# The Kapustin-Witten Equations with Singular Boundary Conditions

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# ABSTRACT

Witten proposed a fasinating program interpreting the Jones polynomial of knots on a 3-manifold by counting solutions to the Kapustin-Witten equations with singular boundary conditions.

In Chapter 1, we establish a gluing construction for the Nahm pole solutions to the Kapustin-Witten equations over manifolds with boundaries and cylindrical ends. Given two Nahm pole solutions with some convergence assumptions on the cylindrical ends, we prove that there exists an obstruction class for gluing the two solutions together along the cylindrical end. In addition, we establish a local Kuranishi model for this gluing picture. As an application, we show that over any compact four-manifold with $S^3$ or $T^3$ boundary, there exists a Nahm pole solution to the obstruction perturbed Kapustin-Witten equations. This is also the case for a four-manifold with hyperbolic boundary under some topological assumptions.

In Chapter 2, we find a system of non-linear ODEs that gives rotationally invariant solutions to the Kapustin-Witten equations in 4-dimensional Euclidean space. We explicitly solve these ODEs in some special cases and find decaying rational solutions, which provide solutions to the Kapustin-Witten equations. The imaginary parts of the solutions are singular. By rescaling, we find some limit behavior for these singular solutions. In addition, for any integer $k$, we can construct a $5|k|$ dimensional family of $C^1$ solutions to the Kapustin-Witten equations on Euclidean space, again with singular imaginary parts. Moreover, we get solutions to the Kapustin-Witten equation with Nahm pole boundary condition over $S^3 \times (0, +\infty)$.

In Chapter 3, we develop a Kobayashi-Hitchin type correspondence for the extended Bogomolny equations on $\Sigma\times$ with Nahm pole singularity at $\Sigma \times \{0\}$ and the Hitchin component of the stable $SL(2, \mathbb{R})$ Higgs bundle; this verifies a conjecture of Gaiotto and Witten. We also develop a partial Kobayashi-Hitchin correspondence for solutions with a knot singularity in this program, corresponding to the non-Hitchin components in the moduli space of stable $SL(2, \mathbb{R})$ Higgs bundles. We also prove the existence and uniqueness of solutions with knot singularities on $\mathbb{C} \times \mathbb{R}^+$. This is joint a work with Rafe Mazzeo.

In Chapter 4, for a 3-manifold $Y$, we study the expansions of the Nahm pole solutions to the Kapustin-Witten equations over $Y \times (0, +\infty)$. Let $y$ be the coordinate of $(0, +\infty)$ and assume the solution convergence to a flat connection at $y \to \infty$, we prove the

sub-leading terms of the Nahm pole solution is $C^1$ to the boundary at $y \to 0$ if and only if $Y$ is an Einstein 3-manifold. For $Y$ non-Einstein, the sub-leading terms of the Nahm pole solutions behave as $y \log y$ to the boundary. This is a joint work with Victor Mikhaylov.

# PUBLISHED CONTENT AND CONTRIBUTIONS

[1] Siqi He, Rotationally Invariant Singular Solutions to the Kapustin-Witten Equations. To appear in Mathematical Research Letters. Available at https://arxiv.org/abs/1510.07706. This paper is entirely my own work.

[2] Siqi He, A Gluing Theorem for the Kapustin-Witten Equations with a Nahm Pole. Preprint. Available at https://arxiv.org/abs/1707.06182.
This paper is entirely my own work.

[3] Siqi He, Rafe Mazzeo, The Extended Bogomolny Equations and Generalized Nahm Pole Solutions. Preprint. Available at https://arxiv.org/abs/1710.10645.
Both authors contributed equally to this project.

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*Chapter 1*

# A GLUING THEOREM FOR THE KAPUSTIN-WITTEN EQUATIONS WITH A NAHM POLE

## 1.1 Introduction

In [63], Witten proposed a gauge theory approach to the Jones polynomial and Khovanov homology. Witten predicted that the coefficients of Jones polynomial should count certain solutions to the Kapustin-Witten equations over $\mathbb{R}^3 \times (0, +\infty)$ with singular boundary conditions on $\mathbb{R}^3 \times \{0\}$. See [25] for a physics approach of this program.

Given a smooth 4-manifold $X$ with boundary, let $P$ denote a principal $SU(2)$ bundle over $X$ and let $\mathfrak{g}_P$ be the adjoint bundle. Let $A$ be a connection over $P$ and $\Phi$ be a $\mathfrak{g}_P$ valued one-form. The Kapustin-Witten equations are:

$$\begin{aligned} F_A - \Phi \wedge \Phi + \star d_A \Phi &= 0, \\ d_A^{\star} \Phi &= 0. \end{aligned} \tag{1.1}$$

When the knot is empty, the singular boundary condition is called the Nahm pole boundary condition and in [45], Mazzeo and Witten proved that there exists a unique Nahm pole solution to (4.1) which corresponds to the Jones polynomial of the empty knot. For a general 4-manifold $X$ with 3-manifold boundary $Z$, we hope to find ways to count the number of solutions to the Kapustin-Witten equations with the Nahm pole boundary condition over the boundary. This might lead to the discovery of some new invariants.

Therefore, a basic question to ask is whether there exists a solution to (4.1) with the Nahm pole boundary condition over a general 4-manifold with boundary? In [27], the author constructed some explicit solutions to the Kapustin-Witten equations over $S^3 \times (0, +\infty)$. Kronheimer [37] constructed some explicit solutions to the Kapustin-Witten equations over $Y^3 \times (0, +\infty)$, where $Y^3$ is any hyperbolic closed 3-manifold.

Following Taubes [53] [54], in order to prove the existence of solutions, we hope to establish a gluing theory for the Kapustin-Witten equations, such that the known Nahm pole model solutions can be glued to general 4-manifolds with boundary to obtain new Nahm pole solutions.

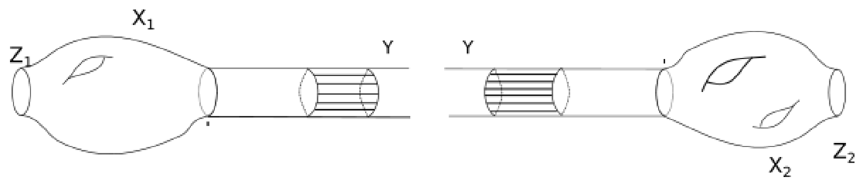


Figure 1.1: Gluing $X_1$, $X_2$ along the cylindrical ends

The main difference in gluing in the Nahm pole case compared to the gluing in the Yang-Mills case and the Seiberg-Witten case is that the Nahm pole boundary is not a classical non-degenerate elliptic boundary condition. However, it is a uniformly degenerate elliptic problem, as studied by R.Mazzeo [41]. We mainly need the analytic tools developed in [41] [45].

For $i = 1, 2$, let $X_i$ be 4-manifolds with boundaries $Z_i$ and infinite cylindrical ends identified with $Y_i \times (0, +\infty)$. Let $(A_i, \Phi_i)$ be solutions to the Kapustin-Witten equations (4.1) over $X_i$ with Nahm pole boundary conditions over $Z_i$ and convergence to flat $SL(2;\mathbb{C})$ connections $(A_{\rho_i}, \Phi_{\rho_i})$ over the cylindrical ends.

If $Y_1 = Y_2$, $(A_{\rho_1}, \Phi_{\rho_1}) = (A_{\rho_2}, \Phi_{\rho_2})$, we can define a new 4-manifold $X^\sharp$ and approximate solutions $(A^\sharp, \Phi^\sharp)$ by gluing together the cylindrical ends. See Figure 1.1, where the shaded parts are glued together.

We prove the following theorem:

**Theorem 1.1.1.** *Under the hypotheses above, if*

*(a)For some* $p_0 > 2$, $\lim_{T\to+\infty} \|(A_i, \Phi_i) - (A_\rho, \Phi_\rho)\|_{L_1^{p_0}(Y_i\times\{T\})} = 0$,

*(b)* $\rho$ *is an acyclic flat* $SL(2;\mathbb{C})$ *connection,*

*then for* $p \geq 2$ *and* $\lambda \in [1 - \frac{1}{p}, 1)$ *and sufficiently large* $T$*, we have:*

*(1) for some constant* $\delta$*, there exists a* $y^{\lambda+\frac{1}{p}} H_0^{1,p}$ *pair* $(a, b) \in \Omega^1_{X^{\sharp T}}(\mathfrak{g}_P) \times \Omega^1_{X^{\sharp T}}(\mathfrak{g}_P)$ *with*

$$\|(a, b)\|_{y^{\lambda+\frac{1}{p}-1} L_1^p} \leq Ce^{-\delta T},$$

*(2) there exists an obstruction class* $h \in H^2_{(A_1,\Phi_1)}(X_1) \times H^2_{(A_2,\Phi_2)}(X_2)$ *such that* $h = 0$ *if and only if* $(A^\sharp + a, \Phi^\sharp + b)$ *is a solution to the Kapustin-Witten equations (4.1).*

In the statement of the theorem, $\rho$ acyclic means that $\rho$ is a regular point in the representation variety, and $H^2_{(A_i,\Phi_i)}$ means the cokernel of the linearization operator of the Kapustin-Witten equations over the point $(A_i, \Phi_i)$. Further, $y^{\lambda+\frac{1}{p}}H_0^{1,p}$ and $y^{\lambda+\frac{1}{p}-1}L_1^p$ are weighted norms which will be precisely introduced in Section 5.

In addition, in Section 8, we also prove a gluing theorem when $\rho$ is reducible with a different weighted norm.

The statement and proof of Theorem 1.1 are analogous to the statement and proof of the gluing theorem for the ASD equation, due to C.Taubes [53], [54]; cf. also [19], [20], [24].

Moreover, for $p \in (2,4)$ and $\lambda \in [1-\frac{1}{p}, 1)$, denote by $\mathcal{M}_i$ the moduli space of solutions to the Kapustin-Witten equations satisfying the assumption (a), (b) in Theorem 1.1 modulo the gauge action. We have the following Kuranishi model for the gluing picture.

**Theorem 1.1.2.** *Let $(A_i, \Phi_i)$ be a connection pair over a manifold $X_i$ with a Nahm pole over $Z_i$. For sufficiently large $T$, there is a local Kuranishi model for an open set in the moduli space over $X^\sharp$:*

*(1) There exists a neighborhood $N$ of $\{0\} \subset H^1_{(A_1,\Phi_1)} \times H^1_{(A_2,\Phi_2)}$ and a map $\Psi$ from $N$ to $H^2_{(A_1,\Phi_1)} \times H^2_{(A_2,\Phi_2)}$.*

*(2) There exists a map $\Theta$ which a homeomorphism from $\Psi^{-1}(0)$ to an open set $V \subset \mathcal{M}_{X^\sharp}$.*

Here $H^k_{(A_i,\Phi_i)}$ is the $k$-th homology associated to the Kuranishi complex of $(A_i, \Phi_i)$ and $\mathcal{M}_{X^\sharp}$ is the moduli space of Nahm pole solutions to the Kapustin-Witten equations over $X^\sharp$. See also the Kuranishi model construction in Seiberg-Witten theory by T. Walpuski and D. Aleksander [17].

As for the model solutions, we don't know whether the obstruction class vanishes or not and right now we don't have any transversality results for the Kapustin-Witten equations. We just consider the obstruction class as a perturbation to the equation. See [18] for the obstruction perturbation for ASD equations. We obtain the following theorem:

**Theorem 1.1.3.** *Let $M$ be a smooth compact 4-manifold with boundary $Y$. Assume $Y$ is $S^3$, $T^3$ or any hyperbolic 3-manifold. When $Y$ is hyperbolic, we assume that the inclusion of $\pi_1(Y)$ into $\pi_1(M)$ is injective. For a real number $T_0$, we can glue $M$ to $Y \times (0, T_0]$ along $\partial M$ and $Y \times \{T_0\}$ to get a new manifold, which we denote as $M_{T_0}$.*

*For $T_0$ large enough, there exists a $SU(2)$ bundle $P$ and its adjoint bundle $\mathfrak{g}_P$ over $M_{T_0}$ such that given any interior non-empty open neighborhood $U \subset M$, we have:*

*(1) There exist $h_1 \in \Omega^2_{M_{T_0}}(\mathfrak{g}_P),\ h_2 \in \Omega^0_{M_{T_0}}(\mathfrak{g}_P)$ supported on $U$,*

*(2) There exist a connection $A$ over $P$ and a $\mathfrak{g}_P$-valued 1-form $\Phi$ such that $(A, \Phi)$ satisfies the Nahm pole boundary condition over $Y \times \{0\} \subset M_{T_0}$ and $(A, \Phi)$ is a solution to the following obstruction perturbed Kapustin-Witten equations over $M_{T_0}$:*

$$\begin{aligned} F_A - \Phi \wedge \Phi + \star d_A \Phi &= h_1, \\ d_A^{\star} \Phi &= h_2. \end{aligned} \tag{1.2}$$

Here is the outline of the paper. In Section 2, we introduce some preliminaries on the Kapustin-Witten equations, including the Kuranishi complex and some examples of the Nahm pole solutions. In Section 3, we introduce a gauge fixing condition and the elliptic system associated to the equations. In Section 4, we study the gradient flow of the Kapustin-Witten equations, and the structure of the linearization operator over $Y \times \mathbb{R}$. In Section 5, we establish the Fredholm theory for the linearization operator over manifolds with boundaries and cylindrical ends. In Section 6, we build up a slicing theorem and Kuranishi model for the Nahm pole solutions. In Section 7, after assuming the solution over cylindrical ends is simple and $L^p_1$ converges to a flat $SL(2;\mathbb{C})$ connection over the cylindrical end for $p > 2$, we prove that the solution will exponentially decay to the $SL(2;\mathbb{C})$ flat connection in the cylindrical ends. In Section 8, we describe the obstruction in the second homology group of the Kuranishi complex to the existence of solutions when gluing along the cylindrical ends. In Section 9, we build up a local Kuranishi model for the gluing picture. In Section 10, we apply the gluing theorem and get some existence results for the Nahm pole solutions to the perturbed equations. In Appendix 1, we introduce the $L^p$ version of Mazzeo's work for a uniformly degenerate elliptic operator. In Appendix 2, we introduce a proof of a Hardy type inequality for the weighted norm which is used to prove a slicing theorem.

### 1.2 Preliminaries of the Kapustin-Witten Equations and the Nahm Pole Boundary condition

In this section, we introduce some preliminaries on the Kapustin-Witten equations and the Nahm pole boundary condition.

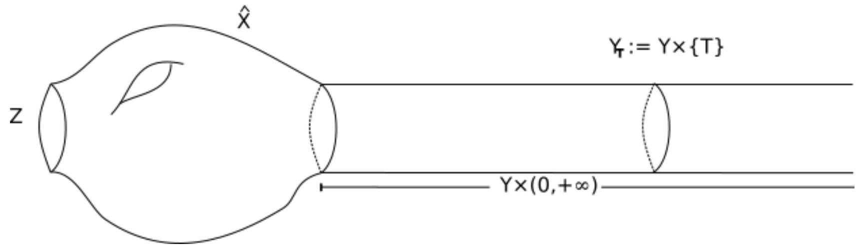


Figure 1.2: The shape of manifold we study

**Kapustin-Witten Map**

Let $\hat{X}$ be a smooth compact connected four-manifold with two connected boundary components $Y$ and $Z$. Take $X$ to be the four-manifold obtained by gluing $\hat{X}$ and $Y \times [0, +\infty)$ along the common boundary $Y$ , that is $X := \hat{X} \cup_Y (Y \times [0, +\infty))$. For any positive real number $T$, we denote by $Y_T$ the slice $Y \times \{T\} \subset X$ and $X(T) := \hat{X} \cup_Y (Y \times (0, T))$. For simplicity, the metric we always consider on $X$ is cylindrical along a neighborhood of $Z$ and is the product metric over $Y \times [T, +\infty)$ for some $T$ big enough. This is illustrated in Figure 1.2:

Now suppose $P$ is an $SU(2)$ bundle over $X$, $\mathfrak{g}_P$ is the associated adjoint bundle and $\mathcal{A}_P$ is the set of all the $SU(2)$ connections on $P$. We define the configuration space as follows: $C_P := \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$, $C_P^{'} := \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)$.

The gauge-equivariant Kapustin-Witten map is the map $KW : C_P \longrightarrow C_P^{'}$:

$$KW(A, \Phi) : = \begin{pmatrix} F_A - \Phi \wedge \Phi + \star d_A \Phi \\ d_A^{\star} \Phi \end{pmatrix}. \tag{1.3}$$

To be more explicit, denote by $\mathcal{G}_P$ the gauge group of $P$. Then, the action of $g \in \mathcal{G}_P$ on $(A, \Phi) \in \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ is given by

$$g(A, \Phi) = (A - (d_A g)g^{-1}, g\Phi g^{-1}).$$

Under this action, the Kapustin-Witten map is gauge equivariant, i.e.,

$$KW(A - (d_A g)g^{-1}, g\Phi g^{-1}) = gKW(A, \Phi)g^{-1}.$$

**Nahm Pole Boundary condition**

In [63], Witten proposed a gauge theoretic approach to Jones polynomial. A key objective of this program is to study the solutions to the Kapustin-Witten equations (4.1) satisfying the Nahm pole boundary condition.

To begin with, we introduce the Nahm pole boundary condition.

Given a 4-manifold $X$, with 3-dimensional boundary $Z$, a $SU(2)$ bundle $P$ over $X$ and the associated adjoint bundle $\mathfrak{g}_P$, for integers $a = 1, 2, 3$, take $\{\mathfrak{e}_a\}$ to be any unit orthogonal basis of $TZ$, the tangent bundle of $Z$, take $\{\mathfrak{e}_a^\star\}$ to be its dual and take $\{\mathfrak{t}_a\}$ to be section of the adjoint bundle $\mathfrak{g}_P$ with the relation $[\mathfrak{t}_a, \mathfrak{t}_b] = 2\epsilon_{abc}\mathfrak{t}_c$. Identify a neighborhood of $Z$ with $Z \times (0, 1)$, denote the boundary of $W$ by $\partial W$ and identify it with $Z \times \{0\}$. We denote by $y$ as the coordinate on $(0, 1)$.

**Definition 1.2.1.** *A connection pair $(A, \Phi) \in \mathcal{C}_P$ over $X$ satisfies the Nahm pole boundary condition if there exist $\{\mathfrak{e}_a\}$, $\{\mathfrak{t}_a\}$ as above such that the expansion of $(A, \Phi)$ in $y \to 0$ of $Z \times (0, 1)$ will be $A \sim A_0 + O(y)$ and $\Phi \sim \frac{\sum_{a=1}^{3} e_a^\star t_a}{y} + O(1)$. In addition, we call $(A, \Phi) \in \mathcal{C}_P$ a Nahm pole solution if $(A, \Phi)$ is a solution to the Kapustin-Witten equations (4.1).*

In fact, a Nahm pole solution to the Kapustin-Witten equation will have more restrictions on the expansion, as pointed out in [45].

**Proposition 1.2.2.** *[45]For a Nahm pole solution $(A, \Phi)$ to the Kapustin-Witten equation, we have*

*(1)* $\Phi_0 = 0$.

*(2) Using $\sum e_a^\star t_a$ to identify $\mathfrak{g}_P|_Y$ with $TY$, $A_0$ is the Levi-Civita connection of $Z$.*

**Examples of Nahm Pole Solutions**

Here are some examples of solutions to the Kapustin-Witten equations satisfying the Nahm pole boundary condition.

**Example 1.2.3.** *(Nahm [51])Nahm pole solutions on $T^3 \times \mathbb{R}^+$. Take the trivial $SU(2)$ bundle and denote $(A, \Phi) = (0, \frac{\sum \mathfrak{t}^i dx^i}{y})$. Then $F_A = 0$ and $\Phi \wedge \Phi = \frac{\sum [\mathfrak{t}^i, \mathfrak{t}^j] dx^i \wedge dx^j}{2y^2}$. In addition, $d_A \Phi = -\frac{\sum \mathfrak{t}^i dy \wedge dx^i}{y^2}$. Therefore, $(A, \Phi)$ is a Nahm pole solution to the Kapustin-Witten equations $F_A - \Phi \wedge \Phi + \star d_A \Phi = 0$.*

**Example 1.2.4.** *Nahm pole solutions on $S^3 \times \mathbb{R}^+$. Equip $S^3$ with the round metric and take $\omega$ be Maurer–Cartan 1-form of $S^3$ and we can write $\omega = g^{-1}dg$ for some suitable function $g : S^3 \to SU(2)$ with $\deg(g) = 1$. Then, let $y$ be the coordinate of*

$\mathbb{R}^+$, *and denote*

$$\begin{aligned}(A_1,\Phi_1)&=(\frac{6e^{2y}}{e^{4y}+4e^{2y}+1}\omega,\frac{6(e^{2y}+1)e^{2y}}{(e^{4y}+4e^{2y}+1)(e^{2y}-1)}\omega),\\(A_2,\Phi_2)&=(\frac{2(e^{4y}+e^{2y}+1)}{e^{4y}+4e^{2y}+1}\omega,\frac{6(e^{2y}+1)e^{2y}}{(e^{4y}+4e^{2y}+1)(e^{2y}-1)}\omega).\end{aligned}\tag{1.4}$$

*Theorem 6.2 in [27] shows that* $(A_1,\Phi_1)$ *and* $(A_2,\Phi_2)$ *are two Nahm-Pole solutions to the Kapustin-Witten equations. In addition, the solutions (4.56) will converge to the unique flat* $SL(2;\mathbb{C})$ *connection in the cylindrical end of* $S^3\times\mathbb{R}^+$.

**Example 1.2.5.** *(Kronheimer [37]) Nahm pole solutions on* $Y^3\times\mathbb{R}^+$*, where* $Y^3$ *is any hyperbolic three manifold.*

*Let* $Y^3$ *be a hyperbolic three manifold equipped with the hyperbolic metric* $h$*. Consider the associated* $PSL(2;\mathbb{C})$ *representation of* $\pi_1(Y)$*. By Culler's theorem [15], this lifts to* $SL(2;\mathbb{C})$ *and determines a flat* $SL(2;\mathbb{C})$ *connection* $\nabla^{flat}$*. Denote by* $\nabla^{lc}$ *the Levi-Civita connection and by* $A^{lc}$ *the connection form. Take* $i\omega:=\nabla^{flat}-\nabla^{lc}$*. Then locally,* $\omega=\sum\mathfrak{t}_i e_i^\star$ *where* $\{e_i^\star\}$ *is an orthogonal basis of* $T^\star Y$ *and* $\{\mathfrak{t}_a\}$ *are sections of the adjoint bundle* $\mathfrak{g}_P$ *with the relation* $[\mathfrak{t}_a,\mathfrak{t}_b]=2\epsilon_{abc}\mathfrak{t}_c$*. We also have* $\star_Y\omega=F_{\nabla^{lc}}$*. Therefore, by the Bianchi identity, we obtain* $\nabla^{lc}(\star_Y\omega)=0$.

*Combining* $F_{flat}=0$ *and the relation* $\nabla^{flat}-\nabla^{lc}=i\omega$*, we obtain* $F_{\nabla^{lc}+i\omega}=0$*. Hence* $F_{lc}=\omega\wedge\omega$*,* $\nabla^{lc}\omega=0$.

*Take* $y$ *to be the coordinate of* $\mathbb{R}^+$ *in* $Y^3\times\mathbb{R}^+$*, set*

$$f(y):=\frac{e^{2y}+1}{e^{2y}-1},$$

*and take*

$$(A,\Phi)=(A^{lc},f(y)\omega).\tag{1.5}$$

*Clearly,* $f(y)\to1$ *as* $y\to+\infty$ *and* $f(y)\sim\frac{1}{y}$ *as* $y\to0$.

*Let us check that the solution satisfies the Kapustin-Witten equations over* $Y^3\times(0,+\infty)$*. We compute*

$$\begin{aligned}&F_{A^{lc}}-\Phi\wedge\Phi=(1-f^2)F_{A^{lc}},\\&d_{A^{lc}}\Phi=d_{A^{lc}}\omega+f'(y)dy\wedge\omega=f'(y)dy\wedge\omega,\end{aligned}$$

*and*

$$d_{A^{lc}}\star_{Y\times(0,+\infty)}(f(y)\omega)=d_{A^{lc}}(f(y)(\star_Y\omega)\wedge dy)=f(y)d_{A^{lc}}(\star\omega)\wedge dy=f(y)(d_{A^{lc}}F_{A^{lc}}\wedge dy)=0.$$

*Combining this with the previous equations and using the relation* $1 - f^2 + f' = 0$*, we see that* $KW(A, \Phi) = 0$*.*

*Since* $f(y) \to 1$ *as* $y \to +\infty$*, (1.5) converges to the* $SL(2;\mathbb{C})$ *flat connection* $\rho$*.*

**Example 1.2.6.** *Nahm Pole solutions on the unit disc* $D^4$*.*

*This is an example from [27] of the Nahm pole solution to the Kapustin-Witten equations (4.1) over a compact manifold with boundary. Identify the quaternions* $\mathbb{H}$ *with* $\mathbb{R}^4$*,* $x = x_1 + x_2 I + x_3 J + x_4 K \in \mathbb{H}$ *and let* $D^4$ *be the unit disc of* $\mathbb{H}$*. Now define:*

$$(A, \Phi) = (Im(\frac{3}{|x|^4 + 4|x|^2 + 1}\bar{x}dx),\ Im(\frac{3(|x|^2+1)}{(|x|^4 + 4|x|^2 + 1)(|x|^2 - 1)}\bar{x}dx)).$$

*It is shown in [27] that this solution is a Nahm pole solution to the Kapustin-Witten equations over* $D^4$*.*

**Example 1.2.7.** *(S.Brown, H.Panagopoulos and M.Prasad [11])Two-sided Nahm pole solutions on* $(-\frac{\pi}{2}, \frac{\pi}{2}) \times T^3$*.*

*Consider the trivial* $SU(2)$ *bundle* $P$ *over* $(-\frac{\pi}{2}, \frac{\pi}{2}) \times T^3$ *and let* $\{\mathfrak{t}_a\}$ *to be elements in* $\mathfrak{g}_P$ *with the relation* $[\mathfrak{t}_a, \mathfrak{t}_b] = 2\epsilon_{abc}\mathfrak{t}_c$ *and* $dx_i$ *to be three orthogonal basis of cotangent bundle of* $T^3$*. Now define:*

$$(A, \Phi) = (0, \frac{1}{cos(y)}dx_1\mathfrak{t}_1 + \frac{1}{cos(y)}dx_2\mathfrak{t}_3 + \frac{sin(y)}{cos(y)}dx_3\mathfrak{t}_3). \tag{1.6}$$

*Then it is easy to check that* $KW(A, \Phi) = 0$*.*

**Remark.** *All these solutions over manifolds with cylindrical ends decay exponentially to flat* $SL(2;\mathbb{C})$ *connections. The* $\Phi$ *terms in these examples do not have a dy component on the cylindrical ends.*

### Kuranishi Complex

Now we will present the Kuranishi complex associated to the Kapustin-Witten equations (4.1). See also [40] some similar computations for the Vafa-Witten equations

Given a connection pair $(A, \Phi) \in \mathcal{C}_P$ satisfying (4.1), the complex associated to $(A, \Phi)$ is:

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d^0_{(A,\Phi)}} \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \xrightarrow{d^1_{A,\Phi}} \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) \to 0, \tag{1.7}$$

where $d^0_{(A,\Phi)}$ is the infinitesimal gauge transformation and $d^1_{(A,\Phi)}$ is the linearization of $KW$ at the pair $(A, \Phi)$.

To be more explicit, denote the Lie algebra of $\mathcal{G}_P$ by $\Omega^0(\mathfrak{g}_P)$. Then, the corresponding infinitesimal action of $\xi \in \Omega^0(\mathfrak{g}_P)$ will be:

$$\begin{aligned} &d^0_{(A,\Phi)}(\xi) : \Omega^0(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P), \\ &d^0_{(A,\Phi)}(\xi) = \begin{pmatrix} -d_A\xi \\ [\xi, \Phi] \end{pmatrix}. \end{aligned} \tag{1.8}$$

The linearization of the Kapustin-Witten equations at a point $(A, \Phi)$ is given by:

$$\begin{aligned} &d^1_{(A,\Phi)} : \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \to \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P), \\ &d^1_{(A,\Phi)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d_A a - [\Phi, b] + \star(d_A b + [\Phi, a]) \\ - \star [a, \star\Phi] + d_A^\star b \end{pmatrix}. \end{aligned} \tag{1.9}$$

The following result about this Kuranishi complex is classical:

**Proposition 1.2.8.** *The connection pair $(A, \Phi) \in C_P$ satisfies $KW(A, \Phi) = 0$ if and only if $\forall \xi \in \Omega^0(\mathfrak{g}_P)$, we have $d^1_{(A,\Phi)} \circ d^0_{(A,\Phi)}(\xi) = 0$.*

*Proof.* The $\Omega^2(\mathfrak{g}_P)$ component of the image of $d^1_{(A,\Phi)} \circ d^0_{(A,\Phi)}(\xi)$ equals:

$$\begin{aligned} &- d_A d_A \xi + [\Phi, [\Phi, \xi]] + \star(d_A[\xi, \Phi] + [\Phi, -d_A\xi]) \\ =& - [F_A, \xi] + [\Phi \wedge \Phi, \xi] - \star[d_A\Phi, \xi] \\ =& - [F_A - \Phi \wedge \Phi + \star d_A\Phi, \xi]. \end{aligned}$$

While the $\Omega^0(\mathfrak{g}_P)$ component is:

$$\begin{aligned} &d_A^\star[\xi, \Phi] - \star[-d_A\xi, \star\Phi] \\ =& - \star d_A[\xi, \star\Phi] + \star[d_A\xi, \star\Phi] \\ =& - [\xi, \star d_A \star \Phi] \\ =& [\xi, d_A^\star\Phi]. \end{aligned}$$

The statement follows immediately. □

Therefore, $d^0_{(A,\Phi)}$ and $d^1_{(A,\Phi)}$ in (1.7) will form a complex. We can define the homology groups:

$$H^0_{(A,\Phi)} = \text{Ker } d^0_{(A,\Phi)}, \; H^1_{(A,\Phi)} = \text{Ker } d^1_{(A,\Phi)}/\text{Im } d^0_{(A,\Phi)}, \text{ and } H^2_{(A,\Phi)} = \text{Coker } d^1_{(A,\Phi)}.$$

We denote the isotropy group of connection pair $(A, \Phi)$ as $\Gamma_{(A,\Phi)} = \{u \in \mathcal{G} | u(A, \Phi) = (A, \Phi)\}$. Recall that $H^0_{(A,\Phi)}$ is the Lie algebra of the stabilizer of $(A, \Phi)$ and $H^1_{(A,\Phi)}$ is the formal tangent space.

The formal dual Kuranishi complex with respect to the $L^2$ norm and Dirichlet boundary condition is:

$$0 \to \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) \xrightarrow{d^{1,\star}_{(A,\Phi)}} \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \xrightarrow{d^{0,\star}_{(A,\Phi)}} \Omega^0(\mathfrak{g}_P) \to 0, \tag{1.10}$$

where

$$\begin{aligned} &d^{1,\star}_{(A,\Phi)} : \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P), \\ &d^{1,\star}_{(A,\Phi)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} d^\star_A \alpha + \star[\Phi, \alpha] - [\Phi, \beta] \\ -\star d_A \alpha + \star[\Phi, \star\alpha] + d_A \beta \end{pmatrix}, \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} &d^{0,\star}_{(A,\Phi)} : \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \to \Omega^0(\mathfrak{g}_P), \\ &d^{0,\star}_{(A,\Phi)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -d^\star_A a + \star[\Phi, \star b] \end{pmatrix}. \end{aligned} \tag{1.12}$$

The Kapustin-Witten map also has the following structure:

**Proposition 1.2.9.** *The map KW has an exact quadratic expansion:*

$$KW(A + a, \Phi + b) = KW(A, \Phi) + d^1_{(A,\Phi)}(a, b) + \{(a, b), (a, b)\},$$

*where*

$$\{(a, b), (a, b)\} = \begin{pmatrix} a \wedge a - b \wedge b + \star[a, b] \\ -\star [a, \star b] \end{pmatrix}. \tag{1.13}$$

*Proof.* We have the following direct computation:

$$\begin{aligned} &F_{A+a} - (\Phi + b) \wedge (\Phi + b) + \star d_{A+a}(\Phi + b) \oplus d^\star_{A+a}(\Phi + b) \\ =&F_A + d_A a + a \wedge a - \Phi \wedge \Phi - [\Phi, b] - b \wedge b \\ &+ \star d_A \Phi + \star d_A b + \star[\Phi, a] + \star[a, b] \\ &\oplus d^\star_A \Phi + d^\star_A b - \star[a, \star\Phi] - \star[a, \star b] \\ =&KW(A, \Phi) + d^1_{(A,\Phi)}(a, b) + \{(a, b), (a, b)\}. \end{aligned}$$

□

## 1.3 Gauge Fixing, Elliptic System and Inner Regularity

Recall that we consider a smooth 4-manifold $X$ with boundary 3-manifold $Z$ and cylindrical ends identified with $Y \times (0, +\infty)$ and with an SU(2) bundle $P$ over X. In this section we will discuss the properties of solutions to the Kapustin-Witten equations (4.1) away from the boundary $Z$.

Suppose that $(A_0, \Phi_0)$ is a fixed reference connection pair in $C_P$ and write $(A, \Phi) = (A_0, \Phi_0) + (a, b)$. Our Sobolev norms used in this section are defined in the usual way: for example, for $a \in \Omega^1(\mathfrak{g}_P)$, we write

$$\|a\|_{L^p_k(X)} := (\sum_{j=0}^{k} \|\nabla^j_{A_0} a\|^p_{L^p(X)})^{\frac{1}{p}},$$

and for a pair $(a, b) \in \Omega^1(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P)$, we write

$$\|(a, b)\|_{L^p_k(X)} := (\|a\|^p_{L^p_k(X)} + \|b\|^p_{L^p_k(X)})^{\frac{1}{p}},$$

for any $1 \leq p \leq \infty$ and non-negative integer $k$.

### Gauge Fixing Condition

For gauge-invariant equations, in order to use elliptic PDE theory, we need to define a suitable gauge fixing condition.

Given a reference connection pair $(A_0, \Phi_0)$, in our situation, we can considered the traditional Coulomb gauge or another gauge differing by lower order terms which is associated by the operator $d^{0,\star}_{(A_0,\Phi_0)}$ in (1.10).

Denote

$$\mathcal{L}^{gf}_{(A_0,\Phi_0)} := d^{0,\star}_{(A_0,\Phi_0)} \tag{1.14}$$

and we have the following definition:

**Definition 1.3.1.** *Let $(A, \Phi) \in C_P$ and denote $(a, b) := (A, \Phi) - (A_0, \Phi_0)$. We say $(A, \Phi)$ is in the Coulomb gauge relative to $(A_0, \Phi_0)$ if $d^\star_{A_0} a = 0$. In addition, we say $(A, \Phi)$ is in the Kapustin-Witten gauge relative to $(A_0, \Phi_0)$ if $(a, b)$ satisfies $\mathcal{L}^{gf}_{(A_0,\Phi_0)}(a, b) = 0$ or*

$$d^\star_{A_0} a - \star[\Phi_0, \star b] = 0.$$

For the Coulomb gauge fixing, there are some known results in [62] for compact manifolds with boundary.

**Proposition 1.3.2.** *[62, Theorem 8.1] Suppose $U$ is a compact submanifold of $X$, $P$ is the $SU(2)$ bundle over $X$. Fixed a reference connection pair $A_0$, there exists a constant $C$ depending on $A_0$ such that if $(A, \Phi) \in \mathcal{C}_P$ and for $p > 2$,*

$$\|A - A_0\|_{L^p_1(U)} \leq C$$

*then there exists a gauge transformation $u \in \mathcal{G}_P$ such that*

$$\begin{aligned} d^{\star}_{A_0}(u(A) - A_0) &= 0, \\ \star\,(u(A) - A_0)|_{\partial U} &= 0. \end{aligned} \tag{1.15}$$

We also prove a simple result on the existence of the Kapustin-Witten gauge representatives, working over a closed base manifold.

**Proposition 1.3.3.** *Let $M$ be a closed 3 or 4-dimensional manifold and let $P$ be an SU(2) bundle over $M$, fix $(A_0, \Phi_0) \in \mathcal{C}_P := \mathcal{A}_P \times \mathfrak{g}_P$. There exists a constant $c(A_0, \Phi_0)$ such that if $(A, \Phi) \in \mathcal{C}_P$ and for some $p > 2$,*

$$\|(A - A_0, \Phi - \Phi_0)\|_{L^p_1(M)} \leq c(A_0, \Phi_0),$$

*then there is a gauge transformation $u \in \mathcal{G}_P$ such that $u(A, \Phi)$ is in the Kapustin-Witten gauge relative to $(A_0, \Phi_0)$.*

*Proof.* Denote $a := A - A_0$ and $b := \Phi - \Phi_0$, so by definition, for $u \in \mathcal{G}_P$, the gauge group action on $(A, \Phi)$ will be:

$$u(A_0 + a) - A_0 = uau^{-1} - (d_{A_0}u)u^{-1},$$

$$u(\Phi_0 + b) - \Phi_0 = u\Phi_0 u^{-1} + ubu^{-1} - \Phi_0.$$

The equation to be solved for $u \in \mathcal{G}_P$, is

$$d^{\star}_{A_0}(uau^{-1} - (d_{A_0}u)u^{-1}) - \star[\Phi_0, \star(u\Phi_0 u^{-1} - \Phi_0)] - \star[\Phi_0, \star ubu^{-1}] = 0. \tag{1.16}$$

We write $u = exp(\chi) = e^{\chi}$ for a section $\chi \in \mathfrak{g}_P$, and define

$$G(\chi, a, b) := d^{\star}_{A_0}(e^{\chi}ae^{-\chi} - (d_{A_0}e^{\chi})e^{-\chi}) - \star[\Phi_0, \star(e^{\chi}\Phi_0 e^{-\chi} - \Phi_0)] - \star[\Phi_0, \star e^{\chi}be^{-\chi}].$$

To solve the equation $G(\chi, a, b) = 0$, we use the implicit function theorem. We extend the domain of $G$ to sections $\chi \in L^p_2(M)$ and bundle valued 1-forms $a, b \in$

$L_1^p(M)$. Since for $p > 2$, $L_2^p(M)$ sections are continuous in 3-dimensions and 4-dimensions, we get $G(\chi, a, b)$ in $L^p(M)$. The derivative of $G$ at $\chi = 0$, $a = 0$, $b = 0$ is

$$\begin{aligned} & DG(\xi, \alpha, \beta) \\ =& d_{A_0}^{\star}\alpha - d_{A_0}^{\star} d_{A_0}\xi - \star[\Phi_0, \star[\xi, \Phi_0]] - \star[\Phi_0, \star\beta] \\ =& - d_{A_0}^{\star} d_{A_0}\xi - \star[\Phi_0, \star[\xi, \Phi_0]] + d_{A_0}^{\star}\alpha - \star[\Phi_0, \star\beta]. \end{aligned}$$

Denote $H(\xi) := -d_{A_0}^{\star} d_{A_0}\xi - \star[\Phi_0, \star[\xi, \Phi_0]]$ and $I(\alpha, \beta) := d_{A_0}^{\star}\alpha - \star[\Phi_0, \star\beta]$, then

$$DG(\xi, \alpha, \beta) = H(\xi) + I(\alpha, \beta).$$

Denote $\mathbb{A}_0 = A_0 + i\Phi_0$ and define

$$d_{\mathbb{A}_0}^{\star}(\alpha + i\beta) := d_{A_0}^{\star}\alpha - \star[\Phi_0, \star\beta] + i(d_{A_0}^{\star}\beta + \star[\Phi, \star\alpha]).$$

Obviously,

$$I(\alpha, \beta) = Re(d_{\mathbb{A}_0}^{\star}(\alpha + i\beta)).$$

If we show that the operator $H$ is surjective to the image of $I$, the implicit function theorem will give a small solution $\chi$ to the equation $G(\chi, a, b) = 0$. Thus we will study the cokernel of the operator $H$.

If $\eta \in \text{Coker } H$, we have

$$\langle H(\xi), \eta \rangle = 0 \text{ for all } \xi,$$

by taking $\xi = \eta$, we obtain

$$\langle H(\eta), \eta \rangle = -\|d_{A_0}\eta\|_{L^2(M)} - \|[\eta, \Phi_0]\|_{L^2(M)}.$$

Therefore, any element $\eta$ in the cokernel of $H$ satisfies $\|d_{A_0}\eta\| = 0$ and $\|[\eta, \Phi_0]\| = 0$, which implies

$$d_{\mathbb{A}_0}(\eta) = 0.$$

If for some $\alpha_0, \beta_0$, $I(\alpha_0, \beta_0)$ is not in the image of $H$, we have $d_{\mathbb{A}_0} I(\alpha_0, \beta_0) = 0$ and we know that:

$$\begin{aligned} 0 =& \langle d_{\mathbb{A}_0} Re(d_{\mathbb{A}_0}^{\star}(\alpha_0 + i\beta_0)), \alpha_0 + i\beta_0 \rangle \\ =& \langle Re(d_{\mathbb{A}_0}^{\star}(\alpha_0 + i\beta_0)), d_{\mathbb{A}_0}^{\star}(\alpha_0 + i\beta_0) \rangle \\ =& \langle Re(d_{\mathbb{A}_0}^{\star}(\alpha_0 + i\beta_0)), Re(d_{\mathbb{A}_0}^{\star}(\alpha_0 + i\beta_0)) \rangle. \end{aligned}$$

By taking the real part of the inner product, we get $I(\alpha_0, \beta_0) = 0$.

Therefore, $H$ is surjective to the image of $I$ and by implicit function theorem, we prove the result. $\square$

**Elliptic System**

Now, we will use the gauge fixing condition to get an elliptic system associated with the Kapustin-Witten equations. This is also considered similarly for the Vafa-Witten equations in [40]. We denote $\mathcal{L}_{(A_0,\Phi_0)} := d^1_{(A_0,\Phi_0)}$. Here the $d^1_{(A_0,\Phi_0)}$ is the linearization of the Kapustin-Witten map in (1.7).

Given $(A_0, \Phi_0) \in C_P$, by Proposition 1.2.9, for $(a, b) \in \Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P)$, the equation $KW(A_0 + a, \Phi_0 + b) = \psi_0$ is equivalent to

$$\mathcal{L}_{(A_0,\Phi_0)}(a, b) + \{(a, b), (a, b)\} = \psi_0 - KW(A_0, \Phi_0).$$

To make the equation elliptic in the interior, it is natural to add the gauge fixing condition $\mathcal{L}^{gf}_{(A_0,\Phi_0)}(a, b) = 0$ or $d^\star_{A_0} a = 0$.

By adding the Kapustin-Witten gauge, we define the Kapustin-Witten operator $\mathcal{D}_{(A_0,\Phi_0)}$:

$$\begin{aligned} &\mathcal{D}_{(A_0,\Phi_0)} : \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \to \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) \\ &\mathcal{D}_{(A_0,\Phi_0)} := \mathcal{L}_{(A_0,\Phi_0)} + \mathcal{L}^{gf}_{(A_0,\Phi_0)}. \end{aligned} \tag{1.17}$$

Denote $\psi = \psi_0 - KW(A_0, \Phi_0)$, then the elliptic system can be rewritten as:

$$\mathcal{D}_{(A_0,\Phi_0)}(a, b) + \{(a, b), (a, b)\} = \psi. \tag{1.18}$$

Similarly, for the Coulomb gauge, we can denote $\hat{\mathcal{D}}_{(A_0,\Phi_0)} := \mathcal{L}_{(A_0,\Phi_0)} + d^\star_{A_0}$, then we get another elliptic system:

$$\hat{\mathcal{D}}_{(A_0,\Phi_0)}(a, b) + \{(a, b), (a, b)\} = \psi. \tag{1.19}$$

**Interior Regularity of the Elliptic System**

Local interior estimates for the elliptic system (1.18) are considered in [22] in the context of $PU(2)$ monopoles.

**Theorem 1.3.4.** *[22] Take a bounded open set $\Omega$ in the interior part of $X$ and let $P$ to be a principal $SU(2)$ bundle over $X$. Suppose that $P|_\Omega$ is trivial and $\Gamma$ is a smooth flat connection. Suppose that $(a, b)$ is an $L^2_1(\Omega)$ solution to the elliptic system (1.18) over $\Omega$ and take the back ground pair $(A_0, \Phi_0) = (\Gamma, 0)$, where $\psi$ is in $L^2_k(\Omega)$ for $k \geq 1$ an integer. There exist an constant $\epsilon = \epsilon(\Omega)$ such that for any precompact open subset $\Omega' \subseteq \Omega$, if $\|(a, b)\|_{L^4(\Omega)} \leq \epsilon$ we have $(a, b) \in L^2_{k+1}(\Omega')$ and there is a* *universal polynomial* $Q_k(x, y)$*, with positive real coefficients, depending at most on* $k, \Omega, \Omega'$ *with* $Q_k(0,0) = 0$ *and*

$$\|(a,b)\|_{L^2_{k+1}(\Omega')} \leq Q_k(\|\psi\|_{L^2_k(\Omega)}, \|(a,b)\|_{L^2(\Omega)}).$$

*In addition, if* $(\psi, \tau)$ *is in* $C^\infty(\Omega)$ *then* $(a, b)$ *is in* $C^\infty(\Omega')$ *and if* $(\psi, \tau) = 0$*, then*

$$\|(a,b)\|_{L^2_{k+1}(\Omega')} \leq C\|(a,b)\|_{L^2(\Omega)}.$$

By the previous theorem, we can get interior regularity for the solutions with Nahm pole boundary conditions:

**Corollary 1.3.5.** *If* $(A, \Phi)$ *is a solution to the Kapustin-Witten equations (4.1) over* $X$*, for* $\Omega \subset$ *a bounded open set, if* $\|(A,\Phi)\|_{L^p_1(\Omega)}$ *is bounded, then for any proper open subset* $\Omega' \subset \Omega$*,* $(A, \Phi)$ *is smooth over* $\Omega'$*.*

*Proof.* Applying Proposition 1.3.2 and Theorem 1.3.4, the corollary comes out immediately.

□

## 1.4 Gradient Flow

In this section, we will discuss the gradient flow associated to Kapustin-Witten equations over a cylinder $X := Y \times \mathbb{R}$. See Taubes [56] for a general computation of the topological twitsted equations. Denote the coordinate in $\mathbb{R}$ as $y$, then we use the product metric on $X$ and the volume form we specify is $\mathrm{Vol}_Y \wedge dy$, where $\mathrm{Vol}_Y$ is a volume form over $Y$. In this section, we denote by $\star_4$ the 4-dimensional Hodge star operator of $\mathrm{Vol}_Y \wedge dy$ and denote by $\star$ the 3 dimensional Hodge star operator with respect to $\mathrm{Vol}_Y$.

### Generalized Gradient Flow Equations

To begin, suppose $P$ is an $SU(2)$ bundle over $X$ and $A$ is a given connection on $P$. Using parallel transport along the slice of $Y$ into $Y \times \mathbb{R}$, we can consider $A$ as a map from $\mathbb{R}$ to connections on $P$. Similarly, the field $\Phi$ can be written as $\Phi = \phi + \phi_y dy$. Here $\phi$ is a map from $\mathbb{R}$ to $\Omega^1(\mathfrak{g}_P)$ and $\phi_y$ is a map from $\mathbb{R}$ to the section of the adjoint bundle $\mathfrak{g}_P$.

If $(A, \Phi) = (A, \phi + \phi_y dy)$ obeys the Kapustin-Witten equations (4.1), then we compute

$$\begin{aligned} F_A - \Phi \wedge \Phi &= -\frac{d}{dy} A dy + F_A + [\phi_y, \phi] dy - \phi \wedge \phi, \\ \star_4 d_A \Phi &= \star d_A \phi_y - \star \frac{d}{dy} \phi + \star d_A \phi \wedge dy, \\ \star_4 d_A \star_4 \Phi &= \frac{d}{dy} \phi_y + \star d_A \star \phi = 0. \end{aligned} \tag{1.20}$$

Thus the Kapustin-Witten equations (4.1) are reduced to the following flow equations:

$$\begin{aligned} &\frac{d}{dy} A - \star d_A \phi - [\phi_y, \phi] = 0, \\ &\frac{d}{dy} \phi - d_A \phi_y - \star (F_A - \phi \wedge \phi) = 0, \\ &\frac{d}{dy} \phi_y - d_A^\star \phi = 0. \end{aligned} \tag{1.21}$$

These gradient flow equations are closely related to the complex Chern-Simons functional. Denote $\mathbb{A} := A + i\phi$, then the complex Chern-Simons functional $\mathrm{CS}^{\mathbb{C}}(\mathbb{A})$ on 3-manifold $Y^3$ is:

$$\mathrm{CS}^{\mathbb{C}}(\mathbb{A}) := \int Tr(\mathbb{A} \wedge d\mathbb{A} + \frac{2}{3} \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}). \tag{1.22}$$

Now, we define the following functional for the flow equations (1.21)

**Definition 1.4.1.** *Use the notation above, the extended Chern-Simons functional* ECS *is denoted as follows:*

$$\mathrm{ECS}(A, \phi, \phi_y) = \frac{1}{2} \mathrm{Im}(\mathrm{CS}^{\mathbb{C}}(\mathbb{A})) + \int_Y Tr(\phi \wedge \star d_A \phi_y), \tag{1.23}$$

*where the* Im *is taking the imaginary part of the complex Chern-Simons functional.*

Then we have the following proposition:

**Proposition 1.4.2.** *Equation (1.21) is the gradient flow for the extended Chern-Simons functional.*

*Proof.* Take $A = A_0 + a$, $\phi = \phi_0 + b$, $\phi_y = (\phi_y)_0 + c$, then the linearization of $\frac{1}{2}\mathrm{Im}(\mathrm{CS}^{\mathbb{C}}(\mathbb{A}))$ is:

$$\int b \wedge (F_{A_0} - \phi_0 \wedge \phi_0) + \int a \wedge d_A\phi.$$

In addition, the linearization of $\int_Y Tr(\phi \wedge \star d_A \phi_y)$ is:

$$\int_Y Tr(b \wedge \star d_{A_0}(\phi_y)_0) + \int_Y Tr(a \wedge \star[(\phi_y)_0, \phi_0]) + \int_Y Tr(c \wedge \star d^{\star}_{A_0}\phi_0).$$

Therefore, the gradient of ECS at $(A_0, \phi_0, (\phi_y)_0)$ is:

$$\nabla\mathrm{ECS}(A_0, \phi_0, (\phi_y)_0) = (-\star d_{A_0}\phi_0 - [(\phi_y)_0, \phi_0],\ -d_{A_0}(\phi_y)_0 - \star(F_{A_0} - \phi_0 \wedge \phi_0),\ -d^{\star}_{A_0}\phi_0), \tag{1.24}$$

where the minus sign is coming from the inner product we take for $s, s' \in \Omega^0(\mathfrak{g}_P)$ is $-\mathrm{Tr}(ss')$.

The result follows immediately. □

In addition, we can compute the Hessian operator $\mathcal{H}_{(A,\phi,\phi_y)}$ of $-$ECS at point $(A, \phi, \phi_y)$:

$$\begin{aligned} &\mathcal{H}_{(A,\phi,\phi_y)} : \Omega^1_Y(\mathfrak{g}_P) \times \Omega^1_Y(\mathfrak{g}_P) \times \Omega^0_Y(\mathfrak{g}_P) \to \Omega^1_Y(\mathfrak{g}_P) \times \Omega^1_Y(\mathfrak{g}_P) \times \Omega^0_Y(\mathfrak{g}_P), \\ &\mathcal{H}_{(A,\phi,\phi_y)} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \star d_A b + \star[\phi, a] + [\phi_y, b] - [\phi, c] \\ \star d_A a - \star[\phi, b] + d_A c - [\phi_y, a] \\ d^{\star}_A b - \star[a, \star\phi] \end{pmatrix}. \end{aligned} \tag{1.25}$$

Then we have the following expansion of the extended Chern-Simons functional $ECS$:

**Proposition 1.4.3.** *For $(a, b, c) \in \Omega^1_Y(\mathfrak{g}_P) \times \Omega^1_Y(\mathfrak{g}_P) \times \Omega^0_Y(\mathfrak{g}_P)$, we have the following expansions: (1) For the* ECS*, we have*

$$\begin{aligned} &\mathrm{ECS}(A + a, \phi + b, \phi_y + c) - \mathrm{ECS}(a, b, c) \\ = &\int_Y (\langle (a, b, c), \nabla\mathrm{ECS}(A, \phi, \phi_y) \rangle - \frac{1}{2}\langle (a, b, c), \mathcal{H}_{(A,\phi,\phi_y)}(a, b, c) \rangle - \{a, b, c\}^3), \end{aligned} \tag{1.26}$$

*where $\nabla\mathrm{ECS}(A, \phi, \phi_y)$ is the gradient of* ECS *defined in (1.24), $\mathcal{H}_{(A,\phi,\phi_y)}$ is the Hessian operator (1.25) and $\{a, b, c\}^3$ are cubic terms of $a, b, c$.*

*To be explicit, if we denote* $\mathbb{B} = a + ib$, *then the cubic terms are*

$$\{a, b, c\}^3 = \frac{1}{3}\mathrm{Im}(\mathbb{B} \wedge \mathbb{B} \wedge \mathbb{B}) + b \wedge \star[a, c]. \tag{1.27}$$

*(2) For* $\nabla\mathrm{ECS}$, *we have*

$$\begin{aligned} &\nabla\mathrm{ECS}(A + a, \phi + b, \phi_y + c) - \nabla\mathrm{ECS}(A, \phi, \phi_y) \\ =& -\mathcal{H}_{(A,\phi,\phi_y)}(a, b, c) - \{a, b, c\}^2, \end{aligned} \tag{1.28}$$

*where*

$$\{a, b, c\}^2 = \begin{pmatrix} \star[a, b] + [c, b] \\ [a, c] + \star(a \wedge a - \star b \wedge b) \\ -\star[a, \star b] \end{pmatrix}. \tag{1.29}$$

*Proof.* By a direct computation, we can verify these results. □

As we have the gauge action, we can formally defined the extended Hession operator for ECS:

The extended Hession operator $\mathcal{EH}$ at the point $(A, \phi, \phi_y)$ is defined as:

$$\mathcal{EH}_{(A,\phi,\phi_y)} : \Omega^1_Y(\mathfrak{g}_P) \times \Omega^1_Y(\mathfrak{g}_P) \times \Omega^0_Y(\mathfrak{g}_P) \times \Omega^0_Y(\mathfrak{g}_P) \to \Omega^1_Y(\mathfrak{g}_P) \times \Omega^1_Y(\mathfrak{g}_P) \times \Omega^0_Y(\mathfrak{g}_P) \times \Omega^0_Y(\mathfrak{g}_P),$$

$$\mathcal{EH}_{(A,\phi,\phi_y)} \begin{pmatrix} a_1 \\ b_1 \\ a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \star d_A b_1 + \star[\phi, a_1] + d_A a_0 - [\phi, b_0] + [\phi_y, b_1] \\ \star d_A a_1 - \star[\phi, b_1] + d_A b_0 + [\phi, a_0] - [\phi_y, a_1] \\ d_A^\star a_1 + \star[b_1, \star\phi] + [\phi_y, b_0] \\ d_A^\star b_1 - \star[a_1, \star\phi] + [\phi_y, a_0] \end{pmatrix}. \tag{1.30}$$

We have the following proposition of these two Hessian operators:

**Proposition 1.4.4.** *(1)* $\mathcal{EH}_{(A,\phi,\phi_y)}(a_1, b_1, 0, b_0) = \mathcal{H}_{(A,\phi,\phi_y)}(a_1, b_1, b_0)$.

*(2)* $\mathcal{EH}$ *and* $\mathcal{H}$ *are self-adjoint operators.*

*Proof.* By a direct computation, we can verify these results. □

In some case of 4-manifold with boundary and cylinderical ends, we can have some simplification of the flow equations (1.21). Let $X$ to be a 4-manifold with boundary $Z$ and cylindrical end which identified with $Y \times (0, +\infty)$ and we denote $y$ to be the coordinate of $(0, +\infty)$.

**Definition 1.4.5.** *Let* $(A, \Phi)$ *to be a solution to the Kapustin-Witten equations (4.1) over* $X$. *Over the cylindrical end* $Y \times (0, +\infty)$, *let* $\phi_y$ *be the* $dy$ *component of* $\phi$. *The solution* $(A, \Phi)$ *is called* ***simple*** *if there exists* $T_0$ *such that the restriction of* $\phi_y$ *over* $Y \times (T_0, +\infty)$ *is zero.*

Here is an identity due to Taubes:

**Lemma 1.4.6.** *[56, Page 36] If* $(A, \phi, \phi_y)$ *satisfies (1.21), we have the following identity:*

$$\frac{1}{2}(-\frac{\partial^2}{\partial y^2}|\phi_y|^2 + d^\star d|\phi_y|^2) + |\frac{\partial}{\partial y}\phi_y| + |d_A\phi_y| + 2|[\phi_y, \phi]|^2 = 0.$$

With this identity, we have an immediate corollary by the maximum principle:

**Corollary 1.4.7.** *Let* $(A, \Phi = \phi + \phi_y dy)$ *be a solution to the Kapustin-Witten equations or equivalently the flow equations (1.21) over* $Y \times I$ *where* $I \subset \mathbb{R}$. *We have the following:*

*(1) Over* $Y \times \mathbb{R}$, *if* $sup_Y|\phi_y|$ *has limit zero in the non-compact directions of* $\mathbb{R}$, *then* $\phi_y = 0$ *over* $Y \times \mathbb{R}$.

*(2) Over* $Y \times (0, +\infty)$, *let* $y$ *be the coordinate of* $(0, +\infty)$. *If* $(A, \Phi)$ *satisfies the Nahm pole boundary condition over* $Y \times \{0\} \subset Y \times (0, +\infty)$ *and converges under* $C^0$ *norm to a flat* $SL(2; \mathbb{C})$ *connection when* $y \to +\infty$, *we have* $\phi_y = 0$.

*Proof.* (1) is an immediately corollary of the previous lemma and maximal principle.

For any $(A, \Phi = \phi + \phi_y dy)$ in the assumption of (2), by the definition of Nahm pole boundary condition and flat $SL(2; \mathbb{C})$ connection, we know

$$\lim_{y\to 0} \sup |\phi_y|_{Y^3\times\{y\}} = 0, \ \lim_{y\to +\infty} \sup |\phi_y|_{Y^3\times\{y\}} = 0.$$

(2) also follows from an application of maximal principal. □

Therefore, we have the following simplification of the gradient flow equation:

**Corollary 1.4.8.** *When* $\phi_y = 0$, *(1.21) reduces to simple gradient flow equations:*

$$\begin{aligned} &\frac{d}{dy}A - \star d_A\phi = 0, \\ &\frac{d}{dy}\phi - \star(F_A - \phi \wedge \phi) = 0, \\ &d_A^\star \phi = 0. \end{aligned} \tag{1.31}$$

*This will be the gradient flow to the functional $Im(CS^{\mathbb{C}}(\mathbb{A}))$ along with the stability condition $d_A^{\star}\phi = 0$.*

**Proposition 1.4.9.** *If $(A, \phi)$ satisfies the first two equations of (1.31):*

$$\frac{d}{dy}A - \star d_A\phi = 0,$$
$$\frac{d}{dy}\phi - \star(F_A - \phi \wedge \phi) = 0,$$

*then $\frac{d}{dy}(d_A \star \phi) = 0$.*

*Proof.* We compute:

$$\begin{aligned}\frac{d}{dy}(d_A \star \phi) &= d \star \frac{d}{dy}\phi + [\frac{d}{dy}A, \star\phi] - [\star\frac{d}{dy}\phi, A]\\ &= d_A(F_A - \phi \wedge \phi) + \star d_A\phi \wedge \star\phi - \star\phi \wedge \star d_A\phi\\ &= 0.\end{aligned}$$

□

**Acyclic Connection of the Characteristic Variety**

Now, consider the behavior of the complex Chern-Simons functional in a neighborhood of an $SL(2;\mathbb{C})$ flat connection. For a 3 manifold $Y^3$, let $\rho : \pi_1(Y^3) \to SL(2;\mathbb{C})$ be an $SL(2;\mathbb{C})$ representation of $Y^3$'s fundamental group and let $\mathbb{A} = A + i\phi$ be the flat $SL(2;\mathbb{C})$ connection associated to $\rho$.

A flat $SL(2;\mathbb{C})$ connection will satisfy the equations:

$$F_A - \phi \wedge \phi = 0,$$
$$d_A\phi = 0.$$

Denote $\mathfrak{g}_P^{\mathbb{C}}$ as the complexification of $\mathfrak{g}_P$, then a flat $SL(2;\mathbb{C})$ connection will bring in a twisted de Rham complex:

$$0 \to \Omega^0(\mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{d_{\mathbb{A}}} \Omega^1(\mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{d_{\mathbb{A}}} \Omega^2(\mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{d_{\mathbb{A}}} \Omega^3(\mathfrak{g}_P^{\mathbb{C}}) \to 0, \tag{1.32}$$

with the homology groups:

$$H^k_{\mathbb{A}} := \frac{\mathrm{Ker}(d_{\mathbb{A}} : \Omega^k(\mathfrak{g}_P^{\mathbb{C}}) \to \Omega^{k+1}(\mathfrak{g}_P^{\mathbb{C}}))}{\mathrm{Im}(d_{\mathbb{A}} : \Omega^{k-1}(\mathfrak{g}_P^{\mathbb{C}}) \to \Omega^k(\mathfrak{g}_P^{\mathbb{C}}))}.$$

In addition, we have the natural identification given by the real part and imaginary part of the bundle:

$$\Omega^k(\mathfrak{g}_P^{\mathbb{C}}) \cong \Omega^k(\mathfrak{g}_P) \oplus \Omega^k(\mathfrak{g}_P).$$

To be explicit, given $a + ib \in \Omega^k(\mathfrak{g}_P^{\mathbb{C}})$, we have the following maps:

$$\begin{aligned}
&d_{\mathbb{A}} : \Omega^k(\mathfrak{g}_P^{\mathbb{C}}) \to \Omega^{k+1}(\mathfrak{g}_P^{\mathbb{C}}),\\
&d_{\mathbb{A}}(a+ib) = d_A a - [\phi, b] + i(d_A b + [\phi, a]),\\
&d_{\mathbb{A}}^{\star} : \Omega^{k+1}(\mathfrak{g}_P^{\mathbb{C}}) \to \Omega^k(\mathfrak{g}_P^{\mathbb{C}}),\\
&d_{\mathbb{A}}^{\star}(a+ib) = d_A^{\star} a - \star[\phi, \star b] + i(d_A^{\star} b + \star[\phi, \star a]).
\end{aligned}$$

**Remark.** *Given $s, s' \in \Omega^0(\mathfrak{g}_P^{\mathbb{C}})$, under the identification of $\Omega^0(\mathfrak{g}_P^{\mathbb{C}}) \cong \Omega^0(\mathfrak{g}_P) \oplus \Omega^0(\mathfrak{g}_P)$, there exist $s_1, s_2, s_1', s_2' \in \Omega^0(\mathfrak{g}_P)$ such that $s = s_1 + is_2$ and $s' = s_1' + s_2'$. The inner product we take is $\langle s, s' \rangle = -\mathrm{Tr}(s\bar{s}') = \langle s_1 s_1' \rangle + \langle s_2 s_2' \rangle$ and $d_{\mathbb{A}}^{\star}$ is the adjoint of $d_{\mathbb{A}}$ with respect to this inner product. This explains the sign of $d_{\mathbb{A}}^{\star}(a+ib)$.*

Now we will discuss the Hessian for the complex Chern-Simons functional.

For the first two equations of (1.31), we have:

$$\begin{aligned}
&\frac{d}{dy}A - \star d_A \phi = 0,\\
&\frac{d}{dy}\phi - \star(F_A - \phi \wedge \phi) = 0.
\end{aligned} \tag{1.33}$$

By a direct computation, we define the Hessian for the functional at an $SL(2;\mathbb{C})$ connection $\mathbb{A} = A + i\phi$ as:

$$\begin{aligned}
&Q_{\mathbb{A}} : \Omega_Y^1(\mathfrak{g}_P^{\mathbb{C}}) \to \Omega_Y^1(\mathfrak{g}_P^{\mathbb{C}}),\\
&Q_{\mathbb{A}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \star d_A b + \star[\phi, a] \\ \star d_A a - \star[\phi, b] \end{pmatrix},
\end{aligned} \tag{1.34}$$

which is a $y$-independent linearization of (1.33).

In addition, it is not hard to see that the Hessian operator is horizontal, which means $Q_{\mathbb{A}}$ can also be considered as an operator:

$$Q_{\mathbb{A}} : \mathrm{Ker}\ d_{\mathbb{A}}^{\star} \to \mathrm{Ker}\ d_{\mathbb{A}}^{\star}.$$

However, it is much eaiser to consider operators with gauge fixing conditions. We define the extended Hessian $\widehat{Q}_{\mathbb{A}}$ as follows:

$$\widehat{Q}_{\mathbb{A}} : \Omega^1_Y(\mathfrak{g}_P)\times\Omega^1_Y(\mathfrak{g}_P)\times\Omega^0_Y(\mathfrak{g}_P)\times\Omega^0_Y(\mathfrak{g}_P)\to\Omega^1_Y(\mathfrak{g}_P)\times\Omega^1_Y(\mathfrak{g}_P)\times\Omega^0_Y(\mathfrak{g}_P)\times\Omega^0_Y(\mathfrak{g}_P),$$
$$\widehat{Q}_{\mathbb{A}}\begin{pmatrix} a_1\\ b_1\\ a_0\\ b_0\end{pmatrix}=\begin{pmatrix} \star d_A b_1+\star[\phi,a_1]+d_A a_0-[\phi,b_0]\\ \star d_A a_1-\star[\phi,b_1]+d_A b_0+[\phi,a_0]\\ d_A^\star a_1+\star[b_1,\star\phi]\\ d_A^\star b_1-\star[a_1,\star\phi]\end{pmatrix}. \tag{1.35}$$

Now we have the following proposition about the Hessian operator:

**Proposition 1.4.10.** *(1) $\widehat{Q}_{\mathbb{A}}=\mathcal{EH}_{(A,\phi,0)}$ and it is a self-adjoint operator.*

*(2) $\widehat{Q}_{\mathbb{A}}$ is an isomorphism if and only if $H^1_{\mathbb{A}}=0$ and $H^0_{\mathbb{A}}=0$.*

*Proof.* (1) is an immediate corollary of Proposition 1.4.4.

For (2), using the Hodge theorem, we can decompose the 1-form as $\Omega^1(\mathfrak{g}_P^{\mathbb{C}})=\mathrm{Ker}\ d^\star_{\mathbb{A}}\oplus\mathrm{Im}\ d_{\mathbb{A}}$. Under this decomposition, the extended Hessian operator can be separated into two parts, which we denote as $\widehat{Q}_{\mathbb{A}}=Q_{\mathbb{A}}\oplus S_{\mathbb{A}}$. Here $S_{\mathbb{A}}$ is defined as follows:

$$S_{\mathbb{A}} : \mathrm{Im}\ d_{\mathbb{A}}\times\Omega^0(\mathfrak{g}_P)\to\mathrm{Im}\ d_{\mathbb{A}}\times\Omega^0(\mathfrak{g}_P),$$
$$S_{\mathbb{A}}\begin{pmatrix} a_1\\ b_1\\ a_0\\ b_0\end{pmatrix}=\begin{pmatrix} d_A a_0-[\phi,b_0]\\ d_A b_0+[\phi,a_0]\\ d_A^\star a_1-\star[\phi,\star b_1]\\ d_A^\star b_1+\star[\phi,\star a_1]\end{pmatrix}. \tag{1.36}$$

By Hodge theory, we know that $\mathrm{Ker}(Q_{\mathbb{A}})=\mathrm{Ker}\ d_{\mathbb{A}}\cap\mathrm{Ker}\ d^\star_{\mathbb{A}}=H^1_{\mathbb{A}}$ and $\mathrm{Ker}(S_{\mathbb{A}})=H^0_{\mathbb{A}}$ □

Therefore, we have the following terminology:

**Definition 1.4.11.** *The flat connection $\mathbb{A}$ is called non-degenerate if $H^1_{\mathbb{A}}$ is zero and acyclic if $H^1_{\mathbb{A}}$ and $H^0_{\mathbb{A}}$ is zero.*

Now, we will discuss the relation of the extended Hessian and the linearization of the Kapustin-Witten map. Let $(A,\Phi)$ be a solution to the Kapustin-Witten equations.

Recall the linearization

$$\begin{aligned} &\mathcal{L}_{(A,\Phi)} : \Omega^1_X(\mathfrak{g}_P)\times\Omega^1_X(\mathfrak{g}_P)\to\Omega^2_X(\mathfrak{g}_P)\times\Omega^0(\mathfrak{g}_P),\\ &\mathcal{L}_{(A,\Phi)}\begin{pmatrix} a\\ \phi\end{pmatrix} = \begin{pmatrix} d_A a-[\Phi,b]+\star(d_A b+[\Phi,a])\\ -\star[a,\star\Phi]+d_A^\star b\end{pmatrix}, \end{aligned} \tag{1.37}$$

as well as the gauge fixing operator

$$\begin{aligned} &\mathcal{L}^{gf}_{(A,\Phi)} : \Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P)\to\Omega^0(\mathfrak{g}_P),\\ &\mathcal{L}^{gf}_{(A,\Phi)}\begin{pmatrix} a\\ b\end{pmatrix} = \ d_A^\star a+\star[b,\star\Phi]. \end{aligned} \tag{1.38}$$

and define the following operator

$$\mathcal{D}_{(A,\Phi)} := \mathcal{L}_{(A,\Phi)}+\mathcal{L}^{gf}_{(A,\Phi)} : \Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P)\to\Omega^2(\mathfrak{g}_P)\times\Omega^0(\mathfrak{g}_P)\times\Omega^0(\mathfrak{g}_P).$$

Let $I$ to be an interval of $\mathbb{R}$ and denote $y$ as the coordinate of $I$. Over $Y\times I$, we have the following identifications:

$$\begin{aligned} \Omega^1_X(\mathfrak{g}_P)&\cong\Omega^0_Y(\mathfrak{g}_P)\oplus\Omega^1_Y(\mathfrak{g}_P),\\ \alpha_0 dy+\alpha_1&\to\alpha_0\oplus\alpha_1,\\ \Omega^2_X(\mathfrak{g}_P)&\cong\Omega^1_Y(\mathfrak{g}_P)\oplus\Omega^1_Y(\mathfrak{g}_P),\\ \alpha_1 dy+\alpha_2&\to\alpha_1\oplus\star\alpha_2. \end{aligned} \tag{1.39}$$

Take $(a,b)\in\Omega^1_X(\mathfrak{g}_P)\times\Omega^1_X(\mathfrak{g}_P)$, under the previous identification, we denote $a = a_0dy+a_1$, $b=b_0dy+b_1$, then we have the following relation of the operator $\mathcal{D}_{(A,\Phi)}$ and the extended Hessian $\mathcal{EH}$:

**Proposition 1.4.12.** *For any connection $(A,\Phi=\phi+\phi_y dy)$ over $Y\times I$, if we choose a gauge such that $A$ don't have $dy$ component, then we have*

$$\mathcal{L}_{(A,\Phi)} = -\frac{d}{dy}+\mathcal{H}_{(A,\phi,\phi_y)},$$

$$\mathcal{D}_{(A,\Phi)} = -\frac{d}{dy}+\mathcal{EH}_{(A,\phi,\phi_y)}.$$

*Proof.* Over $Y\times\mathbb{R}$ with volume form $\mathrm{Vol}_Y\wedge dy$, under the identification (1.39), take $a=a_0dy+a_1$ and $b=b_0dy+b_1$, and we have the following computation:

$$d_A a-[\Phi,b] = (-\frac{d}{dy}a_1+d_A a_0-[\phi,b_0]+[\phi_y,b_1])dy+(d_A a_1-[\phi,b_1]),$$

and

$$d_A b + [\Phi, a] = (-\frac{d}{dy} b_1 + d_A b_0 + [\phi, a_0] - [\phi_y, a_1])dy + (d_A b_1 + [\phi, a_1]).$$

Therefore, we have

$$\begin{aligned}
&d_A a - [\phi, b] + \star_4(d_A b + [\phi, a]) \\
=&(-\frac{d}{dy} a_1 + d_A a_0 - [\phi, b_0] + [\phi_y, b_1] + \star(d_A b_1 + [\phi, a_1]))dy \\
&+ \star(-\frac{d}{dy} b_1 + d_A b_0 + [\phi, a_0] - [\phi_y, a_1] + \star(d_A a_1 - [\phi, b_1])).
\end{aligned}$$

By our assumption, $\Phi$ doesn't have $dy$ component, we have the following computation:

$$\begin{aligned}
&d_A^{\star_4} b - \star_4[a, \star_4\Phi] \\
=&d_A^{\star} b_1 - \star[a_1, \star\Phi] - \frac{d}{dy} b_0 + [\phi_y, a_0] \\
=&d_A^{\star} b_1 + \star[\Phi, \star a_1] - \frac{d}{dy} b_0 + [\phi_y, a_0].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&d_A^{\star_4} a + \star_4[b, \star_4\Phi] \\
=&d_A^{\star} a_1 + \star[b_1, \star\Phi] - \frac{d}{dy} a_0 - [\phi_y, b_0] \\
=&d_A^{\star} a_1 - \star[\Phi, \star b_1] - \frac{d}{dy} a_0 - [\phi_y, b_0].
\end{aligned}$$

The result follows immediately from our computation. □

## 1.5 Fredholm Theory

In this section, we will introduce the Fredholm theory for the Kapustin-Witten equations with Nahm Pole boundary condition over manifold with boundary and cylindrical end. See [19], [23] for the Fredholm theory of manifold with cylindrical end and see also [45],[41], for the Fredholm theory on compact manifold with boundary.

### Sobolev Theory on a Manifold with Boundary

In this subsection, we review the Sobolev theory on a manifold with boundary and cylindrical end.

We begin with a suitable functional space. Take $\hat{X}$ to be a compact four-manifold with two boundary components $Y$ and $Z$. Take $X$ to be the four-manifold gluing $\hat{X}$ and $Y\times[0,+\infty)$ along the common boundary $Y$, $X = \hat{X}\cup_Y Y\times[0,+\infty)$. Denote by $E$ a bundle over $X$ and fix a background connection $\nabla$, let $L^p_k(X,E)$ be the completion of smooth $E$-valued functions on $X$ with respect to the norm

$$\|f\|_{L^p_k} = \Big(\sum_{i=0}^{k}\int_X |\nabla^i f|^p\Big)^{\frac{1}{p}},$$

where $\nabla^k f$ is the symmetric tensor product of $\nabla f$.

For a manifold with boundary, we have the following Sobolev embedding theorem

**Proposition 1.5.1.** *([9, Thm 2.30], [62, Appendix]) For a compact 4-manifold $\hat{X}$ (with boundary), $k\geq l$, $q\geq p$ and the indices $p$, $q$ are related by*

$$k-\frac{4}{p}\geq l-\frac{4}{q},$$

*then there is a constant $C_{\hat{X},p,q}$, such that for any section $f$ of a unitary bundle over $\hat{X}$, we have*

$$\|f\|_{L^q_l}\leq C_{\hat{X},p,q},\|f\|_{L^p_k}.$$

As a manifold with cylindrical ends has finite geometry, we have the parallel Sobolev embedding theorem for a manifold with boundary and cylindrical ends.

**Corollary 1.5.2.** *If $X$ is a 4-manifold with boundary and cylindrical ends, $k\geq l$, $q\geq p$ and the indices p,q are related by*

$$k-\frac{4}{p}\geq l-\frac{4}{q},$$

*then there is a constant $C_{X,p,q}$, such that for any section f of a unitary bundle over X,we have*

$$\|f\|_{L^q_l}\leq \|f\|_{L^p_k}.$$

*Proof.* After identifying the cylindrical ends with $Y\times[0,+\infty)$, we can take open covers $\{U_i\}$ as follows: $U_0 := X/(Y\times[1,+\infty))$ and for $i\geq 1$, $U_i := Y\times(i-1,i+1)$.

Given a function $f$, let $f_i$ be the restriction of the function to the open cover $U_i$, for $p\leq q$, we have the following inequality,

$$\|f\|_{L^q_l}\leq C(\sum_{i=0}^{+\infty}\|f_i\|^q_{L^q_l})^{\frac{1}{q}}\leq C(\sum_{i=0}^{+\infty}\|f_i\|^p_{L^q_l})^{\frac{1}{p}}\leq C(\sum_{i=0}^{+\infty}\|f_i\|^p_{L^p_k})^{\frac{1}{p}}\leq C\|f\|_{L^p_k}.$$

□

**Elliptic Weight and Nahm Pole Model**

In this subsection, we will discuss the elliptic weights and Fredhlom property of the Kapustin-Witten operator, which is first introduced in [45],[41] .

Take $X$ to be a manifold with boundary and cylindrical end, choose a cylindrical neighborhood of $X$ which we will denote as $Y \times (0,1] \subset X$, $Y \times \{0\} = \partial X$.

Now we shall study the action of $\mathcal{L}$ on the weighted Sobolev space, and so we start by giving the definition of these.

Choose a smooth function $y : X \to \mathbb{R}$, which is smaller than 1 and equals the distance function $d(x, \partial X)$ in a neighborhood of $\partial X$.

For any $\lambda \in \mathbb{R}$, we can define the following weighted Sobolev space:

$$y^\lambda L^p(X, E) := \{y^\lambda f | f \in L^p(X, E)\}.$$

It is easy to see that a suitable norm on this space will be:

$$\|f\|_{y^\lambda L^p(X,E)} := (\int_X y^{-\lambda p} |f|^p dx)^{\frac{1}{p}}.$$

Next, we have the following edged Sobolev space which was introduced in [41]. Using a local coordinate on $X$ and let $y$ to be the coordinates locally orthogonal to the boundary, we have

$$H_0^{k,p}(X) = \{f \in L^p(X) | (y\partial_{\vec{x}})^\alpha (y\partial_y)^j f \in L^p(X), \forall j + |\alpha| \leq k\}. \tag{1.40}$$

To be explicit, a suitable norm on this space will be

$$\|f\|_{H_0^{k,p}(X)} := (\int_X \sum_{\forall j+|\alpha| \leq k} |(y\partial_{\vec{x}})^\alpha (y\partial_y)^j f|^p)^{\frac{1}{p}}.$$

We define the weighted edge Sobolev space as follows:

$$y^\lambda H_0^{k,p} = \{f = y^\lambda f_1 | f_1 \in H_0^{k,p}\}. \tag{1.41}$$

Mazzeo and Witten in [45] have the following theorem for the Fredholm property of the Kapustin-Witten operator $\mathcal{D}_{(A,\Phi)}$ with suitable weighted edge Sobolev spaces:

**Theorem 1.5.3.** *[45, Proposition 5.2] Let X be a manifold with boudary and* $(A, \Phi)$ *be a Nahm pole solution to the Kapustin-Witten equations. Let* $\mathcal{D}_{(A,\Phi)}$ *be the*

*Kapustin-Witten operator (*1.17*) to the Nahm pole solution, and suppose that* $\lambda \in (-1, 1)$*, then the operator*

$$\mathcal{D}_{(A,\Phi)} : y^{\lambda+\frac{1}{2}} H_0^{1,2}(X) \to y^{\lambda-\frac{1}{2}} L^2(X)$$

*is a Fredholm operator.*

We also have the following modification of the Theorem for $p \geq 2$ due to R.Mazzeo [42]:

**Theorem 1.5.4.** *[42] Let X be a manifold with boudary and* $(A, \Phi)$ *be a Nahm pole solution to the Kapustin-Witten equations. Let* $\mathcal{D}_{(A,\Phi)}$ *be the Kapustin-Witten operator (*1.17*) to the Nahm pole solution, and suppose that* $\lambda \in (-1, 1)$ *and* $p \geq 2$*, then the operator*

$$\mathcal{D}_{(A,\Phi)} : y^{\lambda+\frac{1}{p}} H_0^{1,p}(X) \to y^{\lambda+\frac{1}{p}-1} L^p(X)$$

*is a Fredholm operator.*

*Proof.* See Appendix 1. □

Now, we will introduce some basic properties of these weighted Sobolev spaces that will be used in this paper.

**Proposition 1.5.5.** $y^l H_0^{k,p} = \{f \in L^p | f \in y^l L^p, \nabla f \in y^{l-1} L^p, \cdots \nabla^k f \in y^{l-k} L^p\}.$

*Proof.* By (1.40), for $g \in H_0^{k,p}$, we know that $g \in L^p, \nabla g \in y^{-1} L^p, \cdots, \nabla^k g \in y^{-k} L^p$. Therefore, by the definition of the weighted edge Sobolev space (1.41), for $f \in y^l H_0^{k,p}$, there exist a $g \in H_0^{k,p}$ such that $f = y^l g$. For any positive integers $m, n \leq k$, by the Leibniz rule, we have

$$\nabla_x^m \nabla_y^n f = \sum_{i=0}^{l} y^{l-i} \nabla_x^m \nabla_y^{n-i} g. \tag{1.42}$$

By the definition of $g$, we have $\nabla_x^m \nabla_y^{n-i} g \in y^{i-m-n} L^p$, therefore $y^{l-i} \nabla_x^m \nabla_y^{n-i} g \in y^{l-m-n} L^p$. So for $m, n$, we have $\nabla_x^m \nabla_y^n f \in y^{l-m-n} L^p$.

Therefore, for any integer $j$ with $0 \leq j \leq k$, we have $\nabla^j f \in y^{l-j} L^p$.

□

In addition, we have the following properties for these spaces.

**Proposition 1.5.6.** (1)*(Different Weight Relation)For any positive integer p, if* $\lambda_2 > \lambda_1$*, then*

$$y^{\lambda_2} L^p(X) \hookrightarrow y^{\lambda_1} L^p(X).$$

*Here the* $\hookrightarrow$ *means the inclusion of Banach space.*

(2)*(Embedding to Usual Sobolev Space)* $y^\lambda H_0^{k,p} \hookrightarrow y^{\lambda-k} L_k^p$,

(3)*(Hölder inequality)* $\forall \lambda, \lambda_1, \lambda_2 \in \mathbb{R}$ *with* $\lambda \leq \lambda_1 + \lambda_2$*, and positive real numbers* $p, q, r$*, we have*

$$\|fg\|_{y^\lambda L^r(X)} \leq \|f\|_{y^{\lambda_1} L^p(X)} \|g\|_{y^{\lambda_2} L^q(X)}.$$

*Proof.* (1) Given a function $f$ over a manifold $X$, we have the following inequality:

$$\begin{aligned}\|f\|_{y^{\lambda_1} L^p} &= (\int y^{-\lambda_1 p} |f|^p)^{\frac{1}{p}} \\ &= (\int y^{-\lambda_2 p} y^{(\lambda_2 - \lambda_1)p} |f|^p)^{\frac{1}{p}} \\ &\leq (\int y^{-\lambda_2 p} |f|^p)^{\frac{1}{p}} (\text{here we use } \lambda_2 > \lambda_1).\end{aligned}$$

(2) Given $f \in y^l H_0^{k,p}$, by Prop 1.5.5, for any $0 \leq j \leq k$, $\nabla^j f \in y^{l-j} L^p$ and by (1), we have the embedding $y^{l-j} L^p \hookrightarrow y^{l-k} L^p$. The result follows immediately.

(3) Given two functions $f$ and $g$ over $X$, we have the following inequality:

$$\begin{aligned}\|fg\|_{y^\lambda L^r(X)} &= (\int f^r g^r y^{-r\lambda})^{\frac{1}{r}} \\ &\leq (\int (f y^{-\lambda_1})^r (g y^{-\lambda_2})^r)^{\frac{1}{r}} \\ &\leq (\int (f y^{-\lambda_1})^p)^{\frac{1}{p}} (\int (g y^{-\lambda_2})^q)^{\frac{1}{q}} \\ &\leq \|f\|_{y^{\lambda_1} L^p(X)} \|g\|_{y^{\lambda_2} L^q(X)}.\end{aligned}$$

□

Using these inequalities, we have the following two corollaries:

**Corollary 1.5.7.** *(1) For any* $\lambda \in \mathbb{R}$, $p \geq 2$*, we have*

$$y^{\lambda + \frac{1}{p}} H_0^{1,p}(X) \hookrightarrow y^{\lambda + \frac{1}{p} - 1} L_1^p(X) \hookrightarrow y^{\lambda + \frac{1}{p} - 1} L^{2p}(X),$$

*(2) For* $\lambda \geq 1 - \frac{1}{p}$*, we have*

$$\|fg\|_{y^{\lambda + \frac{1}{p} + 1} L^p(X)} \leq \|f\|_{y^{\lambda + \frac{1}{p}} L^{2p}(X)} \|g\|_{y^{\lambda + \frac{1}{p}} L^{2p}(X)}.$$

*(3)For $\lambda \geq 1 - \frac{1}{p}$, $p \geq 2$, we have* $\|fg\|_{y^{\lambda+\frac{1}{p}-1}L^p(X)} \leq \|f\|_{y^{\lambda+\frac{1}{p}-1}L^p_1(X)}\|g\|_{y^{\lambda+\frac{1}{p}-1}L^p_1(X)}$.

*Proof.* For (1) this is immediate corollary of Proposition 1.5.6 combining with the Sobolev embedding $L^p_1 \hookrightarrow L^{2p}$ with $p \geq 2$.

(2) For $\lambda \geq 1 - \frac{1}{p}$, we have $\lambda + \frac{1}{p} + 1 \leq \lambda + \frac{1}{p} + \lambda + \frac{1}{p}$. By Proposition 1.5.6, the Holder inequality implies the result.

(3) For $\lambda \geq 1 - \frac{1}{p}$, we have $\lambda + \frac{1}{p} - 1 \leq \lambda + \frac{1}{p} - 1 + \lambda + \frac{1}{p} - 1$. Using Proposition 1.5.6 and Sobolev embedding $L^p_1 \hookrightarrow L^{2p}$, the statement follows immediately. □

**Fredholmness on Infinite Cylinder**

In this section, we introduce the Fredholm theory for the Kapustin-Witten operator $\mathcal{D}_{(A,\Phi)}$ (1.17) over the four-manifold $W := Y^3 \times (-\infty, +\infty)$.

Consider a smooth solution $(A, \Phi)$ to the Kapustin-Witten equations over $W$, which converges in $L^p_1$ norm to acyclic flat $SL(2, \mathbb{C})$ connection over both sides, we have the following proposition:

**Proposition 1.5.8.** *Under the assumption above, the operator $\mathcal{D}_{(A,\Phi)} : L^p_1(W) \to L^p(W)$ is a Fredholm operator.*

*Proof.* By Proposition 1.4.12, we have $\mathcal{D}_{(A,\Phi)} = -\frac{d}{dy} + \mathcal{E}\mathcal{H}$. As we assume that $(A, \Phi)$ converges to acyclic flat connections over both sides, for some $p > 2$, this is a classical result of [39, Theorem 1.3]. See [23, Proposition 2b.1] for the Yang-Mills case and also [38, Proposition 14.2.1] for the $p = 2$ version.

□

**The Kapustin-Witten Operator with Acyclic Limit**

As before, denote by $\hat{X}$ a compact four-manifold with two boundary components $Y$ and $Z$. Take $X$ to be the four-manifold obtained by gluing $\hat{X}$ and $Y \times [0, +\infty)$ along the common boundary $Y$, that is $X = \hat{X} \cup_Y Y \times [0, +\infty)$. Given an $SU(2)$-bundle $P$ over $X$, and a solution $(A, \Phi)$ to the Kapustin-Witten equations (4.1), with Nahm pole boundary condition on $Z$ and which converges in $L^p_1$ norm to acyclic connections over the cylindrical ends for some $p > 2$, we have the following theorem:

**Proposition 1.5.9.** *Under the assumption as above, the Kapustin-Witten operator (*1.17*)*

$$\mathcal{D}_{(A,\Phi)} : y^{\lambda+\frac{1}{p}} H^{1,p}_0(X) \to y^{\lambda-1+\frac{1}{p}} L^p(X)$$

*is a Fredholm operator.*

*Proof.* We will use the parametrix method to prove this theorem. For simplicity, we denote $\mathcal{D}_{(A,\Phi)}$ by $D$ in this proof. To be more explicit, we hope to find two operators

$$P : y^{\lambda-1+\frac{1}{p}} L^p(X) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(X)$$

and

$$R : y^{\lambda+\frac{1}{p}} H_0^{1,p}(X) \to y^{\lambda-1+\frac{1}{p}} L^p(X)$$

such that $S^l(\rho) := DP(\rho) - \rho,\ S^r(\rho) := RD(\rho) - \rho$, are two compact operators.

Choose $U_0 := Y \times (T, +\infty)$ and let $U_1$ be a compact cylindrical neighborhood of $\partial X$. By Proposition 1.5.8, there exist $P_0, R_0$ such that over $U_0$, we have $DP_0(\rho) = \rho$ and $R_0 D(\rho) = \rho$. By the compactness of $\hat{X}$, we can take a finite cover $\{U_i | i = 0 \cdots n\}$. In each open set $U_i$, by Theorem 1.5.4 and the elliptic operator property on the inner open set of the manifold, there exist $P_i, R_i$ and compact operators $S_i^l$ and $S_i^r$ such that $DP_i(\rho) = \rho + S_i^l(\rho),\ R_i D(\rho) = \rho + S_i^r(\rho)$.

Denote $X = \bigcup_{i=0}^n U_i$, take $S_0^l(\rho) := 0$ and $S_0^r(\rho) := 0$. We take a partition of unity $\{\beta_i\}$ to these covers and define operators $P(\rho) := \sum_{i=0}^n \beta_i P_i(\rho)$ and $R(\rho) := \sum_{i=0}^n \beta_i R_i(\rho)$.

We have

$$\begin{aligned} DP(\rho) &= \sum_{i=0}^n \nabla\beta_i \star P_i(\rho) + \sum_{i=0}^n \beta_i DP_i(\rho) \\ &= \sum_{i=0}^n \nabla\beta_i \star P_i(\rho) + \rho + \sum_{i=0}^n \beta_i S_i^l(\rho). \end{aligned} \tag{1.43}$$

Here, $\sum_{i=0}^n S_i^l(\rho)$ is a finite sum of compact operators and it will be compact.

For the terms $\sum_{i=0}^n \nabla\beta_i \star P_i(\rho)$, recall that

$$P_i : y^{\lambda-1+\frac{1}{p}} L^p(X) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(X).$$

In addition, we know $\nabla\beta_i$ is supported over $\hat{X}$. For functions supported on $\hat{X}$, the norm $y^{\lambda+\frac{1}{p}} H_0^{1,p}(X)$ is equivalent to $L_1^p(X)$ and $y^{\lambda-1+\frac{1}{p}} L^p(X)$ is equivalent to $L^p(X)$. By the compactness of the Sobolev embedding of $L_1^p(X)$ into $L^p(X)$, we know that $\sum_{i=0}^n \nabla\beta_i \star P_i(\rho)$ is also a compact operator.

For the right inverse, we have the following computation:

$$RD(\rho) = \sum_{i=0}^{n} \beta_i R_i D(\rho) = \sum_{i=0}^{n} \beta_i(\rho + S_i^r(\rho)) = \rho + \sum_{i=0}^{n} \beta_i S_i^r(\rho). \tag{1.44}$$

Here $\sum_{i=0}^{n} \beta_i S_i^r(\rho)$ is a finite sum of compact operator thus it is a compact operator. To summarize, we proved that $D$ is a Fredholm operator. □

**Reducible Limit Connection**

Given $(A, \Phi)$ a solution to the Kapustin-Witten equations (4.1) over $W := Y \times (-\infty, +\infty)$, take $t$ as the coordinate for $(-\infty, +\infty)$. Assume that $(A, \Phi)$ $L_2^2$ converges to a non-degenerate $SL(2;\mathbb{C})$ flat connection $(A_{\rho_1}, \Phi_{\rho_1})$ when $y \to -\infty$ and a non-degenerate $SL(2;\mathbb{C})$ flat connection $(A_{\rho_2}, \Phi_{\rho_2})$ when $y \to +\infty$. For $i = 1, 2$, if either of $(A_{\rho_i}, \Phi_{\rho_i})$ is reducible, Proposition 1.5.8 is not true since zero can be in the spectrum of the extended Hessian operator $\hat{Q}_{(A_{\rho_i}, \Phi_{\rho_i})}$ (1.35).

Therefore, we hope to use a weight to get rid of the 0 spectrum and we need to introduce the exponential weight in the cylindrical end. For any real positive number $\alpha$ and norm U, given an arbitrary smooth function $h$ which equals $e^{\alpha t}$ over every cylindrical end $[T, +\infty) \times Y$ and $Y \times (-\infty, -T]$ when $T$ is big enough, we can define the weighted norm by

$$\|f\|_{U_\alpha} = \|hf\|_U.$$

To be explicit, for a $f$, we denote $\|f\|_{y^{\lambda+\frac{1}{p}} H^{k,p}_{0,\alpha}} := \|hf\|_{y^{\lambda+\frac{1}{p}} H^{k,p}_{0}}$ and $\|f\|_{y^{\lambda+\frac{1}{p}-1} L^p_\alpha} := \|hf\|_{y^{\lambda+\frac{1}{p}-1} L^p}$.

Our operator $\mathcal{D}_{(A,\Phi)}$ can naturally defined over these weighted spaces:

$$\mathcal{D}_{(A,\Phi),\alpha} : y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(W) \to y^{\lambda+\frac{1}{p}-1} L^p_\alpha(W).$$

We have the following result:

**Proposition 1.5.10.** *Under the assumption as above, we can choose $\alpha$ such that the operator $\mathcal{D}_{(A,\Phi),\alpha} : L^p_{1,\alpha}(W) \to L^p_\alpha(W)$ is a Fredhom operator.*

*Proof.* Considering the operator $\mathcal{D}_{(A,\Phi),\alpha}$ over a tube, acting on the weighted Sobolev space with $e^{\alpha t}$ as weight function. This is equivalent to an operator $D'$ acting on an unweighted space with the relation

$$D' = \mathcal{D}_{(A,\Phi),\alpha} - \alpha.$$

Therefore, when $\alpha$ is not in the spectrum of the extended Hessian operator over the limit flat connections $\hat{Q}_{(A_{\rho_i},\Phi_{\rho_i})}$, this is a classical result of [39, Theorem 1.3]. □

**Sobolev Theory for weighted space**

Recall that we denote by $\hat{X}$ a compact four-manifold with two boundary components $Y$ and $Z$. Take $X$ to be the four-manifold obtained by gluing $\hat{X}$ and $Y\times[0,+\infty)$ along the common boundary $Y$ , that is $X = \hat{X}\cup_Y Y\times[0,+\infty)$. Take an $SU(2)$-bundle $P$ over $X$, and a solution $(A,\Phi)$ to the Kapustin-Witten equations (4.1), with Nahm pole boundary condition on $Z$ which converges to a reducible $SL(2;\mathbb{C})$-connections over the cylindrical ends.

Over this space $X$, for any real number $\alpha$ and norm $U$ fix a smooth weight function $h$ which approximates $e^{\alpha t}$ over the cylindrical end $[T,+\infty)\times Y$. When $T$ is large enough, we can define the weighted norm $U_\alpha$ as:

$$\|f\|_{U_\alpha} := \|hf\|_U.$$

Similarily, we have the Sobolev embedding theorem for the weighted norms.

**Proposition 1.5.11.** *If X is a 4-manifold with boundary and cylindrical ends, $k\geq l$, $q\geq p$ and the indices p,q are related by*

$$k-\frac{4}{p}\geq l-\frac{4}{q},$$

*for any given weighted function, there exists a constant C, such that for any section f of a unitary bundle over X, we have*

$$\|f\|_{L^l_{q,\alpha}(X)}\leq C\|f\|_{L^k_{p,\alpha}(X)}.$$

*Proof.* This is immediate from the definition of the weighted norm and the usual Sobolev embedding theorem. □

In addition, after fixing a weight function, we have the following inequalities for these weighted edge norms:

**Proposition 1.5.12.** *(1) For any $\lambda \in \mathbb{R}$, we have*

$$y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(X) \hookrightarrow y^{\lambda+\frac{1}{p}-1} L^{p}_{1,\alpha}(X),$$

*(2) For $\lambda \geq 1 - \frac{1}{p}$ , $\alpha > 0$, we have the following inequality:*

$$\|fg\|_{y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X)} \leq \|f\|_{y^{\lambda+\frac{1}{p}-1} L^{2p}_\alpha(X)} \|g\|_{y^{\lambda_0-\frac{1}{2}} L^{2p}_\alpha(X)}.$$

*Proof.* The statement in (1) is immediately using Corollary 1.5.7 and the definition of weighted Sobolev space. By Corollary 1.5.7, we know

$$\|fg\|_{y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X)} \leq \|f\|_{y^{\lambda+\frac{1}{p}-1} L^{2p}(X)} \|g\|_{y^{\lambda+\frac{1}{p}-1} L^{2p}_\alpha(X)}.$$

In addition, as we assume $\alpha > 0$, we know that

$$\|f\|_{y^{\lambda+\frac{1}{p}-1} L^{2p}(X)} \leq \|f\|_{y^{\lambda+\frac{1}{p}-1} L^{2p}_\alpha(X)}.$$

The statement in (2) follows immediately. □

**Fredholm Property for the Reducible Limit**

**Proposition 1.5.13.** *Under the assumpution above, there exist $\alpha$ such that the operator*

$$\mathcal{D}_{(A,\Phi),\alpha} : y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(X) \to y^{\lambda-1+\frac{1}{p}} L^p_\alpha(X)$$

*is a Fredhlom operator.*

*Proof.* The main difference between the reducible limit and the acyclic limit is the behavior of the operator $\mathcal{D}_{(A,\Phi)}$ over the cylindrical end. In the reducible case, we use Proposition 1.5.10 over the cylindrical ends to get a parametrix and the results follow similarly as Theorem 1.5.9. □

**The Index**

Now we will give an explicit computation of the index for a manifold with cylindrical end.

For a compact manifold with boundary, in [45], Mazzeo and Witten have a computation of the index of $\mathcal{D}_{(A,\Phi)}$ where $(A, \Phi)$ is a Nahm pole solution to the Kapustin-Witten equations (4.1):

For our case, let $X$ be a manifold with boundary $Z$ and cylindrical end which is identified with $Y \times [0, +\infty)$. We finish a parallel computation for the index of a manifold with boundary and cylindrical end. Under the previous assumption, by Proposition 1.5.9, 1.5.13, we can define the index for these Fredholm operator.

If $(A, \Phi)$ has an acyclic limit, we denote by $\mathrm{Ind}_X(P)$ the index of $\mathcal{D}_{(A,\Phi)}$ in the setting of Theorem 1.5.9:

$$\mathrm{Ind}_X(P) := \dim \ \mathrm{Ker}\mathcal{D}_{(A,\Phi)} - \dim \ \mathrm{Coker}\mathcal{D}_{(A,\Phi)}. \tag{1.45}$$

For compact manifold with Nahm pole boundary condition, Mazzeo and Witten has the following computations:

**Proposition 1.5.14.** *[45, Proposition 4.2] Let X be a compact manifold with boundary, let P be an SU(2) bundle over X, let $(A, \Phi)$ be a solution to the Kapustin-Witten equations with acyclic limit which satisfies the Dirichlet boundary condition over the boundary, then*

$$\mathrm{Ind}P = -3\chi(X).$$

After a modification of their proof, we have the following computation for solutions have irreducible limits:

**Proposition 1.5.15.** *Let X be a manifold with boundary and cylinderical ends, let P be an SU(2) bundle over X, let $(A, \Phi)$ be a solution to the Kapustin-Witten equations with acyclic limit which satisfies the Dirichlet boundary condition over the boundary, then*

$$\mathrm{Ind}P = -3\chi(X).$$

*Proof.* Let $(A, \Phi)$ be a solution to the Kapustin-Witten equation with Dirichlet boundary condition, then the index of the operator $\mathcal{D}_{(A,\Phi)}$ corresponds to the relative boundary condition for the Gauss-Bonnet operator, which the index is equals to $-3\chi(X, \partial X)$ and by Poincare duality, $\chi(X, \partial X) = \chi(X)$.

□

**Proposition 1.5.16** ([45, Proposition 4.3])**.** *Under the same assumption as Proposition 1.5.15, let $(A, \Phi)$ be a solution to the Kapustin-Witten equations (4.1) with the Nahm pole boundary condition over $\partial X$, then*

$$\mathrm{Ind}P = -3\chi(X).$$

*Here $\chi(X)$ is the Euler characteristic of $X$.*

*Proof.* First consider the special case of $(0,1]\times Z$ with the product metric. Let $(A_0,\Phi_0)$ be a connection pair satisfying the Nahm pole boundary condition over $\{0\}\times Z$ and regular over $\{1\}\times Z$. Let $\mathcal{D}_{\mathcal{N},\mathcal{R}}$ be the elliptic operator corresponding to this. By [45] (3.12), as $\mathcal{D}_{\mathcal{N},\mathcal{R}}$ is pseudo skew-Hermitian, we have Ind $\mathcal{D}_{\mathcal{N},\mathcal{R}}$=0.

Now consider a general $(A,\Phi)$ over $X$ satisfying the Nahm pole boundary condition, we choose a tubular neighborhood of $X$ near the boundary and identify it with $Y\times(0,1]$. Let $\mathcal{D}_{\mathcal{N},\mathcal{R}}$ be the restriction of the operator $\mathcal{D}_{(A,\Phi)}$ to $Y\times(0,1]$ and let $\mathcal{D}_{\mathcal{R}}$ be the restriction of $\mathcal{D}_{(A,\Phi)}$ over the complement of $Y\times(0,1]$ in $X$. By a standard excision theorem of index [10, Prop 10.4], we obtain

$$\mathrm{Ind}\ \mathcal{D}_{(A,\Phi)} = \mathrm{Ind}\ \mathcal{D}_{\mathcal{N},\mathcal{R}} + \mathrm{Ind}\ \mathcal{D}_{\mathcal{R}}.$$

By the previous argument, we have Ind $\mathcal{D}_{\mathcal{N},\mathcal{R}} = 0$ and by Proposition 1.5.15, we know Ind $\mathcal{D}_{\mathcal{R}} = \chi(X)$. Thus we have Ind $\mathcal{D}_{(A,\Phi)} = \chi(X)$.

□

If $(A,\Phi)$ is a reducible but non-degenerate limit as in the case of Proposition 1.5.13, we denote by $\mathrm{Ind}_X(P,\alpha)$ the index of $\mathcal{D}_{(A,\Phi),\alpha}$ with respect to weight $\alpha$:

$$\mathrm{Ind}_X(P,\alpha) := \dim\ \mathrm{Ker}\mathcal{D}_{(A,\Phi),\alpha} - \dim\ \mathrm{Coker}\mathcal{D}_{(A,\Phi),\alpha}. \tag{1.46}$$

Take $\alpha^+$ to be a real number that is slightly bigger than 0 and below the positive spectrum of $\widehat{Q}_{\mathbb{A}_\rho}$ and $\alpha^-$ to be a real number that is slightly smaller than 0 and above the negative spectrum of $\widehat{Q}_{\mathbb{A}_\rho}$, then we have two indices:

$$\mathrm{Ind}^+_X(P) := \mathrm{Ind}(P,\alpha^+),\ \mathrm{Ind}^-_X(P) := \mathrm{Ind}(P,\alpha^-). \tag{1.47}$$

For $i=1,2$, suppose $X_i$ is a manifold with boundary $Z_i$ and cylindrical end $Y_i\times[0,+\infty)$ and $Y_1=Y_2$, $P_i$ is an $SU(2)$ bundle over $X_i$. Let $(A_i,\Phi_i)\in \mathcal{C}_{P_i}$ converge to the same $SL(2;\mathbb{C})$ flat connection $\rho$, then we can glue these two manifolds along the common boundary $Y$ to form $X^\sharp$ and a bundle $P^\sharp$. As it converges to the same flat connection $\rho$, we can define a new pair $(A^\sharp,\Phi^\sharp)$ on $P^\sharp$.

We denote by $\mathrm{Ind}_{X^\sharp}(P^\sharp)$ the index of the operator $\mathcal{D}_{(A^\sharp,\Phi^\sharp)}$ on $X^\sharp$.

If $(A_i, \Phi_i)$ both have acyclic limits, then we have the following gluing relation of these indices:

**Proposition 1.5.17.** $\mathrm{Ind}_{X^\sharp}(P^\sharp) = \mathrm{Ind}_{X_1}(P_1) + \mathrm{Ind}_{X_2}(P_2)$.

*Proof.* It is straight forward to apply the same argument in [19, Proposition 3.9] to our case. □

If $(A_i, \Phi_i)$ is reducible but non-degenerate, we can choose $\alpha_1 = -\alpha_2 > 0$ whose absolute value is smaller than the smallest absolute value of eigenvalues of $\widehat{Q}_{\mathbb{A}_\rho}$. We denote $\mathrm{Ind}^+_{X_1}(P_1)$ to be the index corresponding to the weight $\alpha_1$ and $\mathrm{Ind}^-_{X_2}(P_2)$ to be the index corresponding to the weight $\alpha_2$. We have the following Proposition:

**Proposition 1.5.18.** $\mathrm{Ind}_{X^\sharp}(P^\sharp) = \mathrm{Ind}^+_{X_1}(P_1) + \mathrm{Ind}^-_{X_2}(P_2)$.

*Proof.* See [19, Proposition 3.9], the same argument is straight forward in our case. □

Moreover, for the index over the same bundle with small positive and negative weights, we obtain:

**Proposition 1.5.19.** *[19, Proposition 3.10]* $\mathrm{Ind}^+_X(P) - \mathrm{Ind}^-_X(P) = -\dim\ ker\widehat{Q}_{\mathbb{A}_\rho}$.

Now we will do some explicit computation of indices. Consider the model case $W$ to be the 'flask' manifold obtained by gluing a punctured 4-sphere to a tube $S^3 \times \mathbb{R}$. Consider the trivial bundle and the trivial connection over this space, we have the following Lemma for indices:

**Lemma 1.5.20.** *The indices for W are*

$$\mathrm{Ind}^+_W = -6,\ \mathrm{Ind}^-_W = 0.$$

*Proof.* Obviously, $W\sharp W$ is diffeomorphic to $S^4$. In addition, by Proposition 1.5.16, we know the index of operators $\mathcal{D}_{(A,\Phi)}$ over $S^4$ is -6. Therefore, by the gluing relation of the index Proposition 1.5.18, we have

$$\mathrm{Ind}^+_W + \mathrm{Ind}^-_W = -6.$$

In addition, by Proposition 1.5.19, we have

$$\mathrm{Ind}^+_W = \mathrm{Ind}^-_W - 6.$$

Combining these two index formulas, we get the result we want. □

**Corollary 1.5.21.** *If the cylindrical end of X has the form $S^3 \times [0, +\infty)$, we have*

$$\mathrm{Ind}^+_X(P) = -3\chi(X) - 3,\ \mathrm{Ind}^-_X(P) = -3\chi(X) + 3. \tag{1.48}$$

*Proof.* Denote by $\bar{X}$ a smooth compactification of $X$ over the tube $S^3 \times [0, +\infty)$ and denote by $\bar{P}$ the extension of the bundle P.

By Proposition 1.5.18, we have

$$\mathrm{Ind}^+_X(P) + \mathrm{Ind}^-_W = \mathrm{Ind}_{\bar{X}}(\bar{P}),$$

$$\mathrm{Ind}^-_X(P) + \mathrm{Ind}^+_W = \mathrm{Ind}_{\bar{X}}(\bar{P}).$$

In addition, by Proposition 1.5.16, we know that

$$\mathrm{Ind}_{\bar{X}}(\bar{P}) = -3\chi(\bar{X}) = -3\chi(X) - 3.$$

We get the result we want. □

## 1.6 Moduli Theory

In this section, we will introduce the moduli theory for the solutions to the Kapustin-Witten equations (4.1) with Nahm pole boundary condition.

### Framed Moduli Space

In this section, we will give suitable norms to define the moduli space.

Let $X$ to be a smooth 4-manifold with 3-manifold boundary $Z$ and cylindrical end which is identified with $Y \times (0, +\infty)$. Now suppose $P$ is an $SU(2)$ bundle over $X$, $\mathfrak{g}_P$ is the associated adjoint bundle, $\mathcal{A}_P$ is the set of all $SU(2)$ connections on $P$, and $C_P := \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$.

For $i = 1, 2, 3$, fix an orthogonal frame $\{e_i\} \in T^\star Y$, choose a reference connection pair $(A_0, \Phi_0) \in C_P$ to the Kapustin-Witten equations (4.1). We require that $(A_0, \Phi_0)$ satisfies the Nahm pole boundary condition for this frame, thus there exists $\{t_i\} \in \mathfrak{g}_P$ such that the leading expansion of $\Phi_0$ when $y \to 0$ is $\Phi_0 \sim \frac{\sum_{i=1}^3 e_i t^i}{y}$. In addition, fix a flat acyclic $SL(2;\mathbb{C})$ representation $\rho$. If we denote $Y_T := \{T\} \times Y \subset (0, +\infty) \times Y$, we assume that $(A_0, \Phi_0)$ convergence to $\rho$ in $L^p_1$ norm for some $p > 2$.

Now, we will introduce a suitable configuration space. Given real numbers $p, \lambda$ with $p > 2$ and $\lambda \in [1 - \frac{1}{p}, 1)$ , we have the following definition:

**Definition 1.6.1.** *Given a smooth Nahm pole solution* $(A_0, \Phi_0)$*, we define the framed configuration space* $\mathcal{C}^{fr}_{p,\lambda}$ *as follows:*

$$\mathcal{C}^{fr}_{p,\lambda} := \{(A_0, \Phi_0) + (a, b) \mid (a, b) \in y^{\lambda+\frac{1}{p}} H^{1,p}_0(\Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P))\}. \tag{1.49}$$

*Here* $y^{\lambda+\frac{1}{p}} H^{1,p}_0(\Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P))$ *is the completion of smooth 1-forms with respect to the norm* $y^{\lambda+\frac{1}{p}} H^{1,p}_0$.

We have some basic properties of the framed configuration space:

**Proposition 1.6.2.** *(1) Any* $(A, \Phi) \in \mathcal{C}^{fr}_{p,\lambda}$ *satisfies the Nahm pole boundary condition.*

*(2) Assume X has non-vanishing boundary and cylindrical end which identified with* $Y \times (0, +\infty)$*. Let P be an* $SU(2)$ *bundle over it, let* $(A_1, \Phi_1)$ *be a connection pair satisfying the Nahm pole boundary condition, denote* $\frac{\sum_{j=1}^3 e_j t_j^1}{y}$ *is the leading part of* $\Phi_1$*. If* $(A_1, \Phi_1 - \frac{\sum_{j=1}^3 e_j t_j^1}{y}) \in H^{1,p}_0(X)$*, then there exists a global gauge transformation* $g \in \mathcal{G}$ *such that* $g(A_1, \Phi_1) \in \mathcal{C}^{fr}_{p,\lambda}$*.*

*Proof.* For (1), as for $\lambda \in [1 - \frac{1}{p}, 1)$ ,$p > 2$, any differential form which blows-up as $y^{-1}$ is not contained in $y^{\lambda+\frac{1}{p}} H^{1,p}_0$, the result follows immediately.

For (2), for $i = 0, 1$, $(A_i, \Phi_i)$ both satisfies the Nahm pole boundary condition (Definition 1.2.1). Then there exists orthogonal basis $e_j \in T^\star Y$ and $t^i_j \in \mathfrak{g}_P$ for $j = 1, 2, 3$ such that $[t^i_{j_1}, t^i_{j_2}] = \epsilon_{j_1 j_2 j_3} t^i_{j_3}$ where $\epsilon_{j_1 j_2 j_3}$ is the Kronecker symbol of $j_1, j_2, j_3$. In addition, let the asymptotic expansion of $\Phi_i$ at $y = 0$ to be $\frac{\sum_{j=1}^3 e_j t_j^i}{y} + O(y)$. By the commutation relation of $t^i_j$, there exists a $\hat{g} : Z \to SU(2)$ such that $\hat{g}(\frac{\sum_{j=1}^3 e_j t_j^1}{y})\hat{g}^{-1} = \frac{\sum_{j=1}^3 e_j t_j^0}{y}$. By the Hopf theorem, the homotopy type of maps from $Y^3$ to $SU(2)$ is totally determined by the degree. Given $\hat{g} : Z \to SU(2)$, consider $Y_{T_0} = Y \times \{T_0\} \subset X$, we can choose a band to connect $Z$ and $Y$ which is homeomorphic to $Z \sharp Y_{T_0}$. We can choose a map $\hat{g}' : Y_{T_0} \to SU(2)$ whose degree equals minus degree of $g$ and extend these two maps to $Z \sharp Y_{T_0}$ and denote as $\tilde{g}$. Then $\tilde{g}$ has degree zero and can be extended to whole $X$, which we denote as $g$.

By the assumption that $(A_i, \Phi_i)$ are smooth, we have $g(A_1, \Phi_1) - (A_0, \Phi_0) \in y^{\lambda+\frac{1}{p}} H^{1,p}_0$. □

**Remark.** *For a general compact 4-dimensional manifold with boundary, Proposition 1.6.2 is not true. For* $D^4$*, the 4-dimensional unit disc, a choice of frame gives a* *map from* $S^3 \to SO(3)$ *which is* $\pi_3(SO(3))$. *Two frames corresponding to different elements in* $\pi_3(SO(3))$ *can not be globally gauge equivalent.*

As we fixed a base connection $(A_0, \Phi_0)$ to define the framed configuration space, we can also consider the gauge group that preserves the frame.

**Definition 1.6.3.** *The framed gauge group* $\mathcal{G}^{fr}$ *is defined as follows:*

$$\mathcal{G}^{fr} := \{g \in Aut(P) \mid g|_Y = 1\}. \tag{1.50}$$

Given $g \in \mathcal{G}^{fr}$, the action of $g$ on $(A_0, \Phi_0)$ will be

$$g(A_0, \Phi_0) = (A_0 - d_{A_0}g\ g^{-1}, g\Phi_0 g^{-1}). \tag{1.51}$$

Then, we have

$$g(A_0, \Phi_0) - (A_0, \Phi_0) = (-d_{A_0}g\ g^{-1},\ [g, \Phi_0]g^{-1}). \tag{1.52}$$

We consider the following weighted frame gauge group $\mathcal{G}^{fr}_{p,\lambda}$:

$$\mathcal{G}^{fr}_{p,\lambda} = \{g \in \mathcal{G}^{fr} \mid d_{A_0}g\ g^{-1} \in y^{\lambda+\frac{1}{p}} H_0^{1,p}(\Omega^1),\ [g, \Phi_0]g^{-1} \in y^{\lambda+\frac{1}{p}} H_0^{1,p}(\Omega^1)\}. \tag{1.53}$$

For convenience, we denote $d^0(\xi) := d^0_{(A_0,\Phi_0)}(\xi) = (d_{A_0}\xi, [\Phi_0, \xi])$ and we have the following lemma on the weighted frame gauge group $\mathcal{G}^{fr}_{p,\lambda}$.

**Lemma 1.6.4.** $\mathcal{G}^{fr}_{p,\lambda} = \{g \in \mathcal{G}^{fr} \mid \nabla_0 g \in y^{\lambda+\frac{1}{p}} L^p,\ \nabla_0^2 g \in y^{\lambda+\frac{1}{p}-1} L^p,\ [\Phi_0, g] \in y^{\lambda+\frac{1}{p}} L^p,\ \nabla_0[\Phi_0, g] \in y^{\lambda+\frac{1}{p}-1} L^p\}$.

*Proof.* Obviously $\{g \in \mathcal{G}^{fr} \mid d^0 g \in y^{\lambda+\frac{1}{p}} L^p,\ \nabla_0(d^0)g \in y^{\lambda+\frac{1}{p}-1} L^p\} \subset \mathcal{G}^{fr}_{p,\lambda}$. For the other side, we argue as follows: take $\alpha := d_{A_0}g g^{-1}$, then $\alpha \in y^{\lambda+\frac{1}{p}} H_0^{1,p} \hookrightarrow y^{\lambda+\frac{1}{p}-1} L^{2p}$. By $d_{A_0}g = \alpha g$, we have $\nabla_0 g \in y^{\lambda+\frac{1}{p}-1} L^{2p}$ since the pointwise norm of $g$ is 1. In addition, $\nabla_0 d_{A_0} g = (\nabla_0\alpha)g + \alpha\nabla_0 g$. As $\nabla_0\alpha \in y^{\lambda+\frac{1}{p}-1} L^p$ and the pointwise norm of $g$ is 1, we have $\nabla_0\alpha g \in y^{\lambda+\frac{1}{p}-1} L^p$. As $\alpha \in y^{\lambda+\frac{1}{p}-1} L^{2p}$, $\nabla_0 g \in y^{\lambda+\frac{1}{p}-1} L^{2p}$, we have $\alpha\nabla_0 g \in y^{\lambda+\frac{1}{p}-1} L^p$. For $[g, \Phi_0]g^{-1} \in y^{\lambda+\frac{1}{p}} H_0^{1,p}(\Omega^1)$, we have $[g, \Phi_0]g^{-1} \in y^{\lambda+\frac{1}{p}} L^p$. Letting $\beta = [g, \Phi_0]g^{-1}$, then $\beta \in y^{\lambda+\frac{1}{p}} L^p$ implies $\beta g = [g, \Phi_0] \in y^{\lambda+\frac{1}{p}} L^p$. In addition, $\beta \in y^{\lambda+\frac{1}{p}} H_0^{1,p}$ implies $\nabla_0\beta \in y^{\lambda+\frac{1}{p}} L^p$ and $\beta \in y^{\lambda+\frac{1}{p}-1} L^{2p}$. In addition, we have $\nabla_0 g \in y^{\lambda+\frac{1}{p}-1} L^p$, thus $\nabla_0[g, \Phi_0] = \nabla_0(\beta\ g) = (\nabla_0\beta)g + \beta\nabla_0 g \in y^{\lambda+\frac{1}{p}-1} L^p$. Therefore, $\mathcal{G}^{fr}_{p,\lambda} \subset \{g \in \mathcal{G}^{fr} \mid \nabla_0 g \in y^{\lambda+\frac{1}{p}} L^p,\ \nabla_0^2 g \in y^{\lambda+\frac{1}{p}-1} L^p\}$.

□

Thus we can rewrite $\mathcal{G}^{fr}_{p,\lambda}$ as $\mathcal{G}^{fr}_{p,\lambda} = \{g \in \mathcal{G}^{fr} \mid d^0 g \in y^{\lambda+\frac{1}{p}} L^p,\ \nabla_0 d^0 g \in y^{\lambda+\frac{1}{p}-1} L^p\}$.

**Lemma 1.6.5.** *The space* $y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\mathfrak{g}_P)$ *is an algebra and* $y^{\lambda+\frac{1}{p}}H_0^{1,p}(\mathfrak{g}_P)$ *is a module over this algebra.*

*Proof.* For the algebra statement, we only need to prove $u_1u_2 \in y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\mathfrak{g}_P)$, or equivalently, $u_1u_2 \in y^{\lambda+\frac{1}{p}+1}L^p$, $\nabla_0(u_1u_2) \in y^{\lambda+\frac{1}{p}}L^p$, $\nabla_0^2(u_1u_2) \in y^{\lambda+\frac{1}{p}-1}L^p$.

Since $u_i \in y^{\lambda+\frac{1}{p}+1}H_0^{2,p}$, we have $u_i \in y^{\lambda+\frac{1}{p}+1}L^p$, $\nabla_0 u_i \in y^{\lambda+\frac{1}{p}}L^p$ and $\nabla_0^2 u_i \in y^{\lambda+\frac{1}{p}-1}L^p$.

By Proposition 1.5.6, $u_i \in y^{\lambda+\frac{1}{p}+1}H_0^{1,p} \hookrightarrow y^{\lambda+\frac{1}{p}}L_1^p \hookrightarrow y^{\lambda+\frac{1}{p}}L^{2p}$. By Corollary 1.5.7, we have $u_1u_2 \in y^{\lambda+\frac{1}{p}+1}L^p$. In addition, we know $\nabla_0 u_i \in y^{\lambda+\frac{1}{p}}H_0^{1,p} \hookrightarrow y^{\lambda+\frac{1}{p}-1}L^{2p}$ and $u_i \in y^{\lambda+\frac{1}{p}}L^{2p}$. As $\lambda \geq 1-\frac{1}{p}$, we have $\lambda+\frac{1}{p}+\lambda+\frac{1}{p}-1 \geq \lambda+\frac{1}{p}$. By the Hölder inequality in Proposition 1.5.6, we have $\nabla_0 u_1u_2 \in y^{\lambda+\frac{1}{p}}L^p$.

For $u_i \in y^{\lambda+\frac{1}{p}+1}H_0^{2,p}$, as $p > 2$ and $\lambda \geq 1-\frac{1}{p}$, we have $u_i \in y^{\lambda+\frac{1}{p}-1}L_2^p \hookrightarrow C^0$ and $\nabla_0^2 u \in y^{\lambda+\frac{1}{p}-1}L^p$. Therefore, we have $\nabla_0^2 u_1u_2 \in y^{\lambda+\frac{1}{p}-1}L^p$.

The module statement can also be proved in a similar way. □

Fix a base point $p_0 \in X$ and define a system of neighborhoods of the identity in $\mathcal{G}_P$ as

$$U_\epsilon = \{g \in \mathcal{G}^{fr} \mid \|d^0 g\|_{y^{\lambda+\frac{1}{p}}L^p} \leq \epsilon,\ \|\nabla_0 d^0 g\|_{y^{\lambda+\frac{1}{p}-1}L^p} \leq \epsilon,\ |g(p_0)-1| \leq \epsilon\}. \tag{1.54}$$

This topology is independent of the base point $p_0$.

With the previous lemma, we can establish a Lie group structure on $\mathcal{G}^{fr}$:

**Corollary 1.6.6.** *(1)* $\mathcal{G}_{p,\lambda}^{fr}$ *is a Lie group with Lie algebra*

$$Lie(\mathcal{G}_{p,\lambda}^{fr}) = y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\mathfrak{g}_P).$$

*(2)* $\mathcal{G}_{p,\lambda}^{fr}$ *acts smoothly on* $C_{p,\lambda}^{fr}$.

*Proof.* This result follows immediately from Lemma 1.6.4 and Lemma 1.6.5. □

Now, we have the following proposition of the framed configuration space $C_{p,\lambda}^{fr}$ and the framed gauge group $\mathcal{G}_{p,\lambda}^{fr}$:

**Proposition 1.6.7.** *For any* $(A,\Phi) \in C_{p,\lambda}^{fr}$*, we have* $KW(A,\Phi) \in y^{\lambda+\frac{1}{p}-1}L^p(\Omega^2 \oplus \Omega^0)$, *Proof.* By the definition of $C^{fr}_{p,\lambda}$, there exists $(a,b) \in y^{\lambda+\frac{1}{p}}H^{1,p}_0(\Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P))$ such that $(A,\Phi) = (A_0,\Phi_0) + (a,b)$.

By Proposition 1.2.9, we have $KW(A,\Phi) = KW(A_0,\Phi_0) + \mathcal{L}^1(a,b) + \{(a,b),\ (a,b)\}$. Here $\{(a,b),\ (a,b)\}$ is a quadratic term. By theorem 1.5.4, we have $\mathcal{L}^1(a,b) \in y^{\lambda+\frac{1}{p}-1}L^p(\Omega^2\times\Omega^0)$. By the embedding $y^{\lambda+\frac{1}{p}}H^{1,p}_0 \hookrightarrow y^{\lambda+\frac{1}{p}-1}L^p_1 \hookrightarrow y^{\lambda+\frac{1}{p}-1}L^{2p}$, we have $\{(a,b),(a,b)\} \in y^{\lambda+\frac{1}{p}-1}L^p$. □

Now we will study the behavior of the gauge group (1.53) over the cylindrical end. We have the following proposition which describes the limit behavior of the group $\mathcal{G}^{fr}_{p,\lambda}$. We need the hypothesis that $\rho$ is acyclic. Let $(A_\rho,\Phi_\rho)$ be the flat $SL(2;\mathbb{C})$ connection associate to $\rho$.

Recall that $d^0_{(A,\Phi)}(\xi) = (d_A\xi, [\Phi,\xi])$ and $\rho$ acyclic implies Ker $d^0_{(A_\rho,\Phi_\rho)} = 0$ and the connection $d_{A_\rho}$ itself may still be a reducible $SU(2)$ connection.

We have the following lemma over the cylindrical end:

**Lemma 1.6.8.** *Suppose $X$ is a manifold with boundary and cylindrical end which is identified with $Y\times(0,+\infty)$, for $(A_0,\Phi_0)$ a reference connection which $L^p_1$ converges to $(A_\rho,\Phi_\rho)$ over the cylindrical end for $p>2$, then for $T$ is large enough, we have*

*(1) $d^0_{(A_0,\Phi_0)} : L^p_2(Y\times(T-1,T+1)) \to L^p_1(Y\times(T-1,T+1))$ is injective,*

*(2) $\|\xi\|_{L^p_2(Y\times(T-1,T+1))} \le C\|d^0_{(A_0,\Phi_0)}\xi\|_{L^p_1(Y\times(T-1,T+1))}$.*

*Proof.* For convenience, during the proof, we write $L^p_k$ short for $L^p_k(Y\times(T-1,T+1))$.

(1) Denote $(a,b) = (A_\rho,\Phi_\rho) - (A_0,\Phi_0)$, then for any $\xi \in L^p_2$, we have

$$\|d^0_{(A_\rho,\Phi_\rho)}\xi - d^0_{(A_0,\Phi_0)}\xi\|_{L^p_1} \le C(\|[a,\xi]\|_{L^p_1} + \|[b,\xi]\|_{L^p_1}) \le C(\|a\|_{L^p_1} + \|b\|_{L^p_1})\|\xi\|_{L^p_2}.$$

If $\xi \in$ Ker $d^0_{(A_0,\Phi_0)}$, we obtain

$$\|d^0_{(A_\rho,\Phi_\rho)}\xi\|_{L^p_k} \le C(\|a\|_{L^p_1} + \|b\|_{L^p_1})\|\xi\|_{L^p_2}.$$

For $T$ is large enough, $C(\|a\|_{L^p_1} + \|b\|_{L^p_1})$ is smaller than the operator norm of $d^0_{(A_0,\Phi_0)}$, which implies $\xi = 0$.

(2) If the inequality is not true, then there exists a sequence $\{\xi_n\}$ with

$$\|\xi_n\|_{L^p_1} = 1,\ \lim_{n\to\infty}\|d^0_{(A_0,\Phi_0)}\xi_n\|_{L^p_1} = 0,$$

which implies $\|\xi_n\|_{L^p_2}$ is bounded.

Then, $\xi_n$ weak converges to $\xi_\infty$ in $L^p_2$ and strongly converges to $\xi_\infty$ in $L^p_1$, which implies $\|d^0_{(A_0,\Phi_0)}\xi_\infty\|_{L^p_1} = 0$ and $\|\xi_\infty\|_{L^p_1} = 1$. As Ker $d^0_{(A_0,\Phi_0)} = 0$, we have $\xi_\infty = 0$, contradicting $\|\xi_\infty\|_{L^p_1} = 1$.

□

We have the following corollary:

**Corollary 1.6.9.** *If over the cylindrical end,* $(A_0, \Phi_0)$ *converges in* $L^p_1$ *norm to* $(A_\rho, \Phi_\rho)$ *and* $(A_\rho, \Phi_\rho)$ *is an irreducible flat* $SL(2;\mathbb{C})$ *connection, then* Ker $d^0_{(A_0,\Phi_0)} = 0$.

*Proof.* As before, we denote $d^0 := d^0_{(A_0,\Phi_0)}$. By Kato inequality, we know for $\xi \in \Omega^0$, we have the pointwise estimate

$$d|\xi| \leq |d_{A_0}\xi| \leq |d^0(\xi)|.$$

Therefore, $\xi \in \text{Ker} d^0$ implies $|\xi|$ is a constant. In addition, by Lemma 1.6.8, we know $\xi = 0$ over $Y \times (T-1, T+1)$ when $T$ is large enough, therefore, we have $\xi$ is identically zero. □

**Proposition 1.6.10.** *If the limiting connection* $\rho$ *is irreducible, then for* $T$ *is large enough, there is a constant* $C$ *such that for any section* $\xi \in \mathfrak{g}_P$*, with* $d^0\xi \in L^p(Y \times [T, +\infty))$ *and* $\nabla_0 d^0\xi \in L^p(Y \times [T, +\infty))$*, we have*

*(1)* $|\xi| \to 0$ *at the cylindrical end and* $\sup|\xi| \leq C(\|d^0\xi\|_{L^p_1(Y\times[T,+\infty))})$,

*(2) Either* $|g(x) - 1| \to 0$ *or* $|g(x) + 1| \to 0$ *as* $x$ *tends to infinity in X.*

*Proof.* (1) For an integer $k > T + 1$, over a band $B_k := Y \times (k-1, k+1)$, we have

$$\|\xi\|_{C^0(B_k)} \leq \|\xi\|_{L^p_2(B_k)} \leq C\|d^0\xi\|_{L^p_1(B_k)}.$$

The statement follows immediately.

(2) Denote by $g_k$ the restriction of $g$ to the band $B_k$. After identifying different bands with $Y^3 \times (-1, 1)$, we can consider $\{g_k\}$ a sequence of gauge transformation over $Y^3 \times (-1, 1)$ with $\|\nabla_0 d^0 g_k\|_{L^p}$, $\|\nabla_0 g_k\|_{L^p}$ converging to zero. As the pointwise norm of g is always 1, by Rellich lemma, $g_k$ strongly converges to $g_\infty$ in $L^p_1$ which implies $d^0 g_\infty = 0$. By our assumption, we have $g_\infty = \pm 1$. □

Now, we can define a framed quotient space as follows:

**Definition 1.6.11.** *We define $\mathcal{B}^{fr}_{p,\lambda}$ as the following weighted framed moduli space:*

$$\mathcal{B}^{fr}_{p,\lambda} = \mathcal{C}^{fr}_{p,\lambda}/\mathcal{G}^{fr}_{p,\lambda}.$$

In addition, we have the following definition of the moduli space:

**Definition 1.6.12.** *The framed moduli space $\mathcal{M}^{fr,\rho}_{p,\lambda}(X)$ is defined as follows:*

$$\mathcal{M}^{fr,\rho}_{p,\lambda}(X) = \{(A,\Phi) \in \mathcal{C}^{fr}_{p,\lambda} | KW(A,\Phi) = 0\}/\mathcal{G}^{fr}_{p,\lambda}. \tag{1.55}$$

We have the following basic properties of the framed moduli space:

**Proposition 1.6.13.** *(1) For any $(A,\Phi) \in \mathcal{M}^{fr,\rho}_{p,\lambda}$ satisfies the Nahm pole boundary condition.*

*(2) Any $(A,\Phi) \in \mathcal{M}^{fr,\rho}_{p,\lambda}$ converges to $\rho$ in $L^p_1$ norm.*

*Proof.* (1) is an immediate consequences of Proposition 1.6.2. (2) is a consequence of the definition of $\mathcal{C}^{fr}_{p,\lambda}$. □

**Slicing Theorem**

Now we study the local properties of the moduli space and we will assign a suitable norm to the Kuranishi complex (1.7).

Define $(\Omega^0, \lambda, k, p)$ as follows:

$$(\Omega^0, \lambda, k, p) := y^\lambda H^{k,p}_0(\Omega^0(\mathfrak{g}_P)).$$

Here the notation $y^\lambda H^{k,p}_0(\Omega^0(\mathfrak{g}_P))$ means the completion of the smooth sections of $\Omega^0(\mathfrak{g}_P)$ in the norm $y^\lambda H^{k,p}_0$. Similarly, we define

$$(\Omega^1 \times \Omega^1, \lambda, k, p) := y^\lambda H^{k,p}_0(\Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P))$$

and

$$(\Omega^2 \times \Omega^0, \lambda, k, p) := y^\lambda H^{k,p}_0(\Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)).$$

Now we rewrite the Kuranishi complex (1.7) at the point $(A_0, \Phi_0)$ with respect to the new norm as follows:

$$0 \to (\Omega^0, \lambda+1+\frac{1}{p}, 2, p) \xrightarrow{d^0_{(A_0,\Phi_0)}} (\Omega^1\times\Omega^1, \lambda+\frac{1}{p}, 1, p) \xrightarrow{\mathcal{L}_{(A_0,\Phi_0)}} (\Omega^2\times\Omega^0, \lambda+\frac{1}{p}-1, 0, p) \to 0, \tag{1.56}$$

Here we only considered $\lambda \in [1 - \frac{1}{p}, 1)$.

We have the following Proposition for the operator $d^0_{(A_0,\Phi_0)}$:

**Proposition 1.6.14.** *The operator* $d^0_{(A_0,\Phi_0)}$*:*

$$d^0_{(A_0,\Phi_0)} : y^{\lambda+1+\frac{1}{p}} H_0^{2,p}(\Omega^0(\mathfrak{g}_P)) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(\Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P))$$

*is a closed operator.*

*Proof.* see Appendix 2. □

**Corollary 1.6.15.** $y^{\lambda+\frac{1}{p}} H_0^{2,p}(\Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P)) = \mathrm{Im}\ d^0_{(A_0,\Phi_0)} \oplus (\mathrm{Ker}\ d^{0,\star}_{(A_0,\Phi_0)} \cap y^{\lambda+\frac{1}{p}} H_0^{1,p})$

*Proof.* Let $x = (x_1, x_2) \in \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P)$, by definition of $d^0_{(A_0,\Phi_0)}$, we have

$$\langle d^0_{(A_0,\Phi_0)} \xi, x\rangle = \langle d_{A_0}\xi, x_1\rangle + \langle [\Phi_0, \xi], x_2\rangle$$

where $\langle\ ,\ \rangle$ means the $L^2$ inner product.

Integrating by parts, we have

$$\langle d_{A_0}\xi, x_1\rangle = \langle \xi, d^\star_{A_0} x_1\rangle - \int_{\partial X} tr(\xi \wedge \star x_1).$$

As $\xi \in y^{\lambda+\frac{1}{p}} H_0^{2,p}(\Omega^0(\mathfrak{g}_P))$ and $x_1 \in y^{\lambda+\frac{1}{p}} H_0^{1,p}(\Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P))$, we have $x_1|_{\partial X} = 0$ and $\xi|_{\partial X} = 0$. Therefore,

$$\langle d^0_{(A_0,\Phi_0)} \xi, x\rangle = \langle \xi, d^{0,\star}_{(A_0,\Phi_0)} x\rangle.$$

Suppose $x \in \mathrm{Coker}\ d^0_{(A_0,\Phi_0)}$, then for $\forall \xi \in y^{\lambda+\frac{1}{p}+1} H_0^{2,p}$, we obtain $\langle d^0_{(A_0,\Phi_0)}\xi, x\rangle = 0$. As $\lambda > -1$, integrating by parts, we have $\langle \xi, d^{0,\star}_{(A_0,\Phi_0)} x\rangle = 0$. Thus $d^{0,\star}_{(A_0,\Phi_0)} x = 0$. Combining this with Proposition 1.6.14, we finish the proof.

□

Fixe a reference connection pair $(A_0, \Phi_0) \in \mathcal{C}^{fr}_{p,\lambda}$ and $\epsilon > 0$. We set:

$$T_{(A,\Phi),\epsilon} := \{(a, b) \in \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P) \mid d^{0,\star}_{(A,\Phi)}(a, b) = 0, \|(a, b)\|_{y^{\lambda+\frac{1}{p}} H_0^{1,p}} < \epsilon\}. \tag{1.57}$$

Thus we have a natural map $p : T_{(A,\Phi),\epsilon} \to \mathcal{B}^{fr}_{p,\lambda}$, which is induced by the inclusion of $T_{(A,\Phi),\epsilon}$ into $\mathcal{C}^{fr}_{p,\lambda}$ composed with quotienting by the gauge group $\mathcal{G}^{fr}_{p,\lambda}$.

We have the following slicing theorem for the moduli space $\mathcal{B}^{fr}_{p,\lambda}$.

**Theorem 1.6.16.** *Given a point* $(A,\Phi)\in C^{fr}_{p,\lambda}$*, denote by* $[(A,\Phi)]\in\mathcal{B}^{fr}_{p,\lambda}$ *the equivalence class under the projection map. For small* $\epsilon>0$*,*

*(1) if* $(A,\Phi)$ *is irreducible, then* $T_{(A,\Phi),\epsilon}$ *is a homeomorphism to a neighborhood of* $[(A,\Phi)]$ *in* $\mathcal{B}^{fr}_{p,\lambda}$*.*

*(2) if* $(A,\Phi)$ *is reducible, then* $T_{(A,\Phi),\epsilon}/\Gamma_{(A,\Phi)}$ *is a homeomorphism to a neighborhood of* $[(A,\Phi)]$ *in* $\mathcal{B}^{fr}_{p,\lambda}$*.*

*Proof.* Consider the map

$$\begin{aligned} &S: T_{(A,\Phi),\epsilon}\times\mathcal{G}_P/\{\pm1\}\to C_P,\\ &S(A+a,\Phi+b,g)=g(A+a,\Phi+b). \end{aligned}\tag{1.58}$$

The map has derivative at $a=0,b=0,g=1$ as:

$$\begin{aligned} DS:&\text{Ker } d^{0,\star}_{(A,\Phi)}\times\Omega^0(\mathfrak{g}_P)\to\Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P),\\ &(a,b,\xi)\to(a,b)+d^0_{(A,\Phi)}(\xi). \end{aligned}\tag{1.59}$$

By Corollary 1.6.15, we know $DS$ is always surjective.

(1) If $(A,\Phi)$ is irreducible, then $DS$ is injective, by the implicit function theorem, we know for $\epsilon$ small enough, $S$ is a homeomorphism.

(2) If $(A,\Phi)$ is reducible, then $DS$ has kernel $H^0_{(A,\Phi)}$. Let $H^{0\perp}_{(A,\Phi)}$ be the orthogonal of $H^0_{(A,\Phi)}$ with respect to the $L^2$ inner product. Then this time the restriction map

$$S: T_{(A,\Phi),\epsilon}\times\exp(H^{0\perp}_{(A,\Phi)})/\{\pm1\}\to C_P$$

is a local diffeomorphism. In addition, the multiplication map $\Gamma_{(A,\Phi)}\times\exp(H^{0\perp}_{(A,\Phi)})\to\mathcal{G}_P$ at the identity will have derivative 1. Thus for $g\in\mathcal{G}_P$ close to 1, there exist $l\in\Gamma_{(A,\Phi)}$ and $m\in\exp(H^{0\perp}_{(A,\Phi)})$ such that $g=ml$ and the splitting is unique. Therefore, we get a homeomorphism from $T_{(A,\Phi),\epsilon}/\Gamma_{(A,\Phi)}$ to a neighborhood of $[(A,\Phi)]$ in $\mathcal{B}^{fr}_{p,\lambda}$.

□

**Kuranishi Model**

Given $(A_0,\Phi_0)$ a solution to the Kapustin-Witten equations, all the other solutions within the slice $T_{(A,\Phi),\epsilon}$ are given by the set $Z(\Psi)$ of zeros of the map

$$\begin{aligned} &T_{(A,\Phi),\epsilon}\xrightarrow{\Psi} y^{\lambda+\frac{1}{p}-1}L^p(\Omega^2(\mathfrak{g}_P)\times\Omega^0(\mathfrak{g}_P)),\\ &\Psi(a,b)=KW(A_0+a,\Phi_0+b)=\mathcal{L}^1(a,b)+\{(a,b),(a,b)\}. \end{aligned}\tag{1.60}$$

By Theorem 1.5.9, $D\Psi$ is a Fredholm operator and for the homology associated to the Kuranishi complex, we have the following identification:

$$\begin{aligned} H^2_{(A,\Phi)} &\cong Ker\mathcal{L}^\star \cap y^{\lambda+\frac{1}{p}-1}L^p, \\ H^1_{(A,\Phi)} &\cong Ker\mathcal{L} \cap (Kerd^{0,\star}_{(A_0,\Phi_0)} \cap y^{\lambda+\frac{1}{p}}H^{1,p}_0). \end{aligned} \tag{1.61}$$

Therefore, we have the following Kuranishi picture of the moduli space:

**Proposition 1.6.17.** *[18, Proposition 8] For any solution $(A, \Phi)$ with $KW(A, \Phi) = 0$, for $\epsilon$ sufficiently small, there is a map $\rho$ from a neighborhood of the origin in the harmonic space $H^1_{(A,\Phi)}$ to the harmonic space $H^2_{(A,\Phi)}$ such that if $(A, \Phi)$ is irreducible, a neighborhood of $[(A, \Phi)] \in \mathcal{M}^{fr}_{p,\lambda}$ is carried by a diffeomorphism onto*

$$Z(\rho) = \rho^{-1}(0) \subset H^1_{(A,\Phi)}$$

*and if $(A, \Phi)$ is reducible, then a neighborhood of $[(A, \Phi)]$ is modelled on*

$$Z(\rho)/\Gamma_{(A,\Phi)}.$$

## 1.7 Exponential Decay

In this section, we will prove the exponential decay over the cylindrical ends which is identified with $Y^3 \times (0, +\infty)$ with the convergence assumption. We denote $y$ as the coordinate of $(0, +\infty)$. As before, let $\mathrm{Vol}_Y$ be a given volume form of $Y$. We denote $\star_4$ by the 4-dimensional Hodge star operator with respect to the Volume form $\mathrm{Vol}_Y \wedge dy$ and denote $\star$ by the 3 dimension Hodge star operator with respect to $\mathrm{Vol}_Y$.

**Theorem 1.7.1.** *Let $(A, \Phi)$ be a solution to the Kapustin-Witten equations over $Y^3 \times (0, +\infty)$, and $Y_T := Y^3 \times \{T\} \in Y^3 \times (0, +\infty)$ is a slice. For a non-degenerate flat $SL(2;\mathbb{C})$ connection $(A_\rho, \phi_\rho)$ corresponding to the representation $\rho$.*

*Suppose for some $p > 2$, $\lim_{T\to+\infty} \|(A, \Phi) - (A_\rho, \phi_\rho)\|_{L^p_1(Y_T)} = 0$, then there exists a positive number $\delta$, such that $\|(A, \Phi) - (A_\rho, \phi_\rho)\|_{C^\infty(Y\times[T,+\infty))} \leq Ce^{-\delta T}$.*

We follow the ideas in [19].

Over slice $Y_t$, denote $(A, \Phi)|_{Y_t} = (A(t), \phi(t)+\phi_y(t)dy)$, denote $\gamma(t) := (A(t), \phi(t), \phi_y(t))$. Recall the gradient of the extended Chern-Simons function is denoted as $\nabla\mathrm{ECS}$, then the flow equations (1.21) can be rewrote as

$$\frac{d}{dy}\gamma(t) + \nabla\mathrm{ECS}(\gamma(t)) = 0. \tag{1.62}$$

**Definition 1.7.2.** *The analytic energy is defined as* $\mathcal{E}^{an}(t) := \int_{Y_t} |\frac{d}{dy}\gamma(t)|^2 + |\nabla \mathrm{ECS}(\gamma(t))|^2 d\mathrm{Vol}$.

Now we introduce some basic computation related to the analytic energy $\mathcal{E}^{an}(t)$. Let $\mathrm{ECS}(\rho)$ to be $\mathrm{ECS}(A_\rho, \phi_\rho, 0)$ and $\mathrm{ECS}(T)$ to be $\mathrm{ECS}(\gamma(T))$, then we have the following proposition:

**Proposition 1.7.3.** *Over* $Y \times (0, +\infty)$*, under the assumption of Theorem 1.7.1, denote* $(A, \Phi = \phi + \phi_y dy)$ *and denote* $J(T) := \int_T^{+\infty} \mathcal{E}^{an}(t)dt$*, then* $J(T) = 2(\mathrm{ECS}(T) - \mathrm{ECS}(\rho))$.

*Proof.* We have the following computations for the gradient flow equations (1.62):

$$\begin{aligned} 0 &= \int_{Y\times[T,+\infty)} |\frac{d}{dy}\gamma(t) + \nabla \mathrm{ECS}(\gamma(t))|^2 \\ &= \int_{Y\times[T,+\infty)} |\frac{d}{dy}\gamma(t)|^2 + |\nabla \mathrm{ECS}(\gamma(t))|^2 + 2\langle \frac{d}{dy}\gamma(t), \nabla \mathrm{ECS}(\gamma(t))\rangle. \end{aligned} \tag{1.63}$$

By definition of the gradient, we obtain

$$\int_{Y\times[T,+\infty)} 2\langle \frac{d}{dy}\gamma(t), \nabla \mathrm{ECS}(\gamma(t))\rangle = 2(\mathrm{ECS}(\rho) - \mathrm{ECS}(T)). \tag{1.64}$$

Therefore, $J(T) = 2(\mathrm{ECS}(T) - \mathrm{ECS}(\rho))$. □

Let $(A_t, \Phi_t = \phi_t + (\phi_y)_t dy)$ be the restriction of the solution $(A, \Phi)$ to the slice $Y_t$. Take $(a_t, b_t, c_t) = (A_t, \Phi_t, (\phi_y)_t) - (A_\rho, \phi_\rho, 0)$, denote $\mathcal{H}_\rho(a_t, b_t, c_t) := \mathcal{H}_{(A_\rho,\phi_\rho,0)}(a_t, b_t, 0, c_t)$ and $\mathcal{EH}_\rho := \mathcal{EH}_{(A_\rho,\phi_\rho,0)}$. Then we have the following lemma:

**Lemma 1.7.4.** *Under the assumption of Theorem 1.7.1, then for t large enough, there exist positive constant* $C_1$*,* $C_2$ *such that we have the following estimates:*

$$\mathcal{E}^{an}(t) \geq C_1 \|\mathcal{EH}_\rho(a_t, b_t, c_t)\|^2_{L^2(Y_t)}, \tag{1.65}$$

$$\mathrm{ECS}(A_\rho + a_t, \phi_\rho + b_t, c_t) - \mathrm{ECS}(A_\rho, \phi_\rho, 0) \leq C_2 \|\mathcal{EH}_\rho(a_t, b_t, c_t)\|^2_{L^2(Y_t)}, \tag{1.66}$$

where $C_2$ is a number depends on the smallest absolute eigenvalue of $\mathcal{EH}_\rho$.

*Proof.* As we assume $\rho$ is nondegenerate, by Proposition 1.4.10, we have the following estimate $\|(a_t, b_t, c_t)\|_{L^2_1(Y_t)} \leq C \|\mathcal{EH}_\rho(a_t, b_t, c_t)\|_{L^2(Y_t)}$ over the slice $Y_t$.

In addition, by the convergence assumption, $(a_t, b_t, c_t)$ will have small $L^p_1$ norm when $t$ is large enough, by Proposition 1.3.3, in each slice $Y_t$, we can make $(A, \Phi)$ in the Kapustin-Witten gauge relative to $(A_\rho, \Phi_\rho)$:

$$\mathcal{L}^{gf}_{(A_\rho,\Phi_\rho)}(a_t, b_t) = 0.$$

To be explicit, we have

$$d^{\star}_{A_\rho} a_t - \star[\Phi_\rho, \star b_t] = 0. \tag{1.67}$$

Under this gauge, we have

$$\mathcal{H}_\rho(a_t, b_t, c_t) = \mathcal{E}\mathcal{H}_\rho(a_t, b_t, c_t). \tag{1.68}$$

Now, we have enough preparation for proving the estimate.

$$\begin{aligned}\mathcal{E}^{an}(t) &\geq \int_{Y_t} |\nabla \mathrm{ECS}(A_t, \phi_t, (\phi_y)_t)|^2 \\ &\geq \int_{Y_t} |\mathcal{H}_\rho(a_t, b_t, c_t) + \{(a_t, b_t, c_t)^2\}|^2 \\ &\geq \int_{Y_t} |\mathcal{H}_\rho(a_t, b_t, c_t)|^2 + |\{(a_t, b_t, c_t)^2\}|^2 + 2\langle \mathcal{H}_\rho(a_t, b_t, c_t), \{(a_t, b_t, c_t)^2\}\rangle.\end{aligned} \tag{1.69}$$

Over 3 dimensional manifold, we have the Sobolev embedding $L^2_1(Y_t) \to L^r(Y_t)$ for $r \leq 6$.

Then we have

$$\|\{(a_t, b_t, c_t)^2\}^2\|^2_{L^2(Y_t)} \leq C\|(a_t, b_t, c_t)\|^4_{L^4(Y_t)} \leq C\|\mathcal{E}\mathcal{H}_\rho(a_t, b_t, c_t)\|^4_{L^2(Y_t)},$$

$$\int_{Y_t} \langle \mathcal{H}_\rho(a_t, b_t, c_t), \{(a_t, b_t, c_t)^2\}\rangle \leq C\|\mathcal{E}\mathcal{H}_\rho(a_t, b_t, c_t)\|^3_{L^2(Y_t)}.$$

By the convergence assumption, we know $\lim_{t\to+\infty} \|\mathcal{E}\mathcal{H}_\rho(a_t, b_t, c_t)\|_{L^2(Y_t)} = 0$. Thus, for inequality (1.69), $\|\{(a_t, b_t, c_t)^2\}^2\|^2_{L^2(Y_t)}$ and $\int_{Y_t} \langle \mathcal{H}_\rho(a_t, b_t, c_t), \{(a_t, b_t, c_t)^2\}\rangle$ can be absorb by the first term and by choosing $t$ large enough, we get the estimate we want.

For the statement (2), under the gauge fixing condition (1.67), we have the following estimate

$$\begin{aligned}&\int_{Y_t} \langle (a_t, b_t, c_t), \mathcal{H}_\rho(a_t, b_t, c_t)\rangle \\ =&\int_{Y_t} \langle (a_t, b_t, c_t), \mathcal{E}\mathcal{H}_\rho(a_t, b_t, c_t)\rangle \\ \leq&\frac{1}{\delta}\|\mathcal{E}\mathcal{H}_\rho(a_t, b_t, c_t)\|^2_{L^2(Y_t)},\end{aligned} \tag{1.70}$$

where $\delta$ is smallest absolute eigenvalue of the operator $\mathcal{EH}_\rho$ and by the non-degenerate assumption, $\mathrm{Ker}\mathcal{EH}_\rho = 0$ and $\delta$ is bounded blow away from 0.

Thus, we have

$$\begin{aligned}
&\mathrm{ECS}(A_\rho + a_t, \phi_\rho + b_t, c_t) - \mathrm{ECS}(A_\rho, \phi_\rho, 0)\\
\leq& -\frac{1}{2}\int_{Y_t} \langle (a_t, b_t, c_t), \mathcal{H}_\rho(a_t, b_t, c_t)\rangle - \int_{Y_t} \{(a, b, c)^3\}\\
\leq& C\|\mathcal{EH}_\rho(a_t, b_t, c_t)\|^2_{L^2(Y_t)} + C\|\mathcal{EH}_\rho(a_t, b_t, c_t)\|^3_{L^2(Y_t)}\\
\leq& C_2\|\mathcal{EH}_\rho(a_t, b_t, c_t)\|^2_{L^2(Y_t)}.
\end{aligned} \tag{1.71}$$

□

Now we obtain the following proposition:

**Proposition 1.7.5.** *With the assumption above, if $\mathcal{E}^{an}(t)$ is bounded, then there exists a constant $C$ such that*

$$J(T) \leq Ce^{-\delta t}.$$

*Proof.* By Lemma 1.7.4, we have the following:

$$\begin{aligned}
J(t) &= \mathrm{ECS}(A_\rho + a_t, \phi_\rho + b_t, c_t) - \mathrm{ECS}(A_\rho, \phi_\rho, 0)\\
&\leq C_2\|\mathcal{EH}_\rho(a_t, b_t, c_t)\|^2_{L^2(Y_t)}\\
&\leq \frac{C_2}{C_1}\mathcal{E}^{an}(t)\\
&\leq -\frac{C_2}{C_1}\frac{d}{dt}J(t).
\end{aligned} \tag{1.72}$$

Thus, take $\delta = \frac{C_1}{C_2}$, we have:

$$\delta J(t) + \frac{dJ(t)}{dt} \leq 0.$$

From here we get that $J(t) \leq Ce^{-\delta t}$. □

Using these corollaries, we can give the following estimate of the decay of solutions.

**Proposition 1.7.6.** *For all $T$ is large enough that we have $\|(A_t, \Phi_t) - (A_\rho, \Phi_\rho)\|^2_{L^2_k(Y\times[T,+\infty))} \leq Ce^{-\delta T}$.*

*Proof.* Fixing a Kapustin-Witten gauge for $(a_t, b_t, c_t) := (A_t, \phi_t, (\phi_y)_t) - (A_\rho, \phi_\rho, 0)$. By the non-degenerate assumption, we have $\|(a_t, b_t, c_t)\|_{L^2_1(Y_t)} \leq C\|\mathcal{EH}_\rho(a_t, b_t, c_t)\|_{L^2(Y_t)}$.

In addition, by Lemma 1.7.4, for $T$ is large enough, we have

$$\|\mathcal{E}\mathcal{H}_\rho(a_t, b_t)\|^2_{L^2(Y_t)} \leq C\mathcal{E}^{an}(t).$$

Therefore, we compute

$$\begin{aligned}\|(A,\Phi)-(A_\rho,\Phi_\rho)\|_{L^2_1(Y\times[T,+\infty))} =& \int_T^{+\infty} \|(a_t,b_t)\|_{L^2_1(Y_t)} dt\\ \leq& C\int_T^{+\infty} \|\mathcal{E}\mathcal{H}_\rho(a_t,b_t)\|_{L^2(Y_t)} dt\\ \leq& C\int_T^{+\infty} \mathcal{E}^{an}(t)dt\\ \leq& CJ(t).\end{aligned} \tag{1.73}$$

By the exponential decay of $J(t)$, we proved the result for k=1. Take bootstrapping method, we get that the $L^2_k$ norm exponentially decays. □

*Proof of Theorem 1.7.1*: We only need to show that for every integer $k$, we have $\|(a_t, b_t)\|_{C^k} \leq Ce^{-\delta t}$. By the Sobolev embedding for $L^2_{k'}$ and $C^k$, the result follows immediately. □

## 1.8 Constructing Solutions

In this section, we will prove the gluing theorem for the Kapustin-Witten equations with Nahm pole boundary condition.

For $i = 1, 2$, consider $X_i$ to be a 4-manifold with boundary $Z_i$ and infinite cylindrical end identified with $Y_i \times (0, +\infty) \subset X_i$, let $P_i$ to denote a $SU(2)$ bundle and $(A_i, \Phi_i) \in C_{P_i}$ be a solutions to the Kapustin-Witten equations over bundle $P_i$ which approach to a flat connection $\rho_i$ and satisfies the Nahm pole boundary condition on the boundary $Z_i$.

If $Y_1 = Y_2$, we can define a new family of 4-manifolds $X^{\sharp T}$. To be precise, we fix an isometry between $Y_1$ and $Y_2$, we first delete the infinite portions $Y_1 \times [2T, +\infty)$ $Y_2\times$, $[2T, +\infty)$ from the two ends, and then identify $(y, t) \in Y_1 \times (T, 2T)$ with $(y, 2T - t) \in Y_2 \times (T, 2T)$. This is in Figure 1.3 and Figure 1.4:

In addition, if the limit flat connections coincide, we denote $\rho = \rho_1 = \rho_2$, we can fix an identification of these flat bundles and get a new bundle $P^{\sharp T}$ with a natural connection $(A^\sharp, \Phi^\sharp)$, which we will explicitly define in subsection 7.1.

Now we restate our theorem as follows:

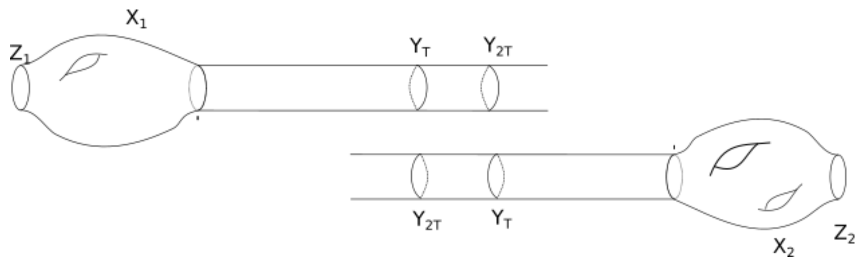


Figure 1.3: Two cylindrical-end manifold $X_1$ and $X_2$ with boundary

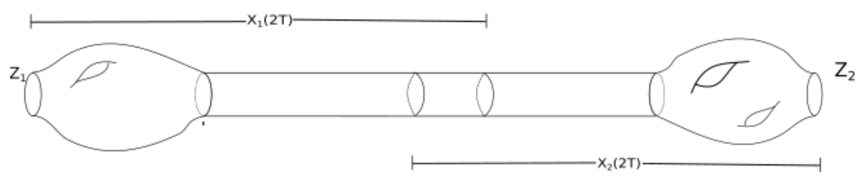


Figure 1.4: $X_1$, $X_2$ glued together to form $X^{\sharp T}$

**Theorem 1.8.1.** *Under the hypotheses above, if*

*(a)* $\lim_{T\to+\infty} \|(A_i, \Phi_i) - (A_{\rho_i}, \Phi_{\rho_i})\|_{L_1^{p_0}(Y_i\times\{T\})} = 0$ *for some* $p_0 > 2$,

*(b)* $\rho$ *is an acyclic* $SL(2;\mathbb{C})$ *flat connection,*

*then for* $p \geq 2$ *and* $\lambda \in [1-\frac{1}{p}, 1)$*, we have:*

*(1) for some constant* $\delta$*, there exists a* $y^{\lambda+\frac{1}{p}}H_0^{1,p}$ *pair* $(a, b) \in \Omega^1_{X^{\sharp T}}(\mathfrak{g}_P) \times \Omega^1_{X^{\sharp T}}(\mathfrak{g}_P)$ *with*

$$\|(a, b)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p} \leq Ce^{-\delta T},$$

*(2) there exists an obstruction class* $h \in H^2_{(A_1,\Phi_1)}(X_1) \times H^2_{(A_2,\Phi_2)}(X_2)$ *such that* $h = 0$ *if and only if* $(A^\sharp + a, \Phi^\sharp + b)$ *is a solution to the Kapustin-Witten equations (4.1).*

We break the proof of this theorem into several parts.

**Approximate Solutions**

Denote by $(a_i, \phi_i) := (A_i, \Phi_i) - (A_{\rho_i}, \Phi_{\rho_i})$, the difference between our solution and the limit flat connections.

Define a new pair $(A_i', \Phi_i') = (A_{\rho_i}, \Phi_{\rho_i}) + \chi(t)(a_i, \phi_i)$, here $\chi(t)$ is a cut off function which equals 0 on the complement of $Y_i \times (T+1, +\infty)$ and 1 on $X_i(T) := X_i \backslash Y_i \times (T, +\infty)$. From this construction, we know that $KW(A_i', \Phi_i')$ is supported on $Y \times (T, T+1)$.

As $(A_1', \Phi_1')$ and $(A_2', \Phi_2')$ agree on the end, we can glue then together to get an approximate solution $(A^\sharp, \Phi^\sharp)$ on $X^{\sharp T}$, and we denote the new bundle as $P^\sharp$. Using the above process, we can define a map:

$$\begin{aligned} &I : C_{P_1} \times C_{P_2} \to C_{P^\sharp} \\ &I((A_1, \Phi_1), (A_2, \Phi_2)) := (A^\sharp, \Phi^\sharp), \end{aligned} \tag{1.74}$$

and this map depends on choice of $T$ and the cut-off function we choose.

We have the following estimate for the approximate solution:

**Proposition 1.8.2.** $\|KW(A^\sharp, \Phi^\sharp)\|_{y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T})} \leq Ce^{-\delta T}$.

*Proof.* By construction, we know $KW(A^\sharp, \Phi^\sharp)$ is only supported on a compact subset of $X^{\sharp T}$ and in this area the weight function is bounded. Combining this with the exponential decay result: Theorem 1.7.1, we get the estimate we want. □

**Gluing Regular Points**

First, we assume that $H^2_{(A_1,\Phi_1)} = 0$ and $H^2_{(A_2,\Phi_2)} = 0$ and the limit flat connection is irreducible.

We use the previous notation from (1.17). Recall that $\mathcal{L}_{(A_i,\Phi_i)}$ is the linearization of the Kapustin-Witten equations, we denote $\mathcal{L}_i := \mathcal{L}_{(A_i,\Phi_i)}$. Recall $\mathcal{L}_i^{gf} := \mathcal{L}^{gf}_{(A_i,\Phi_i)}$ is the Kapustin-Witten gauge fixing operator (1.14). Now, we denote $\mathcal{D}_i := \mathcal{L}_i \oplus \mathcal{L}_i^{gf}$. By Theorem 1.5.9, we get a Fredholm operator over $X_i$:

$$\mathcal{D}_i : y^{\lambda+\frac{1}{p}} H_0^{1,p}(X_i) \to y^{\lambda+\frac{1}{p}-1} L^p(X_i).$$

By assumption, we know $\mathcal{D}_i$ is surjective, then there exists a right inverse

$$Q_i : y^{\lambda+\frac{1}{p}-1} L^p(X_i) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(X_i), \tag{1.75}$$

such that $\mathcal{D}_i Q_i = Id$. Therefore, after restricting the domain of $Q_i$ to the image of $\mathcal{L}_i$, we get a right inverse for $\mathcal{L}_i$ and for simplicity, we still denote the right inverse as $Q_i$ and we obtain $\mathcal{L}_i Q_i = Id$.

Take $\phi_i$ to be a cut off function supported in $X_i(2T)$, with $\phi_i(x) = 1$ on $X_i(T)$ and $\phi_1 + \phi_2 = 1$ on $X^{\sharp T}$. The graph of cut-off function $\phi_1$ is in the following Figure 1.5:

By definition, we chose $\phi_i$ with the estimate $\|\nabla \phi_i\|_{L^\infty(X_i)} \leq \epsilon(T) \leq \frac{C}{T}$.

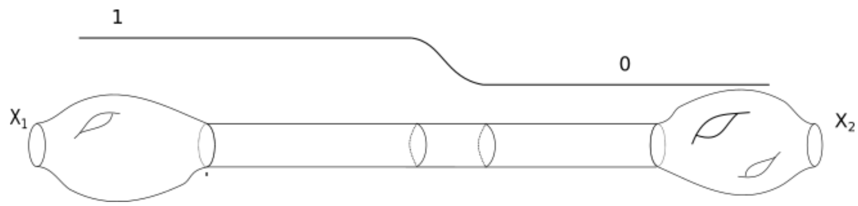


Figure 1.5: The graph of cut-off function $\phi_1$

Take $\xi \in y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$, denote by $\xi_i$ the restriction of $\xi$ on $X_i(2T)$, then we define a new approximate inverse operator $\hat{Q}(\xi) := \phi_1 Q_1(\xi_1) + \phi_2 Q_2(\xi_2)$, which can be written as

$$\hat{Q} : y^{\lambda+\frac{1}{p}-1}L^p(X_i) \to y^{\lambda+\frac{1}{p}}H_0^{1,p}(X_i). \tag{1.76}$$

Denoting $\mathcal{L}^{\sharp T} := \mathcal{L}_{(A^\sharp,\Phi^\sharp)}$ as follows:

$$\mathcal{L}^{\sharp T} : y^{\lambda+\frac{1}{p}}H_0^{1,p}(X_i) \to y^{\lambda+\frac{1}{p}-1}L^p(X_i).$$

After these preparations, we have the following relationship between these two operators:

**Lemma 1.8.3.** *For $\mathcal{L}^{\sharp T}$, $\hat{Q}$ as above, and for $\forall \xi \in y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$, we have*

$$\|\mathcal{L}^{\sharp T}\hat{Q}(\xi) - \xi\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq \epsilon(T)\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}.$$

*Proof.* For $\mathcal{L}^{\sharp T}\hat{Q}(\xi)$, by definition, we have the following computation:

$$\begin{aligned}\mathcal{L}^{\sharp T}\hat{Q}(\xi) &= \mathcal{L}^{\sharp T}(\phi_1 Q_1(\xi_1) + \phi_2 Q_2(\xi_2))\\ &= \nabla\phi_1 \star Q_1(\xi_1) + \nabla\phi_2 \star Q_2(\xi_2) + \phi_1\mathcal{L}^{\sharp T}Q_1(\xi_1) + \phi_2\mathcal{L}^{\sharp T}Q_2(\xi_2)\\ &= \nabla\phi_1 \star Q_1(\xi_1) + \nabla\phi_2 \star Q_2(\xi_2) + \phi_1\mathcal{L}_1 Q_1(\xi_1) + \phi_2\mathcal{L}_2 Q_2(\xi_2)\\ &\quad + \phi_1(\mathcal{L}^{\sharp T} - \mathcal{L}_1)Q_1(\xi_1) + \phi_2(\mathcal{L}^{\sharp T} - \mathcal{L}_2)Q_2(\xi_2).\end{aligned} \tag{1.77}$$

For the term $\nabla\phi_1 \star Q_1(\xi_1) + \nabla\phi_2 \star Q_2(\xi_2)$, we know $\|\nabla\phi\|_{L^\infty} < \epsilon(T)$.

Therefore, we obtain

$$\|\nabla\phi_1 \star Q_1(\xi_1)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq \epsilon(T)\|Q_1(\xi_1)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}.$$

By Proposition 1.5.5, $y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T}) \subset y^{\lambda+\frac{1}{p}}L^p(X^{\sharp T})$, and by Proposition 1.5.6, we obtain $y^{\lambda+\frac{1}{p}}L^p(X^{\sharp T}) \subset y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$, therefore, $y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T}) \subset y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$. In addition, by (1.75), we know

$$\|Q_1(\xi_1)\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T})} \leq C\|\xi_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}.$$

Therefore, we obtain

$$\begin{aligned}\|\nabla\phi_1 \star Q_1(\xi_1)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} &\leq \epsilon(T)\|Q_1(\xi_1)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}\\ &\leq \epsilon(T)\|Q_1(\xi_1)\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T})}\\ &\leq \epsilon(T)C\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})},\end{aligned} \tag{1.78}$$

where the constant $C$ is independent of $T$. Similarily, we have the same estimate for $Q_2$:

$$\|\nabla\phi_2 \star Q_2(\xi_2)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq \epsilon(T)C\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}. \tag{1.79}$$

For the term $\phi_1(\mathcal{L}^{\sharp T} - \mathcal{L}_1)Q_1(\xi_1) + \phi_2(\mathcal{L}^{\sharp T} - \mathcal{L}_2)Q_2(\xi_2)$, by Theorem 1.7.1, we know that the operators $\mathcal{L}^{\sharp T} - \mathcal{L}_1$ and $\mathcal{L}^{\sharp T} - \mathcal{L}_2$ are order zero and the operator norm will exponentially decay as $T \to \infty$ . Therefore, we have

$$\|\phi_1(\mathcal{L}^{\sharp T} - \mathcal{L}_1)Q_1(\xi_1) + \phi_2(\mathcal{L}^{\sharp T} - \mathcal{L}_2)Q_2(\xi_2)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} < \epsilon(T)\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}. \tag{1.80}$$

For the remaining terms, we have

$$\phi_1\mathcal{L}_1Q_1(\xi_1) + \phi_2\mathcal{L}_2Q_2(\xi_2) = \phi_1\xi_1 + \phi_2\xi_2 = \xi. \tag{1.81}$$

Combining all the discussion above, we get the estimate we want.

□

**Proposition 1.8.4.** *There exists an operator* $Q^{\sharp T}$ *with*

$$Q^{\sharp T} : y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T}),$$

*such that* $\mathcal{L}^{\sharp T}Q^{\sharp T} = Id$. *In addition, there exists a constant* $C$ *independent of* $T$ *such that for* $\forall\xi \in y^{\lambda+\frac{1}{p}-1}L^p(X_i)$, *we have*

$$\|Q^{\sharp T}(\xi)\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T})} \leq C\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}.$$

*Proof.* Take $R(\xi) := \mathcal{L}^{\sharp T}\hat{Q}(\xi) - \xi$, by Proposition 1.8.3, we know when $T$ is large enough, the operator norm of $R$ will be very small. Therefore, $R + Id$ is invertible.

Take $Q^{\sharp T} := \hat{Q}(Id + R)^{-1}$ then by definition, we have that $Q^{\sharp T}$ is an operator from $y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$ to $y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T})$ and $\mathcal{L}^{\sharp T}Q^{\sharp T} = Id$.

The operator norm of $R + Id$ is less than three and the operator norm of $\hat{Q}$ is dominated by the operator norm of $Q_1$ plus the operator norm of $Q_2$. Thus, the operator norm of $Q^{\sharp T}$ is independent of $T$.

□

Given an approximate solution $(A^\sharp, \Phi^\sharp)$, for any connection $(A, \Phi)$, write

$$(A, \Phi) = (A^\sharp, \Phi^\sharp) + (a, b).$$

We hope to find suitable $(a, b)$ such that $KW(A, \Phi) = 0$.

By Proposition 1.2.9, we have the following quadratic expansion:

$$KW(A, \Phi) = KW(A^\sharp, \Phi^\sharp) + \mathcal{L}^{\sharp T}(a, b) + \{(a, b), (a, b)\}.$$

We will solve the equations

$$KW(A^\sharp, \Phi^\sharp) + \mathcal{L}^{\sharp T}(a, b) + \{(a, b), (a, b)\} = 0. \tag{1.82}$$

Take $\eta := -KW(A^\sharp, \Phi^\sharp)$ and replace $(a, b)$ by $Q^{\sharp T}(\alpha)$, then the quadratic expansion becomes

$$\eta = \alpha + \{Q^{\sharp T}(\alpha), Q^{\sharp T}(\alpha)\}. \tag{1.83}$$

Now our target is to solve this equation for some $\alpha \in y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T})$.

Take $S(\alpha) := \{Q^{\sharp T}(\alpha), Q^{\sharp T}(\alpha)\}$, we have the following proposition for the operator $S$:

**Proposition 1.8.5.** *For any $\lambda_0 \in [1 - \frac{1}{p}, 1)$, $S$ is an operator:*

$$S : y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T})$$

*satisfying $S(0) = 0$ and for two elements $\alpha, \beta \in L^p(X^{\sharp T})$, there exists a constant $k$ independent of $T$ such that*

$$\|S(\alpha) - S(\beta)\|_{y^{\lambda+\frac{1}{p}-1} L^p} \leq k(\|\alpha\|_{y^{\lambda+\frac{1}{p}-1} L^p} + \|\beta\|_{y^{\lambda+\frac{1}{p}-1} L^p})(\|\alpha - \beta\|_{y^{\lambda+\frac{1}{p}-1} L^p}).$$

*Proof.* First, we prove $S$ is the suitable operator. As $Q^{\sharp T} : y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(X^{\sharp T})$, using the Sobolev embedding $y^{\lambda+\frac{1}{p}} H_0^{1,p}(X^{\sharp T}) \hookrightarrow y^{\lambda+\frac{1}{p}-1} L_1^p(X^{\sharp T})$(Corollary 1.5.7), we can consider $Q^{\sharp T}$ as an operator from $y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T})$ to $y^{\lambda+\frac{1}{p}-1} L_1^p(X^{\sharp T})$.

Denote $\alpha = (\alpha_1, \alpha_2)$ and from the definition of $S$, we have

$$\begin{aligned} S(\alpha) &= \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_2)\} \\ &= \begin{pmatrix} Q^{\sharp T}(\alpha_1) \wedge Q^{\sharp T}(\alpha_1)) - Q^{\sharp T}(\alpha_2) \wedge Q^{\sharp T}(\alpha_2) + \star[Q^{\sharp T}(\alpha_1), Q^{\sharp T}(\alpha_2)] \\ - \star\,[Q^{\sharp T}(\alpha_1), \star Q^{\sharp T}(\alpha_2)] \end{pmatrix}. \end{aligned} \tag{1.84}$$

As the terms appearing in $S(\alpha)$ are quadratic terms, using the Hölder inequality for $\lambda \geq 1 - \frac{1}{p}$ such that $\|fg\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq \|f\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \|g\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})}$(Corollary 1.5.7), we have that $S$ is an operator

$$S : y^{\lambda+\frac{1}{p}} L^p(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}} L^p(X^{\sharp T}).$$

Now, we will show that $S$ has the desired estimate. Denote $\beta = (\beta_1, \beta_2)$, from the definition of $S$, we have the following computation:

$$\begin{aligned} & S(\alpha) - S(\beta) \\ =& \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_2)\} - \{(\beta_1, \beta_2), (\beta_1, \beta_2)\} \\ =& \begin{matrix} Q^{\sharp T}(\alpha_1) \wedge Q^{\sharp T}(\alpha_1) - Q^{\sharp T}(\alpha_2) \wedge Q^{\sharp T}(\alpha_2) + \star[Q^{\sharp T}(\alpha_1), Q^{\sharp T}(\alpha_2)] - \star[Q^{\sharp T}(\beta_1), Q^{\sharp T}(\beta_2)] \\ - \star\,[Q^{\sharp T}(\alpha_1), \star Q^{\sharp T}\alpha_2] + \star[Q^{\sharp T}(\beta_1), \star Q^{\sharp T}\beta_2] \end{matrix} \end{aligned} \tag{1.85}$$

Now we make estimates for each term appearing in $S(\alpha) - S(\beta)$.

$$\begin{aligned} & \|Q^{\sharp T}(\alpha_1) \wedge Q^{\sharp T}(\alpha_1) - Q^{\sharp T}(\beta_1) \wedge Q^{\sharp T}(\beta_1)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\ \leq& \|Q^{\sharp T}(\alpha_1) \wedge (Q^{\sharp T}(\alpha_1) - Q^{\sharp T}(\beta_1))\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\ & + \|(Q^{\sharp T}(\alpha_1) - Q^{\sharp T}(\beta_1)) \wedge Q^{\sharp T}(\beta_1)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\ \leq& \|Q^{\sharp T}(\alpha_1)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \|Q^{\sharp T}(\alpha_1) - Q^{\sharp T}(\beta_1)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \\ & + \|Q^{\sharp T}(\beta_1)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \|Q^{\sharp T}(\alpha_1) - Q^{\sharp T}(\beta_1)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \\ \leq& C\|\alpha_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \|\alpha_1 - \beta_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\ & + C\|\beta_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \|\alpha_1 - \beta_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\ \leq& C(\|\alpha_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} + \|\beta_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})})(\|\alpha_1 - \beta_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}). \end{aligned} \tag{1.86}$$

For another term, we have

$$\begin{aligned}
&\| \star [Q^{\sharp T}(\alpha_1), Q^{\sharp T}(\alpha_2)] - \star[Q^{\sharp T}(\beta_1), Q^{\sharp T}(\beta_2)]\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\
\leq &\|[Q^{\sharp T}(\alpha_1), Q^{\sharp T}(\alpha_2)] - [Q^{\sharp T}(\alpha_1), Q^{\sharp T}(\beta_2)] \\
&[Q^{\sharp T}(\alpha_1), Q^{\sharp T}(\beta_2)] - [Q^{\sharp T}(\beta_1), Q^{\sharp T}(\beta_2)]\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\
\leq &\|[Q^{\sharp T}(\alpha_1), Q^{\sharp T}(\alpha_2-\beta_2)]\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} + \|[Q^{\sharp T}(\alpha_1-\beta_1), Q^{\sharp T}(\beta_2)]\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\
\leq &C\|\alpha_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}\|\alpha_2-\beta_2\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\
&+ \|\beta_2\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}\|\alpha_1-\beta_1\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\
\leq &C(\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} + \|\beta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})})(\|\alpha-\beta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}).
\end{aligned} \tag{1.87}$$

Similarily, we have the following estimate:

$$\begin{aligned}
&\| -\star[Q^{\sharp T}(\alpha_1), \star Q^{\sharp T}(\alpha_2)] + \star[Q^{\sharp T}(\beta_1), \star Q^{\sharp T}(\beta_2)]\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\
\leq &C(\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} + \|\beta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})})(\|\alpha-\beta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}).
\end{aligned} \tag{1.88}$$

Combining the previous computations, we have the result we want. □

We have the following lemma about the operator $S$:

**Lemma 1.8.6.** *([20] Lemma 7.2.23) Let $B$ be a Banach space and let $\| \|_B$ be the norm on $B$. Let $S : B \to B$ be a smooth map on the Banach space $B$ with $S(0) = 0$ and $\|S\xi_1 - S\xi_2\|_B \leq k(\|\xi_1\|_B + \|\xi_2\|_B)(\|\xi_1 - \xi_2\|_B)$, for some constant $k > 0$ and all $\xi_1, \xi_2$ in $B$,then for each $\eta \in B$ with $\|\eta\|_B < \frac{1}{10k}$, there exists a unique $\xi$ with $\|\xi\|_B \leq \frac{1}{5k}$ such that*

$$\xi + S(\xi) = \eta.$$

We now can complete the proof of Theorem 1.1.

*Proof of Theorem 1.1*: Recall we hope to solve the equation (1.83), which is

$$\eta = \alpha + S(\alpha).$$

By Proposition 1.8.5, in Lemma 1.8.6, if we take the Banach space $B$ as $y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$, we know that the operator $S$ satisfies the assumption in Lemma 1.8.6. Therefore, there exists an solution $\alpha$ to equation (1.83) with $\alpha \in y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$.

Let $(a, b) := Q^{\sharp T}(\alpha)$ where $(a, b) \in \Omega^1 \times \Omega^1$, then $(A^\sharp, \Phi^\sharp) + (a, b)$ is a solution to the Kapustin-Witten equations (4.1).

Now we will prove the regularity statement of Theorem 1.1. By Proposition 1.8.4, we have

$$Q^{\sharp T} : y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T}).$$

Therefore, we know $(a,b) \in y^{\lambda+\frac{1}{p}}H_0^{1,p}(X^{\sharp T})$.

As $S$ satisfies

$$\|S(\alpha)-S(\beta)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq k(\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}+\|\beta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})})(\|\alpha-\beta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}).$$

Take $\beta = 0$, we have $\|S(\alpha)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq k\|\alpha\|^2_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}$. By equation (1.83), we have the following estimate:

$$\begin{aligned} \|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq& \|\eta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} + \|S(\alpha)\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \\ \leq& \|\eta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} + k\|\alpha\|^2_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}. \end{aligned} \tag{1.89}$$

WLOG, we can assume $1 - k\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \geq \frac{1}{2}$ and we obtain

$$\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq 2\|\eta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}. \tag{1.90}$$

As in Proposition 1.8.5, we use the estimate $\|Q^{\sharp T}(\alpha)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \leq C\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}$, we have

$$\|(a,b)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \leq C\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})} \leq 2C\|\eta\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}. \tag{1.91}$$

Applying Proposition 1.8.2, we get the estimate we want. □

We can say more about the regularity of solutions we get. Using the equations (1.82), we have the following proposition.

**Proposition 1.8.7.** *For $p > 2$ and $T$ is large enough, suppose $(a,b)$ satisfies the equations (1.82) over $X^{\sharp T}$, then $(a,b)$ is smooth in the interior of $X^{\sharp T}$.*

*Proof.* Fix a interior open set $U \subset X^{\sharp T}$. By (1.91), for any given constant $C$, we can choose $T$ is large enough such that $\|(a,b)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p(X^{\sharp T})} \leq C$. Applying Theorem 1.3.2 over $U$, we get a gauge fixing condition for $(a,b)$. Combing this with equations (1.82) and using the bootstrapping method, we get the regularity we want. □

**Corollary 1.8.8.** *Under the assumption as Theorem* 1.8.1*, if $H^2_{(A_i,\Phi_i)} = 0$ and the limiting flat connection is irreducible, then for $T$ is large enough, there exists a solution to the Kapustin-Witten equations (4.1).*

**Gluing Singular Points in Moduli Space**

In this subsection, we will deal with the singular points $(A_i, \Phi_i)$ with $H^2_{(A_i,\Phi_i)} \neq 0$. As before, we take the norm $y^{\lambda+\frac{1}{p}} H^{1,p}_0(X_i)$ on $\Omega^1_{X_i}(\mathfrak{g}_P)$ and $y^{\lambda+\frac{1}{p}-1} L^p(X_i)$ on $\Omega^0_{X_i}(\mathfrak{g}_P) \oplus \Omega^2_{X_i}(\mathfrak{g}_P)$.

As before, we denote $H^2_{(A_i,\Phi_i)} := (\Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P))/\mathrm{Im}\mathcal{L}_i$. For any $\tau \in \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)$, we denote by $[\tau] \in H^2_{(A_i,\Phi_i)}$ the equivalence class of $\tau$.

We have the following lemma for this cohomology group:

**Lemma 1.8.9.** *Given any bounded open set $U \subset X_i$, for any $\alpha \in H^2_{(A_i,\Phi_i)}$, there exist a $\beta \in \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)$, such that $\beta$ is supported in $U$ and $[\beta] = \alpha$ as cohomology class.*

*Proof.* As the range of $\mathcal{L}_i$ is closed in $y^{\lambda+\frac{1}{p}-1} L^p(\Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)$, we have the following splitting:

$$y^{\lambda+\frac{1}{p}-1} L^p(\Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)) = \mathrm{Im}\mathcal{L}_i \oplus (Ker\mathcal{L}_i^\star \cap y^{\lambda+\frac{1}{p}-1} L^p), \tag{1.92}$$

where $\mathcal{L}_i^\star$ is the $L^2$ adjoint of $\mathcal{L}_i$.

Thus we have the identification $H^2_{(A_i,\Phi_i)} \cong \mathrm{Ker}\mathcal{L}_i^\star \cap y^{\lambda+\frac{1}{p}-1} L^p$. By the classicial unique continuation property of an elliptic operator on the interior [4], for any $\alpha \in \mathrm{Ker}\mathcal{L}_i^\star$, we have $\alpha$ nonvanishing on any interior open set. Denote $l = \dim H^2_{(A_i,\Phi_i)}$, then for an integer $j$, $0 \leq j \leq l$, there exist a basis $\{a_j\} \in H^2_{(A_i,\Phi_i)}$. In addition, we can choose $\{a_j\}$ orthogonal to each other w.r.t the $L^2$ inner product.

In order to prove the lemma, we only need to prove the statement for one of the base $a_j$. We claim that for any fixed $a_j$, there exists a differential form $f \in \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)$ such that $\langle f, a_j \rangle \neq 0$ and $f$ vanishes over the boundary, $f|_{\partial U} = 0$. If not, for any $f \in C^\infty_0(\Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P))$, we have $\langle f, a_j \rangle = 0$. This will imply $a_j$ is identically 0 over an interior open set which contradicts $a_j \in \mathrm{Ker}\mathcal{L}_i^\star$.

By the Gram–Schmidt process and rescaling, we can find a function $g$ which vanishes over $\partial U$, $\langle g, a_j \rangle = 1$ and for $s \neq j$ $\langle g, a_s \rangle = 0$. By the splitting (1.92), we know there exists a $g_0 \in \mathrm{Im}\mathcal{L}_i$, such that $g = g_0 + a_j$. □

By the previous lemma, we know there exists linear operators $\sigma_i$,

$$\sigma_i : H^2_{(A_i,\Phi_i)} \to \Omega^2_{X_i} \oplus \Omega^0_{X_i},$$

such that the operators

$$\mathcal{L}_i \oplus \sigma_i : \Omega^1_{X_i} \oplus \Omega^1_{X_i} \oplus H^2_{(A_i,\Phi_i)} \to \Omega^2_{X_i} \oplus \Omega^0_{X_i}$$

are surjective. By Theorem 1.5.9, we know that $H^2_{(A_i,\Phi_i)}$ is finite dimensional, therefore, we can take the image of $\sigma_i$ to be supported in $X_i(T)$ for $T$ is large enough.

In the notation above, take $H = H^2_{(A_1,\Phi_1)} \oplus H^2_{(A_2,\Phi_2)}$, we can define a map $\sigma$:

$$\sigma = \sigma_1 + \sigma_2 : H \to \Omega^2_{X^{\sharp T}} \oplus \Omega^0_{X^{\sharp T}}.$$

As $\mathcal{L}_i \oplus \sigma_i$ is surjective, there exists an operator $Q_i$, such that $(\mathcal{L}_i \oplus \sigma_i)Q_i = Id$,

$$Q_i : \Omega^2_{X_i} \oplus \Omega^0_{X_i} \to H^2_{(A_i,\Phi_i)} \oplus \Omega^1_{X_i} \oplus \Omega^1_{X_i}. \tag{1.93}$$

Composing $Q_i$ with the projection map into different part of the image, we get operators $\pi_i$ and $P_i$. To be explicit, $Q_i := \pi_i \oplus P_i$ where

$$\pi_i : \Omega^2_{X_i} \oplus \Omega^0_{X_i} \to H^2_{(A_i,\Phi_i)},$$

and

$$P_i : \Omega^2_{X_i} \oplus \Omega^0_{X_i} \to \Omega^1_{X_i} \oplus \Omega^1_{X_i}.$$

Therefore, by definition, for $\forall \xi \in \Omega^2_{X_i} \oplus \Omega^0_{X_i}$, we have

$$\xi = \mathcal{L}_i P_i(\xi) + \sigma_i \pi_i(\xi).$$

As before, we take $\phi_i$ be a cut off function supported in $X_i(2T)$ as in Figure 1.5, with $\phi_i(x) = 1$ on $X_i(T)$ and $\phi_1 + \phi_2 = 1$ on $X^{\sharp T}$. We have the estimate $\|\nabla \phi_i\|_{L^\infty(X_i)} \le \epsilon(T)$.

Given $\xi \in \Omega^2_{X^{\sharp T}} \oplus \Omega^0_{X^{\sharp T}}$, denote by $\xi_i$ the restriction of $\xi$ to $X_i(2T)$, we can define two approximate inverse operators as follows:

Let $\hat{P}(\xi) := \phi_1 P_1(\xi_1) + \phi_2 P_2(\xi_2)$, $\hat{\pi}(\xi) := \phi_1 \pi_1(\xi_1) + \phi_2 \pi_2(\xi_2)$. Similarly, we take $\mathcal{L}^{\sharp T} := \mathcal{L}_{(A^\sharp,\Phi^\sharp)}$, we have the following lemma:

**Lemma 1.8.10.** $\|\mathcal{L}^{\sharp T} \hat{P}(\xi) + \sigma \hat{\pi}(\xi) - \xi\|_{y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T})} \le \epsilon(T) \|\xi\|_{y^{\lambda+\frac{1}{p}-1} L^p(X^{\sharp T})}$.

*Proof.* Compared to Lemma 1.8.3, we have some additional terms in computing $\mathcal{L}^{\sharp T} \hat{P}$.

We have the following computation:

$$\begin{aligned}\mathcal{L}^{\sharp T}\hat{P}(\xi) &= \mathcal{L}^{\sharp T}(\phi_1 P_1\xi) + \mathcal{L}^{\sharp T}(\phi_2 P_2\xi)\\ &= \nabla\phi_1 \star P_1(\xi) + \nabla\phi_2 \star P_2(\xi) + \phi_1\mathcal{L}^{\sharp T}P_1(\xi) + \phi_2\mathcal{L}^{\sharp T}P_2(\xi)\\ &= \nabla\phi_1 \star P_1(\xi) + \nabla\phi_2 \star P_2(\xi) + \phi_1\mathcal{L}_1 P_1(\xi) + \phi_2\mathcal{L}_2 P_2(\xi)\\ &+ \phi_1(\mathcal{L}^{\sharp T} - \mathcal{L}_1)P_1(\xi) + \phi_2(\mathcal{L}^{\sharp T} - \mathcal{L}_2)P_2(\xi).\end{aligned} \tag{1.94}$$

For the terms $\phi_1\mathcal{L}_1 P_1(\xi) + \phi_2\mathcal{L}_2 P_2(\xi)$, we have

$$\begin{aligned}&\phi_1\mathcal{L}_1 P_1(\xi) + \phi_2\mathcal{L}_2 P_2(\xi)\\ =&(\phi_1 + \phi_2)(\xi) + \phi_1\sigma_1\pi_1(\xi) + \phi_2\sigma_2\pi_2(\xi)\\ =&\xi + \sigma\hat{\pi(\xi)}.\end{aligned} \tag{1.95}$$

For the other terms in the final step of (1.94), the estimates are exactly the same as Lemma 1.8.3 and is bounded by $\epsilon(T)\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})}$.

Combining all the arguement above, we get the estimate we want. ◻

Now we can construct the inverse of the operator $\mathcal{L}^{\sharp T}$.

**Corollary 1.8.11.** *For $T$ is large enough, there exist operators $\mathcal{P}^{\sharp T} : \Omega^2_{X^{\sharp T}} \oplus \Omega^0_{X^{\sharp T}} \to \Omega^1_{X^{\sharp T}} \oplus \Omega^1_{X^{\sharp T}}$ and $\pi^{\sharp T} : \Omega^2_{X^{\sharp T}} \oplus \Omega^0_{X^{\sharp T}} \to H$ such that $\forall \xi \in \Omega^2_{X^{\sharp T}} \oplus \Omega^0_{X^{\sharp T}}$, we have*

$$\xi = \mathcal{L}^{\sharp T}\mathcal{P}^{\sharp T}(\xi) + \sigma\pi^{\sharp T}(\xi). \tag{1.96}$$

*In addition, the operator norm of $\mathcal{P}^{\sharp T}$ and $\pi^{\sharp T}$ is bounded independent of $T$.*

*Proof.* By Lemma 1.8.10, denoting $R := (\mathcal{L}^{\sharp T} \oplus \sigma)(\hat{P} \oplus \hat{\pi}) - Id$, we know that when $T$ is large enough, $R$ has operator norm small. Therefore, $Id + R$ is invertible and $Q = (\hat{P} \oplus \hat{\pi})(1 + R)^{-1}$ will be the right inverse of $\mathcal{L}^{\sharp T} \oplus \sigma$. As the image of $Q$ is $H^2_{(A^\sharp,\Phi^\sharp)} \oplus \Omega^1_{X^{\sharp T}} \oplus \Omega^1_{X^{\sharp T}}$, we can take $\mathcal{P}^{\sharp T}$ to be the projection to the $\Omega^1 \oplus \Omega^1$ part of image of $Q$ and $\pi^{\sharp T}$ to be the projection of $H^2$ part of image of $Q$, then by definition, we have

$$\xi = \mathcal{L}^{\sharp T}\mathcal{P}^{\sharp T}(\xi) + \sigma\pi^{\sharp T}(\xi).$$

By classical functional analysis, we know the operator norm of $Id + R$ can be choose to be smaller than 3 and the operator norm of $\hat{P} \oplus \hat{\pi}$ is dominated by $Q_i$ (1.93). Therefore, the operator norm is independent of $T$. ◻

For a pair $(\xi, h)$ with $\xi \in \Omega^2_{X^{\sharp T}} \oplus \Omega^0_{X^{\sharp T}}$ and $h \in H^2$, consider the perturbation equation

$$KW((A^{\sharp}, \Phi^{\sharp}) + \mathcal{P}^{\sharp T}(\xi)) + \sigma(h) = 0. \tag{1.97}$$

Therefore, we have

$$\begin{aligned} &KW(A^{\sharp}, \Phi^{\sharp}) + \mathcal{L}^{\sharp T}\mathcal{P}^{\sharp T}(\xi) + \{\mathcal{P}^{\sharp T}(\xi), \mathcal{P}^{\sharp T}(\xi)\} + \sigma(h) = 0, \\ &KW(A^{\sharp}, \Phi^{\sharp}) + \xi - \sigma\pi^{\sharp T}(\xi) + \{\mathcal{P}^{\sharp T}(\xi), \mathcal{P}^{\sharp T}(\xi)\} + \sigma(h) = 0.\text{(Applying (1.96))} \end{aligned} \tag{1.98}$$

Take $h = \pi^{\sharp T}(\xi)$, we obtain

$$KW(A^{\sharp}, \Phi^{\sharp}) + \xi + \{\mathcal{P}^{\sharp T}(\xi), \mathcal{P}^{\sharp T}(\xi)\} = 0, \tag{1.99}$$

which is the equation (1.83) and it has solution $\xi$. As $\mathcal{P}^{\sharp T}$ is an operator mapping $\Omega^2_{X^{\sharp T}} \oplus \Omega^0_{X^{\sharp T}}$ to $\Omega^1_{X^{\sharp T}} \oplus \Omega^1_{X^{\sharp T}}$, we can define $(a, b) \in \Omega^1_{X^{\sharp T}} \oplus \Omega^1_{X^{\sharp T}}$ by $(a, b) := \mathcal{P}^{\sharp T}(\xi)$. Then if we denote $(A, \Phi) := (A^{\sharp} + a, \Phi^{\sharp} + b)$, $(A, \Phi)$ will solve the equation

$$KW(A, \Phi) + \sigma(h) = 0.$$

By the previous arguments, we get the following corollary, which completes the proof of the second part of Theorem 1.1.

**Corollary 1.8.12.** *For any interior open set U, there exists $(a, b) \in \Omega^1_{X^{\sharp T}} \oplus \Omega^1_{X^{\sharp T}}$ and $h \in H^2_{(A_1,\Phi_1)} \oplus H^2_{(A_2,\Phi_2)}$ solve the equation $KW(A^{\sharp} + a, \Phi^{\sharp} + b) + \sigma(h) = 0$ and satisfy (1) $(A^{\sharp} + a, \Phi^{\sharp} + b)$ is a solution to the Kapustin-Witten equations over $X^{\sharp T}$ if and only if h=0.*

*(2) We have the estimate:*

$$\|(a, b)\|_{y^{\lambda+\frac{1}{p}-1}L^p_1} \leq Ce^{-\delta T}, \ \|\sigma(h)\| \leq Ce^{-\delta T}.$$

*These two constants depend on the choice of the open set and $\delta$ is the positive constant in Proposition 1.8.2.*

*(3) $\sigma(h)$ is supported in $U$.*

*Proof.* The first statement is obvious. For the second statement, by definition, we have

$$\begin{aligned}\|(a,b)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p} &= \|\mathcal{P}^{\sharp T}(\xi))\|_{y^{\lambda+\frac{1}{p}-1}L_1^p} \\ &\leq C\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L_1^p}(\mathcal{P}^{\sharp T} \text{ is bounded}) \\ &\leq C\|KW(A^{\sharp},\Phi^{\sharp})\|_{y^{\lambda+\frac{1}{p}-1}L_1^p}(\text{By (1.90)}) \\ &\leq Ce^{-\delta T}.(\text{ By Proposition 1.8.2})\end{aligned} \tag{1.100}$$

Similarly,

$$\begin{aligned}\|\sigma(h)\|_{y^{\lambda+\frac{1}{p}-1}L_1^p} &= \|\sigma(\pi^{\sharp T}(\xi))\|_{y^{\lambda+\frac{1}{p}-1}L_1^p} \\ &\leq C\|\xi\|_{y^{\lambda+\frac{1}{p}-1}L_1^p}(\pi^{\sharp T} \text{ and } \sigma \text{ are bounded}) \\ &\leq Ce^{-\delta T}.(\text{ By Proposition 1.8.2})\end{aligned} \tag{1.101}$$

The third statement is a direct corollary of lemma 1.8.9. □

Given $(A_i,\Phi_i)$, denote by $\Gamma_i$ the isotropy group of $(A_i,\Phi_i)$, $\Gamma_i = \{g|g(A_i,\Phi_i) = (A_i,\Phi_i)\}$. By Corollary 1.6.9, we know $\Gamma_i = 1$. We will combine the Kuranishi descriptin in Proposition 1.6.17 with the previous construction. Let $N_i \subset H^1_{(A_i,\Phi_i)}$ be a set parametrize a neighborhood of $(A_i,\Phi_i)$ in the moduli space of Nahm pole solutions. If we denote $N := N_1 \times N_2$, then we have the following proposition:

**Proposition 1.8.13.** *For large enough $T$ and small enough $N_i$, given $n \in N$ then we have*

*(1) A family of $y^{\lambda+\frac{1}{p}}H_0^{1,p}$ connections $(A(n),\Phi(n)) + (a(n),b(n))$ parametrized by $N$.*

*(2) There exist a map $\Psi : N \to H^2_{(A_1,\Phi_1)} \times H^2_{(A_2,\Phi_2)}$, such that $(A(n),\Phi(n))+(a(n),b(n))$ satisfies the Kapustin-Witten equations if and only if $\Psi(n) = 0$.*

*(3) Let $\mathcal{M}_{X^{\sharp T}}$ be the moduli space of Nahm pole solutions to the Kapustin-Witten equations over $X$, then there exists a map $\Theta$, whose image is the moduli space $\mathcal{M}_{X^{\sharp T}}$:*

$$\begin{aligned}\Theta : \Psi^{-1}(0) &\to \mathcal{M}_X \\ n &\to (A(n),\Phi(n)) + (a(n),b(n)).\end{aligned} \tag{1.102}$$

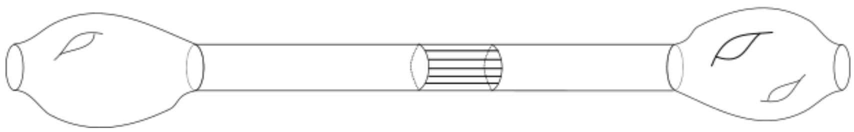

Figure 1.6: The shaded part is $Y \times (-\frac{T}{2}, \frac{T}{2})$

**Gluing for Non-degenerate Limit**

In this section, we will build the gluing theorem for the reducible connection. For simplicity, in this subsection, we only consider the case $H^2_{(A_i,\Phi_i)}(X_i) = 0$. For the $H^2$ non-vanishing case, the result will follows similarly as in subsection 7.3.

As before, we are dealing with manifolds $X_1, X_2$ with cylindrical ends and boundaries as in Figure 1.3. We constructed $X^{\sharp T}$, identified the connecting region with $Y \times (-\frac{T}{2}, \frac{T}{2})$. This will be more precisely shown in Figure 1.6.

For a positive real number $\alpha$, take a smooth weighted function $W_T = e^{\alpha(\frac{T}{2}-|t|)}$ and over a neighborhood of the boundary of $X^{\sharp T}$, let $W_T$ be the distance function to the boundary.

Over the manifolds with boundary and cylindrical ends $X_1$ and $X_2$, we have fixed weighted functions $W_1$ and $W_2$, such that in the connected area $W_1 = e^{\alpha(\frac{T}{2}+t)}$, $W_2 = e^{\alpha(\frac{T}{2}-t)}$ and in the neighborhood of the boundary, $W_1$ and $W_2$ are the distance functions to the boundaries. It is easy to get that in the common area $W_1$, $W_2$ and $W_T$ dominated each other.

On 1-forms of $X^{\sharp T}$, use the norm $y^{\lambda+\frac{1}{p}}H^{1,p}_{0,\alpha}(X^{\sharp T})$ given by the weighted norm given by $H^{1,p}_0(X^{\sharp T})$ and weight function $W_T$. On the 2-forms of $X^{\sharp T}$, use the norm $y^{\lambda+\frac{1}{p}-1}L^p_\alpha(X^{\sharp T})$ given by $L^p(X^{\sharp T})$ and weighted function $W_T$. Respectively, we get $y^{\lambda+\frac{1}{p}}H^{1,p}_{0,\alpha}(X_i)$ and $y^{\lambda+\frac{1}{p}-1}L^p_\alpha(X_i)$ for $X_i$.

By these constructions, we get the following estimate for the approximate solution:

**Proposition 1.8.14.** $\|KW(A^\sharp, \Phi^\sharp)\|_{y^{\lambda+\frac{1}{p}-1}L^p_\alpha(X^{\sharp T})} \leq C(e^{(\alpha-\delta)T})$.

*Proof.* By Theorem 1.7.1, we know the $C^\infty$ norm will decays as $e^{-\delta t}$. In addition, we have the weighte function that equals to $e^{\alpha t}$ in the end. Therefore, we get the decay rate we want. □

Therefore, we can take $\alpha < \delta$ such that the approximate term exponentially decays as $T \to \infty$.

For $i = 1, 2$, denoting $\mathcal{L}_i := \mathcal{L}_{(A_i,\Phi_i)}$, we can regard the operator as

$$\mathcal{L}_{i,\alpha} : y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(X_i) \to y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X_i).$$

For the approximate solution $(A^\sharp, \Phi^\sharp)$, we also have the Fredholm operator $\mathcal{L}^{\sharp T}_\alpha$ for the weighted norm

$$\mathcal{L}^{\sharp T}_\alpha : y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X^{\sharp T}).$$

By our assumption $H^2_{(A_i,\Phi_i)}(X_i) = 0$, we know there exists a right inverse $Q_i$:

$$Q_i : y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X_i) \to y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(X_i),$$

such that $\mathcal{L}_{i,\alpha} Q_i = Id$.

As before, we take $\phi_i$ be a cut off function supported in $X_i(2T)$ as in Figure 1.5, with $\phi_i(x) = 1$ on $X_i(T)$ and $\phi_1 + \phi_2 = 1$ on $X^{\sharp T}$ and we can have the estimate $\|\nabla \phi_i\|_{L^\infty(X_i)} \leq \epsilon(T)$.

Take $\xi \in y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X^{\sharp T})$, denote by $\xi_i$ the restriction of $\xi$ on $X_i(2T)$, then we define a new approximate inverse operator $\hat{Q}_\alpha(\xi) := \phi_1 Q_1(\xi_1) + \phi_2 Q_2(\xi_2)$, which can be written as

$$\hat{Q} : y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X_i) \to y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(X_i).$$

**Right Inverse**

Similarily, we have the following estimate for the operator $\mathcal{L}^{\sharp T}_\alpha$.

**Lemma 1.8.15.** *For $\mathcal{L}^{\sharp T}_\alpha$, $\hat{Q}_\alpha$ as above, and for $\forall \xi \in y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X^{\sharp T})$, we have*

$$\|\mathcal{L}^{\sharp T}_\alpha \hat{Q}_\alpha(\xi) - \xi\|_{y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X^{\sharp T})} \leq \epsilon(T) \|\xi\|_{y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X^{\sharp T})}.$$

*Proof.* After we choose $\alpha < \delta$, we still get the exponential decay result and the proof is exactly the same as Lemma 1.8.3. □

**Proposition 1.8.16.** *There exists an operator $Q^{\sharp T}_\alpha$,*

$$Q^{\sharp T}_\alpha : y^{\lambda+\frac{1}{p}-1} L^p_\alpha(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}} H^{1,p}_{0,\alpha}(X^{\sharp T})$$

*such that $\mathcal{L}^{\sharp T}_\alpha Q^{\sharp T}_\alpha = Id$. In addition, the operator norm of $QS_\alpha$ is independent of T.*

*Proof.* By Lemma 1.8.15, we know $\mathcal{L}^{\sharp T}_\alpha \hat{Q}_\alpha$ has an inverse and we just take $Q^{\sharp T}_\alpha := \hat{Q}_\alpha (\mathcal{L}^{\sharp T}_\alpha \hat{Q}_\alpha)^{-1}$. By definition, we get the inverse we want. For the independence of $T$ from the operator norm, the arguement is exactly the same as Proposition 1.8.4. □

**Existence Theorem**

Over $X^{\sharp T}$, for arbitary $(A,\Phi)$, we denote $(a,b) := (A,\Phi) - (A^\sharp,\Phi^\sharp)$. We have the following expansion for the Kapustin-Witten map:

$$KW(A,\Phi) = KW(A^\sharp,\Phi^\sharp) + \mathcal{L}_\alpha^{\sharp T}(a,b) + \{(a,b),(a,b)\}. \tag{1.103}$$

Take $S_\alpha(a,b) = \{(a,b),(a,b)\}$, then we have the following proposition:

**Proposition 1.8.17.** *For any $\lambda_0 \in [1-\frac{1}{p},1)$, $S_\alpha$ is an operator $S_\alpha : y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T}) \to y^{\lambda+\frac{1}{p}-1}L^p(X^{\sharp T})$ satisfying $S_\alpha(0) = 0$ and for two elements $\beta,\gamma \in y^{\lambda+\frac{1}{p}-1}L^2_\alpha(X^{\sharp T})$, there exists a constant $k$ such that*

$$\|S_\alpha(\beta) - S_\alpha(\gamma)\|_{y^{\lambda+\frac{1}{p}-1}L^p_\alpha} \leq k(\|\beta\|_{y^{\lambda+\frac{1}{p}-1}L^p_\alpha} + \|\gamma\|_{y^{\lambda+\frac{1}{p}-1}L^p_\alpha})(\|\beta-\gamma\|_{y^{\lambda+\frac{1}{p}-1}L^p_\alpha}).$$

*Proof.* The proof is basically the same as the proof of Proposition 1.8.5. We only need to check the Sobolev inequality is still true in the weighted case and this is proved in Proposition 1.5.12. □

Now, we have a parallel theorem to Theorem 1.1:

**Theorem 1.8.18.** *Under the gluing hypotheses in the beginning of the chapter, if*

*(a)* $\lim_{T\to+\infty} \|(A_i,\Phi_i) - (A_{\rho_i},\Phi_{\rho_i})\|_{L_2^{p_0}(Y_i\times\{T\})} = 0$ *for some* $p_0 > 2$,

*(b) $\rho$ is a non-degenerate $SL(2;\mathbb{C})$ flat connection,*

*then for $\lambda_0 \in [1-\frac{1}{p},1)$, there exists a real number $\alpha > 0$ such that we have*

*(1) for some constant $\delta$, there exists a $y^{\lambda_0+\frac{1}{2}}H^{1,p}_{0,\alpha}$ pair $(a,b) \in \Omega^1_{X^{\sharp T}}(\mathfrak{g}_P)\times\Omega^1_{X^{\sharp T}}(\mathfrak{g}_P)$ with*

$$\|(a,b)\|_{y^{\lambda_0-\frac{1}{2}}L^p_{1,\alpha}} \leq Ce^{(\alpha-\delta)T},$$

*(2) there exists an obstruction class $h \in H^2_{(A_1,\Phi_1)}(X_1)\times H^2_{(A_2,\Phi_2)}(X_2)$ such that $h = 0$ if and only if $(A^\sharp + a, \Phi^\sharp + b)$ is a solution to the Kapustin-Witten equations (4.1).*

*Proof.* For the case that $H^2$ vanishes, by Proposition 1.8.17, we know that the opertor $S_\alpha$ satisfies the assumption for Lemma 1.8.6. By Proposition 1.8.14, we know we can choose $\eta$ small enough satifying Lemma 1.8.6. Therefore, by Lemma 1.8.6, there exists a solution to the equation (1.103) and we get a solution to the Kapustin-Witten equations (4.1). The regularity statements of the connections in the theorem will follows by the same way as in Chapter 8.1.

Similarly, we can follow exactly the same as Chapter 1.8.3 and prove the second statement of the theorem.

## 1.9 Local Model for Gluing Picture

In this section, we will give a Kuranishi description of the gluing construction for the Kapustin-Witten equations.

For the description for the anti-self-dual equations, see [19], [20], [53]. In this section, we assume $p \in (2,4)$, $\lambda \in [1-\frac{1}{p}, 1)$ and denote by $q$ the real number satisfying the relationship $1+\frac{4}{q}=\frac{4}{p}$.

**Gauge Fixing Problem**

For $i=1,2$, let $\mathcal{M}_i$ be the moduli space of Nahm pole solutions to the Kapustin-Witten equations over $X_i$ defined in (1.55). Let $N_i$ be pre-compact subsets of the moduli space $\mathcal{M}_i$ such that any element of $N_i$ is regular in the moduli space $\mathcal{M}_i$. To be more explicit, for any $(A_i,\Phi_i)\in N_i$, we have $H^0_{(A_i,\Phi_i)}=0$ and $H^2_{(A_i,\Phi_i)}=0$. By Proposition 1.8.13, we know there exists a map $\Theta_T$ defined as follows:

$$\Theta_T : N_1 \times N_2 \to \mathcal{M}_X. \tag{1.104}$$

We have the following proposition on the map $\Theta$:

**Proposition 1.9.1.** *There exists a $T_0$, such that for any $T > T_0$, we have:*

*(1) For $(A_i,\Phi_i)\in N_i$, let $(A,\Phi):=\Theta_T((A_1,\Phi_1),(A_2,\Phi_2))$, we have $H^2_{(A,\Phi)}=0$,*

*(2) $\Theta_T$ is a diffeomorphism to its image.*

*Proof.* (1) Let $(A^\sharp,\Phi^\sharp)$ be the approximate solution, let $(a,b):=(A,\Phi)-(A^\sharp,\Phi^\sharp)$. By Theorem 1.8.1, we have $\|(a,b)\|_{y^{\lambda+\frac{1}{p}-1}L^p_1}\leq Ce^{-CT}$. Let $\mathcal{L}^1_{(A^\sharp,\Phi^\sharp)}$ ($\mathcal{L}^1_{(A,\Phi)}$ ) be the linearization operator of $(A^\sharp,\Phi^\sharp)$ ($\mathcal{L}_{(A,\Phi)}$). By Proposition 1.8.4, there exists an operator $Q^\sharp : y^{\lambda+\frac{1}{p}-1}L^p \to y^{\lambda+\frac{1}{p}}H^{1,p}_0$ such that $\mathcal{L}^1_{(A^\sharp,\Phi^\sharp)}Q^\sharp = Id$. Therefore, we can choose $T$ big enough such that $\|\mathcal{L}^1_{(A,\Phi)}Q^\sharp - \mathcal{L}^1_{(A^\sharp,\Phi^\sharp)}Q^\sharp\|_{y^{\lambda+\frac{1}{p}-1}L^p}\leq\frac{1}{2}$. This implies that $\mathcal{L}^1_{(A,\Phi)}$ has a right inverse and it is surjective.

(2) By the assumption that $N_i$ is regular, we have $\dim N_i = \mathrm{Ind}P_i$. By Proposition 1.5.17, we have $\mathrm{Ind}P = \mathrm{Ind}P_1 + \mathrm{Ind}P_2$. Let $\mathrm{Im}(\Theta)$ to be the image of $\Theta$. We have $\dim(\mathrm{Im}(\Theta)) = \dim N_1 + \dim N_2$. Therefore, in order to prove $\Theta_T$ is a diffeomorphism, we only need to prove $d\Theta$ is injective. Choose an open subset $U\subset X_1$ which is away

from the gluing part. By Proposition 1.8.7, we know over $U$, $(A,\Phi)$ is $C^1$ close to $(A_1,\Phi_1)$ thus proves that $d\Theta$ is injective. □

Now we will characterize the Nahm pole solutions we found by our gluing construction. Given solutions $(A_i,\Phi_i)$. Let $(A^\sharp,\Phi^\sharp)$ be the approximate solution. Let $d_q^\lambda$ be the metric on the space $\mathcal{B}$ given by

$$d_q^\lambda([(A_1,\Phi_1)],[(A_2,\Phi_2)]) = \inf_{u\in\mathcal{G}} \|(A_1,\Phi_1) - u(A_2,\Phi_2)\|_{y^{\lambda+\frac{1}{p}-1}L^q}. \tag{1.105}$$

Then, we can define an open neighborhood $U(\epsilon)$ of $(A^\sharp,\Phi^\sharp)$ by

$$U_{(A^\sharp,\Phi^\sharp)}(\epsilon) = \{(A,\Phi)\in\mathcal{B} | d_q^\lambda((A,\Phi),(A^\sharp,\Phi^\sharp))| < \epsilon,\ \|KW(A,\Phi)\|_{y^{\lambda+\frac{1}{p}-1}L^p} < \epsilon\}. \tag{1.106}$$

Then we have the following theorem

**Theorem 1.9.2.** *For $\star = 0,1,2$, if $H^\star_{(A_i,\Phi_i)} = 0$, then for small enough $\epsilon$, any point $(A,\Phi)\in U(\epsilon)$ can be represented by the following form $(A,\Phi) = (A^\sharp,\Phi^\sharp) + Q\phi$, where $\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p} \le C\epsilon$ and $Q$ is the right inverse operator defined in Proposition 1.8.4.*

We prove Theorem 1.9.2 by the method of continuation. We need a new interpretation of the operator.

Given $(A_i,\Phi)$ satisfying the assumption of Theorem 1.9.2, let $(A^\sharp,\Phi^\sharp)$ be the approximate solution over $X^{\sharp T}$. In this section, for simplification, we denote $\mathcal{L}$ the linearization operator of $(A^\sharp,\Phi^\sharp)$ and let $Q$ be the right inverse of $\mathcal{L}$. Combining this with the embedding $y^{\lambda+\frac{1}{p}}H_0^{1,p} \hookrightarrow y^{\lambda+\frac{1}{p}-1}L^q$, we have

$$\begin{aligned}&\mathcal{L} : y^{\lambda+\frac{1}{p}}H_0^{1,p}(\Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P)) \to y^{\lambda+\frac{1}{p}-1}L^p(\Omega^2(\mathfrak{g}_P)\times\Omega^0(\mathfrak{g}_P)),\\ &Q : y^{\lambda+\frac{1}{p}-1}L^p(\Omega^2(\mathfrak{g}_P)\times\Omega^0(\mathfrak{g}_P)) \to y^{\lambda+\frac{1}{p}-1}L^q(\Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P)).\end{aligned} \tag{1.107}$$

Let $B\in U_{(A^\sharp,\Phi^\sharp)}(\epsilon)$, then WLOG, we assume $B = (A^\sharp,\Phi^\sharp)+(a,b)$ and consider $B_t$ which is a path of connection pairs defined as follows:

$$B_t := (A^\sharp,\Phi^\sharp) + t(a,b)$$

and we can define the following set $S$:

**Definition 1.9.3.** *Given $\delta$ small enough, define $S\subset[0,1]$ to be the interval of all $t\in[0,1]$ such that there exists gauge transform $u : [0,t]\to\mathcal{G}$ and $\phi : [0,t]\to \Omega^2(\mathfrak{g}_P)\times\Omega^0(\mathfrak{g}_P)$ such that*

(1) $\phi(0) = 0,\ u(0) = 1,$

(2) $u_t(B_t) = (A^\sharp, \Phi^\sharp) + Q(\phi_t)$ *with* $\|\phi_t\|_{y^{\lambda+\frac{1}{p}-1}L^p} < \delta.$

Our target is to prove $S = [0, 1]$. By definition of $S$, we have the following Proposition:

**Proposition 1.9.4.** *$S$ is non empty.*

*Proof.* As $B_0 = (A^\sharp, \Phi^\sharp)$, take $\phi_0 = 0$ and $u(0) = 1$, we know $0 \in S$. □

Now, we are going to prove $S$ is an open set and before the proving, we will need some preparations.

Let $d^0$ to be $d^0_{(A^\sharp,\Phi^\sharp)}$ in the Kuranishi complex (1.7), where for $\xi \in \Omega^0(\mathfrak{g}_P)$, $d^0(\xi) = (-d_{A^\sharp}\xi, [\xi, \Phi^\sharp])$. For any $\xi \in \Omega^0(\mathfrak{g}_P)$ and $\phi \in \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)$, define the operator

$$\begin{aligned} \Pi :&\Omega^0(\mathfrak{g}_P) \times \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P), \\ &(\xi\ ,\ \phi) \to d^0(\xi) + Q(\phi). \end{aligned} \tag{1.108}$$

Let $V_1$ be a norm over $\Omega^0(\mathfrak{g}_P) \times \Omega^2(\mathfrak{g}_P) \times \Omega^0(\mathfrak{g}_P)$ defined as follows:

$$\|(\xi, \phi)\|_{V_1} = \|d^0(\xi)\|_{y^{\lambda+\frac{1}{p}-1}L^q} + \|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}.$$

For $(a, b) \in \Omega^1(\mathfrak{g}_P) \times \Omega^1(\mathfrak{g}_P)$, we define another norm $V_2$ as

$$\|(a, b)\|_{V_2} = \|(a, b)\|_{y^{\lambda+\frac{1}{p}-1}L^q} + \|\mathcal{L}(a, b)\|_{y^{\lambda+\frac{1}{p}-1}L^p}.$$

Then we have the following Proposition:

**Proposition 1.9.5.** *Considering $\Pi$ as operator from $V_1$ to $V_2$:*

$$\Pi : V_1 \to V_2,$$

*we have*

*(1) $\Pi$ is a bounded operator from $V_1$ to $V_2$,*

*(2) There exists a constant $C$ independent of $T$ such that $\|(\xi, \phi)\|_{V_1} \le C\|\Pi(\xi, \phi)\|_{V_2}$.*

*Proof.* (1) We have the following computation for the operator $\Pi$:

$$\begin{aligned}
&\|\Pi(\xi,\phi)\|_{B_2}\\
\leq&\|d^0(\xi)+Q(\phi)\|_{y^{\lambda+\frac{1}{p}-1}L^q}+\|\mathcal{L}\circ d^0(\xi)+\mathcal{L}\circ Q(\phi)\|_{y^{\lambda+\frac{1}{p}-1}L^p}\\
&(\text{Here we use } \mathcal{L}\circ \mathrm{d}^0(\xi)=[\mathrm{KW}(\mathrm{A}^\sharp,\Phi^\sharp),\xi] \text{ and } \mathcal{L}\circ \mathrm{Q}=\mathrm{Id})\\
\leq&\|d^0(\xi)\|_{y^{\lambda+\frac{1}{p}-1}L^q}+\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}+\|[KW(A^\sharp,\Phi^\sharp),\xi]\|_{y^{\lambda+\frac{1}{p}-1}L^p}+\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}\\
\leq&\|d^0(\xi)\|_{y^{\lambda+\frac{1}{p}-1}L^q}+\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}.
\end{aligned}\tag{1.109}$$

(2) Take $\alpha=d^0(\xi)+Q(\phi)$, then we have $\mathcal{L}\alpha=[KW(A^\sharp,\Phi^\sharp),\xi]+\phi$. We have the following estimate:

$$\begin{aligned}
\|d^0(\xi)\|_{y^{\lambda+\frac{1}{p}-1}L^q}&\leq\|\alpha-Q\phi\|_{y^{\lambda+\frac{1}{p}-1}L^q}\\
&\leq\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^q}+\|Q\phi\|_{y^{\lambda+\frac{1}{p}-1}L^q}\\
&\leq\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^q}+\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}.
\end{aligned}\tag{1.110}$$

In addition, by the relation $\mathcal{L}\alpha=[KW(A^\sharp,\Phi^\sharp),\xi]+\phi$, we have

$$\begin{aligned}
\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}&\leq\|\mathcal{L}\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^p}+\|[KW(A^\sharp,\Phi^\sharp),\xi]\|_{y^{\lambda+\frac{1}{p}-1}L^p}\\
&\leq\|\alpha\|_{V_2}+\epsilon\|\xi\|_{C^0}(Here\ we\ use\ Proposition\ 1.8.2)\\
&\leq\|\alpha\|_{V_2}+\epsilon\|d^0(\xi)\|_{y^{\lambda+\frac{1}{p}-1}L^q}\\
&=\|\alpha\|_{V_2}+\epsilon\|\alpha-Q\phi\|_{y^{\lambda+\frac{1}{p}-1}L^q}\\
&\leq\|\alpha\|_{V_2}+\epsilon\|\alpha\|_{y^{\lambda+\frac{1}{p}-1}L^q}+\epsilon\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}.
\end{aligned}\tag{1.111}$$

By taking $\epsilon$ small enough, we get $\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}\leq C\|\alpha\|_{V_2}$.

By definition, $\|(\xi,\phi)\|_{V_1}=\|d^0(\xi)\|_{y^{\lambda+\frac{1}{p}-1}L^q}+\|\phi\|_{y^{\lambda+\frac{1}{p}-1}L^p}$. Combining equations (1.110) and (1.111), we obtain

$$\|(\xi,\phi)\|_{V_1}\leq C\|\alpha\|_{V_2}=C\|\Pi(\xi,\phi)\|_{V_2}.$$

$\square$.

By this estimate, we get an immediate corollary:

**Corollary 1.9.6.** $\Pi$ *is an injective operator.*

**Proposition 1.9.7.** *For* $\star = 0, 1, 2$*, if* $H^{\star}_{(A_i,\Phi_i)} = 0$*, the operator* $\Pi$ *is a surjective operator from* $V_1$ *to* $V_2$

*Proof.* As $Q$ is the inverse of $\mathcal{L}$, by the assumption $H^{\star}_{(A_i,\Phi_i)} = 0$, we know $\text{Ind}\,\Pi = -\text{Ind}\,\mathcal{D}_{(A,\Phi)} = 0$. By Proposition 1.9.5, we know $\Pi$ is injective, thus $\Pi$ is surjective. □

**Proposition 1.9.8.** *S is an open set in* $[0, 1]$.

*Proof.* By Proposition 1.9.7, $\Pi$ is surjective. By the implicit function theorem, we get the result immediately. □

Now, we hope to prove that the set $S$ is a closed set. To begin with, we prove that the condition (2) in Definition 1.9.3 is a closed condition:

**Lemma 1.9.9.** *For suitable* $\delta$ *and* $\epsilon$*, we have* $\|\phi_t\|_{y^{\lambda+\frac{1}{p}-1}L^p} \leq \frac{1}{2}\delta$.

*Proof.* By the relation $u_t(B_t) = (A^\sharp, \Phi^\sharp) + Q(\phi_t)$, we have:

$$KW(u_t(B_t)) = KW(A^\sharp, \Phi^\sharp) + \phi_t + \{Q(\phi_t), Q(\phi_t)\}. \tag{1.112}$$

Therefore, we have

$$\begin{aligned}
\|\phi_t\|_{y^{\lambda+\frac{1}{p}-1}L^p} &\leq \|KW(A^\sharp, \Phi^\sharp)\|_{y^{\lambda+\frac{1}{p}-1}L^p} + \|KW(B_t)\|_{y^{\lambda+\frac{1}{p}-1}L^p} + \|Q(\phi_t)\|^2_{y^{\lambda+\frac{1}{p}-1}L^q} \\
&\quad (Here\ we\ use\ Proposition\ 1.8.2\ and\ definition\ (1.106)) \\
&\leq \epsilon(T) + \epsilon + C^2\|\phi_t\|^2_{y^{\lambda+\frac{1}{p}-1}L^q}.
\end{aligned} \tag{1.113}$$

For $\delta < \frac{1}{2C^2}$, $\epsilon(T) \leq \frac{1}{4}\delta$ and $\epsilon \leq \frac{1}{4}\delta$, we have $\|\phi_t\| \leq \frac{1}{2}\delta$, so the open condition is also closed. □

**Proposition 1.9.10.** *For* $\delta$ *small enough and suitable parameter* $T$ *and* $\epsilon$*, S is a closed set in* $[0, 1]$.

*Proof.* Now is routine to prove the set $S$ is closed. Let assume a sequence $t_i \in S$ with $t_i \to t_0$. For simplification, we denote $B_i := B_{t_i}$ and $\phi_i := \phi_{t_i}$. By the definition of $S$, we have the relationship $u_t(B_i) = (A^\sharp, \Phi^\sharp) + Q(\phi_i)$.

By Lemma 1.9.9, we have the closed condition $\|\phi_t\|_{y^{\lambda+\frac{1}{p}-1}L^p} \leq \frac{1}{2}\delta$. By definition of $B_i$, we have $B_i = (A^\sharp, \Phi^\sharp) + t_i(a, b)$ and $(a, b) \in y^{\lambda+\frac{1}{p}}H^{1,p}_0 \subset y^{\lambda+\frac{1}{p}-1}L^p_1$. We know $B_i$ strongly converges in $y^{\lambda+\frac{1}{p}-1}L^p_1$.

By the uniform bound on the $\phi_i$, the $\phi_i$ converges to a limit $\phi_0$ weakly in $y^{\lambda+\frac{1}{p}-1}L^p$. Define $A_i = (A^\sharp, \Phi^\sharp) + Q(\phi_i)$. $A_i$ is uniformly bounded in $y^{\lambda+\frac{1}{p}}H_0^{1,p} \hookrightarrow y^{\lambda+\frac{1}{p}-1}L_1^p$. Therefore, $A_i$ converges weakly in $y^{\lambda+\frac{1}{p}-1}L_1^p$.

As $u_i$ is a gauge transformation, by the relation $u_i(B_i) = A_i$, we have $du_i = u_iA_i - B_iu_i$. By the boundedness of $A_i$ and $B_i$, we know $u_i$ weakly converges to $u_0$ in $y^{\lambda+\frac{1}{p}-1}L_2^p$. Therefore, by the Sobolev embedding theorem, $u_i$ strongly converges in $y^{\lambda+\frac{1}{p}-1}L_1^p$ to $u_0$. Therefore, we have the relationship $u_0(B_0) = A_0$ which imply $t_0 \in S$. □

We get an immediate corollary from Proposition 1.9.4, Proposition 1.9.8 and Proposition 1.9.10:

**Corollary 1.9.11.** *For the set $S$ in definition 1.9.3, we have* $S = [0, 1]$.

The proof of Theorem 1.9.2 follows immediately.

**Local Model for Regular Moduli Space**

Now, we are able to construct a local model for the gluing picture in the acyclic case without the assumption on $H^1$.

Denote $n_i = \mathrm{Ind}(P_i)$ and we don't assume $n_i = 0$. Denote $\mathcal{M}^\star_{P_i}$ ($\mathcal{M}^\star_P$) to be the moduli space which only consists of solutions to the Kapustin-Witten equations over $X_i$ ($X^{\sharp T}$), which have $H^2 = 0$.

For $i = 1, 2$, given two solutions $(A_i, \Phi_i) \in \mathcal{M}^\star_{P_i}$, there exists an open neighborhood $U_i$ such that we can find functions

$$\chi : U_i \subset \mathcal{M}^\star_{P_i} \to \mathbb{R}^{n_i}$$

which give local coordinates around $(A_i, \Phi_i)$ in the moduli spaces $\mathcal{M}^\star_{P_i}$. Denote

$$U_P(\epsilon) = \{(A, \Phi) \in \mathcal{B} | \exists (A_0, \Phi_0) \in \mathcal{M}^\star_P,\ d_q^\lambda((A, \Phi), (A_0, \Phi_0)) < \epsilon,\ \|KW(A, \Phi)\|_{y^{\lambda+\frac{1}{p}-1}L^p} < \epsilon\}. \tag{1.114}$$

Then by the exponential decay result (Theorem 1.7.1), we know that by choosing suitable compact sets $G_i \subset X_i$ and cut-off functions, we have a natural inclusion $U_i$ into $\mathcal{M}^\star_{P_i}$. Choose $y_i \in Im(\chi_i(U_i))$ and define the cut-down moduli space

$$L = \chi_1^{-1}(y_1) \cap \chi_2^{-1}(y_2) \cap \mathcal{M}^\star_P \subset U_P(\epsilon),$$

which has virtual dimensional 0.

For $T$ is large enough, recall $I : C_{P_1} \times C_{P_2} \to C_P$ is the operator defined in (1.74) that constructs the approximate solution. Denote by $(A_0, \Phi_0) := I(\chi_1^{-1}(y_1), \chi_2^{-1}(y_2))$

the approximate solution constructed by $\chi_1^{-1}(y_1)$ and $\chi_2^{-1}(y_2)$. Then we have the following Proposition. Compare this to Theorem 1.9.2:

**Proposition 1.9.12.** *For $\epsilon$ small enough, there exists a unique solution $(A', \Phi')$ in $L$ such that $U_{(A_0,\Phi_0)}(\epsilon) \cap L = (A', \Phi')$.*

Now, we will define a distance to make a comparision between connection pairs $(A_0, \Phi_0)$ over $X^{\sharp T}$ and $(A_i, \Phi_i)$ over $X_i$.

We can define the norm $d$ as

$$d((A^\sharp, \Phi^\sharp); (A_1, \Phi_1), (A_2, \Phi_2)) = inf_{u \in \mathcal{G}_P} \|(A_0, \Phi_0) - I((A_1, \Phi_1), (A_2, \Phi_2))\|_{L^q(X^{\sharp T})}, \tag{1.115}$$

where the $I$ is the operator that constructs the approximate solutions defined in (1.74).

Summarizing Proposition 1.9.1 and Theorem 1.9.2, we obtain the following statement:

**Theorem 1.9.13.** *Denote by $U_i$ the compact sets of regular points in the moduli space $\mathcal{M}^\star_{P_i}$. There exist $T_0$, $\epsilon_0$ such that for $T > T_0$ and $\epsilon < \epsilon_0$, there exist open neighborhoods $N_i$ of $U_i$ and a map*

$$\Theta : N_1 \times N_2 \to \mathcal{M}^\star_P,$$

*such that*

(1) *$\Theta$ is a diffeomorphism to its image, and the image contains regular points,*

(2) *$d(\Theta((A_1, \Phi_1), (A_2, \Phi_2)); (A_1, \Phi_1), (A_2, \Phi_2))) \leq \epsilon$ for any $(A_i, \Phi_i) \in N_i$,*

(3) *Any connection $(A^\sharp, \Phi^\sharp) \in \mathcal{M}^\star_P$ with $d((A^\sharp, \Phi^\sharp); (A_1, \Phi_1), (A_2, \Phi_2)) \leq \epsilon$ for some $(A_i, \Phi_i) \in N_i$ lies in the image of $\Theta$.*

Now we will have a brief discussion of the local gluing picture in the general case the $H^2$ is non-vanishing. For $(A_i, \Phi_i) \in \mathcal{M}_i$ with $H^2(A_i, \Phi_i)$ non-vanishing, we can do the trick as in Section 1.8 by adding some finite dimensional linear space as the obstruction class and have a similar obstruction type statement as in Theorem 1.9.13. We will precise by state the theorem in general in the next subsection.

### Conclusions

Now, we can summarize what we have proved and state the following theorem

**Theorem 1.9.14.** *Let $(A_i, \Phi_i)$ be connections pairs over manifolds $X_i$ with Nahm poles over $Z_i$, for sufficiently large $T$, there is a local Kuranishi model for an open set in the moduli space over $X^{\sharp T}$:*

(1) *There exists a neighborhood $N$ of $\{0\} \subset H^1_{(A_1,\Phi_1)} \times H^1_{(A_2,\Phi_2)}$ and a map $\Psi$ from $N$ to $H^2_{(A_1,\Phi_1)} \times H^2_{(A_2,\Phi_2)}$.*

(2) *There exists a map $\Theta$ which is a homeomorphism from $\Psi^{-1}(0)$ to an open set $V \subset \mathcal{M}^{\star}_{X^\sharp}$.*

## 1.10 Some Applications

In this section, we will introduce some applications of the gluing theorem 1.8.1.

As for the model solution in Section 2, we don't know whether the obstruction class vanish or not and right now we don't have any transversality result for the Kapustin-Witten equations. We just consider the obstruction class as a perturbation to the equation. See [18] for the obstruction perturbation for ASD equations.

Consider a compact 4-dimensional manifold $X^4$ with a cylindrical end which is identified with $Y^3 \times [0, +\infty)$, given any $SL(2;\mathbb{C})$ representation $\rho$ of $\pi_1(X^4)$:

$$\rho : \pi_1(X^4) \to SL(2;\mathbb{C}),$$

denote by $(A_\rho, \Phi_\rho)$ the $SL(2;\mathbb{C})$ flat connection associated to $\rho$. Then we know $(A_\rho, \Phi_\rho)$ satisfies the following equations:

$$\begin{aligned} &F_{A_\rho} - \Phi_\rho \wedge \Phi_\rho = 0, \\ &d_{A_\rho}\Phi_\rho = 0, \\ &d^{\star}_{A_\rho}\Phi_\rho = 0. \end{aligned} \tag{1.116}$$

Obviously, $(A_\rho, \Phi_\rho)$ is a solution to the Kapustin-Witten equations (4.1).

By gluing the suitable $SL(2;\mathbb{C})$ flat connection, we have the following theorem:

**Theorem 1.10.1.** *Consider a smooth compact 4-manifold $M$ with boundary $Y$. Assume $Y$ is $S^3$, $T^3$ or any hyperbolic 3-manifold. For $Y$ is hyperbolic, we assume the inclusion of $\pi_1(Y)$ into $\pi_1(M)$ is injective. For a real number $T_0$, we can glue $M$ with $Y \times (0, T_0]$ along $\partial M$ and $Y \times \{T_0\}$ to get a new manifold, which denote* *as $M_{T_0} := Y \times (0, T_0) \cup M$. For $T_0$ large enough, there exists an $SU(2)$ bundle $P$ and its adjoint bundle $\mathfrak{g}_P$ over $M_{T_0}$ such that given any interior non-empty open neighborhood $U \subset M$, we have:*

*(1) There exist $h_1 \in \Omega^2_{M_{T_0}}(\mathfrak{g}_P)$, $h_2 \in \Omega^0_{M_{T_0}}(\mathfrak{g}_P)$ supported on $U$,*

*(2) There exists a connection $A$ over $P$ and a $\mathfrak{g}_P$-valued 1-form $\Phi$ such that $(A, \Phi)$ satisfies the Nahm pole boundary condition over $Y \times \{0\} \subset M_{T_0}$ and $(A, \Phi)$ is a solution to the following obstruction perturbed Kapustin-Witten equations over $M_{T_0}$:*

$$\begin{aligned} F_A - \Phi \wedge \Phi + \star d_A \Phi &= h_1, \\ d_A^{\star} \Phi &= h_2. \end{aligned} \tag{1.117}$$

*Proof.* By Example 4.4.3, 1.2.3 and 4.4.4, we know we have model Nahm pole solutions for $Y \times (0, +\infty)$ when Y is $S^3$, $T^3$ or any hyperbolic manifold. Denote the limit of the model solution as $\rho$ which is a flat $SL(2; \mathbb{C})$ connection. Here the model solution for hyperbolic manifold has limit in cylindrical end to a irreducible flat $SL(2; \mathbb{C})$ connection.

Let $M_\infty = Y \times (0, +\infty) \cup M$, choose the flat connection $\rho$ and this will give a solution to the Kapustin-Witten equations over $M_\infty$. For $Y$ hyperbolic, we use the assumption $\pi_1$ injective in order to obtain a flat $SL(2; \mathbb{C})$ connection over $M_\infty$ with limit the irreducible $SL(2; \mathbb{C})$ connection over the cylindrical end coming from the hyperbolic metric.

Applying Theorem 1.8.1 and Theorem 1.8.18, we can glue these two solutions together and by Corollary 1.8.12, we prove the statement for $h_1$, $h_2$. □

In addition, by gluing the model solutions $(A_0, \Phi_0)$ on Example 4.4.3, 1.2.3 and 4.4.4 with themselves, we get the following corollary:

**Corollary 1.10.2.** *For a 3-manifold $Y^3$ equals to $S^3$, $T^3$ or any hyperbolic 3-manifold, for T large enough, there exists a solutions $(A, \Phi)$ over $Y^3 \times (-T, T)$ to the twisted Kapustin-Witten equations*

$$KW(A, \Phi) + h = 0.$$

*Here $(A, \Phi)$ satisfies the Nahm pole boundary condition over $Y^3 \times \{-T\}$ and $Y^3 \times \{T\}$ and $h$ can be choosen to be supported on any interior open set.*

**Appendix 1**

In this Appendix, we will give a brief introduction to the Fredholm theory of uniformly degenerate elliptic operators that is developed in [41] and [45]. We use the notation from [45] for most of the definitions in this Appendix.

Let $M$ be a compact smooth 4-manifold with 3-manifold boundary $Y$ and choose coordinates $(\vec{x}, y)$ near the boundary where $y \geq 0$ and $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. A differential operator $\mathcal{D}_0$ is called uniformly degenerate if for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, in any coordinate chart near the boundary, it has the form

$$\mathcal{D}_0 = \sum_{j+|\alpha|\leq m} A_{j\alpha}(\vec{x}, y)(y\partial_y)^j (y\partial_x)^\alpha, \tag{1.118}$$

where $(y\partial_x)^\alpha = (y\partial_{x_1})^{\alpha_1}(y\partial_{x_2})^{\alpha_2}(y\partial_{x_3})^{\alpha_3}$.

We define the leading term of $\mathcal{D}_0$ in this coordinate chart as

$$\mathcal{D}_0^m := \sum_{j+|\alpha|=m} A_{j\alpha}(\vec{x}, y)(y\partial_y)^j (y\partial_x)^\alpha. \tag{1.119}$$

The operator $\mathcal{D}_0$ is called uniformly degenerate elliptic if $\mathcal{D}_0$ is elliptic at the interior point and if in a neighborhood of the boundary and for (1.119), we replace each $y\partial_{x_i}$ and $y\partial_y$ by variables $\sqrt{-1}k_i$ and $\sqrt{-1}k_4$ and it is invertible when $(k_1, \cdots, k_4) \neq 0$.

There is a model operator over $\mathbb{R}^4_+$, called the indicial operator

$$I(\mathcal{D}_0) = \sum_{j\leq m} A_{j0}(\vec{x}, 0)\lambda^j. \tag{1.120}$$

The indicial root of $I(\mathcal{D}_0)$ is the set of complex numbers $\lambda$ such that $s^{-\lambda} I(\mathcal{D}_0) s^\lambda$ is not invertible.

In [41], Mazzeo works in the class of pseudodifferential operators on $M$ adapted to some particular type of singularity which includes the Nahm pole bounary condition. The class is called 0-pseudodifferential operators. Denote by $\Psi_0^\star(M)$ the elements which are described by the singularity structure of their Schwartz kernels.

Given a pseudodifferential operator $A$, we denote the Schwartz kernel of $A$ as $\kappa_A(y, \vec{x}, y, \vec{x}')$ which is a distribution over $M^2 := M \times M$. We allow $\kappa_A$ to have the standard singularity of pseudodifferential operator along the diagonal $\{y = y', \vec{x} = \vec{x}'\}$ and we will require some special behavior over the boundary of $M^2$, which in coordinates is described as $\{y = 0, y' = 0\}$ and over the intersection of diagonal with the boundary, $\{y = 0, y' = 0, \vec{x} = \vec{x}'\}$.

Let $M_0^2$ be a real blow-up of $M^2$ at the boundary of diagonal, which is constructed by replacing each point in $\{y=0, y'=0, \vec{x}=\vec{x}'\}$ with its inward-pointing normal sphere-bundle. We can describe it in polar coordinates:

$$R = (y^2+(y')^2+|\vec{x}-\vec{x}'|^2)^{\frac{1}{2}}, \omega = (\omega_0, \omega_0^{'}, \hat{\omega}) = (\frac{y}{R}, \frac{y'}{R}, \frac{\vec{x}-\vec{x}'}{R}).$$

Each point at $R=0$ is replaced by a quarter-sphere and $(R,\omega,x')$ can be regarded as a full set of coordinates. $M_0^2$ is a manifold with corners, we call the surface corresponding to $R=0$ the front face. The surfaces corresponding to $\omega_0=0$ and $\omega_0'=0$ are called its left and right faces. We have an obviously blow-down map $\pi: M_0^2 \to M^2$. We say $A \in \Psi_0^{\star}$ if $\kappa_A$ is the push forward of a distribution on $M_0^2$ by the blow-down map $\pi$.

Take a cut-off function $\chi$ over $M_0^2$ which is equals 1 over a small neighborhood of the diagonal set $\{\omega_0=\omega_0', \hat{\omega}=0\}$ and 0 outside of a larger neighborhood. Then $\kappa_A = \kappa_A' + \kappa_A''$, where $\kappa_A' = \chi\kappa_A$ and $\kappa_A'' = (1-\chi)\kappa_A$. Here $\kappa_A'$ supported away from the left and the right faces and has a pseudodifferential singularity of order $m$ along the lift diagonal area. If we factor $\kappa_A' = R^{-4}\hat{\kappa}_A'$, then $\hat{\kappa}_A$ extends smoothly of the front face of $M_0^2$ along the conormal diagonal singularity and $R^{-4}$ only depends on the manifold's dimension that corresponds to the determinant of the blow-down map $\pi$ and $\kappa_A''$ is smooth over the diagonal singularity.

Now we have the following definition of space $\Psi_0^{m,s,a,b}(M)$:

**Definition 1.10.3.** *For any real number $s, a, b$, we denote a psedudifferential operator $A \in \Psi_0^{m,s,a,b}(M)$ if its Schwartz kernel $\kappa_A$ has polyhomogeneous expansion with the terms $R^{-4+s}$ at the front face, $\omega_0^a$ at the left face and $\omega_0^b$ at the right face.*

*We denote $A \in \Psi^{-\infty,a,b}(M)$ if its Schwartz kernels are smooth in the interior and polyhomogeneous at two hypersurfaces ($y=0$ and $y'=0$) of $M^2$.*

In this setting, the identity operator $Id \in \Psi_0^{0,0,\emptyset,\emptyset}$, has zero order over the diagonal and its Schwartz kernel $\delta(y-y')\delta(\vec{x}-\vec{x}')$ is supported over the diagonal which has a trivial expansion at the left and right faces. In polar coordinates, we have the following identification:

$$\delta(y-y')\delta(\vec{x}-\vec{x}') = R^{-4}\delta(\omega_0-\omega_0')\delta(\hat{\omega}), \tag{1.121}$$

and this corresponds to zero in the second superscript.

Now, suppose $P$ is an $SU(2)$ bunlde over $M$ and let $(A,\Phi) \in C_P$ be a Nahm pole solution to the Kapustin-Witten equations. For simplification, let $\mathcal{D} := \mathcal{D}_{(A,\Phi)}$. We denote $\mathcal{D}_0 = y\mathcal{D}$. As pointed out in [45], $\mathcal{D}_0$ is a uniformly degenerate operator of order 1. Choose $p \geq 2$ and $q$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

In [45] Section 5.3, Mazzeo and Witten prove the following result:

**Theorem 1.10.4.** *[45] There exists operators $\mathcal{S} \in \Psi_0^{-1,1,\bar{\lambda},b}(M)$, $R_1 \in \Psi^{-\infty,\bar{\lambda},b}(M)$ and $R_2 \in \Psi^{-\infty,b,\bar{\lambda}}$ such that*

$$\mathcal{D} \circ \mathcal{S} = Id - R_1, \quad \mathcal{S} \circ \mathcal{D} = Id - R_2, \tag{1.122}$$

*where $\bar{\lambda} = 1$ for the case this paper considered and $b \geq 1$.*

In [41], Mazzeo prove the following lemma about the distribution $\Psi_0^{m,s,a,b}$.

**Lemma 1.10.5.** *[41, Theorem 3.25, Remark above Proposition 3.28]*

- *For any real number $\delta$ and $\delta'$, take $A \in \Psi^{-\infty,s,a,b}$ and let $A' = y^{\delta'} A y'^{-\delta}$ be its conjugation, then we have $A' \in \Psi^{-\infty,s+\delta-\delta',a-\delta',b+\delta}$.*
- *For $A \in \Psi^{-\infty,s,a,b}$, if $a > -\frac{1}{p}$, $b > -\frac{1}{q}$, $s \geq 0$, and $u \in L^p$, $v \in L^q$, we have $|\langle Au, v\rangle_{L^2}| \leq \|u\|_{L^p}\|v\|_{L^q}$, which implies that $A$ is a bounded operator from $L^p$ to $L^p$.*

We have the following proposition whose proof is slightly modified from [41] Section 3 due to R. Mazzeo.

**Proposition 1.10.6.** *For any real number $\lambda$ and any $p > 1$, the operator $\mathcal{D}$,*

$$\mathcal{D} : y^{\lambda+\frac{1}{p}} H_0^{1,p}(M) \to y^{\lambda+\frac{1}{p}-1} L^p(M) \tag{1.123}$$

*is a bounded linear operator.*

*Proof.* As $\mathcal{D}$ is a differential operator, the result follows immediately from the definition of $\mathcal{D}$. □

For the operator $\mathcal{S}$, we have the following proposition:

**Proposition 1.10.7.** *For $\lambda \in (-1, 1)$, the operator $\mathcal{S} \in \Psi_0^{-1,1,1,b}(M)$:*

$$\mathcal{S} : y^{\lambda+\frac{1}{p}-1} L^p(M) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(M) \tag{1.124}$$

*is a bounded linear operator.*

*Proof.* Denote the Schwartz kernel of $\mathcal{S}$ as $\kappa_S$. By choosing a cut off function over the diagonal, $\kappa_S = \kappa_S' + \kappa_S''$ where $\kappa_S'$ supported away from the left and the right faces and has a pseudodifferential singularity of order $m$ along the lift diagonal area and $\kappa_S''$ is smooth over the diagonal. Denote $S'$ ($S''$) to be the operator corresponds to the Schwartz kernel $\kappa_S'(\kappa_S'')$.

We first prove that $S' : y^{\lambda+\frac{1}{p}-1}L^p(M) \to y^{\lambda+\frac{1}{p}}H_0^{1,p}(M)$ is bounded. It is sufficient to prove that $S'y^{-1}$ is bounded operator from $y^{\lambda+\frac{1}{p}}L^p(M)$ to $y^{\lambda+\frac{1}{p}}H_0^{1,p}(M)$. We denote $A' := S'y^{-1}$ and now the $A'$ is dilation invariant. Choose a Whitney decomposition of $M$ into a union of boxes $B_i$ whose diameter in $x$ and $y$ directions is comparable to the distance to $\partial M$. For each $B_i$, we can choose an affine map $p_i$ which identifies a standard box $B$ with $B_i$. For $f \in y^{\lambda+\frac{1}{p}-1}L^p(M)$, denote by $f_i$ its restriction to $B_i$. Then $\|f\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(M)}$ and $\sum_i y_i^{-\lambda-\frac{1}{p}}\|f_i\|_{H_0^{1,p}(B_i)}$ are comparable to each other where $y_i$ can be the $y$ coordinate of any points in $B_i$ and same for $\|f\|_{y^{\lambda+\frac{1}{p}-1}L^p}$ and $\sum_i y_i^{-\lambda-\frac{1}{p}}\|f_i\|_{L^p(B_i)}$. We denote $A_i'$ to be the restriction of $A'$ over $B_i$ then we have $p_i^\star(A'f)_i = A_i'(p_i^\star f_i)$. By the approximate dilation invariance, we know $A_i'$ are a uniformly bounded family of psedodifferential operators. Then we have

$$\|A'f\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(M)} \leq C\sum_i y_i^{-\lambda-\frac{1}{p}}\|(A'f)_i\|_{H_0^{1,p}(B_i)} \leq C\sum_i y_i^{-\lambda-\frac{1}{p}}\|p_i^\star(A'f)_i\|_{H_0^{1,p}(B_i)}.$$

The classical $L^p$ theory about pseudodifferential operators of order 1 in every box [58] gives

$$\|p_i^\star(A'f)_i\|_{H_0^{1,p}(B_i)} = \|A_i'p_i^\star f_i\|_{L_1^p(B)} \leq C\|p_i^\star f_i\|_{L^p(B)} = C\|f_i\|_{L^p(B_i)}.$$

Summarizing the discussion above, we get

$$\|A'f\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(M)} \leq C\|f\|_{y^{\lambda+\frac{1}{p}}L^p(M)},$$

thus we get $\|S'f\|_{y^{\lambda+\frac{1}{p}-1}H_0^{1,p}(M)} \leq \|f\|_{y^{\lambda+\frac{1}{p}}L^p(M)}$.

Now, let's consider the operator $S''$, for some $b > 1$. For any integer $k$ and $k'$, we will show $S'' : y^{\lambda+\frac{1}{p}-1}H_0^{k,p}(M) \to y^{\lambda+\frac{1}{p}}H_0^{k',p}(M)$ is a bounded operator. As $S''$ is an infinite smoothing operator over the diagonal, we only need to prove $S''$ is bounded from $y^{\lambda+\frac{1}{p}-1}L^p$ to $y^{\lambda+\frac{1}{p}}L^p$. Denote $A'' = y^{\lambda+\frac{1}{p}}S''y^{1-\frac{1}{p}-\lambda}$, then after the shifting, on the left faces, $A''$ will be polyhomogenous with leading order $b+\lambda+\frac{1}{p}-1$. In order to get bounds of the Schwartz kernel, we require that $b+\lambda+\frac{1}{p}-1 > -\frac{1}{q}$. When $\lambda > -1$, this is automatically satisfied as $b > 1$. The leading order on the right faces will be $\bar{\lambda} - \lambda - \frac{1}{p}$, as $\lambda < 1$ and $\bar{\lambda} = 1$, we automatically get $\bar{\lambda} - \lambda - \frac{1}{p} < -\frac{1}{p}$. By applying the second bullet of lemma 1.10.5, we get $A''$ is bounded from $L^p$ to $L^p$ which implies that $S''$ is bounded from $y^{\lambda+\frac{1}{p}-1}L^p$ to $y^{\lambda+\frac{1}{p}}L^p$. □

For the operator $R_1$, $R_2$, we have the following proposition:

**Proposition 1.10.8.** *For $\lambda \in (-1, 1)$, $i = 1, 2$, and any $\lambda' \leq 1$, the operator $R_i$*

$$R_i : y^{\lambda+\frac{1}{p}}H_0^{k,p}(M) \to y^{\lambda'+\frac{1}{p}}H_0^{k',p}(M) \tag{1.125}$$

*is a bounded for any $k, k'$. In addition,*

$$R_i : y^{\lambda+\frac{1}{p}}L^p \to y^{\lambda+\frac{1}{p}}L^p$$

*is a compact operator.*

*Proof.* We first prove the bounded statement. As $R_1 \in \Psi^{-\infty,1,b}$, it is smooth over the diagonal, we only need to prove that $R_1$ is a bounded operator from $y^{\lambda+\frac{1}{p}}L^p$ to $y^{\lambda'+\frac{1}{p}}L^p$. Using the same trick as the previous proposition, we denote $R_1' = y^{\delta'}R_1y^{-\lambda-\frac{1}{p}}$. In order to get $C^0$ bound of the Schwartz kernel, now we require $\delta' < \frac{1}{p} + 1$ on the left face, which implies $\lambda' \leq 1$, same argument works for $R_2$.

By Arzela-Ascoli theorem, we get that $R_i$ is compact operator. □

**Appendix 2**

Let $X$ be a manifold with boundary $Z$ and cylindrical end with a fixed limit $SL(2;\mathbb{C})$ flat connection, then for any connection pairs $(A_0, \Phi_0)$ satisfying the Nahm boundary condition over $Z$ and converges to $SL(2;\mathbb{C})$ flat connection over the cylindrical end in $L^p_1$ norm for some $p > 2$, we will prove the closeness property of the operator $d^0_{(A_0,\Phi_0)}$. In this appendix, we assume $k \geq 0$ and $\lambda \geq -1$.

We have the following lemma about bounded linear operators between Banach spaces:

**Lemma 1.10.9.** *[62, Appendix E, Lemma E.3] Let $D : X \to Y$ be a bounded operator between Banach spaces.*

*(i) The following are equivalent:*

*• $D$ has a finite dimensional kernel and its image is closed.*

*• There exists a compact operator $K : X \to Z$ to another Banach space $Z$ and a constant $C$ such that*

$$\|u\|_X \leq C(\|Du\|_Y + \|Ku\|_Z) \ \forall u \in X. \tag{1.126}$$

*(ii) The following are equivalent:*

• *D is injectie and its image is closed.*

• *There exists a constant C such that*

$$\|u\|_X \leq C\|Du\|_Y \ \forall u \in X. \tag{1.127}$$

In particular, if a bounded linear operator satisfies (1.126) and it is injective, then it satisfies (3.45).

Consider the operator $d_{A_0}$ associated with the following norms defined as:

$$d_{A_0} : y^{\lambda+\frac{1}{p}+1} H_0^{2,p}(\Omega^0(\mathfrak{g}_P)) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(\Omega^1(\mathfrak{g}_P)). \tag{1.128}$$

By the definition of the norm, $d_{A_0}$ is a bounded linear operator.

Let $\Omega^{odd}(\mathfrak{g}_P)$ be the direct sum of odd differential forms and let $\Omega^{even}(\mathfrak{g}_P)$ be the direct sum of even differential forms. Consider the following operator:

$$\mathcal{K} : y^{\lambda+\frac{1}{p}+1} H_0^{2,p}(\Omega^{even}(\mathfrak{g}_P)) \to y^{\lambda+\frac{1}{p}} H_0^{1,p}(\Omega^{edd}(\mathfrak{g}_P)). \tag{1.129}$$

We denote $\mathcal{K}_0 = y\mathcal{K}$ and we will study the semi Fredholm property of operators $\mathcal{K}$ and $\mathcal{K}_0$.

**Proposition 1.10.10.** *$\mathcal{K}_0$ is a uniformly degenerate elliptic operator and 0 is the only indicial root.*

*Proof.* The statement of uniformly degenerate elliptic operator is obvious. The indicial operator of $\mathcal{K}_0$ is $I(\mathcal{K}_0, \lambda) = A_{10}\lambda$ where $A_{10}$ is an invertible matrix. Thus $I(\mathcal{K}_0, \lambda)$ is not invertible if and only if $\lambda = 0$. □

In [41], Mazzeo proves the following semi Fredholm theory of uniformly degenerate operator:

**Theorem 1.10.11.** *[41, Theorem 6.1] For any $\lambda > 0$, there exist operators G and P such that*

$$G\mathcal{K}_0 = Id - P.$$

*Here G is a bounded operator $G : y^{\lambda+\frac{1}{p}} H_0^{1,p} \to y^{\lambda+\frac{1}{p}} H_0^{2,p}$ and P is a compact operator.*

**Remark.** *As there is only one indicial root, the $\bar{\lambda}$ in the original statement has to be 0. The bounded operator statement and compact operator statement can be proved in a similar way as Proposition 1.10.7 and Proposition 1.10.8.*

An immediately corollary of this theorem is that

**Corollary 1.10.12.** *When $\lambda > 0$, $\mathcal{K}_0$ has finite dimensional kernel and closed range.*

*Proof.* By the previous theorem, as $P$ is a compact operator and if $f \in \mathrm{Ker}\mathcal{K}_0$, we have $Pf = f$. Therefore, the kernel of $P$ is finite dimensional. By Lemma 1.10.9 and the boundness property of $G$, we know $\mathcal{K}_0$ has closed range. □

We have the following proposition:

**Proposition 1.10.13.** *For $\lambda > -1$, the $d_{A_0} : y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\Omega^0(\mathfrak{g}_P)) \to y^{\lambda+\frac{1}{p}}H_0^{1,p}(\Omega^1(\mathfrak{g}_P))$ has finite dimensional kernel and closed range.*

*Proof.* WLOG, we can assume $\mathrm{Ker}d_{A_0}$ is zero and prove the closed range statement. As $\Omega^0(\mathfrak{g}_P)$ is a closed subset of $\Omega^{even}(\mathfrak{g}_P)$, the restriction of $\mathcal{K}_0$ over $\Omega^0(\mathfrak{g}_P)$ which is $yd_{A_0}$ also has closed image. By Lemma 1.10.9, we have the following inequality

$$\|u\|_{y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\Omega^0(\mathfrak{g}_P))} \leq C\|yd_{A_0}u\|_{y^{\lambda+\frac{1}{p}+1}H_0^{1,p}(\Omega^1(\mathfrak{g}_P))}, \tag{1.130}$$

which implies

$$\|u\|_{y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\Omega^0(\mathfrak{g}_P))} \leq C\|d_{A_0}u\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(\Omega^1(\mathfrak{g}_P))}. \tag{1.131}$$

Thus $d_{A_0}$ has closed range. □

**Remark.** *If* Ker $d_{A_0}$*=0, we have $\|\xi\|_{y^{\lambda+\frac{1}{p}+1}} \leq C\|d_{A_0}\xi\|_{y^{\lambda+\frac{1}{p}}}$ which is a gauge theory version of the $L^p$ Hardy inequality over $\mathbb{R}^4_+$ for compact supported functions $u$ and $s = p\lambda + p + 1$:*

$$\left(\int_{\mathbb{R}^4_+} y^{p-s}|\partial_y u|^p\right)^{\frac{1}{p}} \geq \frac{n-1}{p}\left(\int_{\mathbb{R}^4_+} y^{-s}|u|^p\right)^{\frac{1}{p}}. \tag{1.132}$$

Now we have the following proposition:

**Proposition 1.10.14.** *The operator*

$$d^0_{(A_0,\Phi_0)} : y^{\lambda+1+\frac{1}{p}}H_0^{2,p}(\Omega^0(\mathfrak{g}_P)) \to y^{\lambda+\frac{1}{p}}H_0^{1,p}(\Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P))$$

*is a closed operator with finite dimensional kernel.*

Recall the definition of $d^0_{(A_0,\Phi_0)}$ is $d^0_{(A_0,\Phi_0)}(\xi) = (d_{A_0}(\xi), [\Phi_0, \xi])$. Therefore, we obtain Ker $d^0_{(A_0,\Phi_0)} \subset$ Ker $d_{A_0}$ and by Proposition 1.10.13, we know Ker $d^0_{(A_0,\Phi_0)}$ has finite dimension.

Without loss of generality, we assume Ker $d^0_{(A_0,\Phi_0)} = 0$. By Proposition 1.10.13, there exists a constant such that $\|u\|_{y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\Omega^0(\mathfrak{g}_P))} \leq C\|d_{A_0}u\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(\Omega^1(\mathfrak{g}_P))}$. By adding a positive term on the right hand side of the inequality, we have

$$
\begin{aligned}
&\|u\|_{y^{\lambda+\frac{1}{p}+1}H_0^{2,p}(\Omega^0(\mathfrak{g}_P))} \\
\leq &C(\|d_{A_0}u\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(\Omega^1(\mathfrak{g}_P))} + \|[\Phi_0, u]\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}(\Omega^1(\mathfrak{g}_P))}) \\
= &C\|d^0_{(A_0,\Phi_0)}u\|_{y^{\lambda+\frac{1}{p}}H_0^{1,p}}.
\end{aligned}
$$

Applying Lemma 1.10.9, $d^0_{(A_0,\Phi_0)}$ is a closed operator. □

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*Chapter 2*

# ROTATIONALLY INVARIANT SINGULAR SOLUTIONS TO THE KAPUSTIN-WITTEN EQUATIONS

## 2.1 Introduction

In [63], Witten proposed a new physical interpretation of the Jones polynomial and Khovanov homology in terms of counting the solutions of a certain supersymmetric gauge theory in four dimensions and five dimensions. The BPS equations of N=4 twisted super Yang-Mills theory in four dimensions are called the topological twisted equations [34] and play an essential role in this framework.

Let $X$ be a connected, smooth, oriented 4-manifold with a Riemannian metric. Let $P$ be a chosen principle $SU(2)$ bundle over $X$ and let $\mathrm{ad}(P)$ denote the adjoint bundle of $P$. The topological twisted equations are equations for a pair $(A, \phi)$ where A is a connection on $P$ and $\phi$ is a $\mathrm{ad}(P)$ valued 1-form. These equations have the following form:

$$\begin{aligned}(F_A - \phi \wedge \phi + \lambda d_A \phi)^+ &= 0,\\ (F_A - \phi \wedge \phi - \lambda^{-1} d_A \phi)^- &= 0,\\ d_A^\star \phi &= 0.\end{aligned} \tag{2.1}$$

Witten points out that the most interesting case to study is when $\lambda = -1$. In this case, we obtain the following equations, which we call the Kapustin-Witten equations:

$$\begin{aligned}F_A - \phi \wedge \phi - \star d_A \phi &= 0,\\ d_A^\star \phi &= 0.\end{aligned} \tag{2.2}$$

In [34], Kapustin and Witten prove that over a closed manifold, all the regular solutions to the Kapustin-Witten equations are flat $SL(2; \mathbb{C})$ connections. Therefore, the regular solutions to these equations are not so interesting over closed manifolds. However, the Kapustin-Witten equations are interesting over non-compact spaces with singular boundary conditions. Witten's gauge theory approach [63] to the Jones polynomial conjectures that the coefficients of the Jones polynomial of a knot are determined by counting the solutions to the Kapustin-Witten equations with Nahm pole boundary conditions. See also Gaiotto and Witten [25] for an approach to this conjecture. The case of the empty knot is resolved in [44].

In addition, Taubes studied the compactness properties of the Kapustin-Witten equations [57][56]. He shows that there can be only two sources of non-compactness. One is the traditional Uhlenbeck bubbling phenomenon [60] [61], and another is the non-compactness coming from the unboundness of the $L^2$ norm of $\phi$.

Therefore, a natural question to ask is whether the Uhlenbeck bubbling phenomenon can appear for solutions to the Kapustin-Witten equations. In addition, do we have a model solution to the Kapustin-Witten equations.

In this paper, we construct some singular solutions to the Kapustin-Witten equations.

To be more precise, consider the trivial $SU(2)$ bundle $P_0$ over $\mathbb{R}^4$. Denote $x$ to be a point in $\mathbb{R}^4$, after identifying $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and the adjoint bundle $\mathrm{ad}(P_0)$ with imaginary part of the quaternions, we prove the following:

**Theorem 2.1.1.** *for any real number C, the formulas*

$$\begin{cases} A(x) = \mathrm{Im}\Big(\dfrac{3C}{C^2|x|^4 + 4C|x|^2 + 1}\ \bar{x}dx\Big) \\ \phi(x) = \mathrm{Im}\Big(\dfrac{3C(C|x|^2 + 1)}{(C^2|x|^4 + 4C|x|^2 + 1)(C|x|^2 - 1)}\ \bar{x}dx\Big) \end{cases} \tag{2.3}$$

*give solutions to the Kapustin-Witten equations* (4.1) *with the following properties:*

(1) *For $C \neq 0$, the solutions are smooth away from $|x| = \frac{1}{\sqrt{C}}$ and decay to 0 when $|x| \to \infty$.*

(2) *The solutions have instanton number* 0*.*

(3) *When $C \to +\infty$, $|F_A|$ converges to a Dirac measure at x=0.*

(4) *For $C \neq 0$, the pole singularity of $\phi$ at $|x| = \frac{1}{\sqrt{C}}$ cannot be removed by $SU(2)$ gauge transformations.*

In addition, we also prove the following theorem:

**Theorem 2.1.2.** *There exists a family of rotationally invariant solutions to the Kapustin-Witten equations on Euclidean $\mathbb{R}^4$ with instanton number $\pm 1$. These solutions are smooth away from a sphere where the real parts are $C^1$ and the imaginary parts are singular.*

In addition, given an integer $k$, we can generalize the ADHM construction [6] and obtain the following theorem:

**Theorem 2.1.3.** *Given an integer k, there exists a* $5|k|$ *dimensional family of singular solutions to the Kapustin-Witten equations on Euclidean* $\mathbb{R}^4$*. When* $k = \pm 1$*, these include the solutions from Theorem* 1.2*.*

We conjecture that under some non-degeneracy condition, the solutions we obtain in Theorem 2.1.3 have instanton number $k$.

In addition, Witten in [63] suggested to study solutions to (4.1) with a singular boundary condition called the Nahm pole boundary condition. Solutions with the Nahm pole boundary condition play an important role in the gauge theory approach to the Jones polynomial.

In the last chapter, we observe the relation between the singularity which appears in Thm 1.1 and the Nahm pole boundary condition and get the following theorem:

**Theorem 2.1.4.** *There exist two Nahm pole solutions to the Kapustin-Witten on* $S^3 \times (0, +\infty)$*, with instanton number* $\frac{1}{2}$ *and* $-\frac{1}{2}$*.*

In Section 2, we find a system of non-linear ODEs which will give rotationally invariant solutions to the Kapustin-Witten equations. In Section 3, we find a first integral of these ODEs and solve them to obtain the solutions in Theorem 1.1. In Section 4, we prove the rest part of Theorem 1.1. In Section 5, we construct other families of solutions to the Kapustin-Witten equations and prove Theorem 1.2 and Theorem 1.3. In section 6, we build up the relation of our singular solution and Nahm pole.

## 2.2 ODEs from the Kapustin-Witten Equations

### Background

In accordance with the philosophy of the ADHM construction [5][6] for the anti-self-dual equation, we use quaternions to describe the gauge field in $\mathbb{R}^4$. We begin by briefly recalling the elementary properties of quaternions.

We have three elements $I,\ J,\ K$ satisfying the identities: $I^2 = J^2 = K^2 = -1,\ IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J$. A general quaternion $x$ is of the following form:

$$x = x_1 + x_2 I + x_3 J + x_4 K,$$

where $x_1, x_2, x_3, x_4$ are real numbers. After choosing a canonical basis of $\mathbb{R}^4$, we can naturally identify points in $\mathbb{R}^4$ with quaternions. The conjugate quaternion is given by

$$\bar{x} = x_1 - x_2 I - x_3 J - x_4 K$$

and with we have the relation $\overline{xy} = \bar{y}\bar{x}$. In addition, we also know that $x\bar{x} = \bar{x}x = |x|^2 = \sum x_i^2$. For $x = x_1 + x_2 I + x_3 J + x_4 K$, the imaginary part of x is $\mathrm{Im}(x) := x_2 I + x_3 J + x_4 K$. Therefore, the Lie group $SU(2)$ can be identified with the unitary quaternions and the Lie algebra **su**(2) can be identified with the imaginary part of the quaternions.

Using the well known isomorphism of the Lie group $SO(4)$ with $SU(2) \times SU(2)/\sim$, the action of $SO(4)$ on a quaternion $x$ is given by $x \longrightarrow axb$, where $a$, $b$ are unitary quaternions.

In order to find rotationally invariant solutions, we assume that the gauge fields of (2.1) (4.1) have the following form:

$$\begin{aligned} A(x) &: = \mathrm{Im}(f(t)\ \bar{x}dx) \\ \phi(x) &: = \mathrm{Im}(g(t)\ \bar{x}dx) \\ t &: = |x|^2. \end{aligned} \tag{2.4}$$

Here $f(t),\ g(t)$ are real functions with variable $t = |x|^2$. Obviously, $t \geq 0$.

**Remark.** *In the remaining part of the paper, we use $f', g'$ to simplify writing $\frac{df(t)}{dt}$ and $\frac{dg(t)}{dt}$.*

**Proposition 2.2.1.** *$A(x)$ and $\phi(x)$ defined as in* (2.4) *are rotationally invariant up to gauge equivalence.*

*Proof.* It is easy to see that for $a,\ b$ are two unitary quaternions, under the change $x \longrightarrow axb$, we obtain $|axb|^2 = |x|^2$, $A(axb) = \mathrm{Im}(f(t)\ \overline{axb}\ d(axb)) = \bar{b}\mathrm{Im}(f(t)\ \bar{x}dx)b$. Therefore, $A(axb)$ is gauge equivalent to $A(x)$ by a constant gauge transformation. Similarly, we can show $\phi(x)$ is also rotationally invariant up to the same gauge transformation. □

**Basic Properties of Rotationally Invariant Connections**

As the equations (4.1) depend on the metric, we need to be explicit about the metric we choose.

**Definition 2.2.2.** *A metric $g$ on $\mathbb{R}^4$ is called rotationally invariant if in quaternion coordinate $g = h(t)dx \otimes d\bar{x}$. $h(t)$ here is a positive function, $t = |x|^2$.*

**Example 2.2.3.** *The Euclidean metric $dx \otimes d\bar{x}$ and the round metric $\frac{4}{(1+t)^2} dx \otimes d\bar{x}$ on $\mathbb{R}^4$ are both rotationally invariant metrics.*

**Remark.** *In the rest of the paper, all the metrics we considered are rotationally invariant.*

Now, we will introduce some basic properties of connections in (2.4).

**Lemma 2.2.4.** $\mathrm{Im}(\bar{x}dx \wedge \bar{x}dx) = -\frac{1}{2}|x|^2\, d\bar{x} \wedge dx - \frac{1}{2}\, \bar{x}dx \wedge d\bar{x}x.$

*Proof.* Since the wedge product of a real form with itself is zero, we know that

$$\Re(\bar{x}dx) \wedge \Re(\bar{x}dx) = 0.$$

Since

$$\Re(\bar{x}dx) = \frac{\bar{x}dx + d\bar{x}x}{2},$$

we obtain

$$\begin{aligned} 0 =& \Re(\bar{x}dx) \wedge \Re(\bar{x}dx) \\ =& \frac{(\bar{x}dx + d\bar{x}x) \wedge (\bar{x}dx + d\bar{x}x)}{4} \\ =& \frac{\bar{x}dx \wedge \bar{x}dx + \bar{x}dx \wedge d\bar{x}x + t\ d\bar{x} \wedge dx + d\bar{x}x \wedge d\bar{x}x}{4}. \end{aligned}$$

In addition, we have

$$\mathrm{Im}(\bar{x}dx \wedge \bar{x}dx) = \frac{\bar{x}dx \wedge \bar{x}dx + d\bar{x}x \wedge d\bar{x}x}{2}.$$

The plus sign on the right hand side of the above identity is because given two quaternion one forms $\omega_1, \omega_2$, we have $\overline{\omega_1 \wedge \omega_2} = -\bar{\omega}_2 \wedge \bar{\omega}_1$.

The result follows immediately. □

**Lemma 2.2.5.** $\mathrm{Im}(\bar{x}dx) \wedge \mathrm{Im}(\bar{x}dx) = \mathrm{Im}(\bar{x}dx \wedge \bar{x}dx)$.

*Proof.* We calculate that

$$\begin{aligned} & \mathrm{Im}(\bar{x}dx) \wedge \mathrm{Im}(\bar{x}dx) \\ =& \frac{(\bar{x}dx - d\bar{x}x) \wedge (\bar{x}dx - d\bar{x}x)}{4} \\ =& \frac{\bar{x}dx \wedge \bar{x}dx - \bar{x}dx \wedge d\bar{x}x - t\ d\bar{x} \wedge dx + d\bar{x}x \wedge d\bar{x}x}{4} \\ =& \frac{\bar{x}dx \wedge \bar{x}dx + d\bar{x}x \wedge d\bar{x}x}{2} \text{ (by Lemma 2.2.4)} \\ =& \mathrm{Im}(\bar{x}dx \wedge \bar{x}dx). \end{aligned}$$

□

**Lemma 2.2.6.**

$dx \wedge d\bar{x} = -2((dx_1 \wedge dx_2 + dx_3 \wedge dx_4)I + (dx_1 \wedge dx_3 + dx_4 \wedge dx_2)J + (dx_1 \wedge dx_4 + dx_2 \wedge dx_3)K)$

$d\bar{x} \wedge dx = 2((dx_1 \wedge dx_2 + dx_4 \wedge dx_3)I + (dx_1 \wedge dx_3 + dx_2 \wedge dx_4)J + (dx_1 \wedge dx_4 + dx_3 \wedge dx_2)K).$

*Proof.* By direct computation. □

**Remark.** *Given a rotationally invariant metric $g$ (Definition 2.2.2), we can define the Hodge star operator with respect to $g$. Denote $\Omega^{2+}(\Omega^{2-})$ to be the self-dual (anti-self-dual) two-forms with respect to the Hodge star operator. It is easy to see that $dx \wedge d\bar{x} \in \Omega^{2+}$ and $d\bar{x} \wedge dx \in \Omega^{2-}$. Also* $\mathrm{Im}(dx \wedge d\bar{x}) = dx \wedge d\bar{x}$ *and* $\mathrm{Im}(d\bar{x} \wedge dx) = d\bar{x} \wedge dx$.

**Separating Terms in the Topological-Twisted equations**

Since the equations in (2.1) are separated into the self-dual parts and the anti-self-dual parts, we also want to separate our calculation into the self-dual parts and the anti-self-dual parts.

**Lemma 2.2.7.** *For $A(x)$ defined as in* (2.4)*, we have*

$$F_A^+ = -\frac{1}{2}(f' + f^2)\ \bar{x}dx \wedge d\bar{x}x$$

$$F_A^- = (\frac{1}{2}tf' - \frac{1}{2}tf^2 + f)\ d\bar{x} \wedge dx.$$

*Proof.* We calculate that

$$\begin{aligned}
F_A &= dA + A \wedge A \\
&= d\mathrm{Im}(f\ \bar{x}dx) + \mathrm{Im}(f^2\ \bar{x}dx \wedge \bar{x}dx) \\
&= \mathrm{Im}(df\ \bar{x}dx) + \mathrm{Im}(f\ d\bar{x} \wedge dx) + \mathrm{Im}(f^2\ \bar{x}dx \wedge \bar{x}dx) \\
&= \mathrm{Im}((f' + f^2)\ \bar{x}dx \wedge \bar{x}dx) + \mathrm{Im}((f't + f)\ d\bar{x} \wedge dx) \text{ (by } x\bar{x} = |x|^2 = t) \\
&= -\frac{1}{2}(f' + f^2)t\ d\bar{x} \wedge dx + (f't + f)\ d\bar{x} \wedge dx - \frac{1}{2}(f' + f^2)\ \bar{x}dx \wedge d\bar{x}x \text{ (by Lemma 2.2.4)} \\
&= (\frac{1}{2}tf' - \frac{1}{2}tf^2 + f)\ d\bar{x} \wedge dx - \frac{1}{2}(f' + f^2)\ \bar{x}dx \wedge d\bar{x}x.
\end{aligned}$$

The result follows immediately. □

**Lemma 2.2.8.** *For $\phi(x)$ defined as in* (2.4)*, we have*

$$(\phi \wedge \phi)^+ = -\frac{1}{2} g^2\ \bar{x}dx \wedge d\bar{x}x$$

$$(\phi \wedge \phi)^- = -\frac{1}{2} g^2 t\ d\bar{x} \wedge dx.$$

*Proof.* We calculate that

$$\begin{aligned}
\phi \wedge \phi =& \mathrm{Im}(g\ \bar{x}dx) \wedge \mathrm{Im}(g\ \bar{x}dx) \\
=& \mathrm{Im}(g^2\ \bar{x}dx \wedge \bar{x}dx) \text{ (by Lemma 2.2.5)} \\
=& -\frac{1}{2} g^2 t\ d\bar{x} \wedge dx - \frac{1}{2} g^2\ \bar{x}dx \wedge d\bar{x}x. \text{ (by Lemma 2.2.4)}
\end{aligned}$$

□

**Lemma 2.2.9.** *For $(A(x),\ \phi(x))$ defined as in* (2.4)*, we have*

$$(d_A\phi)^+ = -\frac{1}{2}(g' + 2fg)\ \bar{x}dx \wedge d\bar{x}x$$

$$(d_A\phi)^- = (\frac{1}{2} g' t + g - fgt)\ d\bar{x} \wedge dx.$$

*Proof.* We calculate that

$$\begin{aligned}
d_A\phi =& d\phi + A \wedge \phi + \phi \wedge A \\
=& d\mathrm{Im}(g\ \bar{x}dx) + \mathrm{Im}(2fg\ \bar{x}dx \wedge \bar{x}dx) \text{ (by Lemma 2.2.5)} \\
=& \mathrm{Im}(dg\ \bar{x}dx) + \mathrm{Im}(g\ d\bar{x} \wedge dx) + \mathrm{Im}(2fg\bar{x}dx \wedge \bar{x}dx) \\
=& \mathrm{Im}((g' + 2fg)\ \bar{x}dx \wedge \bar{x}dx) + \mathrm{Im}((tg' + g)\ d\bar{x} \wedge dx) \\
=& (-\frac{1}{2}(g' + 2fg)t + (g't + g))\ d\bar{x} \wedge dx - \frac{1}{2}(g' + 2fg)\ \bar{x}dx \wedge d\bar{x}x \\
=& (\frac{1}{2} g' t + g - fgt)\ d\bar{x} \wedge dx - \frac{1}{2}(g' + 2fg)\ \bar{x}dx \wedge d\bar{x}x.
\end{aligned}$$

□

Now, we will discuss the third equation of (2.1).

At first, we have the following identity:

**Lemma 2.2.10.** $x_1\mathrm{Im}(\bar{x}) + x_2\mathrm{Im}(\bar{x}I) + x_3\mathrm{Im}(\bar{x}J) + x_4\mathrm{Im}(\bar{x}K) = 0.$

*Proof.* We calculate that

$$
\begin{aligned}
&x_1\mathrm{Im}(\bar{x}) + x_2\mathrm{Im}(\bar{x}I) + x_3\mathrm{Im}(\bar{x}J) + x_4\mathrm{Im}(\bar{x}K)\\
=&x_1(-x_2I - x_3J - x_4K) + x_2(x_1I + x_3K - x_4J)\\
&+ x_3(x_1J - x_2K + x_4I) + x_4(x_1K + x_2J - x_3I)\\
=&0.
\end{aligned}
$$

□

**Lemma 2.2.11.** *Given a rotational invariant metric $h(t)dx \otimes d\bar{x}$, denote $\star$ to be the Hodge star operator with respect to this metric, we have $d(\mathrm{Im}(\bar{x}) \star dx_1 + \mathrm{Im}(\bar{x}I) \star dx_2 + \mathrm{Im}(\bar{x}J) \star dx_3 + \mathrm{Im}(\bar{x}K) \star dx_4) = 0$.*

*Proof.* By definition, we have $\mathrm{Im}(\bar{x}) = -x_2I - x_3J - x_4K$, $\star(dx_1) = hdx_2 \wedge dx_3 \wedge dx_4$, therefore $(d\mathrm{Im}(\bar{x})) \star dx_1 = 0$. Similarly, we have $(d\mathrm{Im}(\bar{x}I)) \star dx_2 = (d\mathrm{Im}(\bar{x}J)) \star dx_3 = (d\mathrm{Im}(\bar{x}K)) \star dx_4 = 0$.

Therefore,

$$
\begin{aligned}
&d(\mathrm{Im}(\bar{x}) \star dx_1 + \mathrm{Im}(\bar{x}I) \star dx_2 + \mathrm{Im}(\bar{x}J) \star dx_3 + \mathrm{Im}(\bar{x}K) \star dx_4)\\
=&2h^{'}(x_1\mathrm{Im}(\bar{x}) + x_2\mathrm{Im}(\bar{x}I) + x_3\mathrm{Im}(\bar{x}J) + x_4\mathrm{Im}(\bar{x}K)) \qquad (2.5)\\
=&0.
\end{aligned}
$$

□

**Lemma 2.2.12.** *For $(A(x),\ \phi(x))$ defined as in (2.4), we have $A \wedge \star\phi + \star\phi \wedge A = 0$.*

*Proof.* For $A \wedge \star\phi$, we calculate that

$$
\begin{aligned}
&A \wedge \star\phi\\
=&fg\ \mathrm{Im}(\bar{x}dx) \wedge \star\mathrm{Im}(\bar{x}dx)\\
=&fg\ (\mathrm{Im}(\bar{x})dx_1 + \mathrm{Im}(\bar{x}I)dx_2 + \mathrm{Im}(\bar{x}J)dx_3 + \mathrm{Im}(\bar{x}K)dx_4)\wedge\\
&(\mathrm{Im}(\bar{x}) \star dx_1 + \mathrm{Im}(\bar{x}I) \star dx_2 + \mathrm{Im}(\bar{x}J) \star dx_3 + \mathrm{Im}(\bar{x}K) \star dx_4)\\
=&fg\ (\mathrm{Im}(\bar{x})\mathrm{Im}(\bar{x})dx_1 \wedge \star dx_1 + \mathrm{Im}(\bar{x}I)\mathrm{Im}(\bar{x}I)dx_2 \wedge \star dx_2\\
&+ \mathrm{Im}(\bar{x}J)\mathrm{Im}(\bar{x}J)dx_3 \wedge \star dx_3 + \mathrm{Im}(\bar{x}K)\mathrm{Im}(\bar{x}K)dx_4 \wedge \star dx_4).
\end{aligned}
$$

In addition, we calculate that

$$\begin{aligned}
&\star\phi\wedge A\\
=&fg\ \star\mathrm{Im}(\bar{x}dx)\wedge\mathrm{Im}(\bar{x}dx)\\
=&fg\ (\mathrm{Im}(\bar{x})\star dx_1+\mathrm{Im}(\bar{x}I)\star dx_2+\mathrm{Im}(\bar{x}J)\star dx_3+\mathrm{Im}(\bar{x}K)\star dx_4)\wedge\\
&(\mathrm{Im}(\bar{x})dx_1+\mathrm{Im}(\bar{x}I)dx_2+\mathrm{Im}(\bar{x}J)dx_3+\mathrm{Im}(\bar{x}K)dx_4)\\
=&fg\ (\mathrm{Im}(\bar{x})\mathrm{Im}(\bar{x})\star dx_1\wedge dx_1+\mathrm{Im}(\bar{x}I)\mathrm{Im}(\bar{x}I)\star dx_2\wedge dx_2\\
&+\mathrm{Im}(\bar{x}J)\mathrm{Im}(\bar{x}J)\star dx_3\wedge dx_3+\mathrm{Im}(\bar{x}K)\mathrm{Im}(\bar{x}K)\star dx_4\wedge dx_4)\\
=&-A\wedge\star\phi.
\end{aligned}$$

Therefore, we obtain $A\wedge\star\phi+\phi\wedge\star A=0$. □

**Proposition 2.2.13.** *For $(A(x),\ \phi(x))$ defined as in (2.4), for a Hodge star operator correspond to a rotational invariant metric $h(t)dx\otimes d\bar{x}$, we have $d_A\star\phi=0$.*

*Proof.* By definition,

$$d_A\star\phi=d\star\phi+A\wedge\star\phi+\star\phi\wedge A.$$

First, we compute $d\star\phi=0$.

Take $\star_E$ to be the Hodge star operator correspond to the Euclidean metric in $\mathbb{R}^4$, then $\star dx_i=h(t)^2\star_E dx_i$.

By (2.4), we have

$$\begin{aligned}
\phi&=g\ \mathrm{Im}(\bar{x}dx)\\
&=g\ (\mathrm{Im}(\bar{x})dx_1+\mathrm{Im}(\bar{x}I)dx_2+\mathrm{Im}(\bar{x}J)dx_3+\mathrm{Im}(\bar{x}K)dx_4).
\end{aligned}$$

Therefore, we calculate

$$\begin{aligned}
&d\star\phi\\
&=d(gh^2)\ ((\mathrm{Im}(\bar{x})\star_E dx_1+\mathrm{Im}(\bar{x}I)\star_E dx_2+\mathrm{Im}(\bar{x}J)\star_E dx_3+\mathrm{Im}(\bar{x}K)\star_E dx_4))\\
&+g\ d(\mathrm{Im}(\bar{x})\star_E dx_1+\mathrm{Im}(\bar{x}I)\star_E dx_2+\mathrm{Im}(\bar{x}J)\star_E dx_3+\mathrm{Im}(\bar{x}K)\star_E dx_4)\\
&=\frac{\partial(gh^2)}{\partial x_1}\ \mathrm{Im}(\bar{x})dx_1\wedge\star_E dx_1+\frac{\partial(gh^2)}{\partial x_2}\ \mathrm{Im}(\bar{x}I)dx_2\wedge\star_E dx_2\\
&+\frac{\partial(gh^2)}{\partial x_3\,\partial x_4}\mathrm{Im}(\bar{x}J)dx_3\wedge\star_E dx_3+\frac{\partial(gh^2)}{}\ \mathrm{Im}(\bar{x}K)dx_4\wedge\star_E dx_4\ \text{(by Lemma 2.2.11)}\\
&=2(gh^2)'\,(x_1\mathrm{Im}(\bar{x})+x_2\mathrm{Im}(\bar{x}I)+x_3\mathrm{Im}(\bar{x}J)+x_4\mathrm{Im}(\bar{x}K))dx_1\wedge dx_2\wedge dx_3\wedge dx_4\ \text{(by Lemma 2.2.10)}\\
&=0.
\end{aligned}$$

Combining this with Lemma 2.2.12, we finish the proof.

□

**ODEs from the Kapustin-Witten Equations**

Recall that the topological twisted equations (2.1) are equivalent to the following:

$$\begin{aligned} F_A^+ - (\phi \wedge \phi)^+ &= -\lambda (d_A\phi)^+ \\ F_A^- - (\phi \wedge \phi)^- &= \lambda^{-1} (d_A\phi)^- \\ d_A^\star \phi &= 0. \end{aligned}$$

By Proposition 2.2.13, we know that $d_A^\star \phi = 0$ is always satisfied under our assumption (2.4).

Combining Lemma 2.2.7 and 2.2.9, we obtain the following ODEs:

$$\begin{cases} f' + \lambda g' + f^2 - g^2 + 2\lambda f g = 0 \\ t f' - t\lambda^{-1} g' + 2f - 2\lambda^{-1} g + g^2 t - f^2 t + 2t f g \lambda^{-1} = 0. \end{cases} \tag{2.6}$$

To summarize the previous computation, we have the following theorem:

**Theorem 2.2.14.** *Given a solution* $(f(t),\ g(t))$ *to the ODEs* (2.6)*, taking* $A(x) = \mathrm{Im}(f(x)\ \bar{x}dx)$ *and* $\phi(x) = \mathrm{Im}(g(x)\ \bar{x}dx)$ *gives a solution to the topological twisted equations* (2.1).

By some linear transformations, we obtain the following ODEs:

$$\begin{cases} (\lambda + \lambda^{-1}) t f' + 2\lambda f - (\lambda - \lambda^{-1})(t f^2 - t g^2) - 2g + 4 f g t = 0 \\ (\lambda + \lambda^{-1}) t g' + 2\lambda^{-1} g - 2f + (\lambda - \lambda^{-1}) 2 f g t + 2t(f^2 - g^2) = 0. \end{cases} \tag{2.7}$$

Taking $\lambda = -1$, we obtain

$$\begin{cases} t f' + f + g - 2 f g t = 0 \\ t g' + g + f - t(f^2 - g^2) = 0. \end{cases} \tag{2.8}$$

We call the equations (2.8) the **Kapustin-Witten ODEs**.

**Remark.** *The equations* (2.7) *are degenerate at t=0, which means that we may not have the uniqueness theorem for a given initial value. Given a solution* $(f(t),\ g(t))$ *to* (2.7)*, if we assume* $(f(t),\ g(t))$ *is continous near* $t = 0$ *and* $\lim_{t\to 0}(t f'(t),\ t g'(t)) = (0, 0)$*, we can take* $t \to 0$ *in both sides of the equations* (2.7) *and we obtain that* $\lambda f(0) = g(0)$.

## 2.3 Explicit Solutions for ODEs

In this section, we will discuss some properties of the equations (2.7) and explicitly solve the equations (2.7) for some special cases.

**Change of Variables**

We are going to simplify the equations (2.7) by a change of variables.

Take $\tilde{f}(t) := tf(t)$, $\tilde{g}(t) := tg(t)$, then the equations (2.7) become

$$\begin{cases} \frac{\lambda+\lambda^{-1}}{2} t\tilde{f}' = \frac{\lambda-\lambda^{-1}}{2}(\tilde{f}^2 - \tilde{g}^2 - \tilde{f}) + \tilde{g} - 2\tilde{f}\tilde{g} \\ \frac{\lambda+\lambda^{-1}}{2} t\tilde{g}' = \frac{\lambda-\lambda^{-1}}{2}(\tilde{g} - 2\tilde{f}\tilde{g}) + (\tilde{f} - \tilde{f}^2 + \tilde{g}^2). \end{cases} \tag{2.9}$$

**Remark.** *It is easy to see that if* $(f, g)$ *is a solution for some parameter* $\lambda_0$*, then* $(f, -g)$ *is a solution for* $-\lambda_0$*. This is compatible with changing the orientation of the manifold in the topological twisted equation* (2.1).

Taking $u(t) := \tilde{f}(t) - \frac{1}{2}$, $v(t) := \tilde{g}(t)$, we obtain

$$\begin{cases} \frac{\lambda+\lambda^{-1}}{2} tu' = \frac{\lambda-\lambda^{-1}}{2}(u^2 - v^2 - \frac{1}{4}) - 2uv \\ \frac{\lambda+\lambda^{-1}}{2} tv' = -\frac{\lambda-\lambda^{-1}}{2} 2uv - (u^2 - v^2 - \frac{1}{4}). \end{cases} \tag{2.10}$$

In order to obtain an autonomous ODE systems, we take $\tilde{u}(s) := u(e^s)$ and $\tilde{v}(s) := v(e^s)$. We obtain

$$\begin{cases} \frac{\lambda+\lambda^{-1}}{2} \tilde{u}' = \frac{\lambda-\lambda^{-1}}{2}(\tilde{u}^2 - \tilde{v}^2 - \frac{1}{4}) - 2\tilde{u}\tilde{v} \\ \frac{\lambda+\lambda^{-1}}{2} \tilde{v}' = -\frac{\lambda-\lambda^{-1}}{2} 2\tilde{u}\tilde{v} - (\tilde{u}^2 - \tilde{v}^2 - \frac{1}{4}). \end{cases} \tag{2.11}$$

Here $\tilde{u}' := \frac{d\tilde{u}(s)}{ds}$ and $\tilde{v}' := \frac{d\tilde{v}(s)}{ds}$.

**Some Basic Properties**

Even though the equations (2.11) are non-linear, we can find a first integral which can simplify the equations in some special cases.

**Proposition 2.3.1.** *Given* $(\tilde{u}, \tilde{v})$ *a solution to the equations* (2.11) *,* $I(\tilde{u}, \tilde{v}) = \frac{1}{3}\tilde{u}^3 - \tilde{u}\tilde{v}^2 - \frac{1}{4}\tilde{u} - \frac{\lambda-\lambda^{-1}}{2}(\tilde{v}^3 - \tilde{u}^2\tilde{v} + \frac{1}{4}\tilde{v})$ *is a constant.*

*Proof.* We calculate that

$$\begin{aligned}&(\frac{1}{3}\tilde{u}^3 - \tilde{u}\tilde{v}^2 - \frac{1}{4}\tilde{u})' \\ =&\tilde{u}^2\tilde{u}' - \tilde{u}'\tilde{v}^2 - 2\tilde{u}\tilde{v}\tilde{v}' - \frac{1}{4}\tilde{u}' \\ =&\tilde{u}'(\tilde{u}^2 - \tilde{v}^2 - \frac{1}{4}) - 2\tilde{u}\tilde{v}\tilde{v}' \\ =&\frac{\lambda - \lambda^{-1}}{\lambda + \lambda^{-1}}((\tilde{u}^2 - \tilde{v}^2 - \frac{1}{4})^2 + (2\tilde{u}\tilde{v})^2).\end{aligned} \tag{2.12}$$

We calculate that

$$\begin{aligned}&(\tilde{v}^3 - \tilde{u}^2\tilde{v} + \frac{1}{4}\tilde{v})' \\ =&\tilde{v}^2\tilde{v}' - \tilde{v}'\tilde{u}^2 - 2\tilde{v}\tilde{u}\tilde{u}' + \frac{1}{4}\tilde{v}' \\ =&-(\tilde{u}^2 - \tilde{v}^2 - \frac{1}{4})\tilde{v}' - \tilde{u}'(2\tilde{u}\tilde{v}) \\ =&\frac{2}{\lambda + \lambda^{-1}}((\tilde{u}^2 - \tilde{v}^2 - \frac{1}{4})^2 + (2\tilde{u}\tilde{v})^2).\end{aligned} \tag{2.13}$$

The proposition follows immediately. □

Since we would like our solution to exist near $t = 0$, recalling that $\tilde{u}(s) = e^s f(e^s) - \frac{1}{2}$, $\tilde{v}(s) = e^s g(e^s)$, we obtain the following restrictions: $\lim_{s\to-\infty} \tilde{u}(s) = -\frac{1}{2}$ and $\lim_{s\to-\infty} \tilde{v}(s) = 0$. Therefore, combining this with Proposition 2.3.1, we have the following identity:

$$\frac{1}{3}\tilde{u}^3 - \tilde{u}\tilde{v}^2 - \frac{1}{4}\tilde{u} - \frac{\lambda - \lambda^{-1}}{2}(\tilde{v}^3 - \tilde{u}^2\tilde{v} + \frac{1}{4}\tilde{v}) = \frac{1}{12}. \tag{2.14}$$

By Proposition 2.3.1, we can prove the following:

**Proposition 2.3.2.** *For $\lambda \neq \pm 1$, if $f(t)$ does not blow-up in finite time, then $g(t)$ will not blow-up in finite time.*

*Proof.* By the identity (2.14), we have

$$\frac{1}{3}\tilde{u}^3 - \frac{1}{4}\tilde{u} = \frac{\lambda - \lambda^{-1}}{2}(\tilde{v}^3 - \tilde{u}^2\tilde{v} + \frac{1}{4}\tilde{v}) + \tilde{u}\tilde{v}^2 + \frac{1}{12}.$$

If $f(t)$ does not blow-up in finite time, $\tilde{u}$ will also not blow-up in finite time. If $\tilde{v}$ blows-up in finite time then the right hand side of the identity will be unbounded but the left hand side will be bounded, which gives a contradiction. □

Even though the topological twisted equations (2.1) are not conformally invariant, we still have that it is invariant under rescaling by a constant, which leads to the following proposition:

**Proposition 2.3.3.** *If* $(f_0(t), g_0(t))$ *is a solution of the equations* (2.7)*, then for any constant* $C$*,* $(Cf_0(Ct), Cg_0(Ct))$ *are solutions to the equations* (2.7).

*Proof.* By a direct computation. □

**t'Hooft Solution when $\lambda = 0$**

In this subsection, we will prove that we can obtain the t'Hooft solution of Yang-Mills equation from the equations (2.7). By taking $\lambda = 0$, (2.1) becomes

$$\begin{aligned}(F_A - \phi \wedge \phi)^+ &= 0\\ (d_A\phi)^- &= 0\\ d_A^*\phi &= 0.\end{aligned} \tag{2.15}$$

If $\phi = 0$, then we are just considering the anti-self dual equation

$$F_A^+ = 0.$$

By taking $\lambda = 0$, (2.7) becomes

$$\begin{cases} f' + f^2 - g^2 = 0\\ g't + 2g - 2tfg = 0.\end{cases} \tag{2.16}$$

By Theorem 2.12, we know that every solution to the equations (2.16) will give a solution for the $SL(2;\mathbb{C})$ anti-self-dual equation.

If $g = 0$, the equations (2.16) have a solution $(f(t), g(t)) = (\frac{1}{1+t}, 0)$. The corresponding gauge fields are $(A(x) = \mathrm{Im}(\frac{1}{1+|x|^2}\bar{x}dx), \phi(x) = 0)$, which recovers the t'Hooft solution for anti-self-dual equation in [14].

**Remark.** *We can also find a solution using the first integral* $I(\tilde{u}, \tilde{v}) = \tilde{v}\tilde{u}^2 - \frac{1}{3}\tilde{v}^3 - \frac{1}{4}\tilde{v}$*. After some computation, we obtain the solution* $(f(t), g(t)) = (\frac{t}{t^2-1}, \frac{\sqrt{3}}{t^2-1})$ *to* (2.16).

**Explicit Solutions to the Kapustin-Witten ODEs**

Taking $\lambda = -1$, the equations (2.7) become

$$\begin{cases} tf' + f + g - 2fgt = 0 \\ tg' + f + g - t(f^2 - g^2) = 0. \end{cases} \tag{2.17}$$

We can find a solution

$$\begin{cases} f(t) = \dfrac{1}{2t} \\ g(t) = \dfrac{\tan(-\frac{1}{2}\ln(t) + C)}{2t}. \end{cases} \tag{2.18}$$

However, the solution will have so many poles that 0 will be an accumulation point of singularities, which is not what we want. We hope to find a solution which is well-defined near 0.

From the equations (2.10), we obtain the ODEs corresponding to the Kapustin-Witten equations:

$$\begin{cases} tu' = 2uv \\ tv' = u^2 - v^2 - \dfrac{1}{4}. \end{cases} \tag{2.19}$$

Recalling that $u(t) = tf(t) - \frac{1}{2}$ and $v(t) = tg(t)$, we hope to obtain a solution well-defined near $t = 0$. Therefore, we hope to solve (2.19) for the initial value $(u(0) = -\frac{1}{2}, v(0) = 0)$.

By taking $\tilde{u}(s) := u(e^s), \tilde{v}(s) := v(e^s)$, we obtain an autonomous system of ODEs:

$$\begin{cases} \tilde{u}' = 2\tilde{u}\tilde{v} \\ \tilde{v}' = \tilde{u}^2 - \tilde{v}^2 - \dfrac{1}{4} \\ \lim\limits_{s\to-\infty} \tilde{u}(s) = -\dfrac{1}{2} \\ \lim\limits_{s\to-\infty} \tilde{v}(s) = 0. \end{cases} \tag{2.20}$$

By Proposition 2.3.1, we the following identity:

$$\tilde{v}^2\tilde{u} - \frac{1}{3}\tilde{u}^3 + \frac{1}{4}\tilde{u} = -\frac{1}{12}. \tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$\begin{cases} 12\tilde{v}^2\tilde{u} = (2\tilde{u}+1)^2(\tilde{u}-1) \\ \tilde{u}' = 2\tilde{u}\tilde{v} \\ \lim_{s\to-\infty} \tilde{u}(s) = -\frac{1}{2} \\ \lim_{s\to-\infty} \tilde{v}(s) = 0. \end{cases} \tag{2.22}$$

Assuming $\tilde{u} < 0$, take $W(s) := -\tilde{u}(s)$, so we are trying to solve the following ODEs:

$$W(s)' = \pm\frac{1}{\sqrt{3}}\sqrt{W(s)}\sqrt{W(s)+1}(2W(s)-1).$$

We first solve $W(s)' = \frac{1}{\sqrt{3}}\sqrt{W(s)}\sqrt{W(s)+1}(2W(s)-1)$,

Taking $H(s) := \frac{1+4W(s)}{2\sqrt{3}\sqrt{W(s)^2+W(s)}}$, we have the following Lemma:

**Lemma 2.3.4.** $\frac{1}{1-H^2(s)}\ dH(s) = -ds$

*Proof.* We calculate that

$$H'(s) = \frac{(2W(s)-1)W(s)'}{4\sqrt{3}(W(s)^2+W(s))\sqrt{W(s)^2+W(s)}}.$$

In addition, we calculate that

$$1 - H(s)^2 = -\frac{(2W(s)-1)^2}{12(W(s)^2+W(s))}.$$

Combining this with $W(s)' = \frac{1}{\sqrt{3}}\sqrt{W(s)}\sqrt{W(s)+1}(2W(s)-1)$, the result follows immediately. □

By the previous lemma, $\frac{1}{2}\ln(\frac{H(s)+1}{H(s)-1}) = -s + C$. Therefore, $H(s) = \frac{Ce^{-2s}+1}{Ce^{-2s}-1}$. Combining this with $H(s) = \frac{1+4W(s)}{2\sqrt{3}\sqrt{W(s)^2+W(s)}}$, we find $W(s) = \frac{2-3H^2+3H\sqrt{H^2-1}}{2(3H^2-4)}$.

Therefore, we have

$$W(\ln t) = \frac{1}{2}\frac{C^2t^2-2Ct+1}{C^2t^2+4Ct+1}.$$

By definition,

$$f(t) = \frac{\frac{1}{2} - W(\ln t)}{t}.$$

We calculate that

$$f(t) = \frac{3C}{C^2t^2 + 4Ct + 1} \text{ (for any constant C).}$$

Putting this into the equations (2.20) and taking $g(t) = \frac{\tilde{v}(ln(t))}{t}$, we obtain

$$g(t) = \frac{3C(Ct+1)}{(C^2t^2 + 4Ct + 1)(Ct - 1)}.$$

For $W(s)' = -\frac{1}{\sqrt{3}}\sqrt{W(s)}\sqrt{W(s)+1}(2W(s)-1)$, we obtain another solution $(f(t), g(t)) = (\frac{1}{t}\frac{C^2t^2+Ct+1}{C^2t^2+4Ct+1}, -\frac{3C(Ct-1)}{(C^2t^2+4Ct+1)(Ct+1)})$.

To summarize, by solving the equations (2.7) with $\lambda = -1$, we obtain the following proposition:

**Proposition 2.3.5.**

$$\begin{cases} f_1(t) = \dfrac{3C}{(Ct)^2 + 4(Ct) + 1} \\ g(t) = \dfrac{3C(Ct+1)}{((Ct)^2 + 4(Ct) + 1)(Ct - 1)}, \end{cases} \tag{2.23}$$

$$\begin{cases} f_2(t) = \dfrac{1}{t}\dfrac{C^2t^2 + Ct + 1}{C^2t^2 + 4Ct + 1} \\ g(t) = \dfrac{3C(Ct+1)}{((Ct)^2 + 4(Ct) + 1)(Ct - 1)} \end{cases} \tag{2.24}$$

*are two families of solutions to the Kapustin-Witten ODEs* (2.8).

## 2.4 Instanton Number Zero Solutions

In this section, we will give a complete proof of Theorem 1.1.

### Computation of Instanton Numbers

We will now give a formula to compute the instanton number for the rotationally invariant solutions, which will prove property (2) of Theorem 1.1.

**Lemma 2.4.1.** *$\bar{x}dx\wedge d\bar{x}x\wedge\bar{x}dx\wedge d\bar{x}x = 24t^2\ dx_1\wedge dx_2\wedge dx_3\wedge dx_4$, $d\bar{x}\wedge dx\wedge d\bar{x}\wedge dx = -24\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.*

*Proof.* By Lemma 2.2.6, it is just a direct computation. □

Combining this with Lemma 2.2.7, we obtain that

$$\begin{aligned} |F_A^-|^2 &= 6(tf' + 2f - tf^2)^2 \\ |F_A^+|^2 &= 6(f' + f^2)^2 t^2. \end{aligned} \tag{2.25}$$

Since we are considering the solutions over the non-compact space $\mathbb{R}^4$, the instanton number is defined as:

**Definition 2.4.2.** *Given that a connection* $(A(x),\ \phi(x))$ *is a solution to the Kapustin-Witten equations* (4.1)*, if the integral* $\frac{1}{4\pi^2}\int_{\mathbb{R}^4} tr(F_A \wedge F_A)$ *exist, we define the instanton number* $k$ *for* $(A(x),\ \phi(x))$ *is* $k := \frac{1}{4\pi^2}\int_{\mathbb{R}^4} tr(F_A \wedge F_A) \in \mathbb{R}$.

For a rotationally invariant solution as in (2.4), we have a simple formula to compute the instanton number.

**Proposition 2.4.3.** *For a globally defined* $C^1$ *connection* $A(x) = \mathrm{Im}(f(t)\ \bar{x}dx)$ *over* $\mathbb{R}^4$*, by taking* $\tilde{f}(t) := tf(t)$*, the instanton number* $k$ *satisfies:*

$$k = 6\int_0^{+\infty} \tilde{f}(\tilde{f} - 1)\tilde{f}' dt = (2\tilde{f}^3 - 3\tilde{f}^2) \mid_0^{+\infty} .$$

*Proof.* We calculate that

$$\begin{aligned} k =& \frac{1}{4\pi^2}\int_{\mathbb{R}^4} tr(F_A \wedge F_A) \\ =& \frac{1}{4\pi^2}\int_{\mathbb{R}^4} (|F_A^+|^2 - |F_A^-|^2)\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ =& \frac{1}{4\pi^2}\int_{\mathbb{R}^4} (6(f' + f^2)^2 t^2 - 6(tf' + 2f - tf^2)^2)\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ =& \frac{1}{4\pi^2}\int_{\mathbb{R}^4} 24(tf^2 - f)(tf' + f)\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ =& \frac{1}{4\pi^2}\int_{\mathbb{R}^4} 24(\frac{1}{t}\tilde{f}(\tilde{f} - 1)\tilde{f}')\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ =& \frac{1}{4\pi^2} 12\mathrm{Vol}(S^3)\int_0^{+\infty} (\tilde{f}(\tilde{f} - 1)\tilde{f}')\ dt \text{ (by } dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = d\ \mathrm{Vol}_{S^3}\frac{1}{2}tdt) \\ =& 6\int_0^{+\infty} (\tilde{f}(\tilde{f} - 1)\tilde{f}')\ dt \text{ (since } \mathrm{Vol}(S^3) = 2\pi^2) \\ =& (2\tilde{f}^3 - 3\tilde{f}^2) \mid_0^{+\infty} . \end{aligned}$$

□

**Remark.** *The previous formula for the instanton number only works for connections with the specific type* $A(x) = \mathrm{Im}(f(t)\ \bar{x}dx)$*.*

For $A(x) = \mathrm{Im}(f(t)\ xd\bar{x})$, a conjugate form of (2.4), we have the following corollary:

**Corollary 2.4.4.** *For a globally defined* $C^1$ *connection* $A = \mathrm{Im}(f(t)\ \bar{x}dx)$ *over* $\mathbb{R}^4$*, by taking* $\tilde{f}(t) := tf(t)$*, the instanton number k satisfies:*

$$k = -6\int_0^{+\infty} \tilde{f}(\tilde{f}-1)\tilde{f}'dt = (3\tilde{f}^2 - 2\tilde{f}^3)\ |_0^{+\infty}\ .$$

*Proof.* We can calculate in a similar way and obtain that

$$\begin{aligned} |F_A^+|^2 &= 6(tf' + 2f - tf^2)^2 \\ |F_A^-|^2 &= 6(f' + f^2)^2 t^2. \end{aligned} \tag{2.26}$$

By the same computation as in Proposition 2.4.3, we obtain the result. □

**Corollary 2.4.5.** *The solution* $(f_1(t), g(t)) = (\frac{3}{t^2+4t+1}, \frac{3(t+1)}{(t^2+4t+1)(t-1)})$ *to the Kapustin-Witten ODEs* (2.8) *has instanton number zero.*

*The solution* $(f_2(t), g(t)) = (\frac{1}{t}\frac{t^2+t+1}{t^2+4t+1}, \frac{3(t+1)}{(t^2+4t+1)(t-1)})$ *to the Kapustin-Witten ODEs* (2.8) *has instanton number zero.*

*Proof.* Defining $\tilde{f}_1(t) := tf_1(t)$, then $\tilde{f}_1(0) = \tilde{f}_1(+\infty) = 0$. Therefore, by Proposition 2.4.3, we know the instanton number of $A(x) = \mathrm{Im}(f_1(t)\ \bar{x}dx)$ is 0.

Similarly, for $\tilde{f}_2(t) := tf_2(t)$, we have $\tilde{f}_2(0) = \tilde{f}_2(+\infty) = 1$. Therefore, by Proposition 2.4.3, we know the instanton number of $A(x) = \mathrm{Im}(f_2(t)\ \bar{x}dx)$ is 0. □

**Proposition 2.4.6.** *Given that* $(\tilde{f}(t), \tilde{g}(t))$ *is a solution to the equations* (2.9)*, if*

(1) $\lim_{t\to 0} \tilde{f}(t)$ *and* $\lim_{t\to 0} \tilde{g}(t)$ *exist.*

(2) $\lim_{t\to +\infty} \tilde{f}(t)$ *and* $\lim_{t\to +\infty} \tilde{g}(t)$ *exist.*

*Then* $A(x) = \mathrm{Im}(\frac{\tilde{f}(t)}{t}\ \bar{x}dx)$ *is a connection with instanton number 0, 1 or -1.*

*Proof.* After a change of variable and translation, the ODEs (2.9) turn into (2.11). Equation (2.11) is an autonomous system, therefore the limit point must be a equilibrium point of (2.11).

There are two equilibrium points, $(\tilde{u},\tilde{v}) \in \{(\frac{1}{2},0),(-\frac{1}{2},0)\}$ or equivalently $(\tilde{f},\tilde{g}) \in \{(0,0),(1,0)\}$. Therefore, $(\tilde{f}(0),\tilde{g}(0)) \in \{(0,0),(1,0)\}$ and $(\tilde{f}(+\infty),\tilde{g}(+\infty)) \in \{(0,0),(1,0)\}$. By Proposition 2.4.3, we know that $A(x) = \mathrm{Im}(\frac{\tilde{f}(t)}{t}\bar{x}dx)$ can only have instanton number 0, 1 or -1. □

**Bubbling for Instanton Number 0 Singular Solutions to the Kapustin-Witten Equations**

In this subsection, we will prove property (3) of Theorem 1.1: the existence of some bubbling phenomenon for singular solutions.

By previous computation, we know that:

$$\begin{aligned} |F_A^-|^2 &= 6(tf' + 2f - tf^2)^2 \\ |F_A^+|^2 &= 6(f' + f^2)^2 t^2. \end{aligned} \tag{2.27}$$

Consider the solution

$$\begin{cases} f(t) = \dfrac{3}{t^2+4t+1} \\ g(t) = \dfrac{3(t+1)}{(t^2+4t+1)(t-1)}. \end{cases} \tag{2.28}$$

Combining (2.27) and (2.28), we obtain that

$$\begin{aligned} |F_A|^2 =& |F_A^+|^2 + |F_A^-|^2 \\ =& (6(tf' + 2f - tf^2)^2 + 6(f' + f^2)^2 t^2) \\ =& \frac{108(2t^4 + 2t^3 + t^2 + 2t + 2)}{(t^2+4t+1)^4}. \end{aligned}$$

As the curvature norm $|F_A|$ plays an important roles in the Uhlenbeck compactness theorem [20] [60], we also hopes to understand it in the Kapustin-Witten equations.

The graph of $|F_A|(t)$ is depicted in Figure 1:

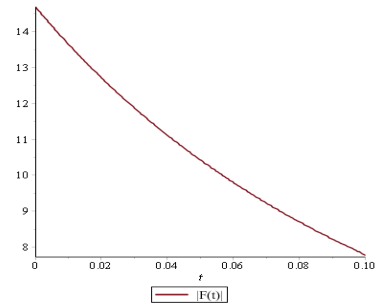
**Proposition 2.4.7.** *$|F_A|(t)$ is decreasing and $|F_A|(0)$ is its maximum.*

*Proof.* A direct computation shows that $\frac{d}{dt}|F_A|^2(t) = -\frac{216(4t^5+5t^4+3t^3+8t^2+19t+15)}{(t^2+4t+1)^5}$, so $|F_A|' < 0$ for all $t \geq 0$. Therefore, $|F_A|$ is decreasing and $|F_A|(0)$ is its maximum. □

By Proposition 3.5, for any constant C, we have the solutions

$$\begin{cases} f^C(t) = \dfrac{3C}{C^2t^2+4Ct+1} \\ g^C(t) = \dfrac{3C(Ct+1)}{(C^2t^2+4Ct+1)(Ct-1)} \end{cases} \tag{2.29}$$

Figure 2.1: Norm of $|F_A|$ as a Function of Radius $t$




to the Kapustin-Witten ODEs.

We define $|F_A^C|(t)$ as the curvature norm for $f^C(t)$, then we have the following proposition:

**Proposition 2.4.8.** (1) $|F_A^C|(t) = C|F_A|(Ct)$.

(2) $\int_{\mathbb{R}^4} |F_A^C|^2(t) d\text{Vol} = \int_{\mathbb{R}^4} |F_A|^2(t) d\text{Vol}$.

*Proof.* For (1), this is an immediately computation. For (2), by the definition of $t = |x|^2$, we get the result immediately. □

**Proposition 2.4.9.** *Let D to be a real number defines as follows:* $D := \int_{\mathbb{R}^4} |F_A^C|^2(t) d\text{Vol}$. *Then we have* $\lim_{C\to+\infty} \frac{1}{D}|F_A^C|(t) = \delta_0$, *where* $\delta_0$ *is the Dirac measure at* 0.

*Proof.* By Proposition 2.4.8, we know that $\frac{1}{D}|F_A^C|(t) = \frac{C}{D}|F_A|(Ct)$ and $\int_{\mathbb{R}^4} \frac{1}{D}|F_A^C|^2(t) d\text{Vol} = 1$. Therefore, the function $\frac{C}{D}|F_A|(Ct)$ is a rescale of $|F_A|(t)$ and has integral 1. By classical approximations to the identity results [52], we obtain $\lim_{C\to+\infty} \frac{1}{D}|F_A^C|(t) = \delta_0$. □

### Non-removability of Singularities for $\phi$ by SU(2) Gauge Transformations

In this subsection, we will prove property (4) of Theorem 1.1. It is suffice to consider the $C = 1$ case.

By Lemma 2.2.9, we obtain that

$$(d_A\phi)^+ = -\frac{1}{2}(g'+2fg)\ \bar{x}dx\wedge d\bar{x}x = \frac{3(t^3+3t+2)}{(t^2+4t+1)(t-1)(t^3+3t^2-3t+1)}\ \bar{x}dx\wedge d\bar{x}x$$

$$(d_A\phi)^- = (\frac{1}{2}g't+g-fgt)\ d\bar{x}\wedge dx = -3\frac{2t^3+3t^2+1}{(t^2+4t+1)(t-1)(t^3+3t^2-3t+1)}d\bar{x}\wedge dx.$$

We calculate that

$$|d_A\phi|^2 = \frac{432(2t^8 + 6t^7 + 5t^6 + 2t^5 + 6t^4 + 2t^3 + 5t^2 + 6t + 2)}{(t^2 + 4t + 1)^2(t - 1)^2(t^3 + 3t^2 - 3t + 1)^2}.$$

Therefore, we know $||d_A\phi||_{L^2}$ is unbounded near $t = 1$.

Since $||d_A\phi||_{L^2}$ is invariant under the $SU(2)$ gauge action, we know that the singularities of $\phi$ can not be removed by $SU(2)$ gauge transformations.

## 2.5 Non-Zero Instanton Number Solutions

### Instanton Number $\pm 1$ Solutions

In this subsection, we are going to give a proof of Theorem 1.2.

First, we are going to give a construction of an instanton number 1 solution.

By Proposition 2.4.3, we know that the instanton number is determined by the limit behavior of our connection $A(x) = \mathrm{Im}(f(t)\ \bar{x}dx)$. In order to construct an instanton number $\pm 1$ solution, we only need to construct a solution with different equilibrium points at $t = 0$ and $t = +\infty$.

By Proposition 2.3.5, taking $C = 1$, we have the following solutions:

$$\begin{cases} f_1(t) = \dfrac{3}{t^2 + 4t + 1} \\ g_1(t) = \dfrac{3(t+1)}{(t^2 + 4t + 1)(t - 1)}, \end{cases} \tag{2.30}$$

$$\begin{cases} f_2(t) = \dfrac{1}{t}\dfrac{t^2 + t + 1}{t^2 + 4t + 1} \\ g_2(t) = \dfrac{3(t+1)}{(t^2 + 4t + 1)(t - 1)}. \end{cases} \tag{2.31}$$

As $g_1(t) = g_2(t)$, we hope to understand the relationship of $f_1(t)$ and $f_2(t)$. The graphs of $f_1(t)$ and $f_2(t)$ are depicted in Figure 2 and obviously, we get $f_1(1) = f_2(1)$.

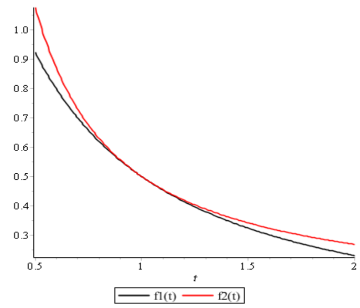
Now, we hope to glue these two solutions to obtain a new solution.

**Proposition 2.5.1.**

$$A(x) = \begin{cases} \mathrm{Im}\Big(\dfrac{3}{t^2 + 4t + 1}\ \bar{x}dx\Big)\ (t \le 1) \\ \mathrm{Im}\Big(\dfrac{1}{t}\dfrac{t^2 + t + 1}{t^4 + 4t + 1}\ \bar{x}dx\Big)\ (t \ge 1) \end{cases} \tag{2.32}$$

$$\phi(t) = \mathrm{Im}\Big(\frac{3(t+1)}{(t-1)(t^2 + 4t + 1)}\ \bar{x}dx\Big) \tag{2.33}$$

Figure 2.2: Graph of $f_1(t)$ and $f_2(t)$.




*is a solution to the Kapustin-Witten Equations* (4.1) *which satisfies the following properties:*

(1) *The solution has instanton number=1.*

(2) $A(x)$ *is* $C^\infty$ *away from* $t = 1$ *and* $C^1$ *at* $t = 1$, $\phi(t)$ *is singular at* $t = 1$.

*Proof.*

$$f(t) = \begin{cases} \dfrac{3}{t^2+4t+1} \ (t \le 1) \\ \dfrac{1}{t}\dfrac{t^2+t+1}{t^4+4t+1} \ (t \ge 1) \end{cases} \tag{2.34}$$

Taking $\tilde{f}(t) = tf(t)$, by a direct computation, we know that $\tilde{f}(0) = 0$, $\tilde{f}(+\infty) = 1$. By Proposition 2.4.3, we know that the instanton number is equal to 1.

Defining $u(t) := \tilde{f}(t) - \frac{1}{2}$, then

$$u(t) = \begin{cases} \dfrac{1}{2}\dfrac{t^2-2t+1}{t^2+4t+1} \ (t \le 1) \\ -\dfrac{1}{2}\dfrac{t^2-2t+1}{t^2+4t+1} \ (t \ge 1). \end{cases} \tag{2.35}$$

By a direct computation, we know that $u(t)$ is a $C^1$ function. Therefore, $A(x)$ is also a $C^1$ connection. □

**Remark.** *By Corollary* 2.4.4*, we know*

$$A(x) = \begin{cases} \mathrm{Im}\Big(\dfrac{3}{t^2+4t+1}\ x d\bar{x}\Big) \ (t \le 1) \\ \mathrm{Im}\Big(\dfrac{1}{t}\dfrac{t^2+t+1}{t^2+4t+1}\ x d\bar{x}\Big) \ (t \ge 1) \end{cases} \tag{2.36}$$

$$\phi(x) = \mathrm{Im}\Big(\frac{3(t+1)}{(t-1)(t^2+4t+1)}\ x d\bar{x}\Big) \tag{2.37}$$

*is a instanton number* $-1$ *solution to the Kapustin-Witten equations.*

**Linear Combination of Solutions**

In this subsection, we aim to generalize the ADHM construction from [5] to obtain higher instanton number solutions to the Kapustin-Witten equations. However, there exists an essential problem to generalizing the instanton number computation method from the anti-self-dual equation case [5]. We conjecture that we will obtain some higher instanton number solutions from this construction.

In view of Corollary 2.4.4, without loss of generality, we can focus on instanton number $k \geq 0$.

Now, let $\lambda_1, \ldots, \lambda_k$ be $k$ real numbers and $b_1, \ldots, b_k$ be $k$ numbers in $\mathbb{H}$. Take $U := (\lambda_1(x - b_1), \ldots, \lambda_k(x - b_k))^T$ and $U^\star$ be the conjugate transpose of $U$. Take $e_0 = 1,\ e_1 = I,\ e_2 = J,\ e_3 = K$, then for any quaternion $b_i$, we can write $b_i = b_{ij}e_j$.

Now we are going to compute an identity which is parallel to $k = 0$ case.

**Lemma 2.5.2.** *For any $g(t) \in C^1$, $d \star \mathrm{Im}(g(|U|^2)\ U^\star dU)=0$*

*Proof.* We calculate that

$$
\begin{aligned}
& d\mathrm{Im}(g(|U|^2)\ \star U^\star dU) \\
=& \sum_{i=1}^{k} dg(|U|^2)\mathrm{Im}(\lambda_i^2(\bar{x} - \bar{b}_i) \wedge \star d(x - b_i)) \text{ (by Lemma 2.2.11)} \\
=& \sum_{i=1}^{k}\sum_{j=1}^{4} dg(|U|^2)\lambda_i^2\mathrm{Im}((\bar{x} - \bar{b}_i)e_j) \wedge \star d(x_j - b_{ij}) \\
=& \sum_{i=1}^{k} \lambda_i^2 \sum_{j=1}^{4} \frac{\partial g}{\partial x_j}\ \mathrm{Im}((\bar{x} - \bar{b}_i)e_j)\ dx_j \wedge \star dx_j \\
=& \sum_{i=1}^{k} \lambda_i^2 g' \sum_{j=1}^{4} \frac{\partial |U|^2}{\partial x_j}\ \mathrm{Im}((\bar{x} - \bar{b}_i)e_j)\ dx_j \wedge \star dx_j \\
=& \sum_{i=1}^{k} \lambda_i^2 g' \sum_{l=1}^{k} \lambda_l^2(x_j - b_{lj}) \sum_{j=1}^{4} \mathrm{Im}((\bar{x} - \bar{b}_i)e_j)\ dx_j \wedge \star dx_j \\
=& 2g'd\mathrm{Vol} \sum_{i=1}^{k}\sum_{l=1}^{k}\sum_{j=1}^{4} \lambda_i^2\lambda_l^2(x_j - b_{lj})\mathrm{Im}((\bar{x} - \bar{b}_i)e_j) \\
=& g'd\mathrm{Vol} \sum_{i=1}^{k}\sum_{l=1}^{k}\sum_{j=1}^{4} \lambda_i^2\lambda_l^2\Big(\sum_{j=1}^{4}(x_j - b_{lj})\mathrm{Im}((\bar{x} - \bar{b}_i)e_j) + \sum_{j=1}^{4}(x_j - b_{ij})\mathrm{Im}((\bar{x} - \bar{b}_l)e_j)\Big).
\end{aligned}
\tag{2.38}
$$

Therefore, in order to show $d\star\mathrm{Im}(g(|U|^2)\ U^\star dU)$=0, we only need to show

$$\sum_{j=1}^{4}(x_j-b_{lj})\mathrm{Im}((\bar{x}-\bar{b}_i)e_j)+\sum_{j=1}^{4}(x_j-b_{ij})\mathrm{Im}((\bar{x}-\bar{b}_l)e_j)=0.$$

By translation, without loss of generality, we can assume $b_i=0$. Then we calculate that

$$\begin{aligned}&\sum_{j=1}^{4}(x_j-b_{lj})\mathrm{Im}((\bar{x})e_j)\\&=\sum_{j=1}^{4}x_j\mathrm{Im}((\bar{x})e_j)-\sum_{j=1}^{4}b_{lj}\mathrm{Im}((\bar{x})e_j)\\&=-\sum_{j=1}^{4}b_{lj}\mathrm{Im}((\bar{x})e_j).\end{aligned}\tag{2.39}$$

For the rest, we calculate that

$$\begin{aligned}&\sum_{j=1}^{4}x_j\mathrm{Im}((\bar{x}-\bar{b}_l)e_j)\\&=-\sum_{j=1}^{4}x_j\mathrm{Im}(\bar{b}_le_j)\\&=-\,(x_1(-b_{l2}I-b_{l3}J-b_{l4}K)+x_2(b_{l1}I+b_{l3}K-b_{l4}J)\\&\quad+x_3(b_{l1}J-b_{l2}K+b_{l4}I)+x_4(b_{l1}K+b_{l2}J-b_{l3}I))\\&=\sum_{j=1}^{4}b_{lj}\mathrm{Im}((\bar{x})e_j).\end{aligned}\tag{2.40}$$

Therefore, we obtain the following identity:

$$\sum_{j=1}^{4}(x_j-b_{lj})\mathrm{Im}((\bar{x}-\bar{b}_i)e_j)+\sum_{j=1}^{4}(x_j-b_{ij})\mathrm{Im}((\bar{x}-\bar{b}_l)e_j)=0.$$

Combining all the things above, we obtain the lemma. □

**Lemma 2.5.3.** *For any* $f(t),\ g(t)\in C^1$*, we have* $\mathrm{Im}(f(|U|^2)\ U^\star dU)\wedge\star\mathrm{Im}(g(|U|^2)\ U^\star dU)=0$.

*Proof.* Since $f(|U|^2)$, $g(|U|^2)$ are real functions, we only need to show that $\mathrm{Im}(U^\star dU)\wedge\star\mathrm{Im}(U^\star dU)=0$.

We calculate that

$$\begin{aligned}
&\mathrm{Im}(U^\star dU) \wedge \star\mathrm{Im}(U^\star dU) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{4} \lambda_i^2 \mathrm{Im}((\bar{x}-\bar{b}_i)e_j)\ dx_j \wedge \sum_{m=1}^{k}\sum_{n=1}^{4} \lambda_m^2 \mathrm{Im}((\bar{x}-\bar{b}_m)e_n)\ \star dx_n \\
&= \sum_{i=1}^{k}\sum_{m=1}^{k}\sum_{j=1}^{4} \lambda_i^2\lambda_m^2 \mathrm{Im}((\bar{x}-\bar{b}_i)e_j)\mathrm{Im}((\bar{x}-\bar{b}_m)e_j)\ dx_j \wedge \star dx_j \\
&= -\star\mathrm{Im}(U^\star dU) \wedge \mathrm{Im}(U^\star dU).
\end{aligned} \tag{2.41}$$

□

**Corollary 2.5.4.** *For any $f(t), g(t) \in C^1$, if $A(x) = \mathrm{Im}(f(|U|^2)\ U^\star dU)$ and $\phi(x) = \mathrm{Im}(g(|U|^2)\ U^\star dU)$, then we have $d_A \star \phi = 0$.*

*Proof.* We have $d_A \star \phi = d\phi + A\wedge\star\phi + \phi\wedge\star A$. This is a direct corollary of Lemma 2.5.2 and Lemma 2.5.3. □

Taking $f_1(t) = \frac{3}{t^2+4t+1}$, $f_2(t) = \frac{1}{t}\frac{t^2+t+1}{t^2+4t+1}$, $g(t) = \frac{3(t+1)}{(t-1)(t^2+4t+1)}$, we have the following proposition:

**Proposition 2.5.5.**

$$A(x) = \begin{cases} \mathrm{Im}\Big(f_1(|U|^2)\ U^\star dU\Big) = \mathrm{Im}\Big(\dfrac{3}{|U|^4+4|U|^2+1}U^\star dU\Big)\ (|U| \le 1) \\ \mathrm{Im}\Big(f_2(|U|^2)\ U^\star dU\Big) = \mathrm{Im}\Big(\dfrac{1}{|U|^2}\dfrac{|U|^4+|U|^2+1}{|U|^4+4|U|^2+1}U^\star dU\Big)\ (|U| \ge 1) \end{cases} \tag{2.42}$$

$$\phi(x) = \mathrm{Im}\Big(g(|U|^2)\ U^\star dU\Big) = \mathrm{Im}\Big(\frac{3(|U|^2+1)}{(|U|^2-1)(|U|^4+4|U|^2+1)}U^\star dU\Big) \tag{2.43}$$

*are solutions to the Kapustin-Witten equations.*

*Proof.* By Corollary 2.5.4, the equation $d_A^\star\phi = 0$ is always satisfied. Therefore, we only need to show that the equation $F_A - \phi\wedge\phi - \star d_A\phi = 0$ is satisfied by ($A(x)$, $\phi(x)$) defined above.

For

$$(A(x), \phi(x)) = (\mathrm{Im}(f(|U|^2)U^\star dU),\ \mathrm{Im}(g(|U|^2)U^\star dU)), \tag{2.44}$$

we observe that all the computations in Section 2 can be finish similarly. To be more precise, replacing every computations in section 2 of $t = |x|^2$ with $|U|^2$, replacing

$x$ with $U$ and $\bar{x}$ with $U^\star$, we can easily obtain the following results comparing to Lemma 2.2.7, Lemma 2.2.8 and Lemma 2.2.9:

$$\begin{aligned}
F_A^+ &= -\frac{1}{2}(f' + f^2)\, U^\star dU \wedge dU^\star U, \\
F_A^- &= (\frac{1}{2}tf' - \frac{1}{2}tf^2 + f)\, dU^\star \wedge dU \\
(\phi \wedge \phi)^+ &= -\frac{1}{2}g^2\, U^\star dU \wedge dU^\star U \\
(\phi \wedge \phi)^- &= -\frac{1}{2}g^2 t\, dU^\star \wedge dU \\
(d_A\phi)^+ &= -\frac{1}{2}(g' + 2fg)\, U^\star dU \wedge dU^\star U \\
(d_A\phi)^- &= (\frac{1}{2}g't + g - fgt)\, dU^\star \wedge dU.
\end{aligned} \tag{2.45}$$

The derivative here is taking the derivative of $|U|^2$.

Therefore, by (2.45), we could get an ODEs comparing to (2.6):

$$\begin{cases}
f(|U|^2)' + \lambda g(|U|^2)' + f(|U|^2)^2 - g(|U|^2)^2 + 2\lambda f(|U|^2)g(|U|^2) = 0, \\
tf(|U|^2)' - t\lambda^{-1}g(|U|^2)' + 2f(|U|^2) - 2\lambda^{-1}g(|U|^2) \\
\quad + g(|U|^2)^2 t - f(|U|^2)^2 t + 2tf(|U|^2)g(|U|^2)\lambda^{-1} = 0.
\end{cases} \tag{2.46}$$

The derivative here is the derivative of $|U|^2$.

Comparing this with (2.7), we are exactly solving the same equations. Therefore, comparing to Proposition 2.5.1, our construction gives solutions to the Kapsutin-Witten equations. □

*Proof of Theorem 1.3.* By our construction, we have the freedom to choose $k$ real numbers $\lambda_1, ..., \lambda_k$ and $k$ quaternions $b_1, ..., b_k$ in $\mathbb{H}$ in Proposition 2.5.5. Therefore, we have a $5k$ dimension family of solutions to the Kapustin-Witten equations. □

### 2.6 Nahm Pole Boundary Solution over $S^3 \times (0, +\infty)$

In this section, we will show that our solutions in Section 4 can provide solutions on $S^3 \times (0, +\infty)$ with Nahm Pole boundary.

**Nahm Pole Boundary condition**

Now, we will discuss what is a Nahm Pole boundary condition to the Kapustin-Witten equations. Give a closed 3-manifold $Y^3$, let $P$ to be a principle $SU(2)$ bundle over $Y^3 \times (0, +\infty)$ and let ad($P$) denote the adjoint bundle of $P$. Give a point $x \in Y^3$, for integer a=1,2,3, let $e_a$ be any orthonormal basis of $T_xY$ and $t_a \in ad(P)$ satisfy the lie algebra relationship $[t_a, t_b] = \epsilon_{abc}t_c$.

From [44] [63], we have the following definition of Nahm Pole boundary condition on $Y^3 \times (0, +\infty)$ and denote $y$ as the coordinate on $(0, +\infty)$.

**Definition 2.6.1.** *A solution* $(A, \phi)$ *to KW equation* (4.1) *over* $Y^3 \times (0, +\infty)$ *satisfies the Nahm pole boundary if there exist orthonormal basis* $e_a$ *such that the Taylor expansion in* $y$ *coordinate nears* $y = 0$ *will be* $\phi \sim \frac{\sum_{a=1}^{3} t_a e_a^\star}{y} + \phi_0 + \dots, A \sim A_0 + y a_1 + \dots$

**Nahm Pole Boundary condition over** $S^3 \times (0, +\infty)$

In this subsection, we will describe the Nahm Pole boundary condition on $S^3$ and show that our solution satisfy the Nahm pole boundary condition.

Now, we first describe the tangent space of $S^3$. Consider $S^3$ as the unit quaternion, $S^3 = \{x = x_1 + x_2 I + x_3 J + x_4 K \in \mathbb{H} | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, and the metric is induced by Euclidean metric on $\mathbb{R}^4$. Fix a point $x \in S^3$, we can identify the tangent space with vectors on $\mathbb{R}^4$, $T_x S^3 = \{v \in \mathbb{R}^4 | < v, x >= 0\}$, here we consider x as a vector space on $\mathbb{R}^4$.

Define three orthnonormal basis

$$\begin{aligned} e_1 &= (-x_2, x_1, -x_4, x_3) \\ e_2 &= (-x_3, x_4, x_1, -x_2) \\ e_3 &= (-x_4, -x_3, x_2, x_1). \end{aligned} \tag{2.47}$$

Obviously, we have $T_x S^3 = span\{e_1, e_2, e_3\}$.

So by the induced metric from the Euclidean metric on $\mathbb{R}^4$, we have the dual unit basis

$$\begin{aligned} e_1^\star &= -x_2 dx_1 + x_1 dx_2 - x_4 dx_3 + x_3 dx_4 \\ e_2^\star &= -x_3 dx_1 + x_4 dx_2 + x_1 dx_3 - x_2 dx_4 \\ e_3^\star &= -x_4 dx_1 - x_3 dx_2 + x_2 dx_3 + x_1 dx_4. \end{aligned} \tag{2.48}$$

In addition, we have

$$\begin{aligned} Im(\bar{x}dx) =&(-x_2dx_1 + x_1dx_2 - x_4dx_3 + x_3dx_4)I \\ &+ (-x_3dx_1 + x_4dx_2 + x_1dx_3 - x_2dx_4)J \\ &+ (-x_4dx_1 - x_3dx_2 + x_2dx_3 + x_1dx_4)K \\ &= e_1^\star I + e_2^\star J + e_3^\star K. \end{aligned} \tag{2.49}$$

Therefore, $(0, \frac{Im(\bar{x}dx)}{2})$ can be consider as a leading term of Nahm pole boundary condition on $S^3$.

Now, we will show that the following singular solutions to the Kapustin-Witten equations over $\mathbb{R}^4$ can be consider as solutions to the Kapustin-Witten equations over $(0, +\infty) \times S^3$.

As the first equation of (4.1) are conformal invariant, consider the solutions

$$\begin{cases} A(x) = \mathrm{Im}\Big(\dfrac{3C}{C^2|x|^4 + 4C|x|^2 + 1}\ \bar{x}dx\Big) \\ \phi(x) = \mathrm{Im}\Big(\dfrac{3C(C|x|^2 + 1)}{(C^2|x|^4 + 4C|x|^2 + 1)(C|x|^2 - 1)}\ \bar{x}dx\Big). \end{cases} \tag{2.50}$$

Using the following conformal transformation,

$$\Psi : (0, +\infty) \times S^3 \to \mathbb{R}^4_{|x| \geq \frac{1}{\sqrt{C}}}$$

$$(y, \omega) \to \frac{1}{\sqrt{C}} e^y \omega.$$

Then the pull back of (2.50) using $\Psi$ gives the following solution on $(0, +\infty) \times S^3$:

$$\begin{cases} A(x) = \dfrac{6}{e^{4y} + 4e^{2y} + 1}\ \displaystyle\sum_{a=1}^{3} t_a e_a^\star \\ \phi(x) = \dfrac{6(e^{2y} + 1)}{(e^{4y} + 4e^{2y} + 1)(e^{2y} - 1)}\ \displaystyle\sum_{a=1}^{3} t_a e_a^\star. \end{cases} \tag{2.51}$$

It is easy to see that when $y \to 0$, $\frac{6(e^{2y}+1)}{(e^{4y}+4e^{2y}+1)(e^{2y}-1)} \sim \frac{1}{y}$ and $y \to +\infty$, the solution exponentially decays.

From section 4, we get another solution to the Kapustin-witten equations (4.1),

$$\begin{cases} A(x) = \mathrm{Im}\Big(\dfrac{1}{|x|^2} \dfrac{C^2|x|^4 + C|x|^2 + 1}{C^2|x|^4 + 4C|x|^2 + 1}\ \bar{x}dx\Big) \\ \phi(x) = \mathrm{Im}\Big(\dfrac{3C(C|x|^2 + 1)}{(C^2|x|^4 + 4C|x|^2 + 1)(C|x|^2 - 1)}\ \bar{x}dx\Big) \end{cases} \tag{2.52}$$

Following the same process, we get another solution to the Kapustin-Witten equations with Nahm Pole and exponentially decays.

Then the pull back of (2.52) using $\Psi$ gives the following solution on $(0,+\infty)\times S^3$:

$$\begin{cases} A(x) = \dfrac{2}{e^{2y}}\dfrac{e^{4y}+e^{2y}+1}{e^{4y}+4e^{2y}+1}\displaystyle\sum_{a=1}^{3} t_a e_a^{\star} \\ \phi(x) = \dfrac{6(e^{2y}+1)}{(e^{4y}+4e^{2y}+1)(e^{2y}-1)}\displaystyle\sum_{a=1}^{3} t_a e_a^{\star}. \end{cases} \tag{2.53}$$

Therefore, we have the following theorem:

**Theorem 2.6.2.**

(1)

$$\begin{cases} A(x) = \dfrac{6}{e^{4y}+4e^{2y}+1}\displaystyle\sum_{a=1}^{3} t_a e_a^{\star} \\ \phi(x) = \dfrac{6(e^{2y}+1)}{(e^{4y}+4e^{2y}+1)(e^{2y}-1)}\displaystyle\sum_{a=1}^{3} t_a e_a^{\star} \end{cases}$$

*is a solution to the Kapustin-Witten equations over* $(0,+\infty)\times S^3$ *with Nahm pole boundary with instanton number* $+\frac{1}{2}$.

(2)

$$\begin{cases} A(x) = \dfrac{2}{e^{2y}}\dfrac{e^{4y}+e^{2y}+1}{e^{4y}+4e^{2y}+1}\displaystyle\sum_{a=1}^{3} t_a e_a^{\star} \\ \phi(x) = \dfrac{6(e^{2y}+1)}{(e^{4y}+4e^{2y}+1)(e^{2y}-1)}\displaystyle\sum_{a=1}^{3} t_a e_a^{\star}. \end{cases}$$

*is a solution to the Kapustin-Witten equations over* $(0,+\infty)\times S^3$ *with Nahm pole boundary with instanton number* $-\frac{1}{2}$.

*Proof.* The computation of instanton number directly follows from Prop. 4.3. □

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*Chapter 3*

# THE EXTENDED BOGOMOLNY EQUATIONS AND GENERALIZED NAHM POLE BOUNDARY CONDITION

This is joint work with Rafe Mazzeo.

## 3.1 Introduction

An intriguing proposal by Witten [63] interprets the Jones polynomial and Khovanov homology of knots on a 3-manifold $Y$ by counting solutions to certain gauge-theoretic equations, see [34], [63], [26] for much more on this. In this picture, the Jones polynomial for a knot $K \subset Y$ is realized by a count of solutions to the Kapustin-Witten equations on $Y \times \mathbb{R}^+$ satisfying a new type of singular boundary conditions. We refer [25], [64], [65] for a more detailed explanation, along with [43], [46] and [28] for the beginnings of the analytic theory for this program. In the absence of a knot, the problem is still of interest and may lead to 3-manifold invariants. When $K = \emptyset$, the singular boundary conditions are called the Nahm pole boundary conditions, while in the presence of a knot, they are called the generalized Nahm pole boundary conditions, or Nahm pole boundary conditions with knot singularities. For simplicity, we usually just refer to solutions with Nahm pole or with Nahm pole and knot singularities.

There are two main sets of technical difficulties in this program. The first arises from the singular boundary conditions, which turn the problem into one of nonstandard elliptic type. These are now understood, see [43], [46]. A more serious difficulty involves whether it is possible to prove compactness of the space of solutions to

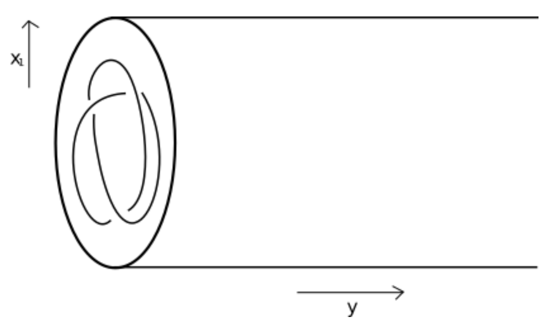


Figure 3.1: A knot placed at the boundary of $Y \times \mathbb{R}^+$

the Kapustin-Witten (KW) equations. An important first step was accomplished by Taubes in [55], [56], but at present there is no understanding about how the Nahm pole boundary conditions interact with these compactness issues.

Gaiotto and Witten [25] proposed the study of a more tractable aspect of this problem. Suppose that we stretch the 3-manifold across a separating Riemann surface $\Sigma$ in a Heegard decomposition of $Y$ which meets the knot transversely. In the limit, $Y$ separates into two components $Y^{\pm}$ and zooming in on the transition region leads to a problem on $\Sigma \times \mathbb{R} \times \mathbb{R}^{+}$ which is independent of the $\mathbb{R}$ direction normal to the separating surface. We are thus led to study the dimensionally reduced problem, called the extended Bogomolny equations, on $\Sigma \times \mathbb{R}^{+}$ with the induced singular boundary condition.

A further motivation for studying the moduli space of solutions of the extended Bogomolny equations on $\Sigma \times \mathbb{R}^{+}$ is provided by the Atiyah-Floer conjecture [8]. In terms of a handlebody decomposition $Y^3 = Y^{+} \cup_{\Sigma} Y^{-}$, the Atiyah-Floer conjecture states that the instanton Floer homology of $Y$ can be recovered from Lagrangian Floer homology of two Lagrangians associated to the handlebodies in the moduli space $\mathcal{M}(\Sigma)$ of flat $SU(2)$ connection of $\Sigma$. These Lagrangians consist of the flat connections which extend into $Y^{+}$ or $Y^{-}$. Another way to view $\mathcal{M}(\Sigma)$ is as the moduli space for the reduction of the anti-selfdual equations to $\Sigma$. One then expects to use Lagrangian intersectional Floer theory to define invariants. We refer to [16], [1] for recent progress on this.

In any case, we are presented with the problem of studying the dimensionally reduced Kapustin-Witten equations on $\Sigma \times \mathbb{R}^{+}$ with generalized Nahm pole boundary conditions. We describe these now; their derivation and further explicit computations appear in Section 2 below. Let $P$ be a principal $SU(2)$ bundle over $\Sigma$, pulled back to $\Sigma \times \mathbb{R}^{+}$, and $\mathfrak{g}_P$ its adjoint bundle. The extended Bogomolny equations are the following set of equations for a connection $A$ on $P$, and $\mathfrak{g}_P$-valued 1- and 0-forms $\phi$ and $\phi_1$, respectively:

$$\begin{aligned} F_A - \phi \wedge \phi &= \star d_A \phi_1, \\ d_A \phi &= \star [\phi, \phi_1], \\ d_A^{\star} \phi &= 0. \end{aligned} \tag{3.1}$$

The knot corresponds in this setting to where the stretched knot crosses $\Sigma$, or in other words, to a set of marked points $\{p_1, \ldots, p_N\}$ on $\Sigma$, see Figure 3.2.

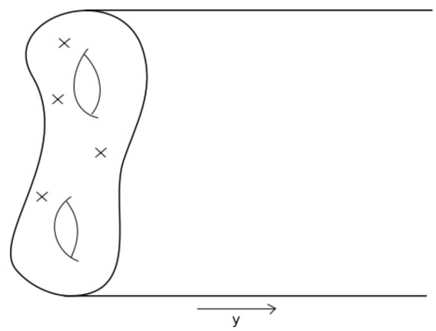


Figure 3.2: $\Sigma \times \mathbb{R}^+$; the 'knots' correspond to points on $\Sigma \times \{0\}$

In the following we the standard linear coordinate $y$ on $\mathbb{R}^+$. Define $\mathcal{M}^{\text{EBE}}_{\text{NP}}$ and $\mathcal{M}^{\text{EBE}}_{\text{KS}}$ to be the moduli spaces of solutions to (3.1) which satisfy the Nahm pole, and generalized Nahm pole, boundary conditions at $y = 0$, and which converge to an $SL(2, \mathbb{R})$ flat connection as $y \to \infty$. For the second of these spaces, we tacitly restrict to the subset of solutions which are compatible with a $SL(2, \mathbb{R})$ structure, as explained more carefully in Section 3. The subscripts NP and KS here stand for 'Nahm pole' and 'knot singularity'. We also write $\mathcal{M}$ for the moduli space of stable $SL(2, \mathbb{R})$ Higgs pairs and recall that $\mathcal{M} = \mathcal{M}^{\text{Hit}} \sqcup \mathcal{M}^{\text{Hit}^c}$, where the first term on the right is the Fuchsian, or Hitchin, component and $\mathcal{M}^{\text{Hit}^c}$ the union of the other components. It is well-known that $\mathcal{M}^{\text{Hit}}$ identified with a finite cover of the Techmüller space for $\Sigma$.

In the spirit of Donaldson-Uhlenbeck-Yau [21],[59], Gaiotto and Witten [25] define maps

$$\begin{aligned} I_{\text{NP}} &: \mathcal{M}^{\text{EBE}}_{\text{NP}} \to \mathcal{M}^{\text{Hit}}, \\ I_{\text{KS}} &: \mathcal{M}^{\text{EBE}}_{\text{KS}} \to \mathcal{M}^{\text{Hit}^c}, \end{aligned} \tag{3.2}$$

which we recall in Section 3. They conjecture that $I_{NP}$ is one-to-one. We prove this here and also describe the map $I_{KS}$. Our main result is:

**Theorem 3.1.1.** *(i) The map* $I_{\text{NP}}$ *is bijection. Explicitly, to every element in the Hitchin component* $\mathcal{M}^{\text{Hit}}$*, there exists a solution to* (3.1) *satisfying the Nahm pole boundary condition. If two solutions to* (3.1) *satisfying these boundary conditions map to the same element in* $\mathcal{M}^{\text{Hit}}$ *under* $I_{\text{NP}}$*, then they are* $SU(2)$*-gauge equivalent.*

*(ii) The map* $I_{\text{KS}}$ *is two-to-one: for every element in the* $\mathcal{M}^{\text{Hit}^c}$*, there exist two solutions to* (3.1) *which satisfy generalized Nahm pole boundary conditions with knot singularities and which are compatible with the* $SL(2, \mathbb{R})$ *structure as* $y \to \infty$*.*

*Any solution to* (3.1) *satisfying these boundary and compatibility conditions is equal, up to* $SU(2)$*-gauge equivalence, with one of these two solutions.*

We define in Section 3 what it means for solutions of (3.1) with knot singularities to be compatible with the $SL(2,\mathbb{R})$ structure as $y \to \infty$. This condition allows (3.1) to be reduced to a scalar equation. There are almost surely solutions to (3.1) which do not satisfy this condition.

The expectation, explained in [63], is that the Jones polynomial should be recovered by counting solutions to the extended Bogomolny equations on $\mathbb{R}^3 \times \mathbb{R}^+$, with a knot singularity at some $K \subset \mathbb{R}^3$. Thus, as a dimensionally reduced version of this problem, we also consider these equations on $\mathbb{C} \times \mathbb{R}^+$:

**Theorem 3.1.2.** *Given any positive divisor* $D = \sum n_i p_i$ *on* $\mathbb{C}$*, there exists a solution to* (3.1) *which has knot singularities of order* $n_i$ *at* $p_i$*. This solution is unique to the scalar equation.*

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### 3.2 Preliminaries

We begin by considering various ways in which the extended Bogomolny equations (3.1) may be interpreted.

#### $S^1$-Invariant Kapustin-Witten Equations

Let $X$ be a smooth 4-manifold with boundary, $P$ an $SU(2)$ bundle over $X$ and $\mathfrak{g}_P$ the adjoint bundle of $P$. If $\widehat{A}$ is a connection on $P$ and $\widehat{\Phi}$ is a $\mathfrak{g}_P$-valued one-form, then the Kapustin-Witten equations for the pair $(\widehat{A}, \widehat{\Phi})$ are

$$\begin{aligned} F_{\widehat{A}} - \widehat{\Phi} \wedge \widehat{\Phi} + \star d_{\widehat{A}} \widehat{\Phi} &= 0, \\ d^{\star}_{\widehat{A}} \widehat{\Phi} &= 0. \end{aligned} \tag{3.3}$$

Consider the special case where $X = S^1 \times Y$ is the product of a circle and a 3-manifold, and where $(\widehat{A}, \widehat{\Phi})$ is an $S^1$ invariant solution to (4.1). We then set

$$\widehat{A} = A + A_1 dx_1, \; \widehat{\Phi} = \phi + \phi_1 dx_1, \tag{3.4}$$

where $A, \phi \in \Omega^1_Y(\mathfrak{g}_P)$ and $A_1, \phi_1 \in \Omega^0_Y(\mathfrak{g}_P)$ are independent of $x_1 \in S^1$. Then (4.1) becomes

$$\begin{aligned} F_A - \phi \wedge \phi - \star d_A\phi_1 - \star[A_1, \phi] &= 0, \\ \star d_A\phi + [\phi_1, \phi] + d_A A_1 &= 0, \\ d_A^\star \phi - [A_1, \phi_1] &= 0. \end{aligned} \tag{3.5}$$

Denoting the quantities on the left of these three qualities by $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$, respectively, we define the expressions

$$\begin{aligned} I_0 &= \int_Y |\mathcal{X}_1|^2 + |\mathcal{X}_2|^2 + |\mathcal{X}_3|^2 \\ I_1 &= \int_Y |F_A - \phi \wedge \phi - \star d_A\phi_1|^2 + |\star d_A\phi + [\phi_1, \phi]|^2 + |d_A^\star\phi|^2, \\ I_2 &= \int_Y |[A_1, \phi]|^2 + |d_A A_1|^2 + |[A_1, \phi_1]|^2, \end{aligned} \tag{3.6}$$

and also, if $Y$ is a 3-manifold with boundary,

$$I_3 = -\int_{\partial Y} \mathrm{Tr}(d_A A_1 \wedge \phi_1) - \int_{\partial Y} \mathrm{Tr}([A_1, \phi_1] \wedge \star\phi).$$

After a straightforward calculation, assuming that all integrations are valid, we have

$$I_0 = I_1 + I_2 + I_3. \tag{3.7}$$

Since $I_0, I_1, I_2$ are all nonnegative, we deduce the

**Proposition 3.2.1.** *If $(A_1, \phi_1)$ satisfies a boundary condition which guarantees that $I_3 = 0$, and if $(A, \phi)$ is irreducible, then $A_1 = 0$ and (3.5) reduces to the equations corresponding to $I_1 = 0$.*

The case of principal interest in this paper is when $Y = \Sigma \times \mathbb{R}^+_y$ and $(\widehat{A}, \widehat{\Phi})$ satisfy the Nahm pole boundary conditions at $y = 0$ and converge as $y \to \infty$ to a flat $SL(2, \mathbb{C})$ connection. The conditions of this proposition are then satisfied. We recall the claim, see [56, Page 36] as well as [28, Corollary 4.7], that for solutions satisfying these boundary conditions, the $dy$ component of $\phi$ vanishes. Results from [43] show that as $y \searrow 0$, $A_1 \sim y^2$ and $\phi_1 \sim \frac{1}{y}$, hence $\star\phi = 0$ at $y = 0$. In addition, $A_1$ and $\phi_1$ both converge to 0 as $y \to \infty$. These facts together imply that $I_3$ vanishes at both $y = 0$ and $y = \infty$, so Proposition 3.2.1 holds.

If an $S^1$-invariant solution satisfies the Nahm pole boundary condition at $y = 0$ and converges to a flat $SL(2, \mathbb{C})$ connection as $y \to \infty$, then the pair $(A, \Phi)$ satisfies the so-called extended Bogomolny equations on $\Sigma \times \mathbb{R}^+$:

$$\begin{aligned} F_A - \phi \wedge \phi &= \star d_A \phi_1, \\ d_A \phi &= \star[\phi, \phi_1], \\ d_A^\star \phi &= 0. \end{aligned} \tag{3.8}$$

Here $A$ is a connection, $\phi \in \Omega^1(\mathfrak{g}_P)$, $\phi_1 \in \Omega^0(\mathfrak{g}_P)$ and the $dy$ component of $\phi$ vanishes.

These equations reduce, when $\phi_1 = 0$, to the Hitchin equations, when $\phi = 0$, to the Bogomolny equations, and when $A = 0$ and $\phi$ is independent of $\Sigma$, to the Nahm equations. Thus one expects that all known techniques for these special cases should be applicable to these hybrid equations as well.

**Hermitian Geometry**

Choose a holomorphic coordinate $z = x_2 + ix_3$ on $\Sigma$ and let $y$ be the linear coordinate on $\mathbb{R}^+$. In these coordinates, define $d_A = \nabla_2\, dx_2 + \nabla_3\, dx_3 + \nabla_y\, dy$ and $\phi = \phi_2\, dx_2 + \phi_3\, dx_3 = \frac{1}{2}(\varphi_z\, dz + \varphi_{\bar{z}}^\dagger\, d\bar{z})$, where $\varphi_z = \phi_2 - i\phi_3$; we also write $\varphi = \varphi_z dz$. Using these, we can rewrite (3.1) in the "three D's" formalism: with $\mathcal{A}_y = A_y - i\phi_1$, set

$$\begin{aligned} \mathcal{D}_1 &= \nabla_2 + i\nabla_3, \\ \mathcal{D}_2 &= \operatorname{ad}\varphi_z = [\varphi_z, \cdot], \\ \mathcal{D}_3 &= \nabla_y - i\phi_1 = \partial_y + \mathcal{A}_y = \partial_y + A_y - i\phi_1. \end{aligned} \tag{3.9}$$

The adjoints of these operators are

$$\begin{aligned} \mathcal{D}_1^\dagger &= -\nabla_2 + i\nabla_3, \\ \mathcal{D}_2^\dagger &= -[\phi_2 + i\phi_3, \cdot], \\ \mathcal{D}_3^\dagger &= -\nabla_y - i\phi_1. \end{aligned} \tag{3.10}$$

The extended Bogomolny equations can then be written in the alternate form

$$[\mathcal{D}_i,\ \mathcal{D}_j] = 0,\ i, j = 1, 2, 3,\ \text{and}\ \sum_{i=1}^{3} [\mathcal{D}_i,\ \mathcal{D}_i^\dagger] = 0. \tag{3.11}$$

We write out the last of these, which is the most intricate. Noting that

$$\begin{aligned} [\mathcal{D}_1,\ \mathcal{D}_1^\dagger] &= [\nabla_2 + i\nabla_3, -\nabla_2 + i\nabla_3] = 2iF_{23}, \\ [\mathcal{D}_2,\ \mathcal{D}_2^\dagger] &= -2i[\phi_2, \phi_3], \\ [\mathcal{D}_3,\ \mathcal{D}_3^\dagger] &= -2i\nabla_y \phi_1, \end{aligned} \tag{3.12}$$

we have

$$\frac{1}{2i}\sum_{k=1}^{3}[\mathcal{D}_k,\ \mathcal{D}_k^\dagger] = F_{23} - [\phi_2, \phi_3] - \nabla_y\phi_1 = 0.$$

As is standard for such equations, cf. [63], the smaller system $[\mathcal{D}_i, \mathcal{D}_j] = 0$ is invariant under the complex ($SL(2;\mathbb{C})$-valued) gauge group $\mathcal{G}_P^{\mathbb{C}}$, while the full system (3.11) is invariant under the unitary gauge group, $\mathcal{D}_i \to g^{-1}\mathcal{D}_i g$, $g \in \mathcal{G}_P$ and the final equation is a real moment map condition. Following the spirit of Donaldson-Uhlenbeck-Yau [21],[59], we thus expect that Hermitian geometric data from the $\mathcal{G}_P^{\mathbb{C}}$-invariant equations play a role in solving the moment map equation.

Suppose that $E$ is a rank 2 Hermitian bundle over $\Sigma \times \mathbb{R}^+$. As we now explain, for any function $f$ and section $s$, $\mathcal{D}_1(fs) = \partial_{\bar{z}}fs + f\mathcal{D}_1 s$, which is a $\partial$-operator in Newlander-Nirenberg sense; $\mathcal{D}_2$ is then a $K_\Sigma$-valued endomorphism of $\mathcal{E}$, while $\mathcal{D}_3$ specifies a parallel transport in the $y$ direction. In terms of these, the equations $[\mathcal{D}_i, \mathcal{D}_j] = 0$ have a nice geometric meaning.

Denote by $E_y := E|_{\Sigma\times\{y\}}$ the restriction of $E$ to each slice $\Sigma \times \{y\}$. Since $\mathcal{D}_1^2 = 0$ is always true for dimensional reasons, the Newlander-Nirenberg theorem gives that $\mathcal{D}_1$ induces a holomorphic structure on $E_y$ for each $y$, i.e., in some gauge, we can write $\mathcal{D}_1 = \bar{\partial}$. A connection $A$ is compatible with this holomorphic structure if $A^{0,1}$ equals $\bar{\partial}$.

Next, $[\mathcal{D}_1, \mathcal{D}_2] = 0$ says that the endomorphism $\varphi$ is holomorphic with respect to this structure, so $(E, \mathcal{D}_1, \varphi)$ is a Higgs pair over each slice. Finally, the equations $[\mathcal{D}_2, \mathcal{D}_3] = 0$, $[\mathcal{D}_1, \mathcal{D}_3] = 0$ show that this family of Higgs pairs is parallel in $y$, i.e., there is a specified identification of these objects at different values of $y$.

Following [21], a data set for our problem consists of a rank two bundle $E$ over $\Sigma \times \mathbb{R}^+$ and a triplet of operators $\Theta = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ on $C^\infty(E)$ satisfying

- $\mathcal{D}_1(fs) = \partial_{\bar{z}}fs + f\mathcal{D}_1 s$, $\mathcal{D}_3(fs) = (\partial_y f)s + f\mathcal{D}_3 s$ for $f \in C^\infty(\Sigma \times \mathbb{R}^+)$ and $s \in C^\infty(E)$;
- $\mathcal{D}_2 = [\varphi, \cdot]$ for some $\varphi \in \Omega^{1,0}(\mathfrak{g}_P)$;
- $[\mathcal{D}_i,\ \mathcal{D}_j] = 0$ for all $i, j$.

Given $(E, \Theta)$, a choice of Hermitian metric $H$ on $E$ determines Hermitian adjoints $\mathcal{D}_i'$ of the operators $\mathcal{D}_i$ by the requirements that for any smooth functions $f$ and sections $s$:

- $\mathcal{D}_1'$ and $\mathcal{D}_3'$ are derivations, i.e., $\mathcal{D}_1'(fs) = (\partial_z f)s + f\mathcal{D}_1 s$, $\mathcal{D}_3'(fs) = (\partial_y f)s + f\mathcal{D}_3 s$, while $\mathcal{D}_2(fs) = f\mathcal{D}_2(s)$;
- $\partial_{\bar{z}} H(s,s') = H(\mathcal{D}_1 s, s') + H(s, \mathcal{D}_1' s')$, $\partial_y H(s,s') = H(\mathcal{D}_3 s, s') + H(s, \mathcal{D}_3' s')$;
- $H(\mathcal{D}_2 s, s') + H(s, \mathcal{D}_2' s') = 0$

The moment map equation in (3.11) can be regarded as an equation for the Hermitian metric $H$. Indeed, setting $\mathcal{D}_y = \frac{1}{2}(\mathcal{D}_3 + \mathcal{D}_3')$, $\mathcal{D}_{\bar{z}} = \mathcal{D}_1$ and $\mathcal{D}_z = \mathcal{D}_1'$, we define a unitary connection $\mathcal{D}_A$, and an endomorphism-valued 1-form $\phi$ and 0-form $\phi_1$ on $(E, \Theta, H)$ by

$$
\begin{aligned}
\mathcal{D}_A(s) &:= \mathcal{D}_1(s)d\bar{z} + \mathcal{D}_1'(s)dz + \mathcal{D}_y(s)dy, \\
[\phi, s] &:= [\mathcal{D}_2, s]dz + [\mathcal{D}_2', s]d\bar{z}, \\
\phi_1 &:= \frac{i}{2}(\mathcal{D}_3 - \mathcal{D}_3').
\end{aligned}
\tag{3.13}
$$

We call $(A, \phi, \phi_1)$ a unitary triplet. Note however that in an arbitrary trivialization of $E$, $(A, \phi, \phi_1)$ may not consist of unitary matrices. We recall a standard result [7] which provides the link between connections in unitary and holomorphic frames. In the following, and later, we refer to parallel holomorphic gauges. These are, as the moniker suggests, holomorphic gauges for each $E_y$ which are parallel with respect to $\mathcal{D}_3$.

**Proposition 3.2.2.** *With $(E, \Theta, H)$ as above, there is a unique triplet $(A, \phi, \phi_y)$ compatible with the unitary structure and with the structure defined by $\Theta$. In other words, in every unitary gauge, $A^\star = -A$, $\phi^\star = \phi$, $\phi_1^\star = -\phi_1$, while in every parallel holomorphic gauge, $\mathcal{D}_1 = \overline{\partial}_E$ and $\mathcal{D}_3 = \partial_y$, i.e., $A^{(0,1)} = A_y - i\phi_1 = 0$.*

*Proof.* With the convention $H(s,s') = \bar{s}^\top H s'$, we compute first in a holomorphic parallel gauge, from the defining equations for the $\mathcal{D}_i'$, that $\bar{\partial} H = (\overline{A^{(1,0)}})^\top H$ and $\partial_y H = H(-A_y - i\phi_1)$, so in this gauge, $A = A^{(1,0)} = H^{-1}\partial H$ and $A_y + i\phi_1 = -H^{-1}\partial_y H$.

Suppose next that we know $H$ with respect to a homolomorphic frame. If $g$ is a complex gauge transformation such that $H = g^\dagger g$, then in the parallel holomorphic gauge,

$$A^{(1,0)} = H^{-1}\partial H = g^{-1}(g^\dagger)^{-1}(\partial_z g^\dagger)g + g^{-1}\partial_z g, \quad A^{(0,1)} = 0. \tag{3.14}$$

If $\widehat{A}$ is the connection form in unitary gauge, then

$$\widehat{A}_z = (g^\dagger)^{-1}\partial_z g^\dagger, \quad \widehat{A}_{\bar{z}} = -(\partial_{\bar{z}} g)g^{-1}, \tag{3.15}$$

and $\widehat{A}_{\bar{z}}^{\dagger} = -\widehat{A}_z$. Thus $g$ transforms the holomorphic form to the unitary one.

Similarily, the same Higgs field in holomorphic and unitary gauge, $\varphi$ and $\phi$, are related by

$$\phi_z = g\varphi g^{-1},\ \phi_{\bar{z}} = (g^{\dagger})^{-1}\bar{\varphi}^{\top}g^{\dagger}. \tag{3.16}$$

For the final component, suppose that $\mathcal{A}_y$ is given in holomorphic gauge. Then in unitary gauge,

$$A_y = \frac{1}{2}((\partial_y g)g^{-1} - (g^{\dagger})^{-1}\partial_y g^{\dagger}),\ \ \phi_1 = \frac{i}{2}((g^{\dagger})^{-1}\partial_y g^{\dagger} + \partial_y g^{\dagger}(g^{\dagger})^{-1}). \tag{3.17}$$

□

We now record some computations in a local holomorphic coordinate chart. Writing $\mathcal{D}_1 = \partial_{\bar{z}} + \alpha$, $\mathcal{D}_1' = \partial_z + A^{(1,0)}$, $\mathcal{D}_3 = \partial_y + \mathcal{A}_y$ and $\mathcal{D}_3' = \partial_y + \mathcal{A}_y'$, we compute:

$$\begin{aligned} A^{(1,0)} &= H^{-1}\partial_z H - H^{-1}(\bar{\alpha})^{\top}H,\\ A &= A^{(1,0)} + \alpha = H^{-1}\partial_z H - H^{-1}\bar{\alpha}^{\top}H + \alpha,\\ \varphi^{\dagger} &= H^{-1}\bar{\varphi}^{\top}H,\\ \mathcal{A}_y' &= H^{-1}\partial_y H - H^{-1}\bar{\mathcal{A}_y}^{\top}H. \end{aligned} \tag{3.18}$$

Thus if $\alpha = \mathcal{A}_y = 0$, and the adjoint operators become

$$\mathcal{D}_1^{\dagger} = -\mathcal{D}_1' = -(\partial_z + H^{-1}\partial_z H),\ \mathcal{D}_2^{\dagger} = -\mathcal{D}_2' = [\varphi^{\dagger},\ ],\ \mathcal{D}_3^{\dagger} = -\mathcal{D}_3' = -\partial_y - H^{-1}\partial_y H. \tag{3.19}$$

Altogether, in a local holomorphic coordinate $z$ for which the metric on $\Sigma$ equals $g_0^2|dz|^2$, and in the holomorphic parallel gauge where $\mathcal{D}_1 = \bar{\partial}$, $\mathcal{D}_3 = \partial_y$, then in local coordinate $(z, y)$, the extended Bogomolny equations (3.11) become

$$-\bar{\partial}_{\bar{z}}(H^{-1}\partial_z H) - g_0^2\partial_y(H^{-1}\partial_y H) + [\varphi_z, \varphi_{\bar{z}}^{\star}] = 0. \tag{3.20}$$

Two sets of data $(E, \Theta)$ and $(E, \widetilde{\Theta})$ are called equivalent if there exists a complex gauge transform $g$ such that $g^{-1}\widetilde{\mathcal{D}}_i g = \mathcal{D}_i$, $i = 1, 2, 3$. A key fact is that $(E, \Theta)$ is completely determined by a Higgs pair $(\mathcal{E}, \varphi)$ over the Riemann surface $\Sigma$.

**Proposition 3.2.3.** *(1) Suppose that* $(E, \Theta)$ *and* $(E, \widetilde{\Theta})$ *are two data sets. If the restrictions of* $\Theta$ *to* $E_y$ *and* $\widetilde{\Theta}$ *to some possibly different* $E_{y'}$ *are complex gauge equivalent, then* $(E, \Theta)$ *and* $(E, \widetilde{\Theta})$ *are equivalent.*

*(2) If $(E, \Theta, H)$ is a solution to the extended Bogomolny equations , and if $g$ is a complex gauge transform, then $(E, \Theta^g)$, where $\Theta^g = (g^{-1}D_1 g, g^{-1}D_2 g, g^{-1}D_3 g), H^g = Hg^{\star_H} g$ is also a solution.*

*Proof.* Since $\mathcal{D}_3$ and $\widetilde{\mathcal{D}}_3$ both define isomorphisms of the Higgs pairs, (1) follows immediately. Then, recalling that $D_i^\dagger$ is the conjugate of $D_i$ with respect to $H$, one may check (2) directly from the definition. □

### 3.3 Boundary Conditions

In this section we introduce boundary conditions for the extended Bogomolny equations over $\Sigma \times \mathbb{R}^+$ at $y = 0$ and as $y \longrightarrow +\infty$.

#### $SL(2, \mathbb{R})$ Higgs-bundles

We impose an asymptotic boundary condition as $y \longrightarrow +\infty$ by requiring that solutions of (3.1) converge to flat $SL(2, \mathbb{R})$ connections. To explain this more carefully, we recall some basic facts about the moduli space of stable $SL(2, \mathbb{R})$ Higgs-bundles, cf. [31], [32].

Consider a Riemann surface $\Sigma$ of genus $g > 1$. A Higgs bundle consists of a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a holomorphic structure on a complex vector bundle $E$ and $\varphi \in H^0(\mathrm{End}(\mathcal{E}) \otimes K)$ is a Higgs field. Let $(\mathcal{E}, \varphi)$ be a rank 2 Higgs bundle such that $\deg E = 0$. It is proved in [31] that once an $SL(2, \mathbb{R})$ structure is fixed, there is an isomorphism $\mathcal{E} \cong L^{-1} \oplus L$, where $L$ is a line bundle with $0 \le \deg L \le g - 1$, in terms of which the Higgs field takes the form

$$\varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \tag{3.21}$$

where $\alpha \in H^0(L^{-2} \otimes K)$ and $\beta \in H^0(L^2 \otimes K)$. When $\deg L = g - 1$ and $L = K^{\frac{1}{2}}$ for one of the $2^{2g}$ square roots of $K$, then we write this canonical form for the Higgs field in the familiar form

$$\varphi = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \tag{3.22}$$

Here 1 is the canonical identity element in $\mathrm{Hom}(L, L^{-1}) \otimes K = \mathrm{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \otimes K = \mathcal{O}$ and $q \in H^0(L^2 \otimes K) = H^0(K^2)$ is a holomorphic quadratic differential. This set of Higgs bundles constitutes the Hitchin component of the $SL(2, \mathbb{R})$ moduli space.

The splittings with $|\deg L| < g - 1$ constitute the non-Hitchin components. Write $k = \deg L$ so that $\deg(L^{-2} \otimes K) = \deg K - 2 \deg L = 2g - 2 - 2k$. Thus when

$0 \leq k < g-1$, the section $\alpha$ has $2g-2-2k$ zeros; these are of course invariant under complex gauge transform.

If $\phi_1 = 0$ in (3.1), or if $D_3 = 0$ in (3.11), we obtain the Hitchin equation

$$F_H + [\varphi, \varphi^\star] = 0, \ \bar{\partial}_A \varphi = 0. \tag{3.23}$$

A rank 2 Higgs pair $(\mathcal{E}, \varphi)$ with $\det(\mathcal{E}) = \mathcal{O}$ is stable if for every $\varphi$-invariant subbundle $S \subset E$, $\deg S < 0$. We say in general that $(\mathcal{E}, \varphi)$ is polystable if it is direct sum of stable Higgs bundle. In the rank 2 case, a polystable Higgs bundle takes the form $(E = L^{-1} \oplus L, \varphi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix})$, but by assumption we shall exclude these.

The solvability of the Hitchin equation (3.23) was analyzed completely in [31].

**Theorem 3.3.1.** *[31] Let $(\mathcal{E}, \varphi)$ be a Higgs pair over $\Sigma$. There exists an irreducible solution $H$ to the Hitchin equations if and only if the Higgs pair is stable, and a reducible solution if and only if it is polystable.*

When $\deg L > 0$, the Higgs pairs $(L^{-1} \oplus L, \varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix})$ are all stable. If $\deg L = 0$, then $L \cong \mathcal{O}$ and $E$ is holomorphically trivial. If $\varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$, then the pair is stable if and only if neither $\alpha$ nor $\beta$ are identically zero. If precisely one of $\alpha$, $\beta$ vanishes, the pair is neither stable nor polystable and the Hitchin equation has no solution. If both $\alpha = \beta = 0$, then the Higgs bundle is polystable and there exist a reducible solution.

In this paper we restrict attention to irreducible solutions. The moduli space of stable $SL(2,\mathbb{R})$-Higgs pairs can then be described as follows:

**Theorem 3.3.2.** *[31] The $SL(2,\mathbb{R})$ Higgs bundle moduli space contains $2g-1$ components, classified by the degree $k$ of the line bundle $L$, $|k| \leq g-1$. The component $\mathcal{M}_k^{SL(2,\mathbb{R})}$ is a smooth manifold of dimension $(6g-6)$ diffeomorphic to a complex vector bundle of rank $(g-1+2k)$ over the $2^{2g}$-fold cover of the symmetric product $S^{2g-2-2k}\Sigma$.*

*Proof.* We sketch the proof. For the $SL(2,\mathbb{R})$ Higgs bundle $(L^{-1} \oplus L, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix})$, the zeroes of $\alpha \in H^0(L^{-2} \otimes K)$ give a divisor $D$ where $\mathcal{O}(D) = L^{-2} \otimes K$, and hence an element of $S^{2g-2-2k}\Sigma$. Then $\beta \in H^0(\Sigma, \mathcal{O}(-D)K^2)$ determines a line bundle.

Note that since we are working with $SL(2,\mathbb{R})$, given $D$ we can only determine $L^2 = \mathcal{O}(-D)K$, but $L$ itself can only be recovered up to the choice of a line bundle $I$ with $I^2 = \mathcal{O}$. There are precisely $2^{2g}$ such choices. □

We recall finally a well-known result:

**Proposition 3.3.3.** *The harmonic metric $H$ corresponding to a stable $SL(2,\mathbb{R})$ Higgs pair splits with respect to the decomposition $E = L^{-1} \oplus L$, $H = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}$.*

A proof appears in [13, Theorem 2.10].

**The Nahm Pole Boundary Condition and Holomorphic Data**

We next recall the Nahm pole boundary condition and its associated Hermitian geometry, following [25].

The starting point is the model solution [63]. Consider a trivial rank 2 bundle $E$ over $\mathbb{C} \times \mathbb{R}^+$. The model Nahm pole solution is

$$A_z = 0,\ \phi_z = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \mathcal{A}_y = -i\phi_1 = \frac{1}{2y}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.24}$$

Under the singular complex gauge transformation, these fields become $g = \begin{pmatrix} y^{-\frac{1}{2}} & 0 \\ 0 & y^{\frac{1}{2}} \end{pmatrix}$ to $\varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_z = 0$ and $\mathcal{A}_y = 0$, i.e., the connection in the $\mathbb{R}^+$ direction transforms to $\partial_y$.

Now, $s = \begin{pmatrix} ay^{-\frac{1}{2}} \\ by^{\frac{1}{2}} \end{pmatrix}$ is an $\mathcal{D}_3$ parallel section of $E$ for any $a, b \in \mathbb{R}$, and indeed is a solution of the full extended Bogomolny equations. A generic solution of this form blows up as $y \to 0$, but there is a well-defined subbundle $L \subset E$, called the **vanishing line bundle**, defined as the space of solutions which tend to 0 as $y \to 0$. For this model solution and line bundle, span $\{\varphi(L), L \otimes K\} = E \otimes K$ at all points.

We say that a solution $(A, \varphi, \phi_1)$ to (3.1) on a rank 2 Hermitian bundle $E$ with determinant zero over $\Sigma$ satisfies the Nahm pole boundary condition if in terms of any local trivialization

$$A_z \sim \mathcal{O}(y^{-1+\epsilon}),\ \varphi = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(y^{-1+\epsilon}),\ \mathcal{A}_y = \frac{1}{2y}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathcal{O}(y^{-1+\epsilon}) \tag{3.25}$$

as $y \to 0$. As described in [43], it is of course necessary to consider fields which lie in some function space, e.g. a weighted Hölder space, and the error estimate $O(y^{-1+\epsilon})$ is interpreted in terms of that norm. The regularity theory in that paper shows that a solution of the extended Bogomolny equations, or indeed of the full Kapustin-Witten system, is then much more regular after being put into gauge.

In exactly the same way as in the model case, this boundary condition defines a line bundle $L \subset E$, and since $\det E = O$, we have $E/L \cong L^{-1}$. On the other hand, $\mathrm{span}\{\varphi(L), L \otimes K\} = E \otimes K$, so that pushing forward $L$ via

$$L \xrightarrow{\varphi} E \otimes K \to (E/L) \otimes K \tag{3.26}$$

shows that $L \cong L^{-1} \otimes K$, i.e., $L \cong K^{\frac{1}{2}}$, and then $E/L \cong K^{-\frac{1}{2}}$. In other words,

$$0 \to K^{\frac{1}{2}} \to E \to K^{-\frac{1}{2}} \to 0. \tag{3.27}$$

In addition, denote $i_1 : \varphi(L) \to E \otimes K$ and $i_2 : L \otimes K \to E \otimes K$, and define:

$$\begin{aligned} i &: \varphi(L) \oplus L \otimes K \to E \otimes K \\ i &= i_1 + i_2. \end{aligned} \tag{3.28}$$

As $\mathrm{span}\{\varphi(L), L \otimes K\} = E \otimes K$, we obtain that $i$ is surjective between two rank two bundles thus isomorphism. Tensoring by $K^{-1}$, we obtain $E \cong K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$.

Under a complex gauge transform, we can then put the Higgs field into the form $\varphi = \begin{pmatrix} t & 1 \\ \beta' & -t \end{pmatrix}$. Setting $g = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$, we compute that $g^{-1}\varphi g = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$. This shows that a $SL(2,\mathbb{R})$ Higgs bundle lies in the Hitchin component of the $SL(2,\mathbb{R})$ Higgs bundle moduli space.

In summary, recalling that $\mathcal{M}^{\mathrm{EBE}}_{\mathrm{NP}}$ is the moduli space of solutions of the extended Bogomolny equations with limit in $SL(2,\mathbb{R})$ and $\mathcal{M}^{\mathrm{Hit}}$ is the Hitchin component of stable $SL(2,\mathbb{R})$ Higgs bundle, we have now explained the map $I_{\mathrm{NP}} : \mathcal{M}^{\mathrm{EBE}}_{\mathrm{NP}} \to \mathcal{M}^{\mathrm{Hit}}$. Gaiotto and Witten [25] conjectured that this map is a bijection, and we show below that this is the case.

**Knot Singularity**

We next define the model knot singularity introduced by Witten in [63], and the modified Nahm pole condition for knots. In the Riemann surface picture, knot singularities correspond to marked points, at which monopoles are wrapped.

Fix coordinates $z = x_2 + ix_3 \in \mathbb{C}$ and $y \in \mathbb{R}^+$ on $\mathbb{C} \times \mathbb{R}^+$. Then, with respect to the Higgs field $\varphi = \begin{pmatrix} 0 & z^n \\ 0 & 0 \end{pmatrix}$ and Hermitian metric $H = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}$, equation (3.20) takes the form

$$-(\Delta + \partial_y^2)u + r^{2n}e^{2u} = 0, \tag{3.29}$$

where $\Delta = \partial_{x_2}^2 + \partial_{x_3}^2$ and $r = |z|$.

Assuming homogeneity in $(z, y)$ and radial symmetry in $z$, Witten [63] obtained the model solution

$$U_n(r, y) = \log\left(\frac{2(n+1)}{(\sqrt{r^2 + y^2} + y)^{n+1} - (\sqrt{r^2 + y^2} - y)^{n+1}}\right). \tag{3.30}$$

To investigate this further, introduce spherical coordinates $(R, \psi, \theta)$,

$$R = \sqrt{r^2 + y^2},\ z = re^{i\theta},\ \sin\psi = \frac{y}{R},\ \cos\psi = \frac{r}{R}.$$

Writing $a = \sqrt{r^2 + y^2} + y$ and $b = \sqrt{r^2 + y^2} - y$, then

$$\frac{a}{R} = 1 + \frac{y}{R} = 1 + \sin\psi,\ \frac{b}{R} = 1 - \frac{y}{R} = 1 - \sin\psi,$$

and hence

$$U_n = -\log y - n\log R + \log\frac{n+1}{S_n(\psi)},$$

where

$$S_n(\psi) = \mathcal{S}_n(a, b) = \sum_{k=0}^{n} a^{n-k}b^k.$$

Note that $U_0 = -\log y$ when $n = 0$, which recovers the model Nahm pole solution. Moreover, $U_n$ is compatible with the Nahm pole singularity in the sense that $U_n \sim -\log y$ as $y \to 0$ for $r \geq \epsilon > 0$.

Defining $g_n = \begin{pmatrix} e^{u_n/2} & 0 \\ 0 & e^{-u_n/2} \end{pmatrix}$, then in unitary gauge

$$A_z = g_n^{-1}\partial g_n,\ A_{\bar{z}} = -(\bar{\partial} g_n)g_n{}^{-1},\ \phi_z = g_n\varphi g_n^{-1},\ \phi_1 = \frac{i}{2}(g_n^{-1}\partial_y g_n + \partial_y g_n g_n^{-1}), \tag{3.31}$$

or explicitly,

$$\begin{aligned}
\phi_z &= \begin{pmatrix} 0 & z^n e^{U_n} \\ 0 & 0 \end{pmatrix} \\
&= \frac{2}{R} \frac{(n+1)\cos^n \psi}{(1+\sin\psi)^{n+1} - (1-\sin\psi)^{n+1}} e^{in\theta} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \frac{1}{R\sin\psi} \frac{(n+1)\cos^n \psi}{S_n(\psi)} e^{in\theta} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\phi_1 &= -U_n' \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} \\
&= \frac{n+1}{R} \frac{(1+\sin\psi)^{n+1} + (1-\sin\psi)^{n+1}}{(1+\sin\psi)^{n+1} - (1-\sin\psi)^{n+1}} \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} \\
A_y &= 0.
\end{aligned} \tag{3.32}$$

Suppose that $s$ is a section with $\mathcal{D}_3 s = 0$. Then for any $a, b \in \mathbb{R}$, $s = \begin{pmatrix} a e^{U_n/2} \\ b e^{-U_n/2} \end{pmatrix}$ is a solution, where $e^{U_n} = (n+1)/(yR^n S_n(\psi))$. As in the Nahm pole case, we can still define a line subbundle $L$ corresponding to parallel sections whose limits as $y \to 0$ vanish; generic parallel sections blow up. However, a new feature here is that $\mathrm{span}(L \otimes K, \varphi(L)) \neq E \otimes K$ precisely at the knot singularities, reflecting the zeroes of $\varphi$.

For any $p \in \Sigma$ we can transport the model solution to $\Sigma \times \mathbb{R}^+$ using the local coordinates $(z, y)$, giving an approximate solution $(A^p, \phi^p, \phi_1^p)$ in a neighborhood of $(p, 0)$. It is convenient

**Definition 3.3.4.** *A solution $(A, \phi, \phi_1)$ to the extended Bogomolny equations satisfies the general Nahm pole boundary condition with knot singularity of order $n$ at $(p, 0) \in \Sigma \times \mathbb{R}^+$ if in a suitable gauge it satisfies*

$$(A, \phi, \phi_1) = (A^p, \phi^p, \phi_1^p) + \mathcal{O}(R^{-1+\epsilon}(\sin\psi)^{-1+\epsilon}) \tag{3.33}$$

*for some $\epsilon > 0$, where $R$ and $\psi$ are the spherical coordiates used above.*

Corresponding to a solution with knot singularity is a set of holomorphic data. Suppose $(A, \phi, \phi_1)$ is a solution with a knot singularity at the points $\{p_j\}$ with orders $n_j$, $j = 1, \cdots, N$. We define the line subbundle $L$ of $E$ and obtain the exact sequence

$$0 \to L \to E \to L^{-1} \to 0. \tag{3.34}$$

Using the asymptotic boundary condition at $y \to +\infty$ and the Milnor-Wood inequality [50], [66], we have $|\deg L| \leq g - 1$.

The knot singularity and Higgs field induce a map

$$P : L \xrightarrow{\varphi} E \otimes K \to L^{-1} \otimes K. \tag{3.35}$$

Regarding $P$ as an element of $H^0(L^{-2} \otimes K)$, we deduce that that there are $2g - 2 - 2\deg L$ marked points, counted with multiplicity.

The data we must specify then consists of the following:

1. An $SL(2;\mathbb{C})$ Higgs bundle with a line subbundle $L$;
2. Marked points $\{p_j\}$ with orders $n_j$;
3. Generic parallel sections of $E$ over $\Sigma \setminus \{p_j\}$ blow up at the rate $y^{-\frac{1}{2}}$;
4. The section $P \in H^0(L^{-2}K)$ in (3.35) has zeroes precisely at $p_j$ of order $n_j$.

Just as for the Nahm pole case, we impose an $SL(2,\mathbb{R})$ structure on the Higgs bundle. The following assumption simplifies the Hermitian geometric data.

**Definition 3.3.5.** *Suppose we have a solution to* (3.1) *which satisfies the general Nahm pole boundary conditions, and assume that the solution converges to an* $SL(2,\mathbb{R})$ *Higgs bundle* $(L^{-1} \oplus L, \varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix})$ *as* $y \to \infty$*. We say that this solution is compatible with the* $SL(2,\mathbb{R})$ *structure at* $y = \infty$ *if either* $L$ *or* $L^{-1}$ *is the vanishing line bundle.*

Merely assuming that the Higgs bundle converges to an $SL(2,\mathbb{R})$ Higgs bundle, as above, is not enough to imply that $L$ is the vanishing line bundle.

**Remark.** *If the exact sequence* (3.34) *splits, the Higgs field may take the slightly more general form* $\varphi = \begin{pmatrix} t & \alpha \\ \beta & -t \end{pmatrix}$*. Such Higgs fields with* $t \neq 0$ *exist, but at present we do not know whether it is possible to solve the extended Bogomolny equations with knot singularity with this data. The vanishing of* $t$ *will play a minor but important technical role below in Proposition 3.3.9, which we need in proving uniqueness theorems later.*

The compatibility of the solution with the $SL(2,\mathbb{R})$ structure is a technical condition that allows us to reduce the Bogomolny equation to a scalar equation. There is one special case where we do not need to assume this compatibility condition. Under the assumption of Definition 3.3.5, denote the vanishing line bundle as $L'$. We then obtain

**Proposition 3.3.6.** *If $L' \neq L$ or $L^{-1}$, then* $\deg L' \leq -|\deg L|$,

*Proof.* The line subbundle $L'$ induces the exact sequence:

$$0 \to L' \to L^{-1} \oplus L \to L'^{-1} \to 0,$$

which defines the holomorphic map $\gamma_1 : L \to L'^{-1}$ and $\gamma_2 : L^{-1} \to L'^{-1}$. Since $L' \neq L$ or $L^{-1}$, we obtain that neither $\gamma_1$ nor $\gamma_2$ equal the identity. In other words, we obtain non-zero elements $\gamma_1 \in H^0(L^{-1} \otimes L'^{-1})$ and $\gamma_2 \in H^0(L \otimes L'^{-1})$. Since $\gamma_1, \gamma_2$ do not have poles, we obtain $\deg(L^{-1} \otimes L'^{-1}) \geq 0$ and $\deg(L \otimes L'^{-1}) \geq 0$, which implies $\deg L' \leq -|\deg L|$. □

Denoting by $N := \sum n_j$ the sum of the orders of the marked points, we conclude the

**Corollary 3.3.7.** *If* $\deg L > 0$ *and* $N < 2g - 2 + 2\deg L$*, then* $L' = L$*.*

*Proof.* Recall that $N = 2g - 2 - 2\deg L'$, and furthermore, if $N < 2g - 2 + 2\deg L$, then $\deg L' > -\deg L$. Proposition 3.3.6 then implies this result. □

**Regularity**

We have defined these boundary conditions both at $y = 0$ and at the knot singular points by requiring the fields $(A, \phi)$ to differ from the corresponding model solutions by an error term, the relative size of which is smaller than the model. In the existence theorems later in this paper this may be all we know about solutions at first. However, to be able to carry out many further arguments it is important to know that, in an appropriate gauge, solutions have much stronger regularity properties. Fortunately there is an appropriate regularity theory available which was developed in [43] in the Nahm pole case and [46] near the knot singularities. We note that in those papers solutions to the full four-dimensional KW system are treated, but those results specialize directly to the present setting, and in fact there are some minor but important strengthenings here which we point out inter alia.

Regularity theory relies on ellipticity, and to turn the extended Bogomolny equations into an elliptic system we must add an appropriate gauge condition. We recall the choice made in [43] for the KW system on a four-manifold and then specialize it in our dimensionally reduced setting. Fix a pair of fields $(\widehat{A}^0, \widehat{\phi}^0)$ on a four-manifold which are either solutions or approximate solutions of KW equations. Then nearby fields can be written in the form $(\widehat{A}, \widehat{\phi}) = (\widehat{A}^0, \widehat{\phi}^0) + (\alpha, \psi)$. The gauge-fixing equation is then

$$d^*_{\widehat{A}^0}\alpha + \star[\widehat{\phi}^0, \star\psi] = 0. \tag{3.36}$$

It is shown in [43] that adjoining (3.36) to the KW equations is elliptic.

Denote by $\mathcal{L}$ the linearization of this system at $(\widehat{A}^0, \widehat{\phi}^0)$. This is a Dirac-type operator with coefficients which blow up at $y = 0$ and $R = 0$ in a very special manner. In the absence of knots, $\mathcal{L}$ is (up to a multiplicative factor) a uniformly degenerate operator, while near a knot it lies in a slightly more general class of incomplete iterated edge operators. These are classes of degenerate differential operators for which tools of geometric microlocal analysis may be applied to construct parametrices, which in turn lead to strong mapping and regularity properties. We refer to [43], [46] for further discussion about all of this and simply state the consequences of this theory here.

Before doing this we first recall that for degenerate elliptic problems it is too restrictive to expect solutions to be smooth up to the boundary. Instead we consider polyhomogeneous regularity. Let $X$ be a manifold with boundary, with coordinates $(s, z)$ near a boundary point, with $s \geq 0$ and $z$ a coordinate in the boundary. We say that a function $u$ is polyhomogeneous at $\partial X$ if

$$u(s, z) \sim \sum_{j=0}^{\infty} \sum_{\ell=0}^{N_j} a_{j\ell}(z) s^{\gamma_j} (\log s)^{\ell}, \quad a_{j\ell} \in C^{\infty}(\partial X).$$

The exponents $\gamma_j$ here is a sequence of (possibly complex) numbers with real parts tending to infinity; importantly, for each $j$, only finitely many factors with (positive integral) powers of $\log s$ can appear. The set of pairs $(\gamma_j, \ell)$ which appear in this expansion is called the index set for this expansion. Denoting this index set by $\mathcal{I}$, we say that $u$ is $\mathcal{I}$-smooth, which emphasizes that this regularity is a very close relative of and satisfactory replacement for ordinary smoothness. Similarly, if $X$ is a manifold with corners of codimension 2, with coordinates $(s_1, s_2, z)$ near a point on the corner, then $u$ is polyhomogeneous if

$$u(s_1, s_2, z) \sim \sum_{i,j=0}^{\infty} \sum_{p,q=0}^{N_{i,j}} a_{ijpq}(z) s_1^{\gamma_i} s_2^{\lambda_j} (\log s_1)^p (\log s_2)^q.$$

In other words, we require the expansion for $u$ to be of product type near the corner. These are all classical expansions with the usual meaning and the corresponding expansions for any number of derivatives hold as well. The reason for introducing this more general notion is precisely because at least in favorable situations, solutions of have this regularity but are not smooth in a classical sense. The important point is that this is a perfectly satisfactory replacement for smoothness up to the boundary and allows one to analyze and manipulate expressions using these 'Taylor series' type expansions.

We first consider the case where there are no knot singularities, but note that this result is a local one and can be applied away from knot singular points. Here the manifold with boundary is simply $\Sigma \times \mathbb{R}^+$ and we use coordinates $(y, z)$.

**Proposition 3.3.8** ([43])**.** *Let $(A, \varphi, \phi_1)$ be a solution to the extended Bogomolny equations near $y = 0$ which satisfies the Nahm pole boundary conditions and is in gauge relative to the model approximate solution. Then these fields are polyhomogeneous with*

$$A = O(1),\ \varphi = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + O(y),\ \phi_1 = \frac{1}{y}\begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} + O(y \log y)$$

This statement incorporates recent work in [29] which provides much more detail about the expansions than is present in [43].

To state the corresponding result in the presence of a knot singularity, we first define the manifold with corners $X$ to be the blowup of $\Sigma \times \mathbb{R}^+$ around each of the knot singular points $(p_j, 0)$. In other words, we replace each $(p_j, 0)$ by the hemisphere $R = 0$ (parametrized by the spherical coordinate variables $(\psi, \theta)$), points of which label directions of approach to that point. The discussion is local near each $p_j$ so we may as well fix coordinates $(R, \psi, \theta)$. The corner of $X$ is defined by $R = \psi = 0$.

**Proposition 3.3.9** ([46])**.** *Let $(A, \phi, \phi_1)$ satisfy the extended Bogomolny equations near $(0, 0)$ as well as the gauge condition relative to the model knot solution $U_n$. Then these fields are polyhomogeneous with the same asymptotics as in the previous proposition when $y \longrightarrow 0$ away from the knot, while*

$$A = A^n + O(R^\epsilon \sin\psi),\ \varphi = \varphi^n + O(R^\epsilon \sin\psi),\ \phi_1 = \phi_1^n + O(R^\epsilon(\sin\psi)\log(\sin\psi))$$

*near the knot. Here $(A^n, \varphi^n, \phi_1^n)$ is the model solution described in §3.3 associated to $U_n$.*

Referring to the language of [46], these rates of decay, i.e., the first exponents in the expansions beyond the initial model terms, are indicial roots of type $II$ and $II'$. The exponent 0 is a possible indicial root of type $II'$, but does not appear in our setting because the $\mathrm{SL}(2,\mathbb{R})$ structure forces $\varphi$ to have no diagonal terms, see Remark 3.3, and it is precisely in these diagonal terms where the exponent 0 might appear in the expansion.

**The Boundary Condition for the Hermitian Metric**

Since we must deal with singularities of the gauge field, it is often simpler to work in holomorphic gauge but consider singular Hermitian metrics. We now describe a boundary condition for the Hermitian metric compatible with the unitary boundary condition defined above. We use the Riemannian metric $g = g_0^2|dz|^2 + dy^2$ on $\Sigma \times \mathbb{R}^+$. The following result is a direct consequence of the previous computations in Section 3.3, 3.3.

**Proposition 3.3.10.** *Consider the Higgs bundle* $(E \cong L^{-1} \oplus L, \varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix})$. *Fix* $p \in \Sigma \times \{0\}$ *and an open set* $U_p$ *containing* $p$. *Let* $H$ *be a polyhomogeneous solution to the Hermitian Extended-Bogomolny Equations* (3.11).

*(1) Suppose that in a local trivilization on* $U_p$, $\varphi|_{U_p} = \begin{pmatrix} 0 & 1 \\ \star & 0 \end{pmatrix}$. *If for some* $\epsilon > 0$,

$$H \sim \begin{pmatrix} y^{-1}(g_0 + \mathcal{O}(y^\epsilon)) & 0 \\ 0 & y(g_0^{-1} + \mathcal{O}(y^\epsilon)) \end{pmatrix} \quad \text{as } y \to 0, \tag{3.37}$$

*then the unitary solution with respect to* $H$ *satisfies the Nahm pole boundary condition near* $p$ *and* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *is the vanishing line bundle in this trivialization.*

*(2) Suppose that in a local trivilization on* $U_p$, $\varphi|_{U_p} = \begin{pmatrix} 0 & z^n \\ \star & 0 \end{pmatrix}$ *(where* $z = 0$ *is the point* $p$*). In spherical coordinates* $(R, \theta, \psi)$, *suppose for some* $\epsilon > 0$,

$$H = \begin{pmatrix} e^{U_n}(1 + \mathcal{O}(R^\epsilon)) & 0 \\ 0 & e^{-U_n}(1 + \mathcal{O}(R^\epsilon)) \end{pmatrix} \text{ as } R \to 0. \tag{3.38}$$

*Then the unitary solution with respect to* $H$ *satisfies the Nahm pole condition with knot singularity at* $p$ *and* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *is the vanishing line bundle in this trivialization. .*

Since we wish to work with holomorphic gauge fields and singular Hermitian metrics, we obtain some restrictions. Let $P$ be an $SU(2)$ bundle and $(A, \phi, \phi_1)$ a solution to the Extended Bogonomy Equations (3.1) with Nahm pole boundary and knot singularities of order $n_j$ at the points $p_j$, $j = 1, \cdots, n$. For each $j$ choose small balls $B_j$ around $p_j$, and also let $B_0$ be a neighborhood of $\Sigma \setminus \{B_1, \cdots, B_k\}$ which does not contain any of the $p_j$. Choosing a partition of unity $\chi_j$ subordinate to this cover, define the approximate solution $u = \sum_{j=0} \chi_j U_{n_j}$ where $U_{n_j}$ is the model solution, and with $U_{n_0} = -\log y$.

**Proposition 3.3.11.** *There exists a Hermitan bundle* $(E, H)$ *such that:*

*(1)* $(H, A^{(0,1)}, \varphi, \mathcal{A}_y)$ *is a solution to the Hermitian Extended Bogomolny equations;*

*(2)* $(A^{(0,1)}, \varphi, \mathcal{A}_y)$ *is bounded as* $y \to 0$*;*

*(3)* $H = \begin{pmatrix} e^u h_{11} & h_{12} \\ h_{21} & e^{-u} h_{22} \end{pmatrix}$*, where* $u$ *is the approximate function above and the* $h_{ij}$ *are bounded.*

*Proof.* We have explained that $(A, \phi, \phi_1)$ is polyhomogeneous, i.e., $(A, \phi, \phi_1) = (A^{p_j}, \phi^{p_j}, \phi_1^{p_j}) + (a, b, c)$ near $p_j$, where $(a, b, c)$ are bounded. Near other points of $\Sigma \times \{0\}$ $(A, \phi, \phi_1)$ is the sum of a Nahm pole and a bounded term. Since $P$ is an $SU(2)$ bundle over $\Sigma \times \mathbb{R}^+$, it is necessarily trivial, so consider the associated rank 2 Hermitian bundle $(E, H_0)$, with $H_0 = \mathrm{Id}$ in some trivialization. Now write $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ where the $h_{ij}$ are bounded. Then $(H_0, A^{(0,1)}, \varphi, \mathcal{A}_y)$, where $\varphi = \phi_z$, $\mathcal{A}_y = A_y - i\phi_1$, is a solution to the Hermitan extended Bogomolny equations (3.11). Consider the complex gauge transform $g = \begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix}$. Since $u$ is compatible with the knot singularity, we obtain a new solution $(H', A^{(0,1)'}, \varphi', \mathcal{A}'_y)$, $H' = H_0 g^\dagger g = \begin{pmatrix} e^u h_{11} & h_{12} \\ h_{21} & e^{-u} h_{22} \end{pmatrix}$, and $A^{(0,1)'}, \varphi', \mathcal{A}'_y$ are all bounded. □

We conclude this section with a brief discussion about the regularity of a harmonic metric which satisfies the boundary conditions described here. Such metrics correspond precisely to the solutions $(A_z, A_y, \varphi, \phi_1)$ of the original extended Bogomolny equations , and for this reason one obvious route to obtain this regularity is to exhibit the direct formula from the set of $A's$ and $\phi's$ to the metric $H$. Another reasonable approach is to simply look at the equation (3.20) defining $H$ and prove the necessarily regularity directly from this equation. In fact, the methods used in [43] and [46] are sufficiently robust that this adaptation is quite straightforward. In the interests of efficiency, we simply state the conclusion:

**Lemma 3.3.12.** *A harmonic metric $H$ which satisfies the boundary conditions discussed above is necessarily polyhomogeneous.*

The terms which appear in the polyhomogeneous expansion of $H$ may be determined by the obvious formal calculations once we know that the expansion actually exists.

## 3.4 Existence of Solutions

We shall prove in this section an existence theorem for the extended Bogomolny equations on $\Sigma \times \mathbb{R}^+$, either without or with knot singularities at $y = 0$. The proofs employ the classical barrier method, which we review briefly.

### Semilinear Elliptic Equations on Noncompact Manifolds

We consider on a Riemannian manifold $(W, g)$ the elliptic equation

$$N(u) := -\Delta u + F(x, u) = 0, \quad F \in C^\infty(W \times \mathbb{R}). \tag{3.39}$$

A $C^2$ function $u^+$ is called a supersolution for this problem if $N(u^+) \geq 0$, while $u^-$ is called a subsolution if $N(u^-) \leq 0$. These are called barriers for the operator. It is often much simpler to construct such functions which are only continuous, and which satisfy the corresponding differential inequalities weakly (either in the distributional or viscosity sense).

**Proposition 3.4.1.** *Suppose that $W$ is a possibly open manifold, and that there exist continuous barriers $u^\pm$ which satisfy $u^- \leq u^+$ everywhere on $W$. Then there exists a solution $u$ to $N(u) = 0$ which satisfies $u^- \leq u \leq u^+$.*

*Proof.* (Sketch) We first assume that $W$ is a compact manifold with boundary. Then $u^\pm$ are bounded functions and we may choose $\lambda > 0$ so that $\partial_u F(x, u) \leq \lambda$ for all numbers $u$ lying in the interval $[u^-(x), u^+(x)]$ for every $x \in W$. The equation can then be written as

$$(\Delta - \lambda)u = \widetilde{F}(x, u) := F(x, u) - \lambda u.$$

We then define a sequence of functions $u_j$, $j = 0, 1, 2, \ldots$, by setting $u_0 = u^-$ and successively solving $(\Delta - \lambda)u_{j+1} = \widetilde{F}(x, u_j)$, and with $u_{j+1}$ equal to some fixed function $\psi$ on $\partial W$ which satisfies $u^-|_{\partial W} \leq \psi \leq u^+|_{\partial W}$. The monotonicity of $\widetilde{F}$ in $u$ and the maximum principle can be used to prove inductively that $u^- = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u^+$. When $W$ is a manifold with boundary we require a version of the maximum principle which holds up to the boundary even for weak solutions; one version appears in [33, Theorem II.1].

It is then obvious that $u_j$ converges pointwise to an $L^\infty$ function $u$, and standard elliptic regularity implies that $u \in C^\infty$ and that $N(u) = 0$.

Now suppose that $W$ is an open manifold. Choose a sequence of compact smooth manifolds with boundary $W_k$ with $W_1 \subset W_2 \subset \cdots$, which exhaust all of $W$. For each $k$, choose a function $\psi_k$ on $\partial W_k$ which lies between $u^-$ and $u^+$ on this boundary, and then find a solution $u_k$ to $N(u_k) = 0$ on $W_k$, $u_k = \psi_k$ on $\partial W_k$. The sequence $u_k$ is uniformly bounded on any compact subset of $W$, so we may choose a sequence which converges (by elliptic regularity) in the $C^\infty$ topology on any compact subset of $W$. The limit function is a solution of $N$ and still satisfies $u^- \le u \le u^+$ on all of $W$. □

We conclude this general discussion by making a few comments about the construction of weak barriers. A very convenient principle is that sub- and supersolutions may be constructed locally in the following sense. Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are two open sets in $W$ and that $w_j$ is a supersolution for $N$ on $\mathcal{U}_j$, $j = 1, 2$. Define the function $w$ on $\mathcal{U}_1 \cup \mathcal{U}_2$ by setting $w = w_1$ on $\mathcal{U}_1 \setminus (\mathcal{U}_1 \cap \mathcal{U}_2)$, $w = w_2$ on $\mathcal{U}_2 \setminus (\mathcal{U}_1 \cap \mathcal{U}_2)$, and $w = \min\{w_1, w_2\}$ on $\mathcal{U}_1 \cap \mathcal{U}_2$. Then $w$ is a supersolution for $N$ on $\mathcal{U}_1 \cup \mathcal{U}_2$. Similarly, the maximum of two (or any finite number) of subsolutions is again a subsolution. In our work below, the individual $w_j$ will typically be smooth, but the new barrier $w$ produced in this way is only piecewise smooth, but is still a sub- or supersolution in the weak sense. We refer to [12, Appendix A] for a proof.

**The Scalar Form of the Extended Bogomolny Equations**

Following the discussion in §3, suppose that $E \cong L \oplus L^{-1}$ and

$$\varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \tag{3.40}$$

When $\deg L = g - 1$, $L = K^{1/2}$ and $\alpha = 1$, we seek a solution of the extended Bogomolny equations which satisfies the Nahm pole boundary condition at $y = 0$, while if $\deg L < g - 1$, then the zeroes of $\alpha$ determine points and multiplicities $p_j$ and $n_j$ on $\Sigma$ at $y = 0$ and we search for a solution which satisfies the Nahm pole boundary condition with knot singularities at these points.

Fix a metric $g = g_0^2|dz|^2 + dy^2$ on $\Sigma \times \mathbb{R}^+$ (where $z = x_2 + ix_3$ is a local holomorphic coordinate on $\Sigma$), and assume also that the solution metric splits as $H = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}$, where $h$ is a bundle metric on $L^{-1}$. We are then looking for a solution to

$$-\Delta_g \log h + g_0^{-2}(h^2\alpha\bar{\alpha} - h^{-2}\beta\bar{\beta}) = 0. \tag{3.41}$$

We simplify this slightly further. Choose a background metric $h_0$ on $L^{-1}$ and recall that its curvature equals $-\Delta_{g_0} \log h_0$. Then writing $h = h_0 e^u$ and calculating the norms of $\alpha$ and $\beta$ in terms of $g_0$ and $h_0$, (3.41) becomes

$$K_{h_0} - (\Delta_{g_0} + \partial_y^2)u + |\alpha|^2 e^{2u} - |\beta|^2 e^{-2u} = 0. \tag{3.42}$$

In the remainder of this paper, we denote by $N(u)$ the operator on the left in (3.42).

An explicit solution to this equation was noted by Mikhaylov in a special case [47]:

**Example 3.4.2.** *Consider the Higgs pair* $(E \cong K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$. *Let* $g_0$ *be the hyperbolic metric on* $\Sigma$ *with curvature* $-2$ *and* $h_0$ *the naturally induced metric on* $K^{-1/2}$, *for which* $K_{h_0} = -1$. *Then restricted to* $\Sigma$*-independent functions,* (3.42) *equals*

$$-1 - \partial_y^2 u + e^{2u} = 0. \tag{3.43}$$

*We seek a solution for which* $u \sim -\log y$ *as* $y \to 0$ *and* $v \to 0$ *as* $y \to \infty$. *The first integral of* (3.43) *is* $u' = -\sqrt{e^{2u} - 2u - 1}$, *and hence the unique solution is*

$$\int_u^\infty \frac{ds}{\sqrt{e^{2s} - 2s - 1}} = y. \tag{3.44}$$

*Note that* $u$ *is monotone decreasing and strictly positive for all* $y > 0$.

*We now describe the precise asymptotics of this solution. If* $u \to \infty$, *then* $s$ *is large; write the denominator as* $e^s\sqrt{1 - (2s+1)e^{-2s}}$, *whence*

$$y = \int_u^\infty e^{-s}(1 + \tfrac{1}{2}(2s+1)e^{-2s} + \ldots)\, ds \sim e^{-u} + \ldots,$$

*so* $u \sim -\log y$. *Similarly, if* $u < \epsilon$ *for some small* $\epsilon$, *then* $e^{2s} - 2s - 1 \sim 2s^2 + \ldots$ *when* $u < s < \epsilon$, *so*

$$u = \int_\epsilon^\infty \frac{ds}{\sqrt{e^{2s} - 2s - 1}} + \int_u^\epsilon (\tfrac{1}{\sqrt{2}s} + \ldots)\, ds = A - \tfrac{1}{\sqrt{2}} \log u + \ldots,$$

*so* $u = Ce^{-\sqrt{2}y} + \ldots$. *Obviously, with only a little more effort, one may develop full asymptotics in both regimes.*

**Limiting Solution at Infinity**

We first consider the simpler problem of finding a solution of the reduction of (3.42) reduced to $\Sigma$, i.e., of

$$K - \Delta u_\infty + |\alpha|^2 e^{2u_\infty} - |\beta|^2 e^{-2u_\infty} = 0, \tag{3.45}$$

where $K = K_{h_0}$ and $\Delta = \Delta_{g_0}$. Without loss of generality, we assume $\deg L \geq 0$ and note that since $\deg L^{-1} \leq 0$, $\int_\Sigma K \leq 0$ (and is strictly negative if the degree of $L$ is positive). A solution to (3.45) is the obvious candidate for the limit as $y \to \infty$ of solutions on $\Sigma \times \mathbb{R}^+$.

**Proposition 3.4.3.** *If $\alpha \not\equiv 0$, which is equivalent to the stability of the pair $(E, \varphi)$, there exists a solution $u_\infty \in C^\infty(\Sigma)$ to* (3.45).

*Proof.* Since this is an equation on $\Sigma$ rather than $\Sigma \times \mathbb{R}^+$, this follows immediately from the existence of solutions to the Hitchin equations [31]. However, we give another proof, at least when $\deg L > 0$, using the barrier method. A proof in the same style when $\deg L = 0$ requires more work so we omit it.

Solve $\Delta w^- = K - \overline{K}$, where $\overline{K} < 0$ is the average of $K$, and set $u^- = w^- - A$ for some constant $A$. Then $K - \Delta u^- + |\alpha|^2 e^{2u^-} - |\beta|^2 e^{-2u^-} \leq \overline{K} + |\alpha|^2 e^{w^- - A}$, which is negative when $A$ is sufficiently large. Thus $u^-$ is a subsolution.

To obtain a supersolution, first modify the background metric $h_0$ by multiplying it by a suitable positive factor so that its curvature $K$ is positive near the zeroes of $\alpha$. Next solve $\Delta w^+ = |\alpha|^2 - B$ where $B$ is the average of $|\alpha|^2$ and set $u^+ = w^+ + A$. Then

$$\begin{aligned} K - \Delta u^+ + |\alpha|^2 e^{2u^+} - |\beta|^2 e^{-2u^+} &= K + B + |\alpha|^2 (e^{2(w^+ + A)} - 1) - |\beta|^2 e^{-2(w^+ + A)} \\ &\geq K + B + 2|\alpha|^2 (w^+ + A) - |\beta|^2 e^{-2(w^+ + A)}. \end{aligned}$$

Away from the zeroes of $\alpha$ this is certainly positive if we choose $A$ sufficiently large. Near these zeroes we obtain positivity using that $K + B > 0$ there and since the final term can be made arbitrarily small. Thus $u^+$ is a supersolution.

Noting that $u^- < u^+$ and applying Proposition 3.4.1, we obtain a solution of (3.45). $\square$

Observe that since it is only the boundary condition, but not the equation, which depends on $y$, this limiting solution is actually a solution of (3.42) on any semi-infinite region $\Sigma \times [y_0, \infty)$, $y_0 > 0$.

**Approximate solutions and regularity near** $y = 0$

As a complement to the result in the previous subsection, we now construct an approximate solution $u_0$ to (3.42) near $\{y = 0\}$. Unlike there, however, we do not find an exact solution, but rather show how to build an initial approximate solution and then incrementally correct it so that it solves (3.42) to all orders as $y \to 0$. In the next subsection we use $u_0$ and $u_\infty$ together to construct global barriers.

We first begin with the simpler situation where there is only a Nahm pole singularity without knots.

**Proposition 3.4.4.** *Let* $L = K^{1/2}$ *and* $\alpha \equiv 1$. *Then there exists a function* $u_0$ *which is polyhomogeneous as* $y \to 0$ *and is such that* $N(u_0) = f$ *decays faster than* $y^\ell$ *for any* $\ell \geq 0$.

*Proof.* We seek $u_0$ with a polyhomogeneous expansion of the form

$$-\log y + \sum_{j,\ell} a_{j\ell}(z) y^j (\log y)^\ell := -\log y + v,$$

where all the coefficients are smooth in $z$, and where the number of $\log y$ factors is finite for each $j$. Rewriting $N(-\log y + v)$ as

$$\left(-\partial_y^2 + \frac{2}{y^2}\right) v + \frac{1}{y^2}(e^{2v} - 2v - 1) - |\beta|^2 y^2 e^{-2v} - \Delta_{g_0} v + K_{h_0}, \tag{3.46}$$

and inserting the putative expansion for $v$ shows that $a_{0\ell} = a_{1\ell} = 0$ for all $\ell$ and $a_{21} = \frac{1}{3}(K_{h_0} - |\beta|^2$, $a_{2\ell} = 0$ for $\ell > 1$, i.e., $v \sim a_{21} y^2 \log y + a_{20} y^2 + O(y^3 (\log y)^\ell)$ for some $\ell$. Inductively we can solve for each of the coefficients $a_{j\ell}$ with $j > 2$ using that

$$\begin{aligned}(-\partial_y^2 &+ 2/y^2) y^j (\log y)^\ell \\ &= y^{j-2} (\log y)^{\ell-2} \Big( (-j(j-1) + 2)(\log y)^2 - \ell(2j-1) \log y - \ell(\ell - 1) \Big).\end{aligned}$$

Note that the coefficient $a_{20}$ is not formally determined in this process and different choices will lead to different formal expansions, and also that there are increasingly high powers of $\log y$ higher up in the expansion.

Now use Borel summation to choose a polyhomogeneous function $u_0$ with this expansion. This has a Nahm pole at $y = 0$ and satisfies $N(u_0) = f = O(y^\ell)$ for all $\ell$, as desired. □

We next turn to the construction of a similar approximate solution to all orders in the presence of knot singularities. To carry this out, we first review a geometric construction from [46] which is at the heart of the regularity theorem quoted in §3.4 for the full extended Bogomolny equations and the analogous result for (3.42) which we describe below.

If $p \in \Sigma$, we define the blowup of $\Sigma \times \mathbb{R}^+$ at $(p, 0)$ to consist of the disjoint union $(\Sigma \times \mathbb{R}^+) \setminus \{(p, 0)\}$ and the hemisphere $S^2_+$, which we regard as the set of inward-pointing unit normal vectors at $(p, 0)$, and denote by $[\Sigma \times \mathbb{R}^+; \{(p, 0)\}]$, or more simply, just $(\Sigma \times \mathbb{R}^+)_p$ There is a blowdown map which is the identity away from $(p, 0)$ and maps the entire hemisphere to this point. This set is endowed with the unique minimal topology and differential structure so that the lifts of smooth functions on $\Sigma \times \mathbb{R}^+$ and polar coordinates around $(p, 0)$ are smooth. We use spherical coordinates $(R, \psi, \theta)$ around this point, so $R = 0$ is the hemisphere and $\psi = 0$ defines the original boundary $y = 0$ away from $R = 0$. This is a smooth manifold with corners of codimension two.

Now fix a nonzero element $\alpha \in H^0(L^{-2}K)$ and denote its divisor by $\sum_{j=1}^N n_j p_j$. For each $j$, choose a small ball $\hat{B}_j$ and a local holomorphic coordinate $z$ so that $p_j = \{z = 0\}$, and write $|\alpha|^2 = \sigma_j^2 r^{2n_j}$ there, with $r = |z|$ and $\sigma_j > 0$. Extend $r$ from the union of these balls to a smooth positive function on $\Sigma \setminus \{p_1, \dots, p_N\}$. By the existence of isothermal coordinates, we write $g_0 = e^{2\phi}\bar{g}_0$ where $\bar{g}_0$ is flat on each $\hat{B}_j$, and set $g = g_0 + dy^2$, $\bar{g} = \bar{g}_0 + dy^2$. Then $\Delta_{g_0} = e^{-2\phi}\Delta_{\bar{g}_0}$ in these balls, and by dilating $\bar{g}_0$, we can assume that $e^{-2\phi} = 1$ at each $p_j$. We denote by $(\Sigma \times \mathbb{R}^+)_{p_1,\dots,p_N}$ the blowup of $\Sigma \times \mathbb{R}^+$ at the collection of points $\{p_1, \dots, p_N\}$.

**Proposition 3.4.5.** *With all notation as above, there exists a function $u_0$ which is polyhomogeneous on $(\Sigma \times \mathbb{R}^+)_{p_1,\dots,p_N}$ and which satisfies $N(u_0) = f$ with $f$ smooth and vanishing to all orders as $y \to 0$ (i.e., at all boundary components of the blowup.*

*Proof.* In a manner analogous to the previous proposition, we construct a polyhomogeneous series expansion for $u_0$ term-by-term, but now at each of the boundary faces of $(\Sigma \times \mathbb{R}^+)_{p_1,\dots,p_N}$.

The initial term of this expansion involves the model solutions $U_n$. Choose nonintersecting balls $\hat{B}_j$ with $B_j \subset\subset \hat{B}_j$ and an open set $\hat{B}_0 \subset \Sigma \setminus \cup_{j=1}^N \overline{B}_j$ so that $\cup_{j=0}^N \hat{B}_j = \Sigma$. Let $\{\chi_j\}$ be a partition of unity subordinate to the cover $\{\hat{B}_j\}$ with $\chi_j = 1$ on $B_j$, $j \geq 1$. We lift each of these functions from $\Sigma$ to the blowup of $\Sigma \times \mathbb{R}^+$. Finally, set

$G_j := U_{n_j} - \log \sigma_j$, where $G_0 := U_0 - \log|\alpha| = -\log y - \log|\alpha|$. Now define

$$\hat{u}_0 := \sum_{j=0}^{N} \chi_j G_j. \tag{3.47}$$

We compute that $N(\hat{u}_0) = f_0$, where $f_0$ is polyhomogeneous and is bounded at the original boundary $\psi = 0$ and has leading term of order $R^{-1}$ at each of the 'front' faces where $R = 0$.

Our goal is to iteratively solve away all of the terms in the polyhomogeneous expansion of $f_0$. This must be done separately at the two types of boundary faces. It turns out to be necessary to first solve away the series at $R = 0$ and after that the series at $\psi = 0$. The reason for doing things in this order is that, as we now explain, the iterative problem that must be solved at the $R = 0$ front faces is global on each hemisphere, and the solution 'spread' to the boundary of this hemisphere, i.e., where $\psi = 0$. By contrast, the iterative problem at the original boundary is completely local in the $y$ directions and may be done uniformly up to the corner where $R = \psi = 0$, so its solutions do not spread back to the front faces.

For simplicity, we assume that there is only one front face, and we begin by considering the model case $(\mathbb{C} \times \mathbb{R}^+)_0$, on which the linearization of (3.42) at $U_n$ can be written

$$L_n = -\partial_R^2 - \frac{2}{R}\partial_R - \frac{1}{R^2}\Delta_{S^2_+} + 2r^{2n}e^{2U_n} = -\partial_R^2 - \frac{2}{R}\partial_R + \frac{1}{R^2}\left(-\Delta_{S^2_+} + T(\psi)\right), \tag{3.48}$$

where the potential equals

$$T(\psi) = \frac{(n+1)^2}{\sin^2\psi S_n(\psi)^2}.$$

In general terms, $L_n$ is a relatively simple example of an 'incomplete iterated edge operator', as explained in more detail in [46], based on the earlier development of this class in [2, 3]. We need relatively little of this theory here and quote from [46] as needed. In the present situation, we can regard $L_n$ as a conic operator over the cross-section $S^2_+$. (It is the fact that this link of the cone itself has a boundary which makes $L_n$ an 'iterated' edge operator.)

The crucial fact is that the operator

$$J = -\Delta_{S^2_+} + T(\psi),$$

induced on this conic link has discrete spectrum. The proof of this is based on the observation that $T(\psi) \sim 1/\psi^2$ as $\psi \to 0$. It can then be shown using standard arguments, cf. [2, 3], that the domain of $J$ as an unbounded operator on $L^2(S^2_+)$ is compactly contained in $L^2$. This implies the discreteness of the spectrum. Another proof which provides more accurate information uses that $J$ is itself an incomplete uniformly degenerate operator, as analyzed thoroughly in [41]. The main theorem in that paper produces a particular degenerate pseudodifferential operator $G$ which invers $J$ on $L^2$. It is also shown there that $G : L^2(S^2_+) \to \psi^2 H^2_0(S^2_+)$ (where $H^2_0$ is the scale-invariant Sobolev space associated to the vector fields $\psi\partial_\psi, \psi\partial_\theta$). The compactness of $\psi^2 H^2_0(S^2_+) \hookrightarrow L^2(S^2_+)$ follows from the $L^2$ Arzela-Ascoli theorem. There is an accompanying regularity theorem: if $(J-\lambda)w = f$ where (for simplicity) $f$ is smooth and vanishes to all orders at $\psi = 0$ and $\lambda \in \mathbb{R}$ (or more generally can be any bounded polyhomogeneous function), then $w$ is polyhomogeneous with an expansion of the form

$$w \sim \sum w_{j\ell}(\theta)\psi^{\gamma_j}(\log\psi)^\ell, \quad w_{j\ell} \in C^\infty(S^1).$$

As usual, there are only finitely many log terms for each exponent $\gamma_j$. These exponents are the indicial roots of the operator $J$, and a short calculation shows that these satisfy $2 = \gamma_0 < \gamma_1 < \dots$. Note that the lowest indiical root equals 2, so solutions all vanish to at least order 2 at $\psi = 0$, which is in accord with our knowledge about the behavior of solutions to the linearization of (3.42) at the model Nahm pole solution $-\log y$.

Denote the eigenfunctions and eigenvalues of $J$ by $\mu_i(\psi, \theta)$ and $\lambda_i$. Since $T(\psi) > 0$, each $\lambda_i > 0$. The restriction of $L_n$ to the $i^{\text{th}}$ eigenspace is now an ODE $L_{n,i} = -\partial_R^2 - 2R^{-2}\partial_R + R^{-2}\lambda_i$. Seeking solutions of the form $R^\delta\mu_i(\psi, \theta)$ leads to the corresponding indicial roots

$$\delta_i^\pm = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\lambda_i},$$

which are the only possible formal rates of growth or decay of solutions to $L_n u = 0$ as $R \to 0$. To satisfy the generalized Nahm pole condition, we only consider exponents greater than $-1$, i.e., the sequence $0 < \delta_1^+ < \delta_2^+ < \dots$. We now conclude the following

**Lemma 3.4.6.** *Suppose that $f \sim \sum f_{j\ell}(\psi, \theta)R^{\gamma_j}(\log R)^\ell$ is polyhomogeneous at the face $R = 0$ on $(\mathbb{C} \times \mathbb{R}^+)_0$, where all $f_{j\ell}$ are polyhomogeneous with nonnegative coefficients at $\psi = 0$ on $S^2_+$. Then there exists a polyhomogeneous function $u$ such that $L_n u = f + h$, where $h$ is polyhomogeneous at $\psi = 0$ and vanishes to all orders as $R \to 0$. At $R \to 0$, $u \sim \sum u_{j\ell}R^{\gamma'_j}(\log R)^\ell$; the exponents $\gamma'_j$ are all of the form*

*$\gamma_i + 2$,$f$ where $\gamma_j$ appears in the list of exponents in the expansion for $f$, or else $\delta_i^+ + \ell$, $\ell \in \mathbb{N}$. Each coefficient function $u_{j\ell}$, as well as the entire solution $u$ itself and the error term $h$, vanish like $\psi^2$ at the boundary $\psi = 0$.*

Using the same result, we may clearly generate a formal solution to our semilinear elliptic equation in exactly the same way. Therefore, using this Lemma, we may now choose a function $\hat{u}_1$ which is polyhomogeneous on $(\Sigma \times \mathbb{R}^+)_{p_1 \dots p_N}$ and such that $N(\hat{u}_0 + \hat{u}_1) = f_1$, where $f_1$ vanishes to all orders at $R = 0$ and is polyhomogeneous and vanishes like $\psi^2$ at $\psi = 0$. The lowest exponent in the expansion for $\hat{u}_1$ equals $\min\{1, \delta_0^+ > 0\}$.

The final step in our construction of an approximate solution is to carry out an analogous procedure at the original boundary $y = 0$ away from the front faces. This can be done almost exactly above. In this case, (3.46) can be thought of as an ODE in $y$ with 'coefficients' which are operators acting in the $z$ variables, so we are effectively just solving a family of ODE's parametrized by z. This may be done uniformly up to the corner $R = \psi = 0$. We omit details since they are the same as before. We obtain after this step a final correction term $\hat{u}_2$ which is polyhomogeneous and vanishes to all orders at $R = 0$, and which satisfies

$$N(\hat{u}_0 + \hat{u}_1 + \hat{u}_2) = f,$$

where $f$ vanishes to all orders at all boundaries of $(\Sigma \times \mathbb{R}^+)_{p_1 \dots p_N}$.

The calculations above are useful not just for calculating formal solutions to the problem, but also for understanding the regularity of actual solutions to the nonlinear equation $N(u) = 0$ which satisfy the generalized Nahm pole boundary conditions with knots. The new ingredient that must be added is a parametrix $G$ for the linearization of $N$ at the approximate solution $u_0$. This operator $G$ is a degenerate pseudodifferential operator for which there is very precise information known concerning the pointwise behavior of the Schwartz kernel. This is explained carefully in [43] for the simple Nahm pole case and in [46] for the corresponding problem with knot singularities. We shall appeal to that discussion and the arguments there and simply state the

**Proposition 3.4.7.** *Let $u$ be a solution to* (3.42) *which is of the form $u = u_0 + v$ where $v$ is bounded as $y \to 0$ (in particular as $\psi \to 0$ and $R \to 0$). Then $u$ is polyhomogeneous at the two boundaries $\psi = 0$ and $R = 0$ of the blowup $(\Sigma \times \mathbb{R}^+)_{p_1,\dots,p_N}$, and its expansion is fully captured by that of $u_0$.*

**Existence of solutions**

We now come to the construction of solutions to (3.42) on the entire space $\Sigma \times \mathbb{R}^+$ which satisfy the asymptotic $SL(2,\mathbb{R})$ conditions as $y \to \infty$ and which also satisfy the generalized Nahm pole boundary conditions with knot singularities at $y = 0$. We employ the barrier method. The main ingredients in the construction of the barrier functions are the approximate solutions $u_0$ and $u_\infty$ obtained above.

We first consider this problem in the simpler case.

**Proposition 3.4.8.** *If $E = K^{1/2} \oplus K^{-1/2}$ and $\varphi = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$, i.e. there are no knot singularities, then there exists a solution $u$ to* (3.42) *which is smooth for $y > 0$, asymptotic to $u_\infty$ as $y \to \infty$, (and which satisfies the Nahm pole boundary condition at $y = 0$).*

*Proof.* Choose a smooth nonnegative cutoff function $\tau(y)$ which equals 1 for $y \leq 2$ and which vanishes for $\tau \geq 3$, and define $\hat{u} = \tau(y)u_0 + (1 - \tau(y))u_\infty$. We consider the operator

$$\widehat{N}(v) = N(\hat{u} + v) = -(\partial_y^2 + \Delta_{g_0})v + e^{2\hat{u}}(e^{2v} - 1) + |\beta|^2 e^{-2\hat{u}}(1 - e^{-2v}) + f,$$

where $f = N(\hat{u})$ is smooth on $\Sigma \times \overline{\mathbb{R}}^+$, vanishes to infinite order at $y = 0$ and vanishes identically for $y \geq 3$.

We now find barrier functions for this equation. Indeed, we compute that if $0 < \epsilon < 1$, then

$$\widehat{N}(Ay^\epsilon) = A\epsilon(1 - \epsilon)y^{\epsilon - 2} + e^{2\hat{u}}(e^{2Ay^\epsilon} - 1) + |\beta|^2 e^{-2\hat{u}}(1 - e^{-2Ay^\epsilon}) + f.$$

The second and third terms on the right are nonnegative because $Ay^\epsilon > 0$, and we can certainly choose $A$ sufficiently large so that the entire right hand side is positive for all $y > 0$.

We can improve this supersolution for $y$ large. Indeed,

$$\widehat{N}(A'e^{-\epsilon y}) \geq -A'\epsilon^2 e^{-\epsilon y} + e^{2\hat{u}}(2A'e^{-\epsilon y}) + |\beta|^2 e^{-2\hat{u}}(1 - e^{-2A'e^{-\epsilon y}}) + f,$$

and if $\epsilon$ is sufficiently small and $A'$ is sufficiently large, then the entire right hand side is positive, at least for $y \geq 1$, say.

We now define $v^+ = \min\{Ay^\epsilon, A'e^{-\epsilon y}\}$. The calculations above show that $v^+$ is a supersolution to the equation. Essentially the same equations show that $v^- = \max\{-Ay^\epsilon, -A'e^{-\epsilon y}\}$ is a subsolution.

We now invoke Propostion 3.4.1 to conclude that there exists a solution $v$ to $\widehat{N}(v) = 0$, or equivalently, a solution $u = \hat{u} + v$ to $N(u) = 0$, which satisfies $|u + \log y| \leq Ay^{\epsilon}$ as $y \to 0$ and $|u - u_\infty| \leq A' e^{-\epsilon y}$ as $y \to \infty$. The regularity theorem for (3.42) shows that this solution is polyhomogeneous at $y = 0$, and hence must have an expansion of the same type as $\hat{u}$, and a similar but more standard argument can be used to produce a better exponential rate of decay as $y \to \infty$. □

**Proposition 3.4.9.** *Let $E = L \oplus L^{-1}$ and $\varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ be a stable Higgs pair, and let $(p_j, n_j)$ be the 'knot data' determined by $\alpha$. Then there exists a solution u t o (3.42) of the form $u = \hat{u} + v$ where $v \to 0$ as $y \to 0$ and as $y \to \infty$.*

*Proof.* We proceed exactly as before, writing

$$\widehat{N}(v) = N(\hat{u} + v) = -(\partial_y^2 + \Delta_{g_0})v + |\alpha|^2 e^{2\hat{u}}(e^{2v} - 1) + |\beta|^2 e^{-2\hat{u}}(1 - e^{-2v}) + f,$$

with $f = N(\hat{u})$ vanishing to all orders as $y \to 0$ and identically for $y \geq 3$. The same barrier functions obviously work in the region $y \geq 3$, and also in the region near $y = 0$ away from the knot singularities.

To construct barriers near a knot $(p, 0)$ of weight $n$, recall the explicit structure of $\hat{u}$ near this point and expand the nonlinear term $e^{2v} - 1$ one step further to write in some small neighborhood of the front face created by blowing up this point

$$\widehat{N}(v) = (-\partial_R^2 - \frac{2}{R}\partial_R + \frac{1}{R^2}(-\Delta_{S^2_+} + \tilde{T}))v + ke^{2U_n}(e^{2v} - 1 - 2v) + |\beta|^2 e^{-2U_n}(1 - e^{-2v}) + f.$$

Here $k$ is a strictly positive function which contains all the higher order terms in the expansion for $\hat{u}$, and $\tilde{T}$ is a slight perturbation of the term $T$ appearing in the linearization $L_n$. Let $\mu_0$ denote the ground state eigenfunction for this operator on $S^2_+$. The corresponding eigenvalue $\lambda_0'$ is a small perturbation of $\lambda_0$, which we showed earlier was strictly greater than 0. Now compute

$$\widehat{N}(AR^{\epsilon}\mu_0(\psi, \theta)) = (\lambda_0' - \epsilon(\epsilon + 1))AR^{\epsilon - 2}\mu_0 + f + E,$$

where $E$ is the sum of the two terms involving $e^{\pm 2U_n}$. As before, since $v \geq 0$ implies $e^{2v} - 1 - 2v \geq 0$ and $1 - e^{-2v}$, we have that $E \geq 0$, and if $\epsilon$ is sufficiently small, then this first term on the right has positive coefficient, and dominates $f$. We have thus produced a local supersolution near $(p, 0)$. The full supersolution is

$$v^+ = \min\{AR^{\epsilon}\mu_0, A' y^{\epsilon/2}, A'' e^{-\epsilon y}\}.$$

We have chosen to use the exponents $\epsilon$ and $\epsilon/2$ in the first two terms here in order to ensure that the first term is smaller than the second in the interior of the front face $R = 0$; indeed, $AR^\epsilon\mu_0 < A'(R\sin\psi)^{\epsilon/2}$ when $R < (A'/A)^{2/\epsilon}(\sin\psi)^{\epsilon/2}$. This means that $v^+$ agrees with $A'y^{\epsilon/2}$ near the original boundary and with $AR^\epsilon\mu_0$ near the other boundaries, and as before, with the exponentially decreasing term when $y$ is large.

A very similar calculation with the same functions produces a subsolution $v^-$. Altogether, we deduce, by Proposition 3.4.1 again, the existence of a solution $u = \hat{u} + v$ to $N(u) = 0$ with the correct asymptotics. □

## 3.5 Uniqueness

In this section, we prove a uniqueness theorem for solutions of the extended Bogomolny equations satisfying the (generalized) Nahm pole boundary condition. This will be phrased in terms of the associated Hermitian metrics. The key to this is the subharmonicity of the Donaldson metric, which we recall in the first subsection.

### The Distance on Hermitian metrics

Suppose that $H$ is a Hermitian metric on a bundle $E$, with compatible data $(A, \phi, \phi_1)$, which satisfies the extended Bogomolny equations. As we have discussed, it is possible to choose a holomorphic gauge which is parallel in the $y$ direction such that $\mathcal{D}_1 = \partial_{\bar{z}}$, $\mathcal{D}_2 = \operatorname{ad}\varphi$, $\mathcal{D}_3 = \partial_y$. In this gauge, the Hermitian metric $H$ determines the gauge fields by

$$\partial^A = \partial + H^{-1}\partial H,\ \varphi^\star = H^{-1}\varphi^\dagger H,\ \partial_y^{\mathcal{A}} = \partial^{A_y} + i\phi_1 = \partial_y + H^{-1}\partial_y H, \qquad (3.49)$$

where of course $\partial$ is the complex differential on $\Sigma$ and in this trivialization $\varphi^\dagger = \varphi^\dagger = \bar{\varphi}^\top$. We can then write the extended Bogomolny equations as

$$\partial_{\bar{z}}(H^{-1}\partial H) + [\varphi^{\star H}, \varphi] + h_0^2\partial_y(H^{-1}\partial_y H) = 0,$$

where $h_0^2|dz|^2$ is the Riemannian metric on $\Sigma$.

Following [21], we define the distance between Hermitian metrics

$$\sigma(H_1, H_2) = \operatorname{Tr}(H_1^{-1}H_2) + \operatorname{Tr}(H_2^{-1}H_1) - 4, \qquad (3.50)$$

and recall from that paper two important properties:

1) $\sigma(H_1, H_2) \geq 0$, with equality if and only if $H_1 = H_2$;

2) A sequence of Hermitian metric $H_i$ converges to $H$ in the usual $C^0$ norm if and only if $\sup_\Sigma \sigma(H_i, H) \to 0$.

**Lemma 3.5.1.** *Suppose that $H_1$ and $H_2$ are both harmonic metrics. Then the complex gauge transform $h := H_1^{-1}H_2$ satisfies*

$$\partial_{\bar{z}}(h^{-1}\partial^{A_1}h) + \partial_y(h^{-1}\partial_y^{A_1}h) + [h^{-1}[\varphi^\star, h], \varphi] = 0. \tag{3.51}$$

*Proof.* In holomorphic gauge,

$$A_2 = H_2^{-1}\partial H_2 = h^{-1}H_1^{-1}\partial H_1 h + h^{-1}\partial h = H_1^{-1}\partial H_1 + h^{-1}\partial^{A_1}h,$$

hence $\partial_{\bar{z}}(H_2^{-1}\partial H_2) - \partial_{\bar{z}}(H_1^{-1}\partial H_1) = \partial_{\bar{z}}(h^{-1}\partial^{A_1}h)$.

Similarly,

$$H_2^{-1}\partial_y H_2 = H_1^{-1}\partial_y H_1 + h^{-1}(\partial_y h + [H_1^{-1}\partial_y H_1, h]) = H_1^{-1}\partial_y H_1 + h^{-1}\partial_y^{\mathcal{A}_y}h.$$

Hence $\partial_y(H_2^{-1}\partial_y H_2) - \partial_y(H_1^{-1}\partial_y H_1) = \partial_y(h^{-1}\partial_y^{\mathcal{A}_y}h)$.

Finally,

$$[\varphi^{\star H_2}, \varphi] - [\varphi^{\star H_1}, \varphi] = [h^{-1}[\varphi^{\star H_1}, h], \varphi].$$

Altogether, we deduce the stated equation from the harmonic metric equations

$$\partial_{\bar{z}}(H_j^{-1}\partial H_j) + [\varphi^{\star H_j}, \varphi] + h_0^2\partial_y(H_j^{-1}\partial_y H_j) = 0, \ j = 1, 2.$$

□

We next show that $\sigma$ is subharmonic.

**Proposition 3.5.2.** *Define $h = H_1^{-1}H_2$ as above, where $H_1$ and $H_2$ satisfy the Extended Bogonomy equation. Then $(\Delta + \partial_y^2)\sigma \geq 0$ on $\Sigma \times (0, +\infty)$.*

*Proof.* We first compute

$$\begin{aligned}
\partial_{\bar{z}}\partial_z \mathrm{Tr}(h) &= \mathrm{Tr}(\partial_{\bar{z}}\partial^{A_1}h) \\
&= \mathrm{Tr}(\partial_{\bar{z}}(hh^{-1}\partial^{A_1}h)) \\
&= \mathrm{Tr}(\partial_{\bar{z}}(h)h^{-1}\partial^{A_1}h) + \mathrm{Tr}(h\partial_{\bar{z}}(h^{-1}\partial^{A_1}h)) \\
&\geq \mathrm{Tr}(h\partial_{\bar{z}}(h^{-1}\partial^{A_1}h)),
\end{aligned} \tag{3.52}$$

since $\mathrm{Tr}(BhB^\star) \geq 0$ for any matrix $B$.

Continuing on,

$$\begin{aligned}
\partial_y^2 \mathrm{Tr}(h) &= \mathrm{Tr}(\partial_y\partial_y^{\mathcal{A}_1}h) \\
&= \mathrm{Tr}((\partial_y h)h^{-1}\partial_y^{\mathcal{A}_1}h) + \mathrm{Tr}(h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h))) \\
&\geq \mathrm{Tr}(h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h))),
\end{aligned} \tag{3.53}$$

where we use $\partial_y = (\partial_y^{\mathcal{A}_1})^\star$ and that $\star$ is the conjugate transpose with respect to $H_1$.

Finally,

$$\begin{aligned} 0 &= \mathrm{Tr}([[\varphi^\star, h], \varphi]) \\ &= \mathrm{Tr}([h, \varphi]h^{-1}[\varphi^\star, h]) + \mathrm{Tr}(h[h^{-1}[\varphi^\star, h], \varphi]). \end{aligned} \tag{3.54}$$

Since $\mathrm{Tr}([h, \varphi]h^{-1}[\varphi^\star, h]) \geq 0$, we obtain $\mathrm{Tr}(h[h^{-1}[\varphi^\star, h], \varphi]) \leq 0$.

Putting these together gives

$$\begin{aligned} (\partial_{\bar{z}}\partial_z + h_0^2\partial_y^2)\mathrm{Tr}(h) &\geq \mathrm{Tr}(h\partial_{\bar{z}}(h^{-1}\partial^{A_1}h) + h_0^2 h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h))) \\ &\geq \mathrm{Tr}(h\partial_{\bar{z}}(h^{-1}\partial^{A_1}h) + h_0^2 h(\partial_y(h^{-1}\partial_y^{\mathcal{A}_1}h + h[h^{-1}[\varphi^\star, h], \varphi]))) \\ &\geq 0, \end{aligned} \tag{3.55}$$

and dividing by $h_0^2$ proves the claim. □

**Asymptotics of the Hermitian metric**

In order to apply the subharmonicity of $\sigma(H_1, H_2)$ from the last subsection, we need to understand the asymptotics of this function near $y = 0$. This, in turn, relies on a detailed examination of the asymptotics of the Hermitian metric.

**Proposition 3.5.3.** *Fix a Higgs pair* $(E \cong L^{-1} \oplus L, \varphi = \begin{pmatrix} t & \alpha \\ \beta & -t \end{pmatrix})$*. For any* $p \in \Sigma$*, choose an open set* $U_p$ *around* $(p, 0)$ *in* $\Sigma \times \mathbb{R}^+$*. Let* $H$ *be a solution to the Hermitian extended Bogomolny equations* (3.11)*; as explained earlier,* $H$ *is polyhomogeneous on* $(\Sigma \times \mathbb{R}^+)_{p_1 \ldots p_N}$ *(where the* $p_j$ *are the zeroes of* $\alpha$*).*

*(1) Suppose in some local trivilization in* $U_p$ *that* $\varphi|_{U_x} = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$*, where* $q$ *is holomorphic. Suppose also that*

$$H = \begin{pmatrix} \mathcal{O}(y^{-1}) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}. \tag{3.56}$$

*Here* $\mathcal{O}(y^s)$ *indicates a polyhomogeneous expansion with lowest order term a smooth multiple of* $y^s$*. Suppose also that* $H$ *satisfies the Nahm pole boundary condition in unitary gauge. Then*

$$H \sim \begin{pmatrix} y^{-1}g_0 + \mathcal{O}(1) & o(1) \\ o(1) & yg_0^{-1} + \mathcal{O}(1) \end{pmatrix}, \tag{3.57}$$

*where $o(1)$ indicates a polyhomogeneous expansion with positive leading exponent.*

*(2) Suppose that in a local trivilization, $\varphi|_{U_p} = \begin{pmatrix} t & z^n \\ q & -t \end{pmatrix}$ where $z = 0$ is the point $p$ and $q$ holomorphic. If, in spherical coordinates*

$$H = \begin{pmatrix} \mathcal{O}(y^{-1}R^{-n}) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \tag{3.58}$$

*then*

$$H = \begin{pmatrix} \mathcal{O}(y^{-1}R^{-n}) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(yR^n) \end{pmatrix} \tag{3.59}$$

*Proof.* We first address (1). Write $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ and consider a gauge transformation $g$ for which $H = g^2$. Then $g^\dagger = g$ and $g = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$ where $a$ and $d$ are real functions and $ad - b\bar{b} = 1$. We then compute

$$\phi_z = g\varphi g^{-1} = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -\bar{b} & a \end{pmatrix} = \begin{pmatrix} bdq - a\bar{b} & -b^2q + a^2 \\ d^2q - \bar{b}^2 & -bdq + a\bar{b} \end{pmatrix}. \tag{3.60}$$

By proposition 3.3.8, the Nahm pole boundary condition requires that

$$bdq - a\bar{b} \sim o(1),\ d^2q - \bar{b}^2 \sim o(1),\ -b^2q + a^2 \sim \frac{g_0}{y} + \mathcal{O}(1). \tag{3.61}$$

By definition, $H = g^2 = \begin{pmatrix} a^2 + b\bar{b} & ab + bd \\ \bar{b}a + \bar{b}d & d^2 + b\bar{b} \end{pmatrix}$. The leading terms of $d^2 + b\bar{b}$ is positive, hence $b$ and $d$ are bounded. Combining this with (3.61) and the relation $ad - b\bar{b} = 1$, we obtain

$$a \sim y^{-\frac{1}{2}} g_0^{\frac{1}{2}},\ d \sim y^{\frac{1}{2}} g_0^{-\frac{1}{2}},\ b = o(y^{\frac{1}{2}}) \tag{3.62}$$

and thus

$$H = \begin{pmatrix} a^2 + b\bar{b} & ab + bd \\ \bar{b}a + \bar{b}d & d^2 + b\bar{b} \end{pmatrix} = \begin{pmatrix} y^{-1}g_0 + o(y^{-1}) & o(1) \\ o(1) & yg_0^{-1} + o(y) \end{pmatrix}. \tag{3.63}$$

As for (2), we compute

$$\begin{aligned} \phi_z = g\varphi g^{-1} &= \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} t & z^n \\ q & -t \end{pmatrix} \begin{pmatrix} d & -b \\ -\bar{b} & a \end{pmatrix} \\ &= \begin{pmatrix} bdq - a\bar{b}z^n + atd + |b|^2 t & -b^2q + a^2z^n - 2bat \\ d^2q - z^n\bar{b}^2 + 2td\bar{b} & -bdq + a\bar{b}z^n - |b|^2 t - adt \end{pmatrix}. \end{aligned} \tag{3.64}$$

By Proposition 3.3.9, the knot singularity implies that

$$bdq - a\bar{b}z^n + atd + |b|^2 t \sim O(1),\ -b^2 q + a^2 z^n - 2ba \sim z^n e^{U_n} + \cdots,\ -bdq + a\bar{b}z^n - |b|^2 t - adt \sim O(1). \tag{3.65}$$

As before, $H = g^2 = \begin{pmatrix} a^2 + b\bar{b} & ab + bd \\ \bar{b}a + \bar{b}d & d^2 + b\bar{b} \end{pmatrix}$ where $d^2 + b\bar{b} \sim O(1)$, so by the same positivity, $d$ and $b$ are both $O(1)$. Next, $e^{U_n} = f(\psi)/yR^n$ where $f$ is regular. From $-b^2q + a^2z^n - 2ba \sim z^n e^{U_n}$ we get $a \sim y^{-\frac{1}{2}} R^{-\frac{n}{2}}$. In addition, since $ab + bd = O(1)$ and $ad - b\bar{b} = 1$, we see that $b \sim y^{\frac{1}{2}} R^{\frac{n}{2}}$, so $d \sim y^{\frac{1}{2}} R^{\frac{n}{2}}$. Altogether, $H$ has the form (3.59). □

**Proposition 3.5.4.** *Suppose $H_j = \begin{pmatrix} p_j & q_j \\ q_j^\dagger & s_j \end{pmatrix}$, $j = 1, 2$, are two solutions which both satisfy the Nahm pole boundary condition at $y = 0$ and have the same limit as $y \to \infty$. Then $H_1 = H_2$.*

*Proof.* By Propositions 3.3.11 and 3.5.3, we see that as $y \to 0$, $p_j \sim y^{-1} g_0 + \cdots$, $s_j \sim y g_0^{-1} + \cdots$, $q_j \sim o(1)$. We claim that this implies that $\sigma(H_1, H_2) \to 0$ as $y \to 0$. First,

$$H_1^{-1} H_2 = \begin{pmatrix} s_1 p_2 - q_1 q_2^\dagger & \star \\ \star & -q_1^\dagger q_2 + p_1 s_2 \end{pmatrix},$$

so

$$\mathrm{Tr}(H_1^{-1} H_2) = s_1 p_2 - q_1 q_2^\dagger - q_1^\dagger q_2 + p_1 s_2 = 2 + o(1). \tag{3.66}$$

The same holds for $\mathrm{Tr}(H_2^{-1} H_1)$. This proves the claim.

We have now see that $\sigma(H_1, H_2)$ is nonnegative and subharmonic, and approaches 0 as $y \to 0$ and also as $y \to \infty$, hence $\sigma(H_1, H_2) \equiv 0$, i.e., $H_1 = H_2$. □

**Proposition 3.5.5.** *Let $H_1$ and $H_2$ be two Hermitian metrics which are both solutions with a knot singularity of degree $n$ at $(p, 0)$. Then there exists a constant $C$ such that $\sigma(H_1, H_2) \le C$ in a neighborhood $U$ of $(p, 0)$.*

*Proof.* Write $H_j = \begin{pmatrix} a_j & b_j \\ b_j^\dagger & d_j \end{pmatrix}$, $j = 1, 2$. By Propositions 3.3.11 and 3.5.3,

$$a_j \sim y^{-1} R^{-n},\ d_j \sim y R^n,\ b_j = o(1),\ b_j^\dagger = o(1).$$

Thus $\mathrm{Tr}(H_1 H_2^{-1}) = a_1 d_2 - b_1 b_2^\dagger - b_1^\dagger b_2 + d_1 a_2 = O(1)$, and similarly, $\mathrm{Tr}(H_2^{-1} H_1) = O(1)$. The result follows immediately. □

We next recall the Poisson kernel of $\Delta_g = \Delta_{g_0} + \partial_y^2$. For any $p \in \Sigma$, $P_p(z, y)$ is the unique function on $\Sigma \times \mathbb{R}^+$ which satisfies $\Delta_g P_q(z, y) = 0$, $P|_{y=0} = \delta_q$, and $P(z, y) \to 1/\text{Area}(\Sigma)$ as $y \to \infty$.

**Theorem 3.5.6.** *Suppose that there exist two Hermitian metrics $H_1$, $H_2$ which are solutions and satisfy the Nahm pole boundary condition with knot singularities at $p_j$ of degree $n_j$, as determined by the component $\alpha$ in the Higgs field $\varphi = \begin{pmatrix} t & \alpha \\ \beta & -t \end{pmatrix}$. Suppose also that $H_1$ and $H_2$ have the same limit as $y \to \infty$. Then $H_1 = H_2$.*

*Proof.* By Proposition 3.5.3, $\sigma(H_1, H_2) \to 0$ as $y \to 0$ and $z \notin \{p_1, \ldots, p_N\}$. Near each $p_j$ there is a neighbourhood $U_j$ where $\sigma(H_1, H_2)|_{U_j} \leq C$.

Now define $Q(z, y)$ to equal the sum of Poisson kernels $\sum_{j=1}^N P_{p_j}(z, y)$. Then for any $\epsilon > 0$, $(\Delta_{g_0} + \partial_y^2)(\sigma(H_1, H_2) - \epsilon Q) \geq 0$, and $\sigma(H_1, H_2) - \epsilon Q \leq 0$ as $y \to 0$ and as $y \to \infty$. This means that $\sigma(H_1, H_2) \leq \epsilon Q$. Since this is true for every $\epsilon > 0$, we conclude that $\sigma(H_1, H_2) \leq 0$, i.e., $H_1 = H_2$. □

### 3.6 Solutions with Knot Singularities on $\mathbb{C} \times \mathbb{R}^+$

We now consider the extended Bogomolny equations on $\mathbb{C} \times \mathbb{R}^+$ with generalized Nahm pole boundary conditions and a finite number of knot singularities.

#### Degenerate Limit

Consider a trivial bundle $E$ over $\mathbb{C} \times \mathbb{R}^+$, as in [63] and [25], the limiting behavior of the classical Jones polynomial indicates that one expects that for solutions of the extended Bogomolny equations on $\mathbb{C} \times \mathbb{R}^+$, $\phi \to 0$ and $\phi_1 \to 0$ as $y \to \infty$. The equation $\mathcal{D}_3 \varphi = 0$ also implies that the conjugacy class of $\varphi$ is independent of $y$, and as argued in these papers, this implies if $Q$ is any invariant polynomial, then $\partial_y Q(\varphi) = 0$, hence that $\varphi$ is necessarily nilpotent.

Based on these heuristic considerations, we consider a trivial rank 2 holomorphic bundle over $\mathbb{C}$ and assume $\varphi = \begin{pmatrix} 0 & p(z) \\ 0 & 0 \end{pmatrix}$. We can assume $p(z)$ is a polynomial as up to a complex gauge transform the equivalent class of the Higgs bundle only depends on the zeros of the upper trangular part of $\varphi$. In general, the vanishing section determined by the line bundle has the form $s = \begin{pmatrix} R(z) \\ S(z) \end{pmatrix}$. Consider the section $K(z) := (s \wedge \varphi(s))(z) = p(z) S(z)^2$ of the determinant bundle, which we can naturally identify with a holomorphic function on $\mathbb{C}$. Its zero set defines a positive divisor $D$.

If the singular monopoles all have order 1, as $K(z) := (s \wedge \varphi(s))(z) = p(z)S(z)^2$, we obtain that $S(z)$ will not have zeros. Up to a complex gauge transform $g = \begin{pmatrix} 1 & -\frac{R}{S} \\ 0 & 1 \end{pmatrix}$, we can assume in the same trivialization, $\varphi = \begin{pmatrix} 0 & p(z) \\ 0 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In general, we can only assume $\varphi = \begin{pmatrix} t & p \\ q & -t \end{pmatrix}$ and the vanishing line bundle correspond to $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with the nilpotent condition that $t^2 + pq = 0$.

Although we expect to be able to solve extended Bogomolny equations with knot singularities corresponding to any divisor, the equation will generally not reduce to a scalar one, except in the special case where $\varphi = \begin{pmatrix} 0 & p(z) \\ 0 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and it gives an $SL(2, \mathbb{R})$ structure. Now the extended Bogomolny equations reduce to

$$-(\Delta + \partial_y^2)v + |p(z)|^2 e^{2v} = 0, \tag{3.67}$$

and we shall search for a solution for which $v \to -C \log y$ as $y \to \infty$.

**Remark.** *It is not enough to simply require that $v \to -\infty$ as $y \to \infty$. Indeed, if $p(z) \equiv 1$, then $z$-independent solutions solve the ODE $-u'' + e^{2u} = 0$. One solution is $-\log y$, but there is an additional family $\log(\frac{C}{\sinh(Cy)})$ for any $C > 0$. These are the only global solutions to this ODE. The solutions in this second family grow like $-Cy$ as $y \to \infty$, and $\phi_1 \to C \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}$. These solutions appear in [36] and is described by Gaiotto and Witten [25] as a real symmetry breaking phenomenon at $y \to \infty$.*

**Existence**

In this section, we will prove the

**Proposition 3.6.1.** *Let $p(z)$ be any polynomial on $\mathbb{C}$ of degree $N_0 > 1$. Then there exists a solution $u$ to (3.67) satisfying the generalized Nahm pole conditions with knot determined by the divisor $D = \sum n_j p_j$ of the polynomial $p$, and which is asymptotic to $-(N_0 + 1) \log R - \log \sin \psi + O(1)$ as $R \to \infty$, uniformly in $(\psi, \theta) \in S^2_+$.*

*Proof.* As before, first construct a function $\hat{u}$ which is an approximate solution to this equation with boundary conditions to all order in all asymptotic regimes, and then use the method of barriers to find a correction term which gives the exact solution.

We first pass to the blowup of $\mathbb{C}\times\mathbb{R}^+$ around the points $(p_j, 0)$, and in an additional step, also take the radial compactification as $R \to \infty$. This gives a compact manifold with corners which we call $\widehat{X}$ for simplicity; there are boundary faces $F_1, \dots, F_N$, each hemispheres corresponding to the blowups at the zeros of $p$, another boundary face $F_\infty$, also a hemisphere, corresponding to the radial compactification at infinity, and the original boundary $B$, which is a disk with $N$ smaller disks removed.

The first step in the construction of $\hat{u}$ is to use the approximate solutions near each of these faces. Around $F_j$, $j = 1, \dots, N$, we use $U_{n_j}$; near $F_\infty$ we use $U_{N_0}$, but now of course with $R \to \infty$ rather than near 0, and finally near $B$ we ue $-\log y$. Pasting these together gives a polyhogeneous function $\hat{u}_0$ on $\widehat{X}$ for which $N(\hat{u}_0) = f_0$ blows up like $1/R_j$ near each $F_j$, decays like $R^{-3}$ near $F_\infty$, and blows up like $-\log y$ near $y = 0$. Here we are denoting the nonlinear operator by $N$ as before.

The second step is to correct the expansions, or equivalently, to solve away the terms in the expansions of $f_0$, at each of these boundary faces. Near each $F_j$ this is done exactly as in the last section. Near $F_\infty$ it is done in a completely analogous manner, solving away the terms of order $R^{-3-j}$ using correction terms of order $R^{-1-j}$. Near $F_j$ we are using the solvability of the operator $J_{n_j}$, while near $F_\infty$ we use the operator $J_{N_0}$. Finally, exactly as before, we solve away the terms in the expansion of the remainder as $y \to 0$ along $B$. This may be done uniformly up to the boundaries of $B$. Taking Borel sums of each of these expansions, there exists a polyhomogeneous function $\hat{u}_1$ on $\widehat{X}$ which satisfies $N(\hat{u}_0 + \hat{u}_1) = f_1$ where $f_1$ vanishes to all orders at every boundary component of $\widehat{X}$. The approximate solution is $\hat{u} = \hat{u}_0 + \hat{u}_1$.

Now write $\widehat{N}(v) = N(\hat{u} + v)$. We expand this as

$$\widehat{N}(v) = -\Delta_{\bar{g}} v + e^{2\hat{u}}|p(z)|^2(e^{2v} - 1) + f_1.$$

We construct a supersolution using the following three constituent functions: first, $R_\infty^{-\epsilon}\mu_0^{N_0}$ near $F_\infty$ (where $\mu_0^{N_0}$ is the ground state eigenfunction for $J_{N_0}$); next, $R_j^{\epsilon}\mu_0^{n_j}$ near $F_j$. Finally, $y^{\epsilon/2}$ near $B$. We then take

$$v^+ = \min\{R_\infty^{-\epsilon}\mu_0^{N_0}, R_1^{\epsilon}\mu_0^{n_1}, \dots, R_N^{\epsilon}\mu_0^{n_N}, y^{\epsilon/2}\}.$$

It is straightforward to check that $\widehat{N}(v^+) \geq 0$. With the obvious changes, we also obtain a function $v^-$ for which $\widehat{N}(v^-) \leq 0$.

Proposition 3.4.1 now implies that there exists a solution $v$ to this equation. By construction, $u = \hat{u} + v$ satisfies all the required boundary conditions. □

As in Section 4, this existence theorem is accompanied by some sharp estimates for the solution $u$.

**Proposition 3.6.2.** *The solution $u$ obtained in the previous proposition is polyhomogeneous on $\widehat{X}$. In particular, it has a full asymptotic expansion as $R \to \infty$, where the leading term is the model solution $U_{N_0}$.*

This, in turn, leads to a uniqueness theorem for the scalar equations:

**Theorem 3.6.3.** *Let $p(z)$ be a polynomial on $\mathbb{C}$ of degree $N_0 > 1$. Suppose that $u_1$ and $u_2$ are two solutions to* (3.67) *satisfying the generalized Nahm pole conditions with knot determined by the zeroes of polynomial $p$ at $y = 0$. Assume also that as $R \to \infty$, $u_i \sim U_{N_0} + R^{-\epsilon}$, $i = 1, 2$. Then $u_1 = u_2$.*

*Proof.* By (3.67),

$$-(\Delta + \partial_y^2)(u_1 - u_2) + |p(z)|^2(e^{2u_1} - e^{2u_2}) = \left(-(\Delta + \partial_y^2) + |p(z)|^2 F(u_1, u_2)\right) w = 0$$

Here $w = u_1 - u_2$ and $F(u_1, u_2) = (e^{2u_1} - e^{2u_2})/(u_1 - u_2)$. By assumption that both $u_1$ and $u_2$ satisfy the same boundary conditions, and using the regularity theory for solutions, we obtain that $\lim_{y\to 0} w = 0$, while by the hypothesis on decay at infinity, $\lim_{R\to\infty} w = 0$ as well. Noting that $F(u_1, u_2) \geq 0$, no matter whether $u_1 < u_2$ or $u_1 \geq u_2$, the maximum principle implies that $w \equiv 0$, i.e., $u_1 \equiv u_2$. □

*Chapter 4*

# THE EXPANSIONS OF THE NAHM POLE SOLUTIONS TO THE KAPUSTIN-WITTEN EQUATIONS

This is joint work with Victor Mikhaylov.

## 4.1 Introduction

Witten [63] proposed a fascinating program for interpreting the Jones Polynomial of knots on a 3-manifold $Y$ by counting singular solutions to the Kapustin-Witten equations. We refer [25], [64], [48], [34], [63], [26] for more detailed explanations, along with [43], [46] and [28] for the beginnings of the analytic background to this program. In the absence of a knot, the singular boundary condition is called the Nahm pole boundary conditions and the counting of singular solutions might lead to new 3-manifold invariants.

Let $P$ be a principal $G$ bundle over $Y \times \mathbb{R}^+$ where $\mathbb{R}^+ = (0, +\infty)$, and let $\mathfrak{g}_P$ be the adjoint bundle. The Kapustin-Witten equations are the following set of equations for a connection $A$ and $\mathfrak{g}_P$-valued 1-form $\Phi$, respectively:

$$\begin{aligned} &F_A - \Phi \wedge \Phi + \star d_A \Phi = 0, \\ &d_A^{\star} \Phi = 0. \end{aligned} \tag{4.1}$$

In [43], Mazzeo and Witten proved a regularity theorem for the Nahm pole solutions. For any Nahm pole solution $(A, \Phi)$ to the Kapustin-Witten equations with a gauge fixing condition, $(A, \Phi)$ is polyhomogeneous along the boundary. Roughly, if we denote $y$ to be the coordinate of $\mathbb{R}^+$ and $x$ the local coordinate of $Y$, $(A, \Phi)$ will have the following expansions:

$$\begin{aligned} A &\sim \omega + \sum_{i=1}^{+\infty} \sum_{p=1}^{r_i} y^i (\log y)^p a_{i,p}, \\ \Phi &\sim y^{-1} e + \sum_{i=1}^{+\infty} \sum_{p=1}^{r_i} y^i (\log y)^p b_{i,p}, \end{aligned} \tag{4.2}$$

where $e : TY \longrightarrow \mathfrak{g}_P$ is the vierbein form thatngives an endormorphism of the tangent bundle $TY$ of $Y$ and adjoint bundle $\mathfrak{g}_P$ and $\omega$ is the Levi-Civita connection of $Y$ under the pullback of $e$. For each $i$, $r_i$ is a finite positive integer and $a_{i,p}$, $b_{i,p}$ are 1-forms independent of $y$ coordinate and smooth in $x$ direction. The choice of the vierbein term $e$ depends on a choice of principle embedding $\rho : \mathfrak{su}(2) \to G$ and we write the image of $\rho$ as $\mathfrak{su}(2)_t$. Let $\Omega^1_Y(\mathfrak{g}_P)$(resp. $\Omega^0_Y(\mathfrak{g}_P)$) be the $\mathfrak{g}_P$-valued 1-form(resp. 0-form) over $Y$, the action of $\mathfrak{su}(2)_t$ will give a decomposition $\Omega^1_Y(\mathfrak{g}_P) = \oplus V_\sigma(\Omega^0(\mathfrak{g}_P) = \oplus\tau_\sigma)$, where the $V_\sigma$ and $\tau_\sigma$ are irreducible modules and $\sigma$ takes values in positive integers.

We have the following descriptions of the polyhomogeneous solutions to the Kapustin-Witten equations:

**Theorem 4.1.1.** *Let $(A, \Phi)$ be a polyhomogeneous Nahm pole solution to the Kapustin-Witten equations over $Y \times \mathbb{R}^+$. In the temperal gauge, we write $A = \omega + a$, $\Phi = \frac{e}{y} + b + \phi_y dy$, where $\frac{e}{y} + b$ and $A$ don't have $dy$ component. Let $a^\sigma$, $b^\sigma$ be the projection of $a, b$ into the irreducible module $V_\sigma$ and let $\phi^\sigma_y$ be the projection of $\phi_y$ into the irreducible module $\tau_\sigma$. Suppose $a^\sigma, b^\sigma, \phi^\sigma_y$ have the expansions*

$$a^\sigma \sim \sum_{i=1}^{+\infty}\sum_{p=0}^{r_i} a^\sigma_{i,p} y^i (\log y)^p,\ b^\sigma \sim \sum_{i=1}^{+\infty}\sum_{p=0}^{r_i} b^\sigma_{i,p} y^i (\log y)^p,\ \phi^\sigma_y \sim \sum_{i=1}^{+\infty}\sum_{p=0}^{r_i} (\phi^\sigma_y)_{k,p} y^k (\log y)^p.$$

*We write $a^\sigma_k := a^\sigma_{k,0}$, $b^\sigma_k := b^\sigma_{k,0}$, $(\phi^\sigma_y)_k := (\phi^\sigma_y)_{k,0}$ and obtain:*

*(1) When $\sigma = 1$, $a^1, b^1, \phi^1_y$ have the expansions with leading terms*

$$\begin{aligned} a^1 &\sim y^2 \log y a^1_{2,1} + y^2 a^1_2 + O(y^{\frac{5}{2}}),\\ b^1 &\sim y \log y b^1_{1,1} + y b^1_1 + O(y^{\frac{5}{2}}),\\ \phi^1_y &\sim y^2 \log y (\phi^1_y)_{2,1} + y^2 (\phi^1_y)_2 + O(y^{\frac{5}{2}}). \end{aligned}$$

*When $\sigma > 1$, $a^\sigma, b^\sigma$ and $\phi^\sigma_y$ have the expansions with leading terms*

$$a^\sigma \sim y^{\sigma+1} a^\sigma_{\sigma+1} + O(y^{\sigma+\frac{3}{2}}),\ b^\sigma \sim y^\sigma b^\sigma_\sigma + O(y^{\sigma+\frac{1}{2}}),\ \phi^\sigma_y \sim y^{\sigma+1}(\phi^\sigma_y)_{\sigma+1} + O(y^{\sigma+\frac{3}{2}}).$$

*(2) The expansions of $a, b$ are determined by the coefficients $a^\sigma_{\sigma+1}$, $b^\sigma_\sigma$, $(\phi^\sigma_y)_{\sigma+1}$. To be explicit, let $(\hat{A}, \hat{\Phi})$ be another solution with the expansion coefficients $\hat{a}^\sigma_{k,p}$, $\hat{b}^\sigma_{k,p}$, $(\hat{\phi}^\sigma_y)_{k,p}$. If for any $\sigma$, $a^\sigma_{\sigma+1} = \hat{a}^\sigma_{\sigma+1}$, $b^\sigma_\sigma = \hat{b}^\sigma_\sigma$ and $(\phi^\sigma_y)_{\sigma+1} = (\hat{\phi}^\sigma_y)_{\sigma+1}$ then $(A, \Phi)$ and $(\hat{A}, \hat{\Phi})$ have the same expansions.*

As the $O(1)$ terms of the $A$'s expansion is the Levi-Civita connection, we can build up the relationship of the geometry of $Y$ and the expansions of the Nahm pole solutions. Under the previous assumptions, we obtain:

**Theorem 4.1.2.** *(1) If $b^1_{1,1} = 0$, then the expansions of $(A, \Phi)$ don't contains "$\log y$" terms.*

*(2) $b^1_{1,1} = 0$ if and only if $Y$ is an Einstein 3-manifold.*

Combining this with the existence results [51], [27], [37], we obtain the following corollary:

**Corollary 4.1.3.** *Over $Y \times \mathbb{R}^+$, there exists a Nahm pole solution whose sub-leading term is smooth to the boundary if and only if $Y$ is an Einstein 3-manifold.*

We can determine the expansions more clearly when $G = SU(2)$ or $SO(3)$:

**Theorem 4.1.4.** *When $G = SU(2)$ or $SO(3)$, under the previous assumptions, let $(A, \Phi = \phi + \phi_y dy)$ be a polyhomogeneous Nahm pole solution, we obtain:*

*(1) $(A, \Phi = \phi + \phi_y dy)$ has the following expansions:*

$$A \sim \omega + \sum_{i=1}^{+\infty} \sum_{p=0}^{i} a_{2i,p} y^{2i} (\log y)^p, \ \phi \sim y^{-1} e + \sum_{i=1}^{+\infty} \sum_{p=0}^{i} b_{2i-1,p} y^{2i-1} (\log y)^p,$$

$$\phi_y \sim \sum_{i=1}^{+\infty} \sum_{p=0}^{i} (\phi_y)_{2i,i} y^{2i} (\log y)^p,$$

*where $e$ is the vierbein form and $\omega$ under the pull back of $e$ is the Levi-Civita connection of $Y$.*

*(2) If $Y$ is an Einstein 3-manifold, then $(A, \Phi = \phi + \phi_y dy)$ has the following expansions:*

$$A \sim \omega + \sum_{i=1}^{+\infty} a_{2i} y^{2i}, \ \Phi \sim y^{-1} e + \sum_{i=1}^{+\infty} \sum_{p=0} b_{2i-1} y^{2i-1}, \ \phi_y \sim \sum_{i=1}^{+\infty} \sum_{p=0} (\phi_y)_{2i} y^{2i}.$$

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### 4.2 The Nahm Pole Solutions

In this section, we will summarize some basic properties of the Nahm pole solutions to the Kapustin-Witten equations, largely following [43], [30], [49].

**The Setup of the Nahm Pole Solutions**

Let $Y$ be a smooth 3-manifold with a Riemannian metric and we write $M := Y \times \mathbb{R}^+$ and $y$ be the coordinate of $\mathbb{R}^+$. We choose the product metric over $M$ and volume form $\mathrm{Vol}_Y \wedge dy$, where $\mathrm{Vol}_Y$ is the volume form of $Y$. Let $P$ be an $G$ bundle over $M$, where $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_P$ be the adjoint bundle, take $A$ be a connection of $P$, and let $\Phi$ be a $\mathfrak{g}_P$ valued 1-form.

Given a principle embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, for an integer $a = 1, 2, 3$ and a point $x \in Y$, take $\{\mathfrak{e}_a\}$ to be any unit orthogonal basis of $T^\star_x Y$, the cotangent bundle of $Y$ and take $\{\mathfrak{t}_a\}$ to be sections of the adjoint bundle $\mathfrak{g}_P$ lie in the image of $\rho$ with the relation $[\mathfrak{t}_a, \mathfrak{t}_b] = \epsilon_{abc}\mathfrak{t}_c$. We write the vierbein form $e := \sum_{i=1}^3 \mathfrak{t}_i \mathfrak{e}_i$, where $e$ gives an endormorphism of the tangent bundle $TY$ to the adjoint bundle $\mathfrak{g}_P$. The definition of the vierbein form $e$ depends on the choice of $\rho$.

**Definition 4.2.1.** *A solution* $(A, \Phi)$ *to* (4.1) *over* $M$ *is a Nahm pole solution if for any point* $x \in Y$*, there exist* $\{\mathfrak{e}_a\}$*,* $\{\mathfrak{t}_a\}$ *as above such that when* $y \to 0$*,* $(A, \Phi)$ *has the following expansions:* $A \sim O(y^{-1+\epsilon})$*,* $\Phi \sim y^{-1}e + O(y^{-1+\epsilon})$*, for some constant* $\epsilon > 0$.

Now, we will introduce some basic terminology of the regularity of a function over manifold with boundary, largely follows from [41]. Let $\vec{x} = (x_1, x_2, x_3)$ to be the local coordinates of $Y$. For any ($\mathfrak{g}_P$-valued) differential form $\alpha$, we say $\alpha$ is conormal if for any $j \geq 0$ and $\vec{k} = (k_1, k_2, k_3)$ with $k_i \geq 0$, $\sup |(y\partial_y)^j \partial_{\vec{x}}^{\vec{k}} \alpha| \leq C_{j\vec{k}}$, where $\partial_{\vec{x}}^{\vec{k}} = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\partial_{x_3}^{k_3}$ and the sup is taken over an open neighborhood of the boundary. We say $\alpha$ is polyhomogeneous if $\alpha$ is conormal and has an asymptotic expansion $\alpha \sim \sum y^{\gamma_j}(\log y)^p \alpha_{jp}(\vec{x})$. Here the exponents $\gamma_j$ lie in some discrete set $E \subset \mathbb{C}$, called the index set of $\alpha$, which has the properties that $\Re\gamma_j \to \infty$ as $j \to \infty$, the powers $p$ of $\log y$ are all non-negative integers, and there are only finitely many log terms accompanying any given $y^{\gamma_j}$. The notation "$\sim$" means $\sup |\alpha - \sum_{j\leq N} y^{\gamma_j}(\log y)^p \alpha_{jp}| \leq y^{\Re\gamma_{N+1}}(\log y)^q$, and the corresponding statements must hold for the series obtained by differentiating any finite number of times.

Now, we will summarize the regularity theorem in [43]:

**Theorem 4.2.2.** *([43, Prop 5.3, Prop 5.9, Section 2.3)] For* $(A, \Phi)$ *a Nahm pole solution to the Kapustin-Witten equations, choose a smooth reference connection* $A_0$ *and denote* $\Phi_0$ *the leading term of* $\Phi$*, if* $(A, \Phi)$ *satisfies the gauge fixing equation*

$$d_{A_0}^{\star_4}(A - A_0) - \star_4[\Phi_0, \star_4(\Phi - \Phi_0)] = 0,$$

*where $\star_4$ is the 4-dimensional Hodge star operator, then $(A, \Phi)$ is polyhomogeneous with the following expansions:*

$$A \sim \omega + \sum_{i=1}^{+\infty} \sum_{p=1}^{r_i} y^i (\log y)^p a_{i,p}, \ \Phi \sim y^{-1} e + \sum_{i=1}^{+\infty} \sum_{p=1}^{r_i} y^i (\log y)^p b_{i,p}. \tag{4.3}$$

*Here for each i, $r_i$ are finite positive integers and $a_{i,p}$, $b_{i,p}$ are 1-forms independent of y coordinate and smooth in x direction.*

**Remark.** *We write $a := A - A_0$, $b := \Phi - \Phi_0$, the statement that $(A, \Phi)$ is polyhomogeneous with the gauge fixing condition $d_{A_0}^{\star_4} a - \star_4[\Phi, \star_4 b] = 0$ is proved in [43, Proposition 5.9] and it also works for many other gauge fixing conditions, for example $A_y = 0$ and $d_{A_0}^{\star} a - \star[\Phi, \star b] = 0$, or even $d_{A_0}^{\star}(A - A_0) = 0$, where $A_y$ is the dy component of A and $\star$ is the Hodge star operator of $Y$. It is straight forward to check that these two gauge fixing conditions will only bring integer expansions. The claim that $(A, \Phi)$ has the leading terms $(\omega + y(\log y)^p a_1, y^{-1} e + y(\log y)^p)$ is proved in [43, Section 2.3].*

For $(A, \Phi)$ a Nahm pole solution over $Y \times \mathbb{R}^+$, under the temperal gauge, we can assume $A$ doesn't have $dy$ component. We write $\Phi = \phi + \phi_y dy$. We have the following well-known vanishing claim for the $\phi_y$ term, for a proof see [56, Page 36], [28, Corollary 4.7].

**Proposition 4.2.3.** *For $(A, \Phi)$ a Nahm pole solution over $Y \times \mathbb{R}^+$, write $\Phi = \phi + \phi_y dy$, then if $\lim_{y \to +\infty} |\phi_y|_{C^0} = 0$, then $\phi_y = 0$.*

It is straightforward to obtain the following:

**Corollary 4.2.4.** *Let $(A, \Phi)$ be a Nahm pole solution convergences to a flat irreducible $G^{\mathbb{C}}$ connection at $y \to \infty$, then $\Phi$ don't have dy component.*

As pointing out in [49], the reducible limit is important to considered, where $\phi_y$ term might appear. In our paper, we don't make any assumption of the limit of the solution at $y \to +\infty$.

### Elementary Representation Theory

Now we will introduce some representation theory of $\mathfrak{su}(2)$. Consider a principal embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, we call the image $\mathfrak{su}(2)_t$ the principal subalgebra. When $\rho$ is a principal embedding, under the action of $\mathfrak{su}(2)_t$, $\mathfrak{g}$ decomposes as a direct sum of irreducible modules $\tau_\sigma$ with dimension $2\sigma + 1$, and we write $\mathfrak{g} = \oplus_\sigma \tau_\sigma$.

Here $\sigma$ are positive integers of which precisely one equals to 1, corresponding to the principle subalgebra. For simple Lie algebra $\mathfrak{g}$, the values of $\sigma$ are precisely compute in [30]. For example, when $G = SU(N)$, the values of $\sigma$ are $1, 2, 3, \cdots, N-1$.

Under the action of $\mathfrak{su}(2)_t$, the decomposition of $\mathfrak{g}$ will automatically induce a decomposition of $\Omega^1_Y(\mathfrak{g}_P)$, which is the $\mathfrak{g}_P$-valued 1-from on $Y$. We write $V_\sigma = \Omega^1_Y(\tau_\sigma)$, and then

$$\Omega^1_Y(\mathfrak{g}_P) = \oplus_\sigma V_\sigma. \tag{4.4}$$

We can define the projection map $\mathcal{P}_\sigma : \Omega^1_Y(\mathfrak{g}_P) \to V_\sigma$ and $\mathcal{P}_\sigma : \Omega^0_Y(\mathfrak{g}_P) \to \Omega^0(\tau_\sigma)$, for $a, b \in \Omega^1_Y(\mathfrak{g}_P)$, we write $a^\sigma := \mathcal{P}_\sigma a$, $b^\sigma := \mathcal{P}_\sigma b$ and $\phi^\sigma_y := \mathcal{P}_\sigma \phi_y$. The Clebsch-Gordan theorem will imply the following proposition:

**Proposition 4.2.5.** *[35] Under the previous assumptions, for $\sigma_1, \sigma_2$ are positive integers, then*

$$\star a^{\sigma_1} \wedge b^{\sigma_2} \in \oplus_{\sigma=|\sigma_1-\sigma_2|}^{\sigma_1+\sigma_2} V_\sigma, \ \star[\phi^{\sigma_1}_y, a^{\sigma_2}] \in \oplus_{\sigma=|\sigma_1-\sigma_2|}^{\sigma_1+\sigma_2} V_\sigma.$$

It is also important to understand the action of the vierbein form $e$. We define a linear operator:

$$\begin{aligned} L :& V_\sigma \to V_\sigma, \\ & a \to \star[e, a], \end{aligned} \tag{4.5}$$

which obeys the following properties:

**Proposition 4.2.6.** *[43, Section 2.3.2] (1) $L$ has three eigenspaces $V^-_\sigma, V^0_\sigma, V^+_\sigma$ with dimension $2\sigma - 1, 2\sigma + 1, 2\sigma + 3$ and eigenvalues $\sigma + 1, 1, -\sigma$. We can write $V_\sigma = V^-_\sigma \oplus V^0_\sigma \oplus V^+_\sigma$.*

*(2) For 1-form $a \in V_\sigma$, $[\star e, a] = 0$ implies $V^0_\sigma$ component of $a$ is zero.*

We also denote $V^\circ := \oplus_\sigma V^\circ_\sigma$, where $\circ \in \{+, -, 0\}$. For an integer $k$, we can define the following operator $\mathcal{L}^\sigma_k(a) := ka + \star[e, a]$ and obtain

**Corollary 4.2.7.** *$\mathcal{L}^\sigma_k$ is an isomorphism for $k \neq (\sigma + 1), -1, -\sigma$.*

Let $\omega$ be a connection on $Y$, for the 3-dimensional differential operator $\star d_\omega$, which acts from the space $\Omega^1_Y(\mathfrak{g}_P)$ to itself, has the following properties:

**Proposition 4.2.8.** *[30, Page 5]$\star d_\omega : V^-_\sigma \to V^0_\sigma$, $V^0_\sigma \to V^-_\sigma \oplus V^0_\sigma \oplus V^+_\sigma$, $V^+_\sigma \to V^0_\sigma \oplus V^+_\sigma$.*

In addition, the vierbein form $e$ will give an identification of $V_1^0$ and $\Omega^0(\tau_1)$. We define the operator

$$\begin{aligned} &\Gamma : \Omega^1(\mathfrak{g}_P) \to \Omega^0(\mathfrak{g}_P), \\ &\Gamma(a) := \star[a, \star e], \end{aligned} \tag{4.6}$$

where $a$ is a 1-form, then we have the following identities:

**Proposition 4.2.9.** *[49, Appendix A] For $a \in V_\sigma = \Omega^1(\tau_\sigma)$, $b \in \Omega^0(\tau_\sigma)$, we have:*

*(1)$\Gamma a = \Gamma(a)^0$, $[e, \Gamma a] = 2(a)^0$, $\Gamma[e, b] = 2b$, where $(a)^0$ means the projection to the $V_\sigma^0$ part under the decomposition $V_\sigma = V_\sigma^+ \oplus V_\sigma^0 \oplus V_\sigma^0$,*

*(2)$[e, d_\omega^\star a] = 2(\star d_\omega a)^0$, $\Gamma(\star d_\omega a) = d_\omega^\star a$, where $(\star d_\omega a)^0$ is the $V_\sigma^0$ part of $\star d_\omega a$.*

With this preparation, we will state an algebraic lemma which will be heavily used in the following parts of the paper:

**Lemma 4.2.10.** *For $a_1, a_2 \in V_\sigma$, $b \in V_\sigma^+$, $c_1, c_2 \in \Omega^0(\tau_\sigma)$ and a positive integer $r$, if they satisfy the following algebraic equations,*

$$\begin{aligned} &(\sigma + 1)a_1 = \star[e, a_1] - [e, c_1], \\ &(\sigma + 1)c_1 = -\Gamma a_1, \\ &r a_1 + (\sigma + 1)a_2 = \star[e, a_2] + [c_2, e] + \star d_\omega b, \\ &r c_1 + (\sigma + 1)c_2 = -\Gamma a_2 + d_\omega^\star b, \end{aligned} \tag{4.7}$$

*then $a_1 = 0$ and $c_1 = 0$.*

*Proof.* Consider the $V_\sigma^+$ part of the first equation, and we obtain $(a_1)^+ = 0$, where $(a_1)^+$ is the $V_\sigma^+$ part of $a_1$. As $-(\sigma + 1)a_2 + \star[e, a_2] \in V^+ \oplus V^0$ and by Lemma 4.2.8, $\star d_\omega b \in V^+ \oplus V^0$, consider the $V_\sigma^-$ part of the third equation, and we obtain $(a_1)^- = 0$.

The only situation left is the $V^0$ part. Consider the $V^0$ part of the first two equations, we obtain

$$\sigma(a_1)^0 = -[e, c_1], \; c_1 = -\frac{1}{\sigma + 1}\Gamma a_1. \tag{4.8}$$

Applying Proposition 4.2.9, we obtain $\sigma(a_1)^0 = \frac{2}{\sigma+1}(a_1)^0$. When $\sigma \neq 1$, then $(a_1)^0 = 0$.

If $\sigma = 1$, then the two equations in (4.8) are equivalent. To be explicit, we obtain $(a_1)^0 = -[e, c_1]$ or $c_1 = -\frac{1}{2}\Gamma a_1$. We consider the $V_\sigma^0$ part of the last two equations, we obtain

$$
\begin{aligned}
r(a_1)^0 + (a_2)^0 &= [c_2, e] + (\star d_\omega b)^0 \\
rc_1 + 2c_2 &= -\Gamma a_2 + d_\omega^\star b.
\end{aligned}
\tag{4.9}
$$

Using $[e,\ ]$ acts on the second equation of (4.9), we obtain

$$
-r(a_1)^0 + 2(a_2)^0 = 2[c_2, e] + 2(\star d_\omega b)^0.
$$

Comparing the coefficients with the first equation of (4.9), we obtain $(a_1)^0 = 0$. □

**Lemma 4.2.11.** *Let $a, \Theta \in V_\sigma^0$, $\phi, \Xi \in \Omega^0(\tau_\sigma)$, let $\lambda$ be a real number such that $\lambda \neq 2$ or $-1$. If they satisfy*

$$
(\lambda - 1)a = \Theta - [e, \phi],\ \lambda\phi = -\Gamma a + \Xi, \tag{4.10}
$$

*then*

$$
a = \frac{1}{\lambda^2 - \lambda - 2}(\lambda\Theta - [e, \Xi]),\ \phi = \frac{1}{\lambda^2 - \lambda - 2}((\lambda - 1)\Xi - \Gamma\Theta). \tag{4.11}
$$

*Specially, if $\Theta = \Xi = 0$, then $a = \phi = 0$.*

*Proof.* Applying $[e, ]$ to the second equation, by Proposition 4.2.9, we compute

$$
\lambda[e, \phi] = -[e, \Gamma a] + \Gamma\Xi = -2a + \Gamma\Xi.
$$

Combining with the first equation, we obtain $a = \frac{1}{\lambda^2-\lambda-2}(\lambda\Theta - [e, \Xi])$. Acting the operator $\Gamma$ to the first equation and combing with the second, we will obtain $\phi = \frac{1}{\lambda^2-\lambda-2}((\lambda - 1)\Xi - \Gamma\Theta)$. The rest follows immediately. □

## 4.3 Leading Expansions of the Nahm Pole Solutions

In this section, we will determined the coefficients of the Kapustin-Witten equations up to several leading terms. For $P$ a principle $G$ bundle over $Y \times \mathbb{R}^+$, $A$ a connection over $P$ and $\Phi$ a $\mathfrak{g}_P$-valued 1-form. We choose a gauge such that $A$ doesn't have $dy$ component. We write $\Phi = \phi + \phi_y dy$.

The Kapustin-Witten equations reduce to the flow equations:

$$
\begin{aligned}
\partial_y A &= \star d_A \phi + [\phi_y, \phi], \\
\partial_y \phi &= d_A \phi_y + \star(F_A - \phi \wedge \phi), \\
\partial_y \phi_y &= d_A^\star \phi.
\end{aligned}
\tag{4.12}
$$

We denote $A = \omega + a,\ \Phi = y^{-1}e + b + \phi_y\ dy$ where $\omega$ is a connection independent of the $\mathbb{R}^+$ direction, $a, b$ are $1-$forms independent of $y$. Recalling theorem 4.2.2, we can assume $a, b$ have expansions with leading order $O(y(\log y)^p)$.

By a straight forward computation, the $O(y^{-2})$ order coefficients of the flow equations (4.12) is $\star e = e \wedge e$, which is automatically satisfied. The $O(y^{-1})$ order coefficients of equations (4.12) reduce to $d_\omega e = 0,\ d_\omega^\star e = 0$. These equations can be understood as:

**Proposition 4.3.1.** *[43, Section 4.1] Under the pull back induced by the vierbein form $e : TY \to \mathfrak{g}_P$, $\omega$ is the Levi-Civita connection of $Y$.*

Under the decomposition of Proposition 4.2.6, we can also decompose the dual $\star F_\omega$ of the Riemannian curvature

$$\star F_\omega = (\star F_\omega)^- + (\star F_\omega)^+, \tag{4.13}$$

where the first term is the curvature scale and the second term is the traceless part of the Ricci tensor, where these determine the curvauture tenser completely for 3-dimensional manifold. In addition, under the decomposition (4.4), $\star F_\omega \in V_1$, which is the principal subalgebra.

**Leading Order Expansion**

Now, we will explicitly study the expansions of the subleading terms. Recall that we write $A = \omega + a,\ \Phi = y^{-1}e + b + \phi_y\ dy$, using (4.12), we obtain the following equations:

$$\begin{aligned}
\partial_y a &= \star d_\omega b + y^{-1} \star [a, e] + y^{-1}[\phi_y, e] + \star[a, b] + [\phi_y, b],\\
\partial_y b &= \star F_\omega - y^{-1} \star [e, b] + \star d_\omega a + d_\omega \phi_y + [a, \phi_y] + \star a \wedge a - \star b \wedge b,\\
\partial_y \phi_y &= d_\omega^\star b - y^{-1} \star [a, \star e] - \star[a, \star b].
\end{aligned} \tag{4.14}$$

$\sigma = 1$

Recalling (4.4), we have the projection map $\mathcal{P}_\sigma : \Omega^1(\mathfrak{g}_P) \to V_\sigma$ and $\mathcal{P}_\sigma : \Omega^0(\mathfrak{g}_P) \to \Omega^0(\tau_\sigma)$. We define $a^\sigma := \mathcal{P}_\sigma a,\ b^\sigma := \mathcal{P}_\sigma b,\ \phi_y^\sigma := \mathcal{P}_\sigma \phi_y$ By Proposition 4.3.1, $\mathcal{P}_1 \star F_\omega = \star F_\omega$. Using $\mathcal{P}_1$ to (4.14), we obtain:

$$\begin{aligned}
\partial_y a^1 &= \star d_\omega b^1 + y^{-1} \star [e, a^1] + y^{-1}[\phi_y^1, e] + \mathcal{P}_1(\star[a, b] + [\phi_y, b]),\\
\partial_y b^1 &= \star F_\omega - y^{-1} \star [e, b^1] + \star d_\omega a^1 + d_\omega \phi_y^1 + \mathcal{P}_1([a, \phi_y] + \star a \wedge a - \star b \wedge b),\\
\partial_y \phi_y^1 &= d_\omega^\star b^1 - y^{-1} \star [a^1, \star e] - \mathcal{P}_1(\star[a, \star b]).
\end{aligned} \tag{4.15}$$

By Theorem 4.2.2, suppose $a^1, b^1, \phi_y^1$ have the following expansions:

$$a^1 \sim \sum_{p=0}^{r_1} a_{1,p}^1 y(\log y)^p + \cdots, \; b^1 \sim \sum_{p=0}^{r_1} b_{1,p}^1 y(\log y)^p + \cdots, \; \phi_y^1 \sim \sum_{p=0}^{r_1} (\phi_y^1)_{1,p} y(\log y)^p + \cdots.$$

For simplification, we also denote $a_1^1 := a_{1,0}^1$ and $b_1^1 := b_{1,0}^1$.

We obtain the following proposition:

**Proposition 4.3.2.** *(a)* $a_{1,p}^1 = (\phi_y^1)_{1,p} = 0$ *for any* $p$,

*(b) For* $p \geq 2$, $b_{1,p}^1 = 0$,

*(c)* $b_{1,1}^1$ *equals to the* $V^+$ *part of* $\star F_\omega$. *Moreover, there exists* $c^+ \in V^+$ *such that we can express* $b_1^1$ *in the following way:*

$$b_1^1 = c^+ + \frac{1}{2}(\star F_\omega)^0 + \frac{1}{3}(\star F_\omega)^-.$$

*Proof.* (a) Consider the $O((\log y)^{r_1})$ order coefficients of the first and third equations of (4.15), we obtain:

$$\begin{aligned} a_{1,r_1}^1 &= \star[e, a_{1,r_1}^1] + [(\phi_y^1)_{1,r_1}, e], \\ (\phi_y^1)_{1,r_1} &= - \star [a_{1,r_1}^1, \star e]. \end{aligned} \tag{4.16}$$

As $[(\phi_y^1)_{1,r_1}, e] \in V^0$, by Proposition 4.2.6, $(a_{1,r_1}^1)^+ = (a_{1,r_1}^1)^- = 0$. Projecting the first equation into $V^0$ part, we obtain $[(\phi_y^1)_{1,r_1}, e] = 0$. Using Proposition 4.2.9, we obtain $(\phi_y^1)_{1,r_1} = \Gamma[(\phi_y^1)_{1,r_1}, e] = 0$. Combing this with $(\phi_y^1)_{1,r_1} = - \star [a_{1,r_1}^1, \star e]$, we obtain $(a_{1,r_1}^1)^0 = 0$. By an induction on the integer $r_1$, (1) is proved.

(b) For $r_1 \geq 2$, consider the $O((\log y)^{r_1})$ and $O((\log y)^{r_1})$ coefficients of the second equation of (4.15), we obtain

$$\begin{aligned} b_{1,r_1}^1 &= - \star [e, b_{1,r_1}^1], \\ r_1 b_{1,r_1}^1 + b_{1,r_1-1}^1 &= - \star [e, b_{1,r_1-1}^1]. \end{aligned} \tag{4.17}$$

The first equation implies $b_{1,r_1}^1 = 0$, where as $b_{1,r_1-1}^1 + \star[e, b_{1,r_1-1}^1] \notin V^+$, from the second equation we obtain $b_{1,r_1}^1 = 0$.

(c) Consider the $O(\log y)$ and $O(1)$ coefficient equations for the "$b$" terms, we obtain

$$\begin{aligned} & b_{1,1}^1 + \star[e, b_{1,1}^1] = 0, \\ & b_{1,1}^1 + b_1^1 + \star[e, b_1^1] = \star F_\omega, \\ & -a_1^1 + \star[e, a_1^1] = 0, \; [a_1^1, \star e] = 0. \end{aligned} \tag{4.18}$$

The first equation implies $b^1_{1,1} \in V^+$. From the second equation, we obtain that $b^1_{1,1}$ is determined by the $V^+$ part of $\star F_\omega$. However, for any $c^+ \in V^+$, $b^1_1 + c^+$ also satisfies

$$b^1_{1,1} + b^1_1 + c^+ + \star[e, b^1_1 + c^+] = \star F_\omega$$

and $b^1_1$ can not be determinant by these algebraic equations. □

Now, suppose the leading orders of $(a, b)$ is $y$ instead of $y(\log y)$, we compute the $O(1)$ coefficients of equations (4.14) : $b^1_1 + \star[e, b^1_1] = \star F_\omega$ and obtain the following theorem:

**Theorem 4.3.3.** *If the sub-leading term $(a, b)$ of a polyhomegenous solution is $C^1$ to the boundary, then $Y$ is an Einstein 3-manifold.*

*Proof.* If $(a, b)$ is $C^1$, we obtain $b^1_{1,1} = 0$. By (4.18), we obtain $b^1_1 + \star[e, b^1_1] = \star F_\omega$. By Proposition 4.2.6, this implies $F_\omega \notin V^+$. In addition, by Proposition 4.3.1, we obtain that $\omega$ is the Levi-Civita connection, thus $F_\omega \in V^- \oplus V^0$ which implies $Y$ is Einstein. □

Now, we will determine the next order of the expansions. We have the following descriptions of the "$y^2$" order coefficients:

**Proposition 4.3.4.** *Under the previous notation, we have :*

*(a) for $p \geq 0$, $b^1_{2,p} = 0$,*

*(b) for $p \geq 2$, $a^1_{2,p} = (\phi^1_y)_{2,p} = 0$,*

*(c) $(a^1_{2,1})^- = 0$, $(a^1_{2,1})^+ = \frac{1}{3}(\star d_\omega b^1_{1,1})^+$ and $(a^1_{2,1})^0 = \frac{1}{3}(\star d_\omega b^1_{1,1})^0$, $(\phi^1_y)_{2,1} = \frac{1}{3} d^\star_\omega b^1_{1,1}$,*

*(d)there exist $c^0 \in V^0$, $c^- \in V^-$, such that we can write*

$$\begin{aligned} a^1_2 &= -\frac{1}{9}(\star d_\omega b^1_{1,1})^+ + \frac{1}{3}(\star d_\omega b^1_1)^+ + c^0 + c^- - \frac{1}{3}(\star d_\omega b^1_{1,1})^0 + (\star d_\omega b^1_1)^0, \\ (\phi^1_y)_2 &= -\frac{1}{2}\Gamma c_0. \end{aligned} \tag{4.19}$$

*Proof.* (a) For $r_2 \geq 0$, consider the $O(y(\log y)^{r_2})$ coefficients of (4.14), we obtain $2b^1_{2,r_2} = - \star [e, b^1_{2,r_2}]$, which implies $b^1_{2,r_2} = 0$. By induction, we obtain $b^1_{2,p} = 0$ for any $p \geq 0$.

(b) Now, we will consider the $\phi_y$ and $a$ parts. For $r_2 \geq 2$, consider the $O(y(\log y)^{r_2})$ and $O((y\log y)^{r_2-1})$ coefficients of (4.14), the quadratic terms don't contribute to this order and we obtain

$$\begin{aligned} 2a^1_{2,r_2} &= \star[e, a^1_{2,r_2}] - [e, (\phi^1_y)_{2,r_2}],\\ 2(\phi^1_y)_{2,r} &= -\star[a^1_{2,r_2}, \star e],\\ ra^1_{2,r_2} + 2a^1_{2,r_2-1} &= \star[e, a^1_{2,r_2-1}] + [(\phi^1_y)_{2,r_2-1}, e],\\ r_2(\phi^1_y)_{2,r_2} + 2(\phi^1_y)_{2,r_2-1} &= -\star[a^1_{2,r_2-1}, \star e]. \end{aligned} \tag{4.20}$$

By Lemma 4.2.10, we obtain $a^1_{2,r_2} = 0$.

For statement (c) and (d), we will first determine the $V^+$ part of the coefficients.

Consider the $O(y\log y)$ coefficients, we obtain

$$\begin{aligned} 2a^1_{2,1} &= \star d_\omega b^1_{1,1} + \star[e, a^1_{2,1}] - [e, (\phi^1_y)_{2,1}],\\ 2(\phi^1_y)_{2,1} &= d^\star_\omega b^1_{1,1} - \Gamma a^1_{2,1}. \end{aligned} \tag{4.21}$$

The $V^+$ projection of the first equation gives $(a^1_{2,1})^+ = \frac{1}{3}(\star d_\omega b^1_{1,1})^+$. As $b^1_{1,1} \in V^+$, $d_\omega b^1_{1,1} \in V^+ \oplus V^0$. The $V^-$ projection of the second equation gives $(a^1_{2,1})^- = 0$.

From the $O(y)$ coefficients, we obtain

$$\begin{aligned} 2a^1_2 + a^1_{2,1} &= \star d_\omega b^1_1 + \star[e, a^1_2] - [e, (\phi^1_y)_2],\\ (\phi^1_y)_{2,1} + 2(\phi^1_y)_2 &= d^\star_\omega b^1_1 - \star[a^1_2, \star e]. \end{aligned} \tag{4.22}$$

The $V^+$ projection of the first equation gives $a^1_2 := \frac{1}{3}(\star d_\omega b^1_1)^+ - \frac{1}{9}(\star d_\omega b^1_{1,1})^+$. We cannot determine the $V^-$ part of the $a^1_2$.

Now, consider the $V^0$ projection of (4.21) and (4.22), we obtain

$$\begin{aligned} (a^1_{2,1})^0 &= (\star d_\omega b^1_{1,1})_0 - [e, (\phi^1_y)_{2,1}],\\ (a^1_2)^0 + (a^1_{2,1})^0 &= (\star d_\omega b^1_1)^0 - [e, (\phi^1_y)_2],\\ (\phi^1_y)_{2,1} + 2(\phi^1_y)_2 &= d^\star_\omega b^1_1 - \Gamma a^1_2. \end{aligned} \tag{4.23}$$

Using $[e, \ ]$ acts on the third equation and combing with the first equation, we obtain

$$\begin{aligned} 2(a^1_2)^0 - (a^1_{2,1})^0 &= 2(\star d_\omega b^1_1)^0 - 2[e, (\phi^1_y)_2] - (\star d_\omega b^1_{1,1})^0,\\ 2(a^1_2)^0 + 2(a^1_{2,1})^0 &= 2(\star d_\omega b^1_1)^0 - 2[e, (\phi^1_y)_2]. \end{aligned} \tag{4.24}$$

Thus, we obtain $(a^1_{2,1})^0 = \frac{1}{3}(\star d_\omega b^1_{1,1})^0$. Using (4.21), we compute

$$(\phi^1_y)_{2,1} = \frac{1}{2}d^\star_\omega b^1_{1,1} - \frac{1}{2}\Gamma a^1_{2,1} = \frac{1}{2}d^\star_\omega b^1_{1,1} - \frac{1}{6}\Gamma(\star d_\omega b^1_{1,1}) = \frac{1}{3}d^\star_\omega b^1_{1,1}. \tag{4.25}$$

Now, we write $(a_2^1)^0 = c^0 + (\star d_\omega b_1^1)^0 - \frac{1}{3}(\star d_\omega b_{1,1}^1)^0$, then we compute

$$\begin{aligned}(\phi_y^1)_2 &= -\frac{1}{2}(\phi_y^1)_{2,1} + \frac{1}{2}d_\omega^\star b_1^1 - \frac{1}{2}\Gamma a_2^1 \\ &= -\frac{1}{6}(d_\omega^\star b_{1,1}^1)^0 + \frac{1}{2}d_\omega^\star b_1^1 - \frac{1}{2}\Gamma a_2^1 \\ &= -\frac{1}{2}\Gamma c^0.\end{aligned} \tag{4.26}$$

□

**For $\sigma > 1$.**

Now, we will study the coefficients of the expansions when $\sigma > 1$.

Under the projection $\mathcal{P}_\sigma : \Omega^1(\mathfrak{g}_P) \to V^\sigma$ of (4.14), as $\mathcal{P}_\sigma \star F_\omega = 0$ for $\sigma \neq 1$, we obtain

$$\begin{aligned}\partial_y a^\sigma &= \star d_\omega b^\sigma + y^{-1} \star [e, a^\sigma] + y^{-1}[\phi_y, e] + \mathcal{P}_\sigma(\star[a,b] + [\phi_y, b]), \\ \partial_y b^\sigma &= -y^{-1} \star [e, b^\sigma] + \star d_\omega a^\sigma + d_\omega \phi_y + \mathcal{P}_\sigma([a,\phi_y] + \star a \wedge a - \star b \wedge b), \\ \partial_y \phi_y^\sigma &= d_\omega^\star b^\sigma - y^{-1} \star [a^\sigma, \star e] - \mathcal{P}_\sigma(\star[a, \star b]).\end{aligned} \tag{4.27}$$

Let $\sigma_1, \cdots, \sigma_N$ be the possible indices of (4.4) with $\sigma_i > 1$. We assume $a^{\sigma_i},\ b^{\sigma_i},\ \phi_y^{\sigma_i}$ have the expansions with leading terms:

$$\begin{aligned}&a^{\sigma_i} \sim \sum_{p=1}^{r_i} a^{\sigma_i}_{\lambda_i,p} y^{\lambda_i}(\log y)^p + \cdots,\ b^{\sigma_i} \sim \sum_{p=1}^{r_i} b^{\sigma_i}_{\lambda_i,p} y^{\lambda_i}(\log y)^p + \cdots, \\ &\phi_y^{\sigma_i} \sim \sum_{p=1}^{r_i} (\phi_y^{\sigma_i})_{\lambda_i,p} y^{\lambda_i}(\log y)^p + \cdots.\end{aligned} \tag{4.28}$$

Relabel these indices, we can assume $1 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. As we only care about the leading asymptotic behaviors, we can assume $\lambda_i \le \sigma_i$.

**Proposition 4.3.5.** *(a) For $\sigma > 1$, the expansions of $a^\sigma$, $b^\sigma$ and $\phi_y^\sigma$ are*

$$a^\sigma \sim y^{\sigma+1} a^\sigma_{\sigma+1} + O(y^{\sigma+\frac{3}{2}}),\ b^\sigma \sim y^\sigma b^\sigma_\sigma + O(y^{\sigma+\frac{1}{2}}),\ \phi_y^\sigma \sim y^{\sigma+1}(\phi_y^\sigma)_{\sigma+1} + O(y^{\sigma+\frac{3}{2}})$$

*where we write $a^\sigma_{\sigma+1} := a^\sigma_{\sigma+1,0}$, $b^\sigma_{\sigma+1} := b^\sigma_{\sigma+1,0}$ and $(\phi_y^\sigma)_{\sigma+1,0} := (\phi_y^\sigma)_{\sigma+1}$.*

*(b) There exist $c^0_\sigma \in V^0_\sigma,\ c^-_\sigma \in V^-_\sigma,\ c^+_\sigma \in V^+_\sigma$, such that we can write the leading coefficients of the expansions of $a^\sigma, b^\sigma,$ and $\phi_y^\sigma$ as*

$$\begin{aligned}&(\phi_y^\sigma)_{\sigma+1} = c^0_{\sigma_i},\ b^\sigma_\sigma = c^+_\sigma, \\ &a^\sigma_{\sigma+1} = c^-_\sigma + \frac{1}{2\sigma+1}(\star d_\omega c^+_\sigma)^+ + (\star d_\omega b^\sigma_\sigma)^0 + [c^0_\sigma, e]\end{aligned} \tag{4.29}$$

*Proof.* We will show $a^{\sigma_i}_{\lambda_i+1,p} = b^{\sigma_i}_{\lambda_i,p} = 0$ for any $\lambda_i < \sigma_i$ and $p \geq 1$. We prove by induction on the index $i$. As when $i = 1$, the proof is exactly the same as the following proof, and we omit the proof for this case.

Suppose for any $i \leq k-1$, $a^{\sigma_i}_{\lambda_i+1,p} = b^{\sigma_i}_{\lambda_i,p} = (\phi^{\sigma_i}_y)_{\lambda_i+1} = 0$ for any $\lambda_i < \sigma_i$ and $p \geq 1$, in other words, $\sigma_i = \lambda_i$. Then for $i = k$, consider the quadratic terms of (4.27), by Cebsch-Gordan coefficients [35] and the induction assumption, for some positive integer $p$, we have

$$\begin{aligned}&\mathcal{P}_\sigma(\star[a,b]+\star[\phi_y,b]) \in O(y^{\sigma_k+1}(\log y)^p),\ \mathcal{P}_\sigma(\star[a,\star b]) \in O(y^{\sigma_k+1}(\log y)^p),\\&\mathcal{P}_\sigma(\star a\wedge a-\star b\wedge b+[a,\phi_y]) \in O(y^{\sigma_k+1}(\log y)^p).\end{aligned} \tag{4.30}$$

Next, we consider the coefficients of $O(y^{\lambda_k-1}(\log y)^p)$ and $O(y^{\lambda_k}(\log y)^p)$. When $\lambda_k \leq \sigma_k$, the quadratic terms will not influence our following discussions.

Consider the $O(y^{\lambda_k-1}(\log y)^{r_k})$ and $O(y^{\lambda_k}(\log y)^{r_k-1})$ order coefficients of the first equation of (4.27), we obtain

$$\lambda_k a^{\sigma_k}_{\lambda_k,r_k} = \star[e, a^{\sigma_k}_{\lambda_k,r_k}] + [(\phi^{\sigma_k}_y)_{\lambda_k,r_k}, e],\ (\phi^{\sigma_k}_y)_{\lambda_k,r_k} = -\Gamma a^{\sigma_k}_{\lambda_k,r_k}. \tag{4.31}$$

Projecting the first equation into $V^+$ and $V^-$ part, for $\lambda_k \leq \sigma_k$ and any $r_k$, $(a^{\sigma_k}_{\lambda_k,r_k})^\pm = 0$.

Consider the $V^0$ part of the equation, we obtain

$$(\lambda_k-1)(a^{\sigma_k}_{\lambda_k,r_k})_0 = [(\phi^{\sigma_k}_y)_{\lambda_k,r_k}, e],\ (\phi^{\sigma_k}_y)_{\lambda_k,r_k} = -\Gamma a^{\sigma_k}_{\lambda_k,r_k}. \tag{4.32}$$

Applying Lemma 4.2.11, we obtain $a^{\sigma_k}_{\lambda_k,r_k} = 0$.

Consider the $O(y^{\lambda_k-1}(\log y)^{r_k})$ and $O(y^{\lambda_k-1}(\log y)^{r_k-1})$ order coefficients of the second equation of (4.27), for $r_k \geq 1$, we obtain

$$\lambda_k b^{\sigma_k}_{\lambda_k,r_k} = -\star[e, b^{\sigma_k}_{\lambda_k,r_k}],\ r_k b^{\sigma_k}_{\lambda_k,r_k} + \lambda_k b^{\sigma_k}_{\lambda_k,r_{k-1}} = -\star[e, b^{\sigma_k}_{\lambda_k,r_{k-1}}]. \tag{4.33}$$

Thus, $\lambda_k = \sigma_k$ and $b^{\sigma_k}_{\lambda_k,r_k} = 0$ for $r_k \geq 1$. The $O(y^{\sigma_k})$ order coefficients give $\sigma_k b^{\sigma_k}_{\sigma_k} = -\star[e, b^{\sigma_k}_{\sigma_k}]$, thus $b^{\sigma_k}_{\sigma_k} \in V^+$ and it might be non-zero.

Now, we can assume that $a^{\sigma_k}, \phi^{\sigma_k}_y$ have the leading expansions

$$a^{\sigma_k} \sim \sum_{p=0}^{r_k} y^{\lambda_i+1}(\log y)^p a^{\sigma_k}_{\lambda_i+1,p} + \cdots,\ \phi^{\sigma_k}_y \sim \sum_{p=0}^{r_k} y^{\sigma_k+1}(\log y)^p(\phi^{\sigma_k}_y)_{\sigma_k+1,p}, \tag{4.34}$$

where for simplification, we also denote the highest order of the "$\log y$" terms as $r_k$.

For $\lambda_k < \sigma_k$ and $r_k \geq 0$ or $\lambda_k = \sigma_k$ and $r_k > 1$, consider the $O(y^{\lambda_k}(\log y)^{r_k})$, $O(y^{\lambda_k}(\log y)^{r_k-1})$ order coefficients of (4.15), we obtain

$$\begin{aligned}
&(\lambda_k+1)a^{\sigma_k}_{\lambda_k+1,r_k} = \star[e, a^{\sigma_k}_{\lambda_k+1,r_k}] + [(\phi^{\sigma_k}_y)_{\lambda_k+1,r_k}, e],\\
&(\lambda_k+1)\phi^{\sigma_k}_{\lambda_k+1,r_k} = -\star[a^{\sigma_k}_{\lambda_k+1,r_k}, \star e],\\
&r_k a^{\sigma_k}_{\lambda_k+1,r_k} + (\lambda_k+1)a^{\sigma_k}_{\lambda_k+1,r_k-1} = \star[e, a^{\sigma_k}_{\lambda_k+1,r_k-1}] + [(\phi_y)^{\sigma_k}_{\lambda_k+1,r_k-1}, e],\\
&r_k(\phi^{\sigma_k}_y)_{\lambda_k+1,r_k} + (\lambda_k+1)(\phi^{\sigma_k}_y)_{\lambda_k+1,r_k-1} = -\star[a^{\sigma_k}_{\lambda_k+1,r_k-1}, \star e].
\end{aligned} \tag{4.35}$$

By Lemma 4.2.10, we obtain $a^{\sigma_k}_{\lambda_k+1,r_k} = (\phi^{\sigma_k}_y)_{\lambda_k+1,r_k} = 0$.

When $\lambda_k = \sigma_k$ and $r_k = 1$, we obtain

$$\begin{aligned}
&(\sigma_k+1)a^{\sigma_k}_{\sigma_k+1,1} = \star[e, a^{\sigma_k}_{\sigma_k+1,1}] + [(\phi^{\sigma_k}_y)_{\lambda_k+1,r_k}, e],\\
&(\lambda_k+1)\phi^{\sigma_k}_{\sigma_k+1,r_k} = -\star[a^{\sigma_k}_{\sigma_k+1,r_k}, \star e],\\
&(\sigma_k+1)a^{\sigma_k}_{\sigma_k+1} + a^{\sigma_k}_{\sigma_k+1,1} = \star d_\omega b^{\sigma_k}_{\sigma_k} + \star[e, a^{\sigma_k}_{\sigma_k+1}],\\
&r_k(\phi^{\sigma_k}_y)_{\sigma_k+1,r_k} + (\lambda_k+1)(\phi^{\sigma_k}_y)_{\sigma_k+1,r_k-1} = -\star[a^{\sigma_k}_{\sigma_k+1,r_k-1}, \star e].
\end{aligned} \tag{4.36}$$

Applying Lemma 4.2.10, we obtain $a^{\sigma_k}_{\sigma_k+1,r_k} = (\phi^{\sigma_k}_y)_{\lambda_k+1,r_k} = 0$ for $r_k \geq 1$. This completes the first half of the proposition.

For the second half of the proposition, we can assume for $\sigma > 1$, we have the expansion

$$a^\sigma \sim y^{\sigma+1}a^\sigma_{\sigma+1} + O(y^{\sigma+\frac{3}{2}}),\ b^\sigma \sim y^\sigma b^\sigma_\sigma + O(y^{\sigma+\frac{1}{2}}),\ \phi^\sigma_y \sim y^{\sigma+1}(\phi^\sigma_y)_{\sigma+1} + O(y^{\sigma+\frac{3}{2}}).$$

Using (4.27), the leading coefficients satisfy the following equations:

$$\begin{aligned}
&(\sigma+1)a^\sigma_{\sigma+1} = \star d_\omega b^\sigma_\sigma + \star[e, a^\sigma_{\sigma+1}] + [(\phi^\sigma_y)_{\sigma+1}, e],\\
&\sigma b^\sigma_\sigma = -\star[e, b^\sigma_\sigma],\\
&(\sigma+1)(\phi^\sigma_y)_{\sigma+1} = d^\star_\omega b^\sigma_\sigma - \star[a^\sigma_{\sigma+1}, e].
\end{aligned} \tag{4.37}$$

The claim follows immediately. □

**Remark.** *Even by Proposition 4.3.5, the leading terms in the expansions of $a^\sigma, b^\sigma$ don't have "$\log y$" terms, the* log *terms might still appear in the rest terms of the expansion. By Proposition 4.2.5, the quadratic terms that come from the expansions of $a^1, b^1$ will contribute to the expansions of $a^\sigma, b^\sigma$.*

**Formal Expressions of Higher Order Terms**

In this section, we will give formal expressions of Higher order terms. We write the expansions of $a, b, \phi_y$ as

$$a \sim \sum a_{k,p} y^k (\log y)^p, \; b \sim \sum b_{k,p} y^k (\log y)^p, \; \phi_y \sim \sum b_{k,p} y^k (\log y)^p$$

and write $a^{\sigma}_{k,p} := \mathcal{P}_{\sigma} a_{k,p}, \; b^{\sigma}_{k,p} := \mathcal{P}_{\sigma} b_{k,p}$. For each order of $k$, there will be only a finite number of $p$ such that $a^{\sigma}_{k,p}, \; b^{\sigma}_{k,p}$ and $(\phi^{\sigma}_y)_{k,p}$ are non-vanishing. We obtain the following proposition:

**Proposition 4.3.6.** *For any integer $\sigma$, $p \geq 0$ and $k \geq \sigma + 1$, $a^{\sigma}_{k+1,p}, \; b^{\sigma}_{k,p}, \; (\phi^{\sigma}_y)_{k,p}$ are determined by $\{a^{\sigma_i}_{\sigma_i+1}, b^{\sigma_i}_{\sigma_i+1}, (\phi^{\sigma_i}_y)_{\sigma_i+1}\}$ for all possible integers $\sigma_i$ in the decomposition* (4.4).

*Proof.* Using (4.15), (4.27), consider the $O(y^k (\log y)^p)$ and $O(y^{k-1} (\log y)^p)$ order coefficients, and we obtain

$$\begin{aligned}
&(k+1) a^{\sigma}_{k+1,p} - \star[e, a^{\sigma}_{k+1,p}] = \star d_{\omega} b^{\sigma}_{k,p} - (p+1) a^{\sigma}_{k+1,p+1} + [(\phi^{\sigma}_y)_{k+1,p}, e] \\
&+ \mathcal{P}_{\sigma} \sum_{k_1+k_2=k, \; p_1+p_2=p} (\star[a_{k_1,p_1}, b_{k_2,p_2}] + [(\phi_y)_{k_1,p_1}, b_{k_2,p_2}]), \\
&k b^{\sigma}_{k,p} + \star[e, b^{\sigma}_{k,p}] = \star d_{\omega} a^{\sigma}_{k-1,p} + d_{\omega} (\phi^{\sigma}_y)_{k-1,p} - (p+1) b^{\sigma}_{k,p+1} \\
&+ \mathcal{P}_{\sigma} \sum_{k_1+k_2=k, \; p_1+p_2=p} \star(a_{k_1,p_1} \wedge a_{k_2,p_2} - b_{k_1,p_1} \wedge b_{k_2,p_2} + [a_{k_1,p_1}, (\phi_y)_{k_2,p_2}]), \\
&(k+1)(\phi^{\sigma}_y)_{k+1,p} = -(p+1)(\phi^{\sigma}_y)_{k+1,p+1} + (d^{\star}_{\omega} b^{\sigma})_{k,p} - \star[a^{\sigma}_{k+1,p}, \star e] \\
&- \mathcal{P}_{\sigma} \sum_{k_1+k_2=k, \; p_1+p_2=p} (\star[a_{k_1,p_1}, \star b_{k_2,p_2}]).
\end{aligned} \tag{4.38}$$

We can write the left hand side of the first equation as $-\mathcal{L}^{\sigma}_{-(k+1)}(a^{\sigma}_{k+1,p})$ and the left hand side of the second equation as $\mathcal{L}^{\sigma}_{k}(b^{\sigma}_{k,p})$.

When $k \geq \sigma + 1$, by Corollary 4.2.7 and Lemma 4.2.11, we see $a^{\sigma}_{k+1,p}, \; b^{\sigma}_{k,p}$ and $(\phi^{\sigma}_y)_{k+1,p}$ are uniquely determinant by an algebraic combination of $a^{\sigma}_{k+1,p+1}$, $(\phi^{\sigma}_y)_{k+1,p+1}$ and lower $y^k$ order terms. Inducting $k$ and $p$, we can complete the proof. □

For the $\log y$ terms appear in the expansions, we have the following Proposition:

**Proposition 4.3.7.** *If $b^1_{1,1} = 0$, then the expansion of $a, b$ don't have "$\log y$" terms. To be explicit, for any $p \neq 0$ and any $\sigma$, we have $a^{\sigma}_{k,p} = b^{\sigma}_{k,p} = 0$.*

*Proof.* By Proposition 4.3.4, if $b^1_{1,1} = 0$, we obtain $a^1_{2,1} = (\phi^1_y)_{2,1} = 0$. Combing this with Proposition 4.3.5, for any $\sigma$, $a^\sigma, b^\sigma$ will not contain "$\log y$" terms in their leading orders of the expansions. We will prove the proposition by induction. Suppose for $k$, for any $\sigma$, the expansions for $a^\sigma, b^\sigma$ up to order $O(y^{k+\frac{1}{2}})$ don't contains "$\log y$" terms. Let $r$ by the largest order that for some $\sigma$, $a^\sigma_{k+1,r}$ or $b^\sigma_{k+1,r}$ non-vanishing. By (4.38), as by our assumption $a^\sigma_{k+1,r+1} = b^\sigma_{k+1,r+1} = 0$, we obtain

$$\begin{aligned}
-\mathcal{L}^\sigma_{-(k+1)}(a^\sigma_{k+1,r}) &= \star\, d_\omega b^\sigma_{k,r} + [(\phi^\sigma_y)_{k+1,r}, e] \\
&\quad + \mathcal{P}_\sigma \sum_{k_1+k_2=k,\ p_1+p_2=p} (\star[a_{k_1,p_1}, b_{k_2,p_2}] + [(\phi_y)_{k_1,p_1}, b_{k_2,p_2}]), \\
\mathcal{L}^\sigma_{k+1}(b^\sigma_{k+1,r}) &= \star\, d_\omega a^\sigma_{k,r} \\
&\quad + \mathcal{P}_\sigma \sum_{k_1+k_2=k+1,\ p_1+p_2=r} \star(a_{k_1,p_1} \wedge a_{k_2,p_2} - b_{k_1,p_1} \wedge b_{k_2,p_2} + [a_{k_1,p_1}, (\phi_y)_{k_2,p_2}]) \\
(k+1)(\phi^\sigma_y)_{k+1,r} &= d^\sigma_\omega b^\sigma_{k,r} - \star[a^\sigma_{k,r}, \star e] - \mathcal{P}_\sigma \sum_{k_1+k_2=k+1,\ p_1+p_2=r} (\star[a_{k_1,p_1}, \star b_{k_2,p_2}])
\end{aligned} \tag{4.39}$$

By the induction assumption, for $r \neq 0$, $b^\sigma_{k,r} = 0$, $a^\sigma_{k,r} = 0$ and the quadratic term in the previous equations vanish. By Corollary 4.2.7 and Lemma 4.2.11, for $r \neq 0$, $a^\sigma_{k+1,r} = b^\sigma_{k+1,r} = (\phi^\sigma_y)_{k+1,r} = 0$. □

Now, we can complete the proof of the first two theorem in Chapter 1:

Proof of Theorem 1.1

*Proof.* The statement (1) follows from Proposition 4.3.2, 4.3.4, 4.3.5. The statement (2) follows from Proposition 4.3.6. □

Proof of Theorem 1.2

*Proof.* The statement follows from Proposition 4.3.4 and 4.3.7. □

**More Restrictions**

In this subsection, we will provide a geometry restriction of the coefficients of the expansions. By Theorem 1.1, we can assume $a, b$ have the expansions

$$a \sim a_{2,1} y^2 \log y + a_2 y^2 + \cdots,\ b \sim b_{1,1} y \log y + b_1 y + \cdots,\ \phi_y \sim (\phi_y)_{2,1} y^2 \log y + (\phi_y)_2 y^2 \tag{4.40}$$

The following identity over closed manifold comes from [34]. For any 4-manifold M with 3-manifold boundary Y, let $P$ be a principle $SU(2)$ bundle and $\mathfrak{g}_P$ its adjoint bundle, $A$ a connection on $P$ and $\Phi$ a $\mathfrak{g}_P$-valued 1-form, denote $I_+ := (F_A - \Phi\wedge\Phi + d_A\Phi)^+$ and $I_- := (F_A - \Phi\wedge\Phi + d_A\Phi)^-$, where $+(-)$ means the (anti-)self-dual parts of the two form over the 4-manifold $M$. We obtain:

**Proposition 4.3.8.**

$$\int_M \mathrm{Tr}(I_+^2 + I_-^2) = \int_M \mathrm{Tr}(F_A\wedge F_A) + \int_Y \mathrm{Tr}(\Phi\wedge d_A\Phi). \tag{4.41}$$

*Proof.* We compute

$$\int_M \mathrm{Tr}(I_+^2 + I_-^2) = \int_M \mathrm{Tr}(F_A^2 + (\Phi)^4 - 2F_A\wedge(\Phi)^2 - d_A\Phi\wedge d_A\Phi).$$

Integrating by parts, we obtain $\int_M \mathrm{Tr}(2F_A\wedge\Phi^2 + d_A\Phi\wedge d_A\Phi) = \int_Y \mathrm{Tr}(\Phi\wedge d_A\Phi)$. In addition, $\int_M \mathrm{Tr}(\Phi^4) = 0$. □

Denote $M = Y\times\mathbb{R}^+$, consider $(A,\Phi)$ a Nahm pole solution and convergence $C^\infty$ to a flat $G^{\mathbb{C}}$ connection at $y\to\infty$. Let $Y_y := Y\times\{y\}\subset Y\times\mathbb{R}^+$. By previous identity, we obtain the following:

$$\lim_{y\to 0}\int_{Y_y}\mathrm{Tr}(\Phi\wedge d_A\Phi) - \int_{Y_\infty}\mathrm{Tr}(\Phi\wedge d_A\Phi) + \int_M \mathrm{Tr}(F_A\wedge F_A) = 0. \tag{4.42}$$

In addition, recall the Chern-Simons functional of a connection $A$ over 3-manifold $Y$ is $C_P(A) := \int_Y \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$ and it satisfies $\int_M \mathrm{Tr}(F_A\wedge F_A) = \int_M dC_P(A) = C_P(A|_{Y_0}) - C_P(A|_{Y_{+\infty}})$.

By Proposition 4.3.1 and the assumption that $(A,\Phi)$ convergence to $G^{\mathbb{C}}$ flat connection at $y = +\infty$, we obtain that $\int_M \mathrm{Tr}(F_A\wedge F_A)$ is determined by the Levi-Civita connection and the limit flat connection. In particular, it is bounded and we denote $k := -\int_M \mathrm{Tr}(F_A\wedge F_A) + \int_{Y_\infty}\mathrm{Tr}(\Phi\wedge d_A\Phi)$. Combing this with (4.42), we have the following relationship:

$$k = \lim_{y\to 0}\int_{Y_y}\mathrm{Tr}(\Phi\wedge d_A\Phi), \tag{4.43}$$

where $k$ is a finite number.

We obtain the following proposition:

**Proposition 4.3.9.** *For $(A = \omega + a, \Phi = y^{-1}e + b)$, a polyhomogeneous solution with expansions as in (4.2), we have*

$$\lim_{y\to 0}\int_{Y_y}\mathrm{Tr}(\Phi\wedge d_A\Phi) = -\lim_{y\to 0}\int_{Y_y}\mathrm{Tr}(e\wedge\star y^{-1}\partial_y a). \tag{4.44}$$

*Proof.* As $\Phi$ doesn't have $dy$-component, under the temporal gauge, we compute:

$$\begin{aligned}\int_{Y_y} \mathrm{Tr}(\Phi \wedge d_A\Phi) &= \int_{Y_y} \mathrm{Tr}(\Phi \wedge \star_4(F_A - \Phi \wedge \Phi)) \\ &= -\int_{Y_y} \mathrm{Tr}(\Phi \wedge \star_4(\Phi^2)) + \int_{Y_y} \mathrm{Tr}(\Phi \wedge \star_4(F_A)),\end{aligned}$$

where $\star_4$ is the 4-dimensional Hodge star of $M$.

For the first term, when $y$ small, as $\phi_y \sim O(y^2 \log y)$, we compute

$$\begin{aligned}\int_{Y_y} \mathrm{Tr}(\Phi \wedge \star_4(\Phi^2)) &= \int_{Y_y} \mathrm{Tr}(\phi \wedge \star_4(\Phi^2)) + \int_{Y_y} \mathrm{Tr}(\phi_y dy \wedge \star_4(\Phi^2)) \\ &= \int_{Y_y} \mathrm{Tr}(\phi \wedge \star[\phi, \phi_y]) + o(1) \\ &= \log y \int_{Y_y} \mathrm{Tr}(e \wedge \star[e, (\phi_y)_{2,1}]) + \int \mathrm{Tr}(e \wedge \star[e, (\phi_y)_2]) + o(1).\end{aligned} \tag{4.45}$$

Using the identity $\star e = e \wedge e$ over $Y$, for any 0-form $c$, we have

$$\mathrm{Tr}(e \wedge \star[e, c]) = \mathrm{Tr}(e^3 c - e \wedge c \wedge e^2) = 0.$$

Thus, $\lim_{y\to 0} \int_{Y_y} \mathrm{Tr}(\Phi \wedge \star_4(\Phi^2)) = 0$.

For the other term, we have

$$\begin{aligned}\int_{Y_y} \mathrm{Tr}(\Phi \wedge \star_4 F_A) &= \int_{Y_y} \mathrm{Tr}(\Phi \wedge \star_4(d_\omega a)) \\ &= -\int_{Y_y} \mathrm{Tr}(\Phi \wedge \star(\partial_y a)) \\ &= -\int_{Y_y} \mathrm{Tr}(y^{-1} e \wedge \star(\partial_y a) + b \wedge \star(\partial_y a)) \\ &= -\int_{Y_y} \mathrm{Tr}(y^{-1} e \wedge \star(\partial_y a)) + o(1).\end{aligned}$$

The last equality is because $b \sim O(y(\log y)^p)$ and $\partial_y a \in O((\log y)^p)$ for some $p$. □

We have the following corollary:

**Corollary 4.3.10.** *For the polyhomogeneous solutions* $(A, \Phi)$*, we obtain:*

*(1)* $\int_{Y_0} \mathrm{Tr}(e \wedge \star a_{2,1}) = 0$*,*

*(2)* $k = 2\int_{Y_0} \mathrm{Tr}(e \wedge \star a_2)$*, where* $Y_0$ *is* $Y \times \{0\} \subset Y \times \mathbb{R}^+$*.*

*Proof.* By previous computation, the non-vainishing terms of $\mathrm{Tr}(y^{-1}e \wedge \star\partial_y a)$ will be

$$\mathrm{Tr}(2\log y e \wedge \star a_{2,1} + e \wedge \star a_{2,1} + 2e \wedge \star a_2).$$

By the polyhomogeneous assumption, we know $a_{2,p} \in C^\infty(Y)$. Combine this with (4.44), we know all singular terms should vanish. Thus $\int_{Y_0} \mathrm{Tr}(e \wedge \star a_{2,1}) = 0$ The only remaining term that contributes to the integral is $\int_{Y_0} \mathrm{Tr}(e \wedge \star a_2)$, which verifies the statement (2). □

## 4.4 The Expansions When $G = SO(3)$ or $SU(2)$

**Formula Expansions**

In this section, we will determine all the rest terms in the expansion of a Nahm pole solution when $G = SU(2)$ or $SO(3)$. When $G = SU(2)$ or $SO(3)$, in the notation of (4.4), we only have $\sigma = 1$ and $\tau_1$ is the only irreducible module. Proposition 4.3.2, 4.3.4 still works for this case. We will give a proof Theorem 1.4 by induction.

For $k \geq 1$, suppose $(a, b, \phi_y)$ satisfies (4.14) and has the following expansions:

$$\begin{aligned} a &\sim \sum_{i=1}^{k}\sum_{p=0}^{i} a_{2i,p}y^{2i}(\log y)^p + \sum_{p=0}^{r_{2k+1}} a_{2k+1,p}y^{2k+1}(\log y)^p + \cdots, \\ b &\sim \sum_{i=1}^{k}\sum_{p=0}^{i} b_{2i-1,p}y^{2i-1}(\log y)^p + \sum_{p=0}^{r_{2k+1}} b_{2k+1,p}y^{2k+1}(\log y)^p + \cdots, \\ \phi_y &\sim \sum_{i=1}^{k}\sum_{p=0}^{i}(\phi_y)_{2i,p}y^{2i}(\log y)^p + \sum_{p=0}^{r_{2k+1}}(\phi_y)_{2k+1,p}y^{2k+1}(\log y)^p + \cdots \end{aligned} \tag{4.46}$$

where "$\cdots$" means the higher order terms. In addition, we denote

$$\begin{aligned} &\mathcal{A}_{2k} := \sum_{i=1}^{k}\sum_{p=0}^{i} a_{2i,p}y^{2i}(\log y)^p, \ \mathcal{B}_{2k} := \sum_{i=1}^{k}\sum_{p=0}^{i} b_{2i-1,p}y^{2i-1}(\log y)^p, \\ &C_{2k} := \sum_{i=1}^{k}\sum_{p=0}^{i}(\phi_y)_{2i,p}y^{2i}(\log y)^p, \end{aligned} \tag{4.47}$$

which are terms in the expansions of $a$ and $b$ which vanish slower than $O(y^{2k+\frac{1}{2}})$.

Recall for any 1-form $\alpha$, we define $\mathcal{L}_{2k+1}(\alpha) = (2k+1)\alpha + \star[e, \alpha]$ and obtain the following proposition:

**Proposition 4.4.1.** *Assume $a, b$ have the expansions in* (4.46)*, let $p, s$ be non-negative integers, and we have:*

*(1)For any p,* $a_{2k+1,p} = (\phi_y)_{2k+1,p} = 0$ *and for integer* $s > k+1$*,* $b_{2k+1,s} = 0$*,*

*(2)For* $k+1 \geq s \geq 0$*,* $b_{2k+1,s} = \mathcal{L}_{2k+1}^{-1}(\Theta_{2k+1}^{s} - (s+1)b_{2k+1,s+1})$*, where* $\Theta_{2k+1}^{s}$ *depends on* $\mathcal{A}_{2k}$ *and* $\mathcal{B}_{2k}$*. If* $\mathcal{A}_{2k}$ *and* $\mathcal{B}_{2k}$ *don't have log terms, for* $s \geq 1$*, we obtain* $b_{2k+1,s} = 0$*.*

*Proof.* Consider the first equation in (4.14), consider the expansion of order $O(y^{2k}(\log y)^{r_{2k+1}})$, the quadratic term $\star[a,b]$ will not contribute as $\mathcal{A}_{2k},\ C_{2k}$ only contains even order terms and $\mathcal{B}_{2k}$ only contains odd order terms. Thus, we obtain

$$\begin{aligned}(2k+1)a_{2k+1,r_{2k+1}} &= \star[e, a_{2k+1,r_{2k+1}}] + [(\phi_y)_{2k+1,r_{2k+1}}, e],\\ (2k+1)(\phi_y)_{2k+1,r_{2k+1}} &= -\Gamma a_{2k+1,r_{2k+1}}.\end{aligned} \tag{4.48}$$

As $k \geq 1$, by Proposition 4.2.6 and Lemma 4.2.11, we obtain $a_{2k+1,r_{2k+1}} = 0$. By induction, we obtain $a_{2k+1,p} = 0$ for any $p$.

For the second equations of (4.14), let $s, s_1, s_2$ be non-negative integers, we define

$$\begin{aligned}\Theta_{2k+1}^{s} :=& \star d_\omega a_{2k,s} + d_\omega(\phi_y)_{2k,s} + \sum_{l=1}^{k}\sum_{s_1+s_2=s}\star a_{2l,s_1}a_{2k-2l,s_2}\\ &- \sum_{l=1}^{k}\sum_{s_1+s_2=s}\star b_{2l-1,s_1}b_{2k-2l-1,s_2} + \sum_{l=1}^{k}\sum_{s_1+s_2=s}\star[a_{2l,s_1},(\phi_y)_{2k-2l,s_2}],\end{aligned} \tag{4.49}$$

where $a_{2k,s},\ b_{2k,s}$ are understood as zero if it don't appears in the coefficients of $\mathcal{A}_{2k}$ and $\mathcal{B}_{2k}$.

Consider the coefficients of order $O(r^{2k}(\log y)^{s})$, and we obtain

$$(2k+1)b_{2k+1,s} = -\star[e, b_{2k+1,s}] + \Theta_{2k+1}^{s} - (s+1)b_{2k+1,s+1}. \tag{4.50}$$

Thus, as $k \geq 1$, we obtain $b_{2k+1,s} = \mathcal{L}_{2k+1}^{-1}(\Theta_{2k+1}^{s} - (s+1)b_{2k+1,s+1})$. Suppose $s = r_{2k+1}$ and $r_{2k+1} > k+1$, then $\Theta_{2k+1}^{r_{2k+1}} = 0$. Thus, $b_{2k+1,r_{2k+1}} = 0$ for $r_{2k+1} > k+1$.

If $\mathcal{A}_{2k}, \mathcal{B}_{2k}$ don't contain $log$ terms and $s \neq 0$, we obtain $\Theta_{2k+1}^{s} = 0$. By induction of $s$, this proves the last claim. □

There is another type of expansions we need to consider: for $k \geq 1$, suppose $(a, b)$

has the following expansions:

$$
\begin{aligned}
a &\sim \sum_{i=1}^{k}\sum_{p=0}^{i} a_{2i,p}y^{2i}(\log y)^p + \sum_{p=0}^{r_{2k+2}} a_{2k+2,p}y^{2k+2}(\log y)^p + \cdots,\\
b &\sim \sum_{i=1}^{k+1}\sum_{p=0}^{i} b_{2i-1,p}y^{2i-1}(\log y)^p + \sum_{p=0}^{r_{2k+2}} b_{2k+2,p}y^{2k+2}(\log y)^p + \cdots,\\
\phi_y &\sim \sum_{i=1}^{k}\sum_{p=0}^{i} (\phi_y)_{2i,p}y^{2i}(\log y)^p + \sum_{p=0}^{r_{2k+2}} (\phi_y)_{2k+2,p}y^{2k+2}(\log y)^p + \cdots,
\end{aligned}
\tag{4.51}
$$

where "$\cdots$" means the higher order terms. Similarly, we define

$$
\begin{aligned}
&\mathcal{A}_{2k+1} := \sum_{i=1}^{k}\sum_{p=0}^{i} a_{2i,p}y^{2i}(\log y)^p,\ \mathcal{B}_{2k+1} := \sum_{i=1}^{k+1}\sum_{p=0}^{i} b_{2i-1,p}y^{2i-1}(\log y)^p,\\
&C_{2k+1} := \sum_{i=1}^{k}\sum_{p=0}^{i} (\phi_y)_{2i,p}y^{2i}(\log y)^p.
\end{aligned}
\tag{4.52}
$$

We obtain the following proposition:

**Proposition 4.4.2.** *Assume $a, b, \phi_y$ have the expansions in* (4.51)*, let $p, s$ be non-negative integers, and we have:*

*(1)For any $p$, $b_{2k+2,p} = 0$ and for integer $s \geq k+2$, $a_{2k+2,s} = (\phi_y)_{2k+2,s} = 0$,*

*(2)For $k+1 \geq s \geq 0$, we can write*

$$
\begin{aligned}
a^+_{2k+2,s} &= \frac{1}{2k+3}((s+1)a^+_{2k+2,s+1} + (\Theta^s_{2k+2})^+),\\
a^-_{2k+2,s} &= \frac{1}{2k}((2+1)a^-_{2k+2,s+1} + (\Theta^2_{2k+2})^-),\\
a^0_{2k+2,s} &= \frac{1}{4k^2+6k}((2k+2)((s+1)a_{2k+2,s+1} + \Theta^s_{2k+2})) - [e, \Xi^s_{2k+2} - (s+1)(\phi_y)_{2k+2,s+1}],\\
(\phi_y)_{2k+2,s} &= \frac{1}{4k^2+6k}((2k+1)(\Xi^s_{2k+2} - (s+1)(\phi_y)_{2k+2,s+1}) - \Gamma((s+1)a_{2k+2,s+1} + \Theta^s_{2k+2})),
\end{aligned}
\tag{4.53}
$$

*$a_{2k+2,s} = -\mathcal{L}^{-1}_{-(2k+2)}(\Theta^s_{2k+2} - (s+1)a_{2k+2,s+1})$, where $\Theta^s_{2k+2}, \Xi_{2k+2}$ depend on $\mathcal{A}_{2k+1}$, $\mathcal{B}_{2k+1}$ and $C_{2k+1}$. If $\mathcal{A}_{2k+1}$ and $\mathcal{B}_{2k+1}$ don't have $\log y$ terms, for $s \geq 1$, we obtain $b_{2k+2,s} = 0$.*

*Proof.* Consider the $O(y^{2k+1}(\log y)^{r_{2k+2}})$ terms of the second equations of (4.14). As $\mathcal{A}_{2k+1}, C_{2k+1}$ only contains even order expansions and $\mathcal{B}_{2k+1}$ only contains odd order expansions, the quadratic terms doesn't contribute and we obtain $(2k+1)b_{2k+2,r_{2k+2}} + \star[e, b_{2k+2,r_{2k+2}}]=0$, which implies $b_{2k+2,r_{2k+2}} = 0$. By induction, we obtain for any $p$, $b_{2k+2,p} = 0$.

For a non-negative integer $s$, write

$$\begin{aligned}\Theta^s_{2k+2} &:= \sum_{l=1}^{k}\sum_{s_1+s_2=s} \star[a_{2l,s_1}, b_{2k+1-2l,s_2}] + \star[(\phi_y)_{2l,s_1}, b_{2k+1-2l,2s}] + \star d_\omega b^s_{2k+1},\\ \Xi^s_{2k+2} &:= \sum_{l=1}^{k}\sum_{s_1+s_2=s} \star[a_{2l,s_1}, \star b_{2k+1-2l,s_2}].\end{aligned} \tag{4.54}$$

We compute the $O(y^{2k+1}(\log y)^s)$ coefficients and obtain

$$\begin{aligned}&(2k+2)a_{2k+2,s} + (s+1)a_{2k+2,s+1} = \Theta^s_{2k+2} + \star[e, a_{2k+2,s}] + [(\phi_y)_{2k+2,s}, e],\\ &(2k+2)(\phi_y)_{2k+2,s} + (s+1)(\phi_y)_{2k+2,s+1} = -\Gamma a_{2k+2,s} + \Xi^s_{2k+2}.\end{aligned} \tag{4.55}$$

As $k \geq 1$, applying Lemma 4.2.11, we obtain (4.53).

Suppose $s = r_{2k+1}$ and $r_{2k+1} > k+2$, then $\Theta^{r_{2k+2}}_{2k+2} = \Xi^{r_{2k+2}}_{2k+2} = 0$. Also by the assumption of the expansion, we have $a_{2k+2,s+1} = (\phi_y)_{2k+2,s+1} = 0$. Thus, $a_{2k+2,r_{2k+2}} = a_{2k+2,r_{2k+2}} = 0$ for $r_{2k+2} \geq k+2$. If $\mathcal{A}_{2k+1}, \mathcal{B}_{2k+1}$ don't contain $log$ terms and $s \neq 0$, we obtain $\Theta^s_{2k+2} = 0$. By induction of $s$, this proves the last claim. □

Now, we will give a proof for Theorem 1.4:

Proof of Theorem 1.4: By Proposition 4.3.2, 4.3.4, 4.4.1, 4.4.2, the result follows immediately. □

**Examples**

Now, we will introduce some known results that verifies our theorem:

**Example 4.4.3.** *[27] Nahm pole solutions on $S^3 \times \mathbb{R}^+$. Equip $S^3$ with the round metric and take $\omega$ be Maurer–Cartan 1-form of $S^3$ and $\omega$ satisfies the following relation $d\omega = -2\omega\wedge\omega$ and $\star\omega = \omega\wedge\omega$. Denote $y$ the coordinate of $\mathbb{R}^+$ and denote*

$$(A,\Phi) = \left(\frac{6e^{2y}}{e^{4y}+4e^{2y}+1}\omega, \frac{6(e^{2y}+1)e^{2y}}{(e^{4y}+4e^{2y}+1)(e^{2y}-1)}\omega\right), \tag{4.56}$$

*[27, Theorem 6.2] shows that $(A,\Phi)$ is a Nahm-Pole solution to the Kapustin-Witten equations. In addition, the solutions (4.56) will converge to the unique flat $SL(2;\mathbb{C})$ connection in the cylindrical end of $S^3 \times \mathbb{R}^+$.*

*The expansions of this solution along $y \to 0$ will be*

$$A \sim (1 - \frac{2}{3}y^2 + \frac{2}{9}y^4 - \frac{4}{135}y^6 + \cdots)\omega, \ \Phi \sim (\frac{1}{y} - \frac{1}{3}y - \frac{1}{45}y^3 + \frac{58}{945}y^5 + \cdots)\omega. \tag{4.57}$$

**Example 4.4.4.** *[37] Nahm pole solutions on $Y \times \mathbb{R}^+$ where $Y$ is any hyperbolic three manifold.*

*Let $Y$ be a hyperbolic three manifold equipped with the hyperbolic metric $h$. Consider the associated $PSL(2;\mathbb{C})$ representation of $\pi_1(Y)$, this lifts to $SL(2;\mathbb{C})$ and determines a flat $SL(2;\mathbb{C})$ connection $\nabla^{flat}$. Denote by $\nabla^{lc}$ the Levi-Civita connection and by $A^{lc}$ the connection form. Take $i\omega := \nabla^{flat} - \nabla^{lc}$. Then locally, $\omega = \sum \mathfrak{t}_i e_i^\star$ where $\{e_i^\star\}$ is an orthogonal basis of $T^\star Y$ and $\{\mathfrak{t}_a\}$ are sections of the adjoint bundle $\mathfrak{g}_P$ with the relation $[\mathfrak{t}_a, \mathfrak{t}_b] = 2\epsilon_{abc}\mathfrak{t}_c$. We also have $\star_Y \omega = F_{\nabla^{lc}}$. Therefore, by the Bianchi identity, we obtain $\nabla^{lc}(\star_Y \omega) = 0$. Combining $F_{flat} = 0$ and the relation $\nabla^{flat} - \nabla^{lc} = i\omega$, we obtain $F_{\nabla^{lc}+i\omega} = 0$. Hence $F_{lc} = \omega \wedge \omega$, $\nabla^{lc}\omega = 0$.*

*Take $y$ to be the coordinate of $\mathbb{R}^+$ in $Y^3 \times \mathbb{R}^+$, now set $f(y) := \frac{e^{2y}+1}{e^{2y}-1}$, and take $(A, \Phi) = (A^{lc}, f(y)\omega)$. We refer [28, Section 2.3] for a record of proof in [37] that this is a Nahm pole solution to the Kapustin-Witten equations.*

*As $A^{lc}$ is independent of $y$, the solution has the following expansions:*

$$A \sim A^{lc}, \ \Phi \sim (\frac{1}{y} + \frac{1}{3}y - \frac{1}{45}y^3 + \frac{2}{945}y^5 + \cdots)\omega. \tag{4.58}$$

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