Self-Gluing formula of the monopole invariant and its application on symplectic structures.

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ABSTRACT

Seiberg-Witten theory has been an important tool in studying a class of 4-manifolds. Moreover, the Seiberg-Witten invariants have been used to compute for simple structures of symplectic manifolds. The normal connected sum operation on 4-manifolds has been used to construct 4-manifolds. In this thesis, we demonstrate how to compute the Seiberg-Witten invariant of 4-manifolds obtained from the normal connected sum operation. In addition, we introduce the application of the formula on the existence of symplectic structures of manifolds given by the normal connected sum.

In Chapter 1, we study the Seiberg-Witten theory for various types of 3- and 4-manifolds. We review the Seiberg-Witten equation and invariants for 4-manifolds with cylindrical ends as well as closed and smooth 4-manifolds . Furthermore, we explain how to compute the Seiberg-Witten invariants for two types of 4-manifolds: the products of a circle and a 3-manifold and sympectic manifolds.

In Chapter 2, we prove that the Seiberg-Witten invariant of a new manifold obtained from the normal connected sum can be represented by the Seiberg-Witten invariant of the original manifolds. In [Tau01], the author has proved the case of the operation along tori. In [MST96], the authors have proved the case of the operation along surfaces with genus at least 2 when the product of the circle and the surface is separating in the ambient 4-manifold. In this thesis, we show the proof of the remaining case.

In Chapter 3, we prove the existence of certain symplectic structures on manifolds obtained from the normal connected sum of two 4-manifolds using the multiple gluing formula stated in Chapter 2. We explain how to construct covering spaces of the manifold and compute the Seiberg-Witten invariant of the covering spaces by the gluing formula. From the relation between the Seiberg-Witten invariants and symplectic structures, we prove the main application.

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Chapter 1

BACKGROUND ON THE SEIBERG-WITTEN THEORY

In this chapter, we briefly review various types of the Seiberg-Witten equations and the invariants. We follow [Mor95], [MST96, Section 2] and [HT99]. The Seiberg-Witten equation gives a relation between a connection of a certain bundle and a section of a certain vector bundle. Before introducing the equation, we shortly review the background knowledge from differential geometry.

1.1 Background from differential geometry

Let X, Y be a smooth manifold. Let TX denote the tangent bundle of X. Suppose that $\pi : E \to X$ is a vector bundle. We define $C^\infty(E)$ to be the space of sections of E. If $f : Y \to X$ is a map, then we define the space

$$f^*E = \{(y, e) \in Y \times E | f(y) = \pi(e)\}$$

and a map $f^*E \to Y$ by $(y, e) \to y$. Then, f^*E is a vector bundle over Y. We call $f^*E \to Y$ pull-back vector bundle of E over Y.

For $\pi : E \to X$, we can construct the following short exact sequence of bundles:

$$0 \to \pi^*E \xrightarrow{i} TE \xrightarrow{\pi^*} \pi^*TX \to 0. \tag{1.1.1}$$

The image of i in TE is considered as the subspace of "vertical" tangent vectors since it vanishe in π^*TX. We can choose a "horizontal" subspace of TE which is complementary to $i(\pi^*E)$. The choice of this horizontal subspace which splits the short exact sequence 1.1.1 is formally called a *connection*.

Definition 1.1.1. *A **connection** on E is a map $\mathbf{A} : TE \longrightarrow \pi^*E$ such that*

- $\mathbf{A} \circ i : \pi^*E \longrightarrow \pi^*E$ *is a identity map*
- *For $\alpha \in \mathbb{C}$, a multiplication map $m_\alpha : E \longrightarrow E$ commutes with $\mathbf{A}$. In other words, $m_\alpha^*\mathbf{A} = \alpha \cdot \mathbf{A}$.*

If the connection is given, then we can differentiate sections of a bundle with respect to the given connection. The concept of this differentiation is called *covariant* *derivative*. The covariant derivative measures how much a section deviated from the "horizontal" subspace. Let $\tilde{\pi}$ be the restriction of the bundle map $\pi^*E \to E$ on each fiber.

Definition 1.1.2. *The covariant derivative $\nabla_{\mathbf{A}} : C^\infty(E) \to C^\infty(T^*X \otimes E)$ is defined as follows: for $\psi \in C^\infty(E)$, $\nabla_{\mathbf{A}}(\psi) : T_xX \to E_x$ is the composition*

$$T_xX \xrightarrow{\psi_*} T_{\psi(x)}E \xrightarrow{\mathbf{A}} (\pi^*E)_{\psi(x)} \xrightarrow{\tilde{\pi}} E_x.$$

Definition 1.1.3. *Suppose that $\pi : E \to B$ is a vector bundle and $\mathbf{A}$ is a connection on E. The curvature 2-form $F_{\mathbf{A}} \in \Omega^2(X, End(E))$ is defined by the 2-form which is equal to $dA + A \wedge A$ in local coordinates.*

Suppose that X is a smooth Riemannian 4-manifold. We have the hodge star operator $\star$ from the set of 2-forms on X, $\Omega^2(X)$, to itself. We call a 2-form F **self-dual** (respectively, **anti-self-dual**) if $\star F = F$ (respectively, $\star F = -F$). The set $\Omega^2(X)$ has a decomposition into $\Omega^2_+(X) \oplus \Omega^2_-(X)$, where $\Omega^2_+(X)$ and $\Omega^2_-(X)$ are the set of self-dual two forms and anti-self-dual two forms respectively.

$Spin^c$-structure

Definition 1.1.4. *Let X be a smooth manifold and G be a Lie group. A principal G bundle over X is a surjective map $\pi : P \longrightarrow X$ and a G action on P satisfying:*

- *The action of G on P respects π, i.e., for $g \in G, p \in P$, $\pi(p) = \pi(p.g)$.*
- *The action of G is free and transitive on $\pi^{-1}(x) \subset P$ for each $x \in X$.*
- *On an open ball $U \subset X$, $\pi^{-1}(U) \subset P$ is diffeomorphic to $U \times G$ through the G-equivariant and fiber-preserving diffeomorphism.*

Moreover, given a vector space V and a representation of G, $\rho : G \longrightarrow Aut(V)$, we define ***the associated vector bundle*** *V_P to be the the space obtained from $(P \times V)$ quotient by the equivalence relation $(p.g, v) \sim (p, \rho(g)v)$*

Suppose that X is a n-dimensional oriented Riemmanian manifold. Let $Fr \longrightarrow X$ be the frame bundle of X. In other words, $Fr \longrightarrow X$ is the principal $SO(n)$-bundle. For $n \geq 3$, the fundamental group of $SO(n)$ is isomorphic to $\mathbb{Z}_2$. Let $Spin(n)$ be the connected double cover of $SO(n)$ and $\pi : Spin(n) \longrightarrow SO(n)$ be the covering map. Let $\psi : Spin(n) \longrightarrow Spin(n)$ be the nontrivial covering transformation of π. Moreover,

$$Spin^c(n) = Spin(n) \times U(1)/\mathbb{Z}_2$$

when a nontrivial element $1 \in \mathbb{Z}_2$ sends (x, y) to $(\psi(x), -y)$. Then, the canonical projection $Spin^c(n) \to SO(n)$ is well-defined.

Definition 1.1.5. *A $Spin^c$-structure on n-dimensional oriented Riemannian manifold X is a principal $Spin^c$-bundle F on X with a map $F \longrightarrow Fr$ such that the following diagram commutes:*

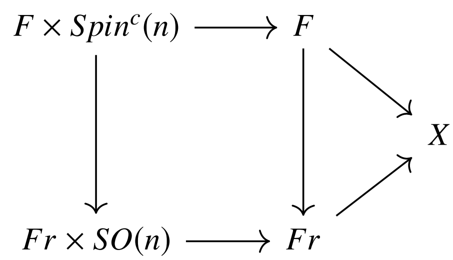

Figure 1.1: Definition of $Spin^c$-structure

We associate the complex vector bundles $S(\tilde{P})$ over X from the representations of $Spin^c(n)$. These associated vector bundles are distinguished from other vector bundles over X through the fiberwise action of T^*X. The action from Clifford multiplication,

$$cl : T^*X \to End(S(\tilde{P})),$$

has the following properties:

- $cl(v)^2 = -|v|^2$
- If $|v| = 1$, then $cl(v)$ is unitary.

Henceforth, we examine more details for 4-dimensional manifolds. Suppose that X is 4-dimensional. Now, we define two $\mathbb{C}^2$-representation, s_+, s_- of $Spin^c(4)$. We have the following relations:

- $SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \text{ for } a, b \in \mathbb{C} \right\}$
- $U(2) \cong SU(2) \times U(1)/\pm 1$

- $SO(4) \cong SU(2) \times SU(2)/\pm 1$
- $Spin(4) \cong SU(2) \times SU(2)$
- $Spin^c(4) \cong (SU(2) \times SU(2) \times U(1)/\pm 1$

We define $s_\pm : Spin^c(4) \longrightarrow Aut(\mathbb{C}^2) \cong U(2)$ to be

$$s_\pm(h_-, h_+, \lambda) = (h_\pm, \lambda).$$

Definition 1.1.6. *With these two representations s_+, s_-, $S_+(\tilde{P})$, $S_-(\tilde{P})$ are two associated $\mathbb{C}^2$ vector bundles to $\tilde{P}$. $S(\tilde{P})$ denotes $S_+(\tilde{P}) \oplus S_-(\tilde{P})$. We call $S(\tilde{P})$* ***the complex spin bundle****. We call sections of these spinor bundles* ***spinors****.*

For $v \in T^*X$, the action $cl(v)$ sends $S_+(\tilde{P})$ to $S_-(\tilde{P})$ and $S_-(\tilde{P})$ to $S_+(\tilde{P})$. More specifically, the action

$$cl : T^*X \otimes S_+(\tilde{P}) \to S_-(\tilde{P})$$

is described as the matrix multiplication. On each fiber, cl is a map from $\mathbb{R}^4 \oplus \mathbb{C}^2$ to $\mathbb{C}^2$. We identify $\mathbb{R}^4 \cong \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \cong \mathbb{R}^2 \right\}$ and

$$cl(x, \psi) = x\psi.$$

Likewise, $cl : T^*X \otimes S_-(\tilde{P}) \to S_+(\tilde{P})$ is defined by

$$cl(x, \psi) = -\bar{x}^T \psi.$$

Definition 1.1.7. *We define the Dirac operator $D_A : C^\infty(S(\tilde{P})) \to C^\infty(S(\tilde{P}))$ when the connection A is given as the composition of the two maps:*

$$C^\infty(S(\tilde{P})) \xrightarrow{\nabla_A} C^\infty(T^*X \oplus S(\tilde{P})) \xrightarrow{cl} C^\infty(S(\tilde{P})).$$

Since cl maps $S_+(\tilde{P})$ to $S_-(\tilde{P})$ and vice versa, the Dirac operator maps $C^\infty(S_+(\tilde{P}))$ to $C^\infty(S_-(\tilde{P}))$ and vice versa. $D_A^\pm$ is the Dirac operator restricted to $C^\infty(S^\pm(\tilde{P}))$.

Remark 1.1.8. *A $Spin^c$-structure $\tilde{P} \in \mathcal{S}_X$ is equivalent with the vector bundle $S(\tilde{P}) = S_+(\tilde{P}) \oplus S_-(\tilde{P})$ with the Clifford action. For $\alpha \in H^2(X, \mathbb{Z})$, let E be the line bundle satisfying that $c_1(E) = \alpha$. The action of α sends S to $S \otimes E$. With this $H^2(X, \mathbb{Z})$ action, $\mathcal{S}_X$ is an affine space modelled on $H^2(X, \mathbb{Z})$.*

1.2 Seiberg-Witten theory for closed 4-manifolds

Let X be a smooth, closed, oriented Riemannian 4-manifold with $Spin^c$-structure and let g be a Riemannian metric on X. Let $\tilde{P}$ be a $Spin^c$-strucure on (X, g). For a unitary connection A on the determinant line bundle of $\tilde{P}$ and a section ψ of $S_+(\tilde{P})$, the Seiberg-Witten equations **SW** associated to a $Spin^c$-structure $\tilde{P}$ are the following:

$$F_A^+ = q(\psi)$$
$$D_A^+(\psi) = 0,$$

where D_A is the Dirac operator and F_A^+ is the self-dual part of curvature 2-form F_A. Here q is a quadratic form from $S_+(\tilde{P})$ to the set of purely imaginary self-dual 2-forms, $\Omega^2_+(X; i\mathbb{R})$. We can describe $q(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2} Id$.

The index of the elliptic system defined from the equations **SW** is given by

$$d(\tilde{P}) = \frac{c_1(\mathcal{L})^2 - (2\chi(X) + 3\sigma(X))}{4}$$

from the Atiyah-Singer index theorem.

For a generic C^∞ self-dual real two-form h on X, we define the perturbed Seiberg-Witten equations $\mathbf{SW_h}$ like the following:

$$F_A^+ = q(\psi) + ih$$
$$D_A^+(\phi) = 0.$$

Let $\mathfrak{m}$ be the set of (A, ψ) satisfying the equation $\mathbf{SW_h}$. The solution set $\mathfrak{m}$ has a $C^\infty(X, S^1)$ action on itself, where $C^\infty(X, S^1)$ is the set of smooth functions from X to S^1. The action is defined as follows: for $g \in C^\infty(X, S^1)$,

$$g(A, \psi) = (A - 2g^{-1}dg, g\psi).$$

This action is called **gauge transformation**. Let the moduli space $\mathcal{M}(\tilde{P}, h)$ be a quotient of the solution space $\mathfrak{m}$ by $C^\infty(X, S^1)$. We call $\mathcal{M}(\tilde{P}, h)$ the moduli space of the $\mathbf{SW_h}$ equation. Moreover, we define $\mathcal{M}^0(\tilde{P}, h)$ to be a quotient of the solution space by $\{\phi \in C^\infty(X, S^1) : \phi(*) = 1\}$, where $* \in X$ is a fixed base-point. When $\mathcal{A}$ is the set of unitary connections of the determinant line bundle of $\tilde{P}$, we define

$$\mathcal{B}(\tilde{P}) := (\mathcal{A} \times C^\infty(S_+(\tilde{P}))/\text{gauge transformation}.$$

We define $b_2^+(X)$ to be the dimension of any maximal subspace of the second cohomology $H^2(X,\mathbb{R})$ on which the intersection form is positive definite. When $b_2^+ > 0$, for generic h, the moduli space $\mathcal{M}(\tilde{P}, h)$ is smooth. For one-parameter family of perturbations h, $\mathcal{M}^0(\tilde{P}, h)$ has a principal circle bundle structure over the moduli space $\mathcal{M}(\tilde{P}, h)$ since we expect no reducibles in the solution space. We remark that we introduce the perturbation term h to make the moduli space smooth.

Now we define the Seiberg-Witten invariant from this moduli space $\mathcal{M}(\tilde{P}, h)$. Orienting the moduli space is equivalent to fixing the orientation of $H^0(X,\mathbb{R}), H^1(X,\mathbb{R})$, $H^2_+(X,\mathbb{R})$. With a properly fixed orientation on the moduli space $\mathcal{M}(\tilde{P}, h)$, we have a principal circle bundle $\mathcal{M}^0(\tilde{P}, h)$ over the moduli space $\mathcal{M}(\tilde{P}, h)$. Let $c \in H^2(\mathcal{M}(\tilde{P}, h))$ be the corresponding Chern class of this circle bundle.

If the dimension of the moduli space is $2l$, which is even, then we define the Seiberg-Witten invariant by $\displaystyle\int_{\mathcal{M}(\tilde{P},h)} c^l$. If the dimension is odd, then we define the invariant to be 0. Moreover, if $b_2^+(X) > 1$, then it is independent from the choice of the metric, i.e., an invariant of X. However, if $b_2^+(X) = 1$, then the invariant depends on both the manifold X and the metric on X.

Therefore, when $b_2^+(X) > 1$, this gives a function

$$SW : \{Spin^c\text{-structures on } X\} \longrightarrow \mathbb{Z}.$$

It is often to use a function on the characteristic classes which amalgamates the information of SW. Let $C(X) \in H^2(X,\mathbb{Z})$ be the subset of characteristic cohomology classes. We recall that characteristic cohomology classes are cohomology classes whose mod two reduction is equal to the second Stiefel-Whitney class. We have $SW_X : C(X) \to \mathbb{Z}$, which is defined by the following:

$$SW_X(k) = \sum_{\substack{Spin^c\text{-structure } s \\ \text{and its determinant line bdundle } L \\ c_1(L)=k}} SW(s).$$

1.3 Seiberg-Witten equation for closed 3-manifolds

We introduce the Seiberg-Witten invariant for 3-manifolds. Let N be a 3-dimensional Riemmanian manifold. Let $\tilde{P_N} \to N$ be a $Spin^c$-structure on N. Since

$$Spin^c(3) \cong SU(2) \times U(1)/\pm 1 \cong U(2),$$

$Spin^c(3)$ has the standard representation on $\mathbb{C}^2$. From this standard representation, there is an associated irreducible complex vector bundle $S(\tilde{P_N})$ unique up to isomorphism. The Clifford action $cl : T^*N \to End(S(\tilde{P}))$ is given by:

$$cl(e_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, cl(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, cl(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where e_1, e_2, e_3 is a standard basis of $\mathbb{R}^3$. With this Clifford action, the Dirac operator D_A for 3-dimensional manifolds is defined in the same way from Definition 1.1.7. The 3-dimensional Seiberg-Witten equations $\mathbf{SW^3}$ for $\tilde{P_N} \to N$ are given by:

$$F_A = q(\psi)$$
$$D_A(\psi) = 0,$$

where A is a unitary connection on the determinant line bundle of $\tilde{P_N}$ and ψ is a section of $S(\tilde{P_N})$.

We have the perturbed Seiberg-Witten equation for 3-manifolds. For any sufficiently small closed real two-form h on N, we define the perturbed Seiberg-Witten equations $\mathbf{SW^3_h}$:

$$F_A = q(\psi) + ih$$
$$D_A(\psi) = 0.$$

$N = S^1 \times C$ **case**

In this subsection, we examine the special case: $N = S^1 \times C$, where C is a oriented surface with genus at least 2. We consider a $Spin^c$-structure $\tilde{P_N}$, whose determinant line bundle $\mathcal{L}$ has a degree $\pm(2 - 2g)$ on every component of C where g is the genus of each component of C.

Proposition 1.3.1. *[MST96, Proposition 5.1., Corollary 5.3.]*

- *If the solution exists on the $Spin^c$-structure $\tilde{P_N}$, then $\tilde{P_N}$ is a pull-back $Spin^c$ structure induced from a $Spin^c$-structure on C.*
- *For a sufficiently small closed real two-form h on N, there is a unique solution to the perturbed Seiberg-Witten equations (SW^3_h) :*

$$F_A = q(\psi) + ih$$
$$D_A(\psi) = 0.$$

This solution represents a smooth point of the moduli space in the sense that its Zariski tangent space is trivial.

We remark that Proposition 1.3.1 is originally proved only for a connected surface C in [MST96]. However, the statement is also true when C is disconnected since the solution restricted to each component satisfies Proposition 1.3.1.

1.4 Seiberg-Witten equation for cylindrical 4-manifolds

In this section, we focus on the case in which a smooth and oriented Riemannian 4-manifold X is orientation-preserving isometric to $I \times N$, where N is a closed oriented three manifold and I is a (possibly infinite) open interval.

We follow [KM07, Chapter II] and [MST96, Section 6]. We fix a $Spin^c$-structure $\tilde{P_N}$ on N. Henceforth, we consider L^2_1-version of the configuration space $C(\tilde{P_N})$. The configuration space $C(\tilde{P_N})$ is the set of pairs (A, ψ) where A is an L^2_1-connection on the determinant line bundle of $\tilde{P_N}$ and ψ is an L^2_1-section of the associated bundle $S(\tilde{P_N})$. $C^*(\tilde{P_N})$ is a subset of $C(\tilde{P_N})$ which consists of only irreducible solutions, i.e., $\psi \neq 0$. The gauge group $\mathcal{G}(\tilde{P_N})$ is the group of L^2_2-maps from N to S^1. Moreover, $\mathcal{B}(\tilde{P_N})$ is the space of $C(\tilde{P_N})$ modulo the gauge group action $\mathcal{G}(\tilde{P_N})$ and $\mathcal{B}^*(\tilde{P_N})$ is the space of $C^*(\tilde{P_N})$ modulo the gauge group.

First, we introduce the Chern-Simons-Dirac functional f on the configuration space of N. We fix a background C^∞-connection A_0 on the complex spin bundle $S(\tilde{P})$. Suppose that $f : C(\tilde{P_N}) \longrightarrow \mathbb{R}$ is defined to be:

$$f(A, \psi) = \int_N F_{A_0} \wedge a + \frac{1}{2} \int_N a \wedge \mathrm{d}a + \int_N \langle \psi, D_A \psi \rangle \mathrm{dvol},$$

where $a = A - A_0$. We remark that the Seiberg-Witten equation of N is equivalent with the equation that the gradient of the Chern-Simons-Dirac functional f vanishes.

Lemma 1.4.1. *[MST96, Lemma 6.4.] We have a natural homomorphism*

$$c : \mathcal{G}(\tilde{P_N}) \longrightarrow H^1(N, \mathbb{Z})$$

defined as follows: for $\sigma \in \mathcal{G}(\tilde{P_N})$, i.e. $\sigma : N \to S^1$, there is an induced map on the first cohomology $\sigma^* : H^1(S^1, \mathbb{Z}) \to H^1(N, \mathbb{Z})$. Let $[g]$ be the fundamental cohomology class of $H^1(S^1, \mathbb{Z})$. Then, $c(\sigma) = \sigma^*([g])$.

Moreover,

$$f(\sigma \cdot (A, \psi)) = f((A, \psi) + 2\pi \langle c(\sigma) \cup c_1(\mathcal{L}), [N] \rangle.$$

Therefore, f descends to

$$f : \mathcal{B}(\tilde{P_N}) \to \mathbb{R}/2\pi\mathbb{Z}.$$

Furthermore, the map c is surjective and its kernel is the component of identity $\mathcal{G}_0(\tilde{P_N})$ inside $\mathcal{G}(\tilde{P_N})$. We define $\tilde{\mathcal{B}}^*(\tilde{P_N})$ by the quotient of $C^*(\tilde{P_N})$ by $\mathcal{G}_0(\tilde{P_N})$. Then, f also descends to a function

$$f : \tilde{\mathcal{B}}^*(\tilde{P_N}) \longrightarrow \mathbb{R}.$$

For the Seiberg-Witten theory on 4-manifolds with boundary, we have a notion of energy, which has a necessary role on the compactness argument of the moduli space. We note that the Chern-Simons-Dirac functional f is related to the topological energy $\mathcal{E}^{top}$ defined in [KM07] in the way that the difference of the functional f on both ends is equal to the topological energy of the solution on the cylinder.

We formulate the Seiberg-Witten equation of $I \times N$. Let t denote I-direction coordinate. The $Spin^c$- structure on $X = I \times N$ is naturally given by $\tilde{P} = I \times \tilde{P_N}$ over $I \times N$. The associated complex spin bundle $S(\tilde{P}) = S(\tilde{P_N}) \oplus S(\tilde{P_N})$, which is a 4-dimensional complex vector bundle. The plus spinor bundle $S_+(\tilde{P})$ is the first summand and the minus bundle $S_+(\tilde{P})$ is the second summand. Moreover, the Clifford action $cl : T^*X \to End(S(\tilde{P}))$ is given by:

$$cl_X(\frac{\partial}{\partial t}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, cl_X(v) = \begin{pmatrix} 0 & -cl(v)^* \\ cl(v) & 0 \end{pmatrix},$$

for $v \in T^*N$.

Definition 1.4.2. *A 4-dimensional $Spin^c$ connection $\mathbf{A}$ on $X = I \times N$ is in* ***temporal gauge*** *if its covariant derivative*

$$\nabla_{\mathbf{A}} = \frac{d}{dt} + \nabla_B$$

for a I-direction-dependent $Spin^c$ connection B on $S(\tilde{P_N})$.

With a covariant derivative ∇_A in temporal gauge and the Clifford action, the Dirac operator $D_{\mathbf{A}}$ for a connection $\mathbf{A}$ is defined from Definition 1.1.7. If we restrict the Dirac operator on the set of sections over the plus spinor bundle, then

$$D^+_{\mathbf{A}} = \frac{\mathrm{d}}{\mathrm{d}t} + D_N.$$

Now, we are ready to define the Seiberg-Witten equation for X. With the $Spin^c$-structure $\tilde{P} = I \times \tilde{P_N} \to I \times N$, we introduce the Seiberg-Witten equations $\mathbf{SW_{cyl}}$ on $I \times N$:

$$F_{\mathbf{A}} = q(\Psi)$$
$$D^+_{\mathbf{A}}(\Psi) = 0,$$

where $\mathbf{A}$ is a unitary connection on the determinant line bundle of $\tilde{P}$ and Ψ is a section of $S_+(\tilde{P})$. The Seiberg-Witten equation is equivalent with the gradient flow equation of the Chern-Simons-Dirac functional in the following way.

Proposition 1.4.3. *[MST96, Proposition 6.6.] We fix an open interval I. If a configuration $(A(t), \psi(t))$ in a temporal gauge for the Spin^c-structure $I \times \tilde{P_N} \to I \times N$ satisfies the Seiberg-Witten equations, then it gives a C^∞-path in $C(\tilde{P_N})$ satisfying the gradient flow equation*

$$\frac{\partial(A, \psi)}{\partial t} = \nabla f(A, \psi).$$

Two solutions to the Seiberg-Witten equations are gauge-equivalent if and only if the paths in $C(\tilde{P_N})$ that they determine in temporal gauges are gauge-equivalent under the action of $\mathcal{G}(\tilde{P_N})$.

We define a finite energy solution on the cylinder.

Definition 1.4.4. *For each solution of the Seiberg-Witten equation over $I \times N$, it has the associated flow line $\gamma : I \to C(\tilde{P_N})$. We call a C^∞-solution of the Seiberg-Witten equation on $I = (a, b) \times N$, where I is possibly infinite whose associated flow line $\gamma : I \longrightarrow C(\tilde{P_N})$ satisfies*

$$\lim_{t_1 \to b, t_2 \to a} f(\gamma(t_1)) - f(\gamma(t_2)) < \infty$$

a finite enery solution on the cylinder.

$N = S^1 \times C$ **case**

In this subsection, we assume that $N = S^1 \times C$ where $g(C) > 1$ and the determinant line bundle of $\tilde{P_N}$ is induced from a line bundle on C whose degree is equal to $\pm(2 - 2g)$. Under this assumption, the moduli space of $I \times N$ has an exponentially decaying property from [MST96, Section 6] because the solution moduli space of the Seiberg-Witten equation for N consists of a single non-degenerate point.

Proposition 1.4.5. *[MST96, Corollary 6.17.] There are positive constants $\epsilon, \delta > 0$ such that for any $T \geq 1$ if $(A(t), \psi(t))$ is a solution to the Seiberg-Witten equations on $[0, T] \times N$ in a temporal gauge and if for each t, $0 \leq t \leq T$, the equivalence class* *of* $(A(t), \psi(t))$ *is within* ϵ *in the* L_1^2*-topology on* $\mathcal{B}^*(\tilde{P_N})$ *of the solution* $[A_0, \psi_0]$ *of the Seiberg-Witten equations on* N*, then the distance from* $[A(t), \psi(t)]$ *to* $[A_0, \psi_0]$ *in the* L_1^2*-topology is at most*

$$d_0 \exp(-\delta t) + d_T \exp(-\delta(T - t)),$$

where d_x *is the* L_1^2*-distance from* $[A(x), \psi(x)]$ *to* $[A_0, \psi_0]$ *when* $x = 0, T$*.*

Let n be a harmonic one-form on C. We introduce the perturbation term from n on the Seiberg-Witten equation on cylindrical manifolds. We define the equation $\mathbf{SW_h}$ on $\mathbb{R} \times N$:

$$\begin{aligned} F_A^+ &= q(\psi) + i(\star n + \mathrm{d}t \wedge n) \\ D_A^+(\psi) &= 0, \end{aligned}$$

where $\star$ is the complex-liner Hodge-star operator on N, dt is the one-form of $\mathbb{R}$-direction and $h = \star n + \mathrm{d}t \wedge n$. We denote the associated Seiberg-Witten equation on N is $SW^3_{\star n}$:

$$\begin{aligned} F_{A(t)} &= q(\psi(t)) + i(\star n) \\ D_{A(t)}(\psi(t)) &= 0. \end{aligned}$$

Moreover, we introduce the perturbed functional f_n corresponding to SW_h.

$$f_n(A, \psi) = \int_N F_{A_0} \wedge a + \frac{1}{2} \int_N a \wedge \mathrm{d}a + \int_N \langle \psi, D_A \psi \rangle \mathrm{dvol} - \int_N i(\star n) \wedge a,$$

where $a = A - A_0$. Analogously, the solution to SW_h in a temporal gauge is equivalent to the gradient flow equation of f_n for the path in $C(\tilde{P_N})$. In addition, a static solution to SW_h is equivalent to a collection of solutions to $SW^3_{\star n}$. This is from [MST96, Claim 6.24]. Moreover, the advantage of the perturbation term n is that the solution of the perturbed equation is always static if the energy of the cylinder $\mathbb{R} \times N$ is finite.

Proposition 1.4.6. *[MST96, Proposition 6.30.] We fix* $K > 0$*. For all sufficiently small non-zero harmonic one-forms n on C and* $h = \star n + dt \wedge n$*, if a solution* $\gamma(t)$ *for* $-\infty < t < \infty$ *to the equation* SW_h *satisfies that* $f_n(\gamma(t))$ *has a finite limit as* $t \to \pm\infty$ *and the difference of these limits is at most* K*, then the solution is static.*

We summarize the description of the static solution in the configuration space from Proposition 1.4.6 and [MST96, Lemma 6.31.]. The critical points of f_n in $\tilde{\mathcal{B}}^*(\tilde{P_N})$

are nondegenerate and form a discrete subset. These critical points map to the solution of SW_n^3 in $\mathcal{B}^*(\tilde{P_N})$ by the natural quotient map.

There exist constants $K > 0$ depending only on N satisfying the following statement: We choose a sufficiently small contractible open neighborhood ν of the critical point for f_n in $\mathcal{B}(\tilde{P_N})$. Let $\gamma(t)$ be a C^1-path in the configuration space $C(\tilde{P_N})$ which solves the gradient flow equation for f_n on $[0, T] \times N$. If $f(\gamma(T)) - f(\gamma(0)) < K$, then the image of γ in $\tilde{\mathcal{B}}^*(\tilde{P_N})$ is a path with endpoints in the same component of the preimage $\bar{\nu} \in \tilde{\mathcal{B}}^*(\tilde{P_N})$ of ν.

Therefore, if the values $\{f(\gamma(t)\}_{t\in[0,T]}$ are included in a sufficiently small interval, then the corresponding path γ is included in the contractible open neighborhood of a solution of SW_n^3 in $\tilde{\mathcal{B}}^*(\tilde{P_N})$. This idea is necessary to prove Theorem 2.2.1 in Chapter 2.

1.5 The exponential decaying property of the cylinder contained in a closed 4-manifold.

Let M be a smooth Riemmanian 4-manifold and N be a smoothly embedded inside M. N has its product neighborhood $[-1, 1] \times N$ inside M. We define the family of Riemannian manifolds $M_s = (M, g_s)$ by varying the metric g_s on $[-1, 1] \times N$. Let $\mathrm{d}t^2, \mathrm{d}\theta^2$ and $\mathrm{d}\sigma^2$ be the usual metric on $[-1, 1], S^1$ and C. We assume that $g|_{[-1,1]\times N} = \mathrm{d}t^2 + \mathrm{d}\theta^2 + \mathrm{d}\sigma^2$ locally. Then, we define the family of functions $\{\lambda_s\}_{s\geq 1}$ from $[-1, 1]$ to $\mathbb{R}$ such that

- λ_s are smooth and identically one on $[-1, -\frac{1}{2}]$ and $[\frac{1}{2}, 1]$,
- $\lambda_s(t) > 0$ on $[-\frac{1}{2}, \frac{1}{2}]$,
- $\int_{-\frac{1}{2}}^{\frac{1}{2}} \lambda_s(t) = s$.

We define $g_s = g$ out of the cylinder $[-1, 1] \times N$ and $g_s = \lambda_s^2(t)\mathrm{d}t^2 + \mathrm{d}\theta^2 + \mathrm{d}\sigma^2$ on the cylinder $[-1, 1] \times N$. This is a way of stretching the cylinder inside M by changing the metric on the cylinder. Let $[-\frac{1}{2}, \frac{1}{2}] \times N \subset M_s$ be T_s. We have a natural isometry from $[0, s] \times N$ to T_s. We remark that $M_s - T_s$ are all diffeomorphic for all $s \geq 1$. Henceforth, $T_s = [0, s] \times N$ denotes the cylinder inside M_s.

We fix a $Spin^c$-structure $\tilde{P}$ on M and let $\tilde{P_N}$ be the restriction of $\tilde{P}$ to N. Now we introduce a family of perturbations h_s of the Seiberg-Witten equation to simplify the solution on the cylinder T_s. Let $\phi_s : M_s \to [0, 1]$ be a smooth function such that:

- ϕ_s is identically 1 on T_s,
- ϕ_s is identically 0 on $M - [-1, 1] \times N$,
- For all s, ϕ_s has the same value on $(M_s - T_s)$.

Let n be a real harmonic one-form on C. We abuse the notation n in a way that also denotes the pull-back 1-form from n on $N = S^1 \times C$. After that, $dt \wedge n$ is a two-form on $[-1, 1] \times N$. Let $\star$ be the Hodge star operator for N from the set of one-forms to two-forms. $\star n$ is a two-form on $N = S^1 \times C$. In addition, we abuse a notation that $\star n$ also denotes the pull-back two-form of $\star n$ on $[-1, 1] \times N$. We define $h_s := \phi_s(\star n + dt \wedge n)$, hence h_s is a self-dual and smooth two-form on $[-1, 1] \times N$. The perturbed Seiberg-Witten equations $\mathbf{SW_{h_s}}$ is given by:

$$F_A^+ = q(\psi) + ih_s$$
$$D_A^+(\psi) = 0.$$

Proposition 1.5.1. *[MST96, Corollary 7.5.] There exists a constant $K > 0$ depending only on M and $\tilde{P}$ such that: For a harmonic one-form $n \neq 0$ on C which is sufficiently small, for $s \geq 1$, and for a solution (A, ψ) to SW_{h_s} on M_s, the restriction of (A, ψ) on the cylinder $T_s = [0, s] \times N$ satisfies that for $t \in [0, s]$, L^2_1-distance from $A(t), \psi(t)$ to a static solution is at most $K \exp(-\delta d(t))$ for $d(t) = \min(t, s - t)$ and δ from Proposition 1.4.5*

1.6 The moduli space of 4-manifolds with cylindrical ends

In this section, we study the perturbed Seiberg-Witten equation on an oriented and smooth 4-manifold X with cylindrical ends $[-1, \infty) \times N$ for a closed 3-manifold N, which is not necessarily connected. Henceforth, the submanifold N inside X denotes $\{0\} \times N \subset X$. Under this assumption, we define a compact and smooth moduli space which consists of solutions to the perturbed Seiberg-Witten equation on X. We mainly follow [MST96, Section 8] and [KM07, Section 24].

We fix a $Spin^c$-structure $\tilde{P}$ on X. The restriction of $\tilde{P}$ to N is denoted by $\tilde{P_N}$. The Seiberg-Witten equation for cylindrical 4-manifolds also works for X. For each C^∞-solution (A, ψ) to the Seiberg-Witten equation on X with respect to $\tilde{P}$, there exists a temporal gauge for $\tilde{P}$ restricted to the cylindrical end satisfying that $[A, \psi]|_{[0,\infty)\times N}$ corresponds to a flow-line $\gamma : [0, \infty) \to C(\tilde{P_N})$, that is a solution of gradient flow equation for f. For the associated flow line γ, if

$$\lim_{t\to\infty} f(\gamma(t)) - f(\gamma(0)) < \infty,$$

then we call the solution (A, ψ) **finite energy solution.**

Suppose that C is an oriented surface which is not necessarily connected. Let $C = \bigsqcup_{\alpha} C_\alpha$, where C_α is a component of C. We assume that each C_α has a genus $g \geq 2$ for all α. We are mainly interested in the case $N = S^1 \times C$. We fix a $Spin^c$-structure $\tilde{P}$ on X. We assume that the determinant line bundle of $\tilde{P}$ restricted to $S^1 \times C_\alpha$ is isomorphic to the pull-back of the line bundle on C_α whose degree is equal to $\pm(2g-2)$ on C_α.

We need two perturbation terms to the equation. The first perturbation term has a form $\star n + dt \wedge n$ on the cylindrical end $[0, \infty) \times N$ to use the exponential decaying property on cylindrical ends Moreover, there exists a set of perturbation terms, which makes the moduli space regular from [KM07, Proposition 24.4.7.]. By adding another perturbation by purely-imaginary two form μ_X^+, we achieve the regularity of the moduli space.

To summarize, for a unitary connection A on the determinant line bundle of $\tilde{P}$ and a section ψ of the plus complex spin bundle $S_+(\tilde{P})$, the perturbed Seiberg-Witten equation $\mathbf{SW}_{\mathbf{h_X}+\mu_\mathbf{X}^+}$ on X is as follows:

$$F_A^+ = q(\psi) + i\phi_X(h) + i\mu_X^+$$

$$D_A^+\psi = 0.$$

- μ_X^+ is a generic and compactly supported self-dual two form.
- n is a harmonic one-form on C and $h = \star n + dt \wedge n$, that is a two-form on the cylindrical end as defined in SW_{h_s}.
- ϕ_X is a C^∞ function which is identically 1 on $[0, \infty) \times S^1 \times C$ and vanishes off of $[-1, \infty) \times S^1 \times C$.

Let $\tilde{\mathcal{M}}(\tilde{P}, n, \mu_X^+)$ be the set of all finite energy solutions to the $SW_{h_X+\mu_X^+}$. The space obtained from $\tilde{\mathcal{M}}(\tilde{P}, n, \mu_X^+)$ quotient by the gauge transformation group which consists of C^∞-change of gauge is denoted by the moduli space $\mathcal{M}(\tilde{P}, n, \mu_X^+)$.

Definition 1.6.1. *For each solution (A, ψ) of the Seiberg-Witten equation on X, we define*

$$c(A) = -\frac{1}{4\pi^2}\int_X F_A \wedge F_A.$$

We call $c(A)$ the Chern integral of the solution.

The Chern integral defines a continuous function from $\mathcal{M}(\tilde{P}, n, \mu_X^+)$ to $\mathbb{R}$. Let $\mathcal{M}_c(\tilde{P}, n, \mu_X^+)$ be the inverse image of one-point set $\{c\}$ for $c \in \mathbb{R}$. We show that $\mathcal{M}_c(\tilde{P}, n, \mu_X^+)$ is compact. The main ingredient of the proof is that the solution on the cylindrical ends with finite energy is always static from Proposition 1.4.6. We remark that without the cylindrical perturbation term n the compactness property may not be true.

Proposition 1.6.2. *We fix c_0. For sufficiently small nonzero harmonic one-form n on C and $c \leq c_0$, $\mathcal{M}_c(\tilde{P}, n, \mu_X^+)$ is compact.*

Proof. The proof is same as the proof of [MST96, Proposition 8.5.] which proves the case that the cylindrical end is connected. □

If $\mathcal{L}$ is the determinant line bundle of $\tilde{P}$, then $c_1(\mathcal{L}) = \frac{1}{2\pi i}[F_A]$. The index of the Fredholm complex is equal to

$$\frac{1}{4}[c(A) - 2\chi(X) - 3\sigma(X)]$$

and the moduli space is the set of zeros of the Fredholm complex. Therefore, the moduli space $\mathcal{M}_c(\tilde{P}, n, \mu_X^+)$ becomes a smooth and compact manifold with the expected dimension $\frac{1}{4}[c(A) - 2\chi(X) - 3\sigma(X)]$.

Exponential decaying property of the moduli spaces.

The other way to investigate the moduli spaces for cylindrical 4-manifolds is to show the analogous results in [MST96, Section 8]. The statement is exactly the same as [MST96, Corollary 8.6], except that we allow that the cylindrical end may be disconnected.

Proposition 1.6.3. *With the notations above, let $N = S^1 \times C$ and X be a complete Riemannian 4-manifold with cylindrical-end isomorphic to $[-1, \infty) \times N$. Let $\tilde{P}$ be a Spin^c-structure whose restriction to N is isomorphic to the pull back from C of a Spin^c structure whose determinant line bundle has degree $\pm(2-2g)$. Then for any c_0 the following holds for any sufficiently small harmonic one form $n \neq 0 \in \Omega^1(C; \mathbb{R})$ and every $c \geq c_0$: There is a constant $T \geq 1$ such that if (A, ψ) is a finite energy solution to the equations SW_h with Chern integral c, then for every $t \geq T$ the restriction $(A(t), \psi(t))$ is within $\exp(-z(t-T))$ in the L_1^2-topology of a solution to the equations $SW_{\star n}$ on N, where the constant z depends only on N. The same result holds when the curvature equation is replaced by*

$$F_A^+ = q(\psi) + ih + i\mu^+$$

for any sufficiently small, compactly supported, self-dual two-form μ^+ on X.

Orientations of the moduli space.

We introduce a way to orient the moduli space $\tilde{\mathcal{M}}(\tilde{P}, n, \mu_X^+)$ for 4-manifolds X with cylindrical end. Let X be a 4-manifold with cylindrical end and let T be a cylindrical neighborhood of infinity in X. We define $H^2(X, T; \mathbb{R})$ to be the maximal subspace of the second cohomology group $H^2_{\geq}(X, T; \mathbb{R})$ whose intersection pairing is positive semi-definite. With notations above, we have the following proposition.

Proposition 1.6.4. *[MST96, Corollary 9.2.] To orient the moduli space of finite energy solutions to the Seiberg-Witten equations on a cylindrical-end manifold X, it suffices to orient $H^1(X, T; \mathbb{R}) \oplus H^2_{\geq}(X, T; \mathbb{R})$.*

Remark 1.6.5. *Let M be a closed manifold containing the cylinder $T = N \times \mathbb{R}$ as a submanifold, where N is a compact 3-manifold inside M. We can orient the moduli space for M by orienting $H^1(M, T; \mathbb{R}) \oplus H^2_{\geq}(M, T; \mathbb{R})$. This is from the same logic in [MST96, p.772]. We use this remark to trace orientations when gluing moduli spaces.*

1.7 Relative Seiberg-Witten invariant for manifolds with two cylindrical ends

Originally, the Seiberg-Witten invariant is defined for closed manifolds; however, it can be extended to 4-manifolds with cylindrical ends, which is called the relative Seiberg-Witten invariant. On the assumption that the moduli space for such a manifold is smooth and compact, the relative invariant is analogously defined as the original Seiberg-Witten invariant. The authors defined the relative invariant for 4-manifolds with a connected cylindrical end $[0, \infty) \times S^1 \times \Sigma$ with a oriented and at least genus 2 surface C in Section 9.2 of [MST96]. We extend the definition to the case where C is disconnected and has two components.

Definition 1.7.1. *Let M be an oriented, complete, Riemannian four manifold with cylindrical ends T isometric to $[0, \infty) \times S^1 \times C$, where $C = C_1 \bigsqcup C_2$ and C_1, C_2 is an oriented, connected surface with $g(C_1) = g(C_2) > 1$. Let $\tilde{P_M}$ be a Spin^c-structures whose restriction on $S^1 \times C_i$ is isomorphic to the pullback of a Spin^c structure on C_i whose determinant line bundle is degree $(2g - 2)$ for $i = 1, 2$. We defined the smooth and compact moduli space $\mathcal{M}_c(\tilde{P_M}, n, \mu^+)$. This is the space of solutions of the Seiberg-Witten equation with a small perturbation n, μ^+ whose Chern integral is equal to c.*

Let c_1 be the first Chern class of the universal circle bundle over this moduli space. When the dimension of the moduli space is $2d$, we define the relative Seiberg-Witten invariant $SW_c(\tilde{P_X})$ by the pairing of ${c_1}^d$ and $\mathcal{M}_c(\tilde{P_M}, n, \mu^+)$. If the dimension is odd, then we define $SW_c(\tilde{P_X}) = 0$. This relative Seiberg-Witten invariant is the same for all n and μ^+.

1.8 Seiberg-Witten Invariant for the product of a circle and closed 3-manifolds

In this section, we discuss the Seiberg-Witten invariant of 4-manifolds $S^1 \times N$, for a compact, oriented and connected 3-manifold N with $b_1(N) > 0$, is related to the Alexander polynomial of the 3-manifold N [MT96]. Before the statement, we need two notations. First, let

$$H(X) = H^2(X, \mathbb{Z})/\text{Tors}$$

be a non-torsion part of the second cohomology $H^2(X, \mathbb{Z})$ for a closed manifold X. Second, we have the Seiberg-Witten invariants $SW : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ for 4-manifolds X. We have the natural quotient map $q : H^2(X, \mathbb{Z}) \to H(X)$. Then, we define

$$\underline{SW}_X = \sum_{z \in H^2(X,\mathbb{Z})} SW_X(z) q(z)$$

in the group ring $\mathbb{Z}[H(X)]$.

Theorem 1.8.1. *[MT96] Let N be a compact, oriented and connected 3-manifold with $b_1(N) > 0$ such that the boundary of N is empty or a disjoint union of tori. Let $p : S^1 \times N \to N$ be a natural projection and $p_* : H(N) \to H(S^1 \times N)$ be the induced homomorphism by p. Obviously, there exists a natural homomorphism $\Phi_2 : \mathbb{Z}[H(N)] \to \mathbb{Z}[H(S^1 \times N)]$ induced by $2p_*$. When $b_1(N) = 1$, $H(N) \cong H_1(N, \mathbb{Z})/Tors \cong \mathbb{Z}$. Let t be a generator of $H(N)$. Then, there exists an element $\xi \in \pm p_*(H(N))$ such that*

$$\underline{SW}_{S^1 \times N} = \begin{cases} \xi \Phi_2(\Delta_N) & \text{if } b_1(N) > 1 \\ \xi \Phi_2((1-t)^{|\partial N| - 2} \Delta_N) & \text{if } b_1(N) = 1. \end{cases}$$

1.9 Seiberg-Witten Invariants for symplectic manifolds (X, ω)

Suppose that X is a closed and smooth 4-manifold with $b_2^+(X) > 1$. In addition, let ω be a symplectic 2-form on X and $\omega \wedge \omega$ gives the orientation on X. In this section, we discuss the Seiberg-Witten invariant of X for the simple $Spin^c$-structures following [Tau94] and [Tau95].

Let $\mathcal{S}_X$ be the set of $Spin^c$-structures on X. From Remark 1.1.8, we know that $H^2(X,\mathbb{Z})$ has an action on $\mathcal{S}_X$ and the action is free and transitive. When X is symplectic, $\mathcal{S}_X$ with naturally fixed base $Spin^c$-structure has one-to-one correspondence with the second cohomology group . More precisely, for an element $e \in H^2(X,\mathbb{Z})$, the corresponding $Spin^c$-structure has the plus spinor bundle $S_+ = E \oplus K^{-1}E$, where the complex line bundle E satisfies that $c_1(E) = e$ and K is the canonical bundle induced from almost complex structure J compatible with ω. Therefore, we regard the Seiberg-Witten invariant as

$$SW_X : H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Theorem 1.9.1. *[Tau95, Theorem 1, 2] Let the first Chern class of the associated almost complex structure on* (X,ω) *be* K_X*. Then,* $SW_X(K_X) = \pm 1$*. Moreover, for* $c \in H^2(X,\mathbb{Z})$ *with* $SW_X(c) \neq 0$,

$$|c \cdot [\omega]| \leq K_X \cdot [\omega].$$

Furthermore, equality holds if and only if $c = \pm K_X$*.*

Chapter 2

SELF-GLUING FORMULA OF THE SEIBERG-WITTEN INVARIANTS.

2.1 Introduction

The Seiberg-Witten monopole invariant for 4-dimensional manifolds was introduced by Witten in 1994 and has been actively studied in various aspects. It has been interesting to show how the invariants of different 4-manifolds are related and how to compute the Seiberg-Witten monopole invariant from such relations.

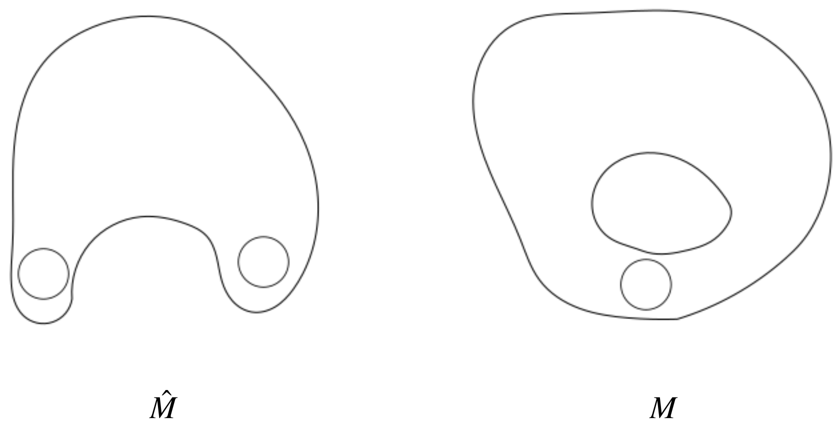

Figure 2.1: Self-Gluing Formula

We have a variety of versions of gluing formulae on 4-dimensional monopole invariants. Suppose that two cylindrical end 4-manifolds X_+, X_- with ends isometric to $[-1, \infty) \times N$ for a 3-manifold Y with $b_1(N) > 0$ are given. Let $X = X_+ \cup_N X_-$ be a closed 4-manifold satisfying $b_2^+(X) \geq 1$. Under this setting, the Seiberg-Witten invariant of X is given by the product of the projection of the Seiberg-Witten invariants of X_+, X_- in [CW03]. We call this formula **Product Formula**. In particular, we consider the case that N is the product of the circle and an oriented 2-surface. When the genus of the surface is at least 2, the similar product formula is proved in [MST96]. When $N = T^3$, the product formula is introduced in [Tau01]. Moreover, in [Tau01], the author proved how the Seiberg-Witten invariants are determined when we glue two isomorphic submanifolds embedded in one connected 4-manifold. We call this formula **Self-Gluing formula**. In Figure 2.1, the black circles denote isomorphic 3-dimensional submanifolds N. We remove an open neighborhood of N from the left manifold and obtain the right manifold after filling the boundaries at infinity. The Self-Gluing formula explains the relation of the Seiberg-Witten invariants of two manifolds in Figure 2.1.

The combination of the product formula and the self-gluing formula enables us to compute the Seiberg-Witten invariants of the manifold obtained from gluing two 4-manifolds along multiple 3-manifolds embedded inside, which is called as **Multiple Gluing Formula**. In this thesis, we generalize the self-gluing formula for the product of the circle and the surface with genus at least 2 in Theorem 2.2.1 and prove the multiple gluing formula in Theorem 2.5.1.

2.2 Setting and Outline

Let M be a closed oriented four-manifold and let $N = S^1 \times C \subset M$, where C is an oriented surface with genus $g > 1$. When N is a separating submanifold inside M, let X and Y be the two components of $M \setminus N$. We fill X, Y by gluing $D^2 \times C$ naturally along the boundaries $S^1 \times C$. $\hat{X}, \hat{Y}$ denote the resulting manifolds. Morgan, Szabo and Taubes showed the product formula on the Seiberg-Witten invariants in [MST96]. In other words, the Seiberg-Witten invariants of $\hat{X}$ and $\hat{Y}$ with properly fixed $Spin^c$-structures determine the Seiberg-Witten invariant of M with the induced $Spin^c$-structures.

In this note, we want to investigate the case when N is non-separating inside M. We consider a neighborhood N, $nbd(N) \cong N \times (0,1) = S^1 \times C \times (0,1)$ inside M. Let $\bar{M} = M \setminus nbd(N)$. Hence, $\bar{M}$ is a smooth 4-manifold with two boundary components $N \times \{0\}$ and $N \times \{1\}$. We fill two boundaries of $\bar{M}$ by a natural map

$$D^2 \times C \hookrightarrow \bar{M}.$$

Let the resulting manifold be $\hat{M}$.

Conversely, we can also obtain M from $\hat{M}$ by the following operation. Let C_1, C_2 be homeomorphic to C. Suppose that there are two smooth embeddings $C_1, C_2 \hookrightarrow M$. In addition, the homology classes $[C_1], [C_2] \in H_2(M, \mathbb{Z})$ have infinite order and $[C_1]^2 = [C_2]^2 = 0$. Moreover, $[C_1] \cdot [C_2] = 0$. In other words, the self-intersection number of C_1, C_2 are both zero and the intersection number of C_1 and C_2 is also zero. These additional conditions allow that there are regular neighborhoods of C_1 and C_2, which are diffeomorphic to $D^2 \times C_1$ and $D^2 \times C_2$ respectively. Let

$$\bar{M} = \hat{M} \setminus (D^2 \times C_1 \sqcup D^2 \times C_2).$$

After that, we glue the two boundaries of $\bar{M}$ by a map

$$S^1 \times C_1 \longrightarrow S^1 \times C_2$$
$$(x, y) \longrightarrow (\bar{x}, f(y)).$$

$\bar{x}$ denotes a complex conjugate of x and $f : C_1 \longrightarrow C_2$ is a diffeomorphism. Consequently, the resulting manifold becomes the original manifold M. In this chapter, we verify the relation between the Seiberg-Witten invariants of M and $\hat{M}$. Now, we are ready to introduce the main theorem of this chapter.

Theorem 2.2.1 (Main Theorem)**.** *Let $M, \hat{M}$ be two closed and oriented four manifolds related as described previously. Suppose that $b_2^+(M) > 1$. It follows that $b_2^+(\hat{M}) > 1$. Suppose that $k \in H^2(M, \mathbb{Z})$ is a characteristic cohomology class satisfying $k|_N = \rho^* k_0$ where $k_0 \in H^2(C; \mathbb{Z})$ satisfies*

$$\langle k_0, [C] \rangle = 2g - 2$$

and $\rho : N \longrightarrow C$ is the natural projection. Let $\mathcal{K}(k)$ be the set of all characteristic classes $k' \in H^2(M; \mathbb{Z})$ satisfying that the restrictions of k and k' on $\bar{M}$ are isomorphic and $k^2 = k'^2$. Similarly, let $\hat{\mathcal{K}}(k)$ be the set of all characteristic classes $\hat{k} \in H^2(\hat{M}; \mathbb{Z})$ with the property that the restrictions of $\hat{k}$ and k on $\bar{M}$ are isomorphic. Then we have the following formula:

$$\sum_{k' \in \mathcal{K}(k)} SW_M(k') = \sum_{\substack{\hat{k} \in \hat{\mathcal{K}}(k), \\ \hat{k}^2 = k^2 - (8g-8)}} SW_{\hat{M}}(\hat{k}).$$

We call this formula **self-gluing formula**. Prior to the proof of the self-gluing formula, we briefly review the product formula [MST96, Theorem 3.1] and a sketch of its proof which gives the idea.

Theorem 2.2.2. *[MST96, Theorem 3.1] Supppose that $b_2^+(\hat{X}), b_2^+(\hat{Y}) \geq 1$. It follows that $b_2^+(M) \geq 1$. Suppose that $k \in H^2(M; \mathbb{Z})$ is a characteristic cohomology class satisfying $k|_N = \rho^* k_0$, where $k_0 \in H^2(C; \mathbb{Z})$ satisfies*

$$\langle k_0, [C] \rangle = 2g - 2.$$

Let k_X and k_Y be the restrictions of k to X and Y. Consider the set $\mathcal{K}(k)$ of all characteristic classes $k' \in H^2(M; \mathbb{Z})$ with the property that $k'|_X = k_X, k'|_Y = k_Y$ and $k'^2 = k^2$. We define $\mathcal{K}_X(k)$ to be all $l \in H^2(X; \mathbb{Z})$ which are characteristic and satisfy $l|_X = k_X$. The set $\mathcal{K}_Y(k)$ is defined analogously. Then for appropriate choices *of orientations of* $H^1(M), H^1(X), H^1(Y)$ *and* $H^2_+(M), H^2_+(X), H^2_+(Y)$ *determining the signs of the Seiberg-Witten invariants, we have*

$$\sum_{k' \in \mathcal{K}(k)} SW_M(k') = (-1)^{b(M,N)} \sum SW_{\hat{X}}(l_1) SW_{\hat{Y}}(l_2), \tag{2.2.1}$$

where $b(M, N) = b_1(X, N) b_2^{\geq}(Y, N)$, *and the sum on the right-hand-side extends over all pairs* $(l_1, l_2) \in \mathcal{K}_X(k) \times \mathcal{K}_Y(k)$ *with the property that*

$$l_1^2 + l_2^2 = k^2 - (8g - 8).$$

It is to be understood in Equation 2.2.1 that the Seiberg-Witten invariant of any manifold with $b_2^+ = 1$ *is the* C^**-negative Seiberg-Witten invariant where* C^* *is the cohomology class Poincare dual to* C.

Remark 2.2.3. *Given the second cohomology class* x *of non-negative square,*

$$SW_X^x : C(X) \to \mathbb{Z}$$

is defined to be C^**-negative Seiberg-Witten invariant.*

We summarize a sketch of the proof of Theorem 2.2.2 which gives a motivation of the proof of the main theorem. First, they define the moduli space of the perturbed Seiberg-Witten equations for one cylindrical end 4 manifolds X and Y. They show the moduli spaces are smooth and compact with the correct dimension, which is given by the index theorem. This is reviewed in Section 1.6. Next, by gluing together two configuration spaces of X and Y, they show that the union of the product of the two moduli spaces with fixed Chern integral is diffeomorphic to the moduli space of the original manifold M. Then, they define the relative Seiberg-Witten invariants for one cylindrical end 4-manifolds, that is reviewed in Section 1.7. The above diffeomorphism between moduli spaces implies that the Seiberg-Witten invariant of M can be represented by the product of the relative Seiberg-Witten invariants of X and Y. Last, the relative Seiberg-Witten invariants of X and Y are equal to the Seiberg-Witten invariants of $\hat{X}$ and $\hat{Y}$ respectively, which implies the statement.

We show that the relative Seiberg-Witten invariants of $\bar{M}$ is equal to the Seiberg-Witten invariant of $\hat{M}$ up to orientations in Section 2.4 by generalizing the gluing theorem of the Seiberg-Witten moduli spaces for 4-manifolds with boundary [MST96, Theorem 9.1.]. From Generalized Product formula, the relative Seiberg-Witten invariant of $\bar{M}$ is equal to the usual Seiberg-Witten invariant of M up to appropriate summations and orientation terms.

Moreover, we express the Seiberg-Witten monopole invariant of 4-manifolds obtained from gluing two 4-manifolds along multiple numbers of certain kinds of 3-manifolds in Section 2.5. It is from the direct application of the product formula in [MST96] and Main Theorem 2.2.1.

2.3 Relation between two moduli spaces over $\bar{M}$ and M.

Suppose that $N = S^1 \times C$ is smoothly embedded inside M. Let g be the chosen metric on M so that the metric g is isometric to the product metric on the neighborhood of N inside M. Let $\bar{M}$ be $M \setminus N$, which is an oriented four manifold with two cylindrical ends $[-\frac{1}{2}, \infty) \times N_1, [-\frac{1}{2}, \infty) \times N_2$, where N_1 and N_2 are homeomorphic to N. We give the Riemannian metric on $\bar{M}$ induced from (M, g).

In Section 1.4, we defined a family of manifold $M_s = (M, g_s)$ for $s \geq 1$ by varying metric g_s. This is equivalent to stretching the cylinder $[-1, 1] \times N$. In addition, we have a different view of stretching the cylinder.

For all $s \geq 1$, let $\bar{M}_s$ be the manifold with boundary obtained from truncating $\bar{M}$ at $\{\frac{s}{2}\} \times N_1, \{\frac{s}{2}\} \times N_2$. Let M_s be the closed Riemmanian four manifold obtained by gluing

$$\left\{\frac{s}{2}\right\} \times N_1 \longrightarrow \left\{\frac{s}{2}\right\} \times N_2$$
$$(z, w) \longrightarrow (\bar{z}, w)$$

from $\bar{M}_s$. A family of manifolds M_s parameterized by $s \in [1, \infty)$ is isometric to (M, g_s) given a family of metrics $\{g_s\}$ on M defined in Section 1.4. Let $N_\pm$ be $\{\frac{s}{2}\} \times N_1, \{\frac{s}{2}\} \times N_2$, respectively. Let T_s be the cylinder inside M_s bounded by N_-, N_+.

We add one remark that any two-form ω on M can be extended to the two-form on M_s in a obvious way so that on the restriction of ω to the cylindrical part T_s the two-form is nonzero and constant.

For all positive e, let $\mathcal{M}_e(\tilde{P}_{\bar{M}}, n, \mu^+_{\bar{M}})$ be the moduli space of finite energy solutions to the perturbed equations with Chern interal e. By choosing sufficiently small and generic n and $\mu^+_{\bar{M}}$, we can arrange that $\mathcal{M}_e(\tilde{P}_{\bar{M}}, n, \mu^+_{\bar{M}})$ is a smooth and compact moduli space.

Let S be the set of isomorphism classes of $Spin^c$ structures $\tilde{P}$ on M with the property that $\tilde{P}|_{\bar{M}} \cong \tilde{P}_{\bar{M}}$. S_e denotes the subset of S, which consists of $Spin^c$ structures whose determinant line bundle $\mathcal{L}$ satisfies $c_1(\mathcal{L})^2 = e$. For $\tilde{P} \in S_e$ with $e \geq 0$, there is the corresponding $Spin^c$ structures $\tilde{P}_s$ over M_s. For large s, $supp(\mu^+_{\bar{M}}) \subset M_s$. Then, for any s sufficiently large, μ^+_s denotes a self-dual form on M_s, which is equal to $\mu^+_{\bar{M}}$.

We define $\mathcal{M}(\tilde{P}_s, h_s, \mu^+)$ to be the moduli space of solutions to the perturbed SW equations $SW_{h_s+\mu^+}$:

$$F_A^+ = q(\psi) + i\phi_s(\star n + dt \wedge n) + i\mu^+.$$

$$D_A(\psi) = 0.$$

- $\phi_s : M_s \longrightarrow [0,1]$ is defined similarly as ϕ_X in $SW_{h_X+\mu^+_X}$.
- $h_s = \phi_s(\star n + dt \wedge n)$.

Theorem 2.3.1. *With the notations and assumptions above, suppose that n is sufficiently small and generic and s sufficiently large. There is a diffeomorphism*

$$\mathcal{M}_e(\tilde{P}_{\bar{M}}, n, \mu^+_{\bar{M}}) \xrightarrow{\Phi} \bigsqcup_{\tilde{P}\in S_e} \mathcal{M}(\tilde{P}_s, n, \mu^+), \tag{2.3.1}$$

determined by gluing the two boundary parts of the solution and deforming slightly so as to be in the solution moduli space.

The proof follows the standard gluing arguments and limiting arguments. The original description for the arguments is from [DK90, Chapter 7]. We describe how to construct Φ and why Φ is bijective.

Proof. We define a map Φ. We first fix $r \geq 1$. For an element $[A, \Psi] \in \mathcal{M}_e(\tilde{P}_{\bar{M}}, n, \mu^+_{\bar{M}})$, we consider the restriction $[A, \Psi]|_{\bar{M}-\bar{M}_r}$ in a temporal gauge. The restricted solution is represented by two curves

$$\gamma_1 : [r, \infty) \to C(\tilde{P_N})$$
$$\gamma_2 : [r, \infty) \to C(\tilde{P_N})$$

at two cylindrical ends.

$\lim_{t\to\infty} \gamma_1(t) = [A_1, \psi_1]$ and $\lim_{t\to\infty} \gamma_2(t) = [A_2, \psi_2]$. Then both $[A_1, \psi_1]$ and $[A_2, \psi_2]$ are solutions of $SW^3_{\star n}$ in $C(\tilde{P_N})$. Therefore, there exists a gauge transformation $g \in \mathcal{G}(\tilde{P_N})$ such that $[A_1, \psi_1] = g \cdot [A_2, \psi_2]$. We can extend this gauge transformation into $[r, \infty) \times N_2$ which is equal to the identity near $\{r\} \times N_2$ and equal to g near the infinite end, hence this gauge transformation extends to the gauge transformation of $\bar{M}$ in a way that is in the component of the identity outside $[r, \infty) \times N_2$.

After the gauge transformation, we can assume that $\lim_{t\to\infty} \gamma_1(t)$ and $\lim t \to \infty\gamma_2(t)$ are equal in $\tilde{\mathcal{B}}^*(\tilde{P_N})$.

We can pick large enough $t \geq r$ such that $f_n(\gamma_1(t))$ and $\lim_{t\to\infty} \gamma_1(t)$ are close enough and $f_n(\gamma_2(t))$ and $\lim_{t\to\infty} \gamma_2(t)$ are also close enough. Therefore, the path γ_1, γ_2 are included in the contractible open ball of the critical point of f_n.

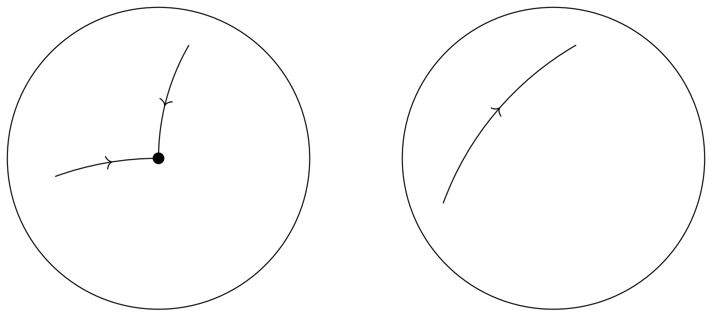

Figure 2.2: Gluing the path in $\tilde{\mathcal{B}}^*(P_N)$.

Schematically, in $\tilde{\mathcal{B}}^*(P_N)$, two curves are approaching as in the left figure of Figure 2.2.

The black dot represents the solution of $SW^3_{\star n}$ in $\tilde{\mathcal{B}}^*(P_N)$ and the circle denotes a contractible open neighborhood. By connecting two curves and perturbing smoothly in the contractible open neighborhood, we can get a solution in $\mathcal{M}(\tilde{P}_t, n, \mu^+)$ for an induced $Spin^c$-structure $\tilde{P}_t$. Since the moduli space $\mathcal{M}_e(\tilde{P}_{\bar{M}}, n, \mu^+_{\bar{M}})$ is compact, for a sufficiently large s, we can define a map Φ for all elements in $\mathcal{M}_e(\tilde{P}_{\bar{M}}, n, \mu^+_{\bar{M}})$. The map is naturally injective from the definition since the value $\Phi([A, \Psi])$ is invariant under perturbations in a contractible open neighborhood.

Furthermore, we show that the map is surjective. For $[A_s, \Phi_s] \in \mathcal{M}(\tilde{P}_s, n, \mu^+)$, the restriction of $[A_s, \Phi_s]$ in the tubular neighborhood of N becomes the curve inside the contractible open neighborhood of the critical point of f_n as in the right figure of Figure 2.2. Therefore, we perturb the path and divide it into two paths approaching to the critical point as the left figure of Figure 2.2. Then, we construct the $[A, \Phi] \in \mathcal{M}_e(\tilde{P}_{\bar{M}}, n, \mu^+_{\bar{M}})$ which maps to $[A_s, \Phi_s]$. □

There is a formula between the relative invariant of $\bar{M}$ and the original invariant of M, followed by Proposition 2.3.1.

Theorem 2.3.2. *With the notations defined in the above section, let S_e be the set of $Spin^c$-structures P on M such that the restriction on $\bar{M}$ is isomorphic to $\tilde{P}_{\bar{M}}$ and its determinant line bundle $\mathcal{L}$ satisfies that $c_1(\mathcal{L})^2 = e$. By orienting $H^1(\bar{M}, T) \oplus H^2_{\geq}(\bar{M}, T)$, we fix the sign for the relative Seiberg-Witten invariant of $\bar{M}$. Moreover, this determines the orientation $H^1(M) \oplus H^2_{\geq}(M)$ which fix a sign of Seiberg-Witten invariant of M. With these fixed orientations, we have the following formula:*

$$\sum_{\tilde{P} \in S_e} SW(\tilde{P}) = SW_e(\tilde{P}_{\bar{M}}).$$

2.4 The Generalized Gluing Theorem for moduli spaces

In [MST96, Theorem 9.1.], the authors explained how to glue two configuration spaces for 4-manifolds with connected cylindrical ends. If we glue two one-cylindrical-end 4-manifolds, then the moduli space of the resulting manifold is represented by the product of the moduli spaces of the two original manifolds. We naturally generalize the gluing theorem to the case that one of the two original manifolds have two cylindrical ends.

Let X denote a 4-manifold with two cylindrical ends and let Y denote a 4-manifold with a cylindrical end. Y has a connected cylindrical end $T = [0, \infty) \times N$, where $N = S^1 \times C$ and C is an oriented, connected surface with genus $g > 1$. Let C_1 and C_2 be homeomorphic to C. X has two cylindrical ends $T_i = [0, \infty) \times N_i$, where $N_i = S^1 \times C_i (i = 1, 2)$. In addition, let $T^s := [0, s] \times N$ and $T_i^s := [0, s] \times N_i$.

We fix $Spin^c$-structures $\tilde{P_X}$ and $\tilde{P_Y}$ whose determinant line bundles restricted to N, N_1, N_2 are all isomorphic to the bundles pulled back from C, C_1, C_2 of a line bundle of degree $(2g - 2)$ on C, C_1, C_2 respectively. As we discussed in Section 2.3, we truncate X, Y at $N_2 \times \{s\}$ and $N \times \{s\}$. By gluing along $N_2 \times \{s\} \subset X$ and $N \times \{s\} \subset Y$, we obtain a new cylindrical-end 4-manifold M_s. Let T'^s denote the cylinder $T_2^s \cup T^s \subset M$. Let S be a set of $Spin^c$-structures $\tilde{P}$ on M_s such that $\tilde{P}|_X = \tilde{P_X}|_{X_s}$ and $\tilde{P}|_Y = \tilde{P_Y}|_{Y_s}$. We have the following diffeomorphism between moduli spaces:

$$\bigsqcup_{c_1 + c_2 = e} \mathcal{M}_{c_1}(\tilde{P_X}, n, \mu_X^+) \times \mathcal{M}_{c_2}(\tilde{P_Y}, n, \mu_Y^+) \xrightarrow{\cong} \bigsqcup_{\tilde{P} \in S} \mathcal{M}_c(\tilde{P}, n, \mu^+). \tag{2.4.1}$$

The same argument in [MST96] can be applied to show that the gluing map induces diffeomorphism.

Theorem 2.4.1 (Generalized Gluing Theorem)**.** *We follow the notations $S, M_s, X, Y, N_1, N_2, N$ defined above. By orienting $H^1(X, T_1 \cup T_2; \mathbb{R}) \oplus H^2_{\geq}(X, T_1 \cup T_2; \mathbb{R})$ and $H^1(Y, T; \mathbb{R}) \oplus$*

$H^2_{\geq}(Y,T;\mathbb{R})$*, we can orient the moduli spaces that appeared on the left hand side. With these choices of orientation, we can determine the orientation of the moduli spaces on the right hand side by orienting* $H^1(M_s,T'^s;\mathbb{R})\oplus H^2_{\geq}(M_s,T'^s;\mathbb{R})$*. With the choices of orientations, we have the following product formula:*

$$\sum_{\tilde{P}\in S} SW_c(\tilde{P}) = (-1)^{b^1(X,T_1\cup T_2)b^2_{\geq}(Y,T)} \sum_{c_1+c_2=c} SW_{c_1}(\tilde{P_X})SW_{c_2}(\tilde{P_Y}).$$

The orientation term $(-1)^{b^1(X,T_1\cup T_2)b^2_{\geq}(Y,T)}$ in Theorem 2.4.1 comes from Remark 1.6.5 and [MST96, Section 9.1.].

In [MST96, Section 9.4], the relative invariant of $D^2\times C$ is computed, where C is a connected and oriented genus $g>1$ surface. The invariant $SW_c(\tilde{P})$ is zero, unless $c=4g-4$. In the case of $c=4g-4$, the relative invariant is equal to 1 with the proper orientation which orients the moduli space positively. Thus, the following corollary comes from Product formula when we glue $D^2\times C$ and X, which is a 4-manifold with cylindrical end $[0,\infty)\times S^1\times C$.

Proposition 2.4.2. *[MST96, Corollary 9.9] Let X be an oriented Riemannian four-manifold with a cylindrical end isometric* $[0,\infty)\times S^1\times C$*, where C is a connected and oriented surface with genus* $g>1$*. Let* $\hat{X}$ *be the closed four manifold obtained by filling in X with* $D^2\times C$*. Then for Spin*^{c}*-structure* $\tilde{P}\longrightarrow\hat{X}$*, satisfying that the determinant line bundle* $\mathcal{L}$ *of* $\tilde{P}$ *has degree* $(2g-2)$ *on* $\{0\}\times C$*, we have*

$$SW(\tilde{P}) = SW_c(\tilde{P}|_X),$$

where

$$c+(4-4g) = \langle c_1(\mathcal{L})^2, [\hat{X}]\rangle.$$

Schematically speaking, if there is an $D^2\times C$ embedded inside the four-manifold, then the relative invariant of the manifold obtained by removing $D^2\times C$ is equivalent to the invariant of the original manifold. Thereafter, we want to prove that even if we remove another $D^2\times C$ inside the resulting manifold, the relative invariant still remains unchanged.

Proposition 2.4.3. *Let X be a compact and oriented* 4 *manifold with two cylindrical ends* $T_i=[0,\infty)\times N_i$*, where* $N_i=S^1\times C_i$ *and* $i=1,2$*. Let* $g(C_1)=g(C_2)=g>1$*. We fill* $\{\infty\}\times N_i$ *by gluing* $D^2\times C_i$*. Let* $\hat{X}$ *be the manifold obtained X by filling two cylindrical ends with* $D^2\times C_i$ *for* $i=1,2$*. Then, for Spin*^{c}*-structures* $\tilde{P}\longrightarrow\hat{X}$*,* *whose determinant line bundle $\mathcal{L}$ restricted to $\{0\}\times C_i$ is a pull-back from the degree $(2g-2)$ line bundle on C, we have the following formula:*

$$SW(\tilde{P}) = SW_c(\tilde{P}|_X),$$

where

$$c + 8 - 8g = \langle c_1(\mathcal{L})^2, [\hat{X}]\rangle.$$

Proof. Let X_i be the manifold obtained from X by filling $\{\infty\}\times N_i$ part. Let P be a $Spin^c$-structure on X whose determinant line bundle on $\{0\}\times C_i$ is the pull-back of degree $(2g-2)$ line bundle on C_i. First, we remark that there is a naturally extended $Spin^c$-structure P' on X_i. We use Theorem 2.4.1 for X and $D^2\times C_1$. Then

$$SW_c(P') = SW_{c_1}(P)$$

with the property that $c_1 + 4 - 4g = c$. Then the statement is followed by Corollary 9.9 of [MST96]. □

With Proposition 2.4.3, we verify the relationship between $\bar{M}$ and $\hat{M}$. We are ready to show the Main Theorem.

Proof of Main Theorem. From Proposition 2.4.3 and Theorem 2.3.2, the main theorem follows naturally. □

2.5 The gluing formula along multiple boundaries whose type is $S^1\times C$.

In this subsection, we assume that X_1, X_2 are compact, oriented, smooth 4-manifolds. Suppose that there are connected, oriented disjoint surfaces $\Sigma_1,\cdots,\Sigma_l \hookrightarrow X_1, X_2$ whose genus are at least 2. Suppose that the intersection number of Σ_i and Σ_j are all zero for all $i, j = 1, 2, \cdots, l$. Suppose that the manifolds X_1', X_2' are obtained from X_1, X_2 by removing the neighborhoods of surfaces $D^2\times\Sigma_i$ for $i = 1, 2, \cdots l$. Then, $\partial X_1' = \partial X_2' = (S^1\times\Sigma_1)\sqcup\cdots\sqcup(S^1\times\Sigma_l)$. We glue the boundaries of X_1', X_2' along natural diffeomorphisms. Then, we call the resulting manifold X. Moreover, we assume that $b_2^+(X_1), b_2^+(X_2), b_2^+(X) > 1$.

Theorem 2.5.1. *We start with the characteristic cohomology class $k\in H^2(X,\mathbb{Z})$ satisfying that $k|_{S^1\times\Sigma_i} = p^*k^i$ where $k^i\in H^2(\Sigma_i,\mathbb{Z})$ satisfies $\langle k^i,[\Sigma_i]\rangle = 2g(\Sigma_i)-2$ and $p : S^1\times\Sigma_i\to\Sigma_i$ is a natural projection for $i = 1, 2, \cdots, l$. Let $k_{X_i'}\in H^2(X_i',\mathbb{Z})$ be the restriction of k on X_i' for $i = 1, 2$. Let $\mathcal{K}(k)$ be the set of all characteristic classes $s\in H^2(X,\mathbb{Z})$ such that $s|_{X_1'} = k_{X_1'}, s|_{X_2'} = k_{X_2'}$. Moreover, we define $\mathcal{K}_{X_i}(k)$*

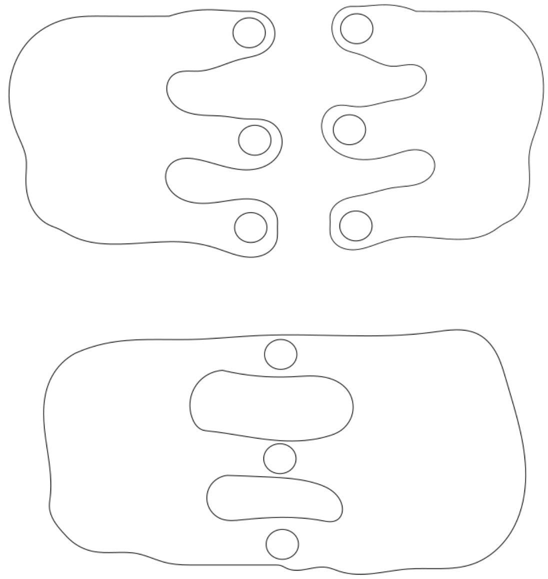

Figure 2.3: Multiple Gluing Formula, $l = 3$.

as the set of all characteristic classes $s \in H^2(X_i, \mathbb{Z})$ *such that* $s|_{X_i'} = k_{X_i'}$ *for* $i = 1, 2$. *With the appropriate choice of orientation,*

$$(-1)^{\star} \sum_{s \in \mathcal{K}(k)} SW_X(s) = \sum SW_{X_1}(s_1) SW_{X_2}(s_2) \tag{2.5.1}$$

where the right hand side sums over $(s_1, s_2) \in \mathcal{K}_{X_1}(k) \times \mathcal{K}_{X_2}(k)$, *satisfying that*

$$s_1^2 + s_2^2 = s^2 - \sum_{i=1}^{l} (8g(\Sigma_i) - 8).$$

Proof. It is easily proved by applying Theorem 2.2.1 and [MST96, Theorem 3.1.] repeatedly. We put the orientation term on the left hand side, which is different from the convention in the original paper [MST96] for the convenience in Section 3.4. □

Remark 2.5.2. *As an analogue of [MST96, Remark 3.2.], two different elements in* $\mathcal{K}(k), \mathcal{K}_{X_1}(k), \mathcal{K}_{X_2}(k)$ *differ by linear combinations of* $[\Sigma_i]^*$, *which is a cohomology class which is dual to the second homology class* $[\Sigma_i]$, *with integer coefficients.*

Chapter 3

APPLICATION ON THE EXISTENCE OF THE SYMPLECTIC STRUCTURE.

3.1 Introduction

Mccarthy and Wolfson defined an operation between two symplectic 4-manifolds, which is called a *symplectic normal connect sum* [MW94]. The symplectic normal connect sum is a construction of a new symplectic 4-manifold from two symplectic manifolds M_1, M_2. Let Σ_1, Σ_2 be symplectic submanifolds embedded in M_1, M_2 respectively. Suppose that Σ_1 has self-intersection number $n \geq 0$ and Σ_2 has self-intersection number $-n \leq 0$. Let $N_1(\Sigma_i), N_2(\Sigma_i)$ be the tubular neighborhoods of Σ_i inside M_i for $i = 1, 2$. Suppose that $N_1(\Sigma_i)$ is contained in the interior of $N_2(\Sigma_i)$. Let $W_i \subset M_i$ be the complements of the interior of $N_1(\Sigma_i)$ inside $N_2(\Sigma_i)$. Let $f : \Sigma_1 \to \Sigma_2$ be a diffeomorphism. Then there exists an orientation preserving diffeomorphism $\bar{f} : W_1 \to W_2$ induced from f such that $\bar{f}(\partial N_2(\Sigma_1)) = \partial N_1(\Sigma_2)$. We glue M_1, M_2 along $\bar{f}$. The resulting manifold is defined to be the symplectic normal connect sum, denoted by $M_1 \#_{\bar{f}} M_2$. It is shown that $M_1 \#_{\bar{f}} M_2$ is symplectic in [MW94]. We call Σ the surface inside $M_1 \#_{\bar{f}} M_2$ came from Σ_1 and Σ_2. There exists a natural converse of the statement.

Question 3.1.1. *If $(M_1 \#_{\bar{f}} M_2, \Sigma)$ is a symplectic pair, then (M_1, Σ_1) and $(M_2, \Sigma_2$ are also symplectic pairs.*

We examine whether the question is true in restricted cases. We consider two simple types of 4-manifolds. Suppose that $M = S^1 \times Y$ for a compact, oriented and connected 3-manifold and that $\Sigma_1 \subset Y \subset S^1 \times Y$ is an incompressible oriented surface with genus $g \geq 2$. Let Σ_2 be a surface homeomorphic to Σ_1. Suppose that X is a Σ_2-bundle over an oriented surface B with positive genus. The self-intersection number of Σ_1 is zero since $\Sigma_1 \subset Y \subset S^1 \times Y$. The self-intersection number of Σ_2 is also zero since its tubular neighborhood is a product from the definition of the fiber bundle. Therefore, we can construct X^Y, which is a normal connect sum of (M, Σ_1) and (X, Σ_2).

Theorem 3.1.2. *When $b_1(Y) = 1$, X^Y has a symplectic form ω and its canonical structure K_ω satisfying $\langle K_\omega, [\Sigma] \rangle = 2g(\Sigma) - 2$ if and only if Y is a surface bundle* *over the circle.*

We note that when $g(\Sigma_1) = g(\Sigma_2) = 1$, the question is true when $M_1 = S^1 \times Y$ where Y is a manifold obtained from S^3 by 0-surgery along a knot K and $\Sigma_1 = S^1 \times K'$ where K' is a dual knot of K inside Y and M_2 is a torus bundle over a surface and Σ_2 is a fiber torus.

3.2 Outline

In this section, we prove Theorem 3.1.2 by following steps. The following techniques might be applied to prove Question3.1.1 for more general cases. For fibered three manifolds Y, Theorem 3.1.2 can be easily shown. We focus on non-fibered manifolds. In Section 3.3, we construct covering spaces of X^Y. We first construct covering spaces $\tilde{X}$ of X and $\tilde{M}$ of M, respectively, and glue multiple copies of $\tilde{X}, \tilde{M}$ in a specific way to get a covering space $\tilde{X^Y}$ over X^Y. In Chapter 1, we introduced the main ingredients to compute the Seiberg-Witten invariant of $\tilde{X^Y}$. If we assume that X^Y has a symplectic structure, then the constructed covering space $\tilde{X^Y}$ also has a symplectic structure. However, we show that $\tilde{X^Y}$ cannot have a symplectic structure due to the obstruction from the Seiberg-Witten invariants in Section 3.4. This completes the proof of Theorem 3.1.2.

3.3 The construction of the covering space $\tilde{X^Y}$.

We first prove that when the surface bundle over the surface is given, arbitrary covering space over the fiber can be extended to a covering space over the total space.

Lemma 3.3.1. *Let Σ be a connected and orientable surface with genus g which is more than 1 and $\tilde{\Sigma}$ be a connected, orientable surface with genus $ng - n + 1$ for a positive integer n. Suppose that a finite n-sheeted normal covering $\rho : \tilde{\Sigma} \longrightarrow \Sigma$ is given. Let X be a Σ-bundle over B. Then, there exists a $\tilde{\Sigma}$-bundle $\tilde{X}$ over an oriented surface $\tilde{B}$ such that there exists a covering $\tilde{\rho} : \tilde{X} \longrightarrow X$ satisfying that the restriction of $\tilde{\rho}$ on the fiber $\tilde{\Sigma}$ is isomorphic to ρ.*

Proof. We fix a basepoint $x \in B \subset X$. Let Σ be a fiber of $x \in X$. Henceforth, $\Gamma := \pi_1(\Sigma, x)$. The surface bundle X corresponds to a short exact sequence of fundamental groups [Got68, page 51]:

$$1 \longrightarrow \Gamma = \pi_1(\Sigma, x) \longrightarrow \pi = \pi_1(X, x) \xrightarrow{q} F = \pi_1(B, x) \longrightarrow 1.$$

Note that q is a quotient map from $\pi_1(X, x)$ to $\pi_1(B, x)$. Let A denote the normal subgroup $\rho_*(\pi_1(\tilde{\Sigma}, \tilde{x}))$ of Γ for a fixed $\tilde{x} \in \rho^{-1}(x)$. Let N be the normalizer of A in π, i.e., $N = \left\{g \in \pi : gAg^{-1} = A\right\}$. $A \trianglelefteq N$ and $\Gamma \trianglelefteq N$ since $\Gamma \trianglelefteq \pi$.

We first show that N/Γ is a surface group. If we show that N/Γ is a finite index subgroup of π/Γ, then it must be a surface subgroup. Let Λ be the set of subgroups of Γ whose index is equal to $|\Gamma/A|$. Clearly, $A \in \Lambda$ and Λ is a finite set since Γ is finitely generated.

π has an action on Λ: for $g \in \pi$ and $H \in \Lambda$,

$$g \cdot H = gHg^{-1} \in \Lambda.$$

This action naturally defines the map $\Phi : \pi \to Perm(\Lambda)$, where $Perm(\Lambda)$ is a permutation group of Λ which is finite. Obviously, $N = \{g \in \pi | g \cdot A = A\}$. π/N is a finite set since $\ker\Phi \leq N$ and $\pi/\ker\Phi = Perm(\Lambda)$ is finite. Therefore, N/Γ is a surface group since it is a finite index subgroup of the surface group π/Γ.

There is a short exact sequence

$$1 \longrightarrow \Gamma/A \hookrightarrow N/A \longrightarrow N/\Gamma \longrightarrow 1. \tag{3.3.1}$$

Lemma 3.3.2. *For a finite group G and a surface group S, suppose that there is a group extension H satisfying*

$$1 \longrightarrow G \hookrightarrow H \longrightarrow S \longrightarrow 1.$$

Then, H always contains a surface subgroup, which is also isomorphic to a proper subgroup of S.

Proof. Let C be the centralizer of G in H. i.e. $C = \{h \in H : hx = xh \text{ for all } x \in G\}$. We show that C is a finite index subgroup of H.

- H has an action on itself defined by conjugation:

  $$g \cdot x = gxg^{-1} \text{ for all } g, x \in H.$$

- Since $G \trianglelefteq H$, $g \cdot G = G$. Therefore, the action gives the homomorphism ϕ from H to $Aut(G)$, the automorphism group of G. Since $Aut(G)$ is a finite group, $\ker\phi$ becomes a finite-index subgroup of H. Moreover, $\ker\phi = C$.

We have the following short exact sequence:

$$1 \longrightarrow C \cap G \longrightarrow C \longrightarrow S' \longrightarrow 1, \tag{3.3.2}$$

where $S' = C/C \cap G$. If we define a map $\Phi : S' \longrightarrow S$ by the natural inclusion, then this map is well-defined. Moreover, Φ is injective. Therefore, $S' = C/C \cap G$ is isomorphic to a finite-index subgroup of $S = H/G$.

It is easily seen that $C \cap G$ is contained in the center of C. Therefore, a short exact sequence 3.3.2 represents a central extension. This central extension corresponds to an element inside $H^2(S', C \cap G)$. Note that if the corresponding element is zero, then the short exact sequence is splittable. $H^2(S', C \cap G)$ is finite and has only torsion elements since $C \cap G$ is finite.

We examine an arbitrary finite-index subgroup T of S'. Let n be $[T : S']$. Suppose that $a \in H^2(S', C \cap G)$ is the corresponding element to the short exact sequence 3.3.1. Then, $na \in H^2(T, C \cap G)$ is the corresponding element of the short exact sequence

$$1 \longrightarrow C \cap G \longrightarrow q^{-1}(T) \longrightarrow T \longrightarrow 1, \tag{3.3.3}$$

where $q^{-1}(T)$ is a subgroup of C.

Since $C \cap G$ is a finite group, there exists a positive integer m such that $ma = 0$. If we use this m to pick the subgroup T, then the short exact sequence 3.3.3 becomes splittable. Hence, the surface group T becomes a subgroup of $q^{-1}(T) \leq C \leq H$. Moreover, $T \leq S' \leq S$. Therefore, the statement is proved. $\square$

We prove Lemma 3.3.1 based on Lemma 3.3.2. We apply Lemma 3.3.2 to the short exact sequence 3.3.1. From the lemma, there exists a surface subgroup H of N/A, which is also a subgroup of π/Γ. Let H be isomorphic to the fundamental group of Σ_h, which is an oriented surface with genus $h \geq 1$. With H, the following diagram $(\star)$ commutes.

$$(\star) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & q^{-1}(H) & \xrightarrow{q} & H & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \pi & \longrightarrow & \pi/\Gamma & \longrightarrow & 1 \end{array}$$

We construct the covering $\tilde{\rho}$. First, we consider a covering space B' over B corresponding to H, that is a subgroup of $\pi/\Gamma = \pi_1(B, x)$. Then, there is a pull-back Σ-bundle X' over B' and the covering map $X' \to X$.

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

Next, we construct a covering space $\tilde{\rho} : \tilde{X} \to X'$ corresponding to $q^{-1}(H)$. Finally, this $(\tilde{X}, \tilde{\rho})$ is a covering space over X. From the $(\star)$ diagram, the restriction of $\tilde{\rho}$ on the fiber is equal to ρ. Therefore, Lemma 3.3.1 is proved. ◻

If we apply Theorem 1.8.1 to $\tilde{Y}$, then

$$\underline{SW}_{S^1 \times \tilde{Y}} = \begin{cases} \xi \Phi_2(\Delta_{\tilde{Y}}) & \text{if } b_1(\tilde{Y}) > 1 \\ \xi \Phi_2((1-t)^{-2}\Delta_{\tilde{Y}}) & \text{if } b_1(\tilde{Y}) = 1. \end{cases} \tag{3.3.4}$$

To make the Seiberg-Witten invariant of $S^1 \times \tilde{Y}$ easy to compute, we choose the covering $\tilde{Y}$ such that $\Delta_{\tilde{Y}}$ is almost trivial. The following three theorems make it easy to deal with the Alexander polynomials of covering spaces.

Proposition 3.3.3. *[FV11, Proposition 3.6] Let N be a 3-manifold and let $\alpha : \pi_1(N) \to G$ be an epimorphism onto a finite group. Let H_G be $H(N_G)$ and H be $H(N)$. Let $\pi_* : H_G \to H$ be the induced map. Then the twisted Alexander polynomials of N and the ordinary Alexander polynomial of N_G satisfy the following relations:*

If $b_1(N_G) > 1$, then

$$\Delta_N^\alpha = \begin{cases} \pi_*(\Delta_{N_G}) & \text{if } b_1(N) > 1 \\ (a-1)^2 \pi_*(\Delta_{N_G}) & \text{if } b_1(N) = 1, im\pi_* = \langle a \rangle \end{cases}$$

If $b_1(N_G) = 1$, then $b_1(N) = 1$ and

$$\Delta_N^\alpha = \pi_*(\Delta_{N_G}).$$

We call $\phi \in H^1(N)$ *fibered* if ϕ is dual to a fiber of a fibration N over S^1. Friedl and Vidussi proved the vanishing theorem of the twisted Alexander polynomial.

Theorem 3.3.4. *[FV13, Theorem 2.3] Let N be a compact, orientable, connected 3–manifold with (possibly empty) boundary consisting of tori. If $\phi \in H^1(N)$ is not fibered, then there exsits an epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G such that*

$$\Delta_{N,\phi}^\alpha = 0.$$

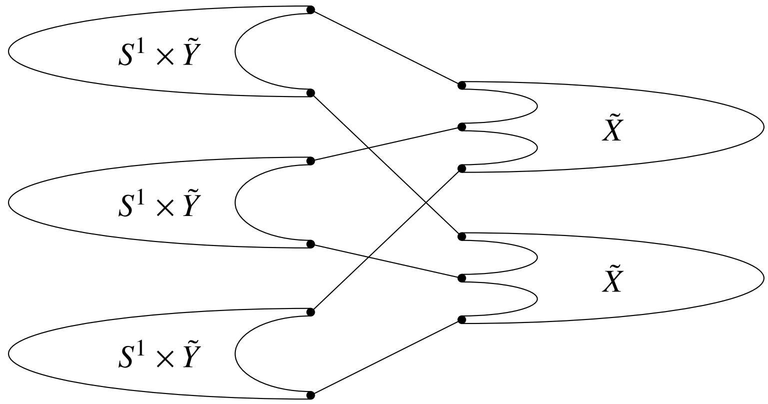

Figure 3.1: Description of $\tilde{X^Y}$ where $l = 2, r = 3$

Proposition 3.3.5. *When $b_1(Y) = 1$ and Y is not fibered, there exists a normal finitely sheeted covering space $\pi : \tilde{Y} \to Y$ such that $\pi_*(\Delta_{\tilde{Y}}) = 0$, where $\pi_* : \mathbb{Z}[H(\tilde{Y})] \to \mathbb{Z}[H(Y)]$ is an induced homomorphism by the covering map π.*

Proof. This statement is straightforward based on Proposition 3.3.3 and Theorem 3.3.4. We can pick an epimorphism $\alpha : \pi_1(Y) \to G$ onto a finite group G such that the covering space $\tilde{Y}$ over Y corresponding to $\ker \alpha$ satisfies that $\pi_*(\Delta_{\tilde{Y}}) = 0 \in \mathbb{Z}[H(Y)]$. □

Remark 3.3.6. *In case of $b_1(Y) > 1$, we cannot have the result in Proposition 3.3.5, however we have a slightly weaker result. From [FV13, Equation (5)],*

$$\Delta^{\alpha}_{N,\phi} = \begin{cases} (t^{div \phi_G} - 1)^2 \phi(\Delta^{\alpha}_N) & \text{if } b_1(N) > 1 \\ \phi(\Delta^{\alpha}_N) & \text{if } b_1(N) = 1. \end{cases}$$

Theorem 3.3.4 asserts that we can pick $\phi \in H^1(N)$ such that $\Delta^{\alpha}_{N,\phi} = 0$. For both cases, $\phi(\Delta^{\alpha}_N) = 0$. Combining with Proposition 3.3.3, $\phi(\pi_*(\Delta_{N_G})) = 0$. We visit this remark later and explain the case $b_1(Y) > 1$.

We have all of the ingredients necessary to construct the covering space of X^Y by gluing several copies of $\tilde{X}$ and $S^1 \times \tilde{Y}$ together.

Lemma 3.3.7. *There exists a finite cover $\tilde{X^Y}$ of X^Y satisfying $b_2^+(\tilde{X^Y}) > 1$.*

Proof. The following construction is an analogue of [Ni17]. We start with a finite normal covering space $\tilde{Y}$ over Y and a covering map $p : \tilde{Y} \to Y$. Then, there is a covering spaces $p : S^1 \times \tilde{Y} \to S^1 \times Y$ defined by

$$(z, y) \to (z^m, p(y))$$

for a fixed positive integer $m > 1$. Henceforth, $\tilde{M}, M$ denote $S^1 \times \tilde{Y}, S^1 \times Y$ respectively. Let l be the number of components of $p^{-1}(\Sigma)$. $l > 1$ since $l \geq m$. Since p is normal, all the components of $p^{-1}(\Sigma)$ are homeomorphic. Let $\tilde{\Sigma}$ be the oriented surface homeomorphic to the component of $p^{-1}(\Sigma)$. Let $p|_{\tilde{\Sigma}} : \tilde{\Sigma} \longrightarrow \Sigma$ be a $n-$fold covering. We apply Lemma 3.3.1 to $p|_{\tilde{\Sigma}}$. Then we get a covering space $\tilde{p} : \tilde{X} \to X$. Let r be the number of components of $\tilde{p}^{-1}(\Sigma)$. Note that r is divisible by $[\pi_1(B) : H] > 1$ with the notation in Lemma 3.3.1, hence $r > 1$.

We take r copies of $S^1 \times \tilde{Y}$ and l copies of $\tilde{X}$. For each copy of $S^1 \times \tilde{Y}$, there are l copies of $\tilde{\Sigma}$. On the other hand, for each copy of $\tilde{X}$, there are r copies of $\tilde{\Sigma}$. We correspond each $\tilde{\Sigma}$ in one copy of $S^1 \times \tilde{Y}$ to $\tilde{\Sigma} \subset \tilde{p}^{-1}(\Sigma)$ for each copy of $\tilde{X}$. For each pair of $\tilde{\Sigma}$ in $\tilde{M}$ and $\tilde{\Sigma}$ in $\tilde{X}$, we remove the neighborhoods of $\tilde{\Sigma}$ and glue along their boundaries $S^1 \times \tilde{\Sigma}$. The gluing maps are all given by a lifting of the gluing map between $S^1 \times Y$ and X. After gluing all, this forms a covering space $X^Y_{r,l}$ over X^Y which retracts onto $K_{r,l}$, a complete bipartite graph. Remark that $X^Y_{r,l}$ is a nlr-fold covering of X^Y. We show that $X^Y_{r,l}$ satisfies the condition. Figure 3.1 describes the case of $l = 2$ and $r = 3$.

We show that $b_2^+(X^K_{r,l}) > 1$. From the definitions of Euler characteristic and signature,

$$\chi(\tilde{X^Y}) = (2b_2^+(\tilde{X^Y}) - 2b_1(\tilde{X^Y}) + 2) - \sigma(\tilde{X^Y}).$$

It is known that $\sigma(X) = (2 - 2g)(2 - 2b)$ where $g(\Sigma) = g$ and $g(B) = b$ and $\chi(S^1 \times \tilde{\Sigma}) = 0$. Therefore, it is easily shown that $\chi(X^Y) = (2 - 2g)(-2b)$. From the Novikov additivity property and $\sigma(S^1 \times \tilde{\Sigma}) = 0$, $\sigma(X^Y) = \sigma(X)$. We cannot specify the exact value of $\sigma(X)$; however, it has a bound

$$\frac{-(b-1)(g-1)}{2} \leq \sigma(X) \leq \frac{(b-1)(g-1)}{2},$$

which is proved in [Kot98]. Therefore,

$$\begin{aligned} b_2^+(\tilde{X^Y}) &= b_1(\tilde{X^Y}) - 1 + \frac{nlr}{2}(\chi(X^Y) + \sigma(X^Y)) \\ &= (l-1)(r-1) - 1 + \frac{nlr}{2}(\chi(X^Y) + \sigma(X^Y)) \\ &> (l-1)(r-1) - 1 + \frac{nlr}{2}(2b(2g-2) - \frac{(b-1)(g-1)}{2}) > 1. \end{aligned}$$

□

3.4 Proof of Theorem 3.1

First, we consider a case in which Y is a surface bundle over S^1. If Y is a surface bundle over S^1, then $S^1 \times Y$ has a symplectic structure according to Friedl and Vidussi [FV11]. X^Y is a normal connected sum of two symplectic 4-manifolds $S^1 \times Y$ and X along symplecticamally embedded surfaces Σ. As a result, (X^Y, ω) is symplectic. Moreover, based on adjuction formula, the canonical symplectic form K_ω satisfies that $\langle K_\omega, [\Sigma]\rangle = 2g - 2$ since $[\Sigma]^2 = 0$ in X^Y.

Before moving to the other case, we define the following notations.

- Let Z be $S^1 \times Y \setminus nbd(\Sigma)$, where $nbd(\Sigma)$ is an neighborhood of Σ, homeomorphic to $D^2 \times \Sigma$.
- Let X' be $X \setminus nbd(\Sigma)$.
- X^Y is obtained from gluing Z and X' along their boundaries.

Second, suppose that Y is not a surface bundle over S^1 and that X^Y has a symplectic structure. Now, by Lemma 3.1, there exists a covering map $p : \tilde{X^Y} \to X^Y$ satisfying the conditions stated in the lemma. If we consider the submanifold $p^{-1}(Z) \subset \tilde{X^Y}$, then $p^{-1}(Z)$ has r components. We pick one component among them and call it M_1'. The boundary of M_1' consists of l copies of $S^1 \times \tilde{\Sigma}$. We fill the boundaries of M_1' by $D^2 \times \tilde{\Sigma}$ trivially. Then, we get a closed manifold M_1 which is homeomorphic to $S^1 \times \tilde{Y}$. Moreover, let $M_2' = \tilde{X^Y} \setminus M_1'$. Then we fill the boundaries of M_2' by $D^2 \times \Sigma$ trivially. We call the resulting manifolds M_2. Remark that M_1 is homeomorphic to $S^1 \times \tilde{Y}$. Conversely, if we do a normal connected sum between M_1, M_2 along l copies of $\tilde{\Sigma}$ repeatedly, then the resulting manifold becomes $\tilde{X^Y}$. The first procedure involves gluing two separated 4-manifolds, while the next involves self-gluing $l - 1$ times. We use the two gluing theorems for M_1, M_2 and $\tilde{X^Y}$. We use Theorem 2.5.1 when a $Spin^c$-structure $\tilde{P}$ is given on $\tilde{X^Y}$.

Henceforth, we say that $K \in H^2(S^1 \times C, \mathbb{Z})$ satisfies the pull-back condition for an orientable surface C when $K|_{S^1 \times C} \in H^2(S^1 \times C, \mathbb{Z}) = \rho^*(k_0)$ where $\rho : S^1 \times C \to C$ is a natural projection and $k_0 \in H^2(C, \mathbb{Z})$ satisfies that $\langle k_0, C\rangle = 2g(C) - 2$. From now on, $H^2(-)$ denotes a cohomology group with integer coefficients if not specified.

Suppose that ω is a symplectic 2-form on X^Y. Let $K_\omega \in H^2(X^Y, \mathbb{Z})$ be the canonical class of the symplectic structure ω on X^Y. Since K_ω is not torsion from the assumption $\langle K_\omega, [\Sigma]\rangle \neq 0$, we can perturb $\omega \in H^2(X^Y, \mathbb{R})$ to be a rational cohomology class and then scale ω properly so that $[\omega] \in H^2(X^Y, \mathbb{Z})$. We show that K_ω satisfies the pull-back condition. Based on adjunction inequality, for any closed curve α which is homologically nontrivial in C, $T^2 = S^1 \times \alpha \in S^1 \times C$, $\langle K_\omega, S^1 \times \alpha \rangle = 0$. Moreover, from the assumption

$$\langle K_\omega, [\Sigma] \rangle = 2g(\Sigma) - 2.$$

Consequently, K_ω satisfies the pull-back condition.

Let Ω be the pull back 2-form of ω on $\tilde{X^Y}$. Let $K_\Omega \in H^2(\tilde{X^Y}, \mathbb{Z})$ be the canonical class of the symplectic structure Ω on $\tilde{X^Y}$. $K_\Omega = p^*(K_\omega)$. Likewise, $K_\Omega \in H^2(S^1 \times \tilde{\Sigma})$ fulfills the pull-back condition on every component. In conclusion, we can use Theorem 3.1.2 for this canonical structure K_Ω. i_1^*, i_2^* denote the natural maps $H^2(\tilde{X^Y}, \mathbb{Z}) \to H^2(M_1')$ and $H^2(\tilde{X^Y}, \mathbb{Z}) \to H^2(M_2')$ induced by inclusion maps $M_1', M_2' \hookrightarrow \tilde{X^Y}$. Henceforth, $K_{M_1'}, K_{M_2'}$ denote $i_1^*(K_\Omega), i_2^*(K_\Omega)$ respectively. We pick $k \in H^2(\tilde{X^Y}, \mathbb{Z})$ which also satisfies the condition that $k|_{S^1 \times \tilde{\Sigma}} = \rho^*(k_0)$. Then, the restriction $(k - K_\Omega)$ on $S^1 \times \tilde{\Sigma}$ becomes zero. Moreover, let $k_{M_1'} = i_1^*(k)$, $k_{M_2'} = i_2^*(k)$. Then, $k_{M_1'} - K_{M_1'}$ and $k_{M_2'} - K_{M_2'}$ vanish on the boundary.

We focus on the element $(k - K_\Omega) \smile [\Omega]$ in $H^4(\tilde{X^Y}, \mathbb{Z}) \cong \mathbb{Z}$. We can decompose the integer corresponding to $(k - K_\Omega) \smile [\Omega]$ in the following way. First, $[F] = PD[(k - K_\Omega)] \in H_2(\tilde{X^Y})$, where a surface $F \in C_2(\tilde{X^Y})$ has no intersection with $S^1 \times \tilde{\Sigma}$. Next, F can be decomposed into $F_1 + F_2$ for $F_1 \in C_2(M_1')$, $F_2 \in C_2(M_2')$. Then,

$$(k - K_\Omega) \smile [\Omega] = PD[F_1] \smile i_1^*[\Omega] + PD[F_2] \smile i_2^*[\Omega]. \tag{3.4.1}$$

The equation 3.4.1 verifies the relationship between $Spin^c$-structures $k \in H^2(\tilde{X^Y}, \mathbb{Z})$ and its restrictions to M_1', M_2'.

We add the equation in Theorem 2.5.1, $(-1)^\star \sum_{s \in \mathcal{K}(k)} SW_X(s) = \sum SW_{M_1}(s_1) SW_{M_2}(s_2)$ for $k \in H^2(\tilde{X^Y})$ satisfying that

1. $k|_{S^1 \times \tilde{\Sigma}} = \rho^*(k_0)$ where $\rho : S^1 \times \tilde{\Sigma} \to \tilde{\Sigma}$ and $k_0 \in H^2(\tilde{\Sigma})$ satisfies that $\langle k_0, [\tilde{\Sigma}] \rangle = 2g(\tilde{\Sigma}) - 2$.
2. $k \smile [\Omega] = K_\Omega \smile [\Omega] \in \mathbb{Z}$
3. $k^2 = K_\Omega^2$.

It is known that only K_Ω satisfies all of the above properties (1)-(3) and $SW \neq 0$. Therefore, the left hand side is exactly equal to $SW(K_\Omega)$ which is $+1$ or -1 from

Theorem 1.9.1. The right hand side becomes $\sum_{(\hat{z_1},\hat{z_2})\in H^2(M_1)\times H^2(M_2)} SW_{M_1}(\hat{z_1})SW_{M_2}(\hat{z_2})$ for $(\hat{z_1},\hat{z_2}) \in H^2(M_1)\times H^2(M_2)$ satisfying

1. $\hat{z_1}^2 + \hat{z_2}^2 = k^2 - (4g-4)l$
2. There exist $[F_1],[F_2] \in H_2(M_1'), H_2(M_2')$ such that
   - $F_1 \cap \partial M_1' = F_2 \cap \partial M_2' = \phi$
   - $(\hat{z_1}|_{M_1'} - K_{M_1'}) = j_1(PD[F_1])$ and $(\hat{z_2}|_{M_2'} - K_{M_2'}) = j_2(PD[F_2])$ where $j_1 : H^2(M_1', \partial M_1') \to H^2(M_1'), j_2 : H^2(M_2', \partial M_2') \to H^2(M_2')$ are natural maps.
3. $[F_1],[F_2]$ determined in (2) satisfy that $PD[F_1] \smile i_1^*[\Omega] + PD[F_2] \smile i_2^*[\Omega] = 0$.

Now we see

$$\sum_{(\hat{z_1},\hat{z_2})\in H^2(M_1)\times H^2(M_2)} SW_{M_1}(\hat{z_1})SW_{M_2}(\hat{z_2}) = \sum_{\hat{z_2}} SW_{M_2}(\hat{z_2})(\sum_{\hat{z_1}} SW_{M_1}(\hat{z_1})). \quad (3.4.2)$$

We want to prove that the inner sum $\sum SW_{M_1}(\hat{z_1})$ in the right hand side becomes zero. Remark that if $SW_{M_1}(\hat{z_1}) \neq 0$, then $\hat{z_1}^2 = 0$. Therefore, the first condition does not need to be considered in the inner sum of Equation 3.4.2. Let Z_m be a set of $\hat{z_1} \in H^2(S^1 \times \tilde{Y})$ such that the corresponding F_1 satisfies $PD[F_1] \smile i_1^*[\Omega] = m$. In other words, we show that $\sum_{\hat{z_1}\in Z_m} SW_{M_1}(\hat{z_1}) = 0$ for all m.

We have the formula for the Seiberg Witten invariants. For $h \in H^2(S^1 \times \tilde{Y})$, $[h]$ denotes the quotient element in $H(S^1 \times \tilde{Y}) \cong H^2(S^1 \times \tilde{Y})/Tors$. If two elements $x, y \in H^2(S^1 \times \tilde{Y})$ satisfy that $[x] = [y]$, then $x \in Z_m$ is equivaltent to $y \in Z_m$ since the result of cup products does not depend on the torsion part. Let

$$\Delta_{\tilde{Y}} = \sum_{h\in H(\tilde{Y})} g_h \cdot h$$

for $g_h \in \mathbb{Z}$. We have the natural projection $p : S^1 \times \tilde{Y} \to \tilde{Y}$ and the induced map $p^* : H(\tilde{Y}) \to H(S^1 \times \tilde{Y}) \cong H(\tilde{Y}) \oplus (\mathbb{Z} \otimes H^1(\tilde{Y})/Tors)$. The image of p^* is included in the first summand.

First, suppose that $b_1(\tilde{Y}) > 1$. Then from Theorem 3.3.4,

$$\underline{SW}_{S^1\times\tilde{Y}} = \xi\Phi_2(\Delta_{\tilde{Y}}).$$

From Kunneth formula, $H^2(S^1 \times \tilde{Y}) = H^2(\tilde{Y}) \oplus (H^1(S^1) \otimes H^1(\tilde{Y}))$. Since $\xi \in \pm p^*(H(\tilde{Y}))$, ξ supports on the first summand of $H(S^1 \times \tilde{Y})$. Conclusively,

$$\underline{SW}_{M_1} = \xi \Phi_2(\Delta_{\tilde{Y}})$$
$$\implies \sum_{h \in H(M_1)} SW_{M_1}(h)[h] = \sum_{h \in H(S^1 \times \tilde{Y})} g_h \cdot (2h + \xi).$$

Therefore, we can summarize the equality:

1. $\sum_{\substack{h \in H^2(S^1 \times \tilde{Y}), \\ [h] = 2l + \xi \in H(S^1 \times \tilde{Y})}} SW(h) = g_l$ for $l \in H(\tilde{Y}) \subset H(S^1 \times \tilde{Y})$.
2. Otherwise, $\sum_{[h]=k} SW(h) = 0$ where k is not represented by $2l + \xi$.

We define $\phi : H^2(S^1 \times \tilde{Y}) \to H(Y)$ to be the composition of trivial quotient maps $H^2(S^1 \times \tilde{Y}) \to H(S^1 \times \tilde{Y}) \to H(\tilde{Y})$ and $\pi_* : H(\tilde{Y}) \to H(Y)$. Therefore,

$$\sum_{h \in \pi_*^{-1}(x)} g_h = 0 \tag{3.4.3}$$

for each $x \in H(Y)$. This is equivalent with

$$\sum_{\substack{h \in H^2(S^1 \times \tilde{Y}) \\ \phi(h) = x}} SW_{S^1 \times \tilde{Y}}(h) = 0 \tag{3.4.4}$$

for each $x \in H(Y)$.

Lemma 3.4.1. *If $\hat{z}, \hat{w} \in H^2(\tilde{Y}) \subset H^2(S^1 \times \tilde{Y})$ satisfy that $\phi(\hat{z}) = \phi(\hat{w})$, then*

$$\hat{z} \in Z_k \iff \hat{w} \in Z_k.$$

Proof. Let F_z, F_w be the 2-chain corrsponding to $\hat{z}, \hat{w}$ respectively. We observe that the covering p restricted to M_1' is isomorphic to the covering $p_1 : S^1 \times \tilde{Y} \to S^1 \times Y$ restricted to M_1'. Let $p' : M_1' \to p(M_1') = Z$. Recall that $Z \cong S^1 \times Y \setminus D^2 \times \Sigma$. The diagram 3.2 commutes. The horizontal maps are covering maps p, p' and the vertical maps are natural inclusion maps.

Recall that $\Omega = p^*[\omega]$. Therefore, $i_1^*[\Omega] = p'^*[\omega|_Z]$. Moreover, we also have the following commuting diagram 3.3.

$$\begin{array}{ccc} \tilde{X^Y} & \xrightarrow{p} & X^Y \\ \uparrow & & \uparrow \\ M_1' & \xrightarrow{p} & Z \end{array}$$

Figure 3.2: Covering diagram

$$\begin{array}{ccc} \tilde{M}_1 & \xrightarrow{p} & S^1 \times Y \\ \uparrow & & \uparrow \\ M_1' & \xrightarrow{p} & Z \end{array}$$

Figure 3.3: Covering diagram

Hence,

$$\begin{aligned} PD[F_z] \smile i_1^*[\Omega] &= \langle [F_z], i_1^*[\Omega] \rangle \\ &= \langle [F_z], p'^*[\omega|_Z] \rangle \\ &= \langle [p'(F_z)], [\omega|_Z] \rangle \\ &= PD[p'(F_z)] \smile [\omega|_Z] \\ &= \langle p'(F_z), [\omega|_Z] \rangle. \end{aligned}$$

The second commutative diagram indicates that $p'(F_z) = p'(F_w)$. Therefore, the statement is true.

□

Finally, based on Lemma 3.4.1, the inner sum of Equation 3.4.2 can be decomposed into the sum of Equation 3.4.4 for some x. Therefore, the inner sum of Equation 3.4.2 becomes zero.

Second, suppose that $b_1(\tilde{Y}) = 1$. Since $b_1(Y) = 1$, $\pi_* : H(\tilde{Y}) \to H(Y)$ is a homomorphism from $\mathbb{Z} \to \mathbb{Z}$. Since this homomorphism is not trivial, π_* is injective. Therefore, $\pi_*(\Delta_{\tilde{Y}}) = 0$ implies that $\Delta_{\tilde{Y}} = 0$. Therefore, $\sum_{[h]=l} SW(h) = 0$ for all $l \in H(S^1 \times \tilde{Y})$. Therefore, the inner sum of the right hand side is also zero.

Therefore, this is a contradiction because Equation 3.3.2 is not true. This implies that X^Y does not have a symplectic structure and hence concludes the proof of Theorem 3.1.2.

Remark 3.4.2. *We add remarks in a more general setting. First, in case of* $b_1(Y) > 1$*, according to Remark 3.3.6, we need to show the corresponding equation to Equation 3.4.3: for each* $m \in \mathbb{Z}$,

$$\sum_{\psi \smile \pi^*(h) = m} g_h = 0$$

for a non-fibered $\psi \in H^1(Y)$*. Moreover, to extend the results to the case of* $b_1(Y) > 1$*, Lemma 3.4.1 is transformed into: for two* $\hat{z}, \hat{w}$ *satisfying that*

$$\psi \smile \pi(\hat{z}) = \psi \smile \pi(\hat{w}),$$

the statement $\hat{z} \in Z_k \iff \hat{w} \in Z_k$ *is true. However, we cannot use the same proof in Lemma 3.4.1 since the* 2*-form* ω *on* X^Y *does not have the unique corresponding* 2*-form on* $S^1 \times Y$.

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