ON THE RECONSTRUCTION PROBLEM IN GRAPH THEORY

Thesis by

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Abstract

The thesis consists of three chapters. The first chapter introduces the basic notions of graph theory and defines vertex-reconstruction and edge-reconstruction problem. The second chapter and third chapter are devoted to the edge-reconstruction of bi-degreed graphs and bipartite graphs respectively.

A bi-degreed graph G is a graph with two degrees $d > \delta$. By elementary arguments we can assume $d = \delta + 1$ and there are at least two vertices of degree δ. Call vertices of degree d "big" vertex and degree δ "small" vertex. Define "symmetric" path of length p S_p to be one with both ends small vertices and all other internal vertices big vertices; define "asymmetric" path of length p A_p to be one with one end a small vertex and all others big vertices. If $s(G)$ is the minimum distance between two small vertices in G, we can show that $s(G)$ is "independent" of G (i.e. it is edge-reconstructable), and that G has at most one nonisomorphic edge-reconstruction H. From this, the concept of "forced move" posed by Dr. Swart is obvious. Using the principle of forced move (and sometimes also "forced edge" posed by Dr. Swart as well), it's easy to derive a few interesting properties, like say G is edge-reconstructable if $s(G)$ is even or if two $S_{s(G)}$'s intersect at an internal vertex, etc. Write s for $s(G)$. When s is odd, consider the concept of s - n-chain, which is n S_s's following from end to end. We can show first s - 3-chain and then s - 2-chain cannot exist. Hence all S_s's are disjoint. Think of S_s's as "lines" in

some geometry. Define two more "distance" functions s_1 and s_2 such that s_1 "represents" the distance from a point to a line and s_2 means the distance between two "skew" lines. With the aid of forced move principle again, we can at last prove every bi-degreed graph with at least four edges is edge-reconstructable.

A bipartite graph G is a graph whose vertex set V is the disjoint union of two sets V_1 and V_2 such that every edge joins V_1 and V_2. By elementary reduction we can assume G to be connected. We define special chains inductively so that it starts at a vertex of minimum degree and always goes to a neighbor or minimum degree. Special chains will be the main tool to prove edge-reconstructability. By G's finiteness, we note they will "terminate" somewhere,and we have three types of termination for them. Let condition A's be that degree sequence of special chain is edge-reconstructable, condition B_i's be that number of special chains is edge-reconstructable (and some more general variations); condition P's be that the "last vertices" of two special chains be not adjacent; we can prove that all A, B_i and P's should hold inductively in an interlocked way. (This is a big task). Then condition P's can be used to prove G's edge-reconstructability for all three types of termination. We can then prove every bipartite graph with at least four edges is edge-reconstructable.

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CHAPTER 1. Reconstruction problem of graph theory, problem definition, fundamentals, and surveys.

Section 1. Graph theory terminologies.

In this thesis, graph theory notations will be principally those of F. Harary [7] unless otherwise mentioned. Fortunately, the notations do not differ too much in literature. (To name a few of graph theory textbooks, see M. Behzad and G. Chartrand [1], C. Berge [2], O. Ore [15], N. Deo [6], etc.)

A graph G consists of a finite nonempty set (vertex set) $V = V(G)$ of p *vertices* together with a prescribed set $E(G)$ (edge set) of q *unordered pairs* of distinct vertices of V. Each pair $e = \{u,v\}$ of vertices in $E(G)$ is an *edge* of G, and e is said to *join* u and v. We write $e = uv$ (or vu equivalently) and say that u and v are *adjacent* vertices (vertex u and edge $e = uv$ are said to be *incident* with each other, as are v and uv. If two distinct edges e and f are incident with each other, they are *adjacent* edges).

It is customary to represent a graph by means of a diagram. The diagram in Fig. 1-1 represents a graph G with $V(G) = \{a,b,c,d,e\}$ and $E(G) = \{ab, bc, ce, be, bd, de\}$.

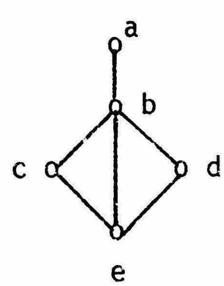


Fig. 1-1

In this graph, we have say b,e are adjacent vertices, but c,d are not.

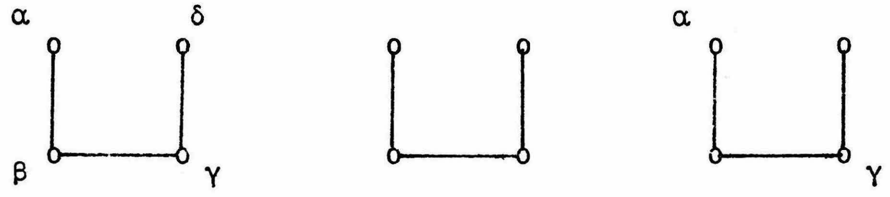
There are several variations of graphs which deserve mention. Note that the definition of a graph permits no *loop,* that is no edge joining a vertex to itself. In a *multigraph,* no loops are allowed but more than one edge can join two vertices. (They are called *multiple edges.)* If both loops and multiple edges are allowed, we have a *pseudograph.* To discriminate, graphs (in the more general sense) without loops or multiple edges will be called *simple* graphs.

A *directed graph* or *digraph* G consists of a finite nonempty set V = V(G) of vertices together with a prescribed set E(G) of *ordered pairs* of distinct vertices. The elements of E(G) are called *directed edges* or *arcs.* (By definition, a digraph is simple, i.e. it contains no loops or multiple arcs.) Our original definition of graphs with edges unordered pairs of distinct vertices will be called *undirected* graphs.

An *infinite* graph G consists of an infinite set V = V(G) or vertices together with a prescribed set E(G) of unordered pairs of distinct vertices. (By definition, an infinite graph is simple and undirected). It is possible that a vertex of G be adjacent to infinitely many vertices (it's easy to construct such an infinite graph, say let Z be the set of all integers, and join an edge for any two distinct integer). If every vertex of (an infinite graph) G is adjacent to only a finite number of vertices, G is called *locally finite.* A graph G with a finite nonempty vertex set V(G) will then be called a *finite* graph.

With the introduction of these various notations, the *graph* defined originally (as in Fig. 1-1) will be a *finite simple undirected* graph for clarity. From now on (and in whole of Chapter 2 and Chapter 3 following), graphs will mean finite simple undirected graphs unless otherwise mentioned.

A graph G is *labeled* (or *vertex-labeled*) if its p vertices are associated with p distinct labels (or names) in a one-to-one manner. A graph G is *unlabeled* if we do not have names for its vertices. A graph G is *partly labeled* if some vertices are labeled and some are not. In Fig. 1-2 below we have the same graph G which is labeled in (a), unlabeled in (b), and partly labeled in (c).



(a) Labeled (b) Unlabeled (c) Partly labeled

Fig. 1-2

We can define *edge-labeled* graphs, *edge-unlabeled* graphs and *partly edge-labeled* graphs in an analogous way.

Two graphs G and H are *isomorphic*, denoted by $G \cong H$, if there exists a one-to-one mapping σ (called an *isomorphism*), from V(G) onto V(H) such that adjacency (and so unadjacency as well) is preserved; i.e. $uv \in E(G)$ if and only if $\sigma(u)\sigma(v) \in E(H)$. The relation "isomorphic to" is easily seen to be an equivalence relation on graphs. We will call H an *isomorph* of G (and vice versa) if G and H are isomorphic. Two isomorphic graphs are considered to be the same graph in a natural way. Two graphs G and H are *non-isomorphic* (and considered as different) if they are not isomorphic; denoted by $G \not\cong H$.

A necessary condition that two (finite) graphs G and H are isomorphic is that they have the same number p of vertices and same number q of edges. Conversely, given two graphs G and H both with p vertices and q edges ($q \leq \binom{p}{2} = \frac{p(p-1)}{2}$ by simple argument), we know that after *finite number* of steps, we can determine if G and H are isomorphic, for say $p!$, the number of permutations of the p vertices would suffice. However, since $p!$ grows very fast, the general problem of determining if two graphs are isomorphic (by an algorithm or not) is convincingly very hard. In Fig. 1-3 below we give three graphs G, H and I with $G \cong H$, $G \not\cong I$.

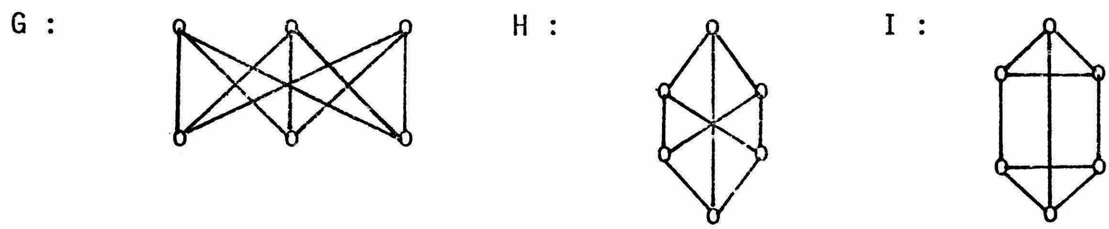


Fig. 1-3

All three graphs here are unlabeled. It is easy to see that G and I are nonisomorphic since I contains a "triangle" (a configuration of three adjacent vertices) but G doesn't.

The isomorphism of G and H is hard by "inspection" only. It would be much easier if we label the graphs as in Fig. 1-4 following:

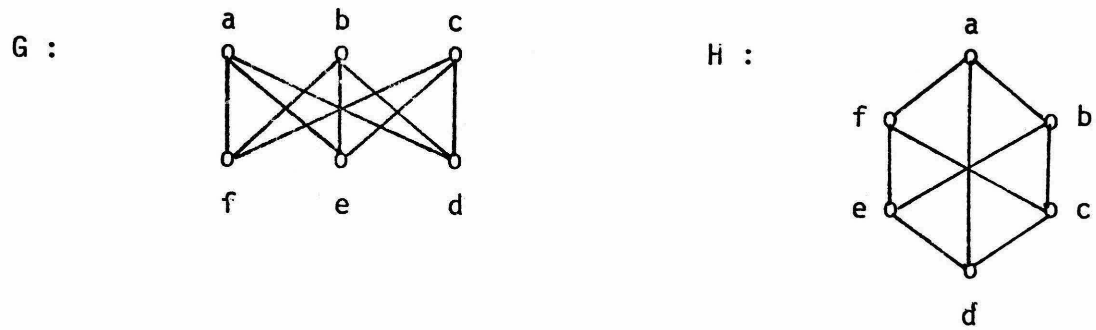
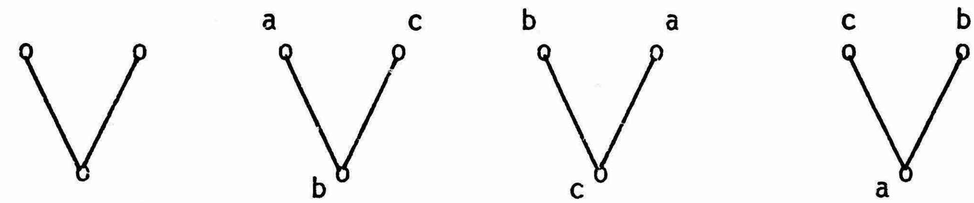


Fig. 1-4

The mapping σ from $V(G)$ to $V(H)$ defined by $\sigma(a) = a$, $\sigma(b) = c$, $\sigma(c) = e$, $\sigma(d) = b$, $\sigma(e) = d$, $\sigma(f) = f$ can be verified to be an isomorphism. This induces a concept called *label-isomorphism*. Given two graphs G and H with same number p of vertices and q of edges; and suppose $u_1, \ldots, u_p$ are labels used to label both graphs, then a one-to-one mapping σ which preserves adjacency (hence an isomorphism) from $V(G)$ onto $V(H)$ is a *label-isomorphism* if $\sigma(u_i) = u_i$, $1 \leq i \leq p$. Denote this by $G \cong_{\ell} H$. It is clear that if two unlabeled graphs are isomorphic, then they are label-isomorphic by some appropriate labelings (although the labeling might be very hard to find). Conversely, if two labeled graphs are label-isomorphic, then their corresponding unlabeled graphs (obtained by "erasing" the labels) are isomorphic. It is conceivable that two labeled graphs may be non-label-isomorphic with the corresponding unlabeled graphs isomorphic however. Label isomorphism (equivalence) classes is then a finer partition of isomorphism classes of graphs. In Fig. 1-5 below we see there are one up to *isomorphism* and three up to *label isomorphism* graphs of three vertices and two edges.

G:



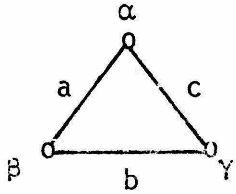
1 unlabeled 3 labeled

Fig. 1-5

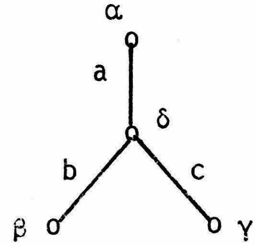
Note that we have six ways to label G by a, b, c and only three label-isomorphism classes (and only one isomorphism class). On p.4 of M. Behzad and G. Chartrand [1], twenty non-label-isomorphic graphs of 4 vertices and 3 edges are shown; among them there are only three isomorphism classes.

We introduce a concept dual to that of isomorphism (or vertex-isomorphism). Two nonempty graphs G and H are *edge-isomorphic*, denoted by $G \cong_e H$, if there exists a one-to-one mapping σ; from E(G) to E(H), such that two edges e and f are adjacent in G if and only if the edges $\sigma(e)$ and $\sigma(f)$ are adjacent in H. (Edge-isomorphism preserves adjacency of edges just as isomorphism preserves adjacency of vertices). However, the roles of edge-isomorphism and isomorphism are not "equal" as hinted by "duality". We see trivially that isomorphic graphs are edge-isomorphic but the converse does not necessarily hold as evidenced by the following nonisomorphic pairs G and H:

G :



H :



Since the edges a, b, c are pairwise adjacent in G and in H, any permutation σ of $\{a,b,c\}$ is an edge-isomorphism, which however cannot "induce" a vertex isomorphism in a natural way since G has 3 but H has 4 vertices. *Edge-isomorphisms* are thus a less natural concept than *isomorphisms*.

So far we have defined *graphs, adjacency, labels,* and *isomorphisms*. Next we will define the important notion of *subgraphs*. A *subgraph* H of G is a (finite, simple, undirected) graph having all its vertices and edges in G, i.e. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given a set S of vertices $\subseteq V(G)$, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S. A subgraph H of G is *vertex-induced* or *induced* if $H = \langle S \rangle$ for some $S \subseteq V(G)$; H is *edge-induced* if $H = \langle F \rangle_e$ for some $F \subseteq E(G)$ and $\langle F \rangle_e$ defined to be the graph whose vertex set consists of those vertices of G incident with at least one edge of F and whose edge set is F. Note that a subgraph need not be vertex-induced or edge-induced.

The *removal of a vertex* v from a graph G results in that subgraph G - v of G consisting of all vertices of G except v and all edges not incident with v. G - v is thus the (vertex-)induced subgraph on $V(G) - \{v\}$. The *removal of an edge* e from a graph G results in that subgraph G - e of G consisting of all vertices of G and all edges except e. G - e is a so called *spanning subgraph,* i.e. it contains all vertices of G; it is an edge-induced graph, and it is maximal with respect to the property of not containing e. The removal of a set of vertices or edges from G is defined by the removal of single elements in succession.

Now we will define a "reverse" operation. The *addition of edge* uv to a graph G where u and v are nonadjacent results in the graph $G + uv$ with the same vertex set and same set of edges with the addition of an edge uv . $G + uv$ is a "supergraph" of G, i.e. G is a subgraph of it. The *addition of a vertex* $x \in V(G)$ results in a graph $G + x$ whose vertex set is the union of $V(G)$ and $\{x\}$ and its edge set is, in addition to those in $E(G)$, all edges of the form xv, $v \in V(G)$. Starting from G, we can define a graph H "recursively" by means of series of additions and/or removals of edges or vertices. For example $G - u - v + wx - za$ may be meaningful. Note further that these operations "commute", say $G - ab + cd = G + cd - ab$.

At this early stage, we are able to state a famous longstanding foremost conjecture in graph theory (since 1941):

Ulam's (reconstruction) conjecture. Let G have p vertices u_i and H p vertices v_i with $p \geq 3$. If $G - u_i \cong H - v_i$ for each i, then $G \cong H$. (see S. Ulam [19]).

This conjecture says that the vertex deleted maximal subgraphs uniquely determines a graph with at least three vertices. This conjecture is false when G has only two vertices. For if G is the graph of two vertices u_1, u_2 and one edge u_1u_2 and H is the graph of two vertices v_1, v_2 without any edge, then $G - u_1 \cong H - v_1$, $G - u_2 \cong H - v_2$ since they are all graphs with one vertex only (and hence no edges at all), which is called *trivial* and denoted by K_1. But $G \not\cong H$, for u_1u_2 areadjacent in G but v_1v_2 aren't in H.

F. Harary reformulated Ulam's Conjecture in the following way:

First come some definitions. A *reconstruction* of a graph G is a graph H such that V(H) = V(G) and $H - v \cong G - v$ for all $v \in V(G)$. G is *reconstructable* (or notationally equivalently *reconstructible*) if every reconstruction of G is isomorphic to G.

(Vertex)-Reconstruction Conjecture (reformulated by F. Harary). Any graph with at least three vertices is reconstructable.

A word of comment. Though G may be *labeled* when we find G - v's, all G - v's are *unlabeled*, otherwise there is no problem.

To get feeling for this problem, it is sometimes helpful to imagine a "deck" of cards on which the vertex-deleted subgraphs of G are drawn, but unlabeled. Presented with such a deck, it is routine to find some graph which produced that deck. The problem confronting the reconstructor is however more demanding. He must show that, regardless of the algorithm used, one necessarily ends up with the same graph.

A good way to know how the reconstruction problem looks is to try reconstructing the graph G in Fig. 1-6 (i.e. finding an algorithm and show there is only one solution).

Note G is labeled but all G - v are unlabeled in Fig. 1-6.

The full generality of (Vertex)-Reconstruction Conjecture seeming intractable anyway, F. Harary later posed the conceptually easier edge-version of Vertex-Reconstruction Conjecture, the *Edge-Reconstruction Conjecture*.

An *edge-reconstruction* of a graph G is a graph H such that E(H) = E(G) and $H - e \cong G - e$ for all $e \in E(G)$. Note that edge-reconstruction is *not* a verbatim reformulation of vertex-reconstruction,

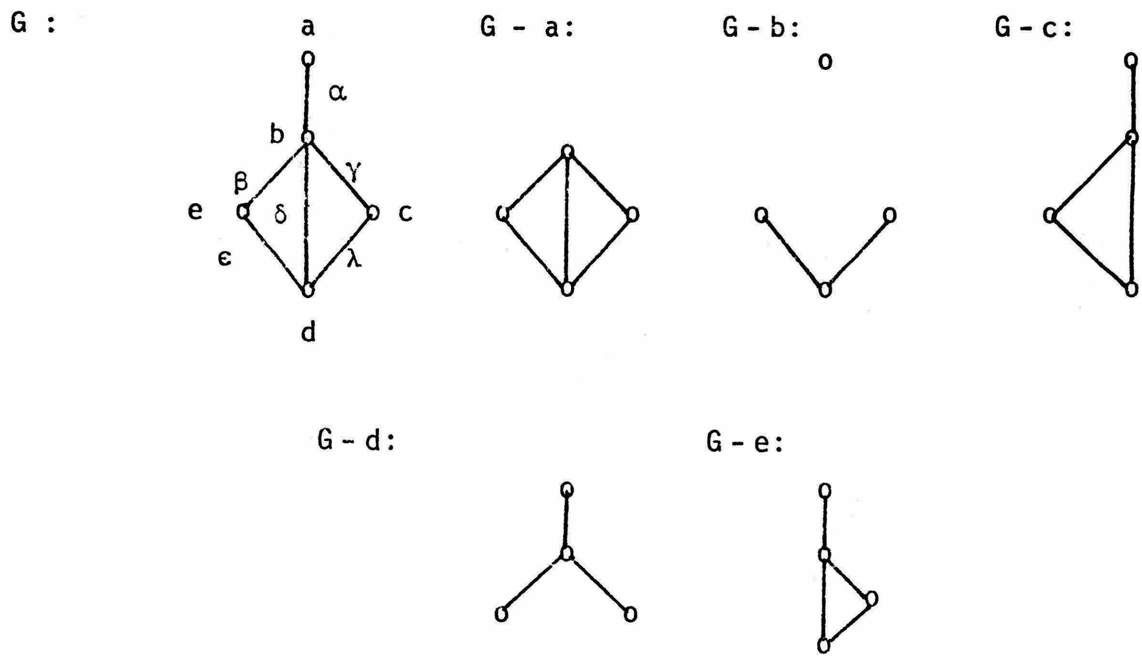


Fig. 1-6

because H is a *vertex-reconstruction* of G if every *vertex-deleted* (maximal) subgraph is *vertex-isomorphic;* while H is an *edge-reconstruction* if every *edge-deleted* (maximal) subgraph is *vertex-isomorphic* (not *edge-isomorphic*!)

A graph G is *edge-reconstructable* if all its edge-reconstructions are isomorphic to G.

Edge-Reconstruction Conjecture. Every graph with at least four edges is edge-reconstructable.

There are two non-edge-reconstructable pairs with two edges and three edges respectively as shown in Fig. 1-7.

To test the muscle on Edge-Reconstruction Problem, the graph G in Fig. 1-6 is good again.

(a) (b)

Fig. 1-7

Though edge-reconstruction problem seems much easier, and more progress has been made, the solved cases are mainly on graphs with simpler topological structures or graphs with "many" edges (compared with number of vertices). Chapter 2 and Chapter 3 of this thesis presents edge-reconstruction of *bi-degreed graphs* and *bipartite graphs* (defined later) with discussions mainly on *degrees* (i.e. the number of edges incident with each vertex), but not too much on *topology*. We will come back to this topic in Section 2.

Let's continue the definitions and terminologies. A *walk* of a graph G is an alternating sequence of vertices and edges v_0, x_1, v_1, ..., v_{n-1}, x_n, v_n beginning and ending with vertices in which each edge is incident with the two vertices immediately preceding and following it. This walk joins v_0 and v_n and may be denoted naturally as $v_0v_1 \ldots v_n$ (edges being evident by context); and called a v_0 - v_n walk. It is *closed* if $v_0 = v_n$ and *open* otherwise. It is a *trail* if all edges are distinct, a *path* if all vertices (and hence all the edges) are distinct. It is a *cycle* if it is closed, all its n vertices are distinct, and $n \geq 3$. The *length* of a walk $v_0v_1 \ldots v_n$ is defined to be n, and it may be called an n-walk. n-paths and n-cycles are defined in a similar way. We denote by C_n the cycle of n vertices (and hence of length n, or an n-cycle), P_n the path of n vertices (and hence of length n-1, and it is an (n-1)-path.). C_3 is often called a *triangle*.

A graph is *connected* if every pair of vertices are joined by a path; *disconnected* if not connected. A maximal connected subgraph of G is a *component* of G.

The *girth* of G is the length of a shortest cycle (if any) in G; the *circumference* the length of any longest cycle. The *distance* $d(u,v)$ between any two vertices u and v is the length of a shortest path joining them if any; otherwise $d(u,v) = \infty$. A shortest u-v path is often called a *geodesic*. The *diameter* of a connected graph is the length of any longest geodesic.

The *degree* of a vertex v in a graph G, denoted deg (v), is the number of edges incident with v. It is trivial to observe that the sum of the degrees of vertices of a graph G is twice the number of edges.

If all vertices of G are of degree r, G is called *regular* of degree r or r-regular. If G has only two degrees δ and d, G is called *bi-degreed*. We have special names for vertices of small degree. A vertex v is *isolated* if $\deg(v) = 0$; it is an *endvertex* if $\deg(v) = 1$.

The *complement* $\overline{G}$ of a graph G also has V(G) as its vertex set, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The *complete graph* K_p has every pair of its vertices adjacent. Thus K_p has $\binom{p}{2}$ edges and is regular of degree $p-1$. The graphs $\overline{K_p}$ are called *totally disconnected*, and are regular of degree 0.

A *bipartite graph* G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins

V_1 with V_2. If G contains every edge joining V_1 and V_2, then G is a *complete bipartite graph*, and G is denoted by $K_{m,n}$ if V_1 and V_2 have respectively m and n vertices. A simple characterization of bipartite graph is that all its cycles are even (see F. Harary [7], p. 18).

A graph is *acyclic* if it has no cycles. A *tree* is a connected acyclic graph. Thus trees are obviously special cases of bipartite graphs. An easy way to recognize a graph as a tree is that G is connected and p, the number of vertices, is equal to q + 1, where q is the number of edges. The *eccentricity* e(v) of a vertex v in a connected graph G is max d(u,v) for all u in G. The *radius* r(G) is the minimum eccentricity of the vertices. Note that the maximum eccentricity is the diameter. A vertex v is a *central vertex* if e(v) = r(G) and the *center* of G is the set of all central vertices. It can be proved that every tree has a center consisting of either one or two adjacent vertices, and trees are called *central* or *bicentral* accordingly.

A *cutvertex* of a graph is one whose removal increases the number of components. A *nonseparable* graph is connected, nontrivial without cutvertices. A *block* of a graph is a maximal nonseparable subgraph. If G is nonseparable, then G itself is called a block. For a connected graph with at least three vertices, we note G is a block if and only if every two vertices of G lie on a common cycle (p. 27 or F. Harary [7]). A block having more than one edge is also *2-connected*, i.e. we have to remove at least two vertices to "disconnect" G.

A graph is said to be *embedded* in a surface S when it is drawn on

S so that no two edges "intersect". A graph is *planar* if it can be embedded in the plane.

Finally, for two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets X_1 and X_2 respectively, the *union* $G = G_1 \cup G_2$ has vertex set $V = V_1 \cup V_2$, edge set $X = X_1 \cup X_2$.

Section 2. Vertex Reconstruction and Edge Reconstruction Conjectures.

In Section 1, we introduced the concepts of Vertex Reconstruction and Edge Reconstruction Conjectures. In this section, several basic lemmas and properties will be stated and/or proved.

We will call a parameter of G *reconstructable* if it takes the same value for all reconstructions of G. Similar definitions hold for *edge-reconstructable* parameters. The first fundamental theorem is due to P. J. Kelly [9].

Lemma 1.1 (Kelly's Lemma).

(a) For any two graphs F and G such that $|V(F)| < |V(G)|$, the number $s(F,G)$ of subgraphs of G isomorphic to F is reconstructable.

(b) For any two graphs F and G such that $|E(F)| < |E(G)|$, the number of subgraphs of G isomorphic to F is edge-reconstructable.

Proof of (a). Each subgraph of G isomorphic to F occurs in exactly $|V(G)| - |V(F)|$ of the subgraphs of $G - v$. Therefore

$$s(F,G) = \sum_{v \in V(G)} \frac{s(F,G - v)}{|V(G)|-|V(F)|}$$

Since the right-hand side of this identity is clearly reconstructable, so, too, is the left-hand side.

Proof of (b) is similar. Q.E.D.

Kelly's Lemma appears to be very useful in general (with a very simple combinatorial proof).

A sequence $d_1, d_2, \ldots, d_p$ of nonnegative integers is called a *degree sequence* of a graph G if the vertices of G can be labeled $v_1, v_2, \ldots, v_p$ so that $\deg(v_i) = d_i$ for all i. Often we express the sequence so that $d_1 \geq d_2 \geq \cdots \geq d_p$.

Corollary 1.1. For any two graphs F and G such that $|V(F)| < |V(G)|$, the number of subgraphs of G which are isomorphic to F, and include a given vertex v, is reconstructable.

Proof of Corollary. This number is just $s(F,G) - s(F,G - v)$ Q.E.D.

Taking $F = K_2$ in Kelly's lemma and in the corollary, we find that the number of edges and the degree sequence, respectively, are reconstructable parameters.

It is now easily seen, as noted by Kelly [9], that regular graphs are reconstructable. Consider a k-regular graph G. Since the degree sequence of G is reconstructable, all reconstructions of G are k-regular. But it is clear that all k-regular reconstructions of G are isomorphic, since each can be obtained (up to isomorphism) from any G-v by adding a vertex and joining it to all the vertices of degree $k - 1$ in G-v. We deduce that all reconstructions of G are isomorphic.

This proof is typical of many on reconstruction in that it splits naturally into two parts, which we shall refer to as "recognizability" and "weak reconstructability." A class $\mathcal{G}$ of graphs is *recognizable* if, for each graph G in $\mathcal{G}$, every reconstruction of G is also in $\mathcal{G}$, and *weakly reconstructable*, if, for each graph G in $\mathcal{G}$, all reconstructions of G that are in $\mathcal{G}$ are isomorphic to G. Thus a class $\mathcal{G}$ is reconstructable if and only if it is recognizable and weakly reconstructable.

The edge-reconstructability of degree sequence, though evident, cannot be proved in a way identical to that of Corollary 1.1.

Lemma 1.2. The degree sequence of a graph with at least four edges is edge-reconstructable.

Proof of Lemma. First, it is a trivial matter to prove the edge-reconstructability of $K_{1,n}$ for $n \geq 4$.

Suppose G has exactly α_0 vertices of degree d_0, α_1 vertices of degree $d_1, \ldots, \alpha_s$ vertices of degree $d_s > 0$ and α_{s+1} vertices of degree $d_{s+1} = 0$ where $d_0 > d_1 > \cdots > d_s > 0$.

Let H be an edge-reconstruction of G. We will show that H satisfies similar conditions.

Let H have β_0 vertices of degree δ_0, β_1 vertices of degree $\delta_1, \ldots, \beta_t$ vertices of degree $d_t > 0$ and β_{t+1} vertices of degree $\delta_{t+1} = 0$, where $\delta_0 > \delta_1 > \cdots > \delta_t > \delta_{t+1} = 0$. We will show that $s = t$ and $\alpha_i = \beta_i$, $d_j = \delta_j$ $\forall$ i, j, $0 \leq i, j \leq s + 1$.

If $d_0 = 1$, then G is union of K_2's plus some $\overline{K}_\ell$. The only non-isomorphic edge-reconstruction H will contain $K_{1,2}$ as a proper subgraph, hence $G \supseteq G - f \cong H - f$ contains $K_{1,2}$, a contradiction.

Now $d_0 > 1$ and by assumption G has exactly α_0 K_{1,d_0}'s as edge-proper subgraphs.

So Kelly's Lemma (Lemma 1.1) applies and H has exactly α_0 K_{1,d_0}'s as subgraphs. So $\delta_0 \geq d_0$. By symmetry, $d_0 \geq \delta_0$ and $d_0 = \delta_0$. But H has exactly $\beta_0 (> o) K_{1,\delta_0} = K_{1,d_0}$'s, so $\beta_0 = \alpha_0$.

Let $\ell = \min(s,t)$ and suppose

$\alpha_o = \beta_o, \alpha_1 = \beta_o, \ldots, \alpha_i = \beta_i$ and

$d_o = \delta_o, d_1 = \delta_1, \ldots, d_i = \delta_i$ for some $0 \leq i < \ell$.

We will see $\alpha_{i+1} = \beta_{i+1}, d_{i+1} = \delta_{i+1}$.

Suppose $d_{i+1} > 1$ first.

The number of $K_{1,d_{i+1}}$'s contained in G is exactly $\sum_{k=0}^{i+1} \alpha_k \binom{d_k}{d_{i+1}}$ = the number of $K_{1,d_{i+1}}$'s in H by Kelly's lemma.

If $\delta_{i+1} < d_{i+1}$ then H would have only

$$\sum_{k=0}^{i} \alpha_k \binom{d_k}{d_{i+1}} \quad K_{1,d_{i+1}}\text{'s} \left(< \sum_{k=0}^{i+1} \alpha_k \binom{d_k}{d_{i+1}}\right).$$

so $\delta_{i+1} \geq d_{i+1}$. By symmetry then, $\delta_{i+1} = d_{i+1}$ and since H has exactly

$$\sum_{k=0}^{i+1} \beta_k \binom{\delta_k}{\delta_{i+1}} = \sum_{k=0}^{i} \alpha_k \binom{d_k}{d_{i+1}} + \beta_{i+1}$$

$K_{1,\delta_{i+1}} = K_{1,d_{i+1}}$'s , $\alpha_{i+1} = \beta_{i+1}$ whenever $d_{i+1} > 1$. (and $d_{i+1} = \delta_{i+1}$).

Similar results hold if $\delta_{i+1} > 1$. So let $d_{i+1} = \delta_{i+1} = 1$. In this case $i + 1 = s = t$. Since $|E(G)| = |E(H)|$ and $\alpha_j = \beta_j$, $0 \leq j \leq i$, we have readily $\alpha_{i+1} = \beta_{i+1}$.

Induction says that $\alpha_i = \beta_i$, $d_j = \delta_j$, $0 \leq i, j \leq \ell$. Suppose $s \neq t$, say $s < t$. Then the number of edges in G is

$$\frac{1}{2} \sum_{i=0}^{s} \alpha_i d_i = \frac{1}{2} \sum_{i=0}^{s} \beta_i \delta_i < \frac{1}{2} \sum_{j=0}^{t} \beta_j \delta_j,$$

which is impossible since $E(G) = E(H)$.

So degree sequence is edge-reconstructable ($H - e \cong G - e$ implies that $V(G) = V(H)$ as well, and so $\alpha_{s+1} = \beta_{s+1}$). Q.E.D.

We can define *edge-recognizability* and *weakly edge-reconstructability* in a similar way.

We end this section by citing some useful concepts from J. A. Bondy and R. L. Hemminger [5].

Let $\mathfrak{F}$ be a class of graphs (that is, a family of graphs closed under isomorphism), and let F and G be graphs such that $F \in \mathfrak{F}$ and $s(F,G) > 0$. A subgraph of G which belongs to $\mathfrak{F}$ is called an $\mathfrak{F}$-subgraph of G; a *maximal $\mathfrak{F}$-subgraph* of G is one which is contained in no other $\mathfrak{F}$-subgraph of G. For instance, when $\mathfrak{F}$ is the class of connected graphs, the maximal $\mathfrak{F}$-subgraphs of G are the components of G. An *(F,G)-chain of length n* is a sequence $(X_0, X_1, \ldots, X_n)$ of $\mathfrak{F}$-subgraphs of G such that $F \cong X_0 \subset X_1 \subset \cdots \subset X_n \subset G$. Two (F,G)-chains are *isomorphic* if they have the same length and corresponding terms are isomorphic graphs. The *rank* of F in G is the length of a longest (F,G)-chain. We state below without proof an interesting result:

Lemma 1.3. (Counting Theorem). Let $\mathcal{G}$ be a recognizable class of graphs, and let $\mathfrak{F}$ be any class of graphs such that, for every G in $\mathcal{G}$, each $\mathfrak{F}$-subgraph of G is (i) vertex-proper; (ii) contained in a unique maximal $\mathfrak{F}$-subgraph of G. Then, for every F in $\mathfrak{F}$ and every G in $\mathcal{G}$, the number $m(F,G)$ of maximal $\mathfrak{F}$-subgraphs of G isomorphic to F is reconstructable.

Counting theorem is generalization of Kelly's Lemma.

Corollary 1.3.1. Disconnected graphs are reconstructable.

Proof of Corollary 1.3.1. A graph G is disconnected if and only if at most one G-v is connected. Therefore, disconnected graphs are recognizable. The counting theorem, with $\mathfrak{F}$ as the class of connected graphs and $\mathcal{G}$ as the class of disconnected graphs, establishes weak reconstructability. Q.E.D.

Corollary 1.3.2. If G is reconstructable and has no isolated vertices, then G is edge reconstructable.

Proof of Corollary 1.3.2. For a graph G without isolated vertices, let $\mathcal{G}$ be the class of all edge reconstructions of G and let $\mathfrak{F}$ be the class of graphs with v - 1 vertices. Since edge reconstructions of G have no isolated vertices, their $\mathfrak{F}$-subgraphs are edge proper and the counting theorem applies. But the maximal $\mathfrak{F}$-subgraphs of G are exactly the vertex-deleted subgraphs of G. It follows that G is edge reconstructable if G is reconstructable. Q.E.D.

Section 3. A very brief survey

This survey does not tend to be complete, nor will it prove anything in detail (with an exception, construction of trees).

The survey paper by J. A. Bondy and R. L. Hemminger [5] summarized more than sixty cases up to 1977. For vertex reconstructions, trees have been treated very deeply (P. J. Kelly [9], F. Harary and E. M. Palmer [8], B. Manvel [13], J. A. Bondy [3] etc.); graphs with cutvertices but no isolated vertices are done by J. A. Bondy [4]; and disconnected graphs were done by almost everyone.

For edge-reconstructions, L. Lovász [12] has proved G is edge-reconstructable if $|E(G)| > \frac{1}{4}|V(G)|(|V(G)| - 1)$; V. Müller [14] has proved G is edge-reconstructable if $2^{|E(G)|-1} > (|V(G)|)!$. J. Lauri [10] did the interesting case that all planar graphs with minimum degree 5 is edge-reconstructable. And in this thesis, we present in full detail the edge-reconstructability of bi-degreed graphs in Chapter 2, and bipartite graphs in Chapter 3.

For digraphs and infinite graphs, counterexamples exist (P. K. Stockmeyer [16], C. Thomassen [18]), and the problem there is to find those reconstructable. The author [11] has proved the (vertex)-reconstructability of some locally-finite trees.

There are many other related reconstruction problems, say reconstructing matrices, reconstructing relationships etc. We finish this chapter by a comparatively short proof of vertex-reconstructability of trees cited from J. A. Bondy and R. L. Hemminger [5].

Theorem 1.1. Trees are reconstructable.

Proof of Theorem 1.1. Trees are recognizable, since a graph G is a tree if and only if G is connected and $|E(G)| = |V(G)| - 1$.

A tree is a path if and only if each degree is at most two. Therefore paths are recognizable, and hence reconstructable.

In a tree which is not a path, every longest path is a vertex-proper subgraph. It follows from Kelly's lemma that the diameter and radius of a tree are reconstructable, and hence that central and bicentral trees are recognizable.

A vertex of a tree is *peripheral* if it is an end of a longest path. Since v is peripheral if and only if deg (v) = 1 and v is in a longest path, the number of peripheral vertices is reconstructable.

A *branch* of a central (bicentral) tree is a maximal subtree in which the central vertex (central edge) is of degree one (is incident with a vertex of degree one). A branch is *radial* if it includes a peripheral vertex of the tree. Note that a bicentral tree has exactly two branches, both of which are radial. A tree is *basic* if it has exactly two branches, just one of which is a path; the path branch is the *stem* and the other branch the *top*.

Now a tree of radius r (and not a path) is basic if and only if it contains no subgraph of one of the three types shown in Fig. 1-8 (where the centers are indicated in black and the distances a and b range between 1 and r - 1). Trees of these types are easily recognizable. (For example, a tree G is of type 1 if and only if it contains a path

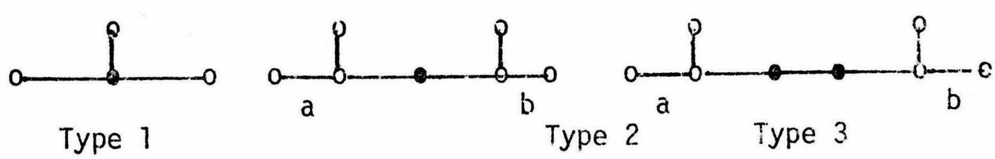


Fig. 1-8

of length $2r = v - 2$ and $r + 2$ paths of length $r + 1$). Therefore, by Kelly's lemma, basic trees are recognizable.

Basic trees are also weakly reconstructable. For let G be a central (bicentral) basic tree. Then all reconstructions of G are isomorphic, since each can be obtained, up to isomorphism, from the bicentral (central) G - v which has a vertex of degree greater than two closest to the central edge (central vertex) by extending a radial path.

It remains to prove that nonbasic trees are reconstructable. Let G be a nonbasic tree, and let F be a basic tree with the same diameter as G. By the counting theorem, the number of maximal basic subtrees of G isomorphic to F is reconstructable. We can use this information to find the radial branches of G as follows. Each non-path radial branch which includes k peripheral vertices of G is the top of p(G)-k maximal basic subtrees of G, where p(G) is the number of peripheral vertices in G. This gives us the non-path radial branches of G (with multiplicities). The number of path radial branches is then p(G) minus the total number of peripheral vertices in the non-path radial branches.

In the central case, it still remains to reconstruct the nonradial branches. But they are just the nonradial branches of a G - v obtained by deleting either a peripheral vertex of a radial branch which includes at least two peripheral vertices, if there is such a branch, or a nonperipheral end vertex of a radial branch, if there is such a vertex; otherwise, all radial branches are paths, and the nonradial branches can be found from a G - v obtained by deleting a peripheral vertex.

Q.E.D.

CHAPTER 2. Edge-reconstruction of bidegreed graphs

Section 1. Introduction.

Recall from Chapter 1 that a graph G is edge-reconstructable if all possible edge-reconstructions of G are isomorphic to G. In this chapter we will investigate the edge-reconstructability of bidegreed graphs, i.e. graphs which have exactly two degrees d and δ with $d > \delta \geq 0$.

The result of this chapter comes out in this way: it was first motivated by J. A. Bondy and R. L. Hemminger [5] as the edge-version of problem 1 in their paper; and then greatly prompted by Edward R. Swart [17] wherefrom a few nice ideas and theorems were used and then generalized. The main result obtained (in Section 5) is:

MAIN THEOREM. Every bidegreed graph G with at least four edges is edge-reconstructable (which solves this problem in full force).

Section 2 introduces elementary results and the useful concept of "forced move" (and "forced edge") by E. R. Swart [17]. In Section 3, the principle of "forced move" is applied by the author to establish a few more interesting "excludable configurations". In Section 4, we investigate the structure of "connection pattern" of "*minimum-distance-paths*", and conclude that they must all be disjoint. The remaining case is then solved by two more "distance functions" in Section 5.

Section 2. Elementary results and inspiration by Swart.

By Lemma 1.2, degree sequence is edge-reconstructable for graphs G with at least four edges. So bidegreed graphs are edge-recognizable, i.e. if H is an edge-reconstruction of a bidegreed graph G with two degrees d and δ, then H is also bidegreed with degrees d and δ.

We immediately observe that there is nothing to do unless $d = \delta + 1$. For if $d \geq \delta + 2$, then

(i) removing a $\delta - \delta$ edge creates two vertices of degree $\delta - 1$,

(ii) removing a $\delta - d$ edge creates a vertex of degree $d - 1 > \delta$ and a vertex of degree $\delta - 1$,

(iii) removing a $d - d$ edge creates two vertices of degrees $d - 1 > \delta$;

and so, edge-reconstructability of degree sequence implies that G can be edge-reconstructed from any of the G-e's. In the above, a $\delta - \delta$ edge means an edge with both ends vertices of degree δ, etc. Henceforth we assume $d = \delta + 1$.

A few more elementary properties can be proved using degree argument:

1. G is edge-reconstructable if $d = 1$ or 2.
2. G is edge-reconstructable if it has just one vertex of smallest degree δ.

For if $d = 1$, then G is disconnected consisting of links plus isolated vertices and is trivially edge-reconstructable; and $d = 2$ (and $\delta = 1$) means G is disjoint union of free standing paths and so presents no difficulty at all. The case G has only one vertex u of smallest degree δ is also easy, for if we delete any edge uv, then in G - uv, u is the only vertex of degree d - 2, and v the only vertex of degree d - 1, so we have only one way to restore the deleted edge: its original position. Henceforth we assume $d \geq 3$ and G have at least two vertices of degree δ.

In investigating this problem, it is a usual practice to restrict consideration to certain subgraphs or "configuration". To illustrate, consider the case when G has two adjacent vertices u,v of smallest degree δ. Then G is easily seen to be edge-reconstructable for u and v are two vertices of degree d - 2 in G - uv and again the only way to restore the deleted edge to get an edge-reconstruction H of G is its original position, otherwise H will have a vertex of degree d - 2 which is impossible. Hence we have only to consider the "petite" subgraph or "configuration" uv, not any "large" graph G-e at this stage. To represent this concept diagramatically, we call

a vertex of degree d a "big" vertex and denote it by o,

a vertex of degree $\delta = d - 1$ a "small" vertex and denote it by x,

a vertex of degree d - 2 a "tiny" vertex and denote it by Δ.

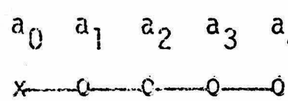
The above argument becomes:

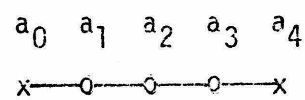
In drawing a configuration as above the structure of the rest of the graph is assumed to be arbitrary - except insofar as it is constrained by the structure of the configuration itself. Moreover it is understoond that we do not mean $deg(u) = deg(v) = 1$ in G, but $deg(u) = deg(v) = d - 1 \geq 1$ in general.

From the above, we see that if a bidegreed G of degrees d and d - 1 contains a configuration uv with $deg(u) = deg(v) = d - 1$, that G is edge-reconstructable. This leads to a new useful concept:

Definition 2.1. *Excludable configuration.* A configuration C is excludable if its existence in G enforces G to be edge-reconstructable. We then see immediately that the edge uv of two adjacent "small" vertices is an excludable configuration.

Let's call A_p a path of length p with one end a vertex of degree δ and all other vertices of degree d; let's also denote by S_p a path of length p with both ends vertices of degree δ and all other vertices of degree d. A_4 and S_4 are depicted below for illustration:

A_4: 

S_4: 

An S_p which starts at a_0, and then passes $a_1, a_2, \ldots, a_{p-1}$ sequentially to stop at a_p, will be denoted as an $S_p\ a_o\ a_1 \ldots a_p$ or simply an $S_p\ a_o - a_p$ if it is immaterial to mention the internal vertices. Similar convention holds for A_q's.

Since G has no isolated vertices $(d \geq 3)$, the two facts that disconnected graphs are vertex-reconstructable and that the vertex-reconstructability of a graph without isolated vertices implies its edge-reconstructability (see [5]) together tell us that G can be assumed to be connected. Hence some S_p's must exist in G for certain p's.

Let $s(G)$ be the minimum of such p's. Clearly, $s(G) \geq 2$ for S_1 is an excludable configuration already mentioned. If H is any edge-reconstruction of G, then $s(H) \geq 2$ for $s(H) = 1$ implies H is edge-reconstructable which in turn implies G is edge-reconstructable. Consider an $S_{s(G)}$ $a_0a_1 \ldots a_{s(G)}$. In $G - a_0a_1$, a_0 is a vertex of degree $d-2$, hence $H \cong G - a_0a_1 + a_0h$, where h may be a_1, $a_{s(G)}$ or some other small vertex. In any case it is readily seen that $s(H) \leq s(G)$. A symmetry argument (since $s(H) \geq 2$) implies immediately $s(G) \leq s(H)$ and so $s(H) = s(G)$ for any edge-reconstruction H of G. From now on, we will write s for $s(G)$ (or $s(H)$). Intuitively, it is the minimum distance between any two vertices of degree $d-1$ in G (or any edge-reconstruction H). s and S_s will be a principal tool to solve our problem in the following.

It is conceivable that a big graph G may have a large number of edge-reconstructions, all nonisomorphic to each other. So it is quite remarkable at this early stage to observe that G can have at most one nonisomorphic edge-reconstruction H. In fact, any edge-reconstruction $H \cong G - a_0a_1 + a_0h$, h is a_1, a_s or some other small vertex by the previous paragraph, where $a_0a_1 \ldots a_s$ is an S_s in G. But if h is not a_1 or a_s, then $s(H) \leq s-1 = s(G)-1$, which is impossible. So $H \cong G - a_0a_1 + a_0a_s$ is the only possible nonisomorphic edge-reconstruction.

If $H \cong G$, then G is edge-reconstructable. If $H \ncong G$, then G is not edge-reconstructable by definition, we will then prove G's edge-reconstructability logically by either deriving a contradiction or proving H is edge-reconstructable (then G is edge-reconstructable since G is also an edge-reconstruction of H) or even that G is edge-reconstructable.

Before going further, we cite a few interesting notations and results from [17].

We notice that in order to restore a missing edge to an edge-deleted subgraph, it is necessary to:

1. Avoid creating a multiple edge.
2. Ensure that the degree sequence is preserved.
3. Avoid creating another configuration already known to be excludable.

Definition 2.2. *Forced edge.* If, in conformity to the three conditions mentioned above, an edge deleted from a given configuration can only be restored to its original position, we refer to it as a forced edge.

Note that edge uv joining two vertices of degree $d-1$ is then also a forced edge. Forced edge is a very useful tool to make lots of configurations excludable. The main idea of introducing excludable configurations is that we will build larger excludable configurations from smaller ones gradually so that at last we have a list big enough to prove edge-reconstructability for every bidegreed graph G.

A concept similar to forced edge, which is also very powerful is:

Definition 2.3. *Forced move.* If any edge deleted from a configuration can be validly replaced in two identical positions in conformity to the three conditions just before Definition 2.2, we will refer to its replacement in the position which differs from its original position as a forced move.

As an example, since $H \cong G - a_0a_1 + a_0a_s$ is the only possible non-isomorphic edge-reconstruction for a given S_s $a_0a_1 \ldots a_s$, the move from a_0a_1 to a_0a_s is a forced move. We will denote this symbolically as $a_0a_1 \to a_0a_s$ or $a_0a_1 \to a_sa_0$.

We note that a forced move always changes an isomorph G' of G to an isomorphs H' of H and vice versa. So if we start at G and execute an odd number of forced moves, we are ending at an isomorph H_k of H (it goes in this way, $G \to H_1 \to G_1 \to H_2 \to G_2 \to \cdots \to G_{k-1} \to H_k$, where G_i's and H_j's are isomorphs of G and H respectively); if furthermore we return to our initial configuration after this odd number of forced moves, then we get $H \cong H_k \equiv G$ since the structure of the rest of the graph is not affected by this sequence of forced moves, and we get a contradiction. Hence follows [17]:

Lemma 2.1. Every configuration which contains a forced edge or which can be recovered by an odd number of forced moves is excludable.

We conclude this section with a simple application of the idea of Lemma 2.1.

Lemma 2.2. G is edge-reconstructable if s is even.

Proof of Lemma. Consider an S_s $a_0a_1 \ldots a_s$ in G. The forced move $a_0a_1 \to a_sa_0$ changes G to $H_1 \cong H$ and the old S_s $a_0a_1 \ldots a_s$ to a new S_s $a_1a_2 \ldots a_sa_0$ while leaving the remaining part of the graph intact. The next forced move $a_1a_2 \to a_0a_1$ changes H_1 to $G_1 \cong G$, the S_s $a_1a_2 \ldots a_sa_0$ to another S_s $a_2a_3 \ldots a_sa_0a_1$. Proceeding in this way, we see that when s is even, $a_{s-1}\,a_s \to a_{s-2}a_{s-1}$ changes $H_{s/2} \cong H$ to $G_{s/2} \cong G$ and the S_s $a_{s-1}a_sa_o \ldots a_{s-2}$ to $a_sa_oa_1 \ldots a_{s-1}$, and $a_sa_o \to a_{s-1}a_s$ changes $G_{s/2} \cong G$ to $H_{s/2+1} \cong H$ and $a_sa_oa_1 \ldots a_{s-1}$ to $a_oa_1 \ldots a_s$.

In $H_{s/2+1}$, we see that the "old" S_s $a_oa_1 \ldots a_s$ in G is returning to its original position while the remaining part of the graph is kept intact through the whole process. Hence $H_{s/2+1}$ is identically G and so $H \cong H_{s/2+1} = G$, a contradiction, and we are done. Q.E.D.

Figure 2-1 illustrates the argument used in Lemma 2.2.

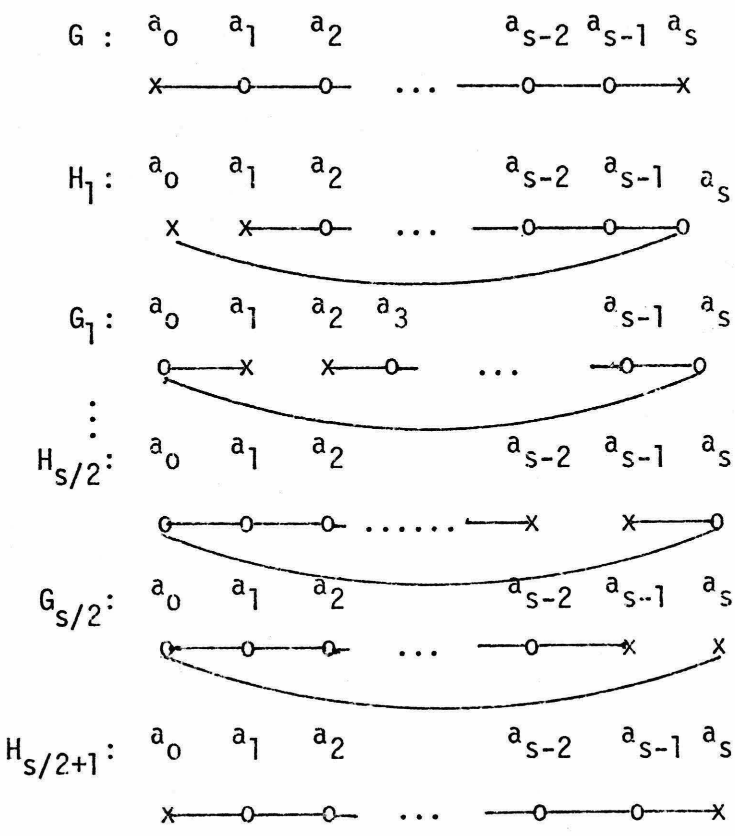


Fig 2-1.

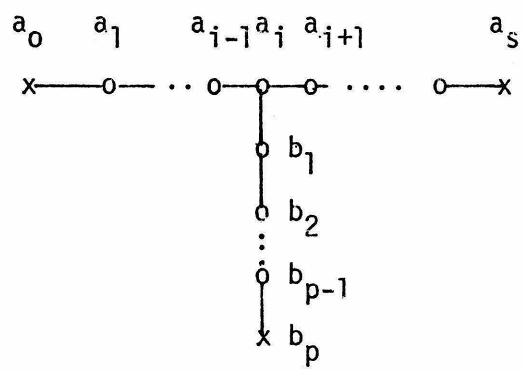
Section 3. Further application of forced-move principle.

In this section, the forced-move principle stated in Section 2 is used to derive several interesting lemmas.

Definition 2.4. T_p-*configuration*. A T_p-configuration C is a configuration consisting of an S_s $a_0a_1 \ldots a_s$ and an A_p $b_0b_1 \ldots b_p$ with $\deg(b_p) = d-1$, $b_0 = a_i$ for some i, $0 < i < s$, and no $b_j = a_k$ for any $0 < j \leq p$, $0 \leq k \leq s$.

Remark. We cannot allow $b_0 = a_0$ or a_s for otherwise G would contain an S_p with $p < s = s(G)$, impossible by the definition of s.

Intuitively a T_p-configuration looks like below:



Lemma 2.3. Every T_p-configuration with $p < s$ is excludable.

Proof of Lemma. If not, then G has the unique nonisomorphic edge-reconstruction H. And $s(G) = s(H)$ says that none of G or H or their isomorphs can contain an S_p. Now applying i consecutive forced-moves ($a_0a_1 \to a_sa_0, a_1a_2 \to a_0a_1, \ldots, a_{i-1}a_i \to a_{i-2}a_{i-1}$) we get an isomorph of G or H which contains S_p as a configuration, contradiction. Q.E.D.

As a special application of Lemma 2.3, we see that two S_s's cannot intersect at the internal "big" vertices. They can be joined at the end "small" vertices however. This simplifies the future configurations very much.

It can be proved that T_p-configurations are excludable for $p \geqq s$, but their proofs are much much harder, exactly that of the proof of the problem in full force.

It is conceivable that the two small vertices a_o and a_s in an S_s $a_o a_1 \ldots a_s$ may also be joined by another S_s $a_o b_1 b_2 \ldots b_{s-1} a_s$. $a_o a_1 \ldots a_s$ and $a_o b_1 \ldots b_{s-1} a_s$ need not be disjoint interiorly, that is, we might have $b_i = a_i$ for some $0 < i < s$ (obviously, we cannot have $b_i = a_j$ for some $0 < i \neq j < s$).

Our next step will be to prove that the above situation cannot happen.

Lemma 2.4. It is impossible that two vertices of degree $d-1$ be joined by two S_s's. (So the configuration of two small vertices joined by two S_s's is excludable.)

Proof of Lemma. We suppose at first that $b_1 \neq a_1$. Let j be the first positive integer such that $b_j = a_j$, then $1 < j \leq s$. Now $a_o b_1 \to a_o a_s$ is a forced move sending G to $H_1 = G - a_o b_1 + a_o a_s \cong H$, for in $G - a_o b_1$, a_o is a vertex of degree $d-2$ (an impossible degree in G or H) and b_1, a_s are two vertices of degree $d-1$ of distance $s-1$ apart. Consider $H_1 - a_o a_1$. In this edge-deleted subgraph, a_o is a vertex of degree $d-2$, and so by H_1's non-edge-reconstructability, some isomorph G_1 of G is equal to $H_1 - a_oa_1 + a_oc$, c is a small vertex in $H_1 - a_oa_1$ other than a_1. If c is not b_1, then in G_1, the S_s $a_1a_2 \ldots a_sa_o$ and A_{j-1} $b_1b_2 \ldots b_j$ form a T_{j-1} excludable by Lemma 2.3, and so $G_1 = H_1 - a_oa_1 + a_ob_1 \cong G$. But now $a_oa_s \to a_oa_1$ is a forced move sending G_1 to $H_2 \cong H$ since a_o ia a vertex of degree $d-2$ and a_1, a_s are two small vertices distance $s-1$ apart in $G_1 - a_oa_s$.

So far the set of three forced moves we used are sequentially:

$$a_ob_1 \to a_oa_s, \quad a_oa_1 \to a_ob_1, \quad a_oa_s \to a_oa_1.$$

It is then obvious that we return to G identically (not just an isomorph) after them, and so $G = H_2 \cong H$, a contradiction.

We have proved the case when $b_1 \neq a_1$. Now let $b_1 = a_1$. Let $i > 1$ be the first integer such that $b_i \neq a_i$. Applying $i-1$ forced moves as in Lemma 2.3 we see that in an isomorph of G or H (depending on i is odd or even), $a_{i-1}a_i \ldots a_sa_o \ldots a_{i-2}$ and $a_{i-1}b_i \ldots b_{s-1}a_sa_o \ldots a_{i-2}$ are two S_s's joining two small vertices a_{i-1} and a_{i-2}, and the condition $b_i \neq a_i$ in this isomorph has the same meaning as $b_1 \neq a_1$ in G. Q.E.D.

Remark. This lemma is proved for G. But the same argument holds for any isomorph of G or H. We will assume this practice throughout.

A similar argument can prove G's edge-reconstructability if G has only two small vertices. Consider the unique S_s $a_oa_1 \ldots a_s$ in G (uniqueness by Lemma 2.4) and consider a vertex $c \neq a_1$ adjacent to a_o. $a_oc \to a_oa_s$ is a forced move sending G to $H_1 \cong H$ since G has only two small vertices. a_0 and c are joined in H_1 by an S_s $a_0c_1c_2 \ldots c_s, c_s = c$ since $s(H_1) = s(G)$. c_1 may be a_1 or not. And $a_0c_1c_2 \ldots c_s$ is an A_s in G. Now it's trivial to observe that $a_0c_1 \to a_0c_s(=a_0c)$ and $a_0a_s \to a_0a_1$ are forced moves, and so the sequence of three moves:

$$a_0c \to a_0a_s, \qquad a_0c_1 \to a_0c, \qquad a_0a_s \to a_0c_1$$

return us to G identically, and we get a contradiction.

Now the topology of interconnections of different S_s's becoming simpler, we may then ask the natural question: Is the number $n(G)$ of S_s's in G edge-reconstructable, in other words, is $n(G) = n(H)$? The affirmative answer is proved by the following lemma:

Lemma 2.5. The number $n(G)$ of S_s's in G is edge-reconstructable.

Proof of Lemma. Before starting to prove, let's make a few intuitive concepts more precise. Recall that a vertex is small if it has the smallest degree δ, and big if it has degree d. A vertex b will be said to "lie on" an S_s or A_s $a_0a_1 \ldots a_s$ if b is some a_i, $0 \leq i \leq s$; $b = a_i$ is an "end" if $i = 0$ or s, "internal" vertex if $0 < i < s$; and in this case we will also say that the S_s or A_s $a_0a_1 \ldots a_s$ "contains" b.

Consider in G a fixed S_s $a_0a_1 \ldots a_s$. Let

$n_G(a_0)$ be the number of S_s's containing a_0 not counting $a_0a_1 \ldots a_s$;

$n_G(a_s)$ be the number of S_s's containing a_s not counting $a_0a_1 \ldots a_s$;

$n_G(a_i)$ be the number of A_s's of the form $a_ib_1b_2 \dots b_s$, with b_s the unique small vertex unequal to a_o or a_s, and $0 < i < s$.

Note that in the A_s $a_ib_1b_2 \dots b_s$, $b_s \neq a_o$ or a_s, no b_j can be equal to some a_k, otherwise Lemma 2.3 enforces G's edge-reconstructability. Let K_G be the set of all other S_s's. Then every S_s in K_G is of the form $b_ob_1 \dots b_s$ with none of b_o or b_s equal to a_o or a_s. And every S_s in K_G is disjoint from $a_oa_1 \dots a_s$ by Lemma 2.3. We see immediately that:

$$n(G) = n_G(a_o) + n_G(a_s) + 1 + |K_G|.$$

Consider now the forced move $a_oa_1 \to a_sa_o$, which transforms G to $H_1 \cong H$. The old S_s $a_oa_1 \dots a_s$ in G becomes a new S_s $a_1a_2 \dots a_sa_o$ in H_1. The $n_G(a_s)$ S_s's containing a_s not counting $a_oa_1 \dots a_s$ become in H_1 $n_G(a_s)$ A_s's containing a_s as a big end (an end which is a big vertex) with the other small end unequal to a_o or a_1. The $n_G(a_1)$ A_s's which contain a_1 as the big end with the small end unequal to a_o or a_s become now $n_G(a_1)$ S_s's containing a_1 as a small end ($a_oa_1 \dots a_s$ exclusive however). It can be seen very easily that the other S_s's or A_s's which have a_i as a big end or which are members of K_G remain intact in this move. (Lemma 2.4. eliminates some annoyance)

Now if we define in H_1 a function n_{H_1} in the same way n_G was defined by considering the S_s $a_1a_2 \dots a_sa_o$, then we see from the previous argument that $n_{H_1}(a) = n_G(a)$ for all vertices a in $a_oa_1 \dots a_s$. Define K_{H_1} in the same way as K_G, we see again:

$$n(H_1) = n_{H_1}(a_1) + n_{H_1}(a_o) + 1 + |K_{H_1}|.$$

Actually, $n(H_1) = n_{H_1}(a_1) + n_{H_1}(a_o) + 1 + |K_G| = n_G(a_1) + n_G(a_o) + 1 + |K_G|$ since K_{H_1} is easily seen to be the same set as K_G. Next, consider the forced move $a_1a_2 \to a_oa_1$, which transforms $H_1 = G - a_oa_1 + a_sa_o$ to $G_1 = H_1 - a_1a_2 + a_oa_1 \cong G$. We can define n_{G_1} and K_{G_1} in a way similar as before and get $n_{G_1}(a) = n_{H_1}(a) = n_G(a)$ for all $a \in a_oa_1 \ldots a_s$ and $K_{G_1} = K_{H_1} = K_G$. Furthermore, since our S_s of consideration is $a_2a_3 \ldots a_sa_oa_1$ this time, we have

$$n(G_1) = n_{G_1}(a_2) + n_{G_1}(a_1) + 1 + |K_G|, \text{ or}$$

$$n(G) = n_G(a_2) + n_G(a_1) + 1 + |K_G|, \text{ since } G_1 \cong G \text{ implies}$$

$n(G_1) = n(G)$.

Similar argument shows that the forced move $a_2a_3 \to a_1a_2$ sends G_1 to $H_2 = G_1 - a_2a_3 + a_1a_2$ with $n_{H_2}(a) = n_G(a)$ for all $a \in a_oa_1 \ldots a_s$, $K_{H_2} = K_G$, and

$$n(H_2) = n_G(a_3) + n_G(a_2) + 1 + |K_G|.$$

Proceeding in this way, we see that

$$\begin{aligned} n(G) &= n_G(a_o) + n_G(a_s) + 1 + |K_G| \\ &= n(G_1) = n_G(a_2) + n_G(a_1) + 1 + |K_G| \\ &= n(G_2) = n_G(a_4) + n_G(a_3) + 1 + |K_G| \\ &\ \ \vdots \end{aligned}$$

$$= n(G_{(s-1)/2}) = n_G(a_{s-1}) + n_G(a_{s-2}) + 1 + |K_G|, \text{ and}$$

$$n(H_1) = n_G(a_1) + n_G(a_o) + 1 + |K_G|$$

$$= n(H_2) = n_G(a_3) + n_G(a_2) + 1 + |K_G|$$

$$\vdots$$

$$= n(H_{(s+1)/2}) = n_G(a_s) + n_G(a_{s-1}) + 1 + |K_G|;$$

where G_i's and H_j's come from the sequence:

$G \to H_1 \to G_1 \to H_2 \to G_2 \to \cdots \to H_{(s-1)/2} \to G_{(s-1)/2} \to G$ resulted from the sequence of forced moves: $a_oa_1 \to a_sa_o$, $a_1a_2 \to a_oa_1$, $a_2a_3 \to a_1a_2$, ..., $a_{s-1}a_s \to a_{s-2}a_{s-1}$, $a_sa_o \to a_{s-1}a_s$.

Adding the $(s+1)/2$ equations for $n(G_i)$'s (with $G = G_o$ say), we get

$$(s+1)\cdot n(G)/2 = \sum_{k=0}^{s} n(a_k) + (1+|K_G|)(s-1)/2;$$

adding the $(s+1)/2$ equations for $n(H_j)$'s, we get

$$(s+1)\cdot n(H_1)/2 = \sum_{k=0}^{s} n(a_k) + (1+|K_G|)(s-1)/2$$

$$= (s+1)\cdot n(G)/2.$$

Hence $n(H) = n(H_1) = n(G)$ as was to be proved. Q.E.D.

Corollary 2.5. Notations as in the proof of Lemma 2.5, we have $n_G(a_o) = n_G(a_2) = \cdots = n_G(a_{s-1})$, and $n_G(a_1) = n_G(a_3) = \cdots = n_G(a_s)$. Also, $n_G(a) = n_{G_i}(a) = n_{H_j}(a)$ for all a in some fixed S_s $a_oa_1 \ldots a_s$, and G_i and H_j are some isomorphs when we do the sequence of forced moves $a_oa_1 \to a_sa_o$, $a_1a_2 \to a_oa_1$, ..., $a_sa_o \to a_{s-1}a_s$.

Proof of Corollary. The second half of the statement was already noted in the proof of lemma. For the first half, we see that $n(G_1) = n(H_1)$ implies $n_G(a_2) + n_G(a_1) + 1 + |K_G| = n_G(a_1) + n_G(a_0) + 1 + |K_G|$, which in turn implies $n_G(a_o) = n_G(a_2)$; $n(G_2) = n(H_2)$ implies by a similar way that $n_G(a_2) = n_G(a_4)$; and so by comparing $n(G_i) = n(H_i)$ for $i \leq i \leq (s-1)/2$, we see easily $n_G(a_o) = n_G(a_2) = n_G(a_4) = \cdots = n_G(a_{s-1})$. The second equality $n_G(a_1) = n_G(a_3) = \cdots = n_G(a_s)$ follows by comparing $n(G_j) = n(H_{j+1})$, $1 \leq j \leq (s-1)/2$. Q.E.D.

Remark. It's conceivable that $n_{G'}(a)$ or $n_{H'}(a) \neq n_G(a)$ for some isomorph G' of G or isomorph H' of H if they do not appear somewhere while doing the forced moves $a_oa_1 \to a_sa_o$, $\cdots$, $a_sa_o \to a_{s-1}a_s$.

The more strict term for $n_G(a)$, $a \in a_oa_1 \ldots a_s$ should be $n_{G,a_oa_1 \ldots a_s}(a)$, and so $n_G(a) = n_{H_1}(a)$ more precisely means $n_{G,a_oa_1 \ldots a_s}(a) = n_{H_1, a_1a_2 \ldots a_sa_o}(a)$. However, since n_G is always defined by implicitly assuming an $a_oa_1 \ldots a_s$, we will write $n_G(a)$ for $n_{G,a_oa_1 \ldots a_s}(a)$ unless it is confusing.

Also, we may write $n_{G,a_o \ldots a_s}(a)$ as $n_{G,a_o-a_s}(a)$ if the internal vertices are irrelevant.

By means of Lemma 2.4, we establish the excludability of a configuration which will be useful in Section 4.

Consider the configuration in Fig. 2-2 below, which consists of an S_s $a_oa_1 \ldots a_{s-1}a_s$ together with an A_s $a_{s-1}e_1e_2 \ldots e_{s-1}a_s$ joining a_{s-1}

to a_s. Some of the e_j's may lie on $a_0a_1 \ldots a_s$. $a_{s-1}a_s \to a_sa_0$ is obviously a forced move, and it gives two S_s's joining a_{s-1} and a_s in $H' = G - a_{s-1}a_s + a_sa_0$; so H' and hence G is edge-reconstructable by Lemma 2.4.

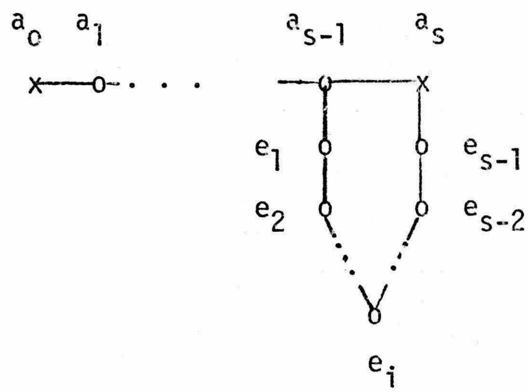


Fig. 2-2

This tells us that a_{s-1} on an S_s $a_0a_1 \ldots a_{s-1}a_s$ cannot lie on an A_s joining it and a_s. (However, it is conceivable that a_{s-1} may lie on an A_s with a_0 the other end). Similar fact holds for a_1.

Furthermore, we note that if in Fig. 2-2, a_{s-1} and a_s are joined by an $A_p, p < s$, instead of A_s, then G is edge-reconstructable; for $H' = G - a_{s-1}a_s + a_sa_0$ contains the obviously excludable S_p.

Section 4. Excludability of s-three-chains, s-three-cycles, and s-two-chains.

Let's review the interconnection structure of S_s's so far. We know that no two S_s's can intersect at an internal vertex (Lemma 2.3), and no two S_s's will have the same two ends (Lemma 2.4); but since S_s's can have one end in common, it is still conceivable that long "chains" (or "cycles") of S_s's joined end to end can exist making the structure still quite intricate.

To investigate this possibility, we have:

Definition 2.5. s-*three-chain*. An ordered quadruple (a,b,c,d) of four distinct small vertices a,b,c,d is called an s-three-chain if a - b, b - c, c - d are all S_s's.

Remark. In the definition above, there is no problem of which S_s joining a and b will be chosen, for there is only one. A permutation of a,b,c,d say (a,c,b,d) need not be an s-three-chain. Also, (d,c,b,a) is an s-three-chain physically the same as (a,b,c,d) but defined as different logically. To rescue this situation, we define an equivalence relation $\sim$ on the set of s-three-chains by letting each equivalence class consist of exactly two elements (a,b,c,d) and (d,c,b,a). By abuse of language, we will write (a,b,c,d) for the class [(a,b,c,d)].

Definition 2.6. s-*three-cycle*. An unordered triple {a,b,c} of three distinct small vertices a,b,c is called an s-three-cycle if a - b, b - c, c - a are all S_s's.

Feelings of s-three-chain and s-three-cycles can be gained by looking at Fig. 2-3(a) and (b) below:

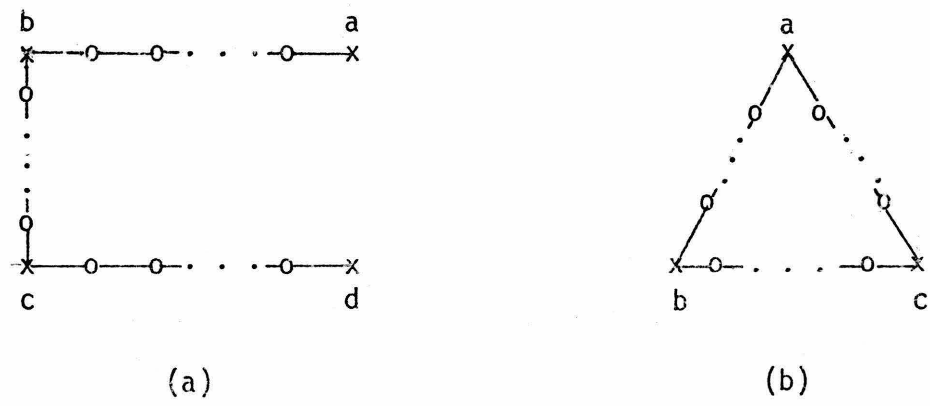


Fig. 2-3. a depiction of (a) s-three-chain and (b) s-three-cycle.

Before investigating them, we will introduce another useful definition which is a generalization of the concept of forced edge.

Definition 2.7. *Forced vertex.* If, in conformity to three conditions mentioned before Definition 2.2, an edge $\alpha\beta$ can only be replaced with one of its end α fixed (i.e. $\alpha\beta$ changed to $\alpha\gamma$ for some γ), we refer to α as a forced vertex (in $G - \alpha\beta$).

As an example, if a is a small vertex, and b is any adjacent vertex, then a is a forced vertex in G - ab. Though not seeming very useful at first sight, the concept of forced vertex is applied easily to establish a forced edge (and hence G's edge-reconstructability): ab can be proved to be a forced edge if we can show that a and b are both forced vertices. For illustration, we see that if a,b are adjacent small vertices, then in G - ab, a and b are both forced vertices, hence establishing ab as a forced edge (cf. Section 2).

Lemma 2.5 proved at the end of Section 3 will be the main tool to prove excludability of s-three-chains and s-three-cycles.

Recall that s is odd by Lemma 2.2. We will divide the proof of excludability of s-three-chains into two parts: $s \geq 7$ and $s = 3$ or 5. The proof for $s \geq 7$ will be stated as Proposition 2.6, due to its big size; and the proof for $s = 3$ or 5 will be stated as Lemma 2.9.

Consider now for $s \geq 7$ an s-three-chain (a,b,c,d) and rewrite $b = b_0$, $c = b_s$ (so that b and c are joined by an S_s $b_0 b_1 \ldots b_s$). We will write $n_G(b_i)$ for $n_{G,b_0 \ldots b_s}(b_i)$ unless some other S_s is used. Now $n_G(b) \geq 1$ since b lies on a-b and b-c. Similarly, $n_G(c) \geq 1$ and we see immediately that $n_G(b_i) \geq 1$ for all $0 \leq i \leq s$ by Corollary 2.5. In particular, $n_G(b_3) \geq 1$ implies b_3 lies on some A_s $b_3 e_1 \ldots e_s$ in G with $e_s \neq b$ or c. Here none of e_j for $1 \leq j < s$ can lie on the s-three-chain if $e_s \neq a$ or d. (However, it is conceivable that $e_s = a$ or d.) We see also that b_4 lies on some A_s $b_4 f_1 \cdots f_s$ in G with $f_s \neq b$ or c. Again, none of f_k for $1 \leq k < s$ can lie on the s-three-chain; and it is still possible that $f_s = a$ or d or e_s when $e_s \neq a$ or d.

We will prove excludability case by case depending on the distinctness of f_s, e_s, a and d. The recognizability that G contains an s-three-chain satisfying a certain case is trivial by looking at $G - bb'$, where b' is adjacent to b on ab.

First comes the most general case:

Case 1. f_s, e_s, a and d are all distinct.

Let's draw the configuration T consisting of the s-three-chain (a,b,c,d) plus the two A_s's $b_3 e_1 \ldots e_s$ and $b_4 f_1 \ldots f_s$ in Fig. 2-4 below.

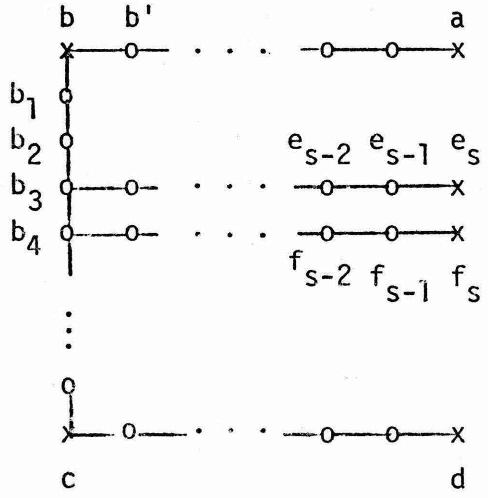
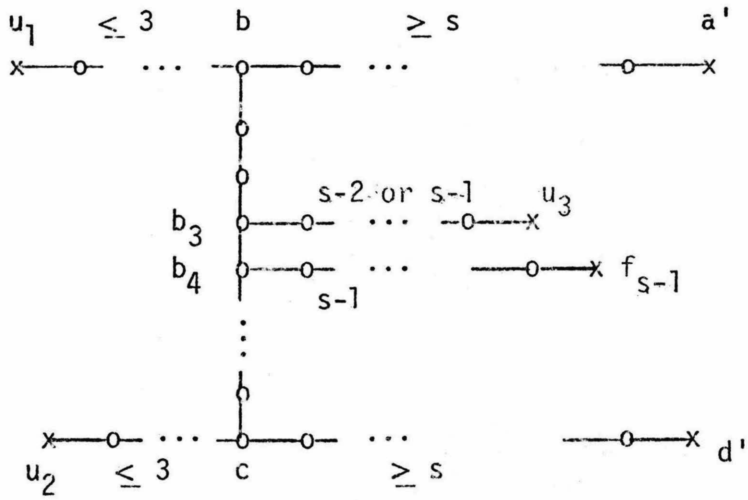


Fig. 2-4 Configuration T used in Case 1 of excludability of s-three-chains for $s \geq 7$.

Note that $f_{s-1} \neq e_{s-1}$, $f_{s-2} \neq e_{s-2}$, otherwise $s \leq 4$ (and it is clear that $e_{s-2} \neq f_{s-1}$, $f_{s-2} \neq e_{s-1}$). $H_\alpha = G - e_{s-1}e_s + e_s b \cong H$ and $H_\beta = G - e_{s-1}e_s + e_s c \cong H$ are the only two possible ways to edge-reconstruct from $G - e_{s-1}e_s$ since e_s is a forced vertex and the S_s $b - c$ and A_{s-1} $b_3 - e_{s-1}$ form a T_{s-1} in $G - e_{s-1}e_s$.

Let's consider H_α first. Denote by T_α the configuration in H_α obtained from T in G by the same kind of operations from which H_α is obtained from H, i.e. $T_\alpha = T - e_{s-1}e_s + e_s b$. (We will assume this "natural" association of graphs and configurations from now on). In $H_\alpha - e_{s-1}e_{s-2}$, e_{s-1} is a forced vertex, and a candidate for edge-reconstruction is $G_{\alpha,1} = H_\alpha - e_{s-1}e_{s-2} + e_{s-1}g$, where $g \neq e_{s-2}$ is any small vertex in $H_\alpha - e_{s-1}e_{s-1}$. (g may be a, e_s, c, d, f_s or something else).

Suppose for now that $g \neq f_s$. Then f_s is a small vertex in $G_{\alpha,1}$ and $b_4 f_1 \ldots f_s$ is still an A_s in $T_{\alpha,1}$. The only possible ways to edge-reconstruct from $G_{\alpha,1} - f_s f_{s-1}$ are $H_{\alpha,2} = G_{\alpha,1} - f_s f_{s-1} + f_s h$, with $h \neq f_{s-1}$ any small vertex in $G_{\alpha,1} - f_s f_{s-1}$. Now our configuration $T_{\alpha,2} = T_{\alpha,1} - f_s f_{s-1} + f_s h = \cdots = T - e_{s-1}e_s + e_s b - e_{s-1}e_{s-2} + e_{s-1} g - f_s f_{s-1} + f_s h$ will have the general look as below:



The picture is self-explanatory. The upper left corner "$u_1 \quad \leq 3 \quad b$" says that the distance of u_1 and b is at most three (u_1 can be f_s, e_{s-1}, e_s or b with distance respectively 3, 2, 1, 0). Also $b_3 - u_3$ is an A_{s-2} if $u_3 = e_{s-2}$, and A_{s-1} if $u_3 = f_s$, u_3 cannot be any other vertex. Note that though u_1 and u_3 can be f_s at different times, they cannot be equal to f_s at the same time.

Consider $T_{\alpha,2} - b_3 b_4$. b_4 lies on an S_{s-1} $b_4 - f_{s-1}$ and an S_p $b_4 - u_2$ with $p \leq s - 4 + 3 < s$. So either b_4 is a forced vertex or $f_{s-1}u_2$ is a replacing edge. Similarly we see that b_3 lies on $b_3 - u_3$ which is an S_{s-1} or S_{s-2} and an S_q $b_3 - u_1$ with $q \leq 3 + 3 \leq 6$, and so either b_3 is a forced vertex or $u_1 u_3$ is a replacing edge. Since u_1, u_2, u_3, f_{s-1} are all distinct in $T_{\alpha,2}$ with none of them equal to b_3 or b_4, we see that b_3 and b_4 are both forced vertices, enforcing $b_3 b_4$ to be a forced edge; so $H_{\alpha,2}$ and hence

G is edge-reconstructable.

We have shown that $s \geq 7$ implies G's edge-reconstructability in the subcase H_α and assuming $g \neq f_s$.

We will now show $g \neq f_s$ for the subcase H_α. Suppose not, and we go back to $G_{\alpha,1} = H_\alpha - e_{s-1}e_{s-2} + e_{s-1}f_s$ (now $g = f_s$). In $T_{\alpha,1} - b_3b_4$, b_3 lies on S_{s-2} $b_3 - e_{s-2}$ and S_4 $b_3 - e_s$, so either b_3 is a forced vertex or e_se_{s-2} is a replacing edge. But $b_4 - c$ is an S_{s-4}, and so since b_4, c, b_3, e_{s-2}, e_s are all distinct, $b_3b_4 \to b_3c$ is a forced move sending $G_{\alpha,1}$ to $H_{\alpha,\gamma}$. In $T_{\alpha,\gamma}$, rewrite $c - d$ as $c_0c_1 \ldots c_s(d = c_s)$, and consider $T_{\alpha,\gamma} - d_sd_{s-1}$. Any possible edge-reconstruction $G_{\alpha,\delta}$ will be $= H_{\alpha,\gamma} - d_sd_{s-1} + d_{s-1}j$, with j any small vertex $\neq d_s$ in $H_{\alpha,\gamma} - d_sd_{s-1}$. Now as in the previous paragraph, in $G_{\alpha,\delta} - b_3c$, b_3 lies on an S_{s-2} $b_3 - e_{s-2}$ or an S_{s-1} $b_3 - d_s$ and also an S_p, $p \leq 3 + 2 = 5 < 7$, and c lies on an S_{s-4} $c - b_4$ or S_{s-3} $c - d_s$ and also an S_{s-1} $c - d_{s-1}$; so as before b_3c is a "forced edge" and we are done, finishing the subcase H_α for Case 1.

The proof for H_β follows in the same vein except when $g = f_s$ (i.e. $G_{\beta,1} = H_\beta - e_{s-1}e_{s-2} + e_{s-1}f_s$). The above argument does not apply since $b_4 - c$ is no longer an S_{s-4} in $T_{\beta,1} - b_3b_4$ (c is not a small vertex now). We proceed by using results for H_α. From $G_{\beta,1} - e_sc$, we can edge-reconstruct some $G_{\beta,1} - e_sc + e_sk \cong H$. If $k \neq b$, then our new graph contains a T_{s-2} (the S_s $b - c$ and A_{s-2} $b_3 - e_{s-2}$) and so G is edge-reconstructable. But $k = b$ implies our isomorph of H contains the excludable configuration T_α as described before, and we are done, completing the proof for Case 1.

Remark. In the proof before, we deleted edges in this order: $e_s e_{s-1}$, $e_{s-1}e_{s-2}$, $f_{s-1}f_{s-2}$. We can prove excludability of T in the same vein if edges are deleted in another way: $f_s f_{s-1}$, $f_{s-1}f_{s-2}$, $e_{s-1}e_s$.

Pausing for a moment, we see that the above type of argument works when $e_s = f_s \neq$ a or d; the proof is even simpler. Some minor change is observed, for example, in $G_{\alpha,1} = H_\alpha - e_{s-1}e_{s-2} + e_{s-1}g$, g may be f_{s-1} now (g couldn't be f_{s-1} when e_s, f_s, a, d are distinct as in case 1). The type of argument leading to that $b_3b_4 \to b_3c$ is a forced move sending $G_{\alpha,1}$ to $H_{\alpha,\gamma}$ is no longer necessary here.

We state this as a variation of Case 1:

Case 1'. $f_s = e_s \neq$ a or d (proof already mentioned).

Next we consider the case when exactly one of f_s or e_s coincides with a or d.

Case 2. $f_s =$ a or d, $e_s \neq$ a or d; or

$e_s =$ a or d, $f_s \neq$ a or d. (still $s \geq 7$ assumed)

Graphically we mean that all the four configurations shown in Fig. 2-5 are excludable.

The proofs of these four subcases being essentially the same, we will do Fig. 2-5 (a) only as an illustration. For simplicity, we assume $e_{s-1} \notin$ c-d. Same practice will hold for Case 3 following. To get feeling for proof, see Lemma 2.12.

Denote the configuration in Fig. 2-5(a) by $U^{(a)}$. To avoid T_{s-1}, the only edges we can replace in $G - e_{s-1}d$ (to get $H' \cong H$) are dc and bd.

If we replace by dc, then Kelly's Lemma (Lemma 1.1) on (s+1)-cycles says that there is a path of length s (easily shown to be an A_s) joining

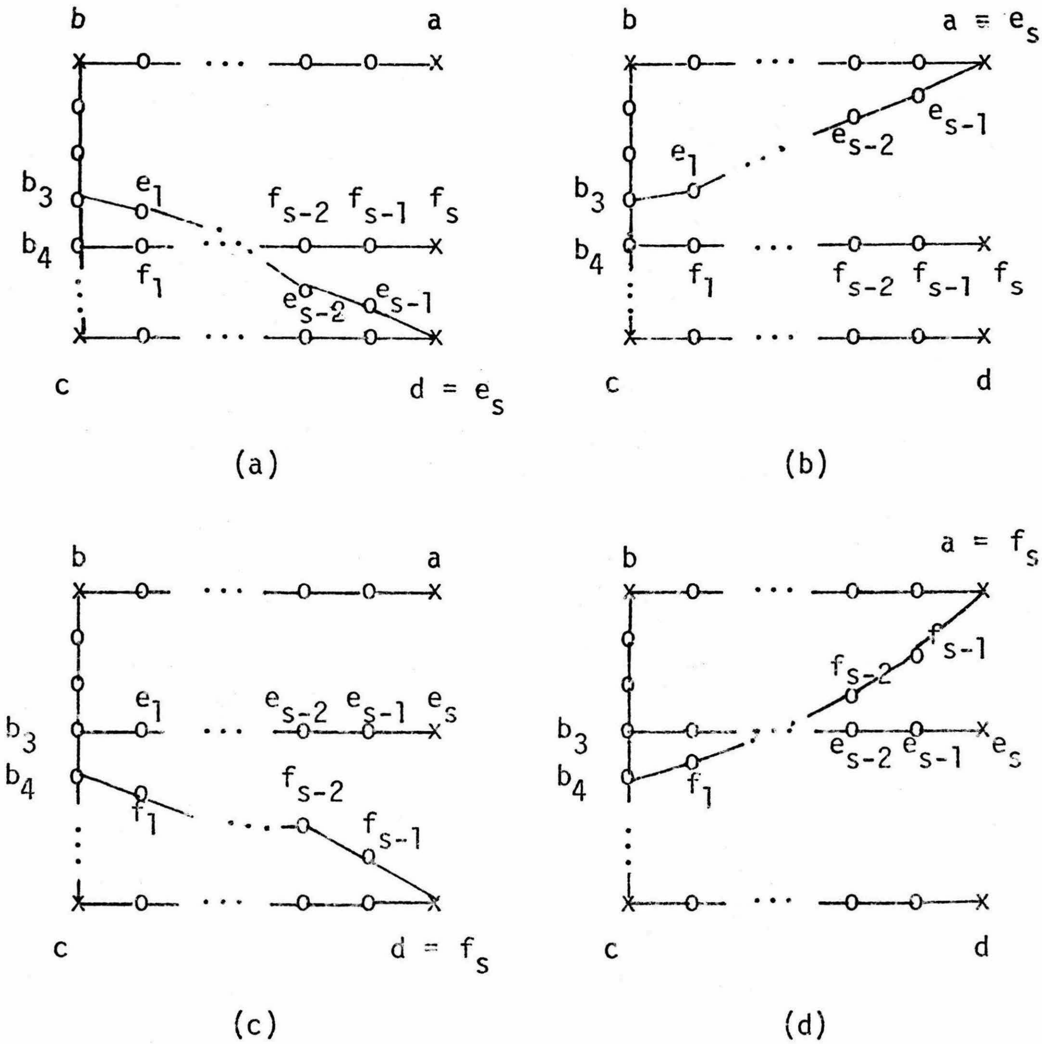


Fig. 2-5

e_{s-1} and d in G not containing $e_{s-1}d$ (so that the A_s e_{s-1}-d followed by edge de_{s-1} is the "old" $(s+1)$-cycle in G, and the A_s d-c followed by the newly added edge dc is the "new" $(s+1)$-cycle in H'). The A_s e_{s-1}-d cannot contain any vertex on $b_0b_1 \dots b_s$, to avoid some possible $T_p, p < s$. Now three forced moves $b_0b_1 \to b_sb_0$, $b_1b_2 \to b_0b_1$ and $b_2b_3 \to b_1b_2$ give us an isomorph of H containing the excludable configuration as in Fig. 2-2.

Now we consider replacing de_{s-1} by db. The only ways to edge-reconstruct from $H' - e_{s-1}e_{s-2}$ are $G_1 = H' - e_{s-1}e_{s-2} + e_{s-1}g$, g some small vertex in the edge-deleted subgraph. We claim that g cannot be f_s. Suppose not, we see that the fact that b_3 lies on an S_{s-2} $b_3 - e_{s-2}$ and S_4 $b_3 - d$ and $b_4 - c$ is an S_{s-4} in $G_1 - b_3b_4$ enforces the move $b_3b_4 \to b_3c$ and gives us $H_2 \cong H$. Consider deleting db in $U_2^{(a)}$, then $db \to db_4$ is the only possibility (to avoid a T_{s-1} or T_{s-2}), which also gives us $G_3 \cong G$. Rewrite $c - d$ as $c_0c_1 \ldots c_s$, with $c = c_0$, $d = c_s$, and delete $c_{s-1}c_s$ in the configuration $U_3^{(a)}$ (which is contained in G_3), the only eligible edge-reconstructions are $H_4 = G_3 - c_{s-1}d + dh$, where h may be a, b, e_{s-2}, e_{s-1} or some other small vertices not on $U_3^{(a)}$. We note now that in $H_4 - b_3c$, b_3 lies on an S_3 $b_3 - b$ or S_4 $b_3 - d$ and also on an S_{s-2} $b_3 - e_{s-2}$ or S_{s-1} $b_3 - d$ (it depends on the value of h, note also that it is impossible that b_3 and d are joined by both an S_4 and an S_{s-1}) such that the other two small vertices are distinct, and c lies on an S_{s-1} $c_0 - c_{s-1}$ and S_4 $c - d$; so it is easy to see that b_3c is the only way to recover a graph, proving our claim that g cannot be f_s.

Returning now to $G_1 = H' - e_{s-1}e_{s-2} + e_{s-1}g$ in the previous paragraph, with the recognition that $g \neq f_s$. We can edge-reconstruct $H'' = G_1 - f_sf_{s-1} + f_sh'$, h' some small vertex $\neq f_{s-1}$ in $G_1 - f_sf_{s-1}$. Similar type of argument as before will show that from $H'' - b_3b_4$, b_3b_4 is the only edge we can replace (hence a "forced edge" in a more general sense) using the fact that $s \geqq 7$ is the minimum distance between any two small vertices. We are now done for the proof of subcase Fig. 2-5 (a) of Case 2. Similar proofs of the other three subcases will be omitted here.

The results obtained in Case 1, Case 1' and Case 2 readily give a new interesting summary-type result which we state as:

Lemma 2.7. G is edge-reconstructable if G contains an s-three-chain (a,b,c,d) with $n_{G,b-c}(b) \geq 3$ or $n_{G,b-c}(c) \geq 3$; here $s \geq 7$.

Proof of Lemma. Consider $n_{G,b-c}(b) \geq 3$ first. Then $n_{G,b-c}(b_4) = n_{G,b-c}(b) \geq 3$, and b_4 is the big end of at least three A_s's in G. It cannot happen that b_4 and a are joined by more than one A_s, for then, in an isomorph of G which is obtained from G by four forced moves $b_0b_1 \to b_sb_0, b_1b_2 \to b_0b_1, b_2b_3 \to b_1b_2, b_3b_4 \to b_2b_3$, we see two small vertices joined by two S_s's, contradictory to Lemma 2.4. Similarly, b_4 and d are joined by at most one A_s in G. So b_4 must lie on at least $3 - 1 - 1 = 1$ A_s $b_4f_1 \dots f_s$ with $f_s \neq a$ or d. Since $n_{G,b-c}(b_3) = n_{G,b-c}(c) \geq 1$, b_3 lies on an A_s $b_3e_1 \dots e_s$. The case $e_s = a$ or d is treated in Case 2, $e_s = f_s$ in Case 1', and $e_s \neq$ any of f_s, a, d in Case 1. In all cases, we see our s-three-chain for $s \geq 7$ is excludable, in other words G is edge-reconstructable if G contains such a configuration.

The case $n_{G,b-c}(c) \geq 3$ is done in a similar way. Q.E.D.

Henceforth we assume that $n_{G,b-c}(b) \leq 2$, $n_{G,b-c}(c) \leq 2$ for any s-three-chain (a,b,c,d). We note that if b_3 lies on an A_s $b_3e_1 \dots e_s$ with $e_s \neq a$ or d, then arguments as in Lemma 2.7 using Case 1, Case 1', Case 2 say that G is edge-reconstructable. So any small end of an A_s with b_3 or b_4 as the big end will be assumed to be a or d. Furthermore, if $n_{G,b-c}(b) = 2$, then $b - d$ should be another S_s, otherwise say $b - e$ is another S_s, then for the s-three-chain (e,b,c,d),

$b_3 - a$ is an A_s with $a \neq e$ or d, and the previous argument works. Similar fact holds for c as well. Furthermore we have $n_{G,a-b}(a) \leq 2$ and $n_{G,c-d}(d) \leq 2$: for if not and say $n_{G,a-b}(a) \geq 3$, then a is joined by $3 - 1 - 1 = 1$ S_s to some small vertex $e \neq b, c,$ or d, and then G is edge-reconstructable by Lemma 2.7 applied to the s-three-chain (e,a,b,c).

We now come to the remaining case of Proposition 2.6.

Case 3. $n_{G,b-c}(b) \leq 2$, $n_{G,b-c}(c) \leq 2$.

Without loss of generality, let $n_{G,b-c}(b) \leq n_{G,b-c}(c)$

Subcase 3. (a) $n_{G,b-c}(c) = 2$, $n_{G,b-c}(b) = 1$ or 2.

The situation is illustrated in Fig. 2-6.

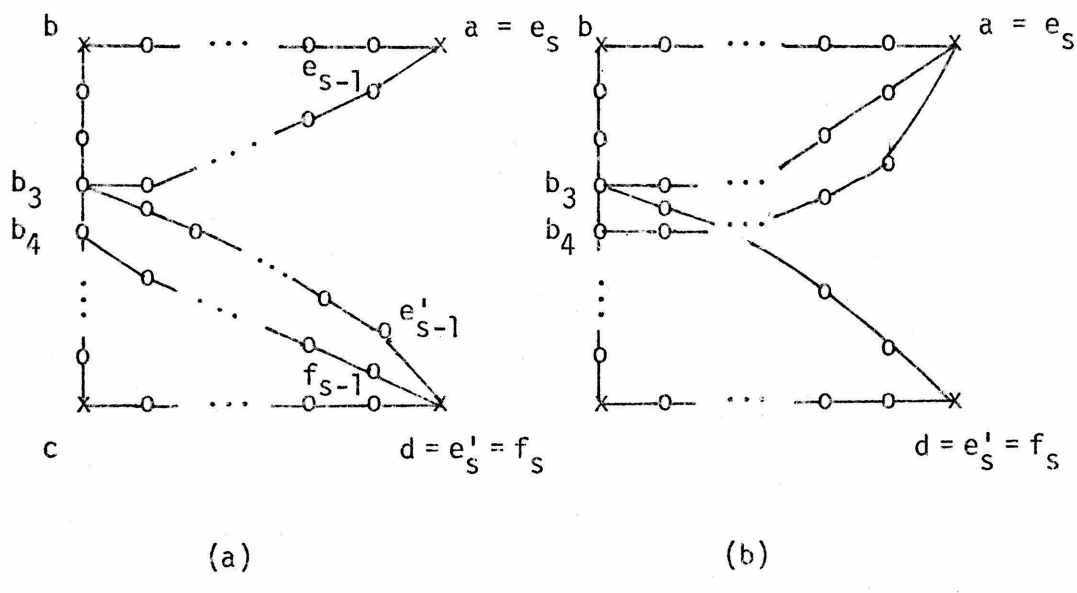


Fig. 2-6

Let b_3 lie on two A_s's $b_3e_1 \ldots e_s$ and $b_3e'_1 \ldots e'_s$ with

$e_s = a$, $e'_s = d$. Let b_4 lie on an A_s $b_4 f_1 \ldots f_s$ with $f_s = a$ or d. (The case $f_s = d$ corresponding to Fig. 2-6 (a), $f_s = a$ to Fig. 2-6(b)). The three A_s's just mentioned are disjoint except intersecting at the ends b_3, a or d; and this can be easily seen in an isomorph G_2 of G where $b_0 b_1 \ldots b_s$ becomes $b_4 b_5 \ldots b_2 b_3$ (applying 4 forced moves to G).

Since proofs are similar, we will do the case for Fig. 2-6 (a) only. $G - a\, e_{s-1}$ can only be edge-reconstructed to $G - a\, e_{s-1} + ab$ or $G - a\, e_{s-1} + ac$. In the former case, e_{s-1} and e_s are joined by an A_s in G and we have an excludable configuration as in Fig. 2-2 in G_2 of last paragraph. Next delete $f_{s-1}d$ in $H' = G - a\, e_{s-1} + ab$. The possible edge-reconstruction is $G' = H' - f_{s-1}\, d + dg$, with g some small vertex. If g is e_{s-1}, then b_3 is joined to d by two different A_s's (namely $b_3 e'_1 \ldots e'_s$ and $b_3 e_1 \ldots e_{s-1}\, d$) in G'. So three forced moves $b_o b_1 \to b_s b_o, b_1 b_2 \to b_o b_1, b_2 b_3 \to b_1 b_2$ will give us an isomorph of H where two small vertices b_3 and d are joined by two different S_s's; this is impossible by Lemma 2.4. (This is a place where we use heavily the fact that $n_G(c) = 2$). If g is not e_{s-1}, then argument as in Case 1 tells us that $b_3 b_4$ is a "forced edge" in G', i.e., after considering all possibilities, $b_3 b_4$ is the only edge we can replace in $G' - b_3 b_4$. ($s \geq 7$ is also used heavily, argument fails if $s = 5$). Hence we are done with subcase 3(a).

Subcase 3. (b). $n_{G,b-c}(b) = n_{G,b-c}(c) = 1$.

Again, we note $n_G(a) \leq 1$, $n_G(d) \leq 1$, otherwise we are done by subcase 3. (a) by some s-three-chain (h,a,b,c). Now we observe that none of a, b, c, d can lie on any s-three-cycle. For if, say, b lies on an s-three-cycle, then $n_G(b) = 1$ enforces the third "side" of this s-three-cycle to be an S_s joining a and c, which in turn implies $n_G(c) \geq 2$, a contradiction.

Let b' be adjacent to b on the S_s a - b. The *forced* move bb' → ba gives some $H' \cong H$. Since $n_{H'}(b') = n_G(b') = n_G(a) \leq 1$ (Corollary 2.5), b' lies on no s-three-cycle in H'. Hence the move bb' → ba has no effect on any existing s-three-cycles, and we see readily that the number of s-three-cycles is edge-reconstructable in this subcase.

As before, let b_3 lie on the A_s $b_3e_1 \ldots e_s$ and b_4 on $b_4f_1 \ldots f_s$. We know e_s and f_s must be either a or d. To save writing in this subcase, we use $G \to H_1 \to G_1 \to H_2 \to G_2 \to \cdots \to H_{(s-1)/2} \to G_{(s-1)/2} \to H_{(s+1)/2}$ when the sequence of forced moves is $b_0b_1 \to b_sb_0, b_1b_2 \to b_0b_1, \ldots,$ etc. Now e_s and f_s cannot be equal, otherwise in $G_2 \cong G$ we have a "new" s-three-cycle $\{b_3, b_4, e_s\}$ and since b, c lies on no s-three-cycle in G, the isomorph G_2 of G has one more s-three-cycle than G has, impossible. Hence we have $e_s = a$, $f_s = d$, or $e_s = d$, $f_s = a$.

Consider $e_s = a$, $f_s = d$ first. b_1 must lie on an A_s $b_1 - h$, $h \neq b,c$. h cannot be a, otherwise H_1 has one more s-three-cycle than G. Reindex $b_1, b_2, \ldots, b_s$ in H_1 by $b_0', b_1', \ldots b_s'$. Now if $h \neq d$, then in H_1, $b_4 = b_3'$ lies on an A_s $b_3 - a$ with $a \neq h,d$ for the s-three-chain (h, b_1, b, a), and so H_1 is edge-reconstructable by the paragraph right before Case 3. Hence h = d.

Observe that it is impossible that $b_i - a$ and $b_{i+1} - a$ (or $b_i - d$

and $b_{i+1}-d$) are both A_s's in G, for otherwise some isomorph of G or H will have one more s-three cycle than G. Now it is clear that b_5-a is an A_s (not b_5-g for any other g).

Let b_2-j be the A_s with a_2 as the big end ($n_G(a_2)=1$) in G. j cannot be d since b_1-d is an A_s, cannot be a since b_3-a is an A_s. In G_2 with reindexing $b_2=b''_0$, $b_3=b''_1$, ..., $b_5=b''_3$, ...; b_5-a is an A_s with $a\neq j,d$ for the s-three-chain (j,a_2,a_1,d) and so G_2 is edge-reconstructable.

Let now $e_s=d$ (and $f_s=a$). From $G-e_{s-1}d$, we have two possible edge-reconstructions $\cong H$, namely $G-e_{s-1}d+dc$ and $G-e_{s-1}d+db$ (so that no T_{s-1} is created). The former is excluded as usual since e_{s-1} and e_s are joined by an A_s in G. For the latter, we see that e_{s-1} lies on two S_s's in $H'=G-e_{s-1}d+db$ since b lies on two S_s's in G. Denote these two by $e_{s-1}-\alpha$, $e_{s-1}-\beta$. Then $e_{s-1}-\alpha$, $e_{s-1}-\beta$ are A_s's in G. We claim that one of α,β say α must be c. Suppose not. Since $n_{G,b-c}(b_3)=1$, b_3 lies on $1+1=2$ S_s's as the small end in H_2, and we still have $n_{H_2,b_3e_1\ldots e_s}(b_3)=1$. But $\alpha,\beta\neq c$ implies $2=n_{H_2,b_3e_1..e_s}(e_{s-1})=n_{H_2,b_3e_1\ldots e_s}(b_3)=1$, a contradiction, proving our claim.

Let the A_s joining c and e_{s-1} be $cg_1\ldots g_s, e_{s-1}=g_s$. Suppose at first that g_1 does not lie on b-c or c-d. Then $G-cg_1$ can be edge-reconstructed to give $G-cg_1+ce$, an isomorph of H. If $e\neq b$ or c, then in H_2 (see p.53) we have a T_{s-1}. If $e=b$ or c, then c and g_1 are joined by an A_s in G, and in H' we have an excludable configuration as in Fig. 2-2. So g_1 must lie on b-c or c-d. If g_1

lies on c-d, then $g_1 = c_1$ (with $cd = c_0c_1 \ldots c_s$). Let $i \geq 1$ be the first integer such that $g_i = c_i$, $g_{i+1} \neq c_{i+1}$; then H' contains a T_{s-i} (S_s c-d and A_{s-i} $c_i - e_{s-1}$), and we are done. Now let g_1 lie on b-c, or $g_1 = b_{s-1}$. Now $G - b_{s-1}b_s + b_sb_0$ has one more s-three-cycle than G ($\{b_{s-1}, b_s, d\}$ is new), a contradiction.

Having done now Case 3 also, we claim to have proved the technical lemma below:

Proposition 2.6. An s-three-chain is excludable if $s \geq 7$.

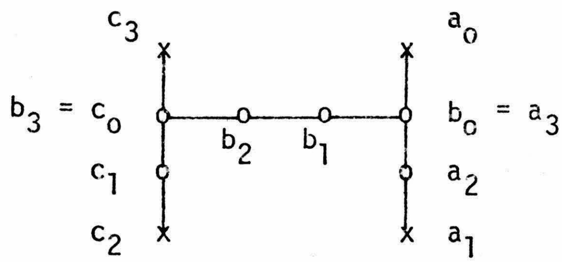
Note that the argument used in proving Proposition 2.6 does not apply when s = 3 or 5, and we need a separate discussion. We will appeal to a result on p. 22 of Swart [17], which is restated here for reference:

Proposition 2.8. (Swart) If a bi-degreed G is not edge-reconstructable, then the *girth* (the shortest length of cycles) of G is ≥ 8.

Lemma 2.9. An s-three-chain is excludable if s = 3 or 5.

Proof of Lemma. Let s = 3 first, and consider a 3-three-chain (a,b,c,d). Rewrite the S_3's a-b, b-c, c-d as $a_0a_1a_2a_3$, $b_0b_1b_2b_3$, $c_0c_1c_2c_3$ respectively with the understanding that $a = a_0$, $b = a_3 = b_0$, $c = b_3 = c_0$, $d = c_3$.

The two forced moves $c_2c_3 \to c_3c_0$, $a_2a_3 \to a_3a_0$ give us in an isomorph G' of G a configuration C which looks like the English letter "H" as shown in the following:



We readily observe that $n_{G',c_2-c_3}(c_0) = n_{G',c_2-c_3}(c_2)$. Now, in the graph $H_0 = G' - a_3a_0 + a_1a_0$, $b_3b_2b_1b_0$ becomes a "new" A_3 at $b_3 = c_0$ ($b_3b_2b_1b_0$ is neither A_3 nor S_3 in G), and we would have $n_{H_0,c_2-c_3}(c_0) = n_{H_0,c_2-c_3}(c_2) + 1$ unless, in G, c_0 is joined to a by an A_3 $c_0 - a_1$ or c_2 is joined to b_0 by an A_3 $b_0 - c_2$. None of the two A_3's in G, namely $c_0 - a_1$ or $c_2 - b_0$, can contain any "big" vertex in (a,b,c,d); otherwise it is easy to find some T_p, $p<3$ in an isomorph of G or H by suitable forced moves.

In G, the first alternative gives us a cycle of length 8: the S_3 $c_0 - a_3$ (i.e. c-b) followed by A_2 $a_3a_2a_1$ and then the A_3 joining a_1 and c_0; there are exactly two small vertices (c_0,a_3) of distance 3. The second alternative that c_2 be joined to b_0 by an A_3 in G gives us also a cycle of length 8 of a similar "description" as above. Since proofs will be identical except changes in notation, we do the first alternative only. But this is trivial now, since $b_2b_3 \to b_3b_0$ is a forced move; and in the new graph ($\cong$ H) the edge b_3b_0 ($=c_0a_3$), followed by A_2 $a_3a_2a_1$ and the A_3 $a_1 - c_0$ is a cycle of length $1 + 2 + 3 = 6 < 8$, contradictory to Proposition 2.8.

Next, let $s = 5$, and consider a 5-three-chain (a,b,c,d). As before, rewrite a-b, b-c, c-d by $a_0 \dots a_5$, $b_0 \dots b_5$, $c_0 \dots c_5$, with $a = a_o$, $b = a_5 = b_o$, $c = b_5 = c_o$, $d = c_5$. Arguing as in the case $s = 3$, we see that cycle of length $5 + 5 + (5 - 1) = 14$ exists by two alternatives, where, say, the first alternative gives in G the cycle by the S_5 c-b followed by A_4 $a_5a_4a_3a_2a_1$ and then the A_5 $a_1 - c$. The two forced moves $a_oa_1 \to a_5a_o$ and $a_1a_2 \to a_oa_1$ now send G to $G_1 \cong G$. Then let e be adjacent to c on the S_5 $c - a_1$ in G_1, $ce \to ca_1$ is a forced move sending G_1 to H_1, wherein the original

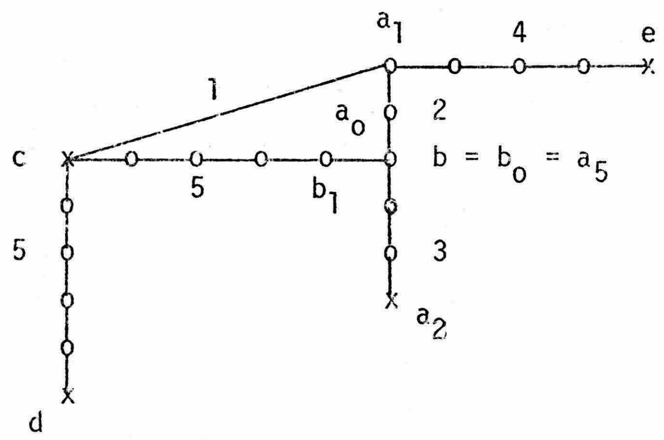


Fig. 2-7

5-three-chain (a,b,c,d) becomes a configuration C' as in Fig. 2-7. In C', we have a cycle of length $1 + 2 + 5 = 8$, which is pretty improved from the starting value 14, but still not good enough : we need some cycle of length < 8.

Let's consider $H_1 - bb_1$. Since $b_1 - c$ is an S_4 and $a_2 - b$ is an S_3 in this subgraph, we have $4 - 1 = 3$ ways of replacing an edge to get $G_2 \cong G$, namely b_1a_2, ca_2, and bc.

If the replacing edge is bc, then in G, $cb\,a_oa_1\,c$ is a cycle of length $4 < 8$, we are done by Proposition 2.8. If it is ca_2, then ca_2 followed by $a_2a_3a_4a_5a_0a_1$ and a_1c is a cycle of length $1 + 5 + 1 = 7 < 8$; and if it is b_1a_2, then b and c are now small vertices on an S_3 ba_0a_1c, a contradiction. Hence we have proved the case $s = 5$ completing the proof of Lemma 2.9. Q.E.D.

We combine the results of Proposition 2.6 and Lemma 2.9 in the following

Proposition 2.10. All s-three-chains are excludable for bi-degreed graph G.

Next we come to the excludability of a closely related configuration, the s-three-cycles. The proof for it is much simpler by the result of Proposition 2.10.

Lemma 2.11. s-three-cycles are excludable.

Proof of Lemma. Let $\{a,b,c\}$ be an s-three-cycle in G. Then we see immediately that $n_{G,a-b}(a) = n_{G,a-c}(a) = 1$, for if not, then from an S_s $a - d$ with $d \neq b,c$, we have an s-three-chain (d,a,b,c), impossible by Proposition 2.10. Similarly $n_{G,b-c}(b) = n_{G,b-c}(b) = 1$, $n_{G,a-b}(c) = n_{G,b-c}(c) = 1$.

Rewrite a-b as $a_0a_1 \ldots a_s$ with $a = a_0, b = a_s$. $n_{G,a_0-a_s}(a_1) = n_{G,a_0-a_s}(b) = 1$ implies that a_1 lies on an A_s $a_1 - d$, $d \neq a,b$. If $d \neq c$, the forced move $a_0a_1 \to a_sa_0$ gives an s-three-chain (d,a_1,a_0,c) in $H' = G - a_0a_1 + a_sa_0$, impossible, and we should have $d = c$. $a_1 - c$ cannot contain any big vertex in $\{a,b,c\}$ by the same move $a_0a_1 \to a_sa_0$.

In particular, if f is adjacent to c on $a_1 - c$, f cannot lie on $\{a,b,c\}$.

Now $G - cf$ can be edge-reconstructed in only two ways to get an isomorph of H and also to avoid a T_{s-1}, namely $G - cf + ca$ and $G - cf + cb$. In both cases, Kelly's Lemma on $(s+1)$-cycles tells us that f is joined to c by an A_s $f - c$ in G. Now $H' = G - a_0a_1 + a_sa_0$ contains an excludable configuration as in Fig. 2-2 and we are done. Q.E.D.

Coming back to the connection pattern of the "minimum-distance-paths" S_s's, we see that no "s - n-cycle" can exist ($n = 2$ by Lemma 2.4, $n = 3$ by Lemma 2.11, $n \geq 4$ by Proposition 2.10), and also no "s - n-chain" can exist for $n \geq 3$ by Proposition 2.10. (Here "s - n-cycle" and "s - n-chain" are defined in a natural way similar to s-three-cycle and s-three-chain). The pattern is simplified greatly, but it still remains the possibility that two S_s's be joined at an end, in other words, "s-two-chain" might exist. To make the notation more precise, we state

Definition 2.9. s-*two-chain*. An ordered triple (a,b,c) of distinct small vertices is an s-two-chain if a-b, b-c are all S_s's.

As in the case of s-three-chains, (c,b,a) and (a,b,c) will be "equivalent" in a natural way, and we will write (a,b,c) to mean $[(a,b,c)]$, the equivalence class of (a,b,c).

Let's now consider an s-two-chain (a,b,c) in G. We have immediately that $n_{G,a-b}(a) = n_{G,b-c}(c) = 0$ since s-three-chain and s-three-cycle are impossible. Write b-a as $b_0b_1 \ldots b_s$ with $b = b_0$, $a = b_s$.

We see $n_G(b_2) = n_G(b_o) \geq 1$ and b_2 lies on an A_s $b_2 d_1 \ldots d_s$. Conceivably $d_1 \ldots d_s$ can intersect b-c at an internal vertex or an end small vertex.

As an aid of proof, let's introduce the concept of an "n-star". An n-star $(\alpha; \beta_1, \ldots, \beta_n)$ or simply an n-star at α is a set of n S_s's $\alpha - \beta_1, \alpha - \beta_2, \ldots, \alpha - \beta_n$ such that $\alpha, \beta_1, \ldots, \beta_n$ are all distinct.

An n-star at α looks like a star with n "arms" all joined at the "center" α. No two $\alpha - \beta_i, \alpha - \beta_j$ can intersect internally by Lemma 2.3. We will now show that for every positive integer n, the number of n-stars is edge-reconstructable. Consider our s-two-chain (a,b,c) again. Since $n_{G,b-a}(b_1) = n_{G,b-a}(a) = 0$, the forced move $b_o b_1 \to b_s b_o$ does not "destroy" any n-star at a nor "create" any n-star at b_1; it does not affect any other n-stars at all (n fixed in the argument), and so the number of n-stars is edge-reconstructable.

The small end d_s of the A_s $b_2 d_1 \ldots d_s$ can coincide with c or not, and we will treat them differently. We now state and prove the lemma on excludability of s-two-chains.

Lemma 2.12. s-two-chains are excludable.

Proof of Lemma. We let G_1 be the graph obtained from G by two forced moves $b_o b_1 \to b_s b_o$, $b_o b_2 \to b_o b_1$ in this lemma.

Let $d_s \neq c$ first. Note now no d_i can be an internal vertex on a-b or b-c. We see $n_{G,b-c}(b) = n_{G,b-c}(b_2) = n_{G_1,b_2-d_s}(b_2)$(true by definition) $= n_{G_1,b_2-d_s}(d_{s-1})$. d_{s-1} cannot be joined by an A_s to d_s to avoid configuration as in Fig. 2-2. The only ways to edge-reconstruct from $G - d_{s-1}d_s$ are $G - d_{s-1}d_s + d_s a$ and $G - d_{s-1}d_s + d_s b$, to prevent T_{s-1}. If the new edge is $d_s a$, an "old" $(n_G(b)+1)$-star at b is "destroyed", and the edge-reconstructability of $(n_G(b)+1)$-stars implies that there is an $(n_G(b)+1)$-star at d_{s-1} in $G - d_{s-1}d_s + d_s a$. But there are only $n_{G_1}(d_{s-1}) = n_G(b)$ A_s's with d_{s-1} as a big end in G_1, d_{s-1} should be joined by an A_s to b in G. In $G - d_{s-1}d_s + d_s a$ again, the fact that there is an $(n_G(b)+1)$-star at d_{s-1} says d_{s-1} lies on another S_s $d_{s-1} - e$ since $n_G(b) \geq 1$, and so we get an s-three-chain (e, d_{s-1}, b, c) if $e \neq c$ or s-three-cycle $\{d_{s-1}, b, c\}$ if $e = c$, a contradiction.

Now consider $H' \cong G - d_{s-1}d_s + d_s b$ (still $d_s \neq c$ assumed). In $H' - b_1 b$, $d_s - b$ is an S_1, $a - b_1$ is an S_{s-1} and there are $4 - 1 = 3$ ways to edge-reconstruct some G', namely replacing by $b_1 d_s$, ad_s and ab. By a discussion on $(n_G(b)+1)$-stars as before, we note that d_{s-1} is the "center " of an $(n_G(b)+1)$-star in H' and hence must be joined to a by an A_s in G.

If the replacing edge from $H' - b_1 b$ is $b_1 d_s$, then G' has $n_G(b)$ S_s's at b and $n_G(b)+1$ S_s's at d_{s-1} and no S_s at d_s while G has $n_G(b)+1$ S_s's at b, no S_s's at d_{s-1}, at most one S_s at d_s (easy to see $n_{G_2}(d_s) = 0$ and the only S_s's d_s can lie on in G are $d_s - a$ and $d_s - b$, $d_s - a$ is impossible by the s-three-chain (d_s, a, b, c)). Since no other S_s's is affected going from G to G', we see $n(G') - n(G) \geq n_G(b) + n_G(b) + 1 - (n_G(b)+1) - 1 = n_G(b) - 1 \geq 0$. $n(G') = n(G)$ then enforces $n_G(b) = 1$ and also that d_s

is joined to b by an S_s at the same time. But since $d_s \neq c$ by assumption, we see immediately that $n_G(b) \geq 2$, contradictory to the fact that $n_G(b) = 1$ just proved.

Next consider replacing by ad_s. The edge-reconstructability of $(n_G(b) + 1)$-stars now enforces the existence of an S_s joining b and b_1 or joining b and d_{s-1} in G'. The former case gives an excludable configuration in G as Fig. 2-2. The latter case gives us an s-three-chain or s-three-cycle in G' since d_{s-1} lies on $n_G(b)+1 \geq 2$ S_s's.

At last, we consider replacing by ab. The edge-reconstructability of $(n_G(b) + 1)$-stars entails that d_{s-1} and b_1 be joined by an S_s in G', or equivalently, a path of length s in G. Consider H' - b_1b_2. In this subgraph, b_2 lies on an S_{s-2} $b_2 - a$ and S_{s-1} $b_2 - d_{s-1}$, and $b_1 - d_s$ is an S_2. So $b_2b_1 \to b_2d_s$ is a "forced move" sending H' to some G_α since all vertices mentioned are distinct. But now in G_α, d_{s-1} lies on $n_G(b) + 2$ S_s's, the number of $(n_G(b)+1)$-stars is then found to be one less than that of G (b_1 cannot be the "center" of $(n_G(b)+1)$-star since $n_G(b) \geq 1$), a contradiction. We are now done for the case $d_s \neq c$.

Next consider the case $d_s = c$. First suppose $d_{s-1} \notin$ the S_s $b - c$ ($d_{s-1} \notin$ the S_s $a - b$ clearly otherwise we have a T_1). Consider G - $d_{s-1}d_s$. d_sb or d_sa must be a replacing edge to avoid a T_{s-1}. If d_sb is a replacing edge, the new graph contains a newly created $(s+1)$-cycle, namely the S_s $b - c$ followed by the edge cb, and Kelly's lemma implies the existence of an A_s joining d_{s-1} and d_s; but then in G_1, we have an excludable configuration as in Fig. 2-2.

If $d_s a$ is a replacing edge, the same argument as the case $d_s \neq c$ works and we are done when $d_{s-1} \notin b-c$.

Now suppose $d_{s-1} \in b-c$. Write $b-c$ as $c_0 c_1 \ldots c_s$ with $b = c_0$, $c = c_s$. Clearly $d_{s-1} = c_{s-1}$. Let $k < s-1$ be the biggest integer such that $d_k \notin c_0 c_1 \ldots c_s$ (we have $d_{k+1} = c_{k+1}, d_{k+2} = c_{k+2}, \ldots, d_{s-1} = c_{s-1}$, $d_s = c_s$ then). Clearly $k \geq 2$ (otherwise we have a cycle of length ≤ 6). Consider $G - d_k d_{k+1}$. Since $c_0 d_{k+1}$ and $d_{k+1} c_s$ are forbidden S_{k+1}, S_{s-k-1} and a, b, d_k are the three "ends" of a forbidden T_k in this subgraph, we see that the only replacing edges are bc and $d_{k+1}a$. In the former case, Kelly's Lemma implies d_k and d_{k+1} are joined by a path of length s and so we can get a configuration as in Fig. 2-2 after number of appropriate forced moves. The latter case would imply that b and d_k are joined by an A_s $b - d_k$ in G. Let b' be adjacent to b on $b - d_k$. Suppose $b' \neq c_1, b_1$. Consider $G - bb' + bd$. If $d \neq a$ or c, then G_1 contains a T_{s-1}. If $d = a$ or c then bb' are joined by an A_s and in $H' = G - d_k d_{k+1} + d_{k+1} a$ we get an excludable configuration. Note b' cannot be c_1 otherwise H' contains a T_{s-1}. Hence $b' = b_1$ is the only possibility. Next consider $G - d_{k-1} d_k$. We can prove that b and d_{k-1} are joined by an A_s ($d_{k-1}c$, say, cannot be a replacing edge since $b - a$ and $b - d_k$ will form some T_p, $p < s$ by the fact $b' = b_1$). If b" is the vertex adjacent to b on the A_s $b - d_{k-1}$, we can show similarly that difficulty presents only when $b'' = b_1$. Then we can consider $G - d_{k-1} d_{k-2}$, show d_{k-2} and b are joined by an A_s, and we can assume the vertex $b^{(3)}$ adjacent to b on the A_s $b - d_{k-2}$ is b_1 again. Proceed in this way, we can at last show that d_1 and b are joined by an A_s

and if $b^{(k)}$ is the vertex adjacent to b on $d_1 - b$, then we can assume $b^{(k)} = b_1$. But now observe that $b_2d_1 \to b_2c$ is a forced move which gives in some $H'' \cong H$ two S_s's which are not disjoint internally ($b - a$ and $b - d_1$ have at least b_1 in common), so H and hence G is edge-reconstructable. We are thus done for the proof of Lemma 2.12.

Q.E.D.

Remark. In the proof of Case 2 and Case 3 of Proposition 2.6 before, we have assumed e_{s-1} or $f_{s-1} \notin a - b$ to simplify discussion; their proofs will be essentially similar to the case $d_s = c$, $d_{s-1} \in b - c$ of Lemma 2.12.

Section 5. Use of some other minimum-distance-functions and proof of the main theorem.

By Lemma 2.12 of Section 4, we know that no two S_s's can intersect at any vertex, whether at a big vertex of degree d or a small vertex of degree δ. The S_s's now have no interconnection patterns and they are hence very "sparsely" distributed in the graph G. We will introduce two new "minimum-distance-functions" to handle this remaining case.

Recall in Section 2 we have proved that G cannot have only one small vertex by degree argument and G cannot have only two small vertices by principle of forced move. So G must have at least three small vertices. We will also assume G to be connected (see Corollaries 1.3.1 and 1.3.2.)

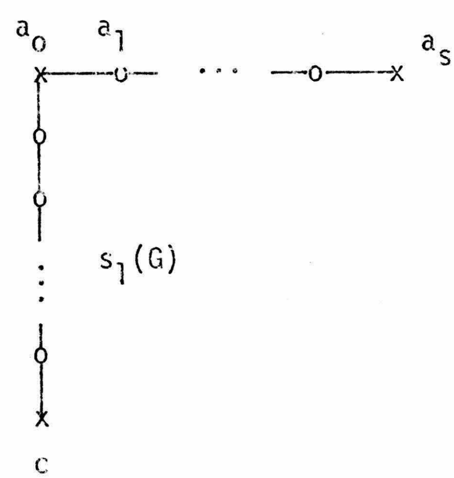
In G, given an S_s $a_0a_1 \ldots a_s$ and a small vertex c which does not lie on any S_s, we may ask : can we define the distance of c from the "line" $a_0a_1 \ldots a_s$ in a natural way? The answer is yes and is quite easy to implement. For G's connectivity tells us that c and a_0 are joined by some path P. Let a_i be the first vertex on the intersection of P and $a_0 \ldots a_s$. Then the "segment" Q of P traversed from c to a_i is disjoint from $a_0 - a_s$ except at a_i, and its length can be naturally thought of as the "distance of c to $a_0 - a_s$ along the path Q". With c and $a_0 - a_s$ fixed, we let Q range over all possible paths joining c and some a_i on $a_0 - a_s$ and also disjoint from $a_0 - a_s$ except at a_i, then the minimum of distance of c to $a_0 - a_s$ along the path Q over all Q's is the "distance" of c from $a_0 - a_s$. We denote it by $\rho_G(c, a_0 - a_s)$.

We define $s_1(G) \equiv \min \rho_G(c, a_0 - a_s)$ with c ranging over all small vertices not lying on any S_s, and $a_o - a_s$ ranging over all S_s's in G. $s_1(G)$ is our first minimum-distance-function to be used.

Now if say the minimum $s_1(G)$ is attained at some small vertex c and S_s $a_o - a_s$ (i.e. $s_1(G) = \rho_G(c, a_o - a_s)$), and the distance of c to $a_o - a_s$ is attained by a path Q joining c to a certain a_i on $a_o - a_s$, then an even number i implies that even number of forced moves will lead us to some $G' \cong G$ and in G' $a_o a_1 \dots a_s$ becomes $a_i a_{i+1} \dots a_{i-1}$ with a path Q of length $s_1(G)$ joining c to a_i and edge-disjoint from $a_i a_{i+1} \dots a_{i-1}$, and an odd number i implies that even number of forced moves will lead us to some $G'' \cong G$ and in G'' $a_o a_1 \dots a_s$ becomes $a_{i+1} a_{i+2} \dots a_i$ with a path Q of length $s_1(G)$ joining c to a_i and edge-disjoint from $a_{i+1} a_{i+2} \dots a_i$.

In any of the two cases just described, we can assume (renaming if necessary) in G that we have the configuration of an S_s $a_o a_1 \dots a_s$ and a small vertex c not on any S_s and a path Q of length $s_1(G)$ joining a_o to c and disjoint from $a_o \dots a_s$ except at a_o (note $s_1(G) = s_1(G') = s_1(G''))$. The situation is drawn as below:

G :



Now, for the unique nonisomorphic edge-reconstruction H of G, we can define $s_1(H)$ in a similar way. For the S_s $a_0a_1 \dots a_s$ just described, the forced move $a_0a_1 \to a_sa_0$ gives us $H' \cong H$, and in H', c is a small vertex not on any S_s's and there is a path joining a_0 and c of distance $s_1(G)$, so we see immediately $s_1(H') \leq d(a_0,c)$ (definition of $s_1(H')) \leq s_1(G)$, or $s_1(H) \leq s_1(G)$. A symmetry argument readily gives $s_1(G) \leq s_1(H)$ and so $s_1(G) = s_1(H)$ and we may denote it by s_1.

We define our second minimum distance function $s_2(G)$ as follows. Given any two S_s's $a_0a_1 \dots a_s$ and $b_0b_1 \dots b_s$ (they are disjoint by Lemma 2.12), we define their distance to be the minimum length of a path Q joining some a_i in $a_0 - a_s$ and b_j in $b_0 - b_s$, such that Q is disjoint from $a_0 - a_s$ and $b_0 - b_s$ except a_i and b_j. Denote this by $\rho_G(a_0 - a_s, b_0 - b_s)$. (This is conceptually the perpendicular distance of two skew lines in space). Define $s_2(G)$ to be the minimum of $\rho_G(a_0 - a_s, b_0 - b_s)$ as $a_0 - a_s, b_0 - b_s$ range over all distinct pairs of S_s's in G.

As in the case of $s_1(G)$, we may assume (by forced moves) that in G we have a configuration consisting of two different S_s's $a_0a_1 \dots a_s$, $b_0b_1 \dots b_s$ and a path Q of length $s_2(G)$ joining a_0 to b_0 which is disjoint from $a_0 - a_s$, $b_0 - b_s$ except at a_0 and b_0. We can define $s_2(H)$ in a similar way. By forced move and symmetry argument we have immediately $s_2(H) = s_2(G)$ and we may denote their common value by s_2. (We define s_2 to be ∞ if there is only one S_s).

We know that s_1 and s_2 are both greater than or equal to $s + 1$ by means of Lemma 2.12. Also recall that T_s is an excludable configuration (since after a certain number of forced moves, T_s becomes an s-two-chain); or equivalently, $n_{G,a_0a_1 \ldots a_s}(a_i) = 0$ for any vertex a_i on an arbitrary S_s $a_0a_1 \ldots a_s$. Before going too far, we will prove a useful result similar to Lemma 2.4 by utilizing Lemma 2.12:

Lemma 2.13. It is impossible that two small vertices be joined by an S_s and also an S_{s+1}.

Proof of Lemma. Suppose not, and let two small vertices a and b be joined by an S_s $c_0c_1 \ldots c_s$ and S_{s+1} $d_0d_1 \ldots d_{s+1}$, with $a = c_0 = d_0$, $b = c_s = d_{s+1}$. Without loss of generality, let $c_1 \neq d_1$ (otherwise applying a certain number of forced moves and we get the same condition in an isomorph of G or H as in Lemma 2.4). $ad_1 \to ab$ is a forced move otherwise the two S_s's $a - b$, $b - d_1$ form an s-two-chain excludable by Lemma 2.12. We see next $ac_1 \to ad_1$ is a forced move otherwise the S_s $a - c_1$ and A_s $b - d_1$ form a T_s. At last $ab \to ac_1$ is a forced move since in the isomorph of G with ab deleted, $b - d_1$ is an S_{s-1}, a is a vertex of degree $d - 2$.

Now the three forced moves $ad_1 \to ab$, $ac_1 \to ad_1$, $ab \to ac_1$ return us to the original graph G, so G is edge-reconstructable by Lemma 2.1, and we are done. Q.E.D.

Corollary 2.13. It is impossible to have a configuration C consisting of an S_{s+1} $a_0a_1 \ldots a_{s+1}$ together with a path $b_0b_1 \ldots b_p$ of length $p \leq s$ joining two adjacent vertices a_i and a_{i+1} for some i, with

$b_o = a_i$, $b_p = a_{i+1}$.

Proof of Corollary. Conceptually the configuration C has the form as in Fig. 2-8 below:

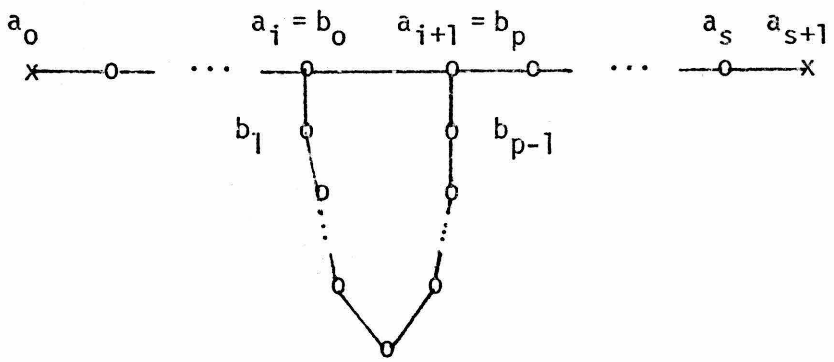


Fig. 2-8

It is conceivable that some b_j for $0 < j < p$ may be an internal vertex of $a_o a_1 \ldots a_s$. Suppose $i = 0$ first. $a_o a_1 \to a_{s+1} a_o$ is clearly a forced move (otherwise an s-two-chain results), which gives an S_s and an S_{s+1} joining a_o and a_1, impossible by Lemma 2.13. Now let $i > 0$. We see $a_o a_1 \to a_{s+1} a_o$ is again a forced move, for if not, then the configuration consisting of an S_s $a_1 - a_s$ and a path of length $p \leq s$ joining a_i and a_{i+1} can be transformed by means of $i - 1$ forced moves to a configuration excludable by Fig. 2-2 or the comment after it. Next, we see $a_1 a_2 \to a_o a_1$ is also a forced move using the same argument as before (when $i > 1$). Proceeding in this way, we see i forced moves will transform our configuration to one satisfying the condition $i = 0$ specified at the beginning (with reindexing of course). Q.E.D.

We will investigate the problem by comparing the values of s_1 and s_2 (in two cases).

Case 1. $s_2 \le s_1$

As remarked before, we assume in G a configuration C consisting of two S_s's $a_0a_1 \ldots a_s$ and $b_0b_1 \ldots b_s$, and also an S_{s_2} $c_0c_1 \ldots c_{s_2}$ (is the Q before) with $c_0 = a_0$, $c_{s_2} = b_0$, and no c_i lies on $a_0a_1 \ldots a_s$ or $b_0b_1 \ldots b_s$ for $0 < i < s_2$.

As a first reduction we will show that s_2 can be assumed to be equal to $s + 1$. So suppose $s_2 > s + 1$ now. Consider $G - c_{s_2}c_{s_2-1}$. If our edge-reconstruction $H' = G - c_{s_2}c_{s_2-1} + c_{s_2}d$, $d \ne a_0, a_s$, then $s_2 = s_2(H') \le s_2-1$ when d lies on an S_s in H' and $s_2 = s_2(H') \le s_1$ (assumption) $\le s_2-1$ when d does not lie on any S_s in H', both of them lead to the impossible inequality $s_2 \le s_2-1$; and so $d = a_0, a_s$. If $d = a$, then no matter whether b lies on an S_s or not in H', we will have $s_2(H') \le s + 1$ (since $s_2 \le s_1$ by assumption) and so $s_2 = s + 1$ since $s_2 \ge s + 1$. If $d = b$, then it is clear that $\rho_{H'}(b_0-b_s, c_0-c_s) = s + 1$ and hence $s + 1 \le s_2(H') \le \rho_{H'}(b_0-b_s, c_0-c_s)$ enforces $s_2 = s_2(H') = s + 1$.

In G, c_s lies on an A_s $c_s - c_0$. If c_s lies also on another c_s-g, with $g \ne a_0, a_s$, then $c_{s+1}c_s \to c_{s+1}c_0$ is a forced move by the excludability of s-two-chains (Lemma 2.12); and if $c_s - c_0$ is the only A_s in G on which c_s is a big end, $c_{s+1}c_s \to c_{s+1}a_s$ is a forced move by Lemma 2.12 and also the edge-reconstructability of number of S_s's (Lemma 2.5). (Note that c_sc_{s+1} is a forced edge if

c_s lies on at least two A_s's c_s-g_1, c_s-g_2 with g_1, g_2, a_0 all distinct (Lemma 2.12 again).)

Consider first the case that c_s lies on an A_s c_s-e, $e \neq a_0$; e may be a_s, b_s or some other small vertex in G (note e cannot be c_{s+1} by Corollary 2.13). Let $e \neq a_s, b_s$ first. Let e' be adjacent to e on the A_s c_s-e and consider G - e'e. If some H" = G - ee' + ef with $f \neq b_0$, then $\rho_{H''}(a_0\text{-}a_s, b_0\text{-}e') \leq s$ when $f \neq a_s$ and $\rho_{H''}(a_0, b\text{-}e') \leq s$ when $f = a_s$; both of these two inequality enforces $s_2 = s$, a contradiction, and so $f = b_0$. But then n(H") = n(G) implies that e' lies on S_s e' - g in H", $g \neq b_0$, or equivalently an A_s e'-g in G, $g \neq b_0$. Now for $H' = G - c_{s+1}c_s + c_{s+1}c_0$, we will have $n_{H',c_s\text{-}e}(e') = 1 \neq 0$, an impossibility, unless $g = a_0$ in which case $n_{H',c_s\text{-}e}(e') = 0$. But then ee' is a forced edge since otherwise either (a_s, a_0, e') or (e', b_0, b_s) will appear as an s-two-chain. We are done when $e \neq a_s$ or b_s.

If $e = b_s$, then b_0 and b_s are two small vertices joined in G by the S_s b_0 - b_s and also an S_{s+1} (b_0c_s followed by the A_s c_s - b_s) and G is edge-reconstructable by Lemma 2.13.

Now let $e = a_s$. Suppose first that a_1 and c_s are not joined by a path of length s in G. Consider $H_1 = G - a_0a_1 + a_sa_0$. In H_1, our original configuration C (mentioned in the beginning of Case 1) becomes a configuration C_1 consisting of the S_s's $a_0a_sa_{s-1} \ldots a_1$ and $b_0b_1 \ldots b_s$, and also an S_{s+1} $c_0c_1 \ldots c_{s+1}$ with $a_0 = c_0$, $b_0 = c_{s+1}$ and no c_i lies on $a_0a_s \ldots a_1$ or $b_0 \ldots b_s$, $0 < i < s + 1$.

Furthermore c_s and a_1 are not joined by an A_s in H_1. Now $c_{s+1}c_s \to c_{s+1}a_1$ is a forced move (Lemma 2.5 & 2.12), and in the new graph, a_0 and c_s are small vertices joined by a new S_s $a_0 - c_s$ and a new S_{s+1} (a_0a_s followed by the A_s $a_s - c_s$), impossible by Lemma 2.13. So a_1 and c_s are joined by a path of length s in G.

Since c_s is joined to a_s by an A_s by assumption, G contains a configuration C' of the same form as C; more explicitly, C' consists of two S_s's $a_sa_{s-1} \cdots a_0$, $b_0 \cdots b_s$ and an S_{s+1} joining a_s and b_0 (b_0c_s followed by the A_s $c_s - a_s$) which is disjoint form $a_s - a_0$ and $b_0 - b_s$ except at the "ends" a_s and b_0. By arguments as in the previous paragraph, a_{s-1} and c_s are joined by a path of length s in G.

Now, in $G - c_sc_{s+1}$, c_{s+1} is a vertex of degree $d-2$ (hence a forced vertex) and $\{a_0,a_s,c_s\}$ is an s-three-cycle, so the replacing edge can only be $c_{s+1}a_0$ or $c_{s+1}a_s$. In either case, we get a configuration excludable by Lemma 2.13 (if the replacing edge is $c_{s+1}a_s$, then in $G - c_sc_{s+1} + c_{s+1}a_s$, a_0 and c_s are joined by the S_s $a_0 - c_s$ and an S_{s+1} formed by a_0a_1 and the A_s $a_1 - c_s$; and when the replacing edge is $c_{s+1}a_0$, a_s and c_s are joined by an S_s and an S_{s+1} in a similar way).

We have done the subcase when c_s lies on some A_s c_s-e, $e \neq a_0$. We now know that a_s is the only small vertex in G which is joined to c_s by an A_s. a_1 cannot be joined to c_s by a path of length s in G, otherwise in $G - a_0a_1 + a_sa_0$, we are returning to the subcase that c_s lies on some A_s $c_s - e$, $e \neq a_0$ (actually $e = a_1$ is a small vertex in the new graph). a_2 may be or may be not joined to c_s by a path of length s in G. Suppose first that a_2 is not joined to c_s by a path of length s. Consider the isomorph G_β of G obtained from G by two forced moves : $a_oa_1 \to a_sa_o$, $a_1a_2 \to a_oa_1$. In $G_\beta - c_{s+1}c_s$, c_{s+1} is a forced vertex of degree $d-2$, and since the S_s $a_1a_oa_s \ldots a_3a_2$ and A_s $a_o - c_s$ together form an excludable T_s, $c_{s+1}a_1$ and $c_{s+1}a_2$ are the only two possible replacing "new" edges. But neither of these is possible since it will make $n(G_\beta) = n(G) - 1$ (since the S_s $a_1 - a_2$ is destroyed and no new S_s $a_1 - c_s$ or $a_2 - c_s$ can be created by assumption that a_1, a_2 are both not joined to c_s by a path of length s by assumption).

Hence assume a_2 is joined by a path of length s to c_s in G. Now $c_{s+1}c_s \to c_{s+1}a_s$ is a forced move, and let $H_\gamma = G - c_{s+1}c_s + c_{s+1}a_s$. We have three edges to replace for $H_\gamma - a_1a_2$, namely a_oa_2, a_1b_o, a_ob_o. If it is a_oa_2, then $n(G_\delta) = n(H_\gamma) - 1$ for $G_\delta = H_\gamma - a_1a_2 + a_oa_2$ since a_1 lies on no A_s's in G ($n_{G,a_o-a_s}(a_1) = 0$ and a_1 is not joined to c_s by a path of length s), and we get a contradiction. If it is a_1b_o, then the new graph contains an s-two-chain (a_o,c_s,a_2), and we are done. So let the replacing edge be a_ob_o. Then $n(G_\delta) = n(H_\gamma) - 1$ for $G_\delta = H_\gamma - a_1a_2 + a_ob_o$ since two S_s's $a_o - c_s$ and $b_o - b_s$ in H_γ are destroyed, but only one S_s $a_2 - c_s$ is created, and we get a contradiction to Lemma 2.5.

Now that we have also done the subcase that c_s does not lie on any A_s $c_s - e$, $e \neq a_o$, we are done with the proof of Case 1.

Case 2. $s_1 < s_2$.

Now in G we can assume the existence of a configuration D consisting of an S_s $a_0a_1 \ldots a_s$ and an S_{s_1} $c_0c_1 \ldots c_{s_1}$ with $c_0 = a_0$, no c_i lies on $a_0a_1 \ldots a_s$, $0 < i \leq s_1$, and c_{s_1} lies on no S_s in G.

As in Case 1, our first reduction will be to show that s_1 can be assumed to be $s + 1$. Consider $G - c_{s_1}c_{s_1-1}$. If $H' = G - c_{s_1}c_{s_1-1} + c_{s_1}d$ is an edge-reconstruction with $d \neq a_0$ or a_s, then $s_1 = s_1(H') \leq s_1 - 1$ if c_{s_1-1} does not lie on any S_s in H' and $s_1 < s_2(H') \leq s_1 - 1$ if c_{s_1-1} lies on an S_s in H'. Both lead to the result that $s_1 < s_1$ and are hence impossible. So $d = a_0$ or a_s. Let $d = a_s$ first. $n(H') = n(G)$ implies that c_{s_1-1} is on an S_s $c_{s_1-1} - e$ in H'. If e isn't a_0, then $\rho_{H'}(a_0, c_{s_1-1} - e) \leq s_1 - 1$ implies $s_1 = s_1(H') \leq s_1 - 1$, a contradiction, and so $e = a_0$. But then $c_0 - c_{s_1-1}$ is an S_s and we have $s_1 = s + 1$ in this case.

Next let $d = a_0$, and suppose $s_1 > s + 1$. We will prove a contradiction. c_{s_1-1} must lie on an S_s $c_{s_1-1} - f$ in H' since $n(H') = n(G)$. If $f = a_s$, then $\rho_G(c_{s_1}, a_0 - a_s) \leq 1 + s$, and we have $s_1 = s + 1$. If $f = c_{s_1}$, then $\rho_G(a_s, c_{s_1-1} - c_{s_1}) \leq 1 + s$, and we have again $s_1 = s + 1$. So let $f \neq a_s, c_{s_1}$. Consider $G - c_{s_1-1}c_{s_1-2}$. In this edge-deleted subgraph, we see that c_{s_1-1} and c_{s_1-2} are two adjacent small vertices, and c_{s_1-2} is a small vertex with the "distance" of c_{s_1-2} and $a_0 - a_s$ equal to $s_1 - 2 < s_1$. To edge-reconstruct some isomorph H'' of H from $G - c_{s_1-1}c_{s_1-2}$, the replacing edge must have one of its end be equal to c_{s_1} or c_{s_1-1}, and the other end be one of c_{s_1-2}, a_o, a_s; and we have $6 - 1 = 5$ subcases to consider, namely $c_{s_1-2}\, c_{s_1}$, $a_o c_{s_1-1}$, $a_o c_{s_1}$, $a_s c_{s_1}$, $a_s\, c_{s_1-1}$. For illustration, let E be the configuration consisting of the S_s $a_o - a_s$, S_{s_1} $c_o - c_{s_1}$, and A_s c_{s_1}-f.

First consider the subcase when the replacing edge is $c_{s_1-2}\, c_{s_1}$. This subcase is trivial for clearly $n(H") = n(G) + 1$ (since $c_{s_1} - f$ is a new S_s and no S_s at c_{s_1} can be "destroyed" by assumption of our configuration D) which leads to a contradiction.

Secondly let the replacing edge be $a_o c_{s_1-1}$. Then $n(H") = n(G)$ says that c_{s_1-2} lies on an S_s $c_{s_1} - g$ in H", g clearly unequal to a_o. Now $c_{s_1} c_{s_1-1} \to c_{s_1} a_o$ is a forced move sending G to H', and we note $\rho_{H'}(g, c_{s_1} - f) \leq s + 1$ enforces $s_1 = s + 1$ if $g \neq f$. But when $g = f$, c_{s_1-1} and f are clearly joined by an S_s and an S_{s+1}, and this possibility is excluded by Lemma 2.13. Hence we have done the subcase when the replacing edge is $a_o c_{s-1}$.

Then we let the replacing edge be $a_o c_{s_1}$.

Since $\rho_{H"}(a_s, c_{s_1-1} - f) \leq 2 + s$, we see $s_1 = s + 2$ by the assumption $s_1 > s + 1$. Next we note we can edge-reconstruct $G' = H" - a_{s-1}a_s + a_s g$. If $g \neq c_{s+1}$ or f, then $\rho_{G'}(a_{s-1}, c_{s+1} - f) \leq s + 1$ implies $s_1(G') = s + 1$, and we are done. Now if $g = c_{s+1}$ or f, then $n(G') = n(G)$ says that a_{s-1} lies on an A_s $a_{s-1} - h$ in G. Note that c_{s-1} must lie on an A_s $c_{s-1} - i$, $i \neq c_{s+1}$, f in H" (otherwise $c_s\ c_{s-1}$ is obviously a forced edge). In H", we have a configuration F as illustrated below in Fig. 2-9. Note $i \neq a_s$, otherwise $s_2 = s + 2 = s_1$.

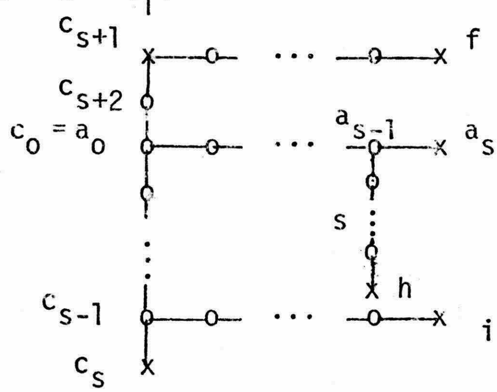


Fig. 2-9

Note h may coincide with c_s or not. If $h = c_s$, then in $G' = H" - a_{s-1}a_s + a_s g$ with $g = c_{s+1}$ or f, $\rho_{G'}(i, a_{s-1} - c_s) \leq 1 + s$ enforces $s_1 = s + 1$. So let $h \neq c_s$. But then it is easy to see that $c_{s+2}\ c_o$ is a forced edge. (Any other way of replacing $c_{s+2}\ c_o$ by a new edge will create a T_s configuration except $c_{s+1}\ c_o$. And replacing by $c_{s+1}\ c_o$ enforces the existence of an A_s c_{s+2}-j in H"; so since the "distance" of j and c_{s+1} - f is $\leq s + 1$, we get $s_1 = s + 1$ again). We are done for the subcase that the replacing edge is $a_o\ c_{s+2}$.

Next, let the new edge be $a_s c_{s_1}$. $\rho_{H"}(a_o, c_{s_1-1} - f) \leq 1 + 1 + s = s + 2$ implies that $s_1 = s + 2$ and so $c_o c_1 \cdots c_{s_1-2}$ is a new S_s in H". We then have $n(H") = n(G) + 1$, a contradiction (note c_{s_1} does not lie on any S_s in G, neither does a_s).

Finally, let's consider the subcase when the replacing edge is $a_s c_{s_1-1}$. Since $n(H") = n(G)$, c_{s_1-2} lies on some S_s $c_{s_1-2} - g$ in H".

g must be a_0 otherwise $\rho_{H''}(a_0, c_{s_1-2} - g) \leq s_1 - 2$ implies $s_1 \leq s_1 - 2$, a contradiction. We then see immediately that $s_1 = s + 2$.

Consider $G - c_2c_3$ now. We can replace new edges by three ways: c_0c_3, c_0c_{s+2} and c_2c_{s+2} since $c_0 - c_2$ is an S_2 and $c_3 - c_{s+2}$ is an S_{s-1} in $G - c_2c_3$. Denote by $H^{(4)}$ the new graph obtained.

Let c_0c_3 be the new edge first. Then $n(H^{(4)}) = n(G)$ enforces that c_2 lie on an S_s $c_2 - h$ in $H^{(4)}$. We see c_2 lies on an A_s $c_2 - h$ in G now. If $h \neq a_s$, we have a configuration in G of the same form as F in Fig. 2-9 by the same kind of argument over there. The argument following Fig. 2-9 then shows that $s_1 = s + 1$.. Now let $h = a_s$, and rewrite $c_2 - a_s$ as $e_0e_1 \ldots e_s$, $c_2 = c_0$, $a_s = e_s$.

Consider $G - c_0c_1$ now. c_0 is clearly a forced vertex and if $H^{(5)} = G - c_0c_1 + c_0j$, $j \neq a_s$, then $\rho_{H^{(5)}}(c_1, c_0 - a_s) \leq 1 + s$ implying $s_1 = s + 1$ by the fact that c_2 and a_s are joined by a path of length s. Hence $j = a_s$ is the only possibility. But then, in $H^{(5)}$, $\rho_{H^{(5)}}(c_{s+2}, c_1 - c_{s+2}) \leq s + 1$ implying that $s_1 = s + 1$, and we are done for the subcase the replacing edge is c_0c_3 (and $s_1 = s + 2$).

Now under the assumption $s_1 = s + 2$ we have seen a few interesting facts. For the configuration D consisting of an S_s $a_0 - a_s$ and an S_{s+1} $c_0 - c_{s+1}$ with $c_0 = a_0$, $c_i \neq$ any a_j for $i > 0$, and c_{s+1} not on any S_s, we see that c_1 cannot lie on any A_s (by Lemma 2.13 or $s_1 > s + 1$). The previous argument also shows that c_2 cannot lie on any A_s in G. (Note also that c_{s+1} must lie on an A_s $c_{s+1} - f$

otherwise $c_{s+1}\, c_{s+2}$ is clearly a forced edge).

Next, let c_0c_{s+2} be the edge replacing c_2c_3 in G. Since $n(H^{(4)}) = n(G)$, c_2 or c_3 must lie on an S_s in $H^{(4)}$. The possibility that c_2 lies on some S_s $c_2 - h$ with $h \neq c_3$ is already excluded. Consider the case that $c_2 - c_3$ is an S_s in $H^{(4)}$. Then c_2 and c_3 are joined by an S_s and also an S_{s+2} $c_3 \cdots c_{s+2}c_0c_1c_2$. Such configuration can be shown to be excludable in a way similar to Lemma 2.5 or Lemma 2.13. More explicitly, let's write the S_s joining c_2 and c_3 by $g_0g_1 \ldots g_s$ and the S_{s+2} by $h_0h_1 \ldots h_{s+2}$ with

$g_0 = h_0 = c_2$, $g_s = h_{s+2} = c_3$. By means of forced moves, we can assume $g_1 \neq h_1$. Then $h_0h_1 \to h_0h_{s+2}$, $g_0g_1 \to h_0h_1$, $g_0g_s \to g_0g_1$ are three forced moves returning us to G. The fact $s_1 = s + 2$ is used twice in the proof. So now assume c_3 is on an A_s $c_3 - i$ in G. The original configuration E in G becomes in $H^{(4)}$ a configuration $E^{(4)}$ in which c_2 is the small vertex of distance $s + 2$ form the S_s $c_3 - i$ Hence c_1 must lie on an S_s $c_1 - j$, $j \neq c_2, c_3, i$ in $H^{(4)}$ (otherwise c_2c_1 is a forced edge). Then j is a true small vertex in G, and we see $\rho_G(j, a_0 - a_s) \leq s + 1$ implies $s_1 = s + 1$. We note that a_3 cannot lie on any A_s in G in the configuration $E \subseteq G$.

At last, we let $c_2\, c_{s+2}$ be the edge replacing c_2c_3 in G. It readily follows that c_4 lies on some A_s $c_4 - j$ in G. In $G - c_3c_4$, $c_0 - c_3$ is an S_3, $c_4 - c_{s+2}$ is an S_{s-2}, and so we have three ways to replace c_3c_4 to get $H^{(5)}$, namely c_0c_4, c_0c_{s+2} and c_3c_{s+2}, if $s > 3$. If it is c_0c_4, then $n(H^{(5)}) = n(G)$ says a_3 lies on some A_s

in G, a situation already excluded. If it is $c_0 c_{s+2}$, then E becomes $E^{(5)}$ with c_3 the small vertex of distance $s + 2$ from the S_s $c_4 - j$, and so c_2 lies on an A_s in G, another impossible situation. If it is $c_3\, c_{s+2}$, then $n(H^{(5)}) = n(G) + 1$, contradiction.

So we are done except the case $s_1 = s + 2$ and $s = 3$. Since the details for this case are rather lengthy we will skip its proof here and leave it in Appendix 2-A.

So far we have finished the "elementary" reduction that s_1 can be assumed to be $s + 1$. Consider our configuration D again, which is described at the beginning of Case 2. $c_s - a_0$ is an A_s in G. c_s may lie on some other A_s $c_s - f$, $f \neq a_0$, or c_s may not.

Subcase 2.(a) c_s lie on an A_s $c_s - a_s$.

Note that c_s cannot lie on more than one A_s's $c_s - f$, $c_s - g$ with f, g, a_0 all distinct, for otherwise $c_s\, c_{s+1}$ is clearly a forced edge. Let's consider the case $f = a_s$ first. The proof will be very similar to that for Case 1. If a_1 and a_s are not joined by an A_s in G, then for $H_1 = G - a_0 a_1 + a_s a_0$, $c_{s+1} c_s \to c_{s+1}\, a_1$ is a forced move since a_1 and c_s are not joined by an A_s in H_1; this then gives two small vertices joined by an S_s and an S_{s+1} in the new graph, impossible by Lemma 2.13. So a_1 and c_s are joined by an A_s in G. Symmetry argument then says that a_{s-1} and c_s are joined by an A_s in G as well. Since the only ways we can "replace" $c_{s+1}\, c_s$ are $c_{s+1}\, a_0$ or $c_{s+1}\, a_s$, we see that our new graph will contain an S_s and an S_{s+1} joining two small vertices a_0 and c_s (a_s and c_s resp.) if the replacing edge is $c_{s+1} a_s$ ($c_{s+1} a_0$ resp.).

We are done when $f = a_s$.

Subcase 2(b) c_s does not lie on any A_s $c_s - f$, $f \neq a_0$, in G.

(The proof essentially the same as in Case 1.)

In particular, we know that c_s and a_s are not *joined* by an A_s. It's obvious that $c_{s+1}c_s \to c_{s+1}a_s$ is a forced move sending G to some H'. If a_1 is joined to c_s by a path of length s (this path cannot contain c_{s+1} otherwise $s_1 \leq s - 1$), then in H' we have c_0 and c_s joined by an S_s and an S_{s+1}, impossible by Lemma 2.13. So a_1 and c_s are not joined by a path of length s. If a_2 and a_s are neither joined by a path of length s, then consider the isomorph G_β of G obtained from G by two forced moves: $a_0a_1 \to a_sa_0, a_1a_2 \to a_0a_1$. Now it is easy to see that $c_{s+1}c_s$ is a forced edge, for c_{s+1} is a forced vertex in $G - c_{s+1}c_s$, we have to replace by $c_{s+1}a_1$ or $c_{s+1}a_2$ to avoid a T_s, and doing any of them will cause the number of S_s's in the new graph to be 1 less than that of G since a_1, a_2 are not joined to c_s by an S_s now.

So a_2 is joined to c_s by a path of length s. Consider $H_\gamma = G - c_{s+1}c_s + c_{s+1}a_s$. In H_γ, we see, as in Case 1, that we have three edges a_0a_2, a_1a_{s+1}, a_0c_{s+1} to replace a_1a_2 (to get a new graph G_δ). If it is a_0a_2, we have $n(G_\delta) = n(H_\gamma) - 1$. If it is a_1c_{s+1}, we get an s-two-chain (a_0, c_s, a_2) in G_δ. If it is $a_0\ c_{s+1}$, we are returning to subcase 2(a) with our new configuration D' now consists of the S_s $a_2 - c_s$, an S_{s+1} joining a_2 and a_1, and a_0 is adjacent to a_1 on the S_{s+1} $a_2 - a_1$ with a_0 and c_s also lying on an A_s. An easy way to see it is considering $a_2 = a'_o$, $c_s = a'_s$, $a_o = c'_s$, $a_1 = c'_{s+1}$. (This argument also holds in Case 1.)

Subcase 2(c) c_s lies on some A_s $c_s - f$, $f \neq a_s$.

Now consider $f \neq a_s$. We can replace c_oc_1 by c_of or c_oc_{s+1} since in $G - c_oc_1$, c_1, f, c_{s+1} are the three ends of a forbidden T_s. We cannot replace by c_of for otherwise in the new graph H', $\rho_{H'}(c_1 - c_{s+1}, a_o - a_s) \leq 1 + s$ enforces $s_2 = s + 1$, contradictory to the assumption (and fact) that $s < s_1 < s_2$. Hence in H', we have a configuration F' as in Fig. 2-10.

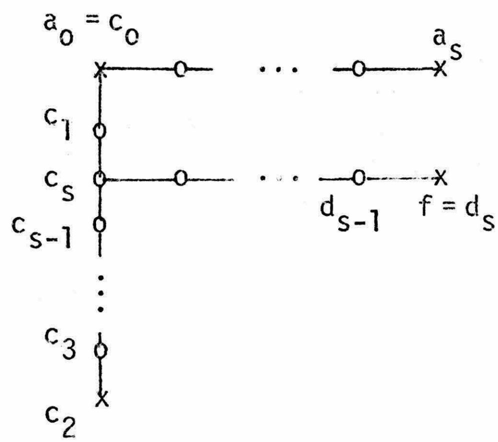


Fig. 2-10

We note f cannot be a_s or a_o by discussion before. f cannot be c_2 since c_2 is a big vertex in G while f is small in G. Rewrite $c_s - f$ as $d_od_1 \ldots d_s$, $c_s = d_o$, $f = d_s$. d_{s-1} cannot be an internal vertex of $a_o - a_s$ or $c_o - c_{s+1}$ by obvious argument.

Consider $H' - d_{s-1}d_s$. Suppose we edge-reconstruct $G'' = H' - d_{s-1}d_s + d_se$, $e \neq a_o, a_s, c_2$. We see immediately that c_oc_1 is a forced edge; for c_0 is obviously a forced vertex, and the new edges should be c_0d_{s-1} or c_0c_2; but for both cases we easily get the unhappy consequence that $s_2 = s$. Hence e must be one of a_0, a_s, or c_2.

For the configuration $D \subset F'$ consisting of an S_s $a_0 - a_s$ and an S_{s+1} $c_0\ c_{s+1} \ldots c_3c_2c_1$ with $a_0 = c_0$, we see that c_{s+1} cannot lie on an A_s $c_{s+1} - g$ in H', $g \neq c_1$. For if $g = a_s$, then we have an excludable configuration by Lemma 2.13. So let $g \neq a_s$. Let g' be adjacent to g on $c_{s+1} - g$. g' cannot lie on $a_0 - a_s$ or $c_0 - c_1$ by trivial argument. We can edge-reconstruct $G'' = H' - gg' + gh$. h must be a_0 or a_s otherwise G'' contains an s-two-chain. If $h = a_s$, then G'' contains a T_s ($a_0 - g'$ as S_s, $c_1 - c_{s+1}$ as A_s). So $h = a_0$. $n(G'') = n(H')$ says that g' lies on an A_s $g' - i$ in H'. Conceivably i may coincide with c_1 or not. From our configuration F' on p. 81, we see c_2 must lie on some A_s $c_2 - j$ in H' by subcase 3(b), j cannot be a_s by subcase 3(a). j may coincide with d_s or not. j cannot be c_1 otherwise c_2c_1 is easily seen to be a forced edge.

First let $i \neq c_1$, $j \neq d_s$. In $H' - c_0\ c_{s+1}$, c_0 is a forced vertex, and we have two T_s's (one has S_s $c_{s+1} - g$, A_s $g' - i$, the other has S_s $c_s \ldots c_1$, A_s $c_2 - j$). So $c_0\ c_{s+1}$ is a forced edge in order to avoid any T_s's. If $i = c_1$ or $j = d_s$, we can show easily that the new graph G_α will have a T_s-configuration or will satisfy $s_2(G_\alpha) = s + 1$, which is impossible. As an illustration, let $i = c_1$, $j = d_s$. In $H' - c_0c_{s+1}$, we have only one T_s (S_s $c_{s+1} - g$, A_s $g' - i$), and we may replace by c_0g or c_0i to avoid a T_s. But then we will have $\rho_{G_\alpha}(a_0 - a_s, c_1 - c_{s+1}) \leq s + 1$ or $\rho_{G_\alpha}(a_0 - a_s, c_{s+1} - g) \leq s + 1$

enforcing $s_2 = s + 1$ which is impossible. So c_{s+1} cannot lie on any A_s $c_{s+1} - g$, $g \neq c_1$ in H'.

Returning to our discussion at the beginning of subcase 3(c). Consider our configuration D consisting of the S_s $a_0 - a_s$ and S_{s+1} $c_0 - c_{s+1}$ again. Recall that c_s must lie on an A_s $c_s - f$ in G with $f \neq a_s$. We have seen that $c_0c_1 \to c_0c_{s+1}$ is a forced move sending G to H' and c_2 lies on an A_s $c_2 - j$ in H'. Since $j \neq c_1$, j is a true small vertex in G; and look at our D again, we see c_2 lies on some c_2-j). Rename j by c_2', f by c_s'. Consider $G - c_1c_2$. In this subgraph c_0c_1 is an S_1, $c_2 - c_{s+1}$ is an S_{s-1} and so c_0c_2, c_0c_{s+1} and c_0c_{s+1} are the only three possible edges to replace c_1c_2. Joining c_0c_2 would enforce, by edge-reconstructability of number of S_s's, that c_1 lies on an S_s $c_1 - c_1'$ in the new graph. Since c_1' is a small vertex in G, c_1 lies on an A_s $c_1 - c_1'$, a situation excluded in the previous paragraph. Joining c_1c_{s+1} would give $\rho(c_0 - c_s, c_2 - c_2') \leq s + 1$ in the new graph implying $s_2 = s + 1$, a contradiction. Hence $c_1c_2 \to c_{s+1}c_0$ is a "forced move" sending G to some H_2 in which our configuration D becomes some D_2 consisting of the S_s $c_2 - c_2'$ and S_{s+1} $c_2c_3 \ldots c_{s+1}c_0c_1$. Repeating the same argument for the configuration D_2 in the graph H_2, we see c_4 will lie on an A_s $c_4 - c_4'$ in H_2 and the forced move $c_3c_4 \to c_1c_2$ will send H_2 to G_4 with D_2 becoming D_4 consisting of the S_s $c_4 - c_4'$ and S_{s+1} $c_4c_5 \ldots c_{s+1}$ $c_0c_1c_2c_3$. Furthermore c_6 would lie on an A_s $c_6 - c_6'$. Proceeding in this way, we see that since s is odd, $(s-1)/2$ forced moves will send us to G_{s-1} or H_{s-1} depending on whether the residue of s modulo 4 is 1 or 3; in this new graph, the "old" configuration D

becomes D_{s-1} consisting of the S_s $c_{s-1} - c'_{s-1}$ and S_{s+1} $c_{s-1}c_sc_{s+1}c_0 \dots c_{s-2}$. The small vertex c'_s in G is still a small vertex in this new graph since $c'_s \neq c_0, c_{s+1}$ ($c'_s \neq c_0$ is the definition of subcase 3(c), $c'_s = c_{s+1}$ would enforce c_sc_{s+1} to be a forced edge by Lemma 2.5 and Lemma 2.12) and thus is unaffected by the sequence of forced moves. But then we again get a situation excluded in the previous paragraph. Note that in the proof, we do not treat the three cases $e = a_0, a_s, c_1$ separately.

Now that we have done subcase 2(c), we have proved Case 2 completely since these three subcases are exhaustive (and mutually exclusive). Combining the results of Case 1 and Case 2, we are ready to state (and claim having proved) the following:

Proposition 2.14. If s-two-chains are excludable, then a bi-degreed graph G (with at least four edges) is edge-reconstructable.

With Lemma 2.12 and Proposition 2.14, we conclude immediately our main theorem:

Theorem 2.1. (MAIN THEOREM OF CHAPTER 2) Every bi-degreed graph G with at least four edges is edge-reconstructable.

Section 6. Brief digression of generalization of methods.

Bi-degreed graphs are a natural "next step" when people have done the trivial regular graphs (of one degree only). And this "next step" is terribly hard to prove. After this is done, one might think: what is the next family of graphs we can do? Tri-degreed graphs might seem a natural approach. Its solution is trivial unless the three degrees are d, $d + 1$, $d + 2$; d, $d + 1$, $d + 3$; or d, $d + 2$, $d + 3$. (See! we have more annoying cases to do). It does not sound trivial to generalize results of bi-degreed graphs to graphs with three, four, five, ... etc. degrees.

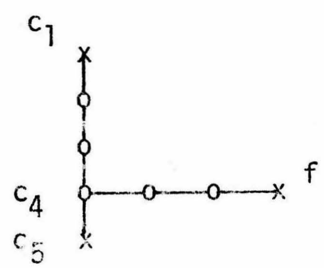
Most of the methods (and concepts) in this chapter however can be generalized to graphs such that its minimum degree δ and the next to minimum degree d differ by 1. For example, *if there exist two vertices of degree* δ *and a path joining them with all "internal" vertices of degree* d, we can then define $s(G)$ in a way as in Section 2, and we can show that G is edge-reconstructable. Under the same assumption, we can show the validity of Lemma 2.3, Lemma 2.5, Proposition 2.6, Lemma 2.11, Lemma 2.12 etc., but not Proposition 2.14 (i.e. s-three-chains can be shown to be excludable, but s_1 and s_2 may be hard to define). Note that G may contain vertices of degree δ and degree $d = \delta + 1$ but no paths joining vertices of degree δ with all internal vertices of degree d .

Appendix 2-A
Proof of the subcase $s_1 = s + 2$, $s = 3$ on p. 79

In $G - c_1c_2$, c_oc_1 is an S_1, c_2, c_5, and f are the three small vertices of a T_3. So we have $6 - 1 = 5$ ways, namely c_oc_2, c_oc_5, c_1f, c_1c_5, c_of, to replace the edge c_1c_2. If it is c_oc_2, we get immediately $s_1 = 4$, and if it is c_1f, we have $s_2 = s_1 = 5$, contrary to our assumption that $5 = s_1 < s_2$. If it is c_oc_5, we will get a contradiction by the same argument leading to an excludable configuration as in Fig. 2-9 (i.e. if we reindex, then some c_2' lies on an A_s in a configuration D' of the same form as D). If it is c_1c_5, then we get a contradiction as in the case of c_oc_{s+2} replacing c_2c_3, in other words, after reindexing, some c_3'' lies in an A_s in some D'' "congruent" to D, an already excluded situation. Hence we are left with the case of c_of replacing c_1c_2. And we see some $H_\mu = G - c_1c_2 + c_of \cong H$ ($c_1c_2 \to c_of$ is then a forced move).

Consider in G again our configuration D (as depicted in Fig. 2-A(a)), $a_oc_1 \to a_of$ is a forced move sending G to H' and D to D' as in Fig. 2-A(b). (Note we cannot replace by a_oc_5 otherwise a configuration as in Fig. 2-9 results; we cannot replace by a_od, $d \notin D$, otherwise H' contains an excludable configuration T as below.

T :



T is easily shown to be excludable since c_4c_5 is obviously a forced edge for otherwise $s_1 = 3 + 1 = 4$, a contradiction.)

In H', apply the arguments in this appendix one paragraph before, say let g be adjacent to f on $c_4 - f$, we see $fg \to a_oc_1$ is a forced move sending H' to G'' (and D' to D'') as in Fig. 2-A(c). (Heuristically, think $c_1' = f$, $c_2' = g$, $f' = c_1$.) Then consider $G'' - a_of$. If fg is the replacing edge, we are returning to D after three forced moves, and so G is edge-reconstructable by Lemma 2.1. (see Fig. 2-A(d)). The only remaining possibility is that we join c_5f. But for this we can prove contradiction easily by looking at $G - c_2c_3$ and consider all possible replacing edges (say some of them will lead to $s_2 = 5$, impossible).

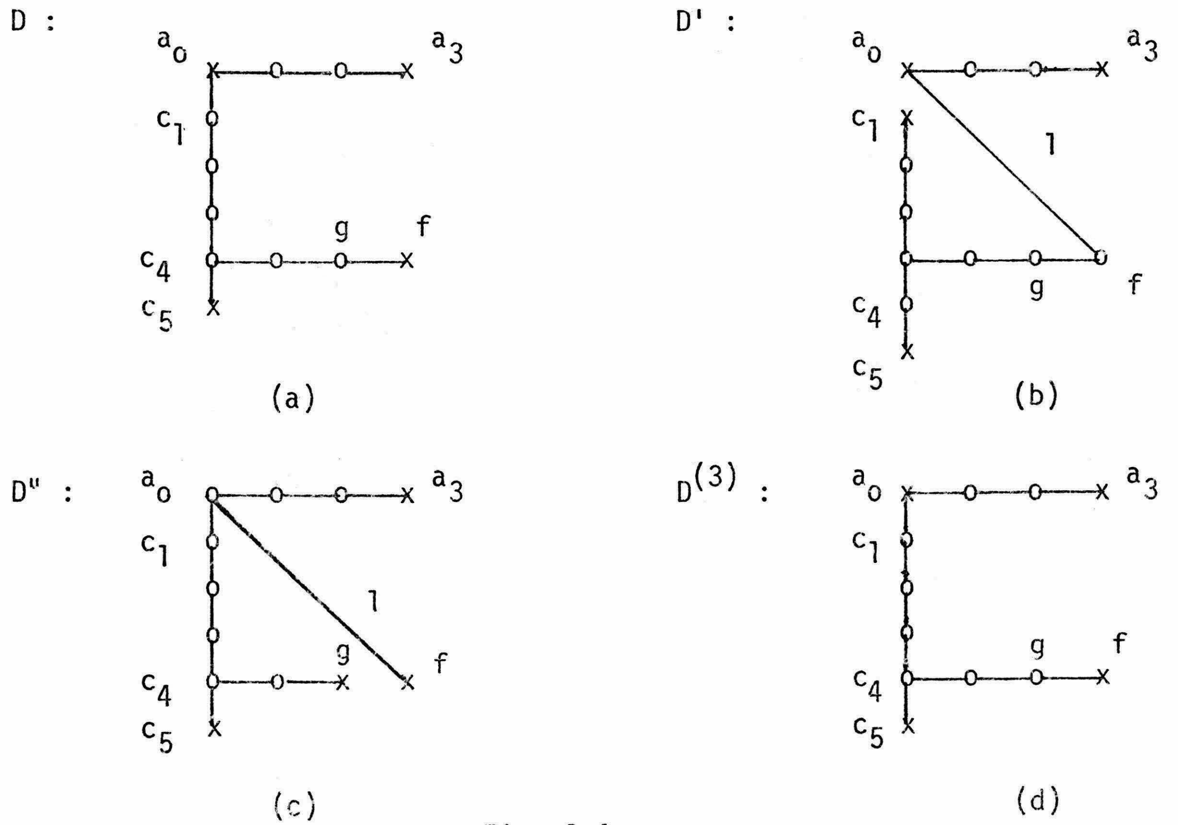


Fig. 2-A

Chapter 3. Edge-reconstruction of bipartite graphs

Section 1. Introduction

In this chapter, we will investigate the edge-reconstructability of bipartite graphs, i.e. graphs G whose vertex set $V(G)$ can be partitioned into two subsets $V_1(G)$ and $V_2(G)$ such that every edge of G joins $V_1(G)$ with $V_2(G)$.

A simple necessary and sufficient condition for a graph to be bipartite is that all its cycles are of even length (see p. 18 of F. Harary [7] for proof). Trees are then special cases of bipartite graphs since they are acyclic. Since the reconstruction problem of trees has been done quite deeply and extensively, it then comes naturally to investigate the (edge-) reconstructability of bipartite graphs. In J. A. Bondy and R. L. Hemminger [5], they pointed out that the reconstruction of bipartite graphs is a challenging open problem and they singled out the edge-version as Problem 9 of their survey paper.

This chapter solves that problem in full force by Theorem 3.1 (in Section 7) stated as follows:

MAIN THEOREM. Every bipartite graph with at least four edges is edge-reconstructable.

As in Chapter 2, we will start to build a list of *excludable configurations* until at last the list is big enough to cover every bipartite graph with at least four edges. Since we have in general more than two kinds of degrees for our graph G, we will use the small circles o to represent vertices; vertices will be labeled by lower case Latin letters a, b, c, ... etc. (with or without subscripts), their degrees denoted by Greek letters α, β, γ, ... etc. or Arabic numerals 1, 2, 3, ... etc. If we want to mention labeling as well as degree, we write the labeling followed by a comma, and then the degree. As an illustration, suppose a vertex a of degree α is joined to a vertex b of degree 4, then we have three different ways to represent them diagramatically as in Fig. 3-1 below:

(a) labelings only

(b) degrees only

(c) labelings and degrees

Fig. 3-1

Section 2. Elementary results

First, we show that bipartite graphs are edge-recognizable. Suppose G is bipartite and H is an edge-reconstruction of G which is not bipartite. We will derive a contradiction.

Note that a graph is bipartite if and only if it contains no odd cycles, we see at once that H contains an odd cycle C_n. If H has an edge ef not on C_n, then H - ef contains C_n and so $G \supseteq G - ef \cong H - ef$ has an odd n-cycle, contradictory to the fact that G is bipartite.

Hence $H = C_n \cup \overline{K}_m$, i.e. H is the disjoint union of an (odd) n-cycle and $m \geq 1$ isolated vertices. Obviously $n \geq 5$ since H has at least four edges. Now all $G - ef \cong H - ef$ are of the form $P_n \cup \overline{K}_m$, the union of a path of length $n - 1$ and m isolated vertices for any edge ef on C_n. Clearly G, nonisomorphic to H, will either be of the form $P_{n+1} \cup \overline{K}_m$ or contain some C_k, k even, as an edge-proper subgraph. The former says some $G - gh \cong P_{n-1} \cup K_2 \cup K_m \not\cong P_{n+1} \cup \overline{K}_m$; and the latter says some $G - gh \cong P_n \cup \overline{K}_m$ contains C_k as a subgraph; both lead to contradiction, and we see bipartite graphs are edge-recognizable.

Next, we will prove that G can be assumed to be connected. Logically, we will show that if all connected bipartite graphs are edge-reconstructable, then all bipartite graphs are edge-reconstructable. (All graphs assumed to have at least four edges). Recall Lemma 1.2., which says that the degree sequence is edge-reconstructable; in particular we know if G has isolated vertices or not. By assumption, we may assume G to be disconnected (and then prove its edge-reconstructability based on the premise that all connected bipartite graphs be edge-reconstructable). Since disconnected graphs are well-known to be vertex-reconstructable and vertex-reconstructable graphs without isolated vertices are edge-reconstructable (Lemma 1.3), we will assume G to have isolated vertices.

Let $G = I \cup \overline{K}_m$, where $m \geq 1$ and I has no isolated vertices. I may be connected or disconnected, and is edge-reconstructable by the last paragraph. Now an edge-reconstruction H is obtained from $H - ef \cong G - ef = (I - ef) \cup \overline{K}_m$ by adding a new edge; ef here is an arbitrary edge. We can write $H = L \cup \overline{K}_p$, where L has no isolated vertices and $p \geq 0$ (Note $I - ef$ may have none, one, or two isolated vertices). By the edge-reconstructability of degree sequence, H must have the same number of isolated vertices (vertices of degree 0) as G has, and so

$p = m$. Now $(I - gh) \cup \overline{K}_m = G - gh \cong H - gh = (L - gh) \cup \overline{K}_m$ for all edges gh of G. Since graph isomorphisms are doing with incidence relationships and have nothing to do with isolated vertices, we have immediately $I - gh \cong L - gh$ for all edges gh in G, and so $I \cong L$ since I is assumed to be edge-reconstructable. So $G = I \cup \overline{K}_m \cong L \cup \overline{K}_m = H$, and we have proved that G can be assumed to be connected.

In particular, the minimum degree $\mu_0(G)$ of G is ≥ 1. Note that the vertex set partition $V(G) = V_1(G) \cup V_2(G)$ for a connected bipartite graph is unique, i.e. well-defined (the partition is not unique for a disconnected bipartite graph by obvious argument). We will say that two vertices a, b are "in the same part" in G if a, b both belong to $V_1(G)$ or both belong to $V_2(G)$; a,b will be "in different part" in G if one of a,b belong to $V_1(G)$ and the other belongs to $V_2(G)$. The same practice will be used for any isomorph or edge-reconstruction of G or edge-deleted subgraphs $G - ef$'s.

Let's say that an edge ab has a *degree type* (α,β) if $\deg(a) = \alpha$, $\deg(b) = \beta$, or $\deg(a) = \beta$, $\deg(b) = \alpha$.

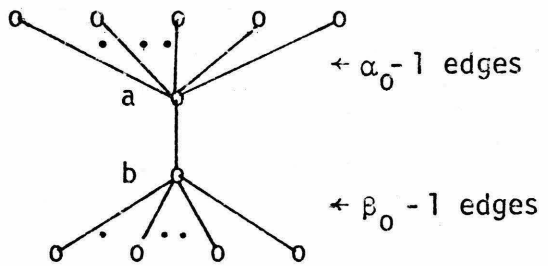
Lemma 3.1. For fixed integers α_0 and β_0, the number of edges of degree type (α_0,β_0) is edge-reconstructable.

Proof of Lemma. Define a partial order "$\leq_G$" on the set of all ordered pairs (γ,δ) which is the degree type of some edge in G such that $(\gamma_1,\delta_1) \leq_G (\gamma_2,\delta_2)$ if and only if $\gamma_1 \leq \gamma_2$, $\delta_1 \leq \delta_2$. We say $(\gamma_1,\delta_1) <_G (\gamma_2,\delta_2)$ if $(\gamma_1,\delta_1) \leq_G (\gamma_2,\delta_2)$ but $(\gamma_1,\delta_1) \neq (\gamma_2,\delta_2)$.

(γ,δ) is a maximal degree type in G if $(\gamma,\delta) \leq_G (\gamma',\delta')$ for (γ,δ) and (γ',δ') degree types of some edges in G implies $(\gamma,\delta) = (\gamma',\delta')$. Since G is finite, degrees are bounded, and maximal degree type (pairs) in G exists.

Let H be an arbitrary edge-reconstruction of G. We can define $\leq_H$ in a similar way. Suppose (α_0,β_0) is a maximal degree type in G at first. Let G have $\lambda > 0$ edges of degree type (α_0,β_0). Then G has exactly λ (edge-)subgraphs T of the form below (by maximality):

T :



If T is G itself, then very elementary argument will show G's edge-reconstructability (ab is clearly a forced edge if $\alpha_0 > 2$, $\beta_0 > 2$. If $\alpha_0 = 1$, we have K_{1,β_0} which was done in the proof of Lemma 1.2. If $\alpha_0 = 2$, ab is again a forced edge). So T is edge-proper in G, and by Kelly's Lemma (Lemma 1.1), H has exactly $\lambda > 0$ subgraphs of the form T. Let $(\alpha',\beta') \geq_H (\alpha_0,\beta_0)$ be of maximal type in H, then H has a subgraph T' of a similar form as T except that we have α' edges incident with a (instead of α_0 edges) and β' edges incident with b. Again, we can assume T' to be edge-proper in H. By Kelly's Lemma again, G has a subgraph of the form T'. Let $(\alpha'',\beta'') \geq_G (\alpha',\beta')$ be of maximal type in G. Now $(\alpha'',\beta'') \geq_G (\alpha',\beta')$ implies $\alpha'' \geq \alpha'$, $\beta'' \geq \beta'$; $(\alpha',\beta') \geq_H (\alpha_0,\beta_0)$ implies $\alpha' \geq \alpha_0$ and $\beta' \geq \beta_0$, so $\alpha'' \geq \alpha_0$, $\beta'' \geq \beta_0$ and $(\alpha'',\beta'') \geq_G (\alpha_0,\beta_0)$. Since (α_0,β_0) is of maximal type in G, we have $(\alpha'',\beta'') = (\alpha',\beta') = (\alpha_0,\beta_0)$. Now that

(α_0, β_0) is of maximal type in H, the fact that H has exactly $\lambda > 0$ subgraphs of the form T is exactly equivalent to that H has exactly $\lambda > 0$ edges of degree type (α_0, β_0) (equivalence not true if (α_0, β_0) is not maximal). We have done the case that (α_0, β_0) is maximal degree type in G.

Now let's do "induction" on the partial order $\leq_G$. We assume that the number of edges of degree type (γ, δ) is edge-reconstructable for every $(\gamma, \delta) >_G (\alpha_0, \beta_0)$. In symbols, let H be an edge-reconstruction of G, let $a_{\gamma,\delta}$ ($b_{\gamma,\delta}$ resp.) > 0 be the number of edges in G (H resp.) of the degree type (γ, δ) with $\gamma \geq \alpha_0$, $\delta \geq \beta_0$ but no equality for both. "Induction" says $a_{\gamma,\delta} = b_{\gamma,\delta}$ for all such (γ, δ)'s. We also see $(\gamma, \delta) >_G (\alpha_0, \beta_0) \Leftrightarrow (\gamma, \delta) >_H (\alpha_0, \beta_0)$ since $a_{\gamma,\delta} > 0 \Leftrightarrow b_{\gamma,\delta} > 0$. The number of subgraphs in G isomorphic to T (with $\deg(a) = \alpha_0$, $\deg(b) = \beta_0$) as mentioned earlier is

$$\sum_{(\gamma,\delta) \geq_G (\alpha_0,\beta_0)} \binom{\gamma - 1}{\alpha_0 - 1} \binom{\delta - 1}{\beta_0 - 1} a_{\gamma,\delta} = \sum_{(\gamma,\delta) >_G (\alpha_0,\beta_0)} \binom{\gamma - 1}{\alpha_0 - 1} \binom{\delta - 1}{\beta_0 - 1} a_{\gamma,\delta} + a_{\alpha_0,\beta_0},$$

where a_{α_0,β_0} is the number of edges of degree-type (α_0, β_0) in G. This number is, by Kelly's Lemma, equal to the number of subgraphs in H isomorphic to T, which in turn is equal to:

$$\sum_{(\gamma,\delta) \geq_H (\alpha_0,\beta_0)} \binom{\gamma - 1}{\alpha_0 - 1} \binom{\delta - 1}{\beta_0 - 1} b_{\gamma,\delta} = \sum_{(\gamma,\delta) >_H (\alpha_0,\beta_0)} \binom{\gamma - 1}{\alpha_0 - 1} \binom{\delta - 1}{\beta_0 - 1} b_{\gamma,\delta}$$

$+ \; b_{\alpha_0,\beta_0}$, where b_{α_0,β_0} is the number of edges of degree-type (α_0, β_0) in H.

Since $a_{\gamma,\delta} = b_{\gamma,\delta}$ for all $(\gamma,\delta) >_G (\alpha_0,\beta_0)$ and $(\gamma,\delta) >_G (\alpha_0,\beta_0)$ iff $(\gamma,\delta) >_H (\alpha_0,\beta_0)$, we see immediately that $a_{\alpha_0,\beta_0} = b_{\alpha_0,\beta_0}$ and we are done for the lemma. Q.E.D.

In Section 3 following, we will define *special chains* as a path or walk with some minimum properties on degrees. The "degree sequence" of such a chain is called *degree type*. Let condition A's and B_0's be respectively that the degree type and the number of special chains (of a certain length) be edge-reconstructable. (With condition B_i's generalizations of B_0's). Let condition P be that the "last vertices" of two special chains cannot be adjacent. We can do inductive proofs of these three conditions in an interlocked way in Section 5 and Section 6; leaving the definitions and elementary cases n = 0, 1, 2, 3 in Section 4. Section 7 then concludes with the proof of main theorem using condition P's. In Section 8, there is a short digression on generalization of proof.

Section 3. Definition of special chains and several basic lemmas.

We will generalize the concept of "minimum distance path" between two small vertices in a bi-degreed graph, or S_s, in Chapter 2. Given a bipartite graph G, we will now define *special n-chains* for $n \geq 0$ recursively.

Recall G can be assumed to be connected by Section 2. Hence $\mu_0(G)$, the minimum degree in G, is ≥ 1. By edge-reconstructability of degree sequences, we have $\mu_0(G) = \mu_0(H)$ for any edge-reconstruction H of G, and we may denote their common value by μ_0. We begin our recursive definition step by step in the following manner:

Step 0. Any vertex of degree μ_0 in G is a *special o-chain* in G. Go to next step.

Step 1. Let $\sigma_1(G) = \{b \in V(G) | b_0 b \in E(G)$ for some b_0 of minimum degree in G, i.e. deg $(b_0) = \mu_0\}$. $\sigma_1(G)$ is non-empty obviously. Let a_1 be a vertex of minimum degree in $\sigma_1(G)$. Symbolically, deg $(a_1) = \min \deg (b), b \in \sigma_1(G)$.

Let a_0 be a vertex of degree equal to μ_0, we call $a_0 a_1$ a *special 1-chain* in G. Denote deg (a_1) by $\mu_1(G)$. Go to next step.

Step 1'. We terminate the recursive defining process if $\mu_1(G) = \mu_0(G)$; otherwise go to next step.

Step 2. Let $\sigma_2(G) = \{b \in v(G) | bb_1 \in E(G)$ for some special 1-chain $b_0 b_1$, $b \neq b_0\}$. $\sigma_2(G)$ cannot be empty since deg $(b_1) > \mu_0 \geq 1$. Let a_2 be a vertex of minimum degree in

$\sigma_2(G)$; i.e. deg (a_2) = min deg (b), $b \in \sigma_2(G)$. Let a_0a_1 be a special 1-chain such that $a_2a_1 \in E(G)$, $a_2 \neq a_0$. We call $a_0a_1a_2$ a *special 2-chain* in G. Denote deg (a_2) by $\mu_2(G)$. Note furthermore that a_0, a_1, a_2 are all distinct. Go to next step.

Step 2'. We terminate the process if $\mu_2(G) = \mu_0(G)$; otherwise go to next step.

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Step n. $(n \geq 3)$ We will get to this step only if we do not terminate at any step k', $0 < k < n$. Suppose we have constructed the sets of all special (n-1)-chains of the form $c_0c_1 \ldots c_{n-1}$ where $c_0, c_1, \ldots, c_{n-1}$ are all distinct, and $\mu_{n-1}(G) \equiv \deg(c_{n-1}) > \mu_0$. Consider $\sigma_n(G) =$ $\{b \in V(G) | bb_{n-1} \in E(G)$ for a special (n-1)-chain $b_0b_1 \ldots b_{n-1}, b \neq b_{n-2}\}$. $\sigma_n(G)$ is nonempty since deg $(a_{n-1}) > \mu_0 \geq 1$. Let a_n be a vertex of minimum degree in $\sigma_n(G)$ (deg (a_n) = min deg (b), $b \in \sigma_n(G)$), and let $a_0a_1 \ldots a_{n-1}$ be a special (n-1)-chain (by recursive definition, they are all distinct) such that $a_{n-1}a_n \in E(G)$, $a_n \neq a_{n-2}$; we call $a_0a_1 \ldots a_n$ a *special n-chain* in G. It may happen that $a_n = a_i$ for some i, $0 \leq i \leq n - 3$. Denote deg (a_n) by $\mu_n(G)$. Go to next step.

Step n'. Terminate if $\mu_n(G) = \mu_0$, but $a_n \neq a_0$; otherwise go to next step.

Step n". Terminate if $a_n = a_0$; otherwise go to next step.

Step $n^{(3)}$. Terminate if $a_n = a_i$, $0 < i \leq n - 3$; otherwise go to next step. In the latter case, we see that $a_0, a_1 \ldots$, a_n are all distinct, so the recursive definition assumption that $a_0, a_1, \ldots, a_{n-1}$ are distinct is justified.

Since G is a finite graph, there is a unique smallest positive integer k such that the process terminates at step k' or step k" or step $k^{(3)}$. Denote this k by $\Omega(G)$. We will say that we have a Type-I (Type-II and Type-III respectively) termination if we terminate at step $\Omega(G)'$ (step $\Omega(G)''$ and step $\Omega(G)^{(3)}$ respectively). Note that the special $\Omega(G)$-chain is a path if we have a Type-I termination. Note also that every graph can have only one type of termination by algorithm of definition.

For any edge-reconstruction H of G, we can define special n-chains in H, $\mu_0(H)$, $\mu_1(H)$, ..., $\mu_n(H)$ and $\Omega(H)$ in an analogous way.

Conceivably for a bipartite graph of large size, we can have an immense number of edge-reconstructions, all nonisomorphic to each other. At this early stage, however, we are unable to establish that G can have at most one edge-reconstruction H as we did for the case of bi-degreed graphs. The problem is convincingly harder.

Remark. The above recursive definition of special n-chains holds good for general graphs, not only bipartite ones. We also see that this is a generalization of the concept of S_s's for bi-degreed graphs. In fact, we have a Type-I termination at step s for bi-degreed graphs. It's impossible that bi-degreed graphs have Type-II or Type-III terminations. For if say, a bi-degreed I has a Type-II termination at step t, then $t < s$ otherwise we have Type-I termination at step s already. This now says that there is a t-cycle passing through a small vertex a in I. Let b be a vertex adjacent to a on this t-cycle. We now see that ab is a forced edge for a is a forced vertex, and for $I - ab + ac$, $c \neq b$, a and b are small vertices of distance $\leq t - 1 < s - 1 < s$, impossible. Next assume that a bi-degreed I has a Type-III termination at step u, then $u < s$. We have a special u-chain $a_0a_1 \ldots a_{u-1}a_u$ with a_0 a small vertex, $a_1, a_2, \ldots a_u$ all big vertices and $a_n = a_i$ for some $0 < i \leq u - 3$. If $i = 1$, then a_0a_1 is a forced edge for otherwise we have a Type-II termination at step $u - 1 < s$ in another edge-reconstruction of I, an impossibility already proved. If $i > 1$, then we have a Type-III termination at step $u - 1 < u$ in an edge-reconstruction of I provided a_0a_1 is not a forced edge. If among the (finite number of) edge-reconstructions of I, we choose J to be one with $\Omega(J)$ the minimum, then starting anew with J, we see readily that b_0b_1 is a forced edge for a special $\Omega(J)$-chain $b_0b_1 \ldots b_{\Omega(J)}$, and we are done. In the argument here, we do not assume the knowledge that I can have at most one nonisomorphic edge-reconstructions.

Given a bipartite graph G, let Σ_G be the (finite) set of all its edge-reconstructions. Clearly $H \in \Sigma_G$ implies that $\Sigma_H = \Sigma_G$. Let $M \in \Sigma_G$ be one edge-reconstruction such that $\Omega(M)$ is the minimum in Σ_G, i.e. $\Omega(M) = \min \Omega(H)$, $H \in \Sigma_G$. Renaming if necessary, we can assume from now on that $\Omega(G) \leq \Omega(H)$ for all $H \in \Sigma_G$. This simple observation will prove fruitful in a few lemmas to come. Note also

$\Omega(G) \geq 1$ by definition.

For a given walk $v_0 v_1 \ldots v_n$ of $n + 1$ vertices in any $H \in \Sigma_G$, we will say that $v_0 v_1 \ldots v_n$ is of *degree type* $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ if $\deg(v_i) = \alpha_i$ in H for all i, $0 \leq i \leq n$. This notion of degree type agrees with the notion of degree type of an edge *on p. 91* when $n = 1$. Consider a special $\Omega(G)$-chain $a_0 a_1 \ldots a_{\Omega(G)}$ in G, which has degree type $(\mu_0(G), \mu_1(G), \ldots \mu_{\Omega(G)-1}(G), \mu_{\Omega(G)}(G))$. We note $\mu_0(G) = \mu_0$; the lowest possible degree for $\mu_i(G)$, $0 < i < \Omega(G)$, is $\mu_0 + 1$; and the lowest possible degree for $\mu_{\Omega(G)}(G)$ is μ_0 if it is a Type-I or Type-II termination; the lowest possible degree for $\mu_{\Omega(G)}(G)$ is $\mu_0 + 1$ if it is a Type-III termination. We will show in two following lemmas that we can exclude "minimal-degree" configurations of the form of special $\Omega(G)$-chain in which the degree of every vertex is as low as possible.

Lemma 3.2. A bipartite graph G is edge-reconstructable if G contains a special $\Omega(G)$-chain $a_0 a_1 \ldots a_{\Omega(G)}$ of degree type $(\mu_0, \mu_0 + 1, \mu_0 + 1, \ldots, \mu_0 + 1, \mu_0 + 1, \mu_0)$, $\mu_0 = \mu_0(G)$. (i.e. We have $\deg(a_0) = \deg(a_{\Omega(G)}) = \mu_0$ in G, and $\deg(a_i) = \mu_0 + 1$ for $0 < i < \Omega(G)$ in G).

Proof of Lemma. If $\mu_0 = 1$, then G is $P_{\Omega(G)+1}$, the path of length $\Omega(G)$, and its edge-reconstructability is trivial. We may assume $\mu_0 > 1$ in this lemma.

Note a_0 and $a_{\Omega(G)}$ may coincide or not (and we have Type-II or Type-I termination correspondingly).

Case 1 of Lemma 3.2. $a_{\Omega(G)} \neq a_0$.

Let $\Omega(G) = 1$ first. Then a_0a_1 is a forced edge since in $G - a_0a_1$, a_0 and a_1 are both forced vertices of degree $\mu_0 - 1 < \mu_0$.

Next, we observe that our graph G can be assumed to be a *block*. Since $\mu_0 > 1$, G has no isolated vertices (vertices of degree 0) or "end-vertices" (vertices of degree 1); a result of J. A. Bondy [4] says that connected graph G having *cut-vertices* but no end-vertices is vertex-reconstructable. So if our bipartite G has cut-vertices, it is vertex-reconstructable and hence edge-reconstructable since it has no isolated vertices. Our graph G then is connected without cut-vertices, hence it is a block. By the characterization of blocks as in p. 27 of F. Harary [7], every two vertices a,b of G lie on a common cycle; in other words every two vertices a and b are joined by two paths disjoint everywhere except at a and b. Note that two vertices a and b of G are in the "same part" $V_1(G)$ (or $V_2(G)$) of a connected bipartite G if and only if a and b are of even distance apart in G, and they are in "different parts" if a and b are of odd distance apart (this can be seen readily by elementary argument and the proof is omitted). So a and b in the same part of G are joined by two paths of even length disjoint everywhere except at a and b.

Now, consider the case $\Omega(G) = 2$. $H' = G - a_0a_1 + a_0a_2 \cong H$ is the only possible non-isomorphic edge-reconstruction since a_0 is a forced vertex and a_1a_2, an edge of degree type (μ_0, μ_0) in $G - a_0a_1$, cannot appear in H by the case $\Omega(G) = 1$ before. But now a_0 and

a_2 are joined both by a path P of even length not passing a_1 (by the discussion of previous paragraph) and the edge a_0a_2, in H', so H' contains an odd cycle and cannot be bipartite, contradiction to the fact that bipartite graphs are edge-recognizable.

So suppose $\Omega(G) \geq 3$. We will first show that $\Omega(G)$ must be odd. Suppose $\Omega(G)$ is even and consider $H' = G - a_0a_1 + a_0b \cong H$ for some vertex $b \neq a_1$ of degree μ_0 in $G - a_0a_1$. b cannot be a_i for $0 < i < \Omega(G)$ since degree of a_i in $G - a_0a_1$ is $\mu_0 + 1$. If $b \neq a_{\Omega(G)}$, then $a_1a_2 \ldots a_{\Omega(G)}$ is a path of length $\Omega(G) - 1 < \Omega(G)$ in H' of degree type $(\mu_0, \mu_0+1, \mu_0+1, \ldots, \mu_0+1, \mu_0)$. Since $\mu_0 + 1$ is the minimum possible degree of a_i, $i > 0$ before termination of special chain, we have $\Omega(H') \leq \Omega(G) - 1$, for H' should have a Type-I termination at step $(\Omega(G) - 1)'$ if it did not terminate at some step k', step k'' or step $k^{(3)}$ for $k < \Omega(G) - 1$ ($a_0a_1 \ldots a_{\Omega(H)}$ is clearly a special $\Omega(H')$-chain by definiton). This is contradictory to our assumption that $\Omega(G) \leq \Omega(H)$ for $H \in \Sigma_G$. So $b = a_{\Omega(G)}$ is the only choice. But a_0 and $a_{\Omega(G)}$ are in the same part of G since $\Omega(G)$ is even, and so they are joined by a path P disjoint from $a_0a_1 \ldots a_{\Omega(G)}$ except at the "ends" a_0 and $a_{\Omega(G)}$. In particular, a_0a_1 is not an edge on P, and P is a subgraph of H'. So H' contains an odd cycle formed by P and $a_{\Omega(G)}a_0$, impossible, and we have shown $\Omega(G)$ must be odd.

It's not absolutely necessary to use G's being bipartite in proving that $\Omega(G)$ must be even. Actually the proof of Lemma 2.2 is still valid if G has more than two degrees with μ_0 and $\mu_0 + 1$ the two lowest degrees, but we don't need it now.

So we see that $\Omega(G)$ must be odd and $a_0a_1 \to a_{\Omega(G)}a_0$ is a "forced move". In particular, we see in this case that G can have at most one nonisomorphic edge-reconstruction H. In $H' = G - a_0a_1 + a_{\Omega(G)}a_0$, $a_1a_2 \ldots a_{\Omega(G)}a_0$ is a path of degree type $(\mu_0, \mu_0+1, \mu_0+1, \ldots, \mu_0+1, \mu_0+1, \mu_0)$ and so clearly $\Omega(H') \leq \Omega(G)$. So $\Omega(G) = \Omega(H)$ since we assumed $\Omega(G) \leq \Omega(I)$ for all $I \in \Sigma_G$ at the outset. In this Case 1 of Lemma 3.2, we may then denote their common value by Ω. The forced move $a_0a_1 \to a_\Omega a_0$ changes the special Ω-chain $a_0a_1 \ldots a_\Omega$ in G to the special Ω-chain $a_1a_2 \ldots a_\Omega a_0$; it increments the indices cyclically by 1, note that the remainders of the graphs are intact during this move. Clearly all the other forced moves of the form $a_1a_{i+1} \to a_{i-1}a_i$ for $0 \leq i \leq \Omega$ have the same effects of incrementing indices cyclically by 1 ($a_{\Omega+1}$ is meant to be a_0, and a_{-1} to be a_Ω). Call them forced moves of the first kind (in this lemma only).

Consider now $G - a_1a_2$. In this subgraph, a_0a_1 is an edge of degree type (μ_0, μ_0), $a_2a_3 \ldots a_\Omega$ is a path of degree type $(\mu_0, \mu_0+1, \mu_0+1, \ldots, \mu_0+1,, \mu_0+1, \mu_0)$. None of these two configurations can exist in any isomorph of H (otherwise $\Omega(H) \leq \Omega(G) - 2$). So $H' = G - a_1a_2 + cd \cong H$, where cd has three possibilities: a_0a_Ω, a_1a_Ω, a_0a_2. Since Ω is odd, a_1 and a_Ω (a_0 and a_2 resp.) lie in the same part of G, and are joined by a path of even length not containing a_1a_2, so in H', we have an odd cycle, a contradiction. Hence $a_1a_2 \to a_\Omega a_0$ is a forced move, and this sends the special Ω-chain $a_0a_1a_2 \ldots a_{\Omega-1}a_\Omega$ in G to $a_2a_3 \ldots a_\Omega a_0 a_1$ in H'; it changes the indices cyclically by 2. Call them forced moves of the second kind.

Suppose we can find two nonnegative integers α and β such that

$$\Omega + 1 = \alpha + 2\beta \text{ , and}$$

$$\alpha + \beta \text{ is odd,}$$

then, applying α forced moves of the first kind and β forced moves of the second kind, the indices of $a_0 a_1 \ldots a_\Omega$ are incremented by $1 \cdot \alpha + 2 \cdot \beta = \Omega + 1$, so $a_0 a_1 \ldots a_\Omega$ is returning to its original position after $\alpha + \beta$ forced moves. But $\alpha + \beta$ is odd, so by Lemma 2.1 (which is true for general graph), we see G is edge-reconstructable.

We now proceed to look for such α and β. We may write $\Omega + 1 = 2^\gamma \delta$, with $\gamma \geq 1$ and δ an odd integer since $\Omega + 1$ is even. If $\gamma > 1$, let $\alpha = 2\delta$, $\beta = (2^{\gamma-1} - 1)\delta$, then $\alpha + 2\beta = 2\delta + (2^\gamma - 2)\delta = 2^\gamma \delta = \Omega + 1$ and $\alpha + \beta$ is odd, being the sum of an even integer and an odd integer; while for $\gamma = 1$, let $\alpha = 0$, $\beta = \delta$, we have $\alpha + 2\beta = 2\delta = \Omega + 1$ and $\alpha + \beta = \delta$ is odd. So we are successful to find α and β's and we are done for Case 1 of this lemma.

Case 2 of Lemma 3.2. $a_{\Omega(G)} = a_0$.

From $G - a_0 a_{\Omega(G)}$, the only ways we can edge-reconstruct nonisomorphic edge-reconstructions are $G - a_0 a_{\Omega(G)} + a_0 b$ for b a vertex of degree μ_0 in G not on the special $\Omega(G)$-chain since a_0 is forced vertex and no a_i is of degree μ_0 in G for $0 < i < \Omega(G)$ (edge-reconstructability of degree sequence implies b should be of degree μ_0 in G). But then for a given b with $H' = G - a_0 a_{\Omega(G)} + a_0 b$, $a_0 a_1 \ldots a_{\Omega(G)-1}$ is a path of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1, \mu_0)$ ($\deg(a_j) = \mu_0 + 1$ for $0 < j < \Omega(G) - 1$) and so $\Omega(H') \leq \Omega(G) - 1$

since we should terminate at step $(\Omega(G) - 1)'$ if we didn't terminate before. This is contradictory to our assumption that $\Omega(G) \leq \Omega(H')$ (since $H' \in \Sigma_G$, the set of edge-reconstructions of G), and G has no nonisomorphic edge-reconstructions; hence G is edge-reconstructable. So we are done with Case 2 as well, completing our proof of Lemma 3.2, Q.E.D.

Remark: The proof of this lemma isn't too hard, if not trivial. We have used heavily the fact that G is bipartite (in Case 1). The corresponding proof for a most general graph would sound intractable, though interesting.. For example, when G is bi-degreed (not necessarily also bipartite), this lemma says S_s is excludable, which takes a whole chapter (proof of edge-reconstructability of bi-degreed graphs) to implement. Since $\Omega(G) = 1$ implies in any case that $\mu_1(G) = \mu_0$ and hence G is edge-reconstructable trivially, we will assume $\Omega(G) \geq 2$ from now on.

We have shown the excludability of "minimal-degree" configurations as mentioned in p. 99 of this Chapter for Type-I and Type-II terminations. We will see the corresponding result holds for Type-III termination as well.

Lemma 3.3. G is edge-reconstructable if G contains a special $\Omega(G)$-chain $a_0a_1 \ldots a_{\Omega(G)}$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1)$ $(\deg(a_i) = \mu_0 + 1$ for $0 < i \leq \Omega(G))$.

Proof of Lemma. This is a Type-III termination with $a_{\Omega(G)} = a_k$, $0 < k < \Omega(G)$. Note that k may vary if we choose a different special $\Omega(G)$-chain $b_0b_1 \ldots b_{\Omega(G)}$ of the same degree type. We may fix a chain

$a_0a_1 \ldots a_{\Omega(G)}$ and hence k in this lemma. Any non-isomorphic edge-reconstruction H of G will have the form $G - a_0a_1 + a_0b$, where b is a vertex of degree μ_0 not on $a_1a_2 \ldots a_{\Omega(G)}$. In H, $a_1a_2 \ldots a_{\Omega(G)}$ is a walk of degree type $(\mu_0, \mu_0+1, \ldots, \mu_0+1)$ $(\deg(a_i) = \mu_0+1$ if $1 < i \leq \Omega(G)$, $\deg(a_1) = \mu_0)$ if $k > 1$, it is of degree type $(\mu_0, \mu_0+1, \ldots, \mu_0+1, \mu_0)$ $(\deg(a_i) = \mu_0+1$ if $1 < i < \Omega(G)$, $a_1 = a_{\Omega(G)}$ with $\deg(a_1) = \mu_0$ in H) if $k = 1$; both lead to $\Omega(H) \leq \Omega(G) - 1$, a contradiction. Q.E.D.

Corollary 3.3.1. G is edge-reconstructable if G contains a path $b_0b_1 \ldots b_\omega$ of degree type $(\mu_0, \mu_0+1, \ldots, \mu_0+1)$ $(\deg(b_0) = \mu_0$, $\deg(b_i) = \mu_0+1$, $1 \leq i \leq \omega)$ with $\omega \geq \Omega(G) - 1 \geq 1$.

Proof of Corollary 3.3.1. Since $\mu_0 + 1$ is the lowest possible degree before termination of constructing special chains, we have immediately $\mu_j(G) = \mu_0 + 1$ for all j, $1 \leq j \leq \Omega(G) - 1$. Now G is edge-reconstructable by Lemma 3.2 if we have a Type-I or Type-II termination, and G is edge-reconstructable by Lemma 3.3 if we have a Type-III termination.

Q.E.D.

Corollary 3.3.2. G is edge-reconstructable if G contains paths $c_0c_1 \ldots c_\alpha$ and $d_0d_1 \ldots d_\beta$ both of degree type $(\mu_0, \mu_0+1, \ldots, \mu_0+1)$ with $\alpha, \beta > 0$ and $c_\alpha = d_\beta$ (we have $\deg(c_0) = \deg(d_0) = \mu_0$ and $\deg(c_i) = \deg(d_j) = \mu_0+1$ for $0 < i \leq \alpha$, $0 < j \leq \beta$).

Proof of Corollary 3.3.2. First suppose $c_0c_1 \ldots c_\alpha$ and $d_0d_1 \ldots d_\beta$ are everywhere disjoint except at $c_\alpha = d_\beta$ (in particular $c_0 \neq d_0$). Now $c_0c_1 \ldots c_\alpha d_{\beta-1}d_{\beta-2} \ldots d_0$ is a path of degree type $(\mu_0, \mu_0+1,$

$\ldots, \mu_0 + 1, \mu_0)$ in G and so we have $\alpha + \beta \geq \Omega(G)$. Then we have a path of length $\alpha + \beta - 1 \geq \Omega(G) - 1$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1)$ and Corollary 3.3.1 applies to say that G is edge-reconstructable. Next, let $c_0 \neq d_0$ and suppose $c_0c_1 \ldots c_\alpha$ and $d_0d_1 \ldots d_\beta$ intersects at somewhere besides $c_\alpha = d_\beta$. Let $\gamma < \alpha$ be the smallest positive integer such that c_γ lies on $d_0d_1 \ldots d_\beta$. Then $c_\gamma = d_\delta$ for a fixed δ, $0 < \delta < \beta$. Applying the previous argument to $c_0c_1 \ldots c_\gamma$ and $d_0d_1 \ldots d_\delta$ we have readily that G is edge-reconstructable.

Let now $c_0 = d_0$. Let $\mu > 0$ be the first positive integer such that $c_\mu \neq d_\mu$ (since $c_0c_1 \ldots c_\alpha$ and $d_0d_1 \ldots d_\beta$ are different). Let $\nu > \mu$ $(\nu \leq \alpha)$ be the first positive integer such that c_ν is some d_ρ, $\mu < \rho \leq \beta$. Now the walk $c_0c_1 \ldots c_\nu d_{\rho-1} \ldots d_\mu c_{\mu-1}$ suggests that $\Omega(G) \leq \mu - 1 + (\nu - (\mu - 1)) + (\rho - (\mu - 1)) = \Delta$ for we will have a Type-III termination at step Δ" (at $c_{\mu-1}$) if not before. Now $c_0c_1 \ldots c_\nu d_{\rho-1} \ldots d_\mu$ is a path of length $\Delta - 1 \geq \Omega(G) - 1$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1)$ and we are done by Corollary 3.3.1.Q.E.D.

Remark. Case 2 of Lemma 3.2, Lemma 3.3, Corollary 3.3.1, and Corollary 3.3.2 are all still valid if G is a general graph (not necessarily bipartite). Note that definitions of special chains and Type-I, II, III termination are still meaningful for general graphs $(\Omega(G) \leq \Omega(H)$ for all $H \in \Sigma_G$ still used in the proof).

Lemma 3.4. Given a positive integer δ, the number of paths of length k of the form $a_0a_1 \ldots a_{k-1}b$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1, \delta)$ (deg $(a_0) = \mu_0$, deg $(a_i) = \mu_0 + 1$ for $1 \leq i \leq k - 1$, deg $(b) = \delta$ in G) is edge-reconstructable for all k, $1 \leq k \leq \Omega(G) - 1$ (when $k = 1$, we mean edges of degree type (μ_0,δ).).

Proof of Lemma. Prove by induction on k. When $k = 1$, we see immediately that number of edges of degree type (μ_0,δ) is edge-reconstructable by Lemma 3.1.

Now suppose the number of paths of length k of the degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ is edge-reconstructable for $1 \leq k \leq \Omega(G) - 2$, we will show that the number of paths of length $k + 1$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ is edge-reconstructable. Note that $\delta \geq \mu_0 + 1$. For the fixed integer δ, G may or may not have a path of length of $k + 1$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$. Let G have such a path $a_0a_1 \ldots a_kb$ at first. From $G - a_0a_1$, we can have a nonisomorphic edge-reconstruction $H = G - a_0a_1 + a_0c$, $c \neq a_1$. c is a vertex of degree μ_0 in G and cannot lie on $a_0a_1 \ldots a_kb$ by degree argument. Let a_1 lie on $\alpha \geq 1$ paths of length k of the degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ with a_1 as the "starting" vertex (i.e. paths are of the form $a_1d_1 \ldots d_{k-1}e$) in H; then edge-reconstructability of paths of length k of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ implies that there are exactly $\alpha \geq 1$ paths of length k of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ "starting" at c in G (paths having the form $cd_1' \ldots d_{k-1}'e'$). Now the "move" $a_0a_1 \to a_0c$ "destroys" exactly $\alpha \geq 1$ paths of length of $k + 1$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ (of the form $a_0a_1d_1 \ldots d_{k-1}e$) and it "creates" exactly α paths of length $k + 1$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ (of the form $a_0cd_1' \ldots d_{k-1}'e'$), so the number of paths of length $k + 1$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta)$ is edge-reconstructable when G has at least one path of this form.

Now suppose G has no such path of length $k + 1$. Suppose some $I \in \Sigma_G$ contains a path $i_o i_1 \ldots i_k j$ of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1, \delta)$. We will have $G \cong I - i_o i_1 + i_o m$ for some $m \neq i$, of degree μ_o in I. Argue as in the previous paragraph, we see that G will have the same number (≥ 1) of paths of length $k + 1$ of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1, \delta)$ as I, and we get a contradiction.

We are done with our induction step and hence proof of our lemma.

Q.E.D.

Corollary 3.4. A bipartite G is edge-reconstructable provided that G contains a special $\Omega(G)$-chain $a_o a_1, \ldots a_{\Omega(G)}$ with $\mu_{\Omega(G)-1}(G) = \mu_o + 1$, $\mu_{\Omega(G)}(G) = \mu_o$ and provided that also $\mu_i(H) = \mu_i(G)$ for all i, $0 \leq i \leq \Omega(G) - 2$, and for all $H \in \Sigma_G$. ($\Omega(G) \leq \Omega(H)$ for all $H \in \Sigma_G$ assumed).

Proof of Corollary. If $a_{\Omega(G)} = a_o$, then for $H = G - a_o a_{\Omega(G)} + a_o b$, $b \neq a_{\Omega(G)}$, we have $\Omega(H) \leq \Omega(G) - 1$ for if we do not have $\Omega(H) \leq \Omega(G) - 2$, then $a_o a_1 \ldots a_{\Omega(G)-1}$ is a special $(\Omega(G) - 1)$-chain of degree type $(\mu_o(G), \mu_1(G), \ldots, \mu_{\Omega(G)-2}(G), \mu_o) = (\mu_o(H), \mu_1(H), \ldots, \mu_{\Omega(G)-2}(H), \mu_o)$ in H (by assumption), and so $\Omega(H) = \Omega(G) - 1$. In any case, we get $\Omega(H) \leq \Omega(G) - 1$, a contradiction to the fact that $\Omega(H) \geq \Omega(G)$.

Now let $a_{\Omega(G)} \neq a_o$. By Lemma 3.2, we can assume some $\mu_i(G) > \mu_o + 1$, $0 < i < \Omega(G)$. Let $k_{G,a_o a_1 \ldots a_{\Omega(G)}}$ be the smallest such i's. $k_{G,a_o a_1 \ldots a_{\Omega(G)}}$ will in general depend on G as well as on $a_o a_1 \ldots a_{\Omega(G)}$.

Let $k_G = \max k_{G,c_o c_1 \ldots c_{\Omega(G)}}$, $c_o c_1 \ldots c_{\Omega(G)}$ a special $\Omega(G)$-chain with $c_0 \neq c_{\Omega(G)}$;and suppose the maximum is attained for the special $\Omega(G)$-chain $d_0 d_1 \ldots d_{\Omega(G)}$. We have $0 < k_G < \Omega(G)$ and G cannot have a path of length k_G of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1)$.

Consider $H = G - d_{\Omega(G)-1}d_{\Omega(G)} + d_{\Omega(G)}e$ for some $e \neq d_{\Omega(G)-1}$ of degree μ_0 in G. e cannot lie on $d_0 d_1 \ldots d_{\Omega(G)-1}$ by simple degree requirement. If $e \neq d_0$, then we get a contradiction by the fact that $\Omega(H) \leq \Omega(G) - 1$, for if $\Omega(H) \geq \Omega(G) - 1$, then as before, $d_0 d_1 \ldots d_{\Omega(G)-1}$ will be a special $(\Omega(G) - 1)$-chain of degree type $(\mu_0(G), \mu_1(G), \ldots, \mu_{\Omega(G)-1}(G), \mu_0) = (\mu_0(H), \mu_1(H), \ldots, \mu_{\Omega(G)-1}(H), \mu_0)$ and $\Omega(H) = \Omega(G) - 1$.

So now let $e = d_0$. In H, $d_{\Omega(G)} d_0 d_1 \ldots d_{k_G - 1}$ is a path of length k_G of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1)$; so with $k = k_G$ and $\delta = \mu_0 + 1$ in the lemma, G must have a path of length k_G of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1)$, a contradiction to what we have observed two paragraphs before. Q.E.D.

Corollary 3.4 will prove to be a useful criterion later. By the way, Corollary 3.4 is true also for any graph for which Lemma 3.2 holds (not necessarily bipartite).

We will prove a lemma more general than Lemma 3.4 in a similar vein.

Lemma 3.5. When $\Omega(G) \geq 3$, then for fixed integers δ and ρ, the number of paths of length k of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1, \delta, \rho)$ are edge-reconstructable for $2 \leq k \leq \Omega(G) - 1$.

Proof of Lemma. When $k = 2$, we mean that the number of paths $a_0 bc$ of length 2 of degree type (μ_0, δ, ρ) are edge-reconstructable. Since

$2 \leq \Omega(G) - 1$, we have immediately that $\delta \geq \mu_0 + 1$, $\rho \geq \mu_0 + 1$. Consider $H = G - a_0b + a_0d$, $d \neq b$. Suppose d is not adjacent to b in G, at first. Let b lie on $\alpha \geq 1$ edges of degree type $(\delta-1,\rho)$ in H (bc is such an edge). Then d must lie on exactly $\alpha \geq 1$ edges of degree type $(\delta-1, \rho)$ in G by edge-reconstructability of edges of such degree type (Lemma 3.1). Now the move $a_0b \to a_0d$ destroys exactly α paths of degree type (μ_0, δ,ρ) containing a_0b and creates exactly α paths of the same degree type starting with a_0d, and we are done for this case. Next, let d be adjacent to b in G. We have then $\delta = \rho + 1$ (d may be c, say). Let b lie on $\beta \geq 0$ edges of degree type (ρ,ρ) in H, then d lies on β edges of degree type (ρ,ρ) in G and the move $a_0b \to a_0d$ destroys $\beta + 1$ paths of the degree type $(\mu_0, \rho + 1,\delta)$ starting with a_0b and creates $\beta + 1$ paths of the same type starting with a_0d. So we are done when $k = 2$.

We then proceed inductively. Assuming it true for k, and we will prove it true for $k + 1$, $2 \leq k \leq \Omega(G) - 2$. Prove in the same way as in Lemma 3.4, we first suppose G has a $(k+1)$-path $a_0a_1 \ldots a_{k-1}bc$ of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1,\delta,\rho)$. If a_1 is the starting vertex of exactly $\alpha \geq 1$ k-paths of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1, \delta,\rho)$ in $H = G - a_0a_1 + a_0d$, $d \neq a_1$, then d is the starting vertex of exactly $\alpha \geq 1$ k-paths of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1, \delta,\rho)$ by induction assumption; and so the "move" (not necessarily a forced move) $a_0a_1 \to a_0d$ destroys α $(k+1)$-paths of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0+1,\delta,\rho)$ starting at a_0 and creates α $(k+1)$-paths of same degree type, and hence we are done. Q.E.D.

Remark. We cannot generalize Lemma 3.5 in a "natural" way for the number of k-paths of degree type $(\mu_0, \mu_0 + 1, \ldots, \mu_0 + 1, \delta_1, \delta_2, \ldots, \delta_n)$ with $n \geq 3$, $n \leq k \leq \Omega(G) - 1$. The difficulty lies in starting the induction, for we do not have any "generalized" version of Lemma 3.1 for n-paths, $n \geq 3$ (we have only the version for $n = 2$, i.e. edges).

We can define *k-chains* $b_o b_1 \ldots b_k$ in a way similar to special k-chains so that $b_o, b_1, \ldots, b_{k-1}$ are all disjoint and b_k may be on $b_o b_1 \ldots b_{k-1}$ or not.

Section 4. Several more technical definitions.

Recall from Section 3 that a special n-chain $a_0a_1 \ldots a_n$ in H $(n \leq \Omega(G))$ has the degree type $(\mu_0(H), \mu_1(H), \ldots, \mu_n(H))$, where $H \in \Sigma_G$ (and $\Omega(G) \leq \Omega(H)$). We would ask naturally: is $(\mu_0(H), \mu_1(H), \ldots, \mu_n(H)) = (\mu_0(G), \mu_1(G), \ldots, \mu_n(G))$? This equality is a necessary condition if $H \cong G$, and so we would expect it to hold to achieve our goal (that $H \cong G$ for every $H \in \Sigma_G$).

Definition 3.1. *Condition* A(n). This condition says that for any $H \in \Sigma_G$, $(\mu_0(H), \mu_1(H), \ldots, \mu_n(H)) = (\mu_0(G), \mu_1(G), \ldots, \mu_n(G))$ for a given n. $(\Omega(G) \leq \Omega(H)$ assumed).

Once Condition A(n) holds true, we can then use μ_i to denote the common values of all $\mu_i(I)$, $I \in \Sigma_G$. We will write simply A(n) to mean Condition A(n) in the following. The same practice holds for any other definitions of this kind. Now the degree type of special n-chains being independent of the graph in which it lies, we may then ask: is the number of special n-chains edge-reconstructable? We state a more general definition in the following:

Definition 3.2. *Condition* $B_i(n)$. This condition says that $N_{n,i}(G)$, the number of chains of degree type $(\mu_0, \mu_1, \ldots, \mu_{n-1}, \mu_n+i)$ in G for $n > 0$, $i \geq 0$, is equal to $N_{n,i}(H)$, the number of chains of the same type in H, for any $H \in \Sigma_G$ $(\Omega(G) \leq \Omega(H)$ assumed).

Condition $B_0(n)$ says that the number of special n-chains is edge-reconstructable.

Clearly A(n) and $B_i(n)$ are necessary conditions when $H \cong G$ for any $H \in \Sigma_G$. We then naturally expect them to hold in our struggle to prove G's edge-reconstructability. Their validity will be a building block for our final goal, the main theorem. Of course some other technical (i.e. artificial) definitions (and their validity) will be required as well.

Now let's see how to show the validity of $A(n)$ for the first few values of n. $A(0)$ is the statement that $\mu_0(H) = \mu_0(G)$ for any $H \in \Sigma_G$ and is true by the edge-reconstructability of degree sequence. Note that $\Omega(G) = 1$ implies that $\mu_1(G) = \mu_0$ and a special 1-chain a_0a_1 is itself a forced edge; so we may assume $\Omega(G) \geq 2$. To prove $A(1)$, consider $G - a_0a_1$ for a special 1-chain a_0a_1 of degree type $(\mu_0, \mu_1(G))$ in G. By edge-reconstructability of degree sequence, $G - a_0a_1$ can only be edge-reconstructed to become some $H = G - a_0a_1 + a_0b$, $b \neq a_1$ is a vertex of degree $\mu_1(G) - 1$ in G. Now, a_0b is an edge of degree type $(\mu_0, \mu_1(G))$ in H, and so the "minimality" of special 1-chain in H implies $\mu_1(H) \leq \mu_1(G)$. Let b_0b_1 be a special 1-chain of degree type $(\mu_0, \mu_1(H))$ in H, then $G \cong H - b_0b_1 + b_0c$ for some $c \neq b_1$. We get as before that $\mu_1(G) \leq \mu_1(H)$. So $\mu_1(H) = \mu_1(G)$ and $A(1)$ is proved (we then can denote their common value by μ_1).

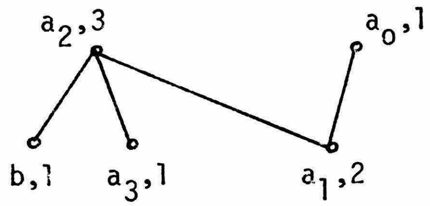
If $\Omega(G) = 2$, then $\mu_2(G) = \mu_0$ by definition of special 2-chain in G (there can be no Type-II or Type-III termination by obvious argument). If $\mu_1 = \mu_0 + 1$, then our special chain in G has the degree type $(\mu_0, \mu_0 + 1, \mu_0)$, and so G is edge-reconstructable by Lemma 3.2. Now let $\mu_1 > \mu_0 + 1$. Consider $G - a_1a_2$, where $a_0a_1a_2$ is a special 2-chain in G. In this subgraph, a_2 is a forced vertex of degree $\mu_0 - 1 < \mu_0$, and a_0a_1 is a forbidden edge of degree type $(\mu_0, \mu_1 - 1)$. So a_1a_2 is a forced edge, since we cannot join a_2a_0 by degree argument

($\mu_1 - 1 > \mu_0$). We may assume $\Omega(G) \geq 3$ from now on.

Now apply Lemma 3.5 (which is true when $\Omega(G) \geq 3$) with $\delta = \mu_1$, $\rho = \mu_2(G)$, $k = 2$, we see that the fact that G has a special 2-chain of degree type $(\mu_0, \mu_1, \mu_2(G))$ implies that H has a 2-path of degree type $(\mu_0, \mu_1, \mu_2\ (G))$ and $\mu_2(H) \leq \mu_2(G)$ for any $H \in \Sigma_G$. Suppose a certain H_0 satisfies $\mu_2(H_0) < \mu_2(G)$, then Lemma 3.5 again implies that G has a 2-path of degree type $(\mu_0, \mu_1, \mu_2(H_0))$ and so $\mu_2(G) \leq \mu_2(H_0) < \mu_2(G)$, a contradiction, and we have proved $A(2)$.

When $\Omega(G) = 3$ we observe again that there can be no Type-II or Type-III terminations by simple argument and so $\mu_3(G) = \mu_0$ and $a_3 \neq a_0$ for a special 3-chain $a_0a_1a_2a_3$ (when $\Omega(G) = 4$, we can have Type-I or Type-II but no Type-III termination; so $\mu_4(G) = \mu_0$ but a_4 may coincide with a_0 for a special 4-chain $a_0a_1a_2a_3a_4$ in G. When $\Omega(G) \geq 5$, we can have Type-III termination as well. The above argument works for bipartite graphs only. For a general non-bipartite graph we may have a Type-III termination when $\Omega(G) = 4$.).

Consider a special 3-chain $a_0a_1a_2a_3$ (actually a path) in G. If $\mu_2 = \mu_0 + 1$, then Corollary 3.4 applies and G is edge-reconstructable. So let $\mu_2 > \mu_0 + 1$ now. Suppose $\mu_0 > 1$ first. As in the second paragraph of proof of Lemma 3.2, we see that G is a *block* if $\mu_0 > 1$, and a_1 and a_3, being in the same "part" of G, are joined by a path of even length not containing a_2a_3. Hence it's impossible that a non-isomorphic edge-reconstruction $H = G - a_2a_3 + a_3a_1$, for otherwise we have an odd cycle in H. H cannot be $G - a_2a_3 + a_3a_0$ since $\mu_2 - 1 > \mu_0$. H cannot be $G - a_3a_2 + a_3b$, $b \neq a_0, a_1$, for then $\mu_2(H) \leq \mu_2 - 1$, a contradiction to A(2). So $\Omega(G) = 3$ implies G's edge-reconstructability when $\mu_0 > 1$. Now consider $\mu_0 = 1$. Argue as above, we see difficulty will present only when $H = G - a_2a_3 + a_3a_1$, in which case $\mu_2 = \mu_1 + 1$. By Lemma 3.1 on edges of degree type $(\mu_1, 1)$ (note $\mu_1 > 1$), we see a_2 must be adjacent to another vertex $b \neq a_3$ of degree 1 in G. Suppose $\mu_1 > 2$. By edge-reconstructability of degree sequence, $H \cong G - a_1a_2 + cd$, where c is a vertex of degree $\mu_1 - 1 > 1 = \mu_0$ in $G - a_1a_2$, and d a vertex of degree $\mu_1 + 1 - 1 = \mu_1$ in the same subgraph. c cannot be a_0 by degree argument. c then must be a_1 otherwise H contains an edge of degree type $(\mu_0, \mu_1 - 1)$ and $\mu_1(H) \leq \mu_1 - 1$, a contradiction. d cannot be a_3 or b by degree argument $(\mu_1 > 1)$. If d isn't a_2, then an isomorph of H $(= G - a_1a_2 + cd)$ contains a path a_3a_2b of degree type $(1, \mu_1, 1)$, which immediately implies $\Omega(H) \leq 2 < 3 = \Omega(G)$, a contradiction to our assumption that $\Omega(G) \leq \Omega(H)$. Finally we let $\mu_1 = 2$ (and $\mu_0 = 1$). G's connectivity implies at once that G is itself a graph as depicted below:



From $G - a_2a_3$, any possible nonisomorphic edge-reconstruction would be P_5 only (since G cannot contain triangle). But P_5 cannot have $K_{1,3}$ as an edge-proper subgraph, which is $G - a_0a_1$. So G is edge-reconstructable in this case as well, and we have proved $\Omega(G) = 3$ implies G's edge-reconstructability.

We note that $\mu_0 = 1$ deserves special treatment since G is no longer a block in this case. We state the fact that $\mu_0 > 1$ implies G is a block in the following lemma for later reference (proved already in second paragraph of Lemma 3.2).

Lemma 3.6. G can be assumed to be a block if $\mu_0 > 1$.

Now let's assume $\Omega(G) \geq 4$ and start to prove A(3). Consider a special 3-chain $a_0a_1a_2a_3$ in G. Let $\mu_2 > \mu_0 + 1$ first. Every $H \cong G - a_1a_2 + cd$ for some c of degree $\mu_1 - 1$ and d of degree $\mu_2 - 1$ in $G - a_1a_2$. If $\mu_1 > \mu_0 + 1$ as well, then a_0 cannot be c or d by degree argument, and a_1 is a forced vertex. The edge-reconstructability of edges of degree type $(\mu_2 - 1, \mu_3(G))$ implies that d is adjacent to a vertex of degree $\mu_3(G)$ in G and so $\mu_3(H) \leq \mu_3(G)$. Consider a special 3-chain $b_0b_1b_2b_3$ in H. We see $\mu_1, \mu_2 > \mu_0 + 1$ still hold and same argument as before says that $\mu_3(G) \leq \mu_3(H)$. So A(3) holds for this subcase. Let $\mu_1 = \mu_0 + 1$ now (still $\mu_2 > \mu_0 + 1$). c can be a_0 or a_1 (and nothing else) by degree argument and d must be adjacent to a vertex of degree $\mu_3(G)$ in G as before. So we get $\mu_3(H) \leq \mu_3(G)$. Symmetry argument then says that $\mu_3(G) = \mu_3(H)$ and A(3) holds.

Now let $\mu_2 = \mu_0 + 1$. Every $H \cong G - a_2a_3 + c'd'$, for some c' of degree μ_0 and d' of degree $\mu_3(G) - 1$ in $G - a_2a_3$. If $\mu_3(G) > \mu_0 + 1$, c' must be a_0 or a_2 otherwise $\Omega(G) \leq 2$ and we see $\mu_3(H) \leq \mu_3(G)$ readily. If $\mu_3(G) = \mu_0 + 1$, then one of c' or d' must be a_0 or a_2 and we see $\mu_3(H) \leq \mu_3(G)$ (which implies $\mu_3(H) = \mu_3(G)$ otherwise $\Omega(H) < 3 < \Omega(G)$). Repeating the same argument for a special

3-chain $b_0b_1b_2b_3$ in H, as before we see $\mu_3(G) \leq \mu_3(H)$ and so A(3) is proved in its full force.

Let's summarize the foregoing results in the following two lemmas.

Lemma 3.7. G is edge-reconstructable if $\Omega(G) \leq 3$.

Henceforth, we may assume $4 \leq \Omega(G)$ $(\leq \Omega(H)$ for all $H \in \Sigma_G)$.

Lemma 3.8. Condition A(n) holds for n = 0, 1, 2, 3.

Next, we investigate the validity of $B_i(n)$ for the first three values of n (n = 1, 2, 3). $B_i(1)$ says that the number of edges of degree type $(\mu_0, \mu_1 + i)$ is edge-reconstructable and this is solved readily by Lemma 3.1. For $B_i(2)$, we apply Lemma 3.5 for $\delta = \mu_1$, $\rho = \mu_2 + i$. We are left with $B_i(3)$ only. Let $\mu_2 > \mu_0 + 1$ first. Let a_2 on a special 3-chain $a_0a_1a_2a_3$ be adjacent to $\alpha \geq 1$ vertices of degree $\mu_3 + i$ other than a_1. If $\mu_1 > \mu_0 + 1$, then a_1 is a forced vertex and any edge-reconstruction $H \cong G - a_1a_2 + a_1d$, $d \neq a_2$. If d isn't adjacent to a_2 in G, then d must be adjacent to α vertices of degree $\mu_3 + i$ in G by Lemma 3.1, and we have "destroyed" α 3-paths of the form $a_0a_1a_2c$ of degree type $(\mu_0, \mu_1, \mu_2, \mu_3 + i)$ and "created" meanwhile α 3-paths of the form a_0a_1de of degree type $(\mu_0, \mu_1, \mu_2, \mu_3 + i)$ while going from G to H, so $B_i(3)$ holds for this situation. If d is adjacent to a_2 in G, then we have $\mu_2 = \mu_3 + i + 1$ and we have "created" $\alpha - 1$ 3-paths of degree type $(\mu_0, \mu_1, \mu_2, \mu_3 + i)$ and "destroyed" meanwhile $\alpha - 1$ 3-paths of the same degree type while going from G to H (the 3-path $a_0a_1a_2d$ is changed to a 3-path $a_0a_1da_2$ of the same degree type). So $B_i(3)$ holds in this case. When $\mu_2 = \mu_0 + 1$, we see in an analogous way as before that when G has 3-chain $a_0a_1a_2c$ of degree type $(\mu_0, \mu_1, \mu_2, \mu_3 + i)$, $H \cong G - a_2c + c'd'$ for some c' of degree μ_0 and d' of degree $\mu_3 + i - 1$ in $G - a_2c$. One end of our new edge must be a_0 or a_2 otherwise $\Omega(H) \leq 2$. If one end is a_2, the other end cannot be a_0 otherwise we have a triangle (3-cycle), and this case is trivial since if say $a_2 = c'$ is adjacent to $\beta \geq 1$ vertices of degree $\mu_3 + i$ in G, the "change" $a_2c \to a_2d'$ gives us in H still $\beta \geq 1$ vertices of degree $\mu_3 + i$ (except that vertex a_3 is replaced by d'), and so the number of 3-paths of degree type $(\mu_0, \mu_1, \mu_2, \mu_3 + i)$ is unchanged. If one end of the new edge is a_0 and a_0 is adjacent to γ vertices of degree $\mu_3 + i$ in G, then the change $a_2c \to a_0d'$ $(d' \neq a_1, a_2, c$ by obvious reasons) destroys β edges of degree type $(\mu_0 + 1, \mu_3 + i)$ and creates $\gamma + 1$ edges of degree type $(\mu_0 + 1, \mu_3 + i)$ and so $\beta = \gamma + 1$ by Lemma 3.1. But clearly the same change $a_2c \to a_0d'$ destroys β 3-paths of the form $a_0a_1a_2e'$ of degree type $(\mu_0, \mu_1, \mu_2, \mu_3 + i)$ and creates $\gamma + 1$ 3-paths of the form $a_2a_1a_0f'$ of the same degree type. No other 3-paths will be affected by this change unless a_0 or a_2 is adjacent to some vertex $b_1 \neq a_1$ of degree μ_1 in G. By the move $a_2c \to a_0d'$ and Lemma 3.1 on the edges of degree type (μ_0, μ_1), we see that if some b_1 of degree μ_1 is adjacent to a_0 in G, then some b_2 of the same degree must be adjacent to a_2 in G and $\mu_3 \leq \mu_1$ in particular.

If $\mu_3 \leq \mu_1 - 2$, consider a special 3-chain $a_0a_1a_2a_3$ in G and delete a_1a_2 from G. In $G - a_1a_2$, a_0a_1 has degree type $(\mu_0, \mu_1 - 1)$, a_2a_3 has degree type (μ_0, μ_3); both cannot happen in any edge-reconstruction. Furthermore, degree argument says that we have to join a vertex of degree μ_0 to a vertex of degree $\mu_1 - 1$ in $G - a_1a_2$; so a_1a_2 is a forced edge in this case. We can now assume $\mu_3 = \mu_1 - 1$ or $\mu_3 = \mu_1$. Note $\mu_1 > \mu_0 + 1$ otherwise we are done by Lemma 3.4.

Let $\mu_3 = \mu_1 - 1$ first. By argument two paragraphs before, $B_i(3)$ holds except when we have $H \cong G - a_2c + a_0d'$. If G contains a 3-path of degree type $(\mu_0, \mu_1, \mu_2, \mu_3 + i)$ with $\mu_3 + i > \mu_1$, then d' cannot be a_3 by degree argument and an isomorph of H contains an edge a_2a_3 of degree type $(\mu_0, \mu_1 - 1)$, a contradiction. So we have to consider only $\mu_3 + i = \mu_1 - 1$ or μ_1 finally. It's impossible that some $c \neq a_1, a_3$ of degree $\mu_1 - 1$ be adjacent to a_2, otherwise a_1a_2 is clearly a forced edge (this is clear if $\mu_1 > \mu_0 + 1$, to avoid an edge of degree type $(\mu_0, \mu_1 - 1)$. If $\mu_1 = \mu_0 + 1$, then we have immediately $\Omega(G) \leq 3$, and Lemma 3.7 says that G is edge-reconstructable).

Our graph G will contain a configuration C as in Fig. 3-2, from which it is easily seen that G can have at most one nonisomorphic edge-reconstruction H. (proof later)

Consider $G - a_0a_1$, we see a_0 is a forced vertex and $B_i(3)$ would be trivial if we can show that the number of 2-paths of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_1 - 1)$ or $(\mu_1 - 1, \mu_0 + 1, \mu_1)$ is edge- reconstructable. To prove this, it suffices to show by induction that the number of 2-paths of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + 1 + k)$ is edge-reconstructable for $0 \leq k \leq \mu_1 - \mu_0 - 1$. Recall $\mu_1 > \mu_0 + 1$. Let def_k represent a 2-path of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + 1 + k)$. For $k = 0$, consider $G - ef_0$. Since $\mu_1 - 1 > \mu_0$ and de is a forbidden edge of degree type $(\mu_0, \mu_1 - 1)$, we see that e is a forced vertex. Let e be adjacent to $\alpha \geq 1$ vertices of degree $\mu_1 - 1$ in G. Going from G to some $H = G - ef_0 + eg$ for some g, we see that the α 2-paths of the form $d'ef_0$ of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + 1)$ become α 2-paths of the form d'eg of the same degree type, and no other 2-path of the same degree type can be created or destroyed otherwise we would have $\mu_1 = \mu_0 + 1$, a contradiction. So the case $k = 0$ is proved. Now assuming the validity for $k \leq \mu_1 - \mu_0 - 2$ and we will show the validity for $k + 1$. Consider def_{k+1} and let e be adjacent to $\beta \geq 1$ vertices of degree $\mu_1 - 1$ in G. In $G - ef_{k+1}$, ed is an edge of degree type $(\mu_0, \mu_1 - 1)$ and f_{k+1} is of degree $\mu_0 + 1 + k < \mu_1 - 1$, so degree argument says that e is a forced vertex. Let $H = G - ef_{k+1} + eg$. Let f_{k+1} be on γ 2-paths of the form ihf_{k+1} with $h \neq e$ of the degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + k + 1)$ in $G - ef_{k+1}$ (and hence in H). g must lie on γ 2-paths of the form i'h'g, $h' \neq e$, of the degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + k + 1)$ in $G - ef_{k+1}$ (and also in G) by induction assumption. But then the move $ef_{k+1} \to eg$ creates γ 2-paths of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + k + 2)$ passing g and destroys γ 2-paths of some type passing f_{k+1}, it changes the β 2-paths of the form $d'ef_{k+1}$ of the degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + k + 2)$ to β 2-paths of the form d'eg of the same type, leaving all other 2-paths of such degree type unaffected. So clearly, the number of 2-paths of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_0 + k + 2)$ is edge-reconstructable, and we are done for the proof of $B_i(3)$ when $\mu_2 = \mu_o +$ $1, \mu_3 = \mu_1 - 1$ (and $\mu_1 > \mu_o + 1$).

What's left in the proof of $B_i(3)$ is the case $\mu_2 = \mu_o + 1$, $\mu_3 = \mu_1$ (and $\mu_1 > \mu_o + 1$ by Lemma 3.4). Consider $G - a_1a_2$ for a special 3-chain $a_oa_1a_2a_3$. Degree argument says that one end of the replacing edge should be a_o or a_1. If one end of the replacing edge is a_o, then the new graph I will have one more edge of degree type $(\mu_o + 1, \mu_1 - 1)$ than G (given by a_ob for some $b \neq a_3$) unless a_2 is adjacent to some vertex of degree $\mu_1 - 1$ in G, which in turn gives an edge of degree type $(\mu_o, \mu_1 - 1)$ in I, a contradiction. Hence a_1 is a "forced vertex", and Lemma 3.1 applied to edges of degree type $(\mu_o, \mu_3 + i)$ easily establishes $B_i(3)$. We have thus done the proof of

Lemma 3.9. $B_i(n)$ are true for $n = 1, 2, 3$, any $i \geq 0$.

Note that the idea and details of proof are pretty simple except the case when $\mu_1 > \mu_o + 1, \mu_2 = \mu_o + 1$ and $\mu_3 = \mu_1 - 1$.

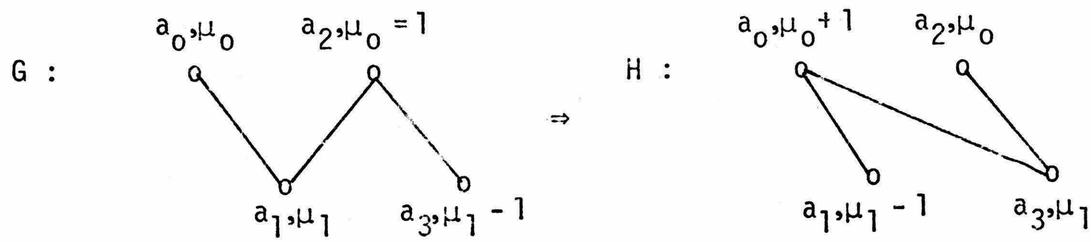


Fig. 3-2

For the proof of Fig. 3-2, note that a_oa_1 and a_2a_3 in $G - a_1a_2$ are both forbidden of degree type $(\mu_o, \mu_1 - 1)$, hence $\mu_1 - 1 > \mu_o$ implies $a_1a_2 \to a_oa_3$ as a forced move. For later reference, we introduce an excludable configuration C' in Fig. 3-3 which occurs very often in practice. To prove the excludibility, note c_2c_3 is a forced edge since $c_0c_1c_2c_3$ and c_4c_5 are both of forbidden degree type and $\mu_1 - 2 \neq \mu_1 - 1$.

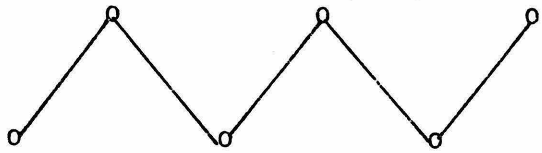


Fig. 3-3

To conclude this section, we will introduce a new technical definition, whose validity for general n will lead to our main theorem in Section 7.

Given n, $0 \leq n \leq \Omega(G) - 2$ (note $\Omega(G) \geq 4$), let $a_0a_1 \ldots a_\alpha$ and $b_0b_1 \ldots b_\beta$ be two special chains of length α and β respectively, with $0 \leq \alpha, \beta \leq n$. If $a_\alpha b_\beta$ is an edge of G $(a_\alpha b_\beta \in E(G))$ and we do not have the annoying situation that $a_0a_1 \ldots a_\alpha$ happens to be $b_0b_1 \ldots b_{\beta-1}$, i.e. $\alpha = \beta - 1$ and $a_i = b_i$ for $0 \leq i \leq \beta - 1$ or the situation that $b_0b_1 \ldots b_\beta$ is $a_0a_1 \ldots a_{\alpha-1}$; then we call this an (α,β)-*coupling* in G of the two special chains of $a_0a_1 \ldots a_\alpha$ and $b_0b_1 \ldots b_\beta$, or simply an (α,β)-coupling if no confusion is caused.

Definition 3.3. *Condition* P(n). For $0 \leq n \leq \Omega(G) - 2$. This condition says that an (α,β)- coupling for $0 \leq \alpha, \beta \leq n$ is an excludable configuration.

Notice that for an (α,β)-coupling in G, it is not necessarily true that $a_0 a_1 \ldots a_\alpha$ and $b_0 b_1 \ldots b_\beta$ are disjoint; they must be distinct however.

Condition P(n) is analogous to the excludability of T_p-configuration in Chapter two (Lemma 2.3).

To give an insight of how P(n) look like (and also to start the induction), we will prove the validity of P(n) here for n = 0, 1, 2.

Lemma 3.10. P(n) is true for n = 0, 1, 2.

Proof of Lemma. We will divide the proof into three cases according to the value of n. Without loss of generality, we may assume $\alpha \geq \beta$.

Case 1 of Lemma 3.10. n = 0.

The only possible (α,β)-coupling is that of an edge $a_0 b_0$ with $\deg(a_0) = \mu_0 = \deg(b_0)$, so $a_0 b_0$ is clearly a forced edge; and P(o) is true trivially.

Case 2 of Lemma 3.10. n = 1.

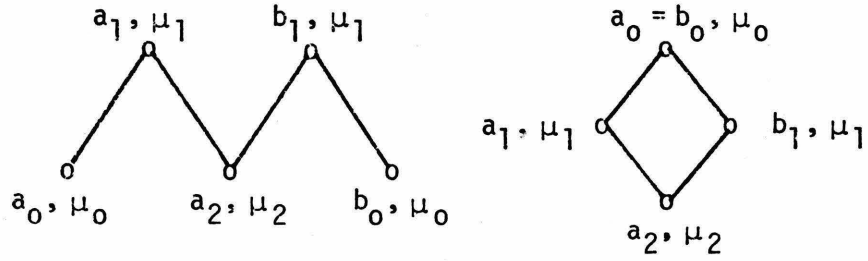
If $\alpha = 0$ then $\beta = 0$, then we are returning to Case 1. So let $\alpha = 1$ now. If $\beta = 0$, then we have $\Omega(G) \leq 2$, and G is clearly edge-reconstructable. So let $\beta = 1$ now. Clearly $a_0 a_1 b_1 b_0$ is a 3-path in G (they are all distinct obviously). If $\mu_1 = \mu_0 + 1$, then we have a path of degree type $(\mu_0, \mu_0 + 1, \mu_0 + 1, \mu_0)$, so $\Omega(G) \leq 3$ and G is edge-reconstructable. Now let $\mu_1 > \mu_0 + 1$. a_1b_1 is clearly a forced edge by degree argument and the fact that $(\mu_0, \mu_1 - 1)$ is a forbidden degree type for edges. We have then done the proof of $P(1)$.

Case 3 of Lemma 3.10. $n = 2$.

We may assume $\alpha = 2$, otherwise we are returning to Case 1 and Case 2. Let $\beta = 0$ first. We have then $\Omega(G) \leq 3$ by the 3-path $a_0a_1a_2b_0$ of degree type $(\mu_0, \mu_1, \mu_2, \mu_0)$, and G is edge-reconstructable by Lemma 3.7. Next consider $\beta = 2$ (the case $\beta = 1$ is much harder and is treated later). We note b_2 is adjacent to a_2. If $b_2 = a_1$, then $b_0b_1b_2$ and a_0 form a (2,0)-coupling for $n = 2$ ($\alpha = 2$, $\beta = 0$) and we are done. So b_2 is distinct from a_0, a_1, a_2. If b_1 is a_2, then a_0a_1 and b_0b_1 form a (1,1) coupling and Case 2 implies G's edge-reconstructability. So we have that $a_0a_1a_2$ and $b_0b_1b_2$ are disjoint and form a "true" (2,2)-coupling. Let $\mu_2 > \mu_0 + 1$ first and consider $G - a_2b_2$. $A(2)$ and the fact that $\mu_2 - 1 > \mu_0$ tell us that we can replace a_2b_2 by a_2b_1, a_1b_2 and a_1b_1 only, to get a nonisomorphic edge-reconstruction H. If the edge replacing a_2b_2 is a_2b_1, then $\mu_2 = \mu_1 + 1$ (and $\mu_0 = 1$ otherwise G is a block and we can show H contains an odd cycle), and since the edge b_0b_1 of degree type (μ_0, μ_1) is changed to a new degree type $(\mu_0, \mu_1 + 1)$ in H, Lemma 3.1 on degree type (μ_0, μ_1) says that b_2 must lie on an edge c_0b_2 of degree type (μ_0, μ_1) in H. We then see that the degree of c_0 in H is μ_0. Since the move $a_2b_2 \to a_2b_1$ neither creates nor destroys any vertex of degree μ_0, the degree of c_0 in G is also μ_0. Now the 3-path $b_0b_1b_2c_0$ in G of degree type $(\mu_0, \mu_1, \mu_2, \mu_0)$ readily establishes that $\Omega(G) \leq 3$ and we are done. The proof when the replacing edge is a_1b_2 is done similar to the case of a_2b_1 by symmetry of configuration (by interchanging a's and b's in the above argument). Now let the replacing edge be a_1b_1. Again $\mu_2 = \mu_1 + 1$ and Lemma 3.1 on edges of degree type (μ_0, μ_1) tell us that one of a_2, b_2, say a_2, must be adjacent to a vertex c_0 of degree μ_0 in H and hence in G. So now $a_0a_1a_2c_0$ gives $\Omega(G) \leq 3$ and we are done.

The case remaining with our (2,2)-coupling is when $\mu_2 = \mu_0 + 1$. If $\mu_1 = \mu_0 + 1$, then G is edge-reconstructable by Lemma 3.2; and so we may assume $\mu_1 > \mu_0 + 1$. Consider $G - b_1b_2$. In this edge-deleted subgraph, $a_0a_1a_2b_2$ is a forbidden 3-path of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_0)$ and b_0b_1 is a forbidden edge of degree type $(\mu_0, \mu_1 - 1)$. Degree argument says that the degree type of the replacing edge must be $(\mu_0, \mu_1 - 1)$ in $G - b_1b_2$. So it can only be b_1a_0 or b_0a_2 besides the trivial replacement b_1b_2 which returns us to G; the latter possibility b_0a_2 can happen only when $\mu_1 = \mu_0 + 2$. If the replacing edge is b_1a_0, then in the new graph H, b_2a_2 is an edge of degree type $(\mu_0, u_0 + 1)$ and we get $\mu_1 = \mu_0 + 1$, a contradiction. If the replacing edge is b_0a_2, then in H, a_0a_1 and b_0a_2 form a (1,1)-coupling and we are done by Case 1. So we have proved the excludability of (2,2)-coupling.

We are left with the possibility that $\beta = 1$. We may assume $b_1 \neq a_1$ otherwise we have an excludable (1,0)-coupling. Depending on $b_0 \neq a_0$ or $b_0 = a_0$, we will have two configurations as depicted in Fig. 3-4 below:



(a) (b)

Fig. 3-4

We will prove their excludability in two subcases following.

Subcase 3(a) of Lemma 3.10. $b_0 \neq a_0$ for a (2,1)-coupling.

We now have a configuration M as in Fig. 3-4 (a). First observe that $\mu_1 > \mu_0 + 1$ otherwise we have $\Omega(G) = 4$ and Corollary 3.4 applies to show G's edge-reconstructability. Next we see that μ_2 must be $\mu_0 + 1$ otherwise we see $a_2b_1 \rightarrow a_1b_1$ is a forced move since in G - a_2b_1, $a_0a_1a_2$ and b_0b_1 are both forbidden by their degree types (and note $\mu_2 - 1 > \mu_0$ now). But then the edge-reconstructability of edges of degree type $(\mu\ ,\mu_2)$ implies that there exists a vertex c_0 in H of degree μ_0 (and hence in G) adjacent to $a_2(\mu_2 = \mu_1 + 1$ now) and we have $\Omega(H) \leq 3$, implying $\Omega(G) \leq 3$ and G is edge-reconstructable.

Now the 3-path $a_0a_1a_2b_1$ of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1)$ tells us that $\mu_3 = \mu_1$ or $\mu_1 - 1$. If $\mu_3 = \mu_1$, then we can edge-reconstruct from G - b_0b_1 by replacing b_0b_1 by b_0a_2 (and $\mu_1 = \mu_0 + 2$, $\mu_0 = 1$ then). But then $B_0(2)$ implies that there exists a special 2-chain $c_0c_1c_2$ in the new graph H with $c_2 = b_1$. The degree of c_0

in G must be μ_0 (i.e. c_0 is not a vertex of degree $\mu_0 + 1$ in G which becomes a vertex of degree μ_0 in H) by obvious argument. c_1 cannot be a_2 otherwise $a_0a_1a_2$ and c_0 form a (2,0)-coupling. Now c_0c_1 and b_0b_1 is a (1,1)-coupling in G and we are done.

So we know that $\mu_3 = \mu_1 - 1$. We have immediately the fact that G can have at most one nonisomorphic edge-reconstruction H by the forced move $d_1d_2 \to d_0d_3$ of a special 3-chain $d_0d_1d_2d_3$.

We will investigate the interconnection pattern of special 3-chains in G for this subcase. Consider two distinct (but not disjoint special 3-chains $d_0d_1d_2d_3$ and $e_0e_1e_2e_3$. The four degrees $\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1$ are all distinct except the possibility that $\mu_1 - 1 = \mu_0 + 1$ when $\mu_1 = \mu_0 + 2$. This excludes the possibility that $e_i = d_j$ for $i \neq j$ except possibly $e_3 = d_2$ or $e_2 = d_3$. But $e_3 = d_2$ (or $e_2 = d_3$) gives us a (2,2)-coupling treated at the beginning of Case 3, and so $e_i = d_j$ only when $i = j$.

Now let $\gamma \geq 0$ be the smallest integer that $e_\gamma = d_\gamma$. We will have $0 \leq \gamma \leq 3$ since $d_0d_1d_2d_3$ and $e_0e_1e_2e_3$ are assumed to be non-disjoint (but still distinct).

Suppose $\gamma = 3$ first. The configuration D connecting $d_0d_1d_2d_3$ and $e_0e_1e_2e_3$ at $d_3 = e_3$ has the general look as in Fig. 3-5(a).

Let's delete e_0e_1 from G. In $G - e_0e_1$, e_0 is a forced vertex of degree $\mu_0 - 1$, and $d_0d_1d_2d_3e_2e_1$ is a 5-path of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1, \mu_0 + 1, \mu_1 - 1)$ which is excludable as configuration D' in Fig. 3-3. So $e_0e_1 \to e_0d_3$ is a forced move sending G

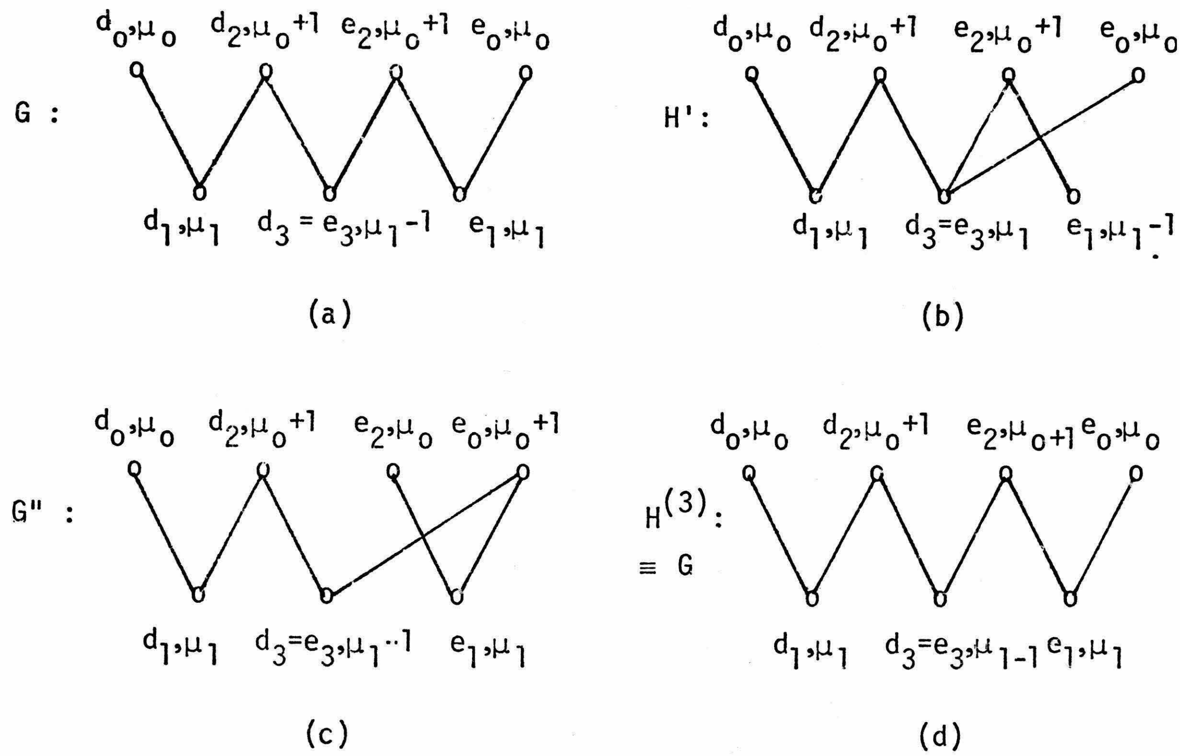


Fig. 3-5

to some $H' \cong H$ as we can see in Fig. 3-5(b). (Note that it's impossible to replace e_oe_1 by e_oe_2 or e_od_2 since then $\mu_1 = \mu_o + 2$, and if the new edge is e_od_2, then d_od_1 and e_od_2 form a (1,1)-coupling in the new graph; while if the new edge is e_oe_2, then $d_3e_2 \rightarrow d_2e_o$ is a forced more and in the last graph we obtained, d_od_1 and d_2d_3 form a (1,1)-coupling.) Since $e_oe_3e_2e_1$ is a special 3-chain of degree type $(\mu_o, \mu_1, \mu_o + 1, \mu_1 - 1)$, $e_2e_3 \rightarrow e_oe_1$ is a forced move sending H' to some $G'' \cong G$ (the configuration D is changed to D'' as depicted in Fig. 3-5(c)). Now in $G'' - e_oe_3$, $d_od_1d_2d_3$ is forbidden of degree type $(\mu_o, \mu_1, \mu_o + 1, \mu_1 - 2)$ and $e_oe_1e_2$ is forbidden of degree type (μ_o, μ_1, μ_o) ; and since we have to replace e_oe_3 by an edge of degree type $(\mu_0, \mu_1 - 2)$ in $G'' - e_0e_3$, the only possible ways are e_2e_3, e_2d_0, e_0d_0, e_2d_2, e_0d_2. If the replacing edge is e_2e_3, then in $H^{(3)} \cong H$, we are returning to our original configuration D (i.e. $D^{(3)} = D$) and so Lemma 2.1 applies to say that G is edge-reconstructable (see Fig. 3-5(d)). The latter four possibilities can happen only when $\mu_0 = 1$ since otherwise G is a block and we would have an odd cycle if we join any one of the four : e_0d_0, e_2d_0, e_0d_2, e_2d_2. If we join e_0d_0 or e_2d_0, then $\mu_1 = \mu_0 + 2$ by degree argument and d_3d_2 is an edge of degree type $(\mu_0, \mu_0 + 1)$ in the new graph $H^{(3)}$, impossible since $\mu_1 > \mu_0 + 1$. If we join e_0d_2 or e_2d_2, then in $H^{(3)}$, we have a 5-path $fe_1gd_2d_1d_0$ of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_0 + 2, \mu_1, \mu_0)$ with $f = e_0$, $g = e_2$ if we join e_2d_2 (and $f = e_2$, $g = e_0$ if we join e_0d_2). Obviously we have $\mu_1 = \mu_0 + 3$ in this case. Now $e_1g \to fd_2$ is a forced move, and in the newly obtained graph gd_2 and d_0d_1 form a (1,1)-coupling. We have now proved $\gamma = 3$ is impossible.

Next, let $\gamma = 2$. This means that $d_0d_1d_2$ and $e_0e_1e_2$ are disjoint except at $d_2 = e_2$. The forced move $d_1d_2 \to d_0d_3$ gives us in the new graph H' that $e_0e_1e_2$ is a 2-path of degree type (μ_0, μ_1, μ_0) and so $\Omega(H') \leq 2$ and we are done (for $\Omega(G) \leq \Omega(H') \leq 2$ implies by Lemma 3.7 that G is edge-reconstructable).

Now, consider $\gamma = 1$. This means $d_0 \neq e_0$, but $d_1 = e_1$. So $d_0d_1e_0$ is a 2-path of degree type (μ_0, μ_1, μ_0) and we see immediately that this case is again impossible.

Finally let $\gamma = 0$. Let $0 < \delta \leq 3$ be the first integer that $d_\delta \neq e_\delta$. If $\delta = 3$, then $d_2 = e_2$ and d_3 and e_3 are two distinct vertices of degree $\mu_1 - 1$ adjacent to d_2 of degree $\mu_0 + 1$. The forced move $d_1d_2 \to d_od_3$ gives in the new graph an edge d_2e_3 of degree type $(\mu_o, \mu_1 - 1)$ which is impossible.

Next, let $\delta = 2$ (still $\gamma = 0$). We have $d_o = e_o$, $d_1 = e_1$ but $d_2 \neq e_2$. d_3 and e_3 may coincide or not. Suppose $d_3 \neq e_3$ at first. The forced move $d_1d_2 \to d_od_3$ gives us in the new graph a 5-path $d_2d_3d_od_1e_2e_3$ of degree type $(\mu_o, \mu_1, \mu_o + 1, \mu_1 - 1, \mu_o + 1, \mu_1 - 1)$, excludable as configuration C' in Fig. 3-3 . Now let $d_3 = e_3$. In G we have a configuration as in Fig. 3-6(a) below.

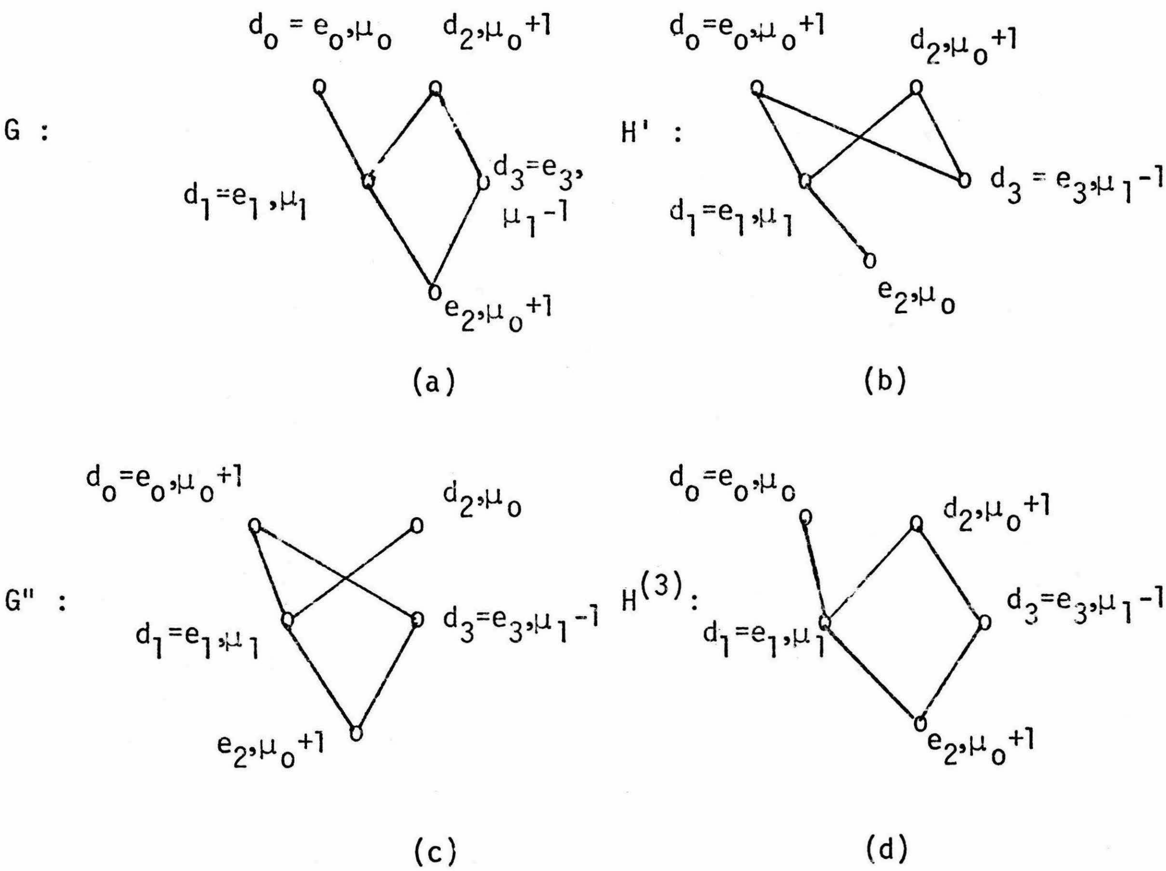


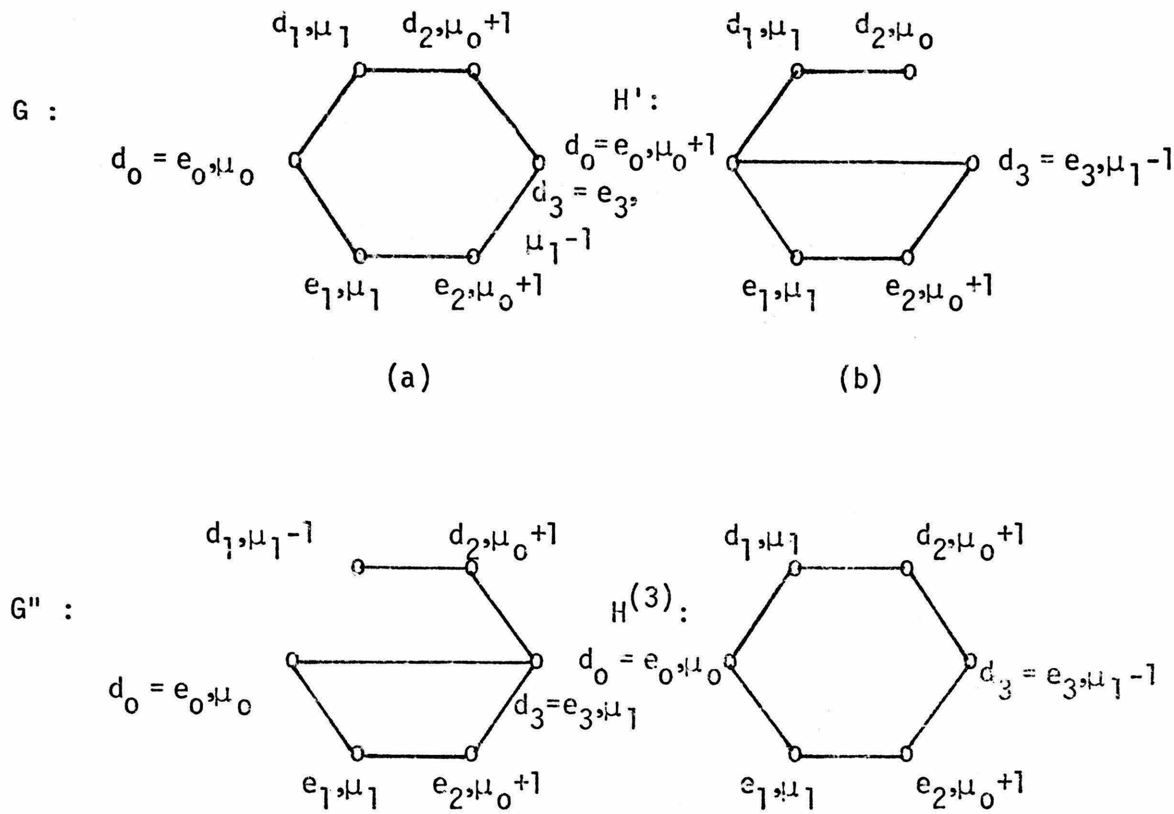
Fig. 3-6

In $G - e_2d_3$, $d_od_1d_2d_3$ and $e_2d_1d_2d_3$ are forbidden of degree type $(\mu_o, \mu_1, \mu_o + 1, \mu_1 - 2)$ and $d_od_1e_2$ is forbidden of degree type (μ_o, μ_1, μ_o) ; so since our replacing edge must be of degree type $(\mu_o, \mu_1 - 2)$ in $G - e_2d_3$, the possibilities are d_od_3, d_od_2, d_oe_2, d_2e_2 (observe that d_od_2 will be possible only if $\mu_1 = \mu_o + 2$ etc.). The latter three will be clearly impossible for a bipartite graph G since they "create" triangles in the new graph H' in an obvious way (so they have to be considered if we want to prove the same lemma for more general graph). After these considerations, $e_2d_3 \to d_od_3$ is a forced move sending G to $H' \cong H$ (see Fig. 3-6(b)).

In $H' - d_2d_3$, $d_2d_1d_o$, $d_2d_1d_od_3$ and $e_2d_1d_od_3$ are all forbidden by degree argument as the previous paragraph; so the replacing edge can be e_2d_3 only (to avoid any triangles again). We see now $d_2d_3 \to e_2d_3$ is a forced move sending H' to $G'' \cong G$. (see Fig. 3-6(c)).

Finally, in $G'' - d_od_3$, $d_od_1d_2$, $d_od_1e_2d_3$, $d_2d_1e_2d_3$ are forbidden by same argument and $d_od_3 \to d_2d_3$ is a forced move sending G'' to $H^{(3)} \cong H$ (Fig. 3-6(d)). We see three forced moves: $e_2d_3 \to d_od_3$, $d_2d_3 \to e_2d_3$, and $d_od_3 \to d_2d_3$ return us to the original configuration in Fig. 3-6(a) (Fig. 3-6(d) and Fig. 3-6(a) are identical), and so Lemma 2.1 applies to say that G is edge-reconstructable. We have proved now that $\delta = 2$ is impossible.

For $\gamma = 0$, we consider at last the case $\delta = 1$. We have now $d_o = e_o$, $d_1 \neq e_1$. d_2 and e_2 may coincide or not. Let $d_2 \neq e_2$ first. Suppose furthermore that $d_3 \neq e_3$ at this moment. This will be proved to be the only possible interconnection pattern later. Next, suppose $d_3 = e_3$ (still $d_1 \neq e_1$, $d_2 \neq e_2$). Our two special 3-chains form a configuration as in Fig. 3-7(a).



(a) (b)

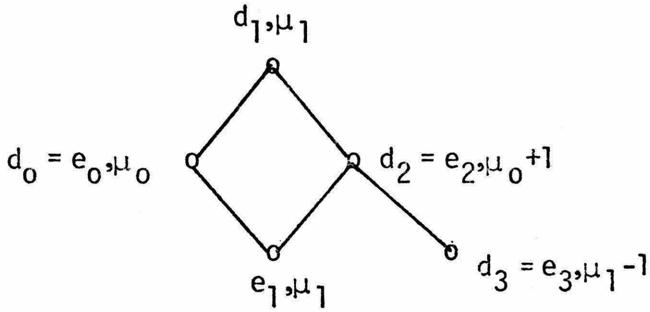
(c) (d)

Fig. 3-7

In $G - d_2d_3$, $d_od_1d_2$ is forbidden of degree type (μ_o, μ_1, μ_o) and $d_oe_1e_2d_3$ is forbidden of degree type $(\mu_o, \mu_1, \mu_o + 1, \mu_1 - 2)$; so degree argument as well as the requirement of no odd cycles in any edge-reconstruction says that $d_2d_3 \to d_od_3$ is a forced move sending G to $H' \cong H$ (Fig. 3-7(b)). Now $d_2d_1d_od_3$ is of degree type $(\mu_o, \mu_1, \mu_o + 1, \mu_1 - 1)$ in H' and so $d_1d_o \to d_2d_3$ is obviously a forced move sending H' to G'' (Fig. 3-7(c)). Finally we observe that d_o is a forced vertex in $G'' - d_0d_3$, and if another end of the replacing edge g doesn't lie on the configuration in Fig. 3-7(c), then $d_0e_1e_2d_3d_2d_1$ is the excludable configuration in Fig. 3-3(b); so g must be e_2, d_2, d_1. To avoid an odd cycle, we see readily that d_1 is the only choice. As in Fig. 3-5 or Fig. 3-6, we see that three forced moves return us to the original graph and so G is edge-reconstructable by Lemma 2.1.

Finally let $d_2 = e_2$ (with $\gamma = 0$, $\delta = 1$). We have $d_1 \neq e_1$. d_3 and e_3 must coincide otherwise the forced move $d_1d_2 \to d_0d_3$ gives an edge d_2e_3 of degree type $(\mu_0, \mu_1 - 1)$. Consider the configuration consisting of the two 3-paths $d_0d_1d_2d_3$ and $e_0e_1e_2e_3$ as below:

G :



We can prove its excludability in a way very similar to that of excludability of the configuration in Fig. 3-6(a). First we note $d_2e_1 \to d_0d_3$ is a forced move sending G to $H' \cong H$ since $e_0e_1e_2e_3$ is a special 3-chain of degree type $(\mu_0, \mu_1, \mu_0+1, \mu_1 - 1)$ in G. Next $d_0d_1 \to d_2e_1$ is a forced move sending H' to $G'' \cong G$ since $d_2d_1d_0e_1$ is of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1)$ in H'. Finally, $d_0d_3 \to d_0d_1$ is a forced move since in $G'' - d_0d_3$, d_0 is a forced vertex, and if the other end g of the replacing edge isn't d_1 or d_2, the new graph $H^{(3)}$ will contain a special 3-chain $d_0e_1d_2d_1$ with $d_3 \neq d_1$ another vertex of degree $\mu_1 - 1$ adjacent to d_2 which is excludable by the forced move $e_1d_2 \to d_0d_1$; if g is d_2, then $H^{(3)}$ contains the triangle $d_0e_1d_2$, so $g = d_1$ is the only choice. We have thus seen that the three forced moves $d_2e_1 \to d_0d_3$, $d_0d_1 \to d_2e_1$ and $d_0d_3 \to d_0d_1$ return us to the original configuration, so G is edge-reconstructable by Lemma 2.1.

We have now investigated all the possibilities of interconnection pattern for non-disjoint special 3-chains of degree type $(\mu_0, \mu_1, \mu_0+1, \mu_1 - 1)$. We found that almost no interconnection pattern exists, i.e. they must be all disjoint except at the starting vertex. Let's state this as a bypassing lemma in proving Lemma 3.10.

Lemma 3.11. All special 3-chains of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1)$ must be disjoint except at the starting vertex.

Let's come back to the configuration M in Fig. 3-4(a). Recall that $\mu_1 > \mu_0 + 1$ and $\mu_2 = \mu_0 + 1$. Rename a_0, a_1, a_2, b_1, b_0 by c_0, c_1, c_2, c_3, c_4 respectively. Note that $c_1c_2 \to c_1c_4$ is a forced move sending the 5-path $c_0c_1c_2c_3c_4$ in G to $c_0c_1c_4c_3c_2$ in some $H' \cong H$. Similarly $c_3c_2 \to c_3c_0$ is a forced move sending $c_0c_1c_2c_3c_4$ to $c_2c_1c_0c_3c_4$ (or $c_4c_3c_0c_1c_2$ which is the same path traced backwards). For simplicity of notation, we will use 0 1 2 3 4 to represent symbolically $c_0c_1c_2c_3c_4$. The forced move $c_1c_2 \to c_1c_4$, or more simply 1 2 → 1 4, will change 0 1 2 3 4 to 0 1 4 3 2. The other forced move 3 2 → 3 0 will change 0 1 2 3 4 to 2 1 0 3 4. Note that 0 1 2 3 4 and 4 3 2 1 0 mean the same path, one is the other traced backwards. We note that the effect of forced moves here is to reverse the order of either the first three digits ("01" "234" to "01" "432", 2 3 4 is reversed to 4 3 2 and 01 is intact) or the last three digits ("012" "34" to "210" "34") when we transform from an isomorph of G to an isomorph of H (and vice versa).

After a moment of reflection, we see that if we start with 0 1 2 3 4 in G, we will have

in G (or isomorphs)	in H (or isomorphs)
0 1 2 3 4	2 1 0 3 4
2 1 4 3 0	4 1 2 3 0
4 1 0 3 2	0 1 4 3 2

provided our forced moves affects only the vertices c_0, c_1, c_2, c_3, c_4 (like $c_1c_2 \to c_1c_4$ or $c_3c_2 \to c_3c_0$ etc.).

Let's look at our $c_0c_1c_2c_3c_4$ (that is 0 1 2 3 4) again. G must have at least one special 3-chain since $\Omega(G) \geq 4$. We will show that c_0 cannot lie on a special 3-chain $c_0' \, c_1' \, c_2' \, c_3'$ with $c_0 = c_0'$ (in the language of previous paragraphs, we will show that 0 cannot lie on a special 3-chain 0' 1' 2' 3' with 0' = 0). Suppose not, and let's consider $G - c_4 \, c_3$. c_4 is a forced vertex in this subgraph and if c_4g is an edge replacing $c_4 \, c_3$, then g cannot be c_2 or c_2' otherwise the new graph has an excludable (1,1)-coupling (given by 01 and 2 4 or 0'1' and 2'4). g must then be c_3' by Lemma 3.11 (or 0 1 2 3 and 0' 1' 2' 3 are two distinct nondisjoint special 3-chains in the new graph). So some $H' = G - c_4c_3 + c_4c_3' \cong H$. Looking at the previous paragraph, we see that if 0 1 2 3 4 is a path in G, then 4 1 0 3 2 or 2 3 0 1 4 is a path in an isomorph of G by appropriate forced moves. This tells us that if 0' 1' 2' 3' 4 is a path in H', then

2' 3' 0' 1' 4 is a path in an isomorph H" of H. Let's delete 1'4 and see what happens. Note that 0 1 2 3 and 0'1'2'3' may intersect somewhere besides 0 = 0'. If they do not intersect anywhere except at 0 = 0', then 1'4 → 3 4 is a forced move otherwise we have an excludable configuration as in Fig. 3-3 . The same argument applies if they intersect at 1 = 1' as well. If they intersect at 2 = 2' (they cannot intersect at 3 = 3' since 3 is a vertex of degree μ_1 and 3' of de-degree μ_1 and 3' of degree μ_1-1 in G), then in H", 2 3 is an edge of degree type $(\mu_0, \mu_1 - 1)$, impossible.

Now in $G' = H" - c_1'c_4 + c_3c_4$, $c_2'c_3'c_0'c_1'$ is a special 3-chain and the forced move $c_3' c_0' \to c_2'c_1'$ returns us to our original configuration consisting of $c_0c_1c_2c_3c_4$ and $c_0'c_1'c_2'c_3'$ while sending us to some $H^{(3)} \cong H$. We then have $G \equiv H^{(3)} \cong H$, a contradiction. Similar argument says that c_4 cannot lie on any special 3-chain.

As an illustration, we depict the case when 0 1 2 3 and 0'1'2'3' intersect at 0 = 0' only in Fig. 3-8 below. The pictures themselves are self-explanatory.

In G, with the fixed 4-path $c_0c_1c_2c_3c_4$ of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1, \mu_0)$, we see that c_4 is a forced vertex in $G - c_3c_4$, and any edge-reconstruction must be of the form $G - c_3c_4 + d_3c_4$ with d_3 lying on a special 3-chain $d_0d_1d_2d_3$ in G by $B_0(3)$. (Note we cannot have $G - c_3c_4 + c_2c_4$ as an edge-reconstruction otherwise we have a (1,1)-coupling and are thus done.). d_3 must not coincide with c_3 since d_3 has degree $\mu_1 - 1$ while c_3 has degree μ_1 in G. It's conceivable that $G - c_3c_4 + d_3c_4$ may be isomorphic to H, the only non-isomorphic edge-reconstruction of G, or even isomorphic to G.

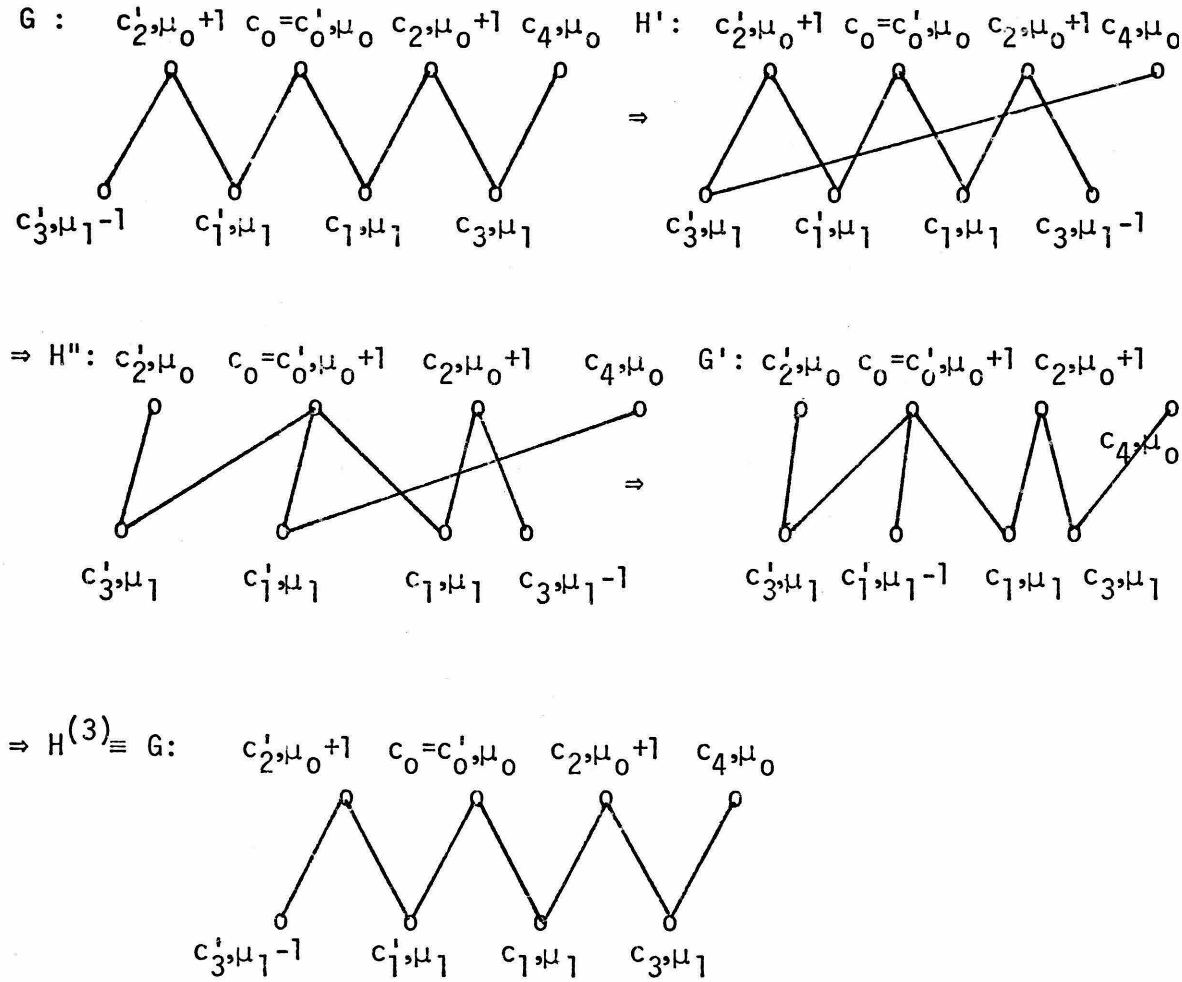


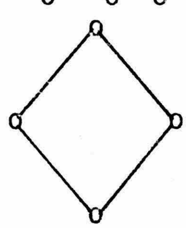
Fig. 3-8

We partition the special 3-chains in G into two classes with respect to the 4-path $c_0c_1c_2c_3c_4$. Call a special 3-chain $d_0d_1d_2d_3$ a Class-1 3-chain if $G - c_3c_4 + d_3c_4$ is isomorphic to H, otherwise a Class-2 3-chain (i.e. when $G - c_3c_4 + d_3c_4 \cong G$). Similar definitions hold for any isomorph of G or H. Class 1 must be nonempty otherwise G is edge-reconstructable (Class 2 can be empty though). Let $n \geq 1$ and $m \geq 0$ be the number of special 3-chains of Class 1 and Class 2 respectively in G. Denote by $0'1'2'3'$, ..., $0^{(n)}1^{(n)}2^{(n)}3^{(n)}$ the n special 3-chains of Class 1 and $0^{(n+1)}1^{(n+1)}2^{(n+1)}3^{(n+1)}$, ..., $0^{(n+m)}1^{(n+m)}2^{(n+m)}3^{(n+m)}$ the m special 3-chains of Class 2 in G (they are all disjoint by Lemma 3.11). Choose $c_0c_1c_2c_3c_4$ among 4-paths of the same degree type in G so that the number of special 3-chains in Class 2 is maximum M. Let C be the configuration in G consisting of (the disjoint union of) $0\ 1\ 2\ 3\ 4$, $0'1'2'3'4'$, ..., $0^{(n)}1^{(n)}2^{(n)}3^{(n)}$, $0^{(n+1)}1^{(n+1)}2^{(n+1)}3^{(n+1)}$, ..., $0^{(n+m)}1^{(n+m)}2^{(n+m)}3^{(n+m)}$. Let $H' = G - 34 + 3^{(i)}4$ for some i, $1 \leq i \leq n$ (i.e. $H' = G - c_3c_4 + c_3^{(i)}c_4$). Since $0^{(i)}1^{(i)}2^{(i)}3^{(i)}4$ is in H', we see as before that $2^{(i)}3^{(i)}0^{(i)}1^{(i)}4$ is in some $H'' \cong H$. Consider $H_\alpha \cong H$ obtained from G by the forced move $c_1^{(i)}c_2^{(i)} \to c_0^{(i)}c_3^{(i)}$ (so $0^{(i)}1^{(i)}2^{(i)}3^{(i)}$ in C becomes $2^{(i)}3^{(i)}0^{(i)}1^{(i)}$). We will see that a Class-2 3-chain $0^{(j)}1^{(j)}2^{(j)}3^{(j)}$ in G, $n + 1 \leq j \leq n + m$, will also be Class 2-chain for H_α as well (i.e. $H_\alpha - c_3c_4 + c_3^{(j)}c_4$ or $H_\alpha - 34 + 3^{(j)}4$ will be isomorphic to H_α and hence H, but not G). This is trivial because first $G - 34 + 3^{(j)}4 = G'$ is isomorphic to G by definition of "Class 2" in G; and secondly $G' - 1^{(i)}2^{(i)} + 0^{(i)}3^{(i)} = H_\beta$ is obviously isomorphic to H; finally we see that $H_\alpha - 34 + 3^{(j)}4 = G - 1^{(i)}2^{(i)} + 0^{(i)}3^{(i)} - 34 + 3^{(j)}4$ is identically equal to $H_\beta = G' - 1^{(i)}2^{(i)} + 0^{(i)}3^{(i)} = G - 34 + 3^{(j)}4 - 1^{(i)}2^{(i)} + 0^{(i)}3^{(i)}$ since all paths involved ($0\ 1\ 2\ 3\ 4$, $0^{(i)}1^{(i)}2^{(i)}3^{(i)}$, $0^{(j)}1^{(j)}2^{(j)}3^{(j)}$) are disjoint. Since $2^{(i)}3^{(i)}0^{(i)}1^{(i)}4$ is in some $H'' \cong H$, we see that $2^{(i)}3^{(i)}0^{(i)}1^{(i)}$ is also a Class-2 3-chain for H_α, and H_α (and hence H which is isomorphic to H) has at least one more element in its Class 2 special 3-chains than G does with respect to the same 4-path $c_0c_1c_2c_3c_4$ common to both graphs. If we start anew in H with a 4-path $d_0d_1d_2d_3d_4$ of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1, \mu_0)$ and define its Class 1 and Class 2 special 3-chains, we will see that an isomorph of G has at least $(M+1) + 1 = M + 2 > M$ special 3-chains of Class 2 with respect to $d_0d_1d_2d_3d_4$, contradiction to the maximality of M defined for 4-paths of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1, \mu_0)$ in G.

Since we have obtained a contradiction finally, we are done with our subcase 3(a).

Subcase 3(b). of Lemma 3.1C. $b_0 = a_0$ for a (2,1)-coupling.

Recall Fig. 3-4(b), which is redrawn here for convenience.



If μ_1 is $\mu_0 + 1$, then a_0a_1 is a forced edge otherwise $a_0b_1a_2a_1$ is an excludable configuration of degree type $(\mu_0, \mu_0 + 1, \mu_0 + 1, \mu_0)$ by Lemma 3.2. We now let $\mu_1 > \mu_0 + 1$. a_0 is obviously a forced vertex and in $G - a_0a_1 + a_0c$, $a_0b_1a_2a_1$ is a newly created special 3-chain of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1)$ (note that if $\mu_3 = \mu_1$ then a_0a_2 is again a forced edge for we cannot join a_2a_0 to edge-reconstruct in order to avoid triangles), and so $B_0(3)$ implies that we have to destroy a special 3-chain by joining a_0d. Degree argument (and principle of avoiding triangles) says that a_o and c lie in G on a special 3-chain $d_o d_1 d_2 d_3$ with $a = d_o$, $d = d_3$. For simplicity, first suppose that none of d_1 or d_2 is any of a_1, a_2, b_1. Consider the configuration consisting of the 4-cycle $a_o a_1 a_2 b_1$ and 3-path $d_o d_1 d_2 d_3$ as in Fig. 3-9(a).

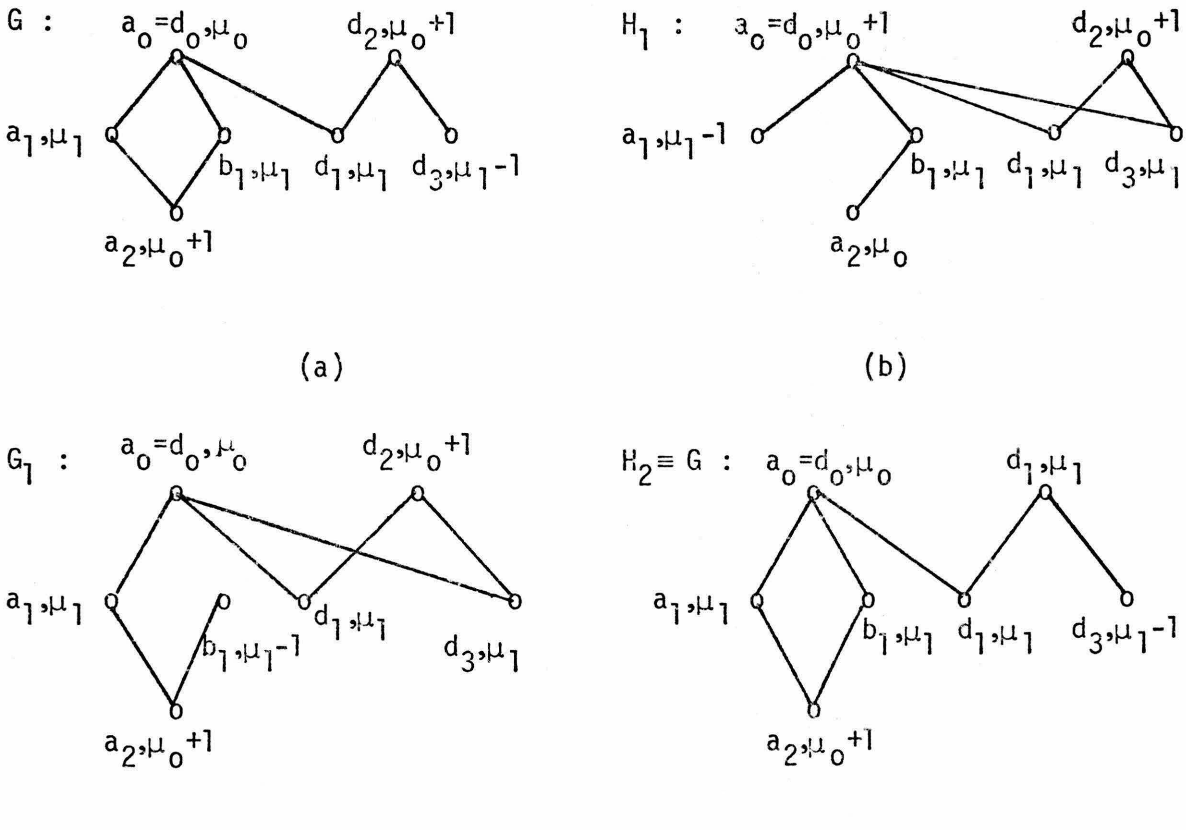


Fig. 3-9

By Lemma 3.11, all special 3-chains are disjoint and there can't be another $d_o' d_1' d_2' d_3'$ with $d_o' = a_o$. We see easily that $a_2 a_1 \to a_o d_3$ is a forced move sending G to some $H_1 \cong H$ (In $G - a_2 a_1$, $a_o a_1$ has degree type $(\mu_o, \mu_1 - 1)$ and $a_o b_1 a_2$ has forbidden degree type

$(\mu_0, \mu_1, \mu_0))$ as seen in Fig. 3-9(b). Next, $a_0b_1 \to a_1a_2$ is a forced move sending H_1 to some $G_1 \cong G$ since $a_1a_0b_1a_2$ is a special 3-chain of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1)$ in H_1 (see Fig. 3-9(c)). Finally we note that $a_0d_3 \to a_0b_1$ is a forced move sending G_1 to some $H_2 \cong H$ since a_0 is a forced vertex and all special 3-chains are disjoint by Lemma 3.11 (so $a_0a_1a_2b_1$ and $d_0d_1d_2d_3$ cannot both be special 3-chains in some edge-reconstruction of G).

Now three forced moves $a_2a_1 \to a_0d_3$, $a_0b_1 \to a_1a_2$, $a_0d_3 \to a_1a_2$ return us to our original configuration G, and we get $H \cong H_2 \equiv G$, a contradiction.

Let's consider then the cases when d_1 or d_2 is one of a_1, a_2, b_1. First suppose d_1 is but d_2 is not. Then d_1 must be one of a_1 or b_1 to avoid triangles. The above argument works except the justification of the forced move $a_0d_3 \to a_0b_1$ is by the fact that the configuration in Fig. 3-3 is excludable. The argument for the case when d_2 is one of a_1, a_2, b_1 but d_1 isn't, follows the same line as the first case when none of d_1, d_2 is a_1, a_2 or b_1. Lemma 3.11 is applied in a different way (so that we don't have $a_0a_1a_2a_3$ and $a_0a_1a_2b_1$ both as special 3-chains). For the case when both d_1 and d_2 are among a_1, a_2, b_1, we must have $d_1 = a_1$, $d_2 = a_2$ or $d_1 = b_1$, $d_2 = a_2$ to avoid triangles. Without loss of generality, let $d_1 = a_1$, $d_2 = a_2$. But now $a_1a_2 \to a_0d_3$ is clearly a forced move which gives us two non-disjoint special 3-chains of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1)$, namely $a_2b_1a_0a_1$ and $a_2d_3a_0a_1$, and this is impossible by Lemma 3.11.

So we have done the proof of our subcase 3(b), hence completing the proof of Case 3, and we are done with the lengthy proof of Lemma 3.10.

Q.E.D.

Section 5. Inductive proof of $A(n)$ and $B_i(n)$

By Lemma 3.8, 3.9 and 3.10 of Section 4, we know that $A(n)$ and $B_i(n)$ are true for $n = 1, 2, 3$, and $P(k)$ is true for $k = 0, 1, 2$. To prove the validity of $A(n)$, $B_i(n)$ and $P(n)$ for a general n, we will do some induction which interlocks these three conditions in a peculiar way. We prove two technical lemmas:

Proposition 3.12. When $\Omega(G) \geq 5$, then for any k, $4 \leq k \leq \Omega(G) - 1$, $A(k)$ and $B_i(k)$ are true for any $i \geq 0$ provided (1) $A(j)$ and $B_i(j)$ are true for any $1 \leq j < k$ and (2) $P(k-1)$ is true.

Proposition 3.13. When $\Omega(G) \geq 5$, then for any m, $3 \leq m \leq \Omega(G) - 2$, $P(m)$ is true if (1) $A(j)$ and $B_i(j)$ are true for any $1 \leq j \leq m$, $i \geq 0$ and (2) $P(m-1)$ is true.

Assuming the validity of Proposition 3.12 and Proposition 3.13, we can prove now an interesting fact:

Proposition 3.14. $A(n)$ and $B_i(n)$ are true for any n, $1 \leq n \leq \Omega(G) - 1$; $P(\alpha)$ is true for any α, $0 \leq \alpha \leq \Omega(G) - 2$.

Proof of Proposition 3.14 (assuming Proposition 3.12 and 3.13).

Assume $\Omega(G) \geq 5$ first. The proof is a folklore one. Suppose $A(n)$ is false for some n, $1 \leq n \leq \Omega(G) - 1$, and let α be the smallest such integer. Then $\alpha \geq 4$ by Lemma 3.8. By Proposition 3.12, either $P(\alpha-1)$ or $B_i(\beta)$ is false for some $1 \leq \beta < \alpha$ ($A(\beta)$ is true by minimality of α). Suppose first $P(\alpha-1)$ is false. Let $\gamma \leq \alpha - 1$ be the smallest integer such that $P(\gamma)$ is false. Then $\gamma \geq 3$ by Lemma 3.10.

Proposition 3.13 says that either $P(\gamma-1)$ is false or $A(j)$ or $B_i(\delta)$ is false for some $1 \leq j, \delta \leq \gamma$. Since $j < \alpha$, the minimality of α and γ say that the only possibility is that $B_i(\delta)$ is false for some δ, $1 \leq \delta \leq \gamma$. Let ϵ be the smallest integer such that $B_i(\epsilon)$ is false. $\epsilon \geq 4$ by Lemma 3.9. Proposition 3.12 again says that either $P(\epsilon-1)$ or $A(\nu)$ or $B_i(\chi)$ is false, some $1 \leq \nu, \chi < \epsilon$, $i \geq 0$. This is impossible since $\epsilon \leq \gamma < \alpha$ and α, γ, ϵ are respectively the smallest integer that A, P and B_i fail; and we get a contradiction. So $A(n)$ is true for any n, $1 \leq n \leq \Omega(G) - 1$.

The validity of $B_i(n)$ for $1 \leq n \leq \Omega(G) - 1$ and $P(m)$, $0 \leq m \leq \Omega(G) - 1$ is done in a similar way (by applying Propositions 3.12 and 3.13).

Heuristic feeling of the interlock induction step of Proposition 3.14 by Proposition 3.12 and Proposition 3.13 can be obtained by the diagram in Fig. 3-10. In that figure, conditions $A(n)$, $B_i(n)$ for any $i \geq 0$ and $P(n)$ are classified as a rank-n condition.

There is only one rank-0 condition $P(0)$ ($A(0)$ is also rank-0 condition, but we don't need it). There is no rank-$(\Omega - 1)$ condition for P, only those for A and B_i's.

For $4 \leq k \leq \Omega(G) - 1$, we see that conditions A and B_i's of rank k are proved by conditions P's, A's and B_i's of smaller rank. For $1 \leq k \leq 3$, their validity is ensured by Lemmas 3.8 and 3.9. For $3 \leq k \leq \Omega(G) - 2$, the condition $P(k)$ is proved by conditions P, A, and B_i's of smaller rank and the conditions $A(k)$ and $B_i(k)$'s (of the same rank).

We are left with the cases $\Omega(G) \leq 4$. But these are readily justified by Lemmas 3.8, 3.9 and 3.10 (actually $\Omega(G)$ can be assumed to be

Rank 0	$P(0)$	Proved by Lemma 3.10			
Rank 1	$P(1)$		$A(1)$	$B_0(1) \ldots B_i(1) \ldots$	proved by Lemmas 3.8 & 3.9
Rank 2	$P(2)$		$A(2)$	$B_0(2) \ldots B_i(2) \ldots$	
Rank 3	$P(3)$		$A(3)$	$B_0(3) \ldots B_i(3) \ldots$	
Rank 4	$P(4)$		$A(4)$	$B_0(4) \ldots B_i(4) \ldots$	
	$\vdots$				
Rank k-1	$P(k-1)$		$A(k-1)$	$B_0(k-1) \ldots B_i(k-1) \ldots$	
Rank k	$P(k)$		$A(k)$	$B_0(k) \ldots B_i(k) \ldots$	
	$\vdots$				
Rank Ω-2	$P(\Omega-2)$		$A(\Omega-2)$	$B_0(\Omega-2) \ldots B_i(\Omega-2) \ldots$	
Rank Ω-1			$A(\Omega-1)$	$B_0(\Omega-1) \ldots B_i(\Omega-1) \ldots$	

Fig. 3-10 Interlock hierarchical structure of Proposition 3.14 (here Ω means $\Omega(G)$)

≥ 4 by Lemma 3.7). Q.E.D.

Proposition 3.14, especially the validity of $P(\alpha)$'s, will be the main tool to prove the edge-reconstructability of G when we have Type-I, Type-II, Type-III terminations respectively (we will prove the main theorem in Section 7). We will prove Proposition 3.12 in this section and Proposition 3.13 in next section (Section 6) in order to complete the proof of Proposition 3.14.

In the following, we will assume $\Omega(G) \geq 5$ and $P(k-1)$, $A(j)$ and $B_i(j)$ are true for any $1 \leq j < k$, $i \geq 0$, and our k satisfies

$4 \leq k \leq \Omega(G) - 1$. We will prove the validity of $A(k)$ and $B_i(k)$.

Note that the validity of $P(k-1)$ implies those of $P(\beta)$'s for any β, $0 \leq \beta < k - 1$, by definition.

Consider a special k-chain $a_0 a_1 \ldots a_k$ in G, $4 \leq k \leq \Omega(G) - 1$. We will divide the proof of Proposition 3.12 into four cases, according to the degree of a_{k-1} and a_{k-2}. Induction assumption says that $\mu_m(G) = \mu_m(H)$ for any $H \in \Sigma_G$ if $0 \leq m \leq k - 1$. However, $\mu_k(G)$ and $\mu_k(H)$ may be different (we want to show they are equal).

The validity of $A(n)$ and $B_i(n)$ seem so trivial that they may be classified as "folklore" theorems. In fact, in Edward R. Swart [17], he conjectured something interesting:

Conjecture of Swart: The number of polygon (i.e. n-cycle) of given degrees for every vertex is edge-reconstructable in a general graph G.

This is a substantial generalization of the well-known fact that the number of n-cycles (so degree of each vertex is assumed to be 2 only) is edge-reconstructable (proof by Kelly's Lemma applied to n-cycles). However, this more general Conjecture of Swart is terribly hard to prove in general graphs. The validity of $B_i(n)$ is trivial if we can have a conjecture similar to that of Swart:

Conjecture. The number of n-paths $a_0 a_1, \ldots a_n$ of degree type $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is edge-reconstructable for any general graph G.

The validity of $A(n)$ is actually a quick corollary of $B_i(n)$. However since the "obvious" conjecture stated above has no obvious proof, we need the validity of $P(\alpha)$'s as an interlock in our induction step.

Case 1 of Proposition 3.12. $\mu_{k-1} > \mu_0 + 1$, $\mu_{k-2} > \mu_0 + 1$.

Consider $G - a_{k-1}a_{k-2}$ for a special k-chain $a_0a_1 \ldots a_k$ in G. In this edge-deleted subgraph, $a_0a_1 \ldots a_{k-2}$ has a forbidden degree type $(\mu_0, \mu_1, \ldots, \mu_{k-2}, \mu_{k-1}-1)$. Let $H = G - a_{k-2}a_{k-1} + cd$ be a non-isomorphic edge-reconstruction of G, where c and d have respectively degrees $\mu_{k-2}-1$ and $\mu_{k-1}-1$ in $G - a_{k-2}a_{k-1}$. Suppose first that a_{k-2} is neither c nor d. By A(k-2), one of c or d must be some a_j, $0 < j < k - 2$ (c or d cannot be a_0 since $\mu_{k-1}-1 > \mu_0$, $\mu_{k-2}-1 > \mu_0$ by assumption of our case). But then $B_0(j)$ implies that in H we should have a special j-chain $b_0b_1 \ldots b_j$ with $b_j = a_{k-1}$ or a_{k-2}. It is easy to see that $b_0b_1 \ldots b_{j-1}$ is a "genuine" special (j-1)-chain in G. (Note that though $b_0b_1 \ldots b_j$ is a genuine special j-chain in H, it is not a special j-chain in G). Now $b_0b_1 \ldots b_{j-1}$ and $a_0a_1 \ldots a_{k-2}$ ($a_0a_1 \ldots a_{k-1}$ resp.) form a (k-2,j-1)-coupling ((k-1,j-1)-coupling) in G if $b_j = a_{k-2}$ ($b_j = a_{k-1}$ resp.), and so P(k-1) says G is edge-reconstructable ($b_0b_1 \ldots b_{j-1}$ cannot be $a_0a_1 \ldots a_{k-3}$ or $a_0a_1 \ldots a_{k-2}$ since otherwise $j = k - 2$ or $k - 1$; note also $b_{j-1}a_{k-1}$ or $b_{j-1}a_{k-2} \in E(G)$). So this case can be excluded.

Hence we may assume one of c, d is a_{k-2}. When $\mu_{k-2} \neq \mu_{k-1}$, then a_{k-2} must be c by degree argument, and when $\mu_{k-2} = \mu_{k-1}$, then it doesn't matter to call a_{k-2} by c or d (i.e. c or d is a "dummy" label here). So we can always assume $c = a_{k-2}$. d may lie on $a_0a_1 \ldots a_{k-3}$ or not. Suppose first that $d = a_j$, $0 < j \leq k - 3$ (d cannot be a_0 since $\mu_{k-1} - 1 > \mu_0$). $B_0(j)$ implies the existence of special $b_0b_1 \ldots b_j$ in H with $b_j = a_{k-1}$. So as before, we have a (k-1,j-1)-coupling in G, and P(k-1) implies G's edge-reconstructability. This case can also be excluded now.

The only case left is that $c = a_{k-2}$ and $d \neq a_j$, for any j, $0 < j \leq k - 3$. Now if a_{k-1} is adjacent to $\alpha \geq 1$ vertices of degree $\mu_k(G)$ in G (a_k is such a vertex), we can show easily by Lemma 3.1 that d is adjacent to exactly $\alpha \geq 1$ vertices of degree $\mu_k(G)$ in H. Let e be such a vertex, then the k - path $a_0 a_1 \ldots a_{k-2}$ d e in H says that $\mu_k(H) \leq \mu_k(G)$ (conceivably e might be some a_j, $0 < j < k - 2$). Note that $\mu_{k-1} = \mu_k(G) + 1$ when $d = a_k$.

Hence we have $\mu_k(H) \leq \mu_k(G)$ for any $H \in \Sigma_G$ when $\mu_{k-1}, \mu_{k-2} > \mu_0 + 1$. The above argument doesn't use the fact that $\Omega(G) \leq \Omega(H)$ for all $H \in \Sigma_G$; so we can use the symmetry argument (starting at some $H \not\cong G$, get an isomorph G' of G from $H - b_{k-2}b_{k-1}$ for some special k-chain $b_0 b_1 \ldots b_k$ in H, and show $\mu_k(G') \leq \mu_k(H)$) and finally conclude that $\mu_k(H) = \mu_k(G)$ for all $H \in \Sigma_G$. A(k) is proved now. The argument of the previous paragraph actually shows the validity of $B_0(k)$ also.

For a fixed $i \geq 0$, suppose G has a k-chain $a_0 a_1 \ldots a_{k-1} b_i$ (b_i may lie on $a_1 \ldots a_{k-3}$) of degree type $(\mu_0, \mu_1, \ldots, \mu_{k-1}, \mu_k + i)$. We will show that $N_{k,i}(H) = N_{k,i}(G)$, i.e. the number of k-chains of such degree type is edge-reconstructable. The proof is essentially the same as that of A(k). Let's sketch it briefly. Consider a nonisomorphic edge-reconstruction $H = G - a_{k-2}a_{k-1} + c\,d$, with the degrees of c and d be respectively $\mu_{k-2} - 1$ and $\mu_{k-1} - 1$ in $G - a_{k-2}a_{k-1}$. If a_{k-2} is neither c nor d, then A(k-2) implies that c or d must be some a_j, $0 < j < k - 2$, and so $B_0(j)$ implies that there exists special $b_0 b_1 \ldots b_j$ in H with $b_j = a_{k-1}$ or a_{k-2}. Since

$b_0b_1 \ldots b_{j-1}$ is clearly a genuine special $(j-1)$-chain in G, $P(k-1)$ implies G's edge-reconstructability then. We then may assume without loss of generality that $c = a_{k-2}$. d may be some a_j, $0 < j < k-2$ or not (say the possibility $d = b_i = a_j$ may occur). But if $d = a_j$, then we can easily find a $(k-1, j-1)$-coupling in G and so G is edge-reconstructable. So $d \neq$ any a_j, $0 \leq j \leq k-1$.

$B_i(k)$ now is a straightforward consequence of Lemma 3.1 on edges of some specified degree type. (To be more precise, details are a little bit different according as d is not adjacent to a_{k-1}, or $d \neq a_k$ is adjacent to a_{k-1} or $d = a_k$; but all of them are trivial to verify).

We have now proved $A(k)$ and $B_i(k)$ when $\mu_{k-1} > \mu_0 + 1$, $\mu_{k-2} > \mu_0 + 1$. We are thus done with Case 1 of Proposition 3.12.

Remark. Case 1 is the only case we have to do if our graph G has no vertices of degree $\mu_0 + 1$, i.e. degree one higher than minimum. This suggests why the edge-reconstructability of bi-degreed graphs deserves special treatment as in Chapter 2 (or more generally, any graph with two "lowest" degrees differing by 1).

Case 2 of Proposition 3.12. $\mu_{k-1} = \mu_0 + 1$, $\mu_{k-2} > \mu_0 + 1$.

Consider again $G - a_{k-1}a_{k-2}$ for a special k-chain $a_0a_1 \ldots a_{k-2}$ $a_{k-1}a_k$ in G. Note that $a_0a_1 \ldots a_{k-2}$ has forbidden degree type $(\mu_0, \mu_1, \ldots, \mu_{k-3}, \mu_{k-2} - 1)$ in $G - a_{k-1}a_{k-2}$, and so any edge-reconstruction H is of the form $G - a_{k-1}a_{k-2} + cd$, with degree of c,d in $G - a_{k-1}a_{k-2}$ respectively equal to μ_0 and $\mu_{k-2} - 1 > \mu_0$. It's conceivable that c may be a_0 in this case which presents more difficulty.

Let's assume $c = a_0$ first and consider $H = G - a_{k-1}a_{k-2} + a_0 d \not\cong G$. Let a_0 be adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G (a_1 is such a vertex), then a_{k-1} is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in H (d cannot be a_{k-1} by degree argument). First let d be nonadjacent to a_{k-1}. We see readily that $\mu_1 \leq \mu_k(G)$. $\mu_1 < \mu_k(G)$ is impossible, otherwise if e is a vertex of degree μ_1 adjacent to a_{k-1} in H, the k-walk $a_0 a_1 \ldots a_{k-1} e$ in G says $\mu_k(G) \leq \mu_1 < \mu_k(G)$, and we have $\mu_k(G) = \mu_1$ for that case. If d is adjacent to a_{k-1} and $d \neq a_k$, then same argument as above says that $\mu_k(G) = \mu_1$; and when $d = a_{k-1}$ we can prove by same type of argument that $\mu_k(G) = \mu_1$ or $\mu_1 - 1$.

As a summary, we see that $\mu_k(G) = \mu_1$ or $\mu_1 - 1$ when $c = a_0$ (i.e. when some $H = G - a_{k-1}a_{k-2} + a_0 d$). Suppose $\mu_k(G) = \mu_1 - 1$ now. This can happen only when $d = a_k$. Since $k \leq \Omega(G) - 1$, we have $\mu_1 - 1 > \mu_0$ or $\mu_1 > \mu_0 + 1$. The 3-path $a_{k-1}a_k a_0 a_1$ in H says that $\mu_3 \leq \mu_1$ and $\mu_2 = \mu_0 + 1$. μ_3 cannot be strictly less than $\mu_1 - 1$ otherwise $b_1 b_2$ is a forced edge for a special 3-chain $b_0 b_1 b_2 b_3$ in G. So $\mu_3 = \mu_1$ or $\mu_1 - 1$. Suppose $\mu_3 = \mu_1$ first. We note $\mu_2 = \mu_0 + 1$, which implies $k \geq 5$. If $k = 5$, then the 5-path $a_4 a_5 a_0 a_1 a_2 a_3$ in H says that $\mu_4 = \mu_2 = \mu_0 + 1$. Now consider $a_0 a_1 \ldots a_5$ again in G. Its degree type is $(\mu_0, \mu_1, \mu_0 + 1, \mu_1, \mu_0 + 1, \mu_1 - 1)(\mu_5(G) = \mu_1 - 1)$. Now $a_3 a_4 \to a_5 a_0$ is a forced move sending G to some nonisomorphic edge-reconstruction I in which $a_0 a_1 a_2 a_3 a_4 a_5$ becomes $a_4 a_5 a_0 a_1 a_2 a_3$; for in $G - a_3 a_4$, $a_0 a_1 a_2 a_3$ is forbidden of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1)$ and $a_4 a_5$ is of degree type $(\mu_0, \mu_1 - 1)$. I is then the unique nonisomorphic edge-reconstruction by forced move. Now it's clear that $a_1a_2 \to a_3a_4$ and $a_5a_0 \to a_1a_2$ are forced moves sending I to $G' \cong G$ and G' to $I' \cong I$ by same arguments. Since we obviously return to our original G after these 3 (an odd number) forced moves, G is edge-reconstructable by Lemma 2.1 and we may assume $k \geq 6$ now.

However, k cannot be six otherwise $\mu_{k-2} = \mu_4 = \mu_0 + 1$, (as seen from $a_5a_6a_0a_1a_2a_3a_4$ in H), contradictory to the assumption of Case 2. We have then $k \geq 7$. Suppose $k = 7$, and we will show G's edge-reconstructability in a similar vein. The 7-path $a_6a_7a_0a_1 \ldots a_5$ in H says that $\mu_4 = \mu_2 = \mu_0 + 1, \mu_3 = \mu_1$. We have $\mu_5 = \mu_5(G) = \deg(a_5)$ in G by degree argument and hence we have readily $\mu_6 = \mu_0 + 1$ (and $\mu_7(G) = \mu_1 - 1$). To simplify the notation, let 0 1 2 3 4 5 6 7 represent $a_0a_1a_2a_3a_4a_5a_6a_7$ (we will follow the same practice in the next few paragraphs) and write 0 1 2 3 4 5 6 7 $\in G$ to mean $a_0a_1 \ldots a_7$ is a 7-path which is a configuration in G. We see 3 4 $\to$ 7 0 is a forced move sending G to a nonisomorphic edge-reconstruction I in which 0 1 2 3 4 5 6 7 becomes 4 5 6 7 0 1 2 3 since in G - 3 4, 0 1 2 3 and 4 5 6 7 are both forbidden of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1 - 1)$. Next we see that 1 2 $\to$ 3 4 is a forced move sending I to G' since in G - 1 2, 4 5 6 7 0 1 and 2 3 are both of forbidden degree type. Now the following diagram is self-explanatory to prove G's edge-reconstructability.

0 1 2 3 4 5 6 7 $\in G$
4 5 6 7 0 1 2 3 $\in I$
2 3 4 5 6 7 0 1 $\in G' \cong G$
0 1 2 3 4 5 6 7 $\in I' \cong I \cong G$.

We may then assume $k \geq 8$ (still $\mu_3 = \mu_1$ and $\mu_k(G) = \mu_1 - 1$ assumed). Clearly $k \neq 8$ by assumption of Case 2, and we have $k \geq 9$. For $k = 2m \geq 9$, the move $a_{2m-1}a_{2m-2} \to a_{2m}a_0$ (possible only if $\mu_0 = 1$) gives $\mu_{2m-2} = \mu_{2m-4} = \mu_{2m-6} = \cdots = \mu_4 = \mu_2 = \mu_0 + 1$, (while $\mu_{2m-3} = \mu_{2m-5} = \cdots = \mu_3 = \mu_1$), contradictory to the assumption of Case 2 that $\mu_{k-2} > \mu_0 + 1$. Let k be an odd integer ≥ 9. We can prove G's edge-reconstructability according as $k \equiv 1 \pmod 4$ or $k \equiv 3 \pmod 4$ as the cases $k = 5, 7$ proved above. By the k-path $a_{k-1}a_k a_0 a_1 \cdots a_{k-2}$, we have $\mu_{k-2} = \mu_{k-4} = \cdots \mu_3 = \mu_1$ and $\mu_{k-3} = \mu_{k-5} = \cdots = \mu_4 = \mu_2 = \mu_0 + 1$ (by the fact $\mu_3 = \mu_1$). We have furthermore $\mu_{k-1} = \mu_1$ and $\mu_k(G) = \mu_1 - 1$. (Using inductive assumption as well).

For $k \equiv 1 \pmod 4$, the following diagram is self-explanatory.

0	1	2	3				4S	4S+1	$\in G$
4S	4S+1	0	1	2	3	$\cdots$	4S-2	4S-1	$\in I$
4S-2	4S-1	4S	4S+1	0	1	$\cdots$	4S-4	4S-3	$\in G' \cong G$
	$\cdots$						$\cdots$		
2	3	4	5				0	1	$\in G^{(S-1)} \cong G$
0	1	2	3				4S	4S+1	$\in I^{(S)} \cong I$

and for $k \equiv 3 \pmod 4$, $k \geq 9$, with $k = 4S + 3$, we note first that $(4S-1)\,4S \to (4S+3)0$ is a forced move so that the k-path $0\,1\,2\,3 \ldots 4S\,(4S+1)\,(4S+2)\,(4S+3)$ in G becomes $4S(4S+1)\,(4S+2)\,(4S+3) \ldots (4S-2)\,(4S-1)$ in I. Now clearly 2S forced moves $((4S-3)\,(4S-2) \to (4S-1)\,4S$, $(4S-5)\,(4S-4) \to (4S-3)\,(4S-2), \ldots, (4S+3)\,0 \to 1\,2)$ gives us $I^{(S)} \cong I$

while returning us to the original configuration 0 1 2 3 ⋯ 4S (4S+1) (4S+2) (4S+3) and Lemma 2.1 applies to show G's edge-reconstructability.

We have now proved that G is edge-reconstructable when $\mu_k(G) = \mu_1 - 1$ and $\mu_3 = \mu_1$. We next assume $\mu_3 = \mu_1 - 1$ (and $\mu_k(G) = \mu_1 - 1$ still holds).

Since $k \geq 4$, the 2-path $a_{k-1}a_ka_o$ in $H = G - a_{k-1}a_{k-2} + a_ka_o$ says $\mu_2 = \mu_o + 1$ (it is easy to see that $\mu_{k-2} = \mu_1$, and so $\mu_1 > \mu_o + 1$). If a_o is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G, then a_{k-1} is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in H and hence adjacent to vertices of degree μ_1 in G(a_{k-2} inclusive). If $\mu_{k-3} > \mu_o + 1$, then note every edge-reconstruction is isomorphic to $G - a_{k-3}a_{k-2} + e\,f$; and we must have one of e,f say e equal to a_{k-3} otherwise P(k-1) as in Case 1 proves G's edge-reconstructability. By $B_o(k-1)$, $a_oa_1 \dots a_{k-3}$ must be the "initial segment" of a special (k-1)-chain $a_oa_1 \dots a_{k-3}$ f g. If g is adjacent to $\beta \geq 0$ vertices of degree μ_1 in the new graph, we can easily see $\beta = \alpha$ by argument above and so $\mu_k(H) \leq \mu_1$. $\mu_k(H)$ cannot be μ_1 by the edge-reconstructability of number of paths of degree type $(\mu_1 - 1, \mu_o + 1, \mu_1 - 1)$. $\mu_k(H)$ cannot be less than $\mu_1 - 1$ otherwise f g is a forced edge in the special 3-chain $a_oa_1 \dots a_{k-3}$ f g h in the new graph, so $\mu_k(H) = \mu_1 - 1 = \mu_k(G)$ if $\mu_{k-3} > \mu_o + 1$.

Then clearly $k \geq 5$ since $\mu_1 > \mu_o + 1$. Suppose a_o is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G, while a_{k-3} is adjacent to β such vertices in G. If $\mu_{k-4} \neq \mu_1$, then a_{k-2} is adjacent to β such vertices in the new graph reconstructed from $G - a_{k-3}\,a_{k-4}$ so $\beta = \alpha$ or $\alpha + 1$; and if we consider $G - a_{k-3}\,a_{k-2}$, we see that all replacing edges lead to $\mu_k(I) = \mu_1 - 1 = \mu_k(G)$ when $\mu_{k-4} \neq \mu_1$ unless the replacing edge is a_0a_{k-4}; in the latter case we get a contradiction since the new graph has one less edge of degree type (μ_0, μ_1) except when $\mu_{k-4} = \mu_1 - 1$. But if $\mu_{k-4} = \mu_1 - 1$, then in $H = G - a_{k-2}a_{k-1} + a_ka_0$ $(c = a_0)$, we have one more 2-path of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_1 - 1)$, contradiction ($a_{k-1}$ cannot be adjacent to two vertices of degree $\mu_1 - 1$ otherwise $a_{k-2}a_{k-1}$ is a forced edge).

Now we have $\mu_{k-4} = \mu_{k-2} = \mu_1, \mu_{k-3} = \mu_{k-1} = \mu_0 + 1$ (we can then prove $k \geq 9$, but this result is not needed). Furthermore, we note that if a_0 is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G, then a_{k-1} is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G (including a_{k-2}), and a_{k-1} is adjacent to (exactly) one vertex of degree $\mu_1 - 1$ in G; and a_{k-3} is adjacent to $\alpha + 1$ vertices of degree μ_1 in G (including a_{k-2} and a_{k-4}). But then for $H = G - a_{k-2}a_{k-1} + a_0a_k$, we have one more edge of degree type (μ_0, μ_1), contradiction.

We have proved that $\mu_k(H) = \mu_k(G)$ (or even more G is edge-reconstructable) when $H = G - a_{k-2}a_{k-1} + a_0a_k$ with $\mu_k(G) = \mu_1 - 1$ ($\mu_3 = \mu_1$ or $\mu_1 - 1$). So A(k) is proved for such case.

Now suppose $\mu_k(G) = \mu_1$ and consider $H = G - a_{k-2}a_{k-1} + a_0d$ again ($c = a_0$ at the beginning of Case 2). If a_0 is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G, then a_{k-1} is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G, and we will have that a_{k-1} is adjacent to $\alpha + 1$ vertices of degree μ_1 in G with $\mu_{k-2} = \mu_1$ or $\mu_1 + 1$ by looking at $G - a_{k-1}a_{k-2}$.

Suppose first $\mu_{k-2} = \mu_1$ $(\mu_k(G) = \mu_1)$. Then $\mu_1 > \mu_0 + 1$. Suppose

$\mu_{k-3} > \mu_o + 1$. Then a replacing edge will give G's edge-reconstructability by P(k-1) unless one end of the edge is a_{k-3}. Say the new edge is $a_{k-3}b_o$. $B_o(k-1)$ says that we have a special (k-1)-chain $a_oa_1 \ldots a_{k-3}\ b_ob_1$ in the new graph I, and b_1 cannot be adjacent to $\alpha + 1$ vertices of degree μ_1 in I, and we have $\mu_k(I) \leq \mu_k(G) = \mu_1$. If $\mu_k(I) \leq \mu_1 - 2$, $d_{k-1}d_{k-2}$ is a forced edge for a special k-chain $d_od_1 \ldots d_k$ in I; and the case $\mu_k(I) = \mu_1 - 1$ (similar to the case $\mu_k(G) = \mu_1 - 1$) is treated before, so $\mu_k(I) = \mu_1 = \mu_k(G)$. We can then assume $\mu_{k-3} = \mu_o + 1$. This argument is the same as that we used for $\mu_k(G) = \mu_1 - 1, \mu_3 = \mu_1 - 1$.

As before we can show that $\mu_{k-4} = \mu_1$ or $\mu_{k-4} = \mu_1 - 1$. If $\mu_{k-4} = \mu_1 - 1$, then by considering $G - a_{k-3}a_{k-2}$, we see that a_oa_{k-4} is the only replacing edge which will give some trouble. From $a_{k-3}a_{k-4}a_oa_1$, we see that $\mu_2 = \mu_o+1$ and $\mu_3 \leq \mu_1$. If $\mu_3 = \mu_1$, we see soon that if k is odd, then $\mu_{k-4} = \mu_{k-6} = \cdots \mu_5 = \mu_3 = \mu_1$, a contradiction to the fact that $\mu_{k-4} = \mu_1 - 1$; and if k is even (then $\mu_o = 1$ and $\alpha = 1$), then $\mu_{k-4} = \mu_{k-6} = \cdots = \mu_2 = \mu_o + 1 = \mu_1 - 1$ implying $\mu_1 = 3$. For the latter case, $a_{k-3}a_{k-2}$ can be easily seen to be a "forced edge" (after eliminating all other trivialities). If $\mu_3 = \mu_1 - 1$, then looking at $G - a_{k-3}a_{k-4}$, we see a_{k-3} is the starting vertex of a special 3-chain $a_{k-3}b_ob_1b_2$ in G with $b_o \neq a_{k-4}$, but then $\mu_k(G) \leq \mu_1 - 1$, contradiction to our assumption that $\mu_k(G) = \mu_1$. So we have shown that $\mu_{k-4} = \mu_1$.

Proceed in this way, we can show that we can assume k is odd, $\mu_2 = \mu_4 = \ldots = \mu_{k-1} = \mu_o + 1, \mu_1 = \mu_3 = \cdots = \mu_{k-4} = \mu_{k-2}$, and $a_1 \ldots a_k$ is "symmetric" in the sense of degrees. Now consider $G - a_1a_2$. The replacing edge must have one of its ends be a_o or a_1. Suppose it is a_1 first, and let the new edge be a_1b. b cannot lie on $a_oa_1 \dots a_k$ since the degree of b in G is μ_o. By $B_o(k-2)$, b_o must be the starting vertex of the same number of special (k-2)-chains in G as a_2 is in H, and so it follows (as in the proof of Lemma 3.3) that a_o is the starting vertex of the same number of k-paths in H of degree type $(\mu_o, \mu_1, \mu_o + 1, \mu_1, \mu_o + 1, \dots, \mu_o + 1, \mu_1)$ as it is in G. So we have in particular that $\mu_k(H) \leq \mu_k(G) = \mu_1$ and so $\mu_k(H) = \mu_k(G)$ since the case when $\mu_k(H)$ is smaller than $\mu_k(G)$ is already treated. Next, let one end of replacing edge be a_o. The edge-reconstructability of 2-paths of degree type establishes a contradiction (for a_2 cannot be adjacent to a vertex of degree $\mu_1 - 1 > \mu_o$) unless one end of the replacing edge is some a_j, $2 < j < k$. Now $B_1(j-2)$ gives a special (j-2)-chain $b_ob_1 \dots b_{j-2}$ with $b_{j-2} = a_1$, $b_o \neq a_2$ (b_o may be a_o) and P(k-1) implies G's edge-reconstructability.

Now let $\mu_{k-2} = \mu_1 + 1$ with $\mu_k(G) = \mu_1$ (μ_1 may be $\mu_o + 1$ here). We can easily prove that $\mu_{k-3} = \mu_o + 1$ and a_{k-3} is adjacent to α vertices of degree μ_1 in G excluding a_{k-4} (that is, if $\deg(a_{k-4}) = \mu_1$, then a_{k-3} is adjacent to $\alpha + 1$ such vertices) by considering $G - a_{k-3}a_{k-4}$. But this is impossible since $\alpha \geq 1$ implies that a_{k-3} is adjacent to some vertex $\neq a_{k-4}$ of degree μ_1 and so $\mu_{k-2} \leq \mu_1$, contradiction.

So far we have proved that if $H = G - a_{k-2}a_{k-1} + a_od$, then $\mu_k(H) = \mu_k(G)$ in all cases. Now consider $H = G - a_{k-2}a_{k-1} + cd$ with $c \neq a_o$. The "P(k-1) type" of argument readily says that c must be a_{k-2}. Lemma 3.1 applied soon says that $\mu_k(H) \leq \mu_k(G)$. Hence we have $\mu_k(H) \leq$

$\mu_k(G)$ in all possibilities without using the fact $\Omega(G) \leq \Omega(H)$ (just that $k \leq \Omega(G) - 1$, $k \leq \Omega(H) - 1$). Symmetry argument can then be applied to say $\mu_k(G) \leq \mu_k(H)$ and so $A(k)$ is proved in complete force for Case 2.

Though its form seems more intricated, the proof of $B_i(k)$ isn't too hard after $A(k)$ is proved. Consider $H = G - a_{k-2}a_{k-1} + cd$ with $c \neq a_0$ first. We have to consider $c = a_{k-2}$ only and Lemma 3.1 immediately implies that $B_i(k)$ holds. Next let $c = a_0$. We know that μ_k can only be $\mu_1 - 1$ or μ_1. Suppose $\mu_k = \mu_1 - 1$ first. We know that μ_3 can only be μ_1 or $\mu_1 - 1$. When $\mu_3 = \mu_1$, we have shown G is edge-reconstructable and there is nothing to worry about $B_i(k)$. Now let $\mu_3 = \mu_1 - 1$. We know $\mu_{k-2} = \mu_1 > \mu_0 + 1, \mu_2 = \mu_0 + 1$. If a_0 is adjacent to $\beta \geq 0$ vertices of degree $\mu_k + i$ in G, then a_{k-1} is adjacent to $\beta \geq 0$ vertices of degree $\mu_k + i$ in H and also in G when $i > 0$ (when $i = o$, a_0 is adjacent to no vertex of degree $\mu_1 - 1$ in G while a_{k-1} is adjacent to exactly one such vertex in G). We see that $\mu_{k-3} > \mu_0 + 1$ implies $B_i(k)$ trivially and we can assume $\mu_{k-3} = \mu_0 + 1$. We have proved that μ_{k-4} must be μ_1 then. But then we can prove a contradiction as before since $H = G - a_{k-2}a_{k-1} + a_0a_k$ has one more edge of degree type (μ_0, μ_1).

For the proof of $B_i(k)$ in Case 2 we are left with the cases $c = a_0$ and $\mu_k(G) = \mu_1$. As before we note μ_{k-2} can be μ_1 or $\mu_1 + 1$, and the latter case leads to contradiction easily. When $\mu_{k-3} > \mu_0 + 1$, $B_i(k)$ is proved trivially and we have $\mu_{k-3} = \mu_0 + 1$ and $\mu_{k-4} = \mu_1$ in a way as in the proof of $A(k)$ for this case; we have k is odd and $\mu_2 = \mu_4 = \cdots = \mu_{k-1} = \mu_0 + 1, \mu_1 = \mu_3 = \cdots = \mu_k$. Consider $G - a_1a_2$.

If one end of the replacing edge is a_1, we have $B_i(k)$ in a straightforward manner (say if $i > 0$, then a_o and a_{k-1} are adjacent to the same number of vertices of degree $\mu_k + i$ for any special k-chain $a_o a_1 \ldots a_k$ in G); and if a_o is one end of the replacing edge we get a contradiction as in the proof of A(k) for this case.

So we have proved A(k) and $B_i(k)$ and are done for Case 2.

Case 3 of Proposition 3.12. $\mu_{k-1} = \mu_{k-2} = \mu_o + 1$.

If all $\mu_j = \mu_o + 1$ for $0 < j \leq k - 3$, then A(k) and $B_i(k)$'s are trivial consequences of Lemma 3.4 (with $\delta = \mu_k(G)$ first, we see $\mu_k(H) \leq \mu_k(G)$; then with $\delta = \mu_k(H)$, we see $\mu_k(G) \leq \mu_k(H)$, so $\mu_k(G) = \mu_k(H)$ for all $H \in \Sigma_G$ and can be represented by μ_k; then with $\delta = \mu_k + i$, we can prove $B_i(k)$).

Let now $m \leq k - 3$ be the largest integer such that $\mu_m > \mu_o + 1$. (Then $\mu_{m+1} = \cdots = \mu_{k-2} = \mu_{k-1} = \mu_o + 1$). As in the proof of Case 2, we will prove the validity of A(k) by proving $\mu_k(H) \leq \mu_k(G)$ for any edge-reconstruction H (without utilizing the fact $\Omega(G) \leq \Omega(H)$).

Consider $H \cong G - a_m a_{m+1} + cd$ with degrees of c and d in $G - a_m a_{m+1}$ respectively equal to μ_o and $\mu_m - 1 > \mu_o$. Suppose $c \neq a_o$ first. Then if $d \neq a_m$, we see by A(m) that d = some a_j, $0 < j < m$; so by $B_i(j)$, $\exists$ special j-chain $b_o b_1 \ldots b_j$ in H with $b_o \neq a_{k+1}$ and $b_j = d$ (since $c \neq a_o$) and so $b_o b_1 \ldots b_{j-1}$ and $a_o a_1 \ldots a_k$ in G implies by P(k-1) that G is edge-reconstrucable. If $d = a_m$ (and $c \neq a_o$) we see by Lemma 3.4 that $\mu_k(H) \leq \mu_k(G)$ (later the same lemma is used to prove $B_i(k)$).

Now let $c = a_o$. d may lie on $a_1 \ldots a_k$ or not. First we will prove that d cannot be some a_j, $m+1 < j < k$. Suppose $d = a_j$, and let $\Delta = j - (m+1) > 0$. We have immediately $\mu_1 = \mu_2 = \cdots = \mu_{\Delta-1} = \mu_o + 1$ (condition void if $\Delta = 1$) and $\mu_\Delta \leq \mu_o + 2$. We have furthermore $\mu_m = \mu_o + 2$. We note μ_Δ can be $\mu_o + 2$ or $\mu_o + 1$. Suppose $\mu_\Delta = \mu_o + 1$, then $a_{m+1}a_{m+2} \ldots a_j$ is a new Δ-chain of degree type $(\mu_o, \mu_1, \ldots, \mu_{\Delta-1}, \mu_\Delta + 1)$ in H, and so $B_1(\Delta)$ says that a "genuine" Δ-chain $b_o b_1 \ldots b_\Delta$ of degree type $(\mu_o, \mu_1, \ldots, \mu_{\Delta-1}, \mu_\Delta + 1)$ in G must be destroyed. Clearly $b_o \neq a_{m+1}$ and $b_\Delta = a_m$. $P(k-1)$ now applies. Hence $\mu_\Delta = \mu_o + 2$. Δ cannot be greater than m for $\mu_m = \mu_o + 2$ but $\mu_s = \mu_o + 1$ for $0 < s < \Delta$. Let's consider $\Delta = m$ now. We have $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = \mu_o + 1, \mu_m = \mu_o + 2, \mu_{m+1} = \cdots = \mu_{2m} = \mu_{2m+1} = \mu_o + 1$ with $j = 2m+1 < k$. In $G - a_{2m}a_{2m+1}$, $a_o a_1 \ldots a_{2m}$ is "symmetric", i.e. the degree type is the same whether we start at a_o or a_{2m}. By degree argument any edge-reconstruction H of G will join two vertices of degree μ_o in $G - a_{2m}a_{2m+1}$, and Lemma 3-4 on the edge-reconstructability of $(k-2m-1)$-paths of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1, \mu_k(G))$ readily gives us $\mu_k(H) \leq \mu_k(G)$ (later the same lemma is used to prove $B_i(k)$ when $A(k)$ is proved). So now we may assume $\Delta < m$.

The path $a_{m+1} \ldots a_j a_o \ldots a_\Delta$ in H readily gives that $\mu_{\Delta+1} = \cdots = \mu_{2\Delta} = \mu_o + 1, \mu_{2\Delta+1} \leq \mu_\Delta = \mu_o + 2$. So $\mu_{2\Delta+1} = \mu_o + 2$ or $\mu_o + 1$ (since $2\Delta + 1 \leq j < k < \Omega(G)$, $\mu_{2\Delta+1}$ cannot be μ_o). Suppose $\mu_{2\Delta+1} = \mu_c + 2 = \mu_\Delta$ at first. Let $m = \alpha\Delta + 1 + \beta$ with $0 \leq \beta < \Delta, \alpha \geq 1$.

α cannot be 1 otherwise $\mu_m = \mu_{\Delta+1+\beta} = \mu_o + 1$ since $\Delta + 1 \leq \Delta + 1 + \beta \leq 2\Delta$, a contradiction since $\mu_m = \mu_o + 2$. So $\alpha \geq 2$. Suppose $\alpha = 2$ and let $\beta = 0$. In H, the path $a_{m+1}a_{m+2} \cdots a_j \ a_o a_1 \cdots a_\Delta \ a_{\Delta+1} \cdots a_{2\Delta+1}$ can be thought of as composed of three segments A B C each of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1, \mu_o + 2)$, $(\mu_o + 1, \ldots, \mu_o + 1, \mu_o + 2)$ and $(\mu_o + 1, \ldots, \mu_o + 1)$ respectively. $a_\Delta a_{\Delta+1} \to a_{2\Delta+1}a_{m+1}$ is a forced move sending H to some nonisomorphic I and A B C becomes C A B. Next $a_j a_o \to a_\Delta a_{\Delta+1}$ is a forced move sending I to $H' \cong H$ and C A B to B C A; finally, $a_{2\Delta+1}a_{m+1} \to a_j a_o$ is a forced move sending H' to $I' \cong I$ and B C A to A B C. Since three forced moves return us to A B C, Lemma 2.1 says H is edge-reconstructable, and hence G is edge-reconstructable. The proof uses the same ideas in Case 2 when $c = a_o$, $\mu_k(G) = \mu_1 - 1$ and $\mu_3 = \mu_1$. Now suppose $\alpha = 2$ and $\beta > 0$. If $\mu_{2\Delta+1} = \mu_o + 1$, the above argument (consider $a_{m+1} \cdots a_j a_o a_1 \cdots a_{2\Delta+1}$ in H) says G is edge-reconstructable. So $\mu_{2\Delta+1} = \mu_o + 1$, and we have readily $\mu_{2\Delta+1+\beta} = \mu_{\Delta+\beta} = \mu_o + 1$ by looking at the path $a_{m+1} \cdots a_j a_o a_1 \cdots a_{2\Delta+1}$ in H again; contradiction, since $\mu_o + 2 = \mu_m = \mu_{2\Delta+1+\beta}$.

So we conclude $\alpha \geq 3$. The general proof now uses the concept of forced-move principle as in Case 2 when $c = a_o$, $\mu_k(G) = \mu_1 - 1$, $\mu_3 = \mu_1$ and also the argument of the previous paragraph. It is quite straightforward and hence is omitted.

Next let's assume $\mu_{2\Delta+1} = \mu_o + 1$ $(= \mu_\Delta - 1)$. In this case, proof proceeds in a way similar to the case for $\mu_3 = \mu_1 - 1$, $\mu_k(G) = \mu_1 - 1$

of Case 1 before (just as the case $\mu_{2\Delta+1} = \mu_o + 2$ is similar to the case $\mu_3 = \mu_1, \mu_k(G) = \mu_1 - 1$).

We now have proved that for $H = G - a_m a_{m+1} + a_o d$, d cannot lie on $a_{m+2} \cdots a_{k-1}$.

We then consider the degree of $\mu_k(G)$. It can be equal to or greater than $\mu_o + 1$.

Subcase 3.(a) of Proposition 3.12. $\mu_k(G) = \mu_o + 1$. (Proof of A(k)).

With $c = a_o$ for $H = G - a_m a_{m+1} + cd$, d can be a_k or not. First suppose $d = a_k$, then $\mu_m = \mu_o + 2$ and for $\chi = k - (m+1) = k - m - 1 \geq 2$, we see $\mu_1 = \mu_2 = \cdots = \mu_{\chi-1} = \mu_o + 1$. Now μ_χ can be $\mu_o + 2$ or $\mu_o + 1$ and it must be $\mu_o + 2$ otherwise $B_1(\chi)$ implies that P(k-1) is applicable as before. Discussing as in the proof that d cannot lie on $a_{m+2} \cdots a_{k-1}$ (two separate cases $\mu_{2\chi+1} = \mu_o + 2$ or $\mu_o + 1$), we get $\mu_k(H) \leq \mu_k(G)$, and so $\mu_k(H) = \mu_o + 1 = \mu_k(G)$ since $k \leq \Omega(H) - 1$.

So we may assume $d \neq a_k$. Let s be the largest integer such that $\mu_s = \mu_o + 1$ and $m > s \geq \chi$ (existence of s guaranteed by above arguments). In $H = G - a_m a_{m+1} + a_o d, B_o(s)$ says that a_{m+1} is the starting vertex of a special s-chain $a_{m+1} b_1 \ldots b_s$ (conceivably b_1 may coincide with a_{m+2}, say). Suppose that $b_1 \neq a_{m+2}$. If a_{m+3} isn't any b's, then from $G - a_{m+2} a_{m+3}$, our new edge-reconstruction will have $\mu_1 = \cdots = \mu_{s+1} = \mu_o + 1$, contradiction to the maximality of s unless a_{m+2} is an end of the replacing edge. The latter case immediately leads to $\mu_k(H) = \mu_o + 1$ except when $a_{m+2} a_o$ is a replacing edge. Then a_{m+3} is the starting vertex of a special s-chain in the new graph and it is easy to get a contradiction from the maximality of s. So a_{m+3} is some $b_j \in b_1 \cdots b_s$. We note that in H, a_{m+1} is the starting vertex of $\alpha \geq 1$ special s-chains if a_o starts $\alpha \geq 1$ such chains in G. Hence d cannot be some b_u (or we can argue as the proof that d cannot $\in a_{m+1} \cdots a_{k-1}$). But then in H $a_{m+1}a_{m+2}a_{m+3}\, b_{j-1}b_{j2} \cdots b_1\, a_{m+1}$ is a chain of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1, \mu_o)$ with length $2 + j \leq 2 + s < 2 + m < \Omega(G)$, a contradiction.

We have done the case $b_1 \neq a_{m+2}$. But it is easy to see why we can assume so, for considering $G - a_{m+1}a_{m+2}$, then $B_o(s)$ would imply a_{m+1} or a_{m+2} is the starting vertex of a special s-chain. The former leads to $b_1 \neq a_{m+2}$, the latter leads to a contradiction by maximality of s.

We have thus done the proof of A(k) for subcase 3(a).

Subcase 3(b) of Proposition 3.12. $\mu_k(G) > \mu_o + 1$ (proof of a(k)).

From $H = G - a_m a_{m+1} + a_o d$, it is immediate that for $\chi = k - m - 1 \geq 2$, we have $\mu_1 = \mu_2 = \cdots \mu_{\chi-1} = \mu_o + 1$ and $\mu_\chi > \mu_o + 1$ (d may be a_k or not here). Note that if a_o is the starting vertex of $\alpha \geq 1$ special $(\chi-1)$-chains in G, then $B_o(\chi-1)$ says that a_{m+1} is the starting vertex of α special $(\chi-1)$-chains in H. Consider $G - a_{m+1} a_{m+2}$. $B_o(\chi-1)$ says that a_{m+2} must be the starting vertex of one special $(\chi-1)$-chain in the new graph (since $\mu_m > \mu_o + 1$), which will imply $\mu_k(G) = \mu_o + 1$, a contradiction.

We have thus done the proof of A(k) for Case 3. We now go through a quick proof of $B_i(k)$ for Case 3. By arguments before, we have to consider only subcase 3(a), i.e. when $\mu_k = \mu_o + 1$. Also we need only consider $H = G - a_m a_{m+1} + a_o d$ with $d \notin a_{m+1} \cdots a_{k-1}$ (looking at the proof that $d \notin a_{m+1} \cdots a_{k-1}$, we can see either $B_i(k)$ hold, or even more, G is edge-reconstructable). Going through the proof again, we see every possibility leads to contradiction or edge-reconstructability of G except when a_{m+2} is an end of the replacing edge and a_o isn't. But for that case $B_i(k)$ is an easy consequence of Lemma 3.3 on $(k-m-2)$-paths of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1, \mu_k + i)$, and so we are done for Case 3.

Case 4 of Proposition 3.12. $\mu_{k-1} > \mu_o + 1, \mu_{k-2} = \mu_o + 1$

If $\mu_j = \mu_o + 1$ for $1 \leq j \leq k-2$, then Lemma 3.5 applied to k-paths of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1, \mu_{k-1}, \mu_k(G))$ says $\mu_k(H) \leq \mu_k(G)$ for any $H \in \Sigma_G$ and then $\mu_k(G) \leq \mu_k(H)$ when ρ in Lemma 3.5 is taken to be $\mu_k(H)$. So $A(k)$ is true for this case. Take $\rho = \mu_k + i$ (and $\delta = \mu_{k-1}$ as before), we see $B_i(k)$ holds as well.

So we can find the largest m, $0 < m < k-2$, such that $\mu_m > \mu_o + 1$ (and $\mu_{m+1} = \ldots = \mu_{k-2} = \mu_o + 1$). Suppose at first that $m = k-3$.

Consider $G - a_{k-3}a_{k-2} + cd$, with degree of c and d respectively equal to μ_o and $\mu_{k-3} - 1 > \mu_o$ in $G - a_{k-3}a_{k-2}$. If $c \neq a_o$, then d must be a_{k-3} otherwise "P(k-1)-type" of argument as the three cases before says G is edge-reconstructable; and $\mu_k(H) \leq \mu_k(G)$ follows from Lemma 3.5 on 2-paths of degree type $(\mu_o, \mu_{k-1}, \mu_k(G))$. Later when we prove that $\mu_k(H) \leq \mu_k(G)$ for all cases and write μ_k for their common value (by symmetry arguments), the same lemma can then be applied to prove $B_i(k)$.

So $c = a_o$, and we see as in Case 2 that $\mu_{k-1} = \mu_1$ or $\mu_1 - 1$;

the latter can happen only when $d = a_{k-1}$. If $\mu_{k-1} = \mu_1 - 1$, then μ_3 may be μ_1 or $\mu_1 - 1$ and we may argue as in Case 2 that $\mu_k(H) \leq \mu_k(G)$ *is true or* even stronger, G is edge-reconstructable. (The argument is essentially the same except some delicate differences in applying different lemmas and also note the number of special 2-chains starting at a_{k-2} in H is the same as the number of special 2-chains starting at a_0 in G). The proof when $\mu_{k-1}=\mu_1$ will follow the same way as in Case 2.

Now let's assume $m < k - 3$. Then the argument will be of the same type as in Case 3 (we have $\mu_{k-2} = \mu_{k-3} = \mu_0 + 1$ say). We can prove for $H = G - a_m a_{m+1} + a_o d$, d cannot lie on $a_{m+1} \cdots a_{k-2}$ as in Case 3. Furthermore we can prove a contradiction as in Subcase 3(b) (since $\mu_{k-1} > \mu_0 + 1$).

So by discussing separately $m = k - 3$ and $m < k - 3$ and then utilizing the same type of proofs as in Case 2 and Case 3, we see that $A(k)$ and $B_i(k)$ of Case 4 can be proved in an "easy" way, completing our proof of Case 4 and hence that of

Proposition 3.12. When $\Omega(G) \geq 5$, then for any k, $4 \leq k \leq \Omega(G) - 1$, $A(k)$ and $B_i(k)$ are true for any $i \geq 0$ provided (1) $A(j)$ and $B_i(j)$ are true for any $1 < j < k$ and (2) $P(k-1)$ is true.

Section 6. Inductive proof of P(k).

In this section we will prove Proposition 3.13 which is inductive proof of P(k). Note in Section 4, we have proved the validity of P(0), P(1), and P(2). The proof of P(2) is extremely hard. Recall in Section 5, we have proved the validity of A's and B_i's based on the inductive assumption of validity of A's, B_i's and P's of lower *rank*.

We now will assume $\Omega(G) \geq 5$ and for a fixed k, $3 \leq k \leq \Omega(G) - 2$, we suppose A(j), $B_i(j)$ and P(k-1) are all true for $1 \leq j \leq k$, $i \geq 0$.

Recall that an (α,β)-coupling is a configuration of a special α-chain $a_0a_1 \ldots a_\alpha$ and a special β-chain $b_0b_1 \ldots b_\beta$ with $a_\alpha b_\beta \in E(G)$ and the degenerate case $a_0a_1 \ldots a_\alpha = b_0b_1 \ldots b_{\beta-1}$ (with $a_i = b_i$, $\alpha = \beta - 1$) or $b_0b_1 \ldots b_\beta = a_0a_1 \ldots a_{\alpha-1}$ is not counted. Recall that P(n) says an (α,β)-coupling with $0 \leq \alpha, \beta \leq n$ is an excludable configuration. Note that P(n) implies P(m) by definition when $n \geq m$.

As our first reduction in proving P(k), we see that we can assume $\alpha \geq \beta$ without loss of generality. Furthermore, α must be k otherwise P(k-1) applies (since $\beta \leq \alpha \leq k - 1$ in that case). We will prove this inductively for β from 0 to k.

But β clearly cannot be zero, otherwise $\Omega(G) \leq k + 1 \leq \Omega(G) - 1$. So the induction is vacuously true at the start, and we may assume $1 \leq \beta \leq k$. We classify (k,β)-couplings according to the degrees of μ_k and μ_β. It is called a (k,β)-coupling of the first kind if $\mu_k > \mu_0 + 1$, $\mu_\beta > \mu_0 + 1$; the second kind if one of μ_k, μ_β is $\mu_0 + 1$ and the other is greater than $\mu_0 + 1$; the third kind if $\mu_k = \mu_\beta = \mu_0 + 1$. We note we have to consider only the first kind when G has no vertices of degree $\mu_o + 1$, (again bi-degreed graphs call attention).

Note furthermore that the cases $a_o \neq b_o$ and $a_o = b_o$ differ in general. When $a_o \neq b_o$, we can assume $a_o a_1 \ldots a_k$ and $b_o b_1 \ldots b_\beta$ are disjoint everywhere (otherwise we may either apply P(k-1) directly or we have a (k,β')-coupling with $\beta' < \beta$ and induction applies). When $a_o = b_o$, then we assume $\gamma > 0$ is the smallest integer such that $a_\gamma \neq b_\gamma$ (then $\gamma \leq \beta$); and $a_\gamma a_{\gamma+1} \ldots a_k$ and $b_\gamma \ldots b_\beta$ must be disjoint everywhere (by same type of argument). The former is less intricated and is usually easier to do; the latter is often harder, but not intractable because it has more "structures" in it (say some cycles).

Case 1 of Proposition 3.13. $\mu_k = \mu_o + 1$, $\mu_\beta = \mu_o + 1$.

So our (k,β)-coupling is of third kind.

Subcase 1(a) of Proposition 3.13. $a_o \neq b_o$

First we note that β can be assumed to be k-1 or k, for if $\beta \leq k - 2$, then $a_o a_1 \ldots a_{k-1}$ and $b_o b_1 \ldots b_\beta a_k$ form a $(k-1, \beta+1)$-coupling which is excludable by P(k-1). Next we observe that G is edge-reconstructable if $a_o a_1 \ldots a_k$ is of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1)$ by Corollary 3.3.2 (for $b_o b_1 \ldots b_\beta$ is of the same type since $\beta \leq k$). Let $k' < k$ be the largest integer such that $\mu_{k'} > \mu_o + 1$. Note $k' < k - 1$ if $\beta = k - 1$, and so k' is also the largest integer $< \beta$ such that $\mu_{k'} > \mu_o + 1$ (when $\beta = k - 1$ or k).

Consider $G - a_{k'}a_{k'+1}$. Any edge-reconstruction is obtained by replacing by an edge ef of degree type $(\mu_o, \mu_{k'} - 1)$ in $G - a_{k'}a_{k'+1}$. Suppose $e = a_o$ first. f must be some vertex in this coupling otherwise $a_{k'+1} \dots a_k$ and $b_o \dots b_\beta$ form a $(k-k'-1,\beta)$-coupling and we are done (if $\beta = k-1$, P(k-1) applies; if $\beta = k$, then $k-k'-1 < \beta$ and induction applies). If f = some b_j, $B_o(j)$ implies existence of special $c_o c_1 \dots c_j$ in the new graph H with $j > 0$, $c_j = a_{k'}$ and we have a $(k',j-1)$-coupling. If f = some a_s, $k'+1 < s \leq k$, then $\mu_{s-k'-1} = \mu_o + 1$ implies a $(k', s-k'-1)$-coupling; and $\mu_{s-k'-1} = \mu_o + 2$ implies a $(k-k'-1,\beta)$-coupling, and we are done when $e = a_o$.

Hence we see that e must be $a_{k'+1}$ or b_o. When $e = a_{k'+1}$, A(k') implies $f \in a_1 \dots a_{k'-1}$ and a "P(k-1)-type" argument works. So $e = b_o$. Again $f \in a_1 \dots a_{k'}$ and "P(k-1)-type" argument says $f = a_{k'}$ is the only possibility. Let $\Delta = k + \beta - 2k' - 1$, we have $\mu_1 = \dots = \mu_\Delta = \mu_o + 1$. We now consider $G - a_{k'+1}a_{k'+2}$ ($a_{k'+2}$ means b_β when $k = k'+1$). There are five ways to replace by a new edge, namely $a_o a_{k'+1}$, $b_o a_{k'+2}$, $a_o a_{k'+2}$, $b_o a_{k'+1}$ and $a_o b_o$. The first two lead to contradiction quickly and the last three imply that $a_{k'+1}$ or $a_{k'+2}$ in the new graph I is the starting vertex of a special Δ-chain by $B_o(\Delta)$. Consider now $H = G - a_{k'}a_{k'+1} + a_{k'}b_o$ or $J = G - b_{k'}b_{k'+1} + b_{k'}a_o$ we can see easily that $\mu_{\Delta+1} = \mu_o+1$. Consider $G - a_{k'+1}a_{k'+2}$ again, we can prove as before that $\mu_{\Delta+2} = \mu_o + 1$ (by $B_o(\Delta+1)$). Proceed in this way, we will get a contradiction finally (say after $k' - \Delta$ steps we prove $\mu_{k'} = \mu_o + 1$), finishing our proof of subcase 1(a).

Subcase 1(b) of Proposition 3.13. $a_o = b_o$.

Let $\gamma > 0$ be the first integer such that $a_\gamma \neq b_\gamma$. As in Subcase 1(a), we note β can be assumed to be $k-1$ or k. But β cannot be k since $a_o = b_o$ and G is bipartite; so β is $k-1$. Let

$k' < k - 1$ be the largest integer such that $\mu_{k'} > \mu_o + 1$. If $k' < \gamma - 1$, then in $G - a_{k'}a_{k'+1}$, $a_{k'+1} \ldots a_k$ and $b_{k'+1} \ldots b_{k-1}$ is a forbidden $(k-\gamma-1, k-\gamma-2)$-coupling, and $a_o a_1 \ldots a_{k'}$ is of forbidden degree type; and so any edge-reconstruction must have a_o or $a_{k'+1} = b_{k'+1}$ as an end of the replacing edge. If it is $a_{k'+1}$, then the other end is some a_j, $0 < j < k' + 1$ by $A(k')$, and a $P(k-1)$-type argument works (i.e. we have a $(k', j-1)$-coupling then). If it is a_o, then the other end is say, some a_j (or b_j), $k' + 1 < j \leq k$. If $\gamma < j < k$, we read from $b_{k'+1} \ldots b_\gamma b_{\gamma+1} \ldots b_j \ldots b_k$ that $\mu_{j-k'+1} = \mu_o + 1$ and so $B_1(j-k'-1)$ says that $a_{k'}$ is the $(j-k'-1)$-st vertex in a special chain in the new graph, so we have a $(k', j-k'-2)$-coupling and G is edge-reconstructable. When $j = k$ and $\mu_{k-k'-1} = \mu_o + 2$, then we have a $(k-k'-1, k-k'-1)$-coupling; while if $\mu_{k-k'-1} = \mu_o + 1$, then we have a $(k', k-k'-2)$-coupling, and G is edge-reconstructable in both cases. The treatment when $k' + 1 < j \leq \gamma$ is similar.

Hence we have $k' \geq \gamma$. We will show that for $H = G - a_{k'}a_{k'+1} + a_o d$, d cannot lie on $a_{k'+1} \ldots a_k$. Let $d = a_i$, $k' + 1 \leq 1 \leq i \leq k$. Then $\mu_1 = \mu_2 = \cdots = \mu_{i-k'-2} = \mu_o + 1$. $\mu_{i-k'-1}$ can be $\mu_o + 1$ or $\mu_o + 2$. If it is $\mu_o + 1$, we will have a $(k', i - k' - 2)$-coupling at d and we are done. So let $\mu_{i=k'-1} = \mu_o + 2$. Note $k' \leq k - 2$. Now delete $a_k a_{k-1}$ from G. Since $k \leq \Omega(G) - 1$, a replacing edge must be $a_k a_o$ or $a_{k-1} a_o$ and hence we have $\mu_{i-k'-1} = \mu_o + 1$ when $i \neq k$. To show that $d \neq a_k$ note that $a_{k'+1}$ is adjacent to a vertex of degree $\mu_o + 1 \neq a_{k'+2}$ and so from $G - a_k a_{k-1}$ we get $\mu_{k=k'-1} = \mu_o + 1$ as well. (The case $\mu_1 = \mu_o + 2$ can be done simply). The fact $a_o = b_o$ is used heavily.

Now if we follow the proof of Subcase 3(a) of Proposition 3.12, we see that we are left with the cases that G is edge-reconstructable. Hence we are done for Case 1.

Case 1' of Proposition 3.13. $\mu_k = \mu_{k-1} = \mu_o + 1, \mu_\beta > \mu_o + 1$; or $\mu_\beta = \mu_{\beta-1} = \mu_o + 1, \mu_k > \mu_o + 1$.

For simplicity of illustration, we will assume $a_o \neq b_o$ (the case $a_o = b_o$ is similar to the corresponding case in Case 1).

First suppose $\mu_k = \mu_{k-1} = \mu_o + 1, \mu_\beta > \mu_o + 1$. If $\beta \leq k - 2$, then our (k,β)-coupling of $a_o a_1 \dots a_k$ and $b_o b_1 \dots b_\beta$ can be interpreted as a $(k-1, \beta+1)$-coupling $a_o a_1 \dots a_{k-1}$ and $b_o b_1 \dots b_\beta a_k$ and so $P(k-1)$ applies. Hence $\beta = k - 1$ or k, which is impossible by degree argument.

Next suppose $\mu_k > \mu_o + 1, \mu_\beta = \mu_{\beta-1} = \mu_o + 1$. As before we can assume $\beta \geq k - 1$. $\beta \neq k$ obviously and so $\beta = k - 1$. If $a_o a_1 \dots a_{k-1}$ is of degree type $(\mu_o, \mu_o + 1, \dots, \mu_o + 1)$ with $\mu_o > 1$, it is easy to see $a_{k-1}a_k$ or $a_{k-2}a_{k-1}$ is a forced edge depending on k is even or odd. The case $\mu_o = 1$ is trivial. So we may assume $k' < k$ be the largest integer such that $\mu_{k'} > \mu_o + 1$. Now the same type of argument as in Subcase 1(a) works and we are done.

Case 1' eliminates some coupling of second kind which "resembles" coupling of third kind.

Case 2 of Proposition 3.13. $\mu_k = \mu_o + 1, \mu_{k-1}, \mu_\beta > \mu_o + 1$; or $\mu_\beta = \mu_o + 1, \mu_k, \mu_{\beta-1} > \mu_o + 1$.

Suppose first $\mu_k = \mu_o + 1, \mu_{k-1}, \mu_\beta > \mu_o + 1$. We may assume

$\beta \geq k - 1$ and β cannot be k by degree argument.

Suppose furthermore $a_o \neq b_o$. Consider $G - a_{k-1}a_k$. $A(k)$ implies that a replacing edge ef should have $e = b_o$ or a_k or a_o and $f \in a_1 \dots a_{k-1}$ or $b_1 \dots b_{k-1}$. By $B_i(j)$ for all j, $1 \leq j \leq k - 1$ and $P(k-1)$, it is easily seen that $a_{k-1}a_k \to a_{k-1}b_o$ is a forced move (i.e. $a_{k-1}b_o$ is the only possible replacing edge). From $a_oa_1 \dots a_{k-1}b_ob_1$, it soon follows that $\mu_{k+1}(H) < \mu_1$ for the new graph H. But $\mu_{k+1}(H)$ cannot be smaller than $\mu_1 - 1$ otherwise c_kc_{k+1} is a forced edge in a special $(k+1)$-chain $c_oc_1 \dots c_{k+1}$ in H. Hence $\mu_{k+1}(H) = \mu_1$ or $\mu_1 - 1$.

Note that $k \geq 3$ now. Call our special k-chain "symmetric" (with respect to degree type) if $\mu_k = \mu_o + 1, \mu_{k-1} = \mu_1, \mu_{k-2} = \mu_2, \dots, \mu_{k-i} = \mu_i, \dots$ etc.; for $1 \leq i \leq k - 1$.

Assume the special k-chain is "non-symmetric" at first. Note that it is impossible that there exists a third special k-chain $c_oc_1 \dots c_k$ such that $c_k = a_k$; for if this is the case, then $c_o \neq a_o$ or b_o say $c_o \neq b_o$, and the forced move $a_{k-1}a_k \to a_{k-1}b_o$ gives in H a k-path $c_oc_1 \dots c_k$ of degree type $(\mu_o, \mu_1, \dots, \mu_{k-1}, \mu_o)$ which in turn implies $\Omega(H) \leq k < \Omega(G)$, a contradiction. Note further that we cannot have a k-path $d_od_1 \dots d_k$ with $d_o = a_k$, $\deg(d_k) = \mu_o$ and $\deg(d_i) = \mu_i, 0 < i < k$ (it has degree type $(\mu_o + 1, \mu_1, \dots, \mu_{k-1}, \mu_o)$), for then in $H' = G - b_{k-1}a_k + b_{k-1}a_o$, we have $\Omega(H') \leq k < \Omega(G)$.

Call the configuration $a_oa_1 \dots a_{k-1}b_ob_1 \dots b_{k-1}a_k$ a $(k,k-1)$-*train* (in H). Clearly a $(k,k-1)$-coupling and a $(k,k-1)$-train is interchangeable by a forced move. Let b_o be a vertex of degree μ_o on the maximum number M of $(k,k-1)$-couplings in G and H, say G. Then b_0 cannot be the starting vertex of a $(k,k-1)$-train otherwise a forced move gives that b_0 lies on $M+1$ $(k,k-1)$-couplings in H. The forced move $a_{k-1}a_k \to a_{k-1}b_0$ now creates one more $(k, k-1)$-train in H without destroying any one. We are done if the number of $(k,k-1)$-trains is edge-reconstructable. This folklore result however is not too trivial. By $B_0(k)$, a_k lies on a k-path $a_k c_1 \ldots c_k$ of degree type $(\mu_0+1, \mu_1, \mu_2, \ldots, \mu_0+1)$ in G. $c_1 \ldots c_k$ is disjoint from the configuration $a_0a_1 \ldots a_kb_{k-1} \ldots b_0$ otherwise a forced move $a_{k-1}a_k \to a_{k-1}b_0$ or $b_{k-1}a_k \to b_{k-1}a_0$ gives contradiction. Consider $G - a_kc_1$. Clearly a_k is a forced vertex. We see $H' = G - a_k c_1 + a_kd$ (d may lie on $c_2 \ldots c_k$) must have the same number of $(k,k-1)$-trains as G has since no $(k,k-1)$-train is affected (otherwise we easily get a contradiction by looking at $H = G - a_{k-1}a_k + a_kb_0$ or $H'' = G - b_{k-1}a_k + b_{k-1}a_0$).

We now can assume our special k-chain is "symmetric". Consider b_{k-2}. If $\mu_{k-2} > \mu_0 + 1$, then from $G - b_{k-1}b_{k-2}$, we easily see that if $\mu_{k+1}(H) = \mu_1$, then any replacing edge entails applicability of P(k-1) except when the replacing edge is $b_{k-2}a_k$, which happens only when $\mu_0 = 1$ and $\mu_1 = \mu_0 + 2 = 3$. But then we have a (k-1, k-1)-coupling and P(k-1) is ready again. So we have $\mu_{k-2} = \mu_0 + 1$. We can again consider if $b_0b_1 \ldots b_{k-2}$ is "symmetric", i.e. we ask if $\mu_{k-2} = \mu_0 + 1$, $\mu_{k-3} = \mu_1, \mu_{k-4} = \mu_2, \ldots, \mu_{k-i} = \mu_i, \ldots$ etc. for $1 \leq i \leq k-2$ are true or not. It's not hard to show that (after all trivial possibilities are eliminated by P(k-1)) $b_{k-1}b_{k-2} \to b_{k-1}b_0$ is a forced move. Define $(k+2, k-3)$-train in a similar way. It is not too hard to show the edge-reconstructability of number of $(k+2, k-3)$-trains and hence the edge-reconstructability of G (the details are more intricated since the two "sides" of the coupling are not of the same length now).

Hence $b_0 b_1 \ldots b_{k-2}$ is "symmetric", i.e. $\mu_j = \mu_{k-j}$, $1 \leq j \leq k-3$. $\mu_{k-2} = \mu_0 + 1$. Combined with the "symmetry" of $b_0 b_1 \ldots b_k$, i.e. $\mu_i = \mu_{k-i}$, $1 \leq i \leq k-1$; we conclude at once that $\mu_1 = \mu_3 = \mu_5 = \ldots, \mu_2 = \mu_4 = \mu_6 = \ldots = \mu_0 + 1$, and k is even (since $\mu_1 > \mu_0 + 1, \mu_k = \mu_0 + 1$). Now it is clear that $b_0 b_1$ is a forced edge since $a_0 a_1 \ldots a_k b_{k-1} \ldots b_1$ of degree type $(\mu_0, \mu_1, \mu_0 + 1, \mu_1, \ldots, \mu_0 + 1, \mu_1, \mu_0 + 1, \mu_1 - 1)$ can be proved to be excludable easily as we did in Case 2 of Proposition 3.12. (with $\mu_k(G) = \mu_1 - 1, \mu_3 = \mu_1$).

Next we consider $\mu_{k+1}(H) = \mu_1 - 1$. As in the previous paragraphs, we can prove $a_0 a_1 \ldots a_k$ is "symmetric", i.e. $\mu_i = \mu_{k-i}$, $1 \leq i \leq k-1$. Consider $G - c_{k-1} c_k$ for a special $(k+1)$-chain $c_0 c_1 \ldots c_{k-1} c_{k+1}$ in G. $c_{k-1} c_k$ is a forced edge unless $c_0 c_{k+1}$ is a replacing edge, in which case, $\mu_{k-1} = \mu_1, \mu_2 = \mu_0 + 1$. So $\mu_{k-2} = \mu_0 + 1$. We can assume $\mu_3 = \mu_1 - 1$ otherwise we are done as in Case 2 of Proposition 3.12. We have furthermore $k \geq 5$. But now $c_{k-1} c_k \to c_0 c_{k+1}$ creates a 2-path $c_{k-3} c_{k-2} c_{k-1}$ of degree type $(\mu_1 - 1, \mu_0 + 1, \mu_1 - 1)$ (by "symmetry", $\mu_{k-3} = \mu_3 = \mu_1 - 1$) while destroying none of the same type, so we get a contradiction.

We have now done the proof of Case 2 for $\mu_k = \mu_0 + 1, \mu_{k-1} > \mu_0 + 1$ and $a_0 \neq b_0$. Let's outline below the ideas when $a_0 = b_0$. Let $\gamma > 0$ be the first integer such that $a_\gamma \neq b_\gamma$. (Note β can only be $k-1$ here). If we delete $a_k b_{k-1}$, difficulty will arise only when a_0 is one end of the replacing edge (otherwise $P(k-1)$ is directly applicable.) Let $H = G - a_k b_{k-1} + a_o d$. If d isn't on the configuration $a_o a_1 \ldots a_k b_{k-1} \ldots b_\gamma$ of the $(k, k-1)$-coupling, we can show a contradiction to $A(k-1)$ (by proving edge-reconstructability of $(k-1)$-paths of degree type $(\mu_o + 1, \mu_1, \mu_2, \ldots, \mu_{k-2}, \mu_{k-1} - 1)$. So d = some a_j or b_j, say a_j. And we can conclude that in H, a_k starts a special j-chain $c_o c_1 \ldots c_j$ with $c_o = a_k$, $c_j = b_{k-1}$. Similarly, we see that in $I = G - a_k b_{-1} + a_o e$, a_k starts a special m-chain $d_o d \ldots d_m$ with $d_o = a_k$, $d_m = a_{k-1}$. If $c_1 \neq a_{k-1}$, we see that in I we have a special m-chain, $m < k$ with $d_m d_o \in E(I)$ and so $\Omega(I) \leq m + 1 < k + 1 = \Omega(G)$, a contradiction. Similar contradiction holds when $d_1 \neq b_{k-1}$. It can be proved that the case $d_1 = b_{k-1}$ and $c_1 = a_{k-1}$ (they have more structure to be considered and hence also more structure to be used) leads to contradiction as well.

Now let's go to the case $\mu_k > \mu_o + 1, \mu_\beta = \mu_o + 1, \mu_{\beta-1} > \mu_o + 1$. Then $\beta < k$. Similar type of argument applies with minor modification and hence proof is omitted.

Case 3 of Proposition 3.13. $\mu_k > \mu_o + 1, \mu_\beta > \mu_o + 1$.

This is a coupling of the first kind. We may have $a_o \neq b_o$ or $a_o = b_o$. When $a_o = b_o$, then let $\gamma > 0$ be the smallest integer such that $a_\gamma \neq b_\gamma$.

First we observe that β can be assumed to be less than k. For if $a_o \neq b_o$ and we consider $G - a_k b_k$; then $A(k)$ says that a new edge ef must have $e \in a_1 \ldots a_k$ and $f \in b_1 \ldots b_k$. When $ef \neq a_k b_k$, then $B_o(j)$ for some j, $0 < j < k$, says that there is a special j-chain $c_0c_1 \ldots c_j$ in the new graph with $c_j = a_k$ or b_k. We then have a $(k,j-1)$-coupling when we have a (k,k)-coupling. So we can assume $\beta < k$ when $a_o \neq b_o$. If $a_o = b_o$, then by "bipartiteness" of G, a_k and b_k must be on the same part of G and $a_kb_k \in E(G)$ is impossible (actually for a general graph we can show that $\beta < k$ by arguments similar to the case $a_o \neq b_o$ above).

Consider $G - a_kb_\beta$ now. A(k) says that a new edge ef must have $e \in a_1 \ldots a_k$ and $f \in b_1 \ldots b_\beta$, and we can find by conditions B_o's of lower rank than k a special $(\delta+1)$-chain $c_oc_o \ldots c_{\delta+1}$ in the new graph H with $c_{\delta+1} = a_k$ or b_β, $0 \leq \delta \leq k-1$. Hence in G we will have a (k,δ)-coupling or (δ,β)-coupling. The latter possibility cannot happen, for $\delta, \beta \leq k-1$ and P(k-1) applies to show G's edge-reconstructability. The former will happen when $c_{\delta+1} = a_k$ and $\delta \geq \beta$ (if $\delta < \beta$, then induction on β says our (k,δ)-coupling is excludable). Also note all three special chains are distinct ($c_oc_1 \ldots c_\delta$ isn't $b_ob_1 \ldots b_\beta$ since $b_\beta a_k \notin E(G)$, but $c_\delta a_k \in E(G)$; $c_oc_1 \ldots c_\delta$ isn't $a_oa_1 \ldots a_{k-1}$ since e must be some a_j, $j < k$, and in H $a_oa_1 \ldots a_j$ isn't a special j-chain). It's conceivable that they may intersect, say $a_o = b_o = c_o$ may happen.

We now note that $c_\delta \notin a_o \ldots a_k$ and $c_\delta \notin b_ob_1 \ldots b_\beta$ since otherwise P(k-1) is applicable. Consider $H = G - c_\delta a_k + ef$, $e \in a_1a_2 \ldots a_k$, $f \in c_oc_1 \ldots c_\delta$. Note we can assume $\mu_\delta > \mu_o + 1$ by results of Case 1' and Case 2 before. Closer investigation on the derivation of the special $(\delta+1)$-chain $c_oc_1 \ldots c_{\delta+1}$ shows that, with the aid of $B_o(\delta+1)$ and $B_o(k)$, we can assume the existence of a special k-chain $c_oc_1 \ldots c_k$ in $H = G - b_\beta a_k + b_\beta c_{\delta+1}$ with $c_{\delta+1} = a_k$ and

$c_k = a_{\delta+1}$. (c_o may be a_o or not). To get $I = G - c_\delta a_k + ef$, all possibilities are easily seen to lead to $P(k-1)$ except when $ef = c_\delta a_{\epsilon+1}$. Again we can find a special k-chain $d_o d_1 \ldots d_k d_{\epsilon+1} \ldots d_k$ in I such that $d_{\epsilon+1} = a_k$, $d_k = a_{\epsilon+1}$; hence $\mu_k = \mu_{\epsilon+1} + 1$ ($\mu_k = \mu_{\delta+1} + 1$ as well). Let $\epsilon \neq \delta$ first. Note $a_{\delta+1} \notin d_o \ldots d_\epsilon$ otherwise $P(k-1)$ is applicable readily (we have $\delta + 1 \leq k - 1$ by the way). In H we see $d_o \ldots d_{\epsilon+1}$ and $c_o \ldots c_\delta$ form an $(\epsilon + 1, \delta)$-coupling with $\epsilon + 1 \leq k - 1$ and we are done. When $\epsilon = \delta$, the above argument still works and we have a special $(\delta+1)$-chain $d_o d_1 \ldots d_{\delta+1}$ distinct form $c_o c_1 \ldots c_{\delta+1}$ in I, we then have a $(\delta+1, \beta)$-coupling with $\delta + 1$, $\beta < k$ unless $d_o d_1 \ldots d_\delta = b_o b_1 \ldots b_\beta$, which can hold only if $\delta = \beta$. In this case we can easily find a $(\delta+1, \delta)$-coupling (or we have a (k, χ)-coupling with $\chi < \delta$ and induction applies), so we are done (looking at $G - a_{k-1} a_k$).

The proof of Proposition 3.13 is now complete. Q.E.D.

Section 7. Proof of Main Theorem

In this section, we will prove the main theorem using Proposition 3.14 as the principal tool which we restate for reference.

Proposition 3.14. $A(n)$ and $B_i(n)$ are true for any n, $1 \leq n \leq \Omega(G) - 1$; $P(\alpha)$ is true for any α, $0 \leq \alpha \leq \Omega(G) - 2$.

Recall that any (bipartite) graph can have exactly one type of termination, namely Type-I, Type-II, Type-III terminations defined in Section 3 of this chapter. In Propositions 3.15, 3.16 and 3.17 following, we will show that each type of termination leads to the edge-reconstructability of G; and so in Theorem 3.1 following we can combine these results and say every bipartite graph with at least four edges is edge-reconstructable.

Proposition 3.15. G is edge-reconstructable if it has a Type-I termination.

Proof of Proposition 3.15. Let $a_0 a_1 \ldots a_{\Omega(G)}$ be a special $\Omega(G)$-chain in G with $a_{\Omega(G)} \neq a_0$. Consider $H = G - a_{\Omega(G)}a_{\Omega(G)-1} + a_{\Omega(G)}a_j$ ($a_{\Omega(G)}$ is a forced vertex by degree argument). $B_0(j)$ implies the existence of a special j-chain $b_0 b_1 \ldots b_j$ in H with $b_j = a_{\Omega(G)-1}$, $0 < j \leq \Omega(G) - 2$. (b_0 may be a_0 say). $a_{\Omega(G)-2}$ cannot lie on $b_0 b_1 \ldots b_{j-1}$ otherwise $P(\Omega(G) - 2)$ works and G is edge-reconstructable. Furthermore $\mu_{\Omega(G)-1} > \mu_1 \geq \mu_0 + 1$ otherwise we have a $(1, \Omega(G) - 2)$-coupling.

Suppose $b_0 \neq a_0$ at first. Then $b_0 b_1 \ldots b_{j-1}$ and $a_0 a_1 \ldots a_{\Omega(G)-2}$ are disjoint otherwise $P(\Omega(G) - 2)$ applies. Consider I =

$G - a_{\Omega(G)-2}a_{\Omega(G)-1} + ef$. If none of $e, f \in b_0b_1 \ldots b_ja_{\Omega(G)}$, then $b_0b_1 \ldots b_j\, a_{\Omega(G)}$ gives a Type-I termination of length $\leq j + 1 \leq \Omega(G)-1$ in I and so $\Omega(I) < \Omega(G)$, a contradiction; and we may assume $e = b_u \in b_0b_1 \ldots b_j$. Similarly $f \in a_0a_1 \ldots a_{\Omega(G)-2}$ since otherwise we have a forbidden degree type. It's impossible that both $e = b_0$ and $f = a_0$ hold since $\mu_{\Omega(G)-1} > \mu_0 + 1$. Now it is easy to find some couplings such that $P(\Omega(G)-2)$ works.

Next let $b_0 = a_0$ and let $\gamma > 0$ be the smallest integer such that $b_\gamma \neq a_\gamma$, then $\gamma \leq j$. If $\mu_{\Omega(G)-2} > \mu_0 + 1$, then we can argue as before and easily see that $P(\Omega(G) - 2)$ works. Difficulty arises only when $\mu_{\Omega(G)-2} = \mu_0 + 1$. If a_0 is adjacent to $\alpha \geq 1$ vertices of degree μ_1 in G, it is easy to see that $a_{\Omega(G)-2}$ is adjacent to α vertices of degree μ_1 in G (by looking at $G - a_{\Omega(G)-2}a_{\Omega(G)-1}$ and note $\mu_{\Omega(G)-1} > \mu_1$). Hence $\alpha = 1$ and $\mu_{\Omega(G)-3} = \mu_1$.

Suppose $\mu_1 > \mu_0 + 1$ at first. Write Ω for $\Omega(G)$ here. If $\mu_{\Omega-4} > \mu_0 + 1$, then it is easy to find couplings satisfying $P(\Omega(G)-2)$ unless $a_{\Omega-4}a_{\Omega-2}$ is a replacing edge (and $\mu_1 = \mu_0 + 2$); but this is impossible since $b_{\gamma-1}b_\gamma \ldots b_{j-1}b_j(=a_{\Omega-1})$ and $a_{\gamma-1}a_\gamma \ldots a_{\Omega-4}a_{\Omega-2}a_{\Omega-1}$ together form an odd cycle. So $\mu_{\Omega-4} = \mu_0 + 1$. From $G - a_{\Omega-1}a_{\Omega-2}$, we see $\mu_2 = \mu_0 + 1$ (for $\mu_{\Omega-1} \geq \mu_1 + 1 \geq \mu_0 + 3$). Consider $G - a_{\Omega-4}a_{\Omega-3}$. Difficulty will arise only when the replacing edge is $a_{\Omega-4}f$, $f \notin a_0a_1 \ldots \ldots a_\Omega$. In that case, we see $a_{\Omega-2}a_{\Omega-1}$ is a forced edge by degree argument (say $a_{\Omega-2}a_{\Omega-3}$ is of forbidden degree type $(\mu_0, \mu_1 - 1)$). So we see $\mu_{\Omega-5} = \mu_1$ or $\mu_1 - 1$ and from $G - a_{\Omega-1}a_{\Omega-2}$, we conclude that $\mu_3 = \mu_1$ or $\mu_1 - 1$. Finally from $G - a_{\Omega-4}a_{\Omega-5}$, we see $B_0(3)$ implies that $\mu_{\Omega-1} = \mu_1$ or μ_1-1, contradictory to the assumption that

$\mu_{\Omega-1} > \mu_1$.

When $\mu_1 = \mu_o + 1$, we can show that $\mu_{\Omega-4} = \Omega_o + 1$. So with $k = \Omega - 2$, $\mu_k = \mu_o + 1$, we are in Case 3 of Proposition 3.12. Hence we can have G's edge-reconstructability unless $a_{\Omega-4}$ is one end of the replacing edge which can be shown to be impossible as previous paragraph.

The above arguments have assumed $\Omega(G) \geq 6$. But it is not too hard to prove that G is edge-reconstructable when $\Omega(G) = 4$ or 5 (prove like what we have done for $\Omega(G) \leq 3$), hence we are done for this Proposition.

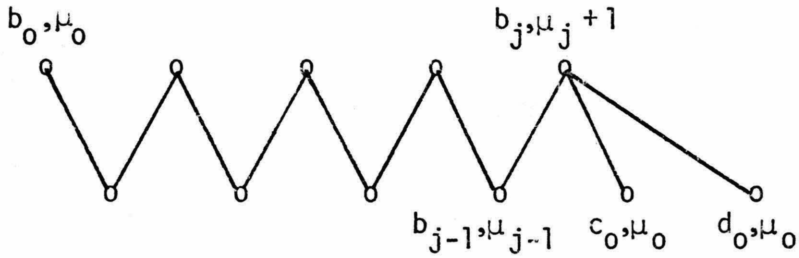
Q.E.D.

Proposition 3.16. G is edge-reconstructable if it has a Type-II termination.

Proof of Proposition 3.16. Again let Ω be a shorthand for $\Omega(G)$. We can assume $\mu_{\Omega-1} > \mu_o + 1$ otherwise any edge-reconstruction $H = G - a_o a_{\Omega-1} + a_o f$, $f \notin a_o a_1 \ldots a_{\Omega-1}$, and $\Omega(H) \leq \Omega(G) - 1$, a contradiction. Consider now $H = G - a_o a_{\Omega-1} + a_o a_j$, $0 < j < \Omega - 1$. $B_o(j)$ implies the existence of a special j-chain $b_o b_1 \ldots b_j$ in H, $b_j = a_{\Omega-1}$.

Let $b_o \neq a_o$ first. Then $b_o b_1 \ldots b_{j-1}$ and $a_o a_1 \ldots a_{\Omega-1}$ can be assumed to be disjoint. If $\mu_{\Omega-2} > \mu_o + 1$, consider $I = G - a_{\Omega-2} a_{\Omega-1} + ef$. If $f \in b_o \ldots b_{j-1}$, then we can assume the existence of special $c_o c_1 \ldots c_k$ in I, $0 < k \leq j - 1$, $c_k = a_{\Omega-1}$. It is then clear that in $H = G - a_o a_{\Omega-1} + a_o a_j$, we have a $(j, k-1)$-coupling "at $a_{\Omega-1}$" ($k \leq j - 1$ is necessary), and we are done. So we can assume $f = a_{\Omega-1}$. But then $A(\Omega-1)$ implies that $e =$ some a_m, $0 < m < \Omega - 2$, and $B_o(m)$ implies at once a $(\Omega - 2, m - 1)$-coupling and so G is edge-reconstructable by $P(\Omega - 2)$.

Now consider $J = G - a_{\Omega-2}a_{\Omega-3} + ef$ again when $\mu_{\Omega-2} = \mu_0 + 1$. If $e = a_0$, then Lemma 3.1 on edges of degree type (μ_0, μ_1) says that $\mu_{\Omega-1} = \mu_1$ and so $b_0b_1 \ldots b_{j-1}$ and $a_0a_{\Omega-1}$ form a $(j-1, 1)$-coupling in G. e cannot be $a_{\Omega-2}$ otherwise we can easily prove a contradiction by $P(\Omega-2)$; and so $e = a_{\Omega-3}$ $(\mu_{\Omega-3} = \mu_0 + 1)$. Then J contains a configuration of the following form:



This can be easily proved to be excludable. (Consider $K = J - c_0b_j + c_0b_i$, $0 < i < j$, we see there exists special $g_0g_1 \ldots g_i$, $g_i = b_j$; but then $g_0g_1 \ldots g_id_0$ gives $\Omega(K) \leq i + 1 < \Omega(G)$.

When $b_0 = a_0$, we see that the above argument still works for this case except that the excludable configuration is changed so that b_0 and c_0 coincide, and $b_0b_j \in E(J)$; the excludability follows in the same vein, and we are done. Q.E.D.

Proposition 3.17. G is edge-reconstructable if it has a Type-III termination.

Proof of Proposition 3.17. Conceivably G can have more than one special Ω-chains all of Type-III terminations. Let k be the smallest integer such that $a_k = a_\Omega$ for some $a_0a_1 \ldots a_\Omega$ special Ω-chain. We will consider the degrees μ_k and $\mu_{\Omega-1}$ in G. (Note $0 < k < \Omega - 1$).

Case 1. of Proposition 3.17. $\mu_k > \mu_o + 1, \mu_{\Omega-1} > \mu_o + 1$.

Consider $G - a_k a_{\Omega-1} + ef = H$. By A(k) we see that $e \in a_1 \ldots a_k$ and (by $A(\Omega-1)$) we can find j, $0 < j < \Omega - 1$, such that a special j-chain $b_o b_1 \ldots b_j$ exists in H with $b_j = a_{\Omega-1}$. b_o may coincide with a_o or not. Clearly $b_{j-1} \notin a_o a_1 \ldots a_{\Omega-2}$ otherwise $P(\Omega-2)$ applies. If $\mu_{\Omega-2} > \mu_o + 1$, consider $G - a_{\Omega-2}a_{\Omega-1}$. In this subgraph, $b_o b_1 \ldots b_{j-1} a_\Omega$ and $a_o a_1 \ldots a_k$ form a forbidden (k,j)-coupling, and so, arguing as in Proposition 3.16, we see G is edge-reconstructable. When $\mu_{\Omega-2} = \mu_o + 1$, we have an excludable configuration in $J = G - a_{\Omega-2}a_{\Omega-3} + a_{\Omega-3}f$ consisting of a special (j-1)-chain $b_o b_1 \ldots b_{j-1}$, a special k-chain $a_o a_1 \ldots a_k$, a vertex $a_{\Omega-2}$ of degree μ_o and three edges $b_{j-1}a_{\Omega-1}$, $a_{\Omega-2}a_{\Omega-1}$, $a_k a_{\Omega-1}$ (this is a "generalization" of the excludable configuration in Proposiiton 3.16).

Case 2. of Proposition 3.17. $\mu_k > \mu_o + 1, \mu_{\Omega-1} = \mu_o + 1$.

Case 3. of Proposition 3.17. $\mu_k = \mu_o + 1, \mu_{\Omega-1} = \mu_o + 1$.

There are nothing to do with these two cases for $a_o a_1 \ldots a_k a_{\Omega-1}$ and $a_o a_1 \ldots a_k a_{k+1} a_{k+2} \ldots a_{\Omega-3} a_{\Omega-2}$ form a $(\Omega - 2, k+1)$-coupling and $P(\Omega - 2)$ applies (Note $k \leq \Omega - 4$ by definiton and $\mu_{k+1} = \Omega_o + 1$ since $\mu_{\Omega-1} = \mu_o + 1$).

Case 4 of Proposition 3.17. $\mu_k = \mu_o + 1, \mu_{\Omega-1} > \mu_o + 1$.

We have obviously $k > 0$. First suppose $\mu_{k-1} > \mu_o + 1$. Consider

$G - a_{k-1}a_k$, we see that difficulty will arise only when $H = G - a_{k-1}a_k + a_{k-1}d$. In H, a_k is a vertex of degree μ_o (hence a forced vertex). Delete $a_k a_{\Omega-1}$ and consider all possibilities to replace by a new edge, we can prove the existence of some special $(j-1)$-chain $b_o b_1 \ldots b_{j-1}$ in G with $b_{j-1}a_{\Omega-1} \in E(G)$ (using some B_o's). Argument as in Case 1 (depending on $\mu_{\Omega-2} > \mu_o + 1$ or $\mu_{\Omega-2} = \mu_o + 1$) shows G is edge-reconstructable.

Now let $\mu_{k-1} = \mu_o + 1$. We can argue as above unless $a_o a_1 \ldots a_{k-1}$ is "symmetric" with respect to degrees, i.e. $\mu_i = \mu_{k-1-i}$, $1 \leq i \leq k-2$. Consider $G - a_k a_{\Omega-1}$. Difficulty arises only when a_o is one end of the replacing edge. But then we have $\mu_1 = \mu_o + 1$ and by symmetry $\mu_{k-2} = \mu_o + 1$. Consider $G - a_k a_{\Omega-1}$ again, we then have $\mu_2 = \mu_o + 1$. By "symmetry" again, $\mu_{k-3} = \mu_o + 1$. Proceeding in this way, we see that $\mu_i = \mu_o + 1$, $1 \leq i \leq k-1$. Now consider $G - a_o a_1$. A nonisomorphic edge-reconstruction will contain a configuration consisting of a special $(k-1)$-path $a_1 \ldots a_k$ of degree type $(\mu_o, \mu_o + 1, \ldots, \mu_o + 1)$ followed by $a_k \ldots a_{\Omega-1}$. Consider $H - a_1 a_2$ for the new graph H again, then the "newer" graph I will contain a special $(k-2)$-path $a_2 \ldots a_k$ followed by $a_k \ldots a_{\Omega-1}$. Proceed in this way, we will get a graph in which a_k is a vertex of degree μ_o as in the previous paragraph. Delete $a_k a_{\Omega-1}$ again, we see G contains a special $(j-1)$-chain $b_o b_1 \ldots b_{j-1}$ with $b_j a_{\Omega-1} \in E(G)$. So arguments as before prove G's edge-reconstructability. We are thus done with the proof of Proposition 3.17.

Q.E.D.

Now that Proposition 3.15, 3.16, 3.17 are all proved, we can then state our main theorem.

Theorem 3.1. (MAIN THEOREM OF CHAPTER 3) Every bipartite graph with at least four edges is edge-reconstructable.

Section 8. Digression on generalization of results

Many concepts and lemmas of this chapter sound easily generalizable to more general graphs, say, that of special n-chains. Many proofs do not use the fact that G is bipartite too heavily; actually just the fact that G doesn't contain triangles. It's conceivable that closer investigation of the proofs might shed light on the general Edge-Reconstruction Problem.

Most lemmas (or propositions) are not too hard when G doesn't have any vertex of degree one higher than minimum ($\mu_0 + 1$ as in the context). This suggests that the results of bi-degreed graphs, or more generally, graphs with two "lowest" degrees differing by one and the methods of bipartite graphs may be combined to prove something. Lemma 3.2 and Corollary 3.4 are very interesting for more general graphs, so are the proofs of A, B_i and p's for $n = 0, 1, 2, 3$.

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