Essays on Investor Beliefs and Asset Pricing

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I dedicate this dissertation to my family.

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ABSTRACT

This dissertation is composed of three chapters addressing the connections between investor beliefs and asset pricing. Specifically, I focus on one prevailing pattern of investor beliefs in the finance literature, return extrapolation. The idea is that investor expectations about future market returns are a positive function of the recent past returns. In this dissertation, I use this concept to understand a number of facts in the asset pricing literature.

Return extrapolation attracts growing attention in the literature, not only because it well explains real-world investors' expectations in the survey, but also because it significantly drives investor demand towards stocks. Therefore, we should anticipate a connection between return extrapolation measurement and the stock market dynamics. However, contrary to the intuition, previous empirical studies fail to document a significant connection. In Chapter 1, "Time-varying Impact of Investor Sentiment", I recover this connection. Specifically, I formally define investors who extrapolate past returns as extrapolators and incorporate their wealth level into analysis. My main finding is that return extrapolation interacts strongly with extrapolators' wealth level in predicting future market returns. Therefore, conditional on extrapolators' wealth level, return extrapolation significantly explains stock market returns.

The return extrapolation concept also raises challenges to the asset pricing models under the rational expectation frameworks. Specifically, rational expectation theories lead to a positive correlation between expectations and future realized returns, whereas return extrapolation indicates a negative correlation. Given this discrepancy, there is a clear demand for a behavioral asset pricing model that can simultaneously explain survey evidence on investor expectations and the classical asset pricing puzzles. In Chapter 2, "Asset Pricing with Return Extrapolation", coauthored with Lawrence Jin, we present a new model of asset prices based on return extrapolation. The model is a Lucas-type general equilibrium framework, in which the agent has Epstein-Zin preferences and extrapolative beliefs. Unlike earlier return extrapolation models, our model allows for a quantitative comparison with the data on asset prices. When the agent's beliefs are calibrated to match survey expectations of investors, the model generates excess volatility and predictability of stock returns, a high equity premium, a low and stable risk-free rate, and a low correlation between stock returns and consumption growth.

In Chapter 3, " 'Dark Matter' of Finance in the Survey ", I investigate another attribute of investor beliefs—tail risk perceptions. Although tail risks play significant roles in explaining asset pricing puzzles, researchers have very limited knowledge about them because tail events are difficult to observe. I use Shiller tail risk survey to empirically investigate tail risk perceptions. In this survey, investors are asked to report their estimated probability for a crash event in the U.S. stock market. However, when using survey data to understand investors' perception of tail risks, there are two fundamental challenges. First, is tail risks survey reliable? Second, to avoid cherry-picking, is there a unified framework to explain different attributes of investor beliefs? My analysis provides positive answers to both questions. First, I show that Shiller tail risk survey is reliable. More importantly, I show that return extrapolation can serve as a unified belief formation framework to explain not only variations in investor expectations but also in tail risk perceptions.

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Chapter 1

TIME-VARYING IMPACT OF INVESTOR SENTIMENT

1.1 Introduction

Whether investor sentiment influences the market has been a long-standing question in the finance literature. From the Great Crash in 1929 to the Internet bubble, from the Nifty Fifty bubble to the 2008 financial crisis, each of these episodes is associated with dramatic changes in asset prices. Traditional finance theories—models in which investors have fully correct beliefs about the asset dynamics and therefore always force the asset prices to the rational present value of expected future cash flows—leave no room for investor sentiment and have considerable difficulty fitting these patterns. However, investor sentiment, which reflects excessive optimism and pessimism in investor beliefs, seems to play a central role in these phenomena. The large gap between the traditional models and the salient market episodes with dramatic asset price movements has made researchers realize that *belief-based* investor sentiment plays an important role in asset pricing dynamics.

Relying on survey evidence, recent studies have highlighted the concept of extrapolation—making forecasts about future returns based on past realized returns—in understanding the dynamics of investor beliefs. Extrapolation implies that investors tend to believe that asset prices continue to increase after a sequence of high returns and fall after a sequence of low returns (Greenwood and Shleifer (2014)), and has been used to account for the excessive optimism and pessimism in the market (**?** and Barberis).[1] As a result, in this paper, I use extrapolation to characterize belief-based investor sentiment. But despite its prevalence in surveys and its importance in investors' portfolio choice, there is no strong empirical evidence that belief-based investor sentiment influences the aggregate stock market. In other words, extrapolation alone leaves the impact of investor sentiment unsolved: for instance, in Greenwood and Shleifer (2014), investor beliefs reported in the survey have only insignificant predictive power over future aggregate stock market returns.

In this paper, I provide a parsimonious approach to reveal the impact of belief-

[1] There might be other forms of extrapolation. I use extrapolation to refer to return extrapolation because according to previous studies, because it better addresses the survey expectation series. There are other studies on extrapolation behavior of investors, for example, Vissing-Jorgensen (2004), Amromin and Sharpe (2013), Koijen et al. (2015), and Kuchler and Zafar (2016).

based investor sentiment. By incorporating the time-varying market impact of extrapolation, which is a notion largely overlooked by the literature, this paper documents and highlights the impact of belief-based investor sentiment on the aggregate market. Specifically, in addition to the basic extrapolation framework, I incorporate the wealth dynamics of investors who extrapolate to proxy for the market impact of extrapolation. In this setting, extrapolation and its market impact together drive the asset price dynamics: when the market impact of extrapolation is high, extrapolation induces irrational demand for the risky asset and, therefore, it leads to mispricing; when the market impact is low, extrapolation simply reflects the recent market dynamics. The interaction between extrapolation and its market impact sheds new light on asset mispricings. More importantly, although investor beliefs alone do not significantly predict future aggregate market returns, they have salient predictive power after conditioning on their market impact—a result that supports both my model implications and my empirical exercise. This conditional predictive power of investor sentiment not only provides direct evidence of the impact of investor beliefs on the aggregate stock market, it also helps to reveal the underlying mechanism of the predictability of returns.

To formalize these arguments, I first develop a continuous-time dynamic equilibrium model that features two types of investors: extrapolators and fundamental investors. On the basis of past price changes, extrapolators form *investor sentiment*, or, equivalently, their perceived expectation about future risky asset returns, and they make an investment choice between a risky and risk-free asset. After consecutive positive price changes in the past, extrapolators become optimistic about future returns, and after a streak of negative price changes, they become pessimistic about future returns. However, the perceived expectations are different from the true ones and, therefore, high or low investor sentiment, in general, cannot continue for long since extrapolators will easily become disappointed by shocks in the future. Consequently, investor sentiment reverts to its mean. Moreover, the major departure from previous extrapolation models is to incorporate the wealth level of extrapolators: all other things being equal, extrapolators have a larger impact on the equilibrium asset prices when their wealth level is higher.

As in earlier models, extrapolators are met in the market by fundamental investors who serve as the counteracting forces and arbitrage against mispricing. When the risky asset is overvalued, fundamental investors short the risky asset and, therefore, correct prices downwards. If the risky asset is undervalued, fundamental investors lean against the wind and push prices upwards. However, their ability to correct mispricing depends on the wealth level of extrapolators: fundamental investors can easily correct mispricing when sentiment-driven wealth is low, while the correction takes longer when the wealth level of extrapolators is high.

This model setting generates the key result: when their wealth level is high, extrapolators drive the asset prices. In this case, high investor sentiment makes the current asset price overvalued, and the future asset price will decline because high investor sentiment will cool down over time. Therefore, investor sentiment negatively predicts future market returns. Conversely, when the wealth level is low, high investor sentiment predicts high future returns because the market is under a price correction. This predictive power of investor sentiment provides direct theoretical support for my conclusion that investor beliefs impact the aggregate stock market.

Moreover, this predictive power supports belief-based explanations of the predictability of returns—that prices temporarily deviate from the level warranted by fundamentals because of the existence of extrapolators, but they revert back as a mispricing correction gradually takes place in the future. Cassella and Gulen (2015) provide empirical support for this explanation. They define the degree of extrapolation (DOX) as the relative weight extrapolators place on recent-versus-distant past returns when they form subjective expectations for future asset returns. When DOX is high and, therefore, when investor beliefs are transitory, mispricings are corrected more quickly and price-scaled variables (such as price-to-dividend ratio) have stronger predictive power. One possible determinant of the variations in DOX is the time-varying consensus level of extrapolation among market participants. In my model, the time-varying wealth level of extrapolators effectively drives the consensus level of extrapolation in the market.

My model matches other salient patterns in the asset pricing literature. For instance, the mean-reversion of investor sentiment naturally generates a negative equity premium when investor sentiment is high, a pattern that is consistent with findings documented by Baron and Xiong (2017). The fact that my model generates a negative correlation between investors' expectations and the subsequent realized returns is consistent with the findings of Greenwood and Shleifer (2014). Moreover, the resulting countercyclical Sharpe ratio is consistent with empirical evidence documented by Lettau and Ludvigson (2010). All these asset pricing patterns seem puzzling under a rational expectations framework.

To empirically test the impact of investor sentiment on the market, I use both the CRSP value-weighted index and Gallup survey data. During the period of December 1996 to September 2011, Gallup asked individual investors to report their expectations of aggregate stock market returns in the next twelve months. Using these responses I measure investors' perceived expectations directly. As a robustness check, I also follow Barberis et al. (2015) and construct an investor sentiment index, Psentiment, that is purely based on extrapolation.[2] Moreover, Gallup survey evidence helps to identify extrapolator groups, which allows me to select a reasonable proxy for the wealth level of extrapolators.

To find reasonable measurements for extrapolators, I focus on the "Households and Non-Profitable Organizations" (HNPO hereafter) sector reported in the "Financial Accounts of the United States". Investors in this sector generally are individual investors who are less sophisticated than institutional investors. Moreover, Yang and Zhang (2017) document, first, that investor sentiment in the survey effectively drives the portfolio position of investors in stocks in the HNPO sector and, second, that such sentiment-driven investment negatively predicts returns in the following quarters.[3] Their analysis effectively indicates that investor beliefs in the HNPO sector are associated with extrapolation. My model therefore uses the total financial assets of the HNPO sector as the proxy for the wealth level of extrapolators.

My empirical tests support the main predictions of my model. Specifically, I construct an interaction term between the Gallup survey expectations and wealth dividend ratios in the HNPO sector, and I use it to predict future market returns. I find a statistically significant and negative coefficient for the interaction term, which indicates that investor sentiment connects closely to market mispricing when the wealth level of extrapolators is high. This result holds over different predictive horizons, ranging from one to six quarters.

Furthermore, following Aiken et al. (1991), I empirically present the predictive pattern of investor sentiment conditional on different wealth levels of extrapolators.When the wealth level of extrapolator is two standard deviation above its mean, one standard deviation increase in investor sentiment measured by the Gallup survey is followed by a significant decrease of 16.2% in future twelve-months returns. This is consistent with my model implication: when extrapolators drive the market,

[2] In my empirical analysis, I use both survey evidence and Psentiment to test my model predictions. If investor beliefs in the survey are mainly driven by extrapolation, then two pieces of parallel evidence should yield similar results for most of my tests. This point is strongly confirmed in my exercise.

[3] Yang and Zhang (2017) use surveys by Gallup, Inc. and the American Association of Individual Investors.

investor sentiment reverts to its mean and leads to a negative predictive sign. This pattern remains valid when Psentiment replaces Gallup. Conversely, when extrapolators' wealth level is two standard deviation below its mean, one standard deviation increase in investor sentiment predicts a striking future market return of 35.9%: as my model implies, when sentiment-driven wealth is low, investor sentiment reflects the market valuation correction and, therefore, it positively predicts future market returns.

Implications for the Literature. This paper belongs to the burgeoning return extrapolation literature. Many works in this field try to understand the role that return extrapolation plays in the aggregate stock market. (Cutler et al. (1990b), De Long et al. (1990), Barberis et al. (2015) and Jin and Sui (2017)). Barberis et al. (2015) use return extrapolation to construct an asset pricing model that can explain central asset pricing facts such as the excess volatility puzzle, the predictability of returns, and the investor belief survey evidence in the data. Jin and Sui (2017) construct a quantitative benchmark of belief-based asset pricing models that can simultaneously explain the equity premium puzzle, the excess volatility puzzle, the predictability of returns, the low correlations between consumption and returns, and investor belief evidence in the surveys. However, all existing studies of return extrapolation ignore the role that the wealth level of extrapolators plays in asset dynamics. By incorporating the wealth dynamics of extrapolators and the counteracting forces from fundamental investors, I document a novel predictive pattern for future stock market returns.

This paper also relates to the idea of limits to arbitrage, which is one of the foundations for the behavioral finance literature. In its pioneer work, Shleifer and Vishny (1997) argue that asset mispricing may exist for a long time because arbitrage activities are limited. This is especially true when investment is delegated to portfolio managers with short investment horizons and when the arbitrage activities face noise trader risk and other risks. Abreu and Brunnermeier (2002) provide an additional argument for limited arbitrage.[4] My paper utilizes the concept of limits to arbitrage in a less direct way. Instead of focusing on the agency issue and risks for the arbitragers, I mainly focus on the belief dynamics and the relative market impact of noise traders. When their wealth level is high, arbitragers effectively face more difficulties correcting the mispricing. In addition, by focusing on extrapolators, I can not only provide a specific belief pattern for noise traders but also document a

[4] For a discussion on limits to arbitrage, see Barberis and Thaler (2003).

salient impact of investor beliefs on the aggregate stock market.

This paper is also relevant to the return predictability literature. Previous studies have suggested that aggregate stock returns are predictable using price-scaled variables, such as the dividend-to-price ratio and the earning-to-price ratio (Fama and French (1988), Campbell and Shiller (1988), and Cochrane (2011)). Some researchers attribute the predictability of stock market returns to variations in investors' required returns. However, behavioral theoretical models attribute the predictability to mispricings induced by investors' biased beliefs (Barberis et al. (2015), Hirshleifer et al. (2015), and Jin and Sui (2017)). Bacchetta et al. (2009) document that the predictability of excess returns is often associated with the predictability of expectational errors, and this conclusion holds true for a broad set of asset classes, including the stock, foreign exchange and bond assets. Moreover, Cassella and Gulen (2015) empirically investigate the extent to which biased beliefs can help explain the observed predictability in the data. In this paper, I connect investor sentiment with return predictability through the time-varying impact of extrapolators and find empirical support for this result.

This paper also contributes to the literature that examines the impact of investor sentiment. Baker and Wurgler (2006) construct the investor sentiment index by directly using the first principal component of important market indicators, such as volume and equity share issuance, and demonstrate that investor sentiment has a large impact on the cross-section of stock returns. Baker and Stein (2004) propose an investor sentiment index based on market liquidity, and they show that it has predictive power for future market returns. Stambaugh et al. (2012), who also use the sentiment index provided by Baker and Wurgler (2006), find that overpricing is more prevalent than underpricing when market-wide sentiment is high. However, most of these studies construct investor sentiment using a "top down" approach, which employs reduced-form variations in investor sentiment over time. By contrast, this paper uses a "bottom up" approach, and it focuses on the belief formation of investors. My focus on the microfoundations of the variation in investor sentiment allows me to shed new light on the dynamic patterns of the investor sentiment index. Like the "top down" literature, my analysis shows that investor sentiment has a large impact on the aggregate stock market.

Finally, this paper sheds light on important patterns in economic activities. First, many studies document a strong cyclicality of debt accumulation. Reinhart and Rogoff (2009) demonstrate that household debt accumulation speeds up during market booms and often leads to severe financial crashes. He and Krishnamurthy (2008) also propose a model that generates a procyclical leverage ratio that is based on financial intermediations. This model also helps generate a procyclical leverage ratio. However, in contrast to previous studies, in my model the procyclicality arises from extrapolation. During market booms, extrapolators become overly optimistic because they extrapolate the past returns and, consequently, they buy more risky assets. Conversely, during recessions, extrapolators become overly pessimistic and, therefore, they have a low leverage level. Second, extrapolation also helps explain the nagative association between household leverage and future consumption growth documented in Mian and Sufi (2009). Extrapolation induces investors to make unreasonable investment decisions that lead to a future decrease in wealth, which pushes down consumption growth rate.

Outline: This paper is organized as follows. In Section 1.2, I document some dynamics of belief-based investor sentiment and identify the group of investors who are more susceptible to extrapolation. In Section 1.3, I construct a behavioral model that incorporates both extrapolation and the wealth dynamics of extrapolators. Then I derive several model predictions about the time-varying impact of investor sentiment. In Section 1.4, I examine these model predictions via both simulation and data analysis. In Section 1.5, I examine the role of extrapolation by considering a rational model as the benchmark. Section 1.6 summarizes results and proposes directions for future research.

1.2 Motivating Facts

The difficulty of investigating the impact of belief-based investor sentiment lies both in how to correctly measure investor beliefs and how to measure its overall market impact. To provide insights into the dynamics of belief-based investor sentiment, I resort to investor expectation surveys which directly asks investors about their beliefs. In my later analysis, I will use two terms—belief-based investor sentiment and survey measurement of investor sentiment—interchangeably whenever there is no confusion. Further, relying on survey measurements of investor sentiment, I identify one specific group of investors who tend to extrapolate so that I can measure the market impact of extrapolators properly.

Investor Sentiment Dynamics

In recent decades, researchers have made progress in understanding investor expectations by analyzing survey evidence. In most of the existing investor surveys, respondents are asked about their expectations on future market returns, ranging from six to twelve months. Compared to other measurements of investor expectations, survey measurements are more direct in extracting investor belief information.

In this paper, I mainly use the Gallup survey which measures individual investors' expectations of the U.S. stock market over the next twelve months.[5] It is conducted monthly between 1996 and 2011, but there are some gaps in later years especially between November 2009 and February 2011 when the survey was discontinued. To extract investor expectations, in each month, Gallup survey asks participants one *qualitative* question: whether they are "very optimistic", "optimistic", "neutral", "pessimistic", "very pessimistic" about stock returns over the next twelve months. With the percentage of each response in the collected survey answers, Gallup reports a *qualitative* investor expectation series to measure investor expectations in the market:

$$Gallup = \%Bullish - \%Bearish, \tag{1.1}$$

where "Bullish" is defined as either "very optimistic" or "optimistic" and "Bearish" is defined as either "pessimistic" or "very pessimistic". This qualitative time series helps us understand the dynamics of investor sentiment in the market. Moreover, Gallup survey also asks more precise *quantitative* questions about investors' perceived expected returns, although only for a shorter sample. Specifically, between September 1998 and May 2003, Gallup asks participants to give an estimate of the percentage return they expect for the stock market over the next year. Therefore, as long as participants in the Gallup survey answer quantitative and qualitative questions in a consistent way, I can effectively get quantitative estimations investor expectation series by rescaling qualitative Gallup investor series with projection method.[6] This projection method also helps me transform qualitative series in other investor expectation surveys to a meaningful quantitative basis.

However, there were two main concerns about the survey data. One concern is that survey evidence is imprecisely measured and thus is noisy. The other is that survey respondents may be confused by the sophisticated questions and therefore could not

[5]I use Gallup survey measurement to measure investor sentiment since Gallup mainly surveys individual investors who are more likely to extrapolate. Moreover, Gallup survey has a quantitative measurement of investor expectations to facilitate my analysis.

[6]Carlson and Parkin (1975) propose a method to generate average expectations from categorical survey data. As pointed in Greenwood and Shleifer (2014), this method has almost no impact on the investor expectation time series.

provide pertinent answers. Fortunately, there are recent developments that show the validity of the investor survey information. Greenwood and Shleifer (2014), among other findings, show that (1) information contained in different surveys reflects similar patterns and (2) the reported investor expectations in the surveys are largely consistent with investors' behaviors (See Figure A.2).[7] Their findings indicate that survey measures of investor expectations are not meaningless noise but represent widely shared beliefs about future returns in the stock market. With these validations from previous literature, I report the following observations to motivate my model.

Observation 1. *Survey measurement of investor sentiment is positively associated with the past returns in the aggregate stock market.*

Observation 1 is the main message in Greenwood and Shleifer (2014), and is a restatement of extrapolation: investors in the surveys over-extrapolate recent returns when forming their expectations in their minds. Formally, I use *extrapolators* to refer to these investors. Therefore, past good returns tend to make extrapolators overly optimistic while past bad returns make them overly pessimistic. For the underlying psychological mechanisms of extrapolation, there are several candidate theories, including representativeness. Kahneman and Tversky (1972) define representativeness as "the degree to which an event (i) is similar in essential characteristics to its parent population, and (ii) reflects the salient features of the process by which it is generated".[8] With representativeness, extrapolators might mistakenly treat a sequence of good (bad) recent returns as a salient feature of the whole distribution of returns, which leads to an over-extrapolation. Although the fact that extrapolators overweight information in the recent returns is commonly documented in recent empirical studies (Amromin and Sharpe (2013), Bacchetta et al. (2009) and Greenwood and Shleifer (2014)), the source of over-extrapolation still remains an important open question.

The belief-based investor sentiment in the surveys is consistent with most of the anecdotal fluctuations of investor sentiment—that investor sentiment rise rapidly

[7]For result one, the authors compare survey sources from the American Association of Individual Investors (AA), Gallup, Graham and Harvey, Investors' Intelligence newsletter expectations, Michigan Survey and Shiller, and document strong correlations between each survey. For result two, the authors examine the reported investor expectations and the investor mutual fund flows, and find two time-series are highly synchronized.

[8]For other references, also see Tversky and Kahneman (1971) and Tversky and Kahneman (1975).

during booms and decrease during crash episodes—in the market. In Figure A.1, I plot the Gallup survey measurement of investor sentiment, ranging from 1996:12 to 2011:09, with the backdrop of shaded NBER recessions. During the "Internet Bubble" episodes in mid-2000, Gallup survey measurement of investor sentiment rises to its peak but drops dramatically after the burst. Similarly, Gallup survey measurement rebounds to the peak before the 2007 financial crisis and declines significantly after the market index fell in 2008.

[Place Table A.1 about here]

To further reveal the close connections between investor sentiment in the survey and extrapolation, I construct a new investor sentiment variable called "Psentiment". Previous studies on the belief patterns in the survey indicate that when forming beliefs on expectations of returns, extrapolators put a decaying weight on the realized returns in the past. For example, Barberis et al. (2015) use the non-linear formula in equation (1.2) to estimate the weighting scheme of extrapolators, with the survey measurements of investor sentiment as the dependent variables and ψ represents for the weighting scheme. $R_{t-(s+1)\Delta t, t-s\Delta t}$ measures the past realized returns within one interval Δt. Using data of quarterly frequency ($\Delta t = 1/4$), they get the estimated ψ of 0.44, implying that the realized returns one year ago are only half as important as the most recent return.

$$\begin{aligned} \text{Expectation}_t &= a + b\Sigma_{s=0}^{\infty}\omega_s R_{t-(s+1)\Delta t, t-s\Delta t}, \\ \omega_s &= \frac{e^{-\psi s \Delta_t}}{\Sigma_{k=0}^{n} e^{-\psi k \Delta_t}}, \end{aligned} \tag{1.2}$$

The constructed investor sentiment variable, Psentiment, is purely based on extrapolation and I use it for robustness check—if investor beliefs in the surveys are mainly driven by extrapolation, then Psentiment and Gallup survey measurement of investor sentiment should yield similar results.

Observation 2. *Survey measurement of investor sentiment tend to revert to its mean: high investor sentiment revises downwards in the future while low investor sentiment revises upwards in the future.*

To empirically test the pattern in Observation 2, I run the following regression:

$$SR_{t+1}[\text{Sent}_{t+N} - \text{Sent}_{t+1}] = c + d\text{Sent}_t + u_t, \tag{1.3}$$

where on the left hand side $SR_{t+1}[\text{Sent}_{t+N} - \text{Sent}_{t+1}]$ measures the investor sentiment revision (SR for short) over future $N - 1$ horizons, Sent_t represents the investor sentiment at time t, and u_t on the right hand side is the corresponding residual at time t. The results for equation 1.3 is reported in Table A.2.

[Place Table A.2 about here]

The negative coefficient d supports the mean-reversion pattern in Observation 2. Using the Gallup survey, one standard deviation increase implies an investor sentiment revision of 14.2% within one quarter and a revision of 62.4% within four quarters. Similarly, with the constructed investor sentiment proxy Psentiment, one standard deviation decrease implies an investor sentiment revision of 6.4% with one quarter and a revision of 32.9% within four quarters. The coefficients remain statistically significant in general.

The reversal property in expectations in Observation 2 is largely due to the fact that investors tend to overreact to the information, and therefore it reflects the excessive optimism and pessimism on expectations of returns in the surveys. When investor sentiment is high, investors perceive higher returns going forward. However, the objective return distributions observed by econometricians remain unchanged—investors get constantly disappointed by future realized returns. As a result, investor sentiment revises downwards in the future. When investor sentiment is low, investors get constantly surprised by future realized returns, and investor sentiment bounce upwards.

Reversal is a general pattern in economic studies. For instance, Greenwood and Hanson (2013) document a systematic reversal in bond spreads. Bordalo et al. (2018) document predictable reversals in the credit market because credit spreads overreact to news. Reversals in investor expectations are also common and are not limited to expectations of stock returns. For instance, Gennaioli et al. (2016) document a salient reversal pattern for CEOs' expectations about their company's earnings growth, and interpret the reversal as a result of over-extrapolating past earnings growth rate. López-Salido et al. (2017) also link reversals to overreaction, and suggest that a period of excessive investor optimism is followed by a reversal in the credit market, a term they refer to as "unwinding of investor sentiment". The reversal patterns capture important properties of investor belief dynamics.

Wealth Level of Extrapolators

The documented patterns in Observations 1 and 2 provide evidence on the dynamics of investor beliefs. However, investor beliefs alone could not provide an answer to the question of how belief-based investor sentiment influences the market: as shown in Table A.3, Gallup survey measurement of investor sentiment does *not* have significant predictive power for future stock market returns. This result seems puzzling at the first glance since previous studies have shown that the extrapolation pattern in surveys reflect market-wide investor expectations—intuitively, the predictive pattern should be strong. One important element that seems missing is the wealth level of extrapolators: after all, extrapolators with extreme sentiment levels but with trivial wealth will have a limited impact on asset dynamics. Therefore, I also examine the wealth level of extrapolators.

To measure the wealth level of extrapolators, it is important to understand which group of investors are more susceptible to extrapolation. In the finance literature, a common way to categorize investors is to divide them into individual investors and institutional investors. Moreover, in the literature, individual investors are often believed to be less sophisticated and more vulnerable to psychological biases, a term formally defined as "dumb money" effect. And empirical evidence is prevailing. For instance, Odean (1999), Barber and Odean (2000) and Barber and Odean (2001) present extensive evidence that individual investors suffer from biased-self attribution, and tend to have wealth-destroying excessive trading. Frazzini and Lamont (2008) use mutual fund flows as a measure of individual investor sentiment for different stocks and find that high sentiment predicts low future returns.[9]

A reasonable proxy for individual investors exists in the Federal Reserve's Z.1 Statistical Release ("Financials Accounts of the United States"), which reports balance sheet information for different sectors of the economy at a quarterly frequency, including the Households and Nonprofit Organizations Sector (HNPO sector). The HNPO sector contains aggregated information about individual investors.

More importantly, as shown in the recent studies, individual investors in the HNPO sector tend to extrapolate. For example, a rigorous examination is reported in Yang

[9]For other studies supporting the "dumb money" effect, see Barber et al. (2008), Sapp and Tiwari (2004).

and Zhang (2017). Specifically, they document that (1) when investor expectations in the surveys are high, investors in the HNPO sector tend to increase their investment in the stock market; and (2) their investment choices negatively predict future stock market returns. These results together show that investors in the HNPO sector can serve as a reasonable proxies for extrapolators. Moreover, in my empirical analysis, I use the total financial wealth to proxy for the wealth level of extrapolators.[10]

Interaction Effect between Investor Sentiment and Wealth Level

Investor sentiment should impact market strongly especially when the wealth level of extrapolators is high. In other words, there should be a strong interaction effect between investor sentiment and the wealth level of extrapolators in explaining asset valuations. To formally test this hypothesis and to motivate my model setting, I run the following regression[11]:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t. \tag{1.4}$$

In my regression, R^e_{t+N} represents the excess returns of the aggregate stock market during the future N months. In addition, I use investor sentiment in the Gallup survey and Psentiment, respectively, to proxy for Sent_t, investor sentiment, and use the total financial assets of HNPO sector at time t as the empirical proxy for the wealth level W_t. Since W_t is non-stationary, I get the series of real dividend D_t from Shiller's website and get the normalized wealth level of W_t/D_t. My sample is at a quarterly frequency and spans from 1996:12 - 2011: 9.[12] For robustness check, I report predictive regressions over one to six quarters.

[Place Table A.4 about here]

The results reported in Table A.4 not only confirm the strong interaction effect between investor sentiment and the wealth level in determining the asset prices, but also point to the usefulness in explaining asset *mispricings*—the strong negative coefficient for the interaction term indicates investor sentiment leads to asset overvaluations and undervaluations when the wealth level is high. When investor

[10]As long as the HNPO sector captures the general properties of extrapolators, it will help understand the impact of investor sentiment on the market.

[11]Alternatively, the market impact of extrapolators can be measured by the relative wealth of extrapolators compared to the price level of the risky asset. The regressions based on this intuition is reported in appendix A.5. The results remains robust.

[12]The Federal Reserve's Z.1 Statistical Release is published at a quarterly frequency. The sample period mainly goes with Gallup surveys.

sentiment is high, the market is highly overvalued due to the high irrational demand from extrapolators and therefore future returns are low; if investor sentiment is low, the future returns are high since the market is undervalued. In addition, compared to the univariate regression using investor sentiment variable, the conditional predictability model has a significantly higher adjusted-R^2. For instance, at the annual horizon, the goodness of fit for the predictive regression using investor sentiment alone is around 0.010, which is small compared to 0.039 in the conditional predictive regression. The increased adjusted-R^2 supports the importance of the interaction effect in explaining the market mispricing. Therefore, I get the following observation:

Observation 3. *Investor sentiment strongly connects with market mispricing through the market impact of extrapolators.*

Observation 3 provides strong evidence for the relation between the market mispricing, investor sentiment and the wealth level of extrapolators. Therefore, for a model that focuses on the impact of investor sentiment and market mispricing, the wealth level of extrapolators should also be incorporated.

1.3 The Behavioral Model

In this section, I build a behavioral model that focuses on the investor sentiment of extrapolators. Specifically, following the patterns of investor sentiment documented in section 1.2, I introduce extrapolation into my model. With extrapolation, the behavioral model essentially captures the overvaluations and undervaluations in the market and shed light on the time-varying impact of investor sentiment on the equilibrium asset price.

The Economy

I consider a continuous-time economy with two types of assets: one risk-free asset with elastic supply curve and a constant rate r, and one risky asset with a fixed per-capita supply of one. Due to extrapolation, extrapolators perceive a biased growth rate of the risky asset, which is different from the true growth rate observed by the outside econometricians.

The risky asset is a claim to the underlying dividend process D_t which, under the true probability measure, follows a geometric Brownian motion and can be generically written as

$$dD_t/D_t = g_D dt + \sigma_D d\omega_t. \quad (1.5)$$

Therefore, the underlying dividend process is governed by the growth rate of g_D and volatility σ_D, which both are positive exogenous parameters. The dividend process is driven by ω_t, a one-dimensional Weiner process under the true probability measure observed by outside econometricians. The equilibrium price P_t for the dividend claim evolves as

$$dP_t/P_t = g_{P,t} dt + \sigma_{P,t} d\omega_t. \quad (1.6)$$

Due to extrapolation, the true growth rate $g_{P,t}$ is different from the perceived growth rate by extrapolators. The growth rate $g_{P,t}$ and volatility term $\sigma_{P,t}$ are both endogenously determined in the equilibrium.

Investors

There are two types of investors in the behavioral model: extrapolators and fundamental investors. Extrapolators are the focus of this paper: their belief formation are subject to psychological heuristics and therefore are misspecified. Fundamental investors, on the other end, serves as the counteracting forces in the market and trade aggressively whenever asset prices deviate from fundamental values. Similarly, I assume that fundamental investors make up a fraction of $1 - \mu$ and extrapolators make up a fraction of μ.

Extrapolative Beliefs

The salient properties about investor sentiment in the surveys—that investors form their expectations based on past realized returns and that their sentiment reverts quickly to its mean—motivate my theoretical settings for investor sentiment. Different from asset pricing models with rational expectations, extrapolators in the behavioral model make systematic errors about future market returns. In order to capture the misspecified beliefs for investors, I propose a mental model.[13] Specifically, I assume extrapolators perceive the following process for the market price, with the perceived growth rate $\hat{g}_{P,t}$ as an affine function of a latent state variable S_t that essentially captures the excessive optimism and pessimism in extrapolators' minds:

[13] A similar model is used in Jin and Sui (2017).

$$dP_t/P_t = \hat{g}_{P,t}dt + \sigma_{P,t}d\omega_t^e,$$
$$\hat{g}_{P,t} \equiv [(1-\theta)\bar{g}_{P,t} + \theta S_t]. \tag{1.7}$$

Here $\bar{g}_{P,t}$ represents the equilibrium growth rate of risky asset in a benchmark economy where each investors have correct beliefs. Parameter θ measures to which extent extrapolators deviate from their beliefs in the rational benchmark. When $\theta = 0$, I get the rational benchmark case with a drift term $\hat{g}_{P,t}$.[14] Notation-wise, any quantities with a hat sign or superscript e represent variables perceived by extrapolators.

Next, I introduce microfoundations for this key latent state variable S_t. Specifically, I assume that from the extrapolators' perspective, the expected growth rate on the risky asset is an affine function of a mental variable $\tilde{\mu}_{P,t}$, and the extrapolators mistakenly believe that this mental variable is governed by a regime-switching process between high and low states μ_H and μ_L, with switching densities χ and λ:

$$\begin{array}{cc} & \begin{array}{cc} \tilde{\mu}_{P,t+dt} = \mu_H & \tilde{\mu}_{P,t+dt} = \mu_L \end{array} \\ \begin{array}{c} \tilde{\mu}_{P,t} = \mu_H \\ \tilde{\mu}_{P,t} = \mu_L \end{array} & \left(\begin{array}{cc} 1 - \chi dt & \chi dt \\ \lambda dt & 1 - \lambda dt \end{array} \right). \end{array} \tag{1.8}$$

Motivated by the fact that investors in the survey over-extrapolate past returns when forming their expectations, I assume that extrapolators in my model update their estimate of the mental variable $\tilde{\mu}_{P,t}$ by looking at the past realized returns. Therefore, μ_H and μ_L reflect extrapolators' subjective perceptions on the risky asset growth rate, and λ and χ represent the speed extrapolators update their perceptions on the risky asset growth rate based on past realized returns. Given this mental model, the latent state variable S_t is the Bayesian inference of the mental variable $\tilde{\mu}_{S,t}$. Formally, I can write the latent state variable as $S_t \equiv \mathbb{E}^e[\tilde{\mu}_{P,t}|\mathscr{F}_t]$, where $\mathscr{F}_t$ is the perceived probability measure based on the filtration of the price process P_t. My later examination on extrapolators' belief structure helps me justify the magnitude of these belief parameters.

Therefore, in my model, investor sentiment corresponds to the perceived growth rate of extrapolators, $\hat{g}_{P,t}$. Throughout my model, I assume that each extrapolator is subject to the identical underlying mental model and the same degree of extrapolation. In other words, latent state variable θ reflects the *consensus* extrapolation

[14] For a detailed solution for $\hat{g}_{P,t}$, see appendix.

level among extrapolators.

By applying the optimal filtering theorem (Liptser and Shiryaev (2013)), I obtain the dynamics for the latent state variable S_t:

$$dS_t = \mu_S dt + \sigma_S d\omega_t^e, \tag{1.9}$$

where

$$\mu_S = \lambda\mu_H + \chi\mu_L - (\chi + \lambda)S_t, \tag{1.10}$$

$$\sigma_S = \sigma_{P,t}^{-1}\theta(\mu_H - S_t)(S_t - \mu_L), \tag{1.11}$$

$$d\omega_t^e = \frac{dP_t}{P_t} - [(1-\theta)\bar{g}_{P,t} + \theta S_t]dt. \tag{1.12}$$

This mental model captures two salient features of investor sentiment in the surveys. First, under the mental model, the latent state variable S_t is driven by the perceived Brownian shock $d\omega_t^e$, which strongly depends on the past realized returns. Increases in past realized returns $\frac{dP_t}{P_t}$ push up the perceived Brownian shock, which further increases the latent state variable S_t and the perceived growth rate of the risky asset $\hat{g}_{P,t}$. Therefore, the mental model naturally leads to an extrapolation pattern: high returns in the past push up investor sentiment and low past returns make extrapolators pessimistic.

Second, the mental model embodies the fact that investor sentiment tends to revert to its mean. When the latent state variable is high, extrapolators mistakenly perceive a high level of growth rate for the risky asset. However, the objective probability measure remain unchanged. Unless extremely good shocks arrive, they will be constantly disappointed by the perceived Brownian shocks $d\omega_t^e$ in the future. As a result, the latent variable will quickly revert downwards. Similarly, if the latent state variable is low and extrapolators perceive a low level of growth rate, extrapolators will meet with relatively large perceived Brownian shocks, pushing up the latent state variable back to its mean.

The underlying mechanism for the reversal of investor sentiment in my model is different from previous “Natural Expectations” models in Fuster et al. (2011). In Fuster et al. (2011), the long-term reversal are exogenously assumed and errors in expectations arise because agents fit a simpler AR(1) model to the data. As a comparison, in my model, the mean-reversal process is endogenous—investors overweight recent information and this overreaction leads to reversals. Such endogenous reversal pattern in investor beliefs helps understand asset mispricings and predictability of returns generated in this model.[15]

Moreover, due to the mental model, the perceived probability measure by extrapolators is different from the objective probability measure observed by outside econometricians.[16] In other words, extrapolators have misspecified but self-consistent beliefs, and they fail to realize that their perception is biased. The true and perceived probability measure can be connected using the following equation:

$$d\omega_t^e = (g_{P,t} - \hat{g}_{P,t})/\sigma_{P,t} dt + d\omega_t. \tag{1.13}$$

Under the probability measure of extrapolators, the perceived dividend process follows

$$dD_t/D_t = \hat{g}_{D,t} dt + \sigma_D d\omega_t^e. \tag{1.14}$$

where $\hat{g}_{D,t}$ is the perceived dividend growth rate by extrapolators.

In my model, $g_{P,t}$, $\hat{g}_{P,t}$ and $\hat{g}_{D,t}$ are endogenous variables determined in the equilibrium. As a comparison, the volatility variables $\sigma_{P,t}$ and σ_D remain unchanged. This is because extrapolators can always calculate the volatilities by calculating the quadratic variations of the stochastic process. All the quantities together constitute two mutually coherent probability measurements.

Finally, it is worth pointing out that, there is a two-way feedback loop in my model: asset prices influence investor expectations due to extrapolation, and changes in investor expectation will in turn drive asset prices. From a theoretical perspective, a model with extrapolation on past returns is usually difficult to solve, since the return process is an endogenous quantity determined in the equilibrium. In this paper, I overcome this difficulty by solving a system of partial differential equations.

Wealth Process of Extrapolators

To better connect with later analyses on the time-varying impact of investor sentiment, I introduce logarithmic utilities for extrapolators to capture the dependence of demand on their wealth level. Moreover, each extrapolator, indexed by superscript

[15] The pattern that investors tend to overweight recent information has been incorporated in other models as well. For instance, Bordalo et al. (2018) focus on investors' belief formation of credit spreads, and proposed a mechanism in which investors overreact to recent news.

[16] The fact that investors have biased beliefs is widely empirically documented in the literature. For example, using option data, Barone-Adesi et al. (2016) document excessive optimism and pessimism for investors in the U.S. stock market.

j, is infinitesimal and therefore acts as price-taker in the market. Specifically, extrapolator j maximizes an additively separable logarithmic utility function under an infinite time horizon and with a time-preference parameter ρ

$$\max_{\{C^j_{t+s}\}_{s\geq 0}} \mathbb{E}^e_t[\int_t^\infty e^{-\rho s} \ln C^j_s ds] \tag{1.15}$$

subject to

$$dW^j_t = -C^j_t dt + rW^j_t dt + \alpha^j_t W^j_t [\hat{g}_{P,t} dt + \frac{D_t}{P_t} dt - r dt + \sigma_{P,t} d\omega^e_t], \tag{1.16}$$

where W^j_t represents the wealth level of extrapolators, C^j_t the optimal consumption choice and α^j_t the optimal risk exposure for extrapolator j. $\mathbb{E}^e_t$ represents the expectation operator under the perceived probability measure, which is biased due to extrapolation.

With the logarithmic utility, extrapolators have wealth-dependent absolute risk aversion: as their wealth declines to zero, extrapolators become infinitely risk-averse. Consequently, as their wealth shrinks, they will liquid the risky asset to prevent their wealth from becoming zero.[17] In addition, the infinitely risk-aversion level at zero wealth level also prevents extrapolators from bankruptcy. Therefore, they can borrow money at risk-free rate irrespective of their wealth level.

Following the standard Merton's approach (Merton (1971)), we get the optimal consumption and portfolio rule for extrapolators:

PROPOSITION 1 *Extrapolators with the objective function in equation (1.15) and wealth process in equation (1.16) will optimally consume and invest at time t according to the following strategies:*

$$C^j_t = \rho W^j_t \tag{1.17}$$

and

$$\alpha^j_t = \frac{\hat{g}_{P,t}(x_t, S_t) + l^{-1}(x_t, S_t) - r}{\sigma^2_{P,t}(x_t, S_t)}, \tag{1.18}$$

[17] The existence of fundamental investors guarantees they can liquidity their risky asset position at any time.

where the price to dividend ratio $l \equiv \frac{P_t}{D_t}$ *and the risky asset volatility* $\sigma_{P,t}$ *both depend on the wealth to dividend ratio* $x_t \equiv \frac{W_t}{D_t}$ *and the latent state variable* S_t*. In other words, the economy depends on state variables* x_t *and* S_t*.*

Proof: See Appendix A.3.

Proposition (1) documents several salient features of logarithmic utilities. First, the optimal consumption strategy for extrapolators is proportional to their wealth level, with adjustment of the time-preference parameters ρ. Second, the optimal risky portfolio choice is myopic, in a sense that extrapolators do not hedge against changes in the future investment set. Therefore, their optimal risky exposure purely depends on the perceived risk premium and the instantaneous volatility rate. Third, the total dollar demand is also proportional to their wealth. Therefore, their wealth level, by and large, determines their market impact on the equilibrium asset prices—a key property that drives my model implications.

In the equilibrium, the risky exposure of extrapolators depends on their perceptions of equity premium, $\hat{g}_{P,t} + l^{-1} - r$. After observing a sequence of high (low) returns, extrapolators become overly optimistic (pessimistic) and increase their risky asset exposure. In cases when their market impact is high, asset tends to become overvalued (undervalued).

More importantly, the optimal strategy in equation (1.18) and the wealth dynamics in equation (1.16) together provide novel insights on how investor sentiment influences the wealth dynamics of extrapolators. When investor sentiment is high, extrapolators perceive high equity premium and therefore lever up to buy risky asset. However, since investor sentiment reverts to its mean, extrapolators demand less risky asset in the future, which leads to decreases in the asset prices and their wealth level, therefore generating negative returns. In the mean-reversion process, the wealth dynamics of extrapolators also play a role: investor sentiment leads investors to take excessive exposure to the risky asset and therefore suffer from wealth decrease. Decline in the wealth level of extrapolators makes the mispricing easier to correct.

Fundamental Investors

Next, I describe the demand function for fundamental investors. I assume that fundamental investors care only about the difference between the current risky asset prices and their fundamental values, hence they construct their risky asset exposure based on the deviation of the asset prices from its fundamental values. Following this assumption, the per capita demand of the risky asset for fundamental investors follows

$$Q_t = (P_{F,t} - P_t)/k, \tag{1.19}$$

where $P_{F,t} = \frac{D_t}{r-g_D}$ is the fundamental value and k is a constant. This linear demand structure is standard in the literature (for example Xiong (2001)), and I provide a micro-foundation for it in the appendix A.1. Intuitively, when $P_{F,t}$ is higher (lower) than its current price, fundamental investors expect profits when asset prices reverts back to the fundamental value P_t. Therefore, they have a strong demand to short (buy) the risky asset.

Fundamental investors are prevailing in the financial market. There are many investors who truly follow strategies that focus on the long-term profit and apply fundamental analysis in their investment. For instance, Abarbanell and Bushee (1997) document that investors incorporate financial statement information and make fundamental analysis when making their investment decision. In addition, there are many mutual funds who set their investment objective as to achieve long-term growth of capital and income, such as the "Fundamental Investor" fund managed by Capital Group. Moreover, such fundamental analysis really brings abnormal returns to their portfolio. For instance, Piotroski (2000) document fundamental analysis can increase annual returns by at least 7.5%.

Several observations about the total dollar demand in equation (1.19) are worth noting. First, for fundamental investors, the total dollar demand is independent of their wealth level, rather, it only depends on the difference between the current risky asset prices and their fundamental values. When asset prices are highly overvalued, fundamental investors anticipate a higher return from holding the risky asset and, as a result, their total dollar demand increases. Conversely, when asset prices are undervalued, fundamental investors lean against the wind and trade actively to push prices upward. Moreover, the extent to which fundamental investors can correct the mispricing also depends on the overall market impact of the extrapolators: if the wealth level of extrapolators is low, fundamental investors can correct mispricing more easily. This demand structure of fundamental investors proves to be useful in generating novel implications on the predictive pattern of investor sentiment.

Second, in my model, fundamental investors act as aggressive arbitragers and they jump into arbitrage activities *whenever* mispricing occurs—different from fully rational investors, fundamental investors in my model are the counteracting forces in the market. In other words, irrespective of their beliefs, fundamental investors have a specific demand function due to unspecified factors such as financial frictions or preference shocks.

Equilibrium

Definition of Equilibrium:

An equilibrium in the behavioral model satisfies the following conditions:

i) extrapolators maximize their objective function in equation (1.15) subject to their wealth process in equation (1.16) under their subjective probability measure induced by extrapolation;

ii) fundamental investors invest in the risky asset according to their demand function in equation (1.18);

iii) risky asset market clears

$$\mu \alpha_t W_t + (1 - \mu) Q_t = P_t. \tag{1.20}$$

To solve for the equilibrium, I face with a fixed-point problem: the optimal demand of extrapolators depends on the instantaneous equity premium and volatilities of the risky asset, which further depends on the total demand in the equilibrium. For the behavioral model, I focus on the symmetric equilibrium where each extrapolator endows with the same level of wealth and follows identical strategies and solve for a fixed-point problem. The equilibrium depends both on the wealth to dividend ratio x_t and the latent state variable S_t. Therefore, I rely on solving a partial differential equation to obtain the solution for the fixed-point problem. As verified later, the price to dividend ratio is monotonically increasing in both x_t and S_t.

After getting the optimal strategy for extrapolators, now I consider a fixed-point problem and solve the Markovian system to get an solution of the price-dividend ratio $l(S_t, x_t)$. To be specific, I focus on an equilibrium, where each extrapolator j follows the same strategies based on the return extrapolation. By using aggregation and the market clearing condition, I can reach the equilibrium as follows:

PROPOSITION 2 *In the symmetric equilibrium, the price-dividend ratio l is a function of current states S_t and x_t, and satisfies the following partial differential equation:*

$$\frac{c_0 l - c_1}{x} = \frac{\hat{g}_{P,t} + l^{-1} - r}{\sigma_{P,t}^2}, \tag{1.21}$$

where σ_P is also a function of S_t and x_t and satisfies

$$\sigma_{P,t} = \frac{(\frac{l_x}{l}x - 1)\sigma_D - \sqrt{(\frac{l_x}{l}x - 1)^2\sigma_D^2 - 4(\frac{l_x}{l}(c_0 l - c_1) - 1)\frac{l_S}{l}\theta(\mu_H - S)(S - \mu_L)}}{2(\frac{l_x}{l}(c_0 l - c_1) - 1)}, \tag{1.22}$$

and $l(S_t, x_t)$ satisfies the following boundary conditions

$$\lim_{x_t \to 0} l = \frac{c_1}{c_0}, \tag{1.23}$$

$$\lim_{x_t \to 0} = \lim_{x \to 0} \frac{\hat{g}_{P,t} + \frac{c_1}{c_0} - r}{\sigma_D^2}, \tag{1.24}$$

and

$$\lim_{x_t \to \infty} l = (r - \hat{g}_{P,t})^{-1}, \tag{1.25}$$

where $c_0 = \frac{k+1-\mu}{k\mu}$ and $c_1 = \frac{1-\mu}{k\mu(r-g_D)}$ are both constants.

In addition, under the perceived probability measure, the dynamics of x_t follow

$$dx_t = \hat{g}_{x,t}(S_t, x_t)dt + \sigma_{x,t}(S_t, x_t)d\omega_t^e, \tag{1.26}$$

where

$$\begin{aligned} \hat{g}_{x,t}(S_t, x_t) &= x_t(r - \rho + \alpha_t[\hat{g}_{P,t} + l^{-1} - r] - \hat{g}_{D,t} + \sigma_D^2 - \alpha_t\sigma_D\sigma_{P,t}) \\ \sigma_{x,t}(S_t, x_t) &= x_t(\alpha_t\sigma_{P,t} - \sigma_D). \end{aligned} \tag{1.27}$$

Proof: See Appendix A.3.

For the boundary conditions, when the wealth to dividend ratio goes to zero, the fundamental investors dominate the market and I get a constant price-dividend ratio in equation (1.23). Moreover, in this case, the partial derivative of l with respect to S, $l_S(S, 0)$, equals to zero because extrapolators' total demand is zero irrespective of the level of the latent state variable S_t. On the other hand, when x_t goes to infinity, in order to clear the market, extrapolators hold a risky asset position close to zero. By the portfolio expression in equation (1.18), we get the expression (1.25). Expression (1.24) follows naturally from equation (1.23).

In solving the ordinary equation in proposition (2), I resort to a projection method with Chebyshev polynomials. Compared to the range of the wealth to dividend ratio of $[0, \infty]$, the required domain for Chebyshev polynomial is $[-1, 1]$. Therefore, I apply the following monotonic transformation for the wealth to dividend ratio

$$z_t = \frac{x_t - \xi}{x_t + \xi}, \tag{1.28}$$

where ξ is a positive constant. Along with the transformed wealth to dividend ratio, I also transform the latent state variable S into a new variable y that lies between $[-1, 1]$:

$$\begin{aligned} y_t =& aS_t + b, \\ a =& \frac{2}{\mu_H - \mu_L}, \quad b = -\frac{\mu_H + \mu_L}{\mu_H - \mu_L}. \end{aligned} \tag{1.29}$$

When the wealth to dividend ratio goes to infinity, the transformed wealth to dividend ratio goes to 1. If the wealth to dividend ratio shrinks to zero, the transformed wealth to dividend ratio goes to -1. Similarly, y_t goes to 1 when S_t goes to its upper bound μ_H, and goes to -1 when S_t goes to its lower bound μ_L.

Calibrated Model Solution

In this section, I report the main numerical solutions. To get a reasonable model solution, I choose both asset and utility parameters that are consistent with the empirical literature. For example, I set g_D to be 1.5% and σ_D to be 10%, which are commonly used in the asset pricing literature. In addition, I set $r = 4\%$ for the risk-free rate. For utility parameters, I choose the time-preference factor $\rho = 2\%$. For other parameters that have no empirical counterparts, such as μ and k, I take a neutral stand and impose a value of $\frac{1}{2}$. For belief parameters, I set $\mu_H = 0.03$, $\mu_L = -0.06$, $\chi = \lambda = 10\%$, $\theta = 0.5\%$. A complete set of parameter values are reported in Table A.6. I report the numerical solutions in Figure A.5.

Price-Dividend Ratio

The upper-left panel reports the price to dividend ratio $l(S_t, x_t)$. In general, $l(S_t, x_t)$ is a monotone function both in the transformed wealth to dividend ratio z_t and the latent state variable y_t. First, for most of z_t levels, a higher latent variable level S_t induces higher total dollar demand and therefore pushes up the price to dividend ratio. When $z_t = 0$, the price to dividend ratio $l(S_t, x_t)$ are constant values of 20:

if extrapolators have no wealth, they will have no market impact on the equilibrium quantities. Second, with a fixed level of latent variable y_t, the dollar demand for the risky asset increases as z_t increases. With given parameters, the solution for $l(S_t, x_t)$ ranges from 20 to 35, which is largely within the reasonable range.

[Place Figure A.5 about here]

Optimal Portfolio Choice

I report the optimal portfolio choice for extrapolators in the lower-right panel. As y_t increases along the axis, extrapolators perceive higher growth rate and therefore increase their position in the risky asset. As z_t increases from -1 to 1, in general, the optimal portfolio decreases in order to meet the market clearing condition. Together, I get the optimal strategies of extrapolators. It is worth noting that, when both z_t and y_t are low and extrapolators have low market impact and become pessimistic about future return growth, they can short the risky asset; in other situations, extrapolators hold the risky asset.

Volatility

The upper-right panel portrays the return volatility σ_P. In most of the cases, σ_P is larger than σ_D. When the wealth to dividend ratio x_t is high and the latent state variable S_t is slightly above its mean, the volatility σ_P reaches its maximum of 17.78%.

Along the latent variable axis, the volatility has a strong hump-shaped pattern due to the belief structure. Specifically, when the latent variable y_t goes to its upper or lower boundary, there is less uncertainty about which regime the underlying state belongs to. On the contrary, when y_t is in the middle region between the high and low bound, the uncertainty increases.

Moreover, increases in z_t in general increases the market impact of extrapolators. As a result, their beliefs have a stronger amplification effect on the exogenous shocks, which explains the increasing volatility along the z_t axis. However, when y_t reaches its maximum or minimum, the volatility is mainly determined by changes in z_t ratio. When y_t is at its maximum, the optimal portfolio is positive but less than one. In this case, a price drop caused by a negative dividend shock leads extrapolators to rebalance their portfolio by purchasing more of the risky asset, which reduces the volatility.[18] Conversely, when y_t is at its minimum, the optimal portfolio is, by and large, negative. In this case, extrapolators short the risky asset. When positive dividend shocks arrive, extrapolators' leverage ratio further decreases and induces them to buy more risky assets, which increases the volatility.

Two Model Predictions

Investor Sentiment and Market Mispricing

In this model, I simultaneously characterize investor sentiment and the wealth level within one unified model, which provides novel insights between investor sentiment and market mispricing. From a static perspective, the market clearing condition in equation (1.20) helps to identify when the market overvaluations and undervaluations would occur. When the wealth level of extrapolators is high, investor sentiment has large impacts on the market: with high investor sentiment, the asset price is largely overvalued; while if investor sentiment is low, the asset price is undervalued.

Investor Sentiment and Return Predictability

The dynamic feature of the behavioral model also sheds light on the connection between investor sentiment and the future asset price dynamics. With higher wealth level, a larger fraction of total asset demand will come from extrapolators. Conditional on high wealth level, high investor sentiment will push asset prices above their fundamental value. However, higher sentiment will revert to its mean quickly, since optimistic investors will more easily get disappointed by the future realized returns. Consequently, both investor sentiment and asset prices will decline in the future, generating negative returns. Conversely, conditional on high wealth level, low investor sentiment predicts a high returns going forward. This together implies a strong negative pattern for investor sentiment when wealth level is high.

Moreover, my model implies a novel predictive pattern of investor sentiment on future returns when the wealth level of extrapolators is low. In this situation, asset prices are mainly driven by fundamental investors and the sentiment of extrapolators reflects the direction of market correction. If extrapolators have high sentiment, extrapolation indicates that the recent returns in the past were high and the market is undergoing an upward correction. Therefore, high sentiment implies good future returns along this upward path. Conversely, if extrapolators have low sen-

[18] Since extrapolation simultaneously make extrapolators less willing to hold the risky asset, the portfolio-rebalancing effect is weaker than that in the rational benchmark model.

timent, extrapolation indicates a downward correction, and fundamental investors will continue shorting risky asset until asset price reaches its fundamental value. Low sentiment predicts negative returns along this path. Together, conditional on the low wealth level of extrapolators, investor sentiment positively predicts future market returns.

Figure A.3 provides intuitive explanations for the positive predictive pattern of investor sentiment when the wealth level of extrapolators is low. If investor sentiment is low, extrapolation indicate that the asset price is undergoing a upward correction. Therefore, the only possible price dynamics is the one that goes up but is below the fundamental value—the wealth level of extrapolators is too low to push asset prices upwards from the fundamental value. Conversely, if investor sentiment is low, then the only possible situation is that the asset price is corrected downwards from overvaluations—the low market impact of extrapolators is not able to push asset prices downwards from the fundamental value.

[Place Figure A.3 about here]

1.4 Model Implications

In this section, I provide analyses for the behavioral model based on simulations. Specifically, I start by checking the belief pattern of extrapolators to see whether it captures the investor sentiment dynamics in the survey. Then I test whether there is a strong connection between investor sentiment and market mispricing, and test the key model implications on the return predictability based on investor sentiment. I also use my model to shed some light on the extant asset pricing patterns in the empirical literature.

Extrapolators' Beliefs

One important prerequisite to investigate the impact of investor sentiment is to understand the dynamics of investor sentiment. In this subsection, I provide justifications for the belief structure in my behavioral model based on model simulations. Specifically, I first compare investor sentiment based on model simulations with that in the Gallup survey. Second, I regress the model-implied investor sentiment on either past twelve month returns or the current log price to dividend ratio, and check whether my model embodies extrapolative expectations. Third, I formally test how much weight extrapolators put on the past realized returns and compare my results to the literature. Last, I test whether the model-implied investor sentiment reverts to its mean.

To simulate my model, I back out a sequence of shocks based on the monthly real dividend data of the S&P 500 index starting from June 1996 to December 2011. This range is consistent with the Gallup series in Greenwood and Shleifer (2014) and therefore can facilitate my comparison between the simulated sentiment series and the survey data. Moreover, I also use these implied shocks for my analyses of the behavioral model, so that I can compare two models within the same background.

Specifically, in order to get the series of shocks, I take the log on the dividend process and then use Ito's lemma to get

$$d\ln D_t = (g_D - \frac{1}{2}\sigma_D^2)dt + \sigma_D d\omega_t. \tag{1.30}$$

Then, I can discretize the equation and back out the shocks using the following formula:

$$\epsilon_t = (\ln D_{t+1} - \ln D_t - (g_D - \frac{1}{2}\sigma_D^2)\Delta_t)/(\sigma_D\sqrt{\Delta_t}). \tag{1.31}$$

Since the real dividend data is at a monthly frequency, I set Δ_t equals to $1/12$. In addition, I set g_D as 1.5% and σ_D as 10%, which are both commonly accepted magnitudes in the asset pricing literature. I take a neutral stand and set the initial sentiment level to be $S_0 = 0$. I also set the initial wealth to dividend ratio x_0 to be 1. Since I have numerically solved the equilibrium, I can easily get the simulated sequence for investor sentiment S_t, price process P_t, the wealth-dividend ratio process x_t as well as $l(S_t, x_t)$, $g_{P,t}(S_t, x_t)$ and $\sigma_P(S_t, x_t)$.

Model-implied Investor Sentiment

Since both my model simulation and Gallup survey are driven by the same underlying shocks to the economy, I can directly compare these two sequences. I plot both series in Figure A.6. From this figure, it is clear that the simulated investor sentiment based on the behavioral model is highly synchronized with the Gallup survey series, and captures the large part of anecdotal fluctuations of investor sentiment in the market. For example, consistent with the Gallup survey, the investor sentiment based on the behavioral model also rises to its peak right before the 2007 financial crisis and drops significantly during the afterward recessions.

[Place Figure A.6 about here]

Extrapolative Belief Structures

Next, I zoom in the extrapolation pattern in my behavioral model. Greenwood and Shleifer (2014) documents that investor expectations about future returns heavily depend on the current price level and past realized returns. Specifically, I regress the perceived expectations of future returns on either the current log price-dividend ratio or the past twelve-month accumulative raw returns, based on the simulated series. I report the regression coefficients, their t-statistics and their R-squared in Table A.7. For robustness check, I report results based on two different types of expectation measures: the expectations of future return growth rate $dP_t/(P_t dt)$ and the expectations of future return growth rate with dividend yield $dP_t/(P_t dt) + l^{-1}$.

Table A.7 indicates the investor belief pattern in my model matches the extrapolation pattern documented in Greenwood and Shleifer (2014). Among others, both in my model and in the survey, subjective expectations on future returns are positively related to the current log price-dividend ratio and the past twelve-month returns. Moreover, the regression coefficients and t-statistics are also close to the regression results based on the Gallup survey. For instance, compared to the regression coefficient of 9.12% and t-statistics of 8.81 from regressions based on Gallup survey, my simulation-based regression yields coefficient of 2.3% and t-statistics of 2.09.

[Place Table A.7 about here]

Extrapolators' Memory Span

Another important dimension of investor belief structure is its memory span, which measures how much weight extrapolators put on the recent returns. In my model, extrapolators' memory span is controlled by the magnitude of belief parameters χ and λ, which determines how fast extrapolators update their beliefs. To formally test extrapolators' memory span, I run the nonlinear regression following Greenwood and Shleifer (2014):

$$\text{Expectation}_t = a + b\Sigma_{s=0}^{\infty}\omega_s R_{t-(s+1)\Delta t, t-s\Delta t}, \tag{1.32}$$
$$\omega_s = \frac{e^{-\psi s \Delta_t}}{\Sigma_{k=0}^{n} e^{-\psi k \Delta_t}}.$$

In Table A.8, I report the regression coefficient a, the intercept b, the adjusted-R^2, and more importantly, the estimated memory span parameter ψ. My simulation is at monthly frequency and therefore I set $\Delta = 1/12$. I impose $n = 600$, which means extrapolators use returns within the past 50 years to form their beliefs. Table A.8 show that the estimated ψ is 0.51. This means a monthly returns three years ago is weighted only 25% as much as the most recent returns. As a comparison, Barberis et al. (2015) obtain an estimator of 0.44 for ψ. This consistency in turn helps justify the belief parameter choice for χ and λ.

It is worth pointing out that there is no consensus on investors' memory spans in the behavioral finance literature. Some studies, including Greenwood and Shleifer (2014) and Kuchler and Zafar (2016), document that investors have short memories and only use recent information in the past few years to form their beliefs. By contrast, other studies such as Malmendier and Nagel (2011), Malmendier and Nagel (2013) and Vanasco et al. (2015) argue that investors form their beliefs based on their life experience—events in the distant past might still have a significant impact in the belief formation process. Explaining this discrepancy is beyond the scope of this paper, but correctly interpreting investors' memory span would help understand the investor beliefs and the asset price dynamics.

[Place Table A.8 about here]

Model-implied Mean-reversion Property of Investor Sentiment

To complete my justification on my belief structure, I check whether investor sentiment in my model reverts to its mean. Consistent with Observation 2, I run the regression in equation (1.3), but replace survey measures with the model-implied investor sentiment.

[Place Table A.9 about here]

In Table A.9 , I report the regression coefficients, t-statistics and the adjusted-R^2. All t-statistics are adjusted with Newey-West correction. In addition, for robustness check, I also report results of forecasting horizons from 1 to 6 quarters. Table A.9 shows that the coefficient of sentiment variable is uniformly negative and statistically significant, which indicates that investor sentiment in my model reverts to its mean.

With the extrapolation pattern in my behavioral model, now I investigate the time-varying impact of investor sentiment.

Impact of Investor Sentiment

The Interaction Effect in the Model

In this section, I test whether the interaction effect in Observation 3 exists in my model. The market clearing condition (1.20) has already provides some clues: sentiment-driven risk exposure α_t and market power W_t together determines the equilibrium asset price. However, to provide a direct comparison, I rely on the model-simulated time series to test the interaction effect in my model.

Following the analysis on Observation 3, I run the regression:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t, \tag{1.33}$$

where R^e_{t+N} represents the future excess returns over the next N months, Sent_t represents model-simulated investor sentiment, and W_t/D_t proxies for the wealth to dividend ratio. These two variables are also the underlying state variables in my model.

[Place Table A.10 about here]

Table A.10 reports the regression results based on the model-simulated data. The coefficient for our interaction term is statistically significant. Therefore, the interaction effect is confirmed in my theoretical model.

Time-varying Impact of Investor Sentiment

Based on my behavioral model, investor sentiment has time-varying impacts on the market. To formally test this model prediction, I empirically test the predictive power of investor sentiment conditional on different market impacts of extrapolators. In section 1.4, I already connect the market impact of extrapolators with mispricing. However, the predictive direction of investor sentiment still remains ambiguous.

To provide a more detailed analysis, I present the predictive coefficient of investor sentiment $b_t = [b + d \times W_t/D_t]$ for future 12-month returns in Table A.11. Specifically, I follow Aiken et al. (1991) and present the predictive pattern of investor sentiment conditional on different wealth levels. In the remaining analysis, I define the high wealth level as two standard deviations above the mean of the wealth to dividend ratio, and the low wealth level as two standard deviations below the mean of the wealth to dividend ratio.

[Place Table A.11 about here]

The results coincide with my theoretical model predictions. When the wealth level of extrapolators is high, one unit increase in Gallup-measured investor sentiment is followed by a significant 16.2% decrease in the future 12-month returns. Replace Gallup survey with Psentiment, the conditional predictive pattern is even stronger: one unit in Psentiment, in general, leads to a tremendous return decline of 26.7% in the next 12 month. If extrapolators have high market power, then they are driving the market and investor sentiment negatively predicts future returns because it reverts to its mean in the future. Such negative predictive pattern is hardly consistent with rational asset pricing models.

Moreover, when the wealth level of extrapolators is low, investor sentiment has a strong positive predictive power. With Gallup survey and Psentiment, the coefficients of the conditional coefficients b_t are 0.359 and 0.479 respectively. At first glance, positive coefficient might be inconsistent with the irrational story of investor sentiment. However, my model indicates they are consistent. The key observation from my model is that, when extrapolators' wealth level is low, the main market power comes from fundamental investors. In this situation, high sentiment indicates the market is going through an upward market correction, leading to positive market returns in the future. Similarly, low sentiment implies that fundamental investors are correcting overvalued asset prices downwards, causing negative market returns.

The regression results based on model simulations in Table A.12 also confirms the conditional predictive pattern of investor sentiment in the model— they are very close to the predictive pattern in the empirical data.

It is worth pointing out that, conditional on the wealth level of extrapolators, investor sentiment has significant predictive power over future market returns. By contrast, without considering the market impact of extrapolation, investor sentiment does not have significant predictive power over future market returns, as shown in Table A.3.

Comparison between Interaction Effect and Degree of Extrapolation

Both my theoretical and empirical results support that the predictive power of investor sentiment on future returns is time-varying. A similar pattern is recently recorded in Cassella and Gulen (2015). In their paper, they define the degree of extrapolation (DOX) as the relative weight extrapolators place on recent-versus-distant past returns when forming their sentiment on future stock market returns.

Their key empirical findings are that DOX is time-varying and that predictability of log price-dividend ratio strongly depends on the DOX. Higher (lower) log price to dividend ratio indicates assets are undervalued (overvalued). When DOX is higher and extrapolators rely more on recent returns, few recent returns will change both their sentiment and irrational demand dramatically, which induces a faster correction of mispricing. These two sets of conditional predictive patterns seem to be fundamentally similar and connected to each other, since both answers one critical question in the asset pricing theory: when will a mispriced asset experience a correction?

[Place Figure A.7 about here]

Specifically, their explanation to this question relies on the aggregate belief transition pattern of extrapolators (DOX), which could either depend on the time-varying participation rate of extrapolators or the time-varying return extrapolation pattern at the individual investor level. With higher DOX, investors easily change their sentiment based on recent returns, which leads to a faster mispricing correction speed and stronger predictive pattern of log price to dividend ratio.

As a comparison, the interaction effect based on my results proves the importance of the time-varying impact of of investor sentiment. When their wealth level is high, investor sentiment leads to higher degree of asset mispricing, which leads to a strong mispricing correction in the future and a significant predictive power of investor sentiment. Two explanations should have overlapped common underlying mechanism.

Given this similarity, a natural hypothesis is that the DOX in their paper might be very correlated with the interaction effect in my model. To check it, I plot both DOX and the interaction effect in Figure A.7. As we can see from the figure, two measurements are indeed highly synchronous. For instance, between 1996 and 2000 and before the Internet bubble bursts, the DOX measurements rise from 0.35 to 0.9. Within the same sample period, the market impact in my model also increases. Therefore, my model provides a formal justification for the time-varying DOX: variations in DOX is largely due to the time-varying impact of extrapolators in the market.

Despite the similarities, there are some new predictions using the interaction term in my model. For example, as I have shown, the time-varying impact of extrapolators implies a novel predictive power of *investor sentiment* on future returns.

Household Leverage and Future Consumption Growth

My model also sheds lights on the recent findings in the household literature. Mian and Sufi (2009) document that household leverage ratio before the Great Depression is a strong and negative predictor of future consumption growth rate. Moreover, they document that the increase in household leverage is largely due to the biased expectations, and belief mistakes lead to a decrease in future income and declines in consumption growth rate.

My model provides a formal justification for this mechanism. In my model, extrapolators form biased beliefs based on past returns, consequently, they become overly optimistic and take excessive leverage. However, their high expectations will not continue for long and consequently, the wealth level of extrapolators in the future will decline. Since the optimal consumption rule is linearly depending on the wealth level, their future consumption rate will also decline as the result.

My model simulation results confirm this. In Table A.13, I report the predictive regression of future consumption growth using current leverage ratio as the predictor. The predictive pattern is significantly negative, and remains stable when forecasting horizons span from 1 to 6 quarters.

Additional Model Implications

Bubbles

Extrapolative beliefs help generate excessive optimism and pessimism in the market, consistent with the anecdotal descriptions of market boom and bust. Specifically, episodes of good returns lead extrapolators to become overly optimistic about the future expected returns, consistent with the high sentiment during booms measured in the survey. Conversely, a sequence of low returns will significantly disappoint extrapolators, leading to a drop in investor sentiment during recessions.

Moreover, what distinguishes this model from others in explaining the bubble phenomena is the role of the wealth dynamics during the bubble and crash episodes. In my model, investor sentiment has large impacts on the dynamics of the wealth process, especially during bubbles when the wealth level of extrapolators is high. High sentiment during bubbles induces investors to lever up, which pushes up the total dollar demand and leads to overvaluations. However, investor sentiment could not last long since extrapolators get disappointed by future shocks, and future dollar demand for the risky asset goes down. The decrease in risky asset price not only makes investors pessimistic but also reduces the overall market impact of extrapolators, which makes fundamental investors easier to correct mispricing and leads to the market crash.

[Place Figure A.8 about here]

Countercyclical Sharpe ratios and Negative Risk Premium

My model with extrapolation naturally generates a procyclical perceived Sharpe ratio, since investor beliefs are mainly driven by recent realized returns in the past: during market booms, investors become optimistic about future returns and the perceived Sharpe ratio increases. Conversely, in the recessions, investors become overly pessimistic and perceive low Sharpe ratio due to low past returns. This procyclicality resonates with the finding in Amromin and Sharpe (2013) where they document a procyclical Sharpe ratios using data from Michigan Surveys of Consumer Attitudes.

Compared to rational expectation models, the behavioral model in my paper has two distinct probability measures. While Sharpe ratio is procyclical under the *perceived* probability measure, my model generates a *countercyclical* Sharpe ratio under the *objective* probability measure. This countercyclicality closely relates to the fact that investor sentiment reverts: with high level of investor sentiment, extrapolators become overly optimistic and perceive high Sharpe ratio, but the Sharpe ratio under objective probability measure is low in the future. This countercyclical Sharpe ratio is consistent with the empirical pattern documented in Lettau and Ludvigson (2010).

Debt Accumulation and Procyclical Leverage

Many studies document the strong cyclicality of debt accumulation. For instance, Reinhart and Rogoff (2009) document that household debt accumulation speeds up during market booms and often leads to severe financial crashes. The extrapolation in my model capture this dynamic pattern for debt. During market booms, extrapolators become overly optimistic by extrapolating the past returns and consequently they take lever up to buy the risky asset. Conversely, during recessions, extrapolators become overly pessimistic and therefore have low leverage level. In addition, consistent with many other models such as He and Krishnamurthy (2008), this model also captures the procyclical leverage. However, different from previous studies, the procyclicality in my model arises from extrapolation.

1.5 The Role of Extrapolation: A Rational Benchmark Model

To correctly evaluate the role of extrapolation, in this section, I impose extrapolators to have fully correct beliefs by setting $\theta = 0$, and compare the model implications with previous behavioral model settings. Such comparison implies that extrapolation is important for understanding investor beliefs and asset price dynamics.

The Economy

In this rational benchmark model, I follow previous setting and consider a continuous-time economy with two types of assets: one risk-free asset and one risky asset. The underlying dividend process D_t for the risky asset still follows a geometric Brownian motion:

$$dD_t/D_t = g_D dt + \sigma_D d\omega_t. \tag{1.34}$$

The dividend process is driven by ω_t, a one-dimensional Weiner process under the true probability measure observed by outside econometricians. The equilibrium price P_t now follows

$$dP_t/P_t = \bar{g}_{P,t} dt + \bar{\sigma}_{P,t} d\omega_t. \tag{1.35}$$

The growth rate $g_{P,t}$ and volatility term $\sigma_{P,t}$ are both endogenously determined in the equilibrium.

Investors

There are two types of investors in the rational benchmark model: extrapolators and fundamental investors. Different from the case of behavioral model, extrapolators have correct beliefs about the asset dynamics and optimize their portfolio choice under the objective probability measure accordingly. Fundamental investors, on the other end, still serves as the counteracting forces in the market and trade aggressively whenever asset prices deviate from fundamental values. Similarly, I assume that fundamental investors make up a fraction of $1 - \mu$ and extrapolators make up a fraction of μ.

Extrapolators

As in the behavioral model, I introduce logarithmic utilities for extrapolators and assume that each of them, indexed by j, is infinitesimal and therefore acts as price-taker in the market. They solve the following optimization problem:

$$\max_{\{C^j_{t+s}\}_{s\geq 0}} \mathbb{E}_t[\int_t^\infty e^{-\rho s} \ln C^j_s ds], \tag{1.36}$$

and subject to budget constraint

$$dW_t^j = -C_t^j dt + rW_t^j dt + \alpha_t^j W_t^j [\bar{g}_{P,t} dt + \frac{D_t}{P_t} dt - r dt + \bar{\sigma}_{P,t} d\omega_t].$$

Most of the properties of the optimal strategies in the behavioral model—that the optimal risky position depends on instantaneous equity premium and volatility, that the investment decision is myopic and ignores the hedging demand—carry over to the rational benchmark case. However, one key difference is that now extrapolators make investment decisions under the true probability measure.

Following the standard Merton's approach, we get the optimal consumption and portfolio rule for extrapolators:

PROPOSITION 3 *In the rational benchmark economy, the extrapolators with the objective function in equation (1.36) and budget constraint in equation (1.37) will optimally consume and invest at time t according to the following strategies:*

$$C_t^j = \rho W_t^j \tag{1.37}$$

and

$$\alpha_t^j = \frac{\bar{g}_{P,t}(x_t) + \bar{l}^{-1}(x_t) - r}{\bar{\sigma}_{P,t}^2(x_t)}, \tag{1.38}$$

where the price to dividend ratio $\bar{l} \equiv \frac{P_t}{D_t}$ depends on the wealth to dividend ratio $x_t \equiv \frac{W_t}{D_t}$. In other words, the equilibrium depends on state variable x_t.

Proof: See Appendix A.2.

Equilibrium

Definition of Equilibrium In this rational benchmark model, the equilibrium satisfies the following conditions:

(i) Rational investors follow the optimal consumption and portfolio choice described in proposition (3).

(ii) Fundamental investors follow their trading strategy defined in equation (1.19).

(iii) The following market clearing condition holds:

$$\mu\alpha W_t + (1-\mu)Q_t = P_t. \tag{1.39}$$

For the rational benchmark model, I still focus on the symmetric equilibrium where each extrapolator is endowed with the same level of wealth and follows identical strategies. The equilibrium depends both on the wealth to dividend ratio x_t and the latent state variable S_t. Therefore, I rely on solving a partial differential equation to obtain the solution for the fixed-point problem. As verified later, the price to dividend ratio is monotonically increasing in both x_t and S_t.

PROPOSITION 4 *In the symmetric equilibrium, the price-dividend ratio $\bar{l}$ is a function of the current states x_t, and satisfies the following partial differential equation:*

$$\frac{c_0\bar{l} - c_1}{x_t} = \frac{\bar{g}_{P,t} + \bar{l}^{-1} - r}{\bar{\sigma}^2_{P,t}}, \tag{1.40}$$

where $\sigma_{P,t}$ is also a function of x_t and satisfies

$$\bar{\sigma}_{P,t} = \sigma_D \frac{1 - \frac{\bar{l}_x}{\bar{l}} x_t}{1 - \frac{\bar{l}_x}{\bar{l}} x_t \alpha_t} \tag{1.41}$$

and $l(x_t)$ satisfies the following boundary conditions:

$$\lim_{x_t \to 0} \bar{l} = \frac{c_1}{c_0}, \tag{1.42}$$

$$\lim_{x_t \to 0} = \lim_{x \to 0} \frac{g_D + \frac{c_1}{c_0} - r}{\sigma^2_D}, \tag{1.43}$$

and

$$\lim_{x_t \to \infty} \bar{l} = (r - \bar{g}_{P,t})^{-1}, \tag{1.44}$$

where $c_0 = \frac{k+1-\mu}{k\mu}$ and $c_1 = \frac{1-\mu}{k\mu(r-g_D)}$ are both constants.

In addition, the dynamics of x_t follows

$$dx_t = g_{x,t}(x_t)dt + \sigma_{x,t} d\omega_t, \tag{1.45}$$

where

$$g_{x,t} = x_t(r - \rho - g_D + \sigma^2_D) + (c_0 l - c_1)(\bar{g}_{P,t} + l^{-1} - r - \sigma_D \bar{\sigma}_{P,t}) \tag{1.46}$$

$$\sigma_{x,t} = (c_0 l - c_1)\bar{\sigma}_{P,t} - x_t \sigma_D. \tag{1.47}$$

Proof: See Appendix A.2.

The volatility pattern in equation (1.41) essentially reflects the wealth effect and the portfolio rebalancing effect, as documented in previous studies (Xiong (2001) and Jin (2015)). When α_t is greater than one, extrapolators borrow money at the risk-free rate r and lever up their positions in the risky asset. Once positive dividend shocks arrive, the increase in the risky asset leads to an increase in their wealth level and hence their absolute risk aversion; at the same time, the leverage ratio of extrapolators is further pushed down, which induce them to buy more risky assets. This wealth effect amplifies the initial dividend shocks, which makes $\bar{\sigma}_{P,t}$ larger than σ_D. Conversely, when α_t is less than one, the portfolio rebalancing effect dominates: once positive dividend shocks arrive, the increase in the risky asset leads to an increase in their wealth level and hence their absolute risk aversion; at the same time, the leverage ratio of extrapolators is further pushed up, which induce them to delever. Together, the portfolio rebalancing effect dampens the initial shocks and make $\bar{\sigma}_{P,t}$ less than σ_D.

In solving the ordinary equation in proposition (4), I still rely on numerical methods. Similarly, I apply the following monotonic transformation:

$$z_t = \frac{x_t - \xi}{x_t + \xi}, \tag{1.48}$$

where ξ is a positive constant.

Calibrated Model for the Benchmark Case

In this section, I solve the rational benchmark model using the projection method. I use the following parameter values: $r = 4\%$, $g_D = 1.5\%$, $\sigma_D = 10\%$, $\rho = 2\%$, $k = 0.5$ and $\mu = 0.5$. The selected asset and utility parameters are reported in the upper two panels in Table A.6.

[Place Table A.6 about here]

[Place Figure A.4 about here]

Price-Dividend Ratio: Benchmark Case

With the above parameters, I report the model solution in Figure A.4. In the upper panel, I report the price-dividend ratio $\bar{l}(x_t)$ as a function of the wealth-to-dividend ratio. As we can see, $\bar{l}(x_t)$ is an monotonically increasing function of x_t. When x_t

equals zero, the risky asset is hold by the fundamental investors and the risky asset price is at its fundamental value, which is relatively low. Increase in the wealth level of extrapolators helps push up the total dollar amount of the risky asset, which in turn increases the price-to-dividend ratio $\bar{l}(x_t)$. With the given parameters, the price to dividend ratio lies between 20 to 40, which is fairly consistent with the empirical moments in the literature.[19]

Optimal Portfolio Weight: Benchmark Case

The portfolio weight α_t for extrapolators is monotonically decreasing in z_t. Recall that the price to dividend ratio positively depends on the wealth to dividend ratio. Therefore, as z_t increases, the dividend yield naturally goes down, which makes the investment set less attractive to the extrapolators. Consequently, extrapolators want to decrease their position in the risky asset, reducing the optimal portfolio α_t.

Volatility: Benchmark Case

I also report the volatility $\bar{\sigma}_{P,t}(x_t)$ in the equilibrium as a function of the wealth to dividend ratio. As discussed before, the reverse-S shaped curve is closely related to extrapolators' portfolio choice α_t. When the transformed wealth-to-dividend ratio z_t is slightly below 1, the wealth fraction they invested in the risky asset grows from zero to slightly above. When α_t is positive but small, faced with exogenous shocks, investors will rebalance their portfolio and dampen the overall volatility. As a result, equilibrium $\bar{\sigma}_{P,t}$ is less than the magnitude of fundamental shocks σ_D and it can be as low as 9.55%. As z_t decreases and the portfolio weight α_t gradually becomes greater than one, a negative dividend shock will cause their leverage ratio to increase and lead extrapolators to reduce their risky asset position, which helps amplify the initial shocks and pushes up volatility $\bar{\sigma}_{P,t}$. When z_t goes down to -1, extrapolators lose their impact and fundamental investors gradually dominate the market, which makes the volatility gradually converge to σ_D.

Extrapolators' Beliefs

Extrapolators in the rational benchmark model have fully correct beliefs about future risky asset returns. As a result, their belief pattern is *not* consistent with the observed dynamics for investor sentiment. To provide a formal justification, I still focus on the model simulation results.

[19] Beeler and Campbell (2012) report the average of log(PD) ratio of around 28.8.

[Place Table A.14 about here]

First, I follow Greenwood and Shleifer (2014) and run the following regression:

$$\mathbb{E}[R_{t+12}] = a_0 + a_1 X_t + \epsilon_t, \tag{1.49}$$

where $\mathbb{E}[R_{t+12}]$ represents the expectation of future twelve-month returns under the true probability measure, X_t represent either past accumulative 12-month returns or the current log price to dividend ratio.

I report the regression results in Table A.14. The belief pattern for extrapolators in the rational benchmark model is not consistent with the extrapolation pattern in the survey: when the current log price to dividend ratio is high, extrapolators with correct beliefs would anticipate that the market is overvalued and then anticipate a decline in the risky asset price.

The Interaction effect

[Place Table A.15 about here]

Moreover, the rational benchmark model does not capture the strong interaction effect between investor sentiment and the wealth level of extrapolators in determining the mispricing. Specifically, I test the Observation 3 by running the same regression in equation (1.4) using the simulated series from the rational benchmark model. The regression results are reported in Table A.15. The coefficients for each of the regressors are all statistically significant but positive because extrapolators always hold correct beliefs.

Household Leverage and Future Consumption Growth

The absence of extrapolation also drives away the negative association between the household leverage and the future consumption growth. In the rational model, households have fully correct beliefs, therefore, they increase their exposure to the risky asset when future returns are high. This leads to a positive correlation between the leverage ratio and future consumption growth rate, which is inconsistent with the pattern documented in Mian and Sufi (2009). As shown in Table A.16, when regressing future consumption growth rate on current leverage ratio, the coefficient α_t is significantly positive.

In summary, with fully correct beliefs, rational models miss most of the important model implications in the behavioral model. Therefore, extrapolation, or equivalently investor beliefs, plays a key role in explaining a set of empirical patterns. The excessive optimism and pessimism induced by extrapolation not only help understand the anecdotal fluctuations in investor sentiment but also helps explain the asset mispricings and consequently the predictability of returns in the aggregate stock market. Moreover, it helps shed light on the household leverage-taking behaviors and corresponding real consequences to the economy. All these facts point to the importance of belief-based investor sentiment in understanding both asset pricing facts and real economic activities.

1.6 Concluding Remarks

In this paper, I have developed a model that focuses on investor sentiment and analyzed its time-varying impact on the market. In my model, there are two types of investors: fundamental investors who trade as aggressive arbitragers and extrapolators who form investor sentiment by over-extrapolating past realized returns. The model implies dynamic connections between investor sentiment and market mispricing through the market impact of extrapolators. When the wealth level of extrapolators is high and therefore have a large impact on the equilibrium, investor sentiment directly causes the market mispricing and negatively predicts future market returns due to the fact that investor sentiment reverts quickly to its mean. Conversely, if the wealth level of extrapolators is low—the situation in which fundamental investors dominate the market and extrapolators have trivial impacts on the market—investor sentiment positively predicts future market returns since it reflects an undergoing market correction.

There are at least two avenues for future work. First, a fully fledged model that simultaneously incorporates extrapolators, fundamental investors and rational investors are in demand. Although intuition tells me that the existence of rational investors will only reinforce my current results of the behavioral model due to their "ride on the bubble" motives, a formal model will help verify my conjecture. Second, my model implies a strong but general connection between investor sentiment and mispricing by the time-varying impact of extrapolators, and I find empirically support using the stock market as an example. However, the connection implied in my model should be a general pattern that applies to a broader set of asset classes. Empirically testing such connections in other asset classes will deepen our understanding of the impact of investor sentiment.

Chapter 2

ASSET PRICING WITH RETURN EXTRAPOLATION

2.1 Introduction

In financial economics, there is growing interest in "return extrapolation", the idea that investors' beliefs about an asset's future return are a positive function of the asset's recent past returns. Models with return extrapolation have two appealing features. First, they are consistent with survey evidence on the beliefs of real-world investors.[1] Second, they show promise in matching important asset pricing facts, such as volatility and predictability in the aggregate market, momentum and reversals in the cross-section, and bubbles (Barberis et al. (2015); Barberis et al. (2017); Hong and Stein (1999)).

One limitation of existing models of return extrapolation, however, is that they can only be compared to the data in a *qualitative* way. Early models, such as Cutler et al. (1990a) and DeLong et al. (1990), highlight the conceptual importance of return extrapolation, but they are not designed to match asset pricing facts quantitatively. Barberis et al. (2015) is a dynamic consumption-based model that tries to make sense of both survey expectations and aggregate stock market prices. However, the simplifying assumptions in the model make it difficult to evaluate the model's fit with the empirical facts. For instance, their model adopts a framework with constant absolute risk aversion (CARA) preferences and a constant interest rate. Under these assumptions, many ratio-based quantities that we study in asset pricing (e.g., the price-dividend ratio) do not have well-defined distributions in the model and therefore do not have properties that match what we observe in the data.

In this paper, we propose a new model of aggregate stock market prices based on return extrapolation that overcomes this limitation. The goal of the paper is to see if the model can match important facts about the aggregate stock market when the agent's beliefs are calibrated to match survey expectations of investors, and to compare the model in a quantitative way to rational expectations models of the stock

[1] Among others, Vissing-Jorgensen (2004), Bacchetta et al. (2009), Amromin and Sharpe (2013), Greenwood and Shleifer (2014), Koijen et al. (2015), and Kuchler and Zafar (2016) document that many individual and institutional investors have extrapolative expectations: they believe that the stock market will continue rising in value after a sequence of high past returns, and that it will continue falling in value after a sequence of low past returns.

market.

We consider a Lucas economy in continuous time with a representative agent. The Lucas tree is a claim to an aggregate consumption process which follows a geometric Brownian motion. Besides the Lucas tree, there are two tradeable assets in the economy: the stock market and an instantaneous riskless asset. The stock market is a claim to an aggregate dividend process whose growth rate is positively correlated with consumption growth. The riskless asset is in zero net supply with its interest rate determined in equilibrium. The representative agent has Epstein-Zin preferences and extrapolative beliefs. She perceives that the expected growth rate of stock market prices is governed by a switching process between two regimes. If recent price growth of the stock market has been high, the agent thinks it is likely that a high-mean price growth regime is generating prices and therefore forecasts high price growth in the future. Conversely, if recent price growth has been low, the agent thinks that it is likely that a low mean-price growth regime is generating prices and therefore forecasts low price growth in the future.

We calibrate the agent's beliefs to match the survey expectations of investors studied in Greenwood and Shleifer (2014). Specifically, we set the belief-based parameters so that, for a regression of the agent's expectations about future stock market returns on past twelve-month returns, the model produces a regression coefficient and a t-statistic that match the empirical estimates from surveys. Our parameter choice also allows the agent's beliefs to match the survey evidence on the relative weight investors put on recent versus distant past returns when forming beliefs about future returns. Overall, the model generates a degree of extrapolative expectations for the agent that matches the empirical magnitude. With the agent's beliefs disciplined by survey data, the model quantitatively matches important facts about the aggregate stock market: it generates significant excess volatility and predictability of stock market returns, a high equity premium, a low and stable interest rate, as well as a low correlation between stock market returns and consumption growth.

We now explain the intuition for the model's implications, starting with excess volatility. The model generates significant excess volatility from the interaction between return extrapolation and Epstein-Zin preferences. Suppose that the stock market has had high past returns. In such a case, return extrapolation leads the agent to forecast high future returns. Under Epstein-Zin preferences, the separation between the elasticity of intertemporal substitution and risk aversion gives rise to a strong intertemporal substitution effect. Therefore, the agent's forecast of high future returns leads her to push up the current price significantly, generating excess volatility.[2]

The mechanism described above for generating excess volatility, together with a strong degree of mean reversion in the agent's expectations about stock market returns, produces the long-horizon predictability of stock market returns that we observe in the data. The agent's beliefs mean-revert, for two reasons. First, by assumption, the agent believes that the expected growth rate of stock market prices tends to switch over time from one regime to the other: the agent believes that her expectations about stock market returns will mean-revert. Second, the agent's return expectations mean-revert *faster* than what she perceives: when the agent thinks that the future price growth is high, future price growth tends to be low endogenously, causing her return expectations to decrease at a pace that exceeds her anticipation. As a result, following periods with a high price-dividend ratio—this is when the high past price growth of the stock market pushes up the agent's expectation about future returns and hence her demand for the stock market—the agent's return expectation tends to revert back to its mean, giving rise to low subsequent returns and hence the predictability of stock market returns using the price-dividend ratio.

Next, we turn to the model's implications for the equity premium. Three factors affect the long-run equity premium *perceived* by the agent. First, because the agent is risk averse, excess volatility causes her to demand a higher equity premium. Second, return extrapolation gives rise to perceived persistence of the aggregate dividend process, which, under Epstein-Zin preferences, is significantly priced, pushing up the perceived equity premium. Finally, the separation between the elasticity of intertemporal substitution and risk aversion helps to keep the equilibrium interest rate low and hence keep the equity premium high. Furthermore, the *true* long-run equity premium is significantly higher than the perceived one. In the model, the agent's beliefs mean-revert faster than what she perceives. Given this, she underestimates short-term stock market fluctuations and hence the risk associated with the stock market. In other words, if an infinitesimal rational agent, one that has

[2] A feedback loop emerges from this mechanism. If current returns are high, that makes the agent think that future returns will also be high, which leads her to push up prices, increasing current returns further, and so on. In general, there is a danger that this feedback loop could "explode". Nonetheless, in the model, we assume the agent believes that the expected growth rate of stock market prices tends to switch over time from one regime to the other; she therefore believes her optimism will decline in the future. As a result, the cumulative impact of the feedback loop on investor expectations and asset prices is finite; the model remains stable. Models like Cutler et al. (1990a) and Barberis et al. (2015) instead introduce fully rational investors in order to counteract the behavioral investors and preserve equilibrium.

the same preferences as the behavioral agent but holds rational beliefs, enters our economy, she would have demanded a higher equity premium: the model produces a true average equity premium that is substantially higher than the perceived equity premium.

Finally, the model generates low interest rate volatility and a low correlation between stock market returns and consumption growth. In the model, the agent separately forms beliefs about the dividend growth of the stock market and about aggregate consumption growth. Here, we assume that the bias in the agent's beliefs about consumption growth derives only from the bias in her beliefs about dividend growth. Given the low observed correlation between consumption growth and dividend growth, the bias in the agent's beliefs about consumption growth is small, consistent with the lack of empirical evidence that investors have extrapolative beliefs about consumption growth. The agent's approximately correct beliefs about consumption growth allow the model to generate low interest rate volatility. They also imply that the agent's beliefs about stock market returns—they co-move strongly with her beliefs about dividend growth—are not significantly affected by fluctuations in consumption growth, giving rise to the low observed correlation between stock market returns and consumption growth.

Although our model is based on return extrapolation, it yields direct implications for cash flow expectations. When the past price growth of the stock market has been high, this has a positive effect not only on the agent's beliefs about future returns, but also on her beliefs about future dividend growth; indeed, her expectations about dividend growth rise at a pace that exceeds her expectations about future returns.[3] Given this, a Campbell-Shiller decomposition using the agent's *subjective* expectations about stock market returns and dividend growth shows that changes in subjective expectations about future dividend growth explain most of the volatility of the price-dividend ratio. This model implication is consistent with the recent empirical findings of de la O and Myers (2017): they find that during periods when the price-dividend ratio of the U.S. stock market is high, investors' expectations of future dividend growth are much higher than their expectations of future stock market returns. As a result, changes in investors' subjective expectations of future dividend growth explain the majority of stock market movements. Importantly, the fact that prices in our model are mainly correlated with cash flow expectations is a consequence of the Campbell-Shiller accounting identity; this statement is about

[3] We provide a detailed explanation of this finding in Sections 2.2 and 2.3.

correlation, not about causality. The agent's *return* expectations determine her cash flow expectations and are the cause of price movements. Given this, our model simultaneously explains the empirical findings of de la O and Myers (2017) on cash flow expectations and the empirical findings of Greenwood and Shleifer (2014) on return expectations. At the same time, the model also explains the empirical findings of Cochrane (2008) and Cochrane (2011) that, under rational expectations, the variation of the price-dividend ratio comes primarily from discount rate variation.

Our model also points to some challenges: when calibrated to the survey expectations data, the model predicts a persistence of the price-dividend ratio that is significantly lower than its empirical value. In other words, to match the empirical persistence of the price-dividend ratio, investors need to form beliefs about future returns based on many years of past returns. However, the available survey evidence suggests that they focus on just the past year or two.

After presenting the model, we compare it to the standard rational expectations models of the aggregate stock market. As with the habit formation model of Campbell and Cochrane (1999), the long-run risks models of Bansal and Yaron (2004) and Bansal et al. (2012), and the rare disasters models of Barro (2006), Gabaix (2012), and Wachter (2013), our model is developed in a Lucas economy with a representative agent. This model structure allows for a direct comparison between our model and models with rational expectations. Here, we focus on the long-run risks models because these are the models most related to ours. We document some different implications.

First, our model differs from the long-run risks models in the way the agent forms expectations. In Bansal and Yaron (2004), dividend growth and consumption growth share a stochastic yet persistent component. High past stock market returns are typically caused by positive shocks to this common component, which, given its persistence, implies high dividend growth and hence high raw returns moving forward. That is, the agent in Bansal and Yaron (2004) has extrapolative beliefs about future raw returns. At the same time, precisely because dividend growth and consumption growth share a persistent component, the comovement between the agent's beliefs about stock market returns—these rationally drive returns—and her beliefs about consumption growth—these determine the interest rate in equilibrium—is high. That is, when the raw returns are high, the interest rate is also high. As a result, the agent does not hold extrapolative beliefs about *excess* returns. In our model, however, the agent extrapolates past stock market returns, but extrapolates past consumption growth much less: the comovement between her beliefs about stock market returns and her beliefs about consumption growth is low. Therefore, the agent has extrapolative beliefs about *both* raw returns and excess returns.

Furthermore, these two models yield different implications for asset prices. Our model produces an equity premium that does not vary significantly with changes in the elasticity of intertemporal substitution. On the contrary, long-run risks models cannot generate a high equity premium with a low elasticity of intertemporal substitution. To see this model difference, we first note that the agent's beliefs in our model are much less persistent than the stochastic component of dividend and consumption growth in Bansal and Yaron (2004), allowing the equilibrium interest rate and hence the equity premium in our model to be less responsive to changes in the elasticity of intertemporal substitution. At the same time, the perceived dividend growth in our model depends more strongly on the agent's beliefs about the price growth of the stock market, pushing up the perceived equity premium; as a comparison, dividend growth in Bansal and Yaron (2004) depends much less on the stochastic growth component. Finally, the true long-run equity premium in our model is above the perceived one, allowing the equity premium to be high even when the elasticity of intertemporal substitution is low.

Our paper adds to a new wave of theories that attempt to understand the role of belief formation in driving the behavior of asset prices and the macroeconomy (Fuster et al. (2011); Gennaioli et al. (2012); Choi and Mertens (2013); Alti and Tetlock (2014); Hirshleifer et al. (2015); Barberis et al. (2015); Jin (2015); Ehling et al. (2015); Vanasco et al. (2015); Pagel (2016); Collin-Dufresne et al. (2016a,b); Greenwood et al. (2016); Glaeser and Nathanson (2017); DeFusco et al. (2017); Bordalo et al. (2018)). Our paper also adds to a growing literature on the source of stock price movements (Cochrane (2008); Cochrane (2011); Chen et al. (2013); de la O and Myers (2017)). Furthermore, it is related to theories of model uncertainty and ambiguity aversion such as Bidder and Dew-Becker (2016). These models typically assume that agents learn about the dynamic properties of the consumption process or the dividend process. Therefore, they are closely linked to the fundamental extrapolation models in the behavioral finance literature, but do not match survey evidence on return expectations. Finally, our paper speaks to the debate between Bansal et al. (2012) and Beeler and Campbell (2012) which focuses on excess predictability: the notion that, in the long-run risks literature, future consumption growth and dividend growth are excessively predicted by current variables such as the price-dividend ratio and the interest rate (see also, Collin-Dufresne et al. (2016b)). Our model does not give rise to excess predictability: return extrapolation in the model only generates *perceived* but not true persistence in consumption growth and dividend growth.

The paper proceeds as follows. In Section 2.2, we lay out the basic elements of the model and characterize its solution. In Section 2.3, we parameterize the model and examine its implications in detail. Section 2.4 provides a comparative statics analysis. Section 2.5 discusses differences between our model and rational expectations models. Section 2.6 further compares the model to a model with fundamental extrapolation, the notion that some investors hold extrapolative expectations about the future dividend growth of the stock market. Section 2.7 concludes and suggests directions for future research. All technical details are in the Appendix.

2.2 The Model

In this section, we first describe the model setup and characterize its solution, and then derive equilibrium quantities that are important for understanding the implications of the model.

Model setup

Asset space.—We consider an infinite-horizon Lucas economy in continuous time with a representative agent. The Lucas tree is a claim to an aggregate consumption process. We assume it is a geometric Brownian motion

$$dC_t/C_t = g_C dt + \sigma_C d\omega_t^C, \tag{2.1}$$

and we denote the price of the Lucas tree at time t as P_t^C.

Besides the Lucas tree, there are two other tradeable assets in the economy; they are the main focus of our analysis. The first asset is the stock market which is a claim to an aggregate dividend process given by

$$dD_t/D_t = g_D dt + \sigma_D d\omega_t^D; \tag{2.2}$$

we denote the price of the stock market at time t as P_t^D.[4] Both ω_t^D and ω_t^C are

[4]Since the aggregate consumption process in the model is exogenous, the dividend payment from the stock market does not further affect consumption. As a result, we can think of the stock market as an asset in zero net supply with a shadow price determined in equilibrium. This is a common assumption adopted by many other consumption-based models such as Campbell and Cochrane

standard Brownian motions. We assume that the instantaneous correlation between dD_t and dC_t is ρ: $\mathbb{E}_t[d\omega_t^D \cdot d\omega_t^C] = \rho dt$. The second asset is an instantaneous riskless asset. This asset is in zero net supply, and its interest rate r_t is determined in equilibrium.

Agent's preferences.—We follow Epstein and Zin (1989, 1991) and assume that the agent has a recursive intertemporal utility

$$U_t = \left[(1 - e^{-\delta dt})C_t^{1-\psi} dt + e^{-\delta dt} \left(\mathbb{E}_t^e[\tilde{U}_{t+dt}^{1-\gamma}]\right)^{(1-\psi)/(1-\gamma)}\right]^{1/(1-\psi)}, \qquad (2.3)$$

where δ is the subjective discount rate, $\gamma > 0$ is the coefficient of relative risk aversion, and $\psi > 0$ is the reciprocal of the elasticity of intertemporal substitution. When ψ equals γ, (2.3) reduces to power utility. The superscript "e" is an abbreviation for "extrapolative" expectations: the certainty equivalence in (2.3) is computed under the representative agent's subjective beliefs, which, as we specify later, incorporate the notion of return extrapolation.

The subjective Euler equation, or the first-order condition, is

$$\mathbb{E}_t^e \left[e^{-\delta(1-\gamma)dt/(1-\psi)} \left(\frac{\tilde{C}_{t+dt}}{C_t}\right)^{-\psi(1-\gamma)/(1-\psi)} \tilde{M}_{t+dt}^{(\psi-\gamma)/(1-\psi)} \tilde{R}_{j,t+dt} \right] = 1. \qquad (2.4)$$

Here $\tilde{M}_{t+dt}$ is the gross return on the optimal portfolio held by the agent from time t to time $t + dt$. In a Lucas economy with a representative agent, the optimal portfolio in equilibrium is the Lucas tree itself, and therefore

$$\tilde{M}_{t+dt} = \frac{\tilde{P}_{t+dt}^C + \tilde{C}_t dt}{P_t^C} = \frac{\tilde{P}_{t+dt}^C + \tilde{C}_{t+dt} dt}{P_t^C} + o(dt). \qquad (2.5)$$

On the other hand, $\tilde{R}_{j,t+dt}$ is the gross return on *any* tradeable asset j in the market from time t to time $t + dt$; as mentioned above, the two tradeable assets we focus on are the stock market and the riskless asset.

Agent's beliefs.—We now turn to the key part of the model: the representative agent's beliefs about stock market returns. According to surveys, real-world investors form beliefs about future stock market returns by extrapolating past returns (Vissing-Jorgensen (2004); Bacchetta et al. (2009); Amromin and Sharpe (2013); Greenwood and Shleifer (2014); Koijen et al. (2015); Kuchler and Zafar (2016)). One natural

(1999) and Barberis et al. (2001).

way to capture this notion of return extrapolation is through a regime-switching model. Specifically, we suppose that the agent believes that the expected growth rate of stock market prices is governed by $(1-\theta)g_D + \theta\tilde{\mu}_{S,t}$, where $\tilde{\mu}_{S,t}$ is a latent variable which switches between a high value μ_H in a high-mean price growth regime H and a low value μ_L ($\mu_L < \mu_H$) in a low-mean price growth regime L with the following transition matrix[5]

$$\begin{array}{c} \\ \tilde{\mu}_{S,t} = \mu_H \\ \tilde{\mu}_{S,t} = \mu_L \end{array} \begin{array}{c} \begin{array}{cc} \tilde{\mu}_{S,t+dt} = \mu_H & \tilde{\mu}_{S,t+dt} = \mu_L \end{array} \\ \begin{pmatrix} 1-\chi dt & \chi dt \\ \lambda dt & 1-\lambda dt \end{pmatrix} \end{array}. \tag{2.6}$$

Here χ and λ are the intensities for the transitions of regime from H to L and from L to H, respectively, and the parameter θ ($0 \leq \theta \leq 1$) controls the extent to which the agent's beliefs are extrapolative: setting θ to zero makes the agent's beliefs fully rational.

Given this *perceived* regime-switching model—this is not the true model—if recent stock market price growth has been high, the agent thinks it is likely that the high-mean price growth regime is generating prices and therefore forecasts high price growth in the future. Conversely, if recent price growth has been low, the agent thinks it is likely that the low-mean price growth regime is generating prices and therefore forecasts low price growth in the future. Formally, at each point in time, the agent computes the expected value of the latent variable $\tilde{\mu}_{S,t}$ given the history of past price growth: $S_t \equiv \mathbb{E}[\tilde{\mu}_{S,t}|\mathcal{F}_t{}^P]$. That is, she applies optimal filtering theory (see, for instance, Liptser and Shiryaev (2013)) and obtains

$$\begin{aligned} dS_t &= (\lambda\mu_H + \chi\mu_L - (\lambda+\chi)S_t)dt + (\sigma_{P,t}^D)^{-1}\theta(\mu_H - S_t)(S_t - \mu_L)d\omega_t^e \\ &\equiv \mu_S^e(S_t)dt + \sigma_S(S_t)d\omega_t^e, \end{aligned} \tag{2.7}$$

where $d\omega_t^e \equiv [dP_t^D/P_t^D - (1-\theta)g_D dt - \theta S_t dt]/\sigma_{P,t}^D$ is a standard Brownian innovation term from the agent's perspective. As a result, she perceives the evolution of the stock market price P_t^D to be

$$dP_t^D/P_t^D = \mu_P^{D,e}(S_t)dt + \sigma_P^D(S_t)d\omega_t^e, \tag{2.8}$$

[5]The models of Barberis et al. (1998), Veronesi (1999), and Jin (2015) also adopt a regime-switching learning structure.

where

$$\mu_P^{D,e}(S_t) = (1-\theta)g_D + \theta S_t. \tag{2.9}$$

The agent's expectation about price growth $\mu_P^{D,e}(S_t)$ is therefore a linear combination of a rational component g_D and a behavioral component S_t; hereafter we call S_t the sentiment variable.

In summary, the evolution of sentiment in (2.7) captures return extrapolation: high past price growth dP_t^D/P_t^D pushes up the perceived shock $d\omega_t^e$, which leads the agent to raise her expectation of the sentiment variable S_t, causing her expectation about future price growth $\mu_P^{D,e}(S_t)$ to rise.[6]

Although the subjective evolution of sentiment (2.7) is derived through optimal learning, the representative agent, it should be emphasized, does *not* hold rational expectations. With rational expectations, the agent will realize in the long run that the regime-switching model (2.6) is incorrect: she can look at the historical distribution of $d\omega_t^e$ and realize that it does not fit a normal distribution with a mean of 0 and a variance of dt. Instead, the agent in our behavioral model always believes that the regime-switching model is correct. In reality, it is possible that investors in the market learn over time that their mental model is incorrect. At the same time, new investors who hold extrapolative beliefs may continuously enter the market. The stable belief system in (2.6) is an analytically convenient way to capture these dynamics. Alternatively, if equations (2.6) and (2.7) represent the *true* data generating process, then the agent does hold rational expectations. In that case, the model becomes a fully rational model with incomplete information.[7] We discuss the predictions of such a model in Section 2.5.

So far we have been focusing on the agent's beliefs about stock market prices. These beliefs also have direct implications for the agent's beliefs about dividend growth. If we write the perceived dividend process as

$$dD_t/D_t = g_D^e(S_t)dt + \sigma_D d\omega_t^e, \tag{2.10}$$

[6] There are many ways to specify the evolution of S_t in order to capture return extrapolation. We derive S_t from a regime-switching model for two reasons. First, such a learning model captures base rate neglect, an important consequence of the representativeness heuristic (Tversky and Kahneman (1974)). To see this, note that the perceived regimes or states, H and L, are not part of the true states of the economy. As a result, assigning positive probability weights to these regimes reflect the bias that the investor neglects the zero base rate associated with such regimes. Second, bounding S_t by a finite range (μ_L, μ_H) reduces the analytical difficulty of solving the model.

[7] Information is incomplete in the sense that the agent does not directly observe the latent variable $\tilde{\mu}_{S,t}$.

we can connect the agent's expectation about dividend growth $g_D^e(S_t)$ to her expectation about stock market price growth $\mu_P^{D,e}(S_t)$. To formally make this connection, we first observe that all the ratio-based quantities in our model (e.g., the price-dividend ratio of the stock market) are a function of the sentiment variable S_t; we can define $f(S_t) \equiv P_t^D/D_t$. We then apply Ito's lemma on both sides of this equation $f(S_t) = P_t^D/D_t$ and match terms to obtain

$$g_D^e(S_t) = \underbrace{(1-\theta)g_D + \theta S_t}_{\text{expectation of price growth}} \quad \underbrace{-(f'/f)\mu_S^e(S_t)}_{\text{expectation of sentiment evolution}} \quad \underbrace{+\sigma_D^2 - \sigma_P^D(S_t)\sigma_D - \tfrac{1}{2}(f''/f)(\sigma_S(S_t))^2}_{\text{Ito correction terms}}, \tag{2.11}$$

where

$$\sigma_P^D(S_t) = \frac{\sigma_D + \sqrt{\sigma_D^2 + 4\theta(\mu_H - S_t)(S_t - \mu_L)(f'/f)}}{2} > \sigma_D. \tag{2.12}$$

Equation (2.11) highlights an "expectations transmission mechanism": it says that the agent's expectation about dividend growth equals the sum of her expectation about stock market price growth, her expectation about how the price-dividend ratio evolves with respect to changes in sentiment, and the Ito correction terms that are related to the agent's risk aversion and the volatility of dividend growth, price growth, and changes in sentiment. In this way, the agent's expectation about price growth affects her expectation about dividend growth.

With the parameter values we specify later, equation (2.11) suggests that the agent's expectation about dividend growth is *more* responsive to changes in sentiment than her expectation about price growth. Under Epstein-Zin preferences, the separation between the elasticity of intertemporal substitution and risk aversion gives rise to a strong intertemporal substitution effect. As a result, when the past price growth has been high, the agent's forecast of high future price growth leads her to push up the current price-dividend ratio, making it a positive function of sentiment. Furthermore, under the regime-switching model, the agent perceives sentiment to be mean-reverting: $\mu_S^e(S_t)$ in (2.7) is a negative function of S_t. This suggests that the agent also perceives the price-dividend ratio to be mean-reverting. Together, these two conditions—the price-dividend ratio is a positive function of sentiment and is perceived to be mean-reverting—imply that the agent anticipates that the price-dividend ratio will decline from a high value when she expects high future price growth. That is, when the agent expects high future price growth, her expectation about dividend growth rises at a pace that exceeds her expectation about future price growth.

To complete the description of the model, we need to further specify the agent's beliefs about consumption growth. To do this, first note that, with a local correlation of ρ between consumption growth and dividend growth, we can rewrite the aggregate consumption process of (2.1) as

$$dC_t/C_t = g_C dt + \sigma_C \left(\rho d\omega_t^D + \sqrt{1-\rho^2} d\omega_t^{\perp} \right), \tag{2.13}$$

where $\omega_t^{\perp}$ is a Brownian motion that is locally uncorrelated with ω_t^D, the Brownian shock on dividends. We then assume that the agent perceives the consumption process as

$$dC_t/C_t = g_C^e(S_t) dt + \sigma_C \left(\rho d\omega_t^e + \sqrt{1-\rho^2} d\omega_t^{\perp} \right). \tag{2.14}$$

That is, we replace the true Brownian shock on dividends $d\omega_t^D$ by the agent's perceived Brownian shock $d\omega_t^e$ and factor the difference between these two Brownian shocks into $g_C^e(S_t)$, the agent's subjective expectation about consumption growth. Conceptually, this amounts to assuming that the bias in the agent's beliefs about consumption growth comes only from the bias in her beliefs about dividend growth. [8] In doing so, we derive the agent's expectation about dividend growth as

$$g_C^e(S_t) - g_C = \rho \sigma_C \sigma_D^{-1} (g_D^e(S_t) - g_D). \tag{2.15}$$

Empirically, the correlation between consumption growth and dividend growth is low—ρ is positive but low—and consumption growth is much less volatile than dividend growth—σ_C is much smaller than σ_D. As a result, (2.15) implies that the bias in the agent's expectation about consumption growth—the difference between $g_C^e(S_t)$ and g_C—is small. This is in keeping with the lack of any evidence that investors have extrapolative beliefs about consumption growth. [9] Moreover, the

[8] For any alternative assumption, one needs to explain why the investor has incorrect beliefs about consumption above and beyond her incorrect beliefs about dividends.

[9] Consistent with the way we model the agent's expectations about consumption growth, Kuchler and Zafar (2016) find that survey expectations are "asset-specific:" respondents who become pessimistic about their employment situation after experiencing unemployment are not pessimistic about other economic outcomes, such as stock prices or interest rates. Similarly, Huang (2016) finds that investors who become optimistic about an industry's future returns after having positive prior investment experience in the industry do not invest heavily in a dissimilar industry.

agent's approximately correct beliefs about consumption growth allow the model to generate low interest rate volatility and a low correlation between consumption growth and stock market returns, both of which are consistent with the data (Campbell (2003); Hansen and Singleton (1982, 1983)).

Model solution

The subjective Euler equation in (2.4) shows that, when pricing the stock market, the gross return from holding the Lucas tree is also part of the pricing kernel. This observation has two implications. First, both the price-dividend ratio $f(S_t) = P_t^D/D_t$ and the wealth-consumption ratio P_t^C/C_t are functions of the sentiment variable S_t; we can define $l(S_t) \equiv P_t^C/C_t$. Second, the two functions f and l are interrelated through Euler equations, so they need to be solved simultaneously. Specifically, using the Euler equation to price the stock market—setting $\tilde{R}_{j,t+dt}$ in (2.4) to the gross return on the stock market—we obtain

$$0 = \begin{bmatrix} -\frac{(1-\gamma)}{1-\psi}\delta - \gamma g_C^e + g_D^e + [(f'/f) + \frac{\psi-\gamma}{1-\psi}(l'/l)]\mu_S^e + \frac{1}{2}[(f''/f) + \frac{\psi-\gamma}{1-\psi}(l''/l)]\sigma_S^2 \\ +\frac{\gamma(\gamma+1)}{2}\sigma_C^2 + \frac{1}{2}\frac{\psi-\gamma}{1-\psi}\frac{2\psi-\gamma-1}{1-\psi}(l'/l)^2\sigma_S^2 - \frac{\gamma(\psi-\gamma)}{1-\psi}\rho\sigma_C\sigma_S(l'/l) - \gamma\rho\sigma_C\sigma_D - \gamma\rho\sigma_C\sigma_S(f'/f) \\ +\frac{\psi-\gamma}{1-\psi}\sigma_D\sigma_S(l'/l) + \frac{\psi-\gamma}{1-\psi}\sigma_S^2(l'/l)(f'/f) + \sigma_D\sigma_S(f'/f) + \frac{\psi-\gamma}{1-\psi}l^{-1} + f^{-1} \end{bmatrix}. \tag{2.16}$$

Similarly, using the Euler equation to price the Lucas tree—setting $\tilde{R}_{j,t+dt}$ in (2.4) to the gross return on the Lucas tree—we obtain

$$0 = \begin{bmatrix} -\frac{1-\gamma}{1-\psi}\delta - (\gamma-1)g_C^e + \frac{\gamma(\gamma-1)}{2}\sigma_C^2 + \frac{1-\gamma}{1-\psi}(l'/l)\mu_S^e + \frac{1-\gamma}{2(1-\psi)}(l''/l)\sigma_S^2 \\ +\frac{1}{2}\frac{1-\gamma}{1-\psi}\frac{\psi-\gamma}{1-\psi}(l'/l)^2\sigma_S^2 + \frac{(1-\gamma)^2}{1-\psi}\rho\sigma_C\sigma_S(l'/l) + \frac{1-\gamma}{1-\psi}l^{-1} \end{bmatrix}. \tag{2.17}$$

Substituting μ_S and σ_S from (2.7), g_D^e and σ_P^D from (2.11) and (2.12), and g_C^e from (2.15) into equations (2.16) and (2.17), we then obtain a system of two ordinary differential equations that jointly determines the evolutions of f and l.[10] The detailed derivation of (2.16) and (2.17) is in the Appendix.

Regarding the boundary conditions for solving the differential equations, note that,

[10]When $\theta = 0$, our model reduces to a fully rational benchmark. In this case, equations (2.16) and (2.17) lead to

$$f = \left[\delta + \psi g_C - g_D - \frac{\gamma(\psi+1)}{2}\sigma_C^2 + \gamma\rho\sigma_C\sigma_D\right]^{-1}, \quad l = \left[\delta + (\psi-1)g_C - \frac{\gamma(\psi-1)}{2}\sigma_C^2\right]^{-1}.$$

in (2.16) and (2.17), the second derivative terms are all multiplied by σ_S, and that σ_S goes to zero as S approaches either μ_H or μ_L. As a result, μ_H and μ_L are both singular points, and therefore no boundary condition is required.

Equations (2.16) and (2.17) cannot be solved analytically. We apply a projection method with Chebyshev polynomials to solve them numerically. We leave the details of the numerical procedure to the Appendix.

Important equilibrium quantities

With the model solution at hand, we derive equilibrium quantities that are important for understanding the model's implications. Specifically, we derive the dynamics of the interest rate, the objective and subjective expectations of stock market returns, and the steady-state distribution of the sentiment variable.

For the interest rate, we use the Euler equation in (2.4) to price the riskless asset—we set $\tilde{R}_{j,t+dt}$ to the gross return on the riskless asset $1 + r_t dt$—and obtain

$$r_t = \tfrac{1-\gamma}{1-\psi}\delta + \gamma g_C^e - \tfrac{\gamma(\gamma+1)}{2}\sigma_C^2 - \tfrac{\psi-\gamma}{1-\psi} \times \begin{bmatrix} (\mu_S^e - \gamma\rho\sigma_C\sigma_S)(l'/l) + \frac{1}{2}\sigma_S^2(l''/l) \\ +\frac{2\psi-\gamma-1}{2(1-\psi)}\sigma_S^2(l'/l)^2 + l^{-1} \end{bmatrix}. \quad (2.18)$$

The interest rate is linked to the agent's time preferences, her subjective expectation about consumption growth, precautionary saving, as well as how the wealth-consumption ratio P_t^C/C_t responds to changes in sentiment.[11]

To understand the risk-return tradeoff in the model, we compute, at each point in time, both the agent's expectation about future stock market returns and the (objectively measured) rational expectation about future stock market returns. From equations (2.8) and (2.9), the log excess return on the stock market from time t to time $t + dt$ is

$$\begin{aligned} r_{t+dt}^{D,e} dt &\equiv \ell\text{n}(P_{t+dt}^D + D_{t+dt}dt) - \ell\text{n}(P_t^D) - r_t dt \\ &= [(1-\theta)g_D + \theta S_t + f^{-1} - \tfrac{1}{2}(\sigma_P^D)^2 - r_t]dt + \sigma_P^D d\omega_t^e. \end{aligned} \quad (2.19)$$

Therefore, the agent's subjective expectation about the log excess return is

$$\mathbb{E}_t^e[r_{t+dt}^{D,e}] = (1-\theta)g_D + \theta S_t + f^{-1} - \tfrac{1}{2}(\sigma_P^D)^2 - r_t, \quad (2.20)$$

[11] When $\theta = 0$, $r = \delta + \psi g_C - \frac{\gamma(\psi+1)}{2}\sigma_C^2$.

and the subjective Sharpe ratio is $[(1-\theta)g_D + \theta S_t + f^{-1} - \frac{1}{2}(\sigma_P^D)^2 - r_t]/\sigma_P^D$.

Next, to compute the rational expectation about the stock market return, we compare (2.2) with (2.10) and obtain a relationship between the true and perceived Brownian shocks

$$d\omega_t^e = d\omega_t^D - (g_D^e(S_t) - g_D)dt/\sigma_D. \tag{2.21}$$

We then substitute (2.21) into (2.19) and derive

$$r_{t+dt}^{D,e}dt = [(1-\theta)g_D+\theta S_t+f^{-1}-\sigma_D^{-1}\sigma_P^D(g_D^e-g_D)-\tfrac{1}{2}(\sigma_P^D)^2-r_t]dt+\sigma_P^D d\omega_t^D. \tag{2.22}$$

As a result, the rational expectation about the log excess return on the stock market is

$$\mathbb{E}_t[r_{t+dt}^{D,e}] = (1-\theta)g_D + \theta S_t + f^{-1} - \sigma_D^{-1}\sigma_P^D(g_D^e - g_D) - \tfrac{1}{2}(\sigma_P^D)^2 - r_t, \tag{2.23}$$

and the objectively measured Sharpe ratio of the stock market return is $[(1-\theta)g_D + \theta S_t + f^{-1} - \sigma_D^{-1}\sigma_P^D(g_D^e - g_D) - \frac{1}{2}(\sigma_P^D)^2 - r_t]/\sigma_P^D$.

All the ratio-based quantities in this model such as the agent's expectation about stock market returns and the interest rate are a function of the sentiment variable S_t. Given this, to provide a statistical assessment of the model's fit with the empirical facts, we also compute the steady-state distribution for the sentiment variable S_t as *objectively* measured by an outside econometrician. To that end, we first obtain the objective evolution of sentiment by substituting the change-of-measure equation (2.21) into the subjective evolution of sentiment in (2.7)

$$dS_t = [\mu_S^e(S_t) + \sigma_D^{-1}\sigma_S(S_t)(g_D - g_D^e(S_t))]dt + \sigma_S(S_t)d\omega_t^D. \tag{2.24}$$

Compared to the subjective evolution of sentiment, the objective evolution exhibits a larger degree of mean reversion: the additional term $\sigma_D^{-1}\sigma_S(S_t)(g_D - g_D^e(S_t))$ in (2.24) is a negative function of sentiment.

Denote the objective steady-state distribution for sentiment as $\xi(S)$. Based on (2.24), we then derive $\xi(S)$ as the solution to the Kolmogorov forward equation (the Fokker-

Planck equation)

$$
\begin{aligned}
0 &= \frac{1}{2}\frac{d^2}{dS^2}\left(\sigma_S^2(S)\xi(S)\right) - \frac{d}{dS}\left([\mu_S^e(S_t) + \sigma_D^{-1}\sigma_S(S_t)(g_D - g_D^e(S_t))]\xi(S)\right) \\
&= (\sigma_S')^2\xi + \sigma_S\sigma_S''\xi + 2\sigma_S\sigma_S'\xi' + \frac{1}{2}\sigma_S^2\xi'' \\
&\quad - [(\mu_S^e)' + \sigma_D^{-1}\sigma_S'(g_D - g_D^e) - \sigma_D^{-1}\sigma_S(g_D^e)']\xi - [\mu_S^e + \sigma_D^{-1}\sigma_S(g_D - g_D^e)]\xi',
\end{aligned}
\tag{2.25}
$$

where σ_S and g_D^e are from (2.7) and (2.11), respectively, and the expressions for σ_S', σ_S'', $(\mu_S^e)'$ and $(g_D^e)'$ are provided in the Appendix. In addition, the steady-state distribution must integrate to one.

2.3 Model Implications

In this section, we examine the implications of the model. We begin by setting the benchmark values for the model parameters. In particular, we calibrate the agent's beliefs to match the survey evidence documented in Greenwood and Shleifer (2014). We then look at two building blocks for the model's implications: a set of important equilibrium quantities, each as a function of sentiment; and the steady-state distribution of sentiment. Finally, we discuss the model's implications for asset prices.

Model parameterization

There are three types of parameters: asset parameters, utility parameters, and belief parameters. For the asset parameters, we set $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$. These values are consistent with those used in Campbell and Cochrane (1999), Barberis et al. (2001), Bansal and Yaron (2004), and Beeler and Campbell (2012).[12] For the utility parameters, we set γ, the coefficient of relative risk aversion, to 10. As pointed out in Bansal et al. (2012) and Bansal and Yaron (2004), the long-run risks literature—a literature that depends significantly on the parameter values of Epstein-Zin preferences for its model implications—typically assigns a value of 10 or below for γ. Bansal and Yaron (2004), for instance, set γ to either 10 or 7.5.[13] For ψ, the reciprocal of the elasticity of intertemporal substitution, there exists a wide range of estimates in the asset pricing literature. The majority of previous papers suggests that ψ should be lower than one, but

[12] The parameter values for g_C and g_D are set such that both $\ell n(C)$ and $\ell n(D)$ grow, on average, at an annual rate of 1.84%; this rate is also used in Barberis et al. (2001).

[13] An estimate of 10 for γ is also the maximum magnitude that Mehra and Prescott (1985) find reasonable.

several other papers argue the opposite.[14] Given this, we set ψ to 0.9, a value that implies an elasticity of intertemporal substitution slightly above one. We explain in Section 2.5 that our model's implications are quantitatively robust even with an elasticity of intertemporal substitution significantly lower than one. Finally, for δ, the subjective discount rate, we assign a value of 2%.

We now turn to the belief parameters. We set μ_H and μ_L, the mean price growth in the high and low regimes, to 15% and −15%, respectively. As we will see later in this section, the probability of the agent's price growth expectations approaching the boundaries of μ_H and μ_L is approximately zero. As a result, the model's implications are not very sensitive to the choice of μ_H and μ_L. Next, we focus on θ, the parameter that controls the extent to which the representative agent is behavioral, and χ and λ, the perceived transition intensities between the high- and low-mean price growth regimes. We calibrate these three parameters to match the survey expectations of investors studied in Greenwood and Shleifer (2014). Specifically, we set $\theta = 0.5$ and $\chi = \lambda = 0.18$ so that the agent's beliefs match survey data along two dimensions.[15] First, for a regression of the agent's expectations about future stock market returns on past twelve-month returns, our parameter choice allows the model to produce a regression coefficient and a t-statistic that match the empirical estimates from surveys. Second, our parameter choice allows the agent's beliefs to match the survey evidence on the relative weight investors put on recent versus distant past returns when forming beliefs about future returns. Below we examine these two dimensions in detail.

Empirically, Greenwood and Shleifer (2014) regress survey expectations about future stock market returns on past twelve-month cumulative raw returns across various survey expectations measures. They find that the regression coefficient is positive and statistically significant. To justify our parameter values for θ, χ, and λ, we want to run the same regression in the context of the model. One caveat, however, is that we are uncertain about what survey respondents think the definition of return is. Does it include the dividend yield or not? Is it a raw return or an excess return? Given this caveat, we examine *four* measures of return expectations: $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]$, the agent's expectation about the percentage return on the stock market, $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$, the agent's expectation about the

[14] See Bansal et al. (2012) for a discussion of this point.

[15] Recall that $\theta = 0$ means the agent is fully rational, whereas $\theta = 1$ means that the agent is fully behavioral. Therefore, 0.5 is a natural default value for θ: it implies that the representative agent is approximately an aggregation of rational and behavioral agents with equal population weights.

percentage return in excess of the interest rate, $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$, the agent's expectation about the price growth of the stock market, and $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t$, the agent's expectation about the price growth in excess of the interest rate. The latter two measures are also plausible candidates because investors may not actively think about the dividend yield when answering survey questions.[16]

[Place Table B.1 about here]

Table B.1 reports the regression coefficient, its t-statistic, the intercept, as well as the adjusted R-squared, when regressing each of the four measures of return expectations described above on either the past twelve-month cumulative raw return or the current log price-dividend ratio, over a sample of 15 years or 50 years. Each reported value—for instance, the regression coefficient—is averaged over 100 trials, with each trial being a regression using monthly data simulated from the model. Here we make two observations. First, the magnitude of the agent's extrapolative beliefs about future stock market returns matches the empirical values suggested by Greenwood and Shleifer (2014). Regressing the agent's expectation about future price growth ($\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$) on the past twelve-month cumulative raw return for a 15-year simulated sample, the regression coefficient is 4.0% with a Newey-West adjusted t-statistic of 8.4. Running the same regression for a 50-year simulated sample, the regression coefficient is 4.0% with a t-statistic of 12.1. As a comparison, for a 5-year sample of data from the Michigan survey, the regression coefficient is 3.9% with a t-statistic of 1.68; for a 15-year sample of data from the Gallup survey, the regression coefficient, after some conversion, is 8% with a t-statistic of 8.81.

Second, by comparing the regression coefficients and the t-statistics across the four measures of return expectations, we find that including the dividend yield in the calculation of return reduces the regression coefficient by about a half, but does not significantly affect the t-statistic. Therefore, even though we model return extrapolation as extrapolating past price growth, the agent also holds extrapolative expectations about the total return. Furthermore, subtracting the interest rate from the expectation of returns only has a small impact on the regression results because of low interest rate volatility. In summary, across all four measures of return expectations, the agent extrapolates past stock market returns when forming expectations about future returns. In Section 2.5, we compare these regression results with those

[16]Hartzmark and Solomon (2017) provide empirical evidence for the idea that investors do not take the dividend yield into account when calculating returns. Barberis et al. (2015) also take this interpretation when calibrating their model parameters to survey expectations.

from the rational expectations models of Bansal and Yaron (2004) and Bansal et al. (2012).

Our parameter choice of θ, χ, and λ is also disciplined by matching the agent's beliefs with the survey evidence on the relative weight of recent versus distant past returns in determining investors' return expectations. Specifically, we estimate the following non-linear least squares regression

$$\text{Expectation}_t = a + b\sum_{j=1}^{n} w_j R^D_{(t-j\Delta t)\rightarrow(t-(j-1)\Delta t)} + \varepsilon_t \tag{2.26}$$

using model simulations, where Expectation_t is the agent's time-t expectation about stock market returns, $R^D_{(t-j\Delta t)\rightarrow(t-(j-1)\Delta t)}$ is the raw return from time $t-j\Delta t$ to $t-(j-1)\Delta t$, and $w_j \equiv e^{-\phi(j-1)\Delta t}\Big/\sum_{l=1}^{n} e^{-\phi(l-1)\Delta t}$. In Equation (2.26), each past realized return is assigned a weight. The weight decreases exponentially the further back we go into the past, and the coefficient ϕ measures the speed of this exponential decline. When ϕ is high, the agent's expectation is determined primarily by recent past returns; when it is low, even distant past returns have a significant impact on the agent's current expectation.

[Place Table B.2 about here]

Table B.2 reports the intercept a, the regression coefficient b, the adjusted R-squared, and most importantly, the parameter ϕ. As before, we examine four expectations measures, $\mathbb{E}^e_t[(dP^D_t+D_t dt)/(P^D_t dt)]$, $\mathbb{E}^e_t[dP^D_t/(P^D_t dt)]$, $\mathbb{E}^e_t[(dP^D_t+D_t dt)/(P^D_t dt)]-r_t$, and $\mathbb{E}^e_t[dP^D_t/(P^D_t dt)]-r_t$. Each reported value is averaged over 100 trials, with each trial being a regression using simulated data with a monthly frequency over either 15 years or 50 years. We set Δt, the time interval for each past return in (2.26), to 1/12 (one month), and we set n, the total number of past returns on the right hand side of (2.26), to 600.[17]

Across the four expectations measures, the estimation of ϕ is stable: it is around 0.42. This value means that a monthly return three years ago is weighted about 25% as much as the most recent return; that is, the agent looks back a couple of years when forming beliefs about future returns. For comparison, Barberis et al. (2015) run the same regression (2.26) using survey data documented in Greenwood and Shleifer (2014); they estimate ϕ at a value of 0.44. We choose the values of θ, χ, and λ such that the model generates about the same estimate of ϕ as surveys.

[17] We choose $n = 600$ because further increasing n has a minimal impact on the estimated values of the parameter ϕ and the adjusted R-squared.

The literature has not reached consensus on the value of ϕ. On the one hand, Greenwood and Shleifer (2014) and Kuchler and Zafar (2016) find that investor expectations depend only on recent returns. On the other hand, Malmendier and Nagel (2011, 2013) and Vanasco et al. (2015) suggest that distant past events may also play an important role when investors form beliefs. Reconciling this discrepancy is beyond the scope of the paper. Here, we provide two possible explanations. First, investors may simultaneously adopt two mechanisms when forming beliefs: one that focuses on recent past events such as daily stock market fluctuations, the other that focuses on infrequent but salient events such as a stock market crash. Second, the horizon over which investors form expectations may affect how far they look back into the past. For instance, the survey expectations data studied in Greenwood and Shleifer (2014) are based on questions that ask investors to forecast stock market returns over the next six to twelve months, which may prompt investors to look back only a couple of years. On the other hand, the equity holdings data studied in Malmendier and Nagel (2011) are based on equity investment decisions that may require investors to forecast equity returns over the next couple of decades; they may therefore examine equity performance over the past few decades.

[Place Table B.3 about here]

We summarize the default parameter values in Table B.3. In Section 2.4, we further provide a comparative statics analysis to examine the sensitivity of the model's implications to changes in these parameter values.

Building blocks

We start with two building blocks for understanding the model's implications. First, we analyze a set of important equilibrium quantities. We then look at the steady-state distribution of sentiment.

Figure B.1 plots the price-dividend ratio of the stock market f, the volatility of stock market returns σ_P^D, the rational expectation about the log excess return $\mathbb{E}[r^{D,e}]$ (the conditional equity premium), and the interest rate r, each as a function of the sentiment variable S.

[Place Figure B.1 about here]

Figure B.1 shows that the model generates substantial excess volatility: the volatility of dividend growth is 11%, while the volatility of stock market returns is typically above 20%. This result stems from the interaction between return extrapolation and Epstein-Zin preferences. With the coefficient of relative risk aversion γ significantly higher than the reciprocal of the elasticity of intertemporal substitution ψ—we set γ to 10 and ψ to 0.9—the intertemporal substitution effect strongly dominates the wealth effect. Given this, when the stock market has had high past price growth, the agent's forecast of high future price growth—this is a result of the agent extrapolating past price growth—leads her to push up the current price, causing the current price growth to rise. When the current price growth is higher, the agent's forecast of future price growth also becomes higher, which leads her to push up the current price, causing the current price growth to further rise, and so on. In other words, a feedback loop emerges from the interaction between beliefs and preferences, giving rise to significant excess volatility.

The mechanism described above for generating excess volatility also allows the model to generate a strong procyclical pattern for the price-dividend ratio of the stock market. With a strong intertemporal substitution effect in the model, the agent's forecast of high future price growth following high past price growth also leads her to push up the current price-dividend ratio. Figure B.1 shows that the price-dividend ratio f is indeed a positive function of sentiment S.

Furthermore, the model generates a strong countercyclical pattern for the true equity premium. Suppose the stock market has had high past price growth. The agent's expectation about future price growth then increases, pushing up the stock market price relative to dividends. Given that sentiment on average tends to revert back to its mean, the price-dividend ratio also tends to mean-revert, leading to low future returns. In addition, the agent's high expectation about future price growth also makes her optimistic about future consumption growth, although to a much lesser extent. This in turn causes the agent to push up the equilibrium interest rate. Together, both a low (rational) expectation of stock market returns and a high interest rate contribute to a low equity premium during high-sentiment periods. Of these two forces, the first one dominates: moving S_t from its mean to the top 25% percentile level causes a total decrease of 9.7% for the equity premium, out of which 9.5% comes from the decrease in the expected log return of the stock market.

[Place Figure B.2 about here]

The second building block for the model's implications is the objectively measured steady-state distribution of sentiment. Figure B.2 plots this steady-state distribution.

Under the true probability measure, sentiment exhibits a strong degree of mean reversion, for two reasons. First, the agent believes that sentiment will naturally mean-revert: with a regime-switching model, the agent believes that the expected growth rate of stock market prices tends to switch over time from one regime to the other. Second, the agent's price growth expectations mean-revert faster than what she perceives: when the agent thinks that the future price growth is high, future price growth tends to be low endogenously, causing her price growth expectations to decrease at a pace that exceeds her anticipation. Overall, the steady-state distribution has a mean of 2.0% and a standard deviation of 2.7%: the chance of sentiment approaching the extreme values of μ_H and μ_L is close to zero.

Model implications for asset prices

The two building blocks—the quantitative relation between important equilibrium quantities and sentiment as well as the steady-state distribution of sentiment—allow us to systematically study the model's implications for asset prices. We begin with examining the long-run properties of stock market prices and returns. Table B.4 reports the model's predictions for six important moments, and compares them side by side with the empirical values. In general, the model matches the facts: it generates significant excess volatility, a high equity premium, a Sharpe ratio similar to the empirical value, an interest rate that has a low level and low volatility, and a price-dividend ratio whose average level is close to the empirical one.

[Place Table B.4 about here]

As explained above, the model generates significant excess volatility from the interaction between return extrapolation and Epstein-Zin preferences. This interaction is quantitatively important. Without return extrapolation, Epstein-Zin preferences alone with $i.i.d.$ dividend growth and consumption growth do not lead to any excess volatility. Without Epstein-Zin preferences—that is, setting both ψ and γ to 10 while keeping all the other parameter values unchanged—return extrapolation alone leads to average return volatility of 13.8%, which implies much less excess volatility compared to the data.

The model also generates a significant equity premium: when measured as the rational expectation of log excess returns, the true long-run equity premium is 4.9%; when measured as the rational expectation of excess returns $\mathbb{E}[(dP_t^D + D_t dt)/(P_t^D dt) - r_t]$, it is 8.6%. In order to understand this model implication, we first note that the model produces a substantial long-run perceived equity premium—this is what the agent thinks she will receive as the average equity premium. When measured as the subjective expectation of log excess returns, the perceived long-run equity premium is 1.6%; when measured as the subjective expectation of excess returns $\mathbb{E}^e[(dP_t^D + D_t dt)/(P_t^D dt) - r_t]$, it is 5.1%. Three factors affect the perceived long-run equity premium. First and most intuitively, excess volatility causes the agent to demand a higher equity premium because she is risk averse. Second, return extrapolation gives rise to *perceived* persistence of both the aggregate dividend process and, to a lesser extent, the aggregate consumption process.[18] Under Epstein-Zin preferences, this perceived persistence is significantly priced, pushing up the perceived equity premium. Finally, the separation between the elasticity of intertemporal substitution and risk aversion keeps the equilibrium interest rate low and hence helps to keep the equity premium high.

[Place Figure B.3 about here]

Furthermore, with incorrect beliefs, the *true* long-run equity premium in the model can be significantly different from the perceived long-run equity premium. We find that the *true* long-run equity premium is significantly higher than the perceived one. In the model, the agent's beliefs mean-revert faster than what she perceives. Given this, she underestimates short-term stock market fluctuations and hence the risk associated with the stock market. In other words, if an infinitesimal rational agent, one that has the same preferences as the behavioral agent but holds rational beliefs, enters our economy, she would have demanded a higher equity premium: the model produces a true average equity premium that is substantially higher than the perceived equity premium. To illustrate this point, Figure B.3 plots the objective (rational) expectation and the agent's subjective expectation about price growth: for S_t less than 3.2%, the objective expectation about price growth is higher than the subjective expectation; for S_t greater than 3.2%, the opposite is true. Because the sentiment distribution has a mean of 2.0% and a standard deviation of 2.7%, Figure B.3 suggests that, about 67% of the time, the rational expectation about price growth is above the subjective expectation. That is, the true price growth is more likely to be higher than the perceived price growth. As a result, the model produces a true average equity premium that is substantially higher than the perceived average equity premium.

[18] The agent is averse to persistent shocks when $\gamma > \psi$; with our choice of parameter values, this condition is satisfied.

During high-sentiment periods, the model produces a *negative* equity premium: the equity premium averaged over the top quartile of the sentiment distribution is −13.05%. In general, rational expectations models—for instance, long-run risks models and habit formation models—do not generate a negative equity premium at any time.[19] In our model, however, subjective expectations and objective expectations of stock market returns differ significantly during high- or low-sentiment periods: when the sentiment level is high, the agent expects high stock market returns moving forward, but precisely because of her incorrect beliefs, future stock market returns are low on average, generating a negative equity premium. This model implication is consistent with the recent empirical findings of Greenwood and Hanson (2013), Baron and Xiong (2017), Cassella and Gulen (2015), and **?**: these papers document that the expected excess return turns negative during high-sentiment periods.

Next, we examine the model's implications for the predictability of stock market returns. Empirically, Campbell and Shiller (1988) and Fama and French (1988) document that a regression of future log excess returns on the current log price-dividend ratio gives a negative and significant regression coefficient. Moreover, the predictive power of the price-dividend ratio is greater when future returns are calculated over longer horizons.

[Place Table B.5 about here]

Table B.5 reports the regression coefficient β_j and the adjusted R-squared for a regression of the log excess return of the stock market from time t to time $t + j$ on the current log price-dividend ratio

$$r^{D,e}_{t\to t+j} = \alpha_j + \beta_j \ell\text{n}(P^D_t / D_t) + \varepsilon_{j,t} \tag{2.27}$$

over various time horizons j. We calculate the regression coefficients and the R-squared using 10,000 years of monthly data simulated from the model, and compare them side by side with the empirical values. Consistent with the data, β_j is negative and its magnitude increases as the time horizon j increases. A strong degree of mean reversion in sentiment, together with the feedback loop described above for generating excess volatility, allows the model to produce the long-horizon

[19]Strictly speaking, a rational expectations model can generate a negative equity premium if the stock market negatively correlates with some other risk factors in the agent's portfolio and therefore serves as a diversification device.

predictability of stock market returns. When the stock market has had high past price growth, the agent's expectation about future price growth increases, pushing up the current price-dividend ratio. Since the agent's expectation—the sentiment variable—tends to revert back to its mean, subsequent returns are low on average, giving rise to a negative regression coefficient in (2.27).

The magnitudes of the regression coefficient and the R-squared generated from the model are broadly consistent with the empirical values. One difference, however, is that in the model, the R-squared begins to decrease as the time horizon j increases beyond three years, whereas in the data, the R-squared keeps rising over longer horizons. To understand this difference, recall that we calibrate the model to the survey expectations by setting θ to 0.5 and setting λ and χ to 0.18: the agent looks back a couple of years when forming beliefs about future returns. Given this parameter choice, the mean reversion of sentiment tends to occur over the first few years. Over longer horizons, no additional mean reversion in the agent's beliefs contributes to the predictability of stock market returns.

We now further investigate the model's implications for the correlation between stock market returns and consumption growth. Empirically, Hansen and Singleton (1982, 1983) document that this correlation is low. Nonetheless, most consumption-based asset pricing models generate a high, if not perfect, correlation between stock market returns and consumption growth. By imposing rational expectations, these models treat the consumption-based pricing kernels as the only source of stock market movements; stock market movements are driven by changes in consumption. As a result, the correlation between stock market returns and consumption growth is high.[20]

[Place Table B.6 about here]

Table B.6 reports both the model-implied values and the empirical values for the correlation between consumption growth and stock market returns. Consistent with the data, the model produces a low correlation: the correlation between annual log consumption growth and annual log excess returns is 0.19 in the model, and similarly it is 0.09 in the data. In comparison, the model of Campbell and Cochrane (1999)

[20]One exception is Barberis et al. (2001). They use "narrow framing", the notion that investors may evaluate financial risks in isolation from consumption risks, to generate a low correlation between stock market returns and consumption growth. Specifically, they use power utility as the agent's preferences over consumption, but use prospect theory developed by Kahneman and Tversky (1979) as the agent's preferences over financial wealth.

generates a correlation of 0.79, a much higher value. In our model, we assume that the bias in the agent's beliefs about consumption growth comes only from the bias in her beliefs about dividend growth. Given the low correlation observed in the data between consumption growth and dividend growth, the bias in the agent's beliefs about consumption growth in small. As a result, the agent's beliefs about stock market returns—they co-move strongly with her beliefs about dividend growth—are not significantly affected by fluctuations in consumption growth, giving rise to the low observed correlation between stock market returns and consumption growth.

Table B.6 shows that the model also generates a small but negative correlation between the current consumption growth and the stock market return in the subsequent period, an implication that is consistent with the data. Recall that consumption growth and dividend growth are weakly but positively correlated. If the current consumption growth is high, dividend growth is also high on average, which leads the agent to push up the current price, increasing the current price growth and the current level of sentiment. In the subsequent period, sentiment reverts towards its mean value, giving rise to a low stock market return.

[Place Figure B.4 about here]

Although our model is based on return extrapolation, it yields direct implications for cash flow expectations. The expectations transmission mechanism described by equation (2.11) suggests that the agent's expectation about dividend growth is more responsive to changes in sentiment than her expectation about price growth. Moreover, the total return equals the sum of the price growth and the dividend yield, and the dividend yield decreases as sentiment increases. Therefore, the agent's expectation about price growth is more responsive to changes in sentiment compared to her expectation about stock market returns.

Figure B.4 plots the agent's expectation about stock market returns, $\mathbb{E}^e[(dP_t^D + D_t dt)/(P_t^D dt)]$, the agent's expectation about price growth, $\mathbb{E}^e[dP_t^D/(P_t^D dt)]$, and the agent's expectation about dividend growth, $\mathbb{E}^e[dD_t/(D_t dt)]$, each as a function of the sentiment variable S. Quantitatively, Figure B.4 suggests that a one-standard deviation (2.7%) increase in sentiment from its mean (2.0%) pushes up the agent's expectation about stock market returns from 7.44% to 8.22%—a small increase of 0.78%—while it pushes up the agent's expectation about dividend growth from −0.04% to 6.48%—a much larger increase of 6.52%. Also, recall from Figure B.1 that a one-standard deviation increase in sentiment from its mean pushes up the price-dividend ratio of the stock market from 19.16 to 21.75. These results together imply that the price-dividend ratio is mainly correlated with the agent's expectation about dividend growth.

To further understand stock market movements, we follow the procedure in Campbell and Shiller (1988) to decompose, in the context of the model, the log price-dividend ratio of the stock market:

$$\ell\text{n}(P_t^D / D_t) \approx \sum_{j=0}^{\infty} \xi^j \left(\Delta d_{(t+j\Delta t)\to(t+(j+1)\Delta t)} - r^D_{(t+j\Delta t)\to(t+(j+1)\Delta t)} \right) - (\ell\text{n}(\xi) + (1-\xi)\zeta)/(1-\xi), \tag{2.28}$$

where ζ is the in-sample average of the annual log dividend-price ratio, $\xi = e^{-\zeta}/(\Delta t + e^{-\zeta})$, $\Delta d_{(t+j\Delta t)\to(t+(j+1)\Delta t)}$ is the log dividend growth from time $t + j\Delta t$ to $t + (j+1)\Delta t$, and $r^D_{(t+j\Delta t)\to(t+(j+1)\Delta t)}$ is the log gross return over the same period. Equation (2.28) says that the movement of price-dividend ratio comes from either the movement of future dividend growth—this is called "cash flow news"—or the movement of future returns—this is called "discount rate news." The standard approach that empirically addresses the relative importance of these two components is to look at future *realized* dividend growth and returns, and compute

$$\begin{aligned} 1 \approx\ & \underbrace{\frac{\mathbb{C}\text{ov}\left(\sum_{j=0}^{\infty} \xi^j \Delta d_{(t+j\Delta t)\to(t+(j+1)\Delta t)}, \ell\text{n}(P_t^D / D_t)\right)}{\mathbb{V}\text{ar}\left(\ell\text{n}(P_t^D / D_t)\right)}}_{CF_{objective}} \\ & + \underbrace{\frac{\mathbb{C}\text{ov}\left(-\sum_{j=0}^{\infty} \xi^j r^D_{(t+j\Delta t)\to(t+(j+1)\Delta t)}, \ell\text{n}(P_t^D / D_t)\right)}{\mathbb{V}\text{ar}\left(\ell\text{n}(P_t^D / D_t)\right)}}_{DR_{objective}}. \end{aligned} \tag{2.29}$$

The first term on the right hand side of (2.29), $CF_{objective}$, is the contribution of changes in cash flow news to stock market movements, and the second term, $DR_{objective}$, is the contribution of changes in discount rate news to stock market movements. By using future realized dividend growth and returns, this approach effectively imposes rational expectations. Most empirical studies that have conducted a Campbell-Shiller decomposition take this approach.

However, in a model with incorrect beliefs, we can further study the relation between the agent's *subjective* expectations and stock market movements by taking the subjective expectations on both sides of (2.28) and computing

$$1 \approx \underbrace{\frac{\mathbb{C}\text{ov}\left(\mathbb{E}^e_t[\sum_{j=0}^{\infty} \xi^j \Delta d_{(t+j\Delta t)\to(t+(j+1)\Delta t)}], \ell\text{n}(P^D_t / D_t)\right)}{\mathbb{V}\text{ar}\left(\ell\text{n}(P^D_t / D_t)\right)}}_{CF_{subjective}} + \underbrace{\frac{\mathbb{C}\text{ov}\left(-\mathbb{E}^e_t[\sum_{j=0}^{\infty} \xi^j r^D_{(t+j\Delta t)\to(t+(j+1)\Delta t)}], \ell\text{n}(P^D_t / D_t)\right)}{\mathbb{V}\text{ar}\left(\ell\text{n}(P^D_t / D_t)\right)}}_{DR_{subjective}}. \quad (2.30)$$

The first term on the right hand side of (2.30), $CF_{subjective}$, is the contribution of changes in subjective expectations about cash flow news to stock market movements, and the second term, $DR_{subjective}$, is the contribution of changes in subjective expectations about discount rate news to stock market movements.

[Place Table B.7 about here]

Table B.7 reports the four coefficients, $CF_{objective}$, $DR_{objective}$, $CF_{subjective}$, $DR_{subjective}$, as well as their corresponding adjusted R-squared. These coefficients and R-squared are calculated using 10,000 years of monthly data simulated from the model. By using future realized dividend growth and stock market returns and therefore imposing rational expectations, we obtain $DR_{objective} = 0.98$ with a R-squared of 0.209 and $CF_{objective} = 0.02$ with a R-squared of 1.2×10^{-4}. This result replicates the empirical finding of the volatility test literature that the variation of the price-dividend ratio comes primarily from discount rate variation (see Cochrane (2008) and Cochrane (2011)).

On the other hand, by relaxing the rational expectations assumption and using the agent's subjective expectations about dividend growth and returns, we obtain $DR_{subjective} = -0.08$ with a R-squared of 0.982 and $CF_{subjective} = 1.08$ with a R-squared of 0.984. This result unveils a very different picture and highlights the importance of expectations data: changes in the agent's subjective expectations about future cash flow news explain the majority of stock market movements. Empirically, de la O and Myers (2017) find that $DR_{subjective} = -0.09$ and $CF_{subjective} = 1.09$. These values match the theoretical values from our model.

Importantly, the fact that prices in our model are mainly correlated with cash flow expectations is a consequence of the Campbell-Shiller accounting identity; this statement is about correlation, not about causality. The agent's *return* expectations determine her cash flow expectations and are the cause of price movements. Given this, our model simultaneously explains the empirical findings of de la O and Myers (2017) on cash flow expectations and the empirical findings of Greenwood and Shleifer (2014) on return expectations. We provide additional discussion about the relation between return expectations and cash flow expectations in the Appendix.

[Place Table B.8 about here]

The model also points to some challenges: when calibrated to the survey expectations data, the model predicts a persistence of the price-dividend ratio that is significantly lower than its empirical value. Table B.8 presents the empirical values and theoretical values for the autocorrelations of asset prices. Empirically, price-dividend ratios are highly persistent at short lags. Nonetheless, the model produces a persistence for the price-dividend ratio that is much *lower* than the empirical value: the autocorrelation of $\ell n(P^D/D)$ with a lag of three years is 0.5 in the data, but it is essentially zero in the model. In the model, the persistence of the price-dividend ratio is driven by the persistence of the agent's beliefs. The available survey evidence suggests that investors focus on just the past year or two when forming beliefs about future returns. Therefore, when calibrated to surveys, the agent's beliefs tend to mean-revert in a couple of years. However, to match the empirical persistence of the price-dividend ratio, the agent's beliefs need to be much more persistent: the agent needs to form beliefs about future returns based on many years of past returns.

We leave a careful reconciliation of the survey expectations about stock market returns and the observed persistence of the price-dividend ratio to future research. One possibility, however, is to develop a framework that allows for the interaction between financial frictions and the agent's beliefs. **?** show that various types of financial frictions are empirically more persistent than investor sentiment. As a result, investor beliefs can affect asset prices through their interaction with financial frictions, making their impact more persistent.

We conclude this section by discussing the role of rational arbitrageurs. The model has a representative agent whose beliefs are biased. One natural question to ask is: if we introduce rational arbitrageurs, to what extent can they counteract the mispricing caused by the behavioral agent and therefore attenuate the significance of the model implications? Developing such a two-agent model is beyond the scope of the paper. However, three observations suggest that our model implications will remain intact after taking rational arbitrageurs into account.

First, in an economy with both rational and behavioral agents who have recursive preferences, the behavioral agents may eventually dominate the market: there is a positive probability that they take up most of the wealth in the economy in the long run. This is a key finding in Borovicka (2016). It suggests that our model's implications can be the limiting implications of a model with both rational and behavioral agents in the initial period. Second, in an economy with heterogeneous beliefs, asset prices are jointly determined by agents' beliefs weighted by their risk tolerances. A positive fundamental shock causes optimists to gain a larger fraction of wealth and increases their risk tolerance relative to pessimists, which in turn gives optimists a greater weight in driving asset prices, pushing asset prices further up. As a result, heterogeneity in investor beliefs can be an additional source of excess volatility; it can further amplify—rather than attenuate—our model implications.[21] Lastly, as pointed out by Barberis et al. (2015), extrapolative expectations are persistent in a dynamic model, which means that the behavioral agents who extrapolate past returns have persistently high demand for the stock market following high stock market returns. The persistence of this demand prevents near-future stock market returns from becoming too low, which reduces the incentive of rational agents to counteract mispricing. In other words, the persistence of extrapolative beliefs limits the impact of rational arbitrageurs on asset prices.

2.4 Comparative Statics

In this section, we examine the sensitivity of the model's implications to changes in parameter values. We focus on parameters that either have dispersed estimates in the literature or cannot be directly observed from the data. Specifically, for the utility parameters, we study how changes in γ, the coefficient of relative risk aversion, and ψ, the reciprocal of the elasticity of intertemporal substitution, affect the equity premium and the volatility of stock market returns. For the belief parameters, we look at how changes in θ affect the equity premium, the volatility of stock market returns, the price-dividend ratio, and the average interest rate. We also examine how changes in θ, χ, and λ affect return predictability and the persistence of the price-dividend ratio.

Utility parameters

Figure B.5 plots the long-run average of the equity premium and the volatility of stock market returns, each as a function of γ or ψ. The coefficient of relative

[21] See Xiong (2013) for more discussion of this amplification mechanism.

risk aversion is positively related to the equity premium but negatively related to the volatility of returns. Lower risk aversion naturally leads the agent to require a lower equity premium for risk compensation; reducing γ from 10 to 5, the model still explains 75% of the observed equity premium. At the same time, lower risk aversion strengthens the feedback loop described earlier since it increases the agent's demand for risky assets. Therefore, it leads to higher return volatility.

[Place Figure B.5 about here]

Within the examined range, changes in the elasticity of intertemporal substitution do not significantly affect the average equity premium and the average volatility of returns. As a result, our model implications are quantitatively robust to changes in ψ. We provide a detailed explanation of this observation in the next section when we compare our model with the model of Bansal and Yaron (2004).

Belief parameters

Belief Parameters.—Figure B.6a plots the long-run average of the equity premium, the volatility of stock market returns, the price-dividend ratio, and the interest rate, each as a function of θ, the parameter that measures the extent to which the agent is behavioral. Setting θ to zero gives the agent fully rational beliefs. In this case, the equity premium is 0.23%; the return volatility equals the fundamental volatility of 11%; the price-dividend ratio stays constant at 135; and the interest rate stays at 2.35%. With $\theta = 0$, the model fails to match the long-run properties of the stock market. Increasing θ from zero allows the feedback loop described above to emerge, which generates excess volatility, pushes up the perceived equity premium, and significantly reduces the price-dividend ratio. At the same time, a higher θ—and the agent's more extrapolative beliefs about stock market returns—leads the agent to perceive a higher persistence in dividend growth, which, under Epstein-Zin preferences, is significantly priced, causing the perceived equity premium to further rise. Finally, a higher θ leads to a true equity premium that is significantly higher than the perceived one.

[Place Figure B.6a about here]

Overall, a 1% increase in θ leads to a 0.09% increase in the equity premium and a 0.29% increase in the volatility of returns. On the other hand, the effect of θ on the interest rate is small. A higher θ increases the extrapolation bias in the agent's beliefs about stock market returns, but does not significantly affect her beliefs about consumption growth, which determine the equilibrium interest rate.

Figure B.6b examines how changes in θ, χ, and λ affect the predictability of stock market returns and the persistence of the price-dividend ratio. Here the predictability of returns is measured by the slope coefficient in a regression of the next year's log excess return on the current log price-dividend ratio; the persistence of the price-dividend ratio is measured by the one-year autocorrelation of log price-dividend ratios.

[Place Figure B.6b about here]

Figure B.6b shows that higher values of θ, χ, and λ lead to stronger predictability of returns and a lower persistence of the price-dividend ratio. Higher values of χ and λ—that is, higher perceived transition intensities between the high- and low-mean price growth regimes—suggest that the agent focuses on a shorter history of past return realizations when forming beliefs about future returns. A higher value of θ has a similar implication: it suggests that the agent exhibits a stronger extrapolation bias, which means that the agent deviates more from a rational agent whose beliefs depend on both recent and distant past returns. In other words, with a higher θ, the agent relies more heavily on recent past returns when forming beliefs about future returns. Therefore, higher values of θ, χ, and λ all lead to a stronger degree of mean reversion in sentiment, which in turn gives rise to stronger predictability of returns and a lower persistence of the price-dividend ratio.

The comparative statics results in Figure B.6b are consistent with recent empirical findings. At the aggregate level, Cassella and Gulen (2015) find that, during periods when investors' expectations about future returns depend on both recent and distant past returns, the price-dividend ratio does not strongly predict the next year's return. Conversely, during periods when investors' expectations depend primarily on recent past returns, the price-dividend ratio strongly predicts the next year's return. In our model, higher values of θ, χ, and λ lead to a higher ϕ, and therefore the agent's expectations about future returns depend more heavily on recent past returns. In the meantime, they also lead to stronger return predictability. Overall, the model implies that, when the agent forms beliefs based on a short history of past returns, the predictability of returns is strong. To give an example, increasing θ from 0.05 to 0.5 changes ϕ from 0.37 to 0.43. At the same time, it changes β_1, the slope coefficient for a regression of the next year's log excess return on the current log price-dividend ratio, from −0.32 to −0.72; the corresponding R-squared increases from 0.001 to 0.13. In the cross-section, Da et al. (2017) show that stocks associated with a larger extrapolation bias—beliefs of forecasters on these stocks depend more strongly on recent past returns—exhibit stronger return reversals. Applying our model to individual stocks gives the same prediction.

2.5 Comparison with Rational Expectations Models

In this section, we provide a quantitative comparison between our model and several models with rational expectations. First, we look at a rational expectations model that is most analogous to our behavioral model, one in which the regime-switching process characterized in Section 2.2 represents the *true* data generating process. We then examine differences between our model and the rational expectations models of Bansal and Yaron (2004) and Bansal et al. (2012): we focus on the long-run risks models because these are the models most related to ours.

The true regime-switching model

The rational expectations model most analogous to our model is the one that assumes the regime-switching process characterized in equations (2.6) and (2.7) is the true data generating process. In this case, the true evolution of the stock market price is

$$dP_t^D/P_t^D = [(1-\theta)g_D + \theta\tilde{\mu}_{S,t}]dt + \sigma_P^D(S_t)d\omega_t^P, \tag{2.31}$$

where ω_t^P is a standard Brownian motion. As with the behavioral model, the agent in this model does not directly observe the latent variable $\tilde{\mu}_{S,t}$. Instead, she uses past stock market prices to form a Bayesian estimate of $\tilde{\mu}_{S,t}$: $S_t = \mathbb{E}[\tilde{\mu}_{S,t}|\mathcal{F}_t{}^P]$. That is, the perceived evolution of stock market price in (2.8) is fully rational. We further assume that the perceived dividend process (2.10) and the perceived consumption process (2.14) are also rational.

By construction, this rational expectations model produces the same equilibrium prices as our behavioral model: the solutions to the differential equations of (2.16) and (2.17) also apply to this model. Nonetheless, the two models have statistical properties that are significantly different. One difference, for instance, lies in the models' implications for the predictability of stock market returns.

[Place Table B.9 about here]

Table B.9 reports the regression coefficient β_j and the adjusted R-squared for a regression of the log excess return of the stock market from time t to time $t + j$ on the current log price-dividend ratio $\ell n(P_t^D/D_t)$ over various time horizons j (one to seven years), now using the true regime-switching model. Table B.9 shows that the model fails to produce the predictability of stock market returns documented in Campbell and Shiller (1988) and Fama and French (1988): both the regression coefficients and the R-squared are close to zero. In contrast, Table B.5 shows that the behavioral model produces the observed predictability of stock market returns.

With rational expectations, the agent's beliefs about stock market returns are on average correct. Therefore, following high past price growth, the agent properly anticipates high future price growth, which pushes down the dividend yield in equilibrium, leading to flat returns in subsequent periods. As a result, future returns do not vary significantly with the current price-dividend ratio, giving rise to the lack of return predictability in the true regime-switching model.

The long-run risks models

Table B.10 reports the regression coefficients and t-statistics when regressing four measures of rational expectations of return—raw return or excess return, with or without dividend yield—either on the past twelve-month return or on the current log price-dividend ratio. The regressions are based on simulated data from Bansal and Yaron (2004). Interestingly, their model generates, to some extent, extrapolative expectations about future raw returns: a regression of the agent's expectations—this is also the rational expectation—about the next twelve-month total return on past twelve-month total return yields a coefficient of 2.5% with a t-statistic of 2.4 for a 15-year simulated sample; the regression coefficient is 3.0% with a t-statistic of 3.8 for a 50-year simulated sample.

[Place Table B.10 about here]

In Bansal and Yaron (2004), dividend growth and consumption growth share a stochastic yet persistent component. High past stock market returns are typically caused by positive shocks to this common component, which, given its persistence, implies high dividend growth and hence high raw returns moving forward. That is, the agent in Bansal and Yaron (2004) has extrapolative beliefs about future raw returns. At the same time, precisely because dividend growth and consumption growth share a persistent component, high dividend growth tends to coincide with high consumption growth, which implies a high interest rate in equilibrium. That is, when raw returns are high, the interest rate is also high. As a result, the agent does not have extrapolative beliefs about future *excess* returns: as shown in Table B.10, regressing the agent's expectation about future excess returns on past returns, the regression coefficient and the t-statistic are both close to zero.

These regression results highlight one fundamental difference between our model and the model of Bansal and Yaron (2004). In our model, the agent extrapolates past returns on the stock market, but extrapolates past consumption growth much less. The comovement between the agent's beliefs about returns and her beliefs about consumption growth is low. On the contrary, in Bansal and Yaron (2004), the comovement between the agent's beliefs about stock market returns—these rationally drive returns—and her beliefs about consumption growth—these determine the interest rate in equilibrium—is high. Therefore, the agent in our model has extrapolative beliefs about *both* raw returns and excess returns, whereas the agent in Bansal and Yaron (2004) has extrapolative beliefs only about raw returns.

Furthermore, as shown in Figure B.5, our model produces an equity premium and return volatility that do not vary significantly with respect to changes in the elasticity of intertemporal substitution. On the contrary, the long-run risks models cannot generate a high equity premium with a low elasticity of intertemporal substitution. For instance, setting the elasticity of intertemporal substitution to 0.5, our model generates an equity premium of 7.1% (measured as the rational expectation of excess returns), while the model of Bansal and Yaron (2004) produces an equity premium between 1% and 2%. Given this contrast, our model does not face the challenge of defending an elasticity of intertemporal substitution that is greater than one.

To understand this model difference, first note that sentiment, the state variable in our model, is much less persistent than the stochastic component of dividend and consumption growth in Bansal and Yaron (2004), allowing the equilibrium interest rate and hence the equity premium in our model to be less responsive to changes in the elasticity of intertemporal substitution. At the same time, the perceived dividend growth in our model depends more strongly on the state variable of sentiment, pushing up the perceived equity premium; as a comparison, dividend growth in Bansal and Yaron (2004) depends much less on the stochastic growth component.[22] Finally, the true average equity premium in our model is above the perceived one, allowing the equity premium to be high even when the elasticity of

[22]Bansal and Yaron (2004) set their "leverage parameter" ϕ to 3.5. In comparison, with return extrapolation, our model effectively sets ϕ to 14.5, a much higher value.

intertemporal substitution is low.

Table B.11 repeats the regression analyses of Table B.10 using the model of Bansal et al. (2012). Compared to the original model of Bansal and Yaron (2004), Bansal et al. (2012) introduces additional time variation in the long-run risks that further reduces the model's ability to generate extrapolative expectations: when regressing the agent's expectation of raw returns on past returns, the coefficient now becomes insignificant.

[Place Table B.11 about here]

We complete our discussion in this section by making two remarks. First, the persistence introduced by the long-run risks models on consumption growth and dividend growth leads to excess predictability, the notion that future consumption growth and dividend growth are excessively predicted by current variables such as the price-dividend ratio and the interest rate (see Beeler and Campbell (2012) and Collin-Dufresne et al. (2016b) for a detailed discussion). Our model does not give rise to excess predictability: return extrapolation in the model only generates *perceived* but not true long-run risks, and therefore the true consumption growth and dividend growth remain unpredictable.

Second, the rational expectations models and our model generate different implications regarding the role of cash flow expectations in understanding stock market movements. In models with rational expectations, the stock market price is mainly correlated with subjective discount rate (return) expectations; it is not significantly correlated with cash flow expectations. In our model, however, the stock market price is mainly correlated with the agent's subjective expectations of future cash flow growth.

2.6 Fundamental Extrapolation

A literature in behavioral finance focuses on fundamental extrapolation, the notion that some investors hold extrapolative expectations about *fundamentals* such as dividend growth or GDP growth (Barberis et al. (1998); Fuster et al. (2011); Choi and Mertens (2013); Alti and Tetlock (2014); Hirshleifer et al. (2015)). In this section, we provide a quantitative comparison between our model and a model with fundamental extrapolation. To facilitate the comparison, we keep the two models almost identical. The only difference is that, in the model with fundamental extrapolation, sentiment is constructed from past dividend growth, whereas in the model with return extrapolation, sentiment is constructed from past price growth. Below we first briefly describe the key assumptions in this fundamental extrapolation model. We then discuss the model's implications.

Model setup

With fundamental extrapolation, the agent believes that the expected growth rate of *dividends*—instead of the expected growth rate of stock market prices in the case of return extrapolation—is governed by $(1-\theta)g_D + \theta\tilde{\mu}_{S,t}$, where $\tilde{\mu}_{S,t}$ is a latent variable that follows a regime-switching process described in Section 2.2. The agent does not directly observe the latent variable $\tilde{\mu}_{S,t}$. Instead, she computes its expected value given the history of past dividend growth: $S_t \equiv \mathbb{E}[\tilde{\mu}_{S,t}|\mathcal{F}_t^D]$. She then applies optimal filtering theory and derives

$$\begin{aligned} dS_t =& (\lambda\mu_H + \chi\mu_L - (\lambda+\chi)S_t)dt + \sigma_D^{-1}\theta(\mu_H - S_t)(S_t - \mu_L)d\omega_t^e \\ \equiv& \mu_S^e(S_t)dt + \sigma_S(S_t)d\omega_t^e, \end{aligned} \tag{2.32}$$

where $d\omega_t^e \equiv [dD_t/D_t - (1-\theta)g_D dt - \theta S_t dt]/\sigma_D$ is a standard Brownian innovation term from the agent's perspective. That is, she perceives the evolution of dividend as

$$dD_t/D_t = g_D^e(S_t)dt + \sigma_D d\omega_t^e, \tag{2.33}$$

where

$$g_D^e(S_t) = (1-\theta)g_D + \theta S_t. \tag{2.34}$$

In other words, the agent's expectation about dividend growth $g_D^e(S_t)$ is a linear combination of a rational component g_D and a sentiment component S_t constructed from past dividend growth. On the other hand, the perceived evolution of the stock market price can be derived as

$$dP_t^D/P_t^D = \mu_P^{D,e}(S_t)dt + \sigma_P^D(S_t)d\omega_t^e, \tag{2.35}$$

where

$$\begin{aligned} \sigma_P^D(S) =& \sigma_D + (f'/f)\sigma_D^{-1}\theta(\mu_H - S)(S - \mu_L), \\ \mu_P^{D,e}(S) =& (f'/f)\mu_S^e + \tfrac{1}{2}(f''/f)\sigma_S^2 + (1-\theta)g_D + \theta S - \sigma_D^2 + \sigma_D\sigma_P^D(S). \end{aligned} \tag{2.36}$$

As before, f is defined as the price-dividend ratio of the stock market.

As with the return extrapolation model, equations (2.16) and (2.17) determine f and

l, the price-dividend ratio and the wealth-consumption ratio, except that μ_S, σ_S, g_D^e, $\mu_P^{D,e}$, and σ_P^D are now from (2.32), (2.34), and (2.36).

Model implications

We first examine the model's implications for investor expectations. Table B.12 reports the regression coefficient, its t-statistic, the intercept, as well as the adjusted R-squared, when regressing the four measures of return expectations on either the past twelve-month cumulative raw returns or the current log price-dividend ratio. With fundamental extrapolation, the regression coefficient on past returns and the t-statistic are both close to zero.

[Place Table B.12 about here]

Suppose past dividend growth has been high. On the one hand, it results in high past returns. On the other hand, fundamental extrapolation leads the agent to expect high dividend growth moving forward, but *not* high returns: following high past dividend growth, the stock market price increases to the extent that the agent's expectation of future returns does not change significantly. A fundamental extrapolation model with a representative agent therefore faces a challenge in explaining survey expectations about returns. In contrast, our return extrapolation model is constructed to explain these survey expectations.[23]

[Place Table B.13 about here]

Table B.13 analyzes the model's fit with the long-run properties of the stock market. Using the same parameters that allow the return extrapolation model to well explain the important moments of the stock market, the fundamental extrapolation model generates lower excess volatility and a much lower equity premium. Quantitatively, fundamental extrapolation generates 74.4% of excess volatility and 37.9% of the equity premium that return extrapolation produces.

This quantitative comparison highlights the importance of the feedback loop described above in matching asset pricing facts. With return extrapolation, the feedback loop emerges because extrapolative beliefs are applied to the stock market

[23] A fundamental extrapolation model with *heterogeneous* agents—for instance, one with both an agent who extrapolates past dividend growth and an agent who is fully rational—can potentially generate extrapolative expectations of returns for the behavioral agent in the model. See the model of Ehling et al. (2015) as an example.

return, a variable that is *endogenously* determined in equilibrium. With fundamental extrapolation, however, the feedback loop is absent because extrapolative beliefs are applied to dividend growth, a variable that is exogenous in the model: high past dividend growth makes the agent optimistic about future dividend growth and therefore pushes up the current price, but the higher price does not further affect the agent's beliefs about future dividend growth.

This feedback loop also points to a methodological contribution of the paper. Equation (2.32) shows that, in a fundamental extrapolation model, sentiment S, the state variable that drives asset prices dynamics, can be exogenously specified without solving the equilibrium; this greatly simplifies the model. On the other hand, with return extrapolation, sentiment S determines—and is endogenously determined by—asset prices. As a result, such a model requires solving beliefs and asset prices simultaneously, and therefore imposes a greater modeling challenge. Our numerical approach to solving a system of differential equations provides a solution to this challenge.

2.7 Conclusion

We build a new return extrapolation model that can be brought to the data quantitatively. With the agent's beliefs calibrated to fit the extrapolative expectations data documented in surveys, the model matches the long-run properties of stock market prices: it generates a high average equity premium, significant excess volatility, a low average interest rate, low interest rate volatility, and a price-dividend ratio whose average level is similar to the empirical one. The model also matches the dynamic behavior of stock market prices: it produces the long-horizon predictability of stock market returns, and generates the observed low correlation between stock market returns and consumption growth. We compare our model to the long-run risks models and find that our model's quantitative implications are more robust to changes in the elasticity of intertemporal substitution.

Our analysis has left several important issues for future work. First, when calibrated with the survey expectations data, the model predicts a persistence of the price-dividend ratio that is significantly lower than its empirical value. To reconcile the survey expectations about stock market returns with the observed persistence of the price-dividend ratio, we need a deeper understanding about how investors form beliefs. Second, the extrapolation framework is closely linked to theories of model uncertainty. A careful investigation of this connection may produce useful insights to both literatures. Finally, our representative-agent model neglects an important channel that affects asset prices: the time-varying fraction of wealth held by behavioral agents. Explicitly incorporating rational agents into our framework may lead to additional implications.

Chapter 3

"DARK MATTER" OF FINANCE IN THE SURVEY

3.1 Introduction

> *Tail risk is economists' version of cosmologists' dark matter. It is unseen and not directly observable but it exerts a force that can change over time and that can profoundly influence markets.*
>
> — Steve Ross (2015)[1]

Tail risks have a profound impact on the asset market and play an important role in answering central asset pricing questions such as the equity premium puzzle, the variance premium puzzle and the volatility smile (Barro (2006), Gabaix (2012), Ross (2015), Wachter (2013)). However, due to the rare occurrences of tail events in the historical data, tail risks are usually difficult to observe or measure. As a result, economists have limited knowledge about how investors form their beliefs about tail risks. In this paper, I use Shiller tail risk survey, a survey that directly asks investors about their perceptions of tail risks in the stock market, to investigate this question.

In understanding investors' belief formation on tail events with survey data, I face two fundamental challenges. On the one hand, recent developments in behavioral finance have proved the usefulness of investor belief surveys in some situations. For example, Greenwood and Shleifer (2014) document that investor expectation surveys of different sources reflect meaningful market-wide *investor expectations*.[2] However, it is unclear whether investor belief surveys also contain effective information on other attributes of investor beliefs, such as investors' perception of tail risks. Therefore, one might naturally ask, is the reported tail risks in the survey really reliable? On the other hand, with the recent development in past decades, researchers have collected different pieces of evidence on investor beliefs, including excessive optimism and pessimism concerning investor expectations (Greenwood

[1] John Campbell also mentioned the term "dark matter" of finance in his 2008 Princeton Finance Lectures.

[2] These investor expectation surveys include: the American Association of Individual Investors Sentiment Survey, Gallup Survey, Investor Intelligence Survey, Shiller Survey, Graham and Harvey Survey, and Michigan Survey. In these surveys, respondents answer questions such as what the percentage of expected returns over the next twelve months is.

and Shleifer (2014)), neglected risks before the financial crisis (Baron and Xiong (2017)), and others. For each piece of evidence, researchers have proposed separate belief formation frameworks, such as overconfidence, return extrapolation, and "this-time-is-different" hypothesis. However, there is no unified belief framework to explain different pieces of extant investor belief evidence. Therefore, a more challenging question is, could we interpret the documented investor belief evidence in one unified framework?

My paper primarily targets these two challenges and provides positive answers to both questions. First, for the reliability of survey evidence, with the documented consistency between investor expectation surveys in the previous literature (Greenwood and Shleifer (2014)), I reinforce its validity by showing the connections between investor expectation surveys and Shiller tail risks survey; indeed, variations in investor expectation surveys are highly consistent with changes in Shiller tail risks survey. Not only is there consistency within each survey type, information between different types of surveys is essentially connected and can serve as cross-validations for each other. Second, I show that two documented investor belief patterns—time-varying excessive optimism and pessimism on investor expectations and the time-varying perceptions of tail risks—can be understood within one belief formation framework. Specifically, the proposed framework for investors' belief is based on return extrapolation: investors believe that stock returns will continue rising after a sequence of good returns and continue falling after a sequence of low returns. As a result, perceived left-tail risks decline as investors become optimistic about future returns during market booms and vice versa, which leads to a strong *countercyclical* pattern. Therefore, I have not only expanded understandings of the tail risks dynamics but reinforced the validity of return extrapolation as a belief framework to interpret a wide range of investor belief patterns.

In order to authenticate the consistency between investor expectation surveys and Shiller tail risks survey, I start with a simple return distribution assumption—the conditional normal—for both investor expectation surveys and Shiller tail risks survey. Therefore, based on the expectation information in the investor expectation surveys, I can effectively back out a sequence of left-tail probabilities and compare it with the reported tail event probability in Shiller tail risks survey. The correlation between two variables, it turns out, is as high as 76%, which indicates that approximately 80% of variations in the reported probabilities in Shiller tail risk surveys can be effectively attributed to changes in investors' perceptions in investor expectation surveys. In backing out the implied left-tail probabilities from investor expectation surveys in the main text, I also use VIX index as the perceived time-varying conditional volatility since VIX is widely believed to reflect market anticipation for future stock market volatilities. However, replacing VIX with a constant volatility does not significantly dampen the close connections between the two types of surveys.

To further test the connections between the two types of surveys, I investigate contemporaneous relations between investor expectation surveys and the reported tail event probability in Shiller surveys. Specifically, I rely on the following hypothesis: if two types of surveys contain consistent information, then when investors perceive a higher expected return, they should simultaneously report lower left-tail probabilities in the tail risks surveys. Regressing the reported left-tail probabilities in Shiller on the contemporaneous investor expectation surveys, I find significant support for this hypothesis, which again demonstrates the connections between the two types of surveys. Moreover, this negative association holds true even after controlling for VIX index and a set of other commonly used variables, such as the default yield and the term yield.

In connecting two types of survey evidence, one main empirical concern is to make reasonable assumptions about the perceived conditional return distributions. For my main text analysis, I adhere to the simple conditional normal distribution assumption. It is worth pointing out that, even under normality assumption with time-varying expectations and volatilities, the unconditional return distribution is highly skewed and fat-tailed, largely consistent with the empirical patterns in the data. Nevertheless, I test alternative return distribution assumptions for robustness check. Specifically, I consider skew-normal distribution and see whether skewness would significantly affect results. It turns out that the close connections between two types of surveys is insensitive to the return distribution assumptions, and introducing skewness only slightly affects the probability level.

With the confirmed consistency between different investor belief surveys, I next make attempts to understand variations in the perceived investor expectations and perceived tail risks with one unified belief formation framework. Motivated by previous studies (Amromin and Sharpe (2013), Greenwood and Shleifer (2014)), return extrapolation serves well to explain the excessive optimism and pessimism on investor expectations and their portfolio choice. A natural conjecture is return extrapolation could also drive the time-varying perceptions of tail risks. I first construct this connection by investigating the determinants of the reported tail risks in the Shiller survey. Specifically, return extrapolation indicates that the current price level—log price to dividend ratio—positively affects investors' perceived expectations of returns. If changes in investors' perceived expectations influence variations in investors' perception of tail risks, then reported tail risks should also depend on the current price to dividend ratio level, with a negative sign. I find significant support for this conjecture, although the perceived left-tail probabilities also positively correlate with the time-varying VIX.

To further link return extrapolation to variations in the perceived tail risks, I first rely on the results of previous literature (Barberis et al. (2015), Greenwood and Shleifer (2014)) and construct an investor expectation measurement, denoted as Psentiment, based on return extrapolation. Specifically, Barberis et al. (2015) estimate how much weight investors put on past realized returns when using return extrapolation to form their expectations of returns, and I directly borrow those coefficients to construct my return extrapolation expectation measurement. With constructed Psentiment, the implied perceived left-tail probability with conditional normality assumption and VIX index explains around 55% of the variations in the reported tail risks in Shiller surveys. Even without information from VIX, it still explains 47% of the total variations. The close connections between return extrapolation and perceived tail risks remain robust even with skew-normal distributions.

As a striking contrast, rational expectations cannot explain the time-varying pattern for the perceived tail risks for the stock market in the survey. Specifically, during market booms, rational expectation models would predict lower expected returns going forward, and therefore a higher tail event risks, which leads to a procyclical pattern of tail risks. Hence, rational expectation models find difficulty in explaining both the time-varying investor expectations (Greenwood and Shleifer (2014)) and the variations in perceived tail risks in the surveys.

Literature Review:

My paper relates to a strand of literature focusing on the validity and usefulness of investor belief surveys. Greenwood and Shleifer (2014) demonstrate a high correlation within investor expectation surveys, thereby indicating that investor expectation surveys are not meaningless noise but rather reflections of widely shared investor beliefs in the market. Amromin and Sharpe (2013) use data obtained from a series of Michigan Surveys of Consumer Attitudes and demonstrate that household investors appear to extrapolate past realized returns when forming their expectations of future returns. Additionally, their reported expectations affect their portfolio choices. Gennaioli et al. (2016) rely on Duke quarterly survey of CFO and show that the actual investment activities are explained quite well by CFO's expectations in the survey. My paper reinforces the validity of survey information by showing the consistency between different types of surveys and that survey information are mutually supportive to each other. Moreover, I show that the reported left-tail probabilities in the survey have strong predictive power for future market performance.

My paper belongs to the burgeoning return extrapolation literature. Many works in this field attempt to understand the role of return extrapolation in driving the behavior of aggregate market performance. (Cutler et al. (1990b), De Long et al. (1990), Barberis et al. (2015), Jin and Sui (2017)). Barberis et al. (2015) use return extrapolation to construct an asset pricing model that can explain central asset pricing facts, such as excess volatility puzzle and the predictability of returns, as well as the investor belief survey evidence in the data. Jin and Sui (2017) construct a quantitative benchmark of belief-based asset pricing models that can simultaneously explain the equity premium puzzle, excess volatility puzzle, predictability of returns, low correlations between consumption and returns, as well as the investor belief evidence in the surveys. However, all existing studies on return extrapolation primarily focus on explaining time-varying investor expectations of returns. In this paper, I show that return extrapolation also helps understand investors' perceptions of tail risks and therefore can serve as a unified framework for explaining a set of investor belief evidence.

This paper resolves a general challenge in the literature, mentioned in Barberis , to simultaneously explain the overestimation and underestimation of tail events during different episodes. For example, people underestimate tail risks in the run-up to the 2008 U.S. financial crisis but, as documented in this paper, overestimate the tail event probabilities during the financial crisis. Baron and Xiong (2017) also document that bank equity investors neglect crash risks during credit expansions. The proposed belief mechanism with return extrapolation in this paper provides a plausible answer. During the market boom before the 2008 U.S. financial crisis, investors extrapolate past good returns and become overly optimistic about future stock market returns; increase in perceived investor expectations naturally reduce the perceived tail risks in the stock market. During the financial crisis, when investors pessimistically perceive low future stock market returns, the perceived tail risks in the stock market increase dramatically. **?** propose a similar mechanism to explain the credit cycles.

This paper also relates to the literature measuring the left-tail risks. Chen et al. (2017) propose a tractable measurement of model fragility and use it to evaluate the asset pricing models with rare disasters in the consumption process. Barone-Adesi et al. (2016) apply behavioral pricing kernel theory to estimate aggregate preference and beliefs from option prices and historical returns. Their estimates pertaining to left-tail risks based on options data are consistent with the Shiller Crash Index used in this paper. Goetzmann et al. (2016) investigate the role of media influence in shaping investors' perceptions of tail risks, and find that recent market declines and adverse market events made salient by the financial press are associated with higher subjective crash probabilities. My paper proposes a belief framework to explain multisets set of investor belief evidence.

3.2 Data

For my empirical analysis in this paper, I mainly draw on two types of investor belief surveys: (1) surveys on investors' perceived tail risks and (2) surveys on investor expectations of future returns. Specifically, for investor tail risks survey, I draw on one question in Shiller survey that targets exclusively on both institutional and individual investors' perceived tail risks. For investor expectation surveys, I rely on five data sources: the Gallup investor survey, the American Association of Individual Investors survey (AA), Investor Intelligence's summary of professional investors' belief (II), Shiller's survey on individual investors' expectations, Graham and Harvey's surveys of CFOs. Although carried out by different institutions, each of five investor expectation surveys all asks participants similar questions about their expectations on future market returns.

It is worth noting that Shiller survey asks both investor expectation question and tail risks question. In my analysis, I use his tail risks question and formally refer to it as Shiller tail risks survey. Meanwhile, I also use his investor expectation questions and formally refer to it as Shiller investor expectation survey. Additionally, all the surveys used in my analysis primarily focus on the U.S. stock market.

Moreover, although two types of investor belief surveys collect different attributes of investor belief information, as long as investor belief surveys are reliable, two belief attributes in each type of surveys should be connected together within a certain belief formation framework. Before I touch on details about the connections between the two types of surveys, I first describe the two types in detail below.

Shiller Tail Risks Survey

Since 1989, Robert Shiller has been surveying individual and institutional investors on their views about U.S. stock market; semi-annually for ten years and then monthly by the International Center for Finance at the Yale School of Management since July, 2001. Relying on a market survey firm, Shiller categorizes investors into two groups: individual investors and institutional investors. During each month, around 300 questionnaires are mailed to investors in each groups. One of the questions in the survey specifically asks investors about their estimation on the probability that a catastrophic event will occur over the next six months. The original question is presented to survey subjects as follows:

"What do you think is the probability of a catastrophic stock market crash in the U.S., like that of October 28, 1929 or October 19, 1987, in the next six months, including the case that a crash occurred in the other countries and spreads to the U.S.? (An answer of 0% means that it cannot happen, an answer of 100% means it is sure to happen.)"

Responses to this question help me to extract investors' perceived left-tail risks. Specifically, the International Center for Finance at Yale School of Management and Shiller construct Stock Market Crash Index for both individual and institutional investor subsamples, measured by the percentage of investors in each groups who report a left-tail probability within next six months of less than 10%. Therefore, high levels in the original Crash index indicate *lower* perceived left-tail probabilities from investors and vice versa. Through out the paper, I use the inverse of the original index, just to keep the relation that higher index refers to higher perceived tail risks. I plot the Crash index in Figure C.1.

The crash index for the subsample of individual investors starts from 1999 while the sample of institutional investors starts from 1990.[3] Despite difference in sample spans and investor groups, two indices are highly synchronized, which indicates that Shiller tail risk survey measurements contain consensus information on investors' perceptions of tail risks. Moreover, the reported tail probability has a strong cyclical pattern—it remains relatively low during bubble episodes such as between 2004 and 2006 and becomes high during financial crisis between 2008 and 2010. Such a trend is also consistent with other important economic variables such as default yield, term yield, and VIX. In fact, as will be clear later, perceived tail risks are

[3]There are a few observations before 1999 for the individual investor subsample.

highly correlated with many important business cycle indicators. Indeed, Shiller Survey has proven to contain useful information not only on investor beliefs but also on asset price dynamics. For example, Barone-Adesi et al. (2013) estimate behavioral pricing kernels using market option data and find them to be highly correlated with Shiller tail risks series. [4]

[Place Figure C.1 about here]

Investor Expectation Surveys

In the past decades, researchers have made progress in understanding investor belief formation mechanism with information based on investor expectation surveys. Among others, Greenwood and Shleifer (2014) demonstrate investor expectation surveys are not meaningless noise but really capture the consensus investor expectations in the market. One of the main purposes of this paper is to demonstrate the consistency between investor expectation surveys and left-tail risks surveys. However, to better connect previous results and put my results into a coherent framework, I also presents some evidence on the consistency within investor expectation surveys. For a more detailed discussion on investor expectation surveys, see Greenwood and Shleifer (2014).

Different from investors' perceptions of tail risks, investor expectation surveys measure another attribute of investor belief information: the perceived expectations of future returns. In this section, I briefly describe the investor expectation surveys, especially two types (*quantitative* and *qualative*) of investor expectation questions.

Gallup

The Gallup survey measures individual investors' expectations of the U.S. stock market over the next twelve months. It is conducted monthly between 1996 and 2011, but there are some gaps in later years, especially between November 2009 and February 2011 when the survey was discontinued.

To extract investor expectations, in each month, Gallup survey asks participants one *qualitative* question: whether they are "very optimistic", "optimistic", "neutral", "pessimistic" or "very pessimistic" about stock returns over the next twelve months.

[4] The reported probabilities in Shiller tail risk survey are high partially because they are subjective probabilities, which reflect both probabilities and risk aversion. To test the robustness of these survey responses, Goetzmann et al. (2016) run an independent experiments on Amazon's Mechanical Turk using similar questions, and find the reported responses remain unchanged.

With the percentage of each response in the collected survey answers, Gallup reports a *qualitative* investor expectation series to measure investor expectations in the market:

$$Gallup = \%Bullish - \%Bearish, \tag{3.1}$$

where “Bullish” is defined as either “very optimistic” or “optimistic” and “Bearish” is defined as either “pessimistic” or “very pessimistic”. This qualitative time series helps us understand the dynamics of investor sentiment in the market.

However, to investigate the connections between investor expectation surveys and Shiller tail risks survey, a *quantitative* estimations of expected returns would better serve my purpose. Fortunately, Gallup survey also asks more precise *quantitative* questions on investors’ perceived expected returns, although only for a shorter sample. Specifically, between September 1998 and May 2003, Gallup asks participants to give an estimate of the percentage return they expect for the stock market over the next year.

Therefore, as long as participants in the Gallup survey answer quantitative and qualitative questions in a consistent way, I can effectively get quantitative estimations investor expectation series by rescaling qualitative Gallup investor series with projection method.[5] Indeed, the reported quantitative Gallup investor expectation series turns out to have a high correlation of 84% with the qualitative investor expectation series within the short sample between September 1998 and May 2003. I make the further assumption that such high correlation between quantitative and qualitative investor expectation series still holds true even beyond the short sample period. Therefore I can get rescaled Gallup quantitative series at any time as long as the Gallup qualitative series is available. This projection method also helps me transform qualitative series in other investor expectation surveys to a meaningful quantitative basis.

American Association of Individual Investors

The American Association of Individual Investors Sentiment Survey is conducted weekly to members of American Association of Individual Investors starting from January 1987. Similar with Gallup, it asks qualitative questions and measures the

[5] Carlson and Parkin (1975) propose a method to generate average expectations from categorical survey data. As pointed in Greenwood and Shleifer (2014), this method has almost no impact on the investor expectation time series.

percentage of individual investors who are bullish, neutral or bearish on the U.S. stock market over the next six months. They report the AA qualitative investor expectation series following the equation (3.1).

Investor Intelligence

"Investor Intelligence" is conducted monthly in 1963, then biweekly through June 1969 when it was shifted to weekly later. Instead of asking investors directly, directors of this survey classify 120 independent financial market newsletters as having "bullish", "neutral" or "bearish" forecasts of returns on the U.S. stock market over the near term. Similarly, they report the II qualitative investor expectation series following the equation in (3.1).

Shiller Investor Expectation Survey

In Shiller's survey question set, one question asks investors whether or not they expect the market to rise over the following year, which essentially is the qualitative investor expectation question. Shiller constructs the one-year individual confidence index which measures the percentage of individual investors who expect the market to rise over the following year. I use the one-year individual confidence index as the qualitative investor expectation series for my later analysis.

Garham and Harvey

John Graham and Campbell Harvey have been surveying chief financial officers (CFOs) of major U.S. corporations quarterly. In their question, they ask CFOs their expectation of stock market returns on the U.S. stock market directly over the next twelve months. The quantitative expectation series is available from October 2000.

Consistency within Investor Expectation Surveys

In using survey data to investigate investor beliefs, one prerequisite is the validity of survey information. If there is no common information among investor belief surveys, then corresponding investigations would be meaningless. For the five investor expectation surveys mentioned above, they should reflect market-wide investor expectations of returns. Fortunately, previous studies have confirmed in point. (Greenwood and Shleifer (2014)): investor expectation surveys are highly correlated with each other.

Specifically, I report the information for the original qualitative investor expectation series in Panel A of Table C.1. Clearly, qualitative investor expectation series are highly correlated with each other. For instance, the correlation between Gallup and AA has a correlation of 0.63. This indicates that investor expectation surveys are not meaningless noise but really contain consistent information on investors' expectations for future stock returns.

[Place Table C.1 about here]

Another general concern for the survey data is whether the reported responses truly reflect investors' real beliefs in their minds. Although unlikely, it is still possible that investors misinterpret the questions and report irrelevant answers. To address this possibility, Greenwood and Shleifer (2014) investigated the reported investor expectations and the mutual fund flows at the aggregate level.

Rescaling

With the reliability of investor expectation surveys, I proceed to investigate the connections between investor expectation surveys and Shiller tail risks survey. To build quantitative connections between investor expectation surveys and left-tail risks survey, I transform each *qualitative* investor expectation series to *quantitative* series by projecting Gallup quantitative investor expectation series onto qualitative investor expectation series from five surveys. Panel B of Table C.1 report the summary statistics for the rescaled series. The average expected return ranges from 0.09 to 0.11 while the standard deviations range from 0.002 to 0.030. I use these rescaled investor expectation series for my analyses.

3.3 Consistency between Two Types of Surveys

With investor expectation surveys, economists have improved understandings on how investors form expectations of returns in their mind—there is a strong extrapolative structure in investors' expectations. However, demonstrating the consistency within investor expectation surveys is merely the first step in showing the validity of the contained information in the survey—after all, investor beliefs have various belief attributes, including expectations, variance, perceived left-tail risks and others. If surveys can only reflect one isolated type of investor belief information but fail to capture variations in other attributes of investor beliefs, then the validity of survey information is greatly undermined. Moreover, proposing belief formation mechanism only based on one attribute of belief could lead to severe over-fitting problem—the proposed belief mechanism might only explain certain types of belief attributes but miss in most of other attributes. Such over-fitting problem essentially lead to the "lack of discipline" critique.

Fortunately, extant surveys also records different attributes of investor beliefs, including investor expectations and perceived left-tail probabilities. The variety of investor belief attributes in the survey provide me an opportunity to answer one "big picture" question on the validity of survey information: is investor belief survey information really reliable?

To address this question, throughout this section, I treat the aggregated survey information as the consensus beliefs among survey participants. For example, an rescaled investor expectation of 10% in the survey effectively reflects a consensus market-wide investor expectation level of 10%. Irrespective of underlying return distribution assumptions, increase in consensus investor expectation should effectively reduces the consensus left-tail probabilities. If investor expectation surveys and tail risks survey truly contain consistent and effective information, there should be strong connections between the consensus information contained in each type of surveys. Therefore, investigating common variations between the consensus information in different investor belief surveys can help me effectively address the validity of survey information.

Interpretations of Shiller Tail Risks Survey

In this section, I start with interpretations of Shiller tail risks survey question under semi-annual frequency. Specifically, I interpret the reported probabilities in Shiller tail risks survey as the probabilities for a catastrophic event within six months. To fully extract information from the Shiller tail risks survey question, I also define the catastrophic tail event as the occurrence of more than 27.81% decline within future six-month horizon. [6]

Figure C.2 provides an intuition for the reported tail event probability of the catastrophic event in investors' mind. At each point in time, investors in the surveys form a perceived distribution for future returns, based on their perceived expected returns, conditional volatility, and other factors. When asked about the probability of the catastrophic event in Shiller tail risks survey, they report their tail event probability based on their perceptions in the minds, which corresponds to the shaded area in

[6] The number 27.81% is the average of the maximum market decline from two defined market downturn episodes in Shiller tail risk question: in 1929 market downturn the maximum decline is -31.95% and in 1987 market downturn, the maximum decline is 23.67%.

this figure. Therefore, this figure essentially provides a connection between investor expectation surveys and Shiller left-tail risk surveys. After all, the shaded area for the perceived tail event probability essentially depends on investors' perceived expectations of returns in the surveys.

[Place Figure C.2 about here]

To facilitate my analysis for this connection, I start with a simple normality assumption for the future return distribution, but will also report the robustness of my results using other types of distributions in the appendix. Specifically, to connect the perceived quantities in the surveys, I mainly focus on the following perceived return dynamics:

$$\Delta P_t / P_t = \mu^e_{P,t}\Delta t + \sigma^e_{P,t}\Delta\omega^e_t, \tag{3.2}$$

where $\mu^e_{P,t}$ and $\sigma^e_{P,t}$ are the annualized perceived expectations and perceived volatility of S&P 500 index, and $\Delta\omega^e_t$ is the perceived Brownian shocks that drives the return dynamics.

Dynamics in equation (3.2) indicates the return distribution at time t follows a conditional normal distribution with mean $\mu^e_{P,t}$ and volatility $\sigma^e_{P,t}$. Naturally, the perceived expectation of returns $\mu^e_{P,t}$ corresponds to the reported probabilities in investor expectation surveys at time t. For the perceived volatility $\sigma^e_{P,t}$, I use the VIX index based on S&P 500 index option at time t from Chicago Board Options Exchange because it reflects market-wide expectations of annualized future 30-day return volatility. [7] Then I can back out the left-tail probability at time t purely based on information in investor expectation survey and VIX using the following formula:

$$\text{Prob}^e(\Delta P_t / P_t < \bar{C}_t) = \Psi\left(\frac{\bar{C}_t - \mu^e_{P,t}\Delta t}{\sigma^e_{P,t}\sqrt{\Delta t}}\right), \tag{3.3}$$

where function Ψ represents the standard normal distribution and $\Delta t = 0.5$ under semi-annual frequency. Despite the conditional normality, the unconditional distribution of return distribution is highly skewed—with perceived expectations of returns being time-varying and positive in general, the unconditional return distribution is left-skewed.

[7] Replacing VIX with realized volatility does not significantly change my results.

The implied left-tail probability based on equation (3.3) builds the connection between two types of surveys. I plot the left-tail probability implied from investor expectation surveys in Figure C.3. All implied left-tail probabilities have strong countercyclical pattern and fluctuate synchronously with the reported left-tail probability from Shiller tail risk survey: investors in both types of survey perceive a rapidly increasing left-tail risks during NBER recessions and perceive a low left-tail risks during market booms such as Internet bubbles episodes around 1999 and the episodes between 2003 to 2006.

[Place Figure C.3 about here]

Correlations between Two Types of Surveys

To investigate the connections between two types of surveys, one natural way is to analyze their correlations: if two surveys reflect consistent information, then when investors report higher expected returns in expectation surveys, they should also report a lower tail event probabilities in Shiller tail risks survey.

To formally check this conjecture, I record correlations between the implied left-tail probability from investor expectation surveys and that from Shiller left-tail risks survey in column one of Table C.2. Despite different survey sources, the correlations are all around 0.60, which means the left-tail probabilities implied by investor expectation survey explain about 60% variations of Shiller tail risk survey. The corresponding p-values are all close to zero, indicating strong positive correlations between the implied crash probabilities and the Shiller tail risks surveys.

[Place Table C.2 about here]

Contemporaneous Connections between Two Types of Surveys

I conclude this section by providing another set of evidence on the consistency between two types of surveys. Based on the proposed belief mechanism in Figure C.2, changes in the investors' perceived expectations should negatively associate with their contemporaneous reported left-tail probabilities. To formally test this pattern, I run the following regression:

$$\text{Crash Index} = a + b \underbrace{\mu^{e}_{P,t}}_{\text{perceived expectations}} + u_t. \tag{3.4}$$

The proposed mechanism implies b should be negative and statistically significant.

Table C.3 and C.4 present results for the above regression. Table C.3 reports the regression results from equation (3.4). To provide a robust test, I report all possible associations between each investor expectation surveys and Shiller left-tail risks survey.

[Place Table C.3 and C.4 about here]

The results indicate that, regardless of the investor subsample I choose, responses in investor expectation surveys are negatively and significantly associated with reported left-tail probabilities within the same period, which is consistent with the proposed belief mechanism in Figure C.1.

3.4 Connections between Return Extrapolation and Perceived Left-tail Probabilities

What is the underlying belief mechanism for the perceived left-tail risks dynamics? With what I have shown so far—variations in the perceived left-tail probabilities are largely connected to changes in the perceived investor expectations—investigating patterns of investor expectations can help understand the left-tail probability dynamics.

In this, several previous empirical studies have stressed the role of extrapolative beliefs in explaining investor expectation variations in the survey and therefore can guide our research. Among others, Greenwood and Shleifer (2014) document that investors expect *excessively* higher returns going forward after observing a sequence of good past returns and vice versa, a belief formation pattern formally defined as *return extrapolation* in the literature. Additionally, previous works also provide evidence on how much weight investors put on the past returns when forming expectations of future stock returns.

Due to the important role of return extrapolation in driving investor expectations, very likely, it could help explain the perceived left-tail risks dynamics. For instance, during market booms, investors observe episodes of good returns and return extrapolation leads investors to become overly optimistic. With elevated perceived expectations, the perceived left-tail probabilities drop significantly and therefore investors neglect left-tail risks. Conversely, when the market is in its bust, a sequence of negative returns makes investors overly pessimistic. With a nosedive in investor expectations, the perceived left-tail probabilities rise rapidly—which accord with the countercyclicality of tail risks I document (FigureC.1).

Connections Based on Correlations

To test the connection between return extrapolation and the perceived left-tail risks dynamics, I first construct an expectation variable, Psentiment, which represents an expectation level based on return extrapolation. With this new expectation variable, I investigate to what extent return extrapolation alone can explain the perceived left-tail risks dynamics.

In constructing the new expectation variable Psentiment, I use the following formula in previous studies:

$$\text{Psentiment}_t = a + b \sum_{j=1}^{n} w_j R^D_{(t-j\Delta t)\to(t-(j-1)\Delta t)} + \varepsilon_t, \tag{3.5}$$

where $w_j = e^{-\lambda(j-1)\Delta_t} / \sum_{l=1}^{n} e^{-\lambda(l-1)\Delta_t}$ represents the decision weight and λ measures the memory decay speed for investors since it determines how much weight to assign when forming expectations of future stock returns. All parameters a, b, λ can be directly estimated from the investor expectation surveys.[8] Here I use the estimators $a = 0.004$, $b = 2.04$ and $\lambda = 0.428$ based on Gallup survey from Barberis et al. (2015) to construct my Psentiment variable. A magnitude of $\lambda = 0.43$ indicates the past returns three years ago is only 25% as important as the most recent returns.

I then back out the implied left-tail probabilities with Psentiment proxying for $\mu^e_{P,t}$. I report the correlation between implied left-tail probabilities based on Psentiment and the left-tail probability measures directly from Shiller tail-risk surveys in Panel A of Table C.5.

[Place Table C.5 about here]

It turns out that return extrapolation explains 40% variations of reported probabilities in Shiller survey. If we use Shiller crash indices which greatly eliminate outliers of reported probabilities, the correlation rises to 55%. Therefore, return extrapolation does a decent job in explaining the perceived left-tail risks dynamics.

Rational expectations models, on the contrary, do not fit into the documented left-tail risks pattern in the survey. Specifically, I use two types of rational expectation

[8]For detailed estimation procedure, see Barberis et al. (2015)

measures: logDP expected returns and ex-post twelve-months returns. The logDP expected returns are the fitted values from the multivariate regression of excess stock market returns over the next twelve months on the log dividend to price ratio, the Treasury-bill rate, default spread and the term spread. The ex-post twelve-months returns are just the accumulative future twelve-months returns in the future. The correlations between implied left-tail probabilities based on rational expectations and the left-tail probability measures directly from Shiller tail-risk surveys are reported in Panel B of Table C.5. Contrary to the results in Panel A, the correlations are in general very low and with an average magnitude of 0.10. For example, the implied left-tail risks based on logDP expected return and reported probability from Shiller tail-risk surveys only have a correlation of 0.09. The maximum correlation is only around 0.11, which is only less than one-third of that in Panel A of Table C.5 based on return extrapolation. The striking contrast between Panel A and B indicates that rational expectation models can hardly explain the dynamics of perceived left-tail probabilities.

Throughout my exercises so far, I have used both time-varying investor expectations and time-varying VIX to back out the perceived left-tail probabilities. One natural question to ask is how much variations return extrapolation alone can explain for left-tail probabilities. To answer this question, I construct the implied left-tail probabilities in equation (3.3) by excluding information from VIX and imposing a constant volatility of 20% for $\sigma^{e}_{P,t}$. The corresponding correlation results are reported in Table C.6.

[Place Table C.6 about here]

The reported results reinforce the connections between return extrapolation and the perceived left-tail probabilities. In Panel A of Table C.6, even without the VIX information, the implied left-tail probabilities from expectations based on either survey or return extrapolation on average explains 45% of the variations in the reported left-tail probabilities. When using Shiller Crash Index, the average correlations rise to an average of 55%. The p-values for each correlation relationships are still identically close to zeros—excluding VIX information does not undermine the underlying connections between return extrapolation and reported left-tail probabilities.

A striking contrast occurs in Panel B of Table C.6 in which I report the correlations between the implied left-tail probabilities based on rational expectations and the reported left-tail probabilities in the surveys. Without the assistance of VIX, now the overall correlations become negative. For example, compared to the previous (maximum) magnitude of 0.62 in Panel A of Table C.6, now the correlations becomes 0.04. Therefore, rational expectations tend to interpret the variations in the reported left-tail probabilities in the wrong directions—they predict high perceived left-tail risks during booms and low perceived left-tail risks during recessions—which are opposite to the patterns in the survey.

Connections Based on Regressions

Guided by previous empirical research (Greenwood and Shleifer (2014)), investor expectations in the surveys are highly correlated with the current log price to dividend ratios.[9] Moreover, correlation analysis in previous sections provides intuitive connections between return extrapolation and left-tail probabilities. If the perceived tail risks in the survey are truly driven by the variations in the investors' perceived expectations, then the reported tail risks should also depend on the current log price to dividend ratio. Therefore, I run the following regression:

$$\text{Crash Index} = a + blog(P_t/D_t) + cVIX_t + dX_t + u_t. \quad (3.6)$$

Here P_t/D_t denotes the price to dividend ratio which is a measure of the price level, VIX_t stands for the VIX index at time t and X_t denotes other variables. I report the regression results in Table C.7 in which all reported standard errors are corrected following the approach in Newey and West (1986).

[Place Table C.7 and C.8 about here]

The reported results are largely consistent with the previous analyses, especially with the proposed belief formation mechanism in Figure C.2. Overall, the reported left-tail probabilities are well explained by both the current $\log(P_t/D_t)$ ratios and VIX index. First, the coefficient of $\log(P_t/D_t)$ is negative: when past price has been high, investors expect higher returns going forward, therefore they perceive a lower left-tail probabilities.[10] Second, the coefficient for the VIX index is positive. This

[9] For most surveys, investor expectations are also closely connected with the past twelve-months returns. However, for Shiller survey, investor expectations are only connected to the current $\log(P_t/D_t)$ ratio (see Greenwood and Shleifer (2014)). Hence, we only use $\log(P_t/D_t)$ ratio here to explain the variations in the reported left-tail probabilities in the Shiller left-tail risks surveys.

[10] A strong pattern documented in Greenwood and Shleifer (2015) which supports the return extrapolation hypothesis.

is because, as VIX_t increases, investors perceive higher volatilities for future returns and hence a higher left-tail probabilities. Moreover, consistent with my previous findings, both time-varying investor expectations and VIX play important roles in determining the perceived left-tail probabilities[11]

In addition to the current price to dividend ratio and VIX index, I experiment with several other proxies related to the left-tail probabilities, including the realized volatilities, current risk free rate, default spread defined as the difference between Moody's Baa and Aaa corporate bond yield, and the term yield which is the difference between 10-year T-bond and three-month T-bill yields. None of these variables except for the default yield, it turns out, has robust explanatory power. Although these variables sometimes have statistically significant explanatory power, they are not stable and eliminated after controlling for the current price to dividend ratio and VIX index.

The results in Table C.5, C.6, C.7 and C.8 are not only broadly consistent with the evidence on return extrapolation, but also expand the role of return extrapolation and deepen understandings on investors belief formation patterns in different dimensions. When stock market prices have been increasing, investors not only become overly optimistic about the expectation of future market returns, they also perceive low left-tail probabilities and thus neglect left-tail risks. Further, these results also reinforce the validity of survey evidence: not only that survey measures of investor expectations are highly correlated across data sources, reported investor left-tail probabilities and investor expectations in the surveys are mutually coherent.

Moreover, the fact I document in this section—that return extrapolation explains both the excessive optimism and pessimism in investor expectations and perceived left-tail risk—raises a higher bar for all other belief formation mechanisms: in order to convincingly explain the observed patterns in the survey, models should both explain dynamics in investor expectations and the variations in the left-tail probabilities.

3.5 Concluding Remarks

Despite the importance of tail risks in explaining central asset pricing facts such as equity premium, option variance premium and volatility smile, economists have

[11]Notice that, when using Shiller Crash Index as a proxy for investor beliefs on tail risks, there is a difference between results from two investor subsamples: institutional investors are less influenced by the current price level but are still influenced by the VIX index. This indicates that institutional investors might be more sophisticated and therefore are less subject to psychological heuristics.

very limited knowledge about investors' perceptions of tail risks. In this paper, I use survey data to measure investors' perceptions on tail event probabilities in the stock market.

Specifically, I make several contributions to understanding investor's perceptions of tail risks using survey data. First, to further relieve concerns on survey information quality, I reinforce its validity by demonstrating consistency between two different types of investor belief surveys; indeed, changes in investor expectation surveys can explain up to 76% variations in Shiller tail risks survey. Therefore, compared to the findings in Greenwood and Shleifer (2014) that surveys contain consistent information on investor expectations, my results effectively provide further cross-validation evidence between different survey types. To argue against one type of survey, researchers need to explain variations both in investor expectation surveys and Shiller tail risks survey.

Second, with improved confidence in the quality of survey data, I demonstrate that investors' perceptions of tail risks and perceived expectations of returns can be largely interpreted within one unified belief formation framework based on return extrapolation. Therefore, return extrapolation can effectively explain the time-varying optimism and pessimism on investor expectations as well as the overestimation and underestimation of tail risks. By explaining investor beliefs in different dimensions within one framework, return extrapolation effectively avoids the "lack of discipline" critique.

There are several directions for future research. First, although return extrapolation seems to be the main driver for the perceived tail risks dynamics, VIX also plays a significant role. Therefore, understanding investors' perceptions of volatilities can help further refine understandings of investors' perceptions of tail risks. Second, due to similarities between perceived tail risks in the stock market and the consumption process, the belief mechanism based on return extrapolation has the potential to help understand the time-varying probabilities in rare disasters literature. A further investigation along this direction seems very interesting.

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Appendix A

APPENDIX TO CHAPTER ONE

A.1 Micro-foundations for Fundamental Investors

To keep the decision problem of fundamental investors simple, I model the fundamental investors as an overlapping generation (OLG) of agents. To explain the OLG timeline in a continuous time setting clearly, I index time as t, $t + \Delta t$, $t + 2\Delta t$ and so on. Each fundamental investor in generation t inherits wealth from the last generation and lives between period t, and $t + \Delta t$. For simplicity, fundamental investors are assumed to have homogeneous wealth levels of W_t^f at the beginning of period t and adjust their portfolio to maximize the exponential utility with respect to their bequest to the next generation, $W_{t+\Delta t}^f$.

One crucial assumption here, however, is that fundamental investors can liquidate their risky asset at the fundamental value $P_{F,t} \equiv \frac{D_t}{r-g_D}$ at $t + \Delta t$.[1] There are multiple ways to justify this assumption. One possible situation is that at the end of each period, fundamental investors can trade with mutual funds who target on fundamental values of assets with a price of $P_{F,t}$.

The excess return from holding the risky asset for fundamental investors is therefore $\frac{P_{F,t}-P_t}{P_t} + \frac{1}{l}$, where $\frac{1}{l}$ is the dividend to price ratio. Moreover, their trade counterparts also suffer from exogenous liquidity shocks, $\tilde{\epsilon} \sim N(1 - r + \frac{1}{l}, \frac{\sigma_\epsilon^2}{P_t})$.[2] The liquidity shock also affects the resale returns of the risky asset price—with a realized shock of ϵ_t, holding risky asset only yields an excess return of $\frac{P_{F,t}-P_t}{P_t} + \frac{1}{l} - \epsilon_t$.

The timeline for fundamental investors is as follows. At the beginning of period t, fundamental investors receive bequest amount of W_t^f and observe the risky asset price, P_t, as well as the fundamental value, $P_{F,t}$. They select their risky asset position to maximize:

$$U(W_{t+\Delta t}^f) = -exp(-\gamma_h W_{t+\Delta t}^f). \tag{A.1}$$

[1]The fundamental value of the risky asset follows the Gordon growth formula.

[2]Such shocks can be motivated by liquidity constraints. Intuitively, the mean value of liquidity shock is positively correlated with interest rate r: higher interest rate elevate the liquidity shock. Also, the variance of ϵ is inversely correlated to the price level of the risky asset, with higher risky asset price associated with lower liquidity risk. At the same time, the liquidity shock is also negatively related to the current price level, measured by the price to dividend ratio l.

with budget constraint

$$W_{t+\Delta t}^f = W_t^f r + \alpha_t^f W_t^f (1 + \frac{P_{t+\Delta t} - P_t}{P_t} + \frac{1}{l} - \epsilon_t - r), \tag{A.2}$$

where γ_h represents the risk aversion coefficient of fundamental investors and α_t^f is the total risky asset demand of fundamental investors. The first order condition gives the optimal portfolio choice:

$$\alpha_t^f = \frac{P_{F,t} - P_t}{\gamma_h \sigma_\epsilon^2 W_t^f}. \tag{A.3}$$

Therefore, the per-capita total dollar demand of fundamental investors is

$$Q_t \equiv \alpha_t^f W_t^f = \frac{P_{F,t} - P_t}{\gamma_h \sigma_\epsilon^2} \equiv \frac{P_{F,t} - P_t}{k}, \tag{A.4}$$

where k is a constant.

A.2 Rational Benchmark Model

Following the standard Merton method, we have the optimal portfolio for the extrapolators with fully correct beliefs:

$$\alpha_t = \frac{\bar{g}_{P,t} + l^{-1} - r}{\bar{\sigma}^2_{P,t}}. \tag{A.5}$$

Using Ito's lemma on both sides of $x_t \equiv \frac{W_t}{D_t}$:

$$\begin{aligned} dx_t = d(W_t/D_t) =& x(dW_t/W_t - dD_t/D_t + (dD/D)^2 - (dW/W)(dD/D)) \\ =& x(r - \rho + \alpha_t(\bar{g}_{P,t} + l^{-1} - r) - g_D + \sigma^2_D - \alpha_t \sigma_D \bar{\sigma}_{P,t})dt \\ &+ x(\alpha_t \bar{\sigma}_{P,t} - \sigma_D)d\omega_t \\ \equiv& g_x dt + \sigma_x d\omega_t. \end{aligned} \tag{A.6}$$

Combined with the geometric form of the dividend process and the logarithmic form utility function, I conjecture that the underlying state of the economy is $x_t \equiv \frac{W_t}{D_t}$, the wealth to dividend ratio. In other words, all equilibrium quantities can be expressed as a function of x_t. For example, the price to dividend can be denoted as $\bar{l}(x_t) = \frac{P_t}{D_t}$. For notational convenience, I denote the dynamics of x_t as:

$$dx_t = g_{x,t}(x_t)dt + \sigma_{x,t} d\omega_t. \tag{A.7}$$

Then I apply Ito's lemma on both sides of $l = P/D$ and get:

$$dl = l_x dx + \frac{1}{2} l_{xx} \sigma^2_x dt = (l_x g_x + \frac{1}{2} l_{xx} \sigma^2_x)dt + l_x \sigma_x d\omega_t, \tag{A.8}$$

and

$$\begin{aligned} d(P_t/D_t) =& dP_t/P_t - dD_t/D_t + (dD_t/D_t)^2 - (dD_t/D_t)(dP_t/dP_t) \\ =& l(\bar{g}_{P,t} - g_D + \sigma^2_D - \sigma_D \bar{\sigma}_{P,t})dt + l(\bar{\sigma}_{P,t} - \sigma_D)d\omega_t. \end{aligned} \tag{A.9}$$

By matching terms:

$$l_x g_x + \frac{1}{2} l_{xx} \sigma^2_x = l(\bar{g}_{P,t} - g_D + \sigma^2_D - \sigma_D \bar{\sigma}_{P,t}), \tag{A.10}$$

and

$$l_x \sigma_x = l(\bar{\sigma}_{P,t} - \sigma_D). \tag{A.11}$$

Then by combining equation (A.6) and (A.10), I can solve for $\bar{g}_{P,t}$:

$$\begin{aligned}\bar{g}_{P,t} =& \frac{l_x x(r - \rho - g_D + \sigma_D^2) + l_x(c_0 l - c_1)(l^{-1} - r - \sigma_D \bar{\sigma}_{P,t})}{l - l_x(c_0 l - c_1)} \\ &+ \frac{1}{2} l_{xx} \sigma_x^2 + l(g_D - \sigma_D^2 + \sigma_D \bar{\sigma}_{P,t}) l - l_x(c_0 l - c_1).\end{aligned} \tag{A.12}$$

Then I can get

$$g_x = x_t(r - \rho - g_D + \sigma_D^2) + (c_0 l - c_1)(\bar{g}_{P,t} + l^{-1} - r - \sigma_D \bar{\sigma}_{P,t}) \tag{A.13}$$

and solve for

$$\bar{\sigma}_{P,t} = \sigma_D + \frac{l_x}{l} x(\alpha_t \bar{\sigma}_{P,t} - \sigma_D). \tag{A.14}$$

With $x_t \alpha_t = c_0 l - c_1$, I have:

$$\begin{aligned}\bar{\sigma}_{P,t} &= \sigma_D \frac{1 - \frac{l_x}{l} x_t}{1 - \frac{l_x}{l}(c_0 l - c_1)}, \\ \sigma_x &= (c_0 l - c_1)\bar{\sigma}_{P,t} - x_t \sigma_D.\end{aligned} \tag{A.15}$$

Combined with the market clearing conditions and the Chebyshev polynomials (for more details, see appendix A.3), I numerically solve this model.

A.3 Behavioral Model

In the behavioral model, the time-varying investment set is driven by two state variables: x_t and S_t. Therefore, using the standard argument in Merton (1971), I define the extrapolators' value function as

$$J(W_t, S_t, x_t) \equiv \max_{\{C_{t+s}\}_{s\geq 0}} \mathbb{E}_t^e[\int_t^\infty e^{-\rho s} \ln C_s ds] \tag{A.16}$$

For the logarithmic form utility function, I guess the corresponding value function has the form

$$J(W_t, S_t, x_t) = 1/\rho \ln(W_t) + j(S_t, x_t), \tag{A.17}$$

Under extrapolators' subjective beliefs, after omitting the subscripts, the Hamilton-Jacobian-Bellman (HJB) equation follows

$$\begin{aligned} \rho J(W, S, x) =& ln(W) + \rho j(S, x) \\ =& \max_{C,\alpha} [\ln C + J_W W[-C/W dt + r dt + \alpha[\hat{g}_{P,t} + l^{-1} - r]dt] \\ &+ 1/2 J_{WW}\alpha^2 W^2 \sigma_{P,t}^2 dt + J_S \mu_S dt + 1/2 J_{SS}\sigma_S^2 dt \\ &+ J_x g_x dt + 1/2 J_{xx}\sigma_x^2 dt + J_{Sx}\sigma_S \sigma_x dt], \end{aligned} \tag{A.18}$$

where J_{WS} and J_{Wx} all equal to zero and are omitted.

By the FOC *w.r.t* C, I have

$$1/C_t - J_W = 0,$$

which implies

$$C_t = \rho W_t. \tag{A.19}$$

Substitute equation (A.19) into the HJB equation (A.18), I have the following optimization problem with respect to α_t:

$$\max_{\alpha} [\ln \rho + J_W W[-\rho dt + r dt + \alpha(\hat{g}_{P,t} + l^{-1} - r)dt] + 1/2 J_{WW}\alpha^2 W^2 \sigma_{P,t}^2 dt \tag{A.20}$$

$$+ J_S \mu_S dt + 1/2 J_{SS}\sigma_S^2 dt + J_x g_x dt + 1/2 J_{xx}\sigma_x^2 dt + J_{Sx}\sigma_S \sigma_x dt].$$

Again, with the FOC $w.r.t$ α

$$J_W W(\hat{g}_{P,t} + l^{-1} - r) + J_{WW}\alpha_t^2 W^2 \sigma_{P,t}^2 = 0,$$

and substitute $J_W = \rho/W_t$ and $J_{WW} = -\rho\frac{1}{W^2}$, I get

$$\alpha_t = \frac{\hat{g}_{P,t} + l^{-1} - r}{\sigma^2_{P,t}}. \tag{A.21}$$

The fundamental investors' linear demand function, and the geometric Brownian form of the dividend process and the logarithmic form of extrapolators' utility function, together indicate a linear relation between the equilibrium price P_t and the dividend process D_t. For analytical convenience, I denote W_t as the aggregate wealth and define $x_t \equiv \frac{W_t}{D_t}$ and $l(S_t, x_t) \equiv \frac{P_t}{D_t}$, which follows:

$$\begin{aligned} dx_t &= g_{x,t}(S_t, x_t)dt + \sigma_{x,t}(S_t, x_t)d\omega_t \\ &= \hat{g}_{x,t}(S_t, x_t)dt + \sigma_{x,t}(S_t, x_t)d\omega^e_t. \end{aligned} \tag{A.22}$$

With extrapolators' optimal strategy, I can solve for the dynamics of x_t as follows:

$$dx_t = x_t(r - \rho + \alpha_t[\hat{g}_{P,t} + l^{-1} - r] - \hat{g}_{D,t} + \sigma^2_D - \alpha_t\sigma_D\sigma_{P,t})dt + x_t(\alpha_t\sigma_{P,t} - \sigma_D)d\omega^e_t, \tag{A.23}$$

which yields

$$\begin{aligned} \hat{g}_{x,t}(S_t, x_t) &= x_t(r - \rho + \alpha_t[\hat{g}_{P,t} + l^{-1} - r] - \hat{g}_{D,t} + \sigma^2_D - \alpha_t\sigma_D\sigma_{P,t}), \\ \sigma_{x,t}(S_t, x_t) &= x_t(\alpha_t\sigma_{P,t} - \sigma_D). \end{aligned} \tag{A.24}$$

To solve for $\hat{g}_{P,t}$ and $\hat{g}_{D,t}$, I consider $l(S_t, x_t) \equiv \frac{P_t}{D_t}$ and apply Ito's lemma on both sides of it:

$$\begin{aligned} RHS = dl &= l_x dx + l_S dS + 1/2 l_{xx}(dx)^2 + 1/2 l_{SS}(dS)^2 + l_{Sx}(dx)(dS) \\ &= (l_x\hat{g}_x + l_S\mu_S + 1/2 l_{xx}\sigma^2_x + 1/2 l_{SS}\sigma^2_S + l_{Sx}\sigma_x\sigma_S)dt + \\ &\quad + (l_x\sigma_x + l_S\sigma_S)\omega^e_t, \end{aligned} \tag{A.25}$$

and

$$LHS = d(\frac{P}{D}) = \frac{P}{D}[\frac{dP}{P} - \frac{dD}{D} + (\frac{dP}{P})^2 - (\frac{dP}{P})(\frac{dD}{D})] \tag{A.26}$$
$$= l\Big((\hat{g}_{P,t} - \hat{g}_{D,t} + \sigma_D^2 - \sigma_D \sigma_{P.t})dt +$$
$$+ (\sigma_{P,t} - \sigma_D)\omega_t^e\Big).$$

By matching terms, I get

$$\sigma_{P,t} = \sigma_D + l^{-1}(l_x \sigma_x + l_S \sigma_S), \tag{A.27}$$

and

$$\hat{g}_{P,t} = \hat{g}_{D,t} - \sigma_D^2 + \sigma_D \sigma_{P.t} + l^{-1}(l_x \hat{g}_x + l_S \mu_S + 1/2 l_{xx} \sigma_x^2 + 1/2 l_{SS} \sigma_S^2 + l_{Sx} \sigma_x \sigma_S). \tag{A.28}$$

Also, by the market clearing condition (1.20) in the main text, I have

$$\mu \alpha_t W_t + (1 - \mu) Q = P_t, \tag{A.29}$$
$$\mu \alpha_t x_t + \frac{1 - \mu}{k(r - g_D)} = \frac{1 - \mu + k}{k} l,$$
$$\alpha_t x_t = \frac{1 - \mu + k}{k\mu} l - \frac{1 - \mu}{\mu k (r - g_D)} \equiv c_0 l - c_1,$$

Then I solve for $\sigma_{P,t}$. substitute (A.24) into (A.27), and use the market clearing condition, I can get

$$\sigma_x = \alpha_t x \sigma_{P,t} - \sigma_D x = (c_0 l - c_1)\sigma_{P,t} - x\sigma_D. \tag{A.30}$$

Substitute the above equation and equation (1.10) into equation (A.27), I get a function for σ_P:

$$(l_x/l(c_0 l - c_1) - 1)\sigma_{P,t}^2 - (l_x/lx - 1)\sigma_D \sigma_{P,t} + l_s/l\theta(\mu_H - S)(S - \mu_L) = 0, \tag{A.31}$$

and we get the expression for σ_P

$$\sigma_{P,t} = \frac{(\frac{l_x}{l}x-1)\sigma_D - \sqrt{(\frac{l_x}{l}x-1)^2\sigma_D^2 - 4(\frac{l_x}{l}(c_0 l - c_1)-1)\frac{l_S}{l}\theta(\mu_H - S)(S-\mu_L)}}{2(\frac{l_x}{l}(c_0 l - c_1)-1)}. \tag{A.32}$$

There are two roots to the equation (A.31) and by taking $x \to 0$, I can easily exclude one of them.

The PDE for function $l(S,x)$ can be obtained by combining the market clearing condition and extrapolators' optimal trading strategy:

$$\alpha_t = \frac{c_0 l - c_1}{x} = \frac{\hat{g}_{P,t} + l^{-1} - r}{\sigma_{P,t}^2}. \tag{A.33}$$

Note that σ_P is a nonlinear function of l, l_x and l_S. To solve it, I apply Chebyshev projection method described in the appendix.

Then further we can get expressions for g_x. First, by combining the drift terms in equation (A.26) and (A.25), I have

$$l(\hat{g}_{P,t} - \hat{g}_{D,t} + \sigma_D^2 - \sigma_D\sigma_{P,t}) = l_x\hat{g}_x + l_S\mu_S + 1/2 l_{xx}\sigma_x^2 + 1/2 l_{SS}\sigma_S^2 + l_{Sx}\sigma_x\sigma_S, \tag{A.34}$$

and with the definition of $\hat{g}_x$ in equation (A.24), I get

$$\begin{aligned}\hat{g}_{D,t} =& \frac{xl_x/l(r-\rho+\alpha_t(\hat{g}_{P,t}+l^{-1}-r)+\sigma_D^2-\sigma_D\sigma_{P,t}\alpha_t)+l_s/l\mu_S+1/2\frac{l_{xx}}{l}\sigma_x^2}{xl_x/l-1} \\ &+ \frac{1/2\frac{l_{ss}}{l}\sigma_s^2+\frac{l_{sx}}{l}\sigma_S\sigma_x-(1-\theta)g_D-\theta S-\sigma_D^2+\sigma_D\sigma_{P,t}}{xl_x/l-1},\end{aligned} \tag{A.35}$$

I can get the expression for the risky asset growth rate under the true probability measure:

$$g_{P,t} = \hat{g}_{P,t} + \sigma_{P,t}/\sigma_D(g_D - \hat{g}_{D,t}). \tag{A.36}$$

Then $\hat{g}_x$ becomes

$$\hat{g}_x = x(r-\rho+\alpha_t(\hat{g}_{P,t}+l^{-1}-r) - \hat{g}_{D,t} + \sigma_D^2 - \alpha_t\sigma_D\sigma_{P,t}). \tag{A.37}$$

For the boundary conditions, when $x_t \to 0$, fundamental investors dominate and risky asset price is mainly driven by fundamental investors' demand; by the market clearing condition (1.20), I have

$$\lim_{x_t \to 0} l = \frac{c_1}{c_0}, \tag{A.38}$$

where $c_0 = \frac{k+1-\mu}{k\mu}$ and $c_1 = \frac{1-\mu}{k\mu(r-g_D)}$ are both constant. Furthermore,

$$\begin{aligned}
\lim_{x \to 0} l_x &= \lim_{x \to 0} \frac{l(S,x) - l(S,0)}{x} \\
&= \lim_{x \to 0} \frac{\alpha x/c_0 + \frac{c_1}{c_0} - \frac{c_1}{c_0}}{x} \\
&= \lim_{x \to 0} \frac{\hat{g}_{P,t} + \frac{c_0}{c_1} - r}{\sigma_D^2} \frac{1}{c_0} \equiv \frac{\alpha(S,0)}{c_0}.
\end{aligned} \tag{A.39}$$

For the case where $x_t \to \infty$, extrapolators dominate and in order to clear the market, α goes to 0; otherwise the asset prices goes to infinity. This means that the conditional excess returns perceived by extrapolators are zero, leading to

$$\lim_{x \to \infty} l = (r - \hat{g}_{P,t})^{-1}. \tag{A.40}$$

Chebyshev polynomial requires a domain of $[-1, 1]$. Therefore, the following transformations maps S and x to new variables that lie between -1 and 1:

$$\begin{aligned}
y &= aS + b, \\
a &= \frac{2}{\mu_H - \mu_L}, \quad b = -\frac{\mu_H + \mu_L}{\mu_H - \mu_L}, \\
z &= \frac{x - \zeta}{x + \zeta},
\end{aligned} \tag{A.41}$$

where ζ is some constant. When $S = \mu_L$,

$$y = \frac{2\mu_L}{\mu_H - \mu_L} - \frac{\mu_H + \mu_L}{\mu_H - \mu_L} = -1; \tag{A.42}$$

when $S = \mu_H$,

$$y = \frac{2\mu_H}{\mu_H - \mu_L} - \frac{\mu_H + \mu_L}{\mu_H - \mu_L} = 1; \tag{A.43}$$

Similarly, when $x = 0$, $z = -1$ and when $x = +\infty$, $z = 1$.

The following equalities prove to be useful:

$$x = \frac{\zeta z + \zeta}{1 - z}, \tag{A.44}$$

and

$$\frac{\partial z}{\partial x} = \frac{(1 - z)^2}{2\zeta}. \tag{A.45}$$

Further, I define $l(S, x) = l(\frac{y-b}{a}, \frac{\zeta(z+1)}{1-z}) \equiv m(y, z)$. Then the corresponding derivatives of $m(y, z)$ follow

$$\begin{aligned} l_x =& \frac{\partial z}{\partial x} m_z = \frac{(1 - z)^2}{2\zeta} m_z, \\ l_S =& \frac{\partial y}{\partial S} m_y = a m_y, \\ l_{xx} =& \frac{(1 - z)^3}{(2\zeta)^2}[(1 - z)m_{zz} - 2m_z] \\ l_{SS} =& a^2 m_{yy}, \\ l_{Sx} =& a\frac{(1 - z)^2}{2\zeta} m_{yz}. \end{aligned} \tag{A.46}$$

Substitute all the terms into the PDE and its boundary conditions, I get

$$\lim_{z \to -1} m(y, z) = \frac{c_1}{c_0}, \tag{A.47}$$

$$\lim_{z \to -1} m_z(y, z) = \frac{\zeta}{2}\frac{1}{c_0}\alpha(S, 0) \equiv \frac{\zeta}{2c_0}\alpha_0, \tag{A.48}$$

$$\lim_{z \to 1} m(y, z) = (r - \hat{g}_{P,t})^{-1} = \frac{1}{r - (1 - \theta)g_D - \frac{\theta}{a}(y - b)}, \tag{A.49}$$

where $\alpha_0 = \frac{(1-\theta)g_D + \frac{\theta}{a}(y-b) + \frac{c_0}{c_1} - r}{\sigma_D^2}$ and $\alpha_{0,y} = \frac{\theta}{a\sigma_D^2}$. I use Chebyshev polynomial to approximate it.

To be specific, I set $m(y, z)$ as follows:

$$m(y, z) \equiv \frac{c_1}{c_0} + \frac{\zeta}{2c_0}(1 + z)\alpha_0 + (1 + z)^2 \sum_{i+j\leq N} a_{ij}T_i(z)T_j(y) \tag{A.50}$$

$$\equiv \frac{c_1}{c_0} + \frac{\zeta}{2c_0}(1 + z)\alpha_0 + (1 + z)^2 v(y, z), \tag{A.51}$$

where $v(y, z) = \sum_{i+j\leq N} a_{ij}T_i(z)T_j(y)$. Here $T_i(z)$ and $T_j(y)$ are Chebyshev Polynomial functions and $a_{i,j}$ are the coefficients to be determined. Moreover,

$$m_z(y, z) = \frac{\zeta}{2c_0}\alpha_0 + 2(1 + z)v(y, z) + (1 + z)^2 v_z(y, z), \tag{A.52}$$

$$m_{zz}(y, z) = 4(1 + z)v_z(y, z) + 2v(y, z) + (1 + z)^2 v_{zz}(y, z), \tag{A.53}$$

$$m_y(y, z) = \frac{\zeta}{2c_0}(1 + z)\alpha_{0,y} + (1 + z)^2 v_y(y, z), \tag{A.54}$$

$$m_{yy}(y, z) = (1 + z)^2 v_{yy}(y, z)(\text{since } w_{yy} = 0), \tag{A.55}$$

$$m_{zy}(y, z) = \frac{\zeta}{2c_0}\alpha_{0,y} + 2(1 + z)v_y(y, z) + (1 + z)^2 v_{zy}(y, z), \tag{A.56}$$

$$\tag{A.57}$$

and boundary conditions (A.38) and (A.39) hold automatically. Here $T_i(z)$ and $T_j(y)$ are Chebyshev polynomials evaluated at z and y respectively. Therefore, I need to minimize

$$\sum_{ij}^{M} \beta_{ij}^1 \Big[\frac{c_0 m(x_i, y_j) - c_1}{x_i} - \frac{\hat{g}_P(x_i, y_j) + m(x_i, y_j)^{-1} - r}{\sigma_P^2(x_i, y_j)}\Big]^2 \tag{A.58}$$
$$+K \sum_j \beta_j^2 \Big[m(y_j, 1) - (r - \hat{g}_{P,t})^{-1}\Big]^2.$$

A.4 Figures and Tables

Figure A.1: Gallup Investor Expectations and S&P 500 Index

In the sample period 1996:10 - 2011:11, I plot both investor expectation index in Gallup survey and the S&P500 index. The blue solid line plots Gallup Index, defined as the percentage difference between bullish investors and bearish investors. The red dashed line plot the S&P500 index. The shaded areas represent NBER recessions.

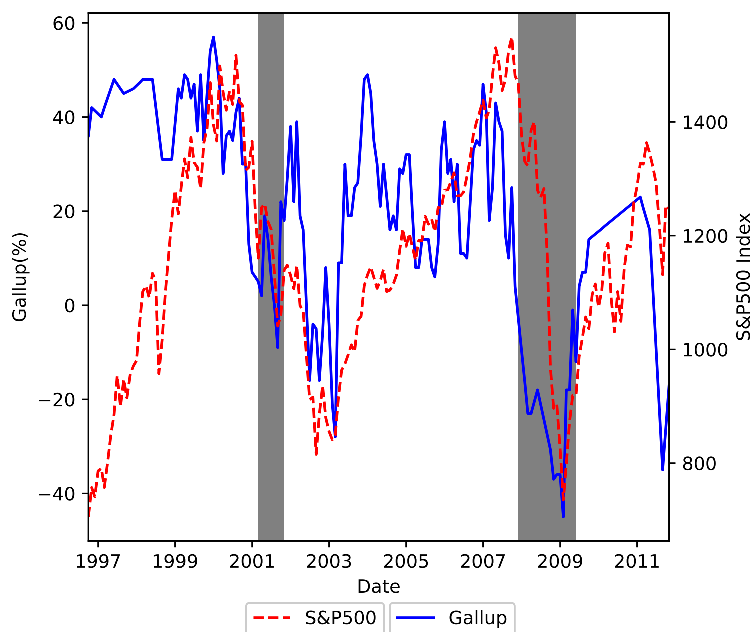


Figure A.2: Gallup Investor Expectations and Household Mutual Fund Flows

In the sample period 1996:10 - 2011:11, I plot both investor expectation index in Gallup survey and the household flows from HNPO sector. The blue solid line plots Gallup Index and red dashed line plots the household mutual fund flows. The shaded areas represent NBER recessions.

fund household inflow vs Gallup survey.png

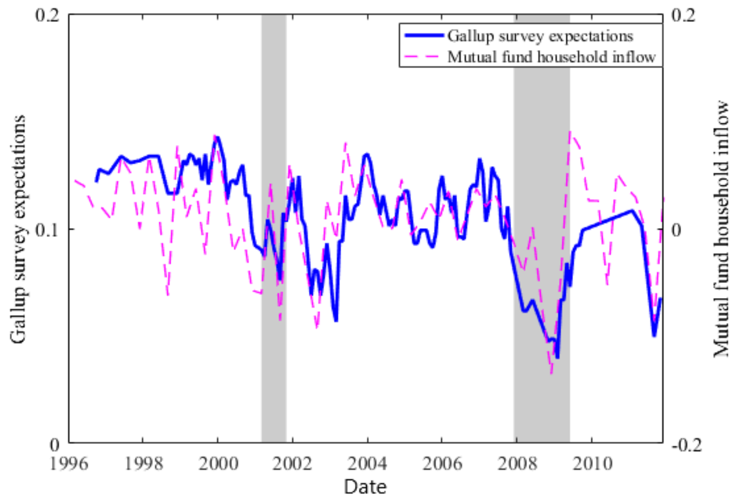


Figure A.3: Predictive Power of Investor Sentiment: Intuition

This figure provides intuition for the positive predictive pattern of investor sentiment when the wealth is low. When the wealth level is low, extrapolators have low market impact and they are very unlikely to push asset prices away from the fundamental values. In this situation, investor sentiment reflects market correction and therefore positively predict future returns.

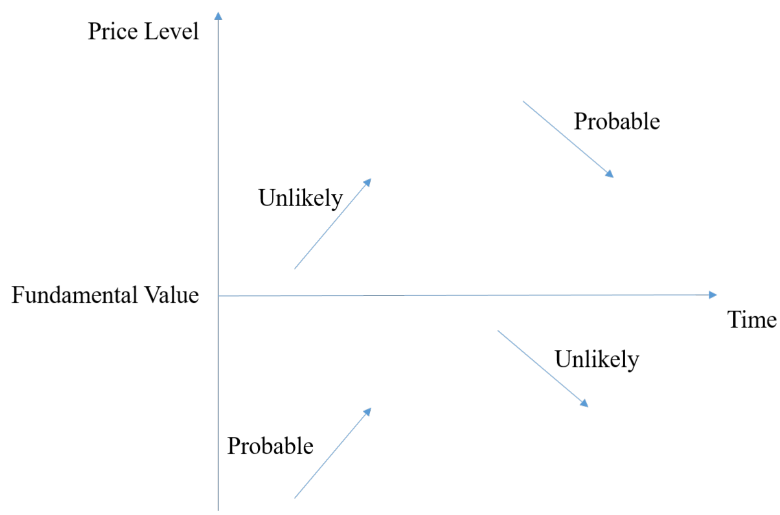


Figure A.4: Calibrated Model Solutions: Rational Benchmark.

This figure plots the price to dividend ratio l, the optimal portfolio α_t, and the return volatility σ_P as functions of both the latent state variable S_t and the transformed wealth to dividend ratio z_t in the rational benchmark model. As z_t goes to -1 (1), the wealth to dividend ratio $\frac{W_t}{D_t}$ goes to zero (infinity). The parameter values are $r = 4\%$, $g_D = 1.5\%$, $\rho = 2\%$, $k = 0.5$, $\mu = 0.5$, $\mu_H = 3\%$, $\mu_L = -9\%$, $\chi = 0.15$, $\lambda = 0.15$.

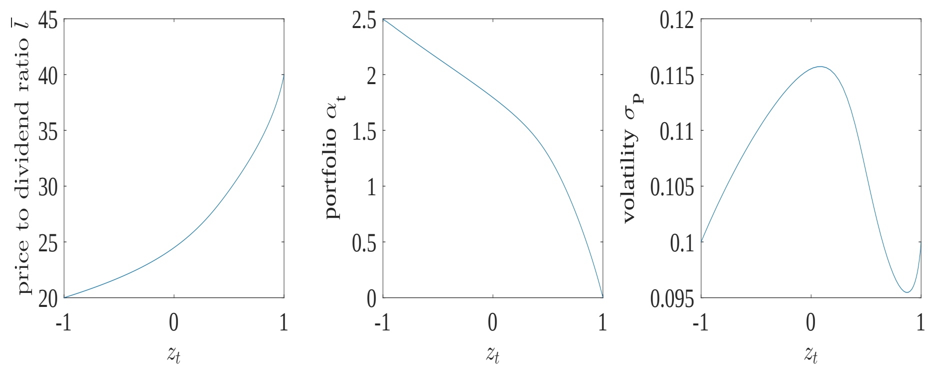


Figure A.5: Calibrated Model Solutions: Behavioral Model.

This figure plots the price to dividend ratio l, the optimal portfolio α_t, and the return volatility σ_P as functions of both the latent state variable S_t and the transformed wealth to dividend ratio z_t in the behavioral model. As z_t goes to -1 (1), the wealth to dividend ratio $\frac{W_t}{D_t}$ goes to zero (infinity). The parameter values are $r = 4\%$, $g_D = 1.5\%$, $\rho = 2\%$, $k = 0.5$, $\mu = 0.5$, $\mu_H = 3\%$, $\mu_L = -9\%$, $\chi = 0.15$, $\lambda = 0.15$.

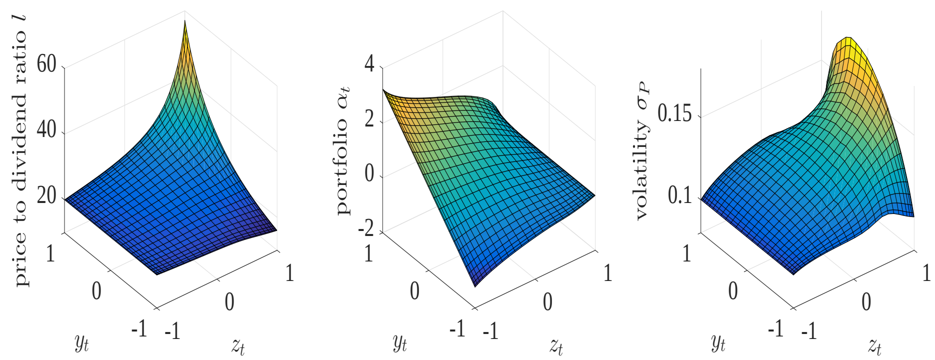


Figure A.6: Gallup Survey and Simulated Investor Sentiment.

In the sample period 1996:10 - 2011:11, I plot both investor expectation index in Gallup survey and investor sentiment based on model simulations. The blue solid line plots Gallup Index and red dashed line plots the investor sentiment based on model simulations. The shaded areas represent NBER recessions. The parameter values are $r = 4\%$, $g_D = 1.5\%$, $\mu_H = 0.03\%$, $\mu_L = -0.06\%$, $\chi = \lambda = 10\%$, $\theta = 0.5\%$, $\rho = 2\%$, $k = 0.5$, $\mu = 0.5$, $\mu_H = 3\%$, $\mu_L = -3\%$.

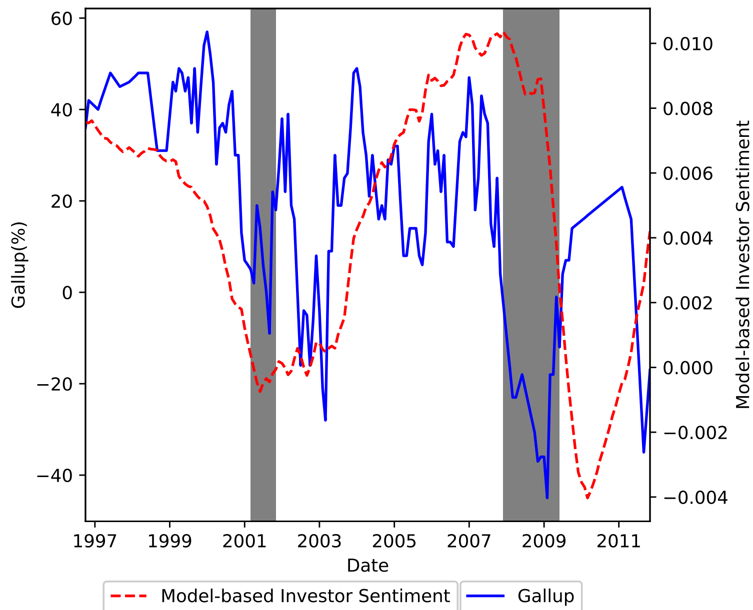
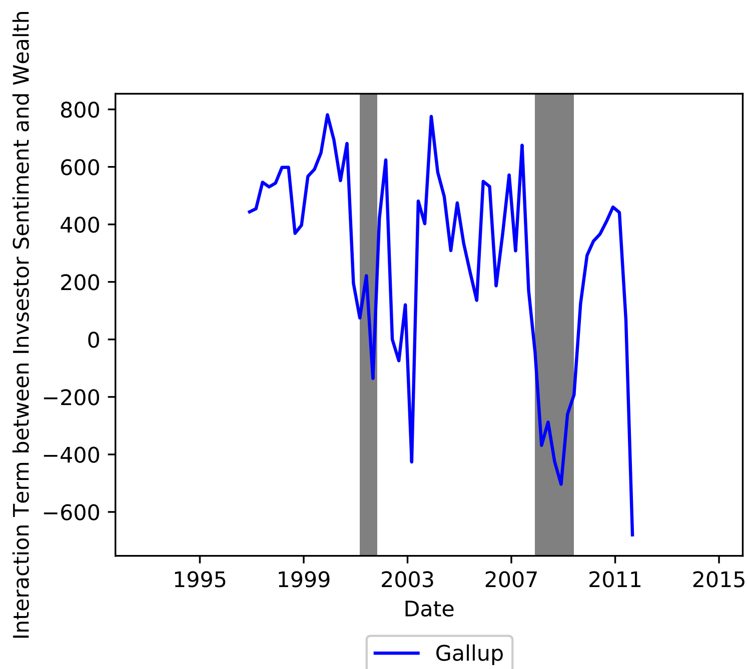


Figure A.7: Interaction between Wealth and Investor Sentiment vs Degree of Extrapolation.

This figure plots Interaction between wealth and investor sentiment (upper panel) vs degree of extrapolation (lower panel). The lower panel is based on Cassella and Gulen (2015). The parameter values are $r = 4\%$, $g_D = 1.5\%$, $\mu_H = 0.03\%$, $\mu_L = -0.06\%$, $\chi = \lambda = 10\%$, $\theta = 0.5\%$, $\rho = 2\%$, $k = 0.5$, $\mu = 0.5$, $\mu_H = 3\%$, $\mu_L = -3\%$.



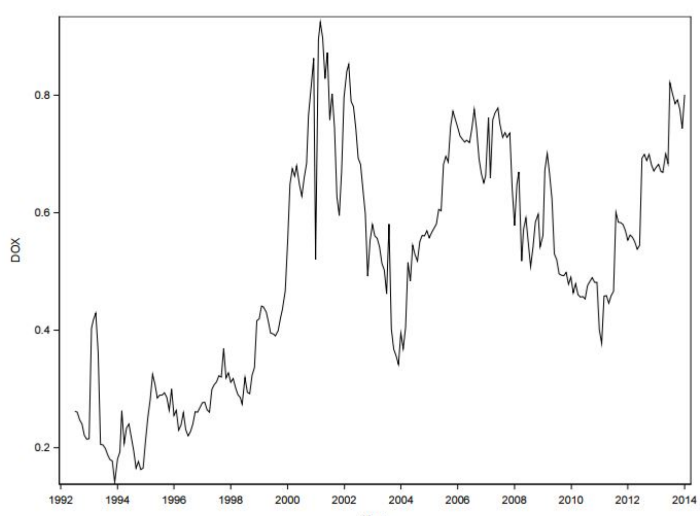


Figure A.8: Perceived Sharpe Ratio and True Sharpe Ratio.

This figure plots the true Sharpe ratio (red dashed line) and perceived Sharpe Ratio (blue solid line) based on model simulation. The parameter values are $r = 4\%$, $g_D = 1.5\%$, $\mu_H = 0.03\%$, $\mu_L = -0.06\%$, $\chi = \lambda = 10\%$, $\theta = 0.5\%$, $\rho = 2\%$, $k = 0.5$, $\mu = 0.5$, $\mu_H = 3\%$, $\mu_L = -3\%$.

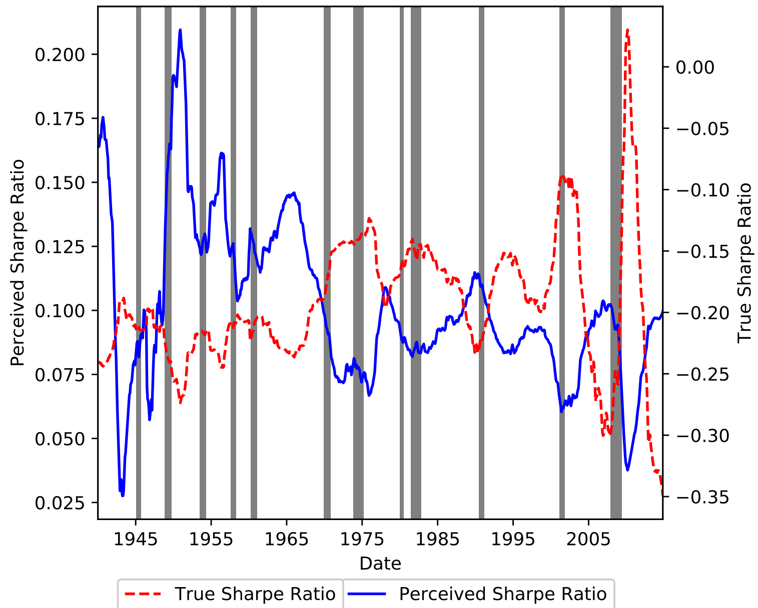


Table A.1: Summary Statistics

This table reports the summary statistics and correlation matrix between key variables. Panel A reports the general information for the Gallup survey measurement of investor sentiment, excess returns over next twelve months, wealth to dividend ratio for the HNPO sector, and the Psentiment variable. Panel B reports the correlation matrix for these key variables.

Statistic	N	Mean	St. Dev.	Min	Max
Gallup Measurement of Investor Sentiment	182	0.107	0.011	0.077	0.124
Excess Return (next 12-months)	912	0.086	0.174	−0.471	0.611
Wealth to dividend ratio	183	0.984	0.599	0.137	2.137
Psentiment	492	0.023	0.011	−0.024	0.047

Panel A: General information

	Gallup_rescaled	CRSPex12	WD	Psentiment
Gallup Measurement of Investor Sentiment	1.00	-0.02	-0.32	0.75
Excess Return (next 12-months)	-0.02	1.00	0.03	-0.12
Wealth to dividend ratio	-0.32	0.03	1.00	-0.28
Psentiment	0.75	-0.12	-0.28	1.00

Panel B: Correlation matrix for the key variables

Table A.2: Forecast Revision of Investor Sentiment

This table reports the results from the following predictive regression of N-months ahead sentiment revision based on current sentiment level:

$$SR_{t+1}[\text{Sent}_{t+N} - \text{Sent}_{t+1}] = c + d\text{Sent}_t + u_t, \quad \text{(A.59)}$$

where on the left hand side $SR_{t+1}[\text{Sent}_{t+N} - \text{Sent}_{t+1}]$ measures the changes in investor sentiment over future N horizons, Sent_t represents the investor sentiment at time t and u_t on the right hand side is the corresponding residual at time t. Panel A reports the regression results based on the Gallup survey. Panel B reports the regression results based on the Psentiment. All standard errors in parenthesis are based on the Newey-West correction (Newey and West (1986)).

Panel A: Forecast Revision of Investor Sentiment: Gallup

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Constant	0.015**	0.033***	0.066***	0.081***
	(0.006)	(0.011)	(0.015)	(0.015)
Gallup	−0.142***	−0.313***	−0.624***	−0.774***
	(0.053)	(0.099)	(0.142)	(0.149)
Observations	179	176	170	164
R^2	0.062	0.132	0.284	0.359
Adjusted R^2	0.057	0.127	0.279	0.355

Panel B: Forecast Revision of Investor Sentiment: Psentiment

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Constant	0.001**	0.004**	0.008***	0.011***
	(0.001)	(0.001)	(0.002)	(0.002)
Psentiment	−0.064**	−0.158***	−0.329***	−0.477***
	(0.026)	(0.053)	(0.084)	(0.086)
Observations	489	486	480	474
R^2	0.031	0.080	0.162	0.235
Adjusted R^2	0.029	0.078	0.160	0.233

Note: *p<0.1; **p<0.05; ***p<0.01

Table A.3: Predictive Regressions on Future Returns

This table reports the results from the following predictive regression:

$$R^e_{t+N} = a_0 + a_1 \text{Sent}_t + \epsilon_t, \tag{A.60}$$

where R^e_{t+N} represents the excess return of CRSP value-weighted index over the next N-month, Sent_t represents quantitative investor sentiment variable measured by Gallup survey. Results for future 1 to 6 quarters are reported in the column 1 to 4. Panel A reports the predictive regression results based on the investor sentiment in Gallup survey. All standard errors in parentheses are adjusted using Newey-West correction. The sample spans from 1996:10 to 2011:11.

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
	(1)	(2)	(3)	(4)
Constant	−0.079	−0.054	0.090	0.372
	(0.134)	(0.255)	(0.409)	(0.450)
Gallup	0.850	0.730	−0.400	−2.819
	(1.201)	(2.292)	(3.711)	(4.230)
Observations	182	182	182	182
R^2	0.011	0.004	0.0005	0.015
Adjusted R^2	0.005	-0.002	-0.005	0.010

Note: *p<0.1; **p<0.05; ***p<0.01

Table A.4: Conditional Predictive Regressions on Future Returns

This table reports the results from the following predictive regression:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t, \tag{A.61}$$

where R^e_{t+N} represents the excess return of CRSP value-weighted index over the next N-month, Sent_t represents investor sentiment variable, W_t represents the total financial asset value of the HNPO sector. Results based on future returns of 1 to 6 quarters are reported in the column 1 to 4. Panel A reports the predictive regression results based on the investor sentiment in Gallup survey. Panel B reports the predictive regression results based on Psentiment. All standard errors in parentheses are adjusted using Newey-West correction. The sample spans from 1996:10 to 2011:11.

Panel A: Predictive Regression based on Gallup

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Gallup	11.200***	23.683***	29.549**	50.552**
	(3.787)	(8.977)	(14.805)	(20.127)
W_t/D_t	0.741***	1.638**	2.081**	3.729***
	(0.265)	(0.716)	(0.996)	(1.410)
Gallup × W_t/D_t	−7.180***	−15.916**	−20.573**	−35.970***
	(2.474)	(6.738)	(8.892)	(13.543)
Constant	−1.150***	−2.420**	−2.942*	−5.183**
	(0.430)	(0.973)	(1.685)	(2.125)
Observations	60	60	60	60
R^2	0.042	0.087	0.088	0.143
Adjusted R^2	-0.009	0.038	0.039	0.097

Panel B: Predictive Regression based on Psentiment

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Psentiment	20.384***	37.181***	54.921***	84.599***
	(3.067)	(5.329)	(16.467)	(21.284)
W_t/D_t	0.023***	0.041***	0.059***	0.098***
	(0.005)	(0.010)	(0.021)	(0.027)
Psentiment×	−1.496***	−2.776***	−4.161***	−6.398***
W_t/D_t	(0.208)	(0.350)	(1.159)	(1.546)
Constant	−0.323***	−0.546***	−0.753**	−1.291***
	(0.090)	(0.162)	(0.340)	(0.421)
Observations	60	60	60	60
R^2	0.179	0.307	0.367	0.524
Adjusted R^2	0.135	0.270	0.333	0.499
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.5: Conditional Predictive Regressions on Future Returns (with Controls)

This table reports the results from the following predictive regression:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + X_t + \epsilon_t, \tag{A.62}$$

where R^e_{t+N} represents the excess return of CRSP value-weighted index over the next N-month, Sent_t represents Gallup survey measurement of investor sentiment, W_t represents the total financial asset value of the HNPO sector. Results for future 1 to 6 quarters are reported in the column 1 to 4. X_t represent the commonly used forecasters for the aggregate future market returns, cay. All standard errors in parentheses are adjusted using Newey-West correction. The sample spans from 1996:10 to 2011:11.

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Gallup	15.277**	30.900***	40.223***	66.621***
	(6.108)	(9.780)	(14.455)	(10.360)
W_t/D_t	0.077**	0.164**	0.195*	0.363***
	(0.033)	(0.070)	(0.113)	(0.064)
Gallup × W_t/D_t	−0.799**	−1.673**	−2.008*	−3.633***
	(0.319)	(0.678)	(1.081)	(0.623)
CAPE	−0.007***	−0.013***	−0.023***	−0.033***
	(0.002)	(0.004)	(0.005)	(0.004)
cay	0.194	1.191	4.062*	4.613
	(0.604)	(1.033)	(2.398)	(3.102)
Constant	−1.312**	−2.694***	−3.314**	−5.764***
	(0.596)	(0.987)	(1.543)	(1.041)
Observations	60	60	60	60
R^2	0.162	0.317	0.493	0.625
Adjusted R^2	0.085	0.254	0.446	0.590
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.6: Parameter Values

Parameter	Variable	Value
Asset parameter:		
Expected dividend growth	g_D	1.5%
Standard deviation of dividend growth	σ_D	10%
Correlation between dD and dC	ρ	0.2
Risk-free rate	r	4%
Utility parameter:		
Subjective discount factor	δ	0.02
Belief parameter:		
Degree of extrapolation	θ	0.5
Transition intensity from H to L	χ	0.10
Transition intensity from L to H	λ	0.10
Return in state H	μ_H	0.03
Return in state L	μ_L	-0.06
Other parameter:		
Demand parameter for fundamental investors	k	0.5
Population fraction	μ	0.5

Table A.7: Simulated Extrapolative Expectations: Behavioral Models

This table reports the results from the following regression focusing on the determinants of investor sentiment, based on the simulations of the behavioral model:

$$\mathbb{E}^e[R_{t+12}]_t = a_0 + a_1 X_t + \epsilon_t, \tag{A.63}$$

where $\mathbb{E}[R_{t+12}]_t$ represents the perceived expected returns at time t, X_t represents either past accumulative 12-month returns (R12) or the current log price to dividend ratio (logPD). Results in column 1 to 2 are based on the perceived expectations of returns with dividend yield. Results in column 1 to 2 are based on the perceived expectations of returns without dividend yield. All standard errors in parentheses are corrected based on the Newey-west approach.

	Dependent variable:			
	Expectations with dividend yield		Expectations without dividend yield	
	(1)	(2)	(3)	(4)
R12	0.023^{**}		0.011^{**}	
	(0.011)		(0.006)	
logPD		0.100^{***}		0.051^{***}
		(0.005)		(0.003)
Constant	-0.050^{***}	-0.328^{***}	-0.016^{***}	-0.160^{***}
	(0.012)	(0.014)	(0.006)	(0.007)
Observations	239	239	239	239
R^2	0.235	0.914	0.221	0.907
Adjusted R^2	0.232	0.914	0.218	0.907
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.8: Memory Structure of Investor Expectations: Behavioral Models

The table reports the memory decay parameter ψ, the intercept a, the regression coefficient b, and the adjusted R-squared, for running the non-linear least squares regression

$$\text{Expectation}_t = a + b \sum_{j=1}^{n} w_j R^D_{(t-j\Delta t)\to(t-(j-1)\Delta t)} + \varepsilon_t,$$

over a sample of 15 years or 50 years, where $w_j = e^{-\psi(j-1)\Delta t}/\Sigma_{l=1}^{n} e^{-\psi(l-1)\Delta t}$. Here $\Delta t = 1/12$ and $n = 600$.

	$\mathbb{E}^e_t[R^D_{t+dt}]$	
ϕ	0.510	0.521
a	0.064	0.064
b	1.15	1.18
R^2	0.78	0.78

Table A.9: Forecast Revision of Investor Sentiment: Simulated Results Based on Behavioral Model

This table reports the results from the following predictive regression based on the simulated series from the behavioral model:

$$SR_{t+1}[\text{Sent}_{t+N} - \text{Sent}_{t+1}] = c + d\text{Sent}_t + u_t, \tag{A.64}$$

where on the left-hand side $SR_{t+1}[\text{Sent}_{t+N} - \text{Sent}_{t+1}]$ measures the changes in investor sentiment over future N horizons, Sent_t represents the investor sentiment at time t and u_t on the right hand side is the corresponding residual at time t. Panel A reports the regression results based on the Gallup survey. Panel B reports the regression results based on the Psentiment. All standard errors in parentheses are again based on the Newey-West correction.

Panel A: Forecast Revision of Investor Sentiment: with Dividend Yield

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Constant	0.001	0.005*	0.018***	0.036***
	(0.001)	(0.003)	(0.006)	(0.009)
Sent	−0.021	−0.090*	−0.332***	−0.666***
	(0.022)	(0.051)	(0.108)	(0.164)
Observations	236	233	227	221
R^2	0.015	0.052	0.173	0.335
Adjusted R^2	0.011	0.048	0.170	0.332

Panel B: Forecast Revision of Investor Sentiment: without Dividend Yield

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Constant	0.0001	0.0004**	0.001***	0.002***
	(0.0001)	(0.0002)	(0.0004)	(0.001)
Sent	−0.022	−0.080**	−0.253***	−0.468***
	(0.015)	(0.034)	(0.061)	(0.083)
Observations	896	893	887	881
R^2	0.017	0.048	0.137	0.242
Adjusted R^2	0.016	0.047	0.136	0.242
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.10: Model-simulated Conditional Predictive Regressions on Future Returns: Behavioral Models

This table reports the results from the following predictive regression:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t, \tag{A.65}$$

where R^e_{t+N} represents the excess return of CRSP value-weighted index over the next N-month, Sent_t represents investor sentiment variable, W_t/D_t represents the wealth to dividend ratio in the model. Results for future 1 to 6 quarters are reported in the column 1 to 4. All standard errors in parentheses are adjusted using Newey-West correction.

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
	(1)	(2)	(3)	(4)
Constant	0.979***	0.955***	0.908***	0.870***
	(0.020)	(0.047)	(0.111)	(0.177)
Sent_t	6.714***	13.330**	24.550**	32.641
	(2.201)	(5.184)	(12.465)	(20.182)
W_t/D_t	0.019	0.043	0.101	0.165
	(0.012)	(0.029)	(0.067)	(0.104)
$\text{Sent}_t \times W_t/D_t$	−3.068**	−6.638**	−14.146**	−21.679**
	(1.194)	(2.749)	(6.275)	(9.789)
Observations	239	239	239	239
R^2	0.420	0.496	0.624	0.713
Adjusted R^2	0.413	0.489	0.619	0.709

Note: *p<0.1; **p<0.05; ***p<0.01

Table A.11: Conditional Predictive Power of Investor Sentiment.

This table reports the conditional predictive power of investor sentiment S_t, $a_1 + a_3 W_t/D_t$, from the following predictive regression:

$$R^e_{t+12} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t, \tag{A.66}$$

where R^e_{t+N} represents the excess return of CRSP value-weighted index over the next N-month, Sent_t represents investor sentiment in Gallup survey, W_t represents the total financial asset value of the HNPO sector. The sample spans from 1996:10 to 2011:11.

Investor Sentiment	Wealth Level	Conditional coefficient $b_t = [b + d \times W_t/D_t]$	t-statistics	p-value
Gallup	2	-0.162	-2.660	0.010
	0	0.098	1.428	0.159
	-2	0.359	2.034	0.047
Psentiment	2	-0.267	-3.915	0.000
	0	0.106	2.108	0.039
	-2	0.479	3.020	0.004

Table A.12: Conditional Predictive Power of Investor Sentiment: Simulations based on the Behavioral Models.

This table reports the conditional predictive power of investor sentiment S_t, $a_1 + a_3 W_t/D_t$, from the following predictive regression:

$$R^e_{t+12} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t, \tag{A.67}$$

where R^e_{t+12} represents the excess return of CRSP value-weighted index over the next 12-month, Sent_t represents investor sentiment, W_t/D_t represents the wealth to dividend ratio in the model.

Investor Sentiment	Wealth Level	Conditional coefficient $b_t = [b + d \times W_t/D_t]$	t-statistics	p-value
Behavioral Model	2	-0.025	-3.421	0.000
	0	-0.000	-0.034	0.973
	-2	0.025	2.885	0.004

Table A.13: Leverage Ratio and Future Consumption Growth: Behavioral Models

This table reports the predictive power of leverage ratio on future consumption growth rate based on behavioral models.

$$\text{Consumption Growth}_{t+N} = a_0 + a_1\alpha_t + \epsilon_t, \tag{A.68}$$

where α_t represents the leverage ratio of extrapolators, and all standard errors in parentheses are adjusted using Newey-West correction.

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
	(1)	(2)	(3)	(4)
α_t	-0.010^{***}	-0.022^{***}	-0.041^{***}	-0.045^{***}
	(0.003)	(0.005)	(0.010)	(0.013)
Constant	0.001^{***}	0.003^{***}	0.006^{***}	0.008^{***}
	(0.0001)	(0.0001)	(0.0002)	(0.0003)
Observations	896	893	887	881
R^2	0.014	0.020	0.021	0.013
Adjusted R^2	0.013	0.019	0.020	0.012

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Table A.14: Simulated Extrapolative Expectations: Rational Benchmark Models

This table reports the results from the following regression focusing on the determinants of investor sentiment, based on the simulations of the rational benchmark model:

$$\mathbb{E}[R_{t+12}]_t = a_0 + a_1 X_t + \epsilon_t, \quad \text{(A.69)}$$

where $\mathbb{E}[R_{t+12}]_t$ represents the perceived expected returns at time t, X_t represents either the past accumulative 12-month returns (R12) or the current log price to dividend ratio (logPD). Results in column 1 to 2 are based on the perceived expectations of returns without dividend yield. Results in column 1 to 2 are based on the perceived expectations of returns with dividend yield. All standard errors in parentheses are corrected based on Newey-west approach.

	Dependent variable:			
	Expectations without dividend yield		Expectations with dividend yield	
	(1)	(2)	(3)	(4)
R12	−0.00003		−0.001	
	(0.0002)		(0.002)	
logPD		−0.006***		−0.049***
		(0.001)		(0.0005)
Constant	0.018***	0.037***	0.022***	0.173***
	(0.0003)	(0.002)	(0.002)	(0.001)
Observations	239	239	239	239
R^2	0.0003	0.913	0.016	0.999
Adjusted R^2	-0.004	0.912	0.012	0.999

Note: *p<0.1; **p<0.05; ***p<0.01

Table A.15: Model-simulated Conditional Predictive Regressions on Future Returns: Rational Benchmark Models

This table reports the results from the following predictive regression of N-months ahead returns based on simulations of the rational benchmark model:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t, \quad \text{(A.70)}$$

where R^e_{t+N} represents the excess return over the next N-month, Sent_t represents investor sentiment variable, W_t/D_t represents the wealth to dividend ratio in the model. Results for future 1 to 6 quarters are reported in the column 1 to 4. All standard errors in parentheses are adjusted using Newey-West correction.

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
	(1)	(2)	(3)	(4)
Constant	1.245***	1.561***	2.307**	3.205**
	(0.163)	(0.392)	(0.954)	(1.630)
Sent_t	−8.969	−20.669	−48.522	−82.221
	(6.303)	(15.138)	(36.854)	(62.965)
W_t/D_t	−0.017**	−0.039**	−0.091**	−0.154*
	(0.008)	(0.019)	(0.046)	(0.079)
$\text{Sent}_t \times W_t/D_t$	0.660**	1.520**	3.596**	6.079**
	(0.307)	(0.726)	(1.744)	(2.994)
Observations	239	239	239	239
R^2	0.047	0.071	0.125	0.176
Adjusted R^2	0.035	0.060	0.114	0.165

Note: *p<0.1; **p<0.05; ***p<0.01

Table A.16: Leverage Ratio and Future Consumption Growth: Rational Models

This table reports the predictive power of leverage ratio on future consumption growth rate based on rational models.

$$\text{Consumption Growth}_{t+N} = a_0 + a_1\alpha_t + \epsilon_t, \tag{A.71}$$

where α_t represents leverage ratio of extrapolators, and all standard errors in parentheses are adjusted using Newey-West correction.

	Dependent variable:			
	WD_growth1 1 Quarter	WD_growth2 2 Quarters	WD_growth3 4 Quarters	WD_growth4 6 Quarters
	(1)	(2)	(3)	(4)
α_t	3.003***	4.500***	6.081***	7.014***
	(0.139)	(0.274)	(0.532)	(0.773)
Constant	0.003***	0.006***	0.012***	0.018***
	(0.0001)	(0.0002)	(0.0004)	(0.001)
Observations	896	893	887	881
R^2	0.341	0.232	0.129	0.086
Adjusted R^2	0.341	0.231	0.128	0.085
Note:				*p<0.1; **p<0.05; ***p<0.01

A.5 Alternative Regressions

In this appendix, I replace W_t/D_t with W_t/P_t to measure the relative impact of extrapolators on the stock market. Results remain robust.

Table A.17: Conditional Predictive Regressions on Future Returns

This table reports the results from the following predictive regression of N-months ahead returns from CRSP value-weighted index:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/P_t + d\text{Sent}_t \times W_t/P_t + \epsilon_t, \tag{A.72}$$

where R^e_{t+N} represents the excess return over the next N-month, Sent_t represents investor sentiment variable, W_t represents the total financial asset value of the HNPO sector. P_t measures the price level. W_t/P_t measures the relative share held by the extrapolators. Results for future 1 to 6 quarters are reported in the column 1 to 4. Panel A reports the predictive regression results based on the investor sentiment in Gallup survey. Panel B reports the predictive regression results based on Psentiment. All standard errors in parentheses are adjusted using Newey-West correction. The sample spans from 1996:10 to 2011:11.

Panel A: Predictive Regression based on Gallup

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Gallup	6.910**	16.227**	25.851**	28.236**
	(3.363)	(7.944)	(12.621)	(13.194)
W_t/P_t	9.168***	23.852***	39.029***	45.375***
	(3.267)	(8.536)	(14.089)	(15.823)
Gallup × W_t/P_t	−76.408***	−204.113***	−332.067***	−382.004**
	(28.250)	(75.413)	(128.641)	(152.968)
Constant	−0.801**	−1.865**	−2.984**	−3.287**
	(0.395)	(0.909)	(1.407)	(1.415)
Observations	60	60	60	60
R^2	0.062	0.147	0.200	0.201
Adjusted R^2	0.012	0.101	0.158	0.159

Panel B: Predictive Regression based on Psentiment

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Psentiment	1.753	4.201*	5.301	3.497
	(1.242)	(2.416)	(4.313)	(5.860)
W_t/P_t	1.010***	2.344***	3.740***	4.121***
	(0.350)	(0.563)	(0.949)	(1.182)
Psentiment × W_t/P_t	−29.961*	−74.290**	−92.760	−63.356
	(17.928)	(35.020)	(59.738)	(82.405)
Constant	−0.044	−0.103**	−0.144	−0.113
	(0.029)	(0.052)	(0.089)	(0.109)
Observations	159	159	159	158
R^2	0.025	0.061	0.079	0.080
Adjusted R^2	0.006	0.043	0.062	0.062
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.18: Conditional Predictive Regressions on Future Returns (With Controls)

This table reports the results from the following predictive regression of N-months ahead returns from CRSP value-weighted index:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/P_t + d\text{Sent}_t \times W_t/P_t + X_t + \epsilon_t, \tag{A.73}$$

where R^e_{t+N} represents the excess return over the next N-month, Sent_t represents Gallup survey measurement of investor sentiment, W_t represents the total financial asset value of the HNPO sector. P_t measures the price level. W_t/P_t measures the relative share held by the extrapolators. Results for future 1 to 6 quarters are reported in the column 1 to 4. X_t represents the commonly used forecasters for the aggregate future market returns, cay. All standard errors in parentheses are adjusted using Newey-West correction. The sample spans from 1996:10 to 2011:11.

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
Gallup	8.563***	19.265***	31.521***	37.765***
	(2.684)	(5.227)	(5.116)	(4.812)
W_t/P_t	8.301***	21.442***	33.901***	39.911***
	(3.035)	(5.749)	(4.725)	(6.872)
Gallup $\times W_t/P_t$	−77.873***	−198.279***	−314.599***	−385.603***
	(28.883)	(50.089)	(33.191)	(74.938)
CAPE	−0.006***	−0.011***	−0.021***	−0.033***
	(0.002)	(0.003)	(0.005)	(0.008)
cay	1.172*	2.919**	6.039***	7.191***
	(0.609)	(1.161)	(1.809)	(2.336)
Constant	−0.758**	−1.774***	−2.803***	−3.032***
	(0.310)	(0.648)	(0.767)	(0.635)
Observations	60	60	60	60
R^2	0.141	0.319	0.536	0.605
Adjusted R^2	0.062	0.256	0.493	0.569

Note: *p<0.1; **p<0.05; ***p<0.01

Table A.19: Model-simulated Conditional Predictive Regressions on Future Returns: Behavioral Models

This table reports the results from the following predictive regression of N-months ahead returns based on model simulations:

$$R^e_{t+N} = a + b\text{Sent}_t + cW_t/P_t + d\text{Sent}_t \times W_t/P_t + \epsilon_t, \tag{A.74}$$

where R^e_{t+N} represents the excess return over the next N-month, Sent_t represents investor sentiment variable, W_t represents the total financial asset value of the HNPO sector. P_t measures the price level. W_t/P_t measures the relative share held by the extrapolators. Results for future 1 to 6 quarters are reported in the column 1 to 4. All standard errors in parentheses are adjusted using Newey-West correction.

	Dependent variable:			
	1 Quarter	2 Quarters	4 Quarters	6 Quarters
	(1)	(2)	(3)	(4)
Sent_t	6.710***	13.400***	24.754**	32.892
	(2.202)	(5.154)	(12.390)	(20.120)
W_t/P_t	0.371	0.864	2.037	3.352
	(0.249)	(0.584)	(1.352)	(2.105)
$\text{Sent}_t \times W_t/P_t$	−62.920***	−136.792**	−291.685**	−445.884**
	(24.376)	(55.709)	(127.197)	(198.796)
Constant	0.980***	0.956***	0.909***	0.872***
	(0.020)	(0.046)	(0.111)	(0.177)
Observations	239	239	239	239
R^2	0.065	0.075	0.121	0.203
Adjusted R^2	0.053	0.064	0.109	0.193
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.20: Conditional Predictive Power of Investor Sentiment.

This table reports the conditional predictive power of investor sentiment S_t, $a_1 + a_3 W_t/D_t$, from the following predictive regression of N-months ahead returns from CRSP value-weighted index:

$$R^e_{t+12} = a + b\text{Sent}_t + cW_t/D_t + d\text{Sent}_t \times W_t/D_t + \epsilon_t, \quad \text{(A.75)}$$

where R^e_{t+12} represents the excess return over the next twelve months, Sent_t represents investor sentiment in Gallup survey, W_t represents the total financial asset value of the HNPO sector. P_t measures the price level. W_t/P_t measures the relative share held by the extrapolators. The sample spans from 1996:10 to 2011:11.

Investor Sentiment	Wealth Level	Conditional coefficient $b_t = [b + d \times W_t/P_t]$	t-statistics	p-value
Gallup	2	-0.054	-2.179	0.034
	0	0.100	1.461	0.149
	-2	0.255	2.018	0.048
Psentiment	2	-0.039	-1.717	0.091
	0	0.008	0.421	0.676
	-2	0.055	1.201	0.235

Table A.21: Conditional Predictive Power of Investor Sentiment: Simulations based on the Behavioral Models.

This table reports the conditional predictive power of investor sentiment S_t, $a_1 + a_3 W_t/P_t$, from the following predictive regression of N-months ahead returns from CRSP value-weighted index:

$$R^e_{t+12} = a + b\text{Sent}_t + cW_t/P_t + d\text{Sent}_t \times W_t/P_t + \epsilon_t, \qquad \text{(A.76)}$$

where R^e_{t+12} represents the excess return over the next twelve months, Sent_t represents investor sentiment, W_t represents the total financial asset value of the HNPO sector. P_t measures the price level. W_t/P_t measures the relative share held by the extrapolators.

Investor Sentiment	Wealth Level	Conditional coefficient $b_t = [b + d \times W_t/P_t]$	t-statistics	p-value
Behavioral Model	2	-0.065	-2.384	0.021
	0	-0.001	-0.067	0.947
	-2	0.064	1.797	0.078

Appendix B

APPENDIX TO CHAPTER TWO

B.1 Derivation of the Differential Equations

For the subjective Euler equation (2.4), setting $\tilde{R}_{j,t+dt}$, the return on the tradeable asset, to the gross return on the stock market, the equation becomes

$$\mathbb{E}_t^e\left[e^{-\delta(1-\gamma)dt/(1-\psi)}\left(\frac{\tilde{C}_{t+dt}}{C_t}\right)^{-\psi(1-\gamma)/(1-\psi)}\left(\frac{\tilde{P}^C_{t+dt}+\tilde{C}_t dt}{P^C_t}\right)^{(\psi-\gamma)/(1-\psi)}\frac{\tilde{P}^D_{t+dt}+\tilde{D}_{t+dt}dt}{P^D_t}\right]=1. \tag{B.1}$$

Using Taylor expansion, (B.1) becomes

$$\begin{aligned}&\mathbb{E}_t^e\left[e^{-\delta(1-\gamma)dt/(1-\psi)}\left(\tilde{C}_{t+dt}\right)^{-\psi(1-\gamma)/(1-\psi)}(\tilde{P}^C_{t+dt})^{(\psi-\gamma)/(1-\psi)}\tilde{P}^D_{t+dt}\left(1+\frac{\psi-\gamma}{1-\psi}\frac{\tilde{C}_{t+dt}}{\tilde{P}^C_{t+dt}}dt+\frac{\tilde{D}_{t+dt}}{\tilde{P}^D_{t+dt}}dt\right)\right]\\&=C_t^{-\psi(1-\gamma)/(1-\psi)}(P^C_t)^{(\psi-\gamma)/(1-\psi)}P^D_t.\end{aligned} \tag{B.2}$$

Rearranging terms gives

$$0=\mathbb{E}_t^e\left[\begin{array}{l}d(\Theta C^{(\psi-\gamma)/(1-\psi)}l^{(\psi-\gamma)/(1-\psi)}Df)+\frac{\psi-\gamma}{1-\psi}\Theta C^{(\psi-\gamma)/(1-\psi)}l^{(2\psi-\gamma-1)/(1-\psi)}Dfdt\\+\Theta C^{(\psi-\gamma)/(1-\psi)}l^{(\psi-\gamma)/(1-\psi)}Ddt\end{array}\right], \tag{B.3}$$

where $\Theta(C,t)\equiv e^{-\delta(1-\gamma)t/(1-\psi)}C^{-\psi(1-\gamma)/(1-\psi)}$. By Ito's lemma, (B.3) leads to

$$0=\mathbb{E}_t^e\left[\begin{array}{l}-\frac{\delta(1-\gamma)}{1-\psi}dt-\gamma(dC/C)+(dD/D)+(df/f)+\frac{\psi-\gamma}{1-\psi}(dl/l)+\frac{\gamma(\gamma+1)}{2}(dC/C)^2\\+\frac{1}{2}\frac{\psi-\gamma}{1-\psi}\frac{2\psi-\gamma-1}{1-\psi}(dl/l)^2-\frac{\gamma(\psi-\gamma)}{1-\psi}(dC/C)(dl/l)-\gamma(dC/C)(dD/D)-\gamma(dC/C)(df/f)\\+\frac{\psi-\gamma}{1-\psi}(dl/l)(dD/D)+\frac{\psi-\gamma}{1-\psi}(dl/l)(df/f)+(df/f)(dD/D)+\frac{\psi-\gamma}{1-\psi}l^{-1}dt+f^{-1}dt\end{array}\right]. \tag{B.4}$$

Using (2.7), (2.10), and (2.14) to further simplify (B.4) gives (2.16).

Setting $\tilde{R}_{j,t+dt}$ in (2.4) to the gross return on the Lucas tree, the subjective Euler equation (2.4) becomes

$$\mathbb{E}_t^e\left[e^{-\delta(1-\gamma)dt/(1-\psi)}\left(\frac{\tilde{C}_{t+dt}}{C_t}\right)^{-\psi(1-\gamma)/(1-\psi)}\left(\frac{\tilde{P}^C_{t+dt}+\tilde{C}_t dt}{P^C_t}\right)^{(1-\gamma)/(1-\psi)}\right]=1. \quad \text{(B.5)}$$

Rearranging terms yields

$$0=\mathbb{E}_t^e\left[d(\Theta C^{(1-\gamma)/(1-\psi)}l^{(1-\gamma)/(1-\psi)})+\tfrac{1-\gamma}{1-\psi}\Theta C^{(1-\gamma)/(1-\psi)}l^{(\psi-\gamma)/(1-\psi)}dt\right]. \quad \text{(B.6)}$$

By Ito's lemma, (B.6) leads to

$$0=\mathbb{E}_t^e\left[\begin{array}{l}-\frac{1-\gamma}{1-\psi}\delta dt-(\gamma-1)(dC/C)+\frac{\gamma(\gamma-1)}{2}(dC/C)^2+\frac{1-\gamma}{1-\psi}(dl/l)\\ +\frac{1}{2}\frac{1-\gamma}{1-\psi}\frac{\psi-\gamma}{1-\psi}(dl/l)^2+\frac{(1-\gamma)^2}{1-\psi}(dC/C)(dl/l)+\frac{1-\gamma}{1-\psi}l^{-1}dt\end{array}\right]. \quad \text{(B.7)}$$

Using (2.7) and (2.14) to further simplify (B.7) gives (2.17). ■

Steady-State Distribution for Sentiment

Below we provide all the terms necessary for solving the Kolmogorov forward equation (2.25). From the expression of σ_S in (2.7)

$$\sigma_S'=\frac{\theta\sigma_P^D(\mu_H+\mu_L-2S)-\theta(\mu_H-S)(S-\mu_L)(\sigma_P^D)'}{(\sigma_P^D)^2},$$

$$\sigma_S''=\frac{\theta(\mu_H-S)(S-\mu_L)\{2[(\sigma_P^D)']^2-\sigma_P^D(\sigma_P^D)''\}}{(\sigma_P^D)^3}-2\theta\frac{\sigma_P^D(\sigma_P^D)'(\mu_H+\mu_L-2S)+(\sigma_P^D)^2}{(\sigma_P^D)^3}. \quad \text{(B.8)}$$

For the expression of μ_S^e in (2.7) and the expression of g_D^e in (2.11)

$$\begin{aligned}(\mu_S^e)'&=-(\lambda+\chi),\\ (g_D^e)'&=\theta-\sigma_D(\sigma_P^D)'-\mu'_S(f'/f)-\mu_S[f''/f-(f')^2/f^2]\\ &\quad-\sigma_S\sigma_S'(f''/f)-\tfrac{1}{2}\sigma_S^2[f'''/f-f'f''/f^2],\end{aligned} \quad \text{(B.9)}$$

where σ_P^D is from (2.11), and $(\sigma_P^D)'$ and $(\sigma_P^D)''$ are

$$(\sigma_P^D)' = \frac{\theta(\mu_H + \mu_L - 2S)(f'/f) + \theta(\mu_H - S)(S - \mu_L)[f''/f - (f')^2/f^2]}{\sqrt{\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)}},$$

$$\begin{aligned}(\sigma_P^D)'' = & -\frac{2\{\theta(\mu_H + \mu_L - 2S)(f'/f) + \theta(\mu_H - S)(S - \mu_L)[f''/f - (f')^2/f^2]\}^2}{[\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)]^{3/2}} \\ & + \frac{-2\theta f'/f + 2\theta(\mu_H + \mu_L - 2S)[f''/f - (f')^2/f^2]}{\sqrt{\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)}} \\ & + \frac{\theta(\mu_H - S)(S - \mu_L)[f'''/f - 3(f'f'')/f^2 + 2(f')^3/f^3]}{\sqrt{\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)}}.\end{aligned} \tag{B.10}$$

■

B.2 Numerical Procedure for Solving the Equilibrium

We use a projection method with Chebyshev polynomials to jointly solve the two differential equations (2.16) and (2.17). The value of the sentiment variable S ranges from μ_L to μ_H, whereas the domain for Chebyshev polynomials is [–1, 1]. Therefore, we transform S to a new state variable z

$$z \equiv aS + b, \qquad \text{where} \quad a = \frac{2}{\mu_H - \mu_L}, \ b = -\frac{\mu_H + \mu_L}{\mu_H - \mu_L}, \tag{B.11}$$

and we define $h(z) \equiv f(S(z))$ and $j(z) \equiv l(S(z))$. Equations (2.16) and (2.17) can be rewritten as

$$0 = \begin{bmatrix} -\frac{(1-\gamma)}{1-\psi}\delta - \gamma g_C^e + g_D^e + [(h'/h) + \frac{\psi-\gamma}{1-\psi}(j'/j)]a\mu_S^e + \frac{1}{2}[(h''/h) + \frac{\psi-\gamma}{1-\psi}(j''/j)]a^2\sigma_S^2 \\ +\frac{\gamma(\gamma+1)}{2}\sigma_C^2 + \frac{1}{2}\frac{\psi-\gamma}{1-\psi}\frac{2\psi-\gamma-1}{1-\psi}(aj'/j)^2\sigma_S^2 - \frac{\gamma(\psi-\gamma)}{1-\psi}\rho\sigma_C\sigma_S(aj'/j) \\ -\gamma\rho\sigma_C\sigma_D - \gamma\rho\sigma_C\sigma_S(ah'/h) \\ +\frac{\psi-\gamma}{1-\psi}\sigma_D\sigma_S(aj'/j) + \frac{\psi-\gamma}{1-\psi}\sigma_S^2a^2(j'/j)(h'/h) + \sigma_D\sigma_S(ah'/h) + \frac{\psi-\gamma}{1-\psi}j^{-1} + h^{-1} \end{bmatrix} \tag{B.12}$$

and

$$0=\begin{bmatrix} -\frac{1-\gamma}{1-\psi}\delta-(\gamma-1)g_C^e+\frac{\gamma(\gamma-1)}{2}\sigma_C^2+\frac{1-\gamma}{1-\psi}(aj'/j)\mu_S^e+\frac{1-\gamma}{2(1-\psi)}(a^2j''/j)\sigma_S^2 \\ +\frac{1}{2}\frac{1-\gamma}{1-\psi}\frac{\psi-\gamma}{1-\psi}(aj'/j)^2\sigma_S^2+\frac{(1-\gamma)^2}{1-\psi}\rho\sigma_C\sigma_S(aj'/j)+\frac{1-\gamma}{1-\psi}j^{-1} \end{bmatrix}. \tag{B.13}$$

We approximate h and j by

$$\hat{h}(z)=\sum\nolimits_{r=0}^{n}a_rT_r(z), \qquad \hat{l}(z)=\sum\nolimits_{r=0}^{m}b_rT_r(z), \tag{B.14}$$

where $T_r(z)$ is the r^{th} degree Chebyshev polynomial of the first kind.[1] The projection method chooses the coefficients $\{a_r\}_{r=0}^n$ and $\{b_r\}_{r=0}^m$ so that the differential equations are *approximately* satisfied. One criterion for a sufficient approximation is to minimize the weighted sum of squared errors

$$\begin{aligned} &\sum\nolimits_{i=1}^{N}\frac{1}{\sqrt{1-z_i^2}}\begin{bmatrix} -\frac{(1-\gamma)}{1-\psi}\delta-\gamma g_C^e+g_D^e+[(\hat{h}'/\hat{h})+\frac{\psi-\gamma}{1-\psi}(\hat{j}'/\hat{j})]a\mu_S^e+\frac{1}{2}[(\hat{h}''/\hat{h}) \\ +\frac{\psi-\gamma}{1-\psi}(\hat{j}''/\hat{j})]a^2\sigma_S^2+\frac{\gamma(\gamma+1)}{2}\sigma_C^2+\frac{1}{2}\frac{\psi-\gamma}{1-\psi}\frac{2\psi-\gamma-1}{1-\psi}(a\hat{j}'/\hat{j})^2\sigma_S^2 \\ -\frac{\gamma(\psi-\gamma)}{1-\psi}\rho\sigma_C\sigma_S(a\hat{j}'/\hat{j})-\gamma\rho\sigma_C\sigma_D-\gamma\rho\sigma_C\sigma_S(a\hat{h}'/\hat{h}) \\ +\frac{\psi-\gamma}{1-\psi}\sigma_D\sigma_S(a\hat{j}'/\hat{j})+\frac{\psi-\gamma}{1-\psi}\sigma_S^2a^2(\hat{j}'/\hat{j})(\hat{h}'/\hat{h})+\sigma_D\sigma_S(a\hat{h}'/\hat{h}) \\ +\frac{\psi-\gamma}{1-\psi}\hat{j}^{-1}+\hat{h}^{-1} \end{bmatrix}_{z=z_i}^2 \\ &+\sum\nolimits_{i=1}^{N}\frac{1}{\sqrt{1-z_i^2}}\begin{bmatrix} -\frac{1-\gamma}{1-\psi}\delta-(\gamma-1)g_C^e+\frac{\gamma(\gamma-1)}{2}\sigma_C^2+\frac{1-\gamma}{1-\psi}(a\hat{j}'/\hat{j})\mu_S^e \\ +\frac{1}{2}\frac{1-\gamma}{1-\psi}\frac{\psi-\gamma}{1-\psi}(a\hat{j}'/\hat{j})^2\sigma_S^2+\frac{(1-\gamma)^2}{1-\psi}\rho\sigma_C\sigma_S(a\hat{j}'/\hat{j})+\frac{1-\gamma}{1-\psi}\hat{j}^{-1} \\ +\frac{1-\gamma}{2(1-\psi)}(a^2\hat{j}''/\hat{j})\sigma_S^2 \end{bmatrix}_{z=z_i}^2, \end{aligned} \tag{B.15}$$

where $\{z_i\}_{i=0}^N$ are the N zeros of $T_N(z)$. By the Chebyshev interpolation theorem, if N is sufficiently larger than n and m, and if the sum of weighted square in (B.15) is sufficiently small, the approximated functions $\hat{h}(z)$ and $\hat{l}(z)$ are sufficiently close to the true solutions.

For the numerical results in the main text, we set $m=40$, $n=40$, $N=400$. We then apply the Levenberg-Marquardt algorithm and obtain a minimized sum of squared errors less than 10^{-11}. The small size of the total error indicates convergence of the numerical solution. The solution is also insensitive to the choice of n, m, or

[1] See Mason and Handscomb (2003) for a detailed discussion of the properties of Chebyshev polynomials.

N. Together, these findings suggest that the numerical solutions are a sufficient approximation for the true h and j functions.

The same numerical procedure is applied to solving the Kolmogorov forward equation (2.25). ■

B.3 Additional Discussion about Return Expectations and Cash Flow Expectations

The direct implication of return extrapolation is that the agent's subjective expectation about the future stock market return

$$\mathbb{E}^e_t[P^D_{t+dt}] = 1 + \mathbb{E}^e_t[r^D_{t+dt}]dt = \mathbb{E}^e_t\left[\frac{P^D_{t+dt}}{P^D_t}\right] + \frac{D_t dt}{P^D_t} \tag{B.16}$$

is a positive function of the stock market's recent past returns. Rearranging terms gives

$$\frac{P^D_t}{D_t} = \frac{1}{\mathbb{E}^e_t[r^D_{t+dt}] - \mathbb{E}^e_t[dP^D_t/(P^D_t dt)]}. \tag{B.17}$$

That is, the current price-dividend ratio is determined by the agent's subjective expectation about the future stock market return $\mathbb{E}^e_t[r^D_{t+dt}]$ and the agent's subjective expectation about future price growth $\mathbb{E}^e_t[dP^D_t/(P^D_t dt)]$. Equation (B.17) does not suggest an explicit role for the agent's expectation about dividend growth in determining the price-dividend ratio.

However, two conditions allow us to link the price-dividend ratio of the stock market to the agent's expectation about dividend growth. First, the law of iterated expectations must hold so that we can iterate forward the Euler equation (2.4) with the stock market as the tradeable asset. Second, the transversality condition must hold so that the economy permits no bubbles.[2] These two conditions allow us to obtain

$$\frac{P^D_t}{D_t} = \mathbb{E}^e_t\left[\int_t^\infty e^{-\delta(1-\gamma)(s-t)/(1-\psi)}\left(\frac{\tilde{C}_s}{C_t}\right)^{-\psi(1-\gamma)/(1-\psi)} \tilde{M}_{t\to s}^{(\psi-\gamma)/(1-\psi)}\left(\frac{\tilde{D}_s}{D_t}\right)ds\right], \tag{B.18}$$

where $\tilde{M}_{t\to s}$ denotes the continuously compounded gross return for holding the Lucas tree from time t to time s ($> t$). Equation (B.18) says that the current price-dividend ratio of the stock market equals the agent's subjective expectation of the

[2] The transversality condition holds in this economy as the stock market price is bounded by a finite range.

sum of discounted future dividend growths.

For an infinitely-lived agent, (B.18) further implies that the agent is aware of the fact that both her expectation about future price growth and her expectation about future returns are linked to her expectation about future dividend growth. The specific relationship between these expectations is discussed in Sections 2.2 and 2.3. ■

B.4 Figures and Tables

The figure plots the price-dividend ratio of the stock market f, the volatility of stock market returns σ_P^D, the rational expectation about the log excess return $\mathbb{E}[r^{D,e}]$ (the conditional equity premium), and the interest rate r, each as a function of the sentiment variable S. The parameter values are g_C = 1.91%, g_D = 2.45%, σ_C = 3.8%, σ_D = 11%, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, δ = 2%, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, μ_H = 15%, and μ_L = −15%.

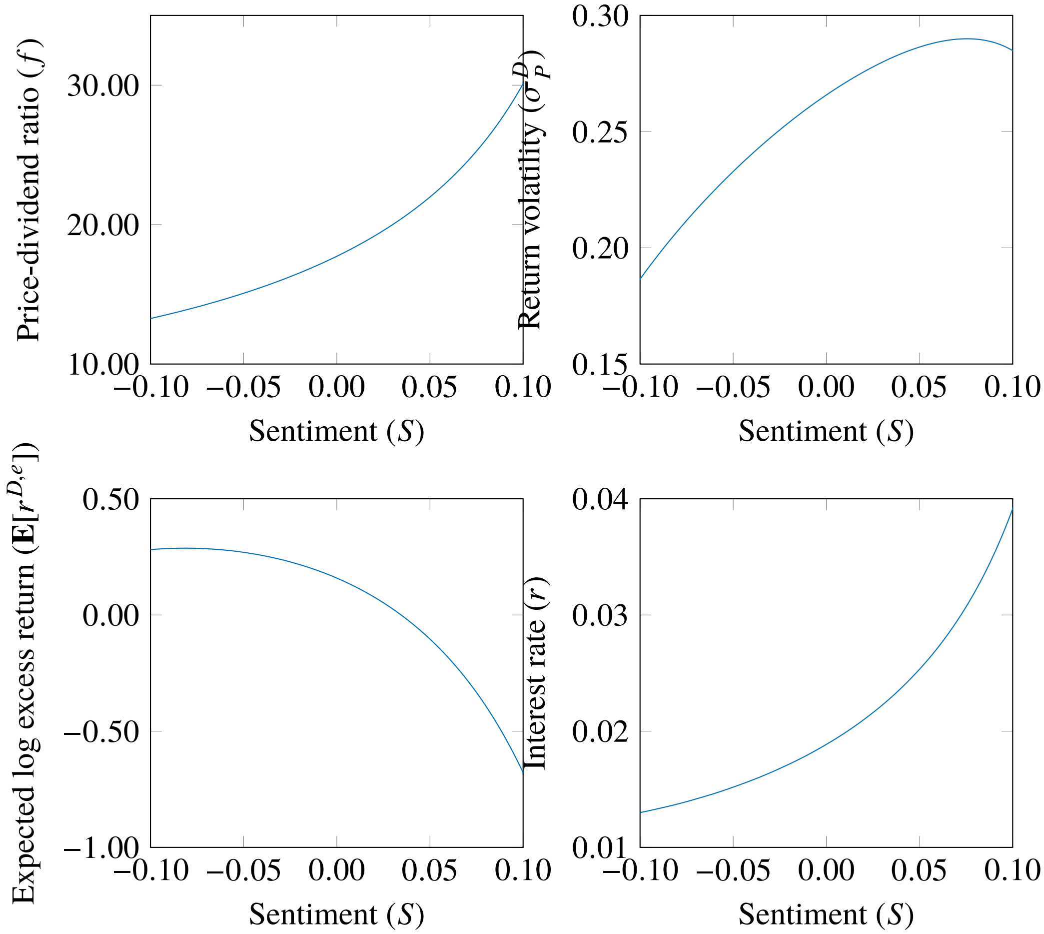


Figure B.1: Important Equilibrium Quantities Each as a Function of Sentiment.

Figure B.2: Objectively Measured Steady-State Distribution of Sentiment.

The figure plots the objective steady-state distribution of sentiment ξ as a function of the sentiment variable S. The parameter values are g_C = 1.91%, g_D = 2.45%, σ_C = 3.8%, σ_D = 11%, ρ = 0.2, γ = 10, ψ = 0.9, δ = 2%, θ = 0.5, χ = 0.18, λ = 0.18, μ_H = 15%, and μ_L = –15%.

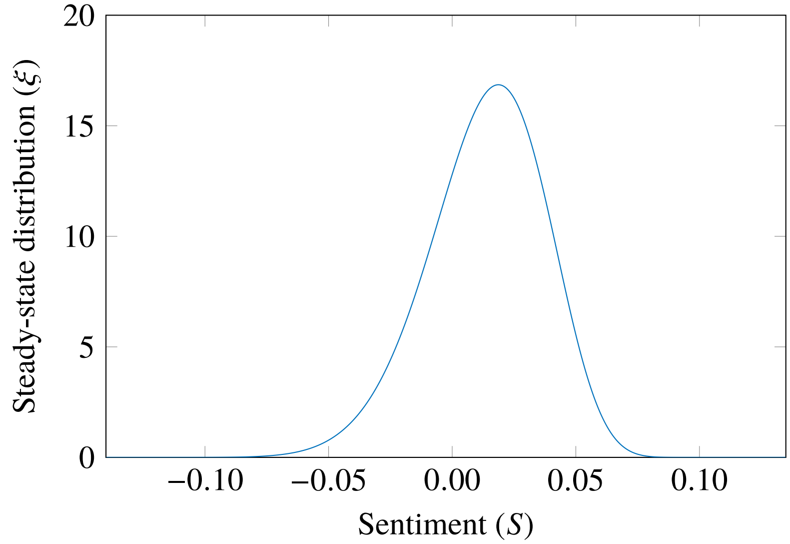


Figure B.3: Objective and Subjective Expectations about Price Growth.

The dashed line plots the objective (rational) expectation about price growth, $\mathbb{E}_t[(dP_t^D)/(P_t^D dt)]$, as a function of the sentiment variable S_t. The solid line plots the agent's subjective expectation about price growth, $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$, as a function of the sentiment variable S_t. The parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$.

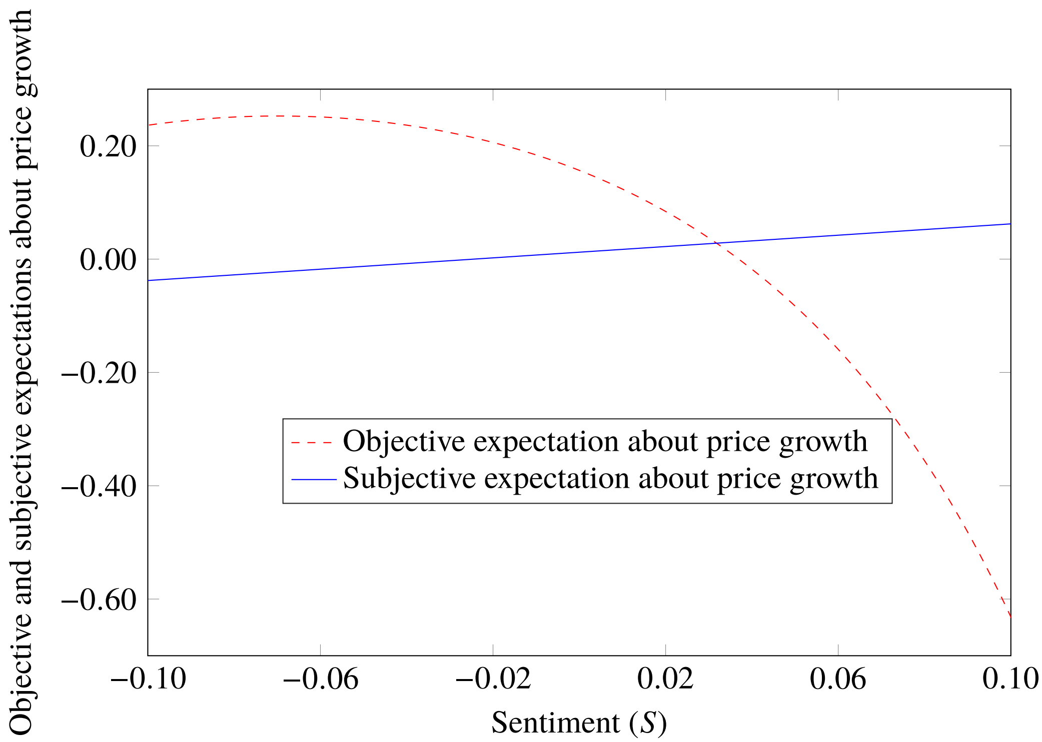


Figure B.4: Agent's Expectations about Stock Market Returns, Price Growth, and Dividend Growth.

The dashed line plots the agent's expectation about stock market returns, $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] = (1-\theta)g_D + \theta S_t + 1/f$, as a function of the sentiment variable S_t. The dotted-dashed line plots the agent's expectation about price growth, $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] = (1-\theta)g_D + \theta S_t$, as a function of the sentiment variable S_t. The solid line plots the agent's expectation about dividend growth, $\mathbb{E}_t^e[dD_t/(D_t dt)]$, as a function of the sentiment variable S_t. The parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$.

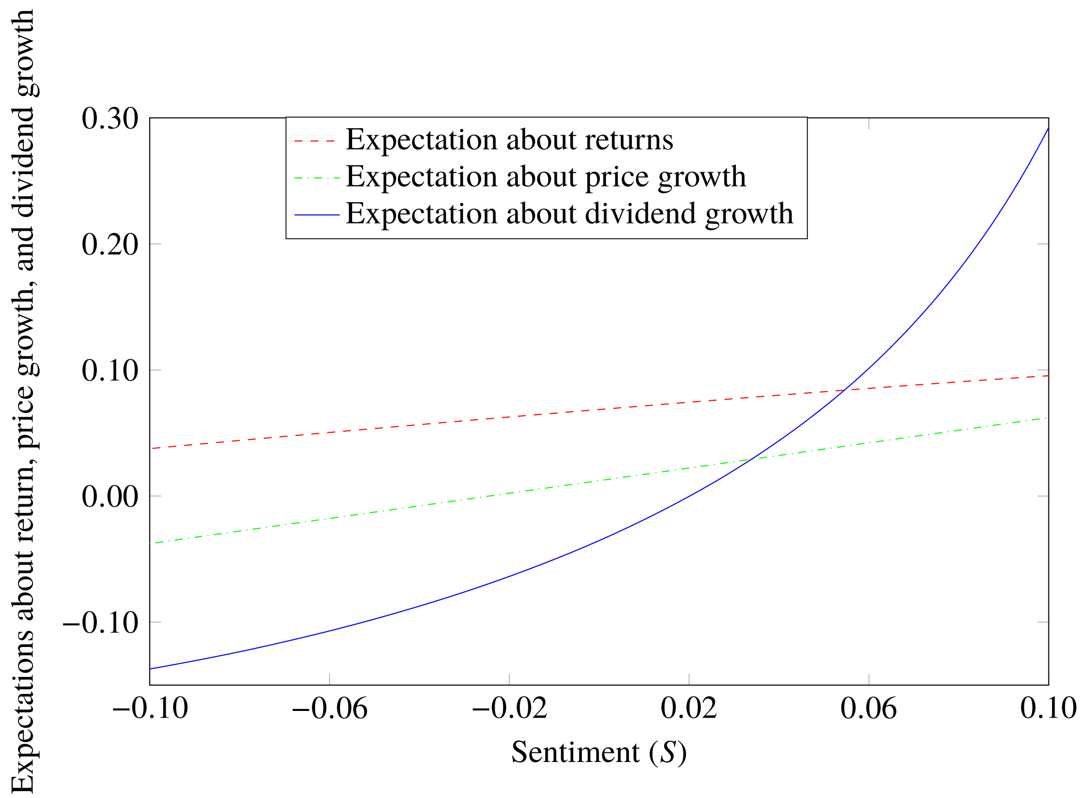


Figure B.5: Comparative Statics: Utility Parameters.

The upper panel plots the average equity premium $\mathbb{E}[r^{D,e}]$ and the average volatility of stock market returns $\sigma(r^{D,e})$, each as a function of γ, the coefficient of relative risk aversion. The lower panel plots the average equity premium $\mathbb{E}[r^{D,e}]$ and the average volatility of stock market returns $\sigma(r^{D,e})$, each as a function of ψ, the reciprocal of the elasticity of intertemporal substitution. The default values for γ and ψ are 10 and 0.9, respectively. The other parameter values are g_C = 1.91%, g_D = 2.45%, σ_C = 3.8%, σ_D = 11%, ρ = 0.2, δ = 2%, θ = 0.5, χ = 0.18, λ = 0.18, μ_H = 15%, and μ_L = −15%.

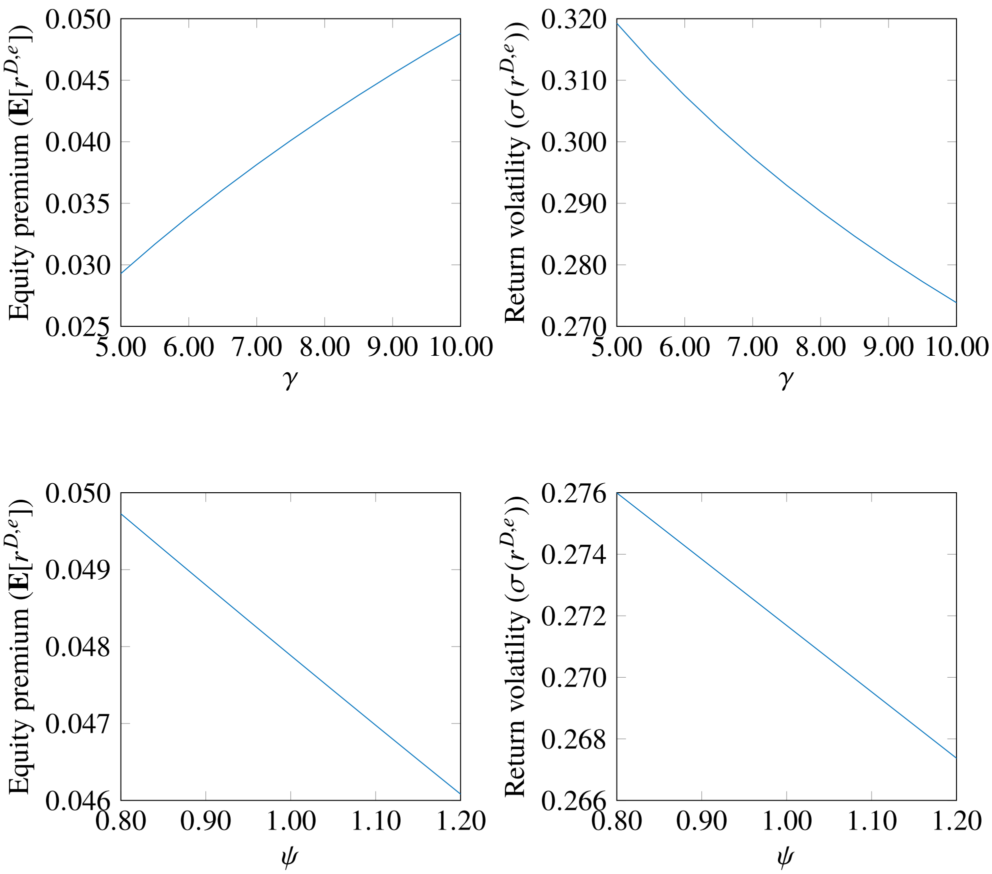


Figure B.6a: Comparative Statics: Belief Parameters (I).

The figure plots the average equity premium $\mathbb{E}[r^{D,e}]$, the average volatility of stock market returns $\sigma(r^{D,e})$, the average price-dividend ratio $\exp(\mathbb{E}[\ell n(P/D)])$, and the average interest rate $\mathbb{E}[r]$ (in percentage), each as a function of θ, the parameter that controls the extent to which the agent is behavioral. The other parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$.

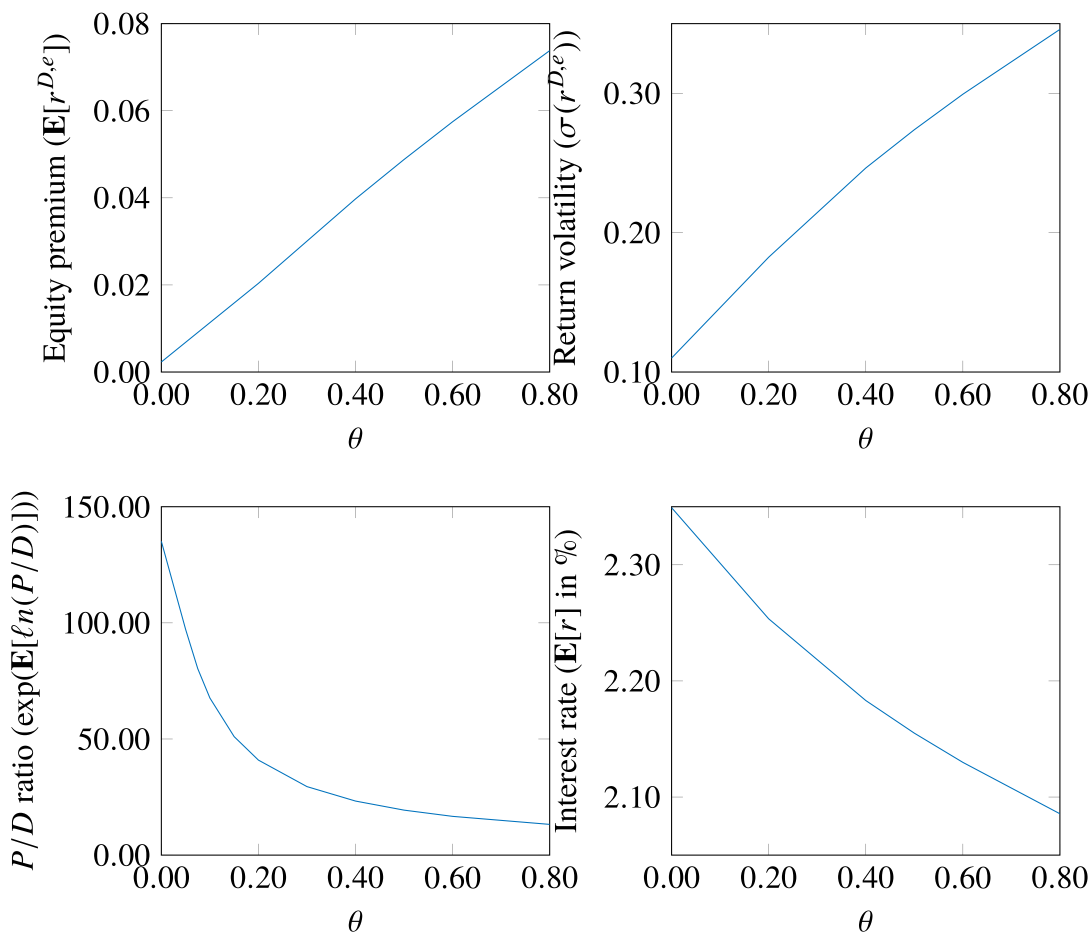


Figure B.6b: Comparative Statics: Belief Parameters (II).

The figure plots the predictability of stock market returns and the persistence of the price-dividend ratio, each as a function of θ, χ, or λ. The predictability of returns is measured by the slope coefficient for a regression of the next year's log excess return on the current log price-dividend ratio. The persistence of the price-dividend ratio is measured by the one-year autocorrelation of log price-dividend ratios. The default values for θ, χ, and λ are 0.5, 0.18, and -0.18, respectively. The other parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\mu_H = 15\%$, and $\mu_L = -15\%$.

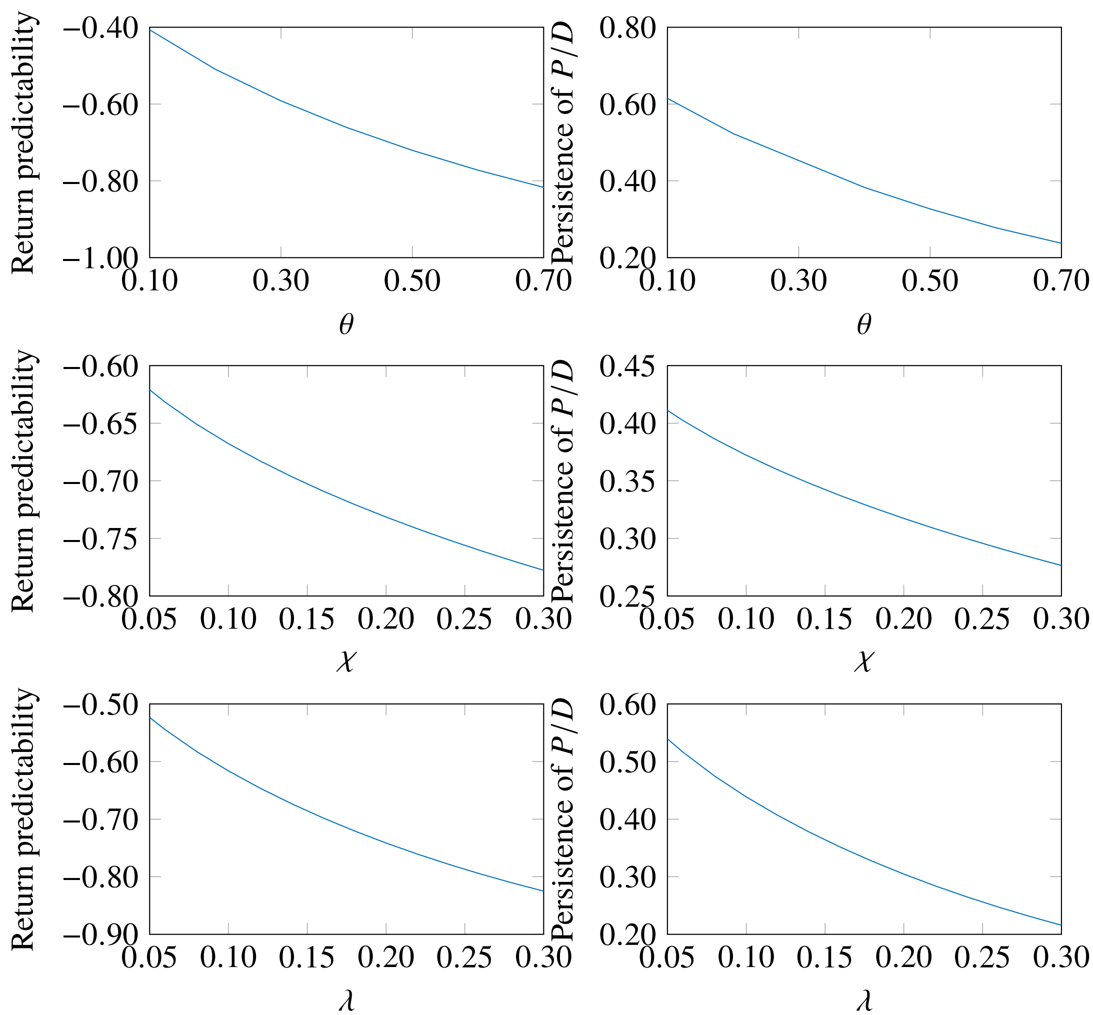


Table B.1: Investor Expectations.

	$\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]$				$\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$			
$R^D_{t-12\to t}$	0.022	0.023			0.040	0.040		
	(8.2)	(11.7)			(8.4)	(12.1)		
$\ell n(P/D)$			0.068	0.069			0.120	0.121
			(29.5)	(39.2)			(36.8)	(48.9)
Constant	0.07	0.07	–0.13	–0.13	0.02	0.02	–0.33	–0.34
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.57	0.54	0.98	0.98	0.58	0.55	0.99	0.99

	$\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$				$\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t$			
$R^D_{t-12\to t}$	0.013	0.013			0.030	0.030		
	(6.6)	(9.0)			(7.8)	(11.1)		
$\ell n(P/D)$			0.039	0.039			0.091	0.091
			(12.4)	(16.6)			(22.3)	(28.7)
Constant	0.05	0.05	–0.06	–0.06	–0.002	–0.002	–0.27	–0.27
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.51	0.47	0.92	0.91	0.56	0.53	0.97	0.97

The table reports the regression coefficient and the t-statistic (in parenthesis), the intercept, as well as the adjusted R-squared, for regressing the agent's expectation about future stock market returns either on the past twelve-month cumulative raw return $R^D_{t-12\to t}$ or on the current log price-dividend ratio $\ell n(P_t/D_t)$, over a sample of 15 years or 50 years. In the top panel, the expectations measure for the first four columns is $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]$, and the expectations measure for the last four columns is $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$. In the bottom panel, the expectations measure for the first four columns is $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$, and the expectations measure for the last four columns is $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t$. Each reported value is averaged over 100 trials, and each trial represents a regression using monthly data simulated from the model. The t-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H =$

15%, and $\mu_L = -15\%$.

Table B.2: Determinants of Investor Expectations.

	$\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]$		$\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$	
ϕ	0.432	0.417	0.428	0.420
a	0.064	0.064	0.004	0.004
b	1.15	1.18	2.04	2.07
Sample size	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.98	0.98	0.99	0.98

	$\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$		$\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t$	
ϕ	0.414	0.408	0.418	0.420
a	0.047	0.047	–0.013	-0.010
b	0.70	0.67	1.58	1.55
Sample size	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.92	0.90	0.97	0.96

The table reports the parameter ϕ, the intercept a, the regression coefficient b, and the adjusted R-squared, for running the non-linear least squares regression

$$\text{Expectation}_t = a + b \sum_{j=1}^{n} w_j R^D_{(t-j\Delta t)\to(t-(j-1)\Delta t)} + \varepsilon_t,$$

over a sample of 15 years or 50 years, where $w_j \equiv e^{-\phi(j-1)\Delta t} \Big/ \sum_{l=1}^{n} e^{-\phi(l-1)\Delta t}$, Δt = 1/12, and n = 600. In the top panel, the expectations measure for the first four columns is $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]$, and the expectations measure for the last four columns is $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$. In the bottom panel, the expectations measure for the first four columns is $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$, and the expectations measure for the last four columns is $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t$. Each reported value is averaged over 100 trials, and each trial represents a regression using monthly data simulated from the model. The t-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameter values are g_C = 1.91%, g_D = 2.45%, σ_C = 3.8%, σ_D = 11%, ρ = 0.2, γ = 10, ψ = 0.9, δ = 2%, θ = 0.5, χ = 0.18, λ = 0.18, μ_H =

15%, and $\mu_L = -15\%$.

Table B.3: Parameter Values.

Parameter	Variable	Value
Asset parameters:		
Expected consumption growth	g_C	1.91%
Expected dividend growth	g_D	2.45%
Standard deviation of consumption growth	σ_C	3.8%
Standard deviation of dividend growth	σ_D	11%
Correlation between dD and dC	ρ	0.2
Utility parameters:		
Relative risk aversion	γ	10
Reciprocal of EIS	ψ	0.9
Subjective discount rate	δ	0.02
Belief parameters:		
Degree of extrapolation	θ	0.5
Perceived transition intensity from H to L	χ	0.18
Perceived transition intensity from L to H	λ	0.18
Upper bound of sentiment	μ_H	0.15
Lower bound of sentiment	μ_L	–0.15

Table B.4: Basic Moments.

Statistic	Theoretical value	Empirical value
Equity premium ($\mathbb{E}[r^{D,e}]$)	4.88%	3.90%
Return volatility ($\sigma(r^{D,e})$)	27.4%	18.0%
Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$)	0.20	0.22
Interest rate ($\mathbb{E}[r]$)	2.16%	2.92%
Interest rate volatility ($\sigma(r)$)	0.33%	2.89%
Price-dividend ratio ($\exp(\mathbb{E}[\ell n(P/D)])$)	19.4	21.1

The table reports six important moments about stock market prices and returns: the long-run average of the equity premium (the rational expectation of log excess return, $\mathbb{E}[r^{D,e}]$), the average volatility of stock market returns (the volatility of log excess return, $\sigma(r^{D,e})$), the Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$), the average interest rate ($\mathbb{E}[r]$), interest rate volatility ($\sigma(r)$), and the average price-dividend ratio of the stock market ($\exp(\mathbb{E}[\ell n(P/D)])$). The theoretical values for these moments are computed over the objectively measured steady-state distribution of sentiment S. The model parameters are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. For the empirical values, five out of six are from Campbell and Cochrane (1999); the empirical value for interest rate volatility is not reported in Campbell and Cochrane (1999), so we report the value from Beeler and Campbell (2012).

Table B.5: Return Predictability Regressions.

Horizon (years)	Theoretical value 10× coefficient	Theoretical value Adjusted R-squared	Empirical value 10× coefficient	Empirical value Adjusted R-squared
1	–7.2	0.13	–1.3	0.04
2	–9.5	0.16	–2.8	0.08
3	–10.1	0.15	–3.5	0.09
5	–10.6	0.13	–6.0	0.18
7	–11.0	0.12	–7.5	0.23

The table reports the regression coefficient β_j and the adjusted R-squared for a regression of the log excess return of stock market from time t to time $t + j$ on the current log price-dividend ratio $\ell n(P_t^D/D_t)$

$$r_{t\to t+j}^{D,e} = \alpha_j + \beta_j \ell n(P_t^D / D_t) + \varepsilon_{j,t},$$

where j = 1, 2, 3, 5, and 7 (years). The theoretical values are calculated using 10,000 years of monthly data simulated from the model. The parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. The empirical values are from Campbell and Cochrane (1999).

Table B.6: Correlation between Consumption Growth and Stock Returns.

Correlation	Theoretical value			Empirical value
	monthly	quarterly	annual	annual
$\mathbb{C}\text{orr}(r^{D,e}_{t\to t+1}, \ell\text{n}(C_{t-1}/C_{t-2}))$	–0.01	–0.02	–0.02	–0.05
$\mathbb{C}\text{orr}(r^{D,e}_{t\to t+1}, \ell\text{n}(C_{t}/C_{t-1}))$	–0.01	–0.03	–0.06	–0.08
$\mathbb{C}\text{orr}(r^{D,e}_{t\to t+1}, \ell\text{n}(C_{t+1}/C_{t}))$	0.20	0.20	0.19	0.09
$\mathbb{C}\text{orr}(r^{D,e}_{t\to t+1}, \ell\text{n}(C_{t+2}/C_{t+1}))$	0.00	–0.00	0.01	0.49
$\mathbb{C}\text{orr}(r^{D,e}_{t\to t+1}, \ell\text{n}(C_{t+3}/C_{t+2}))$	–0.00	–0.00	0.01	0.05

The table reports correlations between log consumption growth and log excess returns of the stock market. The log consumption growth and log excess returns are computed at either a monthly, a quarterly, or an annual horizon. Correlations are either contemporaneous or with a lead-lag structure; for instance, at a monthly frequency, $\mathbb{C}\text{orr}(r^{D,e}_{t\to t+1}, \ell\text{n}(C_{t+2}/C_{t+1}))$ is the correlation between the current monthly log excess return and the log consumption growth in the subsequent month. The theoretical values are calculated using 10,000 years of monthly data simulated from the model. The parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. The empirical values are from Campbell and Cochrane (1999).

Table B.7: Campbell-Shiller Decomposition.

Coefficient	Realized dividend growth and returns: Value	Realized dividend growth and returns: Adjusted R-squared
$DR_{objective}$	0.98	0.209
$CF_{objective}$	0.02	1.2×10^{-4}

Coefficient	Subjective expectations about dividend growth and returns: Value	Subjective expectations about dividend growth and returns: Adjusted R-squared
$DR_{subjective}$	−0.08	0.982
$CF_{subjective}$	1.08	0.984

The table reports the four coefficients, $CF_{objective}$, $DR_{objective}$, $CF_{subjective}$, and $DR_{objective}$, defined in equations (2.29) and (2.30), as well as their corresponding adjusted R-squared. These coefficients and R-squared are calculated using 10,000 years of monthly data simulated from the model. At each point in time, for a given level of sentiment, subjective expectations about dividend growth and returns are calculated as the average values of 100 trials. Each trial is 50 years of monthly simulated data under the agent's expectations with the given initial level of sentiment. For realized dividend growth and returns, both $\sum_{j=0}^{\infty} \xi^j \Delta d_{(t+j\Delta t)\to(t+(j+1)\Delta t)}$ and $\sum_{j=0}^{\infty} \xi^j r^D_{(t+j\Delta t)\to(t+(j+1)\Delta t)}$ are approximated using 50 years of monthly simulated data. From the simulated data, $\xi = 0.9957$. The other parameter values are g_C = 1.91%, g_D = 2.45%, σ_C = 3.8%, σ_D = 11%, ρ = 0.2, γ = 10, ψ = 0.9, δ = 2%, θ = 0.5, χ = 0.18, λ = 0.18, μ_H = 15%, and μ_L = −15%.

Table B.8: Autocorrelations of Log Excess Returns and Log Price-dividend Ratios.

Lag (years)	Theoretical value			Empirical value		
	$\ell n(P^D/D)$	$r^{D,e}$	$\Sigma_{i=1}^{j}\rho(r_t^{D,e}, r_{t-i}^{D,e})$	$\ell n(P^D/D)$	$r^{D,e}$	$\Sigma_{i=1}^{j}\rho(r_t^{D,e}, r_{t-i}^{D,e})$
1	0.33	–0.28	–0.28	0.78	0.05	0.05
2	0.11	–0.09	–0.37	0.57	–0.21	–0.16
3	0.05	–0.02	–0.39	0.50	–0.08	–0.09
5	0.00	–0.01	–0.40	0.32	–0.14	–0.28
7	–0.02	–0.01	–0.41	0.29	0.11	–0.15

The table reports, over various lags j, the autocorrelations of log price-dividend ratios and log excess returns, as well as the partial sum of the autocorrelations of log excess returns. The operator $\rho(x, y)$ computes the sample correlation between variable x and variable y. The theoretical values are calculated using 10,000 years of monthly data simulated from the model; for each month, we compound the next 12 months of log excess returns to obtain an annual log excess return. The parameter values are g_C = 1.91%, g_D = 2.45%, σ_C = 3.8%, σ_D = 11%, ρ = 0.2, γ = 10, ψ = 0.9, δ = 2%, θ = 0.5, χ = 0.18, λ = 0.18, μ_H = 15%, and μ_L = –15%. The empirical values are from Campbell and Cochrane (1999).

Table B.9: Return Predictability Regressions in the True Regime-Switching Model.

Horizon (years)	Theoretical value: 10× coefficient	Theoretical value: 10^3×Adjusted R-squared	Empirical value: 10× coefficient	Empirical value: Adjusted R-squared
1	0.2	0.16	–1.3	0.04
2	0.4	0.31	–2.8	0.08
3	0.5	0.32	–3.5	0.09
5	0.8	0.54	–6.0	0.18
7	0.7	0.34	–7.5	0.23

The table reports the regression coefficient β_j and the adjusted R-squared for a regression of the log excess return of stock market from time t to time $t + j$ on the current log price-dividend ratio $\ell n(P_t^D / D_t)$

$$r_{t\to t+j}^{D,e} = \alpha_j + \beta_j \ell n(P_t^D / D_t) + \varepsilon_{j,t},$$

where $j = 1, 2, 3, 5$, and 7 (years). The theoretical values are calculated using 10,000 years of monthly data simulated from the true regime-switching model described in Section 2.5. The parameter values are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. The empirical values are from Campbell and Cochrane (1999).

Table B.10: Investor Expectations in Bansal and Yaron (2004).

	Expectation of return				Expectation of return w/o divd.			
$R^D_{t-12\to t}$	0.025	0.030			0.025	0.031		
	(2.4)	(3.8)			(2.5)	(4.0)		
$\ell\text{n}(P/D)$			0.068	0.067			0.069	0.067
			(5.6)	(7.6)			(5.8)	(7.6)
Constant	1.06	1.05	0.88	0.88	1.06	1.05	0.88	0.89
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.140	0.142	0.500	0.451	0.136	0.145	0.529	0.451

	Expectation of excess return				Expectation of excess return w/o divd.			
$R^D_{t-12\to t}$	−0.001	0.000			0.000	0.001		
	(−0.1)	(0.1)			(−0.1)	(0.2)		
$\ell\text{n}(P/D)$			−0.008	−0.012			−0.010	−0.011
			(−0.8)	(−1.3)			(−0.8)	(−1.2)
Constant	0.06	0.06	0.08	0.10	0.06	0.06	0.09	0.09
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.026	0.008	0.087	0.054	0.026	0.008	0.093	0.053

The table reports the regression coefficient and the t-statistic (in parenthesis), the intercept, as well as the adjusted R-squared, for regressing four measures of rational expectation of return—raw return or excess return, with or without dividend yield—either on the past twelve-month cumulative raw return or on the current log price-dividend ratio, over a sample of 15 years or 50 years. The conditional expectation of subsequent returns, the dependent variable in each regression, is computed by averaging realized returns across simulations over a twelve-month horizon for a given state of the economy. Each reported value is the estimator median over 1,000 trials, and each trial represents a regression using monthly data simulated from Bansal and Yaron (2004). The t-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameters take their default values from

Tables II and IV of Bansal and Yaron (2004).

Table B.11: Investor Expectations in Bansal et al. (2012).

	Expectation of return				Expectation of return w/o divd.			
$R^D_{t-12\to t}$	0.006	0.010			0.006	0.011		
	(0.7)	(1.3)			(0.7)	(1.3)		
$\ell n(P/D)$			−0.006	−0.041			−0.006	−0.038
			(−0.3)	(−3.1)			(−0.2)	(−2.9)
Constant	1.08	1.08	1.11	1.22	1.08	1.08	1.11	1.21
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.031	0.021	0.129	0.147	0.033	0.020	0.139	0.136

	Expectation of excess return				Expectation of excess return w/o divd.			
$R^D_{t-12\to t}$	−0.003	0.000			−0.004	−0.001		
	(−0.3)	(0.1)			(−0.5)	(−0.1)		
$\ell n(P/D)$			−0.071	−0.095			−0.036	−0.054
			(−5.2)	(−10.7)			(−3.4)	(−5.2)
Constant	0.09	0.08	0.30	0.37	0.08	0.08	0.19	0.24
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.021	0.010	0.432	0.599	0.026	0.008	0.226	0.254

The table reports the regression coefficient and the t-statistic (in parenthesis), the intercept, as well as the adjusted R-squared, for regressing four measures of rational expectation of return—raw return or excess return, with or without dividend yield—either on the past twelve-month cumulative raw return or on the current log price-dividend ratio, over a sample of 15 years or 50 years. The conditional expectation of subsequent returns, the dependent variable in each regression, is computed by averaging realized returns across simulations over a twelve-month horizon for a given state of the economy. Each reported value is the estimator median over 1,000 trials, and each trial represents a regression using monthly data simulated from Bansal et al. (2012). The t-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameters take their default values from Table 1 of Bansal et al. (2012).

Table B.12: Investor Expectations in the Fundamental Extrapolation Model.

	$\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]$				$\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$			
$R^D_{t-12\to t}$	0.004	0.003			0.010	0.010		
	(0.21)	(0.63)			(1.15)	(2.04)		
$\ell n(P/D)$			0.017	0.019			0.040	0.042
			(0.88)	(1.62)			(2.08)	(3.52)
Constant	0.05	0.05	–0.01	–0.02	0.03	0.03	–0.12	–0.13
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.14	0.04	0.24	0.10	0.17	0.08	0.31	0.22

	$\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$				$\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t$			
$R^D_{t-12\to t}$	–0.004	–0.004			0.003	0.002		
	(–0.88)	(–1.13)			(0.07)	(0.42)		
$\ell n(P/D)$			–0.009	–0.007			0.014	0.016
			(–0.49)	(–0.54)			(0.71)	(1.35)
Constant	0.03	0.03	0.06	0.05	0.006	0.006	–0.05	–0.05
Sample size	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.	15 yr.	50 yr.
R^2	0.15	0.05	0.22	0.06	0.14	0.04	0.23	0.08

The table reports the regression coefficient and the t-statistic (in parenthesis), the intercept, as well as the adjusted R-squared, for regressing the agent's expectation about future stock market returns either on the past twelve-month cumulative raw return $R^D_{t-12\to t}$ or on the current log price-dividend ratio $\ell n(P_t/D_t)$, over a sample of 15 years or 50 years. In the top panel, the expectations measure for the first four columns is $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]$, and the expectations measure for the last four columns is $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]$. In the bottom panel, the expectations measure for the first four columns is $\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$, and the expectations measure for the last four columns is $\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t$. Each reported value is averaged over 100 trials, and each trial represents a regression using monthly data simulated from the fundamental extrapolation model described in Section 2.6. The t-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameter values are g_C = 1.91%, g_D = 2.45%, σ_C = 3.8%, σ_D = 11%, ρ = 0.2, γ = 10, ψ = 0.9, δ = 2%, θ = 0.5, χ = 0.18, λ = 0.18, μ_H = 15%, and μ_L = –15%.

Table B.13: Basic Moments in the Fundamental Extrapolation Model.

Statistic	Theoretical value	Empirical value
Equity premium ($\mathbb{E}[r^{D,e}]$)	1.85%	3.90%
Return volatility ($\sigma(r^{D,e})$)	20.4%	18.0%
Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$)	0.08	0.22
Interest rate ($\mathbb{E}[r]$)	2.27%	2.92%
Interest rate volatility ($\sigma(r)$)	0.23%	2.89%
Price-dividend ratio ($\exp(\mathbb{E}[\ell n(P/D)])$)	44.4	21.1

The table reports six important moments about stock market prices and returns: the long-run average of the equity premium (the rational expectation of log excess return, $\mathbb{E}[r^{D,e}]$), the average volatility of stock market returns (the volatility of log excess return, $\sigma(r^{D,e})$), the Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$), the average interest rate ($\mathbb{E}[r]$), interest rate volatility ($\sigma(r)$), and the average price-dividend ratio of the stock market ($\exp(\mathbb{E}[\ell n(P/D)])$). The theoretical values for these moments are computed over the objectively measured steady-state distribution of sentiment S in the fundamental extrapolation model described in Section 2.6. The model parameters are $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. For the empirical values, five out of six are from Campbell and Cochrane (1999); the empirical value for interest rate volatility is not reported in Campbell and Cochrane (1999), so we report the value from Beeler and Campbell (2012).

Appendix C

APPENDIX TO CHAPTER THREE

C.1 Figures and Tables

Figure C.1: Tail Risk Measurements Based on Shiller Tail Risks Survey.

In the sample period 1990:01 - 2016:01, I plot the Shiller Crash Index for the individual and institutional investor subsample, respectively. The shaded areas represent NBER recessions.

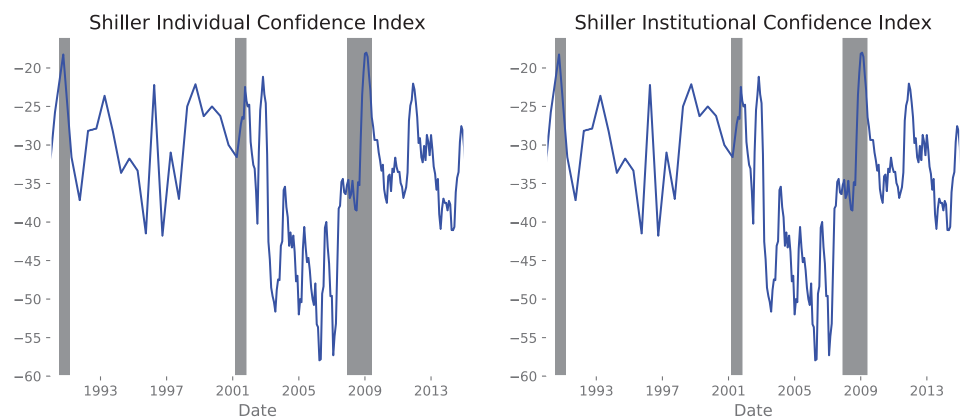


Figure C.2: Connecting Investor Expectations with Perceived Left-tail Risks.

The figure provides an intuitive connection between investor expectations and their perceived left-tail probabilities. (Figure source: Bordalo et al. (2018).)

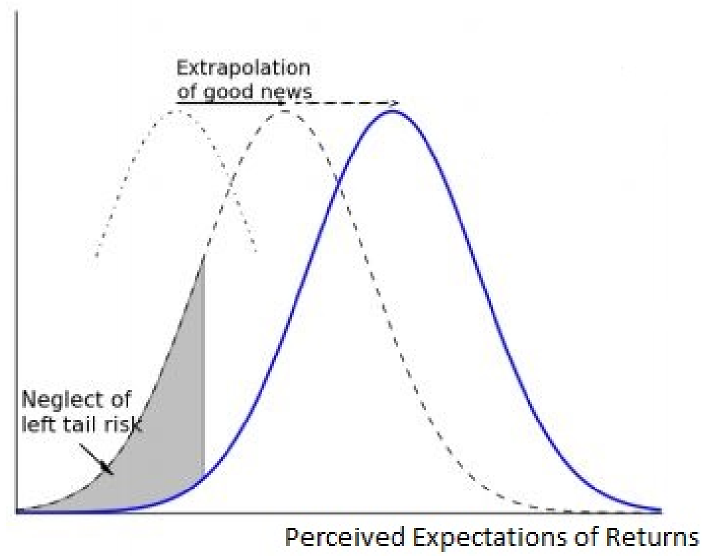


Figure C.3: Implied Left-tail Risks based on Investor Expectations.

In the sample period 1990:01 - 2016:01, I plot four the implied left-tail risks from investor expectation surveys (AA, II, and Gallup) and from return extrapolation. In the first three figures, I plot the implied left-tail risks from three investor expectation surveys. In the lower-left figure, I plot the implied left-tail risks based on return extrapolation. The shaded areas represent NBER recessions.

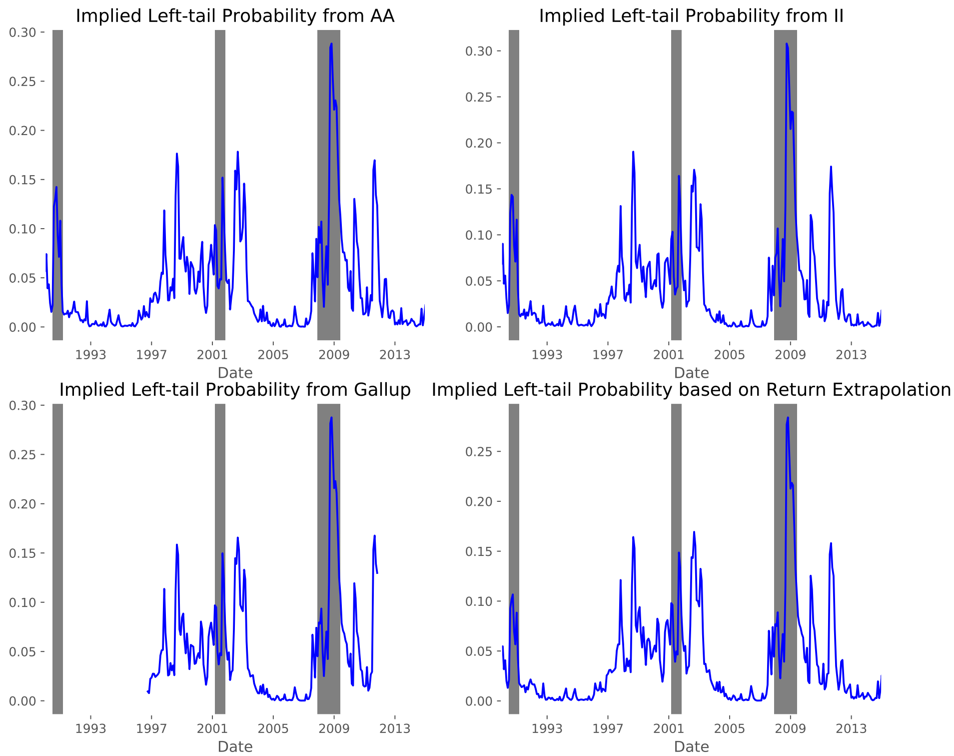


Table C.1: Summary Statistics and Correlations between Investor Expectation Surveys

This table reports the summary statistics and correlation matrix between investor expectation surveys. Panel A reports the summary statistics for the qualitative investor expectations. Panel B reports the summary statistics for the rescaled quantitative investor expectations. Panel C reports the correlations between the qualitative investor expectations.

Statistic	N	Mean	St. Dev.	Min	Max
AA Investor Expectation	331	0.086	0.153	−0.410	0.505
II Investor Expectation	626	13.632	19.794	−49.200	66.640
Gallup Investor Expectation	182	19.362	22.938	−45.000	57.000
Shiller Investor Expectation	183	79.287	7.258	61.940	95.620
Graham & Harvey investor Expectation	35	5.757	1.541	2.180	9.060

Panel A: Summary Statistics for the Qualitative Investor Expectation Series

Statistic	N	Mean	St. Dev.	Min	Max
AA Investor Expectation	331	0.099	0.016	0.047	0.143
II Investor Expectation	626	0.099	0.030	0.004	0.179
Gallup Investor Expectation	182	0.107	0.011	0.077	0.124
Shiller Investor Expectation	164	0.111	0.002	0.106	0.116
Graham & Harvey investor Expectation	35	0.109	0.018	0.067	0.148

Panel B: Summary Statistics for the Rescaled Quantitative Investor Expectation Series

	AA	II	Gallup	Shiller	Graham & Harvey
AA	1.00	0.53	0.63	0.32	0.60
II	0.53	1.00	0.53	0.10	0.74
Gallup	0.63	0.53	1.00	0.53	0.74
Shiller	0.32	0.10	0.53	1.00	0.81
Graham & Harvey	0.60	0.74	0.74	0.81	1.00

Panel C: Correlations between raw investor expectation series

Table C.2: Correlations between the Left-tail Probabilities from Investor Expectation Surveys.

This table shows pairwise correlation coefficients between the implied left-tail probabilities from investor expectation surveys and these directly from Shiller left-tail risks survey. For example, Left-tail Probability based on AA represents the implied left-tail probability based on the rescaled AA investor expectation series. The Shiller crash index is based on the U.S. Crash Index of ICF at Yale, which measures the percentage of investors who report a less than 10% left-tail probabilities.

	Individual Individual Crash Index	Institutional Institutional Crash Index
Left-tail Probability based on AA	0.56***	0.55***
Left-tail Probability based on II	0.55***	0.54***
Left-tail Probability based on Gallup	0.67***	0.69***
Left-tail Probability based on Shiller	0.56***	0.58***
Left-tail Probability based on Graham & Harvey	0.76***	0.76***
Note:		*p<0.1; **p<0.05; ***p<0.01

Table C.3: Contemporaneous Connections between Shiller Left-tail Probability and Investor Expectations: Individual Investors

I estimate time-series regressions of the form:

$$\text{Shiller Individual Crash Index}_t = a + b\text{Survey Sentiment}_t + X_t + u_t, \qquad \text{(C.1)}$$

where Shiller Individual Crash Index$_t$ denotes the Individual Crash Index based on Shiller left-tail questions from the individual investor subsample under the interpretation of annual frequency, Survey Sentiment$_t$ represents survey measurements of investor expectations on future U.S. equity market returns and X_t represents controls including averaged VIX index and the realized $S\&P$ 500 volatility within month t. All survey measurements of investor expectations are rescaled, which can be interpreted in units of nominal stock returns. Panel A shows results for contemporaneous correlations. Panel B shows predictive relations for one-month ahead perceived crash probabilities. Panel C shows predictive relations for three month ahead perceived crash probabilities. All t-statistics are corrected using the approach in Newey and West (1986).

	Dependent variable:									
	Crash Probability from Shiller Survey: Individual Investors									
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
AA	−190.812***					−107.270**				
	(53.248)					(48.919)				
II		−153.325***					−36.006			
		(41.471)					(62.678)			
Gallup			−485.934***					−194.966*		
			(100.201)					(115.515)		
Shiller				−2,006.273***					−1,897.276***	
				(604.249)					(346.645)	
Graham&Harvey					−301.859***					−153.205**
					(26.756)					(73.590)
VIX_agg						0.224**	0.242	0.276*	0.263***	0.437**
						(0.100)	(0.171)	(0.148)	(0.093)	(0.193)
vol_monthly						0.240**	0.240**	0.200	0.258***	0.159*
						(0.096)	(0.105)	(0.124)	(0.099)	(0.084)
Constant	−14.764**	−16.533***	16.125	189.497***	−0.574	−31.976***	−38.881***	−23.972*	167.234***	−29.005***
	(5.740)	(4.959)	(10.397)	(67.790)	(4.200)	(6.667)	(10.221)	(14.529)	(39.337)	(10.817)
Observations	170	171	131	164	35	169	170	131	164	35
R^2	0.168	0.189	0.366	0.311	0.375	0.396	0.357	0.515	0.634	0.653

Note: *p<0.1; **p<0.05; ***p<0.01

Table C.4: Contemporaneous Connections between Shiller Left-tail Probability and Investor Expectations: Institutional Investors

I estimate time-series regressions of the form:

$$\text{Shiller Institutional Crash Index}_t = a + b\text{Survey Sentiment}_t + X_t + u_t, \tag{C.2}$$

where Shiller Institutional Crash Index$_t$ denotes the Institutional Crash Index based on Shiller left-tail questions from the institutional investor subsample under the interpretation of annual frequency, Survey Sentiment$_t$ represents survey measurements of investor expectations on future U.S. equity market returns, and X_t represents controls including averaged VIX index and the realized $S\&P$ 500 volatility within month t. All survey measurements of investor expectations are rescaled which can be interpreted in units of nominal stock returns.

	Dependent variable:									
	Crash Probability from Shiller Survey: Institutional investors									
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
AA	−126.767*					−11.219				
	(65.666)					(74.383)				
II		−157.872***					−50.902			
		(37.404)					(64.026)			
Gallup			−412.324***					129.371		
			(140.506)					(120.671)		
Shiller				−1,502.081**					−1,364.190**	
				(758.031)					(557.957)	
Graham&Harvey					−297.620***					−150.549***
					(40.810)					(47.069)
VIX_agg						0.419***	0.317*	0.604***	0.326**	0.348***
						(0.157)	(0.176)	(0.223)	(0.150)	(0.135)
vol_monthly						0.226	0.263**	0.315**	0.353***	0.332***
						(0.147)	(0.128)	(0.131)	(0.133)	(0.094)
Constant	−23.091***	−18.396***	5.991	130.600	−5.847	−47.045***	−41.185***	−69.783***	102.092*	−35.525***
	(6.409)	(3.932)	(13.565)	(83.372)	(4.791)	(9.680)	(10.186)	(15.179)	(61.698)	(6.938)
Observations	187	188	135	164	35	186	187	135	164	35
R^2	0.054	0.162	0.184	0.126	0.330	0.354	0.362	0.572	0.524	0.693
Note:										*p<0.1; **p<0.05; ***p<0.01

Table C.5: Left-tail Probabilities based on Return Extrapolation and Rational Expectations (with VIX).

This table compares the left-tail probabilities based on return extrapolation and rational expectations, with the information form VIX index. The Shiller crash index is based on the U.S. Crash Index of ICF at Yale, which measures the percentage of investors who report a less than 10% left-tail probabilities.

	Shiller Individual Crash Index	Shiller Institutional Crash Index
Left-tail Probability based on Psentiment	0.55***	0.55***
Note:	*p<0.1; **p<0.05; ***p<0.01	

Panel A: Left-tail Probability based on Return Extrapolation.

	Shiller Individual Crash Index	Shiller Institutional Crash Index
Left-tail Probability based on logDP	0.09	0.36***
Left-tail Probability based on future realized returns	0.18*	0.20**
Note:	*p<0.1; **p<0.05; ***p<0.01	

Panel B: Left-tail Probability based on rational expectations.

Table C.6: Left-tail Probabilities based on return extrapolation and rational expectations (without VIX).

This table compares the left-tail probabilities based on return extrapolation and rational expectations, without the information form VIX index. The Shiller crash index is based on the U.S. Crash Index of ICF at Yale, which measures the percentage of investors who report a less than 10% left-tail probabilities.

	Shiller Individual Crash Index	Shiller Institutional Crash Index
Left-tail Probability based on AA	0.41***	0.24**
Left-tail Probability based on II	0.44***	0.39***
Left-tail Probability based on Gallup	0.61***	0.44***
Left-tail Probability based on Psentiment	0.30***	0.18*
Left-tail Probability based on Shiller	0.56***	0.36***
Left-tail Probability based on Graham \& Harvey	0.62***	0.59***
Note:	$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$	

Panel A. Correlations for implied left-tail probabilities based on surveys (without VIX).

	Shiller Individual Crash Index	Shiller Institutional Crash Index
Left-tail Probability based on logDP	-0.30***	-0.03
Left-tail Probability based on future realized returns	0.04	0.04
Note:	$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$	

Panel B. Correlations for left-tail probabilities based on rational expectations(without VIX).

Table C.7: Determinant of Perceived Left-tail Probabilities: Individual Investors

I estimate time-series regressions of the form:

$$\text{Shiller Individual Crash Index}_t = a + b\log(D_t/P_t) + VIX_t + X_t + u_t, \quad \text{(C.3)}$$

where $\log(D_t/P_t)$ represents log dividend to price ratio of the U.S. equity market returns, VIX_t represents the average of the VIX index within month t, and X_t represents controls including the realized S&P 500 volatility, risk free rate, default yield and term yield during month t. All t-statistics are corrected using the approach in Newey and West (1986).

	Dependent variable:									
	Left-tail Probabilities									
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Past12	-13.305^{*}			-6.685^{*}	4.256	7.305		4.654	4.533	7.368^{*}
	(7.479)			(4.017)	(4.481)	(5.955)		(5.105)	(5.320)	(3.954)
logPD		-22.964^{***}		-21.093^{***}	-19.025^{***}	-17.758^{***}	-17.903^{***}	-14.726^{**}	-14.970^{**}	−4.079
		(7.030)		(6.732)	(5.235)	(5.543)	(5.472)	(6.209)	(6.918)	(5.361)
VIX			0.442^{***}		0.342^{**}	0.273^{**}	0.231^{**}	0.284^{**}	0.288^{**}	0.377^{***}
			(0.100)		(0.150)	(0.124)	(0.114)	(0.121)	(0.131)	(0.116)
SP_vol						0.169	0.096	0.108	0.110	0.280^{***}
						(0.145)	(0.108)	(0.121)	(0.125)	(0.082)
Riskfree								−843.172	−837.061	$-3,329.633^{***}$
								(823.644)	(829.322)	(876.495)
def									−0.220	−0.085
									(2.389)	(2.113)
Term										-4.059^{***}
										(0.701)
Constant	-18.746^{**}	58.186^{**}	-41.381^{***}	57.780^{**}	31.445	21.623	31.964	14.223	15.436	−24.214
	(8.143)	(27.459)	(2.638)	(23.454)	(22.229)	(26.141)	(23.055)	(28.608)	(32.079)	(24.210)
Observations	193	190	192	190	189	189	189	189	189	189
Adjusted R^2	0.095	0.337	0.276	0.357	0.476	0.488	0.476	0.504	0.501	0.593
Note:										*p<0.1; **p<0.05; ***p<0.01

Table C.8: Determinant of Perceived Left-tail Probabilities: Institutional Investors

I estimate time-series regressions of the form:

$$\text{Shiller Institutional Crash Index}_t = a + b\log(D_t/P_t) + VIX_t + X_t + u_t, \qquad \text{(C.4)}$$

where $\log(D_t/P_t)$ represents log dividend to price ratio of the U.S. equity market returns, VIX_t represents the average of the VIX index within month t, and X_t represents controls including the realized S&P 500 volatility, risk free rate, default yield and term yield during month t. All t-statistics are corrected using the approach in Newey and West (1986).

	Dependent variable:									
	Left-tail Probabilities									
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Past12	−16.314***			−13.193**	1.479	5.909		4.513	−0.135	−0.131
	(5.130)			(5.317)	(5.782)	(6.680)		(6.648)	(6.242)	(6.302)
logPD		−13.594		−10.968	−7.984	−7.490	−7.239	−7.035	−10.982	−11.016
		(8.724)		(7.652)	(6.785)	(7.275)	(6.992)	(8.052)	(7.640)	(7.948)
VIX			0.540***		0.518***	0.395**	0.365**	0.412**	0.530***	0.531***
			(0.123)		(0.177)	(0.166)	(0.162)	(0.173)	(0.167)	(0.167)
SP_vol						0.260	0.202	0.186	0.277	0.279
						(0.173)	(0.149)	(0.159)	(0.187)	(0.191)
Riskfree								−664.955	−877.172	−893.259
								(830.918)	(717.608)	(894.496)
def									−6.913**	−6.959**
									(3.341)	(3.247)
Term										−0.040
										(1.402)
Constant	−17.558***	18.600	−45.377***	22.189	−15.247	−23.884	−17.053	−22.350	1.976	2.207
	(5.348)	(33.376)	(3.309)	(26.688)	(25.390)	(28.608)	(28.727)	(33.258)	(32.345)	(33.726)
Observations	210	207	209	207	206	206	206	206	206	206
R^2	0.107	0.112	0.286	0.180	0.345	0.370	0.363	0.383	0.416	0.416
Adjusted R^2	0.103	0.107	0.283	0.172	0.335	0.358	0.353	0.368	0.398	0.395

Note: *p<0.1; **p<0.05; ***p<0.01