MAXIMAL CLIQUES IN GRAPHS ASSOCIATED WITH COMBINATORIAL SYSTEMS

Thesis by

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Abstract.

Maximal cliques in various graphs with combinatorial significance are investigated. The Erdös, Ko, Rado theorem, concerning maximal sets of blocks, pairwise intersecting in s points, is extended to arbitrary t-designs, and a new proof of the theorem is given thereby.

The simplest case of this phenomenon is dealt with in detail, namely cliques of size r in the block graphs of Steiner systems $S(2,k,v)$. Following this, the possibility of nonunique geometrisation of such block graphs is considered, and a nonexistence proof in one case is given, when the alternative geometrising cliques are normal.

A new Association Scheme is introduced for the 1-factors of the complete graph; its eigenvalues are calculated using the Representation Theory of the Symmetric Group, and various applications are found, concerning maximal cliques in the scheme.

Contents.

INTRODUCTION.

A clique in a graph is a set of vertices such that any two are adjacent. By a maximal clique, in this thesis, we shall mean a clique which reaches a prescribed bound. The bound that we shall be most interested in, in Chapters II, III, and IV is that given by a generalisation to t-designs (due to the author) of the Erdös, Ko, Rado Theorem. The graphs that we shall be concerned with in these chapters are the block graphs of the designs, with the blocks of the design as vertices. They contain certain cliques which correspond to the sets of blocks through a point of the design. The afore-mentioned theorem states that in most cases these are the largest cliques in the block graph. We investigate the possibility of other cliques of the same size, paying particular attention in Chapter III to the simplest case, Steiner systems $S(2,k,v)$.

Following this we consider the possibility of nonunique geometrisation of a block graph. This requires the existence of another set of cliques, of the same size as our special cliques, on which can be defined a t-design with the same parameters as the original. We determine a relation between two such sets of geometrising cliques, in the case of 2-designs, and consider in detail an extremal type of clique, which we call a normal clique, and show that an alternative geometrisation by normal cliques of the block graph of a 2-design gives rise to a symmetric design on the points of the original 2-design.

Steiner systems can be viewed as maximal cliques in graphs derived from Association Schemes, the basic theory of which is dealt with in Chapter I. In Chapter V we define an association scheme on the 1-factors of the complete graph on $2n$ vertices, give a method for determining its eigenvalues, and apply this method for the cases $n = 4,5,6$. The work relies heavily on the Representation Theory of the Symmetric Group. We again look at various maximal collections of 1-factors with combinatorial significance.

CHAPTER I.

ASSOCIATION SCHEMES; CLIQUES AND DESIGNS THEREIN; FURTHER EIGENVALUE TECHNIQUES; DEFINITIONS.

(1) Association Schemes.

An association scheme on a set Ω of size v is a partition of the 2-subsets of Ω into m classes or relations $R_1, \ldots\ldots\ldots, R_m$ satisfying

$$\text{(a)} \quad x \in \Omega \Rightarrow \left| \{y : \{y, x\} \in R_i\} \right| = \nu_i .$$

$$\text{(b)} \quad x \neq y \in \Omega, \ \{x, y\} \in R_i$$

$$\Rightarrow \left| \{z : \{z, x\} \in R_j, \ \{z, y\} \in R_k\} \right| = p_{ijk} ,$$

for nonnegative integers ν_i $(i = 1, \ldots\ldots, m)$

and p_{ijk} $(i,j,k = 1, \ldots, m)$.

To each relation R_i we can associate a graph on Ω with adjacency matrix A_i of size v. If we let $A_0 = I$, the identity matrix, we have

$$J = A_0 + A_1 + \ldots\ldots\ldots + A_m ,$$

where J is the $v \times v$ matrix of all 1's.

Furthermore we have

$$A_k A_j = A_j A_k = \sum_{i=0}^{m} p_{ijk} A_i ,$$

where the p_{ijk}'s with one or more subscripts zero are suitably defined. So the A_i's span an associative and commutative algebra $\mathcal{a}$, known as the Bose Mesner algebra of the scheme.

Let V be the v-dimensional vector space over the complex numbers on which the matrices A_i can be said to act. Then there exists an orthogonal decomposition $V = V_0 \oplus V_1 \oplus \ldots\ldots \oplus V_m$ (the eigenspaces of the A_i's, which are simultaneously diagonalisable) such that if E_i denotes the matrix of the orthogonal projection $V \to V_i$, then

$$\mathcal{a} = \text{span}\left\{A_i\right\}_{i=0}^{m} = \text{span}\left\{E_i\right\}_{i=0}^{m} .$$

It is customary to take $V_0 = \text{span}\left\{(1,1,\ldots\ldots,1)^T\right\}$. The V_i's are known as the eigenspaces of the algebra, and $\dim V_i = \mu_i$, say.

(2) <u>Eigenvalues of the scheme, and Delsarte's inequalities</u>.

We have $$A_j = P_j(0)\,E_0 + P_j(1)\,E_1 + \ldots\ldots + P_j(m)\,E_m,$$

for scalars $P_j(k)$ which are known as the eigenvalues of the scheme. It is not difficult to show that the $P_j(k)$'s, together with the μ_i's determine all the parameters of the scheme.

Delsarte (7) develops conditions on the distribution vector of a subset of elements of the scheme in terms of the eigenvalues. Given a subset $Y \subseteq \Omega$, its distribution vector $\underline{a}$ is an $m+1$ dimensional vector with

$$a_0 = 1,$$

$$a_i = \frac{1}{|Y|} \sum_{y,z \in Y} \left| \{\{y , z\}\} \cap R_i \right| , \qquad i = 1 \text{ to } m.$$

Delsarte's inequalities state that

$$\sum_{j=0}^{m} a_j \, P_i(j)/\nu_j \geq 0, \qquad \text{for } i = 0 \text{ to } m,$$

and so a knowledge of the eigenvalues restricts the possible subsets of the scheme.

Let ϕ_Y be the characteristic vector of Y. Then the inequalities are equivalent to the fact that

$$\phi_Y^T E_i \phi_Y \geqslant 0,$$

$$\text{since} \qquad E_i = \frac{\mu_i}{v} \sum_{j=0}^{m} \frac{P_i(j)}{\nu_j} A_j$$

(3) <u>The Johnson and Hamming Schemes.</u>

The Johnson scheme is defined on $\mathcal{P}_k(v)$, the set of all k-subsets of a v-set. There are k nontrivial relations, (as long as $v \geqslant 2k+1$), depending on the intersection of the two sets, i.e.

$$\{A , B\} \in R_i \quad \text{if and only if} \quad |A \cap B| = k - i.$$

These schemes provide a general setting for the theory of statistical designs, and set intersection problems.

The Hamming scheme is defined on $GF(q)^v$, the set of all vectors in a v-dimensional vector space over a finite field. There are v nontrivial relations, depending on the number of nonzero entries in the difference $\underline{v}_1 - \underline{v}_2$, for two vectors in the space. These schemes are of use in the theory of error-correcting codes.

The eigenvalues of the Johnson scheme are given by

$$P_i(j) = \sum_{u=0}^{i} (-1)^u \binom{j}{u}\binom{k-j}{i-u}\binom{v-k-j}{i-u}$$

$$= E_i(j), \text{ an Eberlein polynomial.}$$

$$\mu_i = \binom{v}{i} - \binom{v}{i-1} \qquad , \qquad \nu_i = \binom{k}{i}\binom{v-k}{i}$$

For the Hamming scheme we have

$$P_i(j) = \sum_{u=0}^{j} (-q)^u (q-1)^{i-u} \binom{v-u}{j-u}\binom{j}{u} \quad ,$$

$$\mu_i = (q-1)^i = \nu_i \; .$$

(4) <u>A simple application of eigenvalue methods</u>.

Let us suppose that we have a family $\mathcal{A}$ of k-subsets of a v-set, (where $v \geq 2k+1$), which pairwise intersect in at least one point. Then this is a coclique in the graph corresponding to empty intersection. We have the following bound due to Hoffman (18) on the size α of a coclique in a regular graph, namely

$$\alpha \leq \frac{-v\lambda}{d-\lambda} ,$$

where v is the number of vertices,

d is the valency,

λ is the minimum eigenvalue of the adjacency matrix.

By examining the eigenvalues $P_k(j)$ of the graph, we find

$$\lambda = P_k(1) = (-1)\binom{v-k-1}{k-1}$$

is the smallest eigenvalue of A_k.

$$\Rightarrow |a| \leq \frac{\binom{v}{k}\binom{v-k-1}{k-1}}{\binom{v-k}{k}+\binom{v-k-1}{k-1}} = \frac{\binom{v}{k}}{\left(\frac{v-k}{k}\right)+1} = \binom{v-1}{k-1} .$$

This is the simplest case of the theorem of Erdös, Ko, and Rado (12).

(5) <u>The code-clique theorem.</u>

A subset Y of an m-class association scheme is a clique with respect to relations $\{R_i : i \in N\}$ for some subset N of $\{1,\ldots,m\}$, if its distribution vector satisfies

$$a_i = 0 \quad \text{for all } i \notin N, \quad i \neq 0.$$

It is a code with respect to these relations if

$$a_i = 0 \quad \text{for all } i \in N.$$

Let Y be a code with respect to $\{R_i : i \in N\}$, and Z be a clique with respect to the same relations. Then

$$|Y| . |Z| \leqslant v.$$

For a proof see Delsarte (7).

(6) t-designs in the Johnson scheme.

A $t-(v,k,\lambda)$ design is an ordered pair $(X,\mathcal{B})$, where X is a set of size v, and $\mathcal{B}$ is a family of k-subsets (blocks) of X with the property that any t-subset of X is contained in exactly λ of the blocks of $\mathcal{B}$. To avoid degenerate cases, it is assumed that $0 < t \leqslant k < v$.

Consider $N_{t,k}$ a $\binom{v}{t}$ by $\binom{v}{k}$ matrix with rows indexed by $\mathcal{P}_t(v)$, and columns indexed by $\mathcal{P}_k(v)$, with

$$N_{t,k}(t,B) = \begin{cases} 1 & \text{if} \quad t \subseteq B, \\ 0 & \text{if not.} \end{cases}$$

Then if ϕ_Y is the characteristic vector of the blocks of a t-design,

$$N_{t,k}\,\phi_Y = \lambda \underline{j} \;,$$

where $\underline{j}$ is the $\binom{v}{t}$ column vector of all ones.

i.e $$N_{t,k}\left[\phi_Y - \frac{\lambda}{\binom{v-t}{k-t}}\, j'\right] = 0,$$

where j' is the $\binom{v}{k}$ column vector of all ones.

It can be shown, (7), that the row space of $N_{t,k}$ is $V_0 \oplus V_1 \oplus \dots \oplus V_t$.

Hence, if ϕ_Y is the characteristic vector of a t-design,

$$E_1\phi_Y = E_2\phi_Y = \dots\dots = E_t\phi_Y = 0.$$

(7) The degree of a subset of $\mathcal{P}_k(v)$.

This is defined to be one less than the number of nonzero entries in the distribution vector of the subset. We have the following theorems due to Ray-Chaudhuri and Wilson (25).

Theorem: If a subset Y of $\mathcal{P}_k(v)$ has degree d, then $|Y| \leq \binom{v}{d}$.

Theorem: The number of blocks, $|\mathcal{B}|$, of a 2s-design satisfies

$$|\mathcal{B}| \geq \binom{v}{s} .$$

Proofs of these can be found in (7).

Finally, we have the following;

Theorem: Suppose we have a t-design Y of degree s, with $t \geq 2s - 2$. Then the restriction of the Johnson scheme to Y is an association scheme with s classes.

See Cameron and Van Lint (5).

(8) Interlacing Theorems.

In his thesis, W.Haemers (15) gives a full account of the method of interlacing for the eigenvalues of a square matrix and a square submatrix, and many applications in the theory of graphs and combinatorial structures. We give here one of his fundamental theorems, which we shall make use of in Chapter III.

Definition: Suppose that A and B are square matrices over the complex numbers, of size n and m respectively, with $m \leqslant n$. If

$$\lambda_i(A) \geqslant \lambda_i(B) \geqslant \lambda_{n-m+i}(A) \qquad \text{for all } i = 1 \text{ to } m,$$

then the eigenvalues of B are said to interlace those of A.

If there exists an integer $k > 0$, $k \leqslant m$, such that

$$\lambda_i(A) = \lambda_i(B) \qquad \text{for } i = 1 \text{ to } k,$$

and $\lambda_{n-m+i}(A) = \lambda_i(B) \qquad \text{for } i = k+1 \text{ to } m,$

then the interlacing is said to be tight.

Theorem: Let S be a complex $n \times m$ matrix such that $S^* S = I_m$, (where S^* is the complex conjugate transpose of S). Let A be a Hermitian matrix of size n. Define $B = S^* A S$. Then,

(i) the eigenvalues of B interlace those of A.

(ii) If $\lambda_i(B) \in \{\lambda_i(A), \lambda_{n-m+i}(A)\}$ for some $i \in \{1,\ldots\ldots,m\}$, then there exists an eigenvector $\underline{v}$ of B with eigenvalue $\lambda_i(B)$, such that $S\underline{v}$ is an eigenvector of A with the same eigenvalue.

(iii) If for some $k \in \{1,\ldots\ldots,m\}$, $\lambda_i(A) = \lambda_i(B)$ for all $i \in \{1,\ldots\ldots,k\}$, and $\underline{v}_i$ is an eigenvector for B with value $\lambda_i(B)$, then $S\underline{v}_i$ is an eigenvector of A with eigenvalue $\lambda_i(B)$ for all $i = 1,\ldots,k$.

(iv) If the interlacing is tight, then $SB = AS$.

For our purposes, $S = [I_m, 0]^T$.

(9) <u>Other Definitions</u>.

A Steiner system $S(t,k,v)$ is a t-$(v,k,1)$ design. When $\lambda > 1$, we write a t-(v,k,λ) design as an $S_\lambda(t,k,v)$.

A symmetric block design is a 2-(v,k,λ) design in which the number of blocks $b = v$. By the second theorem in section (7) of this Chapter, $b \geqslant v$, which is Fisher's inequality.

A pairwise balanced design is a pair $(X,\mathcal{B})$, where X is a set of size v and $\mathcal{B}$ is a set of subsets of X, (of no prescribed size), such that any two points of X are contained in exactly one element of $\mathcal{B}$.

The block graph of an $S(2,k,v)$, written $BG[S(2,k,v)]$, or BG when it is known which Steiner system we are talking about, has as vertex set the b blocks of the Steiner system, and as edge set the pairs of incident blocks. Block graphs can be defined for more general designs.

A partial geometry and a strongly regular graph are as defined in Bose (1). A strongly regular graph can also be regarded as a 2 class association scheme.

CHAPTER II.

MAXIMAL CLIQUES IN THE BLOCK GRAPHS OF t-DESIGNS.

(1) An extension of the Erdös, Ko, Rado theorem to t-designs.

In (26) the following theorem is proved:

Theorem: Let $\mathcal{B}$ represent the set of blocks of a t-(v,k,λ) design. Given $0 < s < t \leqslant k$, then there exists a function $f(k,t,s)$ with the following property: suppose there is a set $\mathcal{A} \subseteq \mathcal{B}$ of blocks such that for all $A,B \in \mathcal{A}$, $|A \cap B| \geqslant s$; then if $v \geqslant f(k,t,s)$,

$$|\mathcal{A}| \leqslant b_s = \text{the number of blocks through } s \text{ points.}$$

Furthermore, if $v > f(k,t,s)$, then the only families of blocks reaching this bound are those consisting of all blocks through some s points.

$$\text{If } s < t-1, \text{ then } f(k,t,s) \leqslant s + \binom{k}{s}(k-s+1)(k-s).$$

$$\text{If } s = t-1, \text{ then } f(k,t,s) \leqslant s + (k-s)\binom{k}{s}^2 .$$

It is well known that for any $s \leqslant t$, the number of blocks, b_s, through s points of X, is independent of the choice of these points, and

$$b_s = \lambda \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s} .$$

Extensive use is made of the fact that

$$\binom{n}{r} \Big/ \binom{n-1}{r-1} = n/r \quad .$$

Let $\mathcal{P}_k(v)$ denote the set of all k-subsets of a v-set. Then it may be regarded as a k-(v,k,1) design. So the theorem has the following theorem due to Erdös, Ko , and Rado, (12), as an immediate corollary:

Theorem: Given $0 < s \leqslant k \leqslant v$, then there exists a function $g(k,s)$ with the following property: suppose there is a set $\mathcal{A}$ of k-subsets of a v-set such that for all $A, B \in \mathcal{A}$, $|A \cap B| \geqslant s$; then if $v \geqslant g(k,s)$,

$$|\mathcal{A}| \leqslant \binom{v-s}{k-s},$$ the number of k-subsets containing an s-subset.

Frankl (13) has shown that if $s \geqslant 15$, then,

$$g(k,s) = (k - s + 1)(s + 1) + s,$$

and conjectures that this holds for all s.

Proof of the theorem:

Let $\mathcal{A}$ be a family of blocks satisfying the conditions of the theorem. Let $\mathcal{C}$ be the set of s-subsets which are at the intersection of at least two blocks of $\mathcal{A}$. Let n_p be the number of blocks of $\mathcal{A}$ containing the s-subset p of the family $\mathcal{C}$. Let $|\mathcal{A}| = w$.

Count (p,B) such that $p \in \mathcal{C}$, $p \subseteq B \in \mathcal{A}$, to obtain,

$$\sum_{p \in \mathcal{C}} n_p \leq w \binom{k}{s} .$$

Count (p,B,A) such that $p \in \mathcal{C}$, $p \subseteq B \cap A$, with $A,B \in \mathcal{A}$;

$$\sum_{p \in \mathcal{C}} n_p(n_p - 1) = \sum_{\substack{A \neq B \\ A,B \in \mathcal{A}}} \binom{|A \cap B|}{s} \geq w(w-1).$$

Now if $\mathcal{A}$ is not the set of all blocks through an s-subset, then, for each $p \in \mathcal{C}$, there is some block $B \in \mathcal{A}$, with $p \not\subseteq B$. Any other block $A \in \mathcal{A}$, which contains p, contains at least s points of B. So if d is the maximum number of blocks of $\mathcal{B}$ which contain p and at least s points of B, then $n_p \leq d$. Hence,

$$w(w-1) \leq (d-1) \sum_{p \in \mathcal{C}} n_p \leq (d-1) w \binom{k}{s},$$

$$w - 1 \leq (d-1) \binom{k}{s} \quad \text{and so} \quad w < d \binom{k}{s} .$$

If $d \binom{k}{s} \leq b_s$, we are done.

The following lemma gives an upper bound for d.

Lemma: Let p be an s-subset, and B a block not containing p. Let d be the number of blocks containing p and at least s points of B. Then

(i) if $s \leqslant t/2$ and $v \geqslant k^2 + 2t$,

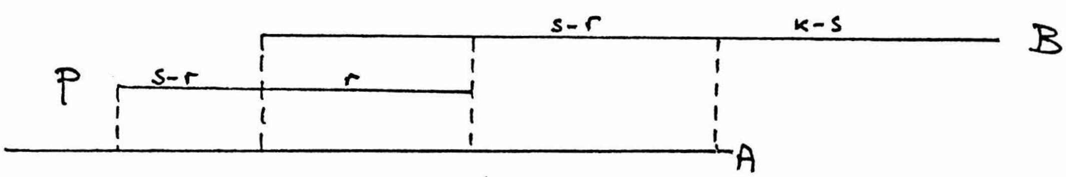
or (ii) if $t/2 < s < t - 1$ and $v \geqslant s + \binom{k}{s}(k - s)$,

then $$d \leqslant (k - s + 1)\lambda\binom{v - s - 1}{t - s - 1}\Big/\binom{k - s - 1}{t - s - 1} \quad ;$$

(iii) if $s = t - 1$,

then $$d \leqslant \binom{k}{s}\lambda \ .$$

Proof of (i): Take an s-subset p and a block B containing r points of p, for some $r < s$.



Let d_r be the number of blocks containing p and at least s points of B.

Then $$d_r \leqslant \binom{k - r}{s - r} b_{2s-r} = \lambda\binom{k - r}{s - r}\binom{v - 2s + r}{t - 2s + r}\Big/\binom{k - 2s + r}{t - 2s + r} = e_r, \text{ say,}$$

since there are $\binom{k - r}{s - r}$ s-subsets of B which contain all points of $B \cap p$, and these, together with the remaining $s - r$ points of p, each determine a family of b_{2s-r} blocks with the required property.

These families have in their union all such blocks. Clearly,

$$e_{r+1}/e_r = (s - r)(v - 2s + 1)/(k - r)(k - 2s + r + 1)$$

$$> (v - 2t)/k^2.$$

So if $v \geq k^2 + 2t$, then $e_{r+1} > e_r$ for all $r < s - 1$,

and so $d_r \leq e_{s-1} = \lambda (k - s + 1) \binom{v - s - 1}{t - s - 1} \Big/ \binom{k - s - 1}{t - s - 1}$.

Hence d is bounded as required.

<u>Proof of (ii) and (iii)</u>:

Let c_j maximum number of blocks containing j points of X.

So if $j \leq t$, then $c_j = b_j$,

but, if j t, then $c_j \leq \lambda$.

Then with d_r as in the proof of case (i),

$$d_r \leq \binom{k - r}{s - r} c_{2s-r} = e_r .$$

So, if $r < 2s - t$, then $d_r \leq \lambda \binom{k - r}{s - r} \leq \binom{k}{s} \lambda$;

if $r = 2s - t$, then $d_r \leq \lambda \binom{k - 2s + t}{2t - 2s}$;

if $r > 2s - t$, then $d_r \leq \lambda \binom{k - r}{s - r} \binom{v - 2s + r}{t - 2s + r} \Big/ \binom{k - 2s + r}{t - 2s + r}$.

It is clear, using the same argument as in (i), that,

$$d = \max d_r \leq \max (e_o, e_{s-1})$$

$$= \max \left(\lambda \binom{k}{s}, \lambda (k - s - 1) b_{s+1}\right),$$

as long as $v \geq k^2 + 2t$.

If $s < t - 1$, then $d \leq \max \left(\lambda \binom{k}{s}, \lambda (v - t)/k\right)$.

So if $v \geq t + \binom{k}{s} k$, then $d \leq \lambda (k - s - 1) b_{s+1}$.

If $s = t - 1$, then $d \leq \max \left(\lambda \binom{k}{s}, \lambda (k - t)\right) = \binom{k}{s} \lambda$.

So ends the proof of the lemma.

We want $d \binom{k}{s} \leq b_s$.

In cases (i) and (ii), this requires

$$\lambda \binom{k}{s} (k - s + 1) \binom{v - s - 1}{t - s - 1} \Big/ \binom{k - s - 1}{t - s - 1} \leq \lambda \binom{v - s}{t - s} \Big/ \binom{k - s}{t - s},$$

i.e. $\binom{k}{s} (k - s + 1)(k - s) \leq (v - s)$,

or $v \geq s + (k - s)(k - s + 1) \binom{k}{s}$, as in the statement of the theorem.

In case (iii) this requires

$$\lambda \binom{k}{s}^2 \leq \lambda\,(v - t + 1)/(k - t + 1)$$

or $$v \geq s + (k - s)\binom{k}{s}^2 .$$

We have strict inequality in $|\mathcal{A}| \leq b_s$, when v satisfies these conditions, and $\mathcal{A}$ does not consist of all blocks containing s points, so the other conclusions of the theorem hold.

(2) Corollary: If $v \geq k^2$, then a t-design, with $t \geq 2$, has disjoint blocks.

Proof: We need only consider 2-designs. Suppose any two blocks of the design intersect. Then $\mathcal{B}$, the set of blocks of the design, satisfies the bound in the proof of the previous theorem, given by

$$|\mathcal{B}| \leq d\binom{k}{s} \qquad \text{where } s = 1.$$

Hence $$\lambda\binom{v}{2} \Big/ \binom{k}{2} \leq \lambda k^2 .$$

If $v \geq k^2$, this is impossible.

We provide another proof of this fact in Chapter IV.

(3) On a paper of Deza, Erdős, and Frankl.

In this paper (11), the authors prove a number of theorems which extend the ideas of the Erdős, Ko, Rado Theorem. They define a (v,M,k)-system

$\mathcal{A} \subseteq \mathcal{P}_k(v)$, such that for any two distinct $A, B \in \mathcal{A}$, $|A \cap B| \in M$.
Let $M = \{m_1, m_2, \ldots\ldots, m_r\}$, $m_1 < m_2 < \ldots\ldots\ldots < m_r < k$.
Then they prove, among other things, the following;

<u>Theorem</u>: If $\mathcal{A}$ is a (v,M,k)-system, $v \geqslant g(k,s)$ for all $s < k$,

and $$|\mathcal{A}| \geqslant c(k) \prod_{i=2}^{r} (v - m_i)/(k - m_i)$$ then

(i) there exists an m_1-subset D of X such that $D \subseteq B$ for all $B \in \mathcal{A}$.

(ii) $(m_2 - m_1) \mid (m_3 - m_2) \mid \ldots\ldots\ldots \mid (m_r - m_{r-1}) \mid (k - m_r)$.

(iii) $\prod_{i=1}^{r} (v - m_i)/(k - m_i) \geqslant |\mathcal{A}|$.

Remarks on this theorem and its applicability to designs:

(a) The condition that v is large compaired to k is very necessary. For consider the tight 4-design $S(4,7,23)$. This is unique, (19), and any two distinct blocks intersect in 1 or 3 points. If we apply the theorem, we get $|\mathcal{B}| \leqslant 22.20/6.4 < 20$, whereas the design has 253 blocks.

(b) The authors remark that equality in (iii) is realizable by the family of hyperplanes of any perfect matroid design.

(c) For the case $|M| = 2$, the designs with this property are called quasi-symmetric and are dealt with in Chapter IV. It is shown there that $v \leq k^2$ for these designs. However, it is also shown that the divisibility conditions for the intersection numbers (as in (ii)) hold.

(d) As was stated in Chapter I, $|\mathcal{A}| \leq \binom{v}{|M|}$, for any (v,M,k)-system. In most cases (iii) is stronger than this, but it must be said again that this only holds for large v.

CHAPTER III.

CLIQUES OF SIZE r IN $BG[S(2,k,v)]$.

(1) Cliques have size at most r for all $v > k^2 - k + 1$.

To an $S(2,k,v)$ we ascribe two more parameters, namely

$$b = \frac{v(v-1)}{k(k-1)} \quad , \qquad r = \frac{v-1}{k-1} \quad ,$$

where b is the number of blocks of the design, and r is the number of blocks through each point of the design.

We make use of the fact that $BG[S(2,k,v)]$ is a strongly regular graph, since $S(2,k,v)$ is a partial geometry. BG has parameters

$$\left[b, \quad k(r-1), \quad (r-2) + (k-1)^2, \quad k^2\right]$$

and so its adjacency matrix, A, satisfies

$$A^2 + (2k - r + 1) A - k(r - k - 1) I = k^2 J$$

Therefore A has eigenvalues

$k(r-1)$	once,
$r - k - 1$	$v - 1$ times,
$-k$	$b - v$ times.

Now we look at the 3-dimensional algebra generated by A, (since $v > k^2 - k + 1$, $b > v$). The second nontrivial idempotent of the algbra, is given by

$$E_2 = \left[I + A\lambda_2/d + \frac{(J - A - I)(-\lambda_2 - 1)}{v - d - 1} \right] \frac{\mu_2}{v}$$

using the notation of Chapter I.

Let ϕ be the characteristic vector of a clique of size w in BG. Then the condition $\phi^T E_2 \phi \geqslant 0$ implies

$$w + \frac{\lambda_2}{d} w(w - 1) \geqslant 0 \qquad \text{or} \qquad w \leq 1 + \frac{d}{-\lambda_2},$$

which in our case gives

$$w \leq r.$$

If equality holds, then $E_2 \phi = 0$, and so

$$\phi - A\frac{\phi}{r - 1} + \frac{(J - A - I)(k - 1)}{-k(r - 1) + (\frac{vr}{k} - 1)}\phi = 0,$$

$$A\phi = (r - k - 1)\phi + k\underline{j} = (r - 1)\phi + k(\underline{j} - \phi),$$

which implies that any vertex of B not in the clique of size r is adjacent to exactly k vertices of the clique. And so, in the terminology of Delsarte (7) this is a completely regular clique.

There are numerous ways of proving this result. Elementary counting methods have been used by Bose (1), and Neumaier (24); the latter would call this a regular clique of size r and nexus k, and shows that if an edge regular graph contains a regular clique of a given size and nexus then any other regular clique must have the same size and nexus. This is of interest in our study because it implies that the block graph of a Steiner system $S(2,k,v)$ can only be the block graph of another Steiner system if it has the same parameters.

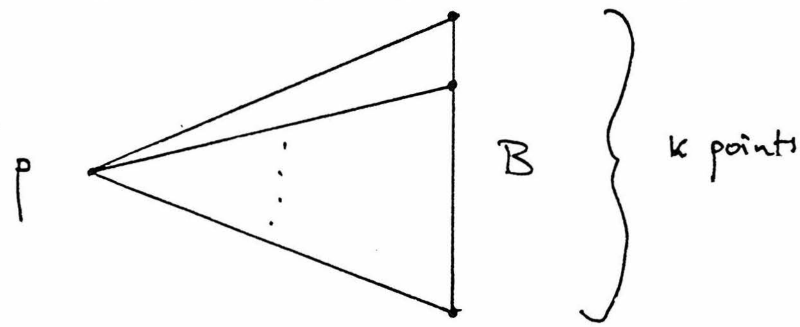
One can also use the interlacing method of Haemers (15) to prove this result but it is very similar to the association scheme method.

(2) <u>The pairwise balanced design on C</u>.

We find the best way to look at this problem is to dualise and to consider the partial geometry on b vertices, with v lines of size r, such that any two lines meet at a unique vertex. BG is then the point graph of this partial geometry. A clique C of size r, not corresponding to the set of r blocks through a point, is then a set of r vertices of the geometry such that any two lie on some line of the geometry, and in fact the restrictions of the lines to the r vertices of C form a pairwise balanced design, except for the fact that this p.b.d. may have repeated blocks of size one.

This p.b.d. has the property that for any two of its blocks, if they have combined size greater than k, they must meet. For suppose not. Then the lines of the partial geometry to which these correspond meet at some vertex outside the clique. But then this vertex is adjacent to more than k vertices of the clique contradicting (1). We shall make repeated use of this property later in this chapter.

We note that for any line L of the p.g. , if $C \neq L$, then $|C \cap L| \leq k$. Otherwise a vertex $x \in L \setminus C$ is adjacent to more than k vertices of C. This can also be seen in the following manner. Let B be a block in a clique C, and p a point of the design not on B.



So if C is not the set of blocks through a point p , C contains at most k of the blocks through p , because there are only k blocks through p which intersect B .

We shall call a line of the partial geometry an i-line with respect to a certain clique C , of size r , if it contains i vertices of the clique. If the clique does not correspond to a line, then $i \leq k$.

Let n_i be the number of i-lines with respect to C.

As can be done for any p.b.d. , we form the following system of equations for the n_i's.

$$\text{(A)} \left\{ \begin{array}{lll} \sum_{i=0}^{k} n_i = v & & \text{counting lines of the p.g.} \\ \sum_{i=0}^{k} i n_i = rk & & \text{counting } (x,L) \text{ , } x \in C, \quad x \in L. \\ \sum_{i=0}^{k} (i-1) i n_i = r(r-1) & & \text{counting } (x_1 , x_2, L), \quad x_1 , x_2 \in C \cap L \end{array} \right.$$

We shall apply these equations to particular cases.

We may rewrite these equations in terms of variables $x_L = |L \cap C|$.
Let h be the number of lines which contain at least one vertex of C.

$$\sum_{L:|L\cap C|>0} 1_L = h,$$

$$\sum_L x_L = rk,$$

$$\sum_L x_L(x_L - 1) = r(r-1) .$$

These equations imply

$$\sum (x_L - \frac{r+k-1}{k})^2 = h - \frac{rk^2}{r+k-1},$$

and so h is a minimum when

$$x_L = \frac{r+k-1}{k} \quad \text{is a constant for each line } L \text{ which}$$

intersects C.

Such a clique C, where any point covered by the r blocks of C lies in a constant number of these blocks, we shall call a normal clique. We have shown that the blocks of a normal clique cover the minimum number of points of the design. It would also be nice to be able to find an upper bound for h, the number of points covered by blocks of C.

Let n be the least integer greater than $(r+k-1)/k$. Then if

C is not normal, some point lies on at least n blocks of C, i.e. some line L of the p.g. intersects C in at least n vertices. But then any vertex $x \in L \setminus C$ lies on at most $k - n$ other lines intersecting C, and so on at least $n - 1$ lines which do not intersect C. There are at least $r - n$ such vertices. So $n_0 \geqslant (n - 1)(r - n)$ and $h \leqslant v - (n - 1)(r - n)$ with equality if and only if $r = n^2 - n + 1$, and the p.b.d. on C is a projective plane of order n.

More generally, if some point lies on m blocks of C, $m > n$, $h \leqslant v - (m - 1)(r - m)$, with equality if $r = m^2 - m + 1$, and the p.b.d. is a projective plane of order m.

This bound for h is not entirely satisfactory but we will see an example of a design later where both the upper and lower bounds are met.

(3) <u>If $r > k^2 - k + 1$, then the only cliques of size r in $BG[S(2,k,v)]$ are those corresponding to r blocks through a point.</u>

Let C be a clique which is not a line of the partial geometry; then any vertex x of C lies on exactly k lines, and any line through x contains at most k vertices of C. So, since any vertex of C lies on a line through x,

$$|C| \leqslant k(k - 1) + 1.$$

This result was known to Bruck (2) and Bose (1), and is also a special case of a theorem of Deza (10).

(4) <u>The case where $|C \cap L| = k$.</u>

Suppose we have a line L of the partial geometry intersecting a clique C of size r in k vertices, (i.e. a point of $S(2,k,v)$ covered by k blocks of the clique).

Consider the subgraph of $BG[S(2,k,v)]$ consisting of the $r - k$ vertices of $L \setminus C$ and the $r - k$ vertices of $C \setminus L$. Now any vertex of $C \setminus L$ is already adjacent to k vertices of L (namely $L \cap C$). So it cannot be adjacent to any vertices of $L \setminus C$. In other words $C \triangle L$ is a pair of disjoint cliques of size $r - k$, with no edges between them. Label the vertices of BG so that the first $r - k$ are vertices of $C \setminus L$ and the next $r - k$ are those of $L \setminus C$. Then $\underline{u} = (\underline{j}_{r-k}, -\underline{j}_{r-k})$ is an eigenvector of the subgraph $C \triangle L$, with eigenvalue $r - k - 1$. But $r - k - 1$ is the second eigenvalue of BG, whereas $C \triangle L$ has this eigenvalue with multiplicity two. Hence we can use part (ii) of the interlacing theorem of Haemers (15), (see Chapter I), which implies that $\underline{u}' = (\underline{j}_{r-k}, -\underline{j}_{r-k}, \underline{0}_{b-2(r-k)})$ is an eigenvector of BG with eigenvalue $r - k - 1$. Hence

$$\sum_{j=1}^{r-k} a_{ij} = \sum_{j=r-k+1}^{2(r-k)} a_{ij}, \quad 2(r-k)+1 \leq i \leq b.$$

i.e. for any vertex $x \notin C \triangle L$, the number of vertices in $C \setminus L$ adjacent to x is equal to the number of vertices in $L \setminus C$ adjacent to x. (Note that this holds trivially for the vertices in $C \cap L$) .

Let a_i be the number of vertices not in $C \cup L$, adjacent to i vertices of $L \setminus C$ and i of $C \setminus L$. Then

$$\sum a_i = b - 2r + k,$$

$$\sum ia_i = (r - k)(k - 1)(r - 1), \quad \text{counting } (x,y),\ x \in L \setminus C$$
$$y \in BG \setminus (L \cup C), \quad x \sim y,$$

$$\sum i(i - 1)a_i = (r - k)(r - k - 1)(k - 1) \quad \text{counting } (x,y,z)$$
$$x,y \in L \setminus C, \quad z \sim x,y.$$

These equations will be applied in (11).

It is possible to obtain similar equations when $|C \cap L| = m < k$, by making use of the fact that the complement of $C \triangle L$ is bipartite. We have never found a use for them, however.

(5) The case $r = k$.

An $S(2,k,k^2 - k + 1)$ is a projective plane of order $k - 1$. The point-block incidence matrix N of a projective plane satisfies

$$N N^T = (k - 1) I + J \qquad \text{and} \qquad NJ = kJ \qquad (@)$$

and so N is a nonsingular matrix of size v.

A well known result of Ryser (see Hall (16)), shows, by simple matrix manipulation, that equations (@) imply

$$N^T N = (k-1) I + J \quad \text{and} \quad N^T J = kJ \ .$$

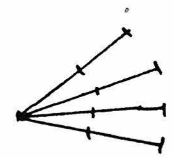
So any two blocks of a projective plane meet in a point. Hence $BG[S(2,k,k^2 - k + 1)]$ is the complete graph on $k^2 - k + 1$ vertices. From now on we exclude the case $r = k$ from our investigations, as the block graph gives no information about the design.

(6) <u>The case $r = k + 1$. ($v = k^2$, $b = k(k+1)$.)</u>

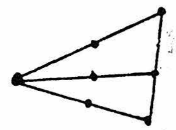
Take a point p and a block B such that $p \notin B$; then there are k blocks through p which intersect B and so one that does not. So every point $p \notin B$ lies on a unique block B' such that $B' \cap B = \phi$. There are k^2 blocks which intersect B. Hence any block B lies in a parallel class of k blocks such that any point lies in exactly one of them. Hence $BG\ S(2,k,k^2)$ is the complete $(k+1)$-partite graph with parts of size k. There are k^{k+1} cliques of size $r = k+1$, consisting of a block from each class; k^2 of them correspond to $k+1$ blocks through a point. From this description it is easy to see how to extend an $S(2,k,k^2)$, an affine plane of order k, to an $S(2,k,k^2+k+1)$, a projective plane, by adding $k+1$ more points which correspond to the parallel classes, and one more block, the block at infinity, containing these points.

(a) $k = 3$, $r = 4$, $v = 9$, $b = 12$.

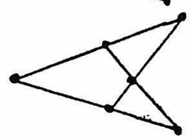
This is the unique affine plane of order 3. Its block graph has cliques of the form



9 of this type,



72 of this type,



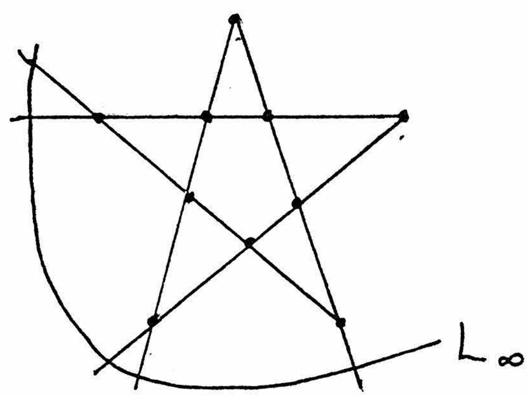
none of this type,

because, if there were, let B be the block through x,y. Then B intersects all four blocks of the clique, a contradiction.

(b) $k = 4$, $r = 5$, $v = 16$, $b = 20$.

The unique affine plane of order 4.

Now it is possible that there are cliques of the form



and they do exist, because all lines meet the line at infinity, L_∞, in distinct points (since they already meet each other), in the extension PG(2,4). Now we have a set of six lines in PG(2,4) such that any point lies in 0 or 2 of them. This is the dual of an oval in PG(2,4) and it is well known that these exist, (see, for example, Biggs and White (0)). In fact it is known that there are 168 of them and they divide into three orbits of size 56 under the action of the group PSL(3,4). The ovals in a given orbit intersect in 0 or 2 points, and by the transitivity of PSL(3,4) each point lies in 16 of these ovals which pairwise intersect in one other point. Returning to the dual set-up we have 16 dual-ovals containing L_∞ such that any two of them have one further line in common. So in our block graph we have 16 5-cliques pairwise intersecting in a vertex of the graph. Since PSL(3,4) is doubly transitive on the lines of PG(2,4), and so transitive on the lines, not L_∞, any block of AG(2,4) is at the intersection of two of our dual-ovals. Hence we have another geometrisation of the block graph. We notice that the sixteen cliques are normal cliques in the sense of section (2).

Now the 5 blocks in the clique corresponding to the dual-oval cover 10 points of AG(2,4), and so there are six remaining points. Any line of PG(2,4) must meet all the lines of the dual-oval, and so must go through 3 of the 15 points covered by it in PG(2,4). Hence it contains two of the remaining 6 points. So these six points form an oval again. In this way we have 16 ovals in PG(2,4), disjoint from L_∞, and in the same orbit under PSL(3,4). So they intersect in 0 or 2 points. Suppose O_1 and O_2 are ovals disjoint from L_∞ and from each other. There are six lines exterior to O_1, one of them L_∞, and five others, each of which contain exactly 2 of the remaining 4 points (S). Consider three of them L_1, L_2, L_3, such that L_1 meets L_2, and L_2 meets L_3 in points inside O_2. Then $L_1 \cap S$ and $L_3 \cap S$ are 2-subsets of S which are both disjoint from $L_2 \cap S$. This implies that $|L_1 \cap L_3| = 2$, a conflict. So $|O_1 \cap O_2| = 2$. Hence the 16 ovals disjoint from L_∞ form a (16,6,2) symmetric design.

Hence their complements in $AG(2,4)$ form a $(16,10,6)$ design.

This example illustrates a number of points we shall develop in Chapter IV. It is interesting to wonder whether for even k, larger than 4, one can produce the same sort of set-up. We shall show in Chapter IV that if it is possible to find a set of $4t^2$ dual-ovals which geometrise the block graph of $AG(2,2t)$, then there exists a $(4t^2,\ 2t^2+t,\ t^2+t)$ symmetric design associated with this construction. Such designs can be obtained from Hadamard matrices (see Hall(16)).

(7) <u>The case $r = k^2 - k + 1$.</u>

Let $n = k - 1$, then $v = n^3 + n^2 + n + 1$ and $S(2,k,v)$ has the same parameters as the points and lines of projective 3-space.

(a) Any clique of size r in BG, not consisting of r blocks through a point, is a subplane.

We have equality in the bound in (3) which implies that any line of the partial geometry intersects C in 0 or k vertices. So the p.b.d. on C has replication number k and block size k and so is a projective plane.

(b) If BG has $n^3 + n^2 + n + 1$ cliques corresponding to subplanes then the Steiner system is the set of points and lines of $PG(3,n)$ and so n must be a prime power.

(i) Any two subplanes have at at most one block in common. If they have more than one, they (as cliques in BG) have an edge x,y in common and this lies on one of the lines L of BG. But BG has $\lambda = (r-2) + (k-1)^2 = 2(k-1)^2 + k - 2$. Let C and D represent the subplanes. Ignoring the vertices of D for the moment, x and y are already joined to the same $2(k-1)^2 + k - 2$ vertices, (since $|C \cap L| = k$). Hence they cannot both be adjacent to any vertices not in C or L. Hence $k^2 - k + 1 \leq 2k - 2$, or $(k-2)(k-1) \leq -1$, clearly impossible.

(ii) Any edge lies in a unique subplane.

Any subplane contains $(n^2 + n + 1)(n^2 + n)/2$ edges,

BG has $(n^2 + n + 1)(n^2 + 1)(n + 1)(n^2 + n)/2$ edges.

There $n^3 + n^2 + n + 1$ subplanes with no edge in common, hence the result. So we have another geometrisation of the block graph.

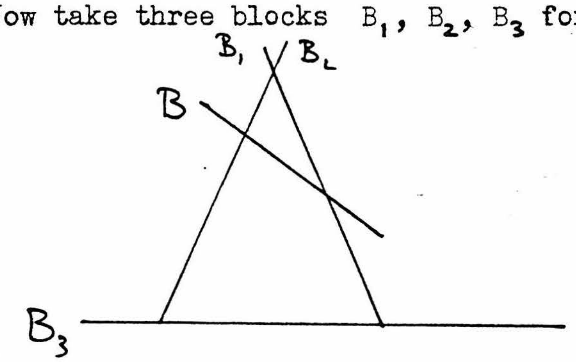
(iii) Any triangle consisting of three blocks intersecting, but not at a fixed point, lies in a unique subplane.

Any subplane contains $n^2(n^2 + n + 1)(n + 1)n/6$ such triangles,

BG has $(n^2 + n + 1)(n^2 + 1)(n^3 + n^2)(n + 1)n/6$ such triangles.

No triangle lies in more than one subplane, hence the result.

Now take three blocks B_1, B_2, B_3 forming a triangle, viz.



and suppose B intersects B_1 and B_2. Then B B_1 B_2 forms another triangle, and so they lie in some clique corresponding to a subplane D, say. Let $B_1 B_2 B_3$ lie in the clique C. Since $|C \cap D| \geqslant |\{B_1, B_2\}| \geqslant 2$, we have $C = D$, and so B intersects B_3. Therefore the blocks of our Steiner system satisfy the Pasch axiom and we have projective 3-space over $GF(n)$ (since 3-space is necessarily Desarguesian).

We have shown that the only BG of a design with these parameters which has nonunique geometrisation is that of the points and lines of projective 3-space. Moreover, there is a symmetric design between the points and planes of the design with parameters $(n^3 + n^2 + n + 1,\ n^2 + n + 1,\ n + 1)$, and the cliques corresponding to subplanes are normal cliques.

(e) Lorimer's construction.

Lorimer (21) has given a method of constructing an $S(2, n+1, n^3+n^2+n+1)$ which is resolvable, contains at least $n^2 + n + 1$ subplanes, and exists whenever there is a projective plane of order n. His method could conceivably give a large number of such designs for each $n > 2$ and will also be of use to us later because of a construction of Denniston (8), and so we will give it in some detail.

Let π be a plane of order n. For each line L of π, let G_L be a set of $n + 1$ permutations of the points of L such that;

(1) $1 \in G_L$,

(2) G_L is sharply transitive on the points of L.

This requires, in effect, the existence of a Latin square of order $n + 1$, and the cyclic group of order $n + 1$ is one possibility.

Let $X = \bigcup_{L \in \pi} G_L$,

assuming the identity in each G_L is the same, and all other permutations are distinct. So $|X| = 1 + n(n^2 + n + 1)$.

Blocks on X are of two types,

(A) $G_L : L \in \pi$,

(B) for $x \notin L$, $B_{x,L} = \{f : f^{-1}(x) \in L\}$ which we call the block determined by x and L.

So the number of blocks is $(n^2 + n + 1)(n^2 + 1)$. It is not difficult to check that these give a design with the required parameters on X. All blocks have size $n + 1$, so we only have to check that any two points lie on some block. Take $f \in G_L$, $g \in G_M$. If $M = L$ then $f, g \in G_L$. If $M \neq L$, let x be the point of intersection of L and M in π. Let N be the line through $f^{-1}(x)$ and $g^{-1}(x)$. Then $f, g \in B_{x,L}$.

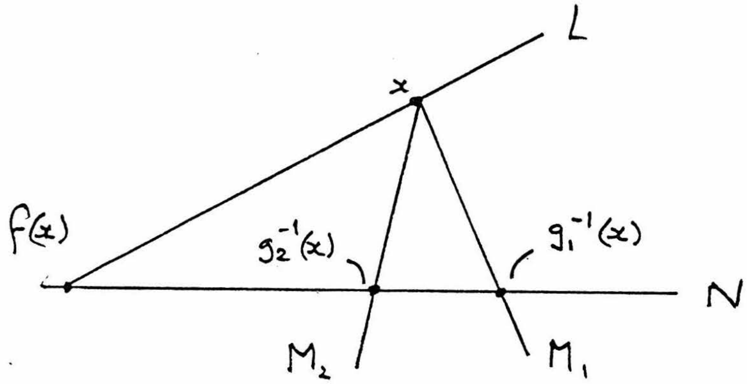
To prove the resolvability of the design we need to define a loop structure on X. First we do this for each L. Take a particular $x_0 \in L$ and define $(fg)(x_0) = f(g(x_0))$ for $f, g \in L$. This makes G_L a loop with 1 as the identity.

Now take $f \in G_L$, $g \in G_M$, $M \neq L$. Let $\{x\} = M \cap L$ and N be the line of π through $f(x)$ and $g^{-1}(x)$. Define fg by

$$(fg)(g^{-1}(x)) = f(x), \qquad \text{so} \quad fg \in G_N .$$

To show that our definition gives a loop on X, we need to show that the maps $g \longrightarrow fg$ and $g \longrightarrow gf$ are bijections. We do the first.

Let $f \in G_L \setminus \{1\}$. $g \longrightarrow fg$ is certainly a bijection on G_L . Suppose $g_1, g_2 \in X \setminus G_L$ and $fg_1 = fg_2$. Let $g_i \in G_{M_i}$ and $L \cap M_i = \{x_i\}$ for $i = 1,2$. Since $fg_1 = fg_2$, these lines must be the same, say N . But N meets L at $f(x_1) = f(x_2)$ and so $x_1 = x_2 = x$ say. So we have the following picture.

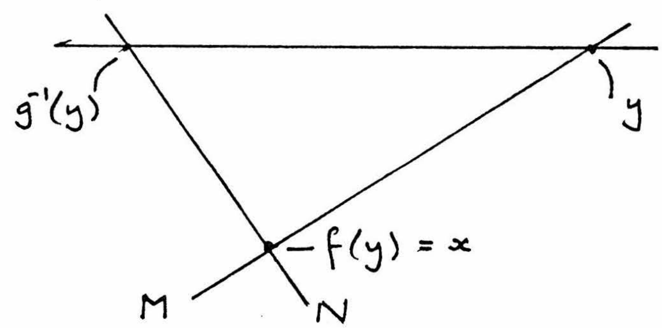


But $fg_1 = fg_2 = h$ and $h(g_2^{-1}(x)) = h(g_1^{-1}(x)) = f(x)$

$\Rightarrow g_2^{-1}(x) = g_1^{-1}(x)$ since G_N is a loop, $\Rightarrow g_2 = g_1$.

The other proof is similar. So X is a loop.

Now we consider the left cosets of the subloops G_L. Let L be a line of π , and $f \in X \setminus G_L$. Say $f \in G_M$, $M \neq L$, $M \cap L = \{y\}$. Put $x = f(y)$.



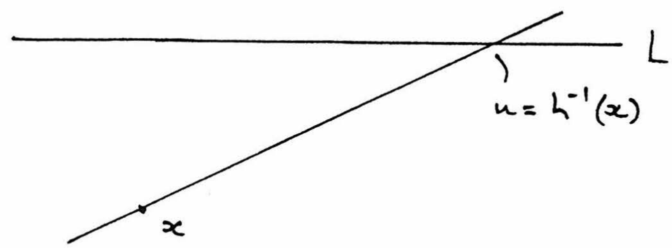
Let $g \in G_L$, and let N be the line through x and $g^{-1}(y)$.

$fg \in G_N$ is such that $fg(g^{-1}(y)) = f(y) = x$.

So $(fg)^{-1}(x) = g^{-1}(y) \in L$.

Therefore $fg \in B_{x,L}$. So $fG_L \subseteq B_{x,L}$.

Conversely let $h \in B_{x,L}$. Then $u = h^{-1}(x) \in L$.



Let $g \in G_L$ be such that $g(u) = y$. Then $h(g^{-1}(y)) = h(u) = x = f(y)$. So $h = fg \in fG_L$. Therefore $fG_L = B_{x,L}$.

Now the collection of left cosets of G_L is a partition of the elements of X, and so the corresponding blocks form a spread. Each block G_L or $B_{x,L}$ lies in one such spread, and so the design is resolvable.

Finally we show that X contains subplanes isomorphic to π. Let $x \in \pi$, and $P \leq X$ contain 1 and every permutation in X which does not fix x. Then if P contains two points from a block of X, it contains the whole block, and P is a plane isomorphic to π.

For suppose $f,g \in G_L$, $f,g \in P$. f,g do not fix x, so $x \in L$, and so for all $h \in G_L$, $h \in P$.

Suppose $f \in G_L \setminus \{1\}$, $g \in G_M - \{1\}$, $f,g \in P$, $L \neq M$, $\Rightarrow x \in L$ and M. Let $f,g \in B_{x,N}$, $x \notin N$, but for all $h \in B_{x,N}$, $h^{-1}(x) \in N$ so $B_{x,N} \leq P$.

The isomorphism $\phi : \pi \rightarrow P \leq X$ is such that $\phi(y)(y) = x$. This implies that X contains $n^2 + n + 1$ planes isomorphic to π, but it may contain more.

The simplest case of this construction is for the plane of order 2. There is only one loop of three elements, and this gives rise to $PG(3,2)$.

The next is that which arises from the plane of order 3. If we use the four permutations $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$ for each loop G_L, we do not get projective 3-space; its blocks are generated by the following base blocks (mod 13), (c.f. Denniston (8)).

$A\ B_0\ C_0\ D_0$

$B_3\ B_{10}\ D_5\ D_6$	$C_9\ C_4\ B_2\ B_5$	$D_1\ D_{12}\ C_6\ C_2$
$B_4\ B_6\ C_7\ D_{11}$	$C_{12}\ C_5\ D_8\ B_7$	$D_{10}\ D_2\ B_{11}\ C_8$
$C_{11}\ D_3\ D_9\ D_{12}$	$D_7\ B_9\ B_1\ B_{10}$	$B_8\ C_1\ C_3\ C_4$

It is invariant under the nonabelian group of order 39 , and contains 39 subplanes, images, under the group, of the points

$$A\quad B_0\quad C_0\quad D_0\quad B_6\quad C_6\quad D_6\quad B_8\quad C_8\quad D_8\quad B_9\quad C_9\quad D_9\ .$$

(one element of the group, of order 3, multiplies suffices by 3). We will make further use of this design.

One might wonder what would happen if some G_L's were given the V_4 loop structure, and some the C_4 (cyclic group) structure.

(8) <u>The case $r = k^2 - k$.</u>

We show that there are no cliques of size $k^2 - k$, except the lines in BG. Suppose there is a set of $k^2 - k$ blocks (C) not all through the same point, and pairwise intersecting. Then each block of C contains $k - 1$ points which are covered by k blocks of C, and one point covered by $k - 1$ blocks of C. But then there must be k of these $(k - 1)$-points. However no two can lie on a block of C . So there is some block $B \notin C$ containing at least two $(k - 1)$-points. Then B is adjacent to $2k - 2$ blocks of C . And $2k - 2 > k$ unless $k = 2$.

(9) Steiner systems with $k = 3$.

We are only concerned with the cases $r = 3,4,6,7$. Our previous analysis has dealt with all these, so we know all about cliques of size r in BG $S(2,3,v)$

(10) Steiner systems with $k = 4$.

We have six cases where we have to look,

$$r = 4, \quad 5, \quad 8, \quad 9, \quad 12, \quad 13,$$

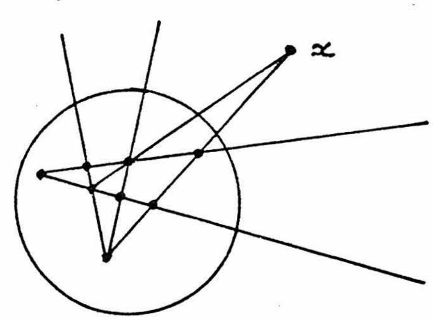
$$v = 13, \quad 16, \quad 25, \quad 28, \quad 37, \quad 40,$$

and we only have to consider the two new cases $r = 8$, and $r = 9$.

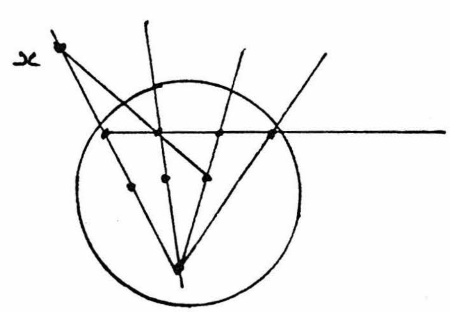
We make repeated use of the special property of the p.b.d. on C described in (2). To save much explanation we make extensive use of diagrams and indicate vertices of BG which are adjacent to more than k vertices of C (supplying our contradictions) by x. Lines through C are shown as smooth lines, where possible, and vertices as $\bullet$.

$r = 8$, $v = 25$.

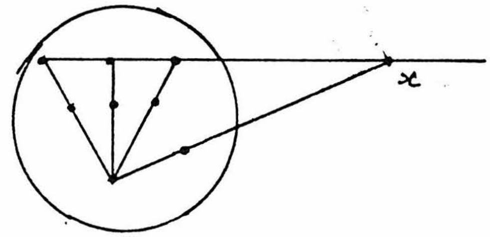
Case $n_4 = 2$.



$n_4 = 1$.



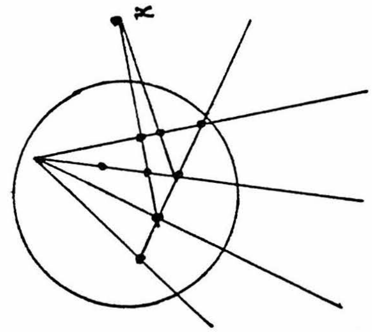
These constructions are the only possible with 4-lines, so $n_4 = 0$.



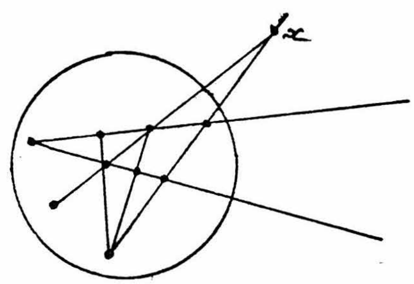
Hence $BG[S(2,4,25)]$ has no other cliques of size 8.

<u>$r = 9$, $v = 28$.</u>

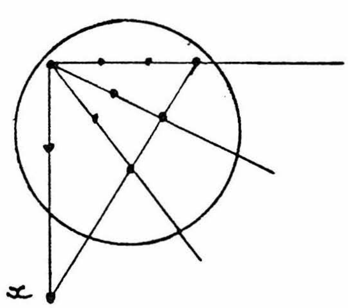
$n_4 = 3$



$n_4 = 2$



$n_4 = 1$



$n_4 = 0$, implies that all vertices lie on four 3-lines, and so the p.b.d. on C is AG(2,3), but this has disjoint blocks of size 3. So BG[S(2,4,28)] has no other cliques of size 9.

(11) Steiner systems with $k = 5$.

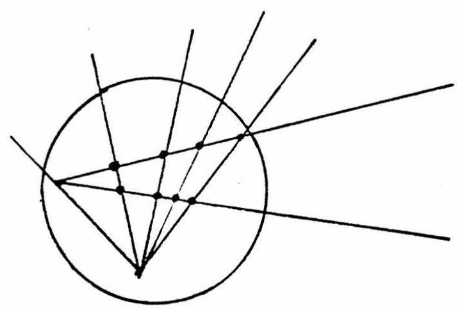
We have eight cases of interest:

$r =$	5,	6,	10,	11,	15,	16,	20,	21,
$v =$	21,	25,	41,	45,	61,	65,	81,	85,

and need only consider the four new cases, $r = 10,11,15,16$.

(a) $r = 10,\ v = 41$.

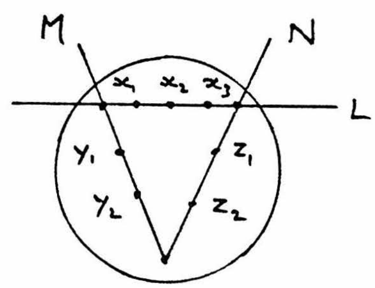
$n_5 = 2,$



Take a 3-line. Then there are only six 2-lines disjoint from it inside the clique, whereas there would have to be seven. There are only two 1-lines and they already meet inside C.

$n_5 = 1,$

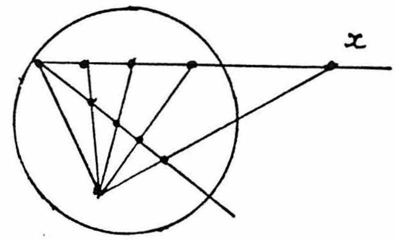
$n_4 = 2,$



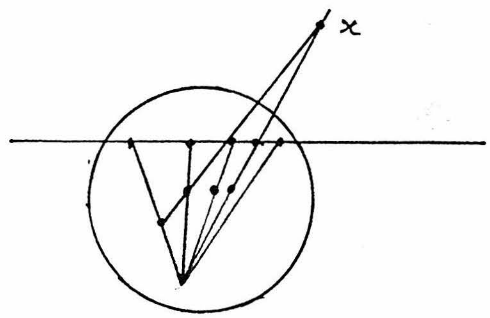
Let x_1, x_2, x_3 be the remaining vertices on L the 5-line. Let y_1, y_2 and z_1, z_2 be the remaining vertices on M,N (4-lines) respectively. Without loss of generality we have lines $\{x_2, z_1, y_2\}$, $\{x_2, z_2, y_1\}$,

$\{x_3, z_2, y_2\}$, $\{x_3, z_1, y_1\}$, (since the line containing $x_1 y_j$ must intersect N and does so in z_1 or z_2). But now the same applies for the linecontaining $x_1 y_1$, but there is no point of N where it can intersect.

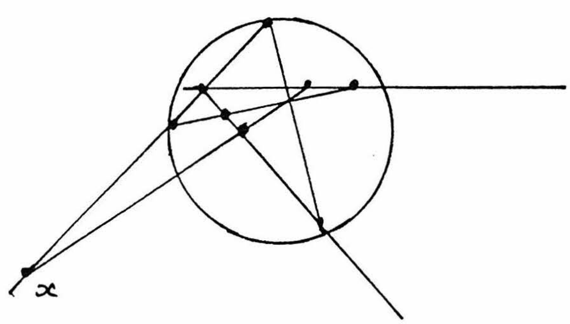
$n_5 = 1$,
$n_4 = 1$,



$n_5 = 1$,
$n_4 = 0$,

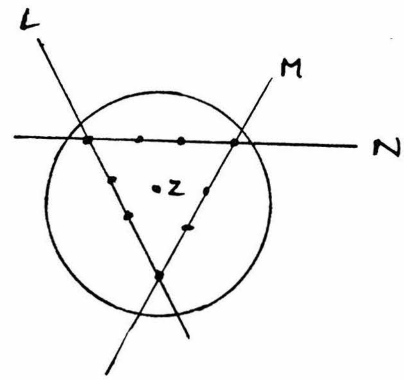


$n_5 = 0$,
$n_4 = 4$,



$n_4 = 3$,

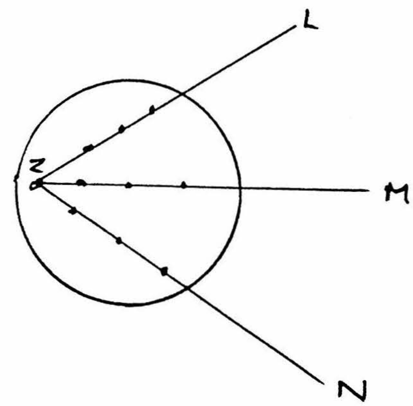
first case,



Any line through z must intersect L, M, and N. There is one line through each of $L \cap M$, $L \cap N$, $M \cap N$, and one other. But this one will have four vertices on it. We have already dealt with the case $n_4 = 4$.

$n_4 = 3$,

second case,



Now any other line must intersect each of L, M, and N. So all other lines, not through z, are 3-lines. If we delete z, the restriction to the nine remaining vertices of the clique must be AG(2,3). But this has disjoint 3-lines.

At this point we can use equations A of section (2).

$$n_0 + n_1 + n_2 + n_3 + n_4 + n_5 = 41,$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 = 50,$$

$$n_2 + 3n_3 + 6n_4 + 10n_5 = 45.$$

If $n_5 = 0$, and $n_4 \leq 2$,

then $n_1 + n_2 \leq 9$, $\Rightarrow n_2 \leq 9$, $\Rightarrow n_3 \geq 8$.

But the maximum number of pairwise intersecting 3-sets is seven.

(b) <u>$r = 16$, $v = 65$</u>.

If $n_5 = 0$, any vertex of C must lie on five 4-lines, (since there are 15 remaining vertices). But then we must have AG(2,4) on the 16 vertices, which is impossible because this has disjoint blocks.

So $n_5 > 0$.

Now we make use of equations B of section (4).

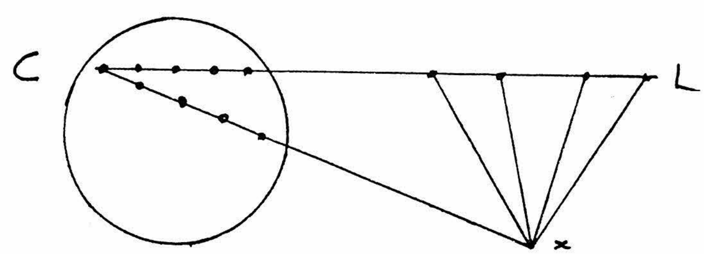
$$a_0 + a_1 + a_2 + a_3 + a_4 = 181,$$

$$a_1 + 2a_2 + 3a_3 + 4a_4 = 660,$$

$$a_2 + 3a_3 + 6a_4 = 880.$$

$$a_1 + a_2 - 2a_4 = -220 \Rightarrow a_4 \geq 110.$$

A 4-vertex can only arise from a 5-line in C in the following manner,



(since four of the lines on which x lies must intersect L outside C).

Each 5-line contains eleven 4-vertices,

$$\Rightarrow n_5 \geq 110/11 + 1 = 11.$$

Each vertex of C can lie on at most three 5-lines,

$$\Rightarrow n_5 \leq 3.16/5 < 10, \text{ a contradiction.}$$

(c) $\underline{r = 15, \quad v = 61}$.

Suppose $n_5 > 0$. Applying equations (B),

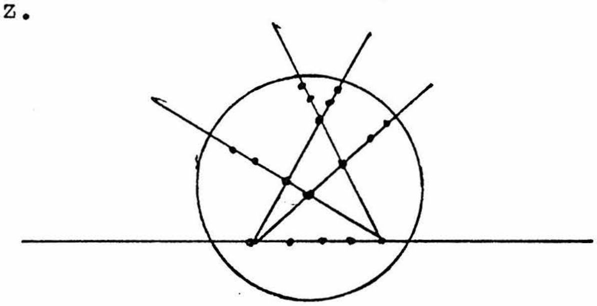
$$a_0 + a_1 + a_2 + a_3 + a_4 = 158,$$
$$a_1 + 2a_2 + 3a_3 + 4a_4 = 560,$$
$$a_2 + 3a_3 + 6a_4 = 720.$$

$\Rightarrow a_4 \geqslant 80, \Rightarrow n_5 \geqslant 9.$

But any vertex lies on at most three 5-lines. So $n_5 \leqslant 15.3/5 = 9$.

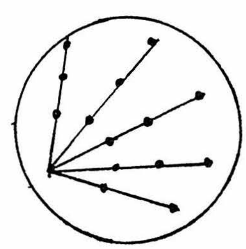
Therefore $n_5 = 9$,

and any vertex lies on exactly three 5-lines. Consider two vertices on a 5-line, viz.



This gives at least seventeen vertices in C.

So $n_5 = 0$.



This is the only possible set-up for the five lines through a vertex. There must then be five 3-lines such that any vertex lies on exactly one of them. This means that there are disjoint 3-lines.

(d) $r = 11, \quad v = 45$.

The most difficult case.

(i) Suppose $n_5 > 0$, i.e. $n_5 = 1$ or 2.

Applying equations (B),

$$a_0 + a_1 + a_2 + a_3 + a_4 = 82,$$

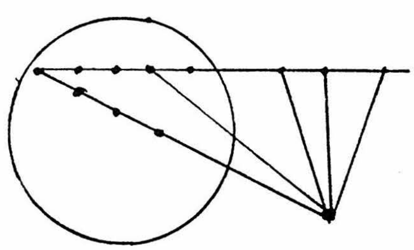
$$a_1 + 2a_2 + 3a_3 + 4a_4 = 240,$$

$$a_2 + 3a_3 + 6a_4 = 240,$$

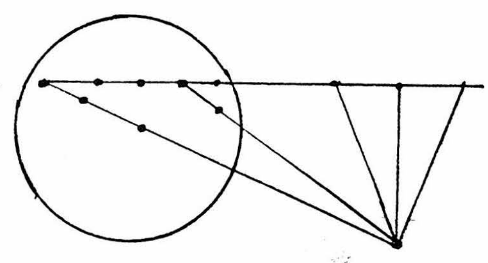
and $a_4 = 0$ or 6 (since $n_5 = 1$ or 2).

If $n_5 = 1$, then $a_4 = 0 \Rightarrow a_1 + a_2 = 0, \quad a_3 = 80, \quad a_0 = 2.$

We have two possible sorts of 3-vertices,



x of these vertices,



y of these vertices.

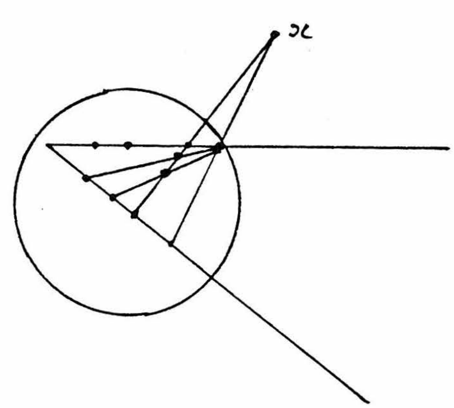
So $n_5 = 1, \quad n_4 = x/7, \quad n_3 = y/8, \quad n_2 = y/9,$

$\Rightarrow 72 \mid y, \quad 7 \mid x,$ and $x + y = 80$.

This is impossible.

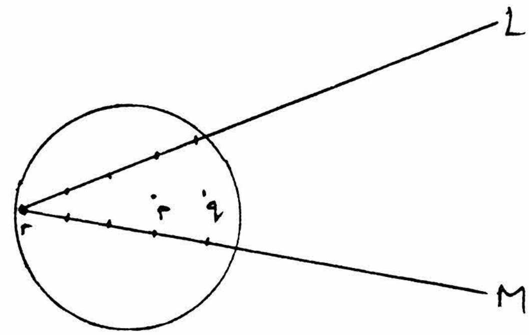
$n_5 = 2,$

$n_4 = 1,$



$n_5 = 2,$

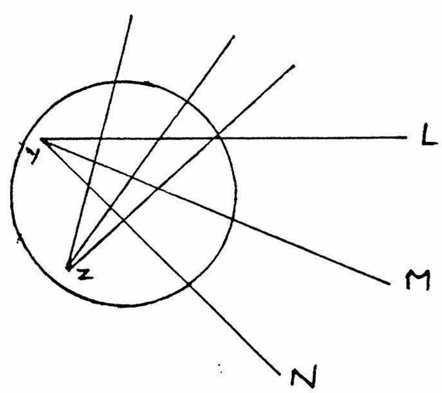
$n_4 = 0,$



Let p and q be the two vertices not in L or M , and let $r = L \cap M$. Put $\{y_i\}_{i=1}^{4}$, $\{z_i\}_{i=1}^{4}$ as the remaining vertices in L,M respectively. Then, without loss $\{y_i , p , z_i\}$ are the 3-lines through p. Now $\{y_1 , q , z_j\}$ must be a 3-line for some j. Suppose, without loss, that $j = 2$. Then $\{y_3 , p , z_3\}$ is disjoint from this.

Hence $n_5 = 0$.

Suppose we have three 4-lines through a vertex y.



Then there is one remaining vertex z and every line through z must intersect L, M, and N. Hence we have another three 4-lines through z. Any line not through y or z must intersect each of L, M, and N, and so must be a 3-line. The restriction of the lines to these nine vertices must therefore be AG(2,3). This has disjoint 3-lines and a pair can be found which do not correspond to any of our 4-lines.

Hence any vertex has at most two 4-lines through it and so $n_4 \leq 5$. Applying equations (A),

$$n_0 + n_1 + n_2 + n_3 + n_4 = 45,$$
$$n_1 + 2n_2 + 3n_3 + 4n_4 = 55,$$
$$n_2 + 3n_3 + 6n_4 = 55,$$

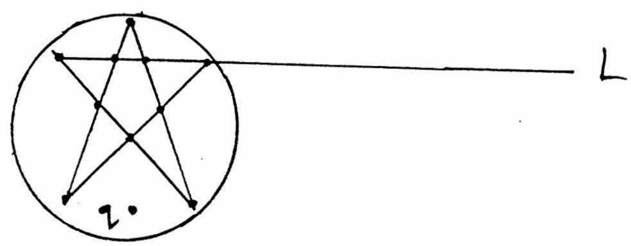
$\Rightarrow \quad n_1 + n_2 = 2n_4$.

If $\quad n_4 \leq 4$, then $\quad n_1 + n_2 \leq 8, \quad n_2 \leq 8,$

$\Rightarrow \quad 3n_3 \geq 23, \quad$ or $\quad n_3 > 7$.

But any two of these 3-lines must intersect in C, which is impossible if there are more than 7.

If $\quad n_4 = 5 \quad$ we have,



One vertex q is not covered by any of the 4-lines. It must lie on five 3-lines. But then one of these cannot intersect L in C.

Hence we know all about maximal cliques of size r in BG S(2,5,v) . We make use of the unique geometrisation of BG[S(2,5,45)] in Chapter V.

(12) Normal Cliques.

In (3) we defined a normal clique C. Any point of an $S(2,k,v)$ containing a normal clique C is covered by 0 or $k_0 = \frac{r+k-1}{k}$ blocks of C.

We must have $k_0 \mid k$. Any line in the block graph is either a 0-line or a k_0-line. Any vertex outside C is adjacent to k vertices of C and so must lie on (k/k_0) k_0-lines.

The k_0-lines form a p.b.d. on C with constant block size, i.e. we have an $S(2,k_0,r)$ on the vertices of the clique.

Reverting to the $S(2,k,v)$, we have that any block not in C must contain (k/k_0) points covered by the blocks of C. Let S be the set of points covered by no blocks of C. Then any block not in C must contain $k - (k/k_0) = n$ of these points, and of course any block in C contains none of them. And so S is a maximal n-arc of the design. We have stumbled on a set-up previously investigated by Morgan (23).

We note that k and k_0 determine the parameters of the design. $k = k_0$ implies that we have a subplane in the design with $r = k^2 - k + 1$.

$k_0 = 2$ implies that we have a $k/2$-arc in $AG(2,k)$ where k is even.

(a) $k_0 = 3$, $k = 6$ requires the existence of an $S(2,6,66)$ which has recently been constructed by Denniston (8), in such a way that the induced design on the 4-arc is the $S(2,4,40)$ given by Lorimer and described in (7c). We give the blocks of the $S(2,6,66)$ below, with suffices generated (mod 13).

$$A\ B_0\ C_0\ D_0\ Y_0\ Z_0$$

$B_3\ B_{10}\ D_5\ D_6\ Y_0\ Y_2$	$C_9\ C_4\ B_2\ B_5\ Y_0\ Y_6$	$D_1\ D_{12}\ C_6\ C_2\ Y_0\ Y_5$
$B_4\ B_6\ C_7\ D_{11}\ Y_0\ Z_1$	$C_{12}\ C_5\ D_8\ B_7\ Y_0\ Z_3$	$D_{10}\ D_2\ B_{11}\ C_8\ Y_0\ Z_9$
$C_{11}\ D_3\ D_9\ D_{12}\ Z_0\ Z_3$	$D_7\ B_9\ B_1\ B_{10}\ Z_0\ Z_3$	$B_8\ C_1\ C_3\ C_4\ Z_0\ Z_9$

$$Y_2\ Y_6\ Y_5\ Z_4\ Z_{10}\ Z_{12}$$

The thirteen blocks $\{Y_2, Y_6, Y_5, Z_4, Z_{10}, Z_{12}\} + i$, pairwise intersect and form our normal clique C covering the 26 points $\{Y_i, Z_i : i = 0,12\}$. There are also at least 39 other maximal cliques in the design corresponding to the 39 subplanes in the $S(2,4,40)$, and the blocks of each of these cover 39 points.

We might ask whether it is possible to construct an $S(2,6,66)$ using $PG(3,3)$ as our 4-arc. It does not seem easy to do this, (9).

To be able to construct an $S(2,k,v)$ with a normal clique, it is necessary to find a set of kr/k_0 spreads in the $S(2,n,v - rk/k_0)$ on the n-arc such that any block lies in k/k_0 of these spreads. We might call this a k/k_0-packing.

(b) $k_0 = 4$, $k = 8$ requires the existence of an $S(2,8,176)$ which has a maximal 6-arc on 126 points. There are at least two known $S(2,6,126)$ and one of these is defined on the 126 isotropic points of a unitary bilinear form on $PG(2,25)$.

We consider this design in some detail. Let $(x,y,z) \in GF(25)^3$ be a representative of a projective point in $PG(2,25)$. Then this point is said to be isotropic if $x\bar{x} + y\bar{y} + z\bar{z} = 0$, where $\bar{x} = x^5$ is the Frobenius automorphism of $GF(25)$. Clearly this is a good definition.

Any line of $PG(2,25)$ can be written in the form $\{(x,y,z): ax+by+cz=0\}$. Let $[a,b,c]$ represent this line. Then by a slight abuse of language we will call a line isotropic if the point (a,b,c) is. Clearly, by this duality between points and lines, there are the same number of isotropic points as lines.

The group $PGU(3,5)$ of automorphisms of $PG(2,25)$ preserving the unitary form is doubly transitive on the isotropic points. Let $u \neq v$ be isotropic points. $\langle u,v\rangle = \sum u_i \bar{v}_i \neq 0$, otherwise we would have a 2-dimensional totally isotropic subspace of a 3-dimensional nonsingular space. So by normalisation we can assume $\langle u,v\rangle = 1$. Let w be such that u,v lie in the line $[\bar{w}]$, i.e. $\langle u,w\rangle = \langle v,w\rangle = 0$. Then u,v and w are independent and span the space. Again $\langle w,w\rangle \neq 0$, otherwise we would have a totally isotropic 2-dimensional subspace. So we have

$$\langle u,u\rangle = \langle v,v\rangle = 0, \quad \langle u,v\rangle = \langle w,w\rangle = 1, \quad \langle u,w\rangle = \langle v,w\rangle = 0.$$

Any pair of distinct isotropic points can be extended to a basis in this manner and there is clearly a unitary mapping taking one basis to another.

In a similar manner $PGU(3,5)$ is transitive on the nonisotropic points and lines.

Let us abbreviate to n-points, -lines for nonisotropic, and i-points, -lines for isotropic.

Let $w \in GF(25)$ be such that $w^{12} = 1$, $w^6 = -1$. Consider the n-line $[1,0,0]$. Suppose it contains an isotropic point $(0,1,z)$. Then this is isotropic if and only if $1 + z^6 = 0 \Leftrightarrow z^6 = -1$,

$$\Leftrightarrow z \text{ is a primitive 12th root of unity,}$$

$$\Leftrightarrow z = w^i, \quad i \in \{1,3,5,7,9,11\}.$$

So any i-line contains 6 i-points.

Consider the i-line $[0,1,w]$. It contains the i-point $(0,1,w)$. Any other point (x,y,z) on the line is such that $y + wz = 0$, or $y = -wz$. Then $x^6 + y^6 + z^6 = x^6 + w^6z^6 + z^6 = x^6$, which is only zero if $x = 0$. So any i-line contains one i-point.

Suppose there are A i-points,lines, and B n-points,lines. Then, since any n-line contains 6 i-points, and any i-point lies on 25 n-lines,

$$6B = 25A \qquad \text{also} \qquad A + B = 651,$$

$$\text{so} \quad A = 126, \quad \text{and} \quad B = 525.$$

By the double transitivity of $PGU(3,5)$ on I, the set of 126 i-points, the restrictions of the 525 n-lines to I form an $S(2,6,126)$.

Furthermore $(1,0,0)$ lies on 20 n-lines whose restrictions are therefore disjoint in I, and the remaining six points of I, not on any of these 20 lines are $(0,1,w^i)$ where $i \in \{1,3,5,7,9,11\}$ and these all lie on the line $[1,0,0]$. So to any n-point there corresponds a spread of 21 blocks of I. Finally, considering the 25 n-points on an i-line, we obtain a 1-packing of I.

However, for our construction we require a 2-packing of 50 spreads, such that no spreads have more than one line in common.

Consider the i-line $x = 2y$, i.e. $[1,-2,0]$.
It contains 25 n-points $(2,1,x)$, $x \neq 0$, and $(0,0,1)$.
$\langle(2,1,x),(2,1,y)\rangle = 0 \iff x\bar{y} = 0 \iff x = 0$ or $y = 0$.
So if p,q are n-points on the same i-line then $\langle p,q\rangle \neq 0$, and so the spreads $S(p)$ and $S(q)$ have no lines in common. So if p and q are n-points such that $\langle p,q\rangle = 0$, then p and q lie on some n-line

$[r]$ and furthermore $q \in [p] \Rightarrow [p] \in S(q)$ and $[q] \in S(p)$ similarly. So $S(p)$ and $S(q)$ have two lines in common. This shows that we cannot find a set of fifty spreads with our required property, which all correspond to n-points. For suppose we could. Let $S(p)$ be one of these, and consider the n-line $[p] \in S(p)$. Then $[p]$ must lie in another spread, say $S(q)$. But then $\langle p,q \rangle = 0$, which implies that $S(p)$ and $S(q)$ have two lines in common.

So to construct an $S(2,8,176)$ from this design we need to find other spreads. I have not been able to learn of the existence of any.

(13) Steiner Systems with $k \geq 6$.

The methods for $k = 3,4,5$ used in earlier sections in this chapter become very long and tedious for $k \geq 6$.

CHAPTER IV.

$BG[S_\lambda(2,k,v)]$ FOR GENERAL λ, AND NONUNIQUE GEOMETRISATION.

(1) $BG[S_\lambda(2,k,v)]$ for $\lambda > 1$.

When $\lambda > 1$ there is no guarantee that the blocks of the design have only two nontrivial intersection numbers. In fact if N is the point-block $(v \times b)$ incidence matrix of the design,

$$N N^T = (r - \lambda)I + \lambda J = \lambda\left[\frac{(v-k)}{(k-1)} I + J\right].$$

$$N^T N = \sum_{i=0}^{k} iB_i \,, \qquad \text{where} \qquad \sum_{i=0}^{k} B_i = J.$$

Considering the design as a subset of the Johnson scheme $J(v,k)$, the B_i's are adjacency matrices of subgraphs of the Johnson relations A_{k-i} and B_k is the $b \times b$ identity matrix (assuming that there are no repeated blocks).

In general we cannot define an association scheme on the blocks of the design, (as we could in Chapter III), but it is possible sometimes, and if the design has only two intersection numbers we are guaranteed a strongly regular graph. This is a special case of a theorem in Chapter I, section(7). Such 2-designs are called quasi-symmetric and were introduced and investigated by Goethals and Seidel (14). We shall consider these first, because of their greater structure.

(2) Quasi-symmetric designs.

We give a proof, found in (5), that the B_i's form a 2-class association scheme. Let $x < y$ be the intersection numbers of the design.

$$N^T N = x(J - I - B) + yB + kI,$$

hence $$B = \frac{N^T N - (k-x)I - xJ}{y - x},$$

and so B has eigenvalues,

$$[rk - (k-x) - xv]/(y-x) \qquad \text{once},$$

$$[r - \lambda - (k-x)]/(y-x) \qquad v - 1 \text{ times},$$

$$-(k-x)/(y-x) \qquad b - v \text{ times}.$$

Therefore B is the adjacency matrix of a strongly regular graph since it generates a 3-dimensional algebra.

Since the eigenvalues are integers, we must have $(y - x) \mid (k - x)$ and $(y - x) \mid (r - \lambda)$. The first of these conditions is similar to the second part of the theorem of Deza, Erdös and Frankl (see Chapter II). We wonder whether this holds more generally for designs.

Suppose $x > 0$, then any two blocks intersect, so we are not going to be able to extend the Erdös, Ko, Rado theorem to this case. However consider a block A, and count (p_1, p_2, B) such that $p_1, p_2 \notin A$, $p_1, p_2 \in B$, $|B \cap A| > 0$.

$$\lambda(v - k)(v - k - 1) \leqslant rk(k - 1)(k - 2) ,$$

which implies that $v \leq k^2 - 1$, and is a very generous bound. So there are no quasi-symmetric designs with $x > 0$ and $v > k^2 - 1$.

If $x = 0$, we look at the set of r blocks through a point p of the design. Any two intersect in y points, and (ignoring the case $y = 1$, which only occurs if it is a Steiner system), they intersect in $y - 1$ points apart from p. So there is a $S_{y-1}(2, \lambda, r)$ on the blocks through p, and therefore,

$$(y - 1)(r - 1) = (\lambda - 1)(k - 1), \quad \text{and} \quad \lambda \leqslant k - 1 \text{ (Fisher's inequality).}$$

$$\Rightarrow v = 1 + r(k - 1)/\lambda = 1 + \left[1 + \frac{(\lambda - 1)(k - 1)}{(y - 1)}\right] \frac{(k - 1)}{\lambda}$$

$$\leqslant k^2 - k + 1.$$

So if $v > k^2 - 1$, there are no quasi-symmetric designs with block-size k on v points.

However we can do better in the case when $x = 0$. We need $y \mid k$. So if $y > 1$ and k is a prime then there are no quasi-symmetric designs of block-size k.

We are interested in families of pairwise intersecting blocks. Using the clique bound of Chapter I,

$$w \leq 1 + d/-\lambda_2 = 1 + \frac{(r-1)ky}{yk} = r ,$$

and so again such families have maximal size r, and by an argument similar to III(1), cliques of size r are regular with nexus

$$\frac{k(k-y)\lambda(\lambda-1)}{(r-\lambda)y(y-1)} .$$

The block graphs of Hadamard 3-designs (these are quasi-symmetric 2-designs) are the complements of a 1-factor, and so it is easy to find families of r pairwise intersecting blocks, in this case.

Note: There are on the whole very few quasi-symmetric designs. We list possible parameter values for $k \leq 12$, $x = 0$, $y > 1$.

$S_3(2,4,8)$ $y = 2$, $S_5(2,6,22)$ $y = 2$, $S_5(2,6,12)$ $y = 3$, $S_4(2,6,21)$ $y = 2$,
$S_7(2,8,16)$ $y = 4$, $S_4(2,9,27)$ $y = 3$, $S_9(2,10,20)$ $y = 5$, $S_6(2,10,70)$ $y = 2$,
$S_{11}(2,12,24)$ $y = 6$, $S_{11}(2,12,57)$ $y = 3$, $S_5(2,12,100)$ $y = 2$, $S_{11}(2,12,112)$ $y = 2$.

(3) Cliques of size r in BG "isomorphic" to the set of r blocks through a point.

The set of r blocks through a point have certain properties when regarded as a clique of size r in the graph with adjacency matrix

$$\sum_{k > i > 0} B_i .$$

For instance, for any block B in the clique C,

$$\sum_{\substack{A \neq B \\ A \in C}} |A \cap B| = \left|\{(p,A) : p \in A \cap B\}\right|$$

$$= (r - 1) + (\lambda - 1)(k - 1).$$

When considering nonunique geometrisation of the block graph we shall be looking for such cliques of size r.

Let x_p be the number of blocks of C containing a particular point p of the design. Let h be the number of points covered by at least one block of C. Then,

$$\sum 1_p = h,$$

$$\sum x_p = rk,$$

$$\sum x_p(x_p - 1) = \sum_{B_1, B_2 \in C} |B_1 \cap B_2| = r\left[(r - 1) + (\lambda - 1)(k - 1)\right],$$

$$\Rightarrow \sum x_p(x_p - \lambda) = r(r - \lambda).$$

Now by analogy to equations A when $\lambda = 1$ in III(2), we have

$$\frac{1}{\lambda^2} \sum \left[x_p - \lambda\left(\frac{(v - 1) + (k - 1)^2}{(k - 1)k}\right)\right]^2 = h - \frac{(v - 1)k^2}{(v - 1) + (k - 1)^2}.$$

So h is a minimum of $\dfrac{(v-1)k^2}{(v-1)+(k-1)^2}$

when $x_p = \lambda\left[\dfrac{(v-1)+(k-1)^2}{(k-1)k}\right]$ for any point p covered by the blocks of C. And so we have again characterized normal cliques.

(4) Nonunique geometrisation in $BG[S_\lambda(2,k,v)]$

$BG[S_\lambda(2,k,v)]$ has a set of v cliques of size r corresponding to the sets of r blocks through points of the design. These cliques have the property that any two of them have λ blocks (vertices) in common, and each edge of B_i is contained in exactly i of them. They are said therefore to geometrise the block graph. We want to know whether we can find another set of v cliques with these properties.

Suppose we can. Then we can form the $v \times b$ incidence matrix N_2 of these cliques against blocks, i.e.

$$N_2(C,B) = \begin{cases} 1 & \text{if } B \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Let N_1 be the point-block incidence matrix of the design. Then

$$N_2^T N_2 = \sum_{i=0}^{k} iB_i = N_1^T N_1 \quad ,$$

$$N_2 N_2^T = (r-\lambda)I + J = \lambda\left[\frac{(v-k)}{(k-1)} I + J\right] = N_1 N_1^T \ .$$

Now consider the $v \times v$ matrix,

$$X = N_1 N_2^T ,$$

and so $X(p,C)$ = the number of blocks of the design which contain p and lie in the clique C.

$$X X^T = N_1 N_2^T N_2 N_1^T = N_1 \left(\sum_{i=0}^{k} iB_i \right) N_1^T$$

$$= N_1 N_1^T N_1 N_1^T = \left((r - \lambda) I + \lambda J \right)^2$$

$$= \lambda^2 \left[\left(\frac{v - k}{k - 1} \right)^2 I + \left(v + \frac{2(v - k)}{k - 1} \right) J \right] .$$

Therefore X satisfies a matrix equation similar to that satisfied by the incidence matrix of a symmetric design. In the particular case when all the new geometrising cliques are normal,

$$X(p,C) = \lambda k_0 = \lambda \left[\frac{(v - 1) + (k - 1)^2}{k(k - 1)} \right] \text{ if } p \text{ is covered by a block of } C.$$

$$= 0 \quad \text{otherwise.}$$

Then $Y = X/k_0\lambda$ is a $(0,1)$-matrix and satisfies

$$Y Y^T = \frac{1}{k_0^2} \left[\left(\frac{v - k}{k - 1} \right)^2 I + \left(v + \frac{2(v - k)}{k - 1} \right) J \right] .$$

In other words, there is a symmetric design governing the incidence between points and cliques, and the parameters of this design depend only on v and k and are independent of λ.

Note: We say a point p and a clique C of a design are incident if there is a block B such that $p \in B \in C$.

Let k_0 and $n = k/k_0$ be given. Then $k = nk_0$, and $r = \lambda(nk_0(k_0 - 1) + 1)$, $v = \lambda(nk_0(k_0 - 1) + 1)(nk_0 - 1) + 1$.

$$\text{Then} \quad Y Y^T = n^2(k_0 - 1)^2 I + \left[n^2(k_0 - 1) + n\right] J .$$

In this instance there is nothing to be gained by applying the Bruck, Ryser, Chowla conditions, because the equation is always satisfied by

$$Y = \frac{1}{\lambda k_0}\left[(r - \lambda) I + \lambda J\right] .$$

(5) Nonunique geometrisation by normal cliques in $BG[S(2,k,v)]$.

We know that for some k, for $r = k + 1$ and $r = k^2 - k + 1$, an $S(2,k,v)$ exists whose block graph is geometrisable in more than one way and the cliques of the second geometrisation are normal with respect to the first and vice versa, (III(6) and (7)). Furthermore in both cases there is a symmetric design between the points and the cliques.

We examine the possibility of such an alternative geometrisation by normal cliques for other values of r (as a function of k) beginning with the case $(2,k,v) = (2,6,66)$. We prove that there is no $S(2,6,66)$ whose block graph has this structure, although as we have seen, there is a design with these parameters containing at least one normal clique.

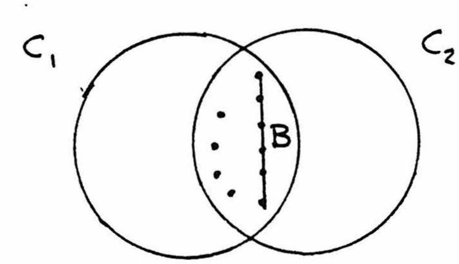
(a) Proposition: Any point-block pair (p,B) such that $p \notin B$, lies in exactly k/k_0 cliques.

Proof: There are k cliques which contain B and each of these contains 0 or k_0 blocks through p. But for any block A through p which meets B, (there are k such), there is a unique clique containing B and A. They meet in a point, so they meet in a clique. So there are exactly k/k_0 cliques containing p and B.

Note: This is the dual of the fact that there are k/k_0 points covered by the blocks of a normal clique C, which lie in any block $B \notin C$.

(b) So, for $k = 2k_0$, consider the k cliques containing a block. Take one of these, C_1 say, and consider the intersections of the $k - 1$ other cliques with this one. They all contain the block, and any other point covered by C_1 is contained in exactly one other clique.

Let us now consider an $S(2,6,66)$ and suppose its block graph has another geometrisation by normal cliques. Then the point-clique incidence is governed by a $(66,26,10)$ symmetric design. So any two cliques have exactly 10 points in common, (i.e. there are 10 points covered by the blocks of both). They also have one block in common, so we have the following.



C_1 and C_2 have the block B in common and four other points x_1, x_2, x_3, x_4. Suppose x_i and x_j lie on a block A which is in C_k ($i \neq j \in \{1,2,3,4\}$, $k = 1$ or 2). Then A must intersect B. But then A contains at least three points of C_1, and so $A \in C_1$. Similarly $A \in C_2$. In which case C_1 and C_2 have two blocks in common, a contradiction. Hence no two of x_1, x_2, x_3, x_4 lie in a block of C_1 or of C_2.

The dual design on the blocks of C_1 is an $S(2,3,13)$. There are 26 3-lines corresponding to the points covered by C_1. Now x_1, x_2, x_3, x_4 lie in no block of C_1, and so correspond to four disjoint 3-lines in the $S(2,3,13)$. The remaining vertex is that corresponding to B. There are 5 cliques apart from C_1 which contain B and any point covered by C_1 lies in exactly one of them. So in the $S(2,3,13)$ each of $C_2, \ldots\ldots, C_6$ gives rise to a set of four disjoint 3-lines which do not contain x_B. In other words the set of twenty 3-lines not containing x_B can be divided into 5 parallel classes of four 3-lines. This can be done for any x_B in C .

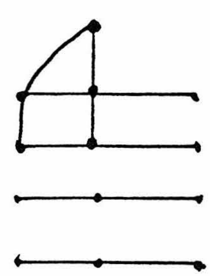
(c) We must now check that neither of the two nonisomorphic $S(2,3,13)$'s (see Hall(16)), have this special resolvability property. One such $S(2,3,13)$ has blocks as given in the first two columns following and the other has the blocks of the second two columns.

A		B	
0 2 8	1 5 4	0 2 8	e 5 4
1 3 9	2 6 5	1 3 5	2 6 5
2 4 t	3 7 6	2 4 t	3 7 6
3 5 e	4 8 7	4 9 1	4 8 7
4 6 w	5 9 8	4 6 w	5 9 8
5 7 0	6 t 9	5 7 0	6 t 9
6 8 1	7 e t	6 8 1	7 e t
7 9 2	8 w e	7 9 2	8 w e
8 t 3	9 0 w	8 t 3	9 0 w
9 e 4	t 1 0	9 e 3	t 1 0
t w 5	e 2 1	t w 5	e 2 1
e 0 6	w 3 2	e o 6	w 3 2
w 1 7	0 4 3	w 1 7	0 w 3

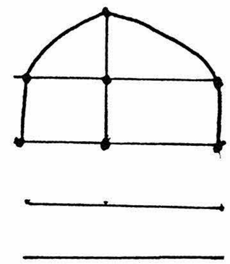
We consider the twenty blocks which do not contain the point 6. It is possible to find parallel classes of four blocks, i.e. 1 3 9, 2 4 t, e 8 w, 0 5 7, in A.

First we need a proposition,

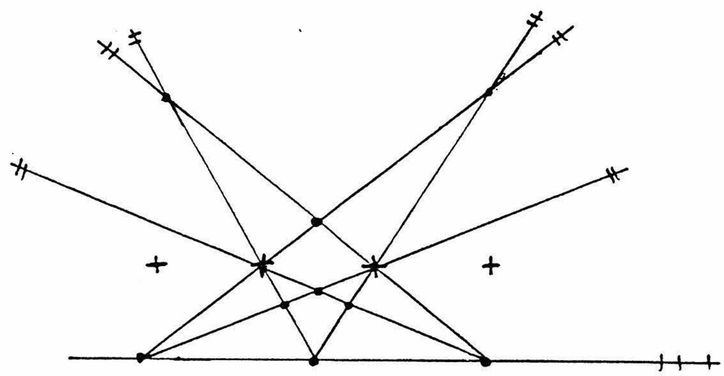
<u>Proposition</u>: Given four disjoint blocks of an $S(2,3,13)$, the configuration below cannot occur.



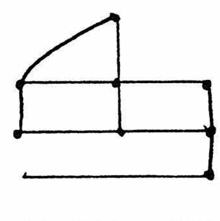
Proof: First possibility;



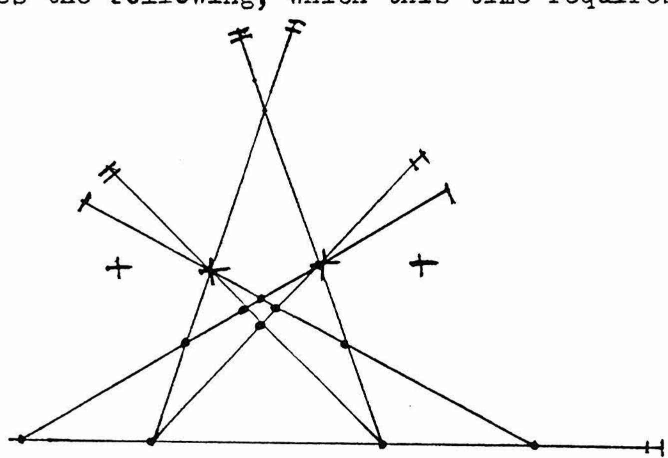
Dualising gives the following set-up, which requires at least 28 dual-points, i.e. 28 blocks in the design.



Second possibility;



Dualising gives the following, which this time requires 27 blocks.



Hence neither of these configurations can occur.

Consider the blocks of A first, and the block 7 9 2. The three blocks through 6 which intersect 7 9 2 are 6 7 $\underline{3}$, 6 9 $\underline{t}$, and 6 2 $\underline{5}$. So 3 , t , and 5 must lie in different blocks of the parallel class containing 7 9 2 , by the proposition. The only block containing 3 , disjoint from 7 9 2 , and not containing 6 , t , or 5 is 3 0 4 . But now all the blocks through t intersect 7 9 2 or 3 0 4 , or contain 5 . Hence there is no parallel class containing 7 9 2.

Consider now the blocks of B , and the block 7 9 2 again. Once more we have that 3 , t and 5 must lie in different blocks of the parallel class not covering 6. The only block containing 3 that will suffice is 3 0 4. As in A , all the blocks through 5 intersect 7 9 2 or 3 0 4 , or contain 6 or t . Hence there is no parallel class containing 7 9 2 and not covering 6 .

So neither S(2,3,13) is resolvable at the point 6 . So we cannot have an S(2,6,66) whose block graph has an alternative geometrisation by normal cliques.

It is possible to prove that a necessary condition for the existence of an S(2,k,v) with nonunique geometrisation by normal cliques of size r , with $k = 2k_0$, is the existence of an $S(2,k_0,r)$ with the special resolvability property, i.e. for any point of the $S(2,k_0,r)$, the blocks not containing that point are divided into k - 1 parallel classes of $(r - 1)/k_0$ blocks.

The next case $k = 8$, $k_0 = 4$, $v = 176$, $r = 25$, requires the existence of a (176,50,14) symmetric design, and a special S(2,4,25). I have not been able to determine whether such an S(2,4,25) exists.

(6) An $S_2(2,8,176)$ whose block graph is geometrisable in two ways.

In her thesis, M.Smith (28), following the work of G.Higman (17), considers the doubly transitive representation of the Higman Sims group on 176 points. It turns out that there are two such representations, and a subgroup of index 176, isomorphic to $P\Sigma U(3,5^2)$, which does not fix a point in one representation, has two orbits, one of size 50 and the other of size 126. If we consider these sets of size 50 we obtain a symmetric block design on the 176 points with parameters (176,50,14).

Furthermore the group has subgroups isomorphic to S_8 (the symmetric group on 8 letters), and these have orbits of size 8 and 168; (this was the method used by G.Higman to construct this representation). If we consider the orbits of size 8, we obtain an $S_2(2,8,176)$ design, and because the stabiliser of each block is S_8, the blocks must intersect in 0,1 or 2 points. (If they intersect in more, the condition that any two points have exactly two blocks through them is violated, by the 8-fold transitivity of the block stabiliser on the points of the block).

The blocks of size 8, called conics by Higman, lie in exactly 8 of the sets of size 50 of the symmetric design, (called quadrics by Higman). So they give an $S_2(2,8,176)$ on the quadrics with dual incidence. Finally any quadric contains exactly 50 conics, any two of which intersect in 1 or 2 points. So we have a new set of 176 cliques of size 50 in $BG[S_2(2,8,176)]$ corresponding to another geometrisation of the block graph. Furthermore the cliques are normal, covering the minimum number, 50, points of the design, and regular, any other block containing exactly two of these points, intersecting 14 of the blocks in one point, and one of them in two.

The structure of the 50 blocks of size 8 intersecting in one or two points is that obtained by taking the Moore graph of valency 7, and constructing 50 blocks as follows. For any vertex x, let B_x be that vertex together with the 7 adjacent vertices. Then clearly,

$$|B_x \cap B_y| = \begin{cases} 2 & \text{if } x \text{ is adjacent to } y, \\ 1 & \text{if not.} \end{cases}$$

CHAPTER V

AN ASSOCIATION SCHEME FOR THE 1-FACTORS OF THE COMPLETE GRAPH, AND CLIQUES AND DESIGNS THEREIN.

(1) Definitions, and the Centraliser Algebra of a Permutation Group.

A 1-factor of K_{2n} is a collection of n edges of the complete graph on $2n$ vertices, such that any vertex lies on a unique edge. A given 1-factor can be identified with the fixed-point-free involution of S_{2n} which interchanges the pairs of points that make up the edges of the 1-factor. The natural action of the symmetric group S_{2n} on the 1-factors of K_{2n} is permutation equivalent to its action by conjugation on its fixed-point-free involutions.

The Centraliser Algebra of a transitive permutation group G on a set Ω of size v, is the set of $v \times v$ matrices over the complex numbers which commute with all the permutation matrices of G. If the group G has rank $m + 1$ on Ω (i.e. the stabiliser of a point has m orbits on the remaining points), and the centraliser algebra is commutative, then the orbits of G on the 2-subsets of Ω form an m-class association scheme, as defined in Chapter I. In fact, using the notation of Wielandt (30), for any orbit Δ of the stabiliser G_1, of the point 1, we can associate the following matrix A_Δ, where,

$$A_\Delta(a,b) = \begin{cases} 1 & \text{if there is a } d \in \Delta,\ g \in G \text{ such that } (1,d)g = (a,b). \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to show that these matrices generate the centraliser algebra of the representation. Furthermore the sum over distinct orbits of the corresponding matrices is the all 1 matrix J. So if the algebra is commutative it can be regarded as an association algebra with m classes.

For each $g \in G$, let $P(g)$ be the $v \times v$ permutation matrix defined by the action of g on Ω.

<u>Theorem A</u>: The centraliser of G on Ω is commutative if the permutation character of G on Ω is the sum of distinct characters of G.

<u>Proof</u>: Let $D_1, D_2, \ldots, D_r$ be the irreducible constituents of the permutation representation on Ω, i.e. for some unitary matrix U, any $v \times v$ permutation matrix $P(g)$ in G has the following decomposition,

$$U^{-1}P(g)U = \begin{bmatrix} D_1 & & & & & & \\ & D_2(g) & & & & & \\ & & \ddots & & & & \\ & & & D_2(g) & & & \\ & & & & \ddots & & \\ & & & & & D_r(g) & \\ & & & & & & \ddots \\ & & & & & & & D_r(g) \end{bmatrix}$$

with 0's outside the blocks on the main diagonal. We write this in the form

$$U^{-1}P(g)U = \Big[D_1(g),\ \underbrace{D_2(g),\ldots,D_2(g)}_{e_2},\ldots\ldots\ldots,\ \underbrace{D_r(g),\ldots,D_r(g)}_{e_r}\Big].$$

Then, if $\mathcal{A}$ is the centraliser algebra,

$$U^{-1}\mathcal{A}U = \left[\mathcal{W}_{e_1}, \mathcal{W}_{e_2} \times I_{f_2}, \ldots\ldots\ldots\ldots, \mathcal{W}_{e_r} \times I_{f_r}\right],$$

where $e_1 = 1$,

$\mathcal{W}_{e_i}$ = the ring of all $e_i \times e_i$ matrices over the complex numbers,

I_{f_i} = the identity matrix of size f_i, the degree of the i-th irreducible,

and $\times$ represents Kronecker product.

Clearly this is commutative if and only if $e_i = 1$ for all i.

For the i-th conjugacy class $\mathcal{C}_i$ of G, define the i-th class matrix,

$$C_i = \sum_{g \in \mathcal{C}_i} P(g) \quad .$$

Then we have;

<u>Theorem B</u>: All the class matrices belong to the centraliser algebra $\mathcal{A}$. They commute with each other. Furthermore, the centraliser algebra is commutative if and only if the class matrices generate $\mathcal{A}$.

<u>Proof</u>:

$$P(h)^{-1} C_i P(h) = \sum_{g \in \mathcal{C}_i} P(h^{-1} gh)$$

$$= \sum_{g \in \mathcal{C}_i} P(g) = C_i, \quad \text{for all } h \in G.$$

$$C_i C_j = \sum_{g \in \mathcal{C}_i} P(g) C_j = \sum_{g \in \mathcal{C}_i} C_j P(g) = C_j C_i \quad .$$

So if the class matrices generate $\mathfrak{A}$, it is commutative.

Conversely, suppose $\mathfrak{A}$ is commutative. Then from theorem A, the representation has distinct constituents.

Consider $[\chi_i(j)]$, the character table of G. ($\chi_i(j)$ = the value of the i-th character on the j-th conjugacy class). It is well known that its columns are orthogonal, and so , as a matrix, it is nonsingular. In particular, if we look at the matrix formed by the rows corresponding to the irreducible constituents in our permutation representation, it has full row rank, $m+1$, and so it has a set of $m+1$ independent columns. Consider this square submatrix, which with suitable renumbering, we can call

$$[\chi_i(j)] \qquad i,j = 1,\ldots\ldots\ldots,m+1.$$

It is nonsingular.

Again, from the basic representation theory,

$$\sum_{g\in\mathcal{C}_j} D_i(g) = \frac{|\mathcal{C}_j|\chi_i(j)}{f_i} I_{f_i} ,$$

and so,

$$U^{-1}C_jU = \left[|\mathcal{C}_j|\chi_1(j),\ \frac{|\mathcal{C}_j|\chi_2(j)}{f_2} I_{f_2},\ldots\ldots\ldots,\frac{|\mathcal{C}_j|\chi_{m+1}(j)}{f_{m+1}} I_{f_{m+1}}\right].$$

The matrix $\left[\dfrac{|\mathcal{C}_j|\chi_i(j)}{f_i}\right] = \left[\dfrac{1}{f_1},\ldots\ldots,\dfrac{1}{f_{m+1}}\right][\chi_i(j)]\left[|\mathcal{C}_1|,\ldots\ldots\ldots,|\mathcal{C}_{m+1}|\right]$

is nonsingular, since $\left[\chi_i(j)\right]$ is.

Therefore the C_d's are independent. But dim $\mathcal{a} = m + 1$. Hence they span $\mathcal{a}$.

Theorems A and B can be found in Wielandt (30).

(2) <u>The character of S_{2n} on the 1-factors of K_{2n}.</u>

We give the proof of a theorem by Thompson (29), proved in a very different manner by Saxl and James, see (27), but perhaps known earlier, concerning the constituents of the permutation representation of S_{2n} on the 1-factors of K_{2n}.

<u>Theorem C</u>: Let χ be the character of this representation, and let $\chi^{(\pi)}$ be the irreducible character of S_{2n} corresponding to the partition π. Then

$$(\chi, \chi^{(\pi)}) = \begin{cases} 1 & \text{if every part of } \pi \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular this representation has rank $p(n)$, the number of partitions of n.

<u>Proof</u>: (due to Thompson)

(a) $(\chi, \chi) = p(n)$ is not difficult; we do this in detail in the next lemma.

(b) We need to show $(\chi, \chi^{(\pi)}) \neq 0$ if every part of π is even. To do this we make use of the methods of the representation theory of the symmetric group over the complex numbers; see, for example, Curtis and Reiner (6).

Let T be a tableau associated with π. Then for each $k = 1$ to $2n$, we have a position (i_k, j_k), where the number k can be said to lie in the tableau.

For each edge $e = \{k, \ell\}$ of K_{2n}, set

$$h_T(e) = |j_k - j_\ell| \quad , \qquad w_T(e) = |i_k - i_\ell| \; .$$

If u is a 1-factor, set

$$h_T(u) = \max_{e \text{ an edge of } u} h_T(e), \qquad w_T(e) = \max_{e \text{ edge of } u} w_T(e)$$

Since every part of π is even, there is a unique $u = u(T)$, such that

$$h_T(u(T)) = 0, \qquad w_T(u(T)) = 1.$$

e.g. for $2n = 6$, $\pi = (4,2)$

$$T = \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & & \end{matrix} \qquad u(T) = a_1 a_2 \,|\, a_3 a_4 \,|\, a_5 a_6 \, .$$

Let C, R be the largest subgroup of S_{2n} which fixes each column, respectively row, of T.

Set $$e(C)^- = |C|^{-1} \sum_{c \in C} sg(c)\, c \, ,$$

$$e(R)^+ = |R|^{-1} \sum_{r \in R} r,$$

$e(T) = e(C)^- e(R)^+$, then $e(T)$ is a principal idempotent of the group algebra QS_{2n}, and

$$\dim \ QV.\, e(T) = (\chi\, ,\, \chi^{(\pi)})\ ,$$

where V is the vector space spanned by all the 1-factors of K_{2n}. We need to show that $V.e(T) \neq 0$. We show that the coefficient of $u(T)$ in $u(T).e(T)$ is positive.

$$\text{Let} \quad f(T) = e(T)\ |C|\ |R| = \sum_{\substack{c\in C\\ r\in R}} sg(c)cr.$$

The coefficient of $u(T)$ in $u(T).f(T)$ is

$$\sum_{\substack{(c,r)\\ c\in C,\ r\in R\\ cr \text{ fixes } u(T)}} sg(c) \qquad .$$

Suppose $c \in C$, $r \in R$, cr fixes $u(T)$.

Then $\quad (u(T))cr = u(T), \qquad u(T).c = u(T).r^{-1}$.

But $\quad w_T(u(T).c) = w_T(u(T)) = 1$, and $\quad h_T(u(T).r^{-1}) = h_T(u(T)) = 0$, (since the functions w_T, h_T are constant on C-orbits, R-orbits, respectively), and so

$$u(T).c = u(T).r^{-1} = u(T).$$

Let $c = c_1.c_2 \ldots\ldots c_{2f-1}.c_{2f}$, where $2f$ is the number of columns of T, and c_i moves only the points in column i. Then, obviously, c_i acts on column i in the same way as c_{i+1} acts on column $i+1$, for i odd. Hence $sg(c_i) = sg(c_{i+1})$. Hence $sg(c) = 1$. We certainly have $u(T) = u(T).1.1$ and so the result follows.

We have included this proof because we generalise it later when considering designs in the scheme.

(3) A useful lemma.

Theorems A and C together imply that the 1-factors of K_{2n} give rise to a scheme with $p(n) - 1$ classes. The main point of Theorem C is that it identifies the eigenspaces of the association scheme with certain irreducible representation subspaces of S_{2n}. We give a proof that the rank of the representation is $p(n)$ which will also be of use later.

Lemma: Let f_1, f_2, f_3 be 1-factors of K_{2n}.

Then 1) For every partition π of n there corresponds a relation between the 1-factors.

2) If $\{f_1, f_2\}$, and $\{f_1, f_3\} \in R(\pi)$ then there exists $\sigma \in S_{2n}$ such that $f_1\sigma = f_1$ and $f_2\sigma = f_3$.

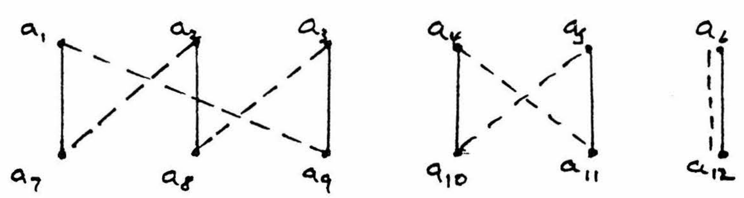
3) There exists $\sigma \in S_{2n}$ such that $f_1\sigma = f_2$ and $f_2\sigma = f_1$.

4) There exists $\sigma \in S_{2n}$ such that σ fixes at least n of the $2n$ points and $f_1\sigma = f_2$.

In fact, if $f_1 + f_2$ (for definition see below) has x components of length 2, and y components of length greater than two, then there are exactly 2^y permutations $\sigma \in S_{2n}$ which fix $n + x$ points and such that $f_1\sigma = f_2$.

Proof: 1) Take any two 1-factors f_1, f_2: their edges taken together, i.e. $f_1 + f_2$, cover every vertex twice and so form a collection of disjoint circuits of even length (including length 2,) and so we obtain a partition 2π of $2n$ into even parts, and so π is a partition of n. Conversely, given a partition π of n, 2π is a partition of $2n$ into even parts, and it is not difficult to find 1-factors f_1, f_2 such that the circuits of $f_1 + f_2$ have lengths the constituents of 2π.

2) Suppose $f_1 + f_2$ and $f_1 + f_3$ have as components circuits of the same lengths, i.e. they both correspond to the same partition of n. Consider first just $f_1 + f_2$, and the longest circuit c_1, of $f_1 + f_2$, of length $2m_1$ say. Then any point in this circuit lies on one edge of f_1 and one edge of f_2. Pick an arbitrary point of the circuit and label it a_1. Then go along the f_1 edge at a_1, and label the adjacent point a_{n+1}. Then go along the f_2 edge at a_{n+1} and label the next point a_2. Continue in this manner until all the points of c_1 have been labelled, with labels $a_1, \ldots, a_{m_1}, a_{n+1}, \ldots, a_{n+m_1}$. Now label the other circuits similarly, so we might have for $n = 6$,



where f_1 edges are continuous and f_2 edges are dotted.

Now produce a similar labelling for $f_1 + f_3$ with labels $b_1, \ldots, b_{2n}$. Then because $f_1 + f_2$ and $f_1 + f_3$ correspond to the same partition of n, there is a $\sigma \in S_{2n}$ such that $a_i \to b_i$, $i = 1$ to $2n$, which fixes f_1 and maps f_2 to f_3.

3) For each circuit c_i of $f_1 + f_2$ choose an orientation $\gamma(c_i)$

of the points of c . Then $\sigma = \prod \gamma(c_i)$ is such that $f_1\sigma = f_2$ and $f_2\sigma = f_1$.

4) In the same way as in 2) we consider a particular circuit c of $f_1 + f_2$ of length $2r$ greater than two, and label the vertices $a_1, \ldots, a_r, a_{n+1}, \ldots, a_{n+r}$. Then $(a_1, a_2, \ldots, a_r)$ and $(a_{n+r}, \ldots, a_{n+1})$ are two such $\sigma \in S_{2n}$ which fix at least r points of c and map the edges of f_1 to those of f_2. Now, clearly, there can be no $\sigma \in S_{2n}$ which maps the edges of f_1 to the edges of f_2 , and fixes two consecutive points of c. So any such σ fixes at most r points of c , and no two consecutive ones, and so any σ which fixes exactly r points of c , fixes alternate points. So any edge of f_1 has one of its points fixed, and so the other point in the edge must go either to the point two to the left in an orientation of c , or to a point two to the right. This determines the action of σ on the circuit. Finally the only way we can have σ mapping f_1 to f_2 and fixing $n + x$ points, is for it to fix each repeated edge and act in one of two ways on each circuit of length greater than two. Hence the result.

1), 2), and 3) are enough to show that we have an association scheme with $p(n) - 1$ classes.

4) will be of use later.

(4) <u>The eigenvalues of the scheme</u>.

For the Johnson and Hamming schemes the eigenvalues are known in terms of the basic parameters and the Krawchuk and Eberlein polynomials. It does not seem possible to provide such a neat closed form for the eigenvalues of our scheme, but we present a method for finding them, making use of theorem B.

Clearly the class matrices here do not form a basis for the centraliser algebra, and there is no unique way of representing the A_i's as linear combinations of the C_j's. However, a canonical expression can be found in terms of the C_j's for only those conjugacy classes whose elements fix at least n points, and in terms of these C_j's the expression is unique. The reason for this is part 4) of the preceeding lemma.

Now the number of conjugacy classes whose elements fix at least n points is exactly $p(n)$, and it turns out that we have a triangular system of equations connecting the C_j's and the A_i's. First we index the A_i's, C_j's, and E_k's by partitions of n as follows:

Let π be a partition of n.

$A(\pi)$ is the adjacency matrix corresponding to the relation $R(\pi)$ such that $\{f_1, f_2\} \in R(\pi)$ if the circuits of $f_1 + f_2$ have as lengths the components of 2π.

$C(\pi)$ is the class matrix corresponding to the conjugacy class in S_{2n} whose cycle lengths are components of π together with n fixed points.

$E(\pi)$ is the projection matrix of V onto $V(2\pi')$, the eigenspace of which is isomorphic as an S_{2n}-module to the Specht module $S^{(2\pi')}$ arising from the partition $2(\pi')$ of $2n$, where π' is the partition conjugate to π. The ordering of the partitions π of n follows that used by D.E.Littlewood (20) in his character tables of the symmetric groups.

The eigenvalues of the C_i's on the irreducible subspaces are well known, and are easily calculated from the character table of S_{2n}, or if this is not available, by the methods of the representation theory of the symmetric group. We have,

$$\lambda_k(C_i) = |\mathcal{C}_i| \chi^k(i) / \chi^k(1) .$$

The main problem is to find a_{ij}'s such that

$$C_i = \sum_{j=1}^{i} a_{ij} A_j .$$

This is done for the cases $n = 4,5,6$, but beyond this point the calculations become very involved.

(5) <u>Computing the a_{ij}'s.</u>

We recall

$C_i(f_1, f_2) =$ the number of $\sigma \in \mathcal{C}_i$ such that $f_1\sigma = f_2$.

So $a_{ij} =$ the number of $\sigma \in \mathcal{C}_i$ such that if $\{f_1, f_2\} \in R_i$,

then $f_1\sigma = f_2$.

For small n, and for some i and j, this is a relatively satisfactory method for calculating the a_{ij}'s. However, the length of these computations increases rapidly with n, and for $n = 5,6$, the following shorter method was used.

The underlying idea is to pick a particular f_1 and to count the number of $\sigma \in \mathcal{C}_i$ such that $\{f_1, f_1\sigma\} \in R_j$.

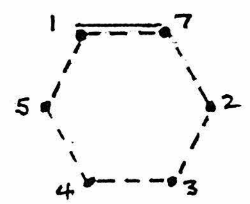
Calling this number b_{ij}, we have

$$a_{ij} = b_{ij}/\nu_j \ .$$

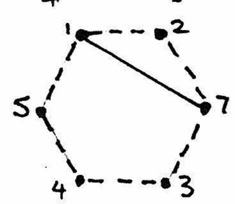
The calculation of the b_{ij}'s falls into two parts. First, we find the number of $\sigma \in \mathcal{C}_i$ which contain certain edges of f_1 in their cycle structure in a prescribed manner, and then we find the number of ways in which we can choose such edges. The best way to illustrate the method is with an example: we do the case where $n = 6$, $\mathcal{C}_7 = \{(123456)\}$, $f_1 = (1,7)(2,8)(3,9)(4,10)(5,11)(6,12)$

First, there is the case where the 6-cycle consists of one point from each edge of the 1-factor. As we have remarked in part 4) of the lemma, this gives rise to $a_{77} = 2$.

Second, there is the case where the 6-cycle contains 2 points from one edge of the 1-factor and four other points, one from each of four edges of the 1-factor. Then we have three subcases,

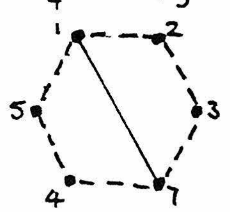


48 σ such that $\{f_1, f_1\sigma\} \in R_6$,



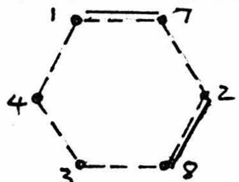
48 R_6,

giving $a_{7,6} = \dfrac{120.6.5.2^4}{2304} = 25$.

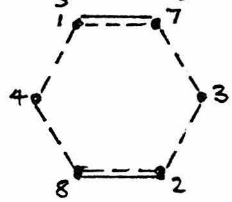


24 R_6,

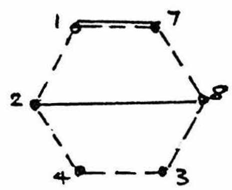
Third, there is the case where the 6-cycle contains two edges of the 1-factor and two other points, one from each of two of the remaining edges of the 1-factor. Then we have 8 subcases.



16 R_3

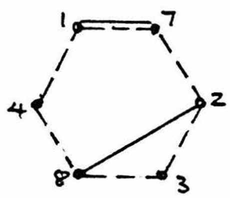


8 R_3



16 R_3

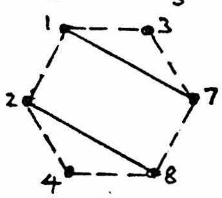
These contribute



32 R_3

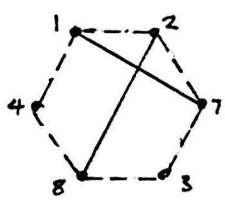
$$\frac{80.15.6.2^2}{720} = 40\ A_3$$

$$\frac{16.15.6.2^2}{180} = 32\ A_4$$

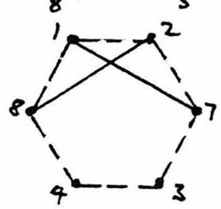


8 R_3

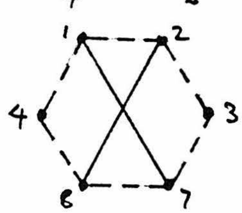
$$\frac{24.15.6.2^2}{160} = 54\ A_2$$



16 R_4

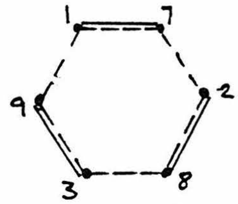


16 R_2

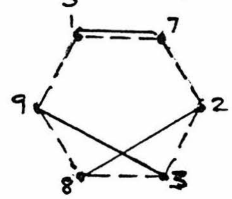


8 R_2

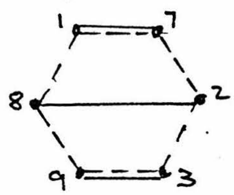
Fourth, there is the case where the 6-cycle contains 3 edges of the 1-factor. We then have 5 subcases.



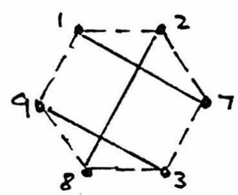
16 R_2



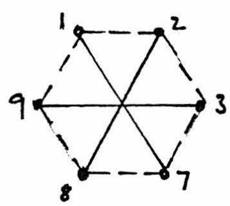
48. R_1



24 R_2



24 R_2



8 R_0

These contribute

$$\frac{48.20}{30} = 32\ A_1$$

$$\frac{64.20}{160} = 8\ A_2$$

$$8.20 = 160\ A_0 \ .$$

From all this we obtain,

$$C_7 = 160\ A_0 + 32\ A_1 + 62\ A_2 + 40\ A_3 + 32\ A_4 + 25\ A_6 + 2\ A_7 \ .$$

(6) <u>The eigenvalues and the a_{ij}'s.</u>

<u>Case $n = 4$.</u> $p(4) = 5$.

(1) $C_0 = A_0 = I$

(12) $C_1 = 4A_0 + 2A_1$

(123) $C_2 = 4A_1 + 2A_2$

(1234) $C_3 = 12A_0 + 2A_1 + 9A_2 + 2A_3$

(12)(34) $C_4 = 18A_0 + 4A_1 + 3A_2 + 4A_4$.

P matrix	1^4	$2,1^2$	$3,1$	4	$2,2$	μ_i
[8]	1	12	32	48	12	1
[6,2]	1	5	-4	8	-2	20
[4^2]	1	2	8	-2	7	14
[$4,2^2$]	1	-1	-2	4	-2	56
[2^4]	1	-6	8	-6	3	14

Case n = 5. p(5) = 7.

(1) $C_0 = A_0$

(12) $C_1 = 5A_0 + 2A_1$

(123) $C_2 = 4A_1 + 2A_2$

(1234) $C_3 = 20A_0 + 2A_1 + 9A_2 + 2A_3$

(12)(34) $C_4 = 30A_0 + 6A_1 + 3A_2 + 4A_4$

(12)(345) $C_5 = 36A_1 + 10A_2 + 8A_3 + 16A_4 + 4A_5$

(12345) $C_6 = 24A_1 + 12A_2 + 16A_3 + 2A_6$.

P matrix	1^5	$2,1^3$	$3,1^2$	$4,1$	$2^2,1$	$2,3$	5	μ_i
[10]	1	20	80	240	60	160	384	1
[8,2]	1	11	26	24	6	-20	-48	35
[6,4]	1	6	-4	-26	11	20	-8	90
[$6,2^2$]	1	3	2	-8	-10	-4	16	225
[$4^2,2$]	1	0	-10	10	5	10	4	252
[$4,2^3$]	1	-4	2	6	-3	10	-12	300
[2^5]	1	-10	20	-30	15	-20	24	42

Case n = 6 . $p(6) = 11$.

(1)	$C_0 = A_0$
(12)	$C_1 = 6A_0 + 2A_1$
(123)	$C_2 = 4A_1 + 2A_2$
(1234)	$C_3 = 30A_0 + 2A_1 + 9A_2 + 2A_3$
(12)(34)	$C_4 = 45A_0 + 8A_1 + 3A_2 + 4A_4$
(12)(345)	$C_5 = 48A_1 + 12A_2 + 8A_3 + 16A_4 + 4A_5$
(12345)	$C_6 = 32A_1 + 12A_2 + 16A_3 + 2A_6$
(123456)	$C_7 = 160A_0 + 32A_1 + 62A_2 + 40A_3 + 32A_4 + 25A_6 + 2A_7$
(12)(3456)	$C_8 = 120A_0 + 64A_1 + 66A_2 + 28A_3 + 24A_4 + 18A_5 + 10A_6 + 4A_8$
(12)(34)(56)	$C_9 = 140A_0 + 36A_1 + 10A_2 + 4A_3 + 8A_4 + 6A_5 + 8A_9$
(123)(456)	$C_{10} = 160A_0 + 22A_2 + 8A_3 + 32A_4 + 8A_5 + 5A_6 + 4A_{10}$.

P matrix	1^6	$2,1^4$	$3,1^3$	$4,1^2$	2,1	2,3,1	5,1	6	2,4	2^3	3^2	μ_i
[12]	1	30	160	720	180	960	2304	3840	1440	120	640	1
[10,2]	1	19	72	192	48	80	192	-384	-144	-12	-64	54
[8,4]	1	12	16	-18	27	24	-144	-48	108	30	-8	275
[6,6]	1	9	-8	-78	33	120	-48	-24	-114	-27	136	132
[8,2^2]	1	9	22	12	-12	-60	-48	96	-24	-12	16	616
[6,4,2]	1	4	-8	-18	3	0	32	16	-4	-2	-24	2673
[4^3]	1	0	-20	30	15	-60	24	0	-60	30	40	462
[6,2^3]	1	0	4	-6	-21	12	24	-48	12	6	16	1925
[4^2,2^2]	1	-3	-8	24	3	0	-24	-12	24	-9	4	2640
[4,2^4]	1	-8	12	-6	3	20	-24	48	-36	6	-16	1485
[2^6]	1	-15	40	-90	45	-120	144	-120	90	-15	40	132

Note: R. Roth has calculated these P-matrices directly by computing the p_{ijk}'s - a monumental task - and the figures we have arrived at independently have each served as useful checks for the other.

(7) Designs in the Scheme.

In analogy with the Johnson and Hamming schemes, we wish to consider t-designs in this association scheme. Let us call a collection Y of 1-factors a t-design, if for any collection of t disjoint edges of K_{2n}, there are a constant number of 1-factors in the set Y which contain these edges.

Consider the matrix $N_{t,m}$ which has columns indexed by all the 1-factors and rows indexed by sets of t disjoint edges, and

$$N_{t,m}(e,f) = \begin{cases} 1 & \text{if the edges of } e \text{ are contained in those of } f. \\ 0. & \text{otherwise.} \end{cases}$$

If ϕ_Y is the characteristic vector of Y, the condition that Y is a t-design can be written

$$N_{t,m}\,\phi_Y = \underline{1}\,,$$

or $$(N_{t,m} - k\,\lambda\, J_{t,m})\,\phi_Y = 0, \quad \text{for some constant } k.$$

So we need to find out which irreducible representation subspaces make up the row space of $N_{t,m}$.

<u>Lemma</u>: Let N be an $n \times m$ matrix over the complex numbers with the property that a group G acts on both the rows and the columns of N. i.e. for all $g \in G$, if $P(g)$ is the permutation induced by g on the rows, and $Q(g)$ that on the columns,

$$P(g)^{-1} N Q(g) = N .$$

Then the row space (over C) and the column space (over C) of N decompose into isomorphic irreducible representation modules of G.

<u>Proof</u>: There exist square matrices A of size m, and B of size n, such that

$$A^{-1} N B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where } r \text{ is the rank of } N.$$

So $\quad A^{-1} P(g) \quad A A^{-1} \quad N B B^{-1} \quad Q(g) B = A^{-1} \quad N B.$

Let $\quad P_1(g) = A^{-1} P(g) A, \qquad Q_1(g) = B^{-1} Q(g) B ,$

$$= \begin{bmatrix} P_{11}(g), & P_{12}(g) \\ P_{21}(g), & P_{22}(g) \end{bmatrix} , \qquad = \begin{bmatrix} Q_{11}(g), & Q_{12}(g) \\ Q_{21}(g), & Q_{22}(g) \end{bmatrix} .$$

Then

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_1(g) = P_1(g) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} ,$$

$$\begin{bmatrix} Q_{11}(g), & Q_{12}(g) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_{11}(g), & 0 \\ P_{21}(g), & 0 \end{bmatrix} .$$

So $\quad Q_{11}(g) = P_{11}(g) \quad$ and $\quad Q_{12}(g) = 0, \quad P_{21}(g) = 0,$ for all $g \in G$.

Hence the row and column spaces decompose into the same irreducibles.

Now consider $N_{t,m}$. It satisfies the conditions of the lemma, with S_{2m} acting on the rows and the columns. Since the row space is contained in the representation module of S_{2m} on the 1-factors, any irreducible constituent $S^{(\pi)}$ of $N_{t,m}$ must correspond to a partition π of $2m$ into even parts. On the other hand, the column space is a submodule of the representation of S_{2m} on $M^{(\mu)}$ (the set of tableaux of type μ), where $\mu = (2^t, 2m - 2t)$, i.e. $2m = 2m - 2t + \underbrace{2 + \ldots\ldots + 2}_{t \text{ times}}$.

We prove the following;

Theorem: Let $\pi = 2\nu$ and $\pi \geq (2^t, 2m - 2t)$ be a partition of $2m$ into even parts. Then $S^{(\pi)} \leq \text{row}(N_{t,m})$, $\text{col}\ (N_{t,m})$.

Proof: We proceed in exactly the same manner as in the proof of Theorem C.

Let T be a tableau corresponding to the partition $\pi = \pi_1 + \ldots\ldots + \pi_k$, where $\pi_1 \geq 2m - 2t$, and $\pi_i \geq 2$ for all $i > 1$.

Let $S \leq R$ be the subgroup that acts as the symmetric group on the first $2m - 2t$ positions in the first row of the tableau.

We define $u(T)$ as in Theorem C,

and $$v(T) = \sum_{s \in S} u(T).s \ .$$

Now $v(T)$ occurs as a vector in the row space of $N_{t,m}$, since it is the characteristic vector of the set of 1-factors containing the t disjoint edges of $u(T)$ which do not contain the letters in the first $2m - 2t$ positions of T.

Again consider the coefficient of $u(T)$ in $v(T).f(T)$, namely,

$$\sum_{\substack{c \in C,\ r \in R \\ u(T).cr \in u(T).S}} sg(c) \quad .$$

If $u(T).cr = u(T).s$ for $s \in S$, then $u(T).c = u(T).sr^{-1}$.

So $w_T(u(T).c) = w_T(u(T)) = 1,\qquad h_T(u(T).sr^{-1}) = h_T(u(T)) = 0.$

Hence $u(T).c = u(T).sr^{-1} = u(T)$,

and the remainder of the argument follows exactly as before.

We have shown that

$$\text{row } N_{t,m} \leq \operatorname*{span}_{\pi_1 \geq m - t.} \{ E(2\pi) \} \quad .$$

<u>Note</u>: Thompson proves this for the case $t = 2$, but not by this method.

Let us look at the case $m = 6$, i.e. the 1-factors of K_{12}. The A_i's can be arranged in order depending on the number of edges that two 1-factors have in common. The E_i's can be arranged in the order in which they first appear in row($N_{t,m}$). We have

(1^6)	$[12]$
$(2,1^4)$	$[10,2]$
$(3,1^3)$	$[8,4]$, $[8,2^2]$
$(4,1^2)$, $(2^2,1^2)$	$[6,6]$, $[6,4,2]$, $[6,2^3]$
$(5,1)$, $(2,3,1)$	$[4^3]$, $[4^2,2^2]$, $[4,2^4]$
(6), $(4,2)$, (3^2), (2^3).	$[2^6]$.

These 1-factor schemes are not metric in the sense of Delsarte, but they bear a strong resemblance to the Johnson scheme, in that it is possible to define a generalised metric on the 1-factors, by saying that two 1-factors are at distance i if they have $n - i$ edges in common. One small difficulty arises in that there is no distance 1. However if d denotes the distance function, then

$$d(f_1, f_2) + d(f_2, f_3) \geq d(f_1, f_3),$$

for any three 1-factors f_1, f_2, f_3.

From what we have just seen there is a similar partial order on the E_i's.

(8) Particular Examples, and Applications.

(a) 1-factors arising from an oval of $n + 2$ points in $PG(2,n)$.

In such a case n must be even, and to each of the $n - 1$ points outside the oval there corresponds a 1-factor of K_{n+2}, determined by the intersections of the lines through that point with the oval, (see Thompson (29)). This set Y of 1-factors of K_{n+2} has the property that no two of them have more than one edge in common, (for if they did, they would correspond to the same exterior point), and so Y is a clique with respect to certain relations. Furthermore any pair of disjoint edges is contained in a unique 1-factor. Therefore Y is a 2-design, and we have, for the characteristic vector ϕ_Y of Y;

$$E_{[n-2,2]}\,\phi_Y = E_{[n-4,2^2]}\,\phi_Y = E_{[n-4,4]}\,\phi_Y = 0.$$

This gives three equalities in Delsarte's inequalities for the distribution vector. However the other inequalities are not very strong in this case.

One can also define two partial geometries on the set of $n^2 - 1$ exterior points depending on whether or not they lie on a secant or an exterior line of the oval. Let us consider the case $n = 10$. Then we get a $(5,11,5)$ partial geometry the dual of which is an $S(2,11,45)$. It has been shown that a projective plane of order 10 has no automorphisms apart from the identity. Suppose we have an oval in the plane. Is it possible for the set of 99 1-factors arising from this oval, there is a permutation $g \in S_{12}$ which fixes the set as a whole. Suppose it was. Then the block graph of the $S(2,11,45)$ would have an automorphism. But if g permutes the lines among themselves, then g could be extended to an automorphism of the plane. Otherwise g would have to map the lines of the block graph to another set of geometrising cliques. But we have shown in III(11) that none exist. So this set of 99 1-factors has no nontrivial automorphisms.

(b) 1-factorisations.

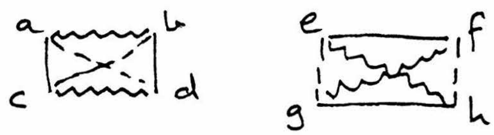
A 1-factorisation of K is a set of $2n - 1$ 1-factors of K_{2n} such that any edge lies in exactly one of them. Cameron (4) has asked for which relations between 1-factors is it possible to construct a 1-factorisation such that any two of the 1-factors are related in the same way, and he gives a list of the known cases. A 1-factorisation of this type can be considered as a clique with respect to this relation. The figures we have calculated add only a little to what was already known.

(c) A proof of the isomorphism of A_8 and $PSL(4,2)$ using the techniques of this thesis.

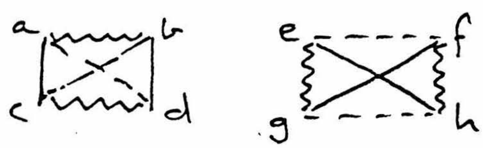
We consider the graph on the 105 1-factors of K , corresponding to the relation (2,2), i.e. any two 1-factors are adjacent if they lie as follows,

Then, from the P-matrix we obtain that the adjacency matrix has least eigenvalue -2 and has constant line sums. We can apply the considerable theory of such graphs, fully recorded in (3), to obtain that G is the line graph of some graph. Furthermore G has the same parameters as the flag graph of a symmetric design with parameters (15,7,3), and so is the flag graph of such a design. G has 30 cliques of size 7, corresponding to the points and blocks of this design. These must be 1-factorisations such that the relations between the seven constituent 1-factors are all (2,2). These are the only such 1-factorisations since the flag graph has only these 30 cliques of size seven. The parallel classes of $AG(3,2)$ give such a 1-factorisation, and this has automorphism group of size 8.7.6.4 Therefore S_8 is transitive on these 30 1-factorisations, and A_8 has two orbits of size 15.

There are 35 partitions of 8 into two lots of 4. To any such partition we can associate nine 1-factors, and three of our 1-factorisations in the same orbit of A_8, these being



under (hgf)



under (hfg)

In this way we obtain a set of 35 triples on the 15 1-factorisations in an orbit of A . The triples have 0 or 1 1-factorisation in common. And so we have an S(2,3,15) on one orbit, and the same S(2,3,15) (corresponding to the same 35 partitions) on the other orbit. And so it is doubly geometrisable. Hence it is projective 3-space over GF(2). A acts as a simple group of automorphisms on it and, by comparing orders we see it must equal PSL(4,2).

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