ALIGNMENT OF INTERSTELLAR GRAINS

Thesis by

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TO MY PARENTS

AND SISTERS --

who have waited

so long

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ABSTRACT

We calculate the alignment of interstellar dust grains with respect to the magnetic field of our galaxy. The alignment is found for several values of magnetic field strength, internal grain temperature, and grain shape. We treat in detail the following processes which affect the alignment: (i) a dissipative magnetic torque due to Davis and Greenstein; (ii) the collisions of the grain with interstellar hydrogen; (iii) the non-zero internal temperature of the grain.

We obtain a Fokker-Planck equation which takes account of these processes, and the solution of this equation provides the probability distribution of grain orientations. The equation is solved for these cases: (i) spherical grains in all fields; (ii) needles, prolate spheroidal grains, nearly-spherical oblate grains, and disks in strong magnetic fields; (iii) needles and nearly-spherical grains in weak fields. Using the distribution of orientations, we calculate the degrees of alignment.

Our results are in mixed agreement with those of E. M. Purcell and in good agreement with the weak-field calculation of C. R. Miller. We find that for the relatively strong magnetic field of 10^{-5} gauss and grain temperature of 10^{o}K, the measures of alignment are smaller than the values obtained from complete orientation of the grains.

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CHAPTER I

INTRODUCTION

Interstellar grains are the small particles of dust which exist between the stars of our galaxy. Although these grains form but a small part of the total mass of our galaxy, they have an important place in astrophysics. For these particles play a role in many interesting problems, which include: the formation of molecules like OH, H_2, or NH_3 on the grain surfaces; the temperature balance of the interstellar gas; and the polarization of light from distant stars. In their review articles, J. M. Greenberg(1) and N. H. Dieter and W. M. Goss(2) provide a more complete list of these questions and discuss several of them. In this paper, we will consider only a portion of one of these problems. The main topic is the polarization of light from distant stars; the portion which we will treat is the orientation of the dust grains in space.

J. S. Hall(3) and W. A. Hiltner(4) discovered the polarization of starlight in 1949. At the time, their discovery was accepted as giving strong evidence that the grains do exist. Indeed, two features of their results were noteworthy: (i) a correlation between the degree of polarization and the magnitude of the absorption of the starlight; (ii) a uniformity of the directions of polarization over large areas of the sky. Thus, (i) showed that absorbing grains of dust could polarize the starlight; while (ii) made it difficult to conceive of anything else which might. For most of the other possible sources of polarization would involve a small region of space, and (ii) made them unlikely prospects.

To explain this polarization, in 1951 L. Davis, Jr., and J. L. Greenstein,[5] hereafter called DG, proceeded in the following manner. They first assumed that a magnetic field exists within our galaxy. Using a model for the dust, in which the grains are bombarded by surrounding hydrogen atoms or ions, DG next proposed that a dissipative magnetic torque acts on the particles. By means of this torque, the magnetic field of our galaxy aligns the grains with respect to the field direction. DG further calculated the distribution of grain orientations which this aligning torque yields. Finally, they[6] used a classical theory of light scattering, due to R. Gans,[7] in order to find the polarization which the partially oriented particles produce. This calculation was one of the first to provide evidence that a magnetic field does exist in our galaxy, and many accepted the treatment as giving a fair idea of the processes at work.

Yet the paper of DG was incomplete in several respects. They made only a rough calculation of the distribution of grain orientations; thus, it was difficult to estimate the field strength needed to produce a given degree of alignment. In addition, the Rayleigh-Gans scattering theory is correct only when dust grains are small compared to the wavelength of the incident light. Since the particles are thought to be of order 10^{-5} cm. in size, this scattering theory is incorrect for visible light. Therefore, DG were unable to accurately predict the polarization which would be produced once the size, composition, and temperature of the grains were specified, together with the magnetic field strength. Thus, it was impossible to be sure if the DG process was correct; neither could the observational data on polarization be used in order to limit the parameters of the grains or to confidently estimate the field strength.

Since the original paper, a fair amount of work has been done. In 1962 C. R. Miller treated the statistical mechanics of the DG process in more detail. Using the same DG model for the grains and the forces acting on them, Miller improved their rough estimate for the distribution of grain orientations. He obtained a Fokker-Planck equation for the alignment of the particles; and he solved this equation for the case of nearly-spherical grains in weak magnetic fields.[8]

Independently of Miller, in 1967 R. V. Jones and L. Spitzer, Jr.,[9] hereafter called JS, obtained a Fokker-Planck equation for the alignment of spheres having a positive internal temperature. They treated this case because in 1956 C. Kittel[10] pointed out that a positive internal temperature would generate fluctuations of magnetization in the grain, tending to disorient the particle. JS solved their Fokker-Planck equation for spherical grains in an arbitrary magnetic field; they gave a rough treatment of nearly-spherical grains; and they treated in detail the possibility of new grain compositions in order to permit alignment in weak magnetic fields.

In 1968[11] and 1969[12] J. M. Greenberg published review articles on the status of the interstellar grain problem. In both papers he treated the case when the magnetic field has an irregular direction; the result is that the qualitative effect on the polarization is the same as the effect of incompletely aligned grains. He also summarized[13] his microwave analogue experiments on the scattering produced by particles of any size--especially particles of size equal to or larger than the wavelength of the incident radiation.

In 1969 E. M. Purcell[14] used a Monte-Carlo computer calculation to simulate the history of a single grain. His computer program generated random collisions of the grain with surrounding gas molecules, random evaporation of atoms from the surface of the grain, and the systematic DG alignment mechanism. Using this program, he found the alignment for several grain shapes and grain temperatures.

After 1950 the galactic field itself became an accepted fact. The techniques of radio astronomy were used in order to measure the field strength, and Greenberg[15] quoted typical values of $2\text{-}5\times10^{-6}$ gauss. Both Greenberg and JS stated that the DG process may demand magnetic fields an order of magnitude larger. The reason is that the polarization data apparently require the grains to be substantially aligned; this result would demand magnetic fields strong relative to the effects of the gas collisions. Thus, the strong field case must be considered in treating the grain alignment. However, the question of how large a field is needed by the DG process remains unsettled.

This paper will extend Miller's work on alignment: it will treat the strong field case, consider non-spherical grains, and deal with the effect of positive grain temperature in a fashion somewhat different from that of JS. Starting with Miller's Fokker-Planck equation, we will add extra terms to describe the grain temperature. The enlarged equation will be solved for the following cases: (i) an exact solution for spherical grains in arbitrary magnetic fields; (ii) an approximate solution for needles, disks, and nearly-spherical grains in strong magnetic fields; (iii) an approximate solution for needles and nearly-spherical grains in weak fields. Finally, we will compare results with those of JS and Purcell.

Our work will not calculate the polarization to be expected, nor will it treat the observational data. In principle, once the scattering properties and the alignment are known for the grain, then the polarization can be found; Greenberg[16] has described the procedure elsewhere. In addition, we will not give a detailed discussion of the galactic field strength, nor will we consider any grain composition different from the one treated by DG.

The discussion proceeds as follows. In Chapter II, we first introduce the variables which describe the orientation of the grain in space. Next, in terms of these variables, we define a probability density which provides the distribution of orientations for the grain. Using this probability function, we obtain the parameters which measure the degree of alignment for the grain. In order to find a differential equation for the probability function, we introduce and briefly discuss the Fokker-Planck equation of statistical mechanics.

Now, to each physical process which affects the grain's orientation, there corresponds a set of terms in the Fokker-Planck equation. Therefore, each of these processes is treated in turn. To find the terms due to the steady aligning torque, we discuss the DG mechanism. To find the terms due to collisions of the grain with surrounding gas atoms, we present Miller's results for these quantities. His detailed derivation is given in an appendix.

Our contribution to the alignment problem begins in the fifth section of Chapter II, where we conclude the procedure of adding terms to the Fokker-Planck equation. The effects of the grain's internal temperature are discussed, and the relevant terms are added so as to obtain the enlarged Fokker-Planck equation for alignment. In Chapter III this equation is solved for the various cases mentioned. In Chapter IV the parameters which measure grain alignment are calculated. Chapter V discusses all these results and concludes our work.

CHAPTER II

THE FOKKER PLANCK EQUATION FOR GRAIN ALIGNMENT

1. A Set of Variables to Characterize the Grain Alignment

We first introduce variables to describe the orientation of the grain in space, which is shown in Figure 1:

$\underset{\sim}{A}$ is the symmetry axis of the grain

$\hat{A}$ is a unit vector along $\underset{\sim}{A}$

$\underset{\sim}{B}$ is the magnetic field

$\underset{\sim}{J}$ is the angular momentum of the grain

β is the angle between $\underset{\sim}{J}$ and $\underset{\sim}{B}$

θ is the angle between $\underset{\sim}{J}$ and $\underset{\sim}{A}$

φ is the angle between $\underset{\sim}{B}$ and $\underset{\sim}{A}$

Ψ is the angle between the plane of $\underset{\sim}{B}$ and $\underset{\sim}{J}$ and the plane of $\underset{\sim}{A}$ and $\underset{\sim}{J}$. (1)

All symbols used in this paper are listed in an appendix.

Let us briefly consider the free body motion of the grain. The angular momentum, $\underset{\sim}{J}$, remains constant. Since the particle has rotational symmetry, the axis $\underset{\sim}{A}$ rotates uniformly about $\underset{\sim}{J}$, the angle between them staying fixed. Thus, β and θ are constant, while Ψ increases uniformly. Since the orientation of $\underset{\sim}{J}$ about $\underset{\sim}{B}$ is random, we need no azimuthal angle for $\underset{\sim}{J}$; in addition, all calculations are averaged over Ψ.

We next present variables to describe the alignment of the grains:

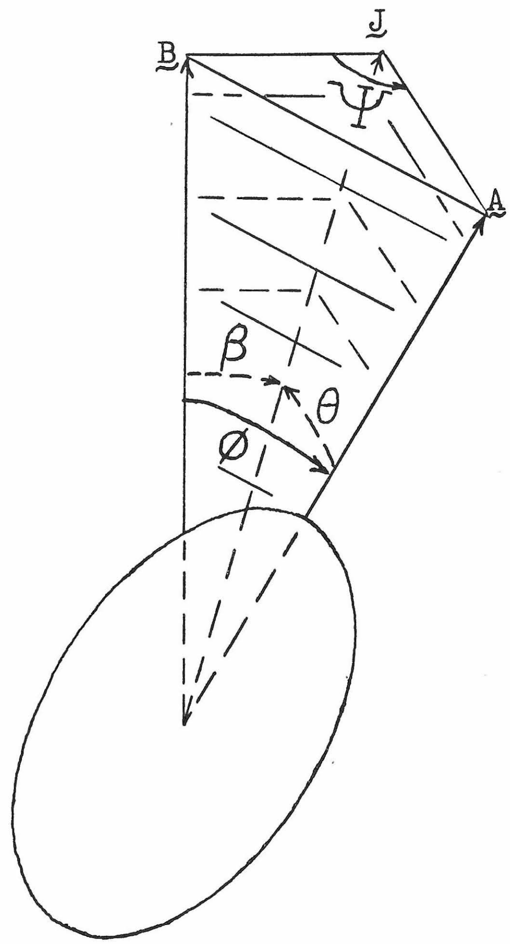


Figure 1

The Orientation Angles

B is the magnetic field

J is the angular momentum of the grain

A is the symmetry axis of the grain

$$r = \cos\beta ,$$

$$\rho = \cos\theta ,$$

$$-1 \leqq (r, \rho) \leqq 1 . \qquad (2)$$

Let the probability density of orientations be

$$W(r, \rho) = W_e(r, \rho) + W_1(r, \rho) . \qquad (3)$$

In this equation $W(r, \rho)$ is the fraction of the grains for which r lies between r and $r + dr$ and ρ lies between ρ and $\rho + d\rho$; W_e is the equilibrium density in the absence of a magnetic field. In general, W and W_1 also depend upon the size of the grain, while W_e does not. However, since we only treat the case for which the grains all have the same dimensions, we do not include the particle size as a variable in defining W. Moreover, W is not a function of Ψ because all calculations are averaged over that angle.

Finally, to measure the grain alignment, we use the convenient numbers

$$F = -\int_{-1}^{1}\int_{-1}^{1} W_1(r, \rho) \cdot (\cos^2\varphi)\, dr\, d\rho , \qquad (4)$$

$$Q_A = \frac{3}{2}\left[\int_{-1}^{1}\int_{-1}^{1} W(r, \rho) \cdot (\cos^2\varphi)\, dr\, d\rho\right] - \tfrac{1}{2} , \qquad (5)$$

$$= \frac{3}{2}\left(\frac{1}{3} - F\right) - \frac{1}{2} = -\frac{3}{2} F ,$$

$$Q_J = \frac{3}{2}\left[\int_{-1}^{1}\int_{-1}^{1} W(r, \rho) \cdot (\cos^2\beta)\, dr\, d\rho\right] - \tfrac{1}{2} . \qquad (6)$$

The quantity F was used by DG,[17] and the quantities Q_A and Q_J were used by Purcell.[18] Both Q_A and F measure the alignment of the symmetry axis, while Q_J measures the alignment of the angular momentum. The factor of $\frac{1}{3}$ in equation (5) is the average value of $(\cos^2\varphi)$ when W is equal to W_e, since W_e is a random distribution of orientations.

We note that if the Gans theory is valid, then F, Q_A, and Q_J are the only quantities needed - along with the optical properties of the grain - in order to compute the polarization. Any other scattering theory requires the complete description of the grain orientations contained in W. In Chapter IV we calculate the values of Q_A and Q_J for the various cases of interest after we obtain W.

2. The Fokker-Planck Equation

In order to find W, we may apply the Fokker-Planck equation, which is a parabolic, or diffusion, type of differential equation treated in statistical mechanics. The Fokker-Planck equation is often used for situations in which a probability function depends on variables which are themselves subject to random changes. Such is the case for the grain, for which the random changes arise from two main sources: (i) the collisions of the particle with surrounding hydrogen atoms or ions; (ii) the effects of its non-zero temperature, which are discussed in a later section. There are extensive treatments of the Fokker-Planck equation in the works of S. Chandrasekhar[19] and N. Wax.[20] We will briefly discuss the equation along the lines of Chandrasekhar's presentation.

Let $(x_1, \ldots, x_n)$ be the set of variables of interest, and let $W(x_1, \ldots, x_n, t)$ be its probability distribution at time t. Thus, $W\,dx_1\,dx_2 \ldots dx_n$ is the probability that the $i^{\underline{th}}$ variable is in the range x_i to $x_i + dx_i$ at time t, for $i = 1, \ldots, n$. During a small time interval Δt, let the $i^{\underline{th}}$ variable change by an amount Δx_i.

We now assume that the processes which cause this change Δx_i can be separated into two portions. One portion yields a "steady" rate of change, $\dot{x}_i$, which is due to some known external force which may depend on x_i. The change due to this process is

$$(\Delta x_i)_{steady} = \dot{x}_i \Delta t. \tag{7}$$

The other portion is written

$$(\Delta x_i)_{fluct} = \delta x_i \tag{8}$$

and is a fluctuating change for which we have only statistical knowledge. The total change in x_i during the time interval Δt is

$$\Delta x_i = \dot{x}_i \Delta t + \delta x_i , \qquad i = 1, \ldots, n. \tag{9}$$

This separation of $\dot{x}_i \Delta t$ from δx_i is justified under the following conditions: there must exist time intervals Δt during which δx_i undergoes many fluctuations, while $\dot{x}_i \Delta t$ is small. In other words, given two successive times, t_0 and $t_0 + \Delta t$, the "steady" forces are strongly correlated at the two instants, while the "random" forces are uncorrelated. Such is the case for the grain. We will find that the steady DG alignment process requires time scales of order 10^6 years; on the other hand, the average time between collisions of the particle with hydrogen atoms is of order 30 minutes.

We must next describe the δx_i in equation (9) by some transition probability Θ. We will consider several possibilities. In the first instance, we assume that Θ is independent of $\dot{x}_i$ because the $\dot{x}_i$ have been separated from the δx_i. Thus, let $\Theta(x_1, \ldots, x_n; \delta x_1, \ldots, \delta x_n;$

$;\Delta t)\ d(\delta x_1) \ldots d(\delta x_n)$ be the probability that a change will occur in x_i by an amount between $\dot{x}_i\,\Delta t + \delta x_i$ and $\dot{x}_i\,\Delta t + \delta x_i + d(\delta x_i)$, $i = 1, \ldots, n$, during time Δt if the current values of the variables are $x_1, \ldots, x_n$. Define the expectation values E_i and E_{ij}, in terms of first and second moments of Θ:

$$\langle \delta x_i \rangle = \int \ldots \int \delta x_i\ \Theta(x_m; \delta x_m; \Delta t)\ d(\delta x_1) \ldots d(\delta x_n)$$

$$= E_i\,\Delta t + O[(\Delta t)^2] \quad , \tag{10}$$

$$\langle \delta x_i\ \delta x_j \rangle = \int \ldots \int \delta x_i\ \delta x_j\ \Theta(x_m;\ \delta x_m; \Delta t)$$

$$d(\delta x_1) \ldots d(\delta x_n)$$

$$= E_{ij}\,\Delta t + O[(\Delta t)^2] \quad . \tag{11}$$

Our notation indicates that we expect these moments to be proportional to Δt.

Let the third moments and all higher ones be proportional to higher powers of Δt. The situation is now precisely the one treated by Chandrasekhar,[21] and the Fokker-Planck equation for W is

$$\frac{\partial W}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(WE_i) + \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(WE_{ij}) - \sum_i \frac{\partial}{\partial x_i}(W\dot{x}_i). \tag{12}$$

The separation of Δx_i into $\dot{x}_i\,\Delta t$ and δx_i is somewhat arbitrary. For example, the terms in (12) resulting from the $\dot{x}_i$ and δx_i contributions may be combined into one $E_i^{(new)}$, while still keeping Θ independent of $\dot{x}_i$. We then find that

$$E_i^{(new)}\Delta t = \langle \Delta x_i \rangle = \dot{x}_i \Delta t + E_i^{(old)} \Delta t$$

$$E_i^{(new)} = \dot{x}_i + E_i^{(old)} \quad . \tag{13}$$

Thus, the first and last terms in equation (12) would be combined into one. For E_{ij} we obtain

$$\begin{aligned} E_{ij}^{(new)}\Delta t &= \langle \Delta x_i \Delta x_j \rangle \\ &= \langle \dot{x}_i \dot{x}_j \rangle (\Delta t)^2 + \Delta t \langle \dot{x}_i \delta x_j + \dot{x}_j \delta x_i \rangle + \langle \delta x_i \delta x_j \rangle \\ &= (\Delta t)^2 \left[\dot{x}_i \dot{x}_j + \dot{x}_i E_j^{(old)} + \dot{x}_j E_i^{(old)} \right] + E_{ij}^{(old)} \Delta t . \end{aligned} \tag{14}$$

Since terms are only kept to order Δt, this yields

$$E_{ij}^{(new)} = E_{ij}^{(old)} \quad , \tag{15}$$

and the second term of equation (12) remains unchanged.

It is also possible to treat Θ as a "complete" transition probability, including both "steady" and "fluctuation" effects. The result would be to separate E_i and E_{ij} into several contributions. Thus, we would find that

$$E_i^{(new)} = E_i^{(steady)} + E_i^{(collisions)} + E_i^{(pos.\ temp.)} \quad , \tag{16}$$

and similarly for E_{ij}. In equation (16) $E_i^{(steady)}$ is the portion due to the DG process; $E_i^{(collisions)}$ is the contribution due to collisions of the grain with surrounding hydrogen; and $E_i^{(pos.\ temp.)}$ is the part due to the effects of the grain's positive internal temperature. The final equation for W would be the same as equation (12).

The terms E_i and E_{ij} in equations (12) and (13) are often expressed as diffusion coefficients, and we now turn to finding them. Each physical process affecting the grain's orientation in space will contribute its own set of diffusion coefficients to the right-hand side of equation (12). We will consider these processes in turn, starting with the Davis-Greenstein mechanism. We will then treat the effects of collisions and finally consider the grain's non-zero internal temperature.

3. Diffusion Coefficients Due to the Davis-Greenstein Process

We will calculate the relevant terms in the Fokker-Planck equation after a brief discussion of the Davis-Greenstein mechanism. Our work follows that of DG.

We begin with the following assumptions:

(i) A magnetic field exists within our galaxy. The field is essentially uniform and constant over distances of astronomical units and times on the order of days.

(ii) The grains are spheroids of revolution of order 10^{-5} cm. in size. This form is chosen because it is the simplest non-spherical shape, which is needed in order to produce polarization. The size is obtained from the data on extinction of the starlight.

(iii) The particles are formed mainly of ice with enough impurities to be weakly paramagnetic. Other authors have proposed different compositions, but we only treat the "dirty-ice" model.

(iv) Hydrogen surrounds and bombards the grains. The hydrogen temperature is 100°K for the gaseous H I regions and $10^{4\circ}$K for the ionized H II regions. For H I regions, and for a grain density of

1 gm/cm^3, these assumptions imply that the particle has an angular speed of order 10^5 rad/sec.

Next, let us consider the motion of the grain in somewhat more detail than was done at the beginning of this chapter. Assume that the grain's angular velocity is $\underset{\sim}{\omega}$ and that the aligning torque is weak. Therefore, during a time interval of duration $1/\omega$, the rotationally symmetric grain is almost a free body. If the grain were truly free, it would behave as follows:

(i) The angular momentum would remain constant.

(ii) The symmetry axis $\underset{\sim}{A}$ would rotate uniformly around $\underset{\sim}{J}$, taking Ψ through 2π radians in each cycle. One such cycle of Ψ is called a nutation.

(iii) The angle θ between $\underset{\sim}{J}$ and $\underset{\sim}{A}$ would remain constant.

Since the particle is not really free, the effect of the small aligning torque is to change β, θ, and J^2 by a small amount during each nutation.

Consider the situation in the rest frame of the particle. From this point of view, the magnetic field varies sinusoidally, so that

$$\underset{\sim}{B} = \underset{\sim}{B}_0 \cos \omega t \ . \tag{17}$$

This oscillating field induces a magnetization $\underset{\sim}{M}$ in the grain, where

$$\underset{\sim}{M} = \underset{\sim}{B}_0 \, (\chi' \cos \omega t + \chi'' \sin \omega t) \ . \tag{18}$$

In this equation the particle's magnetic susceptibility is assumed to be complex, with χ' and χ'' being its real and imaginary parts. The term with χ'' measures the small amount by which $\underset{\sim}{M}$ is out of phase with $\underset{\sim}{B}$. Now let an average of $\underset{\sim}{M}$ be taken over one nutation. Following DG,[22] we find that one result of this averaging is a component $\underset{\sim}{M}_p$, which is normal to $\underset{\sim}{B}$, and which has the value

$$\underset{\sim}{M}_p = (\chi''/\omega)(\underset{\sim}{\omega} \times \underset{\sim}{B}) \quad . \tag{19}$$

This component $\underset{\sim}{M}_p$ generates a dissipative torque given by

$$\frac{d\underset{\sim}{J}}{dt} = V\,(\underset{\sim}{M}_p \times \underset{\sim}{B}) \quad , \tag{20}$$

where V is the grain's volume and $\underset{\sim}{J}$ is its angular momentum. This torque is the aligning agent of DG,[23] for its effect is to tend to orient $\underset{\sim}{J}$ parallel to $\underset{\sim}{B}$. Both DG[24] and Purcell[25] considered the value of

$$(\chi''/\omega) = (2.5 \times 10^{-12})/T_i \quad , \tag{21}$$

where T_i is the internal temperature of the grain. From the extended discussion of Greenberg[26] for T_i, we find that typical values are 10^oK.

Define, further, the variables

I = the moment of inertia of the grain about $\underset{\sim}{A}$

γI = the moment of inertia of the grain about an axis normal to $\underset{\sim}{A}$

$$D = (\chi''/\omega)(V/I\gamma) \quad . \tag{22}$$

If we consider the rates of change of β, θ and J^2 due to the aligning torque, and if we average them over the grain's motion, then we find from DG[27] that

$$\frac{d\beta}{dt} = -DB^2 \sin\beta \cos\beta \,(\gamma\cos^2\theta + \sin^2\theta) \quad , \tag{23}$$

$$\frac{d\theta}{dt} = DB^2(\gamma - 1) \sin\theta \cos\theta \left(1 - \frac{1}{2} \sin^2\beta\right) \quad , \tag{24}$$

$$\frac{d(J^2)}{dt} = -2\, DB^2 J^2 \sin^2\beta \,(\gamma \cos^2\theta + \sin^2\theta) \quad . \tag{25}$$

For a spherical grain of density 1 gm/cm^3, $T_i = 10^o$K, radius 10^{-5}cm, we find that $D = 6 \times 10^{-3}$ sec^{-1} gauss^{-2}. If $B = 10^{-5}$ gauss, then $DB^2 = 6 \times 10^{-13}$ sec^{-1}, so that the characteristic time for the torque to act is of order 10^5 yr.

There also is a rotation of $\underset{\sim}{J}$ around $\underset{\sim}{B}$ which is called precession. This precession is due to the χ' term in equation (18). From DG,[28] we find that $\chi' \sim 10^4 \chi''$, so that the precession is of order 10^4 times faster than the alignment, yet still slower than the nutation. In all of our calculations, we will average over the angle Ψ and the orientation of $\underset{\sim}{J}$ around $\underset{\sim}{B}$.

Finally, we turn to calculating the contribution of the DG process to the right hand side of equation (12), the Fokker-Planck equation. We use as our variables x_i the same ones that Miller[29] did:

$$\begin{aligned} \mu &= J \cos\theta , & -\infty < \mu < \infty \\ \eta &= J \cos\beta , & -\infty < \eta < \infty \\ \zeta &= J^2 , & 0 < \zeta < \infty \end{aligned} \tag{26}$$

Miller used these quantities because they are convenient for treating the effects of collisions of the grain with surrounding hydrogen atoms.

The DG process will contribute to the right hand side of equation (12)

$$R^{(DG)} = -\sum_{i} \frac{\partial}{\partial x_i}(W\dot{x}_i) \quad . \tag{27}$$

If we use equation (26) and the relation

$$\dot{J} = \frac{1}{2J}\frac{d}{dt}(J^2) \quad , \tag{28}$$

and if we further use equations (23), (24), and (25), we find

$$\begin{aligned}\dot{\mu} &= \dot{J}\cos\theta - J\dot{\theta}\sin\theta \\ &= DB^2\left\{\frac{3}{2}(\gamma-1)\frac{\eta^2\mu^3}{\zeta^2} - \frac{1}{2}(\gamma-1)\frac{\mu^3}{\zeta} \right. \\ &\qquad \left. + \left[1-\frac{1}{2}(\gamma-1)\right]\frac{\eta^2\mu}{\zeta} - \left[1+\frac{1}{2}(\gamma-1)\right]\mu\right\}\end{aligned} \tag{29}$$

$$\dot{\eta} = \dot{J}\cos\beta - J\dot{\beta}\sin\beta = 0 \quad , \tag{30}$$

$$\begin{aligned}\dot{\zeta} &= \frac{d}{dt}(J^2) \\ &= DB^2\left\{-2(\gamma-1)\mu^2 - 2\zeta + 2(\gamma-1)\frac{\eta^2\mu^2}{\zeta} + 2\eta^2\right\}\end{aligned} \tag{31}$$

Since η is the projection of $\underset{\sim}{J}$ on $\underset{\sim}{B}$, and since the aligning torque leaves this component constant, equation (30) is to be expected. Thus, we obtain that

$$R^{(DG)} = \frac{\partial}{\partial\mu}(\dot{\mu}W) - \frac{\partial}{\partial\zeta}(\dot{\zeta}W) \quad . \tag{32}$$

4. <u>Diffusion Coefficients Due to Collisions of the Grain with Hydrogen Atoms or Ions</u>

Miller found the diffusion coefficients due to collisions of the grain with surrounding hydrogen atoms, or ions. These collisions produce some of the random changes which affect the variables μ, η and ζ, in addition to the systematic effects of the DG process. In this section we summarize Miller's argument and his results. We repeat his detailed derivation in an appendix, and we add a section to treat the positive internal temperature of the grain. The reader is invited to consult this appendix.

The collision terms in equation (12) may be written

$$R^{(c)} = -\sum_i \frac{\partial}{\partial x_i}\left[WE_i^{(c)} - \frac{1}{2}\sum_j \frac{\partial}{\partial x_j}\left\{WE_{ij}^{(c)}\right\}\right] , \tag{33}$$

where c represents "collisions." If we write the term in brackets as $L_i^{(c)}$, then this equation becomes

$$R^{(c)} = -\frac{\partial}{\partial\mu} L_\mu^{(c)} - \frac{\partial}{\partial\eta} L_\eta^{(c)} - \frac{\partial}{\partial\zeta} L_\zeta^{(c)} . \tag{34}$$

In order to obtain the coefficients $E_i^{(c)}$ and $E_{ij}^{(c)}$, Miller proceeded as follows. He assumed that a single atom-grain collision occurs quickly enough to produce an impulse $\delta \underset{\sim}{J}$ of angular momentum. This impulse is the δx_i term used in equations (10) and (11). The effect of $\delta \underset{\sim}{J}$ is to change $\underset{\sim}{J}$, β, and θ, but not the particle's orientation space. The grain's reorientation follows from its nutation about the new $\underset{\sim}{J}$.

Next, Miller found $\delta \underset{\sim}{J}$ due to a single collision, considering elastic and inelastic impacts. By assigning an effective mass m^+ to the hydrogen atom, both types of collision could be treated together. Miller's "elastic" collision was one in which all components of the atom's initial velocity are reversed. In a standard elastic collision, only the velocity component normal to the grain surface is reversed, while the component parallel to the grain surface is unchanged. This standard collision is not treated because it is more difficult than

Miller's version. Finally, we add a section to Miller's calculation of m^+ for an inelastic collision in order to consider the grain's non-zero internal temperature.

Miller went on to assume that the surrounding gas atoms have a Maxwell distribution of velocities at temperature T, and that this distribution gives the transition probability Θ. The relevant variables of integration were the surface of the grain and the velocities of the hydrogen atoms. By integrating the vector $\delta \underset{\sim}{J}$ and the tensor $(\delta \underset{\sim}{J})(\delta \underset{\sim}{J})$ over these variables and Θ, Miller found $E_i^{(c)}$ and $E_{ij}^{(c)}$.

We need the following quantities to express $L_i^{(c)}$:

$$m = \text{mass of the hydrogen atom} \tag{35a}$$

$$T = \text{temperature of the surrounding gas} \tag{b}$$

$$T_i = \text{internal temperature of the grain} \tag{c}$$

$$c^2 = (2kT/m) \tag{d}$$

$$n_H = \text{number of hydrogen atoms/cm}^3 \tag{e}$$

$$m^+ = \text{effective mass of a hydrogen atom in its collision with the grain}$$

$$= \begin{cases} m \text{ for an elastic collision of Miller's type} \\ \frac{1}{2} m(1 + \sqrt{T_i/T}\,) \text{ for an inelastic collision} \end{cases}$$

$$g = \pi^{-1/2}\, n_H\, m^+ c \tag{g}$$

$$2a\epsilon = \text{length of } \underset{\sim}{A}\text{, the axis of symmetry of the grain} \tag{h}$$

$$2a = \text{length of a diameter normal to } \underset{\sim}{A} \tag{i}$$

h, α = parameters depending on the grain shape and arising from the integrations over the surface of the particle

$$h = \pi a^4 \left\{ 1 + \frac{1}{2} \frac{\epsilon^2}{\epsilon^2 - 1} + \left[2 - \frac{1}{2} \frac{\epsilon^2}{\epsilon^2 - 1} \right] \cdot \frac{\epsilon^2}{|\epsilon^2 - 1|^{1/2}} \cdot \begin{pmatrix} \sin^{-1} \\ \sinh^{-1} \end{pmatrix} \frac{|\epsilon^2 - 1|^{\frac{1}{2}}}{\epsilon} \right\} \tag{j}$$

$$a h = \pi a^4 \left\{ 1 + \epsilon^2 - \frac{1}{4} \frac{\epsilon^4}{\epsilon^2 - 1} + \left[2 + \frac{\epsilon^2}{\epsilon^2 - 1} \right] \cdot \frac{1}{4} \frac{\epsilon^4}{|\epsilon^2 - 1|^{1/2}} \cdot \begin{pmatrix} \sin^{-1} \\ \sinh^{-1} \end{pmatrix} \frac{|\epsilon^2 - 1|^{\frac{1}{2}}}{\epsilon} \right\} \tag{k}$$

In these equations for h and $a h$, $\sin^{-1}$ is used for a prolate spheroid ($\epsilon > 1$), and $\sinh^{-1}$ is used for an oblate spheroid ($\epsilon < 1$). A plot of a is shown in Figure 2. For special grain shapes, a has the values

$$\begin{array}{ll} \alpha, a = 1 & \text{disk} , \\ a = 1 & \text{sphere} , \\ a \cong 1 + \frac{2}{5}(\epsilon^2 - 1) & \text{nearly-spherical grain} , \\ a \cong \frac{1}{2}\epsilon^2 + 1/3 & \text{needle} . \end{array} \tag{l}$$

From the defining equation (22) γ may be found for a grain of uniform density, so that

$$\gamma = (1/2)(1 + \epsilon^2) \quad . \tag{35m}$$

If a subscript on W denotes a partial derivative with respect to that variable, then the L terms in equation (34) are

$$L_\mu = -ghm^+c^2 \left[\left(\frac{\mu}{m^+c^2 I} + \frac{\mu}{2\zeta} \right) W + \frac{1}{2} W_\mu + \frac{1}{2} \frac{\mu\eta}{\zeta} W_\eta + \mu W_\zeta \right] \tag{36}$$

$$\begin{aligned} L_\eta = -ghm^+c^2 \Bigg\{ & \left[\frac{1}{2}(1-a)\frac{\mu^2\eta}{\zeta^2} + \frac{1}{2} a \frac{\eta}{\zeta} + \frac{a}{m^+c^2 I\gamma}\eta + \left(1 - \frac{a}{\gamma}\right) \cdot \frac{1}{m^+c^2 I} \frac{\mu^2\eta}{\zeta} \right] W + \frac{1}{2}\frac{\mu\eta}{\zeta} W_\mu \\ & + \left[\frac{1}{4}(a+1) + \frac{1}{4}(a-1)\left(\frac{\eta^2}{\zeta} + \frac{\mu^2}{\zeta} - 3\frac{\mu^2\eta^2}{\zeta^2} \right) \right] W_\eta + \left[a\eta + (1-a)\frac{\mu^2\eta}{\zeta} \right] W_\zeta \Bigg\} \end{aligned} \tag{37}$$

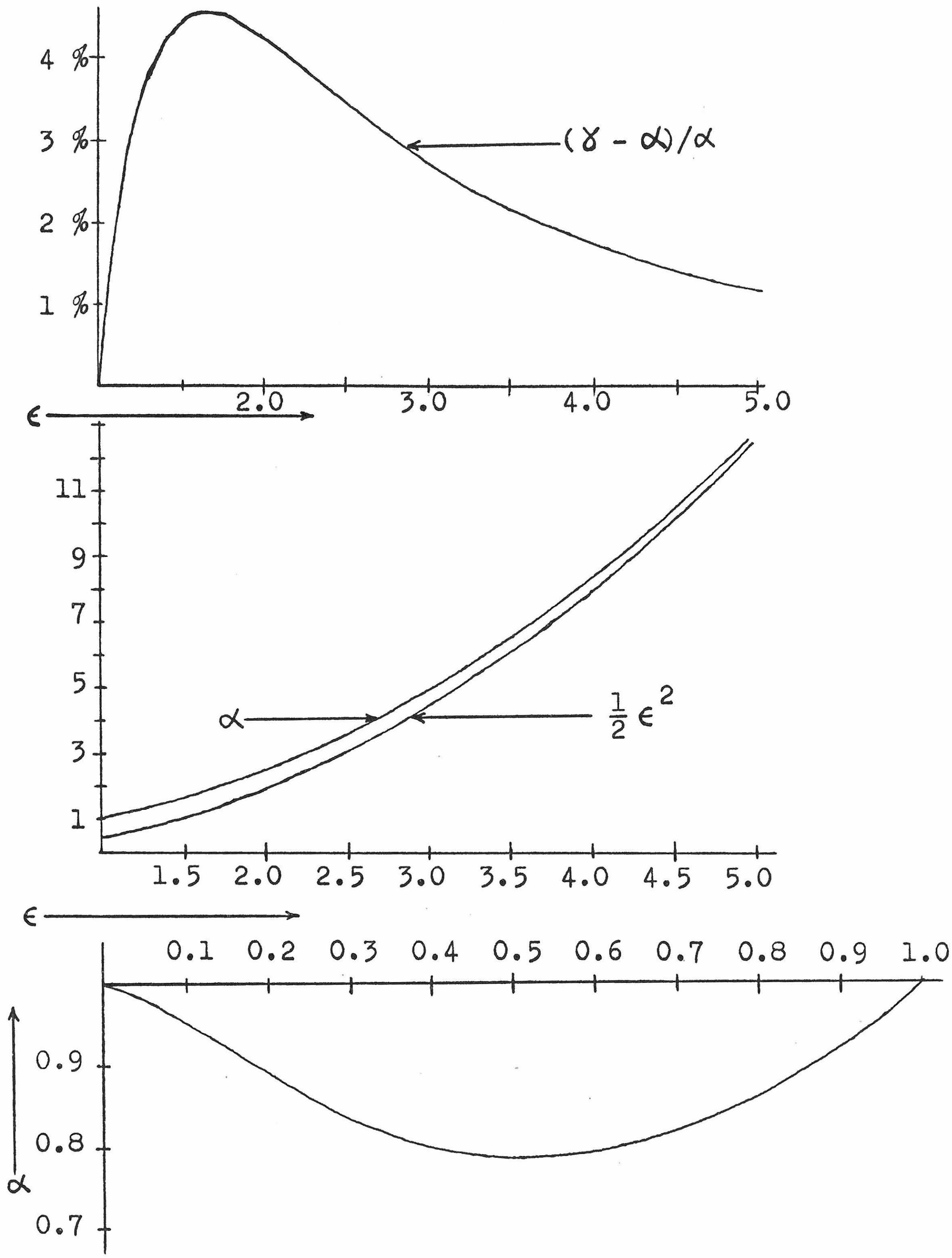


Figure 2

Plots of α and $(\gamma - \alpha)/\alpha$

$$L_\zeta = -ghm^+c^2\left\{\left[a+(1-a)\frac{\mu^2}{\zeta}+\frac{2(\gamma-a)\mu^2+2a\zeta}{m^+c^2 I\gamma}\right]W+\mu W_\mu\right.$$
$$\left.+\left[a\eta+(1-a)\frac{\mu^2\eta}{\zeta}\right]W_\eta+2\left[a\zeta+(1-a)\mu^2\right]W_\zeta\right\} \qquad (38)$$

5. Diffusion Coefficients Due to the Non-Zero Internal Temperature of the Grain

If the grain's internal temperature, T_i, is non-zero, then random changes are generated in the variables μ, η, and ζ. This source of random effects is independent both of the DG alignment process and of the grain-atom collisions. We will now discuss the effects of T_i and then derive the corresponding diffusion coefficients for the Fokker-Planck equation.

(a) Quantitative Effects of the Internal Temperature

(i) The T_i Parameter and Its Effects

Let us first consider the quantity T_i itself. There are several processes which may heat the grain and affect T_i, including: collisions with gas atoms, the dissipative torque of DG, bombardment by low energy cosmic rays, and energy absorption from the interstellar radiation field. According to Greenberg,[26] the mechanism involving the interstellar radiation field dominates in fixing T_i. Since this process is largely independent of others affecting the grain, we may treat the quantity T_i as a free parameter in our calculations.

Next, consider the effects of T_i. If T_i is non-zero, then the grain's magnetization, $\underset{\sim}{M}$, fluctuates--as Kittel[10] first noted. We assume that these fluctuations, $\Delta \underset{\sim}{M}$, occur randomly, both in magnitude and direction. In particular, $\Delta \underset{\sim}{M}_p$, the fluctuating component normal to the field, generates a torque $V(\Delta \underset{\sim}{M}_p \times \underset{\sim}{B})$; this torque produces fluctuations $\delta \underset{\sim}{J}$ in the angular momentum $\underset{\sim}{J}$.

We now assume - as in Section (4) - that the fluctuations $\delta \underset{\sim}{J}$, due to T_i, are the δx_i terms to be used in equations (10) and (11). We will next find which components of $\delta \underset{\sim}{J}$ have non-zero average values, showing which diffusion coefficients are important. To find these expectation values, we consider a simple problem for which we know the probability distribution of grain orientations. We then write the Fokker-Planck equation for this problem: by working backwards from our known solution, we obtain the diffusion coefficients. In part (c), we transform these quantities to the μ, η, ζ coordinate system.

(ii) The Non-Zero Moments of $\delta \underset{\sim}{J}$

To find $\delta \underset{\sim}{J}$, let us suppose that X and Y are two orthogonal and equivalent directions in space normal to $\underset{\sim}{B}$. Let $(\Delta M)_X$ and $(\Delta M)_Y$ be the components of $\Delta \underset{\sim}{M}$ along X and Y, so that these two components represent $\Delta \underset{\sim}{M}_p$. Since $\Delta \underset{\sim}{M}_p$ is assumed random, the symmetry of the situation requires that the average values are

$$\langle(\Delta M)_X\rangle = \langle(\Delta M)_Y\rangle = 0 \quad . \tag{39}$$

We also assume that $(\Delta M)_X$ is uncorrelated with $(\Delta M)_Y$, so that

$$\langle(\Delta M)_X \, (\Delta M)_Y\rangle = 0 \quad . \tag{40}$$

Therefore, of the averages which determine the Fokker-Planck coefficients, the only non-zero averages we expect to find are those for

$\langle(\Delta M)_X{}^2\rangle$ and $\langle(\Delta M)_Y{}^2\rangle$. Since the fluctuations, $\delta \underset{\sim}{J}$, are proportional to the torque, $V(\Delta \underset{\sim}{M}_p \times \underset{\sim}{B})$, we find

$$(\delta J)_X \propto (\Delta M)_Y \quad , \tag{41}$$

$$(\delta J)_Y \propto (\Delta M)_X \quad . \tag{42}$$

Since $\langle(\Delta M)_X{}^2\rangle$ and $\langle(\Delta M)_Y{}^2\rangle$ are the only relevant non-zero average values of $\Delta \underset{\sim}{M}$, $\langle(\delta J)_X{}^2\rangle$ and $\langle(\delta J)_Y{}^2\rangle$ are the only non-zero average values of $\delta \underset{\sim}{J}$. In addition, these two quantities are equal because X and Y are equivalent directions in space. Thus, our problem of finding $\delta \underset{\sim}{J}$ is reduced to obtaining $\langle(\delta J)_X{}^2\rangle$ or $\langle(\delta J)_Y{}^2\rangle$.

(iii) A Simplified Physical Situation

Let the grain be set spinning and assume that the gas is removed. The only dominant processes left to work are the steady torque of DG and the thermal fluctuations. In addition, suppose that the particle is constrained to rotate about an axis along X, so that only non-zero component of $\underset{\sim}{J}$ is J_X. We may write a Fokker-Planck equation for this simplified system in which $\underset{\sim}{J}$ becomes the variable of interest, x, in equations (9) through (12). For the component J_X, equation (9) becomes

$$(\Delta J)_X = K_X \, \Delta t + (\delta J)_X \quad . \tag{43}$$

In this equation Δt is a time interval during which many fluctuations $(\delta J)_X$ occur; K_X represents the effect of the steady torque of Davis and Greenstein; and $(\Delta J)_X$ is the total change in J_X.

From the discussion following equation (9), we may put the Fokker-Planck equation into the form

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial J_X}(WE_X) + \frac{1}{2}\frac{\partial^2}{\partial J_X^2}(WE_{XX}) \quad . \tag{44}$$

In this equation, E_X is given by equation (10) as

$$E_X = \lim_{\Delta t \to 0} \langle (\Delta J)_X \rangle / \Delta t \quad , \tag{45}$$

and since $\langle (\delta J)_X \rangle = 0$ for this case, we find

$$E_X = K_X \quad . \tag{46}$$

From DG,[30] K_X for a spheroidal grain is given by

$$E_X = K_X = -\frac{\chi''}{\omega} VB^2 \frac{J_X}{I}\left(\frac{\gamma \cos^2\theta + \sin^2\theta}{\gamma}\right) \quad , \tag{47}$$

$$E_X = -\frac{\chi''}{\omega} VB^2 \frac{J_X}{I_X} \quad , \tag{48}$$

where

$$I_X = I\left(\frac{\gamma}{\gamma\cos^2\theta + \sin^2\theta}\right) \quad . \tag{49}$$

In addition, E_{XX} is given by equation (11) as

$$E_{XX} = \lim_{\Delta t \to 0} \langle (\delta J)_X^2 \rangle / \Delta t \quad , \tag{50}$$

so that our problem of finding $\langle (\delta J)_X^2 \rangle$ reduces to obtaining E_{XX}.

(iv) Determination of $\langle (\delta J)_X^2 \rangle$.

Next, consider the situation at equilibrium. We have

$$\frac{\partial W}{\partial t} = 0 \quad , \tag{51}$$

and W must be the Maxwell-Boltzmann distribution for the temperature, T_i, describing the system. Since the total energy is J_x^2/I_x, the Boltzmann distribution law gives, from DG,[31]

$$W_i = \text{const} . \exp\left[-\frac{J_X^2}{2kT_i I_X}\right] . \tag{52}$$

Let us set

$$E_{XX} = E_o , \tag{53}$$

substitute equations (48), (51), and (52) into (44), and solve for E_o. The result is

$$W_i E_o = C_1 J_X + C_2 + 2 \int W_i E_X \, dJ_X , \tag{54}$$

where C_1 and C_2 are constants of integration. In order that E_o be well behaved for large J_X, for which W_i is small, we set $C_1 = 0$. We may simplify equation (54) further by deducing from equation (52) that

$$W_i E_X = \frac{\chi''}{\omega} VB^2 kT_i \frac{\partial W_i}{\partial J_X} , \tag{55}$$

and integrating equation (54) by parts. The result is

$$E_o = 2kT_i\left(\frac{\chi''}{\omega}\right)VB^2 + W_i^{-1}\left[C_2 - 2kT_i VB^2 \int W_i \frac{\partial}{\partial J_X}\left(\frac{\chi''}{\omega}\right) dJ_X\right] . \tag{56}$$

If we define ω_X by the relation

$$J_X = I_X \, \omega_X , \tag{57}$$

where I_X is given by equation (49), then we obtain

$$E_o = 2kT_i\left(\frac{\chi''}{\omega}\right)VB^2 + W_i^{-1}\left[C_2 - 2kT_iVB^2\int W_i\frac{\partial}{\partial\omega_X}\left(\frac{\chi''}{\omega}\right)d\omega_X\right] \qquad (58)$$

In equation (58) we take C_2 to be a constant of integration, and we define $(\partial/\partial\omega_X)(\chi''/\omega)$ for negative ω_X so as to give the correct symmetry properties to the indefinite integral. Equation (58) is the value of E_o in the case that (χ''/ω) is allowed to depend on ω_X. For this case E_o depends on ω through the first and second terms; it varies with the grain shape through the W_i factor in the second term. We will not treat the problem of a more complicated dependence of (χ''/ω) on ω.

For our cases of interest, (χ''/ω) is independent of ω, as DG[24] and Purcell[25] noted, so that the second term in equation (58) vanishes. The result now is

$$E_o = 2kT_i(\chi''/\omega)VB^2 \quad , \qquad (59)$$

which agrees with that of Jones and Spitzer for a spherical grain.[32] However, this equation is true for a grain of any shape, and not only for a sphere. Since J_X is absent, we conclude that this value of E_o is correct even when the constraint on rotation about the X direction is removed.

(b) Qualitative Effects of the Internal Temperature

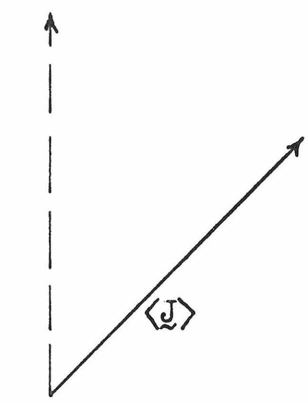
The orientation of $\langle \underset{\sim}{J} \rangle$ with respect to $\underset{\sim}{B}$ is pictured schematically in Figure 3. Let $\langle J_X \rangle$ be defined as in the last section, while $\langle J_B \rangle$ is the average value of the component parallel to $\underset{\sim}{B}$. If $\underset{\sim}{B}$ is zero, then only the grain-atom collisions affect the orientation. These collisions yield an isotropic distribution in space, so that there is no net alignment. In Figure 3(a), we represent this situation schematically by the vector shown. Suppose that T_i is zero and $\underset{\sim}{B}$ is not; then, in addition to the gas collisions, the Davis-Greenstein mechanism is active, removing energy from the rotational modes normal to $\underset{\sim}{B}$. The rate of this energy loss is, from DG,[33]

$$\frac{dR_o}{dt} = -V(\chi''/\omega)(\underset{\sim}{\omega} \times \underset{\sim}{B})^2 \quad , \tag{60}$$

where R_o is the rotational energy of the grain. Thus, $\langle J_X \rangle$ decreases, while $\langle J_B \rangle$ does not, and $\langle \underset{\sim}{J} \rangle$ is aligned toward $\underset{\sim}{B}$.

Since the thermal fluctuations produce a mean square contribution, $\langle (\delta J)_X^2 \rangle$, angular momentum is sent into the rotational modes normal to $\underset{\sim}{B}$. If the gas temperature, T, is equal to the grain temperature, T_i, the system of grains plus gas molecules must be in thermodynamic equilibrium--yielding no net orientation. Thus, when $T_i = T$, the DG alignment process is balanced by the thermal fluctuations. Because the orientation is known when $T_i = 0$, we may conclude that the DG process orients $\langle \underset{\sim}{J} \rangle$ toward $\underset{\sim}{B}$ so long as $T_i < T$. The degree of alignment decreases as T_i approaches T.

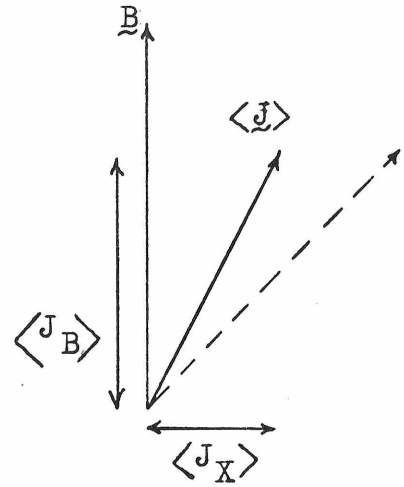
For $T_i > T$ the fluctuations continue to send angular momentum into the rotational modes normal to $\underset{\sim}{B}$. Thus, $\langle \underset{\sim}{J} \rangle$ is aligned away from $\underset{\sim}{B}$, and this tendency becomes more pronounced as the grain gets hotter. The behavior of $\underset{\sim}{J}$ fixes that of the symmetry axis $\underset{\sim}{A}$, since DG[23] state that the long axis of the grain tends to become perpendicular to $\underset{\sim}{J}$. These qualitative features will be shown in more detail in the next chapter when we solve the Fokker-Planck equation.



(a)

$B = 0$

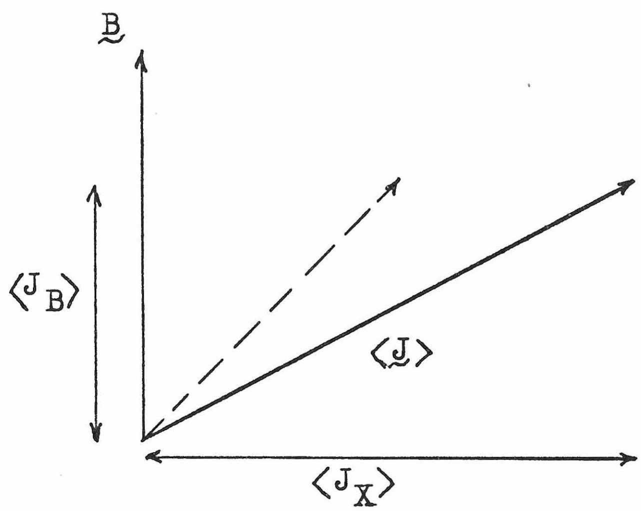
No net alignment



(b)

$T_i < T$, $\langle J_X \rangle < \langle J_B \rangle$

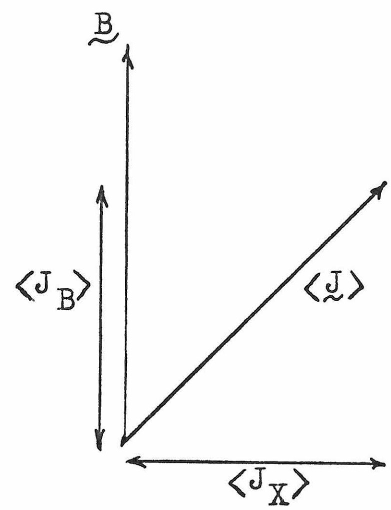
Alignment by Davis-Greenstein process



(c)

$T_i > T$, $\langle J_X \rangle > \langle J_B \rangle$

Orientation of $\underset{\sim}{J}$ away from $\underset{\sim}{B}$



(d)

$T_i = T$, $\langle J_X \rangle = \langle J_B \rangle$

No alignment

Figure 3

Alignment of $\underset{\sim}{J}$

(c) The Diffusion Coefficients

From the discussion in part (a) of this chapter, the only non-zero moments due to the fluctuations are

$$E_{XX} = E_{YY} = E_o = 2kT_i(\chi''/\omega)VB^2 \quad . \tag{61}$$

We need only change these moments to Miller's μ, η, ζ coordinate system. Let the grain be oriented as in Figure 4 with the field $\underset{\sim}{B}$ along the Z_1 axis and the plane of $\underset{\sim}{J}$ and $\underset{\sim}{B}$ the Y-Z_1 plane. If $\hat{B}$ is a unit vector along $\underset{\sim}{B}$, and $\hat{A}$ a unit vector along $\underset{\sim}{A}$, then

$$\hat{B}_X = \hat{B}_Y = 0 \quad ,$$

$$\hat{B}_{Z_1} = 1 \quad , \tag{62}$$

$$J_X = 0 \quad ,$$

$$J_Y = -J\sin\beta, \quad J_{Z_1} = J\cos\beta = \eta \quad , \tag{63}$$

$$\hat{A}_X = \sin\Psi \sin\theta$$

$$\hat{A}_Y = \cos\Psi \sin\theta \cos\beta - \cos\theta \sin\beta$$

$$\hat{A}_{Z_1} = \cos\Psi \sin\theta \sin\beta + \cos\theta \cos\beta \quad . \tag{64}$$

Since

$$\mu = \underset{\sim}{J}.\hat{A}$$

$$\eta = \underset{\sim}{J}.\hat{B}$$

$$\zeta = \underset{\sim}{J}.\underset{\sim}{J} \quad , \tag{65}$$

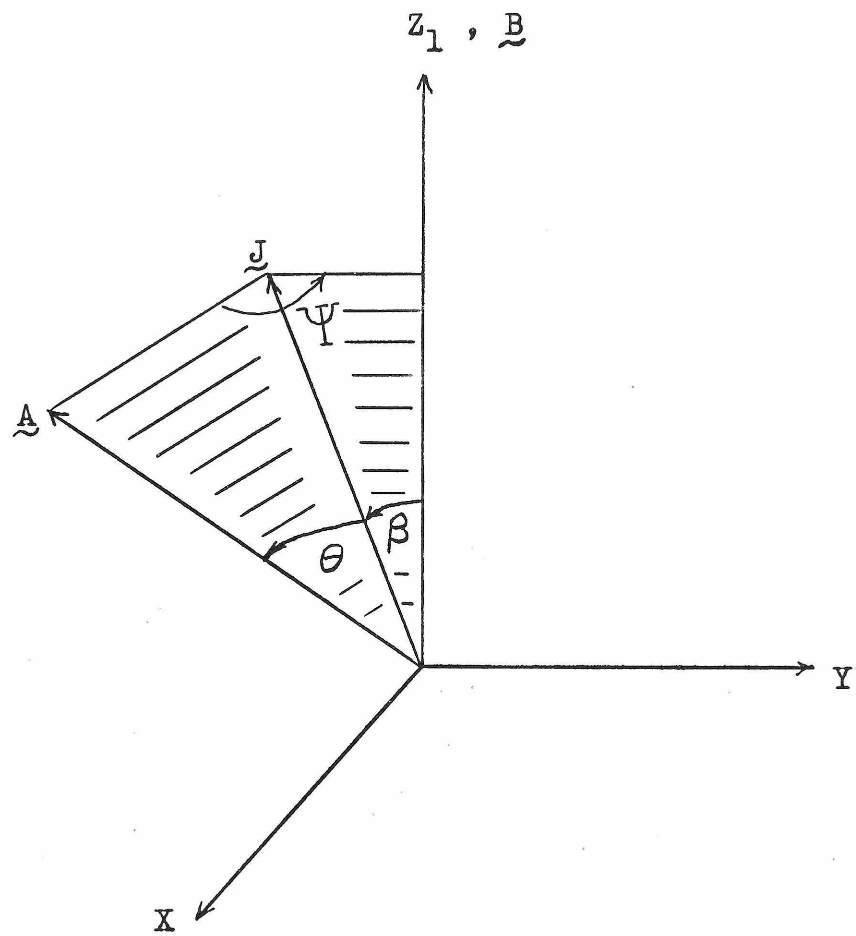


Figure 4

Orientation of the X, Y, Z_1 Coordinate System

$$\delta\mu = \hat{A}_X \, \delta J_X + \hat{A}_Y \, \delta J_Y + \hat{A}_{Z_1} \, \delta J_{Z_1}$$

$$\delta\eta = \hat{B}_X \, \delta J_X + \hat{B}_Y \, \delta J_Y + \hat{B}_{Z_1} \, \delta J_{Z_1}$$

$$\begin{aligned} \delta\zeta &= (\underset{\sim}{J} + \delta\underset{\sim}{J})^2 - \underset{\sim}{J}^2 \\ &= 2(J_X \, \delta J_X + J_Y \, \delta J_Y + J_{Z_1} \, \delta J_{Z_1}) \\ &\quad + (\delta J)_X^2 + (\delta J)_Y^2 + (\delta J)_{Z_1}^2 \quad , \end{aligned} \tag{66}$$

where all second order terms have been kept. Thus, we find

$$\begin{aligned} \delta\mu &= \sin\Psi \sin\theta \, \delta J_X + (\cos\Psi \sin\theta \cos\beta - \cos\theta \sin\beta) \delta J_Y \\ &\quad + (\cos\Psi \sin\theta \sin\beta + \cos\theta \cos\beta) \delta J_{Z_1} \\ \delta\eta &= \delta J_{Z_1} \\ \delta\zeta &= 2(-J \sin\beta \, \delta J_Y + J \cos\beta \, \delta J_{Z_1}) + (\delta J)_X^2 \\ &\quad + (\delta J)_Y^2 + (\delta J)_{Z_1}^2 \quad . \end{aligned} \tag{67}$$

If we take the expectation values of these quantities, we find that the only non-zero moments are

$$\begin{aligned} E_\zeta &= 2E_o \\ E_{\mu\mu} &= \left(\frac{1}{2} + \frac{1}{2}\frac{\mu^2}{\zeta} + \frac{1}{2}\frac{\eta^2}{\zeta} - \frac{3}{2}\frac{\mu^2 \eta^2}{\zeta^2}\right) E_o \\ E_{\zeta\zeta} &= 4\zeta \left(1 - \frac{\eta^2}{\zeta}\right) E_o \\ E_{\mu\zeta} &= 2\mu \left(1 - \frac{\eta^2}{\zeta}\right) E_o \quad . \end{aligned} \tag{68}$$

Thus, the extra terms due to the grain's temperature are

$$R^{(T)} = E_o\Big\{W_{\mu\mu}\left(\frac{1}{4} + \frac{1}{4}\frac{\mu^2}{\zeta} + \frac{1}{4}\frac{\eta^2}{\zeta} - \frac{3}{4}\frac{\mu^2\eta^2}{\zeta^2}\right)$$

$$+ W_\mu\left(\frac{\mu}{\zeta} - \frac{\mu\eta^2}{\zeta^2}\right) + 2(\zeta - \eta^2)W_{\zeta\zeta} + \left(4-2\frac{\eta^2}{\zeta}\right)W_\zeta$$

$$+ 2\mu\left(1 - \frac{\eta^2}{\zeta}\right)W_{\mu\zeta} + W\left(\frac{1}{2\zeta} + \frac{1}{2}\frac{\eta^2}{\zeta^2}\right)\Big\} \tag{69}$$

6. Final Form of the Alignment Equation

The Fokker-Planck equation for the alignment process now takes the form

$$\frac{\partial W}{\partial t} = R^{(DG)} + R^{(c)} + R^{(T)} \quad , \tag{70}$$

where the terms in R are given by equations (32), (34), (36)-(38), and (69). We may check the algebra to this point by setting $T_i = T$ and substituting

$$W = \text{const.} \times \frac{1}{\sqrt{\zeta}} \exp\{-[\zeta + (\gamma-1)\mu^2]/(2I\gamma k T)\} \tag{70a}$$

into equation (70). Since this W is the Maxwell-Boltzmann solution of DG[31] in the μ, η, ζ system, we do find that $\frac{\partial W}{\partial t} = 0$, as expected.

It has proven more convenient to solve this equation using a different set of variables from Miller's μ, η, ζ set. We therefore introduce the variables

$$r = \cos\beta = \frac{\eta}{\sqrt{\zeta}} \, , \quad -1 \leqq r \leqq 1 \quad , \tag{71a}$$

$$s = (J\cos\theta)/\sqrt{m^+c^2I\gamma} \, , \quad -\infty < s < \infty \, , \tag{b}$$

$$= \mu/\sqrt{m^+c^2I\gamma} \quad ,$$

$$z = (J^2 \sin^2\theta)/(m^+ c^2 I\gamma) = \frac{\zeta - \mu^2}{m^+ c^2 I\gamma}, \quad 0 \leqq z < \infty \quad , \tag{c}$$

$$\tau = (z + s^2)^{1/2} = \frac{\sqrt{\zeta}}{\sqrt{m^+ c^2 I\gamma}} , \quad 0 \leqq \tau < \infty \quad , \tag{d}$$

$$W = \exp[-(z + \gamma s^2)] \, f(r, s, z) \quad , \tag{e}$$

$$b = \frac{DB^2(m^+ c^2 I\gamma)}{ghm^+ c^2} = \frac{\chi''}{\omega} \frac{VB^2}{gh} \quad , \tag{f}$$

$$\epsilon_o = E_o/(ghm^+ c^2) \quad , \tag{g}$$

$$\frac{\epsilon_o}{b} = T_i/(\frac{m^+}{m})T \quad . \tag{71h}$$

The r, s, z coordinate system is useful for the case when the magnetic field is weak, and we regard τ as a dependent variable. The parameter b compares the effects of the magnetic field with those of the gas collisions. The parameter ϵ_o compares the effects of the temperature fluctuations with those of the gas collisions. In defining the function f, we have factored from W the Maxwell-Boltzmann solution of Davis and Greenstein.[31] This solution is valid for the case in which $B=0=E_o$, or $b = 0 = \epsilon_o$. If we take equation (70), set $(\partial W/\partial t)=0$ for the steady-state solution, make the above changes of variable, and let a subscript on f mean a partial derivative with respect to that variable, we obtain

$$\left[\frac{(a+1)z + 2as^2}{4\tau^4}\right] \cdot \left[(1-r^2)f_{rr} - 2r\,f_r\right] + \frac{1}{2} f_{ss} - \gamma s f_s$$

$$+ 2a\,z\,f_{zz} + 2a(1-z)f_z$$

$$+ \frac{b}{\tau^2}\Big\{\big[\tfrac{1}{2}(\gamma+3)z - 2z^2(1-r^2) + (\gamma+1)s^2 - 2\gamma^2 s^4(1-r^2)$$

$$-(\gamma-1)r^2 s^2 + \tfrac{1}{2}(\gamma-1)r^2 z - (\gamma+1)^2 s^2 z - (\gamma^2-6\gamma+1)r^2 s^2 z\big]f$$

$$-r(1-r^2)(z+\gamma s^2)f_r + \tfrac{1}{2} s f_s\big[2\gamma(1-r^2)s^2 + \{\gamma(1+r^2)+(1-3r^2\}z\big]$$

$$+ z f_z\big[\gamma s^2(1-3r^2)+s^2(1+r^2)+2z(1-r^2)\big]\Big\}$$

$$+\epsilon_o \Big\{\big[\tfrac{1}{4}(1+r^2) + \tfrac{1}{4}\frac{s^2}{\tau^2}(1-3r^2)\big]f_{ss} + \big[-\frac{s^2 z}{\tau^2}(1-3r^2) + 2z(1-r^2)\big]f_{zz}$$

$$+ \frac{r^2}{2\tau^2}(1-r^2)f_{rr} - 2\frac{rz}{\tau^2}(1-r^2)f_{rz} - \frac{rs}{\tau^2}(1-r^2)f_{rs}$$

$$+ \frac{sz}{\tau^2}(1-3r^2)f_{sz} + \big[-\frac{1}{\tau^2}(1-3r^2)(z+\gamma s^2) - \gamma(1+r^2)\big]sf_s$$

$$+\big[\tfrac{3}{2} - \tfrac{1}{2}r^2 - \tfrac{1}{2}\frac{s^2}{\tau^2}(1-3r^2) - 4z(1-r^2) - 2(\gamma-1)(1-3r^2)\frac{s^2 z}{\tau^2}\big]f_z$$

$$+\big[\tfrac{1}{2}\frac{r}{\tau^2}(1-3r^2) + 2(\gamma-1)\frac{rs^2}{\tau^2}(1-r^2) + 2r(1-r^2)\big]f_r$$

$$+\frac{1}{\tau^2}\big[-\tfrac{1}{2}(\gamma+3)z + 2z^2(1-r^2) - (\gamma+1)s^2 + 2\gamma^2 s^4(1-r^2)$$

$$+(\gamma-1)r^2 s^2 - \tfrac{1}{2}(\gamma-1)r^2 z + (\gamma+1)^2 s^2 z + (\gamma^2-6\gamma+1)r^2 s^2 z\big]f\Big\} = 0 \;. \tag{72}$$

By direct substitution we see that $f=1$, which is the Maxwell-Boltzmann solution, satisfies this equation for the cases $b = 0 = \epsilon_o$ and for $b = \epsilon_o$ (or $T_i = \frac{m^+}{m} T$). We shall solve this equation in the next chapter for other values of the parameters.

In the coordinate system

$$
\begin{aligned}
\tau &= (z+s^2)^{\frac{1}{2}} = (\sigma^2+\nu^2)^{\frac{1}{2}} = J/\sqrt{m^+c^2 I\gamma}\ , && 0 \le \tau < \infty \\
\sigma &= \tau r = \tau \cos\beta\ , && -\infty < \sigma < \infty \\
\nu &= \tau(1-r^2)^{\frac{1}{2}} = \tau \sin\beta\ , && -\infty < \nu < \infty \\
\rho &= \frac{s}{\tau} = \cos\theta\ , && -1 \le \rho \le 1\ . \qquad (74)
\end{aligned}
$$

equation (72) becomes

$$
\begin{aligned}
&\left\{\frac{1}{2} + \frac{1}{2}(\alpha-1)[(1-\rho^2) - \frac{1}{2}\frac{\nu^2}{\tau^2}(1-3\rho^2)]\right\} f_{\sigma\sigma} - [1+(\alpha-1)-(\alpha-\gamma)\rho^2]\sigma f_\sigma \\
&+\left\{\frac{1}{2}(1+\epsilon_o) + \frac{1}{4}(\alpha-1)[(1+\rho^2) + \frac{\nu^2}{\tau^2}(1-3\rho^2)]\right\} f_{\nu\nu} \\
&+ \{\frac{1}{2}(1+\epsilon_o) + \frac{1}{4}(\alpha-1)(1+\rho^2)\}\frac{1}{\nu} f_\nu \\
&+ \{(b-2\epsilon_o-1) + (\gamma-1)[(b-2\epsilon_o)\rho^2-1] - (\alpha-\gamma)(1-\rho^2)\}\nu f_\nu \\
&+ \frac{(1-\rho^2)}{2\tau^2}\left[1 + \epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2} + (\alpha-1)\rho^2\right] f_{\rho\rho} \\
&- \left\{1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2} - \frac{1}{2}(\alpha-1)(1-3\rho^2) - (b-2\epsilon_o)(\gamma-1).\tau^2(1-\rho^2)(1-\frac{1}{2}\frac{\nu^2}{\tau^2}) \right. \\
&\qquad \left. - (\alpha-\gamma).\tau^2(1-\rho^2)\right\}\frac{\rho}{\tau^2} f_\rho \\
&+ \frac{1}{2}(\alpha-1)\frac{\sigma\nu}{\tau^2}(1-3\rho^2) f_{\sigma\nu} - (\alpha-1)\frac{\sigma\rho}{\tau^2}(1-\rho^2) f_{\sigma\rho} - (\alpha-1)\frac{\nu\rho}{\tau^2}(1-\rho^2) f_{\nu\rho} \\
&+ (b-\epsilon_o)\left\{2(1-\nu^2) + (\gamma-1)\left[(1-\rho^2) - \frac{1}{2}\frac{\nu^2}{\tau^2}(1-3\rho^2) - 2(\gamma-1)\tau^2\rho^2(1-\rho^2)\right.\right. \\
&\qquad \left.\left. + (\gamma-1)\nu^2\rho^2(1-3\rho^2) - 4\nu^2\rho^2\right]\right\} f = 0\ .
\end{aligned}
$$

(72i)

CHAPTER III

SOLUTIONS OF THE ALIGNMENT EQUATION

In this chapter we will solve equation (72). The only exact solution to be obtained is for spherical grains in an arbitrary magnetic field. The equation will next be solved approximately for needles nearly-spherical grains, and disks in strong magnetic fields. Finally, the equation will be solved approximately for needles and nearly-spherical grains in weak fields.

For all of these cases, we seek a solution to equation (72) such that

$$W = \left\{\exp[-(z + \gamma s^2)]\right\} f(r, s, z) \tag{71}$$

is a well-behaved probability function. This means that W is everywhere well-behaved, finite, positive, integrable, and that W approaches zero for z, $s \to \infty$. We expect that W may behave as a δ-function for b or $\epsilon_o \to \infty$, but for finite b and ϵ_o, W should have no singularities. The normalization is chosen so that the integral of W over all of phase space is unity. This normalization will be found in Chapter IV, where we calculate the measures of alignment.

Finally, we note that it is possible to prove that the function W is unique. This means that if W satisfies the Fokker-Planck equation for $(\partial W/\partial t) = 0$, and if W is a well-behaved probability function as defined above, then W is unique. The theorem of A. H. Gray is stated and proved elsewhere.[39] This uniqueness property allows us to solve each case by whatever method is most convenient and to be sure that the solution obtained is correct. It eliminates the need to treat the general solutions of the various differential equations that will be considered; any solution which satisfies the previous requirements is the unique solution.

1. <u>The Sphere</u>

For the sphere, we find that

$$\alpha = \gamma = 1 \quad . \tag{73}$$

We also will use the variables

$$\begin{aligned}
\tau &= (z + s^2)^{1/2} = J/(m^+c^2 I\gamma)^{1/2} \ , & 0 < \tau < \infty \\
&= (\sigma^2 + \nu^2)^{1/2} \\
\sigma &= \tau r = \tau \cos\beta \ , & -\infty < \sigma < \infty \\
\nu &= \tau(1-r^2)^{1/2} = \tau \sin\beta \ , & -\infty < \nu < \infty \\
\rho &= \frac{s}{\tau} = \cos\theta & -1 \leqq \rho \leqq 1 \quad .
\end{aligned} \tag{74}$$

when σ, ν, ρ will be regarded as independent variables and τ as dependent. The τ variable is dimensionless and represents J in units of $\sqrt{m^+c^2 I\gamma}$, while σ and ν are the components of τ parallel and normal to B. This σ, ν, τ, ρ coordinate frame is convenient for the cases of the sphere in all fields and of the other shapes in strong field. If the above variables are substituted into equation (72i), the result is

$$\frac{1}{2} f_{\sigma\sigma} - \sigma f_\sigma + \frac{1}{2} f_{\nu\nu} + [\frac{1}{2\nu} + (b-1)\nu]f_\nu + (2b-2b\nu^2)f$$

$$+ \frac{1}{2\tau^2}[(1-\rho^2)f_{\rho\rho} - 2\rho f_\rho] + \epsilon_0\left\{\frac{1}{2} f_{\nu\nu} + (\frac{1}{2\nu} - 2\nu)f_\nu\right.$$

$$\left. + \frac{1}{4\tau^2}(1 + \frac{\sigma^2}{\tau^2})[(1-\rho^2)f_{\rho\rho} - 2\rho f_\rho] + (-2+2\nu^2)f\right\} = 0 . \qquad (75)$$

In this equation, f is defined by

$$f = W_{MB}^{-1} W = [\exp(\sigma^2 + \nu^2)] W \quad , \qquad (76)$$

where W_{MB} is the Maxwell-Boltzmann solution, and a subscript on f means a partial derivative with respect to that variable.

We note that the sphere has no dynamically defined symmetry axis. Thus, we may take the symmetry axis to be an arbitrary marking located anywhere on the sphere. All values of $\rho = \cos\theta$ are now expected to be equally probable, so that

$$f_\rho = 0 \quad . \qquad (77)$$

In addition, W_{MB} describes the distribution if the gas acts alone. Now only the gas affects σ - the component of $\underset{\sim}{J}$ parallel to $\underset{\sim}{B}$. Therefore, we expect that W_{MB} gives the complete distribution of σ, which would not occur in f. Thus, we test the assumption that

$$f_\sigma = 0 \quad . \qquad (78)$$

With these assumptions, our equation for f becomes

$$\frac{1}{2}(1 + \epsilon_o)f_{\nu\nu} + \frac{1}{2}(1 + \epsilon_o)\frac{1}{\nu} f_\nu + (b - 2\epsilon_o - 1)\nu f_\nu$$

$$+ 2(b - \epsilon_o)(1 - \nu^2)f = 0 \quad . \qquad (79)$$

We now find a well-behaved solution consistent with these assumptions. It is

$$f = \exp\left\{[-(b-\epsilon_o)/(1+\epsilon_o)]\,\nu^2\right\} \quad . \tag{80}$$

The unnormalized probability is, therefore,

$$W = \exp\left\{-(\sigma^2 + \nu^2) - \frac{(b - \epsilon_o)}{(1 + \epsilon_o)}\,\nu^2\right\} \quad ,$$

$$= \exp\left\{-\sigma^2 - \nu^2\left(\frac{1 + b}{1+\epsilon_o}\right)\right\} \tag{81}$$

For the case that $b >> 1$ and $b >> \epsilon_o$ (or $T >> T_i$), W approaches a δ-function in ν. This shows that for a strong field and a cold grain, $\underset{\sim}{J}$ is aligned toward $\underset{\sim}{B}$, since $\nu \to 0$ implies $\beta \to 0$. As T_i approaches T, $\epsilon_o \to b$, and the alignment in ν decreases. For $T_i = T$ (or $\epsilon_o = b$), $W = W_{MB}$, and there is no alignment in ν regardless of how strong is the magnetic field. For $T_i > T$ (or $\epsilon_o > b$), we find that the orientation reverses in ν.

If all the constants are put in, then

$$\sigma^2 = \frac{J^2\cos^2\beta}{m^+(2kT/m)I} = \frac{J^2\cos^2\beta}{2Ik(m^+T/m)}$$

$$\nu^2 = \left(\frac{1 + b}{1 + \epsilon_o}\right) = \frac{J^2\sin^2\beta}{2Ik(m^+T/m)}\left(\frac{1 + b}{1 + \epsilon_o}\right)$$

$$= \frac{J^2\sin^2\beta}{2Ik}\;\frac{1}{(m^+T/m)}\left[\frac{1 + b}{1 + b\left(\frac{T_i}{T}\right)\left(\frac{m}{m^+}\right)}\right] \tag{82}$$

The distribution in σ is Maxwellian at a temperature T_{eff} given by

$$T_{eff} = \left(\frac{m^+}{m}\right)T \quad . \tag{83}$$

The (m^+/m) factor measures the energy lost in heating the grain because the grain-atom collisions are inelastic. Thus, some of the energy transferred by the collisions does not go into the grain's rotational modes. From the discussion of the grain-atom collision given in the appendix, we have

$$\left(\frac{m^+}{m}\right) = \frac{1}{2}\left(1+\sqrt{\frac{T_i}{T}}\right) \quad . \tag{84}$$

When $T_i = T$, $m^+ = m$, and the effective rotational temperature for σ is just T. When $T_i = 0$, $T_{eff} = \frac{1}{2}\,T$; for $T_i > T$, $T_{eff} > T$, so that the grain's rotational energy for σ is transferred to the gas.

The distribution in ν is Maxwellian at a temperature T_ν:

$$\begin{aligned}\left(\frac{m^+}{m}\right)T_{av} &= \left(\frac{m^+}{m}\right)T\left[\frac{1+b(T_i/T)(m/m^+)}{1+b}\right] = T_\nu \\ &= \frac{(m^+/m)T + bT_i}{1+b} \\ &= \frac{n_H\,(m^+/m)T + b_oT_i}{n_H + b_o} \quad ,\end{aligned}$$

$$b_o = n_H b \tag{85}$$

For $b = 0$, $T_\nu = (m^+/m)T = T_{eff}$; for $n_H = 0$, corresponding to removal of the gas, $T_\nu = T_i$, while T_{eff} is undetermined because T is undefined. Equation (85) is the same result as was found by

Jones and Spitzer,[34] except for the (m^+/m) factor. If we let $T \to 0$ with $n_H \neq 0$, this means that a low-temperature ideal gas surrounds the grain and absorbs its energy, making $T_{av} < T_i$.

2. The Strong-Field Case

(i) Prolate Grains

In this section we will solve equation (72) for prolate grains in strong magnetic fields, starting with nearly-spherical particles and then treating needles. The parameter b, defined in equation (71), is much greater than unity, while ϵ_o is regarded as a free parameter since it depends on the internal temperature, T_i. The ratio of the semiaxes, ϵ, is greater than unity: ϵ is slightly larger than unity for a nearly spherical grain and much greater than unity for the needle.

For a grain of uniform density, the ratio of the moments of inertia is

$$\gamma = \frac{1}{2}(1 + \epsilon^2) \quad . \tag{35}$$

Let us define an additional parameter

$$\delta = (\epsilon^2 - 1) > 0 \quad , \tag{86}$$

where $\delta << 1$ for nearly-spherical particles and $\delta >> 1$ for needles. We may obtain the shape factor a by expanding equation (35) in powers of δ and keeping the dominant terms. The results are

$$a \cong 1 + \frac{2}{5}\delta \;, \quad \delta << 1, \text{ nearly-spherical grain} \tag{87}$$

$$a \cong \frac{1}{2}\epsilon^2 + \frac{1}{3} \;, \quad \delta >> 1 \text{, needle} \quad . \tag{88}$$

Equation (88) suggests that we try

$$a \cong \gamma = 1 + \frac{1}{2}\delta = \frac{1}{2}\epsilon^2 + \frac{1}{2} \tag{89}$$

for all prolate values of ϵ. If we check this approximation, we find that it is less accurate than equation (87) for $\delta << 1$ and equation (88) for $\delta >> 1$. However, the relative deviation of γ from a is no greater than 5% for all values of $\epsilon \geq 1$, and neither of equations (87) and (88) has that accuracy over the whole range. Therefore, we will use equation (89) for all prolate grains. Figure 2 has a plot of $(\gamma-\alpha)/\alpha$.

Let us begin with the nearly-spherical case. We might expect the behavior of the angular momentum alignment to be similar to that of the sphere. If we assume $b >> 1$ and $b >> \epsilon_o$ (or $T >> T_i$) - that is, strong field and cold grain - then we find from equation (81) for the sphere that

$$\langle \nu \rangle \sim 1/\sqrt{b} \quad . \tag{90}$$

This means that W is small except where $\nu \leqslant 1/\sqrt{b}$. We will tentatively assume the same properties for the solution to the nearly-spherical case.

In addition, the Maxwell-Boltzmann solution is

$$W_{MB} = \exp[-(z + \gamma s^2)]$$

$$= \exp[-\sigma^2 - \nu^2 - (\gamma-1)\tau^2 \cos^2\theta] \quad . \tag{91}$$

For the case of the sphere, the function f is $\exp[-b\,\nu^2]$ when b is non-zero. We therefore look for a term $\exp[-b(\gamma-1)\tau^2 \cos^2\theta]$ in the function f for the nearly-spherical grain. Thus, define the scale changes

$$N = \sqrt{b}\;\nu = \sqrt{b}\;\tau\;\sin\beta \quad , \qquad -\infty < N < \infty$$

$$P = \sqrt{b(\gamma-1)}\;\cos\theta = \sqrt{\tfrac{1}{2}b\delta}\;\rho \; , \qquad -\sqrt{\tfrac{1}{2}b\delta} < P < \sqrt{\tfrac{1}{2}b\delta} \quad , \tag{92}$$

where $b\delta >> 1$. The coefficient $\sqrt{b}$ is a scale factor for ν , and the coefficient $\sqrt{\frac{1}{2}b\delta}$ is a scale factor for $\rho = \cos\theta$. We expect to find that

$$\langle\nu\rangle_{strong} \sim 1/\sqrt{b} \quad ,$$

$$\langle\rho\rangle_{strong} = \langle\cos\theta\rangle_{strong} \sim 1/\sqrt{b(\gamma-1)} \quad , \tag{93}$$

which are assumptions that must be justified by the solution.

We now turn to solving equation (72). Set $\alpha \cong \gamma$ and change variables to σ, ν, τ, and ρ as defined in equation (74). The result is a long expression, equation (72a), which is given in the appendix. We have no need for it here because we only desire those terms which dominate in strong fields. The equation contains derivatives of the function f with respect to ν and ρ, along with other variations.

If we change variables to N and P, then

$$f_\nu = \sqrt{b}\; f_N\,, \qquad f_{\nu\nu} = b\, f_{NN}$$

$$f_\rho = \sqrt{\tfrac{1}{2}b\delta}\; f_P\,, \quad f_{\rho\rho} = (\tfrac{1}{2}b\delta)f_{PP}\;. \tag{94}$$

Next, we substitute these derivatives into equation (72a), yielding terms of order b, bδ, δ, 1, 1/b, and smaller quantities. We choose the terms of order b and bδ (bδ >> 1) as the dominant ones and ignore the rest. Although δ << 1 for a nearly-spherical grain, the only assumption made here is that δ << b. This procedure allows us to treat the needle so long as b >> δ >> 1. The value of δ only determines which of b and bδ is larger. Thus, the dominant terms yield

$$b\left\{\frac{1}{2}(1+\frac{1}{4}\delta+\epsilon_o)f_{NN}+\frac{1}{2}(1+\frac{1}{4}\delta+\epsilon_o)\frac{1}{N}f_N+(1-2\frac{\epsilon_o}{b}-\frac{1}{b})N f_N + 2(1-\frac{\epsilon_o}{b})(1-\frac{N^2}{b})f\right\}$$
$$+b\delta\left\{\frac{1}{4\tau^2}(1+\epsilon_o)f_{PP}+\frac{1}{2}P(1-2\frac{\epsilon_o}{b})f_P+\frac{1}{2}(1-\frac{\epsilon_o}{b})f\right\}$$
$$+\text{ terms of order } \delta,\ 1,\ 1/b,\ \text{etc.} = 0\quad. \tag{95}$$

Although equation (95) is only accurate to order bδ and b, we have kept smaller terms for convenience. As δ → 0, we obtain equation (79) for the sphere. Since the variables in equation (95) have been separated, we may treat the two groups of terms in succession. The terms involving N yield

$$\frac{1}{2}(1+\frac{1}{4}\delta+\epsilon_o)f_{NN}+\frac{1}{2}(1+\frac{1}{4}\delta+\epsilon_o)\frac{1}{N}f_N+(1-2\frac{\epsilon_o}{b}-\frac{1}{b})N f_N + 2(1-\frac{\epsilon_o}{b})(1-\frac{N^2}{b})f = 0\quad. \tag{96}$$

An approximate solution

$$f(N) = \exp\left[-\left(\frac{1-\epsilon_o/b}{1+\epsilon_o+\frac{1}{4}\delta}\right)N^2\right] \tag{97}$$

when substituted into equation (96), leaves a residual term

$$\frac{1}{2}\,\frac{\delta}{b}\left(\frac{1-\epsilon_o/b}{1+\epsilon_o+\frac{1}{4}\delta}\right)N^2\, f \quad . \tag{97a}$$

Since this quantity is much smaller than the terms of interest, we may take equation (97) as giving the N-dependence of f.

The terms involving P yield

$$\frac{1}{4\tau^2}(1+\epsilon_o)f_{PP} + \frac{1}{2}\,P(1-2\,\frac{\epsilon_o}{b})f_P + \frac{1}{2}(1-\frac{\epsilon_o}{b})f = 0 \quad , \tag{98}$$

with the approximate solution

$$f(P) = \exp\left[-\,\frac{(1-\epsilon_o/b)}{1+\epsilon_o}\,\tau^2\,P^2\right] \tag{99}$$

Again if we substitute this back into equation (95), we find a residual term

$$\epsilon_o\delta\left(\frac{1-\epsilon_o/b}{1+\epsilon_o}\right)\tau^2\,P^2\,f \quad . \tag{100}$$

In the appendix we discuss this term and other residues in more detail, and we show that they are all smaller than the quantities of interest.

Thus, our solution for f is

$$\begin{aligned} f &= f(P).f(N) \\ &= \exp\left[-\,\frac{(1-\epsilon_o/b)}{1+\epsilon_o}\,\tau^2\,P^2 - \frac{(1-\epsilon_o/b)}{1+\epsilon_o+\frac{1}{4}\delta}\,N^2\right] \\ &= \exp\left[-\,\frac{(b-\epsilon_o)}{1+\epsilon_o}(\gamma-1)\tau^2\cos^2\theta - \frac{(b-\epsilon_o)}{1+\epsilon_o+\frac{1}{4}\delta}\,\tau^2\sin^2\beta\right] \; , \end{aligned} \tag{101}$$

which yields for the probability distribution

$$W = f \exp[-\tau^2 - (\gamma-1)\tau^2 \cos^2\theta]$$
$$= \exp\left\{-\tau^2 \cos^2\beta - \tau^2 \sin^2\beta \frac{\left(1+b+\frac{1}{4}\delta\right)}{\left(1+\epsilon_o+\frac{1}{4}\delta\right)} - (\gamma-1)\left(\frac{1+b}{1+\epsilon_o}\right)\tau^2\cos^2\theta\right\}. \tag{102}$$

Since the main approximation in our derivation is that $b \gg \delta$, this solution is also valid for a needle in an extremely strong field ($b \gg \delta \gg 1$). We will also use this result for the case $\delta \gg b \gg 1$, although the solution would not be rigorously correct. To solve this case would require that we keep all the terms of order δ in equation (72a), which would be difficult. We discuss this case of $\delta \gg b \gg 1$ further in the appendix.

We summarize below our detailed analysis of the accuracy of equation (101) given in the appendix:

$$\text{If} \quad \left\{\begin{matrix} \epsilon_o \ll b \quad (T_i \ll T) \\ \text{or} \\ b^2 \gg \epsilon_o \gg b \end{matrix}\right\} \text{ and } \{b \gg \delta\}, \quad \text{then equation (101)}$$

is numerically accurate to terms of order $\frac{1}{b}$.

On the other hand, if (i) $\epsilon_o \sim b$, (ii) $\epsilon_o \gg b^2$, or (iii) $\delta \gg b$, then the solution may be numerically inaccurate but is always qualitatively correct. The justifications for not finding more accurate solutions in these cases are the following: for cases (i) and (iii) the alignment is too small to justify further effort; and for case (ii) the grain is too hot to be of interest.

Finally, our solution for W does provide the alignment to be expected. As $b \to \infty$ and for $\varepsilon_o \ll b$ ($T_i \ll T$), both $\langle \sin\beta \rangle$ and $\langle \cos\theta \rangle$ become small, so that $\langle \beta \rangle \to 0$ and $\langle \theta \rangle \to \frac{\pi}{2}$. For a warm grain, $\varepsilon_o \gg b$ ($T_i \gg T$), so that $\langle \tau \sin\beta \rangle$ becomes large and $\langle \beta \rangle \to \frac{\pi}{2}$, reversing the orientation.

(ii) Oblate Nearly-Spherical Grains

For these particles the shape factors take the values

$$\varepsilon < 1 , \quad \delta = (\varepsilon^2 - 1) < 0, \quad \alpha \cong 1 + \frac{2}{5}\delta , \quad \gamma = 1 + \frac{1}{2}\delta . \tag{103}$$

We put

$$\delta_1 = -\delta > 0 \quad , \qquad \delta_1 \ll 1 \quad , \tag{104}$$

and proceed to solve equation (72). Since the symmetry axis is expected to align toward $\underset{\sim}{B}$, for large fields $\sin\theta$, rather than $\cos\theta$, becomes small. Thus we use

$$\lambda = \sin\theta = (1-\rho^2)^{\frac{1}{2}} \tag{105}$$

as our angular variable in place of $\rho = \cos\theta$. If equation (72i) is transformed to the σ, ν, τ, λ system, the result is a long expression, equation (72b), given in the appendix. In order to find the dominant terms in strong field, we use the variable

$$q = \sqrt{b\delta_1} \sin\theta = \sqrt{b\delta_1}\, \lambda \quad , \tag{106}$$

which is the analogue of P for the prolate case; in this equation b is large enough so that $b\delta_1 \gg 1$.

If σ, N, τ, and q are used in equation (72b), there results a set of terms of magnitude b, $b\delta_1$, 1, δ_1, and so forth. The equation becomes

$$b\left\{\frac{1}{2}(1+\epsilon_o-\frac{2}{5}\delta_1)f_{NN}+\frac{1}{2}(1+\epsilon_o-\frac{2}{5}\delta_1)\frac{1}{N}f_N\right.$$

$$+\left[(1-2\frac{\epsilon_o}{b})(1-\frac{1}{2}\delta_1)-\frac{1}{b}\right]Nf_N+2\left.\left[(1-\frac{\epsilon_o}{b})(1-\frac{1}{2}\delta_1)-\frac{N^2}{b}(1-\frac{\epsilon_o}{b})f\right]\right\}$$

$$+b\delta_1\left\{\frac{1}{2\tau^2}(1+\epsilon_o)f_{qq}+\frac{1}{2\tau^2}(1+\epsilon_o)\frac{1}{q}f_q+\frac{1}{2}(1-2\frac{\epsilon_o}{b})q\,f_q+(1-\frac{\epsilon_o}{b})f\right\}$$

$$+\text{ terms of order } 1,\ \delta_1,\ \frac{1}{b},\ \text{etc.} = 0\ . \tag{107}$$

As $\delta_1 \to 0$, the second group of terms vanishes, and we are again left with equation (79) for the sphere. Since $b\delta_1 << b$ and the variables are separated, we consider the N and q variations in succession.

The equation for the N dependence is

$$\frac{1}{2}(1+\epsilon_o-\frac{2}{5}\delta_1)f_{NN}+\frac{1}{2}(1+\epsilon_o-\frac{2}{5}\delta_1)\frac{1}{N}f_N+\left[(1-2\frac{\epsilon_o}{b})(1-\frac{1}{2}\delta_1)-\frac{1}{b}\right]Nf_N$$

$$+2\left[(1-\frac{\epsilon_o}{b})(1-\frac{1}{2}\delta_1)-\frac{N^2}{b}(1-\frac{\epsilon_o}{b})\right]f=0\ , \tag{108}$$

with the approximate solution

$$f(N)=\exp\left[-\frac{(1-\epsilon_o/b)}{(1+\epsilon_o)}(1-\frac{1}{2}\delta_1)N^2\right]\ . \tag{109}$$

If this solution is substituted back into equation (108), the residual term is, for $c_1 = (1-\epsilon_o/b)(1-\frac{1}{2}\delta_1)/(1+\epsilon_o)$,

$$\text{residue} = 2\delta_1\, c_1\left\{\frac{2}{5}-N^2\left[\frac{2}{5}c_1+\frac{1}{2b}(1+2\epsilon_o)\right]\right\}f\ , \tag{110}$$

which is much smaller than the terms of interest.

For the q dependence the equation is

$$(1+\epsilon_o)f_{qq} + (1+\epsilon_o)\frac{1}{q}f_q + \tau^2(1-2\frac{\epsilon_o}{b})q\,f_q + 2\,\tau^2(1-\epsilon_o/b)f = 0, \tag{111}$$

with the approximate solution

$$f(q) = \exp\left[-\frac{1}{2}\frac{(1-\epsilon_o/b)}{1+\epsilon_o}\tau^2 q^2\right] . \tag{112}$$

Again, there is a residual term if this solution is substituted into equation (111). In the appendix we discuss this residual term and others. The conclusions are quite similar to those obtained for prolate grains. Therefore, if $\epsilon_o \ll b$ ($T_i \ll T$) or $b^2 \gg \epsilon_o \gg b$, our solution in equation (113) below is numerically accurate to terms of order $\frac{1}{b}$. Otherwise, our solution is qualitatively correct but may be numerically inaccurate. Since the maximum value of δ_1 is $\delta_1 = 1$ for a disk, we see that $\delta_1 \ll b$ for all oblate grains. The requirement for grain alignment is that $b\delta_1 \gg 1$.

The solution for f is

$$\begin{aligned} f &= f(N).f(q) \\ &= \exp\left[-\frac{(1-\epsilon_o/b)}{1+\epsilon_o}(1-\frac{1}{2}\delta_1)N^2 - \frac{1}{2}\frac{(1-\epsilon_o/b)}{1+\epsilon_o}\tau^2 q^2\right] \\ &= \exp\left[-\frac{(b-\epsilon_o)}{1+\epsilon_o}\gamma\tau^2\sin^2\beta - \frac{1}{2}\frac{(b-\epsilon_o)}{1+\epsilon_o}\delta_1\tau^2\sin^2\theta\right], \end{aligned} \tag{113}$$

and the distribution function is

$$\begin{aligned} W &= f.\exp\{-\tau^2[(1-\frac{1}{2}\delta_1)+\frac{1}{2}\delta_1\sin^2\theta]\} \\ &= \exp\left\{-\tau^2\gamma\cos^2\beta - \tau^2\gamma\sin^2\beta.(\frac{1+b}{1+\epsilon_o}) - \frac{1}{2}\delta_1\tau^2\sin^2\theta(\frac{1+b}{1+\epsilon_o})\right\}. \end{aligned} \tag{114}$$

For $\epsilon_o \ll b$ ($T_i \ll T$) and $b \gg 1$, both $\langle \tau \sin\beta \rangle$ and $\langle \tau \sin\theta \rangle$ decrease, meaning that $\langle \beta \rangle$ and $\langle \theta \rangle$ decrease for increasing b - as predicted by the Davis-Greenstein process. However, we will find in Chapter IV and V that the alignment does not become perfect, no matter how strong is the field. For $\epsilon_o \gg b$ ($T_i \gg T$) and $b \gg 1$, $\langle \tau \sin\beta \rangle$ increases for increasing b and the alignment is reversed.

(iii) The Disk

For the disk, the relevant shape factors are

$$\epsilon = 0, \quad \gamma = \frac{1}{2}, \quad \delta_1 = 1, \quad \alpha = 1 \quad . \tag{115}$$

If we substitute these values into equation (72i) and change the ρ dependence to a variation with

$$\lambda = (1-\rho^2)^{\frac{1}{2}} = \sin\theta \quad , \tag{105}$$

then the result is equation (72c) in the appendix. To find the dominant terms, we make the scale changes $N = \sqrt{b}\, \nu$ and

$$Q = \sqrt{b}\, \lambda = \sqrt{b} \sin\theta \quad . \tag{116}$$

In this equation Q is the analogue of $q = \sqrt{b\delta_{\frac{1}{2}}}\, \lambda$ for the case of oblate nearly-spherical grains. The resulting equation is

$$\begin{aligned} b\{ &\frac{1}{2}(1+\epsilon_o) f_{NN} + \frac{1}{2}(1+\epsilon_o)\frac{1}{N} f_N + \frac{1}{2}(1-2\frac{\epsilon_o}{b} - \frac{1}{b}) N f_N \\ &+ 2(1-\frac{\epsilon_o}{b}) f - \frac{1}{2}\frac{N^2}{b}(1-\frac{\epsilon_o}{b}) f \\ &+ \frac{1}{2\tau^2}(1+\epsilon_o) f_{QQ} + \frac{1}{2\tau^2}(1+\epsilon_o)\frac{1}{Q} f_Q + \frac{1}{2}(1-2\frac{\epsilon_o}{b} - \frac{1}{b}) Q f_Q \\ &- \frac{1}{2}\frac{\tau^2 Q^2}{b}(1 - \frac{\epsilon_o}{b}) f \} + \text{terms of order } 1, \frac{1}{b}, \text{ etc.} = 0 . \end{aligned} \tag{117}$$

In equation (117) we have kept all of the important terms of order ϵ_o; the other terms of magnitude ϵ_o are considered in the appendix. We have also kept terms of order unity for convenience. Let us rewrite equation (117) as

$$b\left\{\frac{1}{2}(1+\epsilon_o)(f_{NN} + \frac{1}{\tau^2} f_{QQ}) + \frac{1}{2}(1+\epsilon_o)(\frac{1}{N} f_N + \frac{1}{\tau^2}\frac{1}{Q} f_Q)\right.$$
$$+ \frac{1}{2}(1-2\frac{\epsilon_o}{b} - \frac{1}{b})(N f_N + Q f_Q) - \frac{1}{2b}(1-\frac{\epsilon_o}{b})(N^2+\tau^2 Q^2)f$$
$$\left. + 2(1-\frac{\epsilon_o}{b})f\right\} + \ldots = 0 \quad . \tag{118}$$

We see from equation (118) that the variation of f with N is quite similar to that of f with Q. Thus, we consider

$$\frac{1}{2}(1+\epsilon_o)(f_{NN} + \frac{1}{N} f_N) + \frac{1}{2}(1-2\frac{\epsilon_o}{b} - \frac{1}{b})N f_N + (1-\frac{\epsilon_o}{b})(1 - \frac{N^2}{2b})f = 0 \,, \tag{119}$$

which has the well-behaved solution

$$f(N) = \exp\left\{-\frac{1}{2}\frac{(1-\epsilon_o/b)}{1+\epsilon_o} N^2\right\} \quad . \tag{120}$$

Thus, the solution for f is

$$f = \exp\left\{-\frac{1}{2}\frac{(1-\epsilon_o/b)}{1+\epsilon_o} N^2\right\} \cdot \exp\left\{-\frac{1}{2}\frac{(1-\epsilon_o/b)}{1+\epsilon_o}\tau^2 Q^2\right\} \tag{121}$$

$$= \exp\left\{-\frac{1}{2}\frac{(b-\epsilon_o)}{1+\epsilon_o}\tau^2(\sin^2\beta + \sin^2\theta)\right\} \tag{122}$$

We consider the residual terms in the appendix and show that they may be neglected.

Equation (122) is the same as equation (113) for the oblate nearly-spherical grain if we use the values $\gamma = \frac{1}{2}$ and $\delta_1 = 1$ for the disk. Since equations (113) and (114) are valid for $\delta_1 \ll 1$ and $\delta_1 = 1$, we will use them for all of the other intermediate values of δ_1. We make no claim that this is the correct solution for all oblate spheroids - only that it should give the qualitative behavior. Thus, we use equation (114) for all oblate spheroids ($0 < \delta_1 \leq 1$). The orientation in strong fields is qualitatively the same as in the nearly-spherical case, although the alignment becomes more pronounced as the disk shape is approached. Finally, we show in the appendix that equation (122) is numerically accurate to order $\frac{1}{b}$ for $\epsilon_o << b$ ($T_i << T$) and $\epsilon_o >> b$ ($T_i >> T$), and that it is qualitatively correct for $\epsilon_o \sim b$.

3. The Weak Field Case

When the magnetic field is weak, then $b << 1$. Although ϵ_o might, in principle, take on any values, we will only treat the cases for which $\epsilon_o \leqslant b$. The reason is that if $\epsilon_o >> b$, then we are unable to make the approximations which allow us to solve equation (72) more easily.

We expect the distribution function W to be close to the Maxwell-Boltzmann solution, meaning that f is near unity. For example, if equation (81) for the sphere is expanded with $b << 1$, $\epsilon_o \leqslant b$, then the result is

$$f \cong 1 - (b-\epsilon_o)\nu^2 \quad . \tag{123}$$

Let us therefore try to solve for f in a perturbation series using powers of b, so that

$$f \cong 1 + b\,\psi + \text{terms of higher order in b.} \tag{124}$$

If this form for f is put into equation (72), the terms to first order in b are

$$\left[\frac{(a+1)z + 2as^2}{4\tau^4}\right]\left[(1-r^2)\psi_{rr} - 2r\,\psi_r\right] + \frac{1}{2}\psi_{ss} - \gamma s\,\psi_s$$

$$+ 2a\,z\,\psi_{zz} + 2a(1-z)\psi_z$$

$$+ \frac{1}{\tau^2}\left(1-\frac{\epsilon_o}{b}\right)\Big[\tfrac{1}{2}(\gamma+3)z - 2z^2(1-r^2) + (\gamma+1)s^2 - 2\gamma^2 s^4(1-r^2)$$

$$- (\gamma-1)r^2s^2 + \tfrac{1}{2}(\gamma-1)r^2z - (\gamma+1)^2 s^2 z - (\gamma^2-6\gamma+1)r^2s^2z\Big] = 0 \quad . \tag{125}$$

For convenience, we will solve for ψ with $\epsilon_o = 0$, since equation (125) indicates that

$$\psi(T_i > 0) = \left(1 - \frac{\epsilon_o}{b}\right) \cdot \psi(T_i = 0) \quad . \tag{126}$$

Now the derivatives in r form the differential operator for the Legendre polynomials. Therefore, let us write the inhomogeneous portion of equation (125) in terms of the first two Legendre polynomials, which are

$$P_o(r) = 1 \quad , \qquad P_2(r) = \tfrac{1}{2}(3r^2-1) \quad . \tag{127}$$

The result is

$$\left[\frac{(a+1)z + 2as^2}{4\tau^4}\right]\left[(1-r^2)\psi_{rr} - 2r\,\psi_r\right] + \frac{1}{2}\psi_{ss} - \gamma s\,\psi_s + 2az\,\psi_{zz}$$

$$+ 2a(1-z)\psi_z + \left[\tfrac{2}{3}(\gamma+2) - \tfrac{4}{3}(z + \gamma^2 s^2)\right] \cdot P_o$$

$$+ \frac{1}{\tau^2}\left[\tfrac{4}{3}z^2 + \tfrac{4}{3}\gamma^2 s^4 - \tfrac{2}{3}(\gamma-1)s^2 + \tfrac{1}{3}(\gamma-1)z - (\tfrac{2}{3}\gamma^2 - 4\gamma + \tfrac{2}{3})s^2 z\right]P_2 = 0. \tag{128}$$

Having separated the r variable from s and z, we now assume that

$$\psi(r, s, z) = \sum_{\ell=0}^{\infty} P_\ell(r) Q_\ell(s, z) \quad , \tag{129}$$

where the P_ℓ are the Legendre polynomials. Not all of the Q_ℓ will be needed in measuring the alignment of the grains. The degree of alignment will involve the quantities

$$\langle \cos^2\beta \rangle = \tfrac{1}{2} \int_{-1}^{1} dr \cdot r^2 \int_{-\infty}^{\infty} ds \int_0^{\infty} dz\, W(r, s, z)$$

$$\langle \cos^2\varphi \rangle = \tfrac{1}{2} \int_{-1}^{1} dr \int_{-\infty}^{\infty} ds \int_0^{\infty} dz \left(\tfrac{3}{2} r^2 \rho^2 - \tfrac{1}{2} r^2 - \tfrac{1}{2}\rho^2 + 1\right) W(r, s, z) \quad . \tag{130}$$

The r terms in these integrals will include only P_o and P_2 integrated over the range of orthogonality for the Legendre polynomials. Therefore, only the terms in P_o and P_2 from equation (129) will yield non-zero integrals, so that only Q_o and Q_2 need be found. From the equation

$$(1-r^2) P_{\ell, rr} - 2r\, P_{\ell, r} = -\ell(\ell+1) P_\ell \quad ,$$

we obtain

$$\Bigg[\sum_\ell P_\ell \cdot \Big\{ \Big[\frac{(\alpha+1)z + 2\alpha s^2}{4\tau^4} \Big] \Big[-\ell(\ell+1) Q_\ell \Big] + \tfrac{1}{2} Q_{\ell, ss} - \gamma s\, Q_{\ell, s}$$
$$+ 2\alpha (z Q_{\ell, zz} + Q_{\ell, z} - z\, Q_{\ell, z} \Big\} \Bigg] + P_o \cdot \Big[\tfrac{2}{3}(\gamma+2) - \tfrac{4}{3}(z + \gamma^2 s^2) \Big]$$
$$+ \frac{P_2}{\tau^2} \cdot \Big[\tfrac{4}{3} z^2 + \tfrac{4}{3}\gamma^2 s^4 - \tfrac{2}{3}(\gamma-1)s^2 + \tfrac{1}{3}(\gamma-1)z - (\tfrac{2}{3}\gamma^2 - 4\gamma + \tfrac{2}{3}) \Big] = 0 \, . \tag{131}$$

Since the P_ℓ are linearly independent in r, we may set this equation equal to zero for each ℓ. For $\ell = 0$, the result is

$$\frac{1}{2} Q_{0,ss} - \gamma s\, Q_{0,s} + 2\alpha\left[z\, Q_{0,zz} + Q_{0,z} - z\, Q_{0,z}\right]$$

$$+ \frac{2}{3}(\gamma+2) - \frac{4}{3} z - \frac{4}{3}\gamma^2 s^2 = 0 \quad . \tag{132}$$

An acceptable solution to this equation is the particular solution

$$Q_0 = -\frac{2}{3}\left(\frac{z}{\alpha} + \gamma s^2\right) \quad . \tag{133}$$

The homogeneous solution diverges for large values of s and z faster than W_{MB} converges. Since the particular solution does not have this problem, it is acceptable while the homogeneous solution is not.

For $\ell = 2$, the equation for Q_2 is

$$[-3(\alpha+1)z - (6\alpha s^2)]\, Q_2 + \tau^4[Q_{2,ss} - 2\gamma s\, Q_{2,s} + 4\alpha z\, Q_{2,zz} + 4\alpha(1-z)Q_{2,z}]$$

$$+ \left[\frac{8}{3} z^3 + \frac{2}{3}(\gamma-1)z^2 - \frac{2}{3}(\gamma-1)s^2 z + \left(-\frac{4}{3}\gamma^2 + 8\gamma + \frac{4}{3}\right)s^2 z^2\right.$$

$$\left. + \left(\frac{4}{3}\gamma^2 + 8\gamma - \frac{4}{3}\right)s^4 z - \frac{4}{3}(\gamma-1)s^4 + \frac{8}{3}\gamma^2 s^6\right] = 0 \quad . \tag{134}$$

We will now solve equation (134) for the various geometries, that is, needles and nearly-spherical grains. We do not treat the disk because we were unable to separate variables for that case.

(i) The Needle

For the needle, both α and γ are large, and $\alpha \cong \gamma$. From the Maxwell-Boltzmann solution

$$W_{MB} = \exp[-(z+\gamma s^2)] = \exp\{-\tau^2[1+(\gamma-1)\cos^2\theta]\} \quad ,$$

we find that

$$\langle\cos\theta\rangle \sim (1/\sqrt{\gamma-1}) \ll 1 \quad ,$$

meaning that W_{MB} is small except where $\cos\theta \lesssim (1/\sqrt{\gamma-1})$, so that $\langle\theta\rangle$ is nearly 90°. Therefore, we obtain for the needle

$$s = \tau\cos\theta \sim (1/\sqrt{\gamma}) \ ,$$

$$z = \tau^2 \sin^2\theta \cong \tau^2 \quad . \tag{135}$$

It will be convenient to solve equation (134) using s and z, but after the solution is found, we will set $z \cong \tau^2$. Let us define

$$S = \sqrt{\gamma}\, s \quad , \tag{136}$$

substitute into equation (134), and choose the dominant terms in γ. The result is

$$-3Q_2 + z[Q_{2,SS} - 2S\,Q_{2,S} + 4z\,Q_{2,zz} + 4(1-z)Q_{2,z}]$$
$$+ \frac{2}{3}z(1-2S^2) + \text{terms of order } \gamma^{-1} \text{ or smaller} = 0 \ . \tag{137}$$

Assume that

$$Q_2 = C(S)Z_2(z) \ ,$$

and obtain

$$-3C\,Z_2 + z\Big[Z_2(C_{SS} - 2\,S\,C_S) + C(4z\,Z_{2,zz} + 4[1-z]Z_{2,z})\Big]$$
$$-\frac{1}{3}z(-2+4\,S^2) = 0 \quad . \tag{138}$$

The derivatives in S form the differential operator for the Hermite polynomials, and $(-2 + 4\,S^2)$ is the second Hermite polynomial. Therefore, we choose

$$C = -2 + 4\,S^2 \quad , \tag{139}$$

which allows us to factor the S-dependence from equation (138). We obtain for Z_2 the equation

$$z^2 Z_{2,zz} + z(1-z)Z_{2,z} - (z + \tfrac{3}{4})Z_2 - \tfrac{1}{12}\,z = 0 \quad . \tag{140}$$

If we set

$$Z_2(z) = z^{\sqrt{3}/2}\; u_1(z) \quad , \tag{141}$$

then we find for $u_1(z)$ the equation

$$z\,u_{1,zz} + [(\sqrt{3} + 1) - z]\,u_{1,z} - (\tfrac{\sqrt{3}}{2} + 1)u_1 = \tfrac{1}{12}\,z^{-\sqrt{3}/2} \,. \tag{142}$$

Let us write the parameters of this equation in the form

$$a_1 = j+1, \quad b_1 = 2j+1 \;, \quad j = \sqrt{3}/2 \quad . \tag{143}$$

Now consider the homogeneous terms in equation (142). They form the confluent hypergeometric equation, which has two linearly independent solutions. One solution[35] is $M(a_1, b_1, z)$ which is well-behaved at the origin and diverges as $z \to \infty$. The other solution is $U(a_1, b_1, z)$ which is well-behaved as $z \to \infty$ but has a singularity at the origin. Thus, we see that both homogeneous solutions to equation (142) are unacceptable.

To solve the inhomogeneous equation, we will use a Green's function method. Equation (142) may be written in the form

$$\frac{d}{dz}\left[z^{(\sqrt{3}+1)}e^{-z}u_{1,z}\right]-\left(\frac{\sqrt{3}}{2}+1\right)e^{-z}z^{\sqrt{3}}u_1=\frac{1}{12}z^{\sqrt{3}/2}e^{-z}\,. \tag{144}$$

Thus, the equation for the Green's function $G(z,z')$ is

$$\frac{d}{dz}\left[z^{(\sqrt{3}+1)}e^{-z}\frac{dG}{dz}\right]-\left(\frac{\sqrt{3}}{2}+1\right)e^{-z}z^{\sqrt{3}}G=\delta(z-z')\;; \tag{145}$$

if $z\neq z'$, this is the confluent hypergeometric equation with solutions $M(a_1,b_1,z)$ and $U(a_1,b_1,z)$. Let us write G in the form

$$G(z,z')=\begin{cases}A_1\,M(a_1,b_1,z)\,U(a_1,b_1,z') & 0<z<z'\\ A_1\,M(a_1,b_1,z')\,U(a_1,b_1,z) & z'<z<\infty\,.\end{cases} \tag{146}$$

If we integrate equation (145) over a small region near z', $z'-\epsilon_1<z<z'+\epsilon_1$ with ϵ_1 small, then we obtain

$$\begin{aligned}\left.\frac{dG}{dz}\right|_{z'+\epsilon_1}-\left.\frac{dG}{dz}\right|_{z'-\epsilon_1}&=e^{z'}(z')^{-2j-1}\\ &=A_1\cdot\text{Wronskian}\,(M,U)\big|_{z'}\,.\end{aligned} \tag{147}$$

If the Wronskian is evaluated,[35] then the result for A_1 is

$$A_1=-\frac{\Gamma(j+1)}{\Gamma(2j+1)}=-\frac{\Gamma(1+\sqrt{3}/2)}{\Gamma(1+\sqrt{3})} \tag{148}$$

Therefore, the solution for u_1 is

$$\begin{aligned}u_1=-\frac{1}{12}\frac{\Gamma(j+1)}{\Gamma(2j+1)}\Big\{&U(a_1,b_1,z)\int_0^z dz'\,M(a_1,b_1,z')e^{-z'}(z')^j\\ &+M(a_1,b_1,z)\int_z^\infty dz'\,U(a_1,b_1,z')e^{-z'}(z')^j\Big\}\end{aligned}$$

$$j=\sqrt{3}/2\,,\quad a_1=j+1,\quad b_1=2j+1, \tag{149}$$

and thus,

$$Q_2 = (-2+4\gamma s^2)\, z^{\sqrt{3}/2}\, u_1(z) \quad . \tag{149a}$$

(ii) Nearly-Spherical Grains

In this section we will treat both the prolate and oblate shapes because the alignment is small in weak fields, so that the two cases are quite similar. Our procedure is different from that of Miller, and we will discuss these differences in detail in an appendix. The shape factors are given by

$$\alpha \cong 1 + \frac{2}{5}\delta \;, \quad \delta = \epsilon^2 - 1 \;, \quad |\delta| << 1 \;, \; \gamma = 1 + \frac{1}{2}\delta \;, \quad \gamma^2 \cong 1+\delta \;. \tag{150}$$

If $\delta = 0$, then we are back to the sphere and

$$(Q_2)_{\text{sphere}} = \frac{2}{3}(z+s^2) \;. \tag{151}$$

Thus, we set

$$Q_2 = \frac{2}{3}(z+s^2) + \delta K \quad , \tag{152}$$

Substitute all of the parameters into equation (134), and collect the terms of first order in δ. The result is

$$\tau^4\left[K_{ss} - 2s\,K_s + 4z\,K_{zz} + 4(1-z)K_z\right] - 6\tau^2 K$$
$$- \frac{16}{15}z^3 + \frac{3}{5}z^2 - \frac{4}{5}s^2z^2 - \frac{3}{5}s^2 z - \frac{6}{5}s^4 + \frac{8}{5}s^4 z + \frac{4}{3}s^6 = 0 \;. \tag{153}$$

Next, choose

$$K = -\frac{4}{15}z + \frac{1}{3}s^2 + \Lambda(s,z) \quad , \tag{154}$$

and substitute this into equation (153). The equation for Λ is

$$\tau^2[\Lambda_{ss}-2s\Lambda_s + 4z\,\Lambda_{zz} + 4(1-z)\Lambda_z] - 6\Lambda + \frac{9}{5}(z-2s^2) = 0. \qquad (155)$$

If we change variables to

$$\tau = (z+s^2)^{\frac{1}{2}} \ , \quad \rho = \frac{s}{\tau} = \cos\theta \ ,$$

then we obtain

$$\tau^2\,\Lambda_{\tau\tau} + 2\tau(1-\tau^2)\Lambda_\tau + (1-\rho^2)\Lambda_{\rho\rho} - 2\rho\Lambda_\rho - 6\Lambda + \frac{9}{5}\tau^2(1-3\rho^2) = 0 \ . \qquad (156)$$

Let

$$\Lambda = N_o(\tau)\,R_1(\rho) \ ,$$

so that

$$\frac{1}{N_o}\left[\tau^2\,N_{o,\,\tau\tau} + 2\tau(1-\tau^2)N_{o,\,\tau}\right] + \frac{1}{R_1}\left[(1-\rho^2)R_{1,\,\rho\rho} - 2\rho R_{1},_\rho\right]$$
$$- 6 + \frac{9}{5}\tau^2(1-3\rho^2)\cdot\frac{1}{N_oR_1} = 0 \ . \qquad (157)$$

The derivatives in ρ form the differential operator for the Legendre polynomials, and the factor $(1-3\rho^2)$ is $-2\,P_2(\rho)$. Thus, we try

$$R_1 = P_2(\rho) \ ,$$

substitute into equation (157), and obtain

$$\tau^2\,N_{o,\,\tau\tau} + 2\tau(1-\tau^2)N_{o,\,\tau} - 12N_o - \frac{18}{5}\tau^2 = 0 \ , \qquad (158)$$

so that this procedure allows us to factor out the ρ-dependence. If we set

$$N_o(\tau) = \tau^3 X_1(p) \quad , \qquad p = \tau^2 \quad , \tag{159}$$

then the equation for X_1 is

$$p X_{1,pp} + (\frac{9}{2} - p)X_{1,p} - \frac{3}{2} X_1 = \frac{9}{10} p^{-3/2} \quad . \tag{160}$$

The homogeneous terms yield the confluent hypergeometric equation, and neither homogeneous solution is acceptable for the same reasons as given in the analysis for the needle. We solve for the inhomogeneous solution by the same Green's function method as before and find that

$$X_1(p) = -\frac{12}{175}\Big\{U(\tfrac{3}{2}, \tfrac{9}{2}, p) \int_o^p M(\tfrac{3}{2}, \tfrac{9}{2}, p')\cdot(p')^2 e^{-p'} dp' + M(\tfrac{3}{2}, \tfrac{9}{2}, p) \int_p^\infty U(\tfrac{3}{2}, \tfrac{9}{2}, p')\cdot(p')^2 e^{-p'} dp'\Big\}$$

$$\Lambda = \tau^3 X_1(p)\cdot\frac{1}{2}(3\rho^2 - 1), \quad p = \tau^2, \quad \tau = (z+s^2)^{\frac{1}{2}}, \quad \rho = \frac{s}{\tau}$$

$$Q_2 = (\frac{2}{3})(z+s^2) + \delta\Big[-\frac{4}{15} z + \frac{1}{3} s^2 + \Lambda\Big] \quad . \tag{161}$$

CHAPTER IV

MEASURES OF GRAIN ALIGNMENT

We will now use the distribution functions calculated in Chapter III in order to find the degree of grain alignment. The measures of grain orientation will be found analytically for the case of strong magnetic field and numerically for the case of weak field. The alignment in strong field will be determined for prolate and oblate spheroids at all temperatures.

The measures of alignment will be Purcell's quantities

$$Q_J = \frac{3}{2}\langle\cos^2\beta\rangle - \frac{1}{2} \quad ,$$

$$Q_A = \frac{3}{2}\langle\cos^2\varphi\rangle - \frac{1}{2} \quad ,$$

where the averages are taken over the distribution function W. The function W is normalized so that its integral over all of phase space is unity. Thus, we require that

$$\int_0^\infty d\tau \cdot \tau^2 \int_{-1}^{1} \left(\frac{2\pi\, dr}{4\pi}\right) \int_{-1}^{1} \left(\frac{2\pi\, d\rho}{4\pi}\right) \cdot W_{norm}(\tau, \rho, r) = 1 \quad , \qquad (162)$$

where

$$W_{norm} = \frac{W}{N}$$

is the normalized distribution, W is one of the distributions we have found in Chapter III, and N is the normalization. In equation (162) W is averaged over the solid angles corresponding to $r = \cos\beta$ and $\rho = \cos\theta$. In addition, we have, after integrating over τ, that

$$\langle \cos^2\beta \rangle = \int_{-1}^{1} \frac{1}{2}\, dr \cdot r^2 \int_{-1}^{1} \frac{1}{2}\, d\rho \cdot W_{norm}(r,\rho) \quad , \tag{162a}$$

$$\langle \cos^2\varphi \rangle = \int_{-1}^{1} \frac{1}{2}\, dr \int_{-1}^{1} \frac{1}{2}\, d\rho \cdot \left(\frac{1}{2} - \frac{1}{2} r^2 - \frac{1}{2}\rho^2 + \frac{3}{2} r^2 \rho^2\right) W_{norm}(r,\rho) \; . \tag{162b}$$

1. The Sphere in all Fields

Equation (81) for the sphere may be written

$$W = \exp\left\{-\tau^2\left[\cos^2\beta + \frac{1}{\xi^2}\sin^2\beta\right]\right\} = e^{\left\{-\tau^2\left[r^2 + \frac{1}{\xi^2}(1-r^2)\right]\right\}} , \tag{163}$$

where

$$\xi^2 = \frac{1+\epsilon_o}{1+b} = \frac{T_{av}}{T} = \frac{1+b(T_i/T)}{1+b} \cong \frac{T_i}{T} \text{ for large } b \; . \tag{164}$$

If W is integrated over τ, the normalized distribution in β is

$$W_\beta(\beta) = \frac{1}{4\pi}\, \xi\left[\xi^2 \cos^2\beta + \sin^2\beta\right]^{-3/2} \; ; \tag{165}$$

when Q_J is evaluated, the result is

$$Q_J = \begin{cases} \dfrac{1}{2(1-\xi^2)}\left[2 + \xi^2 - \dfrac{3\xi}{\sqrt{1-\xi^2}} \tan^{-1}\left(\dfrac{\sqrt{1-\xi^2}}{\xi}\right)\right], & \xi < 1 \\[2ex] \dfrac{1}{2(1-\xi^2)}\left[2 + \xi^2 - \dfrac{3\xi}{\sqrt{\xi^2-1}} \ln\left(\xi + \sqrt{\xi^2-1}\right)\right], & \xi > 1 \; . \end{cases} \tag{166}$$

As expected, $Q_A = 0$ for all ξ , and $Q_J = 0$ for $\xi = 1$.

2. The Strong-Field Case

(i) Prolate Grains

The distribution function for a prolate grain is given by

$$W = \exp\left\{-\tau^2\left[\cos^2\beta + \sin^2\beta\left(\frac{1+b+\frac{1}{4}\delta}{1+\epsilon_o+\frac{1}{4}\delta}\right) + \cos^2\theta\cdot(\gamma-1)\left(\frac{1+b}{1+\epsilon_o}\right)\right]\right\} . \tag{102}$$

Although this distribution is inaccurate for weak fields, it does provide an answer for the weak field case. Thus, we will assume that equation (102) gives some qualitative idea of the distribution for intermediate and weaker fields, although we make no claim that it is accurate in these cases.

Let us define the quantities

$$\begin{aligned} \xi_o^2 &= \frac{1+\epsilon_o + \frac{1}{4}\delta}{1+b+\frac{1}{4}\delta} , \\ A_1^2 &= (\xi_o^2 - 1) , \\ B_1^2 &= \frac{\xi_o^2}{\xi^2}(\gamma-1) , \end{aligned} \tag{167}$$

and integrate W over τ; the result is

$$\begin{aligned} W_{norm}(r,\rho) &= \int_0^\infty W_{norm}\,\tau^2\,d\tau \\ &= \frac{1}{N_i}\left[A_1^2 r^2 + B_1^2 \rho^2 + 1\right]^{-3/2} \\ &= \frac{1}{N_i} W(r,\rho) . \end{aligned} \tag{168}$$

Here, $W(r,\rho)$ is the distribution over angles, with $r = \cos\beta$ and $\rho = \cos\theta$, and N_i is the normalization constant. To find N, we consider two cases: $\xi_o < 1$ and $\xi_o > 1$. Therefore, we obtain

$$
\begin{aligned}
N_1 &= \int_{-1}^{1} \frac{2\pi\, dr}{4\pi} \int_{-1}^{1} \frac{2\pi\, d\rho}{4\pi}\, W(r, \rho) \\
&= \frac{1}{A_1 B_1} \ln \left\{ \frac{(1-A_1^2+B_1^2)^{\frac{1}{2}} + A_1 B_1}{\sqrt{(1-A_1^2)(1+B_1^2)}} \right\}
\end{aligned}
$$

$$
\xi_o < 1 \quad (\text{or } T_i < T), \quad A_1^2 = 1 - \xi_o^2 \quad ; \tag{169}
$$

and similarly we find

$$
\begin{aligned}
N_2 &= \frac{1}{A_2 B_1} \tan^{-1} \left\{ \frac{A_2 B_1}{\left[1 + A_2^2 + B_1^2\right]^{\frac{1}{2}}} \right\} \\
\xi_o &> 1 \quad (\text{or } T_i > T), \quad A_2^2 = \xi_o^2 - 1 \quad .
\end{aligned} \tag{170}
$$

Equation (168) is different from the distribution Jones and Spitzer[36] assumed for a non-spherical grain. In their distribution r and ρ are completely uncoupled, so that

$$
W_{JS}(r, \rho) = \frac{\xi}{4\pi} \left[(\xi^2-1)r^2+1\right]^{-\frac{3}{2}} \cdot \frac{\gamma^{\frac{1}{2}}}{4\pi} \left[(\gamma-1)\rho^2+1\right]^{-\frac{3}{2}} \quad . \tag{171}
$$

Equation (168) permits such a separation only for ξ and γ close to unity. The integration over τ is what couples the variation of r with that of ρ. The values of $\langle \cos^2\beta \rangle$ are found to be

$$
\langle \cos^2\beta \rangle_1 = \frac{1}{A_1^2} - \frac{1}{N_1 A_1^3} \tan^{-1} \left[\frac{A_1}{\sqrt{1-A_1^2+B_1^2}} \right], \quad T_i < T \tag{172}
$$

$$
\langle \cos^2\beta \rangle_2 = \frac{1}{A_2^2} + \frac{1}{N_1 A_2^3} \cdot \ln \left\{ \frac{A_2 + (1 + A_2^2 + B_1^2)^{\frac{1}{2}}}{\sqrt{1+B_1^2}} \right\}, \quad T_i > T. \tag{173}
$$

Finally, the values of $\langle \cos^2\varphi \rangle$ are

$$
\langle \cos^2\varphi \rangle_1 = \frac{1}{2N_1 A_1^2 B_1^2} \left[(1-A_1^2+B_1^2)^{\frac{1}{2}} - N_1(1-A_1^2)(1+B_1^2)\right], \quad T_i < T \tag{174}
$$

$$\langle \cos^2\varphi \rangle_2 = \frac{1}{2N_2 A_2^2 B_1^2}\left[N_2(1+A_2^2)(1+B_1^2)-(1+A_2^2+B_1^2)\right] , \quad T_i > T . \tag{175}$$

The quantities Q_J and Q_A may be found from these equations, and we plot Q_A in Figure 5 of Chapter V.

(ii) Oblate Grains

The distribution function for an oblate grain is

$$W = \exp\left\{-\tau^2\left[\gamma \cos^2\beta + \gamma \sin^2\beta\left(\frac{1}{\xi^2}\right) + \frac{1}{2}\,\delta_1 \sin^2\theta\left(\frac{1}{\xi^2}\right)\right]\right\} \tag{114}$$

$$= \exp\left\{-\frac{\tau^2}{\xi^2}\left[\gamma(\xi^2-1)r^2 - (1-\gamma)\rho^2+1\right]\right\} , \tag{176}$$

$$\frac{1}{2} \le \gamma \le 1 \quad , \quad \xi^2 = \frac{1+b(T_i/T)}{1+b} \cong \frac{T_i}{T} \text{ for large b} .$$

The distribution over angles is

$$W_{norm}(r, \rho) = \int_0^\infty d\tau .\, \tau^2 .\, W_{norm}(\tau, r, \rho)$$

$$= \frac{1}{N_i}\left[\gamma(\xi^2-1)-(1-\gamma)\rho^2+1\right]^{-3/2} , \tag{177}$$

where the normalization constants are

$$N_3 = \frac{1}{\xi_3} \cdot \frac{1}{\sqrt{1-\gamma}} \tan^{-1}\left[\xi_3 \frac{\sqrt{1-\gamma}}{\sqrt{\gamma-\xi_3^2}}\right], \quad \xi < 1, \quad \xi_3^2 = \gamma(1-\xi^2) , \tag{178}$$

$$N_4 = \frac{1}{\xi_4} \cdot \frac{1}{\sqrt{1-\gamma}} \ln\left\{\frac{\sqrt{\xi_4^2+\gamma} + \xi_4\sqrt{1-\gamma}}{\sqrt{\gamma(1+\xi_4^2)}}\right\} , \quad \xi > 1, \quad \xi_4^2 = \gamma(\xi^2-1) . \tag{179}$$

The values of $\langle \cos^2\beta \rangle$ are

$$\langle\cos^2\beta\rangle = \frac{1}{N_i}\int_{-1}^{1}\frac{1}{2}dr.r^2\int_{-1}^{1}\frac{1}{2}d\rho\, W(r,\rho)$$

$$\langle\cos^2\beta\rangle_3 = \frac{1}{\xi_3^2} - \frac{1}{N_3}\frac{1}{3\,\xi_3^3}\sin^{-1}\frac{\xi_3}{\sqrt{\gamma}}\ , \quad \xi < 1 \qquad (180)$$

$$\langle\cos^2\beta>4 = -\frac{1}{\xi_4^2} + \frac{1}{N_4}\cdot\frac{1}{\xi_4^3}\, \ln\left\{\frac{1}{\sqrt{\gamma}}\left[\xi_4+\sqrt{(\gamma+\xi_4^2)}\right]\right\},\ \xi > 1 \qquad (181)$$

Finally, the expressions for $\langle\cos^2\varphi\rangle$ are

$$\langle\cos^2\varphi\rangle = \frac{1}{N_i}\int_{-1}^{1}\frac{1}{2}dr\int_{-1}^{1}\frac{1}{2}d\rho\left(\frac{1}{2}-\frac{1}{2}r^2-\frac{1}{2}\rho^2+\frac{3}{2}r^2\rho^2\right)W(r,\rho) \qquad (182)$$

$$\langle\cos^2\varphi\rangle_3 = \frac{1}{8N_3}\left\{-\frac{4}{(1-\gamma)}\cdot\frac{1}{\xi_3^2}\left[\sqrt{\gamma-\xi_3^2}\ -N_3\gamma(1-\xi_3^2)\right]\right\},\ \xi < 1 \qquad (183)$$

$$\langle\cos^2\varphi\rangle_4 = \frac{1}{8N_4}\left\{\frac{4}{(1-\gamma)}\cdot\frac{1}{\xi_4^2}\left[\sqrt{\gamma+\xi_4^2}\ -N_4\gamma(1+\xi_4^2)\right]\right\},\ \xi > 1\ . \qquad (184)$$

The quantity Q_A corresponding to these equations is plotted in Figure 6 of Chapter V.

3. The Weak-Field Case

For weak fields ($b \ll 1$), the distribution function is

$$W = f\exp[-(z+\gamma s^2)]$$

$$= f\exp\{-\tau^2[1+(\gamma-1)\cos^2\theta]\}$$

$$= f\exp\{-\tau^2[1+\frac{1}{2}\delta\rho^2]\}\ ,$$

$$f = 1+(b-\varepsilon_o)\psi$$

$$\psi = -\frac{2}{3}\left(\frac{z}{\alpha}+\gamma s^2\right)+\frac{1}{2}(3r^2-1).Q_2(s,z)\ , \qquad (185)$$

and Q_2 was found in Chapter III. We will use both the (τ, ρ) and the (s, z) coordinate systems for the necessary integrations; the relation between their differentials is

$$ds\,dz = 2\,\tau^2\,d\tau d\rho \quad . \tag{186}$$

Thus, the normalization for W becomes

$$N = \frac{1}{2}\int_{\text{limits}} dr\,dsdz\;W(r, s, z)$$
$$= \frac{1}{2}\int_{-1}^{1} dr \int_{-\infty}^{\infty} ds \int_{0}^{\infty} dz.\,\{\exp[-(z+\gamma s^2)]\}\,.\,[1+(b-\varepsilon_o)\psi]$$
$$= \frac{1}{2}\,\frac{\sqrt{\pi}}{\sqrt{\gamma}}\,\{1 - \frac{1}{3}(b-\varepsilon_o)(\frac{2}{a} + 1\}$$
$$\frac{1}{N} \cong 2\,\frac{\sqrt{\gamma}}{\sqrt{\pi}}\,\{1 + \frac{1}{3}(b-\varepsilon_o)(\frac{2}{a} + 1)\} \tag{187}$$

The factor involving $Q_2(s, z)$ in N vanishes when we integrate over r. In addition, we assume that $b << 1$ and $\varepsilon_o \leqslant b$.

Let us now find $\langle\cos^2\beta\rangle$. The integral is

$$\langle\cos^2\beta\rangle = \frac{1}{N}\cdot\frac{1}{2}\int_{-1}^{1}\frac{1}{2}\,dr.\,r^2\int_{-\infty}^{\infty} ds \int_{0}^{\infty} dz.\,\{\exp[-(z+\gamma s^2)]\}.\,\{1+(b-\varepsilon_o)\psi\}$$
$$= \frac{1}{3} + \frac{2}{15}\,\frac{\sqrt{\gamma}}{\sqrt{\pi}}(b-\varepsilon_o)\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dz.\,\{\exp[-(z+\gamma s^2)]\}.\,Q_2(s, z)\,. \tag{188}$$

The factor of $\frac{1}{3}$ is the value obtained from a totally random orientation of the grains. We see that the deviation from randomness only involves the function Q_2. Thus,

$$Q_J = \frac{3}{2}\langle\cos^2\beta\rangle - \frac{1}{2}$$
$$= \frac{1}{5}\,\frac{\sqrt{\gamma}}{\sqrt{\pi}}\,(b-\varepsilon_o)\int_{-\infty}^{\infty} ds \int_{0}^{\infty}\{\exp[-(z+\gamma s^2)\}.\,Q_2(s, z)dz\;. \tag{189}$$

For $\langle\cos^2\varphi\rangle$ the result is

$$\langle\cos^2\varphi\rangle = \frac{1}{N}\cdot\frac{1}{2}\int_{-1}^{1}\frac{1}{2}dr\int_{-\infty}^{\infty}ds\int_{0}^{\infty}dz.\left(\frac{3}{2}r^2\rho^2-\frac{1}{2}r^2-\frac{1}{2}\rho^2+\frac{1}{2}\right)W(r,s,z)$$

$$= \frac{1}{8N}\int_{-1}^{1}dr\int_{-\infty}^{\infty}ds\int_{0}^{\infty}dz\,(1-r^2-\rho^2+3r^2\rho^2).\{\exp[-(z+\gamma s^2)]\}.\{1+(b-\epsilon_o)\psi\}$$

$$= \frac{1}{2}-\frac{1}{2}\langle\cos^2\beta\rangle$$

$$+\frac{1}{5}\frac{\sqrt{\gamma}}{\sqrt{\pi}}(b-\epsilon_o)\int_{-\infty}^{\infty}ds\int_{0}^{\infty}dz.\rho^2.\{\exp[-(z+\gamma s^2)]\}.Q_2(s,z)\ , \qquad (190)$$

so that

$$Q_A = \frac{3}{2}\langle\cos^2\varphi\rangle - \frac{1}{2}$$

$$= -\frac{1}{2}Q_J + \frac{3}{20}\frac{\sqrt{\gamma}}{\sqrt{\pi}}(b-\epsilon_o)\int_{-\infty}^{\infty}\int_{0}^{\infty}dz.\rho^2.\{\exp[-(z+\gamma s^2)]\}.Q_2(s,z)ds. \qquad (191)$$

It now remains to evaluate these quantities by using the functions Q_2 found in Chapter III.

(i) Nearly-Spherical Grains

For these particles Q_2 is given by equation (161):

$$Q_2 = \frac{2}{3}(z+s^2) + \delta\left[-\frac{4}{15}z+\frac{1}{3}s^2+\Lambda\right]\ ,$$

$$|\delta| \ll 1,\quad p=\tau^2,\quad \tau=(z+s^2)^{\frac{1}{2}}\ ,\quad \rho=\frac{s}{\tau}=\cos\theta\ ,$$

$$\Lambda = \tau^3 X_1(p).\frac{1}{2}(3\rho^2-1)\ ,$$

$$X_1(p) = -\frac{12}{175}\Big\{U(\tfrac{3}{2},\tfrac{9}{2},p)\int_0^p M(\tfrac{3}{2},\tfrac{9}{2},p^1).(p^1)^2e^{-p^1}dp^1$$

$$+M(\tfrac{3}{2},\tfrac{9}{2},p)\int_p^\infty U(\tfrac{3}{2},\tfrac{9}{2},p^1).(p^1)^2e^{-p^1}dp^1\Big\} \qquad (161)$$

We evaluate Q_J and Q_A by integrating over (τ, ρ) space. Let us write

$$I(\rho, \delta) = \int_0^\infty dp . p^2 . X_1(p). \exp\{-p[1+\tfrac{1}{2}\delta\rho^2]\} \quad , \tag{192}$$

so that Q_J and Q_A become

$$Q_J = (b-\epsilon_o)\left\{\frac{1}{5} - \frac{4}{75}\delta + \frac{1}{10}\frac{\delta}{\sqrt{\pi}}\int_{-1}^{1} d\rho . (3\rho^2-1). I(\rho, \delta)\right\} , \tag{193}$$

$$Q_A = (b-\epsilon_o)\left\{-\frac{7}{750}\delta - \frac{1}{20}\frac{\delta}{\sqrt{\pi}}\int_{-1}^{1} d\rho . (3\rho^2-1). I(\rho, \delta)\right.$$

$$\left. + \frac{3}{20}\frac{\delta}{\sqrt{\pi}}\int_{-1}^{1} d\rho . \rho^2 . (3\rho^2-1). I(\rho, \delta)\right\} \quad . \tag{194}$$

The factor of $\frac{1}{5}$ in Q_J is the contribution due to a spherical grain. These integrals must be evaluated numerically, and we will treat in the next chapter the quantity

$$Q_A^{(1)} = Q_A/(b-\epsilon_o) \quad . \tag{195}$$

(ii) The Needle

For the needle Q_2 is given by equation (149):

$$Q_2 = (-2 + 4\gamma s^2)\, z^{\sqrt{3}/2}\, u_1(z) \quad ,$$

$$u_1(z) = -\frac{1}{12}\frac{\Gamma(a_1)}{\Gamma(b_1)}\left\{U(a_1, b_1, z)\int_0^\infty dz^1\, M(a_1, b_1, z^1)e^{-z^1}(z^1)^j\right.$$

$$\left. + M(a_1, b_1, z)\int_z^\infty dz^1\, U(a_1, b_1, z^1)e^{-z^1}(z^1)^j\right\} ,$$

$$j = \frac{\sqrt{3}}{2} \quad , \quad a_1 = j+1, \quad b_1 = 2j+1 \quad . \tag{149}$$

We now change variables to the (τ, ρ) coordinate system and make the approximations of equation (135) for the needle:

$$z = \tau^2 \sin^2\theta \cong \tau^2 = p \quad . \tag{135}$$

Further, define

$$\rho_1 = \sqrt{\gamma - 1}\ \rho = \sqrt{\tfrac{1}{2}\delta}\ \rho\,, \qquad \delta >> 1\ , \tag{196}$$

so that

$$(-2+4\gamma s^2) = -2 + (4 + \frac{8}{\delta})\tau^2 \rho_1^2 \cong -2 + 4\tau^2 \rho_1^2\ , \tag{197}$$

$$\sqrt{\gamma}\, d\rho \cong d\rho_1\ . \tag{198}$$

Equation (189) for Q_J now becomes

$$Q_J = \frac{1}{5}\frac{1}{\sqrt{\pi}}(b-\epsilon_o)\int_{-\sqrt{\frac{1}{2}\delta}}^{\sqrt{\frac{1}{2}\delta}} d\rho_1 \int_o^\infty 2\tau^2 d\tau . Q_2 . \exp[-\tau^2(1+\rho_1^2)], \tag{199}$$

and equation (191) for Q_A becomes

$$Q_A = -\frac{1}{2}Q_J + \frac{1}{\delta}\,\frac{3}{5}\,\frac{1}{\sqrt{\pi}}(b-\epsilon_o)\int_{-\sqrt{\frac{1}{2}\delta}}^{\sqrt{\frac{1}{2}\delta}} d\rho_1 . \rho_1^2 \int_o^\infty dp . p^{\frac{1}{2}} . e^{-p(1+\rho_1^2)} Q_2 . \tag{200}$$

Since the second term is of order $\frac{1}{\delta}$ times the first, we obtain

$$Q_A \cong -\frac{1}{2}Q_J\ . \tag{201}$$

If we substitute for Q_2 from equation (149), then

$$\frac{Q_J}{(b-\epsilon_o)} = \frac{1}{5}\,\frac{1}{\sqrt{\pi}}\int_{-\sqrt{\frac{1}{2}\delta}}^{\sqrt{\frac{1}{2}\delta}} d\rho_1 \int_o^\infty dp . p^{\frac{1}{2}}\, e^{-p(1+\rho_1^2)} . (-2+4\rho_1^2 p) . p^{\sqrt{3}/2}\, u_1(p)\ , \tag{202}$$

and this integral must be evaluated numerically.

CHAPTER V

DISCUSSION AND CONCLUSION

We now possess the measures of alignment for the various cases. In this chapter we discuss these results, noting their features. We next compare our conclusions with those of Miller and Purcell. Finally, we briefly consider the field strength and grain temperature.

1. Discussion of Results

Let us first recall some of the quantities of interest - any other symbols needed may be found in the appendix. Thus,

Q_A = a measure of axial alignment for the grain

$$= \frac{3}{2} \langle \cos^2\varphi \rangle - \frac{1}{2}$$

$$= -\frac{3}{2} F \tag{5}$$

φ = angle between the grain symmetry axis and the magnetic field

T = temperature of the gas

T_i = internal temperature of the grain

ϵ = ratio of the grain semi axes

$\epsilon > 1$ for a prolate spheroid

$\epsilon < 1$ for an oblate spheroid

$\epsilon = 1$ for a sphere

δ = $\epsilon^2 - 1$

b = a parameter which compares the effects of the magnetic field with those of the gas collisions

$$= \frac{\chi''}{\omega} \frac{VB^2}{gh}$$

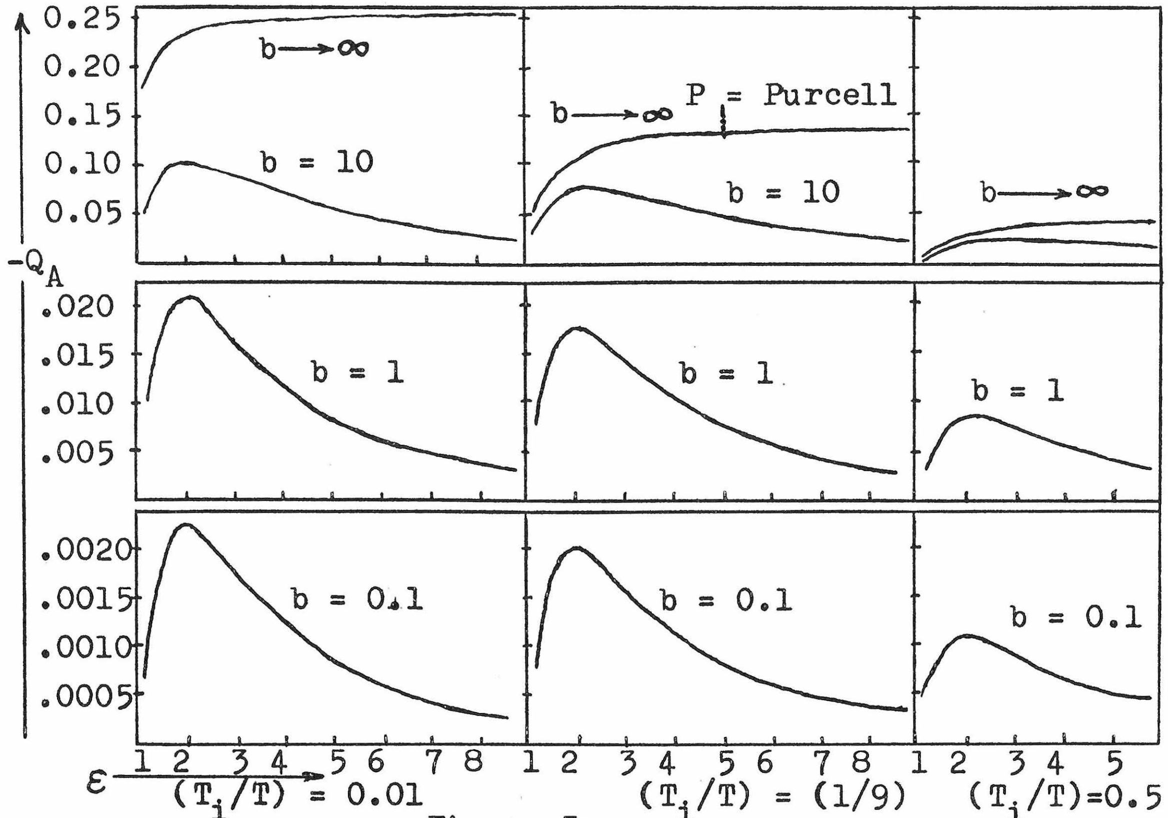


Figure 5a

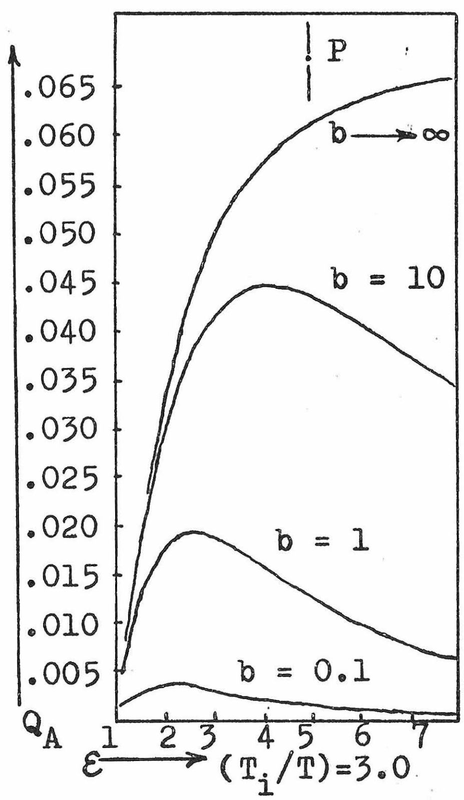


Figure 5b

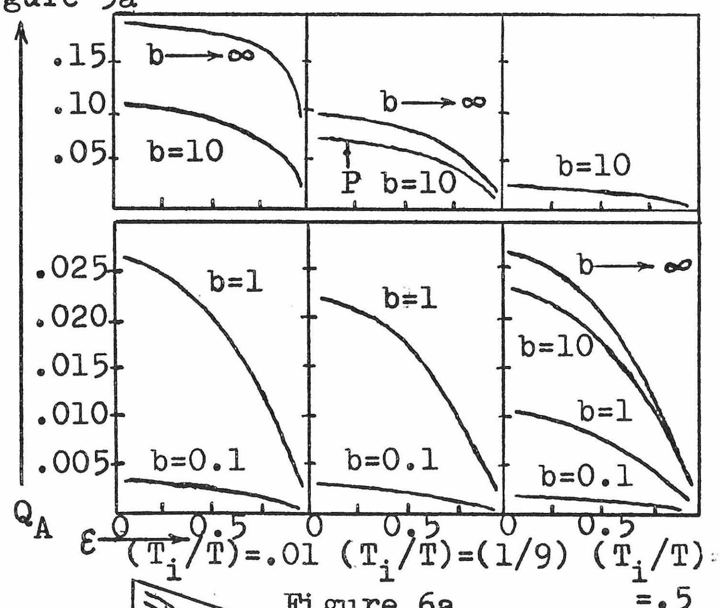


Figure 6a

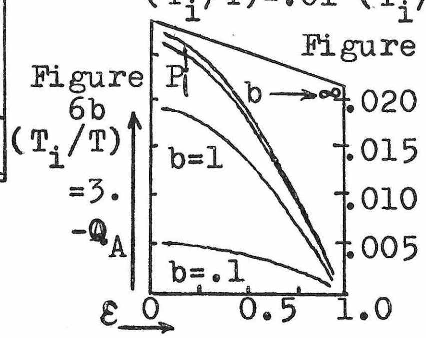


Figure 6b

Figures 5&6

Plots of Q_A

vs.

b, (T_i/T), ε

ϵ_o = a parameter which compares the effects of the internal temperature with those of the gas collisions

$= b\, T_i/(\frac{m^+}{m})\, T$

Since Q_A is directly related to the polarization only if the Rayleigh-Gans scattering theory is correct, we will regard Q_A as simply a convenient measure of the grain's axial alignment with the magnetic field.

Figures 5 and 6 present a plot of Q_A as a function of ϵ for several values of B and (T_i/T). All numerical values were calculated by computer. The interesting cases are for $(T_i/T) < 1$, and they are treated in Figure 5a for prolate grains and Figure 6a for oblate grains. Figures 5b and 6b treat only the case $(T_i/T) = 3$ in order to show that the alignment reverses for $T_i > T$. The points marked "P" are the values calculated by Purcell, together with his claimed uncertainties; they are discussed later. We note that $|Q_A|$ increases as b increases and that $Q_A \to 0$ as $\epsilon \to 1$.

Now consider Figure 5a, which shows the alignment for prolate grains and $(T_i/T) < 1$. For b = 0.1, 1.0, and 10.0, the graphs of $|Q_A|$ rise to a maximum at $\epsilon \cong 2$ and then decrease as ϵ becomes larger. This behavior is caused by the factor of $\exp\left\{-\tau^2 \sin^2\beta . \left(\frac{1+b+\frac{1}{4}\delta}{1+\epsilon_o+\frac{1}{4}\delta}\right)\right\}$ in equation (102). If b is fixed and δ increases, then the term $(1+b+\frac{1}{4}\delta)/(1+\epsilon_o+\frac{1}{4}\delta)$ decreases - thus $\tau \sin\beta$ increases on the average, and $|Q_A|$ ultimately decreases.

Physically, as ϵ increases, so do the particle's volume and surface area. The volume effect increases the magnetic torque, while the surface effect allows more collisions with gas atoms to occur.

These two processes compete and apparently yield the curious graph of $|Q_A|$ for finite b. If b is extremely large, then δ/b is small, and the graphs show only a monotonic increase of $|Q_A|$ with ϵ. In our calculation, we took $b = 10^{40}$ as our "infinite" value; we see that the maximum value of $|Q_A|$ for $b = 10$ is roughly $\frac{1}{2}$ the value for $b \to \infty$.

The temperature effect is also interesting. The increase of internal temperature from $(T_i/T) = 0.01$ to $(T_i/T) = \frac{1}{9}$ substantially affects $|Q_A|$ only for $b \to \infty$; the finite values of b show just a small decline in $|Q_A|$. However, the increase from $(T_i/T) = \frac{1}{9}$ to $(T_i/T) = \frac{1}{2}$ is quite substantial in its effect on $|Q_A|$. Thus, these graphs seem to favor $(T_i/T) \leqslant 0.1$ in order to retain a fair degree of alignment. Finally, the ultimate value of Q_A for complete alignment is -0.5. This number is only approached for extremely large b and extremely small (T_i/T). For example, $b = 10^{40}$ and $(T_i/T) = 10^{-8}$ yield $Q_A \cong -0.42$, and we find that $|Q_A + \frac{1}{2}|$ is proportional to $[\log(b/\epsilon_o)]^{-1}$.

Next, consider Figure 6a, which shows the alignment for oblate grains and $(T_i/T) < 1$. Here, the graphs all increase monotonically as $\epsilon \to 0$, which is a different behavior from that of the prolate grains. The reason is that for oblate grains the relevant factor in equation (114) is $\exp\left\{-\gamma\tau^2 \sin^2\beta . \left(\frac{1+b}{1+\epsilon_o}\right)\right\}$, and the term $(1+b)/(1+\epsilon_o)$ has no dependence on shape. Apparently, the competition between surface and volume effects in the oblate case has different results from those of the prolate case.

We also note that the temperature effects are the same for the oblate as for the prolate grains. For finite b the values of $|Q_A|$ for the two grain types are comparable at all of the temperatures given. Finally, we observe that for $b \to \infty$ and $(T_i/T) \to 0$, the maximum value of Q_A is 0.25. This is less than the value of $Q_A = 1.0$ which would be expected from complete alignment of the disks. We will not try to explain this surprising behavior on physical grounds: we only will note that this result follows from the mathematical solution. However, we again state that for finite values of b, oblate and prolate grains are comparably aligned.

2. Comparison with Purcell's Results

E. M. Purcell[14] wrote a computer program which simulates the history of a single grain. He assumed that a hydrogen atom which strikes the grain remains there, and that other atoms evaporate from the grain surface randomly. He considered two possibilities: (i) evaporation of the atoms at the temperature of the grain; (ii) evaporation at temperature of the gas. The first case is expected to be a more realistic assumption, while the second case applies if the gas atoms collide elastically with the grain - which Jones and Spitzer assumed in their article.

Purcell's calculation seems valid for the strong field case. He called his measure of the field strength δ, and we find that

$$\delta_{\text{Purcell}} = b.\left(\frac{h}{aV}\right) \qquad (203)$$

$$= 2b \text{ for spheres.}$$

For convenience, we will assume $\delta_{\text{Purcell}} = b$. Purcell found that for $\delta = 1$, the value of Q_A reaches saturation, and this occurs for our

			1	2	3	4	5	6	7	8	9
				Q_A				Q_J			
	ε	$\frac{T_i}{T}$	JS	T_i		T		T_i		T	
				P	MG	P	MG	P	MG	P	MG
1	5.0	$\frac{1}{9}$	-0.067	-0.119 +0.008	-0.114	-0.149 +0.009	-0.138	+0.257 +0.018	0.242	+0.316 +0.020	0.291
2	5.0	3	+0.72	+0.041 +0.004	0.045	+0.068 +0.004	0.061	-0.139 +0.014	-0.107	-0.211 +0.012	-0.147
3	0.2	$\frac{1}{9}$	+0.016		0.077	0.061 +0.007	0.097				
4	0.2	3	-0.017		-0.020	-0.023 +0.003	-0.026				
5	10.	0.192		-0.130 +0.012	-0.087		-0.108	+0.267 +0.024	0.177		0.220
6	5.0	0.192		-0.086 +0.009	-0.084		-0.105	+0.186 +0.021	0.181		0.224
7	2.5	0.192		-0.065 +0.006	-0.073		-0.093	+0.176 +0.014	0.194		0.239
8	0.4	0.192		+0.032 +0.005	0.050		+0.065	+0.138 +0.011	0.165		0.198
9	0.1	0.192		+0.057 +0.008	0.055		+0.072	+0.195 +0.014	0.152		0.181

Table 1

A Comparison With Purcell's Calculated Values

The JS column contains Q_A values which Purcell calculated based on the analysis of Jones and Spitzer

The T_i columns contain values calculated for evaporation at the grain temperature

The T columns contain values calculated for evaporation at the gas temperature

Purcell's values (P) are for his $\delta = 1$. Our values (MG) are for $b \longrightarrow \infty$

$b \to \infty$. If we compare Purcell's results with our own, we find the values in Table 1. We have numbered the rows and columns; the empty spaces are quantities which Purcell did not calculate. The other columns are as described below the table. For the T_i columns we calculated Q_A and Q_J assuming that the atom-grain collisions are inelastic; for the T columns we calculated these quantities assuming elastic collisions. The differences between the two cases are seen to be quite substantial, and these contrasts exist for the following reason. From equations (35) and (71) we obtain

$$\frac{\varepsilon_o}{b} = \left(\frac{T_i}{T}\right)\left(\frac{m}{m^+}\right) \quad , \tag{71}$$

$$\frac{m^+}{m} = \begin{cases} 1 & \text{for an elastic collision} \\ \frac{1}{2}\left(1 + \sqrt{T_i/T}\right) & \text{for an inelastic collision} \quad . \end{cases} \tag{35}$$

Thus, the values of ε_o for the two cases can be fairly different. Since the distribution functions found in Chapter III depend exponentially on ε_o, the effect on Q_A can be substantial.

Our rough treatment - in the appendix - of an inelastic atom-grain collision should be equivalent to Purcell's case (i). The only difference is that we assume the same atom to collide with the grain and evaporate from its surface, while Purcell assumed that different atoms take part in each event. We assume that when an average is taken over all collisions and evaporations, the two viewpoints should yield the same results.

If we compare our answers with Purcell's, the agreement is mixed. Consider first the T values of Q_A in columns 4 and 5, which are also plotted in Figures 5 and 6. The numbers in rows 1, 2, and 4 agree quite well, while the pair in row 3 disagree. The two T_i pairs in rows 1 and 2, columns 2 and 3 agree well - yet in columns 6 and 7 the pair[(1,6),(1,7)] agree and the pair [(2,6),(2,7)] disagree. Similarly, the pair [(1,8),(1.9)] agree while [(2.8),(2.9)] disagree. For rows 5-9, columns 2 and 3, three of the pairs agree and two disagree. Similar remarks hold for rows 5-9, columns 6 and 7.

Thus, from 20 pairs of values, 8 of the 11 Q_A pairs and 4 of the 9 Q_J pairs show good agreement within Purcell's claimed uncertainties. The remainder show varying amounts of disagreement. In addition, the trends of the Q_J values disagree. Purcell's Q_J values decrease as $\epsilon \to 1$, while our values increase - reaching a maximum at $\epsilon = 1$ for the sphere. Since we do not know the details of Purcell's calculation, we are unable to account for the curious disagreements.

However, we can explain the discrepancies between the values found by Jones and Spitzer, listed in column 1, and those of Purcell. The distribution function of Jones and Spitzer is equation (171). If we believe that our equation (168) is the more accurate one, then we see that the two equations are similar only for $\epsilon \cong 1$ and $(\frac{1+b}{1+\epsilon_o}) \cong 1$. Since the numbers in column 1 were found for cases which violate these conditions, they are in disagreement with other results.

We may conclude that Purcell's computer program seems to enhance the effect of the magnetic field. The reasons are the following: (i) 12 out of 20 of his results for $\delta = 1$ agree with ours for $b \to \infty$; (ii) the order of magnitude of his other 8 values corresponds to that for large b; (iii) he obtained saturation effects for $\delta = 1$ - that is, if he took δ larger than unity, Q_A did not increase. Finally, although the disagreements are puzzling, it is of some comfort that there is fair agreement on the Q_A values - since the methods of solution are so different.

3. Results for Nearly-Spherical Grains in Weak Fields: A Comparison with Miller

When we evaluate Q_A numerically on Caltech's IBM 360-75 computer, using equations (161), (192), and (194) for this case, we obtain the following values in Table 2 for $\varepsilon_o = 0$:

Table 2

δ	Q_A/b	$-\frac{2}{3} Q_A/b\delta = (F/b\delta)$
0.03	7.148×10^{-4}	0.0159
0.06	1.416×10^{-3}	0.0157
0.09	2.103×10^{-3}	0.0156
0.12	2.717×10^{-3}	0.0154
0.15	3.440×10^{-3}	0.0153

In his thesis C. R. Miller found the alignment for zero temperature, nearly-spherical grains in weak magnetic fields. Though his analysis contains an error, its numerical effect is small for this case; the details are in the appendix. Using a different method from our own, Miller[37] obtained the result that

$$(F_{Miller})/b\delta = 0.0161 \quad . \tag{203a}$$

The agreement between our values and Miller's result is noteworthyespecially as δ approaches zero, the value for a sphere. Our result for F shows an interesting drift as δ increases; thus, F seems to have a small non-linear dependence on δ even when we ignore terms of order δ^2 in our solution of equation (134).

4. Remarks on the Field Strength and Grain Temperature

Our measure for the field strength is the parameter b. If all the constants are substituted, then for a sphere we obtain

$$b = \left(\frac{\chi''}{\omega}\right) \frac{VB^2}{gh} = (10.3) \frac{B_o^2}{a_o \, T_i^o \, n_H^o \left(\frac{m^+}{m}\right) \sqrt{T^o}} \left(\frac{\chi''}{\omega}\right)^o \tag{204}$$

In this equation, the parameters have the values

$$B = (B_o \times 10^{-5}) \text{ gauss},$$

$$T_i = (T_i^o \times 10) \, {}^oK \quad ,$$

$$a = (a_o \times 10^{-5}) \text{ cm.} \quad ,$$

$$n_H = (n_H^o \times 1) \quad \text{H atom/cm}^3 ,$$

$$T = (T^o \times 100) \, {}^oK \quad ,$$

$$\frac{\chi''}{\omega} = \left[\left(\frac{\chi''}{\omega}\right)^o \times (2.5 \times 10^{-12})\right] \cdot \frac{1}{T_i} \quad . \tag{205}$$

Thus, if $B_o = T_i^o = \ldots = 1$, then a field of 10^{-5} gauss may be considered "strong" since $b \cong 10$, while a field of 10^{-6} gauss may be considered "weak" since $b \cong 0.1$. Because b depends on B^2, b is sensitive to changes in the field strength.

For the case of a spheroid, the only factor in equation (204) which changes is (V/h). We find that

$$\left(\frac{V}{h}\right)_{sphere} = \frac{1}{2a} = \frac{0.5}{a} \quad , \qquad (206)$$

$$\left(\frac{V}{h}\right)_{needle} = \frac{16}{9\pi} \cdot \frac{1}{a} = \frac{0.567}{a} \quad , \qquad (207)$$

$$\left(\frac{V}{h}\right)_{disk} = \left(\frac{4}{3}\epsilon\right) \cdot \frac{1}{a} \to 0 \quad \text{as } \epsilon \to 0 \quad . \qquad (208)$$

Thus, for a mathematical disk, $\left(\frac{V}{h}\right) \to 0$. However, from Table 3, we see that $\epsilon = \frac{1}{3}$, or certainly $\epsilon = \frac{1}{5}$, is close enough to a true disk for the purpose of finding Q_A.

Table 3

	Q_A					
	$(T_i/T) = 0.01$		$(T_i/T) = 1/9$		$(T_i/T) = 0.5$	
ϵ	b = 1	b = 10	b = 1	b = 10	b = 1	b = 10
0.05	2.62×10^{-2}	10.38×10^{-2}	2.22×10^{-2}	7.18×10^{-2}	1.02×10^{-2}	2.29×10^{-2}
0.2	2.54	10.21	2.15	7.03	0.99	2.22
0.35	2.36	9.82	1.99	6.68	0.91	2.06
0.5	2.07	9.15	1.74	6.08	0.78	1.80

Thus, $\left(\frac{V}{h}\right)_{disk} \cong \frac{4}{15}\,\frac{1}{a}$, so that all of the geometrical factors in b are reasonably close together.

Most estimates of the grain's internal temperature do not allow T_i to become much smaller than 10 $^{\circ}$K. Of course, (T_i/T) is just as important as T_i itself. We see from Figures 5a and 6a that

$(T_i/T) = 0.1$ is about as large as the ratio can get before $|Q_A|$ decreases substantially. Therefore, if the gas temperature, T, may increase for any reason, then T_i can increase as well. However, the parameter b would then decrease according to equation (204). It is also interesting to note that for prolate grains the best alignment occurs for $\epsilon \cong 2$, while for oblate grains substantial alignment still may occur for $\epsilon = \frac{1}{3}$. Thus, extreme spheroidal grain shapes are unnecessary.

Our approximations in Chapter III are adequate for $b = 10$, $(T_i/T) = 0.1$. If we accept the corresponding values of the other parameters (B_o, a_o, etc. all equal to unity), then the problem is whether the polarization data are consistent with values of $|Q_A| \leqslant 0.1$. If the polarization data demand much greater values of $|Q_A|$, there are severe problems with the alignment mechanism of Davis and Greenstein. If the data permit values of $|Q_A| \leqslant .1$, then some of the parameters in equation (205) could be determined more accurately. It will be of interest to see how this question is ultimately settled.

APPENDIX

A. Detail of Prolate Grains in Strong Field

We set $\alpha \cong \gamma = 1 + \frac{1}{2}\delta$ in equation (72i) and obtain

$$
\begin{aligned}
&\left\{\frac{1}{2} + \frac{1}{4}\delta\left[(1-\rho^2) - \frac{1}{2}\frac{\nu^2}{\tau^2}(1-3\rho^2)\right]\right\} f_{\sigma\sigma} - (1+\frac{1}{2}\delta)\sigma f_{\sigma} \\
&+\left\{\frac{1}{2}(1+\epsilon_o) + \frac{1}{8}\delta\left[(1+\rho^2) + \frac{\nu^2}{\tau^2}(1-3\rho^2)\right]\right\} f_{\nu\nu} \\
&+\{\frac{1}{2}(1+\epsilon_o) + \frac{1}{8}\delta(1+\rho^2)\}\frac{1}{\nu} f_{\nu} \\
&+\{(b-2\epsilon_o-1) + \frac{1}{2}\delta[(b-2\epsilon_o)\rho^2-1]\}\nu f_{\nu} \\
&+\frac{1}{2\tau^2}(1-\rho^2)\left[1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2} + \frac{1}{2}\delta\rho^2\right] f_{\rho\rho} \\
&-\frac{\rho}{\tau^2}\left[1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2} - \frac{1}{4}\delta(1-3\rho^2) - \frac{1}{2}\delta(b-2\epsilon_o)\cdot\tau^2(1-\rho^2)(1-\frac{1}{2}\frac{\nu^2}{\tau^2})\right] f_{\rho} \\
&+\frac{1}{4}\delta\frac{\sigma\nu}{\tau^2}(1-3\rho^2)f_{\sigma\nu} - \frac{1}{2}\delta\frac{\sigma\rho}{\tau^2}(1-\rho^2)f_{\sigma\rho} - \frac{1}{2}\delta\frac{\nu\rho}{\tau^2}(1-\rho^2)f_{\nu\rho} \\
&+(b-\epsilon_o)\left\{2(1-\nu^2) + \delta\left[\frac{1}{2}(1-\rho^2) - \frac{1}{4}\frac{\nu^2}{\tau^2}(1-3\rho^2) - \frac{1}{2}\delta\tau^2\,{}^2\rho^2(1-\rho^2)\right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{4}\delta\nu^2\,{}^2\rho^2(1-3\rho^2) - 2\nu^2\rho^2\right]\right\} f = 0 \quad . \qquad (72a)
\end{aligned}
$$

We make the scale changes $N = \sqrt{b}\,\nu$, $P = \sqrt{b(\gamma-1)}\,\rho = \sqrt{\frac{1}{2}b\delta}\,\rho$, and find that the dominant terms are

$$
\begin{aligned}
&b\delta\left\{\frac{1}{4\tau^2}(1+\epsilon_o - \frac{1}{2}\frac{\epsilon_o}{b}\frac{N^2}{\tau^2} - 2\frac{\epsilon_o}{b\delta}P^2)f_{PP}\right. \\
&\quad \left. + \frac{1}{2}P\left[(1-2\frac{\epsilon_o}{b}) - \frac{\epsilon_o}{b\delta}\cdot\frac{1}{\tau^2}\right]f_P + \frac{1}{2}(1 - \frac{\epsilon_o}{b})f\right\} \\
&+b\left\{\frac{1}{2}(1+\epsilon_o+\frac{1}{4}\delta)f_{NN} + \frac{1}{2}(1+\epsilon_o+\frac{1}{4}\delta)\frac{1}{N}f_N + (1-2\frac{\epsilon_o}{b} - \frac{1}{b})Nf_N\right. \\
&\quad \left. + 2(1 - \frac{\epsilon_o}{b})(1 - \frac{N^2}{b})f\right\} = 0 \quad . \qquad (95a)
\end{aligned}
$$

This equation is different from equation (95) of the main text because here we have kept all terms of order ϵ_o and $\epsilon_o\delta$. Thus, it is valid for all values of ϵ_o.

The discussion for the N dependence proceeds as in the text and needs no elaboration. It remains to consider the terms in P.

We may write the P dependence in the form

$$b\delta\left[\frac{1}{4\tau^2}(1+\epsilon_o)f_{PP} + \frac{1}{2}P(1-\frac{\epsilon_o}{b})f_P + \frac{1}{2}(1-\frac{\epsilon_o}{b})f\right]$$

$$+\left[(-\frac{1}{2}\epsilon_o\,\delta\,\frac{N^2}{\tau^2} - 2\,\epsilon_o P^2)f_{PP} + \frac{1}{2}P(-\epsilon_o\delta - \frac{\epsilon_o}{\tau^2})f_P\right] = 0. \qquad (209)$$

If we set

$$f(P) = \exp(-A\,\tau^2\,P^2) \quad , \quad A = \frac{1-\epsilon_o/b}{1+\epsilon_o} \quad ,$$

then the first group of terms vanishes, and the second group yields as its residue

$$\left[-\frac{1}{2}\epsilon_o\delta(4A^2\tau^2N^2P^2 - 2AN^2 - 2A\,\tau^2P^2)\right.$$

$$\left. - \epsilon_o P^2(8A^2\tau^4P^2 - 4A\tau^2 - A)\right]f \quad . \qquad (210)$$

Thus, the terms in the second group are of order

$$\epsilon_o A,\ \epsilon_o\delta\,A,\ \epsilon_o\,A^2,\ \epsilon_o\,\delta\,A^2\,,$$

while those in the first group, for which the solution is exact, are of magnitude

$$b\delta,\ \epsilon_o\,b\delta,\ A\,b\delta,\ A^2b\delta,\ A^2\epsilon_o\,b\delta;\ \epsilon_o\delta,\ \epsilon_o\delta A \quad .$$

Now suppose $\epsilon_o \ll b$, which includes $\epsilon_o \ll 1$ and $\epsilon_o \sim 1$. Then we find that $A \sim 1$, and the largest terms in the first group are of order $b\delta$ and $\epsilon_o b\delta$. The terms in the second group are of order ϵ_o and $\epsilon_o\delta$, and since $b\delta >> 1$, these residues are much smaller than the dominant terms in the first group.

Next, suppose that $\epsilon_o >> b$. Then we find that $A \sim \frac{1}{b}$, and the largest terms in the first group are now of order $\epsilon_o\delta$, $b\delta$, and $\epsilon_o b\delta$. The largest terms in the second group are of order $\frac{\epsilon_o}{b}$ and $\frac{\epsilon_o\delta}{b}$. Unless ϵ_o becomes extremely large, i.e. $\epsilon_o >> b^2$, these residues may still be ignored. If ϵ_o is so large, then the grain is extremely hot. If the grain does not evaporate altogether at such temperatures, then we know that it will be strongly aligned. This qualitative behavior is already predicted by our solution in the text. We will use this solution for the case of extremely hot grains, even though the solution may be numerically inaccurate.

If $\epsilon_o \sim b$, then we find again that $A \sim \frac{1}{b}$. However, our original assumption that $\langle \nu \rangle \sim \frac{1}{\sqrt{b}}$ and $\langle \rho \rangle \sim \frac{1}{\sqrt{b\delta}}$ fails since actually $\langle \nu \rangle \sim 1/\sqrt{b-\epsilon_o}$ and $\langle \rho \rangle \sim 1/\sqrt{(b-\epsilon_o)\delta}$. Thus, as $\epsilon_o \rightarrow b$, other terms in equation (72a) become important. These other terms cannot change the qualitative behavior of the solution, since we know that

$$f = 1 \quad \text{for} \quad b = \epsilon_o \quad ,$$

and our solution for f does have this correct behavior. Since the alignment is quite weak for $\epsilon_o \sim b$, we will not try to obtain a more accurate solution.

Therefore, we may summarize the results as follows:

For $\epsilon_o << b$ ($T_i << T$) and for $b^2 >> \epsilon_o >> b$, our solution for f - equation (101) - is quite accurate. Thus,

$$f = \exp\left[-\frac{(1-\epsilon_o/b)}{1+\epsilon_o}\tau^2 P^2 - \frac{(1-\epsilon_o/b)}{1+\epsilon_o+\frac{1}{4}\delta}N^2\right] \cdot \left\{1 + \text{terms of order } \frac{1}{b} + \ldots\right\} . \tag{211}$$

For $\epsilon_o \sim b$ and for $\epsilon_o >> b^2$, our solution is qualitatively correct. We have not improved its accuracy because for $\epsilon_o >> b^2$ the grain is too hot for the problem to be of further interest.

Let us finally obtain some idea of how accurate equation (101) is for $\delta >> b >> 1$. Suppose, first, that $b = 0$ and $\delta >> 1$: then equation (95) takes the form

$$(\text{terms of order } \delta) + (\text{terms of order } 1) + \ldots = 0 \quad . \tag{102a}$$

We know that the solution to this equation for $b = 0$ is $f = 1$.

Next, suppose that $\epsilon_o = 0$ - for simplicity - and that $b << 1$, $b\delta >> 1$: then equation (95) becomes

$$(\text{terms of order } \delta) + (\text{terms of order } b\delta) + (\text{terms of order } 1) + (\text{terms of order } b) + \ldots = 0 \quad . \tag{102b}$$

In this equation, the terms of order $b\delta$ and of b are the same as in equation (95) with $\epsilon_o = 0$; only their size relative to the other terms is smaller. The terms of order $b\delta$ provide the alignment in ρ as before: more importantly, however, the terms in b provide again the alignment in ν - since the alignment disappears without them. We may divide equation (102b) by δ and obtain that

$$f(\nu) \cong 1 + \text{terms of order } \frac{b}{\delta} + \ldots \quad , \tag{102c}$$

which shows that the alignment is quite weak.

Finally, suppose that $\delta >> b >> 1$. The only change is that in equation (102b) the terms of order b become larger than those of order 1. The solution still has the same form as equation (102c) because $\frac{b}{\delta} << 1$.

Now consider equation (97): we observe that it has the correct qualitative behavior in all cases, even though it may be numerically inaccurate for $\delta >> b$. Since the alignment is small for this case, we will not attempt to find a more accurate solution. Thus, we use equation (102) for all cases although it may be numerically inaccurate for $\delta >> b$.

If $\epsilon_o \neq 0$, we obtain the summary of results given in the main text of the thesis.

B. Detail of Oblate, Nearly-Spherical Grains in Strong Field.

We set $a \cong 1 + \frac{2}{5}\delta$, $\delta_1 = -\delta > 0$, $\gamma = 1 + \frac{1}{2}\delta = 1 - \frac{1}{2}\delta_1$ in equation (72i) and change the ρ-dependence to a dependence on λ, with

$$\lambda = (1-\rho^2)^{\frac{1}{2}} = \sin\theta \quad . \tag{105}$$

The resulting equation, with $\delta_1 << 1$, is

$$\left[\frac{1}{2} - \frac{1}{10}\delta_1(2\lambda^2+2\frac{\nu^2}{\tau^2} - 3\lambda^2\frac{\nu^2}{\tau^2})\right]f_{\sigma\sigma} - \left[1-\delta_1(\frac{1}{2}-\frac{1}{10}\lambda^2)\right]\sigma f_\sigma$$

$$+\left\{\frac{1}{2}(1+\epsilon_o) - \frac{1}{10}\delta_1\left[2(1-\frac{\nu^2}{\tau^2}) - \lambda^2(1-3\frac{\nu^2}{\tau^2})\right]\right\}f_{\nu\nu}$$

$$+\left[\frac{1}{2}(1+\epsilon_o) - \frac{1}{10}\delta_1(2-\lambda^2)\right]\frac{1}{\nu}f_\nu$$

$$+\left[(b-2\epsilon_o-1)(1-\frac{1}{2}\delta_1) + \frac{1}{2}\delta_1\lambda^2(b-2\epsilon_o-\frac{1}{5})\right]\nu f_\nu$$

$$+\left\{\frac{1}{2}(1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2}) - \frac{1}{5}\delta_1(1-\lambda^2)\right\}\cdot\frac{1}{\tau^2}(1-\lambda^2)f_{\lambda\lambda}$$

$$+\left\{\frac{1}{2}(1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2} - \frac{2}{5}\delta_1)\right\}\cdot\frac{1}{\tau^2}\frac{1}{\lambda}f_\lambda$$

$$+\left\{-(1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2}) + \frac{4}{5}\delta_1 + \frac{1}{2}\delta_1\tau^2\left[(b-2\epsilon_o)(1-\frac{1}{2}\frac{\nu^2}{\tau^2}) - \frac{1}{5}\right]\right\}\frac{1}{\tau^2}\lambda f_\lambda$$

$$+\left\{-\frac{3}{5}\delta_1 - \frac{1}{2}\delta_1\tau^2\left[(b-2\epsilon_o)(1-\frac{1}{2}\frac{\nu^2}{\tau^2}) - \frac{1}{5}\right]\right\}\cdot\frac{1}{\tau^2}\lambda^3 f_\lambda$$

$$+(b-\epsilon_o)\left\{2(1-\nu^2) - \delta_1(\frac{1}{2}\lambda^2 - 2\nu^2 + \frac{1}{2}\frac{\nu^2}{\tau^2} + 2\nu^2\lambda^2 - \frac{3}{4}\frac{\lambda^2\nu^2}{\tau^2})\right\}f = 0\ . \qquad (72b)$$

In equation (72b), we have ignored terms of order δ_1^2.

We make the scale changes $N = \sqrt{b}\,\nu$, $q = \sqrt{b\delta_1}\,\lambda$ and find that the dominant terms are

$$b\{\frac{1}{2}(1+\epsilon_o - \frac{2}{5}\delta_1)f_{NN} + \frac{1}{2}(1+\epsilon_o - \frac{2}{5}\delta_1)\frac{1}{N}f_N + [(1-2\frac{\epsilon_o}{b})(1-\frac{1}{2}\delta_1) - \frac{1}{b}]Nf_N$$
$$+ 2(1-\frac{\epsilon_o}{b})(1-\frac{N^2}{b} - \frac{1}{2}\delta_1)f\}$$

$$+b\delta_1\left\{\frac{1}{2\tau^2}(1+\epsilon_o - \frac{1}{2}\frac{\epsilon_o}{b}\frac{N^2}{\tau^2} - \frac{\epsilon_o}{b\delta_1}q^2)f_{qq} + \frac{1}{2\tau^2}(1+\epsilon_o - \frac{1}{2}\frac{\epsilon_o}{b}\frac{N^2}{\tau^2})\frac{1}{q}f_q\right.$$
$$\left. + \frac{1}{2}(1-2\frac{\epsilon_o}{b} - \frac{\epsilon_o}{b\delta_1}\cdot\frac{1}{\tau^2})qf_q + (1-\frac{\epsilon_o}{b})f\right\} = 0\ . \qquad (107b)$$

This equation is different from equation (107) of the main text because here we have kept all terms of order ϵ_o and $\epsilon_o\,\delta_1$.

If equation (107b) is compared with its counterpart, equation (95a) in the previous appendix for prolate grains, it is seen that the two equations are quite similar. Therefore, we can expect that the residual terms are similar - though not identical - and that the conclusions regarding the accuracy of equation (113) are the same. The important point is that the order of magnitude of the residues compared to that of the dominant terms is the same in this case as in the prolate case. Thus, we may conclude that:

(i) for $\epsilon_o \ll b$ $(T_i \ll T)$ and for $b^2 \gg \epsilon_o \gg b$, our solution for f - equation (113) - is accurate to terms of order $\frac{1}{b}$;

(ii) for $\epsilon_o \sim b$ and for $\epsilon_o \gg b^2$, our solution is only qualitatively correct.

Since $\delta_1 = 1 - \epsilon^2$ and the smallest value of ϵ is $\epsilon = 0$, the largest possible value for δ_1 is $\delta_1 = 1$. Thus, $\delta_1 \ll b$ and we do not have the same situation with the prolate grains in which $\delta \gg b$ can occur.

C. Detail of Disk in Strong Field

We set $a = 1$, $\gamma = \frac{1}{2}$ in equation (72i) and change the ρ dependance to a variation with λ, where

$$\lambda = (1-\rho^2)^{\frac{1}{2}} = \sin\theta \quad . \tag{105}$$

The resulting equation for f is

$$\frac{1}{2} f_{\sigma\sigma} - \frac{1}{2}(1+\lambda^2)\sigma f_\sigma$$

$$+\frac{1}{2}(1+\epsilon_o)f_{\nu\nu} + \frac{1}{2}(1+\epsilon_o)\frac{1}{\nu} f_\nu + \frac{1}{2}(1+\lambda^2)(b-2\epsilon_o-1)\nu f_\nu$$

$$+\frac{1}{2\tau^2}(1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2})(1-\lambda^2)f_{\lambda\lambda} + \frac{1}{2\tau^2}(1+\epsilon_o - \frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2})\frac{1}{\lambda} f_\lambda$$

$$+\left[-\frac{1}{\tau^2}(1+\epsilon_o-\frac{1}{2}\epsilon_o\frac{\nu^2}{\tau^2})+\frac{1}{2}(b-2\epsilon_o-1)(1-\lambda^2)-\frac{1}{4}(b-2\epsilon_o).\frac{\nu^2}{\tau^2}(1-\lambda^2)\right]\lambda f_\lambda$$

$$+(b-\epsilon_o)\left[2-\frac{1}{2}\nu^2-\frac{1}{2}\lambda^2-\frac{1}{2}\frac{\nu^2}{\tau^2}-\frac{1}{2}\tau^2\lambda^2+\frac{3}{4}\frac{\lambda^2\nu^2}{\tau^2}-\frac{3}{4}\lambda^2\nu^2+\lambda^4(\frac{1}{2}\tau^2-\frac{3}{4}\nu^2)\right]f = 0. \tag{72c}$$

We make the scale changes $N = \sqrt{b}\,\nu$, $Q = \sqrt{b}\,\lambda = \sqrt{b}\,\sin\theta$ and find that the dominant terms are

$$b\Big\{\frac{1}{2}(1+\epsilon_o)f_{NN} + \frac{1}{2}(1+\epsilon_o)\frac{1}{N}f_N + \frac{1}{2}(1-2\frac{\epsilon_o}{b}-\frac{1}{b})N f_N - \frac{1}{2}\frac{N^2}{b}(1-\frac{\epsilon_o}{b})f$$

$$+\frac{1}{2\tau^2}(1+\epsilon_o)f_{QQ} + \frac{1}{2\tau^2}(1+\epsilon_o)\frac{1}{Q}f_Q + \frac{1}{2}(1-2\frac{\epsilon_o}{b}-\frac{1}{b})Q f_Q$$

$$-\frac{1}{2}\frac{\tau^2 Q^2}{b}(1-\frac{\epsilon_o}{b})f + 2(1-\frac{\epsilon_o}{b})f\Big\}$$

$$-\frac{1}{2}(\frac{N^2}{\tau^2}+Q^2)(1-\frac{\epsilon_o}{b})f - \frac{\epsilon_o}{2\tau^2}(\frac{1}{2}\frac{N^2}{\tau^2}+Q^2)f_{QQ} - \frac{\epsilon_o}{4\tau^4}N^2\frac{1}{Q}f_Q$$

$$-\frac{\epsilon_o}{\tau^2}Q f_Q = 0 \quad . \tag{117c}$$

Equation (117c) is different from equation (117) of the text because it contains all terms of order ϵ_o.

If we set

$$f = \exp\Big\{-\frac{1}{2}\frac{(1-\epsilon_o/b)}{1+\epsilon_o)}(N^2+\tau^2Q^2)\Big\}\ , \quad A = \frac{1}{2}\frac{(1-\epsilon_o/b)}{1+\epsilon_o}\ , \tag{122}$$

then the first group of terms vanishes, and the second group of terms leaves a residue given by

$$\text{residue} = \left[-A\frac{N^2}{T^2} - A(1-2\epsilon_o)Q^2 - A^2\epsilon_o N^2 Q^2 - 2A^2\epsilon_o T^2 Q^4\right] f \quad (212)$$

We find that the order of magnitude of terms in the first group is

$$b,\ \epsilon_o,\ Ab,\ \epsilon_o A b,\ A^2 b,\ \epsilon_o A^2 b \quad ,$$

while the order of magnitude of terms in the second group is

$$A,\ \epsilon_o A,\ \epsilon_o A^2 \quad .$$

Thus, the "dominant" terms in the first group are always larger by a factor of b than those in the second group for all values of ϵ_o.

If $\epsilon_o \sim b$, however, other terms in equation (72c) may become important and equation (117c) might no longer be an accurate approximation. Equation (122) does have the correct qualitative behavior that $f = 1$ for $\epsilon_o = b$, and we will use it even though it may be numerically inaccurate for $\epsilon_o \sim b$.

Therefore, we may conclude the following:

For $\epsilon_o \ll b$ or $\epsilon_o \gg b$, equation (122) is accurate to terms of order $\frac{1}{b}$.

For $\epsilon_o \sim b$, equation (122) is qualitatively correct but may be numerically inaccurate.

Since $\delta_1 = 1$ for the disk, we do not have the complications involving this parameter which we have in the prolate case.

D. Detail of Miller's Calculation for Weak Field

In his thesis C. R. Miller obtained the Fokker-Planck equation (70) without the terms $R^{(T)}$. In the second part of his work, Miller solved this equation for nearly-spherical grains in weak magnetic fields. His procedure was to first change variables from the original μ, η, ζ set to

$$r = \cos\beta = \eta\zeta^{-\frac{1}{2}}$$

$$\rho = \cos\theta = \mu\,\zeta^{-\frac{1}{2}}$$

$$p = J^2/(m^+c^2 I\,\gamma) = \zeta/(m^+c^2 I\,\gamma) \quad . \tag{213}$$

The variables r, ρ, and p are the same quantities used elsewhere in our work. For purposes of comparison, we note that

$$\text{our } \rho = \cos\theta = \text{Miller's "s"} \quad ,$$

$$\text{our } p = J^2/(m^+ c^2\, I\,\gamma) = \text{Miller's "q"} \quad , \tag{214}$$

while $r = \cos\beta$ is the same in both works.

If the terms $R^{(T)}$ are removed from equation (70), $\frac{\partial W}{\partial t}$ is set equal to zero, the above changes of variable are made, and a subscript on f means a partial derivative with respect to that variable, then the resulting equation is

$$\frac{1}{p}\left\{\frac{(1+a)}{2a}(1-\rho^2)f_{\rho\rho} - \frac{(1+a)}{2a}2\rho\, f_p + \frac{(1+a)}{2a}(1-r^2)f_{rr} - \frac{(1+a)}{2a}2r\, f_r\right.$$

$$\left. + 4p^2\, f_{pp} + \frac{2(1+2a)}{a}\, p\, f_p + \left[\left(\frac{\gamma}{a}+2\right)p - p^2\right] f\right\}$$

$$+ (1-a)\left\{\frac{4}{a}\rho(1-\rho^2)f_{p\rho} + \frac{1}{2ap}(1-3\rho^2+2\rho^4)f_{\rho\rho} + \left[\frac{3\rho^3}{ap} - \frac{2\rho}{ap}\right]f_\rho\right.$$

$$\left. -\rho^2\frac{(1-r^2)}{2ap}f_{rr} + \frac{\rho^2 r}{ap}f_r + \frac{4p\rho^2}{a}f_{pp}\right\}$$

$$+ p\rho^2\left(1 - \frac{\gamma^2}{a}\right)f$$

$$+ \frac{2}{a}b\left\{\left[-\frac{1}{2}(\gamma-1)^2(\rho^2+\rho^4)(r^2+1)p + 2(\gamma-1)^2r^2\rho^4p + p(r^2-1)(1+2(\gamma-1)\rho^2)\right.\right.$$

$$\left. + \frac{1}{2}(\gamma-1)(r^2+1)(\rho^2+1) - 2(\gamma-1)r^2\rho^2 + 2\right]f$$

$$+ 2(1-r^2)\left[1+(\gamma-1)\rho^2\right]pf_p - (1-r^2)\left[1+(\gamma-1)\rho^2\right]rf_r$$

$$\left. + \frac{1}{2}(\gamma-1)(1+r^2)(1-\rho^2)\rho\, f_\rho\right\} = 0 \quad . \tag{215}$$

When Miller obtained this equation, he made an error in finding the coefficient of $(1-a)f_\rho$. Instead of

$$\left[\frac{3\rho^3}{ap} - \frac{2\rho}{ap}\right] ,$$

Miller obtained

$$\frac{\rho^3}{ap} \quad .$$

Although this error may be serious for other cases - such as non-spherical grains - for nearly-spherical grains this error was not serious for reasons to be given shortly. Miller solved this equation as a problem in perturbation theory, regarding the first group of terms as his "unperturbed" equation. All the other terms in this equation are small for weak fields and nearly-spherical grains.

The terms in the first group contain the differential operators for Legendre polynomials in ρ and r and for Laguerre polynomials in p. Miller used these functions as a complete orthonormal set of eigenfunctions and expanded his solution in terms of them. Thus, he wrote f in the form

$$f = \sum_{\ell, m, n} P_\ell(\rho)\, P_m(r) \cdot y_n(p) \quad , \tag{216}$$

obtained $y_n(p)$ in terms of Laguerre polynomials, and then solved for the contribution which each perturbing term in equation (215) would make. He found that only $\ell = m = 2$ was needed to find the contribution to F, and we came to the same conclusion in deriving equation (194) for Q_A. Miller then evaluated numerically the contribution from the p variables and obtained

$$F = 0.0161\,(\gamma-1)\,\frac{DB^2}{Q} \quad , \tag{217}$$

where

$$\text{Miller's } Q^{-1} = \frac{2}{\alpha}\, b . \frac{1}{(DB^2)} \quad ,$$

$$\frac{DB^2}{Q} = \frac{2}{\alpha}\, b \quad . \tag{215}$$

The reason that Miller's error was harmless is that the $(1-\alpha)f_\rho$ term is independent of b; hence, to the accuracy of Miller's approximation, its contribution to F is negligible. Miller's eigenfunction expansion is equivalent to a Green's function method of solution because the infinite sum over the eigenfunctions can be written as a Green's function. Since our solution was substantially different in algebraic detail from Miller's, the agreement of the final results is encouraging. Miller's thesis may be consulted for further details about his solution.

E. Miller's Derivation of the Diffusion Coefficients

Due to Collisions

We will obtain equations (34)-(38) as follows: First, we will recall the variables μ, η, ζ used by Miller. Next, we will find $\delta \underset{\sim}{J}$ due to a single collision, using a coordinate system suited to the grain. We will then find Θ, the transition probability to be used. After that, the moments E_i and E_{ij} of $\delta \underset{\sim}{J}$ will be found. Finally, these moments will be transformed from the grain coordinates to the μ, η, ζ coordinates by using

$$\mu = \underset{\sim}{J} . \hat{A}$$

$$\eta = (\underset{\sim}{J} . \underset{\sim}{B})/B$$

$$\zeta = \underset{\sim}{J} . \underset{\sim}{J} \qquad , \tag{65}$$

which are scalar equations and true in any coordinate system. Only terms to lowest order in the ratio (atom mass/grain mass) will be kept. Our treatment follows Miller's work very closely.

1. A Set of Variables

We must choose a set of variables x_i and δx_i to use in the Fokker-Planck equation (12). For our variables x_i, we use the same ones that Miller did:

$$\mu = J\cos\theta \quad , \quad -\infty < \mu < \infty$$

$$\eta = J\cos\beta \quad , \quad -\infty < \eta < \infty$$

$$\zeta = J^2 \quad , \quad 0 < \zeta < \infty \quad . \tag{26}$$

Now consider a single atom-grain collision: the event occurs quickly enough to produce an impulse $\delta \underset{\sim}{J}$ of angular momentum. This impulse is the δx_i which Miller used in equation (12); $\delta \underset{\sim}{J}$ will change $\underset{\sim}{J}$, β, and θ, but not the particle's orientation in space. The grain's reorientation follows from its nutation about the new $\underset{\sim}{J}$.

2. Effect of a Single Collision

Let $\underset{\sim}{U}$ = velocity of hydrogen atom before impact , (219a)

$\underset{\sim}{U}_1$ = velocity of hydrogen atom after impact , (b)

m = mass of the atom , (c)

m_i = mass of the grain , (d)

$\underset{\sim}{\omega}$ = angular velocity of the grain , (e)

$\underset{\sim}{R}$ = position vector from the origin of the grain's coordinates to the point of impact ,

then the angular momentum lost by the atom is that delivered to the grain, or

$$\delta \underset{\sim}{J} = m\,\underset{\sim}{R} \times (\underset{\sim}{U} - \underset{\sim}{U}_1) \tag{220}$$

For an elastic collision, Miller assumed that the atom reverses all components of its velocity relative to the grain surface. In a standard elastic collision, only the velocity component normal to the grain surface is reversed - while the parallel component is unchanged. Miller's type of elastic collision is much easier to treat than the standard one.

Therefore, let us define two frames of reference: (i) the rest frame of the grain surface, which is the primed frame; (ii) the rest frame of the grain center of mass, which is unprimed. The incoming atom velocity is

$$(\underset{\sim}{U}')_{\text{elastic}} = \underset{\sim}{U} - (\underset{\sim}{\omega} \times \underset{\sim}{R}) \quad , \tag{221}$$

and the outgoing atom velocity is

$$\begin{aligned}(\underset{\sim}{U}_1')_{\text{elastic}} &= -(\underset{\sim}{U}')_{\text{elastic}} \\ &= -[\underset{\sim}{U} - (\underset{\sim}{\omega} \times \underset{\sim}{R})] \quad .\end{aligned} \tag{222}$$

Thus, we find in the unprimed frame that

$$\begin{aligned}(\underset{\sim}{U}_1)_{\text{elastic}} &= (\underset{\sim}{U}_1')_{\text{elastic}} + (\underset{\sim}{\omega} \times \underset{\sim}{R}) \\ &= -\underset{\sim}{U} + 2(\underset{\sim}{\omega} \times \underset{\sim}{R}) \quad ,\end{aligned} \tag{223}$$

$$(\delta \underset{\sim}{J})_{\text{elastic}} = 2m\underset{\sim}{R} \times (\underset{\sim}{U} - \underset{\sim}{\omega} \times \underset{\sim}{R}) \quad . \tag{224}$$

Now consider an inelastic collision and suppose that the grain is at zero temperature. We assume that the atom hits, sticks, and then is thrown off with the local velocity of the grain surface. Or, we assume that for $T_i = 0$, the case Miller treated,

$$(\underset{\sim}{U}_1')_{\text{inelastic}} = 0 \quad ,$$

so that

$$(\underset{\sim}{U}_1)_{\text{inelastic}} = (\underset{\sim}{\omega} \times \underset{\sim}{R}) \quad . \tag{225}$$

Next, suppose that the grain temperature equals the gas temperature. From the second law of thermodynamics, we know that no net flow of energy can occur between the gas and the grain. One easy way to fulfill this requirement is simply to set the outgoing velocity equal and opposite to the incoming velocity - or, for $T_i = T$,

$$(\underset{\sim}{U}_1')_{\text{inelastic}} = +(\underset{\sim}{U}_1')_{\text{elastic}} \quad . \tag{226}$$

Although this assumption is unrealistic, we assume that when an average is taken over all atom-grain collisions, then the final results will be qualitatively correct.

Let us now consider the case for $T_i \neq T$. Suppose that in the primed frame the atom strikes the grain with an average energy of $\frac{1}{2}$ kT, so that

$$\frac{1}{2} m \langle (\underset{\sim}{U}')^2 \rangle = \frac{1}{2} kT \quad . \tag{227}$$

We will discuss this assumption shortly. Next, we assume that the atom leaves the grain, in exactly the opposite direction, with an average energy of $\frac{1}{2} kT_i$, so that

$$\frac{1}{2} m \langle (\underset{\sim}{U}_1')^2 \rangle = \frac{1}{2} kT_i \quad , \tag{228}$$

and, if we take root mean square values, then

$$(\underset{\sim}{U}_1')_{\text{inelastic}} = -\sqrt{\frac{T_i}{T}}\, \underset{\sim}{U}' \quad . \tag{229}$$

Equation (227) is only approximately correct since we should write

$$\frac{1}{2}\, m\,\langle(\underset{\sim}{U})^2\rangle = \frac{1}{2}\, kT = \frac{1}{2}\, m\,\langle[\underset{\sim}{U}' + (\underset{\sim}{\omega}\times\underset{\sim}{R})]^2\rangle \quad . \tag{230}$$

However, we find that

$$\frac{\omega R}{U} \sim \frac{m}{m_i} \quad , \tag{231}$$

where m_i is the grain mass, so that we may ignore this small effect. Equation (229) is also unrealistic, but it appears to have some of the correct qualitative features. The equation is consistent with our previous assumptions for $T_i = 0$ and $T_i = T$, and it has the proper temperature dependence. We only regard equation (229) as a rough estimate which we hope will yield the correct qualitative behavior.

Therefore, we may write

$$\begin{aligned}(\underset{\sim}{U}_1')_{\text{inelastic}} &= -\sqrt{\frac{T_i}{T}}\;\underset{\sim}{U}' \\ &= -\sqrt{\frac{T_i}{T}}\,(\underset{\sim}{U} - \underset{\sim}{\omega}\times\underset{\sim}{R}) \quad ,\end{aligned} \tag{229}$$

so that

$$(\underset{\sim}{U}_1)_{\text{inelastic}} = (\underset{\sim}{U}_1')_{\text{inelastic}} + (\underset{\sim}{\omega}\times\underset{\sim}{R}) \quad ,$$

$$(\underset{\sim}{U} - \underset{\sim}{U}_1)_{\text{inelastic}} = \left(1+\sqrt{\frac{T_i}{T}}\right)(\underset{\sim}{U} - \underset{\sim}{\omega}\times\underset{\sim}{R}) \quad , \tag{232}$$

$$(\delta\underset{\sim}{J})_{\text{inelastic}} = m(1 + \sqrt{T_i/T})\,\underset{\sim}{R}\times(\underset{\sim}{U} - \underset{\sim}{\omega}\times\underset{\sim}{R}) \tag{233}$$

We may combine equations (224) and (233) by writing

$$\delta\underset{\sim}{J} = 2m^+\,\underset{\sim}{R}\times(\underset{\sim}{U} - \underset{\sim}{\omega}\times\underset{\sim}{R}) \quad , \tag{234}$$

where m^+ is an effective mass for the collision model, so that

$$m^{+} = \begin{cases} m \text{ for an elastic collision} \\ \frac{1}{2} m \left(1 + \sqrt{\frac{T_i}{T}}\right) \text{ for an inelastic collision} \end{cases} \quad . \qquad (35f)$$

3. The Transition Probability

Let an element $d\Sigma$ of the grain's surface during a time dt collide with $N \, d\underset{\sim}{U} \, d\Sigma \, dt$ hydrogen atoms with velocities between $\underset{\sim}{U}$ and $\underset{\sim}{U} + d\underset{\sim}{U}$. If dt << (average time between collisions), then $\Theta = N \, d\underset{\sim}{U} \, d\Sigma \, dt$ also gives the probability of a collision during dt. Although equations (7)-(12) use a time Δt >> (average time during collisions) >> dt, Miller[38] showed that using dt to find E_i and E_{ij} is equally correct. Thus,

$$E_i \, dt = dt \int (\delta J_i) N \, d\underset{\sim}{U} \, d\Sigma \quad , \qquad i = x, y, Z \quad , \qquad (235)$$

$$E_{ij} \, dt = dt \int (\delta J_i)(\delta J_j) N \, d\underset{\sim}{U} \, d\Sigma \, , \qquad i, j = x, y, Z \, , \qquad (236)$$

where the x, y, Z coordinate system is fixed in the grain, with its orientation to be specified later; $d\underset{\sim}{U} = dU_x \, dU_y \, dU_Z$; and the integrations are taken over all of the grain surface and atom velocities.

Relative to a point on the grain surface, the atom velocity is $\underset{\sim}{U} - (\underset{\sim}{\omega} \times \underset{\sim}{R})$. For a collision to occur, the normal component of this velocity must be toward the grain. If this requirement is combined with a Maxwell distribution of atom velocities, then

$$\Theta = N \, d\underset{\sim}{U} \, d\Sigma \, dt = \begin{cases} -A \, e^{-mU^2/2kT} \, \underset{\sim}{n} \cdot [\underset{\sim}{U} - (\underset{\sim}{\omega} \times \underset{\sim}{R})] \, d\underset{\sim}{U} \, d\Sigma \, dt \\ \qquad \text{for } \underset{\sim}{n} \cdot [\underset{\sim}{U} - (\underset{\sim}{\omega} \times \underset{\sim}{R})] \le 0 \\ 0 \qquad \text{for } \underset{\sim}{n} \cdot [\underset{\sim}{U} - (\underset{\sim}{\omega} \times \underset{\sim}{R}) > 0 \end{cases} \quad . \qquad (237)$$

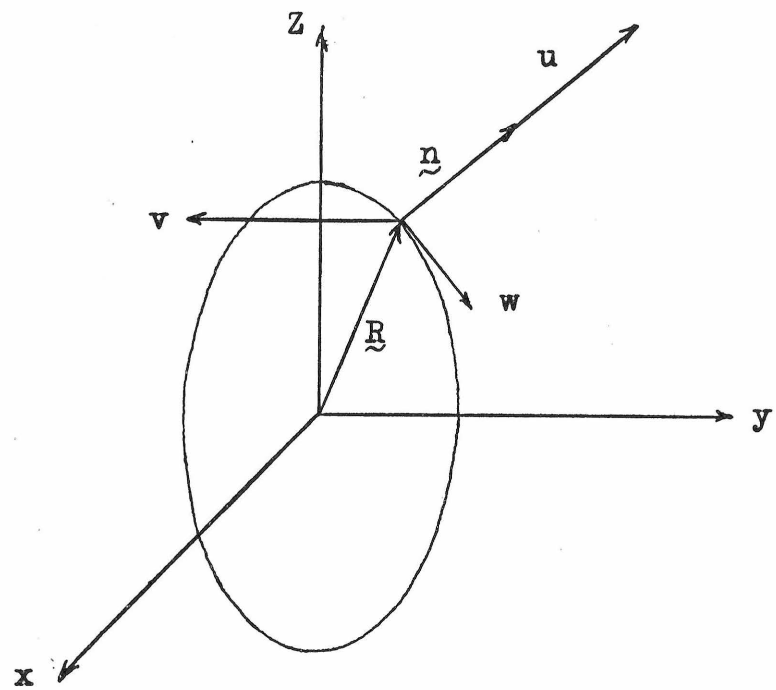


Figure 7

The u, v, w components of $\underset{\sim}{\Omega}$ which are used in the integrations over velocity

Here, $\underset{\sim}{n}$ is a unit vector outward and normal to the grain surface, and A is a constant to normalize the atom density to n_H/cm^3, or

$$n_H = \int A\left(e^{-m\, U^2/2\, kT}\right) d\underset{\sim}{U}$$

$$A = \pi^{-3/2}\, n_H\, c^{-3} \qquad . \qquad (238)$$

In addition, c is the hydrogen atom characteristic velocity, given by

$$c = (2kT/m)^{\frac{1}{2}} \qquad . \qquad (239)$$

4. <u>Orders of Magnitude of Terms and Velocity Integrations</u>

To estimate the order of magnitude of terms in equations (235) and (236), we may expect that a term involving $\underset{\sim}{\omega}$ and $\underset{\sim}{R}$ will be of order $\omega R/c$. Then $kT \sim \frac{1}{2} I\, \omega^2 \sim \frac{1}{2} m_i\, R^2\, \omega^2$, so that

$$(\omega R/c) \sim (2KT/m_i)^{\frac{1}{2}} \Big/ (2kT/m)^{\frac{1}{2}} \sim (m/m_i)^{\frac{1}{2}} \qquad , \qquad (240)$$

and only terms of lowest order in this ratio will be kept.

To treat the limits which equation (237) sets on the velocity integration, introduce the velocity $\underset{\sim}{\Omega}$, where

$$\underset{\sim}{\Omega} = \underset{\sim}{U}/c \qquad . \qquad (241)$$

At each point on the grain surface, attach a coordinate system for $\underset{\sim}{\Omega}$ as shown in Figure 7. The components u, v, and w of $\underset{\sim}{\Omega}$ are oriented so that the u-axis is along $\underset{\sim}{n}$; the v-axis is normal to the $\underset{\sim}{n}$, Z plane and is parallel to the x-y plane; and the w axis is in the $\underset{\sim}{n}$, Z plane. With these coordinates, $\underset{\sim}{n}.\underset{\sim}{U} = c\, u$, so that the limits on $\underset{\sim}{U}$ are

$$-\infty < v,\ w < \infty \quad \text{and} \quad -\infty < u \le \underset{\sim}{n}.(\underset{\sim}{\omega} \times \underset{\sim}{R})/c \qquad .$$

5. The First Moments E_i

We denote the vector for the first three moments by

$$\underset{\sim}{E_J} = (E_x, E_y, E_Z) \tag{242}$$

$$= \int (\delta \underset{\sim}{J})\, N\, d\, \underset{\sim}{U}\, d\Sigma \quad . \tag{243}$$

If we substitute N from equation (237) and ($\delta\ \underset{\sim}{J}$) from equation (234), then

$$-\pi^{3/2}\, \underset{\sim}{E_J}/(2m^+ n_H c^2) = \oint\!\!\oint_{\text{surface}} d\Sigma \int_{v,w=-\infty}^{\infty} dv\, dw \int_{u=-\infty}^{0} (I)\, du$$

$$+ \oint\!\!\oint_{\text{surface}} d\Sigma \int_{v,w=-\infty}^{\infty} dv\, dw \int_{0}^{u_o} (I)\, du \quad , \tag{244}$$

where $u_o = \underset{\sim}{n} \cdot (\underset{\sim}{\omega} \times \underset{\sim}{R})/c$, and the integrand (I) is

$$(I) = \{u(\underset{\sim}{R} \times \underset{\sim}{\Omega}) + (1/c)(u_o)[\underset{\sim}{R} \times (\underset{\sim}{\omega} \times \underset{\sim}{R})] - u_o(\underset{\sim}{R} \times \underset{\sim}{\Omega})$$

$$- (1/c)\, u\, \underset{\sim}{R} \times (\underset{\sim}{\omega} \times \underset{\sim}{R})\} \exp[-(u^2+v^2+w^2)] \quad . \tag{245}$$

Consider the first integral in equation (244). If the first term in (I) is integrated over velocity space, the result will be

$$\iint_{v,w=-\infty}^{\infty} dv\, dw \int_{u=-\infty}^{0} du\, u\, (\underset{\sim}{R} \times \underset{\sim}{\Omega}) \exp[-(u^2 + v^2 + w^2)]$$

$$= \underset{\sim}{R} \times \int u\, \underset{\sim}{\Omega}\, d\Omega \exp[-(u^2 + v^2 + w^2)] \quad .$$

The integrals for the v and w components here will give zero. Only the u component will be left, giving a constant times $\underset{\sim}{R} \times \underset{\sim}{n}$. But,

$$\oint\!\!\oint_{\text{surface}} d\Sigma\, (\underset{\sim}{R} \times \underset{\sim}{n}) = -\iiint_{\text{volume}} dV (\nabla \times \underset{\sim}{R}) = 0 \quad ,$$

since $\nabla \times \underset{\sim}{R} = 0$, so that the first term gives no contribution. As the second term is of higher order in (m/m_i) than the third or fourth terms, we neglect it and are left with, if $\Omega^2 = u^2 + v^2 + w^2$,

$$\text{1st integral} = -\iint_{\text{surface}} d\Sigma \left\{u_o(\underset{\sim}{R}\times\underset{\sim}{n}) \oint\!\!\oint_{v,w=-\infty}^{\infty} dv\, dw \int_{u=-\infty}^{0} du\, u \exp(-\Omega^2)\right.$$

$$\left. + (1/c)\, \underset{\sim}{R} \times (\underset{\sim}{\omega}\times\underset{\sim}{R}) \iint_{v,w=-\infty}^{\infty} dv\, dw \int_{u=-\infty}^{0} du\, u \exp(-\Omega^2)\right\}$$

$$= \frac{\pi}{2c} \oint\!\!\oint_{\text{surface}} d\Sigma\, [(\underset{\sim}{R}\times\underset{\sim}{n})(\underset{\sim}{R}\times\underset{\sim}{n})\cdot\underset{\sim}{\omega} + \underset{\sim}{R} \times (\underset{\sim}{\omega}\times\underset{\sim}{R})] \quad . \tag{246}$$

Now consider the second integral of equation (244). In it the maximum value of $|u|$ is $|u_o| \sim (m/m_i)^{\frac{1}{2}} << 1$, so that $\exp(-u^2) = 1$ + second-order terms. If the terms in the integrand (I) are now treated, we see that the second integral only contributes to third order in $(\omega R/c)$ and may therefore be neglected. Thus, $-\pi^{3/2}\, \underset{\sim}{E_J}/(2m^+ n_H c^2)$ is given by equation (246).

The integrand for equation (246) may be written in terms of a dyadic. Thus, if U_1 is the unit dyad,

$$[(\underset{\sim}{R}\times\underset{\sim}{n})(\underset{\sim}{R}\times\underset{\sim}{n})\cdot\underset{\sim}{\omega} + \underset{\sim}{R}\times(\underset{\sim}{\omega}\times\underset{\sim}{R})]$$

$$= [R^2 U_1 + (\underset{\sim}{R}\times\underset{\sim}{n})(\underset{\sim}{R}\times\underset{\sim}{n}) - (\underset{\sim}{R})(\underset{\sim}{R})]\cdot\underset{\sim}{\omega} \quad , \tag{247}$$

as may be seen by expanding in the u, v, w system. If Φ is the grain's inertia tensor, and Φ^{-1} its inverse, then $\underset{\sim}{\omega} = \Phi^{-1}\underset{\sim}{J}$. If G and g are defined by

$$G = \oint\!\!\oint_{\text{surface}} d\Sigma\, [R^2 U_1 + (\underset{\sim}{R}\times\underset{\sim}{n})(\underset{\sim}{R}\times\underset{\sim}{n}) - (\underset{\sim}{R})(\underset{\sim}{R})] \quad , \tag{248}$$

$$g = \pi^{-\frac{1}{2}} n_H m^+ c \quad , \tag{249}$$

then we obtain

$$E_{\underset{\sim}{J}} = - g G \Phi^{-1} \underset{\sim}{J} \quad . \tag{250}$$

6. <u>The Second Moments E_{ij}</u>

We denote the symmetric matrix of the six second moments by

$$E_{\underset{\sim}{JJ}} = \begin{bmatrix} E_{xx} & E_{xy} & E_{xZ} \\ E_{yx} & E_{yy} & E_{yZ} \\ E_{Zx} & E_{Zy} & E_{ZZ} \end{bmatrix} \tag{251}$$

$$= \int (\delta \underset{\sim}{J})(\delta \underset{\sim}{J}) \, N \, d\underset{\sim}{U} \, d\Sigma \quad , \tag{252}$$

where $(\delta \underset{\sim}{J})(\delta \underset{\sim}{J})$ is the dyadic or its matrix representation. If equations (237) and (234) are substituted into equation (252), there results

$$-\pi^{3/2} E_{\underset{\sim}{JJ}}/[4 n_H c^3 (m^+)^2] = \int I_1 \, du \, dv \, dw \, d\Sigma \ , \tag{253}$$

where the region of integration is the same as for $E_{\underset{\sim}{J}}$, and

$$I_1 = \{(\underset{\sim}{R} \times \underset{\sim}{\Omega})(\underset{\sim}{R} \times \underset{\sim}{\Omega}) + (1/c^2)[\underset{\sim}{R} \times (\underset{\sim}{\omega} \times \underset{\sim}{R})][\underset{\sim}{R} \times (\underset{\sim}{\omega} \times \underset{\sim}{R})]$$
$$- (1/c)(\underset{\sim}{R} \times \underset{\sim}{\Omega})[\underset{\sim}{R} \times (\underset{\sim}{\omega} \times \underset{\sim}{R})] - (1/c)[\underset{\sim}{R} \times (\underset{\sim}{\omega} \times \underset{\sim}{R})](\underset{\sim}{R} \times \underset{\sim}{\Omega})\}$$
$$\{u - (1/c)\underset{\sim}{n} \cdot (\underset{\sim}{\omega} \times \underset{\sim}{R})\} \exp[-(u^2+v^2+w^2)] \quad . \tag{254}$$

The term of lowest order is $u(\underset{\sim}{R} \times \underset{\sim}{\Omega})(\underset{\sim}{R} \times \underset{\sim}{\Omega})\exp(-\Omega^2)$. Any terms which are odd functions of v or w will integrate to zero. If such odd terms are left out, the matrix which represents this dyadic is

$$(\underset{\sim}{R}\times\underset{\sim}{\Omega})(\underset{\sim}{R}\times\underset{\sim}{\Omega}) = \begin{bmatrix} (R_v^2 w^2 + R_w^2 v^2) & -R_u R_v w^2 & -R_u R_w v^2 \\ -R_u R_v w^2 & (R_w^2 u^2 + R_u^2 w^2) & -R_v R_w u^2 \\ -R_v R_w v^2 & -R_v R_w u^2 & (R_u^2 v^2 + R_v^2 u^2) \end{bmatrix}, \quad (255)$$

where R_u, R_v, and R_w are the components of $\underset{\sim}{R}$ along the u, v, w axes. If this expression is multiplied by u $\exp[-(u^2+v^2+w^2)]$ and integrated over the range of velocity variables $-\infty < v,\ w < \infty$, $-\infty < u \le 0$, the result is

$$\int I_1 \, du\, dv\, dw = -\frac{\pi}{4} \begin{bmatrix} (R_v^2 + R_w^2) & -R_u R_v & -R_u R_w \\ -R_v R_u & (2R_w^2 + R_u^2) & -2R_v R_w \\ -R_w R_u & -2R_w R_v & (2R_v^2 + R_u^2) \end{bmatrix}. \quad (256)$$

For the same reasons as given with $\underset{\sim}{E}_J$, the integral over the region $0 < u < u_o = \underset{\sim}{n}\cdot(\underset{\sim}{\omega}\times\underset{\sim}{R})/c$ may be neglected; thus $-\pi^{3/2}\underset{\sim\sim}{E}_{JJ}/[4n_H c^3(m^+)^2]$ is given by the integral of equation (256) over the surface of the grain.

By expanding in the u, v, w coordinate system, we may show that equation (256) equals the dyadic $-(\pi/4)[R^2 U_1 + (\underset{\sim}{R}\times\underset{\sim}{n})(\underset{\sim}{R}\times\underset{\sim}{n}) - (\underset{\sim}{R})(\underset{\sim}{R})]$. Therefore,

$$\underset{\sim\sim}{E}_{JJ} = g\, m^+ c^2 G \,, \quad (257)$$

where g and G are defined in equations (248) and (249).

7. The Moments E_i and E_{ij} for Spheroids

Equations (250) and (257) are valid for an arbitrarily shaped particle. To treat spheroids, we may take the Z-axis as the axis of rotational symmetry for the grain. By considering the off-diagonal terms, we find that G is diagonal in this frame. Thus,

$$G = \begin{bmatrix} \alpha h & 0 & 0 \\ 0 & \alpha h & 0 \\ 0 & 0 & h \end{bmatrix} , \tag{258}$$

where we have written the two equal terms as αh , and G is determined by the two constants α and h.

This coordinate system will also diagonalize the inertia tensor, which may be written

$$\Phi = \begin{bmatrix} \gamma I & 0 & 0 \\ 0 & \gamma I & 0 \\ 0 & 0 & I \end{bmatrix} , \quad \Phi^{-1} = \begin{bmatrix} 1/\gamma I & 0 & 0 \\ 0 & 1/\gamma I & 0 \\ 0 & 0 & 1/I \end{bmatrix} . \tag{259}$$

If we now put equations (258) and (259) into (250) and (257), we obtain

$$E_x = -g \frac{h}{I} \frac{\alpha}{\gamma} J_x$$

$$E_y = -g \frac{h}{I} \frac{\alpha}{\gamma} J_y$$

$$E_Z = -g \frac{h}{I} J_Z , \tag{260}$$

and

$$E_{xx} = g m^{*} c^2 \alpha h$$

$$E_{yy} = g m^{*} c^2 \alpha h$$

$$E_{ZZ} = g m^{*} c^2 h$$

$$E_{xy} = E_{xZ} = E_{yZ} = 0 . \tag{261}$$

We may now perform the surface integrals of equation (248). If the spheroid diameter along its rotation axis is $2a\epsilon$, and its perpendicular diameter is $2a$, then we find

$$h = \pi a^4 \left\{ 1 + \frac{1}{2} \frac{\epsilon^2}{\epsilon^2 - 1} + \left[2 - \frac{1}{2} \frac{\epsilon^2}{\epsilon^2 - 1} \right] \frac{\epsilon^2}{|\epsilon^2 - 1|^{\frac{1}{2}}} \begin{pmatrix} \sin^{-1} \\ \sinh^{-1} \end{pmatrix} \left(\frac{|\epsilon^2 - 1|^{\frac{1}{2}}}{\epsilon} \right) \right\} , \tag{262}$$

$$\alpha h = \pi a^4 \left\{ 1 + \epsilon^2 - \frac{1}{4} \frac{\epsilon^4}{(\epsilon^2 - 1)} + \left[2 + \frac{\epsilon^2}{\epsilon^2 - 1} \right] \left[\frac{1}{4} \frac{\epsilon^4}{|\epsilon^2 - 1|^{\frac{1}{2}}} \right] \begin{pmatrix} \sin^{-1} \\ \sinh^{-1} \end{pmatrix} \left(\frac{|\epsilon^2 - 1|^{\frac{1}{2}}}{\epsilon} \right) \right\} . \tag{263}$$

In these equations, $\sinh^{-1}$ is to be used for an oblate spheroid, $\epsilon < 1$, and $\sin^{-1}$ for a prolate spheroid, $\epsilon > 1$. A plot of α is shown in Figure 2. For the nearly-spherical case, when $\epsilon \cong 1$, we have

$$h \cong \pi a^4 \, [8/3 + (16/15)(\epsilon^2 - 1)]$$

$$\alpha h \cong \pi a^4 \, [8/3 + (32/15)(\epsilon^2 - 1)]$$

$$\alpha \cong 1 + \frac{2}{5}(\epsilon^2 - 1) \quad . \tag{35ℓ}$$

For the disk and sphere, $\alpha = 1$; for the needle

$$\alpha \cong (1/2)\epsilon^2 + (1/3) \quad . \tag{35ℓ}$$

The moments of inertia can be found if the grain is of uniform density. They are

$$I = (2/5) \, M a^2 \quad ,$$

$$\gamma I = (1/5) \, M a^2 (1 + \epsilon^2) \quad , \tag{264}$$

$$\gamma = \frac{1}{2}(1 + \epsilon^2) \quad . \tag{35m}$$

Thus, for the needle, $\alpha \cong \gamma - \frac{1}{6}$.

8. <u>The Moments E_i and E_{ij} for the Fokker Planck Equation</u>

To express the moments in equations (55) and (56) in terms of the orientation variables μ, η, and ζ, we use equation (25) ,

$$\mu = \underset{\sim}{J} . \hat{A}$$

$$\eta = \underset{\sim}{J} . \hat{B}$$

$$\zeta = \underset{\sim}{J} . \underset{\sim}{J} \qquad , \qquad (65)$$

where $\hat{B}$ is a unit vector in the direction of $\underset{\sim}{B}$. Since $\hat{A}$ and $\hat{B}$ remain fixed in a collision and only $\underset{\sim}{J}$ changes,

$$\delta\mu = \hat{A}_x \, \delta J_x + \hat{A}_y \, \delta J_y + \hat{A}_Z \, \delta J_Z$$

$$\delta\eta = \hat{B}_x \, \delta J_x + \hat{B}_y \, \delta J_y + \hat{B}_Z \, \delta J_Z$$

$$\delta\zeta = (\underset{\sim}{J} + \delta\underset{\sim}{J})^2 - \underset{\sim}{J}^2$$

$$= 2(J_x \, \delta J_x + J_y \, \delta J_y + J_Z \, \delta J_Z)$$

$$+ (\delta J_x)^2 + (\delta J_y)^2 + (\delta J_Z)^2 \qquad (265)$$

To find the components of $\hat{A}$, $\hat{B}$, and $\underset{\sim}{J}$ in terms of μ, η, and ζ, orient the grain as shown in Figure 8: the symmetry axis $\hat{A}$ is along Z, as previously; and now the y-axis is in the $\underset{\sim}{J}$, $\hat{A}$ plane. To fix the direction of $\underset{\sim}{B}$, we need ψ, the nutational angle, and all quantities will be averaged over ψ. Thus,

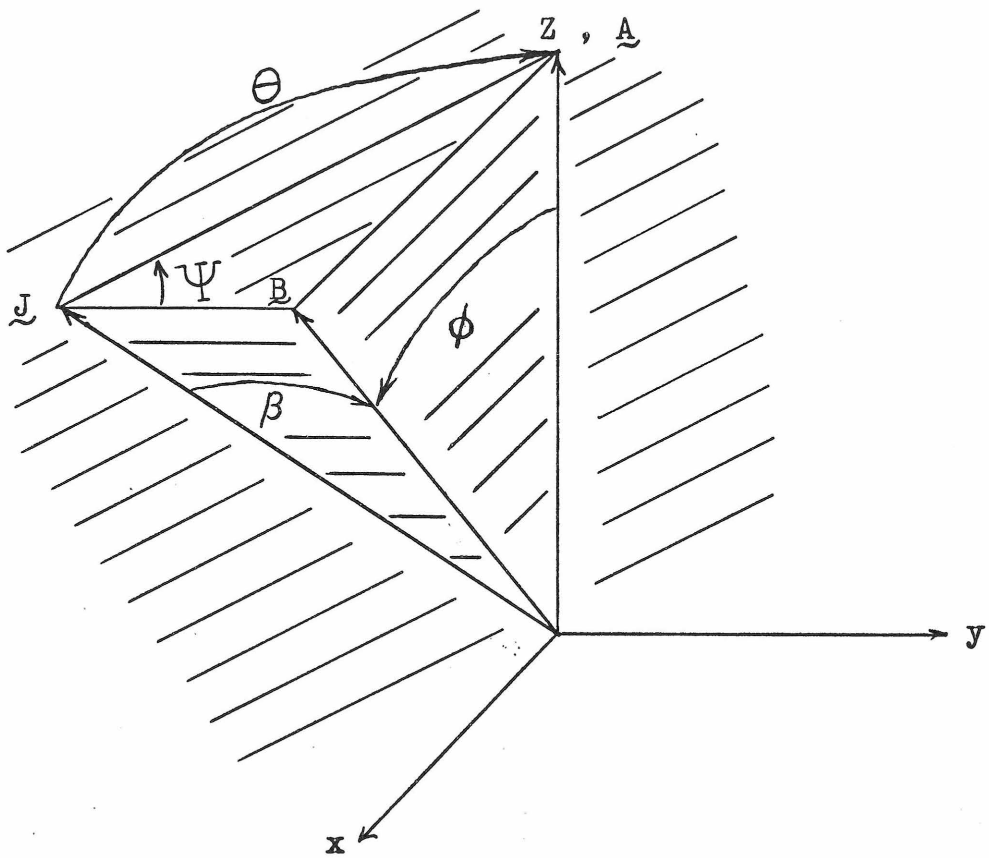


Figure 8

Orientation of the x, y, Z Coordinate System in Space

$$\hat{A}_x = \hat{A}_y = 0$$
$$\hat{A}_Z = 1 \qquad , \tag{266}$$

$$J_x = 0$$

$$J_y = J\sin\theta$$

$$J_Z = J\cos\theta = \mu \qquad , \tag{267}$$

$$\hat{B}_x = \sin\psi \sin\beta$$

$$\hat{B}_y = \cos\psi \sin\beta \cos\theta - \cos\beta \sin\theta$$

$$\hat{B}_Z = \cos\psi \sin\beta \sin\theta + \cos\beta \cos\theta \qquad . \tag{268}$$

From equation (55), we obtain

$$E_x = 0$$
$$E_y = \frac{gh}{I\gamma} a\, J \sin\theta$$
$$E_Z = \frac{gh}{I} \mu \qquad , \tag{269}$$

and from equation (265)

$$\delta\mu = \delta J_Z$$

$$\delta\eta = \hat{B}_x \delta J_x + \hat{B}_y \delta J_y + \hat{B}_Z \delta J_Z$$

$$\delta\zeta = 2J(-\sin\theta\, \delta J_y + \cos\theta\, \delta J_Z)$$

$$+ (\delta J_x)^2 + (\delta J_y)^2 + (\delta J_Z)^2 \qquad . \tag{270}$$

We may now obtain the moments in the μ, η, ζ coordinate system. Equation (68) provides $\delta\mu$ $\delta\eta$, $\delta\zeta$, $(\delta\mu)^2$, $(\delta\eta)^2$, $(\delta\zeta)^2$,

$(\delta\mu\delta\eta)$, $(\mu\delta\zeta)$, and $(\delta\eta\delta\zeta)$; by using equations (260), (261), (268), and (269), we find

$$E_\mu = -\frac{gh}{I}\mu$$

$$E_\eta = -\frac{gh}{I}\eta\left[\frac{a}{\gamma} - \frac{a}{\gamma}\frac{\mu^2}{\zeta} + \frac{\mu^2}{\zeta}\right]$$

$$E_\zeta = \frac{gh}{I}\left[m^+ c^2 I(1+2a) - \frac{2\mu^2(\gamma-a) + 2a\zeta}{\gamma}\right]$$

$$E_{\mu\mu} = gh\, m^+ c^2$$

$$E_{\eta\eta} = gh\, m^+ c^2\left[\frac{1}{2}(a+1) + \frac{1}{2}(a-1)\left(\frac{\eta^2}{\zeta} + \frac{\mu^2}{\zeta} - 3\frac{\mu^2\eta^2}{\zeta^2}\right)\right]$$

$$E_{\zeta\zeta} = 4\, gh\, m^+ c^2[a\,\zeta + (1-a)\mu^2]$$

$$E_{\mu\eta} = gh\, m^+ c^2\frac{\mu\eta}{\zeta}$$

$$E_{\mu\zeta} = 2\, gh\, m^+ c^2 \mu$$

$$E_{\eta\zeta} = 2\, gh\, m^+ c^2\left[a\eta + (1-a)\frac{\mu^2\eta}{\zeta}\right] \quad . \tag{271}$$

These are the moments due to collisions which must be put into equation (33). When this substitution is done, we obtain equations (36), (37), and (38).

F. Effects of Fluctuations in the Galactic Field

According to arguments by J. R. Jokipii and E. N. Parker,[40] the galactic field itself fluctuates. They proposed a stochastic model of the field in order to explain the escape of cosmic rays from the galaxy. In their model, the lines of force do a random walk, and the length scale of these fluctuations is of order 100 pc. To obtain a time scale, we may use the Alfvén velocity $\sim (B/\sqrt{\rho})$ for a characteristic speed. When $B \sim 10^{-5}$ gauss and $\rho \sim 1$ H atom/cm^3, we find a time scale of order 10^{13}-10^{14} sec - which is about the same order as the time required to align the grain.

However, this time scale will not adversely affect the grain dynamics. The aligning torque is obtained by averaging over a single nutation of the grain $\sim 10^{-5}$ sec; thus, the field fluctuations here have no effect. The particle's precession is altered very little, since it occurs $\sim 10^4$ times faster than the alignment - as was shown in the discussion following equation (25). Thus, these field fluctuations affect the particle only in its long-term alignment. The grain is oriented with respect to the field and follows the field direction.

Greenberg[11, 12] treated the case when the field fluctuates more rapidly than the alignment takes place. The result is that the qualitative effect on the polarization is the same as the effect of incompletely aligned grains. At present, it is unclear if such field fluctuations do occur.

G. A Summary of Coordinate Systems and Changes of Variable

(1) The μ, η, ζ system was used for treating the effects of collisions of the grain with surrounding hydrogen.

$$\begin{aligned} \mu &= J\cos\theta \quad , \quad -\infty < \mu < \infty \\ \eta &= J\cos\beta \quad , \quad -\infty < \eta < \infty \\ \zeta &= J^2 \quad , \quad 0 < \zeta < \infty \end{aligned} \tag{26}$$

(2) The X, Y, Z_1 system was used for treating the effects of non-zero T_i .

X and Y are axes fixed in space with X and Y $\perp \underset{\sim}{B}$, X $\perp$ Y, and $Z_1 \parallel \underset{\sim}{B}$. See Figure 4.

(3) The r, s, z system was used for treating the effects of weak fields. We transformed $(\mu, \eta, \zeta) \rightarrow (r, s, z)$ with τ a dependent variable.

$$
\begin{aligned}
r &= \cos\beta = \eta\,\zeta^{-\frac{1}{2}} &&, \quad -1 \le r \le 1 \\
s &= \frac{J\cos\theta}{\sqrt{m^+ c^2 I\gamma}} = \frac{\mu}{\sqrt{m^+ c^2 I\gamma}} &&, \quad -\infty < s < \infty \\
z &= \frac{J^2 \sin^2\theta}{m^+ c^2 I\gamma} = \frac{\zeta - \mu^2}{m^+ c^2 I\gamma} &&, \quad 0 \le z < \infty \\
\tau &= (z+s^2)^{\frac{1}{2}} &&, \quad 0 \le \tau < \infty \quad .
\end{aligned}
\tag{71}
$$

(4) The σ, ν, ρ; τ, λ system was used for treating the strong field case. We transformed $(r, s, z) \rightarrow (\sigma, \nu, \rho)$ with τ a dependent variable.

$$
\begin{aligned}
\sigma &= \tau\cos\beta = \tau r &&, \quad -\infty < \sigma < \infty \\
\nu &= \tau\sin\beta = \tau(1-r^2)^{\frac{1}{2}} &&, \quad -\infty < \nu < \infty \\
\rho &= \cos\theta = \frac{s}{\tau} &&, \quad -1 \le \rho \le 1 \\
\tau &= (\sigma^2 + \nu^2)^{\frac{1}{2}} &&, \quad 0 \le \tau < \infty \quad .
\end{aligned}
\tag{74}
$$

We transformed $\rho \rightarrow \lambda$ for oblate grains

$$
\lambda = (1-\rho^2)^{\frac{1}{2}} = \sin\theta \quad , \quad -1 \le \lambda \le 1 \quad . \tag{105}
$$

(5) The x, y, Z frame was used in order to orient the grain in space while treating the effects of collisions. Thus,

x, y, and Z are axes fixed in the grain as shown in Figure 8.

(6) The u, v, w coordinate system was used for the velocity integrations. See Figure 7 and the discussion following equation (241).

LIST OF SYMBOLS

a, α	shape factor
β	angle between $\underset{\sim}{J}$ and $\underset{\sim}{B}$
γ	ratio of moments of inertia
Γ	$m^+ c^2 I \gamma$
δ	$\begin{cases} (\epsilon^2-1) \\ \text{increment notation} \end{cases}$
δ_1	$(1-\epsilon^2)$
Δ	increment notation
ϵ	ratio of grain semiaxes
	$\epsilon > 1$ prolate grain
	$\epsilon < 1$ oblate grain
	$\epsilon = 1$ sphere
ϵ_o	$E_o/(ghm^+c^2) = b(T_i/T)(m/m^+)$
ζ	J^2
η	$J \cos\beta$
θ	angle between $\underset{\sim}{J}$ and $\underset{\sim}{A}$
Θ	transition probability
λ	$\sin\theta$
Λ	a function of s and z
μ	$J \cos\theta$
ν	$J \sin\beta/\sqrt{\Gamma} = \tau \sin\beta$
ξ	$\sqrt{T_{av}/T} = \sqrt{(1+\epsilon_o)/(1+b)}$
ξ_o	$\sqrt{(1+\epsilon_o+\frac{1}{4}\delta)/(1+b+\frac{1}{4}\delta)}$
ρ	$\cos\theta$
τ	$(\sigma^2+\nu^2)^{\frac{1}{2}} = (z+s^2)^{\frac{1}{2}} = J/\sqrt{\Gamma}$

LIST OF SYMBOLS (Continued)

φ	angle between $\underset{\sim}{B}$ and $\underset{\sim}{A}$
Φ	inertia tensor of grain
χ	magnetic susceptibility
χ'	real part of χ
χ''	imaginary part of χ
ψ	$(f-1)/b$ for weak fields
Ψ	angle between $\underset{\sim}{J}$, $\underset{\sim}{A}$ and $\underset{\sim}{J}$, $\underset{\sim}{B}$ planes
$\underset{\sim}{\omega}$	angular velocity of grain
Ω	speed = U/c
σ	$J \cos\beta/\sqrt{\Gamma} = \tau \cos\beta$
Σ	surface of grain
$\wedge$	unit vector
$\underset{\sim}{A}$	grain symmetry axis
$\underset{\sim}{B}$	magnetic field of galaxy
C	a function of S in weak field
D	$(\chi''/\omega)(V/I\gamma)$
E	$\left\{\begin{array}{l}\text{expectation values}\\ \text{moments of } \Delta J\end{array}\right.$
E_o	$2\,k\,T_i(\chi''/\omega)VB^2$
F	distribution integral
G	shape factor matrix
H	$\left\{\begin{array}{l}\text{hydrogen}\\ \text{Hermite polynomial}\end{array}\right.$
I	longitudinal moment of inertia
γI	transverse moment of inertia
$\underset{\sim}{J}$	angular momentum of grain

LIST OF SYMBOLS (Continued)

K	a function of s and z
K_X	steady effects of DG torque
L	probability current
$\underset{\sim}{M}$	magnetization
$M(a_1, b_1, z)$	confluent hypergeometric function
N	$\left\{ \begin{array}{l} \sqrt{b}\,\nu \\ \text{number of atoms striking grain} \end{array} \right.$
N_o	a function of τ
N_i	normalizations for cases i
P	$\sqrt{b(\gamma-1)}\,\rho = \sqrt{\frac{1}{2}b\delta}\,\rho$
Q	$\sqrt{b}\,\sin\theta$
Q_A	measure of axial alignment
Q_J	measure of $\underset{\sim}{J}$ alignment
$\underset{\sim}{R}$	radius vector
$R^{()}$	right-hand side of equation
R_o	rotational kinetic energy
S	$\sqrt{\gamma}\,s$
T	temperature of gas
T_i	internal temperature of grain
T_{av}	$T(1+\varepsilon_o)/(1+b)$
T_{eff}	$T(m^+/m)$
$\underset{\sim}{U}$	velocity of H atom
$U(a_1, b_1, z)$	confluent hypergeometric function
U_1	unit dyad
V	volume of grain
W	distribution function

LIST OF SYMBOLS (Continued)

W_{MB}	Maxwell-Boltzmann distribution
W_o	W_{MB} for no gas
X	an axis fixed in space $\perp \underset{\sim}{B}$
Y	an axis fixed in space $\perp \underset{\sim}{B}$
X_1	a function of p
Z	an axis fixed in the grain
Z_1	an axis fixed in space $\parallel \underset{\sim}{B}$
Z_2	a function of z
a	transverse semiaxis
$a\epsilon$	semiaxis of symmetry
a_1	$j+1 = \sqrt{3}/2 + 1$
b_1	$2j+1 = \sqrt{3} + 1$
b	$(\chi''/\omega)\, VB^2/gh$
b_o	$n_H\, b$
c	$\sqrt{2kT/m}$
f	$W_{MB}^{-1}\, W$
g	$\pi^{-\frac{1}{2}}\, n_H\, m^+ c$
h	shape factor
i	subscript
j	$\sqrt{3}/2$, subscript
k	Boltzmann's constant
ℓ	subscript
m	mass of H atom, subscript
m_i	mass of grain

LIST OF SYMBOLS (Continued)

$\underset{\sim}{n}$	unit vector
n_H	# H atoms/cm^3
p	τ^2
q	$\sqrt{b\delta_1}\ \lambda = \sqrt{b\delta_1}\ \sin\theta$
r	$\cos\beta$
s	$J \cos\theta/\sqrt{\Gamma} = \tau \cos\theta$
t	time
u, v, w	velocity integration coordinates
x_i	variables used in defining Fokker Planck equation
x, y	axes fixed in grain
z	$J^2 \sin^2\theta/\Gamma = \tau^2 \sin^2\theta$

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