The Volume of Tubes in Homogeneous Spaces

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Ralph Elwood Howard

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ABSTRACT

Let $\tilde{M}$ ($\dim(\tilde{M}) = m + n$) be an oriented Riemannian manifold and M a compact oriented submanifold of $\tilde{M}$. The tube $M(r)$ of radius r about M is the set of points p that can be joined to M by a geodesic of length r meeting M perpendicularly. We give a formula for the volume of $M(r)$ in the case $\tilde{M}$ is a naturally reductive Riemannian homogeneous space (this includes all Riemannian symmetric spaces) and M is such that for each point p of M there is a totally geodesic submanifold of $\tilde{M}$ of dimension complementary to M through p and perpendicular to M at p.

To be more specific,

$$\operatorname{vol}(M(r)) = \sum_{j=0}^{n} \int_M h_j(p,r)\Omega_M(p).$$

Here h_j is a function of the point $p \in M$ and the real number r. Also $h_j(p,r)$ is a homogeneous polynomial of degree j in the components of the second fundamental form of M in $\tilde{M}$.

CONTENTS

1. Introduction

Let M be a submanifold of the Riemannian manifold $\widetilde{M}$. Then a fundamental problem in the geometry of submanifolds is to give invariants of the pair $(M,\widetilde{M})$ that relates the geometry of M to that of $\widetilde{M}$. One such invariant is the volume of the tube $M(r)$, of radius r, about M in $\widetilde{M}$.

In the case where $\widetilde{M}$ is a Euclidean space of dimension $n + m$ and M is compact of dimension n, Hermann Weyl [13] proved that

$$\mathrm{vol}(M(r)) = \sum_{0 \leq 2k \leq n} c_{2k,n,m} r^{m-1+2k} \int_M h_{2k}(p)\Omega_M(p)$$

where $h_{2k}(p)$ is a polynomial of degree $2k$ in the components of the second fundamental form (or of the Weingarten map) of M in $\widetilde{M}$. It is also possible to express $h_{2k}(p)$ as a polynomial of degree k in the components of the curvature tensor of M.

The invariants h_{2k} just defined have proven to be useful in geometry. For example, the first proof of the Gauss-Bonnet theorem for manifolds of dimension greater than two was given by Allendoerfer and Weil [1], and used Weyl's formula. Another example where the invariants h_{2k} are important is the Kinematic formula of Chern [3] and Federer [6]. This shows that it is of some interest to compute the volume of tubes for more general pairs $(M,\widetilde{M})$ and see if invariants similar to the h_{2k} defined by Weyl can be defined.

The results of this paper show that in the case M is a "symmetrically embedded" submanifold of a naturally reductive Riemannian homogeneous space $\widetilde{M}$ (definitions below) then it is possible to define, for each real t and each integer k with $0 \leq k \leq \dim M$ a function

$p \to h_k(p,t;M,\widetilde{M})$ on M such that a formula for the volume of the tube $M(r)$ analogous to Weyl's holds. Specifically, if $n = \dim M$, then

$$\mathrm{vol}(M(r)) = \sum_{k=0}^{n} \int_M h_k(p,r;M,\widetilde{M})\, \Omega_M(p).$$

The function h_k is a polynomial of degree k in the components of the second fundamental form of M in $\widetilde{M}$.

In section 2 those standard results on the geometry of manifolds which will be needed later are given. For the most part, the exposition follows that of Kobayashi and Nomizu [8].

In section 3, we give formulas to compute the curvature and Jacobi fields of a Riemannian manifold M in terms of the curvature and torsion of a connection on M that preserves the metric of M and has the same geodesics as the Riemannian connection of M. It is also shown there is a bijective correspondence between such connections and the smooth 3-forms on M. The results of this section seem to be new, however it is possible they are only of interest when the connection in question is the canonical connection of a naturally reductive Riemannian homogeneous space. In this case they are well known.

Sections 4 and 5 are both expository. Section 4 gives the results on the geometry of submanifolds needed in the sequel. Section 5 gives the results on Riemannian homogeneous spaces that are needed. The calculations of section 3 are used here.

Section 6 contains the main results of this paper. First the notion of a symmetrically embedded submanifold of a naturally reductive Riemannian homogeneous space is defined (definition 6.1). Proposition 6.2 then gives a geometric interpretation of what being symmetrically embedded means. The volume of a tube about a compact symmetrically embedded submanifold is then computed. It is the introduction of the fields of linear maps $\overline{S}(t;U)$, $\overline{C}(t;U)$, and $\overline{S}^{\perp}(t;U)$ along geodesics normal to the submanifold which allows the calculation to be done. These linear maps can also be used to compute the Weingarten map of the tube. However, this calculation is not done here.

The results of section 7 are algebraic. The basic problem is to expand $\det(A+B)$ into a sum by something resembling the bionomial theorem. This was done by Flanders [5]. He uses the universal properties of tensor products in his definition of what is written here as $A*B$. This makes comparison with formulas in classical notation hard. The calculations needed to compare the two are done in detail here.

In section 8, the algebraic results of section 7 are used to expand the function $h(p,t)$ of the tube formula of theorem 6.14 into terms $h_k(p,t) = h_k(p,t;M,\widetilde{M})$ homogeneous of degree k in the components of the Weingarten map of M in $\widetilde{M}$. The functions $h_k(p,t;M,\widetilde{M})$ are then the natural generalization of the invariants defined by Weyl. It is also shown that, if $\widetilde{M}$ is a symmetric space then $h_k(p,t;M,\widetilde{M})$ vanishes for k odd.

In section 9 the classical results of Steiner [11] on parallel surfaces are generalized to hypersurfaces in a naturally reductive Riemannian homogeneous space.

In the case where M is a symmetrically embedded submanifold of a symmetric space $\widetilde{M}$, it is possible to express the linear maps $C(t;U)$, $S(t;U)$, and $S^{\perp}(t;U)$ needed in the tube formula explicitly in terms of the Lie algebra of a transitive group of isometries of $\tilde{M}$. This is done in section 10.

In section 11 a formula relating the invariants $h_{2k}((p_1,p_2),t;M_1 \times M_2,\tilde{M}_1 \times \tilde{M}_2)$ to the invariants of the pairs $(M_1,\tilde{M}_1)$ and $(M_2,\tilde{M}_2)$ is given. This generalizes the corresponding result for the invariants given by Weyl in the case $\tilde{M}_1$ and $\tilde{M}_2$ are Euclidean. This gives more evidence that the invariants introduced here are reasonable generalizations of Weyl's invariants.

In the last section some examples are give.

2. Connections on the frame bundle of a manifold.

All manifolds will be assumed to be Hausdorff, paracompact and of class C^∞. If a manifold is not connected, then all connected components are assumed to have the same dimension. The word "smooth" applied to either manifolds or maps will mean "of class C^∞". If M is a manifold, then TM (also written as $T(M)$) will be the tangent bundle of M and TM_p (or $T(M)_p$) will be the tangent space to M at p. If $f : M \to N$ is a smooth map between manifolds, then $f_{*p} : TM_p \to TN_{f(p)}$ is the derivative of f at p. The characterization of tensor fields as objects multilinear over the ring of smooth functions of a manifold will be used (see [8] vol. 1, page 26).

For the rest of this chapter, fix some manifold M of dimension n and a real vector space $\mathfrak{m}$ of the same dimension as M. We now define the *bundle of linear frames* over M, or, more briefly the *frame bundle* of M. For each p in M, let $L(M)_p$ be the set of all linear isomorphisms of $\mathfrak{m}$ onto TM_p. An element of $L(M)_p$ will be called a frame at p. The frame bundle $L(M)$ is the disjoint union of the $L(M)_p$ with p in M. For each p in M, the set $L(M)_p$ is called the *fibre* of $L(M)$ over p. A map $\pi : L(M) \to M$ is defined by taking all elements of $L(M)_p$ to p. This map is called the *projection* of $L(M)$ onto M. Let $GL(\mathfrak{m})$ be the group of all linear automorphisms of the vector space $\mathfrak{m}$ with its usual structure as a Lie group. Then there is a natural right action of $GL(M)$ on $L(M)$ by

$$(u,a) \mapsto u \circ a,$$

where $u \in L(M)$ and $a \in GL(\mathfrak{m})$. We now wish to make $L(M)$ into a smooth manifold in such a way that the projection π and the action of

$GL(\mathfrak{m})$ on $L(M)$ are smooth. By way of notation, for each open subset U of M let $L(U) = \pi^{-1}(U)$.

Definition 2.1. Let U be an open subset of M. Then a *moving frame over* U is a function

$$e : U \to L(M)$$

such that:

(1) $\pi \circ e$ = identity on U;

(2) If e_p is the value of e at p, then for all v in $\mathfrak{m}$, the function $p \to e_p(v)$ is a smooth vector field on U.

Remark. Let $\varphi : U \to \mathfrak{m}$ be a diffeomorphism of the open subset U of M with the open subset $\varphi(U)$ of $\mathfrak{m}$. Then, under the standard identification of tangent spaces to $\mathfrak{m}$ with $\mathfrak{m}$, the function $\varphi : U \to L(M)$ defined by

$$e_p = (\varphi_{p*})^{-1}$$

is a moving frame over U. Therefore every point of M is in the domain of some moving frame.

Proposition 2.2. There is a unique structure of a differential manifold on $L(M)$ such that:

(1) The projection $\pi : L(M) \to M$ is smooth;

(2) the right action of $GL(\mathfrak{m})$ on $L(M)$ given above is smooth;

(3) every moving frame $e : U \to L(M)$ over some open subset U of M is a smooth function.

Outline of the proof. If the three conditions of the proposition hold, then it is straightforward to check that, for each moving frame

$e : U \to L(M)$ over some open subset U of M, the map φ_e from $U \times GL(m)$ onto $\pi^{-1}(U)$ given by

$$\varphi_e(p,a) = e_p \circ a$$

is a diffeomorphism. This determines the smooth structure of $L(M)$ in the open subset $\pi^{-1}(U)$ of $L(M)$. By the remark before the proposition, $L(M)$ is covered by such sets. Thus, the smooth structure on $L(M)$ is unique, provided it exists.

Let $e_j : U_j \to L(M)$ $j = 1, 2$ be two moving frames over the open subsets U_1, U_2 of M. Then it is not hard to check that

$$\varphi_{e_1}^{-1} \circ \varphi_{e_2} : (U_1 \cap U_2) \times GL(m) \to (U_1 \cap U_2) \times GL(m)$$

is a diffeomorphism. Therefore, the maps φ_e, where e is a moving frame, can be used to define an atlas for $L(M)$. This finishes the proof.

The proof of the following is left to the reader.

Proposition 2.3. With notation as above,

(1) The dimension of $L(M)$ is $n^2 + n$;

(2) the projection π is a submersion (that is, π_{*u} is surjective for all u in $L(M)$);

(3) each fibre $L(M)_p$ is a closed embedded submanifold of $L(M)$ diffeomorphic to $GL(m)$ and the action of $GL(m)$ on the fibre $L(M)$ is simply transitive;

(4) the tangent space to a fibre $L(M)_p$ at a frame u is the kernel of π_{*u}.

We now define the class of geometric objects on which most of our calculations will be done. If G is a closed subgroup of $GL(\mathfrak{m})$, then G also has a right action on $L(M)$ in an obvious way.

Definition 2.4. Let G be a closed subgroup of $GL(\mathfrak{m})$. Then a G-*structure* on M (also called a *reduction of* $L(M)$ *to* G) is an embedded submanifold P of $L(M)$ such that;

(1) The restriction of the projection π to P is a submersion of P onto M;

(2) for each p in M the *fibre* P_p, defined to be $P_p = L(M)_p \cap P$, is an embedded submanifold of P such that the action of G on P_p is simply transitive.

Some elementry facts about G-structures are given in the following.

Proposition 2.5. Let P be a G-structure on M; then,

(1) The dimension of P is $\dim(M) + \dim(G)$.

(2) Each fibre of P is diffeomorphic to G.

(3) If $\pi : P \to M$ is the projection, then, for each p in M and u in the fibre P_u,

$$T(P_p)_u = \text{kernel } (\pi_{*u}).$$

(4) Each point of M has an open neighborhood U and a moving frame $e : U \to L(M)$ defined on U such that $e_p \in P$ for all p in U. (Such moving frames are called *sections of* P *over* U.)

Proof. The first three parts are easy.

Because π is a submersion of P onto M the implicit function theorem lets us find a smooth function $e : U \to P$, defined in an open neighborhood of any given point of M, with $\pi \circ e$ = identity on U. It is not hard to verify that e is a section of P over U.

Examples. (1) It is clear that $L(M)$ is a $GL(\mathfrak{m})$ structure on M.

(2) Recall that a Riemannian metric on M is an assignment of an inner product $\langle\ ,\ \rangle_p$ on each tangent space TM_p to M, in such a way that if X and Y are smooth vector fields on M, then the function

$$p \mapsto \langle X(p), Y(p)\rangle_p$$

is smooth. Put an inner product $(\ ,\ )$ on $\mathfrak{m}$ and let $O(\mathfrak{m})$ be the group of all automorphisms of this inner product. Thus $O(\mathfrak{m})$ is isomorphic as a Lie group to the group of all $n \times n$ real orthogonal matrices. Let M have a Riemannian metric $\langle\ ,\ \rangle$. Then, for each p in M, let $O(M)_p$ be the set of all isometries of $\mathfrak{m}$ onto TM_p. Define $O(M)$ to be the union of the $O(M)_p$ with p in M. Then it can be verified that $O(M)$ is an $O(\mathfrak{m})$-structure on M, called the *bundle of orthogonal frames* of M.

Conversely, given an $O(\mathfrak{m})$-structure P on M we can define a Riemannian metric on M by

$$\langle X,Y\rangle_p = (u^{-1}X, u^{-1}Y)$$

where u is any element of P_p and $(\ ,\ )$ is the inner product on $\mathfrak{m}$; this inner product is well-defined because any two frames in P_p are related by the right action of an element of $O(\mathfrak{m})$. Then P will be the bundle of orthogonal frames for this Riemannian metric. Thus, giving an $O(\mathfrak{m})$-structure on M is the same as giving a Riemannian metric on M.

(3) Suppose M is a complex analytic manifold of complex dimension m (and thus real dimension $n = 2m$). Recall that this means that M has an atlas $\{(\varphi_\alpha, U_\alpha) : \alpha \in A\}$ such that for each $\alpha \in A$ φ_α is a diffeomorphism of the open subset U_α of M onto the open subset $\varphi_\alpha(U_\alpha)$ of $\mathbb{C}^m$ so that for each pair $\alpha, \beta \in A$ the function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is holomorphic. If M is such a manifold, then each tangent space TM_p, to M has the structure of a complex vector space. Multiplication of a tangent vector $X \in TM_p$ by a complex scalar a can be described as follows: Choose a chart $(\varphi_\alpha, U_\alpha)$ from the defining atlas of M with $p \in U_\alpha$, then

$$aX = (\varphi_\alpha)_{*p}^{-1}(a(\varphi_\alpha)_{*p}X).$$

This can easily be checked to be well-defined by using that if $\alpha, \beta \in A$ and $p \in U_\alpha \cap U_\beta$, then $(\varphi_\alpha \circ \varphi_\beta^{-1})_{*\varphi_\beta(p)}$ is complex linear. Now assume that $\mathfrak{m}$ is a complex vector space. For each p in M, let $C(M)_p$ be the set of all complex linear isomorphisms of $\mathfrak{m}$ onto TM_p, and let $C(M)$ be the union of all of the $C(M)_p$ with p in M. If $GL(\mathbb{C}, \mathfrak{m})$ is the group of all complex linear automorphisms of $\mathfrak{m}$, then $C(M)$ is a $GL(\mathbb{C}, \mathfrak{m})$ structure on $\mathfrak{m}$ called the *bundle of holomorphic frames* over M.

(4) If M is a complex analytic manifold, then a Hermitian metric $\langle \, , \rangle$ on M is a choice of a Hermitian inner product $\langle \, , \rangle_p$ on each tangent space TM_p to M such that for all smooth vector fields X, Y on M the complex valued function

$$p \mapsto \langle X(p), Y(p) \rangle_p$$

is smooth. Assume that $\mathfrak{m}$ is a complex vector space with Hermitian inner product (,) and that $U(\mathfrak{m})$ is the group of all complex linear automorphisms of (,). Then it is possible to define a $U(\mathfrak{m})$-structure $U(M)$ on M in a way that should be clear from the last two examples. This $U(\mathfrak{m})$-structure is called the *bundle of unitary frames over* M.

We now record some facts we will need about $GL(\mathfrak{m})$ and its closed subgroups. Let $\mathfrak{gl}(\mathfrak{m})$ be the Lie algebra of all linear endomorphisms of $\mathfrak{m}$ with Lie bracket given by

$$[A,B] = AB - BA.$$

Then $\mathfrak{gl}(\mathfrak{m})$ is the Lie algebra of $GL(\mathfrak{m})$.

For any $A \in \mathfrak{gl}(\mathfrak{m})$ define e^A by its power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} .$$

Then every continuous homomorphism from the group of additive real numbers to $GL(\mathfrak{m})$ is of the form

$$t \mapsto e^{tA}$$

for some A in $\mathfrak{gl}(\mathfrak{m})$.

If G is a closed subgroup of $GL(\mathfrak{m})$ then, by the "closed subgroup theorem" of E. Cartan ([12] Theorem 3.42, page 11), G is an embedded submanifold of $GL(\mathfrak{m})$. Let 1 be the identity element of $GL(\mathfrak{m})$. Then $\mathfrak{g}$, the tangent space to G at 1, is the Lie algebra of G and is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{m})$. By parallel translating the tangent space to G at 1 to the origin (zero element) we can and often will view elements of $\mathfrak{g}$ as linear transformations on $\mathfrak{m}$. It should also be noted that $\mathfrak{g}$ is the set of all A in $\mathfrak{gl}(m)$ such that e^{tA} is in G for all real t.

The *adjoint representation* of G on $\mathfrak{g}$ is given by $a \mapsto Ad(a)$ where

$$Ad(a)A = aAa^{-1}.$$

It is easy to check that

$$e^{Ad(a)A} = ae^{A}a^{-1}.$$

<u>Convention 2.6</u>. Unless stated otherwise, for the rest of this chapter "P" will denote some fixed G-structure on M where G is some fixed closed subgroup of $GL(m)$ with Lie algebra $\mathfrak{g}$.

<u>Definition 2.7</u>. (1) For each a in G define right translation by a on P by

$$r_a(u) = ua$$

(2) For each A in $\mathfrak{g}$ define the *fundamental vector field* A^* on P by

$$A^*(u) = \frac{d}{dt}\Big|_{t=0} ue^{tA}$$

<u>Proposition 2.8</u>. (1) The flow of the vector field A^* is $r_{e^{tA}}$. (For the definition of the flow, or local 1-parameter group generated by a vector field see [12] 1.49 Definitions, page 39.)

(2) For $A \in \mathfrak{g}$ and $a \in G$

$$r_{a*}A^* = (Ad(a^{-1})A)^*.$$

(3) The map $A \mapsto A^*$ is a Lie algebra homomorphism of $\mathfrak{g}$ into the Lie algebra of all smooth vector fields on P.

(4) For each u in P the map $A \mapsto A^*(u)$ is injective.

(5) For each u in P the tangent space to the fibre $P_{\pi u}$ at u is

$$T(P_{\pi u})_u = \{A^*(u) : A \in \mathfrak{g}\}.$$

<u>Proof</u>. (1) It is easy to check that

$$r_{e^{tA}} \circ r_{e^{sA}} = r_{e^{(t+s)A}}.$$

The result now follows from the definition of a flow.

(2) The tangent vector to the curve $t \mapsto ue^{tA}$ at $t = 0$ is $A^*(u)$. Therefore

$$\begin{aligned} r_{a_*}A^*(u) &= \frac{d}{dt}\Big|_{t=0} r_a(ue^{tA}) \\ &= \frac{d}{dt}\Big|_{t=0} ua\,a^{-1}e^{tA}a \\ &= \frac{d}{dt}\Big|_{t=0} ua\,e^{tAd(a^{-1})A} \\ &= (Ad(a^{-1})A)^*(ua). \end{aligned}$$

This proves (2).

(3) For each u in P define a map $\sigma_u : G \to P$ by $\sigma_u(a) = ua$. The tangent vector to the curve $t \mapsto e^{tA}$ at $t = 0$ is A; therefore,

$$\begin{aligned} \sigma_{u*1}A &= \frac{d}{dt}\Big|_{t=0} \sigma_u e^{tA} \\ &= \frac{d}{dt}\Big|_{t=0} ue^{tA} \\ &= A^*(u). \end{aligned}$$

The map σ_{u*1} is linear, which shows $A \mapsto A^*(u)$ is linear for all u. It follows that $A \mapsto A^*$ is linear.

Let $\mathcal{L}_{A^*}$ be the Lie derivative with respect to A^* (see [12], pages 69 and 70 for the definition of Lie derivative and for a proof of the equality $\mathcal{L}_X Y = [X,Y]$). Using (2) and the fact that the flow of A^* is $r_{e^{tA}}$, we have

$$[A^*,B^*](u) = (\mathcal{L}_{A^*}B^*(u)$$

$$= \frac{d}{dt}\Big|_{t=0} r_{e^{-tA}*} B^*(ue^{tA})$$

$$= \frac{d}{dt}\Big|_{t=0} (Ad(e^{tA})B)^*(u).$$

We have just shown the map $C \mapsto C^*(u)$ to be linear from $\mathfrak{g}$ to $T(P)_u$. Therefore, if $t \mapsto C_t$ is any smooth curve in $\mathfrak{g}$ it follows that

$$\frac{d}{dt}(C_t)^* (u) = (\frac{d}{dt}C_t)^* (u).$$

This yields

$$[A^*,B^*](u) = \frac{d}{dt}\Big|_{t=0} (Ad(e^{tA})B)^* (u)$$

$$= (\frac{d}{dt}\Big|_{t=0} Ad(e^{tA})B)^* (u)$$

$$= [A,B]^* (u).$$

This completes the proof that $A \mapsto A^*$ is a Lie algebra homomorphism.

(4) Let $A \in \mathfrak{g}$ and $u \in P$ with $A^*(u) = 0$. Then because the flow of A^* is $r_{e^{tA}}$ it follows that

$$r_{e^{tA}}(u) = ue^{tA}$$
$$= u$$

for all real t. The action of G on fibres is simply transitive; therefore $e^{tA} = 1$ for all t. This implies $A = 0$. This along with linearity of the map $A \mapsto A^*(u)$, proves (4).

(5) By (3) and (4) we see that $\{A^*(u) : A \in \mathfrak{G}\}$ is a linear space of the same dimension as G. The vector space $T(P_{\pi u})_u$ is also of this dimension. Thus to show the two are equal it is enough to show the first is a subspace of the second. If $a \in G$ then it is clear that $\pi \circ r_a = \pi$. Consequently

$$\begin{aligned} \pi_{*u} A^*(u) &= \frac{d}{dt}\Big|_{t=0} \pi u e^{tA} \\ &= \frac{d}{dt}\Big|_{t=0} \pi u \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} \{A^*(u) : A \in \mathfrak{G}\} &\subseteq \mathrm{Kernel}(\pi_{\alpha u}) \\ &= T(P_{\pi u})u. \end{aligned}$$

This finishes the proof.

<u>Definition 2.9</u>. Vectors tangent to some fibre P_p of P will be called *vertical*.

<u>Remark</u>. It will be convenient to use the formalism of vector valued differential forms. The following list of definitions is given so as to fix our conventions on what constants are used in the definitions of exterior derivative and wedge product. Let V be a real vector space. Then a V-*valued* r-*form* ω on M is a smooth assignment for each p in M of an r-linear alternating function ω_p on TM_p with values in V. When $r = 0$, ω is defined to be a smooth function with values in V. In the case $V = \mathbb{R}$, ω is just called an r-form. The *exterior derivative* $d\omega$ of ω is the V-valued $(r+1)$-form given on smooth

vector fields $X_0, \ldots, X_r$ by

$$d\omega(X_0,\ldots,X_r) = \sum_{0\le i\le r} (-1)^i X_i\omega(X_0,\ldots\hat{X}_i,\ldots,X_r)$$

$$+ \sum_{0\le i<j\le r} (-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r).$$

where ^ means the term is omitted. For $r = 0$ and 1 this becomes

$$d\omega(X) = X\omega,$$

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

If ω is a V-valued r-form, θ is a W-valued s-form and $\langle\ ,\ \rangle$ a bilinear function on $V \times W$ with values in the vector space S then the *wedge product* of ω and θ is the s-valued $(r+s)$-form given by

$$\langle\omega\overset{\wedge}{,}\theta\rangle(X_1,\ldots,X_{r+s})$$

$$= \frac{1}{r!s!}\sum_{\sigma}(-1)^{\sigma}\langle\omega(X_{\sigma(1)},\ldots,X_{\sigma(r)}),\theta(X_{\sigma(r+1)},\ldots,X_{\sigma(r+s)})\rangle$$

where the sum is over all permutations σ of the set $\{1,\ldots,r+s\}$ and $(-1)^{\sigma}$ is the sign of the permutation σ. It can be checked that

$$d\langle\omega\overset{\wedge}{,}\theta\rangle = \langle d\omega\overset{\wedge}{,}\theta\rangle + (-1)^r\langle\omega\overset{\wedge}{,}d\theta\rangle.$$

In the case in which both V and W are the real numbers and $\langle\ ,\ \rangle$ is multiplication of real numbers, we just write $\omega\wedge\theta$ for $\langle\omega\overset{\wedge}{,}\theta\rangle$.

Definition 2.10. A *connection* on P is a smooth $\mathfrak{g}$-valued one-form ω on P that satisfies the following two conditions;

(1) The value of ω on vertical vectors is given by

$$\omega_u(A^*(u)) = A$$

for all A in $\mathfrak{g}$ and u in P.

(2) ω transforms under the action of $a \in G$ by

$$r_a^* \omega = \mathrm{Ad}(a^{-1})\omega.$$

<u>Definition 2.11</u>. If ω is a connection on P then for each $u \in P$ let

$$H_u = \mathrm{kernel}(\pi_{*u}).$$

Then H_u is called the *space of horizontal vectors* at u or more briefly the horizontal space at u.

<u>Proposition 2.12</u>. Let $\{H_u : u \in P\}$ be the set of all horizontal vectors, for the connection ω on P. Then,

(1) $\{H_u : u \in P\}$ is a smooth distribution on M.

(2) For all $a \in G$ and $u \in P$

$$r_{a*}H_u = H_{au}$$

(3) For all $u \in P$

$$T(P)_u = H_u \oplus T(P_{\pi u})_u \qquad \text{(direct sum)}.$$

Conversely, let $\{H_u : u \in P\}$ satisfy (1), (2) and (3) and define ω to be the $\mathfrak{g}$-valued one-form on P given by $\omega_u(A^*(u)) = A$ for A in $\mathfrak{g}$ and $\omega_u(X_u) = 0$ if X_u is in H_u. Then ω is a connection on P and the horizontal spaces defined by ω are $\{H_u : u \in P\}$.

<u>Proof</u>. See proposition 1.1 on page 64 of vol. 1 of [8].

<u>Remark</u>. A connection is often defined to be a smooth distribution $\{H_u : u \in P\}$ satisfying (1), (2) and (3) of the last proposition. Then ω is defined as above and is called the connection form of the connection.

(2) Let $\bar{\omega}$ be a connection on $L(M)$. Then $\bar{\omega}$ is the extension of a connection on P if and only if, for each u in P, the space $\bar{H}_u$ of horizontal vectors determined by $\bar{\omega}$ at u is tangent to P.

Proof. The first part is a special case of proposition 6.1 on page 61 of vol. 1 of [8]. The second part is straightforward.

Remark. Some of the definitions below, such as parallel translation along a curve or the curvature and torsion tensors on M, can be given in terms of either a connection ω on P or the extended connection on $L(M)$. It will be left to the reader to show these definitions are independent of which of these two connections is used.

Definition 2.14. Let ω be a connection on P and $c : (\alpha,\beta) \to M$ be any piecewise smooth curve. Then a piecewise smooth curve $\hat{c} : (\alpha,\beta) \to P$ is called a *horizontal lift* of c if and only if $\pi \circ \hat{c} = c$ and $\hat{c}'(t)$ is horizontal for all t.

Proposition 2.15. Let ω be a connection on P, $c : (\alpha,\beta) \to M$ a piecewise smooth curve, $t_0 \in (\alpha,\beta)$ and $u \in P_{c(t_0)}$. Then there is a unique horizontal lift $\hat{c}$ of c to P with $\hat{c}(t_0) = u_0$. If $a \in G$, then the horizontal lift $\gamma : (\alpha,\beta) \to P$ with $\gamma(t_0) = u_0 a$ is given by $\gamma(t) = \hat{c}(t)a$.

Proof. This follows from proposition 3.1, page 69 of [8].

Definition 2.16. Let ω be a connection on P, $c : (\alpha,\beta) \to M$ a piecewise smooth curve and $t_1, t_2 \in (\alpha,\beta)$. Then *parallel translation* along c from $TM_{c(t_1)}$ to $TM_{c(t_2)}$ is defined by

$$\tau_{t_2}^{t_1} = \hat{c}(t_2)\hat{c}(t_1)^{-1}$$

where $\hat{c}$ is any horizontal lift of c to P. Clearly $\tau_{t_2}^{t_1}$ is a linear isomorphism of $TM_{c(t_1)}$ onto $TM_{c(t_2)}$.

By the last proposition any other horizontal lift of c is of the form $t \mapsto \hat{c}(t)a$. It follows that parallel translation is independent of the choice of the horizontal lift of c. It is also easy to check that if t_1, t_2, t_3 are in (α,β) then

$$\tau_{t_3}^{t_2}\,\tau_{t_2}^{t_1} = \tau_{t_3}^{t_1} .$$

Definition 2.17. Let Y be a smooth vector field defined on some open subset U of M and $X(p)$ a tangent vector to M at $p \in U$. Choose a smooth curve $c: (-\varepsilon,\varepsilon) \to U$ for some $\varepsilon > 0$ with $c'(0) = X(p)$. Then define

$$\nabla_{X(p)}Y = \frac{d}{dt}\Big|_{t=0}\tau_0^t Y(c(t))$$

where $\tau_0^t : TM_{c(t)} \to TM_{c(0)}$ is the parallel translation along c defined by the connection ω.

Remarks. (1) For all $t \in (-\varepsilon,\varepsilon)$ the vector $\tau_0^t Y(c(t))$ is in the finite dimensional vector space TMp. The derivative $\frac{d}{dt}\tau_0^t Y(t)$ is computed as the tangent vector to a curve in a vector space.

(2) The vector $\nabla_{X(p)}Y$ is independent of the choice of the curve c with $c'(0) = X(p)$. See pages 114 and 115 of vol. 1 of [8].

(3) To compute $\nabla_{X(p)}Y$ it is enough to know the values of Y along any curve c that fits $X(p)$ in the sense of the definition.

Proposition 2.18. The map $(X(p),Y) \mapsto \nabla_{X(p)}Y$ defined above satisfies the following five relations:

(1) $\nabla_{X_1(p)+X_2(p)}Y = \nabla_{X_1(p)}Y + \nabla_{X_2(p)}Y$.

(2) $\nabla_{cX(p)}Y = c\nabla_{X(p)}Y$ for all real c.

(3) $\nabla_{X(p)}(Y_1+Y_2) = \nabla_{X(p)}Y_1 + \nabla_{X(p)}Y_2$.

(4) $\nabla_{X(p)}(bY) = b(p)\,\nabla_{X(p)}Y + (X(p)b)\,Y(p)$ for all smooth real valued b with the same domain as Y.

(5) If X and Y are smooth vector fields on the open subset U of M, then so is $p \mapsto \nabla_{X(p)}Y$.

Proof. See proposition 1.1, page 114 of vol. 1 of [8].

Definition 2.19. Let $\mathcal{D}$ be the set of all pairs $(X(p),Y)$ where Y is a smooth vector field on some open subset of M and $X(p)$ is a vector tangent to M at some point p in the domain of Y. Then a function $(X(p),Y) \mapsto \nabla_{X(p)}Y$ defined on $\mathcal{D}$ and satisfying the five conditions of 2.18 is called a *covariant derivation* on M. If ∇ is defined from a connection ω then ∇ is called the covariant derivation of ω.

Proposition 2.20. (1) Two connections on P with the same covariant derivation are equal.

(2) Every covariant derivation on M is the covariant derivation of a (unique by (1)) connection on $L(M)$.

Proof. See proposition 7.5, page 143 of vol. 1 of [8].

We now describe parallel translation in terms of the covariant derivation of a connection.

Definition 2.21. Let ∇ be the covariant derivation of the connection ω on P, and $c:(\alpha,\beta) \to M$ a smooth curve. Then a vector field

$t \mapsto Y(t)$ along c is called *parallel* if and only if

$$(\nabla_{c'(t)}Y)(t) = 0$$

for all t in (α,β).

Proposition 2.22. Let ω be a connection with covariant derivation ∇ on P, $c:(\alpha,\beta) \to M$ a smooth curve, and $t_0 \in (\alpha,\beta)$. If τ is the parallel translation defined along c by ω, then every parallel vector field $t \mapsto Y(t)$ along c is of the form

$$Y(t) = \tau_t^{t_0} Y_0$$

for some Y_0 in $TM_{c(t_0)}$. Therefore, for every Y_0 in $TM_{c(t_0)}$ there is a unique parallel field $t \mapsto Y(t)$ along c with $Y(t_0) = Y_0$. The vector $Y(t)$ is called the *parallel translate of* Y_0 *along* c to $c(t)$.

Proof. If $Y(t) = \tau_t^{t_0} Y_0$ then for any t_1 in (α,β)

$$\begin{aligned}
\nabla_{c'(t_1)}Y(t) &= \frac{d}{dt}\Big|_{t=t_1} \tau_{t_1}^{t} Y(t) \\
&= \frac{d}{dt}\Big|_{t=t_1} \tau_{t_1}^{t} \tau_t^{t_0} Y_0 \\
&= \frac{d}{dt}\Big|_{t=t_1} \tau_{t_1}^{t_0} Y_0 \\
&= 0.
\end{aligned}$$

Therefore $Y(t)$ is parallel. Let $t \mapsto Y(t)$ be parallel along c and let $X_1, \ldots, X_n$ be a basis of $TM_{c(t_0)}$. Define fields $x_1(t), \ldots, x_n(t)$ along c by

$$X_0(t) = \tau_t^{t_0} X_j.$$

Then we have just shown each $X_j(t)$ is parallel along c. The map $\tau_t^{t_0}$ from $TM_{c(t_0)}$ to $TM_{c(t)}$ is a linear isomorphism, therefore $X_1(t), \ldots, X_u(t)$ is a basis of $TM_{c(t)}$ for all t in $TM_{c(t)}$. Whence,

$$Y(t) = \sum_{i=1}^{n} y_i(t)\, X_i(t)$$

for some smooth functions $y_1, \ldots, y_n$ on (α,β). By proposition 2.18, we have

$$\begin{aligned} 0 &= \nabla_{c'(t)} Y(t) \\ &= \sum_{i=1}^{n} y_i'(t)\, X_i(t) + \sum_{i=1}^{n} y_i(t)\, \nabla_{c(t)}\, X_i(t) \\ &= \sum_{i=1}^{n} y_i'(t)\, X_i(t). \end{aligned}$$

This shows $y_i' = 0$, so each y_i is constant. Consequently,

$$\begin{aligned} Y(t) &= \sum_{i=1}^{n} y_i(t_0)\, X_i(t) \\ &= \sum_{i=1}^{n} y_i(t_0)\, \tau_t^{t_0} X_i \\ &= \tau_t^{t_0}\Big(\sum_{i=1}^{n} y_i(t_0)\, X_i\Big) \\ &= \tau_t^{t_0}\, Y(t_0). \end{aligned}$$

This finishes the proof (with $Y_0 = Y(t_0)$).

The next several definitions are devoted to defining the curvature and torsion forms on P and the corresponding curvature and torsion tensors on M.

Definition 2.23. The *canonical form* θ on P is the $\mathfrak{m}$-valued one-form on P given by

$$\theta_u(X) = u^{-1}\pi_{*u}X.$$

Remark. The canonical form θ is defined independently of any connection on P and the kernel of θ_u is the space of vertical vectors at u.

Proposition 2.24. If θ is the canonical form on P then θ transforms under the action of G on P by

$$r_a^*\theta = a^{-1}\theta.$$

Proof. Straightforward.

Definition 2.25. Let α be a k-form on P with values in some vector space V. Then the *covariant differential* $D\alpha$ of α defined by the connection ω on P is the V-valued $k+1$ form given by

$$(D\alpha)(X_1,\ldots,X_{k+1}) = d\alpha(hX_1,\ldots,hX_{k+1})$$

where d is exterior derivative and $X = hX + vX$ is the decomposition of X into its horizontal component hX and its vertical component vX defined by the connection ω.

Definition 2.26. Let ω be a connection on P and D the covariant differential defined by ω. Then:

(1) The *torsion form* Θ of ω is the $\mathfrak{m}$-valued two-form given by

$$\Theta = D\theta.$$

(2) The *curvature form* Ω of ω is the $\mathfrak{g}$-valued two-form given by

$$\Omega = D\omega.$$

The proof of the next proposition is straightforward.

Proposition 2.27. The torsion form Θ and the curvature form Ω of a connection ω on P transform under the action of G by

$$r_a^* \Theta = a^{-1}\Theta,$$

$$r_a^* \Omega = Ad(a^{-1})\,\Omega$$

for a in G.

Definition 2.28. Let ω be a connection on P. Then, for each $p \in M$, $X \in TM_p$ and $u \in P_p$ we define the *horizontal lift* $\hat{X}(u)$ of X to u by letting $\hat{X}(u)$ be the unique horizontal vector at u with

$$\pi_{*u}\hat{X}(u) = X.$$

Remark. It is easy to check using 2.12 (2) that $r_{a*}\hat{X}(u) = \hat{X}(ua)$.

Definition 2.29. Define the *torsion tensor* T and the *curvature tensor* R of a connection ω on P by

$$T_p(X,Y) = u(\Theta_u(\hat{X}(u),\hat{Y}(u)))$$

$$R_p(X,Y)Z = u(\Omega_u(\hat{X}(u),\hat{Y}(u))u^{-1}Z)$$

where $X,Y,Z \in TM_p$, $\pi u = p$ and $\hat{X}(u)$, $\hat{Y}(u)$ are the horizontal lifts of X and Y to P.

Elementary calculations using proposition 2.27 and the remark preceding the definition show that the definitions are independent of the choice of u with $\pi u = p$.

Proposition 2.30. The two tensors T and R defined above are related to the covariant derivation ∇ of the connection ω by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where X, Y, Z are smooth vector fields defined on some subset of M.

Proof. This is theorem 5.1, page 133, vol. 1 of [8].

We now define the *covariant derivatives* of $(\nabla_X T)$ and $(\nabla_X R)$ in the usual way, which is by requiring the product rule to hold, i.e.,

$$(\nabla_X T)(Y,Z) = \nabla_X(T(Y,Z)) - T(\nabla_X Y,Z) - T(Y,\nabla_X Z),$$

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W$$
$$-R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W$$

where Y, Z, W are smooth vector fields on some open subset of M.

Proposition 2.31. Let T be the torsion tensor and R the curvature tensor of a connection ω on P. Then the following hold:

First Bianchi Identity.

$$\mathfrak{S}(R(X,Y)Z) = \mathfrak{S}(T(T(X,Y),Z + (\nabla_X T)(Y,Z))$$

Second Bianchi Identity.

$$\mathfrak{S}((\nabla_X R)(Y,Z) + R(T(X,Y),Z)) = 0$$

where $\mathfrak{S}$ is cyclic sum over X, Y and Z.

Proof. This is theorem 5.3, page 135 of vol. 1 of [8].

Definition 2.32. Let ω be a connection on P with covariant derivation ∇. Then a smooth curve $g : (\alpha,\beta) \to M$ is a *geodesic* of ω (or of ∇) if and only if $t \mapsto g'(t)$ is a parallel vector field along g. That is, g is a geodesic of ω if and only if

$$\nabla_{g'(t)} g'(t) = 0$$

for all t in (α,β).

Definition 2.33. Let ω be a connection on P and v a vector in $\mathfrak{m}$. Then the *basic vector field* $B(v)$ on P determined by $v \in \mathfrak{m}$ is defined by letting $B(v)_u$ be the unique horizontal vector at u with

$$\pi_{*u} B(v)_u = u(v).$$

An equivalent definition is

$$B(v)_u = \widehat{u(v)}(u).$$

Proposition 2.34. A curve $g : (\alpha,\beta) \to M$ is a geodesic for the connection ω on P if and only if g is of the form $\pi \circ \gamma$, where $\gamma : (\alpha,\beta) \to P$ is an integral curve of one of the basic vector fields $B(v)$. Consequently, for each tangent vector $X(p)$ to M there is a unique geodesic g defined in a maximal connected neighborhood of zero in the real numbers $\mathbb{R}$ with $y(0) = p$ and $y'(0) = x(p)$.

Proof. See proposition 6.3 and theorem 6.4 on page 139 of vol. 1 of [8].

Definition 2.35. Let ω be a connection on P then the *exponential map* determined by ω is defined as follows. For $X \in TM_p$ write $t \mapsto \exp_p(tX)$ for the unique geodesic with

$$\exp_p(oX) = p$$

$$\frac{d}{dt}\Big|_{t=0} \exp_p(tX) = X.$$

Then the exponential map from TM_p to M is the function

$$X \mapsto \exp_p(X) = \exp_p(1 \cdot X).$$

This is defined in a neighborhood of zero in TM_p.

We will need to take derivatives of the exponential map. This task is reduced to computations with ordinary differential equations by the following definition and proposition.

Definition 2.36. Let $g:(a,b) \to M$ be a geodesic for a connection with covariant derivation ∇. Then

(1) A vector field $Y(t)$ along g is a *Jacobi field* along g if and only if it is a solution to the *Jacobi equation*

$$\nabla^2_{g'(t)}Y(t) + \nabla_{g'(t)}(T(Y(t),g'(t))) + R(Y(t),g'(t))g'(t) = 0$$

along g. Here T and R are the torsion and curvature tensor of ∇.

(2) A *variation of* g *through geodesics* is a smooth function $\alpha:(-\varepsilon,\varepsilon) \times (a,b) \to M$ (for some $\varepsilon > 0$) such that $\alpha(0,t) = g(t)$ and for all $s \in (-\varepsilon,\varepsilon)$ the map $t \mapsto \alpha(s,t)$ is a geodesic.

Proposition 2.37. Let $g:(a,h) \to M$ be a geodesic for a connection with covariant derivation ∇.

Then:

(1) A Jacobi field Y along g is determined by the values of $Y(t0)$ and $(\nabla_{g'(t)}Y)(t_0)$ for any $t_0 \in (a,b)$ and these values can be specified arbitrarily.

(2) If $\alpha:(-\varepsilon,\varepsilon) \times (a,b) \to M$ is a variation of g through geodesics then $t \mapsto \frac{\partial\alpha}{\partial s}(0,t)$ is a Jacobi field along g.

Proof. (1) The Jacobi equation is a homogeneous linear second order ordinary differential equation; therefore, (1) follows from standard results.

(2) See theorem 1.2, page 64 of vol. 2 of [8].

Proposition 2.38. Let ∇ and ∇' be covariant derivations on M. For smooth vector fields X and Y on M, let

$$C(X,Y) = \nabla_X Y - \nabla'_X Y.$$

Then C is a tensor field of type (1,2) (called the *difference tensor* of ∇ and ∇'). The covariant derivations ∇ and ∇' have the same geodesics if and only if C is alternating.

Proof. See proposition 1.5 on page 271 of vol. 2 of [10].

We now turn to connections on Riemannian manifolds.

Proposition 2.39. Let M be a Riemannian manifold with metric $\langle\ ,\ \rangle$ and let O(M) be the bundle of orthogonal frames over M. If ω is a connection on L(M) then the following are equivalent:

(1) ω is the extension of some connection on O(M).

(2) Parallel translation along any smooth curve in M is an isometry between tangent spaces of M.

(3) If ∇ is the covariant derivation of ω and X, Y, Z are smooth vector fields on M then

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Proof. The equivalence of (1) and (2) is the content of proposition 1.5 on page 117 of vol. 1 of [8].

Suppose (2) holds and let Y,Z be smooth vector fields on M. Let X be any tangent vector to M and choose a smooth curve $c : (-\varepsilon,\varepsilon) \to M$ such that $c'(0) = X$. Let τ be the parallel translation along c. Choose an orthonormal basis $e_1, \ldots, e_n$ of $TM_{c(0)}$ and let $e_j(t) = \tau_t^0 e_j$. Because τ_t^0 is an isometry, $e_1(t), \ldots, e_n(t)$ is an orthonormal basis of $TM_{e(t)}$, for all t. There are smooth functions $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ on $(-\varepsilon,\varepsilon)$ with

$$Y(c(t)) = \sum_{i=1}^{n} y_i(t)\ e_j(t),$$

$$Z(c(t)) = \sum_{j=1}^{n} z_j(t)\ e_j(t).$$

Therefore,

$$\begin{aligned}
&c'(t)\langle Y(c(t)),Z(c(t))\rangle \\
&= \frac{d}{dt}\langle \sum_{i=1}^{n} y_i(t)\ e_i(t),\ \sum_{j=1}^{n} z_j(t)\ e_j(t)\rangle \\
&= \frac{d}{dt} \sum_{k=1}^{n} y_k(t)\ z_k(t) \\
&= \sum_{i=1}^{n} y_k'(t)\ z_k(t) + \sum_{k=1}^{n} y_k(t)\ z_k'(t) \\
&= \langle \sum_{i=1}^{n} y_k'(t)\ e_i(t),\ \sum_{j=1}^{n} z_j(t)\ e_j(t)\rangle \\
&\quad + \langle \sum_{i=1}^{n} y_i(t)\ e_i(t),\ \sum_{j=1}^{n} z_j'(t)\ e_j(t)\rangle \\
&= \langle \nabla_{c'(t)} Y(C(t)),Z(c(t)))\rangle \\
&\quad + \langle Y(c(t)),\ \nabla_{c'(t)} X(c(t))\rangle.
\end{aligned}$$

Noting that $c'(0) = X$ shows (2) implies (3).

Now assume (3) holds. Let $c : [a,b] \to M$ be a smooth curve and τ the parallel translation along c. Let Y,Z be vectors in $TM_{c(a)}$. Then

$$\frac{d}{dt}\langle \tau_t^a Y, \tau_t^a Z\rangle = c'(t)\langle \tau_t^a Y, \tau_t^a Z\rangle$$

$$= \langle \nabla_{c'(t)} \tau_t^a Y, \tau_t^a Z \rangle$$

$$+ \langle \tau_t^a Y, \nabla_{c'(t)} \tau_t^a Z \rangle$$

$$= 0.$$

Therefore $\langle \tau_t^a Y, \tau_t^a Z \rangle$ is constant as a function of t. This shows $\langle \tau_b^a Y, \tau_b^a Z \rangle = \langle Y, Z \rangle$, whence τ_b^a is an isometry of TM_a with TM_b. Thus (3) implies (2).

Definition 2.40. A connection on a Riemannian manifold that satisfies the three conditions of the last proposition is called *metric preserving*.

Proposition 2.41 (Fundamental lemma of Riemannian Geometry). Every Riemannian manifold has a unique metric preserving connection with vanishing torsion.

Remark. This connection is called the *Riemannian* connection or the *Levi-Civita connection*.

Proof. See theorem 2.2 on page 158 of vol. 1 of [8].

Definition 2.42. Let M be a Riemannian manifold. Then the *geodesics* of M are the geodesics of the Riemannian connection on M. The *curvature tensor* of M is the curvature tensor of the Riemannian connection. If R is the curvature tensor of M and P is a two-dimensional subspace of some tangent space TM_p then the *sectional curvature of* M *at* P is

$$K(P) = \langle R(X,Y)Y, X \rangle$$

where X,Y is any orthonormal basis of P. An easy calculation shows this is independent of the choice of the basis X,Y.

Proposition 2.44. If M is a Riemannian manifold with metric $\langle\ ,\ \rangle$ and curvature tensor R then for all X, Y, Z, W tangent to M at some point

(1) $\langle R(X,Y)Z,W\rangle + \langle Z,R(X,Y)W\rangle = 0.$

(2) $\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle.$

Proof. See proposition 2.1 on page 201 of vol. 1 of [8].

Remark. (1) of the last proposition tells us that for each $X, Y \in TM_p$ the linear map $R(X,Y)$ on TM_p is skew-symmetric with respect to the inner product $\langle\ ,\ \rangle_p$.

Definition 2.45. Let M be a Riemannian manifold with metric $\langle\ ,\ \rangle$ and $c: [a,b] \to M$ a smooth curve. Then the *length* of c is defined to be the number

$$L(c) = \int_a^b \|c'(t)\| dt$$

where

$$\|c'(t)\| = \sqrt{\langle c'(t),c'(t)\rangle}\ .$$

If p and q are points of M then the *distance from* p *to* q in M is defined to be the infimum of the set of numbers $L(c)$ where c is a curve from p to q.

Proposition 2.46. The geodesics in a Riemannian manifold locally are the curves of minimum length, in the sense that every point of M has an open neighborhood U such that any two points p and q of U can be joined by a unique geodesic contained in U and the length of this geodesic is the distance between p and q.

Proof. See proposition 3.6 on page 116 of vol. 1 of [8].

3. Connections preserving the metric and geodesics of a Riemannian manifold.

It will be convenient to speak of both a connection on the frame bundle $L(M)$ and of its covariant derivation as a connection. Because of the bijective correspondence between covariant derivations and connections on $L(M)$ given by proposition 2.20, this should not lead to any confusion. For the rest of this section "M" will denote a Riemannian manifold with metric $\langle\ ,\ \rangle$.

Definition 3.1. A connection with covariant derivative D will be called a *geometric connection* if and only if D preserves the metric of M and has the same geodesics as the Riemannian connection on M.

We will refer to D and not its connection as the geometric connection. Examples of geometric connections will be given below.

Proposition 3.2. Let D be a geometric connection on the Riemannian manifold M. Let T be the torsion tensor and B the curvature tensor of D. Let R be the curvature tensor of the Riemannian connection ∇ on M. Then, for all smooth vector fields X, Y, Z on M:

(1) The connections D and ∇ are related by

$$\nabla_X Y = D_X Y - \frac{1}{2} T(X,Y).$$

(2) The torsion tensor T of D satisfies

$$\langle T(X,Y),Z\rangle + \langle Y,T(X,Z)\rangle = 0.$$

(Thus the map $Y \mapsto T(X,Y)$ is skew-symmetric.)

(3) $R(X,Y)Z = B(X,Y)Z$

$$- \frac{1}{2}(D_X T)(Y,Z) + \frac{1}{2}(D_Y T)(X,Z) - \frac{1}{2} T(T(X,Y),Z)$$
$$+ \frac{1}{4} T(X,T(Y,Z)) - \frac{1}{2} T(Y,T(X,Z))$$

(4) $R(X,Y)Y = B(X,Y)Y + \frac{1}{2}(D_Y T)(X,Y) - \frac{1}{4} T(T(X,Y),Y)$.

(5) The sectional curvatures of M can be computed by

$$\langle R(X,Y)Y,X\rangle = \langle B(X,Y)Y,X\rangle + \frac{1}{4}\|T(X,Y)\|^2.$$

Proof. (1) Let $C(X,Y) = D_X Y - \nabla_X Y$ be the difference tensor of D and ∇. The connections D and ∇ have the same geodesics; therefore, proposition 2.38 yields that $C(X,Y)$ is alternating. Whence,

$$\begin{aligned} T(X,Y) &= D_X Y - D_Y X - [X,Y] \\ &= \nabla_X Y + C(X,Y) - \nabla_Y X + C(Y,X) - [X,Y] \\ &= (\nabla_X Y - \nabla_Y X - [X,Y]) + 2C(X,Y) \\ &= 2C(X,Y), \end{aligned}$$

where we have used that ∇ has vanishing torsion. This shows

$$C(X,Y) = \frac{1}{2} T(X,Y)$$

and proves (1).

For (2) we use that both ∇ and D are metric preserving. For any smooth vector fields X, Y, Z

$$\begin{aligned} X\langle Y,Z\rangle &= \langle D_X Y,Z\rangle + \langle Y,D_X Z\rangle \\ &= \langle \nabla_X Y + \frac{1}{2} T(X,Y),Z\rangle + \langle Y,\nabla_X Y + \frac{1}{2} T(X,Z)\rangle \\ &= \langle \nabla_X Y,Z\rangle + \langle Y,\nabla_X Z\rangle + \frac{1}{2}(\langle T(X,Y),Z\rangle + \langle Y,T(X,Z)\rangle) \\ &= X\langle Y,Z\rangle + \frac{1}{2}(\langle T(X,Y),Z\rangle + \langle Y,T(X,Z)\rangle). \end{aligned}$$

Therefore,

$$\langle T(X,Y),Z\rangle + \langle Y,T(X,Z)\rangle = 0.$$

(3) Let $X(p)$, $Y(p)$, $Z(p)$ be vectors tangent to M at some point p. Extend these to smooth commuting vector fields X, Y, Z defined on a

neighborhood of p. Then

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z,$$

$$B(X,Y)Z = D_X D_Y Z - D_Y D_X Z,$$

$$T(X,Y) = D_X Y - D_Y X.$$

Now compute

$$\begin{aligned}
R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z \\
&= D_X(D_Y Z - \tfrac{1}{2}\,T(Y,Z)) - \tfrac{1}{2}\,T(X, D_Y Z - \tfrac{1}{2}\,T(Y,Z)) \\
&\quad - D_Y(D_X Z - \tfrac{1}{2}\,T(X,Z)) + \tfrac{1}{2}\,T(Y, D_X Z - \tfrac{1}{2}\,T(X,Z)) \\
&= D_X D_Y Z - \tfrac{1}{2}(D_X T)(Y,Z) - \tfrac{1}{2}\,T(D_X Y, Z) \cancel{- \tfrac{1}{2}\,T(Y, D_X Z)} \\
&\quad \cancel{- \tfrac{1}{2}\,T(X, D_Y Z)} + \tfrac{1}{4}\,T(X, T(Y,Z)) \\
&\quad - D_Y D_X Z + \tfrac{1}{2}(D_Y T)(X,Z) + \tfrac{1}{2}\,T(D_Y X, Z) \cancel{+ \tfrac{1}{2}\,T(X, D_Y Z)} \\
&\quad \cancel{+ \tfrac{1}{2}\,T(Y, D_X Z)} - \tfrac{1}{4}\,T(Y, T(X,Z)) \\
&= (D_X D_Y Z - D_Y D_X Z) - \tfrac{1}{2}(D_X T)(Y,Z) + \\
&\quad - \tfrac{1}{2}\,T(D_X Y - D_Y X, Z) \\
&\quad + \tfrac{1}{4}\,T(X, T(Y,Z)) - \tfrac{1}{4}\,T(Y, T(X,Z)) \\
&= B(X,Y)Z - \tfrac{1}{2}(D_X T)(Y,Z) + \tfrac{1}{2}(D_Y T)(X,Z) - \tfrac{1}{2}\,T(T(X,Y),Z) \\
&\quad + \tfrac{1}{4}\,T(X, T(Y,Z)) - \tfrac{1}{4}\,T(Y, T(X,Z)).
\end{aligned}$$

Evaluation at p finishes the proof of (3).

(4) Set $Z = Y$ in (3) to get

$$\begin{aligned}
R(X,Y)Y &= B(X,Y)Y - \tfrac{1}{2}(D_X T)(Y,Y) + \tfrac{1}{2}(D_Y T)(X,Y) - \tfrac{1}{2}\,T(T(X,Y),Y) \\
&\quad + \tfrac{1}{4}\,T(X, T(Y,Y)) - \tfrac{1}{4}\,T(Y, T(X,Y)).
\end{aligned}$$

But $T(Y,Y) = 0$ and

$$(D_X T)(Y,Y) = D_X(T(Y,Y)) - (T(D_X Y,Y) + T(Y,D_X Y))$$

$$= 0.$$

Consequently,

$$R(X,Y)Y = B(X,Y)Y + \frac{1}{2}(D_Y T)(X,Y) - \frac{1}{2} T(T(X,Y),Y)$$

$$+ \frac{1}{4} T(T(X,Y),Y)$$

$$= B(X,Y)Y + \frac{1}{2}(D_Y T)(X,Y) - \frac{1}{4} T(T(X,Y),Y).$$

This proves (4).

To prove (5), use (4) to get

$$\langle R(X,Y)Y,X\rangle = \langle B(X,Y)Y,X\rangle + \frac{1}{2}\langle (D_Y T)(X,Y),X\rangle$$

$$- \frac{1}{4}\langle T(T(X,Y),Y),X\rangle.$$

By (2) $\langle T(X,Y),X\rangle = 0$, whence

$$\langle (D_Y T)(X,Y),X\rangle = Y\langle T(X,Y),X\rangle - \langle T(D_Y X,Y),X\rangle$$

$$- \langle T(X,D_Y Y),X\rangle - \langle T(X,Y),D_Y X\rangle$$

$$= 0 + \langle T(Y,D_Y X),X\rangle - 0 + \langle D_Y X,T(Y,X)\rangle$$

$$= 0,$$

where (2) has been used in this calculation.

Also by (2)

$$\langle T(T(X,Y),Y),X\rangle = -\langle T(Y,T(X,Y)),X\rangle$$

$$= \langle T(X,Y),T(Y,X)\rangle$$

$$= -\langle T(X,Y),T(X,Y)\rangle$$

$$= -\|T(X,Y)\|^2.$$

The above expression for $\langle R(X,Y)Y,X\rangle$ thus reduces to $\langle B(X,Y)Y,X\rangle + \frac{1}{4}\|T(X,Y)\|^2$.

This finishes the proof.

Proposition 3.3. Let $g:[a,b] \to M$ be a geodesic and let $U(t) = g'(t)$ be the tangent vector field along g. Then the Jacobi field $t \mapsto X(t)$ along g defined by

(1) $(\nabla_U)^2 X + R(X,U)U = 0 \quad X(a) = X_0, (\nabla_U X)(a) = X_1$

can be defined in terms of the geometric connection D by

(2) $(D_U)^2 X + D_U(T(X,U)) + B(X,U)U = 0$

$$X(a) = X_1, \quad (D_U X)(a) = X_1 + \frac{1}{2} T(U,X_0)$$

where R is the curvature tensor of ∇ and T is the torsion and B the curvature tensor of D.

Proof. By (1) of the last proposition

$$\begin{aligned}(\nabla_U)^2 X &= D_U(D_U X - \tfrac{1}{2} T(U,X)) - \tfrac{1}{2} T(U, D_U X - \tfrac{1}{2} T(U,X)) \\ &= (D_U)^2 X - \tfrac{1}{2} D_U(T(U,X)) - \tfrac{1}{2} T(U,D_U X) + \tfrac{1}{4} T(U,T(U,X)).\end{aligned}$$

Using (4) of the last proposition and that $D_U U = 0$ we find

$$\begin{aligned}(\nabla_U)^2 X + R(X,U)U &= (D_U)^2 X - \tfrac{1}{2} D_U(T(U,X)) - \tfrac{1}{2} T(U,D_U X) \\ &\quad + \tfrac{1}{4} T(U,T(U,X)) + B(X,U)U + \tfrac{1}{2}(D_U T)(X,U) - \tfrac{1}{4} T(T(X,U),U) \\ &= (D_U)^2 X + \tfrac{1}{2} D_U(T(X,U)) + \tfrac{1}{2} T(X,D_U U) + \tfrac{1}{2} T(D_U X,U) \\ &\quad + \tfrac{1}{2}(D_U T)(X,U) + B(X,U)U \\ &= (D_U)^2 X + \tfrac{1}{2} D_U(T(X,U)) + \tfrac{1}{2} D_U(T(X,U)) + B(X,U)U \\ &= (D_U)^2 X + D_U(T(X,U)) + B(X,U)U.\end{aligned}$$

Also note

$$X(a) = X_0,\ (\nabla_U X)(a) = (D_U X)(a) - \frac{1}{2} T(U,X(a)) = X_1$$

if and only if

$$X(a) = X_0,\ (D_U X)(a) = X_1 + \frac{1}{2} T(U,X_0).$$

This finishes the proof.

The rest of this section is devoted to proving there is a bijective correspondence between the geometric connections on M and the smooth three-forms on M.

Lemma 3.4. Let T be a smooth tensor field of type (1,2) on M such that, for all X, Y, Z tangent to M at some point, the following hold

(1) $T(X,Y) + T(Y,X) = 0$

(2) $\langle T(X,Y),Z\rangle + \langle Y,T(X,Z)\rangle = 0.$

Then the connection D defined by

$$D_X Y = \nabla_X Y + \frac{1}{2} T(X,Y),$$

where ∇ is the Riemannian connection is a geometric connection with T as its torsion tensor. Thus there is a bijective correspondence between the geometric connections on M and the tensor fields of type (1,2) satisfying (1) and (2).

Proof. Because T is alternating it follows from proposition 2.38 that D has the same geodesics as ∇. The following computation shows that D is metric preserving.

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$
$$= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \tfrac{1}{2}\langle T(X,Y)Z\rangle + \tfrac{1}{2}\langle Y, T(X,Z)\rangle$$
$$= \langle \nabla_X Y + \tfrac{1}{2}T(X,Y), Z\rangle + \langle Y, \nabla_X Z + \tfrac{1}{2}T(X,Z)\rangle$$
$$= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle .$$

Therefore D is geometric. That T is the torsion tensor of D now follows from proposition 3.2 part (1). This finishes the proof.

Lemma 3.5. For every smooth three-form α on M there is a unique smooth tensor field T_α of type (1,2) satisfying (1) and (2) of the last lemma with

$$\alpha(X,Y,Z) = \langle T_\alpha(X,Y), Z\rangle .$$

Moreover, every smooth tensor field T of type (1,2) satisfying (1) and (2) of the last proposition is T_α for some smooth three-form α.

Proof. It is easy to see there is a unique tensor field T_α of type (1,2) with

$$\alpha(X,Y,Z) = \langle T_\alpha(X,Y), Z\rangle .$$

Then $\alpha(X,Y,Z) + \alpha(Y,X,Z) = 0$ implies (1) and $\alpha(X,Y,Z) + \alpha(X,Z,T) = 0$ implies (2) of 3.4.

If T is a tensor field of type (1,2) satisfying (1) and (2) of 3.4 then define

$$\alpha(X,Y,Z) = \langle T(X,Y), Z\rangle .$$

Then α is alternating in X and Y by 3.4 (1), and alternating in Y and Z by 3.4 (2). Therefore α is a three-form on M and it is clear that $T = T_\alpha$.

<u>Proposition 3.6</u>. Let ∇ be the Riemannian connection on M. Then, using the notation of the last lemma, there is a bijective correspondence between the geometric connections D on M and the smooth 3-forms on M given by

$$D_X Y = \nabla_X Y + \frac{1}{2} T_\alpha(X,Y).$$

<u>Proof</u>. This follows immediately from the last two lemmas.

4. Some geometry of submanifolds.

In this section we record some of the facts we need about submanifolds of Riemannian manifolds. Let $\tilde{M}$ be a Riemannian manifold of dimension $m + n$ with metric $\langle\ ,\ \rangle$, and M be an embedded submanifold of $\tilde{M}$ of dimension n. It will be assumed M has the induced metric from $\tilde{M}$. The metric on M will also be denoted by "$\langle\ ,\ \rangle$". The following notation will be used:

$\tilde{\nabla}$ = Riemannian connection on $\tilde{M}$;

∇ = Riemannian connection on M;

$\tilde{R}$ = curvature tensor on $\tilde{M}$;

R = curvature tensor on M;

$T^{\perp}M$ = normal bundle of M in $\tilde{M}$.

Definition 4.1. Let $p \in M$ and $\xi(p) \in T^{\perp}M_p$ then the *Weingarten map* $A(\xi(p)) : TM_p \to TM_p$ is given by

$$A(\xi(p))X = \text{orthogonal projection of } \tilde{\nabla}_X\xi \text{ onto } TM_p,$$

where ξ is any local extension of $\xi(p)$ to a smooth section of $T^{\perp}M$.

Remarks. (1) Let X be a smooth vector field on X and ξ a smooth section of $T^{\perp}M$. Then an elementary calculation shows that the map

$$(X,\xi) \to (\text{orthogonal projection of } \tilde{\nabla}_X\xi \text{ onto } TM)$$

is bilinear over the smooth functions on M, whence $A(\xi(p))$ is independent of the extension of $\xi(p)$ to ξ.

(2) The above definition differs by a sign from the usual definition. This choice of sign purges latter formulas of enough factors of -1 to justify it.

Proposition 4.2. With notation as above, for any smooth vector fields

X, Y on M and smooth section ξ of $T^{\perp}M$ the following hold:

(1) $\nabla_X Y$ = orthogonal projection of $\tilde{\nabla}_X Y$ onto TM.

(2) $\langle A(\xi)X,Y\rangle = -\langle\tilde{\nabla}_X Y,\xi\rangle = \langle A(\xi)Y,X\rangle$.

Thus $A(\xi(p))$ is a self-adjoint map on TM_p.

(3) Let $e_1, \ldots, e_m$ be on orthonormal basis of $T^{\perp}M_p$. Then, for X, Y, Z, W in TM_p

$$\langle\tilde{R}(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle$$
$$+ \sum_{j=1}^{m} (\langle A(e_j)X,Z\rangle\langle A(e_j)Y,W\rangle - \langle A(e_j)X,W\rangle\langle A(e_j)Y,Z\rangle).$$

Proof. See [10] where(1) follows from the last formula on page 46, and (2) and (3) follow from formulas on page 51.

It will be convenient to restate (3).

If V is any finite dimensional real vector space with inner produce $\langle\ ,\ \rangle$ then $\wedge^2(V)$ is also an inner product space with the inner product on $\wedge^2(V)$, also denoted by $\langle\ ,\ \rangle$, given by

$$\langle X\wedge Y, Z\wedge W\rangle = \det\begin{bmatrix}\langle X,Z\rangle & \langle Y,Z\rangle \\ \langle X,W\rangle & \langle Y,W\rangle\end{bmatrix}$$
$$= \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle.$$

Any linear endomorphism A of V determines a linear endomorphism $\wedge^2(A)$ of $\wedge^2(V)$ given on decomposable elements by

$$\wedge^2(A)(X\wedge Y) = (AX)\wedge(AY).$$

Let R be the curvature tensor at some point p of M. Then, as R(X,Y) is an alternating function of X and Y, R induces a linear endomorphism $\wedge(R)$ of $\wedge^2 TM_p$ by

$$\langle\wedge(R)(X\wedge Y), Z\wedge W\rangle = \langle R(X,Y)Z,W\rangle.$$

Usually R and $\wedge(R)$ are both written as simply "R". When $\wedge(R)$ is to be referred to, we will say "view R as a linear map on $\wedge^2 TM$".

Proposition 4.3 (Equation of Gauss). View R as a linear map on $\wedge^2 TM$ and $\tilde{R}$ as a linear map on $\wedge^2 T\tilde{M}$. Let P_p be the orthogonal projection of $\wedge^2 T\tilde{M}_p$ onto its subspace $\wedge^2 TM_p$. Then, for any orthonormal basis $e_1, \ldots, e_m$ of $T^{\perp}M_p$

$$P_p\tilde{R}_p - R_p = \sum_{i=1}^{n} \wedge^2(A(e_j)).$$

Proof. This is a restatement of (3) of the last proposition.

Definition 4.4. The *excess tensor* H_p of M in $\tilde{M}$ at $p \in M$ is the linear endomorphism of $\wedge^2 TM_p$ given by

$$H_p = P_p\tilde{R}_p - R_p$$

where $\tilde{R}_p$ is viewed as a linear map on $\wedge^2 T\tilde{M}_p$, R is viewed as a linear map on $\wedge^2 TM_p$ and P is the orthogonal projection of $\wedge^2 T\tilde{M}_p$ onto $\wedge^2 TM_p$.

We will be interested in product submanifolds of product manifolds. We recall the definitions. Let M_1, M_2 be Riemannian manifolds. Let $\langle\ ,\ \rangle_i$ be the metric on M_i. If $\rho_i : M_1 \times M_2 \to M_i$ is projection then define the *product metric* $\langle\ ,\ \rangle$ on $M_1 \times M_2$ by

$$\langle X,Y\rangle = \langle \rho_{1*}X, \rho_{1*}Y\rangle_1 + \langle \rho_{2*}X, \rho_{2*}Y\rangle_2.$$

The proof of the following is straightforward and is left to the reader.

Proposition 4.5. Let ∇_i be the covariant derivation of the Riemannian connection on M_i, and R_i be the curvature of the Riemannian connection on M_i. Then the covariant derivation ∇ and the curvature

R of the Riemannian connection on $M_1 \times M_2$ are defined by

$$\langle \nabla_X Y, Z \rangle = \langle \nabla_{\rho_{1*}X} \rho_{1*}Y, \rho_{1*}Z \rangle_1 + \langle \nabla_{\rho_{2*}X} \rho_{2*}Y, \rho_{2*}Z \rangle_2 ,$$

$$\langle R(X,Y)Z,W \rangle = \langle R_1(\rho_{1*}X, \rho_{1*}Y)\rho_{1*}Z, \rho_{1*}W \rangle_1$$

$$+ \langle R_2(\rho_{2*}X, \rho_{2*}Y)\rho_{2*}Z, \rho_{2*}W \rangle_2 .$$

In the first equation X, Y, Z are smooth vector fields on $M_1 \times M_2$ so that $\rho_{i*}X$, $\rho_{i*}Y$, $\rho_{i*}Z$ are vector fields on M_i, for $i = 1, 2$; in the second equation, X, Y, Z, W can be any vectors tangent to $M_1 \times M_2$ at some point.

Proposition 4.6. Let M_i be a submanifold of $\tilde{M}_i$ and let A_i be the Weingarten map of M_i in $\tilde{M}_i$ for $i = 1, 2$. Then the Weingarten map of $M_1 \times M_2$ in $\tilde{M}_1 \times \tilde{M}_2$ is defined by

$$\langle A(U)X,Y \rangle = \langle A_1(\rho_{1*}U)\rho_{1*}X, \rho_{1*}Y \rangle_1 + \langle A_2(\rho_{2*}U)\rho_{2*}X, \rho_{2*}Y \rangle_2$$

where X is tangent to $M_1 \times M_2$, U is normal to $M_1 \times M_2$, and Y is tangent to $\tilde{M}_1 \times \tilde{M}_2$ at some point of $M_1 \times M_2$.

Proof. A straightforward calculation using the last proposition.

5. Riemannian homogeneous spaces.

Let M be a connected Riemannian manifold, and $\mathcal{I}(M)$ be the group of isometries of M. That is, $\mathcal{I}(M)$ is the group of all diffeomorphisms of M whose derivatives preserve the length of tangent vectors. We give $\mathcal{I}(M)$ the compact-open topology. For each p in M let $\mathcal{I}(M)_p$ be the subgroup of $\mathcal{I}(M)$ consisting of those isometries which fix p. The subgroup $\mathcal{I}(M)_p$ is called the *isotropy subgroup* of $\mathcal{I}(M)$ at p. The following is well known.

Proposition 5.1. If $\mathcal{I}(M)$ is the isometry group of the connected Riemannian manifold M then:

(1) $\mathcal{I}(M)$ is a Lie transformation group on M. (That is $\mathcal{I}(M)$ has the structure of a Lie group and the map $(a,p) \to ap$ from $\mathcal{I}(M) \times M$ to M is smooth).

(2) Each isotropy subgroup $\mathcal{I}(M)_p$ is compact.

(3) If M is compact then so is $\mathcal{I}(M)$.

(4) If $g \in \mathcal{I}(M)_p$ then g is the identity.

Proof. For the first three see [8], vol. 1, page 239, theorem 3.4. The last part follows easily from the formula $g(\exp_p(X)) = \exp_p(g_{*p}X)$. This formula is clear as exp is defined in terms of the Riemannian metric and g preserves the metric.

The manifold M will be called a *Riemannian homogeneous space* if and only if $\mathcal{I}(M)$ is transitive on M. Since it is not always easy to work with the full group of isometries we make the following:

Convention 5.2. For the rest of this section, we assume that M is a Riemannian homogeneous space and that G is a closed subgroup of the group of isometries of M such that

(1) G is transitive on M; and

(2) Each isotropy subgroup

$$G_p = \{g \in G : g(p) = p\}$$

is a compact subgroup of G.

The following will also be useful.

Notation 5.3. For the rest of this section we fix some point o in M and call it the *origin* of M. Also let $H = \{g \in G : g(o) = o\}$ be this isotropy subgroup of G at the origin.

$$\mathfrak{m} = TM_o = \text{tangent space to } M \text{ at the origin.}$$

Then the frame bundle $L(M)$ of M can be assumed to have as its fibre $L(M)_p$ over p the set of linear isomorphisms of $\mathfrak{m}$ onto TM_p. With this convention it follows that:

Proposition 5.4. The map $g \mapsto g_{*o}$ is a diffeomorphism of G onto a closed embedded submanifold of $L(M)$. Call the image of G under this map $G(M)$. Then $G(M)$ is an H-structure over M in the sense of definition 2.4. The fibre $G(M)_p$ over $p = g(o)$ is the image of the coset gH.

Proof. See chapter X of volume 2 of [8].

Convention 5.5. We will, when convenient, identify G with $G(M)$ via the diffeomorphism of the last proposition and use this identification to move the algebraic structure of G over to $G(M)$. The identity element of G goes over to the identity map on $\mathfrak{m}$. The tangent space to $G(M)$ at the identity will be written as $\mathfrak{g}$, and be assumed to have its usual structure as a Lie algebra. Let $\mathfrak{h}$ be the tangent space to H at the identity. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and, by proposition 2.8, part (5), h is also the space of vertical vectors at 1. To make the notation look like that of section 2 the exponential map from $\mathfrak{g}$ to G will be written as $A \mapsto e^A$.

Definition 5.6. For $g \in G$ let L_g be left translation on $G = G(M)$. That is,

$$L_g(x) = gx.$$

Proposition 5.7. For $A \in h$ the fundamental vector field determined by A on G(M) is

$$A^*(g(o)) = L_{g*} A.$$

Proof. This is an easy computation

$$\begin{aligned} A^*(g(o)) &= \frac{d}{dt}\Big|_{t=o} g(o)e^{tA} \\ &= \frac{d}{dt}\Big|_{t=0} L_g e^{tA} \\ &= L_{g*} A. \end{aligned}$$

Proposition 5.8. There is a subspace $\mathfrak{m}_o$ of $\mathfrak{g}$ such that

(1) $\mathfrak{g} = \mathfrak{m}_o \oplus h$ (direct sum)

(2) $\mathfrak{m}_o$ is invariant under the adjoint action of H on $\mathfrak{g}$.

Proof. See page 199 of volume 2 of [8].

Convention 5.9. We now fix some $\mathfrak{m}_o$ as in proposition 5.8. If $\pi : G(M) \to M$ is the projection then

$$\pi_{*1}\Big|_{\mathfrak{m}_o} \mathfrak{m}_o \to \mathfrak{m} = TM_o$$

is easily seen to be a linear isomorphism. *From now on* $\mathfrak{m}_o$ *will be identified with* $\mathfrak{m}$ *by this isomorphism.*

Definition 5.10. For any vector A in $\mathfrak{g}$ let A_{h} be the h-component and $A_{\mathfrak{m}}$ the $\mathfrak{m}$ component of A relative to the splitting of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{m} \oplus h$. Then

(1) Define a $\mathfrak{m}$-valued one-form θ on $G(M)$ by

$$\theta_g(X) = (L_{g_*^{-1}} X)_{\mathfrak{m}}.$$

(2) Define an h-valued one-form ω on $G(M)$ by

$$\omega_g(X) = (L_{g_*^{-1}} X)_{h}.$$

Proposition 5.11. The form θ is the canonical form on $G(M)$ and ω is a metric-preserving connection on $G(M)$. This connection will be called the *canonical connection* on M.

Proof. By definition the value of the canonical form at $X \in TG(M)_g$ is $g_*^{-1}\ \pi_{*g}\ X$.

But $g^{-1} \circ \pi = \pi \circ L_{g^{-1}}$; therefore,

$$\begin{aligned} \theta_g(X) &= g_*^{-1}\ \pi_{*g}\ X \\ &= \pi_{*1} L g^{-1}{}_* \ X \\ &= (L g^{-1}{}_* \ X)_{\mathfrak{m}}. \end{aligned}$$

The last line holds because the convection 5.9 makes π_{*1} into the projection of $\mathfrak{g}$ onto $\mathfrak{m}$.

It follows directly from proposition 5.7 that $\omega_g(A^*(g)) = A$ for every fundamental vector field A^* on $G(M)$. Let $a \in H$, $g \in G(M)$ and $X \in TG(M)_g$.

Then

$$\begin{aligned}(r_a^*\omega)_g(X) &= \omega_{ga}(r_{a*}X) \\ &= (L_{a^{-1}g^{-1}*}r_{a*}X)_h \\ &= (L_{a^{-1}*}L_{g^{-1}*}r_{a*}X)h \\ &= (L_{a^{-1}*}r_{a*}(L_{g^{-1}*}(L_{g^{-1}*}X))_h \\ &= (Ad(a^{-1})(L_{g^{-1}*}X))_h \\ &= Ad(a^{-1}((L_{g^{-1}*}X)_h) \\ &= Ad(a^{-1})\ \omega_g(X),\end{aligned}$$

where we have used the following facts:

$$\begin{aligned}L_{g^{-1}}r_a &= r_a L_{g^{-1}}, \\ Ad(a^{-1}) &= L_{a^{-1}*}r_{a*}, \\ (Ad(a^{-1})Y)_h &= Ad_{(a^{-1})}(Y_h).\end{aligned}$$

The last of these holds because $\mathfrak{m}$ is $Ad(H)$ invariant. This completes the proof that ω is a connection.

Because G is a group of isometries of M the H-structure $G(M)$ is a submanifold of $O(M)$, the bundle of orthogonal frames on M. Therefore ω can be extended to a connection on $O(M)$. Proposition 2.39 now yields that ω is metric preserving. This finishes the proof.

Proposition 5.12. Let ω be the canonical connection on $G(M)$ and θ the canonical form. Let $X_{\mathfrak{m}}$ and $Y_{\mathfrak{m}}$ be as in 5.10. Then

(1) The torsion form of ω is given at $\mathfrak{g} = TG(M)_1$ by

$$\Theta(X,Y) = -[X_{\mathfrak{m}},Y_{\mathfrak{m}}]_{\mathfrak{m}}.$$

(2) The curvature form of ω is given at $\mathfrak{g} = TG(M)_1$ by

$$\Omega(X,Y) = -[X_{\mathfrak{m}},Y_{\mathfrak{m}}]_{h}.$$

(3) The torsion tensor of ω is given on $TM_0 = \mathfrak{m}$ by

$$T_0(X,Y) = -[X,Y]_{\mathfrak{m}}.$$

(4) The curvature tensor of ω is given on $TM_0 = \mathfrak{m}$ by

$$B_0(X,Y)Z = -[[X,Y]_h,Z].$$

Proof. If X is a left invariant vector field on $G(M)$ (that is $L_{g*}X = X$ for all $g \in G$) then it follows directly from the definitions that $\theta(X)$ and $\omega(X)$ are constants on M. If X is a left invariant vector field on $G(M)$ then let $X_{\mathfrak{m}}$ be the left invariant extension $X(1)_{\mathfrak{m}}$ and likewise for X_h. Then for left invariant vector fields X,Y

$$\begin{aligned}\Theta(X,Y) &= d\theta(X_{\mathfrak{m}},Y_{\mathfrak{m}}) \\ &= X_{\mathfrak{m}}\theta(Y_{\mathfrak{m}}) - Y_{\mathfrak{m}}\theta(X_{m}) - \theta([X_{\mathfrak{m}},Y_{\mathfrak{m}}]) \\ &= 0 - 0 - \theta([X_{\mathfrak{m}},Y_{\mathfrak{m}}]) \\ &= -\theta([X_{\mathfrak{m}},Y_{\mathfrak{m}}]).\end{aligned}$$

As the point $1 \in G(M)$ this reduces to (1).

A similar calculation proves (2).

The convention 5.9 shows that a vector in $TM_0 = \mathfrak{m}$ is its own horizontal lift to 1 in $G(M)$. Putting this into the definition of the torsion tensor and using (1) yields

$$T_o(X,Y) = 1_{*o}(\Theta(X,Y))$$
$$= -[X,Y]_{\mathfrak{m}}.$$

Part (4) follows from (2) the same way (3) followed from (1). This completes the proof.

<u>Proposition 5.13</u>. For the canonical connection on M the geodesics through o are the curves $t \mapsto \pi e^{tX}$ where X is in $\mathfrak{m}$. Parallel translation along the geodesic $t \mapsto \pi e^{tX}$ from o to πe^{tX} is given by $(e^{tX})_{*1}$.

<u>Proof</u>. It is easy to check that the left invarient vector fields X on $G(M)$ with $X(1)$ in $\mathfrak{m}$ are the basic vector fields on $G(M)$ (see definition 2.33). Therefore the integral curves of the basic vectors that pass through 1 are the curves $t \mapsto e^{tX}$ where X is in $\mathfrak{m}$. The first statement of the proposition now follows from proposition 2.34. The curve $t \mapsto e^{tX}$ is horizonal so the second part follows from the definition of parallel translation.

<u>Proposition 5.14</u>. Let D be the covariant derivation of the canonical connection. Then D, T (the torsion tensor) and B (the curvature tensor) are all invariant under G. If S is any tensor field on M invariant under G then $DS = 0$. Thus $DT = 0$ and $DB = 0$.

<u>Proof</u>. It is clear that ω is invariant under G. Each of D, T and B is defined in terms of ω and therefore they are also invariant.

Let $X \in \mathfrak{m}$. Define a vector field $\tilde{X}$ on M by

$$\tilde{X}(p) = \frac{d}{dt}\Big|_{t=0} e^{tX}(p).$$

The flow of this vector field is clearly $\alpha_t(p) = e^{tX}(p)$. Therefore S

is invariant under the flow of $\tilde{X}$ and thus the Lie derivative of S with respect to $\tilde{X}$ is zero. But $(e^{tX})_*$ is parallel translation along the geodesic $t \mapsto \pi e^{tX}$. Thus

$$\begin{aligned}(D_X S)_0 &= (D_{\tilde{X}} S)_0 \\ &= (\mathcal{L}_{\tilde{X}} S)_0 \\ &= 0.\end{aligned}$$

This shows DS vanishes at the origin of M. But DS is G invariant and G is transitive, so DS vanishes everywhere. This completes the proof.

Definition 5.15. The natural connection on M is *naturally reductive* if and only if it has the same geodesics as the Riemannian connection on M.

Because the natural connection on M is metric preserving we see that D is naturally reductive if and only if it is geometric in the sense of section 3.

Proposition 5.16. The canonical connection on M is naturally reductive if and only if, for all X, Y and Z in $\mathfrak{m}$,

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y,[X,Z]_{\mathfrak{m}}\rangle = 0.$$

Proof. If the canonical connection is naturally reductive then the above equation follows from proposition 3.2 (2) and proposition 5.12 (3). To prove the converse, note that by 5.12 (3) the above equation can be rewritten as

$$\langle T_0(X,Y),Z\rangle + \langle Y,T_0(X,Z)\rangle = 0$$

where T_0 is the torsion tensor of the canonical connection at 0. Let D be the covariant derivation of the canonical connection. Then define a new covariant derivation δ on smooth vector fields X and Y by

$$\delta_X Y = D_X Y - \frac{1}{2} T(X,Y).$$

Then a straightforward calculation shows that δ is metric preserving and torsion free. Thus $\delta = \nabla$, the covariant derivation of the Riemannian connection. But then the difference tensor of D and ∇ is alternating, so D and ∇ have the same geodesics by proposition 2.38. This finishes the proof.

Proposition 5.17. Assume the canonical connection on M is naturally reductive and that D is its covariant derivation. Let T be the torsion tensor and B the curvature tensor of D. Let R be the curvature tensor of ∇. Then

(1) For smooth vector fields X and Y on M

$$\nabla_X Y = D_X Y - \frac{1}{2} T(X,Y)$$

(2) $$\langle T(X,Y),Z\rangle + \langle Y,T(X,Z)\rangle = 0$$

(3) $$R(X,Y)Z = B(X,Y)Z - \frac{1}{2} T(T(X,Y)Z) + \frac{1}{4} T(X,T(Y,Z)) - \frac{1}{2} T(Y,T(X,Z))$$

(4) $$R(X,Y)Y = B(X,Y)Y - \frac{1}{2} T(T(X,Y),Y)$$

(5) $$\langle R(X,Y)Y,Y\rangle = \langle B(X,Y)Y,X\rangle + \frac{1}{4}\|T(X,Y)\|^2.$$

Proof. The connection D on M is geometric, therefore this proposition is just proposition 3.2 plus the extra information that $DT = 0$.

Proposition 5.18. Assume the canonical connection on M is naturally reductive and let T, B and R be as in the last proposition. For any vector $U \in TM_p$ define linear endomorphisms T_U, B_U and R_U by

$$T_U(X) = T(X,U),$$

$$B_U(X) = B(X,U)U,$$

$$R_U(X) = R(X,U)U.$$

Then

$$R_U + B_U - \frac{1}{2} T_U^2,$$

both R_U and B_U are symmetric and T_U is skewsymmetric.

Proof. That $R_U = B_U - \frac{1}{2} T_U^2$ is (4) of the last proposition. The skew-symmetry follows from (2) of the last proposition. The Ricci identity (proposition 2.44 (2)) shows R_U is symmetric. The square of a skewsymmetric map is symmetric therefore $B_U = R_U + \frac{1}{2} T_U^2$ is the sum of symmetric maps and thus symmetric.

Proposition 5.19. With notation as in the last proposition, if $g : (\alpha,\beta) \to M$ is a geodesic and $U(t) = g'(t)$ is the tangent along g then the initial value problems

(1) $(\nabla_U)^2 X + R_U X = 0 \quad X(t_0) = X_0, \quad (\nabla_U X)(t_0) = X_1$

(2) $(D_U)^2 X + T_U(D_U X) + B_U X = 0$

$$X(t_0) = X_0 (D_U X)(t_0) = X_1 - \frac{1}{2} T_U X_0$$

define the same Jacobi field along g.

Proof. This is proposition 3.3, where we also use that $(D_U T) = 0$ and $D_U U = 0$ so that $D_U(T(X,U)) = T(D_U X,U)$.

Definition 5.20. A submanifold N of a Riemannian manifold M is *totally geodesic* if and only if every geodesic of N in the induced metric is a geodesic of M.

Proposition 5.21. Let M be a naturally reductive Riemannian homogeneous space and $p \in M$. Let S be a vector subspace of TM_p. Then there is a totally geodesic submanifold N of M passing through p with $TN_p = S$ if and only if for all X, Y, Z in S both $T(X,Y)$ and $B(X,Y)Z$ are in S. (Here T is the torsion tensor and B the curvature tensor of the cononical connection on M).

Proof. See [2], theorem 3.2, page 57.

The following defines a class of Riemannian manifolds that has been very much studied.

Definition 5.22. A Riemannian manifold is a *symmetric space* if and only if it is a naturally reductive Riemannian homogeneous space in which the Riemannian connection equals the canonical connection.

Proposition 5.23. If M is a symmetric space, then, with the notation of proposition 5.17,

$$T = 0, \quad B = R.$$

Proof. Clear from proposition 5.17.

6. Geometry of symmetrically embedded submanifolds of naturally reductive Riemannian homogeneous spaces.

In this section the following notation will be maintained. First $\tilde{M}$ will be an oriented naturally reductive Riemannian homogeneous space of dimension $m + n$. Then M will be an oriented submanifold of $\tilde{M}$ of dimension n. Because most of what follows is local, M will be assumed compact with smooth (possibly empty) boundry.

D = covariant derivation of the canonical connection on $\tilde{M}$,

$\tilde{T}$ = torsion tensor of D,

$\tilde{B}$ = curvature tensor of D,

$\tilde{\nabla}$ = Riemannian connection on $\tilde{M}$,

∇ = Riemannian connection on M,

$\tilde{R}$ = curvature tensor of $\tilde{\nabla}$,

R = curvature tensor of ∇.

Define, for each $U \in TM_p$, linear maps from TM_p to itself by

$$\tilde{T}_U(X) = \tilde{T}(X,U),$$

$$\tilde{B}_U(X) = \tilde{B}(X,U)U,$$

$$\tilde{R}_U(X) = \tilde{R}(X,U)U.$$

Definition 6.1. The submanifold M is *symmetrically embedded* in $\tilde{M}$ if and only if for all $p \in M$ and $U \in T^{\perp}M_p$ the vector space $T^{\perp}M_p$ is stable under both $\tilde{B}_U$ and $\tilde{T}_U$.

Examples of symmetrically embedded submanifolds of homogeneous spaces will be given after the following proposition.

Proposition 6.2. The following are equivalent for a submanifold M of $\tilde{M}$:

(1) M is symmetrically embedded in $\tilde{M}$.

(2) For all $p \in M$ and $U \in T^{\perp}M_p$ the vector space TM_p is stable under both $\tilde{T}_U$ and $\tilde{B}_U$.

(3) For all $p \in M$ there is a totally geodesic submanifold N of $\tilde{M}$ passing through p with $TN_p = T^{\perp}M_p$.

Proof. A symmetric or skew-symmetric linear map on an inner product space stabilizes a subspace if and only if it stabilizes its orthogonal complement. The map $\tilde{B}_U$ is symmetric and the map $\tilde{T}_U$ is skew-symmetric by proposition 5.18. This proves the equivalence of (1) and (2).

By proposition 5.21, if (3) holds then for all $p \in M$ and $U, X \in T^{\perp}M_p$,

$$\tilde{B}_U(X) = \tilde{B}(X,U)U \in TM_p,$$

$$\tilde{T}_U(X) = \tilde{T}(X,U) \in TM_p.$$

Therefore (3) implies (1).

To finish it is enough to show (1) implies (3). By proposition 5.21 it is enough to show that if M is symmetrically embedded in $\tilde{M}$ and $X, Y, Z \in T^{\perp}M_p$ then $\tilde{B}(X,Y)Z \in TM_p$. Therefore suppose M is symmetrically embedded in $\tilde{M}$ and that $X, Y, Z \in T^{\perp}M_p$. Then

$$\tilde{B}(X,Y)Z + \tilde{B}(X,Z)Y = \tilde{B}(X,Y+Z)(Y+Z) - \tilde{B}(X,Y)Y = \tilde{B}(Y,Z)Z \in T^{\perp}M_p.$$

Combining the fact that $DT = 0$ with the first Bianchi identity (proposition 2.31) yields

$$\begin{aligned}\tilde{B}(X,Y)Z + \tilde{B}(Y,Z)X + \tilde{B}(Z,X)Y &= \tilde{T}(\tilde{T}(X,Y)Z) + \tilde{T}(\tilde{T}(Y,Z),X) + \tilde{T}(\tilde{T}(Z,X),Y) \\ &= \tilde{T}_Z\tilde{T}_YX + \tilde{T}_X\tilde{T}_ZY + \tilde{T}_Y\tilde{T}_XZ \in T^{\perp}M_p.\end{aligned}$$

Adding these gives

$$(\widetilde{B}(X,Y)Z + \widetilde{B}(Y,Z)X + \widetilde{B}(Z,X)Y) + (\widetilde{B}(X,Y)Z + \widetilde{B}(X,Z)Y)$$

$$= 2\widetilde{B}(X,Y)Z + \widetilde{B}(Y,Z)X + (\widetilde{B}(Z,X)Y + \widetilde{B}(Y,Z)X)$$

$$= 2\widetilde{B}(X,Y,)Z + \widetilde{B}(Y,Z)X \in T^{\perp}M_p.$$

Doing the permutation $X \mapsto Y$, $Y \mapsto X$, $Z \mapsto Z$ in $\widetilde{B}(X,Y)Z + \widetilde{B}(X,Z)Y$ shows

$$\widetilde{B}(Y,X)Z + \widetilde{B}(Y,Z)X = -\widetilde{B}(X,Y)Z + \widetilde{B}(Y,Z)X \in T^{\perp}M_p.$$

Therefore

$$3\widetilde{B}(X,Y)Z = (2\widetilde{B}(X,Y)Z + \widetilde{B}(Y,Z)X) - (-\widetilde{B}(X,Y)Z + \widetilde{B}(Y,Z)X) \in T^{\perp}M_p.$$

This finishes the proof.

Examples. (1) M is called a *hypersurface* of $\widetilde{M}$ if the codimension of M in $\widetilde{M}$ is one. If p is a point of M then there is a geodesic of $\widetilde{M}$ passing through p and perpendicular to M. By (3) of the last proposition this shows all hypersurfaces of any naturally reductive homogeneous spaces are symmetrically embedded.

(2) Let $\widetilde{M}$ be a space of constant curvature κ. Then by definition $\widetilde{T} = 0$ and $\widetilde{B} = \widetilde{R}$ is given by

$$\widetilde{B}(X,Y)Z = \kappa(\langle Z,Y\rangle X - \langle Z,X\rangle Y).$$

Thus

$$\widetilde{B}_U(X) = \kappa(\langle U,U\rangle X - \langle U,X\rangle U).$$

From this it is easy to check that every submanifold of $\widetilde{M}$ is symmetrically embedded.

(3) Let $\widetilde{M}$ be a complex analytic manifold of constant holomorphic curvature. Then calculations that will be done later show that every complex submanifold of $\widetilde{M}$ is symmetrically embedded.

Other examples will ge given later.

Convention 6.3. From now on M will be assumed to be a symmetrically embedded submanifold of $\tilde{M}$.

Recall that $\tilde{D}$ and $\tilde{\nabla}$ have the same geodesics and therefore the same exponential map. The common exponential map for these two connections will be denoted by exp. The following notation will be used to study the image of $S^{\perp}M$ under the exponential map.

Definition 6.4. (1) Let $\pi : S^{\perp}M \to M$ be the bundle projection.

(2) For $U \in S^{\perp}M$ let

$$g(t;U) = \exp_p(tU).$$

(3) Set $U(t) = g'(t;U)$.

(4) $\mathfrak{J}(t;U)$ = D-parallel translate of $TM_{\pi U}$ along $g(\cdot;U)$ to $g(t;U)$.

(5) $\mathfrak{h}(t;U)$ = Orthogonal complement of the span of $U(t)$ and $\mathfrak{J}(t;U)$ in $T\tilde{M}_{g(t;U)}$.

Proposition 6.5. Let $\mathbb{R}U(t)$ be the span of the vector $U(t)$ in $T\tilde{M}_{g(t;U)}$. Then each of $\mathfrak{J}(t;U)$, $\mathfrak{h}(t;U)$ and $\mathbb{R}U(t)$ is parallel along $g(\cdot;U)$ and $T\tilde{M}_{g(t;U)}$ is the orthogonal direct sum of these spaces. Also $\mathfrak{J}(t;U)$, $\mathfrak{h}(t;U)$ and $\mathbb{R}U(t)$ are all stable under all three of the linear maps $\tilde{T}_{U(t)}$, $\tilde{B}_{U(t)}$ and $\tilde{R}_{U(t)}$.

Proof. The field of spaces $\mathfrak{J}(t;U)$ is D-parallel along $g(\cdot;U)$ by definition. The spaces $\mathbb{R}(t)$ are D-parallel along $g(\cdot;U)$, because $g(\cdot;U)$ is a geodesic and $U(t)$ is its tangent vector. Therefore $\mathfrak{h}(t;U)$ is also D-parallel along $g(\cdot;U)$, as it is the orthogonal complement of D-parallel spaces and D is metric preserving.

Because $\mathfrak{J}(t;U)$, $\mathfrak{h}(t;U)$ and $\mathbb{R}U(t)$ are D-parallel along $g(\cdot;U)$ to show $TM_{g(t;U)}$ is the orthogonal sum of the three it is enough to show for a particular value of t. At $t = 0$ this is easily checked.

Since we are assuming M is symmetrically embedded in $\widetilde{M}$ it follows from proposition 6.2 that both $\mathfrak{J}(0;U) = TM_p$ and

$$T^{\perp}M_{\pi U} = (\mathfrak{h}(0;U) \oplus \mathbb{R}U(0))$$

are stable under both $\widetilde{B}_U$ and $\widetilde{T}_U$.

But

$$\widetilde{B}_U(U) = \widetilde{B}(U,U)U = 0,$$

$$\widetilde{T}_U(U) = \widetilde{T}(U,U) = 0.$$

Therefore $\mathbb{R}U(0)$ is also stable under both $\widetilde{B}_U$ and $\widetilde{T}_U$. But $\widetilde{B}_U$ is symmetric and $\widetilde{T}_U$ is skew-symmetric. Therefore the orthogonal complement of U in $T^{\perp}M_{\pi U}$, which is $\mathfrak{h}(0;U)$, is also stable under both $\widetilde{B}_U$ and $\widetilde{T}_U$. This shows $\mathfrak{J}(t;U)$. $\mathfrak{h}(t;U)$ and $\mathbb{R}U(t)$ are stable under $\widetilde{B}_{U(t)}$ and $\widetilde{T}_{U(t)}$ when $t = 0$. But as all of these are D-parallel along $g(\cdot;U)$ it follows that $\mathfrak{J}(t;U)$, $\mathfrak{h}(t;U)$ and $\mathbb{R}U(t)$ are stable under $\widetilde{B}_{U(t)}$ and $\widetilde{T}_{U(t)}$. That the three subspaces of $T\widetilde{M}_{g(t;U)}$ in question are stable under $\widetilde{R}_{U(t)}$ follows from the equation $\widetilde{R}_{U(t)} = \widetilde{B}_{U(t)} - \frac{1}{2}(\widetilde{T}_{U(t)})^2$ which is given in proposition 5.18. This completes the proof.

<u>Definition 6.6</u>. Let

$$T_U(t) = \widetilde{T}_{U(t)}\Big|_{\mathfrak{J}(t;U)},$$

$$B_U(t) = \widetilde{B}_{U(t)}\Big|_{\mathfrak{J}(t;U)},$$

$$R_U(t) = \tilde{R}_{U(t)}\Big|_{\mathfrak{J}(t;U)},$$

$$T_U^\perp(t) = \tilde{T}_{U(t)}\Big|_{\mathfrak{n}(t;U)},$$

$$B_U^\perp(t) = \tilde{B}_{U(t)}\Big|_{\mathfrak{n}(t;U)},$$

$$R_U^\perp(t) = \tilde{R}_{U(t)}\Big|_{\mathfrak{n}(t;U)}.$$

Then define linear maps

$$\overline{S}(t;U) : \mathfrak{J}(t;U) \to \mathfrak{J}(t;U),$$

$$\overline{C}(t;U) : \mathfrak{J}(t;U) \to \mathfrak{J}(t;U),$$

$$\overline{S}^\perp(t;U) : \mathfrak{n}(t;U) \to \mathfrak{n}(t;U),$$

as the unique solutions to the initial value problems:

(1) $$(\tilde{\nabla}_{U(t)})^2\, \overline{S}(t;U) + R_U(t)\, \overline{S}(t;U) = 0$$

$$\overline{S}(0;U) = 0, \quad (\tilde{\nabla}_{U(t)}\overline{S})(0;U) = (id)\big|_{TM_p},$$

$$(\tilde{\nabla}_{U(t)})^2\, \overline{C}(t;U) + R_U(t)\, C(t;U) = 0$$

$$\overline{C}(0,U) = (id)_{TM_p}, (\tilde{\nabla}_{U(t)}\overline{C})(0;U) = 0,$$

$$(\tilde{\nabla}_{U(t)})^2\, \overline{S}^\perp(t;U) + R_U(t)\, \overline{S}^\perp(t;U) = 0$$

$$\overline{S}^\perp(0;U) = 0, \quad (\tilde{\nabla}_{U(t)}\overline{S}^\perp)(0;U) = (id)\big|_{\mathfrak{n}(0;U)}.$$

By proposition 5.19 these can also be defined by

(1') $$(D_{U(t)})^2\, \overline{S}(t;U) + T_U(t)(D_{U(t)}\overline{S})(t;U) + B_U(t)\, \overline{S}(t;U) = 0$$

$$\overline{S}(0;U) = 0, \quad (D_{U(t)}\overline{S})(0;U) = (id)_{TM},$$

$$(D_{U(t)})^2\, \overline{C}(t;U) + T_U(t)(D_{U(t)}\overline{C})(t1U) + B_U(t)\, \overline{C}(t;U) = 0$$

$$\overline{C}(0;U) = (id)_{TM_p}, (D_U\overline{C})(t;U) = -\frac{1}{2}\, T_U(0),$$

$$(D_{U(t)})^2\,\overline{S}^{\perp}(t;U) + T_U^{\perp}(t)(D_{U(t)}\overline{S}^{\perp})(t;U) + B_U^{\perp}(t)\,\overline{S}^{\perp}(t;U) = 0$$

$$\overline{S}^{\perp}(0;U) = 0; \quad (D_{U(t)}\overline{S}^{\perp})(0;U) = (id)_{\mathfrak{h}(0;U)}.$$

Remarks. (1) if $t \mapsto X(t)$ is any D-parallel vector field along $g(\cdot;U)$ with $X(0)$ in $\mathfrak{J}(0;U)$ then both $t \mapsto \overline{S}(t;U)X(t)$ and $t \mapsto \overline{C}(t;U)X(t)$ are Jacobi fields along $g(\cdot;U)$. In this case $\overline{S}(t;U)X(t)$ and $\overline{C}(t;U)X(t)$ are both in $\mathfrak{J}(t;U)$ for all t. A similar statement is true for $t \mapsto \overline{S}^{\perp}(t;U)X(t)$ when $X(t)$ is a D-parallel vector field along $g(\cdot;U)$ with $X(0)$ in $\mathfrak{h}(0;U)$. These facts follow directly from the definitions.

(2) If the differential equations defining $\overline{S}$, $\overline{C}$ and $\overline{S}^{\perp}$ are written with respect to D-parallel fields, then the differential equations in (1') have constant coefficients.

Definition 6.7. For each number r, define a map

$$f_r : S^{\perp}M \to \widetilde{M}$$

by

$$f_r(U) = \exp_{\pi U}(rU).$$

The image $M(r)$ by f_r is the *tube of radius* r about M.

We now compute the derivative of f_r. If $U \in S^{\perp}M$ and $p = \pi U$, then the fibre $S^{\perp}M_p$ is an embedded submanifold of the total space $S^{\perp}M$. Thus, the tangent $T(S^{\perp}M_p)_U$ to the fibre can be viewed as a subspace of the tangent space $T(S^{\perp}M)_U$ to $S^{\perp}M$. But the sphere $S^{\perp}M_p$ is also embedded in the vector space $T^{\perp}M_p$ as the set of all vectors of unit length. Therefore, the tangent space $T(S^{\perp}M_p)_U$ to the fibre can be identified with the set of all vectors in TM_p that are orthogonal to U. But this is $\mathfrak{h}(0;U)$. Thus $\mathfrak{h}(0;U)$ can be identified with a subspace $T(S^{\perp}M)_U$. Under this identification it is easy to check that $\mathfrak{h}(0;U)$ is the kernel of π_{*U}.

<u>Lemma 6.8</u>. Consider $\mathfrak{h}(0;U)$ as a subspace of $T(S^{\perp}M)_U$ as above. If $X \in \mathfrak{h}(0;U)$ and $X(t)$ is the D-parallel translation of X along $g(\cdot;U)$ to $g(t;U)$ then

$$(f_r)_{*U}X = \bar{S}^{\perp}(r;U)X(r).$$

<u>Proof</u>. Without loss of generality we may assume X is a unit vector. Then define a curve by

$$c(s) = \cos(s)\, U + \sin(s)\, X.$$

Because U and X are orthogonal vectors, this is a curve from the reals to $S^{\perp}M$. Clearly $C(0) = U$ and $C'(0) = X$. Therefore,

$$\begin{aligned}(f_{r*})_U\, X &= \left.\frac{d}{ds}\right|_{s=0} f_r(c(s)) \\ &= \left.\frac{d}{ds}\right|_{s=0} \exp_p(rc(s)),\end{aligned}$$

where $p = \pi U$. Define $\alpha(s,t)$ by

$$\alpha(s,t) = \exp_p(tc(s)).$$

Then $f_r(c(s)) = \alpha(s,r)$ so that

$$(f_r)_{*U}\, X = \frac{\partial\alpha}{\partial s}(0,r).$$

Clearly (see definition 3.26) $\alpha(s,t)$ is a variation of $\alpha(0,t) = \exp_p(tU) = g(t;U)$ through geodesics. Thus, by proposition 2.37, the vector field $\frac{\partial\alpha}{\partial s}(0,t)$ along $g(\cdot;U)$ is a Jacobi field. But $\bar{S}^{\perp}(t;U)X(t)$ is also a Jacobi field along $g(\cdot;U)$. Thus, to prove the lemma it is enough to prove $\frac{\partial\alpha}{\partial s}(0,t)$ and $\bar{S}^{\perp}(t;U)X(t)$ have the same initial conditions at $t = 0$. (See proposition 2.37). We now compute

$$\frac{\partial\alpha}{\partial s}(0,0) = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_p(0)$$

$$= 0$$

$$= \bar{S}^{\perp}(0;U)X(0).$$

The covariant derivation $\tilde{\nabla}$ has no torsion and the vector fields $\frac{\partial\alpha}{\partial s}$, $\frac{\partial\alpha}{\partial t}$ commute thus,

$$\tilde{\nabla}_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial s} = \tilde{\nabla}_{\frac{\partial\alpha}{\partial s}} \frac{\partial\alpha}{\partial t}.$$

This yields

$$(\tilde{\nabla}_{U(t)} \frac{\partial\alpha}{\partial s})(0,0) = (\tilde{\nabla}_{\frac{\partial\alpha}{st}} \frac{\partial\alpha}{\partial s})(0,0)$$

$$= (\tilde{\nabla}_{\frac{\partial\alpha}{\partial s}} \frac{\partial\alpha}{\partial t})(0,0)$$

$$= \left.\tilde{\nabla}_{\frac{\partial\alpha}{\partial s}}\right|_{s=0} \left.\frac{\partial\alpha}{\partial t}\right|_{t=0} \exp_p(tc(s))$$

$$= \left.\tilde{\nabla}_{\frac{\partial\alpha}{\partial s}}\right|_{s=0} c(s)$$

$$= c'(0)$$

$$= X$$

$$= \tilde{\nabla}_{U(t)}\,{}_{t=0}\, \bar{S}^{\perp}(t;U)X(t).$$

This finishes the proof.

<u>Definition 6.9</u>. Let A be the Wiengarten map of M in $\tilde{M}$ (see definition 4.1). Then, for each $U \in S^{\perp}M$, let $A(t;U)$ be the D-parallel translate of $A(U)$ along $g(\cdot;U)$ to $g(t;U)$. Therefore $A(t;U)$ is a linear transformation on $\mathfrak{J}(t;U)$.

Lemma 6.10. Let $\hat{X} \in T(S^{\perp}M)_U$, $X = \pi_{*U}\hat{X}$, and $X(t)$ be the D-parallel translate of X along $g(\cdot;U)$ to $g(t;U)$. Then

$$(f_r)_{*U}\,\hat{X} = (\bar{C}(r;U) + \bar{S}(r;U)A(r;U))\;X(r)$$

$$+ \text{(an element of } \mathfrak{h}(r;U)).$$

Proof. Choose a smooth curve $\xi : (-\varepsilon,\varepsilon) \to S^{\perp}M$ from some neighborhood $(-\varepsilon,\varepsilon)$ of 0 such that $\xi(0) = U$ and $\xi'(0) = \hat{X}$. Set $p = \pi U$ and $c = \pi \circ \xi$. Then $c : (-\varepsilon,\varepsilon) \to M$ is a smooth curve with $c(0) = p$ and $c'(0) = \pi_{*U}\xi'(0) = X$. Also

$$(f_r)_{*U}\,\hat{X} = \frac{d}{ds}\bigg|_{s=0} f_r(\xi(s))$$

$$= \frac{d}{ds}\bigg|_{s=0} \exp_{c(s)}(r\xi(s)).$$

Define $\alpha(s,t) = \exp_{c(s)}(t\xi(s))$. Then the last equation can be written as

$$(f_r)_*\,\hat{X} = \frac{\partial\alpha}{\partial s}(0,r).$$

But, as in the last lemma, $\alpha(s,t)$ is a variation of $\alpha(0,t) = g(t;U)$ through geodesics and thus $\frac{\partial\alpha}{\partial s}(0,t)$ is a Jacobi field along $g(t;U)$. We now find its initial conditions.

$$\frac{\partial\alpha}{\partial s}(0,0) = \frac{\partial}{\partial s}\bigg|_{s=0} \exp_{c(s)}(0)$$

$$= \frac{\partial}{\partial s}\bigg|_{s=0} c(s)$$

$$= c'(0)$$

$$= X.$$

The curve ξ can be viewed as a section of $S^{\perp}M$ and, thus, of $T^{\perp}M$ along c.

Therefore,

$$\tilde{\nabla}_{\frac{\partial\alpha}{\partial s}\big|_{s=0}} \xi(s) = \tilde{\nabla}_{c'(0)}\, \xi(s)$$

$$= \tilde{\nabla}_X \xi(s)$$

$$= A(\xi(0))X + (\text{element of } \mathfrak{n}(0;U))$$

$$= A(U)X + (\text{element of } \mathfrak{n}(0;U)).$$

This yields,

$$(\tilde{\nabla}_{U(t)}\tfrac{\partial\alpha}{\partial s})(0,0) = (\tilde{\nabla}_{\frac{\partial\alpha}{\partial t}} \tfrac{\partial\alpha}{\partial s})(0,0)$$

$$= (\tilde{\nabla}_{\frac{\partial\alpha}{\partial s}} \tfrac{\partial\alpha}{\partial t})(0,0)$$

$$= \tilde{\nabla}_{\frac{\partial\alpha}{\partial s}\big|_{s=0}} \frac{\partial}{\partial t}\Big|_{t=0} \exp_{c(s)}(t\xi(s))$$

$$= \tilde{\nabla}_{\frac{\partial}{\partial s}\big|_{s=0}} \xi(s)$$

$$= A(U)X + (\text{element of } \mathfrak{n}(0;U)).$$

Where, as in the last lemma, we have used the facts that $\tilde{\nabla}$ is without torsion and that $\frac{\partial\alpha}{\partial s}$ and $\frac{\partial\alpha}{\partial t}$ commute.

Let $J(t)$ be the vector field along $g(\cdot;U)$ defined by

$$J(t) = (\overline{C}(t;U) + \overline{S}(t;U)A(t;U))X(t).$$

Then J is a Jacobi field and from the definitions of $\overline{C}$ and $\overline{\xi}$

$$J(0) = X(0) = X, \quad (\tilde{\nabla}_U J)(0) = A(0;U)X(0) = A(U)X.$$

Thus, if $Y(t) = \frac{\partial}{\partial s}(0,t) - J(t)$, then Y is a Jacobi field along $g(\cdot;U)$ with $Y(0) = 0$ and $(\tilde{\nabla}_U Y)(0) \in \mathfrak{n}(0;U)$. Hence, $Y(t)$ is in

$\mathfrak{h}(t;U)$ for all t. This, together with the expression for $(f_r)_{*U}\hat{X}$ in terms of $\frac{\partial\alpha}{\partial s}$, completes the proof.

We now give each fibre $S^{\perp}M_p$ of $S^{\perp}M$ its volume form $\Omega_{S^{\perp}M_p}$ as a unit sphere in $T^{\perp}M_p$. If Ω_M is the volume form on M and $\pi : S^{\perp}M \to M$ is the bundle projection, then a volume form $\Omega_{S^{\perp}M}$ is defined on $S^{\perp}M$ by

$$\Omega_{S^{\perp}M}(U) = \Omega_{(S^{\perp}M)_p}(U) \wedge (\pi^*\Omega_M)(U)$$

where $p = \pi U$. We choose the orientations so that Fubini's theorem holds with the following choice of signs

$$\int_{S^{\perp}M} f(U)\Omega_{S^{\perp}M}(U) = \int_M\Big(\int_{S^{\perp}M_p} f(U)\Omega_{S^{\perp}M_p}\Big)\Omega_M(p),$$

where f is any compactly supported continuous function.

Proposition 6.11. Let $r \in \mathbb{R}$ and $U \in S^{\perp}M$. Assume

(*) $$\det(\overline{C}(r;U) + \overline{S}(r;U)A(r;U))\det(\overline{S}^{\perp}(r;U)) \neq 0.$$

Then $(f_r)_{*U}$ is injective, and thus f_r maps some neighborhook K of U in $S^{\perp}M$ into a hypersurface $K(r)$ of $\tilde{M}$. The tangent space to $K(r)$ at $f_r(U)$ is

$$T(K(r))_{f_r(U)} = \mathfrak{J}(r;U) \oplus \mathfrak{h}(r;U).$$

If $\Omega_{K(r)}$ is the volume element on $K(r)$, then

$$f_r^*\,\Omega_{K(r)} = \det(\overline{C}(r;U) + \overline{S}(r;U)A(r;U))\det(\overline{S}^{\perp}(f;U))\Omega_{S^{\perp}M}(U).$$

Proof. Let $X_1, \ldots, X_{m-1}$ be an oriented orthonormal basis of $T(S^{\perp}M_p)_U = \mathfrak{h}(0;U)$ (with $p = \pi U$) and $Y_1, \ldots, Y_n$ be an oriented orthonormal basis of $T(S^{\perp}M)_U$. Choose elements $\hat{Y}_1, \ldots, \hat{Y}_n$ of

$T(S^{\perp}M)_U$ with $\pi_{*U}\hat{Y}_i = Y_i$ $i = 1, \ldots, n$. By the last two lemmas

$$(f_r)_{*U}\, X_i = \overline{S}^{\perp}(r;U)X_i(r)$$

$$(f_r)_*\, Y_j = (\overline{C}(r;U) + \overline{S}(r;U)A(r;U))Y_j(r) + Z_j$$

where $X_i(t)$ is the parallel field along $g(\cdot;U)$ with $X_i(0) = X_0$; and $Y_j(t)$ is the parallel field along $g(\cdot;U)$ with $Y_j(0) = Y_j$ and Z_j is an element of $\mathfrak{h}(r;U)$. The condition (*) easily implies that $S^{\perp}(r;U)X_i(r)$ for $1 \leq \wedge \leq m - 1$ is a basis of $\mathfrak{h}(r;U)$, and that

$$(\overline{C}(r;U) + \overline{S}(r;U)A(r;U))Y_j(r) \qquad 1 \leq j \leq n,$$

is a basis of $\mathfrak{J}(r;U)$. Therefore

$$(f_r)_{*U}X_i,\quad (f_r)_{*U}Y_j \qquad 1 \leq i \leq m - 1,\quad 1 \leq j \leq n$$

is a basis of $\mathfrak{J}(r;U) \oplus \mathfrak{h}(r;U)$. This proves $(f_r)_{*U}$ is injective.

The statements that U has a neighborhood K mapped into a hypersurface $K(r)$ of $\widetilde{M}$ and that the tangent space to this hypersurface is as claimed now follow from the implicit function theorem.

It is now easy to check that

$$(f_r)_{*U}X_1 \wedge \cdots \wedge (f_r)_{*U}X_{m-1} \wedge (f_r)_{*U}\hat{Y}_1 \wedge \cdots \wedge (f_r)_{*U}\hat{Y}_N$$

$$= \det(\overline{C}(r;U) + \overline{S}(r;U)A(r;U))\det(\overline{S}^{\perp}(r;U))X_1(r)\wedge\cdots\wedge X_{m-1}(r)\wedge Y_1(r)\wedge\cdots\wedge Y_n(r).$$

But as $X_1 \wedge \cdots \wedge X_{m-1} \wedge \hat{Y}_1 \wedge \cdots \wedge \hat{Y}_n$ is dual to $\Omega_{S^{\perp}M}$ (that is, $\Omega_{S^{\perp}M}(X_1,\ldots,X_{m-1},\hat{Y}_1,\ldots,\hat{Y}_n) = 1$) the given formula for $f_r^*\,\Omega_{K(r)}$ holds. This completes the proof.

<u>Corollary 6.12</u>. If the condition (*) holds for all U in $S^{\perp}M$, then the volume of the tube $M(r)$ of radius r about M is

$$\mathrm{vol}(M(r)) = \int_{S^\perp M} \det(\overline{C}(r;U) + \overline{S}(r;U)A(r;U))\det(\overline{S}^\perp(r;U))\Omega_{S^\perp M}(U).$$

Proof. Clear from the last proposition.

Convention 6.13. From here on, the volume of the tube $M(r)$ will be defined by the formula of the last corollary, even when the condition (*) of proposition 6.11 does not hold.

The following result restates what we said above without having to compute any parallel translations.

Theorem 6.14. Let M be a compact symmetrically embedded submanifold of $\widetilde{M}$ with smooth boundary. For each U in $S^\perp M$, set $p = \pi U$, and,

$$T_U = \widetilde{T}_U\big|_{TM_p},$$

$$B_U = \widetilde{B}_U\big|_{TM_p},$$

$$T_U^\perp = \widetilde{T}_U\big|_{T^\perp M_p},$$

$$B_U^\perp = \widetilde{B}_U\big|_{T^\perp M_p}.$$

Define linear maps

$$S(t;U),\ C(t;U) : TM_p \to TM_p,$$

and

$$S^\perp(t;U) : T^\perp M_p \to T^\perp M_p,$$

by the initial value problems

$$S''(t;U) + T_U S'(t;U) + B_U S(t;U) = 0$$

$$S(0;U) = 0,\ S'(0;U) = (\mathrm{id})_{TM_p},$$

$$C''(t;U) + T_U C'(t;U) + B_U C(t;U) = 0$$

$$C(0;U) = (id)_{TM_p}, \quad C'(0;U) = -\frac{1}{2} T_U,$$

$$(S^{\perp})''(t;U) + T_U^{\perp}(S^{\perp})'(t;U) + B_U^{\perp}(S^{\perp})(t;U) = 0$$

$$S^{\perp}(0;U) = 0, \quad (S^{\perp})(0;U) = (id)_{T^{\perp}M_p}.$$

Let $h: M \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h(p,t) = \frac{1}{t}\int_{S^{\perp}M_p} \det(C(t;U) + S(t;U)A(U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U),$$

where A is the Weingarten map of M in $\tilde{M}$. Then the volume of the tube $M(r)$ of radius r about M is

$$\mathrm{vol}(M(r)) = \int_M h(p,r)\Omega_M(p).$$

Remarks. (1) The derivatives, denoted as primes, are to be taken in the usual sense of a function from the real numbers to a finite dimensional real vector space.

(2) It should be noted that $T_U^{\perp} \neq T_U^{\perp}(0)$ as $T_U^{\perp}$ has as its domain $T^{\perp}M_p$, while $T_U^{\perp}(0)$ has $\mathfrak{h}(0;U)$ for its domain.

Proof of the theorem. Let $\tau(t;U)$ be D-parallel translation along $g(\cdot;U)$ from p to $g(t;U)$. Then, because $T_{U(t)}$ and $B_{U(t)}$ are D-parallel along $g(t;U)$, we have

$$S(t;U) = \tau(t;U)^{-1}|_{\mathfrak{J}(t;U)}\overline{S}(t;U)\tau(t;U)|_{TM_p},$$

$$C(t;U) = \tau(t;U^{-1}|_{\mathfrak{J}(t;U)}\overline{C}(t;U)\tau(t;U)|_{TM_p},$$

$$A(U) = \tau(t;U)^{-1}|_{\mathfrak{J}(p;U)}A(t;U)\tau(t;U)|_{TM_p}.$$

Therefore,

$$\det(C(t;U) + S(t;U)A(U)) = \det(\overline{C}(t;U) + \overline{S}(t;U)A(t;U)),$$

and likewise,

$$S^{\perp}(t;U)|_{\mathfrak{n}(0;U)} = \tau(0;U)^{-1}|_{\mathfrak{n}(t;U)}\overline{S}^{\perp}(0;U)\tau(0;U)|_{\mathfrak{n}(0;U)}.$$

However, we have $T^{\perp}M_p = \mathfrak{n}(0;U) \oplus \mathbb{R}U$. Therefore it only remains to compute $S^{\perp}(t;U)$ on $\mathbb{R}U$. Let $X(t) = tU$. Then $X'(t) = U$ and $X''(t) = 0$; also $T_U^{\perp}(X'(t)) = 0$ as $T_U^{\perp}(U) = 0$, and $B_U^{\perp}(X(t)) = 0$ as $B_U^{\perp}(U) = 0$. Thus, $X(t)$ is a solution to

$$X''(t) + T_U^{\perp}(X'(t)) + B_U^{\perp}(X(t)) = 0$$

$$X(0) = 0, \quad X'(0) = U.$$

But $S^{\perp}(t;U)U$ is also a solution to this initial value problem. Therefore

$$S^{\perp}(t;U)U = tU.$$

Using this with what we know about $S^{\perp}(t;U)|_{\mathfrak{n}(0(U)}$ yields

$$\det(S^{\perp}(t;U)) = \det(S^{\perp}(t;U)|_{\mathfrak{n}(0;U)}) \det(S^{\perp}(t;U)|_{\mathbb{R}U})$$

$$= \det(\overline{S}^{\perp}(t;U))t,$$

whence

$$\det(\overline{C}(t;U) + \overline{S}(t;U)A(t;U)) \det(\overline{S}^{\perp}(t;U))$$

$$= \frac{1}{t} \det(C(t;U) + S(t;U)A(U)) \det(S^{\perp}(t;U).$$

The result now follows from corollary 6.12 or convention 6.13 and Fubini's theorem.

7. Some multilinear algebra.

The results of this section are inessential variants of the algebraic results in [5]. What is here written as "$A * B$" is written in Flanders as "AB". His definition of $A * B$ differs from that given here; instead, he uses proposition 7.2 as its definition.

If W is a real vector space, then end(W) will be the algebra of all linear endomorphisms of W. Throughout this section V will be an n-dimensional real vector space, $\Lambda^k(V)$ will be the k-th exterior power of V and S_ℓ is the group of all permutations of $\{1,\ldots,\ell\}$. If σ is a permutation, then $(-1)^\sigma$ will denote the sign of σ.

Definition 7.1. If $A \in \mathrm{end}(\Lambda^a(V))$ and $B \in \mathrm{end}(\Lambda^b(V))$ then let $A * B$ be the endomorphism of $\Lambda^{a+b}(V)$ defined on decomposable elements by

$$(A*B)(x_1\wedge\cdots\wedge x_{a+b})$$

$$= \frac{1}{a!b!} \sum_{\sigma\in S_{a+b}} (-1)^\sigma A(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(a)}) \wedge B(x_{\sigma(a+1)}\wedge\cdots\wedge x_{\sigma(a+b)}).$$

If α is a real valued alternating b-form on V, then α can be viewed as a linear functional on $\Lambda^k(V)$ by

$$\alpha(x_1\wedge\cdots\wedge x_k) = \alpha(x_1,\ldots,x_k).$$

Conversely it is clear that every linear functional on $\Lambda^k(V)$ is of this form, for some α. Let $e_1, \ldots, e_n$ be a basis of V. Then $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $i_1, \ldots, i_k$ range over all k-tubles of positive integers with $1 \le i_1 < \cdots < i_k \le n$ is a basis of $\Lambda^k(V)$. Therefore, our remarks about linear functionals tell us that every element of $\mathrm{end}(\Lambda^k(V))$ can be written as

$$A = \sum_{i_1<\cdots<i_k} \alpha_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

where each $\alpha_{i_1\cdots i_k}$ is a real valued alternating k-form on V. This means A is given on decomposable elements by

$$A(\chi_1\wedge\cdots\wedge\chi_k) = \sum_{i_1<\cdots<i_k} \alpha_{i_1\ldots i_k}(\chi_1,\ldots,\chi_k) e_{i_1}\wedge \cdots \wedge e_{i_k}.$$

Proposition 7.2. Let $e_1, \ldots, e_n$ be a basis of V and

$$A = \sum_{i_1<\cdots<i_a} \alpha_{i_1\ldots i_a} e_{i_1} \wedge \cdots \wedge e_{i_a} \in \operatorname{end}(\wedge^a(V))$$

$$B = \sum_{j_1<\cdots<j_b} \beta_{j_1\ldots j_b} e_{j_1} \wedge \cdots \wedge e_{j_b} \in \operatorname{end}(\wedge^b(V)).$$

Then

$$A * B = \sum_{\substack{i_1<\cdots<i_a \\ j_1<\cdots<j_k}} \alpha_{i_1\ldots i_a} \wedge \beta_{j_1\ldots j_k} e_{i_1} \wedge \cdots \wedge e_{i_a} \wedge e_{j_1} \wedge \cdots \wedge e_{j_b}$$

Proof. Let $\chi_1, \ldots, \chi_{a+b} \in V$. Then,

$$(A*B)(\chi_1\wedge\cdots\wedge\chi_{a+b}) = \frac{1}{a!b!} \sum_{\sigma\in S_\ell} (-1)^\sigma A(\chi_{\sigma(1)}\wedge\cdots\wedge\chi_{\sigma(a)}) \wedge B(\chi_{\sigma(a+1)}\wedge\cdots\wedge\chi_{\sigma(a+b)})$$

$$= \frac{1}{a!b!} \sum_{\sigma\in S_\ell} (-1)^\sigma \Big(\sum_{i_1<\cdots<i_a} \alpha_{i_1\ldots i_a}(\chi_{\sigma(1)},\ldots,\chi_{\sigma(a)}) e_{i_1}\wedge\cdots\wedge e_{i_a} \Big) \wedge$$

$$\Big(\sum_{j_1<\cdots<j_b} \beta_{j_1\ldots j_b}(\chi_{\sigma(a+1)},\ldots,\chi_{\sigma(a+b)}) e_{j_1}\wedge\cdots\wedge e_{j_b} \Big)$$

$$= \sum_{\substack{i_1<\cdots<i_a \\ j_1<\cdots<j_b}} \Big(\frac{1}{a!b!} \sum_{\sigma\in S_\ell} (-1)^\sigma \alpha_{i_1\ldots i_a}(\chi_{\sigma(1)},\ldots,\chi_{\sigma(a)}) \beta_{j_1\ldots j_b}(\chi_{\sigma(a+1)},\ldots,\chi_{\sigma(a+b)}) \Big)$$

$$e_{i_1} \wedge \cdots \wedge e_{i_a} \wedge e_{j_1} \wedge \cdots \wedge e_{j_b}$$

$$= \sum_{\substack{i_1<\cdots<i_a \\ j_1<\cdots<j_b}} (\alpha_{i_1\ldots i_a} \wedge \beta_{j_1\ldots j_b})(\chi_1,\ldots,\chi_{a+b}) e_{i_1} \wedge \cdots \wedge e_{i_a} \wedge e_{j_1} \wedge \cdots \wedge e_{j_b}.$$

This finishes the proof.

Proposition 7.3. Let $A \in \text{end}(\wedge^a(V))$, $B \in \text{end}(\wedge^b(V))$ and $C \in \text{end}(\wedge^c(V))$. Then the map

$$(A,B) \mapsto A * B$$

is bilinear, and

$$A * B = B * A$$

$$(A*B)*C = A*(B*C).$$

Proof. That $A * B$ is a bilinear function of (A,B) is clear. To prove the other two statements, we use the last proposition. Let $e_1, \ldots, e_n$ be a basis of V and

$$A = \sum_{i_1 < \cdots < i_a} \alpha_{i_1 \ldots i_a} e_{i_1} \wedge \cdots \wedge e_{i_a}$$

$$B = \sum_{j_1 < \cdots < j_b} \beta_{j_1 \ldots j_b} e_{j_1} \wedge \cdots \wedge e_{j_b}.$$

Then

$$(A*B) = \sum_{\substack{i_1 < \cdots < i_a \\ j_1 < \cdots < j_b}} \alpha_{i_1 \ldots i_a} \wedge \beta_{j_1 \ldots j_b} e_{i_1} \wedge \cdots \wedge e_{i_a} \wedge \cdots \wedge e_{j_b}$$

$$= (-1)^{ab}(-1)^{ab} \sum_{\substack{i_1 < \cdots < i_a \\ j_1 < \cdots < j_b}} \beta_{j_1 \ldots j_b} \wedge \alpha_{i_1 \ldots i_a} e_{j_1} \wedge \cdots \wedge e_{j_b} \wedge e_{j_1} \wedge \cdots \wedge e_{i_a}.$$

$$= (B*A).$$

The associativity of $*$ follows from proposition 7.2 and the associativity of $\wedge$ by a similar calculation. This completes the proof.

Recall that if $A \in \text{end}(V)$, then $\wedge^k(A)$ is the linear endomorphism of $\wedge^k(V)$ given on decomposable elements by

$$\wedge^k(A)(\chi_1 \wedge \cdots \wedge \chi_k) = (A\chi_1) \wedge \cdots \wedge (A\chi_k).$$

Definition 7.4. If $A \in \mathrm{end}(\wedge^a(V))$ then define $A^{*k} \in \mathrm{end}(\wedge^{ak}(V))$ by

$$A^{*k} = A * A * \cdots * A \quad (k \text{ factors}).$$

Proposition 7.5. If $A, B, A_1, \ldots, A_k \in \mathrm{end}(V)$, then

(1) $$(A_1 * \cdots * A_k)(\chi_1 \wedge \cdots \wedge \chi_k) = \sum_{\sigma \in S_k} (-1)^\sigma A_1\chi_{\sigma(1)} \wedge \cdots \wedge A_k\chi_{\sigma(k)}$$

$$= \sum_{\sigma \in S_k} A_{\sigma(1)}\chi_1 \wedge \cdots \wedge A_{\sigma(k)}\chi_k;$$

(2) $A^{*k} = k! \wedge^k(A)$;

(3) $$\wedge^k(A+B) = \sum_{j=0}^{k} \wedge^j(A) * \wedge^{k-j}(B), \qquad (\text{where } \wedge^\circ(A) = 1)$$

$$\det(A+B) = \sum_{j=0}^{n} \wedge^j(A) * \wedge^{n-j}(B);$$

(4) $$(BA_1) * (BA_2) * \cdots * (BA_k) = \wedge^k(B) \circ (A_1 * \cdots * A_k),$$

$$(A_1B) * (A_2B) * \cdots * (A_kB) = (A_1 * \cdots * A_k) \circ \wedge^k(B).$$

Proof. To show (1) we use induction. Let $\mathrm{perm}(a_1,\ldots,a_k)$ be the group of permutations on $\{a_1,\ldots,a_k\}$. Assume (1) holds for (k-1).

Then

$$(A_1 * \cdots * A_k)(\chi_1 \wedge \cdots \wedge \chi_k) = ((A_1 * \cdots * A_{k-1}) * A_k)(\chi_1 \wedge \cdots \wedge \chi_k)$$

$$= \frac{1}{(k-1)}\frac{1}{1!} \sum_{\sigma \in S_k} (-1)^\sigma (A_1 * \cdots * A_{k-1})(\chi_{\sigma(1)} \wedge \cdots \wedge \chi_{\sigma(k-1)}) \wedge A_k\chi_{\sigma(k)}$$

$$= \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^\sigma \sum_{\theta \in \mathrm{perm}(\sigma(1),\ldots,\sigma(k-1))} (-1)^\theta (A_1\chi_{\theta\sigma(1)} \wedge \cdots \wedge A_{k-1}\chi_{\theta\sigma(k-1)}) \wedge A_k\chi_{\sigma(k)}$$

$$= \frac{(k-1)!}{(k-1)!} \sum_{\rho \in S_k} (-1)^\rho A_1\chi_{\rho(1)} \wedge \cdots \wedge A_k\chi_{\rho(k)}$$

$$= \sum_{\rho \in S_k} (-1)^\rho A_1\chi_{\rho(1)} \wedge \cdots \wedge A_k\chi_{\rho(k)}.$$

The second line of (1) follows from the first by a change of variable. Now (2) follows from (1) by letting $A_1 = A_2 = \cdots = A_k = A$. For (3) we remark that $*$ is commutive and associative, so that $(A+B)^{*k}$ can be expanded by the binomial theorem.

$$\begin{aligned}
\wedge^k(A+B) &= \frac{1}{k!}(A+B)^* \\
&= \frac{1}{k!}\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} A^{*j} * B^{*k-j} \\
&= \sum_{j=0}^{k} \left(\frac{1}{j!} A^{*j}\right) * \left(\frac{1}{(k-j)!} B^{*(k-j)}\right) \\
&= \sum_{j=0}^{k} \wedge^j(A) * \wedge^{k-j}(B).
\end{aligned}$$

The second line of (3) follows from the first and that $\det(A) = \wedge^n(A)$. To prove (4) we use (1).

$$\begin{aligned}
&(BA_1) * \cdots * (BA_k)(x_1\wedge\cdots\wedge x_k) \\
&= \sum_{\sigma\in S_k} (-1)^\sigma (BA_1 x_{\sigma(1)}) \wedge \cdots \wedge (BA_k x_{\sigma(k)}) \\
&= \wedge^k(B)\Big(\sum_{\sigma\in S_k} (-1)^\sigma A_1 x_{\sigma(1)}\wedge\cdots\wedge A_k x_{\sigma(k)}\Big) \\
&= \wedge^k(B) \circ (A_1*\cdots*A_k)(x_1\wedge\cdots\wedge x_k).
\end{aligned}$$

The second line of (4) follows by a similar calculation. This completes the proof.

Remark. It follows from (3) that $\wedge^k(A) * \wedge^{n-k}(I)$ is $\sigma_k(A)$, the k-th elementary symmetric function in the eigenvalues of A. To see this, note that (3) implies

$$\det(\chi I + A) = \sum_k \sigma_k(A)\, \chi^{n-k}$$

$$= \sum_k \wedge^k(A) * \wedge^{n-k}(I)\, \chi^{n-k}.$$

Definition 7.6. Let $e_1, \ldots, e_n$ be a basis of V and $A \in \operatorname{end}(\wedge^k(V))$. Then the *components* $A^{j_1 \ldots j_k}_{i_1 \ldots i_k}$ *of* A *in the basis* $e_1, \ldots, e_n$ are defined by

$$Ae_{i_1} \wedge \cdots \wedge e_{i_k} = \frac{1}{k!} \sum_{j_1, \ldots, j_k} A^{j_1 \ldots j_k}_{i_1 \ldots i_k}\, e_{j_1} \wedge \cdots \wedge e_{j_k}$$

where $A^{j_1 \ldots j_k}_{i_1 \ldots i_k}$ is an alternating function of $i_1, \ldots, i_k$ and also of $j_1, \ldots, j_k$.

If we restrict ourselves to increasing sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$, then the components of A in the basis $e_1, \ldots, e_n$ of V are components of the matrix of A in the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ of $\wedge^k(V)$. It follows that

$$\operatorname{tr}(A) = \sum_{i_1 < \cdots < i_k} A^{i_1 \ldots i_k}_{i_1 \ldots i_k}$$

$$= \frac{1}{k!} \sum_{i_1 \ldots i_k} A^{i_1 \ldots i_k}_{i_1 \ldots i_k}$$

We will write $\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ for the component of $\wedge^k(I)$, the identity map on $\wedge^k(V)$. The components of $\wedge^k(I)$ are the same for any choice of basis of V. It is easy to check that $\delta^{i_1 \ldots i_k}_{j_1 \ldots j_k}$ vanishes unless $i_1, \ldots, i_k$ are all distinct and the sets $\{i_1, \ldots, i_k\}$ are the same. In this case, its value is the sign of the permutation taking each i_ℓ to j_ℓ for $1 \leq \ell \leq k$.

Proposition 7.7. If $A \in \mathrm{end}(\wedge^a(V))$, then

$$A * \wedge^{n-a}(I) = \mathrm{tr}(A).$$

Proof. Let $A_{i_1 \cdots i_a}^{j_1 \cdots j_a}$ be the components of A in the basis $e_1, \ldots, e_n$ of V. Then,

$$A * \wedge^{n-a}(I)(e_1 \wedge \cdots \wedge e_n) = \frac{1}{a!(n-a)!} \sum_{\sigma \in S_n} (-1)^\sigma A(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(a)})$$

$$\wedge\, e_{\sigma(a+1)} \wedge \cdots \wedge e_{\sigma(n)}$$

$$= \frac{1}{a!a!(n-a)!} \sum_{\sigma \in S_n} (-1)^\sigma \sum_{i_1 \cdots i_a} A_{\sigma(1) \cdots \sigma(0)}^{i_1 \cdots i_a}\, e_{i_1} \wedge \cdots \wedge e_{i_a} \wedge e_{\sigma(a+1)}$$

$$\wedge \cdots \wedge e_{\sigma(n)}$$

$$= \frac{1}{a!a!(n-a)!} \sum_{\sigma \in S_n} (-1)^\sigma \sum_{\substack{\{i_1, \ldots, i_a\} \\ = \{\sigma(1), \ldots, \sigma(a)\}}} A_{\sigma(1) \cdots \sigma(a)}^{i_1 \cdots i_a} e_{i_1} \wedge \cdots \wedge e_{i_a} \wedge e_{\sigma(a+1)}$$

$$\wedge \cdots \wedge e_{\sigma(n)}$$

$$= \frac{a!}{a!a!(n-a)!} \sum_{\sigma \in S_n} (-1)^\sigma A_{\sigma(1) \cdots \sigma(a)}^{\sigma(1) \cdots \sigma(a)} e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(a)} \wedge e_{\sigma(a+1)}$$

$$\wedge \cdots \wedge e_{\sigma(n)}$$

$$= \frac{1}{a!(n-a)!} \sum_{\sigma \in S_n} A_{\sigma(1) \cdots \sigma(a)}^{\sigma(1) \cdots \sigma(a)} e_1 \wedge \cdots \wedge e_n$$

$$= \frac{1}{a!} \sum_{i_1 \cdots i_a} A_{i_1 \cdots i_a}^{i_1 \cdots i_a} e_1 \wedge \cdots \wedge e_n$$

$$= \mathrm{tr}(A)\, e_1 \wedge \cdots \wedge e_n.$$

We now relate our formulas to those in the literature.

Proposition 7.8. Let $H \in \mathrm{end}(\wedge^2(V))$. Then

(1) $$H^{*k}(x_1\wedge\cdots\wedge x_{2k}) = \frac{1}{2^k}\sum_{\sigma\in S_k}(-1)^\sigma H(x_{\sigma(1)}\wedge x_{\sigma(2)})$$

$$\wedge\cdots\wedge H(x_{\sigma(2k-1)}\wedge x_{\sigma(2k)})$$

(2) If $H^{ij}_{k\ell}$ are the components of H in the basis $e_1, \ldots, e_n$ of V then

$$H^{*k} * \wedge^{n-2k}(I) = \mathrm{tr}(H^{*k})$$

$$= \frac{1}{4^k}\sum_{\substack{i_1\cdots i_{2k}\\ j_1\cdots j_{2k}}} \delta^{i_1\cdots i_{2k}}_{j_1\cdots j_{2k}} H^{j_1j_2}_{i_1i_2} H^{j_3j_4}_{i_3i_4}\cdots H^{j_{2k-1}j_{2k}}_{i_{2k-1}i_{2k}}.$$

Proof. We show (1) by induction.

$$H^{*k}(x_1\wedge\cdots\wedge x_{2k}) = H^{*(k-1)} * H(x_1\wedge\cdots\wedge x_{2k})$$

$$= \frac{1}{(2k-2)!2!}\sum_{\sigma\in S_{2k}}(-1)^\sigma H^{*(k-1)}(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(2k-2)})\wedge H(x_{\sigma(2k-1)}\wedge x_{\sigma(2k)})$$

$$= \frac{1}{(2k-2)!2!}\sum_{\sigma\in S_{2k}}(-1)^\sigma \frac{1}{2^{k-1}}\sum_{\theta\in \mathrm{perm}(\sigma(1),\ldots,\sigma(2k-2))}(-1)^\theta H(x_{\theta\sigma(1)}\wedge x_{\theta\sigma(2)})$$

$$\wedge\cdots\wedge H(x_{\theta\sigma(2k-3)}\wedge x_{\theta\sigma(2k-2)})\wedge H(x_{\sigma(2k-1)}\wedge x_{\sigma(2k)})$$

$$= \frac{1}{2^k}\sum_{\sigma\in S_{2k}}(-1)^\sigma H(x_{\sigma(1)}\wedge x_{\sigma(2)})\wedge\cdots\wedge H(x_{\sigma(2k-1)}\wedge x_{\sigma(2k)}).$$

This proves (1). To prove (2), we use (1) to find the components of H^{*k}.

$$H^{*k}\, e_{i_1} \wedge \cdots \wedge e_{i_{2k}}$$

$$= \frac{1}{2^k} \sum_{\sigma \epsilon \mathrm{perm}(i_1,\ldots i_{2k})} (-1)^{\sigma} H(e_{\sigma(i_1)} \wedge e_{\sigma(i_2)}) \wedge \cdots \wedge H(e_{\sigma(i_{2k-1})} \wedge e_{\sigma(i_{2k})})$$

$$= \frac{1}{2^k} \sum_{j_1,\ldots,j_{2k}} \delta^{j_1\ldots j_{2k}}_{i_1\ldots i_{2k}} H(e_{j_1} \wedge e_{j_2}) \wedge \cdots \wedge H(e_{j_{2k-1}} \wedge e_{j_{2k}})$$

$$= \frac{1}{2^k}\frac{1}{2^k} \sum_{j_1,\ldots,j_{2k}} \delta^{j_1\ldots j_{2k}}_{i_1\ldots i_{2k}} \Big(\sum_{\alpha_1,\alpha_2} H^{\alpha_1\alpha_2}_{j_1j_2} e_{\alpha_1} \wedge e_{\alpha_2} \Big)$$

$$\wedge \cdots \wedge \Big(\sum_{\alpha_{2k-1},\alpha_{2k}} H^{\alpha_{2k-1}\alpha_{2k}}_{j_{2k-1}j_{2k}} e_{\alpha_{2k-1}} \wedge e_{\alpha_{2k}} \Big)$$

$$= \frac{1}{4^k} \sum_{\alpha_1,\ldots,\alpha_{2k}} \Big(\sum_{j_1,\ldots,j_{2k}} \delta^{j_1\ldots j_{2k}}_{i_1\ldots i_{2k}} H^{\alpha_1\alpha_2}_{j_1j_2} H^{\alpha_3\alpha_4}_{j_3j_4} \cdots H^{\alpha_{2k-1}\alpha_{2k}}_{j_{2k-1}j_{2k}} \Big) e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{2k}}$$

Therefore

$$(H^{*k})^{\alpha_1\ldots\alpha_{2k}}_{i_1\ldots k_{2k}} = \frac{(2k)!}{4^k} \sum_{j_1\ldots j_{2k}} \delta^{j_1\ldots j_{2k}}_{i_1\ldots i_{2k}} H^{\alpha_1\alpha_2}_{j_1j_2} \cdots H^{\alpha_{2k-1}\alpha_{2k}}_{j_{2k-1}j_{2k}}$$

so by the last proposition

$$(H^{*k}) * \wedge^{n-2k}(I) = \mathrm{tr}(H^{*k})$$

$$= \frac{1}{(2k)!} \sum_{i_1,\ldots,i_{2k}} (H^{*k})^{i_1\ldots i_{2k}}_{i_1\ldots i_{2k}}$$

$$= \frac{1}{4^k} \sum_{\substack{i_1,\ldots,i_{2k}\\ j_1,\ldots,j_{2k}}} \delta^{j_1\ldots j_{2k}}_{i_1\ldots i_{2k}} H^{i_1i_2}_{j_1j_2} \cdots H^{i_{2k-1}i_{2k}}_{j_{2k-1}j_{2k}}.$$

This completes the proof.

Proposition 7.9. Consider $\mathbb{R}^m$ with its standard inner product and let A be a linear map from $\mathbb{R}^m$ into $\mathrm{end}(V)$. Then for any orthonormal basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$ define

$$H = \sum_{i=1}^{m} \wedge^2(A(e_i)) \in \mathrm{end}(\wedge^2(V)).$$

Then H is independent of the choice of orthonormal basis, and

$$\int_{S^{m-1}} \wedge^k(A(u))\Omega_{S^{m-1}}(u) = 0 \qquad \text{for } k \text{ odd}$$

and

$$\int_{S^{m-1}} \wedge^{2k}(A(u))\Omega_{S^{m-1}}(u) = \frac{\mathrm{vol}\ (S^{m-1})}{k!m(m+2)\cdots(m+2k-2)} H^{*k}.$$

Proof. The independence of H from the choice of orthonormal basis follows from the second integral formula with $k = 1$. This is because the left side is independent of the basis. The first integral formula is clear, as $\wedge^k(A(u))$ is an odd function of u and the integral of an odd function over the sphere S^{m-1} is zero. To prove the second integral formula we need;

Lemma. If $\alpha = (\alpha_1,\ldots,\alpha_m)$ is a multi-index (that is each α_j is a nonnegative integer) then

$$\int_{S^{m-1}} u^{2\alpha}\ \Omega_{S^{m-1}}(u) = \frac{(2\alpha)!}{m(m+2)\cdots(m+2|\alpha|-2)2^{|\alpha|}\alpha!} \mathrm{vol}(S^{m-1}).$$

Here $u^{2\alpha} = u_1^{2\alpha_1} u_2^{2\alpha_2} \cdots u_m^{2\alpha_m}$

(where $u = (u_1,u_2,\ldots,u_m)$) and

$$\alpha! = \alpha_1!\ \alpha_2! \cdots \alpha_m!\ ,$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m.$$

Proof. If $\int_{\mathbb{R}^m} \chi^{2\alpha} e^{-\chi\cdot\chi}\, \Omega_{\mathbb{R}^m}(\chi)$ is integrated in polar coordinated (and recalling that $\int_0^\infty t^a e^{-t^2} dt = \frac{1}{2}\Gamma(\frac{a+1}{2}))$ we find

$$\int_{\mathbb{R}^m} \chi^{2\alpha} e^{-\chi\cdot\chi}\, \Omega_{\mathbb{R}^m}(\chi)$$

$$= \int_0^\infty \int_{S^{m-1}} (ru)^{2\alpha} e^{-r^2}\, \Omega_{S^{m-1}}(u) r^{m-1}\, dr$$

$$= \int_0^\infty r^{2|\alpha|+m-1} e^{-r^2} dr \int_{S^{m-1}} u^{2\alpha}\, \Omega_{S^{m-1}}(u)$$

$$= \frac{1}{2}\Gamma(|\alpha| + \frac{m}{2}) \int_{S^{m-1}} u^{2\alpha}\, \Omega_{S^{m-1}}(u)$$

$$= \frac{1}{2}\, \frac{m(m+2)\cdots(m+2|\alpha|-2)}{2^{|\alpha|}}\, \Gamma(\frac{m}{2}) \int_{S^{m-1}} u^{2\alpha}\, \Omega_{S^{m-1}}(u).$$

But this integral can also be computed using Fubini's theorem:

$$\int_{\mathbb{R}^m} \chi^{2\alpha} e^{-\chi\cdot\chi} dx = \int_{\mathbb{R}^m} \chi_1^{2\alpha_1} \cdots \chi_m^{2\alpha_m} e^{-\chi_1^2} \cdots e^{-\chi_m^2} dx_1 \cdots dx_m$$

$$= \prod_{i=1}^m \int_{-\infty}^\infty t^{2\alpha_i} e^{-t^2} dt$$

$$= \prod_{i=1}^m \Gamma(\frac{2\alpha_i - 1}{2})$$

$$= \prod_{i=1}^m (\frac{(2\alpha_i)!}{4^{\alpha_i}\, \alpha_i!})\, \Gamma(\frac{1}{2}))$$

$$= \frac{(2\alpha)!}{4^{|\alpha|}(\alpha!)}\, \Gamma(\frac{1}{2})^m.$$

By equating these two expressions for $\int_{\mathbb{R}^m} \chi^{2\alpha} e^{-\chi\cdot\chi} \Omega_{\mathbb{R}^m}(\chi)$, when $\alpha = (0,\ldots,0)$ we see that

$$\mathrm{vol}(S^{m-1}) = \frac{2\Gamma(\frac{1}{2})^m}{\Gamma(\frac{m}{2})} .$$

Now, for any α, equate the two expressions for the integral and use the formula for $\mathrm{vol}(S^{m-1})$ to finish the proof of the lemma.

We now finish the proof of proposition 7.9. Using the multi-index notation of the lemma, the multinomial theorem can be written as

$$(\chi_1+\cdots+\chi_m)^\ell = \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} \chi_1^{\alpha_1} \chi_2^{\alpha_2} \cdots \chi_m^{\alpha_m} .$$

Now let $e_1, \ldots, e_m$ be an orthonormal basis of $\mathbb{R}^m$. We then write elements of S^{m-1} as $u = u_1e_1 + \cdots + u_me_m$, where $u_1^2 + \cdots + u_m^2 = 1$. Let $u = (u_1,\ldots,u_m)$; then the multinomial theorem and 7.5 (2) imply

$$\int_{S^{m-1}} \Lambda^{2k}(A(v))\Omega_{S^{m-1}}(v)$$

$$= \frac{1}{(2k)!} \int_{S^{m-1}} (A(u_1e_1+\cdots+u_me_m))^{*2k}\Omega_{S^{m-1}}(u)$$

$$= \frac{1}{(2k)!} \int_{S^{m-1}} (u_1A(e_1)+\cdots+u_mA(e_m))^{*2k}\Omega_{S^{m-1}}(u)$$

$$= \frac{1}{(2k)!} \sum_{|\beta|=2k} \frac{(2k)!}{\beta!} \int_{S^{m-1}} u^\beta\Omega_{S^{m-1}}(u)A(e_1)^{*\beta_1} * \cdots * A(e_m)^{*\beta_m}.$$

If $\beta = (\beta_1, \ldots, \beta_m)$ and any β_j is odd, then

$$\int_{S^{m-1}} u^\beta \, \Omega_{S^{m-1}}(u) = 0$$

by summetry. Using this fact and the lemma yields

$$\int_{S^{m-1}} \wedge^{2k}(A(v))\Omega_{S^{m-1}}(v)$$

$$= \sum_{|\alpha|=k} \frac{1}{(2\alpha)!} \int_{S^{m-1}} u^{2\alpha} \, \Omega_{S^{m-1}}(u) \, A(e_1)^{*2\alpha_1} * A(e_2)^{*2\alpha_1} * \cdots * A(e_m)^{*2\alpha_m}$$

$$= \sum_{|\alpha|=k} \frac{1}{(2\alpha)!} \frac{(2\alpha)! \; \mathrm{vol}(S^{m-1})}{m(m+2)\cdots(m+2|\alpha|-2)2^{|\alpha|}\alpha!} \; A(e_1)^{*2\alpha_1} * \cdots * A(e_m)^{*2\alpha_m}$$

$$= \frac{\mathrm{vol}(S^{m-1})}{m(m+2)\cdots(m+2k-2)2^k k!} \sum_{|\alpha|=k} \frac{k!}{\alpha!} A(e_1)^{*2\alpha_1} * \cdots * A(e_m)^{*2\alpha_m}$$

$$= \frac{\mathrm{vol}(S^{m-1})}{m(m+2)\cdots(m+2k-2)2^k k!} (A(e_1)^{*2} + \cdots + A(e_m)^{*2})^{*k}$$

$$= \frac{\mathrm{vol}(S^{m-1})}{m(m+2)\cdots(m+2k-2)2^k k!} \, 2^k(\wedge^2(A(e_1)) + \cdots + \wedge^2(A(e_m)))^{*k}$$

$$= \frac{\mathrm{vol}(S^{m-1})}{m(m+2)\cdots(m+2k-2)k!} \, H^{*k}.$$

This finishes the proof.

8. The tube formula.

In this section the algebraic results of the last section are used to restate theorem 6.14.

Theorem 8.1. Let M, $\widetilde{M}$, $C(t;U)$, $S(t;U)$, $S^{\perp}(t;U)$ be as in theorem 6.14. For each k with $0 \leq k \leq n$, define $h_k : M \times \mathbb{R} \to \mathbb{R}$ by

$$h_k(p,t) = \frac{1}{t}\int_{S^{\perp}M_p} \wedge^k(S(t;U)A(U)) * \wedge^{n-k}(C(t;U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U).$$

Then, the volume of the tube $M(r)$ of radius r about M is

$$\mathrm{vol}(M(r)) = \sum_{k=0}^{n}\int_M h_k(p,r)\Omega_M(p).$$

Proof. By theorem 6.14

$$\mathrm{vol}(M(r)) = \int_M h(p,r)\Omega_M(p)$$

where

$$h(p,t) = \frac{1}{t}\int_{S^{\perp}M_p} \det(C(t;U) + S(t;U)A(U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U)$$

$$= \frac{1}{t}\sum_{k=0}^{n}\int_{S^{\perp}M_p} \wedge^k(S(t;U)A(U)) * \wedge^{n-k}(C(t;U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U)$$

$$= \sum_{k=0}^{n} h_k(p,t).$$

In this computation we have used proposition 7.5 (3) to expand $\det(C(t;U) + S(t;U)A(U))$.

Remarks. (1) We can use the formula

$$\wedge^k(S \circ A) = \wedge^k(S) \circ \wedge^k(A)$$

to rewrite the formula for $h_k(p,t)$ as

$$h_k(p,t) = \frac{1}{t}\int_{S^\perp M_p} (\wedge^k(S(t;U)) \circ \wedge^k(A(U)) * \wedge^{n-k}(C(t;U))\det(S^\perp(t;U))\Omega_{S^\perp M_p}(U).$$

This shows that $h_k(p,t)$ is a linear function of the map $U \mapsto \wedge^k(A(U))$.

(2) Both $S(t,U)$ and $S^\perp(t;U)$ vanish to order one at $t = 0$. Thus, for any U, $\wedge^k(S(t;U)A(U))$ vanishes to order at least k at $t = 0$, and $\det(S^\perp(t;U))$ vanishes to order m at $t = 0$. Therefore, it is easy to see that $h_k(p,t)$ vanishes to order at least $m + k - 1$, for all p in M.

The above formula becomes simpler if $\tilde{M}$ is a symmetric space.

<u>Theorem 8.2</u>. If $\tilde{M}$ is an oriented symmetric space, and M is a compact symmetrically embedded submanifold of $\tilde{M}$ with smooth boundary, then, for each $U \in S^\perp M$ let

$$R_U = \tilde{R}_U\big|_{TM_p},$$

$$R_U^\perp = \tilde{R}_U\big|_{T^\perp M_p},$$

where $\tilde{R}$ is the curvature tensor of the Riemannian connection of M.

Define

$$C(t;U),\ S(t;U) : TM_p \to TM_p \qquad (p = \pi U),$$

$$S^\perp(t;U) : T^\perp M_p \to TM_p,$$

by

$$S''(t;U) + R_U S(t;U) = 0 \qquad S(0;U) = 0,\ S''(0;U) = (id)_{TM_p},$$

$$C''(t;U) + R_U C(t;U) = 0 \qquad C(0,U) = (id)_{TM_p},\ C''(0;U) = 0,$$

$$(S^\perp)''(t;U) + R_U^\perp S(t;U) = 0 \qquad S^\perp(0;U) = 0,\ (S^\perp)''(0;U) = (id)_{T^\perp M_p}.$$

$$h_{2k}(p,t) = \frac{1}{t}\int_{S^{\perp}M_p} \wedge^{2k}(S(t;U)A(U))*\wedge^{n-2k}(C(t;U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U).$$

Then the volume of the tube $M(r)$ of radius r about M is

$$\mathrm{vol}(M(r)) = \sum_{k=0}^{[\frac{n}{2}]} \int_M h_{2k}(p,r)\Omega_M(p)$$

where $[\frac{n}{2}]$ is the greatest integer in $\frac{n}{2}$.

Proof. In a symmetric space, $\tilde{T} = 0$ and $\tilde{B} = R$ by proposition 5.21. Therefore, by 8.1, it is enough to show that if $\tilde{M}$ is symmetric, then $h_k(p,t)$ vanishes for k odd.

Note that

$$\begin{aligned} \tilde{R}_{-U}(X) &= \tilde{R}(X,-U)(-U) \\ &= \tilde{R}_U(X). \end{aligned}$$

This shows $R_U = R_{(-U)}$ and $R^{\perp}_{(-U)} = R^{\perp}_{(-U)}$. It then follows from the defining equations of $C(t;U)$, $S(t;U)$ and $S^{\perp}(t;U)$ that all three are even functions of U. But $A(U)$ is a linear function of U and thus an odd function of U. Thus,

$$\wedge^{k}(S(t;U)A(U))*\wedge^{n-k}(C(t;U))\det(S^{\perp}(t;U))$$

is an odd function of U for k odd. The integral of an odd function over the sphere $S^{\perp}M_p$ is zero. This shows $h_k(p,t)$ vanishes for odd k and finishes the proof.

9. Parallel hypersurfaces.

In this section we will use the notation of section 6 with the extra condition that $m = 1$. Then M is a hypersurface of $\tilde{M}$. We assume that M is compact and oriented with smooth boundary. If this is the case, it is possible to choose a smooth unit normal field U along M. For each p in M the vector space $T^{\perp}M_p$ is one-dimensional and therefore contains exactly two vectors of unit length. Therefore,

$$S^{\perp}M_p = \{U(p), -U(p)\}.$$

Define the *parallel hypersurface at a distance* r from M by

$$PM(r) = \{\exp_p(rU(p)) : p \in M\}.$$

It is then clear that the tube $M(r)$ of radius r about M is the union of $PM(r)$ and $PM(-r)$.

Proposition 9.1. With notation as in 8.1,

$$\mathrm{vol}(PM(r)) = \sum_{k=0}^{n} \int_M h_k^{+}(p,r)\Omega_M(p),$$

where

$$h_k^{+}(p,r) = \Lambda^k(S(r;U(p))A(U(p))) * \Lambda^{n-k}(C(r;U(p))).$$

Proof. If M is an oriented hypersurface then $S^{\perp}M$ is the disjoint union of

$$S^{+}M = \{U(p) : p \in M\}$$

and

$$S^{-}M = \{-U(p) : p \in M\}.$$

If $S^{\perp}M$ is replaced by $S^{+}M$, then all the results of section 6 go through as before, except that we will be computing the volume of $PM(r)$

rather than $M(r)$. The same holds true of theorem 8.1. Because $T^{\perp}_{U(p)}(U(p))$ and $B^{\perp}_{U(p)}(U(p))$ both vanish, the initial value problem defining $S^{\perp}(t;U(p))$ becomes

$$(S^{\perp})''(t;U(p)) = 0 \qquad S^{\perp}(0;U(p)) = 0,\ (S^{\perp})'(0;U(p)) = (id).$$

Therefore $S^{\perp}(t;U(p))U(p) = tU(p)$. This yields

$$\det(S^{\perp}(t;U(p))) = t.$$

Because $S^{+}M_p$ has only the one point $U(p)$, integration over $S^{+}M_p$ is just evaluation at this point. Theorem 8.1 now yields

$$\mathrm{vol}(PM(r)) = \sum_{k=0}^{n} \int_M f_k(p,r)\Omega_M(p),$$

where

$$f_k(p,t) = \frac{1}{t}\int_{S^{+}M_p} \Lambda^k(S(t;U)A(U)) * \Lambda^{n-k}(C(t;U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U)$$

$$= \Lambda^k(S(r;U(p))A(U(p))) * \Lambda^{n-k}(C(r;U(p)))$$

$$= h_k^{+}(p,r).$$

This completes the proof.

Remark. In the case $\tilde{M}$ is the Euclidean space of dimension $n + 1$, then both $\tilde{B}$ and $\tilde{T}$ vanish. Using this in the definitions of $c(t;U)$ and $S(t;U)$ shows

$$C(t;U(p)) = (id)_{TM_p}$$

$$S(t;U(p)) = t(id)_{TM_p}.$$

Whence

$$h_k^{+}(p,t) = s^k\Lambda^k(A(U(p))) * \Lambda^{n-k}(I)$$

$$= s^k tr(\Lambda^k(A(U(P)))$$

$$= s^k\sigma_k(A(U(p))$$

where $\sigma_k(A(U(p)))$ is the k-th elementary symmetric function in the eigenvalues of $A(U(p))$. This follows from the remark after the proof of 7.5. This yields

$$\mathrm{vol}(PM(r)) = \sum_{k=0}^{n} r^k \int_M \sigma_k(A(U(p)))\Omega_M(p),$$

a formula due to Steiner, [11].

<u>Proposition 9.2</u>. If M is a hypersurface of the symmetric space $\tilde{M}$ then the volume of $M(r)$, the tube of radius r about M, is

$$\mathrm{vol}(M(r)) = \sum_{0\le 2k\le n} \int_M h_{2k}(p,r)\Omega_M(p)$$

where

$$h_{2k}(p,r) = \frac{2^{k+1}}{(2k)!}(\wedge^{2k}(S(t;U(p))\circ H_p^{*k})*\wedge^{n-2k}(C(t;U(p))).$$

Here H is the excess tensor of M in $\tilde{M}$ defined in definition 4.4. This shows each h_{2k} is a linear function of H^{*k} and that $\mathrm{vol}(M(r))$ only depends on the excess tensor of M in $\tilde{M}$, but is otherwise independent of the embedding of M in $\tilde{M}$.

<u>Proof</u>. By theorem 8.2,

$$\mathrm{vol}(M(r)) = \sum_{0\le 2k\le n} \int h_{2k}(p,r)\Omega_M(p),$$

where

$$h_{2k}(p,t) = \frac{1}{t}\int_{S^{\perp}M_p} \wedge^{2k}(S(t;U)A(U))*\wedge^{n-2k}(C(t;U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U)$$

$$= \wedge^{2k}(S(t;U(p))A(U(p)))*\wedge^{n-2k}(C(t;U(p))$$

$$+ \wedge^{2k}(S(t;-U(p))A(-U(p)))*\wedge^{n-2k}(C(t;-U(p))$$

$$= 2(\wedge^{2k}(S(t;U(p)))\circ\wedge^{2k}(A(U(p))))*\wedge^{n-2k}(C(t;U(p))).$$

We have used the facts that $\det(S^{\perp}(t;U)) = t$, that integration over $S^{\perp}M_p$ is the sum of the evaluations at $U(p)$ and $-U(p)$, and that $S(t;U)$, $C(t;U)$ and $\wedge^{2k}(A(U))$ are even functions of U. In the case at hand, the excess tensor is given by

$$H_p = \wedge^2(A(U(p))).$$

Set $A = A(U(p))$. Then, by proposition 7.5 (2) we have

$$\begin{aligned}\wedge^{2k}(A) &= \frac{1}{(2k)!} A^{*2k} \\ &= \frac{1}{(2k)!}(A^{*2})^{*k} \\ &= \frac{1}{(2k)!}(2\wedge^2(A))^{*k} \\ &= \frac{2^k}{(2k)!} H^{*k}.\end{aligned}$$

Putting this into the above formula for h_{2k} yields the result.

<u>Remark</u>. If $\widetilde{M}$ is not a symmetric space then it is easily seen from the differential equations defining $C(t,U)$ and $S(t;U)$ that they are not even functions of U. Therefore there is no reason to expect the last proposition to hold in any space other than a symmetric space.

10. An algebraic reformulation for symmetric spaces.

Let $\tilde{M}$ be an oriented symmetric space. Let G be a transitive group of isometries of $\tilde{M}$ satisfying the two conditions of convention 5.2. Let o be the origin of $\tilde{M}$, and H be the subgroup of all elements of G that fix o. Let $\tilde{\mathfrak{m}}$ be the tangent space to $\tilde{M}$ at o. Then, as in proposition 5.8 and convention 5.9, we identify $\tilde{\mathfrak{m}}$ with a subspace of $\mathfrak{g}$ (the Lie algebra of G) so that $\tilde{\mathfrak{m}}$ is invariant under the adjoint action of H, and

$$\mathfrak{g} = \tilde{\mathfrak{m}} \oplus h$$

where h is the Lie algebra of G. If $\mathfrak{m}$ is a vector subspace of $\tilde{\mathfrak{m}}$, then denote by "$\mathfrak{m}^{\perp}$" the orthogonal complement of $\mathfrak{m}$ in $\tilde{\mathfrak{m}}$.

Definition 10.1. A *second order germ of a manifold* (or briefly a second order germ) is a pair $(\mathfrak{m},A)$ where $\mathfrak{m}$ is a vector subspace of $\tilde{\mathfrak{m}}$ and A is linear map from $\mathfrak{m}^{\perp}$ to the symmetric linear maps on $\mathfrak{m}$. The dimension of $(\mathfrak{m},A)$ is defined to be the dimension of $\mathfrak{m}$. The linear map A is called the Weingarten map of $(\mathfrak{m},A)$. Two second order germs $(\mathfrak{m}_1,A_1)$ and $(\mathfrak{m}_2,A_2)$ will be considered *equivalent* if and only if there is an element a in H so that

$$a_{*o}\,\mathfrak{m}_1 = \mathfrak{m}_2$$

$$a_{*o}(A_1(Y)X) = A_2(a_{*o}Y)a_{*o}\,X$$

for all X in $\mathfrak{m}_1$ and Y in $\mathfrak{m}_1^{\perp}$.

Definition 10.2. If M is a submanifold of $\tilde{M}$ and $p \in M$, then the *second order germ* $(\mathfrak{m},A)$ *of* M at p will now be defined. Choose any element g in G with $g(p) = 0$. Then $\mathfrak{m} = T(gM)_o$, and A is the Weingarten map for the manifold gM at o. It is clear that different choices of g with $g(p) = o$ give equivalent second order germs in the sense of the last definition.

Definition 10.3. Let $(\mathfrak{m}, A)$ be a second order germ, $\tilde{R}$ the curvature tensor of $\tilde{M}$ at o viewed as a linear map on $\Lambda^2 T\tilde{M}_o$ and P the orthogonal projection from $\Lambda^2 T\tilde{M}_o$ onto $\Lambda^2(\mathfrak{m})$. Then the *curvature tensor* R of $(\mathfrak{m}, A)$ is defined to be

$$R = P\tilde{R} - \sum_{i=1}^{n} \Lambda^2(A(e_j))$$

where $e_1, \ldots, e_m$ is any orthonormal basis of $\mathfrak{m}^\perp$. The *excess tensor* H of $(\mathfrak{m}, A)$ is defined to be

$$H = \sum_{j=1}^{m} \Lambda^2(A(e_j)) = P\tilde{R} - R$$

Remark. Let M be a submanifold of $\tilde{M}$ passing through o whose tangent space at o is $\mathfrak{m}$ and whose Weingarten map at o is A. Then proposition 4.3 and definition 4.4 imply that the curvature of M at o, viewed as a linear map on $\Lambda^2 TM_o$, is the same as the curvature of the second order germ $(\mathfrak{m}, A)$.

Definition 10.4. The second order germ $(\mathfrak{m}, A)$ is said to be *symmetrically embedded* if and only if, for all X and U in $\mathfrak{m}^\perp$, the vector $\tilde{R}(X,U)U$ is also in $\mathfrak{m}^\perp$.

Remark. It is easy to check that a submanifold M of $\tilde{M}$ is symmetrically embedded if and only if its second order germ at each of its points is symmetrically embedded.

Definition 10.5. Let $(\mathfrak{m}, A)$ be a symmetrically embedded second order germ. Define for all $U \in \mathfrak{m}^\perp$ linear maps $R_{\mathfrak{m},U} : \mathfrak{m} \to \mathfrak{m}$, $R^\perp_{\mathfrak{m},U} : \mathfrak{m}^\perp \to \mathfrak{m}^\perp$

by

$$R_{\mathfrak{m},U} = \tilde{R}_U\big|_{\mathfrak{m}}$$

$$R_{\mathfrak{m},U} = \tilde{R}_U\big|_{\mathfrak{m}^\perp}.$$

Now define $S_{\mathfrak{m}}(t;U)$, $C_{\mathfrak{m}}(t;U) : \mathfrak{m} \to \mathfrak{m}$ and $S_{\mathfrak{m}}^\perp(t;U) : \mathfrak{m} \to \mathfrak{m}^\perp$ by the initial value problems:

$$S_{\mathfrak{m}}''(t;U) + R_{\mathfrak{m},U}S_{\mathfrak{m}}(t;U) = 0 \qquad S_{\mathfrak{m}}(0;U) = 0,\ \ S_{\mathfrak{m}}'(0,U) = (id)_{\mathfrak{m}},$$

$$C_{\mathfrak{m}}''(t;U) + R_{\mathfrak{m},U}C_{\mathfrak{m}}(t;U) = 0 \qquad C_{\mathfrak{m}}(0;U) = (id)_{\mathfrak{m}},\ \ C_{\mathfrak{m}}'(0;U) = 0,$$

$$(S_{\mathfrak{m}}^\perp)''(t;U) + R_{\mathfrak{m},U}S_{\mathfrak{m}}^\perp(t;U) = 0 \qquad S_{\mathfrak{m}}^\perp(0;U) = 0,\ \ (S_{\mathfrak{m}}^\perp)'(0;U) = (id)_{\mathfrak{m}}.$$

Proposition 10.6. Let M be a symmetrically embedded submanifold of $\tilde{M}$ and $(\mathfrak{m},A)$ the second order germ of M at $p \in M$. Then the function $h_{2k}(p,t)$ of theorem 8.2 can be computed by

$$h_{2k}(p,t) = \frac{1}{t}\int_{S^\perp M_p} \Lambda^{2k}(S_{\mathfrak{m}}(t;U)A(U)) * \Lambda^{n-2k}(C_{\mathfrak{m}}(t;U))\det(S_{\mathfrak{m}}^\perp(t;U))\Omega_{S^\perp\mathfrak{m}}(U).$$

Here $S^\perp\mathfrak{m}$ is the unit sphere of $\mathfrak{m}^\perp$.

Proof. By definition there is a $g \in G$ with $g(p) = o$ and such that $\mathfrak{m} = T(gM)_o$ and A is the Weingarten map of gM at o. Let A_1 be the Weingarten map of M at p. Then, because g is an isometry of $\tilde{M}$, we see for all $U \in T^\perp M_p$, that $g_{*p}U \in T^\perp(gM)_o$, and

$$A(g_{*p}U) = g_{*p}\big|_{TM_p} A_1(U)(g_{*p})^{-1}\big|_{\mathfrak{m}},$$

$$R_{\mathfrak{m},U} = g_{*p}\big|_{TM_p} R_U(g_{*p})^{-1}\big|_{\mathfrak{m}},$$

$$R_{\mathfrak{m},U}^\perp = g_{*p}\big|_{T^\perp M_p} R_U^\perp(g_{*p})^{-1}\big|_{\mathfrak{m}^\perp}.$$

Set $P_1 = g_{*p}\big|_{TM_p}$ and $P_2 = g_{*p}\big|_{T^\perp M_p}$. It then follows from the initial value problems defining the linear maps involved that,

$$S_{\mathfrak{m}}(t;P_2U) = P_1S(t;U)P_1^{-1}$$

$$C_{\mathfrak{m}}(t;P_2U) = P_1C(T;U)P_1^{-1}$$

$$S_{\mathfrak{m}}^{\perp}(t;P_2U) = P_2S^{\perp}(t;U)P_2^{-1}.$$

This shows $\det(S_{\mathfrak{m}}^{\perp}(t;P_2U)) = \det(S^{\perp}(t;U))$. We now use proposition 7.5 (4) to compute

$$\wedge^{2k}(S_{\mathfrak{m}}(t;P_2U)A(P_2U)) * \wedge^{n-2k}(C(t;P_2U))$$

$$= \wedge^{2k}(P_2S(t;U)A_1(U)P_2^{-1}) * \wedge^{n2-k}(P_2C(t;U)P_2^{-1})$$

$$= \wedge^{n}(P_2)\ \wedge^{2k}(S(t;U)A_1(U)) * \wedge^{n-2k}(C(t;U))\ \wedge^{n}(P_2^{-1})$$

$$= \wedge^{2k}(S(t;U)A_1(U)) * \wedge^{n-2k}(C(t;U)).$$

The function $h_{2k}(p,t)$ is then given by

$$h_{2k}(p,t) = \frac{1}{t}\int_{S^\perp M_p} \wedge^{2k}(S(t;U)A_1(U)) * \wedge^{n-2k}(C(t;U))\det(S^{\perp}(t;U))\Omega_{S^\perp M_p}(U)$$

$$= \frac{1}{t}\int_{S^\perp M_p} \wedge^{2k}(S_{\mathfrak{m}}(t;P_2U))A(P_2U)) * \wedge^{n-2k}(C_{\mathfrak{m}}(t;P_2U))\det(S_{\mathfrak{m}}^{\perp}(t;P_2U))\Omega_{S^\perp M_p}(U).$$

The map $U \mapsto P_2U$ is an isometry of $S^\perp M_p$ with $S^\perp\mathfrak{m}$. The result thus follows by a change of variables in the integral.

We now compute $C_{\mathfrak{m}}(t;U)$, $S_{\mathfrak{m}}(t;U)$ and $S_{\mathfrak{m}}^{\perp}(t;U)$ in terms of the Lie algebra $\mathfrak{g}$. for $X \in \mathfrak{g}$ define a linear map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}(X)Y = [X,Y].$$

The map $X \mapsto \mathrm{ad}(X)$ is called the adjoint representation of $\mathfrak{g}$. It is a Lie algebra homomorphism of $\mathfrak{g}$ into the Lie algebra of all derivations of $\mathfrak{g}$.

Proposition 10.7. Let $(\mathfrak{m},A)$ be a symmetrically embedded second order germ, and $U \in \mathfrak{m}^{\perp}$. Then $R_{\mathfrak{m},U} : \mathfrak{m} \to \mathfrak{m}$ and $R^{\perp}_{\mathfrak{m},U} : \mathfrak{m}^{\perp} \to \mathfrak{m}^{\perp}$ are given by

$$R_{\mathfrak{m},U} = -\mathrm{ad}(U)^2|_{\mathfrak{m}}$$

$$R^{\perp}_{\mathfrak{m},U} = -\mathrm{ad}(U)^2|_{\mathfrak{m}^{\perp}}.$$

Also if $\cosh(t\ \mathrm{ad}(U))$ and $\mathrm{ad}(U)^{-1}\sinh(t\ \mathrm{ad}(U))$ are defined by their power series, that is

$$\cosh(t\ \mathrm{ad}(U)) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}(\mathrm{ad}(U))^{2k},$$

$$\mathrm{ad}(U)^{-1}\sinh(t\ \mathrm{ad}(U)) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(\mathrm{ad}(U))^{2k},$$

then

$$S_{\mathfrak{m}}(t;U) = \mathrm{ad}(U)^{-1}\sinh(t\ \mathrm{ad}(U))|_{\mathfrak{m}}$$

$$C_{\mathfrak{m}}(t;U) = \cosh(t\ \mathrm{ad}(U))|_{\mathfrak{m}},$$

$$S_{\mathfrak{m}}(t;U) = \mathrm{ad}(U)^{-1}\sinh(t\ \mathrm{ad}(U))|_{\mathfrak{m}^{\perp}}.$$

Proof. By proposition 5.23, the torsion tensor T of $\tilde{M}$ is zero and the curvature tensor is the same as that of the canonical connection. Therefore, by proposition 5.12 (3), for X,Y in $\tilde{\mathfrak{m}}$,

$$0 = T(X,Y) = -[X,Y]_{\tilde{\mathfrak{m}}}.$$

Thus $[X,Y] \in h$. Using this in 5.12 (4) yields, for $X, Y, Z \in \tilde{\mathfrak{m}}$,

$$\tilde{R}(X,Y)Z = -[[X,Y]_h,Z]$$

$$= -[[X,Y],Z].$$

So, if $X, U \in \tilde{\mathfrak{m}}$, then

$$\tilde{R}_U(X) = \tilde{R}(X,U)U$$
$$= -[[X,U],U]$$
$$= -[U,[U,X]]$$
$$= -\mathrm{ad}(U)^2 X.$$

This proves the statements about $R_{\mathfrak{m},U}$ and $R^{\perp}_{\mathfrak{m},U}$.

From the formula $R_{\mathfrak{m},U} = -\mathrm{ad}(U)^2|_{\mathfrak{m}}$, it is easy to check that $\cosh(t\ \mathrm{ad}(U))|_{\mathfrak{m}}$ is a solution to the initial value problem defining $C_{\mathfrak{m}}(t;U)$. The other formulas are proved in the same way.

<u>Corollary 10.8</u>. Let $(\mathfrak{m},A)$ be a symmetrically embedded second order germ and $U \in \mathfrak{m}^{\perp}$. Then, for any real number a,

$$C_{\mathfrak{m}}(t;aU) = C_{\mathfrak{m}}(at;U),$$
$$S_{\mathfrak{m}}(t;aU) = \frac{1}{a} S_{\mathfrak{m}}(at;U)$$
$$S^{\perp}_{\mathfrak{m}}(t;aU) = \frac{1}{a} S_{\mathfrak{m}}(at;U).$$

<u>Proof</u>. By the formulas of the last proposition

$$S(t;aU) = \mathrm{ad}(aU)^{-1} \sinh(t\ \mathrm{ad}(AU))$$
$$= \frac{1}{a} \mathrm{ad}(U)^{-1} \sinh((at)\mathrm{ad}(U))$$
$$= \frac{1}{a} S(at;U).$$

The other two equations are proved in the same way.

11. Tubes in product manifolds.

For $\alpha = 1, 2$ let M_α be a compact oriented symmetrically embedded submanifold of dimension n_α in the oriented symmetric space $\tilde{M}_\alpha$ of dimension $n_\alpha + m_\alpha$. Let $(h_\alpha)_{2k}(p,t)$ be the function given by theorem 8.2, so that

$$\mathrm{vol}(M_\alpha(r)) = \sum_{0\le 2k\le n_\alpha} \int_M (h_\alpha)_{2k}(p,r)\Omega_M(p).$$

In this section we prove

Theorem 11.1. The submanifold $M = M_1 \times M_2$ is a symmetrically embedded submanifold of $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$. Let $n = n_1 + n_2$ and

$$\mathrm{vol}(M(r)) = \sum_{0\le 2k\le n} \int_M h_{2k}((p,q),r)\Omega_M((p,q))$$

where $h_{2k}((p,q),t)$ is as in 8.2.

Then

$$h_{2k}((p,q),t) = t \sum_{0\le j\le k} \int_0^{\pi/2} (h_1)_{2j}(p,t\cos\theta)(h_2)_{2(k-j)}(q,t\sin\theta)d\theta$$

with the convention that $(h_\alpha)_{2\ell} = 0$ if $2\ell > n_\alpha$. Therefore,

$$\mathrm{vol}(M(r)) = r\int_0^{\pi/2} \mathrm{vol}(M_1(r\cos\theta))\mathrm{vol}(M_2(r\sin\theta))d\theta.$$

Proof. Assume the formula for $h_{2k}((p,q),t)$.

Then

$$\mathrm{vol}(M(r)) = \sum_{0\le 2k\le n} \int_0 h_{2k}((p,q),r)\Omega_M((p,q))$$

$$= \sum_{0\le 2k\le n} r\int_M \sum_{0\le j\le k} \int_0^{\pi/2}(h_1)_{2j}(p,r\cos\theta)(h_2)_{2(k-j)}(q,r\sin\theta)d\theta\,\Omega_{M_1\times M_2}(p,q)$$

$$= r\int_0^{\pi/2}\int_{M_1\times M_2}\Big(\sum_{0\le 2k\le n_1}(h_1)_{2k}(p,r\cos\theta)\Big)\Big(\sum_{0\le 2j\le n_2}(h_2)(q,r\sin\theta)\Big)\Omega_{M_1\times M_2}(p,q)\,d\theta$$

$$= r\int_0^{\pi/2}\Big(\int_{M_1}\sum_{0\le 2k\le n_1}(h_1)_{2k}(p,r\cos\theta)\Omega_{M_1}(p)\Big)\Big(\int_{M_2}\sum_{0\le 2j\le n_2}(h_2)_{2j}(q,r\sin\theta)\Omega_{m_2}(q)\Big)d\theta$$

$$= r\int_0^{\pi/2}\mathrm{vol}(M_1(r\cos\theta))\,\mathrm{vol}(M_2(r\sin\theta))\,d\theta .$$

The proof that the formula for $h_{2k}(p,t)$ holds will be done in a series of lemmas. It will be more convenient to work with the second order germs of submanifolds than with the submanifolds thenselves. Let $p \in M_1$, $q \in M_2$, and $(\mathfrak{m}_1, A_1)$ be the second order germ of M_1 at p, and $(\mathfrak{m}_2, A_2)$ the second order germ of M_2 at q. Let $\tilde{\mathfrak{m}}_\alpha$ be the space to $\tilde{M}_\alpha$ at 0, its origin. Then, as in the last section, there is a decomposition

$$\tilde{\mathfrak{m}}_\alpha = \mathfrak{m}_\alpha \oplus \mathfrak{m}_\alpha^\perp .$$

Let $\tilde{\mathfrak{m}}$ be the tangent space to $\tilde{M}$ at $(0,0)$. Then we can assume that $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_1^\perp$ and $\mathfrak{m}_2^\perp$ are subspaces of $\tilde{\mathfrak{m}}$ in the natural way. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then the orthogonal complement to $\mathfrak{m}$ in $\tilde{\mathfrak{m}}$ is $\mathfrak{m}^\perp = \mathfrak{m}_1^\perp \oplus \mathfrak{m}_2^\perp$.

Convention 11.2. The letter U always denotes elements of $\mathfrak{m}_1^\perp$ and the letter V will always denote elements of $\mathfrak{m}_2^\perp$.

Define a linear map A from $\mathfrak{m}^\perp$ to the symmetric linear maps on $\mathfrak{m}$ by

$$A(U)|_{\mathfrak{m}_1} = A_1(U),$$
$$A(U)|_{\mathfrak{m}_2} = 0,$$
$$A(V)|_{\mathfrak{m}_1} = 0,$$
$$A(V)|_{\mathfrak{m}_2} = A_2(V).$$

Lemma 11.3. The second order germ of $M = M_1 \times M_2$ at (p,q) is $(\mathfrak{m}, A)$.

Proof. It can be assumed that p is the origin of $\widetilde{M}_1$ and that A_1 is the Weingarten map for M_1 at 0. Similar assumptions are made for M_2 and q. If $\rho_\alpha : \widetilde{M} \to \widetilde{M}_\alpha$ is projection, then our identification of $\widetilde{\mathfrak{m}}_\alpha$ with a subspace of $\widetilde{\mathfrak{m}}$ identifies the derivative $\rho_{\alpha*(0,0)}$ with orthogonal projection of $\widetilde{\mathfrak{m}}$ onto $\widetilde{\mathfrak{m}}_\alpha$. It is clear that the tangent space to $M_1 \times M_2$ at $(0,0)$ is $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Let B be the Weingarten map of M at $(0,0)$. Then because $\rho_{\alpha*}$ is orthogonal projection, proposition 4.6 becomes

$$\langle B(U+V)(X_1+X_2), Y_1 + Y_2\rangle$$

$$= \langle A_1(U)X_1, Y_1\rangle + \langle A_2(V)X_2, Y_2\rangle$$

where $X_1, Y_1 \in \mathfrak{m}_1$ and $X_2, Y_2 \in \mathfrak{m}_2$. This shows $B = A$, and finishes the proof.

Lemma 11.4. Let $\widetilde{R}$, $\widetilde{R}_1$ and $\widetilde{R}_2$ be the curvature tensors of $\widetilde{M}$, $\widetilde{M}_1$ and $\widetilde{M}_2$ respectively. Then for all U, V, all four of $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_1^\perp$ and $\mathfrak{m}_2^\perp$ are stable under $\widetilde{R}_{(U+V)}$, and

$$\widetilde{R}_{(U+V)}\big|_{\mathfrak{m}_1} = (\widetilde{R}_1)_U\big|_{\mathfrak{m}_1},$$
$$\widetilde{R}_{(U+V)}\big|_{\mathfrak{m}_2} = (\widetilde{R}_2)_V\big|_{\mathfrak{m}_2},$$
$$\widetilde{R}_{(U+V)}\big|_{\mathfrak{m}_1^\perp} = (\widetilde{R}_1)_U\big|_{\mathfrak{m}_1^\perp},$$
$$\widetilde{R}_{(U+V)}\big|_{\mathfrak{m}_2^\perp} = (\widetilde{R}_2)_V\big|_{\mathfrak{m}_2^\perp}.$$

This shows that $M = M_1 \times M_2$ is symmetrically embedded in $\widetilde{M}$.

Proof. Let $X_1, Y_1 \in \widetilde{\mathfrak{m}}_1$, $X_2, Y_2 \in \widetilde{\mathfrak{m}}_2$. Using the notation of the last lemma proposition 4.5 yields

$$\langle \tilde{R}_{(U+V)}(X_1+X_2), Y_1 + Y_2\rangle$$

$$= \langle \tilde{R}(X_1+X_2, U+V)(U+V), Y_1 + Y_2\rangle$$

$$= \langle \tilde{R}_1(X_1, U)U, Y_1\rangle + \langle \tilde{R}_2(X_2, V)V, Y_2\rangle$$

$$= \langle (\tilde{R}_1)_U X_1, Y_1\rangle + \langle (\tilde{R}_2)_V X_2, Y_2\rangle.$$

The result now follows easily.

Let $C(t;U+V)$, $C_1(t;U)$ and $C_2(t;V)$ be defined for M, M_1, and M_2 respectively as in theorem 8.2. Make analogous definitions for S, S_1, S_2 and $S^\perp$, $S_1^\perp$, $S_2^\perp$.

Lemma 11.5.

$$C(t;U+V) = C_1(t;U) \oplus C_2(t;V),$$
$$S(t;U+V) = S_1(t;U) \oplus C_2(t;V),$$
$$S^\perp(t,U+V) = S_1^\perp(t;U) \oplus S_2^\perp(t;V),$$

where the notation means

$$C(t;U+V)|_{\mathfrak{m}_1} = C_1(t;U),$$
$$C(t;U+V)|_{\mathfrak{m}_2} = C_2(t;V),$$

etc.

Proof. Using 11.4 it is easy to check that $C_1(t;U) \oplus C_2(t;V)$ satisfies the differential equation defining $C(t;U+V)$. The other cases are similar.

Lemma 11.6. $\det(S^\perp(t;U+V)) = \det(S_1^\perp(t;U))\det(S_2^\perp(t;V))$.

Proof. Clear from 11.5.

Lemma 11.7. $S(t;U+V)A(U+V)$

$$= (S_1(t;U)A_1(U)) \oplus (S_2(t;V)A_2(V)).$$

Proof. This also follows from 11.5 (and the definition of A).

It is possible to view $C_1(t;U)$ as a linear map on $\mathfrak{m}$ by extending $C_1(t;U)$ from $\mathfrak{m}_1$ to $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ by having $C_1(t;U)|_{\mathfrak{m}_2} = 0$. Using a similar convention for $C_2(t;V)$ lets us write

$$C(t;U+V) = C_1(t;U) + C_2(t;U).$$

This convection will be used in the following few lemmas.

Lemma 11.8.

$$\wedge^{2k}(S(t;U+V)A(U+V)) * \wedge^{n-2k}(C(t;U+V))$$

$$= \sum_{0\le i\le 2k} \wedge^{i}(S_1(t;U)A_1(U)) * \wedge^{n_1-i}(C_1(t;U)) * \wedge^{2k-i}(S_2(t;V)A(V)) * \wedge^{n_2-2k+i}(C_2(t;V)).$$

Proof. Let $S = S(t;U+V)$, $A = A(U+V)$, $C = C(t;U+V)$, $S_1 = S_1(t;U)$, etc. Then the last few lemmas and 7.5 (3) yield

$$\wedge^{2k}(SA) * \wedge^{n-2k}(C)$$

$$= \wedge^{2k}(S_1A_1 + S_2A_2) * \wedge^{n-2k}(C_1 + C_2)$$

$$= \sum_{\substack{0\le i\le 2k \\ 0\le j\le n-2k}} (\wedge^{i}(S_1A_1) * \wedge^{j}(C_1)) * (\wedge^{2k-i}(S_2A_2) * \wedge^{n-2k-j}(C_2)).$$

The linear maps S_1A_1 and C_1 take values in a vector space of dimension n_1. Therefore, if $i + j > n_1$, it follows that

$$\wedge^{i}(S_1A_1) * \wedge^{j}(C_1) = 0.$$

Likewise, if $(2k-i) + (n-2k-j) = n_1 + n_2 - (i+j) > n_2$, then

$$\wedge^{2k-i}(S_2A) * \wedge^{n-2k-j}(C_2) = 0.$$

Consequently, the only nonvanishing terms have $i + j = n_1$. Replacing j by $n_1 - i$ and summing on i yields the lemma.

<u>Lemma 11.9</u>. For $0 \leq i \leq 2k \leq n$ let

$$H_{2k,i}(t)$$

$$= \frac{1}{t}\int_{S^{\perp}\mathfrak{m}} \wedge^{i}(S_1(t;U)A_1(U)) * \wedge^{2k-i}(S_2(t;V)A_2(V)) * \wedge^{n_1-i}(C_1(t;U))$$

$$* \wedge^{n_2-2k+i}(C_2(t;V))\det(S_1^{\perp}(t;U))\det(S_2^{\perp}(t;V))\Omega_{S^{\perp}\mathfrak{m}}(U+V).$$

Then

$$h_{2k}(t) = \sum_{i=0}^{2k} H_{2k,i}(t).$$

<u>Proof</u>. This is lemma 11.8 substituted into the definition of $h_{2k}(t)$.

<u>Lemma 11.10</u>. If f is a continuous real valued function on $S^{\perp}\mathfrak{m}$, then

$$\int_{S^{\perp}\mathfrak{m}} f(U+V)\Omega_{S^{\perp}\mathfrak{m}}(U+V)$$

$$= \int_0^{\pi/2}\int_{S^{\perp}\mathfrak{m}_1}\int_{S^{\perp}\mathfrak{m}_2} f(\cos(\theta)U+\sin(\theta)V)\Omega_{S^{\perp}\mathfrak{m}_1}(U)\Omega_{S^{\perp}\mathfrak{m}_2}(V)\cos^{m_1-1}(\theta)\sin^{m_2-1}(\theta)d\theta.$$

<u>Proof</u>. Let $S_\alpha^{\perp} = S^{\perp}\mathfrak{m}_\alpha$ $\alpha = 1, 2$ and $S^{\perp} = S^{\perp}\mathfrak{m}$. Put the product metric on $[0,\pi/2] \times S_1^{\perp} \times S_2^{\perp}$ and define $\varphi : [0,\pi/2] \times S_1^{\perp} \times S_2^{\perp} \to S^{\perp}$ by

$$\varphi(\theta,u,v) = \cos\theta u + \sin\theta v.$$

We now compute the pullback of the volume from $\Omega_{S^{\perp}}$ to $[0,\pi/2] \times S_1^{\perp} \times S_2^{\perp}$.

Let $(\theta,u,v) \in [0,\pi/2] \times S_1^{\perp} \times S_2^{\perp}$. Let $u_1, \ldots, u_{m_1}$ be an orthonormal basis of $\mathfrak{m}_1^{\perp}$ with $u_1 = u$ and let $v_1, \ldots, v_{m_2}$ be an orthonormal basis of $\mathfrak{m}_2^{\perp}$ with $v_1 = v$. Then $\frac{\partial}{\partial\theta}$, $u_2, \ldots, u_{m_1}$, $v_2, \ldots, v_{m_2}$ is an orthonormal basis of the tangent space to

$[0, \pi/2] \times S_1^\perp \times S_2^\perp$ at (θ, u, v), and

$$\varphi_* \frac{\partial}{\partial\theta} = -\sin\theta\, u_1 + \cos\theta\, v_1,$$

$$\varphi_* u_i = \cos\theta\, u_i \qquad 2 \le i \le m,$$

$$\varphi_* v_j = \sin\theta\, v_j \qquad 2 \le j \le m_2.$$

Therefore,

$$\varphi_* \frac{\partial}{\partial\theta} \wedge \varphi_* u_2 \wedge \cdots \wedge \varphi_k u_{m_1} \wedge \varphi_* v_2 \wedge \cdots \wedge \varphi_* v_{m_2}$$

$$= \cos^{m_1-1}(\theta)\sin^{m_2-1}(\theta)(-\sin(\theta)u_1 + \cos(\theta)v_2)$$

$$\wedge u_2 \wedge \cdots \wedge u_{m_1} \wedge v_2 \wedge \cdots \wedge v_{m_2}.$$

The length of this vector is an element of $\wedge^{m_1+m_2-1}(\mathfrak{m}^\perp)$ is $\cos^{m_1-1}(\theta)\sin^{m_2-1}(\theta)$. This shows

$$\varphi^* \Omega_{S^\perp} = \Omega_{S_1^\perp} \wedge \Omega_{S_2^\perp} \wedge \cos^{m_1-1}(\theta)\sin^{m_2-1}(\theta)d\theta.$$

The function φ is surjective. It is also injective outside of a set of measure zero. Therefore

$$\int_{S^\perp} f\Omega_{S^\perp} = \int_{[0,\pi/2]\times S_1^\perp \times S_2^\perp} \varphi^*(f\Omega_{S^\perp}).$$

Using the form of $\varphi^*\Omega_{S^\perp}$ completes the proof.

<u>Lemma 11.12</u>. Let c and s be real numbers. Then

$$\wedge^i(S_1(t;cU)A_1(cU)) * \wedge^{2k-i}(S_2(t;sV)A_2(sV))$$

$$* \wedge^{n_1-i}(C_1(t;cU)) * \wedge^{n_2-2k+i}(C_2(t;sV))$$

$$\det(S_1^\perp(t;cU))\det(S_2^\perp(t;sV))$$

$$= \frac{1}{c^{m_1} s^{m_1}} \Lambda^1 (S_1(ct;U)A_1(U)) * \Lambda^{2k-i}(S_2(st;V)A_2(V))$$

$$* \Lambda^{n_1-1}(C_1(ct;U)) * \Lambda^{n_2-2k+i}(C_2(st;V))$$

$$\det(S_1^{\perp}(ct,U))\det(S_2^{\perp}(st;V)).$$

Proof. This is a consequence of corollary 10.7.

We can now finish the proof of the theorem. If $H_{2k,i}(t)$ is as in 11.9, then we use the last two lemmas to rewrite $H_{2k,i}(t)$ as

$$H_{2k,i}(t)$$

$$= \frac{1}{t} \int_0^{\pi/2} \int_{S^{\perp}\mathfrak{m}_1} \int_{S^{\perp}\mathfrak{m}_2} \Lambda^i (S_1(t\cos\theta;U)A_1(U)) * \Lambda^{2k-i}(S_2(t\sin\theta;V)A_2(V))$$

$$* \Lambda^{n_1-i}(C_1(t\cos\theta;U)) * \Lambda^{n_2-2k+i}(C_2(t\sin\theta;V)$$

$$\det(S_1^{\perp}(t\cos\theta;U))\det(S_2^{\perp}(t\sin\theta;V))\Omega_{S^{\perp}\mathfrak{m}_1}(U)$$

$$\Omega_{S^{\perp}\mathfrak{m}_2}(V) \frac{d\theta}{\cos(\theta)\sin\theta}$$

$$= t \int_0^{\pi/2} \Big(\frac{1}{t\cos\theta} \int_{S^{\perp}\mathfrak{m}_1} \Lambda^i (S_1(t\cos\theta;U)A_1(U)) * \Lambda^{n_1-i}(C_1(t\cos\theta;U))$$

$$\det(S_1^{\perp}(t\cos\theta;U))\Omega_{S^{\perp}\mathfrak{m}_1}(U)\Big)$$

$$\Big(\frac{1}{t\sin\theta} \int_{S^{\perp}\mathfrak{m}_2} \Lambda^{2k-i}(S_2(t\sin\theta;V)) * \Lambda^{n_2-2k+i}(C_2(t\cos\theta;V))$$

$$\det(S_2^{\perp}(t\sin\theta;V))\Omega_{S^{\perp}\mathfrak{m}_2}(V)\Big)d\theta.$$

If i is odd then the integrand for the integral over $S^{\perp}\mathfrak{m}_1$ is an odd function of U and thus reduces to zero. For $H_{2k,2i}(t)$, we use the definition of $(h_\alpha)_{2i}$ to see that

$$H_{2k,2i}(t) = t\int_0^{\pi/2} (h_1)_{2i}(t\cos\theta)(h_2)_{2(k-i)}(t\sin\theta)d\theta.$$

The theorem now follows from lemma 11.9.

12. Examples.

We first consider the case where $\tilde{M}$ is the complete simply connected manifold of constant curvature K of dimension $n + m$. Then every submanifold of $\tilde{M}$ is symmetrically embedded. Let M be a compact oriented submanifold of $\tilde{M}$ of dimension n. Using the notation of theorem 8.2, and the form of the curvature tensor for $\tilde{M}$ given in example (2) following the proof of proposition 6.2, we see that if $U \in S^{\perp}M_p$, $V \in T^{\perp}M_p$ and $X \in TM_p$, then

$$R_U(X) = KX$$

$$R_U^{\perp}(X) = KX - \langle U,V \rangle U.$$

Define two real valued functions c, s on $\mathbb{R}$ by

$$c''(t) + Kc(t) = 0 \qquad c(0) = 1, \quad c'(0) = 0,$$

$$s''(t) + Kc(t) = 0 \qquad s(0) = 0, \quad s'(0) = 1.$$

Using the initial value problems defining $C(t,U)$ and $S(t;U)$ we see

$$C(t;U) = c(t)(id)_{TM_p},$$

$$S(t;U) = s(t)(id)_{TM_p}.$$

We now compute $\det(S^{\perp}(t;U))$.

Note that $R_U^{\perp}(U) = 0$, and

$$S^{\perp}(t;U) = tU.$$

If $V \in T^{\perp}M_p$ and V is perpendicular to U, then $R_U^{\perp}(V) = KV$. Thus,

$$S^{\perp}(t;U)V = s(t)V,$$

and it follows that

$$\det(S^{\perp}(t;U)) = ts(t)^{m-1}.$$

Let A be the Weingarten map of M in $\widetilde{M}$, and let H be the excess tensor of M in $\widetilde{M}$. The integral formula of proposition 7.9 can now be used to compute the function $h_{2k}(p,t)$ of theorem 8.2.

$$h_{2k}(p,t) = \frac{1}{t}\int_{S^{\perp}M_p} \wedge^{2k}(S(t;U)A(U)) * \wedge^{n-2k}(C(t;U))\det(S^{\perp}(t;U))\Omega_{3^{\perp}M_p}(U)$$

$$= s(t)^{m+2k-1}c(t)^{n-2k}\int_{S^{\perp}M_p} \wedge^{2k}(A(U)) * \wedge^{n-2k}(I)\Omega_{S^{\perp}M_p}(U)$$

$$= s(t)^{m+2k-1}c(t)^{n-2k}\left(\int_{S^{\perp}M_p} \wedge^{2k}(A(U))\Omega_{S^{\perp}M_p}(U)\right) * \wedge^{n-2k}(I)$$

$$= \frac{s(t)^{m+2k-1}c(t)^{n-2k}\mathrm{vol}(S^{m-1})}{k!m(m+2)\cdots(m+2k-1)} H^{*k} * \wedge^{n-2k}(I).$$

If the curvature tensor of $\widetilde{M}$ at p is viewed as a linear map on $\wedge^2 T\widetilde{M}_p$, then it has the form

$$\widetilde{R} = -K\wedge^2(id_{T\widetilde{M}p}).$$

Let I be the identity map on TM_p and view the curvature tensor of M at p as a linear map on $\wedge^2 TM_p$. Then, by proposition 4.5 the excess tensor H of M at p is given by

$$H = R + K\wedge^2(I)$$

$$= R + \frac{K}{2} I^{*2}.$$

Here we have used $I^{*j} = j!\wedge^j(I)$. This is also used in the following calculation.

$$H^{*k} * \Lambda^{n-2k}(I) = \frac{1}{(n-2k)!} (R + \frac{K}{2} I^{*2})^{*k} * I^{*(n-2k)}$$

$$= \frac{1}{(n-2k)!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (\frac{K}{2})^{k-j} R^{*j} * I^{*n-j}$$

$$= \frac{k!}{(n-2k)!} \sum_{j=0}^{k} \frac{(n-j)!}{j!(k-j)!} (\frac{K}{2})^{k-j} R^{*j} * \Lambda^{n-j}(I)$$

$$= \frac{k!}{(n-2k)!} \sum_{j=0}^{k} \frac{(n-j)!}{j!(k-j)!} (\frac{K}{2})^{(k-j)} \operatorname{tr}(R^{*j}).$$

The last line of the above follows from proposition 7.8.

The following integral invariants of a Riemannian manifold were introduced by Hermann Weyl [13].

Definition 12.1. If M is a compact oriented Riemannian manifold with smooth boundary and R is the curvature tensor of M viewed as a linear map on $\Lambda^2 TM$ then for each k with

$$0 \leq 2k \leq \dim(M)$$

set

$$w_{2k}(M) = \int_M \operatorname{tr}(R^{*k}) \Omega_M .$$

Then the following (also due to Weyl) holds.

Proposition 12.2. If $\tilde{M}$ is the complete simply connected Riemannian manifold of dimension $n + m$ and M is a compact oriented submanifold of $\tilde{M}$ of dimension n then the volume of $M(r)$, the tube of radius r about M, is given by

$$\operatorname{vol}(M(r))$$

$$= s(r)^{m-1} \operatorname{vol}(S^{m-1}) \sum_{0 \leq 2k \leq n} \frac{s(r)^{2k} c(t)^{n-2k}}{(n-2k)! m(m+2) \cdots (m+2k-2)} \sum_{j=0}^{k} \frac{(n-j)!}{j!(k-j)!} (\frac{K}{2})^{(k-j)} w_{2j}(M)$$

where

$$c''(t) + Kc(t) = 0 \qquad c(0) = 1, \quad c'(0) = 0$$

$$s''(t) + Ks(t) = 0 \qquad s(0) = 0, \quad s'(0) = 1.$$

Proof. This follows from theorem 8.2 by using the above expression for $H^{*k} * \Lambda^{n-2k}(I)$ in the formula given for $h_{2k}(p,t)$.

We now turn to complex manifolds of constant holomorphic curvature. Let $\tilde{M}$ be a complex manifold of complex dimension $n + m$. Recall from example (3) following proposition 2.5 that each tangent space $T\tilde{M}_p$ to $\tilde{M}$ is a complex vector space. Let

$$J_p : T\tilde{M}_p \to T\tilde{M}_p$$

be the linear map on $T\tilde{M}_p$ induced by multiplication by $\sqrt{-1}$. It will be assumed that $\tilde{M}$ has a Riemannian metric $\langle\ ,\ \rangle$ such that

$$\langle JX,JY\rangle = \langle X,Y\rangle$$

for all X,Y tangent to $\tilde{M}$ at the same point. A Hermitian metric $(\ ,\ )$ is then given on each tangent space by

$$(X,Y) = \langle X,Y\rangle + \langle X,JY\rangle \sqrt{-1}.$$

The manifold $\tilde{M}$ is said to have constant holomorphic curvature K if its curvature tensor is given by

$$\tilde{R}(X,Y)Z = K((X,Y)Z - (Y,X)Z - (Y,X)Z + (Z,Y)X);$$

in this case,

$$\begin{aligned}\tilde{R}_U(X) &= \tilde{R}(X,U)U \\ &= K((U,U)X + (X,U)U - Z(U,X)U).\end{aligned}$$

Let $\tilde{M}$ be the complete simply connected space of constant holomorphic curvature K. Then $\tilde{M}$ is known to be a Riemannian symmetric space (See [8] volume II, example 10.5, page 273 and example 10.7, page 282). If K is positive then $\tilde{M}$ is complex projective space. Let M be a compact complex submanifold of $\tilde{M}$ with smooth boundary. Then, for each p in M, both TM_p and $T^{\perp}M_p$ are complex subspaces of $\tilde{TM}_p$. If $U \in S^{\perp}M_p$, then

$$\tilde{R}_U(X) = K((U,U)X + (X,U)U - 2(U,X)U)$$

$$= KX.$$

This shows that M is symmetrically embedded in $\tilde{M}$. It also shows that

$$R_U = K(id_{TM_p}).$$

So, if we again define functions $c(t)$, $s(t)$ by the differential equations

$$c''(t) + Kc(t) = 0 \qquad c(0) = 1, \quad c'(0) = 0,$$

$$s''(t) + K(s(t) = 0 \qquad s(0) = 0, \quad s'(0) = 1,$$

then

$$C(t;U) = c(t)(id_{TM_p}),$$

$$S(t;U) = s(t)(id_{TM_p}).$$

If $Y \in T^{\perp}M_p$ and $(Y,U) = 0$ then

$$R^{\perp}_U(Y) = KY.$$

Thus $S^{\perp}(t;U)Y = s(t)Y$.

Assume that M has complex dimension n. Then the set of $Y \in T^{\perp}M_p$ with $(Y,U) = 0$ has real dimension $2(m-1)$. As before, $R^{\perp}_U(U) = 0$ so $S^{\perp}(t;U) = tU$. Finally, note that

$$R^{\perp}_U(JU) = K((U,U)JU + (JU,U)U - 2(U,JU)U)$$

$$= 4KJU;$$

therefore,

$$S^{\perp}(t;U)JU = \frac{1}{2}\, s(2t)JU.$$

Combining these, we obtain

$$\det(S^{\perp}(t;U)) = \frac{t}{2}\, s(t)^{2(m-1)} s(2t).$$

We can now use the integral formula of proposition 7.9 to compute $h_{2k}(p)t)$.

Let A be the Weingarten map for M in $\tilde{M}$ and H the excess tensor of M in $\tilde{M}$. Then we have

$$h_{2k}(p,t) = \frac{1}{t}\int_{S^{\perp}M_p} \wedge^{2k}(S(t;U)A(U))*\wedge^{2(n-k)}(C(t;U))\det(S^{\perp}(t;U))\Omega_{S^{\perp}M_p}(U)$$

$$= \frac{S(2t)}{2}\, s(t)^{2(m+k-1)}c(t)^{2(n-k)} \int_{S^{\perp}M_p} \wedge^{2k}(A(U))\Omega_{S^{\perp}M_p} * \wedge^{2(n-k)}(I)$$

$$= \frac{s(2ts(t)^{2(m+k-1)}c(t)^{2(n-k)}}{2(k!)(2m)(2m+2)\cdots(2m+2k-2)}\, \mathrm{vol}(S^{2m-1})\; H^{*k} * \wedge^{2(n-k)}(I)$$

$$= \frac{s(2t)s(t)^{2(m+k-1)}c(t)^{2(n-k)}}{2^{k+1}(k!)\; m(m+1)\cdots(m+k-1)}\, \mathrm{vol}(S^{2m-1})\; \mathrm{tr}(H^{*k}).$$

This yields the following proposition due to R. Wolf ([14]) and F. J. Flaherty ([4]).

<u>Proposition 12.3</u>. With notation as above the volume of the tube $M(r)$ about M is

$$\mathrm{vol}(M(r)) = \frac{s(2t)s(t)^{2(m-1)}}{2}\, \mathrm{vol}(S^{2m-1}) \sum_{k=0}^{n} \frac{s((t)^{2(k-1)}c(t)^{2(n-k)}}{2^k m\,(m+1)\cdots(m+k-1)} \int_M \mathrm{tr}(H^{*k})\Omega_M.$$

Therefore $vol(\widetilde{M})$ only depends on the excess tensor of M in $\widetilde{M}$.

As a last example we do a hypersurface in a space of constant holomorphic curvature. To this end let $\widetilde{M}$ be the space of constant holomorphic curvature and complex dimension n. Suppose that M is a hypersurface of $\widetilde{M}$. Let $p \in M$ and $U \in S^{\perp}M_p$. Then the vector $J(U)$ is perpendicular to U and thus tangent to M at p. Define

P_1 = Orthogonal projection of TM_p onto orthogonal complement of JU in TM_p.

P_2 = Orthogonal projection of TM_p onto span of JU.

If $X \in TM_p$ and X is perpendicular to JU, then $(X,U) = 0$. Thus

$$R_U(X) = KX,$$

and so

$$C(t;U)X = c(t)X, \quad S(t;U)X = s(t)X.$$

As above,

$$R_U(JU) = 4KJU;$$

therefore

$$C(t;U)JU = c(2t)JU,$$
$$S(t;U)JU = \frac{1}{2}\, s(2t)JU.$$

These facts together yield

$$C(t;U) = c(t)P_1 + c(2t)P_2,$$

$$S(t;U) = s(t)P_1 + \frac{1}{2}\, s(2t)P_2.$$

Because P_2 has rank one it follows that $\wedge^j(P_2) = 0$ for $j \geq 2$.

Whence

$$\wedge^{2n-1-k}(C(t;U)) = c(t)^{2n-1-k}\wedge^{2n-1-k}(P_1)$$
$$+ c(t)^{2n-k-2}c(2t)P_2 * \wedge^{2n-k-2}(P_1).$$

We choose a smooth unit normal along M and let A be the corresponding Weingarten map. Then P_2A also has rank one; thus

$$\wedge^k(S(t;U)) = s(t)^k\wedge^k(P_1A)$$
$$+ \frac{1}{2}\, s(2t)s(t)^{k-1}(P_2A) * \wedge^{k-1}(P_1A).$$

But P_2 and P_2A both have the same one-dimensional range, and thus $P_2 * (P_2A) = 0$. Therefore, using the notation of proposition 9.1, we have

$$h_k^+(p,t) = \wedge^k(S(t;U)A) * \wedge^{2n-1-k}(C(t;U))$$
$$= s(t)^k c(t)^{2n-1-k}\wedge^k(P_1A) * \wedge^{2n-1-k}(P_1)$$
$$+ \frac{1}{2}\, s(2t)s(t)^{k-1}c(t)^{2n-1-k}(P_2A) * \wedge^{k-1}(P_1)$$
$$+ s(t)^k c(2t)\, c^{2n-2-k}(t)\wedge^k(P_1A) * P_2 * \wedge^{2n-2-k}(P_1).$$

Choose A so that JU is one of its eigenvectors with eigenvalue a_1 and let $a_2, \ldots, a_{2n-1}$ be the other eigenvalues of A. Then let $\sigma_k(a_2,\ldots,a_{2n-1})$ be the k-th element symmetric function in $a_2, \ldots, a_{2n-1}$. Then

$$\wedge^k(P_1A) * \wedge^{2k-1-k}(P_1) = \sigma_k(a_2,\ldots,a_{2n-1}).$$

However, it is not hard to show that, if $K \neq 0$, then $s(t)^k c(t)^{2n-1-k}$ is linearly independent of $s(2t)s(t)^{k-1}c(t)^{2n-1-k}$ and $s(t)^k c(2t)\, c(t)^{2n-2-k}$. Therefore, we can compute $\sigma_k(a_2,\ldots,a_{2n-1})$ from $h_k^+(p,t)$. But this is independent of a_1 so $h_k^+(p,t)$ is not a function of the k-th element symmetric function of A. The best that can be proved is that $h_k^+(p,t)$ is a linear function of $\wedge^k(A)$.

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