\ell \geq 4$, given by the differential operator

$$\begin{aligned}\mathcal{D}^{(1)}_{\ell^-} =& 8\left(-3\Delta^2 + 9\Delta + \ell^2 + \ell + 24\right)\mathcal{D}^-_{0,0,0,1} - 16(\Delta-4)(\Delta+1)\mathcal{D}^-_{0,0,0,2} \\ &- 16\left(\Delta^4 - 6\Delta^3 - \Delta^2 + 30\Delta + \Delta^2\ell^2 - 3\Delta\ell^2 - 4\ell^2 + \Delta^2\ell - 3\Delta\ell - 4\ell\right)\mathcal{D}^-_{0,0,0,3} \\ &+ 16\left(\ell^2 + \ell + 6\right)\mathcal{D}^-_{0,1,0,1} + 8(\ell-\Delta)(\Delta+\ell+1)\mathcal{D}^-_{0,1,0,2} \\ &+ 8\left(\Delta^4 - 6\Delta^3 - 9\Delta^2 + 54\Delta + 44\right)\mathcal{D}^-_{0,1,0,3} \\ &+ 4\left(\Delta^4 - 6\Delta^3 - 7\Delta^2 + 48\Delta + \Delta^2\ell^2 - 3\Delta\ell^2 + \Delta^2\ell - 3\Delta\ell + 72\right)\mathcal{D}^-_{0,2,0,1} \\ &+ 4\left(\Delta^6 - 9\Delta^5 + 13\Delta^4 + 57\Delta^3 - 86\Delta^2 - 120\Delta + \Delta^2\ell^4\right. \\ &\qquad -3\Delta\ell^4 + 2\Delta^2\ell^3 - 6\Delta\ell^3 + 2\Delta^4\ell^2 - 12\Delta^3\ell^2 - 11\Delta^2\ell^2 \\ &\qquad \left.+87\Delta\ell^2 + 40\ell^2 + 2\Delta^4\ell - 12\Delta^3\ell - 12\Delta^2\ell + 90\Delta\ell + 40\ell\right)\mathcal{D}^-_{0,0,1,1} \\ &- 4(\Delta-3)\Delta\left(\Delta^2 - 3\Delta + \ell^2 + \ell - 16\right)\mathcal{D}^-_{0,0,1,2} + 8(\ell-\Delta)(\Delta+\ell+1)\mathcal{D}^-_{0,0,1,3} \\ &+ 16\left(\Delta^2 - 3\Delta + \ell^2 + \ell - 10\right)\mathcal{D}^-_{0,1,1,1} + 8\left(\Delta^2 - 3\Delta + \ell^2 + \ell - 16\right)\mathcal{D}^-_{1,0,0,1}.\end{aligned} \tag{G.21}$$

[2] We need the square since there are left and right three-point structures.

Structures for $\langle TTO_\ell \rangle$ for odd ℓ There exists a unique parity-odd tensor structure for $\langle TTO_\ell \rangle$ for odd $\ell \geq 5$, given by the differential operator

$$\begin{aligned}
\mathcal{D}^{(1)}_{\ell^-} = & -4(\Delta-2)(\Delta-1)\left(\Delta^2-3\Delta-3\ell^2-3\ell+32\right)\mathcal{D}^-_{0,0,0,1} \\
& +8(\ell-3)(\ell-1)(\ell+2)(\ell+4)\mathcal{D}^-_{0,0,0,2} \\
& +8\ell(\ell+1)\left(-6\Delta^2+18\Delta+\ell^4+2\ell^3+\Delta^2\ell^2-3\Delta\ell^2-11\ell^2\right. \\
& \qquad \left.+\Delta^2\ell-3\Delta\ell-12\ell+12\right)\mathcal{D}^-_{0,0,0,3} \\
& -8\left(-\Delta^4+6\Delta^3-25\Delta^2+48\Delta+\ell^4+2\ell^3+\Delta^2\ell^2-3\Delta\ell^2\right. \\
& \qquad \left.-11\ell^2+\Delta^2\ell-3\Delta\ell-12\ell-4\right)\mathcal{D}^-_{0,1,0,1} \\
& +4(\Delta-2)(\Delta-1)(\ell-\Delta)(\Delta+\ell+1)\mathcal{D}^-_{0,1,0,2} \\
& -4(\Delta-2)(\Delta-1)\left(\ell^4+2\ell^3-21\ell^2-22\ell+84\right)\mathcal{D}^-_{0,1,0,3} \\
& -2\left(\ell^6+3\ell^5+\Delta^2\ell^4-3\Delta\ell^4-15\ell^4+2\Delta^2\ell^3-6\Delta\ell^3-35\ell^3-17\Delta^2\ell^2\right. \\
& \qquad \left.+51\Delta\ell^2+54\ell^2-18\Delta^2\ell+54\Delta\ell+72\ell-144\right)\mathcal{D}^-_{0,2,0,1} \\
& -2\ell(\ell+1)\left(-2\Delta^4+12\Delta^3+82\Delta^2-300\Delta+\ell^6+3\ell^5+2\Delta^2\ell^4-6\Delta\ell^4\right. \\
& \qquad -13\ell^4+4\Delta^2\ell^3-12\Delta\ell^3-31\ell^3+\Delta^4\ell^2-6\Delta^3\ell^2-23\Delta^2\ell^2+96\Delta\ell^2 \\
& \qquad \left.+20\ell^2+\Delta^4\ell-6\Delta^3\ell-25\Delta^2\ell+102\Delta\ell+36\ell+64\right)\mathcal{D}^-_{0,0,1,1} \\
& +2(\ell-3)(\ell-2)(\ell+3)(\ell+4)\left(\Delta^2-3\Delta+\ell^2+\ell\right)\mathcal{D}^-_{0,0,1,2} \\
& -4(\Delta-2)(\Delta-1)(\ell-\Delta)(\Delta+\ell+1)\mathcal{D}^-_{0,0,1,3} \\
& -8(\Delta-2)(\Delta-1)\left(\Delta^2-3\Delta+\ell^2+\ell-10\right)\mathcal{D}^-_{0,1,1,1} \\
& +4(\Delta-2)(\Delta-1)\left(\Delta^2-3\Delta+\ell^2+\ell-16\right)\mathcal{D}^-_{1,0,0,1}.
\end{aligned} \tag{G.22}$$

G.2 Conformal generators

The conformal generators act on a local operator $\mathcal{O}(w,z)$ (with spin degrees of freedom encoded by the polarization vector w) of scaling dimension Δ as

$$D\cdot\mathcal{O}(w,x) = (x\cdot\partial+\Delta)\mathcal{O}(w,x), \tag{G.23}$$

$$P_\mu\cdot\mathcal{O}(w,x) = \partial_\mu\mathcal{O}(w,x), \tag{G.24}$$

$$\begin{aligned}
K_\mu\cdot\mathcal{O}(w,x) = & (2x_\mu x^\sigma - x^2\delta^\sigma_\mu)\partial_\sigma\mathcal{O}(w,x)+2\Delta x_\mu\mathcal{O}(w,x) \\
& -2x^\sigma(w_\sigma\frac{\partial}{\partial w^\mu}-w_\mu\frac{\partial}{\partial w^\sigma})\mathcal{O}(w,x)
\end{aligned} \tag{G.25}$$

$$M_{\mu\nu}\cdot\mathcal{O}(w,x) = \left(x_\nu\partial_\mu - x_\mu\partial_\nu + w_\nu\frac{\partial}{\partial w^\mu} - w_\mu\frac{\partial}{\partial w^\nu}\right)\mathcal{O}(w,x). \tag{G.26}$$

Here D, P, K, and M are the dilatation, translation, special conformal, and rotation generators respectively.

G.3 Details on the numerics

In this appendix we give specific details on how the bounds in this paper are obtained from the crossing equations (8.47)-(8.49) and the conformal block decomposition (8.82).

First, we organize the crossing equations (8.47)-(8.49) in a single vector equation

$$\vec{F}_{TTTT} = 0. \tag{G.27}$$

The conformal block decomposition (8.82) then induces a decomposition of the vector $\vec{F}_{TTTT}$,

$$\vec{F}_{TTTT} = \vec{F}_{\mathbf{1}} + \frac{1}{C_T}\Theta^{ab}\vec{F}_{T,ab} + \sum_{(\Delta,\rho)\in S} M^{ab}_{\Delta,\rho}\vec{F}_{\Delta,\rho,ab} = 0. \tag{G.28}$$

Here we have explicitly specified that the summation is over some assumed set of dimensions and spins S. This equation has to be satisfied in any theory whose spectrum of operators is a subset of S. For example, when we say that we impose a gap $\Delta^{\min}_{\text{even}}$ in the parity-even scalar sector, we choose

$$\begin{aligned} S =& \{(\Delta, \boldsymbol{\ell}^+)|\Delta \geq \ell+1, \ell = 2k \geq 2\} \cup \\ & \{(\Delta, \boldsymbol{\ell}^-)|\Delta \geq \ell+1, \ell \geq 4\} \cup \\ & \{(\Delta, \mathbf{2}^-)|\Delta \geq 3\} \cup \\ & \{(\Delta, \mathbf{0}^+)|\Delta \geq \Delta^{\min}_{\text{even}}\} \cup \\ & \{(\Delta, \mathbf{0}^-)|\Delta \geq \tfrac{1}{2}\}. \end{aligned} \tag{G.29}$$

Given a choice of S, we then study two questions:

1. **Feasibility:** Does the system (G.28) have a solution for some θ?
2. **Optimization:** What is the minimal (maximal) value of C_T for a given θ?

Feasibility: To answer the feasibility question, we look for a vector $\vec{\alpha}$ such that

$$\vec{\alpha}\cdot\vec{F}_{\mathbf{1}} = 1, \tag{G.30}$$

$$\vec{\alpha}\cdot\vec{F}_{T} \geq 0, \tag{G.31}$$

$$\vec{\alpha}\cdot\vec{F}_{\Delta,\rho} \geq 0, \quad \forall(\Delta,\rho)\in S. \tag{G.32}$$

Clearly, if such $\vec{\alpha}$ is found, then there cannot be a solution to (G.28), since positive-semidefiniteness of $M_{\Delta,\rho}$, Θ and $C_T > 0$ imply

$$\vec{\alpha} \cdot \vec{F}_{TTTT} \geq 1. \tag{G.33}$$

We then conclude that CFTs with the spectral assumption S do not exist. As usual, this conclusion is rigorous for any Λ, given that the equations (G.30)-(G.32) are satisfied to a sufficient precision. If such an $\vec{\alpha}$ cannot be found, we cannot conclude anything and the spectral assumption S is formally "allowed" by our bounds.

Optimization: Let us start with the case that we want to find a lower bound on C_T for a given θ. Suppose that we have found a vector $\vec{\alpha}$ such that

$$\vec{\alpha} \cdot \vec{F}_{\mathbf{1}} = -1, \tag{G.34}$$

$$\vec{\alpha} \cdot \vec{F}_{\Delta,\rho} \geq 0, \quad \forall (\Delta, \rho) \in S. \tag{G.35}$$

It then follows from $\vec{F}_{TTTT} = 0$ that

$$-1 + \frac{1}{C_T} \vec{\alpha} \cdot (\Theta^{ab} \vec{F}_{T,ab}) \leq 0, \tag{G.36}$$

and thus

$$C_T \geq \vec{\alpha} \cdot (\Theta^{ab} \vec{F}_{T,ab}). \tag{G.37}$$

We then search for an $\vec{\alpha}$ which maximizes

$$\vec{\alpha} \cdot (\Theta^{ab} \vec{F}_{T,ab}) \tag{G.38}$$

subject to (G.34) and (G.35) in order to find the optimal bound. Again, the bounds are rigorous for every Λ.

If our goal is to find an upper bound on C_T, we replace (G.34) with

$$\vec{\alpha} \cdot \vec{F}_{\mathbf{1}} = +1, \tag{G.39}$$

which then analogously implies

$$C_T \leq -\vec{\alpha} \cdot (\Theta^{ab} \vec{F}_{T,ab}). \tag{G.40}$$

We again look for such $\vec{\alpha}$ which maximizes (G.38) in order to find the optimal bound.

Numerical implementation: To search for the vectors α we use the semidefinite solver `SDPB` [35]. In section 8.3 we explained how to obtain rational approximations of the $\langle TTTT\rangle$ conformal blocks required by `SDPB` starting from rational approximations of scalar conformal blocks arising from their pole expansions [36, 49].

These approximations are controlled by the integral parameter κ defined in [35]. The blocks become exact in the limit $\kappa \to \infty$; the convergence is exponential. In practice we use a finite value of κ and check that our results don't change if κ is increased. Another approximation that we have to make is the truncation to a finite range of spins in constraints (G.32) and (G.35). Again, we choose a sufficiently large cutoff and check that the results are independent of it.

Below we list κ, the spin cutoff, and the relevant `SDPB` parameters that we used in calculations for various values of Λ (all figures except figure 8.2 correspond to $\Lambda = 19$):

Λ	≤ 11	13	15	17	19
κ	20	24	24	24	24
spins	≤ 25	≤ 30	≤ 36	≤ 42	≤ 42
`precision`	832	832	832	832	1024
`findPrimalFeasible`	False	False	False	False	False
`findDualFeasible`	False	False	False	False	False
`detectPrimalFeasibleJump`	False	False	False	False	False
`detectDualFeasibleJump`	False	False	False	False	False
`dualityGapThreshold`	10^{-10}	10^{-10}	10^{-10}	10^{-10}	10^{-10}
`primalErrorThreshold`	10^{-30}	10^{-30}	10^{-30}	10^{-30}	10^{-30}
`dualErrorThreshold`	10^{-30}	10^{-30}	10^{-30}	10^{-30}	10^{-30}
`initialMatrixScalePrimal`	10^{20}	10^{20}	10^{20}	10^{20}	10^{20}
`initialMatrixScaleDual`	10^{20}	10^{20}	10^{20}	10^{20}	10^{20}
`feasibleCenteringParameter`	0.1	0.1	0.1	0.1	0.1
`infeasibleCenteringParameter`	0.3	0.3	0.3	0.3	0.3
`stepLengthReduction`	0.7	0.7	0.7	0.7	0.7
`choleskyStabilizeThreshold`	10^{-120}	10^{-120}	10^{-120}	10^{-120}	10^{-180}
`maxComplementarity`	10^{100}	10^{100}	10^{100}	10^{100}	10^{100}

The exclusion plot in figure 8.8 requires testing only feasibility so we set `findPrimalFeasible` and `findDualFeasible` to True. For the scalar bound in figure 8.8 we used the parameters of [35] with $\Lambda = 35$. The stress-tensor conformal blocks as well as the code used for their generation and setting up SDPB are available upon request.