A Symphony of Supersymmetry and Geometry: Invariants, Dualities and Chiral Rings

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To my family and Fan Zhang.

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ABSTRACT

The present dissertation discusses aspects of supersymmetric quantum field theory, whose main themes are two-folded. First, we explore connections between superconformal theories in various dimensions and geometric invariants. Such correspondence arises from compactification of string theory or M-theory, which encodes geometric quantities into physical observables. Second, we study in detail the chiral rings and their quantum corrections in certain supersymmetric gauge theory. The goal is to shed some light on the hitherto mysterious electric-magnetic dualities.

We first consider M5 brane on the product manifold $L(k,1) \times M_3$, where $M_3 = L(p,1)$. Compactification on $L(p,1)$ gives rise to three dimensional theory $T[L(p,1)]$ whose partition function, according to 3d-3d correspondence, is equivalent to Chern-Simons invariants with complex gauge group on $L(p,1)$. We test the statement in Chapter 2 by taking $k = 0$ and calculating the supersymmetric index. We find a full agreement between two seemingly distinct quantities. In particular, when $p = 1$, we see the familiar S^3 partition function of Chern-Simons theory arises from the index of a free theory.

We then move on in Chapter 3 to consider $M_3 = S^1 \times \Sigma$, and twisted compactification on general Riemann surface Σ with tame punctures. The twisted partition function of lens space theory $T[L(k,1)]$ on $S^1 \times \Sigma$ computes the graded dimension of the Hilbert space after geometrically quantizing Hitchin moduli space $\mathcal{M}_H$, dubbed as "tame Hitchin characters" or "equivariant Verlinde formula". We show that this quantity can be computed from the "Coulomb branch index" of the class $\mathcal{S}$ theory $T[\Sigma]$ on $L(k,1) \times S^1$. The gauge groups on two sides of the equivalence are naturally G and the Langlands dual group ${}^L G$. We check explicitly the relation for $G = SU(2)$ or $SO(3)$. We also consider more general case where G is $SU(N)$ or $PSU(N)$ and show that the $SU(N)$ equivariant Verlinde formula can be derived using field theory via (generalized) Argyres-Seiberg duality.

As a further application, in Chapter 4 we use Coulomb branch indices of Argyres-Douglas theories on $S^1 \times L(k,1)$ to quantize moduli spaces $\mathcal{M}_H$ of wild/irregular Hitchin systems. We obtain the "wild Hitchin characters", and observe that the characters can always be written as a sum over fixed points in $\mathcal{M}_H$ under the $U(1)$ Hitchin action, and a limit of them can be identified with matrix elements of the

modular transform ST^kS in certain vertex operator algebras. The appearance of vertex operator algebras, which was known previously to be associated with Schur operators but not Coulomb branch operators, is somewhat surprising.

The BPS spectrum of superconformal theories probe the geometry of Hitchin moduli space. Conversely, physical data of superconformal theories can be read off from Hitchin moduli space as well. We study this dictionary in Chapter 5 for general Argyres-Douglas theories and obtain a refined classification. We also discuss the S-duality of these theories, and find that the weakly coupled descriptions are given by the degeneration limit of auxiliary Riemann sphere with marked points.

Finally, in Chapter 6, we analyze classical and quantum chiral ring relations of four dimensional $\mathcal{N} = 1$ adjoint SQCD with superpotential turned on for the adjoint field. In particular, for the mass deformed theory we obtain the complete on-shell vacuum expectation value for various gauge invariant chiral operators and find non-trivial gaugino condensations. We argue that the solution of the chiral ring is in one-to-one correspondence with supersymmetric vacua, provided that an additional Konishi anomaly equation is included.

PUBLISHED CONTENT AND CONTRIBUTIONS

The majority of the present dissertation is adapted from the following publication [1–5]. Except [5], which is a single-authored paper incorporated as Chapter 6, the other papers are outcomes of active collaboration between authors.

For the original research paper [1] (Chapter 2), and also [2] (Chapter 3), [3] (Chapter 4) and [4] (Chapter 5), I was intimately involved and largely responsible for performing explicit calculations, programming, and writing up notes and drafts. Ideas within these projects were equally contributed from all authors.

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[5] Ke Ye. "On the Chiral Ring and Vacua of $\mathcal{N} = 1$ Adjoint SQCD" (2017). arXiv: 1706.02723 [hep-th].

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Chapter 1

INTRODUCTION

M5-bane compactifications provide a natural framework for constructing low dimensional quantum field theories. In such construction, the world volume usually consists of internal compact manifold M, in the form of $\mathbb{R}^d \times M$, and the effective dimensions are reduced in the infrared. In most of the cases, preservation of supersymmetry is required (in my own views, this is for the purpose of simplification), which imposes strong constraints on M. It is then conjectured that geometric data of M characterizes the effective theory $T[M;G]$ on $\mathbb{R}^d$ with G having Lie algebra of ADE type.

One then wishes to establish a precise dictionary between physical observables and the geometry. On the physics side, the most inclusive quantities is the partition function, $Z(T[M])$. This is computed by replacing flat, non-compact Euclidean spacetime $\mathbb{R}^d$ with a compact curved manifold C. The spectrum of $T[M_3]$ is discretized, and the power of supersymmetric localization enables exact calculation of Z. On the geometry side, the problem of identifying the proper geometric quantities on M that can be associated to $Z(T[M])$ gets harder. Such quantities are often expressible in terms of partition functions of yet another quantum field theory $T[C]$ on M, which may or may not depend on the metric of M. The equality

$$Z_C(T[M]) = Z_M(T[C]) \tag{1.1}$$

follows from reversing orders of compactification.

Progress has been made in the past decade in searching for concrete examples of (1.1). The work of Alday, Gaiotto and Tachikawa [1] discovered what was later known as AGT relation, where M is taken to be an arbitrary Riemann surface Σ with tame (regular) punctures, and C is the four sphere S^4. The corresponding $T[\Sigma]$ is the $\mathcal{N} = 2$ superconformal theories (SCFTs) of class $\mathcal{S}$ introduced in [2], and $T[S^4]$ is equivalent to the Toda theory [3]. Replacing S^4 by $S^3 \times S^1$, one essentially replaces Nekrasov partition function by the superconformal index. The latter is independent of marginal couplings of $T[\Sigma]$, so we could tune these couplings to zero and enumerate letters of a free theory. This implies that the theory on Σ does not rely on the complex structure, hence a topological theory (TQFT). The associativity of TQFT is verified in [4], and in the special case of Schur limit the TQFT is explicitly identified to be q-deformed 2d Yang-Mills theory [5].

If we take M to be three-dimensional manifolds instead of Riemann surfaces, then we enter the realm of 3d-3d correspondence, first developed in [6–9]. It is generally believed that the theory $T[M_3]$ does not depend on the metric of M_3, and is completely specified by the topology and the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. One conjectures that the partition function of 3d $\mathcal{N} = 2$ theory $T[M_3; G]$ is the same as the partition function of $G_{\mathbb{C}}$ Chern-Simons theory on M_3, which, as a TQFT, computes topological invariants of the three manifold. In particular, the supersymmetric vacua of $T[M_3; G]$ shall match the $G_{\mathbb{C}}$ flat connections on M_3.

Unlike $M = \Sigma$, where several examples have been found, the 3d-3d correspondence has suffered from inconsistencies since its birth and not many examples are known. The theory $T_{\mathrm{DGG}}[M_3]$ proposed in [8] systematically misses branches of flat $G_{\mathbb{C}}$ connections. The problem was partially rectified in [10]; however, even the very first example for S^3 partition function for $SU(2)$ Chern-Simons theory [11]

$$Z_{\mathrm{CS}}[S^3; SU(2), k] = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \tag{1.2}$$

is not reproduced in the 3d-3d dictionary.

Moreover, when M_3 is the simplest Seifert manifold $S^1 \times \Sigma$, little work is done that relates 3d-3d correspondence to the better established story in class $\mathcal{S}$ theories. Specifically, if the Chern-Simons partition functions are produced on $S^1 \times \Sigma$, how do they arise from the four dimensional superconformal theories?

Settling the above two questions has several important consequences. Technically speaking, it provides a physical way of computing and studying Chern-Simons invariants. Mathematically speaking, Chern-Simons theory on $S^1 \times \Sigma$ is equivalent to geometrically quantizing Hitchin moduli space $\mathcal{M}_H$ on Σ, which is the space of solutions modulo gauge transformation to the partial differential equations on Σ [12]:

$$\begin{aligned} F_A + \left[\varphi, \varphi^\dagger\right] &= 0, \\ \bar{\partial}_A \varphi &= 0. \end{aligned} \tag{1.3}$$

Here F_A is the curvature two-form of $A = A_z dz + A_{\bar{z}} d\bar{z}$ valued in the adjoint bundle of the principle G-bundle P, and $\bar{\partial}_A$ is the $(0,1)$ part of the covariant derivative d_A. Finally, $\varphi \in \Gamma(\Sigma, \mathrm{ad}(P) \otimes_{\mathbb{C}} K)$ is called the *Higgs field*. One could then use the correspondence to understand the geometry and topology of the moduli space.

Finally, on the physical level the geometry can be conveniently used to describe dynamics of 3d $\mathcal{N} = 2$ theories.

The first theme of the present dissertation is to summarize the attempts made to fill in the gap mentioned above. The strategy is to consider the M5 brane geometry

$$L(k,1) \times M_3\,, \tag{1.4}$$

for $M_3 = L(p,1)$ or $S^1 \times \Sigma$ for various Riemann surfaces Σ with punctures. Here $L(k,1)$ is the $r = 1$ specialization of the lens space $L(k,r)$, defined by a quotient of S^3:

$$(z_1, z_2) \to (e^{2\pi i/k} \cdot z_1, e^{2\pi i r/k} \cdot z_2), \qquad |z_1|^2 + |z_2|^2 = 1. \tag{1.5}$$

In Chapter 2 (based on [13]), we take $k = 0$ and compute the partition function $Z_{S^1 \times S^2}(T[L(p,1)])$, which is also known as the superconformal index. We show that it correctly produces Chern-Simons invariants on $L(p,1)$, and in particular when $p = 1$, how the familiar S^3 partition (1.2) (more precisely, two copies of it) is obtained through a free theory. For large p, we find that the index of $T[L(p,1)]$ becomes a constant independent of p. In addition, we study $T[L(p,1)]$ on the squashed three-sphere S^3_b. This enables us to see clearly, at the level of partition function, to what extent $G_{\mathbb{C}}$ complex Chern-Simons theory can be thought of as two copies of Chern-Simons theory with compact gauge group G.

On general M_3 other than S^3, there is no way to preserve supersymmetry by simply deforming the supersymmetry algebra. Therefore, one needs to topologically twist the theory. The twisted partition function of $T[L(k,1);G]$ on $M_3 = S^1 \times \Sigma$ in fact computes an extension of the celebrated Verlinde formula [14], which is called "equivariant Verlinde formula" in [15]. It is an index formula on the Hitchin moduli space $\mathcal{M}_H$ that is organized as $U(1)$ character, and thus throughout later chapters we will sometimes also call it the "Hitchin character". It can be thought of as the graded dimension formula for the Hilbert space after geometrically quantizing $\mathcal{M}_H$, or as the partition function of a TQFT on Σ — the G/G WZW model with adjoint chiral multiplet. In Chapter 3 (based on [16]), we show that the equivariant Verlinde formula with tame punctures on Σ is identical to the Coulomb branch limit of the superconformal index of class $\mathcal{S}$ theory $T[\Sigma; G]$. A subtlety here is that the Coulomb branch index calculates ${}^L G$ equivariant Verlinde formula. When G is not simply connected, we provide a recipe of computing the index of $T[\Sigma, G]$ as a summation over the indices of $T[\Sigma, \widetilde{G}]$ with non-trivial background 't Hooft fluxes, where $\widetilde{G}$ is the universal cover of G. This is a powerful relation, as the superconformal index is much easier to compute with punctures, and is straightforward to generalize.

Then, in Chapter 4 (based on [17]) we adopt the above relation and consider Riemann sphere S^2 with wild/irregular punctures. Physically, it engineers the general Argyres-Douglas theories [18], a class of strongly coupled, non-Lagrangian superconformal theories in four dimensions. We expect that the Coulomb branch index computes a graded dimension of the Hilbert space after quantizing wild Hitchin moduli space. Mathematically, the wild moduli spaces are extremely hard to define, and their precise geometric structures are even more difficult to analyze. The Coulomb branch index manifests these structures in an incredibly simple way, and allows one to see the fixed points under $U(1)$ Hitchin action in $\mathcal{M}_H$. As a bi-product, we observe that the fixed points on $\mathcal{M}_H$ are in one-to-one correspondence with the highest weight representation of certain vertex operator algebras (VOAs). These non-unitary VOAs were introduced in [19], and initially related Higgs branch of the four dimensional SCFT. What is surprising here is that the Coulomb branch operators also know these VOAs at the level of representation.

Physical spectrum can be utilized to understand $\mathcal{M}_H$ with wild punctures, and conversely one may use the wild punctures to classify the theory. The idea was first systematically explored in [20]. In particular, the spectral curve of the Higgs field in Hitchin system is identified with the Seiberg-Witten curve of the $\mathcal{N} = 2$ theory. In Chapter 5 (based on [21]), we use algebraic techniques to classify irregular punctures of $\mathfrak{g}$ = ADE type, which in turn classify the general Argyres-Douglas theory. We then proceed to analyze the S-duality of these strongly coupled theories and find that, similar to class $\mathcal{S}$ theories, the S-duality may be represented by the various degeneration limits of an auxiliary Riemann sphere with marked points on it.

Chapter 3 to Chapter 5 focus on four-dimensional $\mathcal{N} = 2$ theories, and because of the larger amount of supersymmetry, many physical observables such as branches of vacua and BPS spectrum can be computed exactly. Allowed interactions between supermultiplets are highly constrained, thus restricting the space of theories. For theories with a lower amount of supercharges, for instance $\mathcal{N} = 1$ theories in four dimensions, much richer dynamics are expected. Due to its intrinsic complication, previous studies rely mostly on semi-classical analysis, and sometimes produce suspicious results such as a-theorem violation. Even for slightly more involved matter content beyond SQCD studied by Seiberg [22], the conjectured electric-magnetic dualities are problematic and are not rigorously tested.

Most of the confusion boils down to the question of quantum chiral rings. For instance, the change of chiral rings under renormalization group (RG) flow may count for the violation of a-theorem; electric-magnetic dualities could be verified once the quantum vacua are taken into consideration on both sides.

Therefore, the second theme of the present dissertation is to initiate a systematic study on quantum chiral rings of four dimensional $\mathcal{N} = 1$ SQCD with adjoint chiral multiplet (ASQCD). The theory falls into Arnold's ADE classification [23], and is the simplest generalization to the ordinary SQCD. In Chapter 6 (based on [24]), we discuss the A_{N-1} series, and write down six Konishi anomaly equations that give a set of recursion relations for mass deformed theory. We prove that the solution to these chiral ring relation is in one-to-one correspondence with the supersymmetric vacua. Massless limit is also examined.

Finally, in Chapter 7, open questions and potential future works are discussed. Although string theory and supersymmetry do not find its residence in connecting with real world experiment, in my own opinion their significance lies in the formal aspects of mathematics and physics. For the former, they provide a new, profound framework in unifying distinct realms of mathematics and conveying new insights into algebra and geometry; for the latter, they set up playgrounds of toy models that help in understanding the structures of quantum field theory and gravity beyond perturbation theory. It will be my everlasting pleasure that the current dissertation may contribute at least a little to either aspect.

Chapter 2

CHERN-SIMONS INVARIANTS AND 3D-3D CORRESPONDENCE

2.1 The statement of the correspondence

Let us recall the Chern-Simons theory with complexified gauge group $G_{\mathbb{C}}$ [25]. Let $\mathcal{A}$ be a one form valued in $\mathfrak{g}_{\mathbb{C}}$. The action is given by

$$\begin{aligned} S =& \frac{\tau}{8\pi}\int \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \\ &+\frac{\overline{\tau}}{8\pi}\int \mathrm{Tr}\left(\overline{\mathcal{A}}\wedge d\overline{\mathcal{A}}+\frac{2}{3}\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\right), \end{aligned} \tag{2.1}$$

where $\overline{\mathcal{A}}$ is the complex conjugate of $\mathcal{A}$. τ and $\overline{\tau}$ are holomorphic and anti-holomorphic couplings, and are expanded as

$$\tau = k+\sigma, \quad \overline{\tau} = k-\sigma, \qquad k\in\mathbb{Z}. \tag{2.2}$$

As mentioned in the introduction, the 3d-3d correspondence is an elegant relation between 3-manifolds and three-dimensional field theories [6–9]. The general spirit is that one can associate a 3-manifold M_3 with a 3d $\mathcal{N}=2$ superconformal field theory $T[M_3;G]$, obtained by compactifying the 6d (2,0) theory on M_3

$$\begin{array}{c} \text{6d (2,0) theory on } M_3 \\ \rightsquigarrow \\ \text{3d } \mathcal{N}=2 \text{ theory } T[M_3]. \end{array} \tag{2.3}$$

In this procedure, the 6d theory is topologically twisted along M_3 to preserve $\mathcal{N}=2$ supersymmetry. As a consequence, the 3d $\mathcal{N}=2$ theory $T[M_3;G]$ only depends on the topology of M_3 and the simply-laced Lie algebra $\mathfrak{g}=\mathrm{Lie}G$ that labels the 6d theory[1].

There are two very fundamental relations between M_3 and $T[M_3]$. Firstly, the moduli space of supersymmetric vacua of $T[M_3;G]$ on $\mathbb{R}^2\times S^1$ is expected to be

[1] The theory doesn't depend on small deformations of the metric, but could, in principle, depend on a set of discrete variables, and we already know that a choice of "framing" will change $T[M_3]$. In fact, based on current evidence, it is tempting to conjecture that the topology of cM_3 and the choice of framing completely determine $T[M_3]$.

homeomorphic to the moduli space of flat $G_{\mathbb{C}}$-connections on M_3:

$$\mathcal{M}_{\text{SUSY}}(T[M_3;G]) \simeq \mathcal{M}_{\text{flat}}(M_3;G_{\mathbb{C}}). \tag{2.4}$$

Second, the partition function of $T[M_3]$ on lens space $L(k,1)$ should be equal to the partition function of complex Chern-Simons theory on M_3 at level k [26, 27]:

$$Z_{T[M_3;G]}[L(k,1)_b] = Z_{\text{CS}}^{(k,\sigma)}[M_3;G_{\mathbb{C}}]. \tag{2.5}$$

The level of complex Chern-Simons theory has a real part k and an "imaginary part" σ, and σ is related to the squashing parameter b of lens space $L(k,1)_b = S^3_b/\mathbb{Z}_k$ by

$$\sigma = k\cdot\frac{1-b^2}{1+b^2}. \tag{2.6}$$

For $k=0$, $L(k,1) = S^1\times S^2$, and the equation (2.5) maps the superconformal index of $T[M_3]$ to partition function of complex Chern-Simons theory at level $(0,\sigma)$ [9]

$$\text{Index}_{T[M_3;G]}(q) = \text{Tr}\,(-1)^F q^{\frac{E+j_3}{2}} = Z_{\text{CS}}^{(0,\sigma)}[M_3;G_{\mathbb{C}}]. \tag{2.7}$$

In [15], a candidate for the 3d theory $T[L(p,1)]$ was proposed and studied[2,3]:

$$T[L(p,1);G] \quad = \quad \boxed{\begin{array}{c}\text{3d } \mathcal{N}=2\ G \text{ super-Chern-Simons theory at level } p\\ +\text{ adjoint chiral multiplet } \Phi\end{array}}. \tag{2.8}$$

This theory was used to produce Verlinde formula, the partition function of Chern-Simons theory on $S^1\times\Sigma$, along with its "complexification" — the "equivariant Verlinde formula" or "Hitchin character". Therefore, one may wonder whether this theory could also give the correct partition function of Chern-Simons theory on S^3 in (1.2) and its complex analog:

$$Z_{\text{CS}}[S^3;SL(2,\mathbb{C}),\tau,\bar{\tau}] = \sqrt{\frac{4}{\tau\bar{\tau}}}\sin\left(\frac{2\pi}{\tau}\right)\sin\left(\frac{2\pi}{\bar{\tau}}\right). \tag{2.9}$$

[2]More precisely, the Chern-Simons-adjoint theory is the UV CFT that can flow to numerous different IR theories labelled by different relevant deformations, and $T[L(p,1)]$ is expected to be one of them. The brane system giving rise to $T[L(p,1)]$ only allows deformations that is compatible with $R(\Phi)=2$. The UV description, together with this assignment of R-charge for Φ, is adequate for computing any SUSY-protected quantities associated with $T[L(p,1)]$. Therefore, to avoid clutter, we will not distinguish the IR SCFT $T[L(p,1)]$ and its UV description. Still, it is an interesting question to determine the exact relevant deformation that leads to the correct IR theory. One expects that accidental symmetries will play an important role in the RG flow.

[3]As lens space $L(p,1)$ has trivial cotangent bundles, $T[L(p,1)]$ is the same regardless of whether one twists along $L(p,1)$.

Indeed, according to the general statement of the 3d-3d correspondence, $T[L(p,1)]$ needs to satisfy

$$Z_{T[L(p,1);G]}[L(k,1)_b] = Z_{\mathrm{CS}}^{(k,\sigma)}[L(p,1);G_{\mathbb{C}}] \tag{2.10}$$

and

$$\mathrm{Index}_{T[L(p,1);G]}(q) = \mathrm{Tr}\,(-1)^F q^{\frac{E+j_3}{2}} = Z_{\mathrm{CS}}^{(0,\sigma)}[L(p,1);G_{\mathbb{C}}]. \tag{2.11}$$

And if we take $p = 1$, the above relation states that the index of $T[S^3]$ should give the S^3 partition function of complex Chern-Simons theory. Even better, as there is a conjectured duality [28, 29] relating this theory to free chiral multiplets, one should be able to obtain (1.2) and (2.9) by simply computing the index of a free theory! This relation, summarized in diagrammatic form below,

$$\boxed{\begin{array}{c}\text{Chern-Simons}\\ \text{theory on } S^3\end{array}} \overset{\text{3d-3d}}{\longleftrightarrow} \boxed{\begin{array}{c}\text{Index of}\\ T[S^3]\end{array}} \overset{\text{duality}}{\longleftrightarrow} \boxed{\begin{array}{c}\text{free chiral}\\ \text{multiplets}\end{array}} \tag{2.12}$$

will be the subject of section 2.2. We start section 2.2 by proving the duality (at the level of superconformal index) in (2.12) for $G = U(N)$ and then "rediscover" the S^3 partition function of $U(N)$ Chern-Simons theory from the index of N free chiral multiplets. Then in section 2.3 we go beyond $p = 1$ and study theories $T[L(p,1)]$ with higher p. We check that the index of $T[L(p,1)]$ gives precisely the partition function of complex Chern-Simons theory on $L(p,1)$ at level $k = 0$. In addition, we discover that index of $T[L(p,1)]$ has some interesting properties. For example, when p is large,

$$\mathrm{Index}_{T[L(p,1);U(N)]} = (2N-1)!! \tag{2.13}$$

is a constant that only depends on the choice of the gauge group. In the rest of section 2.3, we study $T[L(p,1)]$ on S_b^3 and use the 3d-3d correspondence to give predictions for the partition function of complex Chern-Simons theory on $L(p,1)$ at level $k = 1$.

2.2 Chern-Simons theory on S^3 and free chiral multiplets

According to the proposal (2.8), the theory $T[S^3]$ is $\mathcal{N} = 2$ super-Chern-Simons theory at level $p = 1$ with an adjoint chiral multiplet. If one takes the gauge group to be $SU(2)$, this theory was conjectured by Jafferis and Yin to be dual to a free $\mathcal{N} = 2$ chiral multiplet [28]. The Jafferis-Yin duality has been generalized to higher rank groups by Kapustin, Kim and Park [29]. For $G = U(N)$, the statement of the duality is:

$$T[S^3] = \boxed{\begin{array}{c}U(N)_1 \text{ super-Chern-Simons theory}\\ +\text{ adjoint chiral multiplet}\end{array}} \overset{\text{duality}}{\longleftrightarrow} \boxed{\begin{array}{c}N \text{ free chiral}\\ \text{multiplets}\end{array}}. \tag{2.14}$$

In [15], a similar duality was discovered[4]:

$$T[L(p,1)] = \boxed{\begin{array}{c} U(N)_p \text{ super-Chern-Simons theory} \\ + \text{ adjoint chiral multiplet} \end{array}} \quad \overset{\text{duality}}{\longleftrightarrow} \quad \boxed{\begin{array}{c} \text{sigma model to} \\ \text{vortex moduli space } \mathcal{V}_{N,p} \end{array}} . \tag{2.15}$$

Here,

$$\mathcal{V}_{N,p} \;\cong\; \left\{(q,\varphi)\middle|\zeta \cdot \mathrm{Id} = qq^\dagger + [\varphi,\varphi^\dagger]\right\} /U(N), \tag{2.16}$$

with q being an $N \times p$ matrix, φ an $N \times N$ matrix and $\zeta \in \mathbb{R}^+$ the "size parameter," was conjectured to be the moduli space of N vortices in a $U(p)$ gauge theory [30]. For $p = 1$, it is a well known fact that (see, e.g. [31])

$$\mathcal{V}_{N,1} \simeq \mathrm{Sym}^N(\mathbb{C}) \simeq \mathbb{C}^N. \tag{2.17}$$

And a power-counting argument implies that, in the IR of the 3d sigma model, the Kähler metric on $\mathcal{V}_{N,1}$ will flow to the flat one. This completes the proof of the "appetizer duality" and its $U(N)$ generalizations proposed in [28] and [29].

In particular, at the level of the superconformal index, one has

$$\text{index of } T[S^3;U(N)] \;=\; \text{index of } N \text{ free chirals}. \tag{2.18}$$

Combining (2.18) with the 3d-3d correspondence, one concludes that the index of the free theory equals the S^3 partition function of Chern-Simons theory. This is what we will explicitly verify in this section.

Chern-Simons theory on the three-sphere. The partition function of $U(N)$ Chern-Simons theory on S^3 is

$$Z_{\mathrm{CS}}\left(S^3;U(N),k\right) = \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left[\sin\frac{\pi j}{k+N}\right]^{N-j}. \tag{2.19}$$

For $N = 2$, this gives back (1.2) for $SU(2)$ (modulo a factor coming from the additional $U(1)$). It is convenient to introduce

$$q = e^{\frac{2\pi i}{k+N}}, \tag{2.20}$$

the variable commonly used for the Jones polynomial, and express (2.19) as (mostly) a polynomial in $q^{1/2}$ and $q^{-1/2}$:

$$Z_{\mathrm{CS}}\left(S^3;U(N),k\right) = C \cdot (\ln q)^{N/2} \prod_{j}^{N-1} \left[q^{j/2} - q^{-j/2}\right]^{N-j}. \tag{2.21}$$

[4]In [15], the adjoint chiral is usually assumed to be massive, which introduces an interesting "equivariant parameter" β. Here we are more concerned with the limit where that parameter is zero.

Here C is a normalization factor that does not depend on q and such factors will be dropped in many later expressions without comment.

One can easily obtain the partition function for $GL(N,\mathbb{C})$ Chern-Simons theory by noticing that it factorizes into two copies of (2.19) at level $k_1 = \tau/2$ and $k_2 = \overline{\tau}/2$

$$Z_{\rm CS}\left(S^3; GL(N,\mathbb{C})\right) = (\ln q \ln \bar{q})^{N/2} \prod_{j=1}^{N-1} \left[q^{j/2} - q^{-j/2}\right]^{N-j} \left[\bar{q}^{-j/2} - \bar{q}^{j/2}\right]^{N-j} . \tag{2.22}$$

Here, in slightly abusive use of notation (*cf.* (2.20)),

$$q = e^{\frac{4\pi i}{\tau}}, \quad \bar{q} = e^{\frac{4\pi i}{\overline{\tau}}} . \tag{2.23}$$

Notice that the quantum shift of the level $k \to k + N$ in $U(N)$ Chern-Simons theory is absent in the complex theory [25, 32, 33]. Although (2.22) is almost a polynomial, it contains "$\ln q$" factors. So, at this stage, it is still somewhat mysterious how (2.22) can be obtained as the index of any supersymmetric field theory.

In (2.22) the level is arbitrary and the $k = 0$ case is naturally related to superconformal index of $T[S^3]$ (2.11). For $k = 0$,

$$q = e^{\frac{4\pi i}{\sigma}}, \quad \bar{q} = e^{-\frac{4\pi i}{\sigma}} = q^{-1}, \tag{2.24}$$

and

$$Z^{(0,\sigma)}_{\rm CS}\left(S^3; GL(N,\mathbb{C})\right) = (\ln q)^N \prod_{j=1}^{N-1} \left[(1-q^j)(1-q^{-j})\right]^{N-j} . \tag{2.25}$$

This is the very expression that we want to reproduce from the index of free chiral multiplets.

Index of a free theory. The superconformal index of a 3d $\mathcal{N} = 2$ free chiral multiplet only receives contributions from the scalar component X, the fermionic component $\overline{\psi}$ and their ∂_+ derivatives. If we assume the R-charge of X to be r, then the R-charge of $\overline{\psi}$ is $1 - r$ and the superconformal index of this free chiral is given by

$$\mathcal{I}_r(q) = \prod_{j=0}^{\infty} \frac{1 - q^{1-r/2+j}}{1 - q^{r/2+j}}. \tag{2.26}$$

In the j-th factor of the expression above, the numerator comes from fermionic field $\partial^j\overline{\psi}$ while the denominator comes from bosonic field $\partial^j X$. Here q is a fugacity variable that counts the charge under $\frac{E+j_3}{2} = R/2 + j_3$, and it is the expectation of the

3d-3d correspondence [9] that this q is mapped to the "q" in (2.25), which justifies our usage of the same notation for two seemingly different variables. Now the only remaining problem is to decide what the R-charges for the N free chiral multiplets are.

The UV description of theory $T[L(p,1)]$ has an adjoint chiral multiplet Φ and in general one has the freedom of choosing the R-charge of Φ. Different choices give different IR fix points which form an interesting family of theories. As was argued in [15] using brane construction, the natural choice — namely the choice that one should use for the 3d-3d correspondence — is $R(\Phi) = 2$. For example, in order to obtain the Verlinde formula, it is necessary to choose $R(\Phi) = 2$ while other choices give closely related yet different formulae. As the N free chirals in the dual of $T[S^3; U(N)]$ are directly related to $\mathrm{Tr}\,\Phi$, $\mathrm{Tr}\,\Phi^2$, ..., $\mathrm{Tr}\,\Phi^N$, the choice of their R-charges should be

$$r_m = R(X_m) = 2m, \text{ for } m = 1, 2, \ldots, N. \tag{2.27}$$

The index for this assignment of R-charges — out of the unitarity bound — contains negative powers of q. However, this is not a problem at all because the UV R-charges are mixed with the $U(N)$ flavor symmetries, and q counts a combination of R- and flavor charges.

One interesting property of the index of a free chiral multiplet (2.26) is that it will vanish due to the numerator of the $(m-1)$-th factor:

$$1 - q^{m - r_m/2} = 0. \tag{2.28}$$

However, there is a very natural way of regularizing it and obtaining a finite result. Namely, we multiply the q-independent normalization coefficient $(r_m/2 - m)^{-1}$ to the whole expression and turn the vanishing term above into

$$\lim_{r_m \to 2m} \frac{1 - q^{m - r_m/2}}{r_m/2 - m} = \ln q. \tag{2.29}$$

And this is exactly how the "$\ln q$" factors on the Chern-Simons theory side arise. With this regularization

$$\mathcal{I}_{2m}(q) = \ln q \prod_{j=1}^{m-1} \left[\left(1 - q^{-j}\right)\left(1 - q^{j}\right) \right], \tag{2.30}$$

and the $2m-1$ factors come from the fermionic fields $\overline{\psi}_m$, $\partial\,\overline{\psi}_m$, ..., $\partial^{2m-2}\,\overline{\psi}_m$. The contribution of $\partial^{2m-1+l}\,\overline{\psi}_m$ will cancel with the bosonic field $\partial^l X$ as they have the same quantum number. The special log term comes from the field $\partial^{m-1}\overline{\psi}_m$, which has exactly $R + 2j_3 = 0$.

Then it is obvious that

$$\text{Index}_{T[S^3;U(N)]} = \prod_{m=1}^{N} \mathcal{I}_{2m}(q) = (\ln q)^N \prod_{j=1}^{N-1} \left[(1-q^j)(1-q^{-j})\right]^{N-j} \tag{2.31}$$

is exactly the partition function of complex Chern-Simons theory on S^3 (2.25). For example, if $N = 1$,

$$\text{Index}_{T[S^3;U(1)]} = \mathcal{I}_2(q) = \ln q. \tag{2.32}$$

For $N = 2$,

$$\text{Index}_{T[S^3;U(2)]} = \mathcal{I}_2(q) \cdot \mathcal{I}_4(q) = (\ln q)^2 \, (1-q^{-1})(1-q). \tag{2.33}$$

To get the renowned S^3 partition function of the $SU(2)$ Chern-Simons theory, we just need to divide the $N = 2$ index by the $N = 1$ index and take the square root:

$$\sqrt{\frac{\text{Index}_{T[S^3;U(2)]}}{\text{Index}_{T[S^3;U(1)]}}} = \sqrt{\mathcal{I}_4(q)} = -i \cdot (\ln q)^{1/2} \left(q^{1/2} - q^{-1/2}\right). \tag{2.34}$$

For compact gauge group $SU(2)$, we substitute in

$$q = e^{\frac{2\pi i}{k+2}} \tag{2.35}$$

and up to an unimportant normalization factor, (2.34) is exactly

$$Z_{\text{CS}}(S^3; SU(2), k) = \sqrt{\frac{2}{k+2}} \sin\frac{\pi}{k+2}. \tag{2.36}$$

As almost anything in a free theory can be easily computed, one can go beyond index and check the following relation

$$Z_{N \text{ free chirals}}(L(k,1)_b) = Z_{CS}^{(k,\sigma)}(S^3; U(N)). \tag{2.37}$$

The left-hand side can be expressed as a product of double sine functions [34] and with the right choice of R-charges it becomes exactly the right-hand side, given by (2.19). As this computation is almost identical for what we did with index, we omit it here to avoid repetition.

Before ending this section, we comment on deforming the relation (2.12). In the formulation of $T[L(p,1)]$ in (2.8), there is a manifest $U(1)$ flavor symmetry that can be weakly gauged to give an "equivariant parameter" β. And the partition function of $T[L(p,1);\beta]$ should be related to β-deformed complex Chern-Simons theory studied in [15]:

$$Z_{T[L(p,1);\beta]}(L(k,1)) = Z_{\beta\text{-CS}}(L(p,1);k). \tag{2.38}$$

When $p = 1$, this $U(1)$ flavor symmetry of $T[S^3;U(N)]$ is expected to be enhanced to a $U(N)$ flavor symmetry (or at least $U(1)^N$—the part that is compatible with the choice of R-symmetry) that is only visible in the dual description with N free chiral multiplets. Then one can deform $T[S^3]$ by adding N equivariant parameters $\beta_1, \beta_2, \ldots, \beta_N$. It is interesting to ask whether the Chern-Simons theory on S^3 naturally admits such an N-parameter deformation and whether one can have a more general relation,

$$\text{Index}_{T[S^3]}(q;\beta_1,\beta_2,\ldots,\beta_N) = Z_{\text{CS}}(S^3;q,\beta_1,\beta_2,\ldots,\beta_N). \tag{2.39}$$

As Chern-Simons theory on S^3 is dual to closed string on the resolved conifold [35, 36], it would also be interesting to understand whether similar deformation of the closed string amplitudes F_g exists.

In the next section, we will be considering $L(p,1)$ with $p > 1$. Notice that, analogous to the $p = 1$ case, $\mathcal{V}_{N,p}$ has $SU(p)\times U(1)$ isometry with the $SU(p)$ part being hidden in the Chern-Simons-matter description of $T[L(p,1)]$. It is also interesting to see what the role played by the fugacities of the $SU(p)$ is.

On a separate issue, the existence of hidden symmetries, either $U(N)$ for $p = 1$ or $SU(p)$ for $p > 1$, shows that accidental symmetries will arise and affect the RG flow of the Chern-Simons-adjoint theory. Therefore, understanding the flow and its IR fixed point will pose an interesting challenge.

2.3 3d-3d correspondence for lens spaces

In the previous section, we focused on $T[S^3]$ and found that it fits perfectly inside the 3d-3d correspondence. This theory is the special $p = 1$ limit of a general class (2.8) of theories $T[L(p,1)]$ proposed in [15]. In this section, we will test this proposal and see whether it stands well with various predictions of the 3d-3d correspondence. There are several tests to run on the proposed lens space theories (2.8). The most basic one is the correspondence between moduli spaces (2.4) that one can formulate classically without doing a path integral:

$$\mathcal{M}_{\text{SUSY}}\left(T[L(p,1);U(N)]\right) \simeq \mathcal{M}_{\text{flat}}\left(L(p,1);GL(N,\mathbb{C})\right). \tag{2.40}$$

And our first task in this section is to verify that this is indeed an equality.

$\mathcal{M}_{\text{SUSY}}$ vs. $\mathcal{M}_{\text{flat}}$

The moduli space of flat H-connections on a three manifold M_3 can be identified with the character variety:

$$\mathcal{M}_{\text{flat}}(M_3; H) \simeq \text{Hom}(\pi_1(M_3), H)/H. \tag{2.41}$$

As $\pi_1(L(p,1)) = \mathbb{Z}_p$, this character variety is particularly simple. For example, if we take $H = U(N)$ or $H = GL(N,\mathbb{C})$ — the choice between $U(N)$ or $GL(N,\mathbb{C})$ does not even matter — this space is a collection of points labelled by Young tableaux with size smaller than $N \times p$. This is in perfect harmony with the other side of the 3d-3d relation where the supersymmetric vacua of $T[L(p,1); U(N)]$ on $S^1 \times \mathbb{R}^2$ are also labelled by Young tableaux with the same constraint [15]. We will now make this matching more explicit.

If we take the holonomy along the S^1 Hopf fiber of $L(p,1)$ to be A, then

$$\mathcal{M}_{\text{flat}}(L(p,1); GL(N,\mathbb{C})) \simeq \{A \in GL(N,\mathbb{C}) | A^p = \text{Id}\}/GL(N,\mathbb{C}). \tag{2.42}$$

First we can use the $GL(N,\mathbb{C})$ action to cast A into Jordan normal form. But in order to satisfy $A^k = \text{Id}$, A has to be diagonal, and each of its diagonal entries a_l has to be one of the p-th roots of unity:

$$a_l^p = 1, \text{ for all } l = 1, 2, \ldots, N. \tag{2.43}$$

One can readily identify this set of equations with the $t \to 1$ limit of the Bethe ansatz equations that determine the supersymmetric vacua of $T[L(p,1); U(N)]$ on $S^1 \times \mathbb{R}^2$ [15]:

$$e^{2\pi i p \sigma_l} \prod_{m \neq l} \left(\frac{e^{2\pi i \sigma_l} - t e^{2\pi i \sigma_m}}{t e^{2\pi i \sigma_l} - e^{2\pi i \sigma_m}} \right) = 1, \quad \text{for all of } l = 1, 2, \ldots, N. \tag{2.44}$$

For $t = 1$, this equation is simply

$$e^{2\pi i p \sigma_l} = 1, \text{ for } l = 1, 2, \ldots, N. \tag{2.45}$$

And this is exactly (2.43) if one makes the following identification:

$$a_l = e^{2\pi i \sigma_l}. \tag{2.46}$$

Of course this relation between a_l and σ_l is more than just a convenient choice. It can be derived using the brane construction of $T[L(p,1)]$. In fact, it just comes from the familiar relation in string theory between holonomy along a circle and positions of D-branes after T-duality. Indeed, in the above expression, the a_I's on the left-hand side label the $U(N)$-holonomy along the Hopf fiber, while the σ_I's on the right-hand side are coordinates on the Coulomb branch of $T[L(p,1)]$ after reduction to 2d, which exactly correspond to positions of N D2-branes.

$G_{\mathbb{C}}$ Chern-Simons theory from G Chern-Simons theory. The fact that $\mathcal{M}_{\text{flat}}$ is a collection of points is important for us to compute the partition function of complex Chern-Simons theory. Although there have been many works on complex Chern-Simons theory and its partition functions, starting from [25, 37] to perturbative invariant in [32, 38], state integral models in [27, 39, 40] and mathematically rigorous treatment in [41–43], what usually appear are certain subsectors of complex Chern-Simons theory, obtained from some consistent truncation of the full theory. In general, the *full* partition function of complex Chern-Simons theory is difficult to obtain, and requires proper normalization to make sense of. Some progress has been made toward understanding the full theory on Seifert manifolds in [15] using topologically twisted supersymmetric theories. However, if $\mathcal{M}_{\text{flat}}(M_3; G_{\mathbb{C}})$ is discrete and happens to be the same as $\mathcal{M}_{\text{flat}}(M_3; G)$, then one can attempt to construct the full partition function of the $G_{\mathbb{C}}$ Chern-Simons theory on M_3 from the G Chern-Simons theory. The procedure is the following. One first writes the partition function of the G Chern-Simons theory as a sum over flat connections:

$$Z^{\text{full}} = \sum_{\alpha\in\mathcal{M}} Z_\alpha. \tag{2.47}$$

And because the action of the $G_{\mathbb{C}}$ Chern-Simons theory (2.1) is simply two copies of the G Chern-Simons theory action at level $k_1 = \tau/2$ and $k_2 = \overline{\tau}/2$, one would have

$$Z_\alpha(G_{\mathbb{C}}; \tau, \overline{\tau}) = Z_\alpha\left(G; \frac{\tau}{2}\right) Z_\alpha\left(G; \frac{\overline{\tau}}{2}\right), \tag{2.48}$$

if $\mathcal{A}$ and $\overline{\mathcal{A}}$ were independent fields. So, one would naively expect

$$Z^{\text{full}}(G_{\mathbb{C}}; \tau, \overline{\tau}) = \sum_{\alpha\in\mathcal{M}} Z_\alpha\left(G; \frac{\tau}{2}\right) Z_\alpha\left(G; \frac{\overline{\tau}}{2}\right). \tag{2.49}$$

But as $\mathcal{A}$ and $\overline{\mathcal{A}}$ are not truly independent, (2.49) is in general incorrect and one needs to modify it in a number of ways. For example, as mentioned before, the quantum shift of the level τ and $\overline{\tau}$ in $G_{\mathbb{C}}$ Chern-Simons theory is zero, so for $Z_\alpha(G)$ on the right-hand side, one needs to at least remove the quantum shift $k \to k + \check{h}$ in G Chern-Simons theory, where $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$. There may be other effects that lead to relative coefficients between contributions from different flat connections α, and the best one could hope for is

$$Z^{\text{full}}(G_{\mathbb{C}};\tau,\overline{\tau}) = \sum_{\alpha\in\mathcal{M}} e^{iC_\alpha} Z'_\alpha\left(G;\frac{\tau}{2}\right) Z'_\alpha\left(G;\frac{\overline{\tau}}{2}\right), \tag{2.50}$$

where

$$Z'_\alpha\left(G;\frac{\tau}{2}\right) = Z_\alpha\left(G;\frac{\tau}{2}-\check{h}\right). \tag{2.51}$$

One way to see that (2.49) is very tenuous, even after taking care of the level shift, is by noticing that the left-hand side and the right-hand side behave differently under a change of framing. If the framing of the three-manifold is changed by s units, the left-hand side will pick up a phase factor

$$\exp\left[\varphi_{\mathbb{C}}^{\text{fr.}}\cdot s\right] = \exp\left[\frac{\pi i(c_L-c_R)}{12}\cdot s\right]. \tag{2.52}$$

Here c_L and c_R are the left- and right-moving central charges of the hypothetical conformal field theory that lives on the boundary of the complex Chern-Simons theory [25]:

$$(c_L,c_R) = \dim G\cdot\left(1-\frac{2\check{h}}{\tau},1+\frac{2\check{h}}{\overline{\tau}}\right). \tag{2.53}$$

The right-hand side of (2.49) consists of two copies of the Chern-Simons theory with compact gauge group G, so the phase from change of framing is

$$\exp\left[\varphi^{\text{fr.}}\cdot s\right] = \exp\left[\frac{\pi i}{12}\left(\frac{\tau/2-\check{h}}{\tau/2}+\frac{\overline{\tau}/2-\check{h}}{\overline{\tau}/2}\right)\dim G\cdot s\right]. \tag{2.54}$$

The two phases are in general different:

$$\varphi_{\mathbb{C}}^{\text{fr.}} - \varphi^{\text{fr.}} = \frac{2\pi i\dim G}{12}. \tag{2.55}$$

So (2.49) has no chance of being correct at all and the minimal way of improving it is to add the phases, C_α, as in (2.50), which also transform under change of framing.

It may appear that the expression (2.50) is not useful unless one can find the values of the C_α's. However, as it turns out, for $k=0$ (or equivalently $\tau=-\overline{\tau}$), all of the C_α's are constant, and (2.50) without the C_α's gives the correct partition function[5]. This may be closely related to the fact that for $k=0$,

$$c_L-c_R = -2\check{h}\dim G\left(\frac{1}{\tau}+\frac{1}{\overline{\tau}}\right) = 0. \tag{2.56}$$

[5]"Correct" in the sense that it matches the index of $T[L(p,1)]$.

Superconformal index

We have shown that the proposal (2.8) for $T[L(p,1)]$ gives the right supersymmetric vacua and we shall now move to the quantum level and check the relation between the partition functions:

$$\mathrm{Index}_{T[L(p,1);U(N)]}(q) = Z_{CS}\left(L(p,1); GL(N,\mathbb{C}), q\right). \tag{2.57}$$

We have already verified this for $p = 1$ in the previous section. Now we consider the more general case with $p \geq 1$.

The superconformal index of a 3d $\mathcal{N} = 2$ SCFT is given by [44]

$$\mathcal{I}(q, t_i) = \mathrm{Tr}\left[(-1)^F e^{-\gamma(E-R-j_3)} q^{\frac{E+j_3}{2}} t^{f_i}\right]. \tag{2.58}$$

Here, the trace is taken over the Hilbert space of the theory on $\mathbb{R} \times S^2$. Because of supersymmetry, only BPS states with

$$E - R - j_3 = 0 \tag{2.59}$$

will contribute. As a consequence, the index is independent of γ and only depends on q and the flavor fugacities, t_i. For $T[L(p,1)]$, there is always a $U(1)$ flavor symmetry and we can introduce at least one parameter t. When this parameter is turned on, on the other side of the 3d-3d correspondence, complex Chern-Simons theory will become the “deformed complex Chern-Simons theory”. This deformed version of Chern-Simons theory was studied on geometry $\Sigma \times S^1$ in [15] and will be studied on more general Seifert manifolds in [45]. However, because in this chapter our goal is to *test* the 3d-3d relation (as opposed to using it to study the deformed Chern-Simons theory), we will usually turn off this parameter by setting $t = 1$, and compare the index $\mathcal{I}(q)$ with the partition function of the *undeformed* Chern-Simons theory, which is only a function of q, as in (2.25).

Viewing the index as the partition function on $S^1 \times_q S^2$ and using localization, (2.58) can be expressed as an integral over the Cartan $\mathbb{T}$ of the gauge group G [46]:

$$\mathcal{I} = \frac{1}{|\mathcal{W}|}\sum_m \int \prod_j \frac{dz_j}{2\pi i z_j} e^{-S_{\mathrm{CS}}(m)} q^{\epsilon_0/2} e^{ib_0(h)} t^{f_0} \exp\left[\sum_{n=1}^{+\infty} \frac{1}{n}\,\mathrm{Ind}(z_j^n, m_j; t^n, q^n)\right]. \tag{2.60}$$

Here $h, m \in \mathfrak{t}$ are valued in the Cartan subalgebra. Physically, e^{ih} is the holonomy along S^1 and is parametrized by z_i, which are coordinates on $\mathbb{T}$.

$$m = \frac{i}{2\pi}\int_{S^2} F \tag{2.61}$$

is the monopole number on S^2 and takes value in the weight lattice of the Langlands dual group ${}^LG$. $|\mathcal{W}|$ is the order of the Weyl group and the other quantities are

$$\begin{aligned} b_0(h) &= -\frac{1}{2}\sum_{\rho\in\mathfrak{R}_\Phi} |\rho(m)|\,\rho(h), \\ f_0 &= -\frac{1}{2}\sum_{\rho\in\mathfrak{R}_\Phi} |\rho(m)|\, f, \\ \epsilon_0 &= \frac{1}{2}\sum_{\rho\in\mathfrak{R}_\Phi}(1-r)\,|\rho(m)| - \frac{1}{2}\sum_{\alpha\in\mathrm{ad}(G)} |\alpha(m)|\,, \\ S_{\mathrm{CS}} &= ip\,\mathrm{tr}\,(mh), \end{aligned} \tag{2.62}$$

and

$$\begin{aligned} \mathrm{Ind}\,(e^{ih_j} = z_j, m_j; t; q) = &- \sum_{\alpha\in\mathrm{ad}(G)} e^{i\alpha(h)} q^{|\alpha(m)|} \\ &+ \sum_{\rho\in\mathfrak{R}_\Phi}\left[e^{i\rho(h)}t\,\frac{q^{|\rho(m)|/2+r/2}}{1-q} - e^{-i\rho(h)}t^{-1}\frac{q^{|\rho(m)|/2+1-r/2}}{1-q}\right] \end{aligned} \tag{2.63}$$

is the “single particle" index. $\mathfrak{R}_\Phi$ is the gauge group representation for all matter fields. Using this general expression, the index of $T[L(p,1);U(N)]$ can be expressed in the following form:

$$\begin{aligned} \mathcal{I}(q,t) = \sum_{m_1\geqslant\cdots\geqslant m_N\in\mathbb{Z}} \frac{1}{|\mathcal{W}_m|}\int\prod_j \frac{dz_j}{2\pi i z_j}\prod_i^N (z_i)^{pm_i}\prod_{i\neq j}^N t^{-|m_i-m_j|/2} q^{-R|m_i-m_j|/4}\left(1-q^{|m_i-m_j|/2}\frac{z_i}{z_j}\right) \\ \prod_{i\neq j}^N \frac{\left(\frac{z_j}{z_i}t^{-1}q^{|m_i-m_j|/2+1-R/2};q\right)_\infty}{\left(\frac{z_i}{z_j}tq^{|m_i-m_j|/2+R/2};q\right)_\infty}\times\left[\frac{(t^{-1}q^{1-R/2};q)_\infty}{(tq^{R/2};q)_\infty}\right]^N. \end{aligned} \tag{2.64}$$

Here we used the q-Pochhammer symbol $(z;q)_n = \prod_{j=0}^{n-1}(1-zq^j)$. $\mathcal{W}_m \subset \mathcal{W}$ is the stabilizer subgroup of the Weyl group that fixes $m \in \mathfrak{t}$ and R stands for the R-charge of the adjoint chiral multiplet and will be set to $R = 2$ — the choice that gives the correct IR theory.

In the previous section, we have found the index for $T[S^3]$ to be exactly equal to the S^3 partition function of Chern-Simons theory. There, we used an entirely different method by working with the dual description of $T[L(p,1);U(N)]$, which is a sigma model to the vortex moduli space $\mathcal{V}_{N,p}$. For $p = 1$, this moduli space is topologically $\mathbb{C}^N$ and the index of the sigma model is just that of a free theory. For $p \geq 2$, such a simplification will not occur and the index of the sigma model is much harder to compute[6]. In contrast, the integral expression (2.64) is *easier* to compute with larger p than with $p = 1$, because fewer topological sectors labelled by the monopole number m contribute. As we will see later, when p is sufficiently large, only the sector $m = (0, 0, \ldots, 0)$ gives non-vanishing contribution. So the two approaches of computing the index have their individual strengths and are complementary to each other.

Now, one can readily compute the index for any $T[L(p, 1); G]$ and then compare $\mathcal{I}(q, t = 1)$ with the partition function of the complex Chern-Simons theory on $L(p, 1)$. We will first do a simple example with $G = SU(2)$, to illustrate some general features of the index computation.

Index of $T[L(p, 1); SU(2)]$. We will start with $p = 1$ and see how the answer from section 2.2 arises from the integral expression (2.64). In this case, (2.64) becomes

$$\begin{aligned}\mathcal{I} &= \sum_{m\in\mathbb{Z}} \int \frac{dz}{4\pi i z} e^{ihm} q^{-2|m|} \left(1 - q^{|m|} e^{ih}\right)^2 \left(1 - q^{|m|} e^{-ih}\right)^2 \prod_{k=0}^{+\infty} \frac{1 - q^{k+1-R/2}}{1 - q^{k+R/2}} \\ &= \sum_{m\in\mathbb{Z}} \int \frac{dz}{4\pi i z} z^m q^{-2|m|} \left(1 + q^{2|m|} - z q^{|m|} - z^{-1} q^{|m|}\right)^2 [(R-2)\ln q] \\ &= \sum_{m\in\mathbb{Z}} \int \frac{dz}{4\pi i z} z^m \left(q^{2|m|} + q^{-2|m|} + 4 - 2\left(z + \frac{1}{z}\right)\left(q^{|m|} + \frac{1}{q^{|m|}}\right) + \left(z^2 + \frac{1}{z^2}\right)\right) \\ &\qquad \times [(R/2 - 1)\ln q]\,.\end{aligned} \tag{2.65}$$

As in section 2.2, the index will be zero if we naively take $R = 2$ because of the $1 - q^{1-r/2}$ factor in the infinite product. When $R \to 2$, the zero factor becomes

$$1 - q^{1-R/2} = 1 - \exp[(1 - R/2)\ln q] \approx (R/2 - 1)\ln q. \tag{2.66}$$

As in section 2.2, we can introduce a normalization factor $(R/2 - 1)^{-1}$ in the index to cancel the zero, making the index expression finite.

The integral in (2.65) is very easy to do and the index receives contributions from three different monopole number sectors

$$\mathcal{I} = \frac{1}{2} \ln q \, (\mathcal{I}_{m=0} + \mathcal{I}_{m=\pm 1} + \mathcal{I}_{m=\pm 2}), \tag{2.67}$$

[6] In general, it can be written as an integral of a characteristic class over $\mathcal{V}_{N,p}$ that one can evaluate using the Atiyah-Bott localization formula. Similar computations were done in two dimensions in, *e.g.*, [6] and [47].

with

$$\mathcal{I}_{m=0} = \int \frac{dz}{2\pi i z}\left(q^0 + q^{-0} + 4\right) = 6, \tag{2.68}$$

$$\mathcal{I}_{m=\pm1} = -2\sum_{m=\pm1}\int \frac{dz}{2\pi i z} z^m \left(q^{|m|} + q^{-|m|}\right)\left(z + \frac{1}{z}\right) = -4(q + q^{-1}), \tag{2.69}$$

and

$$\mathcal{I}_{m=\pm2} = \sum_{m=\pm2}\int \frac{dz}{2\pi i z} z^m \left(z^2 + \frac{1}{z^2}\right) = 2. \tag{2.70}$$

So the index is

$$\begin{aligned}\mathcal{I} &= \frac{1}{2}\ln q\left(6 - 4(q + q^{-1}) + 2\right)\\ &= -2\ln q\left(q^{1/2} - q^{-1/2}\right)^2.\end{aligned} \tag{2.71}$$

Modulo a normalization constant, this is in perfect agreement with results in section 2.2. Indeed, the square root of (2.71) is identical to (2.34) and reproduces the S^3 partition function of the $SU(2)$ Chern-Simons theory,

$$Z_{\text{CS}}(S^3; SU(2), k) = \sqrt{\frac{2}{k+2}}\sin\frac{\pi}{k+2}, \tag{2.72}$$

once we set

$$q = e^{\frac{2\pi i}{k+2}}. \tag{2.73}$$

It is very easy to generalize the result (2.71) to arbitrary p. For general p, the index is given by

$$\begin{aligned}\mathcal{I} &= \frac{1}{2}\ln q\sum_{m\in\mathbb{Z}}\int \frac{dz}{2\pi i z} z^{pm}\\ &\times\left(q^{2|m|} + q^{-2|m|} + 4 - 2\left(q^{|m|} + q^{-|m|}\right)\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right)\right).\end{aligned} \tag{2.74}$$

The only effect of p is to select monopole numbers that contribute. For example, if $p = 2$, only $m = 0$ and $m = \pm1$ contribute to the index and we have

$$\mathcal{I}^{p=2} = \frac{1}{2}\ln q\,(\mathcal{I}_{m=0} + \mathcal{I}^{p=2}_{m=\pm1}) = \frac{1}{2}\ln q\,(6 + 2) = 4\ln q. \tag{2.75}$$

If $p > 2$, only the trivial sector is selected, and

$$\mathcal{I}(p > 2) = \frac{1}{2}\ln q\;\mathcal{I}_{m=0} = 3\ln q. \tag{2.76}$$

This is a general feature of indices of the "lens space theory", and we will soon encounter this phenomenon with higher rank gauge groups.

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$U(2)$	$2(1-q)(1-q^{-1})$	4	3	3	3	3
$U(3)$	$6(1-q)^2(1-q^2)$ $(1-q^{-1})^2(1-q^{-2})$	$28-6q^{-2}$ $-8q^{-1}-8q$ $-6q^2$	$23+2q^{-1}+2q$	16	15	15
$U(4)$	$24(1-q)^3(1-q^2)^2$ $(1-q^3)(1-q^{-1})^3$ $(1-q^{-2})^2(1-q^{-3})$	$504+$ $84q^{-4}-96q^{-3}$ $-80q^{-2}-160q^{-1}$ $-160q-80q^2$ $-96q^3+84q^4$	$204-30q^{-3}$ $-48q^{-2}-24q^{-1}$ $-24q-48q^2$ $-30q^3$	$188+10q^{-2}$ $+24q^{-1}+24q$ $+10q^2$	$121+$ $2q^{-1}+2q$	108
$U(5)$	$120(1-q)^4(1-q^2)^3$ $(1-q^3)^2(1-q^4)$ $(1-q^{-1})^4(1-q^{-2})^3$ $(1-q^{-3})^2(1-q^{-4})$	$12336+$ $120q^{-10}+192q^{-9}$ $-1080q^{-8}+48q^{-7}$ $+120q^{-6}+3792q^{-5}$ $-2016q^{-4}-1296q^{-3}$ $-3312q^{-2}-2736q^{-1}$ $-2736q-3312q^2$ $-1296q^3-2016q^4$ $+3792q^5+120q^6$ $+48q^7-1080q^8$ $+192q^9+120q^{10}$	$3988+$ $180q^{-6}+388q^{-5}$ $-294q^{-4}-932q^{-3}$ $-584q^{-2}-752q^{-1}$ $-752q-584q^2$ $-932q^3-294q^4$ $+388q^5+180q^6$	$2144-$ $240q^{-4}-320q^{-3}$ $-320q^{-2}-192q^{-1}$ $-192q-320q^2$ $-320q^3-240q^4$	$1897+$ $70q^{-3}+192q^{-2}$ $352q^{-1}+352q$ $+192q^2+70q^3$	$1188+$ $14q^{-2}+40q^{-1}$ $40q+14q^2$

Table 2.1: The superconformal index of the "lens space theory" $T[L(p,1),U(N)]$, which agrees with the partition function of $GL(N,\mathbb{C})$ Chern-Simons theory at level $k=0$ on lens space $L(p,1)$.

The test for 3d-3d correspondence. We list the index of $T[L(p,1);U(N)]$, obtained using *Mathematica*, in table 2.1. Due to limitation of space and computational power, it contains results up to $N=5$ and $p=6$. The omnipresent $(\ln q)^N$ factors are dropped to avoid clutter, and after this every entry in table 2.1 is a Laurent polynomial in q with integer coefficients. Also, when the gauge group is $U(N)$, monopole number sectors are labeled by an N-tuple of integers $m=(m_1,m_2,\ldots,m_N)$ and a given sector can only contribute to the index if $\sum m_i=0$.

From the table, one may be able to recognize the large p behavior for $U(3)$ and $U(4)$ similar to (2.75) and (2.76). Indeed, it is a general feature of the index $\mathcal{I}_{T[L(p,1);U(N)]}$ that fewer monopole number sectors contribute when p increases. In order for a monopole number $m=(m_1,\ldots,m_N)$ to contribute,

$$|pm_i|\leq 2N-2 \tag{2.77}$$

needs to be satisfied for all m_i. For large $p > 2N-2$, $\mathcal{I}$ only receives a contribution from the $m = 0$ sector and becomes a constant:

$$\mathcal{I}(U(N), p > 2N-2) = \mathcal{I}_{m=(0,0,0,\ldots,0)} = (2N-1)!!\,. \tag{2.78}$$

For $p = 2N-2$, the index receives contributions from two sectors[7]:

$$\mathcal{I}(U(N), p = 2N-2) = \mathcal{I}_{m=(0,0,0,\ldots,0)} + \mathcal{I}_{m=(1,0,\ldots,0,-1)} = [(2N-1)!! + (2N-5)!!]\,. \tag{2.79}$$

While the $\ln q$ factors (that we have omitted) are artifacts of our scheme of removing zeros in $\mathcal{I}$, the constant coefficient $(2N-1)!!$ in (2.78) is counting BPS states. Then one can ask a series of questions: 1) What are the states or local operators that are being counted? 2) Why is the number of such operators independent of p when p is large?

Partition functions Z_{CS} of the complex Chern-Simons theory on Lens spaces can also be computed systematically. Please see appendix A for details of the method we use. For $k = 0$, $G_{\mathbb{C}} = GL(N, \mathbb{C})$, the partition functions on $L(p,1)$ only depend on $q = e^{4\pi i/\tau}$ as $\overline{q} = e^{4\pi i/\overline{\tau}} = q^{-1}$. After dropping a $(\ln q)^N$ factor as in the index case, it is again a polynomial. We have computed this partition function up to $N = 5$ and $p = 6$ and found a perfect agreement with the index in table 2.1.

From the point of view of the complex Chern-Simons theory, this large p behavior (2.78) seems to be even more surprising — it predicts that the partition functions of the complex Chern-Simons theory on $L(p,1)$ at level $k = 0$ are constant when p is greater than twice the rank of the gauge group. One can then ask 1) why is this happening? And 2) what is the geometric meaning of this $(2N-1)!!$ constant?

$T[L(p,1)]$ on S^3_b

In previous sections, we have seen that the superconformal index of $T[L(p,1)]$ agrees completely with the partition function of the complex Chern-Simons theory at level $k = 0$ given by (2.50) with trivial relative phases $C_\alpha = 0$:

$$Z(G_{\mathbb{C}}; \tau, \overline{\tau}) = \sum_{\alpha \in \mathcal{M}} Z'_\alpha\left(G; \frac{\tau}{2}\right) Z'_\alpha\left(G; \frac{\overline{\tau}}{2}\right), \tag{2.80}$$

for $G = U(N)$. But for more general k, one can no longer expect this to be true. We will now consider the S^3_b partition function of $T[L(p,1)]$, which will give the

[7]Here, double factorial of a negative number is taken to be 1.

partition function of the complex Chern-Simons theory at level [26]

$$(k, \sigma) = \left(1, \frac{1-b^2}{1+b^2}\right). \tag{2.81}$$

And we will examine for which choices of N and p that setting all phases $C_\alpha = 0$ becomes a mistake, by comparing the S_b^3 partition function of $T[L(p,1)]$ to the "naive" partition function (2.80) of the complex Chern-Simons theory at level $k = 1$ on $L(p,1)$.

There are two kinds of squashed three-spheres breaking the $SO(4)$ isometry of the round S^3: the first one preserves $SU(2) \times U(1)$ isometry while the second one preserves $U(1) \times U(1)$ [48]. However, despite the geometry being different, the partition functions of 3d $\mathcal{N} = 2$ theories that one gets are the same [48–51]. In fact, as was shown in [52, 53], three-sphere partition functions of $\mathcal{N} = 2$ theories only admit a one-parameter deformation. We will choose the "ellipsoid" geometry with the metric

$$ds_3^2 = f(\theta)^2 d\theta^2 + \cos^2\theta d\phi_1^2 + \frac{1}{b^4}\sin^2\theta d\phi_2^2, \tag{2.82}$$

where $f(\theta)$ is arbitrary and does not affect the partition function of the supersymmetric theory.

Using localization, partition function of a $\mathcal{N} = 2$ gauge theory on such an ellipsoid can be written as an integral over the Cartan of the gauge group [48, 50]. Consider an $\mathcal{N} = 2$ Chern-Simons-matter theory with gauge group being $U(N)$. A classical Chern-Simons term with level k contributes

$$Z_{\text{CS}} = \exp\left(\frac{i}{b^2}\frac{k}{4\pi}\sum_{i=1}^{N}\lambda_i^2\right) \tag{2.83}$$

to the integrand. The one-loop determinant of $U(N)$ vector multiplet, combined with the Vandermonde determinant, gives

$$Z_{\text{gauge}} = \prod_{i<j}^{N}\left(2\sinh\frac{\lambda_i - \lambda_j}{2}\right)\left(2\sinh\frac{\lambda_i - \lambda_j}{2b^2}\right). \tag{2.84}$$

A chiral multiplet in the representation $\mathfrak{R}$ gives a product of double sine functions:

$$Z_{\text{matter}} = \prod_{\rho\in\mathfrak{R}} s_b\left(\frac{iQ}{2}(1-R) - \frac{\rho(\lambda)}{2\pi b}\right), \tag{2.85}$$

where $Q = b + 1/b$, R is the R-charge of the multiplet and the double sine function is defined as

$$s_b(x) = \prod_{p,q=0}^{+\infty}\frac{pb + qb^{-1} + \frac{Q}{2} - ix}{pb^{-1} + qb + \frac{Q}{2} + ix}. \tag{2.86}$$

Then we can express the S_b^3 partition function of $T[L(p,1)]$ using the UV description in (2.8) as

$$Z(T[L(p,1),U(N)],b)=\frac{1}{N!}\int\prod_i^N\frac{d\lambda_i}{2\pi}\exp\left(-\frac{i}{b^2}\frac{p}{4\pi}\sum_{i=1}^N\lambda_i^2\right)\times\prod_{i<j}^N\frac{4}{\pi^2}\left(\sinh\frac{\lambda_i-\lambda_j}{2}\right)^2\left(\sinh\frac{\lambda_i-\lambda_j}{2b^2}\right)^2, \tag{2.87}$$

which is a Gaussian integral. We list our results in table 2.2 and 2.3. A universal factor

$$\left(\frac{b}{ip}\right)^{N/2}\pi^{-N(N-1)} \tag{2.88}$$

is dropped in making these two tables.

If one compares results in table 2.2 and 2.3 with partition functions of complex Chern-Simons theory naively computed using (2.49), one will find a perfect agreement for $p=1$ once the phase factor

$$\exp\left[\frac{\pi i(c_L-c_R)}{12}\cdot(3-p)\right] \tag{2.89}$$

from the change of framing is added[8]. This agreement is not unexpected because for $p=1$, $\mathcal{M}_{\text{flat}}$ consists of just a single point and there are no such things as relative phases between contributions from different flat connections. Even for $p=2$, the naive way (2.49) of computing partition function of complex Chern-Simons theory seems to be still valid modulo an overall factor. However, starting from $p=3$, the two sides start to differ significantly. See table 2.4 for a comparison between the S_b^3 partition function of $T[L(p,1)]$ and the "naive" partition function of the complex Chern-Simons theory on $L(p,1)$ for $G=U(2)$. Recently, Blau and Thompson studied partition functions of complex Chern-Simons theory on general Seifert manifolds [55], and it is a very interesting problem to check whether their results, when specialized to $L(p,1)$, agree with the prediction of the 3d-3d correspondence using $T[L(p,1)]$.

[8]The complex Chern-Simons theory obtained from the 3d-3d correspondence is naturally in "Seifert framing", as the $T[L(p,1)]$ we used is obtained by reducing M5-brane on the Seifeit S^1 fiber of $L(p,1)$ in [15]. However, the computation in appendix A is in "canonical framing" and differs from Seifert framing by $(3-p)$ units [54].

p	$U(2)$	$U(3)$	$U(4)$
1	$2e^{-2i\pi b^2-\frac{2i\pi}{b^2}}$ $\left(1-e^{\frac{2i\pi}{b^2}}\right)\left(1-e^{2i\pi b^2}\right)$	$6e^{-8i\pi b^2-\frac{8i\pi}{b^2}}\left(1-e^{\frac{2i\pi}{b^2}}\right)^3\left(1+e^{\frac{2i\pi}{b^2}}\right)$ $\left(1-e^{2i\pi b^2}\right)^3\left(1+e^{2i\pi b^2}\right)$	$24e^{-20i\pi b^2-\frac{20i\pi}{b^2}}\left(1-e^{\frac{2i\pi}{b^2}}\right)^6\left(1+e^{\frac{2i\pi}{b^2}}\right)^2$ $\left(1+e^{\frac{2i\pi}{b^2}}+e^{\frac{4i\pi}{b^2}}\right)\left(1-e^{2i\pi b^2}\right)^6$ $\left(1+e^{2i\pi b^2}\right)^2\left(1+e^{2i\pi b^2}+e^{4i\pi b^2}\right)$
2	$2-2e^{-\frac{i\pi}{b^2}}-2e^{-i\pi b^2}$ $+2e^{-i\pi b^2-\frac{i\pi}{b^2}}$	$2e^{-4i\pi(b^2+b^{-2})}$ $\left(1-e^{\frac{2i\pi}{b^2}}\right)\left(1-e^{2i\pi b^2}\right)$ $\Big(-6e^{\frac{i\pi}{b^2}}+3e^{\frac{2i\pi}{b^2}}-6e^{i\pi b^2}+3e^{2i\pi b^2}$ $-4e^{i\pi(b^2+b^{-2})}+3e^{2i\pi(b^2+b^{-2})}$ $-6e^{i\pi\left(b^2+2b^{-2}\right)}-6e^{i\pi\left(2b^2+b^{-2}\right)}+3\Big)$	$8e^{-10i\pi\left(b^2+b^{-2}\right)}\left(1-e^{\frac{2i\pi}{b^2}}\right)^2\left(1-e^{2ib^2\pi}\right)^2$ $\Big(3-9e^{\frac{i\pi}{b^2}}+9e^{\frac{2i\pi}{b^2}}-6e^{\frac{3i\pi}{b^2}}+9e^{\frac{4i\pi}{b^2}}-9e^{\frac{5i\pi}{b^2}}$ $+3e^{\frac{6i\pi}{b^2}}-9e^{ib^2\pi}+9e^{2ib^2\pi}-6e^{3ib^2\pi}$ $+9e^{4ib^2\pi}-9e^{5ib^2\pi}+3e^{6ib^2\pi}-9e^{i\pi\left(b^2+b^{-2}\right)}$ $+27e^{2i\pi\left(b^2+b^{-2}\right)}-4e^{3i\pi\left(b^2+b^{-2}\right)}+27e^{4i\pi\left(b^2+b^{-2}\right)}$ $-9e^{5i\pi\left(b^2+b^{-2}\right)}+3e^{6i\pi\left(b^2+b^{-2}\right)}-27e^{i\pi\left(b^2+2b^{-2}\right)}$ $+27e^{2i\pi\left(b^2+2b^{-2}\right)}-6e^{3i\pi\left(b^2+2b^{-2}\right)}-6e^{i\pi\left(b^2+3b^{-2}\right)}$ $+9e^{2i\pi\left(b^2+3b^{-2}\right)}-27e^{i\pi\left(b^2+4b^{-2}\right)}-9e^{i\pi\left(b^2+5b^{-2}\right)}$ $-9e^{i\pi\left(b^2+6b^{-2}\right)}-18e^{i\pi\left(2b^2+3b^{-2}\right)}+9e^{2i\pi\left(2b^2+3b^{-2}\right)}$ $-27e^{i\pi\left(2b^2+5b^{-2}\right)}-18e^{i\pi\left(3b^2+2b^{-2}\right)}+9e^{2i\pi\left(3b^2+2b^{-2}\right)}$ $-18e^{i\pi\left(3b^2+4b^{-2}\right)}-6e^{i\pi\left(3b^2+5b^{-2}\right)}-18e^{i\pi\left(4b^2+3b^{-2}\right)}$ $-27e^{i\pi\left(4b^2+5b^{-2}\right)}-27e^{i\pi\left(5b^2+2b^{-2}\right)}-6e^{i\pi\left(5b^2+3b^{-2}\right)}$ $-27e^{i\pi\left(5b^2+4b^{-2}\right)}-9e^{i\pi\left(5b^2+6b^{-2}\right)}-9e^{i\pi\left(6b^2+5b^{-2}\right)}$ $-27e^{i\pi\left(2b^2+b^{-2}\right)}+27e^{2i\pi\left(2b^2+b^{-2}\right)}$ $-6e^{3i\pi\left(2b^2+b^{-2}\right)}-6e^{i\pi\left(3b^2+b^{-2}\right)}+9e^{2i\pi\left(3b^2+b^{-2}\right)}$ $-27e^{i\pi\left(4b^2+b^{-2}\right)}-9e^{i\pi\left(5b^2+b^{-2}\right)}-9e^{i\pi\left(6b^2+b^{-2}\right)}\Big)$
3	$2-2e^{-\frac{2i\pi}{3b^2}}-2e^{-\frac{2}{3}i\pi b^2}$ $-e^{-\frac{2i\pi}{3}(b^2+b^{-2})}$	$-3e^{-\frac{8i\pi}{3}\left(b^2+b^{-2}\right)}\times$ $\Big(4e^{\frac{2i\pi}{3b^2}}+2e^{\frac{2i\pi}{b^2}}+2e^{\frac{8i\pi}{3b^2}}$ $+4e^{\frac{2}{3}i\pi b^2}+2e^{2i\pi b^2}+2e^{\frac{8}{3}i\pi b^2}$ $-8e^{\frac{2i\pi}{3}\left(b^2+b^{-2}\right)}+4e^{2i\pi\left(b^2+b^{-2}\right)}$ $-2e^{\frac{8i\pi}{3}\left(b^2+b^{-2}\right)}+8e^{\frac{2i\pi}{3}\left(b^2+3b^{-2}\right)}$ $-4e^{\frac{2i\pi}{3}\left(b^2+4b^{-2}\right)}$ $+4e^{\frac{2i\pi}{3}\left(3b^2+4b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(4b^2+3b^{-2}\right)}$ $+8e^{\frac{2\pi i}{3}\left(3b^2+b^{-2}\right)}-4e^{\frac{2\pi i}{3}\left(4b^2+\pi b^{-2}\right)}+1\Big)$	$-6e^{-\frac{20i\pi}{3}\left(b^2+b^{-2}\right)}\left(1-e^{\frac{2i\pi}{b^2}}\right)\left(1-e^{2ib^2\pi}\right)$ $\Big(1+6e^{\frac{2i\pi}{3b^2}}+5e^{\frac{2i\pi}{b^2}}+8e^{\frac{8i\pi}{3b^2}}+3e^{\frac{4i\pi}{b^2}}+4e^{\frac{14i\pi}{3b^2}}$ $+6e^{\frac{2}{3}ib^2\pi}+5e^{2ib^2\pi}+8e^{\frac{8}{3}ib^2\pi}+3e^{4ib^2\pi}$ $+4e^{\frac{14}{3}ib^2\pi}-18e^{\frac{2i\pi}{3}\left(b^2+b^{-2}\right)}-2e^{\frac{4i\pi}{3}\left(b^2+b^{-2}\right)}$ $+25e^{2ipi\left(b^2+b^{-2}\right)}-28e^{\frac{8i\pi}{3}\left(b^2+b^{-2}\right)}-2e^{\frac{10i\pi}{3}\left(b^2+b^{-2}\right)}$ $+9e^{4i\pi\left(b^2+b^{-2}\right)}-4e^{\frac{14i\pi}{3}\left(b^2+b^{-2}\right)}-4e^{\frac{4i\pi}{3}\left(b^2+2b^{-2}\right)}$ $+15e^{2i\pi\left(b^2+2b^{-2}\right)}+30e^{\frac{2i\pi}{3}\left(b^2+3b^{-2}\right)}-24e^{\frac{2i\pi}{3}\left(b^2+4b^{-2}\right)}$ $+18e^{\frac{2i\pi}{3}\left(b^2+6b^{-2}\right)}-12e^{\frac{2i\pi}{3}\left(b^2+7b^{-2}\right)}$ $+24e^{\frac{4i\pi}{3}\left(2b^2+3b^{-2}\right)}+2e^{\frac{2i\pi}{3}\left(2b^2+5b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(2b^2+7b^{-2}\right)}$ $+24e^{\frac{4i\pi}{3}\left(3b^2+2b^{-2}\right)}+40e^{\frac{2i\pi}{3}\left(3b^2+4b^{-2}\right)}+20e^{\frac{2i\pi}{3}\left(3b^2+7b^{-2}\right)}$ $+40e^{\frac{2i\pi}{3}\left(4b^2+3b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(4b^2+5b^{-2}\right)}-20e^{\frac{2i\pi}{3}\left(4b^2+7b^{-2}\right)}$ $+2e^{\frac{2i\pi}{3}\left(5b^2+2b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(5b^2+4b^{-2}\right)}-4e^{\frac{2i\pi}{3}\left(5b^2+7b^{-2}\right)}$ $+12e^{\frac{2i\pi}{3}\left(6b^2+7b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(7b^2+2b^{-2}\right)}+20e^{\frac{2i\pi}{3}\left(7b^2+3b^{-2}\right)}$ $-20e^{\frac{2i\pi}{3}\left(7b^2+4b^{-2}\right)}-4e^{\frac{2i\pi}{3}\left(7b^2+5b^{-2}\right)}+12e^{\frac{2i\pi}{3}\left(7b^2+6b^{-2}\right)}$ $-4e^{\frac{4i\pi}{3}\left(2b^2+b^{-2}\right)}+15e^{2i\pi\left(2b^2+b^{-2}\right)}+30e^{\frac{2i\pi}{3}\left(3b^2+b^{-2}\right)}$ $-24e^{\frac{2i\pi}{3}\left(4b^2+b^{-2}\right)}+18e^{\frac{2i\pi}{3}\left(6b^2+b^{-2}\right)}-12e^{\frac{2i\pi}{3}\left(7b^2+b^{-2}\right)}\Big)$

Table 2.2: The S_b^3 partition function of $T[L(p,1),U(N)]$. In this table p ranges from 1 to 3.

p	$U(2)$	$U(3)$
4	$2-2e^{-\frac{i\pi}{2b^2}}-2e^{-\frac{1}{2}i\pi b^2}-2e^{-\frac{i\pi}{2}\left(b^2+b^{-2}\right)}$	$-2e^{-2i\pi\left(b^2+b^{-2}\right)}\times$ $\Big(-3-2e^{\frac{i\pi}{2b^2}}+2e^{\frac{3i\pi}{2b^2}}+3e^{\frac{2i\pi}{b^2}}-2e^{\frac{1}{2}i\pi b^2}+2e^{\frac{3}{2}i\pi b^2}+3e^{2i\pi b^2}+4e^{\frac{i\pi}{2}\left(b^2+b^{-2}\right)}$ $+4e^{\frac{3i\pi}{2}\left(b^2+b^{-2}\right)}-3e^{2i\pi\left(b^2+b^{-2}\right)}+4e^{\frac{i\pi}{2}\left(b^2+3b^{-2}\right)}-6e^{\frac{i\pi}{2}\left(b^2+4b^{-2}\right)}$ $+6e^{\frac{i\pi}{2}\left(3b^2+4b^{-2}\right)}+6e^{\frac{i\pi}{2}\left(4b^2+3b^{-2}\right)}+4e^{\frac{i\pi}{2}\left(3b^2+b^{-2}\right)}-6e^{\frac{i\pi}{2}\left(4b^2+b^{-2}\right)}\Big)$
5	$2-2e^{-\frac{2i\pi}{5b^2}}-2e^{-\frac{2}{5}i\pi b^2}+2\cos\frac{4\pi}{5}e^{-\frac{2i\pi}{5}\left(b^2+b^{-2}\right)}$	$6-12e^{-\frac{2i\pi}{5b^2}}+12e^{-\frac{6i\pi}{5b^2}}-6e^{-\frac{8i\pi}{5b^2}}-12e^{-\frac{2}{5}i\pi b^2}$ $+12e^{-\frac{6}{5}i\pi b^2}-6e^{-\frac{8}{5}i\pi b^2}+4\left(\cos\frac{8\pi}{5}+e^{\frac{4i\pi}{5}}\right)e^{-\frac{2i\pi}{5}(4b^2+b^{-2})}$ $4\left(\cos\frac{8\pi}{5}+2\cos\frac{4\pi}{5}\right)e^{-\frac{2i\pi}{5}(b^2+4b^{-2})}+8\left(\cos\frac{4\pi}{5}+2\cos\frac{2\pi}{5}\right)e^{-\frac{2i\pi}{5}\left(b^2+b^{-2}\right)}$ $+8\left(\cos\frac{12\pi}{5}+2\cos\frac{6\pi}{5}\right)e^{-\frac{6i\pi}{5}\left(b^2+b^{-2}\right)}+2\left(\cos\frac{16\pi}{5}+2\cos\frac{8\pi}{5}\right)$ $\times e^{-\frac{8i\pi}{5}\left(b^2+b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(b^2+3b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(b^2-3+3b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(b^2+3+3b^{-2}\right)}$ $-8e^{-\frac{2i\pi}{5}\left(3b^2+b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(3b^2+4b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(3b^2-6+4b^{-2}\right)}$ $-8e^{-\frac{2i\pi}{5}\left(3b^2-3+b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(3b^2+3+b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(3b^2+6+4b^{-2}\right)}$ $-4e^{-\frac{2i\pi}{5}\left(4b^2+3b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(4b^2-6+3b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(4b^2+6+3b^{-2}\right)}$
6	$2-2e^{-\frac{i\pi}{3b^2}}-2e^{-\frac{1}{3}i\pi b^2}+e^{-\frac{i\pi}{3}(b^2+b^{-2})}$	$e^{-\frac{4i\pi}{3}\left(b^2+b^{-2}\right)}\times$ $\Big(-12e^{\frac{i\pi}{3b^2}}-6e^{\frac{i\pi}{b^2}}-6e^{\frac{4i\pi}{3b^2}}-12e^{\frac{1}{3}i\pi b^2}-6e^{i\pi b^2}-6e^{\frac{4}{3}i\pi b^2}-8e^{\frac{i\pi}{3}\left(b^2+b^{-2}\right)}$ $+4e^{i\pi\left(b^2+b^{-2}\right)}+6e^{\frac{4i\pi}{3}\left(b^2+b^{-2}\right)}+8e^{\frac{i\pi}{3}\left(b^2+3b^{-2}\right)}+12e^{\frac{i\pi}{3}\left(b^2+4b^{-2}\right)}$ $-12e^{\frac{i\pi}{3}\left(3b^2+4b^{-2}\right)}-12e^{\frac{i\pi}{3}\left(4b^2+3b^{-2}\right)}+8e^{\frac{i\pi}{3}\left(3b^2+b^{-2}\right)}+12e^{\frac{i\pi}{3}\left(4b^2+b^{-2}\right)}-3\Big)$

Table 2.3: The S_b^3 partition function of $T[L(p,1),U(N)]$. This table, with p ranging from 4 to 6, is the continuation of the previous table 2.2. Due to the limitation of space, only partition functions for $U(2)$ and $U(3)$ are given.

p	S_b^3 partition function of $T[L(p,1);U(2)]$	"naive" partition function of $GL(2,\gamma)$ Chern-Simons theory
1	$2-2q^{-1}-2\overline{q}^{-1}+2(q\overline{q})^{-1}$	$2-2q^{-1}-2\overline{q}^{-1}+2(q\overline{q})^{-1}$
2	$2+2q^{-\frac{1}{2}}+2\overline{q}^{-\frac{1}{2}}+2(q\overline{q})^{-\frac{1}{2}}$	$2i(2+2q^{-\frac{1}{2}}+2\overline{q}^{-\frac{1}{2}}+2(q\overline{q})^{-\frac{1}{2}})$
3	$2+\left(1-\sqrt{3}i\right)q^{-\frac{1}{3}}+\left(1-\sqrt{3}i\right)\overline{q}^{-\frac{1}{3}}+\frac{1}{2}\left(1+\sqrt{3}i\right)(q\overline{q})^{-\frac{1}{3}}$	$2+\left(1-3\sqrt{3}i\right)\overline{q}^{\frac{1}{3}}+\left(1-3\sqrt{3}i\right)q^{\frac{1}{3}}+\frac{1}{2}\left(1+3\sqrt{3}i\right)(q\overline{q})^{\frac{1}{3}}$
4	$2-2iq^{-\frac{1}{4}}-2i\overline{q}^{-\frac{1}{4}}+2(q\overline{q})^{-\frac{1}{4}}$	$8i(q\overline{q})^{\frac{1}{2}}\left(1+iq^{\frac{1}{4}}+i\overline{q}^{\frac{1}{4}}+(q\overline{q})^{\frac{1}{4}}\right)$
5	$2-2e^{\frac{2\pi i}{5}}q^{-\frac{1}{5}}-2e^{\frac{2\pi i}{5}}\overline{q}^{-\frac{1}{5}}+2\cos\frac{4\pi}{5}e^{\frac{4\pi i}{5}}(q\overline{q})^{-\frac{1}{5}}$	$q\overline{q}\left(2-2\left(e^{\frac{3\pi i}{5}}+2e^{\frac{4\pi i}{5}}\right)\overline{q}^{\frac{1}{5}}-2\left(e^{\frac{3\pi i}{5}}+2e^{\frac{4\pi i}{5}}\right)q^{\frac{1}{5}}\right.$ $\left.+\left(1+2e^{\frac{\pi i}{5}}+3e^{\frac{2\pi i}{5}}-4e^{\frac{3\pi i}{5}}-4e^{\frac{4\pi i}{5}}\right)(q\overline{q})^{\frac{1}{5}}\right)$
6	$2-\left(1+\sqrt{3}i\right)q^{-\frac{1}{6}}-\left(1+\sqrt{3}i\right)\overline{q}^{-\frac{1}{6}}-\frac{1}{2}\left(1-\sqrt{3}i\right)(q\overline{q})^{-\frac{1}{6}}$	$6i(q\overline{q})^{\frac{3}{2}}\left(2+(-1+i\sqrt{3})q^{\frac{1}{6}}+(-1+i\sqrt{3})\overline{q}^{\frac{1}{6}}+\frac{1}{2}\left(1+i\sqrt{3}\right)(q\overline{q})^{\frac{1}{6}}\right)$

Table 2.4: The comparison between the S_b^3 partition function of $T[L(p,1),U(2)]$ and the "naive" partition function of the $GL(2,\mathbb{C})$ Chern-Simons theory, obtained by putting together two copies of the $U(2)$ Chern-Simons theory using (2.80), on lens space $L(p,1)$ in "Seifert framing." Notice that when p increases, the difference between the two columns becomes larger and larger.

Chapter 3

THE COULOMB BRANCH INDEX AND THE EQUIVARIANT VERLINDE FORMULA

3.1 Connection to four dimensional SCFTs

In this chapter, we pick $M_3 = \Sigma \times S^1$ for Σ a Riemann surface with punctures. Recall the general M5 brane configuration introduced in Chapter 2,

$$\begin{array}{rl} \text{space-time:} & L(k,1)_b \times T^*M_3 \times \mathbb{R}^2 \\ & \qquad\qquad\quad \cup \\ N \text{ fivebranes:} & L(k,1)_b \times \quad M_3 \end{array} \tag{3.1}$$

If one reduces along the squashed lens space $L(k,1)_b$, one obtains complex Chern-Simons theory at level k on M_3 [26]. In this simple case where $M_3 = S^1 \times \Sigma$, the system is extremely interesting and can be used to gain a lot of insight into complex Chern-Simons theory. For example, the partition function of the 6d $(2,0)$-theory on this geometry gives the "equivariant Verlinde formula", which can be identified with the dimension of the Hilbert space of the complex Chern-Simons theory at level k on Σ:

$$Z_{\text{M5}}(L(k,1) \times \Sigma \times S^1, \beta) = \dim_\beta \mathcal{H}_{\text{CS}}(\Sigma, k). \tag{3.2}$$

Here β is an "equivariant parameter" associated with a geometric $U(1)_\beta$ action whose precise definition will be reviewed in section 3.2. The left-hand side of (3.2) has been computed in several ways in [15] and [13], and each gives unique insight into the equivariant Verlinde formula, the complex Chern-Simons theory, and the 3d-3d correspondence in general. In this chapter, we will add to the list yet another method of computing the partition of the system of M5-branes by relating it to superconformal indices of class $\mathcal{S}$ theories.

The starting point is the following observation. For $M_3 = \Sigma \times S^1$, the setup (3.1) looks like:

$$\begin{array}{rl} N \text{ fivebranes:} & L(k,1)_b \times \quad \Sigma \quad \times S^1 \\ & \qquad\qquad\quad \cap \\ \text{space-time:} & L(k,1)_b \times T^*\Sigma \times S^1 \times \mathbb{R}^3 \,, \end{array} \tag{3.3}$$

which is already very reminiscent of the setting of lens space superconformal indices of class $\mathcal{S}$ theories [56–60]:

$$\begin{array}{rcccccc} N \text{ fivebranes:} & L(k,1)\times S^1 & \times & \Sigma & & & \\ & & & \cap & & & \\ \text{space-time:} & L(k,1)\times S^1 & \times & T^*\Sigma & \times & \mathbb{R}^3 & . \\ & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & \\ \text{symmetries:} & SO(4)_E & & U(1)_N & & SU(2)_R & \end{array} \tag{3.4}$$

In this geometry, one can turn on holonomies of the symmetries along the S^1 circle in a supersymmetric way and introduce three "universal fugacities" (p, q, t). Then the partition function of M5-branes in this geometry is the lens space superconformal index of the 4d $\mathcal{N} = 2$ theory $T[\Sigma]$ of class $\mathcal{S}$:

$$Z_{\text{M5}}(L(k,1)\times S^1 \times \Sigma, p, q, t) = \mathcal{I}(T[\Sigma], p, q, t), \tag{3.5}$$

where we have adopted the following convention for the index[1]:

$$\mathcal{I}(p, q, t) = \mathrm{Tr}\,(-1)^F p^{\frac{1}{2}\delta_{1+}} q^{\frac{1}{2}\delta_{1-}} t^{R+r} e^{-\beta''\tilde{\delta}_{1\dot{-}}}. \tag{3.6}$$

As the left-hand sides of (3.2) and (3.5) are closely related, it is very tempting to ask whether the equivariant Verlinde formula for a Riemann surface Σ, parametrized by $\beta \in \mathbb{R}$, can actually be embedded as a one-parameter family inside the three-parameter space of superconformal indices of the theory $T[\Sigma]$. The goal of this chapter is to give strong evidence for the following proposal:

$$\boxed{\begin{array}{c}\text{equivariant Verlinde formula}\\ \text{at level } k \text{ on } \Sigma \text{ for group } G\end{array}} \quad = \quad \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, {}^L G] \text{ on } L(k,1)\times S^1\end{array}}, \tag{3.7}$$

where the Coulomb branch index is the one-parameter family obtained by taking $p, q, t \to 0$ while keeping $\mathrm{t} = pq/t$ fixed.

To clarify the proposed relation (3.7), we first give a few remarks:

1. When we fixed Σ, G and $k \in \mathbb{Z}$, both sides depend on a real parameter and the identification between them is given by $\mathrm{t} = e^{-\beta}$.

[1] In the literature there are several other conventions in use. The other two most commonly used conventions for universal fugacities are (ρ, σ, τ) which are related to our convention via $p = \sigma\tau, q = \rho\tau, t = \tau^2$, and (t, y, v) with $t = \sigma^{\frac{1}{6}}\rho^{\frac{1}{6}}\tau^{\frac{1}{3}}, y = \sigma^{\frac{1}{2}}\rho^{-\frac{1}{2}}, v = \sigma^{\frac{2}{3}}\rho^{\frac{2}{3}}\tau^{-\frac{2}{3}}$.

2. We will assume $\mathfrak{g} = \text{Lie}\, G$ is of type ADE (modulo possible abelian factors), as $T[\Sigma, {}^L G]$, with ${}^L G$ being the Langlands dual group of G, is not yet defined in the literature when $\mathfrak{g}$ is not simply-laced. Then we have $\mathfrak{g} = {}^L\mathfrak{g}$.

3. When G is simple but not simply-connected, the left-hand side of (3.7) is only defined when k annihilates $\pi_1(G)$ (under the natural $\mathbb{Z}$-action on this abelian group), and the proposal is meant for these values of k.

4. When ${}^L G$ is simple but not simply-connected, the theory $T[\Sigma, {}^L G]$ is not yet defined. Denote the universal cover of ${}^L G$ (which equals the universal cover of G as $\mathfrak{g}$ is of type ADE) as $\widetilde{G}$. We will interpret the Coulomb index of $T[\Sigma, {}^L G]$ as a summation of indices of $T[\Sigma, \widetilde{G}]$ with insertion of all possible 't Hooft fluxes valued in $\pi_1({}^L G)$. The insertion is along the 2d surface $S^1 \times S^1_{\text{Hopf}} \subset S^1 \times L(k, 1)$, where S^1_{Hopf} is the Hopf fiber of the lens space $L(k, 1)$.[2] We will give a concrete argument using string theory for the A_{N-1} series by starting with $\mathfrak{g} = \mathfrak{u}(N)$, and show that this summation naturally arises when we decouple the abelian $\mathfrak{u}(1)$ factor.

5. Conceptually, the reason why G appears on the left of (3.7) while ${}^L G$ appears on the right can be understood as follows. The left-hand side of (3.7) can be viewed as certain B-model partition function of the Hitchin moduli space $\mathcal{M}_H(\Sigma, G)$ [12] (see also (1.3)). Mirror symmetry will produce the Hitchin moduli space associated with the dual group $\mathcal{M}_H(\Sigma, {}^L G)$ [61, 62], and as we will argue in later sections, the corresponding A-model partition function of $\mathcal{M}_H(\Sigma, {}^L G)$ can be identified with the right-hand side of (3.7).

To further illustrate (3.7), we will present the simplest example where $k = 1$ and G is simply connected. The equivariant Verlinde formula formula can be obtained using the TQFT structure studied in [63]

$$\dim_\beta \mathcal{H}_{\text{CS}}(\Sigma, G_{\mathbb{C}}, k = 1) = \frac{|\mathcal{Z}(G)|^g}{\left[\prod_{i=1}^{\text{rank}\, G} (1 - \mathfrak{t}^{d_i})^{h_i}\right]^{g-1}}, \tag{3.8}$$

where $|\mathcal{Z}(G)|$ is the order of the center of group G, d_i's are degrees of the fundamental invariants of $\mathfrak{g} = \text{Lie}\, G$, and h_i's are the dimension of the space of d_i-differentials on Σ. The reader may have already recognized that (3.8) is exactly the Coulomb

[2] Another natural definition of the partition function of $T[\Sigma, {}^L G]$ is as the summation over only fluxes valued in $H^2(L(k, 1), \pi_1({}^L G)) = \mathbb{Z}_k \otimes \pi_1({}^L G)$, which is a subgroup of $\pi_1({}^L G)$. If one takes this as the definition, then (3.7) is correct when k also annihilates $\pi_1({}^L G)$.

branch index of $T[\Sigma, G]$ on $L(k=1,1)=S^3$ times $|\mathcal{Z}(G)|^g$. As we will explain in great detail later, the $|\mathcal{Z}(G)|^g$ factor comes from summation over 't Hooft fluxes, which are labeled precisely by elements in $\mathcal{Z}(G) \simeq \pi_1(^L G)$. The g power morally originates from the fact that there are g "independent gauge nodes" in the theory $T[\Sigma, G]$ (*i.e.* one copy of G for each handle of Σ). So (3.8) agrees with the Coulomb index of $T[\Sigma, ^L G]$.

For $k > 1$, the relation (3.7) becomes more non-trivial, and each flux sector gives generally different contribution. Even if one sets $\mathrm{t}=0$, the identification of Verlinde algebra with the algebra of allowed 't Hooft fluxes in $T[\Sigma, G]$ is novel.

This chapter is organized as follows. In section 3.2, we examine more closely the two fivebranes systems (3.1) and (3.4), and give arguments supporting the relation (3.7) between the equivariant Verlinde formula and the Coulomb branch index. In section 3.3, after reviewing basic facts and ingredients of the index, we verify our proposals by reproducing the already known $SU(2)$ equivariant Verlinde algebra from the Coulomb branch indices of class $\mathcal{S}$ theories on the lens space. We will see that after an appropriate normalization, the TQFT algebras on both sides are exactly identical, and so are the partition functions. In section 3.4, we will use the proposed relation (3.7) to derive the $SU(3)$ equivariant Verlinde algebra from the index of $T[\Sigma, SU(3)]$ computed via the Argyres-Seiberg duality. Careful analysis of the results reveals interesting geometry of the Hitchin moduli space $\mathcal{M}_H(\Sigma, SU(3))$.

3.2 Equivariant Verlinde algebra and Coulomb branch index

One obvious difference between the two brane systems (3.1) and (3.4) is that the S^1 factor appears on different sides of the correspondence. From the geometry of (3.1), one would expect that

$$\begin{array}{c}\text{equivariant Verlinde formula}\\ \text{at level } k \text{ on } \Sigma\end{array} = \begin{array}{c}\text{Partition function of}\\ T[\Sigma\times S^1] \text{ on } L(k,1)\end{array}. \tag{3.9}$$

In particular, there should be no dependence on the size of the S^1, so it is more natural to use "3d variables":

$$t = e^{L\beta-(b+b^{-1})L/r}, \quad p = e^{-bL/r}, \quad q = e^{-b^{-1}L/r}. \tag{3.10}$$

Here, L is the size of the S^1 circle, b is the squashing parameter of $L(k,1)_b$, r measures the size of the Seifert base S^2, and β parametrizes the "canonical mass deformation" of the 3d $\mathcal{N}=4$ theory (in our case $T[\Sigma\times S^1]$) into 3d $\mathcal{N}=2$. The latter is defined as follows on flat space. The 3d $\mathcal{N}=4$ theory has R-symmetry

$SU(2)_N \times SU(2)_R$ and we can view it as a 3d $\mathcal{N} = 2$ theory with the R-symmetry group being the diagonal subgroup $U(1)_{N+R} \subset U(1)_N \times U(1)_R$ with $U(1)_N$ and $U(1)_R$ being the Cartans of $SU(2)_N$ and $SU(2)_R$ respectively. The difference $U(1)_{N-R} = U(1)_N - U(1)_R$ of the original R-symmetry group is now a flavor symmetry $U(1)_\beta$ and we can weakly gauge it to introduce real masses proportional to β. It is exactly how the "equivariant parameter" in [15], denoted by the same letter β, is defined.[3]

In [15], it was observed that much could be learned about the brane system (3.1) and the Hilbert space of complex Chern-Simons theory by preserving supersymmetry along the lens space $L(k,1)$ in a different way, namely by doing partial topological twist instead of deforming the supersymmetry algebra. Geometrically, this corresponds to combining the last $\mathbb{R}^3$ factor in (3.3) with $L(k,1)$ to form $T^*L(k,1)$ regarded as a local Calabi-Yau 3-fold with $L(k,1)_b$ being a special Lagrangian submanifold:

$$\begin{array}{rccccc} N \text{ fivebranes:} & L(k,1)_b & \times & \Sigma & \times S^1 \\ & \cap & & \cap & \\ \text{space-time:} & T^*L(k,1)_b & \times & T^*\Sigma & \times S^1 \\ & \circlearrowleft & & \circlearrowleft & \\ \text{symmetries:} & U(1)_R & & U(1)_N & . \end{array} \tag{3.11}$$

In this geometry, $U(1)_N$ acts by rotating the cotangent fiber of Σ, while $U(1)_R$ rotates the cotangent fiber of the Seifert base S^2 of the lens space.[4] This point of view enables one to derive the equivariant Verlinde formula as it is now the partition function of the *supersymmetric* theory $T[L(k,1),\beta]$ on $\Sigma \times S^1$.

Although the geometric setting (3.11) appears to be different from the original one (3.1), there is substantial evidence that they are related. For example, the equivariant Verlinde formula can be defined and computed on both sides and they agree. Namely, the partition function in the twisted background (3.11) is given by the partition function of $T[L(k,1)]$ on Σ, while the partition function under

[3] More precisely, the dimensionless combination βL is used. And from now on, we will rename $\beta_{\text{new}} = \beta_{\text{old}} L$ and $r_{\text{new}} = r_{\text{old}}/L$ to make all 3d variables dimensionless.

[4] Note, $U(1)_N$ is always an isometry of the system whereas the $U(1)_R$ is only an isometry in certain limits where the metric on $L(k,1)$ is singular (*e.g.* when $L(k,1)$ is viewed a small torus fibered over a long interval). However, if we are only interested in questions that have no dependence on the metric on $L(k,1)$, we can always assume the $U(1)_R$ symmetry to exist. For example, the theory $T[L(k,1)]$, or in general $T[M_3]$ for any Seifert manifolds M_3 should enjoy an extra flavor symmetry $U(1)_\beta = U(1)_N - U(1)_R$.

the background (3.1) is given by an equivariant integral over the Hitchin moduli space, and they are proven to be equal in [63]. Moreover, the modern viewpoint on supersymmetry in curved backgrounds is that the deformed supersymmetry is an extension of topological twisting; see *e.g.*, [64]. Therefore, one should expect that the equivariant Verlinde formula formula at level k could be identified with a particular slice of the four-parameter family of 4d indices (k, p, q, t) (or in 3d variables (k, β, b, r)). And this particular slice should have the property that the index has no dependence on the geometry of $L(k, 1)_b$. Since $T[L(k, 1)]$ is derived in the limit where $L(k, 1)$ shrinks, one should naturally take the $r \to 0$ limit for the superconformal index. In terms of the 4d parameters, that corresponds to

$$p, q, t \to 0. \tag{3.12}$$

This is known as the Coulomb branch limit. In this particular limit, the only combination of (k, p, q, t) independent of b and r that one could possibly construct is

$$\mathrm{t} = \frac{pq}{t} = e^{-\beta}, \tag{3.13}$$

and this is precisely the parameter used in the Coulomb branch index. Therefore, one arrives at the following proposal:

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } U(N)_k \text{ on } \Sigma\end{array}} = \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, U(N)] \text{ on } L(k,1)\times S^1\end{array}}. \tag{3.14}$$

This relation should be more accurately viewed as the natural isomorphism between two TQFT functors:

$$Z_{\mathrm{EV}} = Z_{\mathrm{CB}}. \tag{3.15}$$

At the level of partition function on a closed Riemann surface Σ, it is the equality between the equivariant Verlinde formula and the Coulomb index of $T[\Sigma]$:

$$Z_{\mathrm{EV}}(\Sigma) = Z_{\mathrm{CB}}(\Sigma). \tag{3.16}$$

Going one dimension lower, we also have an isomorphism between the Hilbert spaces of the two TQFTs on a circle:

$$\mathcal{H}_{\mathrm{EV}} = Z_{\mathrm{EV}}(S^1) = \mathcal{H}_{\mathrm{CB}} = Z_{\mathrm{CB}}(S^1). \tag{3.17}$$

As these underlying vector spaces set the stages for any interesting TQFT algebra, the equality above is the most fundamental and needs to be established first. We now show how one can canonically identify the two seemingly different Hilbert spaces $\mathcal{H}_{\mathrm{EV}}$ and $\mathcal{H}_{\mathrm{CB}}$.

$\mathcal{H}_{\mathbf{EV}}$ vs. $\mathcal{H}_{\mathbf{CB}}$

In the equivariant Verlinde TQFT, operator-state correspondence tells us that states in $\mathcal{H}_{\text{EV}}$ are in one-to-one correspondence with local operators. Since these local operators come from codimension-2 "monodromy defects" [65] (see also [66] in the context of 3d-3d correspondence) in $T[L(k,1)]$ supported on the circle fibers of $\Sigma \times S^1$, they are labeled by

$$\mathbf{a} = \text{diag}\{a_1, a_2, a_3, \ldots, a_N\} \in \mathfrak{u}(N) \tag{3.18}$$

together with a compatible choice of Levi subgroup $\mathfrak{L} \subset U(N)$. In the equivariant Verlinde TQFT, one only needs to consider maximal defects with $\mathfrak{L} = U(1)^N$ as they are enough to span the finite-dimensional $\mathcal{H}_{\text{EV}}$. The set of continuous parameters $\mathbf{a}$ is acted upon by the affine Weyl group W_{aff} and therefore can be chosen to live in the Weyl alcove:

$$1 > a_1 \geq a_2 \geq \ldots \geq a_N \geq 0. \tag{3.19}$$

In the presence of a Chern-Simons term at level k, gauge invariance imposes the following integrality condition:

$$e^{2\pi i k\,\mathbf{a}} = \mathbf{1}. \tag{3.20}$$

We can then define

$$\mathbf{h} = k\mathbf{a} \tag{3.21}$$

whose elements are now integers in the range $[0, k)$. The condition (3.20) is also the condition for the adjoint orbit

$$\mathcal{O}_{\mathbf{h}} = \{ghg^{-1} | g \in U(N)\} \tag{3.22}$$

to be quantizable. Via the Borel-Weil-Bott theorem, quantizing $\mathcal{O}_{\mathbf{h}}$ gives a representation of $U(N)$ labeled by a Young tableau $\vec{h} = (h_1, h_2, \ldots, h_N)$. So, we can also label the states in $\mathcal{H}_{\text{EV}}(S^1)$ by representations of $U(N)$ or, more precisely, integrable representations of the loop group of $U(N)$ at level k. In other words, the Hilbert space of the equivariant Verlinde TQFT is the same as that of the usual Verlinde TQFT (better known as the G/G gauged WZW model). This is, of course, what one expects as the Verlinde algebra corresponds to the $\mathrm{t} = 0$ limit of the equivariant Verlinde algebra, and the effect of t is to modify the algebra structure without changing $\mathcal{H}_{\text{EV}}$. In particular, the dimension of $\mathcal{H}_{\text{EV}}$ is independent of the value of t.

One could also use the local operators from the dimensional reduction of Wilson loops as the basis for $\mathcal{H}_{\text{EV}}(S^1)$. In pure Chern-Simons theory, the monodromy defects are the same as Wilson loops. In $T[L(k,1),\beta]$ with β turned on, these two types of defects are still linearly related by a transformation matrix, which is no longer diagonal. One of the many reasons that we prefer the maximal monodromy defects is because, under the correspondence, they are mapped to more familiar objects on the Coulomb index side. To see this, we first notice that the following brane system

$$\begin{array}{rccccc} N \text{ fivebranes:} & L(k,1)_b \times & \Sigma & \times S^1 & & \\ & & \cap & & & \\ \text{space-time:} & L(k,1)_b \times & T^*\Sigma & \times S^1 & \times \mathbb{R}^3 & \\ & & \cup & & & \\ n\times N \text{ “defect” fivebranes:} & L(k,1)_b \times & T^*|_{p_i}\Sigma & \times S^1 & & \end{array} \tag{3.23}$$

gives n maximal monodromy defects at $(p_1, p_2, \ldots, p_n) \in \Sigma$. If one first compactifies the brane system above on Σ, one obtains the 4d $\mathcal{N}=2$ class $\mathcal{S}$ theory $T[\Sigma_{g,n}]$ on $L(k,1)_b \times S^1$. This theory has flavor symmetry $U(N)^n$ and one can consider sectors of the theory with non-trivial flavor holonomies $\{\exp[\mathbf{a}_i], i=1,2,\ldots,n\}$ of $U(N)^n$ along the Hopf fiber. The $L(k,1)$-Coulomb branch index of $T[\Sigma_{g,n}]$ depends only on $\{\mathbf{a}_i, i=1,2,\ldots,n\}$ and therefore states in the Hilbert space $\mathcal{H}_{\text{CB}}$ of the Coulomb branch index TQFT associated to a puncture on Σ are labeled by a $U(N)$ holonomy $\mathbf{a}$. (Notice that, for other types of indices, the states are in general also labeled by a continuous parameter corresponding to the holonomy along the S^1 circle and the 2d TQFT for them is in general infinite-dimensional). As the Hopf fiber is the generator of $\pi_1(L(k,1)) = \mathbb{Z}_k$, one has

$$e^{2\pi i k\mathbf{a}} = \text{Id}. \tag{3.24}$$

This is exactly the same as the condition (3.20). In fact, we have even used the same letter $\mathbf{a}$ in both equations, anticipating the connection between the two. What we have found is the canonical way of identifying the two sets of basis vectors in the two Hilbert spaces

$$\begin{array}{ccc} \mathcal{H}_{\text{EV}}^{\otimes n} & & \mathcal{H}_{\text{CB}}^{\otimes n} \\ \Cup & & \Cup \\ \boxed{\begin{array}{c}\text{Monodromy defects on } \Sigma_{g,n}\times S^1 \\ \text{in } GL(N,\mathbb{C})_k \text{ complex Chern-Simons theory}\end{array}} & = & \boxed{\begin{array}{c}\text{Flavor holonomy sectors} \\ \text{of } T[\Sigma_{g,n}\times S^1, U(N)] \text{ on } L(k,1)\end{array}} \end{array}. \tag{3.25}$$

And, of course, this relation is expected as both sides are labeled by flat connections of the Chan-Paton bundle associated to the coincident N "defect" M5-branes in (3.23). Using the relation (3.25), henceforth we identify $\mathcal{H}_{\text{EV}}$ and $\mathcal{H}_{\text{CB}}$.

The statement for a general group

The proposed relation (3.7) between the $U(N)$ equivariant Verlinde formula and the Coulomb branch index for $T[\Sigma, U(N)]$ can be generalized to other groups. First, one could consider decoupling the center of mass degree of freedom for all coincident stacks of M5-branes. However, there are at least two different ways of achieving this. Namely, one could get rid of the $\mathfrak{u}(1)$ part of $\mathbf{a}$ by either

1. subtracting the trace part from $\mathbf{a}$:

$$\mathbf{a}_{\text{SU}} = \mathbf{a} - \frac{1}{N}\text{tr}\,\mathbf{a}, \tag{3.26}$$

2. or forcing $\mathbf{a}$ to be traceless by imposing

$$a_N = -\sum_{i}^{N-1} a_i \tag{3.27}$$

   to get

$$\mathbf{a}_{\text{PSU}} = \text{diag}(a_1, a_2, \ldots, a_{N-1}, -\sum_{i}^{N-1} a_i). \tag{3.28}$$

Naively, one may expect the two different approaches to be equivalent. However, as we are considering lens space index, the global structure of the group comes into play. Indeed, the integrality condition (3.20) becomes different:

$$e^{2\pi i k\cdot \mathbf{a}_{\text{SU}}} \in \mathbb{Z}_N = \mathcal{Z}(SU(N)) \tag{3.29}$$

while

$$e^{2\pi i k\cdot \mathbf{a}_{\text{PSU}}} = \mathbf{1} = \mathcal{Z}(PSU(N)). \tag{3.30}$$

Here $PSU(N) = SU(N)/\mathbb{Z}_N$ has trivial center but a non-trivial fundamental group. As a consequence of having different integrality conditions, one can get either Verlinde formula for $SU(N)$ or $PSU(N)$. In the first case, the claim is

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } SU(N)_k \text{ on } \Sigma\end{array}} = \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, PSU(N)] \text{ on } L(k,1)\times S^1\end{array}}. \tag{3.31}$$

The meaning of $T[\Sigma, PSU(N)]$ and the way to compute its Coulomb branch index will be discussed shortly. On the other hand, if one employs the second method to decouple the $U(1)$ factor, one finds a similar relation with the role of $SU(N)$ and $PSU(N)$ reversed:

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } PSU(N)_k \text{ on } \Sigma\end{array}} = \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, SU(N)] \text{ on } L(k,1)\times S^1\end{array}}. \tag{3.32}$$

Before deriving these statements, we first remark that they are all compatible with (3.7) for general G, which we record again below:

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } G_k \text{ on } \Sigma\end{array}} = \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, {}^L G] \text{ on } L(k,1)\times S^1\end{array}}, \tag{3.33}$$

since ${}^L U(N) = U(N)$ and ${}^L SU(N) = PSU(N)$. This general proposal also gives a geometric/physical interpretation of the Coulomb index of $T[\Sigma, G]$ on $L(k, 1)$ by relating it to the quantization of the Hitchin moduli space $\mathcal{M}_H(\Sigma, {}^L G)$. In fact, one can make a even more general conjecture for all 4d $\mathcal{N} = 2$ superconformal theories (not necessarily of class $\mathcal{S}$):

$$\boxed{\begin{array}{c}L(k,1) \text{ Coulomb index}\\ \text{of a 4d } \mathcal{N}=2 \text{ superconformal theory } \mathcal{T}\end{array}} \overset{?}{=} \boxed{\begin{array}{c}\text{Graded dimension of Hilbert space}\\ \text{from quantization of } (\widetilde{\mathcal{M}}_{\mathcal{T}}, k\omega_I)\end{array}}. \tag{3.34}$$

Here, $\widetilde{\mathcal{M}}_{\mathcal{T}}$ is the SYZ mirror [67] of the Coulomb branch $\mathcal{M}_{\mathcal{T}}$ of $\mathcal{T}$ on $\mathbb{R}^3 \times S^1$. Indeed, $\mathcal{M}_{\mathcal{T}}$ has the structure of a torus fibration:

$$\begin{array}{ccc}\mathbf{T}^{2d} & \hookrightarrow & \mathcal{M}_{\mathcal{T}}\\ & & \downarrow\\ & & \mathcal{B}\end{array}. \tag{3.35}$$

Here $\mathcal{B}$ is the d-(complex-)dimensional Coulomb branch of $\mathcal{T}$ on $\mathbb{R}^4$, $\mathbf{T}^{2d}$ is the 2d-torus parametrized by the holomonies of the low energy $U(1)^d$ gauge group along the spatial circle S^1 and the expectation values of d dual photons. One can perform T-duality on $\mathbf{T}^{2d}$ to obtain the mirror manifold[5] $\widetilde{\mathcal{M}}_{\mathcal{T}}$

$$\begin{array}{ccc}\widetilde{\mathbf{T}}^{2d} & \hookrightarrow & \widetilde{\mathcal{M}}_{\mathcal{T}}\\ & & \downarrow\\ & & \mathcal{B}\end{array}. \tag{3.36}$$

[5] In many cases, the mirror manifold $\widetilde{\mathcal{M}}_{\mathcal{T}} = \mathcal{M}_{\mathcal{T}'}$ is also the 3d Coulomb branch of a theory $\mathcal{T}'$ obtained by replacing the gauge group of $\mathcal{T}$ with its Langlands dual. One can easily see that $\mathcal{T}'$ obtained this way always has same 4d Coulomb branch B as $\mathcal{T}$.

The dual torus $\widetilde{\mathbf{T}}^{2d}$ is a Kähler manifold equipped with a Kähler form ω, which extends to ω_I, one of the three Kähler forms $(\omega_I, \omega_J, \omega_K)$ of the hyper-Kähler manifold $\widetilde{\mathcal{M}}_{\mathcal{T}}$. Part of the R-symmetry that corresponds to the $U(1)_N - U(1)_R$ subgroup inside the $SU(2)_R \times U(1)_N$ R-symmetry group of $\mathcal{T}$ becomes a $U(1)_\beta$ symmetry of $\widetilde{\mathcal{M}}_{\mathcal{T}}$.

Quantizing $\widetilde{\mathcal{M}}_{\mathcal{T}}$ with respect to the symplectic form $k\omega_I$ yields a Hilbert space $\mathcal{H}(\mathcal{T}, k)$. Because $\widetilde{\mathcal{M}}_{\mathcal{T}}$ is non-compact, the resulting Hilbert space $\mathcal{H}(\mathcal{T}, k)$ is infinite-dimensional. However, because the fixed point set of $U(1)_\beta$ is compact and is contained in the nilpotent cone (= the fiber of $\tilde{\mathcal{M}}_{\mathcal{T}}$ at the origin of $\mathcal{B}$), the following graded dimension is free of any divergences and can be computed with the help of the equivariant index theorem

$$\dim_\beta \mathcal{H}(\mathcal{T}, k) = \sum_{m=0}^{\infty} \mathrm{t}^m \dim \mathcal{H}^m(\mathcal{T}, k) = \int_{\widetilde{\mathcal{M}}_{\mathcal{T}}} \mathrm{ch}(\mathcal{L}^{\otimes k}, \beta) \wedge \mathrm{Td}(\widetilde{\mathcal{M}}_{\mathcal{T}}, \beta). \quad (3.37)$$

Here $\mathrm{t} = e^{-\beta}$ is identified with the parameter of the Coulomb branch index, $\mathcal{L}$ is a line bundle whose curvature is ω_I, and $\mathcal{H}^m(\mathcal{T}, k)$ is the weight-m component of $\mathcal{H}(\mathcal{T}, k)$ with respect to the $U(1)_\beta$ action. In obtaining (3.37), we have used the identification $\mathcal{H}(\mathcal{T}, k) = H^*(\widetilde{\mathcal{M}}_{\mathcal{T}}, \mathcal{L}^{\otimes k})$ from geometric quantization.[6]

Now let us give a heuristic argument for why (3.37) computes the Coulomb branch index. The lens space $L(k, 1)$ can be viewed as a torus fibered over an interval. Following [33, 69, 70] and [71], one can identify the Coulomb branch index with the partition function of a topological A-model living on a strip, with $\mathcal{M}_{\mathcal{T}}$ as the target space. The boundary condition at each end of the strip gives a certain brane in $\mathcal{M}_{\mathcal{T}}$. One can then apply mirror symmetry and turn the system into a B-model with $\widetilde{\mathcal{M}}_{\mathcal{T}}$ as the target space. Inside $\widetilde{\mathcal{M}}_{\mathcal{T}}$, there are two branes $\mathfrak{B}_1$ and $\mathfrak{B}_2$ specifying the boundary conditions at the two endpoints of the spatial interval. The partition function for this B-model computes the dimension of the Hom-space between the two branes:

$$Z_{\text{B-model}} = \dim \mathrm{Hom}(\mathfrak{B}_1, \mathfrak{B}_2). \quad (3.38)$$

Now $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are objects in the derived category of coherent sheaves on $\widetilde{\mathcal{M}}_{\mathcal{T}}$ and the quantity above can be computed using the index theorem. The equivariant version is

$$Z_{\text{B-model},\beta} = \dim_\beta \mathrm{Hom}(\mathfrak{B}_1, \mathfrak{B}_2) = \int_{\widetilde{\mathcal{M}}_{\mathcal{T}}} \mathrm{ch}(\mathfrak{B}_1^*, \beta) \wedge \mathrm{ch}(\mathfrak{B}_2, \beta) \wedge \mathrm{Td}(\widetilde{\mathcal{M}}_{\mathcal{T}}, \beta). \quad (3.39)$$

[6] One expects the higher cohomology groups to vanish, since $\mathcal{L}$ is ample on each generic fiber $\widetilde{\mathbf{T}}^{2d}$. For Hitchin moduli space, the vanishing of higher cohomology for $\mathcal{L}^{\otimes k}$ is proven in [63, 68].

We can choose the duality frame such that $\mathfrak{B}_1 = \mathcal{O}$ is the structure sheaf. Then $\mathfrak{B}_2$ is obtained by acting $T^k \in SL(2,\mathbb{Z})$ on $\mathfrak{B}_1$. A simple calculation shows $\mathfrak{B}_2 = \mathcal{L}^{\otimes k}$. So the Coulomb branch index indeed equals (3.37), confirming the proposed relation (3.34) (see also Chapter 4 for a test of this relation for many Argyres-Douglas theories).

$SU(N)$ **vs.** $PSU(N)$**.** Now let us explain why (3.31) and (3.32) are expected. Both orbits, $\mathcal{O}_{\mathbf{a}_{\text{SU}}}$ and $\mathcal{O}_{\mathbf{a}_{\text{PSU}}}$, are quantizable and give rise to representations of $\mathfrak{su}(N)$. However, as the integrality conditions are different, there is a crucial difference between the two classes of representations that one can obtain from $\mathbf{a}_{\text{SU}}$ and $\mathbf{a}_{\text{PSU}}$. Namely, one can get all representations of $SU(N)_k$ from $\mathcal{O}_{\mathbf{a}_{\text{SU}}}$ but only representations[7] of $PSU(N)_k$ from $\mathcal{O}_{\mathbf{a}_{\text{PSU}}}$. This can be directly verified as follows.

For either $\mathbf{a}_{\text{SU}}$ or $\mathbf{a}_{\text{PSU}}$, quantizing $\mathcal{O}_{\mathbf{a}}$ gives a representation of $SU(N)$ with the highest weight[8]

$$\vec{\mu} = (h_1 - h_N, h_2 - h_N, \ldots, h_{N-1} - h_N) \equiv k(a_1 - a_N, a_2 - a_N, \ldots, a_{N-1} - a_N) \pmod N. \tag{3.41}$$

The corresponding Young tableau consists of $N-1$ rows with $h_i - h_N$ boxes in the i-th row. The integrality condition (3.29) simply says that $\vec{\mu}$ is integral. With no other constraints imposed, one can get all representations of $SU(N)$ from $\mathbf{a}_{\text{SU}}$. On the other hand, the condition (3.30) requires the total number of boxes to be a multiple of N,

$$\sum_{i=1}^{N-1} \mu_i = N \cdot \sum_{i=1}^{N-1} a_i \equiv 0 \pmod N, \tag{3.42}$$

restricting us to these representations of $SU(N)$ where the center $\mathbb{Z}_N$ acts trivially. These are precisely the representations of $PSU(N)$.

What we have seen is that in the first way of decoupling $U(1)$, one arrives at the equivariant Verlinde algebra for $SU(N)_k$, while the second option leads to the $PSU(N)_k$ algebra. Then what happens on the lens space side?

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[7] In our conventions, representations of $PSU(N)_k$ are those representations of $SU(N)_k$ invariant under the action of the center. There exist different conventions in the literature and one is related to ours by $k' = \lfloor k/N \rfloor$. Strictly speaking, when $N \nmid k$, the 3d Chern-Simons theory is not invariant under large gauge transformation and doesn't exist. Nonetheless, the 2d equivariant Verlinde algebra is still well defined and matches the algebra from the Coulomb index side.

[8] Sometimes it is more convenient to use a different convention for the highest weight

$$\vec{\lambda} = (h_1 - h_2, h_2 - h_3, \ldots, h_{N-1} - h_N) \equiv k \cdot (a_1 - a_2, a_2 - a_3, \ldots, a_{N-1} - a_N) \pmod N. \tag{3.40}$$

$T[\Sigma, SU(N)]$ vs. $T[\Sigma, PSU(N)]$. In the second approach of removing the center, the flavor $U(N)$-bundles become well-defined $SU(N)$-bundles on $L(k,1)$ and decoupling all the central $U(1)$'s on the lens space side simply means computing the lens space Coulomb branch index of $T[\Sigma, SU(N)]$. So we arrive at the equivalence (3.32) between $PSU(N)_k$ equivariant Verlinde algebra and the algebra of the Coulomb index TQFT for $SU(N)$. On the other hand, in the first way of decoupling the $U(1)$, the integrality condition

$$e^{2\pi i k\cdot \mathbf{a}} = 1 \tag{3.43}$$

is not satisfied for $\mathbf{a}_{\mathrm{SU}}$. And as in (3.29), the right-hand side can be an arbitrary element in the center $\mathbb{Z}_N$ of $SU(N)$. In other words, after using the first method of decoupling the central $U(1)$, the $U(N)$-bundle over $L(k,1)$ becomes a $PSU(N) = SU(N)/\mathbb{Z}_N$-bundle. Another way to see this is by noticing that for $\exp[2\pi i \mathbf{a}] \in \mathcal{Z}(SU(N))$,

$$\mathbf{a}_{\mathrm{SU}} = \mathbf{a} - \frac{1}{N}\mathrm{tr}\,\mathbf{a} = 0. \tag{3.44}$$

This tells us that the $U(1)$ quotient done in this way has collapsed the $\mathbb{Z}_N$ center of $U(N)$, giving us not a well-defined $SU(N)$-bundle but a $PSU(N)$-bundle. Therefore, it is very natural to give the name "$T[\Sigma, PSU(N)]$" to the resulting theory living on $L(k,1)\times S^1$, as the class $\mathcal{S}$ theory $T[\Sigma, G]$ doesn't currently have proper definition in the literature if G is not simply-connected.

For a general group G, one natural definition of the path integral of $T[\Sigma, G]$ on $L(k,1)\times S^1$ is as the path integral of $T[\Sigma, \widetilde{G}]$ with summation over all possible 't Hooft fluxes labeled by $\pi_1(G) \subset \mathcal{Z}(\tilde{G})$ along $L(k,1)$, where $\widetilde{G}$ is the universal cover of G (see *e.g.* [72, Section 4.1] for nice explanation from the 6d viewpoint). This amounts to summing over different topological types of G-bundles over $L(k,1)$, classified by $H^2(L(k,1), \pi_1(G)) = \pi_1(G)\otimes \mathbb{Z}_k$.

Although this is a valid definition, it is not the right one for (3.7) to work for general k. This is clear from the quantization condition (3.29), which tells us that, in order to get the $SU(N)$ Verlinde algebra, the Lens index of $T[\Sigma, PSU(N)]$ should be interpreted in the following way: in the process of assembling Σ from pairs of pants and cylinders, we should sum over 't Hooft fluxes in the *full* fundamental group $\pi_1(PSU(N)) = \mathbb{Z}_N$, as opposed to $\mathbb{Z}_N\otimes\mathbb{Z}_k$, in the $T[\Sigma, SU(N)]$ theory for each gauge group associated with a cylinder. But in general, $\mathbb{Z}_N\otimes\mathbb{Z}_k$ is only a proper subgroup of $\mathbb{Z}_N$, unless N divides k.

However, general flux backgrounds can be realized by inserting surface operators (which we will refer to as "flux tubes") with central monodromy whose

Levi subgroup is the entire group [65]. In the spatial directions, the flux tube lives on a $S^1 \subset L(k,1)$ that has linking number 1 with the Hopf fiber. So we can choose this S^1 to be a particular Hopf fiber S^1_{Hopf} . The amount of flux is labeled by an element in $\pi_1(G) \subset \mathcal{Z}(\widetilde{G})$. Geometrically, this construction amounts to removing a single Hopf fiber from $L(k,1)$, leading to compactly supported cohomology $H^2_c(L(k,1)\backslash S^1_{\text{Hopf}}, \mathbb{Z}) = \mathbb{Z}$ that is freely generated. Then $H^2_c\left(L(k,1)\backslash S^1_{\text{Hopf}}, \pi_1(G)\right) = \pi_1(G)$, and the flux can take value on the whole $\pi_1(G)$.

When G is a group of adjoint type (*i.e.* $\mathcal{Z}(G)$ is trivial), we will call the index of $T[\Sigma, G]$ defined this way the "full Coulomb branch index" of $T[\Sigma, \widetilde{G}]$, which sums over *all* elements of $\pi_1(G) = \mathcal{Z}(\widetilde{G})$. As it contains the most information about the field theory, it is also the most interesting in the whole family associated to the Lie algebra $\mathfrak{g}$. This is not at all surprising as on the other side of the duality, the $\widetilde{G}$ equivariant Verlinde algebra involves all representations of $\mathfrak{g}$ and is the most interesting one among its cousins.

As for the A_{N-1} series that we will focus on in the rest of this chapter, we will be studying the correspondence (3.31) between the $SU(N)$ equivariant Verlinde algebra and the Coulomb index of $T[\Sigma, PSU(N)]$. But before going any further, we will first address a common concern that the reader may have. Namely, charge quantization appears to be violated in the presence of these non-integral $SU(N)$ holonomies. Shouldn't this suggest that the index is just zero with a non-trivial flux background? Indeed, for a state transforming under the fundamental representation of $SU(N)$, translation along the Hopf fiber of $L(k,1)$ k times gives a non-abelian Aharonov-Bohm phase

$$e^{2\pi i k \mathbf{a}_{\text{SU}}}. \tag{3.45}$$

Since the loop is trivial in $\pi_1(L(k,1))$, one would expect this phase to be trivial. However, in the presence of a non-trivial 't Hooft flux, (3.45) is a non-trivial element in the center of $SU(N)$. Then the partition function with insertion of such an 't Hooft operator is automatically zero. However, this is actually what one must have in order to recover even the usual Verlinde formula in the $\mathfrak{t} = 0$ limit. As we will explain next, what is observed above in the $SU(2)$ case is basically the "selection rule" saying that in the decomposition of a tensor product

$$\text{(half integer spin)} \otimes \text{(integer spin)} \otimes \ldots \otimes \text{(integer spin)} \tag{3.46}$$

there is no representation with integer spins! What we will do next is to use Dirac quantization conditions in $T[\Sigma, PSU(N)]$ to derive the selection rule above and analogous rules for the $SU(N)$ Verlinde algebra.

Verlinde algebra and Dirac quantization

The Verlinde formula associates to a pair of pants a fusion coefficient f_{abc} which tells us how to decompose a tensor product of representations:

$$R_a \otimes R_b = \bigoplus_c f_{ab}{}^c R_c. \tag{3.47}$$

Equivalently, this coefficient gives the dimension of the invariant subspace of three-fold tensor products

$$\dim \mathrm{Inv}(R_a \otimes R_b \otimes R_c) = f_{abc}. \tag{3.48}$$

Here, upper and lower indices are related by the "metric"

$$\eta_{ab} = \dim \mathrm{Inv}(R_a \otimes R_b) = \delta_{a\bar{b}}, \tag{3.49}$$

which is what the TQFT associates to a cylinder.

In the case of $SU(N)$, the fusion coefficients f_{abc} are zero whenever a selection rule is not satisfied. For three representations labeled by the highest weights $\vec{\mu}^{(1)}, \vec{\mu}^{(2)}, \vec{\mu}^{(3)}$ in (3.41) the selection rule is

$$\sum_{i=1}^{N-1} (\mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)}) \equiv 0 \pmod N. \tag{3.50}$$

This is equivalent to the condition that $\mathbb{Z}_N$ acts trivially on $R_a \otimes R_b \otimes R_c$. Of course, when this action is non-trivial, it is easy to see that there can't be any invariant subspace.

Our job now is to reproduce this rule on the Coulomb index side via Dirac quantization. We start with the familiar case of $SU(2)$. The theory $T_2 = T[\Sigma_{0,3}, SU(2)]$ consists of eight 4d $\mathcal{N} = 2$ half-hypermultiplets transforming in the tri-fundamental of the $SU(2)_a \times SU(2)_b \times SU(2)_c$ flavor symmetry. The holonomy $(H_a, H_b, H_c) \in U(1)^3$ of this flavor symmetry along the Hopf fiber is given by a triple(m_a, m_b, m_c) with

$$H_I = e^{2\pi i m_I/k}, \quad I = a, b, c. \tag{3.51}$$

The Dirac quantization requires that the Aharonov-Bohm phase associated with a trivial loop must be trivial. So, in the presence of the non-trivial holonomy along the Hopf fiber, a physical state with charge (e_a, e_b, e_c) needs to satisfy

$$H_a^{ke_a} H_b^{ke_b} H_c^{ke_c} = e^{2\pi i \sum_{I=a,b,c} e_I m_I} = 1, \tag{3.52}$$

or, equivalently,

$$\sum_{I=a,b,c} e_I m_I \in \mathbb{Z}. \tag{3.53}$$

When decomposed into representations of $U(1)^3$, the tri-fundamental hypermultiplet splits into eight components:

$$(\mathbf{2},\mathbf{2},\mathbf{2}) \to \bigoplus_{\text{All } \pm} (\pm 1, \pm 1, \pm 1). \tag{3.54}$$

Therefore, one needs to satisfy eight equations

$$\pm m_a \pm m_b \pm m_c \in \mathbb{Z}. \tag{3.55}$$

For individual m_I, the condition is

$$m_I \in \frac{\mathbb{Z}}{2}, \tag{3.56}$$

which is the same as the relaxed integrality condition (3.29) for $SU(2)$. This already suggests that the condition (3.29) is the most general one and there is no need to relax it further. Indeed, m_i is the "spin" of the corresponding $SU(2)$ representation and we know that all allowed values for it are integers and half-integers.

Besides the individual constraint (3.56), there is an additional one:

$$m_a + m_b + m_c \in \mathbb{Z}\,, \tag{3.57}$$

which is precisely the "selection rule" we mentioned before. Only when this rule is satisfied could R_{m_c} appear in the decomposition of $R_{m_a} \otimes R_{m_b}$.

We then proceed to the case of $SU(N)$. When $N = 3$ the theory T_3 doesn't have a Lagrangian description but is conjectured to have E_6 global symmetry [73]. And the matter fields transform in the 78-dimensional adjoint representation of E_6 [74–76] which decomposes into $SU(3)^3$ representations as follows

$$\mathbf{78} = (\mathbf{3},\mathbf{3},\mathbf{3}) \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}) \oplus (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{8}). \tag{3.58}$$

The $\mathbf{8}$ is the adjoint representation of $\mathfrak{su}(3)$ and, being a representation for both $SU(3)$ and $PSU(3)$, imposes no additional restriction on 't Hooft fluxes. So we only need to understand the quantization condition in the presence of a tri-fundamental matter $(\mathbf{3},\mathbf{3},\mathbf{3})$. A natural question, then, is whether it happens more generally, *i.e.*,

$$\begin{matrix}\text{Dirac quantization condition} \\ \text{for the } T_N \text{ theory}\end{matrix} \quad = \quad \begin{matrix}\text{Dirac quantization condition} \\ \text{for a tri-fundamental matter.}\end{matrix} \tag{3.59}$$

This imposes on the T_N theory an interesting condition, which is expected to be true as it turns out to give the correct selection rule for $SU(N)$ Verlinde algebra.

Now, we proceed to determine the quantization condition for the tri-fundamental of $SU(N)^3$. We assume the holonomy in $SU(N)^3$ to be

$$(H_a, H_b, H_c), \tag{3.60}$$

where

$$H_I = \exp\left[\frac{2\pi i}{k}\mathrm{diag}\{m_{I1}, m_{I2}, \ldots, m_{IN}\}\right]. \tag{3.61}$$

The tracelessness condition looks like

$$\sum_{j=1}^{N} m_{Ij} = 0 \quad \text{for all } I = a, b, c. \tag{3.62}$$

We now have N^3 constraints given by

$$m_{aj_1} + m_{bj_2} + m_{cj_3} \in \mathbb{Z} \quad \text{for all choices of } j_1, j_2, \text{ and } j_3. \tag{3.63}$$

Using (3.62), one can derive the individual constraint for each $i = a, b, c$[9]:

$$\mathbf{m}_I \equiv \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right) \cdot \mathbb{Z} \pmod{\mathbb{Z}}. \tag{3.64}$$

This is exactly the same as (3.29). There is only one additional "selection rule" that needs to be satisfied:

$$\sum_{I=a,b,c} \sum_{j=1}^{N-1} (m_{Ij} - m_{IN}) \equiv 0 \pmod{N}, \tag{3.65}$$

which coincides with (3.50). Therefore, we have demonstrated the equivalence between the Dirac quantization condition of the tri-fundamental and the selection rules in the $SU(N)$ Verlinde algebra. Since the argument is independent of the value of $\mathfrak{t}$, the same set of selection rules also applies to the equivariant Verlinde algebra.

Beside pairs of pants, one needs one more ingredient to build a 2d TQFT — the cylinder. It can be used to glue punctures together to build general Riemann surfaces. Each cylinder corresponds to a free 4d $\mathcal{N} = 2$ vector multiplet. Since all of its components transform under the adjoint representation, it does not alter

[9] In this chapter, bold letters like $\mathbf{m}$ are used to denote an element in the Cartan subalgebra of $\mathfrak{g}$. They are sometimes viewed as a diagonal matrix and sometimes a multi-component vector. The interpretation should be clear from the context.

the individual constraints (3.64). However, the holonomies associated with the two punctures need to be the inverse of each other as the two flavor symmetries are identified and gauged. So the index of $T[\Sigma_{0,2}, SU(N)]$ gives a diagonal "metric"

$$\eta_{ab} \sim \delta_{a\bar{b}}. \tag{3.66}$$

The proportionality constant is $\mathfrak{t}$ dependent and will be determined in later sections.

We can also derive the the Dirac quantization condition for $T[\Sigma_{g,n}, PSU(N)]$. We use m_{Ij} to label the j-th component of the $U(1)^N$ holonomy associated to the I-th puncture. Then the index or any kind of partition function of $T[\Sigma_{g,n}, SU(N)]$ is zero unless

1. each $\vec{m}_I$ satisfies the individual constraint (3.64), and
2. an additional constraint analogous to (3.65),

$$\sum_{I=1}^{n}\sum_{j=1}^{N-1}(m_{Ij} - m_{IN}) \equiv 0 \pmod N\,, \tag{3.67}$$

   is also satisfied.

To end this section, we will explain how the additional numerical factor in (3.8) in the introduction arises from non-trivial 't Hooft fluxes. For $G = SU(N)$, one has

$$Z_{\text{EV}}(\Sigma, k=1, \mathfrak{t}) = N^g \cdot \left[\frac{1}{\prod_{i=1}^{\text{rank}\, G}(1-\mathfrak{t}^{i+1})^{2i+1}}\right]^{g-1}. \tag{3.68}$$

Here we are only concerned with the first factor N^g which is the $k = 1$ Verlinde formula for $SU(N)$

$$Z_{\text{EV}}(\Sigma, k=1, \mathfrak{t}=0) = N^g. \tag{3.69}$$

We now derive this result on the index side.

Consider the twice-punctured torus, obtained by gluing two pairs of pants. Let (a_1, a_2, a_3) and $(b_1, b_2, b_3) \in \mathbb{Z}_N^3$ label the 't Hooft fluxes corresponding to all six punctures. We glue a_2 with b_2, and a_3 with b_3 to get $\Sigma_{1,2}$. Then we have the following set of constraints:

$$a_2 b_2 = 1,\ a_3 b_3 = 1, \tag{3.70}$$

and

$$a_1 a_2 a_3 = 1,\ b_1 b_2 b_3 = 1. \tag{3.71}$$

From these constraints, we can first confirm that

$$a_1 b_1 = 1, \tag{3.72}$$

which is what the selection rule (3.67) predicts. Then there is a free parameter a_2 that can take arbitrary values in $\mathbb{Z}_N$. So in the $\mathfrak{t} = 0$ limit, the Coulomb index TQFT associates to $\Sigma_{1,2}$

$$Z_{\text{CB}}(\Sigma_{1,2}, SU(N), \mathfrak{t} = 0) = N\delta_{a_1,\bar{b}_1}. \tag{3.73}$$

We can now glue $g-1$ twice-punctured tori to get

$$Z_{\text{CB}}(\Sigma_{g-1,2}\,, SU(N)\,, \mathfrak{t} = 0) = N^{g-1}\delta_{a_1,\bar{b}_{g-1}}. \tag{3.74}$$

Taking trace of this gives[10]

$$Z_{\text{CB}}(\Sigma_{g,0}\,, SU(N)\,, \mathfrak{t} = 0) = N^g. \tag{3.75}$$

Combining this with the $\mathfrak{t}$ dependent part of (3.8), we have proved that, for $k = 1$, the equivariant Verlinde formula is the same as the full Coulomb branch index.

We will now move on to cases with more general k to perform stronger checks.

3.3 A check of the proposal

In this section, we perform explicit computation of the Coulomb branch index for the theory $T[\Sigma_{g,n}, PSU(2)]$ in the presence of 't Hooft fluxes (or half-integral flavor holonomies). We will see that after taking into account a proper normalization, the full Coulomb branch index nicely reproduces the known $SU(2)$ equivariant Verlinde algebra. First, we introduce the necessary ingredients of 4d $\mathcal{N} = 2$ superconformal index on $S^1 \times L(k,1)$ for a theory with a Lagrangian description.

The lens space index and its Coulomb branch limit

The lens space index of 4d $\mathcal{N} = 2$ theories is a generalization of the ordinary superconformal index on $S^1 \times S^3$, as $S^3 = L(1,1)$ [78]. For $k > 1$, $L(k,1)$ has a nontrivial fundamental group $\mathbb{Z}_k$, and a supersymmetric theory on $L(k,1)$ tends to have a set of degenerate vacua labeled by holonomies along the Hopf fiber. This feature renders the lens space index a refined tool to study the BPS spectra of the

[10] What we have verified is basically that the algebra of $\mathbb{Z}_N$ 't Hooft fluxes gives the $SU(N)$ Verlinde algebra at level $k = 1$, which is isomorphic to the group algebra of $\mathbb{Z}_N$. Another TQFT whose Frobenius algebra is also related to the group algebra of $\mathbb{Z}_N$ is the 2d $\mathbb{Z}_N$ Dijkgraaf-Witten theory [77]. However, the normalizations of the trace operator are different so the partition functions are also different.

superconformal theory; for instance it can distinguish between theories with gauge groups that have the same Lie algebra but different topologies (*e.g.* $SU(2)$ versus $SO(3)$ [79]). Moreover, as it involves not only continuous fugacities but also discrete holonomies, lens space indices of class $\mathcal{S}$ theories lead to a very large family of interesting and exotic 2d TQFTs [59, 60, 78].

The basic ingredients of the lens space index are indices of free supermultiplets, each of which can be conveniently expressed as a integral over gauge group of the plethystic exponential of the "single-letter index", endowed with gauge and flavor fugacities. This procedure corresponds to constructing all possible gauge invariant multi-trace operators that are short with respect to the superconformal algebra.

In particular, for a gauge vector multiplet the single-letter index is

$$f^V(p,q,t,m,k) = \frac{1}{1-pq}\left(\frac{p^m}{1-p^k} + \frac{q^{k-m}}{1-q^k}\right)(pq + \frac{pq}{t} - 1 - t) + \delta_{m,0}, \tag{3.76}$$

where m will be related to holonomies of gauge symmetries. For a half-hypermultiplet, one has

$$f^{H/2}(p,q,t,m,k) = \frac{1}{1-pq}\left(\frac{p^m}{1-p^k} + \frac{q^{k-m}}{1-q^k}\right)(\sqrt{t} - \frac{pq}{\sqrt{t}}). \tag{3.77}$$

In addition, there is also a "zero point energy" contribution for each type of field. For a vector multiplet and a half hypermultiplet, they are given by

$$\begin{aligned} I_V^0(p,q,t,\mathbf{m},k) &= \prod_{\alpha\in\Delta^+}\left(\frac{pq}{t}\right)^{-[\![\alpha(\mathbf{m})]\!]_k + \frac{1}{k}[\![\alpha(\mathbf{m})]\!]_k^2}, \\ I_{H/2}^0(p,q,t,\mathbf{m},\widetilde{\mathbf{m}},k) &= \prod_{\rho\in\mathfrak{R}}\left(\frac{pq}{t}\right)^{\frac{1}{4}\left([\![\rho(\mathbf{m},\widetilde{\mathbf{m}})]\!]_k - \frac{1}{k}[\![\rho(\mathbf{m},\widetilde{\mathbf{m}})]\!]_k^2\right)}, \end{aligned} \tag{3.78}$$

where $[\![x]\!]_k$ denotes remainder of x divided by k. The boldface letters $\mathbf{m}$ and $\widetilde{\mathbf{m}}$ label holonomies for, respectively, gauge symmetries and flavor symmetries[11]; they are chosen to live in the Weyl alcove and can be viewed as a collection of integers $m_1 \geq m_2 \geq \cdots \geq m_r$.

Now the full index can be written as

$$\begin{aligned} \mathcal{I} = \sum_{\mathbf{m}} I_V^0(p,q,t,\mathbf{m}) I_{H/2}^0(p,q,t,\mathbf{m},\widetilde{\mathbf{m}}) \int \prod_i \frac{dz_i}{2\pi i z_i}\Delta(z)_{\mathbf{m}} \\ \times \exp\left(\sum_{n=1}^{+\infty}\sum_{\alpha,\rho}\frac{1}{n}\left[f^V(p^n,q^n,t^n,\alpha(\mathbf{m}))\alpha(z) + f^{H/2}(p^n,q^n,t^n,\rho(\mathbf{m},\widetilde{\mathbf{m}}))\rho(z,F)\right]\right). \end{aligned} \tag{3.79}$$

[11] As before, the holonomies are given by $e^{2\pi i\mathbf{m}/k}$.

Here, to avoid clutter, we only include one vector multiplet and one half-hypermultiplet. Of course, in general one should remember to include the entire field contents of the theory. Here, F stands for the continuous flavor fugacities and the z_i's are the gauge fugacities; for $SU(N)$ theories one should impose the condition $z_1 z_2 \dots z_N = 1$. The additional summation in the plethystic exponential is over all the weights in the relevant representations. The integration measure is determined by $\mathbf{m}$:

$$\Delta_{\mathbf{m}}(z_i) = \prod_{i,j;m_i=m_j} \left(1 - \frac{z_i}{z_j}\right), \tag{3.80}$$

since a nonzero holonomy would break the gauge group into its stabilizer.

In this chapter we are particularly interested in the Coulomb branch limit, *i.e.* (3.12) and (3.13). From the single letter index (3.76) and (3.77) we immediately conclude that $f^{H/2} = 0$ identically, so the hypermultiplets contributes to the index only through the zero point energy. As for f^V, the vector multiplet gives a non-zero contribution $pq/t = \mathrm{t}$ for each root α that has $\alpha(\mathbf{m}) = 0$. So the zero roots (Cartan generators) always contribute, and non-zero roots can only contribute when the gauge symmetry is enhanced from $U(1)^r$, *i.e.* when $\mathbf{m}$ is at the boundary of the Weyl alcove. This closely resembles the behavior of the "metric" of the equivariant Verlinde algebra, as we will see shortly.

More explicitly, for $SU(2)$ theory, the index of a vector multiplet in the Coulomb branch limit is

$$I_V(\mathrm{t}, m, k) = \mathrm{t}^{-[\![2m]\!]_k + \frac{1}{k}[\![2m]\!]_k^2} \left(\frac{1}{1-\mathrm{t}}\right)\left(\frac{1}{1+\mathrm{t}}\right)^{\delta_{[\![2m]\!],0}}, \tag{3.81}$$

while for tri-fundamental hypermultiplet the contribution is

$$I_{H/2}(\mathrm{t}, m_1, m_2, m_3, k) = \prod_{s_i=\pm} (\mathrm{t})^{\frac{1}{4}\sum_{i=1}^{3}\left([\![m_i s_i]\!]_k - \frac{1}{k}[\![m_i s_i]\!]_k^2\right)}, \tag{3.82}$$

where all holonomies take values from $\{0, 1/2, 1, 3/2, \dots k/2\}$.

Unsurprisingly, this limit fits the name of the "Coulomb branch index." Indeed, in the case of $k = 1$, the index receives only contributions from the Coulomb branch operators, *i.e.* a collection of "Casimir operators" for the theory [58] (*e.g.* $\mathrm{Tr}\,\phi^2$, $\mathrm{Tr}\,\phi^3$, ..., $\mathrm{Tr}\,\phi^N$ for $SU(N)$, where ϕ is the scalar in the $\mathcal{N} = 2$ vector multiplet). We see here that a general lens space index also counts the Coulomb branch operators, but the contribution from each operator is modified according to the background holonomies.

Another interesting feature of the Coulomb branch index is the complete disappearance of continuous fugacities of flavor symmetries. Punctures are now only parametrized by discrete holonomies along the Hopf fiber of $L(k, 1)$. This property ensures that we will obtain a *finite-dimensional* algebra.

Then, to make sure that the algebra defines a TQFT, one needs to check associativity, especially because non-integral holonomies considered here are novel and may cause subtleties. We have checked by explicit computation in $\mathfrak{t}$ that the structure constant and metric defined by lens space index do satisfy associativity, confirming that the "Coulomb branch index TQFT" is indeed well-defined. In fact, even with all p, q, t turned on, the associativity still holds order by order in the expansion in terms of fugacities.

Equivariant Verlinde algebra from Hitchin moduli space

As explained in greater detail in [15], the equivariant Verlinde TQFT computes an equivariant integral over $\mathcal{M}_H$, the moduli space of Higgs bundles. In the case of $SU(2)$, the relevant moduli spaces are simple enough and one can deduce the TQFT algebra from geometry of $\mathcal{M}_H$. For example, one can obtain the fusion coefficients from $\mathcal{M}_H(\Sigma_{0,3}, \alpha_1, \alpha_2, \alpha_3; SU(2))$. Here the α_i's are the ramification data specifying the monodromies of the gauge field [65] and take discrete values in the presence of a level k Chern-Simons term. Since in this case the moduli space is just a point or empty, one can directly evaluate the integral. The result is as follows.

Define $\lambda = 2k\alpha$ whose value is quantized to be $0, 1, \ldots, k$. Let

$$\begin{aligned} d_0 &= \lambda_1 + \lambda_2 + \lambda_3 - 2k, \\ d_1 &= \lambda_1 - \lambda_2 - \lambda_3, \\ d_2 &= \lambda_2 - \lambda_3 - \lambda_1, \\ d_13 &= \lambda_3 - \lambda_1 - \lambda_2, \end{aligned} \tag{3.83}$$

and moreover

$$\Delta\lambda = \max(d_0, d_1, d_2, d_3), \tag{3.84}$$

then

$$f_{\lambda_1\lambda_2\lambda_3} = \begin{cases} 1 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even and } \Delta\lambda \le 0, \\ \mathfrak{t}^{-\Delta\lambda/2} & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even and } \Delta\lambda > 0, \\ 0 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is odd.} \end{cases} \tag{3.85}$$

On the other hand, the cylinder gives the trace form (or "metric") of the algebra

$$\eta_{\lambda_1\lambda_2} = \{1-\mathrm{t}^2, 1-\mathrm{t}, \ldots, 1-\mathrm{t}, 1-\mathrm{t}^2\}. \tag{3.86}$$

Via cutting-and-gluing, we can compute the partition function of the TQFT on a general Riemann surface $\Sigma_{g,n}$.

Matching two TQFTs

So far we have introduced two TQFTs: the first one is given by equivariant integration over Hitchin moduli space $\mathcal{M}_H$, the second one is given by the $L(k,1)$ Coulomb branch index of the theory $T[\Sigma, PSU(2)]$. It is easy to see that the underlying vector space of the two TQFTs are the same, confirming in the $SU(2)$ case the more general result we obtained previously:

$$Z_{\mathrm{EV}}(S^1) = Z_{\mathrm{CB}}(S^1). \tag{3.87}$$

We can freely switch between two different descriptions of the same set of basis vectors, by either viewing them as integrable highest weight representations of $\widehat{su}(2)_k$ or $SU(2)$ holonomies along the Hopf fiber. In this section, we only use highest weights λ as the labels for puncture data, and one can easily translate them into holonomies via $\lambda = 2m$.

Then, one needs to compare the algebraic structure of the two TQFTs and may notice that there are apparent differences. Namely, if one compares I_V and $I_{H/2}$ with η and f in (3.85) and (3.86), there are additional factors coming from the zero point energy in the expressions on the index side. However, one can simply rescale states in the Hilbert space on the Coulomb index side to absorb them.

The scaling required is

$$|\lambda\rangle = \mathrm{t}^{\frac{1}{2}\left([\![\lambda]\!]_k - \frac{1}{k}[\![\lambda]\!]_k^2\right)}|\lambda\rangle'. \tag{3.88}$$

This makes I_V exactly the same as $\eta^{\lambda\mu}$. After rescaling, the index of the half-hypermultiplet becomes

$$I_{H/2} \Rightarrow f'_{\lambda_1\lambda_2\lambda_3} = \mathrm{t}^{-\frac{1}{2}\sum_{i=1}^{3}\left([\![\lambda_i]\!]_k - \frac{1}{k}[\![\lambda_i]\!]_k^2\right)} I_{H/2}(\mathrm{t}, \lambda_1, \lambda_2, \lambda_3, k), \tag{3.89}$$

and this is indeed identical to the fusion coefficient $f_{\lambda\mu\nu}$ of the equivariant Verlinde algebra, which we show as follows. If we define

$$\begin{aligned} g_0 &= m_1 + m_2 + m_3 = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3), \\ g_1 &= m_1 - m_2 - m_3 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3), \\ g_2 &= m_2 - m_1 - m_3 = \frac{1}{2}(\lambda_2 - \lambda_1 - \lambda_3), \\ g_3 &= m_3 - m_1 - m_2 = \frac{1}{2}(\lambda_3 - \lambda_1 - \lambda_3), \end{aligned} \tag{3.90}$$

then our pair of pants can be written as

$$\begin{aligned} f'_{\lambda_1\lambda_2\lambda_3} =& \mathrm{t}^{\frac{1}{2k}([\![g_0]\!]_k[\![-g_0]\!]_k+[\![g_1]\!]_k[\![-g_1]\!]_k+[\![g_2]\!]_k[\![-g_2]\!]_k+[\![g_2]\!]_k[\![-g_2]\!]_k)} \\ &\times \mathrm{t}^{-\frac{1}{2k}(\lambda_1(k-\lambda_1)+\lambda_2(k-\lambda_2)+\lambda_3(k-\lambda_3))}. \end{aligned} \tag{3.91}$$

Now we can simplify the above equation further under various assumptions of each g_i. For instance if $0 < g_0 < k$ and $g_i < 0$ for $i = 1, 2, 3$, then

$$f'_{\lambda_1\lambda_2\lambda_3} = 1. \tag{3.92}$$

If on the other hand, $g_0 > k$ and $g_i < 0$ for $i = 1, 2, 3$, which means $\max(g_0 - k, g_1, g_2, g_3) = g_0 - k$, then

$$f'_{\lambda_1\lambda_2\lambda_3} = \mathrm{t}^{g_0-k}, \tag{3.93}$$

this is precisely what we obtained by (3.85).

Therefore, we have shown that the building blocks of the two TQFTs are the same. And by the TQFT axioms, we have proven the isomorphism of the two TQFTs. For example, they both give t-deformation of the $\widehat{su}(2)_k$ representation ring; at level $k = 10$ a typical example is

$$|3\rangle \otimes |3\rangle = \frac{1}{1-\mathrm{t}^2}|0\rangle \oplus \frac{1}{1-\mathrm{t}}|2\rangle \oplus \frac{1}{1-\mathrm{t}}|4\rangle \oplus \frac{1}{1-\mathrm{t}}|6\rangle \oplus \frac{\mathrm{t}}{1-\mathrm{t}}|8\rangle \oplus \frac{\mathrm{t}^2}{1-\mathrm{t}^2}|10\rangle. \tag{3.94}$$

For closed Riemann surfaces, we list partition functions for several low genera and levels in table 3.1. And this concludes our discussion of the $SU(2)$ case.

3.4 $SU(3)$ equivariant Verlinde algebra from the Argyres-Seiberg duality

In the last section, we have tested the proposal about the equivalence between the equivariant Verlinde algebra and the algebra from the Coulomb index of class $\mathcal{S}$ theories. Then one would ask whether one can do more with such a correspondence

	$k=1$	$k=2$	$k=3$	$k=4$
$g=2$	$\frac{4}{(1-t^2)^3}$	$\frac{2}{(1-t^2)^3}(5t^2+6t+5)$	$\frac{4}{(1-t^2)^3}(4t^3+9t^2+9t+5)$	$\frac{1}{(1-t^2)^3}(16t^4+49t^3$ $+81t^2+75t+35)$
$g=3$	$\frac{8}{(1-t^2)^6}$	$\frac{4}{(1-t^2)^6}(9t^4+28t^3$ $+54t^2+28t+9)$	$\frac{8}{(1-t^2)^6}(8t^6+54t^5+159t^4$ $+238t^3+183t^2+72t+15)$	$\frac{1}{(1-t^2)^6}(64t^8+384t^7+1793t^6$ $+5250t^5+8823t^4+8828t^3$ $+5407t^2+1890t+329)$
$\forall g$	$2\left(\frac{2}{(1-t^2)^3}\right)^{g-1}$	$\left(\frac{2(1-t)^2}{(1-t^2)^3}\right)^{g-1}$ $+2\left(\frac{2(1+t)^2}{(1-t^2)^3}\right)^{g-1}$	$2\left(\frac{5+9t+9t^2+4t^3-\sqrt{5+4t}(1+5t+t^2)}{(1-t^2)^3}\right)^{g-1}+$ $2\left(\frac{5+9t+9t^2+4t^3+\sqrt{5+4t}(1+5t+t^2)}{(1-t^2)^3}\right)^{g-1}$	$\left(\frac{(3+t)(1-t)^2}{(1-t^2)^3}\right)^{g-1}+2\left(\frac{4}{1-t^2}\right)^{g-1}$ $+\left(\frac{4(3+t)(1+t)^3}{(1-t^2)^3}\right)^{g-1}$

Table 3.1: The partition function $Z_{\text{EV}}(T[L(k,1),SU(2)],t) = Z_{\text{CB}}(T[\Sigma_g,PSU(2)],t)$ for genus $g=2,3$ and level $k=1,2,3,4$.

and what are its applications. For example, can one use the Coulomb index as a tool to access geometric and topological information about Hitchin moduli spaces? Indeed, the study of the moduli space of Higgs bundles poses many interesting and challenging problems. In particular, doing the equivariant integral directly on $\mathcal{M}_H$ quickly becomes impractical when one increases the rank of the gauge group. However, our proposal states that the equivariant integral could be computed in a completely different way by looking at the superconformal index of familiar SCFTs! This is exactly what we will do in this section—we will put the correspondence to good use and probe the geometry of $\mathcal{M}_H(\Sigma, SU(3))$ with superconformal indices.

The natural starting point is still a pair of pants or, more precisely, a sphere with three "maximal" punctures (for mathematicians, three punctures with full-flag parabolic structure). The 4d theory $T[\Sigma_{0,3}, SU(3)]$ is known as the T_3 theory [80], which is first identified as an $\mathcal{N}=2$ strongly coupled rank-1 SCFT with a global E_6 symmetry[12] [73]. In light of the proposed correspondence, one expects that the Coulomb branch index of the T_3 theory equals the fusion coefficients $f_{\lambda_1\lambda_2\lambda_3}$ of the $SU(3)$ equivariant Verlinde algebra.

Argyres-Seiberg duality and Coulomb branch index of T_3 theory

A short review. As the T_3 theory is an isolated SCFT, there is no Lagrangian description, and currently no method of direct computation of its index is known in the literature. However, there is a powerful duality proposed by Argyres and Seiberg [76] that relates a superconformal theory with Lagrangian description at

[12] In the following we will use the name "T_3 theory" and "E_6 SCFT" interchangeably.

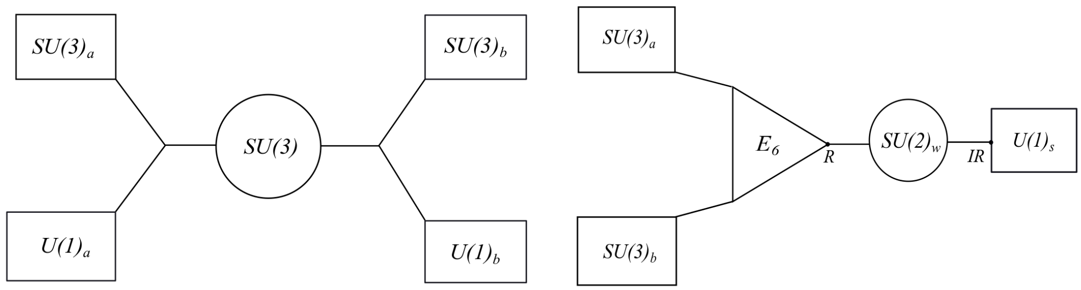
infinite coupling to a weakly coupled gauge theory obtained by gauging an $SU(2)$ subgroup of the E_6 flavor symmetry of the T_3 SCFT.

To be more precise, one starts with an $SU(3)$ theory with six hypermultiplets (call it theory A) in the fundamental representation $3\square \oplus 3\overline{\square}$ of the gauge group. Unlike its $SU(2)$ counterpart, the $SU(3)$ theory has the electric-magnetic duality group $\Gamma^0(2)$, a subgroup of $SL(2,\mathbb{Z})$. As a consequence, the fundamental domain of the gauge coupling τ has a cusp and the theory has an infinite coupling limit. As argued by Argyres and Seiberg through direct analysis of the Seiberg-Witten curve at strong couplings, it was shown that the theory can be naturally identified as another theory B obtained by weakly gauging the E_6 SCFT coupled to an additional hypermultiplet in fundamental representation of $SU(2)$. There is much evidence supporting this duality picture. For instance, the E_6 SCFT has a Coulomb branch operator with dimension 3, which could be identified as the second Casimir operator $\mathrm{Tr}\,\phi^3$ of the dual $SU(3)$ gauge group. The E_6 theory has a Higgs branch of $\dim_{\mathbb{C}} \mathcal{H} = 22$ parametrized by an operator $\mathbb{X}$ in adjoint representation of E_6 with Joseph relation [74]; after gauging $SU(2)$ subgroup, two complex dimensions are removed, leaving the correct dimension of the Higgs branch for the theory A. Finally, Higgsing this $SU(2)$ leaves an $SU(6)\times U(1)$ subgroup of the maximal E_6 group, which is the same as the $U(6) = SU(6) \times U(1)$ flavor symmetry in the A frame.

In [2], the Argyres-Seiberg duality is given a nice geometric interpretation. To obtain theory A, one starts with a 2-sphere with two $SU(3)$ maximal punctures and two $U(1)$ simple punctures, corresponding to global symmetry $SU(3)_a \times SU(3)_b \times U(1)_a \times U(1)_b$, where two $U(1)$ are baryonic symmetry. In this setup, the Argyres-Seiberg duality relates different degeneration limits of this Riemann surface; see figure 3.1 and 3.2.

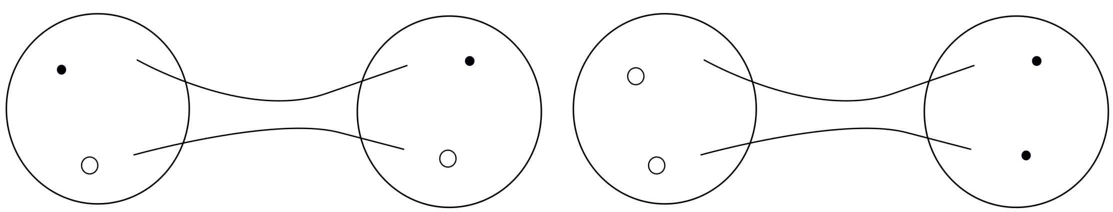
The Argyres-Seiberg duality gives access to the superconformal index for the E_6 SCFT [75]. The basic idea is to start with the index of theory A and, with the aid of the inversion formula of elliptic beta integrals, one identifies two sets of flavor fugacities and extracts the E_6 SCFT index by integrating over a carefully chosen kernel. It was later realized that the above procedure has a physical interpretation, namely the E_6 SCFT can be obtained by flowing to the IR from an $\mathcal{N} = 1$ theory which has Lagrangian description [81]. The index computation of the $\mathcal{N} = 1$ theory reproduces that of [75], and the authors also compute the Coulomb branch index in the large k limit.

Here we would like to obtain the index for general k. In principle, we could start with



(a) (b)

Figure 3.1: Illustration of Argyres-Seiberg duality. (a) The theory A, which is an $SU(3)$ superconformal gauge theory with six hypermultiplets, with the $SU(3)_a \times U(1)_a \times SU(3)_b \times U(1)_b$ subgroup of the global $U(6)$ flavor symmetry. (b) The theory B, obtained by gauging an $SU(2)$ subgroup of the E_6 symmetry of T_3. Note that in the geometric realization the cylinder connecting both sides has a regular puncture R on the left and an irregular puncture IR on the right.



(a) (b)

Figure 3.2: Illustration of geometric realization of Argyres-Seiberg duality for T_3 theory. The dots represent simple punctures while circles are maximal punctures. (a) The theory A, which is an $SU(3)$ superconformal gauge theory with six hypermultiplets, is pictured as two spheres connected by a long tube. Each of them has one simple and two maximal punctures. (b) The theory B, which is obtained by gauging an $SU(2)$ subgroup of the flavor symmetry of the theory T_3. This gauge group connects a regular puncture and an irregular puncture.

the $\mathcal{N} = 1$ theory described in [81] and compute the Coulomb branch index on lens space directly. However, a direct inversion is more intuitive here due to simplicity of the Coulomb branch limit, and can be generalized to arbitrary T_N theories. In the next subsection we outline the general procedure of computing the Coulomb branch index of T_3.

Computation of the index. To obtain a complete basis of the TQFT Hilbert space, we need to turn on all possible flavor holonomies and determine when they correspond to a weight in the Weyl alcove. For the T_3 theory each puncture has $SU(3)$ flavor symmetry, so we can turn on holonomies as $\mathbf{h}^* = (h_1^*, h_2^*, h_3^*)$ for $* = a, b, c$ with constraints $h_1^* + h_2^* + h_3^* = 0$. The Dirac quantization condition tells us that

$$h_i^r + h_j^s + h_k^t \in \mathbb{Z} \tag{3.95}$$

for arbitrary $r, s, t \in \{a, b, c\}$ and $i, j, k = 1, 2, 3$. This means there are only three classes of choices modulo $\mathbb{Z}$, namely

$$\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right), \text{ or } \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right), \text{ or } (0, 0, 0) \quad (\text{mod } \mathbb{Z}). \tag{3.96}$$

Furthermore, the three punctures either belong to the same class (for instance, all are $(1/3, 1/3, -2/3) \pmod{\mathbb{Z}}$) or to three distinct classes. Recall that the range of the holonomy variables are also constrained by the level k, so we pick out the Weyl alcove as the following:

$$D(k) = \{(h_1, h_2, h_3) | h_1 \geq h_2, h_1 \geq -2h_2, 2h_1 + h_2 \leq k\}, \tag{3.97}$$

with a pictorial illustration in figure 3.3.

As we will later identify each holonomy as an integrable highest weight representation for the affine Lie algebra $\widehat{su}(3)_k$, it is more convenient to use the label (λ_1, λ_2) defined as

$$\lambda_1 = h_2 - h_3, \quad \lambda_2 = h_1 - h_2. \tag{3.98}$$

They are integers with $\lambda_1 + \lambda_2 \leq k$ and (λ_1, λ_2) lives on the weight lattice of $su(3)$. The dimension of the representation with the highest weight (λ_1, λ_2) is

$$\dim R_{(\lambda_1, \lambda_2)} = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2). \tag{3.99}$$

Next we proceed to compute the index in the Coulomb branch limit. As taking the Coulomb branch limit simplifies the index computation dramatically, one can easily

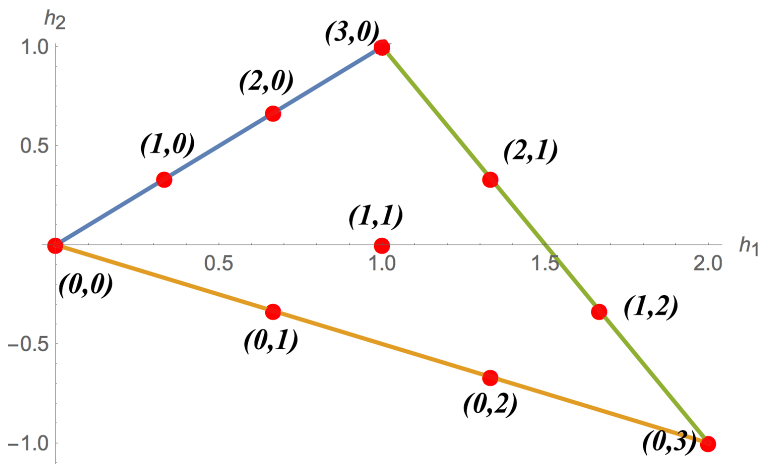


Figure 3.3: The Weyl alcove for the choice of holonomy variables at level $k = 3$. The red markers represent the allowed points. The coordinates beside each point denote the corresponding highest weight representation. The transformation between flavor holonomies and highest weight is given by (3.98).

write down the index for theory A[13]:

$$\begin{aligned} &\mathcal{I}_A(\mathrm{t}, \widetilde{\mathbf{m}}_a, \widetilde{\mathbf{m}}_b, n_a, n_b) \\ &\quad = \sum_{\mathbf{m}} I_{H/2}(\mathrm{t}, \mathbf{m}, \widetilde{\mathbf{m}}_a, n_a) \int \prod_{i=1}^{2} \frac{dz_i}{2\pi i z_i} \Delta(z)_{\mathbf{m}} I_V(\mathrm{t}, z, \mathbf{m}) I_{H/2}(\mathrm{t}, -\mathbf{m}, \widetilde{\mathbf{m}}_b, n_b), \end{aligned} \tag{3.100}$$

where $\mathbf{m}_a, \mathbf{m}_b$ and n_a, n_b denote the flavor holonomies for $SU(3)_{a,b}$ and $U(1)_{a,b}$ respectively. It is illustrative to write down what the gauge integrals look like:

$$\begin{aligned} I_V(\mathrm{t}, \mathbf{m}) &= \int \prod_{i=1}^{2} \frac{dz_i}{2\pi i z_i} \Delta(z)_{\mathbf{m}} I_V(\mathrm{t}, z, \mathbf{m}) \\ &= I_V^0(\mathrm{t}, \mathbf{m}) \times \begin{cases} \frac{1}{(1-\mathrm{t}^2)(1-\mathrm{t}^3)}, & m_1 \equiv m_2 \equiv m_3 \pmod{k}, \\ \frac{1}{(1-\mathrm{t})(1-\mathrm{t}^2)}, & m_i \equiv m_j \neq m_k \pmod{k}, \\ \frac{1}{(1-\mathrm{t})^2}, & m_1 \neq m_2 \neq m_3 \pmod{k}. \end{cases} \end{aligned} \tag{3.101}$$

Except for the zero point energy $I_V^0(\mathrm{t}, \mathbf{m})$ the rest looks very much like our "metric"

[13] In [81] the authors try to compensate for the non-integral holonomies of n_a and n_b by shifting the gauge holonomies $\mathbf{m}$. In contrast, our approach is free from such subtleties because we allow non-integral holonomies for all flavor symmetries as long as the Dirac quantization condition is obeyed.

for the $SU(3)$ equivariant Verlinde TQFT. Moreover,

$$I_{H/2}(\mathbf{m}, \widetilde{\mathbf{m}}_a, n_a) = \prod_{\psi \in R_\Phi} \mathrm{t}^{\frac{1}{4}\left([\![\psi(\mathbf{m},\widetilde{\mathbf{m}}_a,n_a)]\!]_k - \frac{1}{k}[\![\psi(\mathbf{m},\widetilde{\mathbf{m}}_a,n_a)]\!]_k^2\right)}, \tag{3.102}$$

where for a half-hypermultiplet in the fundamental representation of $SU(3)\times SU(3)_a$ with positive $U(1)_a$ charge we have

$$\psi_{ij}(\mathbf{m}, \widetilde{\mathbf{m}}_a, n_a) = \mathbf{m}_i + \widetilde{\mathbf{m}}_{a,j} + n_a. \tag{3.103}$$

Now we write down the index for theory B. Take the $SU(3)_a \times SU(3)_b \times SU(3)_c$ maximal subgroup of E_6 and gauge $SU(2)$ subgroup of the $SU(3)_c$ flavor symmetry. This leads to the replacement

$$\{h_{c,1}, h_{c,2}, h_{c,3}\} \to \{w + n_y, n_y - w, -2n_y\}, \tag{3.104}$$

where n_y denotes the fugacity for the remaining $U(1)_y$ symmetry, and n_s is the fugacity for $U(1)_s$ flavor symmetry rotating the single hypermultiplet. We then write down the index of theory B as

$$\mathcal{I}_B(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_y, n_s) = \sum_w C^{E_6}(\mathbf{h}_a, \mathbf{h}_b, w, n_y) I_V(\mathrm{t}, w) I_{H/2}(-w, n_s)\,, \tag{3.105}$$

where $I_V(\mathrm{t}, w)$ is given by (3.81) with substitution $m \to w$, and $w = 0, 1/2, \ldots, k/2$. Argyres-Seiberg duality tells us that

$$\mathcal{I}_A(\mathrm{t}, \widetilde{\mathbf{m}}_a, \widetilde{\mathbf{m}}_b, n_a, n_b) = \mathcal{I}_B(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_y, n_s)\,, \tag{3.106}$$

with the following identification of the holonomy variables:

$$\begin{aligned} &\widetilde{\mathbf{m}}_a = \mathbf{h}_a, \quad \widetilde{\mathbf{m}}_b = \mathbf{h}_b; \\ &n_a = \frac{1}{3}n_s - n_y, \quad n_b = -\frac{1}{3}n_s - n_y. \end{aligned} \tag{3.107}$$

On the right-hand side of the expression (3.105) we can view the summation as a matrix multiplication with w and n_s being the row and column indices respectively. Then we can take the inverse of the matrix $I_{H/2}(-w, n_s)$, $I^{-1}_{H/2}(n_s, w')$, by restricting the range[14] of n_s to be the same as w and multiply it to both sides of (3.105). This moves the summation to the other side of the equation and gives:

$$\boxed{C^{E_6}(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, w, n_y, k) = \sum_{n_s} \frac{1}{I_V(\mathrm{t}, w)} \mathcal{I}_A(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_a, n_b, k) I^{-1}_{H/2}(n_s, w)}\,. \tag{3.108}$$

[14]As long as it satisfies the Dirac quantization condition, we do not have to know what the range of n_s should be. For example, $n_s = 0, 1/2, \ldots, k/2$ is a valid choice.

We now regard $C^{E_6}(\mathfrak{t}, \mathbf{h}_a, \mathbf{h}_b, \mathbf{h}_c, k)$ as the fusion coefficient of the 2d equivariant Verlinde algebra, and have checked the associativity. Moreover, let us confirm that the index obtained in this way is symmetric under permutations of the three $SU(3)$ flavor fugacities, and the flavor symmetry group is indeed enhanced to E_6. First of all, we have permutation symmetry for three $SU(3)$ factors at, for instance, level $k = 2$:

$$C^{E_6}\left(\frac{2}{3}, \frac{2}{3}, 0, 0, \frac{4}{3}, -\frac{2}{3}\right) = C^{E_6}\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}, -\frac{2}{3}, 0, 0\right) = \cdots = C^{E_6}\left(\frac{4}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0\right) = \frac{1+\mathfrak{t}^4}{1-\mathfrak{t}^3}. \tag{3.109}$$

To show that the index C^{E_6} is invariant under the full E_6 symmetry, one needs to show that the two $SU(3)$ factors, combined with the $U(1)_y$ symmetry, enhance to an $SU(6)$ symmetry. The five Cartan elements of this $SU(6)$ group can be expressed as the combination of the fluxes [81]:

$$\left(h_1^a - n_y, h_2^a - n_y, -h_1^a - h_2^a - n_y, h_1^b + n_y, h_2^b + n_y\right). \tag{3.110}$$

Then the index should be invariant under the permutation of the five Cartans. Note the computation is almost the same as in [81] except that not all permutations necessarily exist—an allowed permutation should satisfy the charge quantization condition. Restraining ourselves from the illegal permutations, we have verified that the global symmetry is enlarged to E_6.

Finally, at large k our results reproduce these of [81], as can be checked by analyzing the large k limit of the matrix $I^{-1}_{H/2}(n_s, w)$. Indeed, at large k the matrix $I_{H/2}(w, n_s)$ can be simplified as

$$I_{H/2} = \mathfrak{t}^{\frac{1}{2}(|w+n_s|+|-w+n_s|)} = \begin{pmatrix} 1 & 0 & \mathfrak{t} & 0 & \mathfrak{t}^2 & 0 & \dots \\ 0 & \sqrt{\mathfrak{t}} & 0 & \mathfrak{t}^{\frac{3}{2}} & 0 & \mathfrak{t}^{\frac{5}{2}} & \\ \mathfrak{t} & 0 & \mathfrak{t} & 0 & \mathfrak{t}^2 & 0 & \\ 0 & \mathfrak{t}^{\frac{3}{2}} & 0 & \mathfrak{t}^{\frac{3}{2}} & 0 & \mathfrak{t}^{\frac{5}{2}} & \\ \mathfrak{t}^2 & 0 & \mathfrak{t}^2 & 0 & \mathfrak{t}^2 & 0 & \\ 0 & \mathfrak{t}^{\frac{5}{2}} & 0 & \mathfrak{t}^{\frac{5}{2}} & 0 & \mathfrak{t}^{\frac{5}{2}} & \\ \vdots & & & & & & \ddots \end{pmatrix}. \tag{3.111}$$

Upon inversion it gives

$$I_{H/2}^{-1}=\begin{pmatrix} \frac{1}{1-\mathrm{t}} & 0 & -\frac{1}{1-\mathrm{t}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sqrt{\mathrm{t}}(1-\mathrm{t})} & 0 & -\frac{1}{\sqrt{\mathrm{t}}(1-\mathrm{t})} & 0 & 0 & \\ -\frac{1}{1-\mathrm{t}} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}(1-\mathrm{t})} & 0 & -\frac{1}{\mathrm{t}(1-\mathrm{t})} & 0 & \\ 0 & -\frac{1}{\sqrt{\mathrm{t}}(1-\mathrm{t})} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}^{\frac{3}{2}}(1-\mathrm{t})} & 0 & -\frac{1}{\mathrm{t}^{\frac{3}{2}}(1-\mathrm{t})} & \\ 0 & 0 & -\frac{1}{\mathrm{t}(1-\mathrm{t})} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}^2(1-\mathrm{t})} & 0 & \\ 0 & 0 & 0 & -\frac{1}{\mathrm{t}^{\frac{3}{2}}(1-\mathrm{t})} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}^{\frac{5}{2}}(1-\mathrm{t})} & \\ \vdots & & & & & & \ddots \end{pmatrix}. \tag{3.112}$$

Here w goes from $0, 1/2, 1, 3/2, \cdots$. For a generic value of w only three elements in a single column can contribute to the index[15]. For large k the index of vector multiplet becomes

$$I_V(w)=\mathrm{t}^{-2w}\left(\frac{1}{1-\mathrm{t}}\right), \tag{3.113}$$

and we get

$$C^{E_6}(\mathrm{t},\mathbf{h}_a,\mathbf{h}_b,w,n_y)=\mathrm{t}^w\,\Big[(1+\mathrm{t})\mathcal{I}_A(\mathrm{t},\mathbf{h}_a,\mathbf{h}_b,n_y,w,k) \\ -\mathrm{t}\,\mathcal{I}_A(\mathrm{t},\mathbf{h}_a,\mathbf{h}_b,n_y,w-1,k)-\mathcal{I}_A(\mathrm{t},\mathbf{h}_a,\mathbf{h}_b,n_y,w+1,k)\Big], \tag{3.114}$$

which exactly agrees with [81].

$SU(3)$ equivariant Verlinde algebra

Now with all the basic building blocks of the 2d TQFT at our disposal, we assemble the pieces and see what interesting information could be extracted.

The metric of the TQFT is given by the Coulomb branch index of an $SU(3)$ vector multiplet, with a possible normalization factor. Note that the conjugation of representations acts on a highest weight state (λ_1, λ_2) via

$$\overline{(\lambda_1,\lambda_2)}=(\lambda_2,\lambda_1), \tag{3.115}$$

[15]By “generic” we mean the first and the second column are not reliable due to our choice of domain for w. It is imaginable that if we take w to be a half integer from $(-\infty, +\infty)$, then such “boundary ambiguity" can be removed. But we refrain from doing this to have weights living in the Weyl alcove.

and the metric $\eta^{\lambda\mu}$ is non-vanishing if and only if $\mu = \bar{\lambda}$. Let

$$N(\lambda_1, \lambda_2, k) = \mathrm{t}^{-\frac{1}{k}([\![\lambda_1]\!]_k[\![-\lambda_1]\!]_k + [\![\lambda_2]\!]_k[\![-\lambda_2]\!]_k + [\![\lambda_1+\lambda_2]\!]_k[\![-\lambda_1-\lambda_2]\!]_k)}, \tag{3.116}$$

and we rescale our TQFT states as

$$(\lambda_1, \lambda_2)' = N(\lambda_1, \lambda_2, k)^{-\frac{1}{2}}(\lambda_1, \lambda_2). \tag{3.117}$$

Then the metric η takes a simple form (here we define $\lambda_3 = \lambda_1 + \lambda_2$):

$$\eta^{(\lambda_1,\lambda_2)\overline{(\lambda_1,\lambda_2)}} = \begin{cases} \frac{1}{(1-\mathrm{t}^2)(1-\mathrm{t}^3)}, & \text{if } [\![\lambda_1]\!]_k = [\![\lambda_2]\!]_k = 0, \\ \frac{1}{(1-\mathrm{t})(1-\mathrm{t}^2)}, & \text{if only one } [\![\lambda_i]\!]_k = 0 \text{ for } i = 1,2,3, \\ \frac{1}{(1-\mathrm{t})^2}, & \text{if all } [\![\lambda_i]\!]_k \neq 0. \end{cases} \tag{3.118}$$

Next we find the "pair of pants" $f_{(\lambda_1,\lambda_2)(\mu_1,\mu_2)(\nu_1,\nu_2)}$, from the normalized Coulomb branch index of E_6 SCFT:

$$f_{(\lambda_1,\lambda_2)(\mu_1,\mu_2)(\nu_1,\nu_2)} = (N(\lambda_1,\lambda_2,k)N(\mu_1,\mu_2,k)N(\nu_1,\nu_2,k))^{\frac{1}{2}} C^{E_6}(\mathrm{t},\lambda_1,\lambda_2;\mu_1,\mu_2;\nu_1,\nu_2;k). \tag{3.119}$$

Along with the metric we already have, they define a t-deformation of the $\widehat{su}(3)_k$ fusion algebra. For instance we could write down at level $k = 3$:

$$\begin{aligned}(1,0)\otimes(1,0) = {} & \frac{1+\mathrm{t}+\mathrm{t}^3}{(1-\mathrm{t})(1-\mathrm{t}^2)(1-\mathrm{t}^3)}(0,1) \oplus \frac{1+2\mathrm{t}^2}{(1-\mathrm{t})(1-\mathrm{t}^2)(1-\mathrm{t}^3)}(2,0) \\ & \oplus \frac{\mathrm{t}(2+\mathrm{t})}{(1-\mathrm{t})(1-\mathrm{t}^2)(1-\mathrm{t}^3)}(1,2).\end{aligned} \tag{3.120}$$

Using dimensions to denote representations, the above reads

$$\begin{aligned}\mathbf{3}\times\mathbf{3} = {} & \frac{1+\mathrm{t}+\mathrm{t}^3}{(1-\mathrm{t})(1-\mathrm{t}^2)(1-\mathrm{t}^3)}\bar{\mathbf{3}} + \frac{1+2\mathrm{t}^2}{(1-\mathrm{t})(1-\mathrm{t}^2)(1-\mathrm{t}^3)}\mathbf{6} \\ & + \frac{\mathrm{t}(2+\mathrm{t})}{(1-\mathrm{t})(1-\mathrm{t}^2)(1-\mathrm{t}^3)}\overline{\mathbf{15}}.\end{aligned} \tag{3.121}$$

When $\mathrm{t} = 0$, it reproduces the fusion rules of the affine $\widehat{su}(3)_k$ algebra, and $f_{\lambda\mu\nu}$ becomes the fusion coefficients $N^{(k)}_{\lambda\mu\nu}$. These fusion coefficients are worked out combinatorically in [82–84]. We review details of the results in appendix B.

With pairs of pants and cylinders, one can glue them together to get the partition function on a closed Riemann surface, which gives the $SU(3)$ equivariant Verlinde formula: a $\mathfrak{t}$-deformation of the $SU(3)$ Verlinde formula. For genus $g = 2$, at large k, one can obtain

$$\begin{aligned}
&\dim_\beta \mathcal{H}_{\mathrm{CS}}(\Sigma_{2,0}; SL(3,\mathbb{C}), k)\\
&= \frac{1}{20160}k^8 + \frac{1}{840}k^7 + \frac{7}{480}k^6 + \frac{9}{80}k^5 + \frac{529}{960}k^4 + \frac{133}{80}k^3 + \frac{14789}{5040}k^2 + \frac{572}{210}k + 1\\
&+ \left(\frac{1}{2520}k^8 + \frac{1}{84}k^7 + \frac{17}{120}k^6 + \frac{17}{20}k^5 + \frac{319}{120}k^4 + \frac{15}{4}k^3 + \frac{503}{2520}k^2 - \frac{1937}{420}k - 3\right)\mathfrak{t}\\
&+ \left(\frac{1}{560}k^8 + \frac{9}{140}k^7 + \frac{31}{40}k^6 + \frac{39}{10}k^5 + \frac{727}{80}k^4 + \frac{183}{20}k^3 + \frac{369}{140}k^2 - \frac{27}{70}k + 1\right)\mathfrak{t}^2\\
&+ \ldots,
\end{aligned} \tag{3.122}$$

and the reader can check that the degree zero piece in $\mathfrak{t}$ is the usual $SU(3)$ Verlinde formula for $g = 2$ [85]:

$$\begin{aligned}
&\dim \mathcal{H}(\Sigma_{g,0}; SU(3), k)\\
&\quad = \frac{(k+3)^{2g-2}6^{g-1}}{2^{7g-7}} \sum_{\lambda_1,\lambda_2} \left(\sin\frac{\pi(\lambda_1+1)}{k+3} \sin\frac{\pi(\lambda_2+1)}{k+3} \sin\frac{\pi(\lambda_1+\lambda_2+2)}{k+3}\right)^{2-2g},
\end{aligned} \tag{3.123}$$

expressed as a polynomial in k.

For a 2d TQFT, the state associated with the "cap" contains interesting information, namely the "cap state" tells us how to close a puncture. Moreover, there are many close cousins of the cap. There is one type which we call the "central cap" that has a defect with central monodromy with the Levi subgroup being the entire gauge group (there is no reduction of the gauge group when we approach the singularity). For $SU(3)$ equivariant Verlinde algebra, besides the "identity-cap" the central cap also includes "ω-cap" and "ω^2-cap," and the corresponding TQFT states are denoted by $|\phi\rangle_1, |\phi\rangle_\omega$ and $|\phi\rangle_{\omega^2}$. One can also insert on the cap a minimal puncture (gauge group only reduces to $SU(2)\times U(1)$ as opposed to $U(1)^3$ for maximal punctures) and the corresponding states can be expressed as linear combinations of the maximal puncture states which we use as the basis vectors of the TQFT Hilbert space.

The cap state can be deduced from f and η written in (3.119) and (3.118), since closing a puncture on a three-punctured sphere gives a cylinder. In algebraic language,

$$f_{\lambda\mu\phi} = \eta_{\lambda\mu}. \tag{3.124}$$

One can easily solve this equation, obtaining

$$|\phi\rangle_1 = |0,0\rangle - \mathrm{t}(1+\mathrm{t})|1,1\rangle + \mathrm{t}^2|0,3\rangle + \mathrm{t}^2|3,0\rangle - \mathrm{t}^3|2,2\rangle. \tag{3.125}$$

For other two remaining caps, by multiplying[16] ω and ω^2 on the above equation (3.125), we obtain

$$\begin{aligned} |\phi\rangle_\omega &= |k,0\rangle - \mathrm{t}(1+\mathrm{t})|k-2,1\rangle + \mathrm{t}^2|k-3,0\rangle + \mathrm{t}^2|k-3,3\rangle - \mathrm{t}^3|k-4,2\rangle, \\ |\phi\rangle_{\omega^2} &= |0,k\rangle - \mathrm{t}(1+\mathrm{t})|1,k-2\rangle + \mathrm{t}^2|0,k-3\rangle + \mathrm{t}^2|3,k-3\rangle - \mathrm{t}^3|2,k-4\rangle. \end{aligned} \tag{3.126}$$

When closing a maximal puncture using $|\phi\rangle_\omega$, we have a "twisted metric" $\eta'_{\lambda\mu}$ which is non-zero if and only if $(\mu_1,\mu_2) = (\lambda_1, k-\lambda_1-\lambda_2)$. When closing a maximal puncture using $|\phi\rangle_{\omega^2}$, we have another twisted metric $\eta''_{\lambda\mu}$ which is non-zero if and only if $(\mu_1,\mu_2) = (k-\lambda_1-\lambda_2, \lambda_2)$. When there are insertions of central monodromies on the Riemann surface, it is easier to incorporate them into twisted metrics instead of using the expansion (3.126).

For minimal punctures, the holonomy is of the form $(u,u,-2u)$, modulo the action of the affine Weyl group, where u takes value $0, 1/3, 2/3, \ldots, k-2/3, k-1/3$. We can use index computation to expand the corresponding state $|u\rangle_{U(1)}$ in terms of maximal punctures. After scaling by a normalization constant

$$\mathrm{t}^{\frac{1}{2}\left([\![3u]\!]_k - \frac{1}{k}[\![3u]\!]_k^2\right)}, \tag{3.127}$$

the decomposition is given by the following:

(1). $\langle 0,0\rangle - \mathrm{t}^2\langle 1,1\rangle$, if $k=u$ or $u=0$;

(2). $\langle 3u,0\rangle - \mathrm{t}\langle 3u-1,2\rangle$, if $k > 3u > 0$;

(3). $\langle 3u,0\rangle - \mathrm{t}^2\langle 3u-2,1\rangle$, if $k = 3u$;

(4). $\langle 2k-3u, 3u-k\rangle - \mathrm{t}\langle 2k-3u-1, 3u-k-1\rangle$, if $3u/2 < k < 3u$;

(5). $\langle 0, 3u/2\rangle - \mathrm{t}^2\langle 1, 3u/2-2\rangle$, if $k = 3u/2$;

(6). $\langle 0, 3k-3u\rangle - \mathrm{t}\langle 2, 3k-3u-1\rangle$, if $u < k < 3u/2$.

[16] More precisely, we multiply holonomies with these central elements and translate the new holonomies back to weights.

The above formulae have a natural $\mathbb{Z}_2$-symmetry of the form $C \circ \psi$, where

$$\psi : (u, k) \to (k - u, k), \tag{3.128}$$

and C is the conjugation operator that acts linearly on Hilbert space:

$$C : (\lambda_1, \lambda_2) \to (\lambda_2, \lambda_1), \quad (\lambda_1, \lambda_2) \in \mathcal{H}. \tag{3.129}$$

This $\mathbb{Z}_2$ action sends each state in the above list to itself. Moreover, it is interesting to observe that when $\mathfrak{t} = 0$, increasing u from 0 to k corresponds to moving along the edges of the Weyl alcove ($c.f.$ figure 3.3) a full cycle. This may not be a surprise because closing a maximal puncture actually implies that one only considers states whose $SU(3)$ holonomy (h_1, h_2, h_3) preserves at least $SU(2) \subset SU(3)$ symmetry, which are precisely the states lying on the edges of the Weyl alcove.

From algebra to geometry

This TQFT structure reveals a lot of interesting geometric properties of moduli spaces of rank 3 Higgs bundles. But as the current chapter is a physics one, we only look at a one example — but arguably the most interesting one—the moduli space $\mathcal{M}_H(\Sigma_{0,3}, SU(3))$. In particular this moduli space was studied in [86, 87] and [88] from the point of view of differential equations. Here, from index computation, we can recover some of the results in the mathematical literature and reveal some new features for this moduli space. In particular, we propose the following formula for the fusion coefficient $f_{\lambda\mu\nu}$:

$$f_{(\lambda_1,\lambda_2)(\mu_1,\mu_2)(\nu_1,\nu_2)} = \mathfrak{t}^{k\eta_0}\left(\frac{k\mathrm{Vol}(\mathcal{M})+1}{1-\mathfrak{t}} + \frac{2\mathfrak{t}}{(1-\mathfrak{t})^2}\right) + \frac{Q_1(\mathfrak{t})}{(1-\mathfrak{t}^{-1})(1-\mathfrak{t}^2)} + \frac{Q_2(\mathfrak{t})}{(1-\mathfrak{t}^{-2})(1-\mathfrak{t}^3)}. \tag{3.130}$$

This ansatz comes from Atiyah-Bott localization of the equivariant integral done in similar fashion as in [15]. The localization formula enables us to write the fusion coefficient f in (3.119) as a summation over fixed points of the $U(1)_H$ Hitchin action. In (3.130), η_0 is the moment map[17] for the lowest critical manifold $\mathcal{M}$. When the undeformed fusion coefficients $N^{(k)}_{\lambda\mu\nu} \neq 0$, one has

$$k\mathrm{Vol}(\mathcal{M}) + 1 = N^{(k)}_{\lambda\mu\nu}, \quad \eta_0 = 0. \tag{3.131}$$

[17]Recall the $U(1)_H$ Hitchin action is generated by a Hamiltonian, which we call η—not to be confused with the metric, which will make no appearance from now on. η is also the norm squared of the Higgs field.

Numerical computation shows that $Q_{1,2}(\mathrm{t})$ are individually a sum of three terms of the form

$$Q_1(\mathrm{t}) = \sum_{i=1}^{3} \mathrm{t}^{k\eta_i}, \quad Q_2(\mathrm{t}) = \sum_{j=4}^{6} \mathrm{t}^{k\eta_j}, \tag{3.132}$$

where η_i are interpreted as the moment maps at each of the six higher fixed points of $U(1)_H$.

The moduli space $\mathcal{M}$ of $SU(3)$ flat connections on $\Sigma_{0,3}$ is either empty, a point or $\mathbb{C}\mathbf{P}^1$ depending on the choice of (λ, μ, ν) [89], and when it is empty, the lowest critical manifold of η is a $\mathbb{C}\mathbf{P}^1$ with $\eta_0 > 0$ and we will still use $\mathcal{M}$ to denote it. The fixed loci of $\mathcal{M}_H(\Sigma_{0,3}, SU(3))$ under $U(1)$ action consist of $\mathcal{M}$ and the six additional points, and there are Morse flow lines traveling between them. The downward Morse flow coincides with the nilpotent cone [90]—the singular fiber of the Hitchin fibration, and its geometry is depicted in figure 3.4. The Morse flow carves out six spheres that can be divided into two classes. Intersections of $D_i^{(1)} \bigcap D_i^{(2)}$ are denoted as $P_{1,2,3}^{(1)}$, and at the top of these $D_i^{(2)}$'s there are $P_{1,2,3}^{(2)}$. We also use $P_1, \ldots, P_6$ and $D_1, \ldots, D_6$ sometimes to avoid clutter. The nilpotent cone can be decomposed into

$$\mathcal{N} = \mathcal{M} \cup D_i^{(1)} \cup D_j^{(2)}, \tag{3.133}$$

which gives an affine E_6 singularity (IV* in Kodaira's classification) of the Hitchin fibration. Knowing the singular fiber structure, we can immediately read off the Poincaré polynomial for $\mathcal{M}_H(\Sigma_{0,3}, SU(3))$:

$$\mathcal{P}_{\mathrm{r}} = 1 + 7\mathrm{r}^2, \tag{3.134}$$

which is the same as that given in [87].

To use the Atiyah-Bott localization formula, we also need to understand the normal bundle to the critical manifolds. For the base, the normal bundle is the cotangent bundle with $U(1)_H$ weight 1. Its contribution to the fusion coefficient is given by

$$t^{k\eta_0} \int_{\mathcal{M}} \frac{\mathrm{Td}(\mathbb{C}\mathbf{P}^1) \wedge e^{k\omega}}{1 - e^{-\beta + 2\omega'}} = \mathrm{t}^{k\eta_0} \left(\frac{k\mathrm{Vol}(\mathcal{M}) + 1}{1 - \mathrm{t}} + \frac{2\mathrm{t}}{(1 - \mathrm{t})^2} \right). \tag{3.135}$$

For the higher fixed points, the first class $P^{(1)}$ has normal bundle $\mathbb{C}[-1] \oplus \mathbb{C}[2]$ with respect to $U(1)_H$, which gives a factor

$$\frac{1}{(1 - \mathrm{t}^{-1})(1 - \mathrm{t}^2)} \tag{3.136}$$

multiplying $e^{k\eta_{1,2,3}}$. For the second class $P^{(2)}$, the normal bundle is $\mathbb{C}[-2] \oplus \mathbb{C}[3]$ and we instead have a factor

$$\frac{1}{(1-\mathrm{t}^{-2})(1-\mathrm{t}^{3})}. \tag{3.137}$$

In this chapter, we won't give the analytic expression for the seven moment maps and will leave (3.130) as it is. Instead, we will give a relation between them:

$$\begin{aligned} 2k &= 6(N^{(k)}_{\lambda\mu\nu} - 1) + 3k(\eta_1 + \eta_2 + \eta_3) + k(\eta_4 + \eta_5 + \eta_6) \\ &= 6k\mathrm{Vol}(\mathcal{M}) + 3k(\eta_1 + \eta_2 + \eta_3) + k(\eta_4 + \eta_5 + \eta_6). \end{aligned} \tag{3.138}$$

This is verified numerically and can be explained from geometry. Noticing that the moment maps are related to the volume of the D's:

$$\begin{aligned} &\mathrm{Vol}(D_1) = \eta_1, \quad \mathrm{Vol}(D_2) = \eta_2, \quad \mathrm{Vol}(D_3) = \eta_3, \\ &\mathrm{Vol}(D_4) = \frac{\eta_4 - \eta_1}{2}, \quad \mathrm{Vol}(D_5) = \frac{\eta_5 - \eta_2}{2}, \quad \mathrm{Vol}(D_6) = \frac{\eta_6 - \eta_3}{2}. \end{aligned} \tag{3.139}$$

The factor 2 in the second line of (3.139) is related to the fact that $U(1)_H$ rotates the $D^{(2)}$'s twice as fast as it rotates the $D^{(1)}$'s. Then we get the following relation between the volume of the components of $\mathcal{N}$:

$$\mathrm{Vol}(\mathbf{F}) = 6\mathrm{Vol}(\mathcal{M}) + 4\sum_{i=1}^{3}\mathrm{Vol}(D_i) + 2\sum_{i=4}^{6}\mathrm{Vol}(D_j). \tag{3.140}$$

Here F is a generic fiber of the Hitchin fibration and has volume

$$\mathrm{Vol}(\mathbf{F}) = 2. \tag{3.141}$$

The intersection form of different components in the nilpotent cone gives the Cartan matrix of affine E_6. Figure 3.5 is the Dynkin diagram of $\widehat{E}_6$, and coefficients in (3.140) are Dynkin labels on the corresponding node. These numbers tell us the combination of D's and $\mathcal{M}$ that give a null vector $\mathbf{F}$ of $\widehat{E}_6$.

Comments on T_N theories

The above procedure can be generalized to arbitrary rank, for all T_N theories, if we employ the generalized Argyres-Seiberg dualities. There are in fact several ways to generalized Argyres-Seiberg duality [2, 80, 91]. For our purposes, we want no punctures of the T_N to be closed under dualities, so we need the following setup [2].

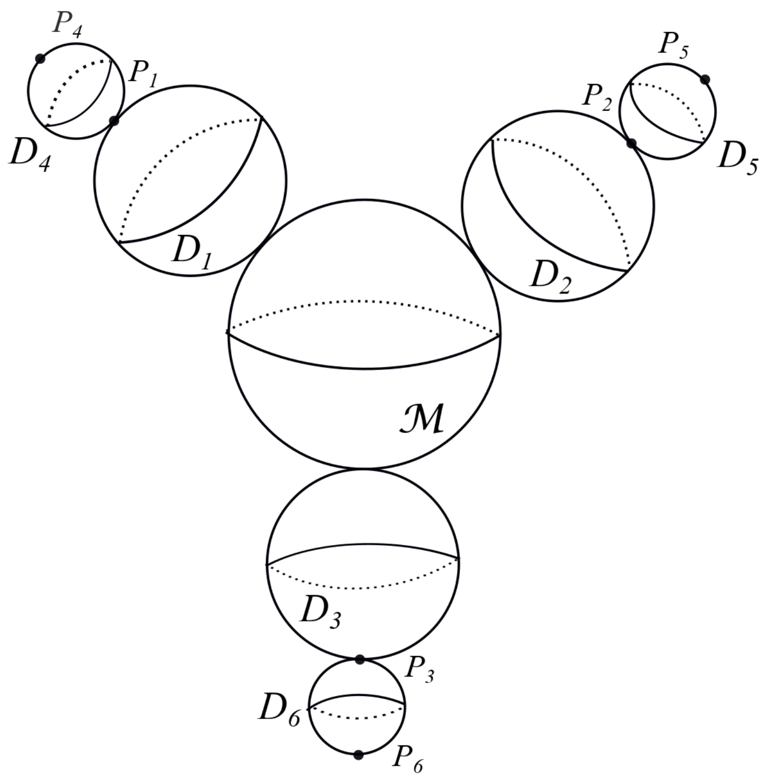


Figure 3.4: The illustration of the nilpotent cone in $\mathcal{M}_H(\Sigma_{0,3}, SU(3))$. Here $\mathcal{M}$ is the base $\mathbb{C}\mathbf{P}^1$, $D_{1,2,3}$ consist of downward Morse flows from $P_{1,2,3}$ to the base, while $D_{4,5,6}$ include the flows from $P_{4,5,6}$ to $P_{1,2,3}$.

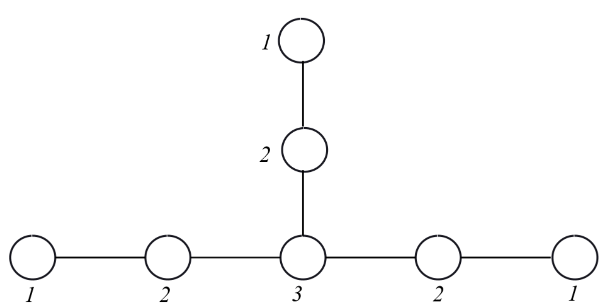


Figure 3.5: The affine $\widehat{E}_6$ extended Dynkin diagram. The Dynkin label gives the multiplicity of each node in the decomposition of the null vector.

We start with a linear quiver gauge theory A' with $N-2$ nodes of $SU(N)$ gauge groups, and at each end of the quiver we associate N hypermultiplets in the fundamental representation of $SU(N)$. One sees immediately that each gauge node is automatically superconformal. Geometrically, we actually start with a punctured Riemann sphere with two full $SU(N)$ punctures and $N-1$ simple punctures. Then, the $N-1$ simple punctures are brought together and a hidden $SU(N-1)$ gauge group becomes very weak. In our original quiver diagram, such a procedure of colliding $N-1$ simple punctures corresponds to attaching a quiver tail of the form $SU(N-1)-SU(N-2)-\cdots-SU(2)$ with a single hypermultiplet attached to the last $SU(2)$ node. See figure 3.6 for the quiver diagrams and figure 3.7 for the geometric realization.

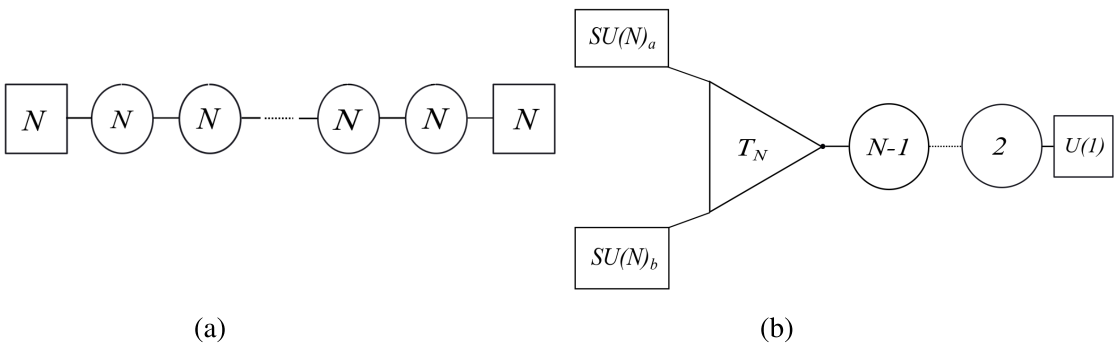


Figure 3.6: Illustration of generalized Argyres-Seiberg duality for the T_N theories. (a) The theory A', which is a linear quiver gauge theory with $N-2$ $SU(N)$ vector multiplets. Between each gauge node there is a bi-fundamental hypermultiplet, and at each end of the quiver there are N fundamental hypermultiplets. In the quiver diagram we omit the $U(1)^{N-1}$ baryonic symmetries. (b) The theory B' is obtained by gauging an $SU(N-1)$ subgroup of the $SU(N)^3$ flavor symmetry of T_N, giving rise to a quiver tail. Again the $U(1)$ symmetries are implicit in the diagram.

Here we summarize briefly how to obtain the lens space Coulomb index of T_N. Let $\mathcal{I}^N_{A'}$ be the index of the linear quiver theory, which depends on two $SU(N)$ flavor holonomies $\mathbf{h}_a$ and $\mathbf{h}_b$ (here we use the same notation as that of $SU(3)$) and $N-1$ $U(1)$-holonomies n_i where $i=1,2,\ldots,N-1$. In the infinite coupling limit, the dual weakly coupled theory B' emerges. One first splits the $SU(N)_c$ subgroup of the full $SU(N)^3$ flavor symmetry group into $SU(N-1)\times U(1)$ and then gauges the $SU(N-1)$ part with the first gauge node in the quiver tail. As in the T_3 case there is a transformation:

$$\left(h^c_1, h^c_2, \cdots, h^c_N\right) \to (w_1, w_2, \cdots w_{N-2}, \tilde{n}_0). \tag{3.142}$$

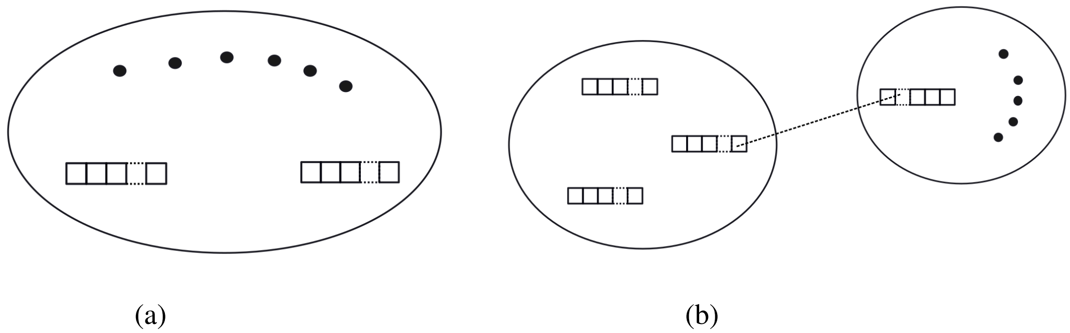


Figure 3.7: Illustration of the geometric realization of generalized Argyres-Seiberg duality for T_N theories. (a) The theory A' is obtained by compactifying 6d $(2,0)$ theory on a Riemann sphere with two maximal $SU(N)$ punctures and $N-1$ simple punctures. (b) The theory B', obtained by colliding $N-1$ simple punctures, is then the theory that arises from gauging a $SU(N-1)$ flavor subgroup of T_N by a quiver tail.

After the $SU(N-1)$ node, there are $N-2$ more $U(1)$ symmetries, and we will call those associated holonomies $\tilde{n}_j$ with $j = 1, 2, \ldots, N-2$. Again there exists a correspondence as in the T_3 case:

$$(n_1, n_2, \ldots, n_{N-1}) \to (\tilde{n}_0, \tilde{n}_1, \ldots, \tilde{n}_{N-2}). \tag{3.143}$$

Then the Coulomb branch index of the theory B' is

$$\mathcal{I}^N_{B'}(\mathbf{h}^a, \mathbf{h}^b, \tilde{n}_0, \tilde{n}_1, \ldots, \tilde{n}_{N-2}) = \sum_{\{w_i\}} C^{T_N}(\mathbf{h}^a, \mathbf{h}^b, w_1, w_2, \cdots w_{N-2}, \tilde{n}_0)\mathcal{I}_T(w_i; \tilde{n}_1, \ldots, \tilde{n}_{N-2}), \tag{3.144}$$

where $\mathcal{I}_T$ is the index of the quiver tail:

$$\begin{aligned}\mathcal{I}_T(w_i; \tilde{n}_1, \ldots, \tilde{n}_{N-2}) = &\sum_{\{w_i^{(N-2)}\}} \sum_{\{w_i^{(N-3)}\}} \cdots \sum_{\{w_i^{(2)}\}} I^V_{N-1}(w_i) I^H_{N-1,N-2}(w_i, w_j^{(N-2)}, \tilde{n}_1) I^V_{N-2}(w_i^{(N-2)}) \\ &\times I^H_{N-2,N-3}(w_i^{(N-2)}, w_j^{(N-3)}, \tilde{n}_2) I^V_{N-3}(w_i^{(N-3)}) \times \ldots \\ &\times I^V_2(w_i^{(2)}) I^H_{2,1}(w_i^{(2)}, \tilde{n}_{N-2}).\end{aligned} \tag{3.145}$$

Now we can view $\mathcal{I}_T$ as a large matrix $\mathfrak{M}_{\{w_i\},\{\tilde{n}_j\}}$, and in fact it is a square matrix. Although the set $\{\tilde{n}_j\}$ appears to be bigger, there is an affine Weyl group $\widehat{A}_{N-2}$ acting on it. From the geometric picture, one can directly see the $A_{N-2} = S_{N-2}$ permuting the $N-2$; and the shift $n_i \to n_i + k$, which gives the same holonomy in $U(1)_i$, enlarges the symmetry to that of $\widehat{A}_{N-2}$. After taking quotient by this symmetry, one requires $\{\tilde{n}_j\}$ to live in the Weyl alcove of $\mathfrak{su}(N-1)$, reducing the cardinality of the set $\{\tilde{n}_j\}$ to that of $\{w_i\}$. Then one can invert the matrix $\mathfrak{M}_{\{w_i\},\{\tilde{n}_j\}}$ and obtain the index C^{T_N}, which in turn gives the fusion coefficients and the algebra structure of the $SU(N)$ equivariant TQFT.

The metric of the TQFT coming from the cylinder is also straightforward even in the $SU(N)$ case. It is always diagonal and only depends on the symmetry reserved by the holonomy labeled by the highest weight λ. For instance, if the holonomy is such that $SU(N) \to U(1)^n \times SU(N_1) \times SU(N_2) \times SU(N_l)$, we have

$$\eta^{\lambda\bar{\lambda}} = \frac{1}{(1-\mathrm{t})^n} \prod_{j=1}^{l} \frac{1}{(1-\mathrm{t}^2)(1-\mathrm{t}^3)\dots(1-\mathrm{t}^{N_j})}. \tag{3.146}$$

This can be generalized to arbitrary group G. If the holonomy given by λ has stabilizer $G' \subset G$, the norm square of λ in the G_k equivariant Verlinde algebra is

$$\eta^{\lambda\bar{\lambda}} = P(BG', \mathrm{t}). \tag{3.147}$$

Here $P(BG', \mathrm{t})$ is the Poincaré polynomial[18] of the infinite-dimensional classifying space of G'. In the "maximal" case of $G' = U(1)^r$, we indeed get

$$P\left(BU(1)^r, \mathrm{t}\right) = P\left(\left(\mathbb{C}\mathbf{P}^\infty\right)^r, \mathrm{t}\right) = \frac{1}{(1-\mathrm{t})^r}. \tag{3.148}$$

[18]More precisely, it is the Poincaré polynomial in variable $\mathrm{t}^{1/2}$. But as $H^*(BG, \mathbb{C})$ is zero in odd degrees, this Poincaré polynomial is also a series in t with integer powers.

Chapter 4

ARGYRES-DOUGLAS THEORIES, WILD HITCHIN CHARACTERS AND VERTEX OPERATOR ALGEBRAS

4.1 Generalization to wild punctures

In Chapter 3, we have proposed a relation linking the quantization of a large class of hyper-Kähler manifolds and BPS spectra of superconformal theories[1]

$$\boxed{\begin{array}{c}\text{Space of Coulomb BPS states of}\\ \text{4d } \mathcal{N}=2 \text{ SCFT } \mathcal{T} \text{ on } L(k,1)\end{array}} = \boxed{\begin{array}{c}\text{Hilbert space from}\\ \text{quantization of } ({}^L\mathcal{M}_{\mathcal{T}}, k\omega_I)\end{array}}. \tag{4.1}$$

Here, the hyper-Kähler space ${}^L\mathcal{M}_{\mathcal{T}}$ is the mirror of the Coulomb branch $\mathcal{M}_{\mathcal{T}}$ of $\mathcal{T}$ on $\mathbb{R}^3 \times S^1$, with ω_I being one of the three real symplectic structures, and "Coulomb BPS states" refer to those which contribute to the superconformal index in the Coulomb branch limit [58]. Each side of (4.1) admits a natural grading, coming from the $U(1)_r \subset SU(2)_R \times U(1)_r$ R-symmetry of the 4d $\mathcal{N}=2$ SCFT, and the proposal (4.1) is a highly non-trivial isomorphism between two graded vector spaces.

This relation was studied in Chapter 3 [16] for theories of class $\mathcal{S}$ [2, 20]. For a given Riemann surface Σ, possibly with regular singularities (or "tame ramifications"), and a compact simple Lie group G, the Coulomb branch $\mathcal{M}_{\mathcal{T}}$ of the theory $T[\Sigma, G]$ compactified on S^1 is the Hitchin moduli spaces $\mathcal{M}_H(\Sigma, G)$ [92–94], whose mirror ${}^L\mathcal{M}_{\mathcal{T}}$ is given by $\mathcal{M}_H(\Sigma, {}^L G)$ associated with the Langlands dual group ${}^L G$ via the geometric Langlands correspondence [61, 65, 95, 96], and the $U(1)_r$ action on it becomes the so-called Hitchin action [12]. Quantizing the Hitchin moduli space gives the Hilbert space of complex Chern-Simons theory $\mathcal{H}(\Sigma, {}^L G_{\mathbb{C}}; k)$, whose graded dimension — the *Hitchin character*[2] — is given by the "equivariant Verlinde formula" proposed in [15] and later proved in [63, 68]. We have verified relation (4.1) by matching the lens space Coulomb index of class $\mathcal{S}$ theories and the Hitchin characters,

$$\mathcal{I}_{\text{Coulomb}}(T[\Sigma, G]; L(k,1) \times S^1) = \dim_{\mathfrak{t}} \mathcal{H}(\Sigma, {}^L G_{\mathbb{C}}; k). \tag{4.2}$$

In the present chapter, we further explore the connection in (4.1) for a wider class of 4d $\mathcal{N}=2$ theories including the A_1 Argyres-Douglas (AD) theories. In the process,

[1] See (3.34). We have stated the proposal here at the categorified level.

[2] The graded dimension (see (4.39)) is the same as the character of the $U(1)$ Hitchin action, lifted from $\mathcal{M}_H$ to acting on $\mathcal{H}$, and hence the name "Hitchin character."

we introduce another player into the story, making (4.1) a triangle,

$$\begin{array}{ccc} \text{Coulomb index of } \mathcal{T} & \longleftrightarrow & \text{quantization of } {}^L\mathcal{M}_{\mathcal{T}} \\ & \searrow\!\!\!\!\nwarrow \qquad\qquad \nearrow\!\!\!\!\swarrow & \\ & \text{vertex operator algebra } \chi_{\mathcal{T}} & \end{array} \tag{4.3}$$

where the vertex operator algebra $\chi_{\mathcal{T}}$ is associated with the 4d $\mathcal{N} = 2$ theory $\mathcal{T}$ à la [19]. We observe that fixed points on $\mathcal{M}_{\mathcal{T}}$ under $U(1)_r$ are in bijection with highest-weight representations of $\chi_{\mathcal{T}}$[3] , and in addition the $\mathfrak{t} \to \exp(2\pi i)$ limit of the Hitchin character can be expressed in terms modular transformation matrix of those representations. The appearance of the VOA is anticipated from the geometric Langlands program, as the triangle above can be understood as an analogue of the "geometric Langlands triangle" formed by A-model, B-model and $\mathcal{D}$-modules for general $\mathcal{M}_{\mathcal{T}}$. However, the role of the VOA $\chi_{\mathcal{T}}$ in the counting of *Coulomb* branch BPS states is somewhat unexpected, since the VOA is related to the Schur operators of $\mathcal{T}$ [19, 97–99], which contains the Higgs branch operators but not the Coulomb branch operators at all! The current chapter shows that, the Coulomb branch index is related to $\chi_{\mathcal{T}}$ through modular transformations.

Argyres-Douglas theories form a class of very interesting 4d $\mathcal{N} = 2$ strongly-interacting, "non-Lagrangian" SCFTs. They were originally discovered by studying singular loci in the Coulomb branch of $\mathcal{N} = 2$ gauge theories [100–102], where mutually non-local dyons become simultaneously massless. The hallmarks of this class of theories are the fixed values of coupling constants and the fractional scaling dimensions of their Coulomb branch operators. Like the class $\mathcal{S}$ theories, Argyres-Douglas theories can also be engineered by compactifying M5-branes on a *Riemann sphere* $\Sigma = \mathbb{C}\mathbf{P}^1$, but now with irregular singularities — or "wild ramifications" [18, 103, 104]. Their Coulomb branch $\mathcal{M}_H(\Sigma, G)$ on $\mathbb{R}^3 \times S^1$ and their mirrors $\mathcal{M}_H(\Sigma, {}^L G)$ are sometimes called *wild Hitchin moduli spaces*. The study of these spaces and their role in the geometric Langlands correspondence (see *e.g.* [105] and references

[3]In the physics literature — and also in this chapter — "chiral algebra" and "vertex operator algebra" (VOA) are often used interchangeably, while in the math literature, the two have different emphasis on, respectively, geometry and representation theory. The "highest-weight representations of $\chi_{\mathcal{T}}$" here denotes a suitable subcategory, closed under modular transform, of the full category of modules of vertex operator algebra. The precise statement will be clear in Section 4.5.

therein) is a very interesting subject and under active development. Over the past few years, much effort has been made to give a precise definition of the moduli space, and analogues for many well-known theorems in the unramified or tamely ramified cases were only established recently (see [106–108], as well as the short survey [88] and references therein). In this chapter, relation (4.1) enables us to obtain the *wild Hitchin characters* for many moduli spaces. Just like their cousins in the unramified or tamely ramified cases [15], wild Hitchin characters encode rich algebraic and geometric information about $\mathcal{M}_H$, with some of the invariants $\mathcal{M}_H$ being able to be directly read off from the formulae. This enables us to make concrete predictions about the moduli space.

For instance, the $L(k, 1)$ Coulomb index of the original Argyres-Douglas theory [100], which in the notation of [18] will be called the (A_1, A_2) theory, is given by

$$\mathcal{I}_{(A_1,A_2)} = \frac{1 - \mathrm{t}^{-\frac{1}{5}} - \mathrm{t}^{\frac{1}{5}} + \mathrm{t}^{\frac{k}{5}}}{(1 - \mathrm{t}^{\frac{6}{5}})(1 - \mathrm{t}^{-\frac{1}{5}})}, \tag{4.4}$$

and it is easy to verify that it agrees with the wild Hitchin character of the mirror of the Coulomb branch ${}^L\mathcal{M}_{(A_1,A_2)} = {}^L\mathcal{M}_{2,3}$ (the precise meaning of this notation will be clarified shortly),

$$\dim_{\mathrm{t}} \mathcal{H}({}^L\mathcal{M}_{2,3}) = \frac{1}{(1 - \mathrm{t}^{\frac{2}{5}})(1 - \mathrm{t}^{\frac{3}{5}})} + \frac{\mathrm{t}^{\frac{k}{5}}}{(1 - \mathrm{t}^{\frac{6}{5}})(1 - \mathrm{t}^{-\frac{1}{5}})}, \tag{4.5}$$

with the two terms coming from the two $U(1)$ fixed points. And the two fixed points correspond to the two highest weight representations of the non-unitary $(2, 5)$ Virasoro minimal model — famously known as the Lee-Yang model — via a detailed dictionary which will be provided in later sections.

This chapter is organized as follows: In Section 4.2, we first briefly recall how the wild Hitchin moduli space $\mathcal{M}_H$ arises from brane geometry and how it is related to general Argyres-Douglas theories. We then proceed to describe $\mathcal{M}_H$, introduce the $U(1)$ Hitchin action on it and discuss its geometric quantization.

In Section 4.3, we obtain the Coulomb branch indices of Argyres-Douglas theories, expressed as integral formulae. We follow the prescription in [109–111] by starting with $\mathcal{N} = 1$ Lagrangian theories that flow to Argyres-Douglas theories in the IR. The TQFT structure for the index is presented in Appendix C.1.

In Section 4.4, we present the wild Hitchin characters, decomposed into summations over the fixed points. Using the character formulae we explore the geometric properties of the moduli space. Confirmation from direct mathematical computation is given in Appendix E. We then study the large-k limits of the wild Hitchin characters, giving a physical interpretation of some fixed points in $\mathcal{M}_H$ as massive vacua on the Higgs branch of the 3d mirror theory. We also study the symmetry mixing upon dimensional reduction, following [112]. Further details are given in Appendix C.2 and D.

In Section 4.5, we study the relation between Hitchin characters and VOAs, and demonstrate that a limit of wild Hitchin characters can be identified with matrix elements of the modular transformation ST^kS. Further, we check the correspondence between the fixed points on $\mathcal{M}_H$ and the highest-weight modules for various examples.

4.2 Wild Hitchin moduli space and Argyres-Douglas theories

We recall that in Chapter 3 and [15, 16], the problem of quantizing the Hitchin moduli space was studied using the following brane set-up

$$\begin{array}{rcccccc} \text{fivebranes:} & L(k,1)\times S^1 & \times & \Sigma & & \\ & & & \cap & & \\ \text{space-time:} & L(k,1)\times S^1 & \times & T^*\Sigma & \times & \mathbb{R}^3 \\ & \circlearrowright & & \circlearrowright & & \circlearrowright \\ \text{symmetries:} & SO(4)_E & & U(1)_N & & SU(2)_R \end{array} \tag{4.6}$$

We will first review how the Hitchin moduli space arises from this geometry, and how adding irregular singularities to Σ leads to a relation between the general Argyres-Douglas theories and wild Hitchin systems.

Hitchin equations from six dimensions

Hitchin moduli spaces were first introduced to physics in the context of string theory and its dimensional reduction in the pioneering work of [92–94] in the past century, and were highlighted in the gauge theory approach to geometric Langlands program [65, 96, 105]. In our brane setting (4.6), which is closely related to the system studied in detail in [20], one can first reduce the M5-branes on the S^1 to obtain D4-branes, whose world-volume theory is given by the 5d $\mathcal{N}=2$ super-Yang-Mills theory. We consider theories with gauge group G of type ADE. In addition to the gauge fields, this theory also contains five real scalars Y^I with $I = 1, 2, \ldots, 5$, corresponding to the motion of the branes in the five transverse directions. Further topological twisting along Σ enables us to identify $\varphi(z) = Y^1 + iY^2$ as a $(1,0)$-form on Σ with respect to the complex structure of Σ. As a consequence, the BPS equations in the remaining three space-time dimensions are precisely the Hitchin equations (1.3). Regarded as a sigma model, the target space of the three-dimensional theory is identified with the Hitchin moduli space $\mathcal{M}_H(\Sigma, G)$ — solutions to the Hitchin equations modulo gauge transformations.

One can allow the Riemann surface Σ to have a finite number of marked points $\{p_1, p_2, \ldots, p_s\}$ for $s \geq 0$. In the neighborhood of each marked point p_i, the gauge connection and the Higgs field take the asymptotic form:

$$\begin{aligned} &A \sim \alpha d\theta, \\ &\varphi \sim \left(\frac{u_n}{z^n} + \frac{u_{n-1}}{z^{n-1}} + \ldots \frac{u_1}{z} + \text{regular} \right) dz. \end{aligned} \tag{4.7}$$

Here $\alpha \in \mathfrak{g}$ and $u_i \in \mathfrak{g}_{\mathbb{C}}$ are collectively called the *ramification data*,[4] and they are fixed in defining $\mathcal{M}_H$ to ensure that the moduli space is symplectic (more precisely, gauge-invariant combinations of them are fixed). When the order of the pole is $n = 1$, we call the puncture *tame* or *regular*. From the M-theory geometry, adding a regular puncture corresponds to the insertion of a set of defect M5-branes placed at the point p_i of Σ, occupying the four spacetime dimensions as well as the cotangent space at $p_i \in \Sigma$. Set-up (4.6) becomes

$$\begin{array}{rcccc} \text{fivebranes:} & L(k,1)_b \times & \Sigma & \times S^1 & \\ & & \cap & & \\ \text{space-time:} & L(k,1)_b \times & T^*\Sigma & \times S^1 & \times \mathbb{R}^3 \\ & & \cup & & \\ \text{“defect” fivebranes:} & L(k,1)_b \times & T^*|_{p_i}\Sigma & \times S^1 & . \end{array} \tag{4.8}$$

The defect fivebranes give rise to a codimension-two singularity in the 6d (2,0) theory and introduce a flavor symmetry of the effective 4d theory $T[\Sigma, G]$ [2, 113]. If u_1 is nilpotent, then the flavor symmetry is given by the commutant subgroup of the nilpotent embedding $\mathfrak{su}(2) \to \mathfrak{g}$; if u_1 is semi-simple, the flavor symmetry is explicitly broken by mass deformations [2, 114]. The ramification data α and u_1 is acted upon by the affine Weyl group of G, and the conjugacy class of the

[4]We use the convention that elements in $\mathfrak{g} = \mathrm{Lie}\, G$ are anti-Hermitian.

monodromy in the complexified gauge connection $\mathcal{A}_z = A_z + i\varphi$ is an invariant of the ramification data.

When $n > 1$ the puncture will be called *wild* or *irregular*, which will play a central role in the present chapter. The leading coefficient matrix u_n is allowed to be either semi-simple or nilpotent as in the tame case. However, now the monodromy of $\mathcal{A}_z$ around p_i needs to be supplemented by more sophisticated data — the Stokes matrices — to fully characterize the irregular puncture [115] (see $e.g.$ [105] for more detail and explicit examples).

The Hitchin moduli space $\mathcal{M}_H(\Sigma, G)$ with fixed local ramification data is hyper-Kähler, admitting a family of complex structures parametrized by an entire $\mathbb{C}\mathbf{P}^1$. There are three distinguished ones (I, J, K), and the corresponding symplectic forms are denoted as $\omega_I, \omega_J, \omega_K$. The complex structure I is inherited from the complex structure of the Riemann surface Σ, over which $\bar{\partial}_A$ defines a holomorphic structure on E, and the triple $(E, \bar{\partial}_A, \varphi)$ parametrizes a *Higgs bundle* on Σ. This is usually referred to as the *holomorphic* or *algebraic* perspective. Alternatively, one can also employ the *differential geometric* point of view, identifying $\mathcal{M}_H$ as the moduli space of flat $G_{\mathbb{C}}$-connections on Σ with the prescribed singularity near the puncture, and the complex structure J comes from the complex structure of $G_{\mathbb{C}}$. There is also the *topological* perspective, viewing $\mathcal{M}_H$ as the character variety $\mathrm{Hom}(\pi_1\Sigma, G_{\mathbb{C}})$, with boundary holonomies in given conjugacy classes (and with inclusion of Stokes matrices in the wildly ramified case). Non-abelian Hodge theory states that the three constructions give canonically isomorphic moduli spaces [12, 116–118]. In the wild case, the isomorphism between the Hitchin moduli space $\mathcal{M}_H$ and moduli space of flat $G_{\mathbb{C}}$-connections was proved in [106, 119], while [106] proved the isomorphism between $\mathcal{M}_H$ and moduli space of Higgs bundles, thus establishing the equivalence of first two perspectives. The wild character variety was later constructed and studied in [107, 108, 120, 121]. In this chapter, we will mainly adopt the holomorphic perspective but will occasionally switch between the three viewpoints as each offers unique insights into $\mathcal{M}_H$.[5]

For later convenience, we shall use below a different but equivalent formulation of Hitchin equations (1.3). Fix a Riemann surface Σ and a complex vector bundle E.

[5] In general, physical quantities know about the full moduli *stack*, where all Higgs bundles including the *unstable* ones are taken into account, as the path integral sums over all configurations. However, for co-dimension reasons, all wild Hitchin characters we will consider are the same for stacks and for spaces. In the tame or unramified cases, there can be differences, and working over the stack is usually preferable. See [63, Sec. 5] for more details.

Given a Higgs bundle $(\bar{\partial}_E, \varphi)$, *i.e.* a holomorphic structure on E and a Higgs field, we additionally equip E with a Hermitian metric h. Then there exists a unique *Chern connection* D compatible with the Hermitian metric whose $(0,1)$ part coincides with $\bar{\partial}_E$. The Hitchin equations are then equations for the Hermitian metric h:

$$\begin{aligned} F_D + \left[\varphi, \varphi^{\dagger_h}\right] &= 0, \\ \bar{\partial}_E\, \varphi &= 0, \end{aligned} \tag{4.9}$$

where $\varphi^{\dagger_h} = h^{-1}\varphi^\dagger h$ is the Hermitian conjugation of the Higgs field. The previous version of Hitchin equations, (1.3), is in the "unitary gauge" where the Hermitian metric is identity. The two conventions are related by a gauge transformation $g \in G_{\mathbb{C}}$ such that

$$g^{-1} \circ \bar{\partial}_E \circ g = \bar{\partial}_{A_u}, \quad g^{-1} \cdot \varphi \cdot g = \varphi_u, \quad g^\dagger \cdot h \cdot g = \mathrm{Id}_u\,, \tag{4.10}$$

where the subscript u indicates unitary gauge.

The moduli space $\mathcal{M}_H$ admits a natural map known as the Hitchin fibration [122],

$$\begin{aligned} \mathcal{M}_H &\to \mathcal{B}, \\ (E, \varphi) &\mapsto \det(x dz - \varphi), \end{aligned} \tag{4.11}$$

where $\mathcal{B}$ is commonly referred to as the *Hitchin base* and generic fibers are abelian varieties. As explained in [20], $\mathcal{B}$ can be identified with the Coulomb branch of the theory $T[\Sigma, G]$ on $\mathbb{R}^4$, and the curve $\det(x dz - \varphi) = 0$ with the Seiberg-Witten curve of $T[\Sigma, G]$.

The Hitchin action. There is a $U(1)$ action on the Hitchin moduli space $\mathcal{M}_H$. As emphasized in [15] and Chapter 3, the existence of the $U(1)$ Hitchin action gives us control over the infinite-dimensional Hilbert space arising from quantizing $\mathcal{M}_H$ in both the unramified or tamely ramified case,[6] and we will also focus in this chapter on the wild Hitchin moduli spaces $\mathcal{M}_H$ that admit similar $U(1)$ actions.

We first recall that in the unramified case, the Hitchin action on the moduli space is given by

$$(A, \varphi) \mapsto (A, e^{i\theta}\varphi). \tag{4.12}$$

On the physics side, it coincides with the $U(1)_r$ symmetry of the 4d $\mathcal{N} = 2$ SCFT $T[\Sigma, G]$. A similar action also exists for Σ with tame ramifications, provided the

[6] Occasionally, it is also useful to talk about the complexified $\mathbb{C}^*$-action, and we will refer to both as the "Hitchin action."

singularities are given by

$$\begin{aligned} A &\sim \alpha d\theta, \\ \varphi &\sim \text{nilpotent}. \end{aligned} \tag{4.13}$$

However, near an irregular singularity, φ acquires an higher order pole (4.7) and the action (4.12) has to rotate the u_i's. As the definition of the $\mathcal{M}_H$ depends on ramification data, this $U(1)$ action does not act on the moduli space — it will transform it into different ones. One can attempt to partially avoid this problem by setting $u_1, u_2, \ldots, u_{n-1}$ to be zero[7] — similar to the case with tame ramifications — but u_n has to be non-zero in order for the singularity to be irregular.

The way out is to modify (4.12) such that it also rotates the z coordinate by, *e.g.*,

$$z \mapsto e^{\frac{i\theta}{n-1}} z. \tag{4.14}$$

To have this action well-defined globally on Σ highly constrains the topology of the Riemann surface, only allowing $\mathbb{C}\mathbf{P}^1$ with one wild singularity, or one wild and one tame singularities.[8] Interestingly, the $U(1)$ Hitchin action on $\mathcal{M}_H$ exists whenever $T[\Sigma, G]$ is superconformal,

$$\boxed{\begin{array}{c}\mathcal{M}_H(\Sigma,G) \text{ admits} \\ U(1) \text{ Hitchin action}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}T[\Sigma,G] \text{ is a} \\ \text{4d } \mathcal{N}=2 \text{ SCFT}\end{array}}. \tag{4.15}$$

This is because superconformal invariance for $T[\Sigma, G]$ implies the existence of $U(1)_r$ symmetry which define a $U(1)$ action on $\mathcal{M}_H$. All possible choices for wild punctures of ADE type on the Riemann sphere are classified in [18, 104], and the resulting theories $T[\Sigma, G]$ are called "general Argyres-Douglas theories", which we will review in the next subsection. In Section 4.2, we will get back to geometry again to give a definition of the wild Hitchin moduli space and describe more precisely the $U(1)$ action on it.

General Argyres-Douglas theories

In this section we take $G = SU(2)$, and moreover assume that the irregular singularity lies at $z = \infty$ (the north pole) on the Riemann sphere. Another regular puncture can also be added at $z = 0$ (the south pole).

[7] More generally, we should choose their values such that the $U(1)$-action on them can be cancelled by gauge tranformations.

[8] We will focus on such Σ and the moduli spaces $\mathcal{M}_H$ associated with them. Henceforth, by "wild Hitchin moduli space", we will be usually referring to these particular $\mathcal{M}_H$, where the $U(1)$ action exists.

Near $z = \infty$, there can be two types of singular behaviors for the Higgs field φ; the leading coefficient can be either semisimple or nilpotent.[9] A semisimple pole looks like

$$\varphi(z) \sim z^{n-2} dz \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \cdots \tag{4.16}$$

with $n > 1$ an integer. For a nilpotent pole, it cannot be cast into this form by usual gauge transformations. But if we are allowed to use a local gauge transformation that has a branch cut on Σ, we can still diagonalize it into (4.16), but now with $n \in \mathbb{Z} + 1/2$. We will not allow such gauge transformation globally in the definition of the moduli space $\mathcal{M}_H$ since it creates extra poles at $z = 0$, but (4.16) is still useful conceptually in local classifications. For example, one can read off the correct $U(1)$ action on z,

$$z \mapsto e^{-\frac{i\theta}{n-1}}. \tag{4.17}$$

In [18], a puncture is called type I if n is integral, and type II if n half-odd. We will use the notation $I_{2,K}$ for the singularity with $K = 2(n-2)$ and the subscript "2" is referring to the $SL(2, \mathbb{C})$ gauge group.

The (A_1, A_{K-1}) **series.** If there is only one irregular singularity $I_{2,K}$ at the north pole, (4.16) will only have non-negative powers of z. This kind of solution describes the (A_1, A_{K-1}) Argyres-Douglas theory in the notation of [18]. Historically, this class of theories was discovered from the maximally singular point on the Coulomb branch of $\mathcal{N} = 2$ $SU(K)$ pure Yang-Mills theory [100, 102]. The Seiberg-Witten curve (or the spectral curve from the Higgs bundle point of view) takes the form

$$x^2 = z^K + v_2 z^{K-2} + \cdots + v_{K-1} z + v_K. \tag{4.18}$$

The Seiberg-Witten differential $\lambda = x dz$ has scaling dimension 1, from which we can derive the scaling dimensions for v_i,

$$[v_i] = \frac{2i}{K+2}. \tag{4.19}$$

For $i > (K+2)/2$, the scaling dimensions of the v_i's are greater than 1, and they are the expectation values of Coulomb branch operators. When K is even, there is

[9] If the leading coefficient is not nilpotent, it can always be made semisimple by a gauge transformation. Also, notice that an semisimple element of $\mathfrak{sl}(2, \mathbb{C})$ is automatically regular.

a mass parameter at $i = (K+2)/2$. The rest with $i < (K+2)/2$ are the coupling constants that give rise to $\mathcal{N} = 2$ preserving deformations

$$\Delta W \sim v_i \int d^4x\, \widetilde{Q}^4 O_i \tag{4.20}$$

for Coulomb branch operator O_i associated to v_{K+2-i}, where $\widetilde{Q}^4$ denotes the product of the four supercharges that do not annihilate O_i. Such deformation terms are also consistent with the pairing $[v_i] + [v_{K+2-i}] = 2$. If we promote all the couplings to the background chiral superfields, one can assign a $U(1)_r$ charge to them, which is equal to their scaling dimensions.[10]

The coupling constants and mass term parametrize deformations of $\mathcal{M}_H$, thus not all v_i's are part of the moduli. Moreover, to have a genuine $U(1)$ action on $\mathcal{M}_H$ itself, the v_i's with $i \leq (K+2)/2$ ought to be set zero in the spectral curve in (4.18). On the other hand, those v_i's with $i > (K+2)/2$ are allowed to be non-zero, and in fact they parametrize the Hitchin base $\mathcal{B}$. In what follows we denote this wild Hitchin moduli space as $\mathcal{M}_{2,K}$, and its Langlands dual as ${}^L\mathcal{M}_{2,K}$. The parameter a in (4.16) can be scaled away but the parameter $\alpha \in \mathrm{Lie}(\mathbb{T})$ corresponding to the monodromy of the gauge connection at the singularity enters as part of the definition of the moduli space $\mathcal{M}_{2,K}(\alpha)$. As argued in [105, Sec. 6], this monodromy has to vanish for odd K, but can be non-zero when K is even.[11] On the physics side, this agrees with the fact that the (A_1, A_{K-1}) theory has no flavor symmetry when K is odd, and generically a $U(1)$ symmetry when K is even [123]. This phenomenon is quite general, and works in the case with tame ramifications as well,

$$\boxed{\begin{array}{c}\text{Monodromy parameters}\\ \text{for the moduli space } \mathcal{M}_H(\Sigma)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{flavor symmetries}\\ \text{for the theory } T[\Sigma]\end{array}}. \tag{4.21}$$

The (A_1, D_{K+2}) series. If Σ also has a regular puncture on the south pole in addition to the irregular $I_{2,K}$ at the north pole, we will get the (A_1, D_{K+2}) Argyres-Douglas theory in the notation of [18]. Originally, this class of theories was discovered at the "maximal singular point" on the Coulomb branch of the $SO(2K+4)$ super-Yang-Mills theory [102].

[10]Our convention here for the $U(1)_r$ charge differs from the usual one as $r_{\text{usual}} = -r$. In our convention, $U(1)_r$ charge for chiral BPS operators will be the same as scaling dimensions. Notice that one can formally assign $U(1)_r$ charge to z as well; the value will turn out to be minus the scaling dimension $-[z]$.

[11]Had the puncture been tame, such monodromy would be required to the zero to have a non-empty moduli space. However, in the wild case, due to Stokes phenomenon, α can take non-zero values. Now, e^{α} is a "formal monodromy," and the real monodromy, which is required to be the identity, is a product of e^{α} with Stokes matrices.

AD theory	order of pole of φ at $z = \infty, 0$	moduli space $\mathcal{M}_H$	$\dim_{\mathbb{C}} \mathcal{M}_H$
(A_1, A_{2N})	$(2N+1)/2, 0$	$\mathcal{M}_{2,2N+1}$	$2N$
(A_1, A_{2N-1})	$N, 0$	$\mathcal{M}_{2,2N}$	$2N-2$
(A_1, D_{2N+1})	$(2N-1)/2, 1$	$\widetilde{\mathcal{M}}_{2,2N-1}$	$2N$
(A_1, D_{2N})	$N-1, 1$	$\widetilde{\mathcal{M}}_{2,2N-2}$	$2N-2$

Table 4.1: Summary of A_1 Argyres-Douglas theories, the order of singularities of the Higgs fields, the corresponding wild Hitchin moduli spaces and their dimensions.

To accommodate the regular puncture, the Higgs field should behave as

$$\varphi(z) \sim z^{n-2} dz \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + \cdots + \frac{dz}{z} \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}. \tag{4.22}$$

Consequently, the Seiberg-Witten curve is

$$x^2 = z^K + v_1 z^{K-1} + \cdots + v_{K-1} z + v_K + \frac{v_{K+1}}{z} + \frac{m^2}{z^2} \tag{4.23}$$

with the same expression for the scaling dimensions in (4.19) except that i now takes value from 1 up to $K+1$. The parameter m has the scaling dimension of mass, and it is identified as a mass parameter for the $SU(2)$ flavor symmetry associated with the regular puncture. Once again, we will turn off all the coupling constants and masses in the spectral curve since they describe deformations of the Coulomb branch moduli. Around the irregular puncture, the monodromy parameter $\alpha_1 \in \mathrm{Lie}(\mathbb{T})$ of the gauge connection A can be non-trivial. Moreover, it may not agree with the monodromy α_2 around the regular puncture. Similar to the (A_1, A_{K-1}) case, $\alpha_1 = 0$ when K is odd, and can be turned on when K is even. The corresponding moduli spaces, denoted as $\widetilde{\mathcal{M}}_{2,K}(\alpha_1, \alpha_2)$, and their Langlands dual ${}^L\widetilde{\mathcal{M}}_{2,K}(\alpha_1, \alpha_2)$ depend on those α's.

Geometry of the wild Hitchin moduli space

We have argued that the wild Hitchin moduli space can be realized as the Coulomb branch vacua of certain Argyres-Douglas theories compactified on a circle. They are summarized in Table 4.1. In accordance with the physics construction, we will now turn to a pure mathematical description of the moduli space.

A mathematical definition of these moduli spaces depends on the singular behavior of the Higgs field φ near irregular singularities, as in [105, 106]. When K is even, the moduli spaces $\mathcal{M}_{2,K}$ and $\widetilde{\mathcal{M}}_{2,K}$ are described in [106]. Consequently, we turn to the case where $K = 2N + 1$ is odd. The corresponding Higgs bundle moduli space is described in [124], and we here describe the corresponding Hitchin moduli space. To motivate the definition of $\mathcal{M}_{2,2N+1}$, note that in this case, the leading coefficient matrix (4.7) is nilpotent, which slightly differs from that of [106]. However, one can diagonalize the Higgs field near the irregular singularity by going to the double cover of the disk centered at infinity (a “lift"), so that locally the Higgs field looks like

$$\varphi \sim u'_N z^{N+\frac{1}{2}} + \dots \tag{4.24}$$

with u'_N regular semi-simple. This polar part of the Higgs field is not single-valued, so we futher impose a gauge transformation across the branch cut [105]

$$g_e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.25}$$

In our definition of $\mathcal{M}_{2,2N+1}$, the local picture at the infinity follows from an equivariant version of the local picture of [106] on the ramified disk with respect to the $\mathbb{Z}_2$-change of coordinate $w \to -w$ for $w^2 = z$. The ramification “untwists” the twisted Cartan so the local model is still diagonal, as in [106].

Two perspectives on solutions of Hitchin's equations appear in Section 4.2, and we use both in the following definition. A solution of Hitchin equations is a triple of $(\bar{\partial}_E, \varphi, h)$ consisting of a holomorphic structure, Higgs field, and Hermitian metric satisfying (4.9). Alternatively, a solution of Hitchin equations in unitary gauge (*i.e.* $h = \mathrm{Id}$) is a pair (A, φ) consisting of a unitary connection d_A and Higgs field φ satisfying (1.3). We use the notation φ for the Higgs field in both perspectives for simplicity.

Next we describe the relevant data needed to specify the moduli space $\mathcal{M}_{2,2N+1}$.

Fixed Data: Take $\mathbb{C}\mathbf{P}^1$ with a marked point at ∞. Fix a complex vector bundle $E \to \mathbb{C}\mathbf{P}^1$ of degree 0 with a trivialization of $\mathrm{Det}E$, the determinant bundle. Let $\bar{\partial}_E$ be a holomorphic structure on E which induces a fixed holomorphic structure on $\mathrm{Det}E$. Let h be a Hermitian metric on E which induces a fixed Hermitian structure on $\mathrm{Det}E$.

At ∞, we allow an irregular singularity, and fix the following data:

$$D_{\text{model}} = d + \varphi_{\text{model}} + \varphi^{\dagger}_{\text{model}}, \tag{4.26}$$

where

$$\varphi_{\text{model}} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \frac{\mathrm{d}u}{u^{K+3}}. \tag{4.27}$$

(To explain the power appearing, note that if u is the holomorphic coordinate on the ramified double cover of the disk at 0, *i.e.* $u^{-2} = z$, then $u^{-(2N+4)}\mathrm{d}u = z^{N+\frac{1}{2}}\mathrm{d}z$.)

Definition of the moduli space, $\mathcal{M}_{2,2N+1}$: Given a triple $(\bar{\partial}_E, \varphi, h)$, denote the lift of the unitary pair (A, φ) by

$$(\widetilde{A}, \widetilde{\varphi}) = l \cdot (A, \varphi). \tag{4.28}$$

A triple $(\bar{\partial}_E, \varphi, h)$ is in $\mathcal{M}_{2,2N+1}$ if it is a solution of Hitchin equations on $\mathbb{C}\mathbf{P}^1$ and on a neighborhood of ∞ the associated flat connection $\tilde{D} = \widetilde{A} + \widetilde{\varphi} + \widetilde{\varphi}^\dagger$ differs from the local model in (4.26) by a deformation allowed by [106]. Moreover, we say that $(\bar{\partial}_E, \varphi, h)$ and $(\bar{\partial}'_E, \varphi', h')$ are gauge equivalent if there is some unitary gauge transformation g by which (A, φ) and (A', φ') are gauge equivalent, and g lifts to an allowed gauge transformation on the ramified disk around ∞. More precisely, the lift $\tilde{g} = l' \circ g \circ l^{-1}$ must be an allowed unitary gauge transformation, in the perspective of [106], from $l \cdot (A, \varphi)$ to $l' \cdot (A', \varphi')$ on the ramified disk around ∞. The moduli space $\widetilde{\mathcal{M}}_{2,2N-1}$ can be defined similarly.

With the above definitions, it is expected that the symplectic form ω_I on $\mathcal{M}_{2,K}$ and $\widetilde{\mathcal{M}}_{2,K}$ can be expressed just as that in [12]:

$$\omega_I = \frac{i}{\pi} \int \operatorname{Tr}\left(\delta A_z \wedge \delta A_{\bar{z}} - \delta\varphi \wedge \delta\varphi^\dagger\right). \tag{4.29}$$

There is a $U(1)$ action on the moduli space $\mathcal{M}_{2,K}$ and $\widetilde{\mathcal{M}}_{2,K}$, by composing the rotation of Higgs field with a rotation of the Riemann sphere. It is defined as:

$$\begin{aligned} z &\xrightarrow{\rho_\theta} e^{-i\frac{2}{2+K}\theta} z, \\ \varphi &\rightarrow e^{i\theta}\rho_\theta^* \varphi, \\ A &\rightarrow \rho_\theta^* A. \end{aligned} \tag{4.30}$$

We say (A, φ) is fixed by the $U(1)$ action if for all θ, the rotated solution is gauge equivalent to the unrotated one. This $U(1)$ action is expected to be Hamiltonian with moment map μ such that

$$d\mu = \iota_V \omega_I, \tag{4.31}$$

where V is the vector field generated by the $U(1)$ action. At the fixed points of the $U(1)$ action, there is evidence that this moment map agrees with the following quantity [124]:

$$\mu = \frac{i}{2\pi}\int \mathrm{Tr}\left(\varphi\wedge\varphi^\dagger - \mathrm{Id}\cdot|z|^K dz\wedge d\bar{z}\right). \tag{4.32}$$

In Appendix E, we compute the weights of the $U(1)$ action at the fixed points. Practically, rather than working with the Hitchin moduli space, we may instead work with the Higgs bundle moduli space diffeomorphic to $\mathcal{M}_{2,K}$ or $\widetilde{\mathcal{M}}_{2,K}$. In the case $\mathcal{M}_{2,2N+1}$, the corresponding Higgs bundle moduli space $\mathcal{M}^{\mathrm{Higgs}}_{2,2N+1}$ is rigorously described in [124]. For the other moduli spaces, we provide a general set-up of the definition for the Higgs bundle moduli space, and leave a rigorous treatment to future work. Unsurprisingly, the fixed data for the Higgs bundle moduli space is the same as the fixed data for the Hitchin moduli space. On the ramified double cover of the disk at ∞ with coordinate $u = z^{-1/2}$, the local model for the Higgs field is

$$\varphi_{\mathrm{model}} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}\frac{\mathrm{d}u}{u^{K+3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z^{K/2}\mathrm{d}z, \tag{4.33}$$

as in (4.26). Additionally, the monodromy at ∞ on the ramified double cover at ∞ is trivial when K is odd, but otherwise a free parameter. The monodromy is algebraically encoded in the data of a filtration structure of the holomorphic vector bundle $\mathcal{E} = (E, \bar{\partial}_E)$ at ∞. The filtered vector bundle of $\mathcal{E}$ and the filtration structure at ∞ are denoted as $\mathcal{P}_\bullet\mathcal{E}$.

A pair $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ consisting a filtered bundle $\mathcal{P}_\bullet\mathcal{E}$ and meromorphic Higgs field φ with pole at ∞ (with no additional compatibility conditions) is in the Higgs bundle moduli space $\mathcal{M}^{\mathrm{Higgs}}_{2,K}$ if there is a holomorphic lift to the ramified disk in which $\psi^*(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ is "unramifiedly good" (in the sense of [125]), *i.e.*,

$$\psi^*\varphi = \varphi_{\mathrm{model}} + \text{holomorphic terms} \tag{4.34}$$

and $\psi^*(\mathcal{P}_\bullet\mathcal{E})$ is the trivial filtration. In $\mathcal{M}^{\mathrm{Higgs}}_{2,K}$, $(\mathcal{P}_\bullet\mathcal{E}, \varphi)$ and $(\mathcal{P}_\bullet\mathcal{E}', \varphi')$ are identified if there is a isomorphism $\eta : \mathcal{P}_\bullet\mathcal{E} \to \mathcal{P}_\bullet\mathcal{E}'$ of $\mathcal{P}_\bullet\mathcal{E}$ and $\mathcal{P}_\bullet\mathcal{E}'$ as filtered vector bundles such that $\varphi' = \eta^{-1}\circ\varphi\circ\eta$.

Quantization of $\mathcal{M}_H$

One of the major goals of this chapter is to study the quantization of wild Hitchin moduli spaces,

$$(\mathcal{M}_H(\Sigma, G), k\omega_I) \quad \rightsquigarrow \quad \mathcal{H}(\Sigma, G, k). \tag{4.35}$$

The quantization problem takes as input the symplectic manifold $(\mathcal{M}_H(\Sigma, G), k\omega_I)$ — the "phase space," and aims to produce a space of quantum states — the "Hilbert space." In this particular case, the resulting space $\mathcal{H}(\Sigma, G, k)$ can be interpreted as the Hilbert space of complex Chern-Simons theory at real level k on Σ, with the complex connection developing singularities near the punctures.

Using the standard machinery of geometric quantization of Kähler manifolds, one can identify the Hilbert space with holomorphic sections of a "prequantum line bundle"

$$\mathcal{H}(\mathcal{M}_H(\Sigma, G), k\omega_I) = H^0(\mathcal{M}_H, \mathcal{L}^{\otimes k}). \tag{4.36}$$

Here $\mathcal{L}$ denotes the determinant line bundle over $\mathcal{M}_H$ whose curvature is cohomologous to ω_I,

$$c_1(\mathcal{L}) = [\omega_I]. \tag{4.37}$$

For all quantization problems, a very interesting question is to find the dimension of the resulting Hilbert space. In the present case, the dimension of $\mathcal{H}$ can be formally written as an integral over $\mathcal{M}_H$,[12]

$$\dim\, H^0(\mathcal{M}_H, \mathcal{L}^{\otimes k}) = \chi(\mathcal{M}_H, \mathcal{L}^{\otimes k}) = \int_{\mathcal{M}_H} e^{k\omega_I} \wedge \mathrm{Td}(\mathcal{M}_H). \tag{4.38}$$

In the above expression, we used the vanishing of higher cohomology groups[13] to rewrite the dimension as an Euler characteristic, and then used index theorem to express it as an integral over the moduli space.

Just like their unramified or tamely ramified cousins, the wild Hitchin moduli spaces are also non-compact and would give rise to infinite-dimensional Hilbert spaces after quantization. This is seen quite clearly from the integral in (4.38), which diverges due to the non-compactness of $\mathcal{M}_H$.

However, as the $U(1)$ Hitchin action is Hamitonian (in particular it preserves ω_I), it also acts on the Hilbert space $\mathcal{H}$. Then the dimension of $\mathcal{H}$ can be refined to the *graded* dimension, defined as the character of the $U(1)$ action,

$$\dim_{\mathrm{t}} \mathcal{H} = \sum_n \dim \mathcal{H}_n \mathrm{t}^n. \tag{4.39}$$

[12]We use integrals for pedagogical reasons. $\mathcal{M}_H$ generically is not a manifold, and should be viewed as a stack.

[13]The vanishing theorem for unramified and tamely ramified cases was proved in [68] and [63], and the vanishing is expected to hold also in the wild case — morally, because of the Kodaira vanishing along the fibers of the Hitchin map.

Here $\mathfrak{t}$ is the fundamental character of $U(1)$, and $\mathcal{H}_n$ is the subspace of $\mathcal{H}$ where $U(1)$ acts with eigenvalue n. In [15], this Hitchin character was computed in the unramified or tamely ramified case, and was found to be given by a Verlinde-like formula, known as the "equivariant Verlinde formula." The word "equivariant" comes from the fact that the Hitchin character can also be written as an integral, similar to (4.38), but now in the $U(1)$-equivariant cohomology of $\mathcal{M}_H$,

$$\dim_{\mathfrak{t}} \mathcal{H}(\Sigma, G, k) = \chi_{U(1)}(\mathcal{M}_H, \mathcal{L}^{\otimes k}) = \int_{\mathcal{M}_H} e^{c_1(\mathcal{L}^{\otimes k}, \beta)} \wedge \mathrm{Td}(\mathcal{M}_H, \beta). \tag{4.40}$$

Here, the second quantity is the equivariant Euler characteristic of $\mathcal{L}^{\otimes k}$ which is then expressed as an integral over $\mathcal{M}_H$ via the equivariant index theorem. This integral will actually converge, but we will need to first briefly review the basics of equivariant cohomology and introduce necessary notation. We will be very concise and readers unfamiliar with this subject may refer to [126] for a more pedagogical account.

Let V be the vector field on $\mathcal{M}_H$ generated by the $U(1)$ action; we pick β to be the degree-2 generator of the equivariant cohomology of $H^\bullet_{U(1)}(\mathrm{pt})$ and is related to $\mathfrak{t}$ by $\mathfrak{t} = e^{-\beta}$. Using the Cartan model for equivariant cohomology, we define the equivariant exterior derivative as

$$\widehat{\delta} = \delta + \beta\iota_V \tag{4.41}$$

with $\widehat{\delta}^2 = 0$ over equivariant differential forms. One can then define the equivariant cohomology as

$$H^\bullet_G(\mathcal{M}_H) = \ker\widehat{\delta}/\mathrm{im}\,\widehat{\delta}. \tag{4.42}$$

For an equivariant vector bundle, one can also define the equivariant characteristic classes. For example, the equivariant first Chern class of $\mathcal{L}$ is now

$$c_1(\mathcal{L}, \beta) = \widetilde{\omega}_I := \omega_I - \beta\mu. \tag{4.43}$$

And one can verify that it is equivariantly closed

$$\widehat{\delta}\,\widetilde{\omega}_I = 0. \tag{4.44}$$

Similarly, one can define the equivariant Todd class $\mathrm{Td}(\mathcal{M}_H, \beta)$ of the tangent bundle of $\mathcal{M}_H$.

Now we can see that the integral in (4.40) has a very good chance of being convergent as $e^{c_1(\mathcal{L},\beta)}$ contains a factor $e^{-\beta\mu}$ which suppresses the contribution from large Higgs fields. Further, one can use the Atiyah-Bott localization formula to write (4.40) as a summation over fixed points of the Hitchin action,

$$\int_{\mathcal{M}_H} e^{c_1(\mathcal{L}^{\otimes k},\beta)} \wedge \mathrm{Td}(\mathcal{M}_H,\beta) = \sum_{F_d} e^{-\beta k \mu(F_d)} \int_{F_d} \frac{\mathrm{Td}(F_d) \wedge e^{k\omega_I}}{\prod_i^{\mathrm{codim}_{\mathbb{C}} F_d} (1 - e^{-x_i - \beta n_i})}, \quad (4.45)$$

where F_d is a component of the fixed points, and $x_i + \beta n_i$ are the equivariant Chern roots of the normal bundle of F_d with n_i being the eigenvalues under the $U(1)$ action. For a Hitchin moduli space, there is finitely many F_d's and each of them is compact, so the localization formula provides a way to compute the Hitchin character. To use the above expression, one must understand the fixed points and their ambient geometry — something that is typically challenging. This makes the relation (4.1) very useful, since it suggests that the Hitchin character, along with all the non-trivial geometric information about $\mathcal{M}_H$ that it encodes, can be obtained in a completely different (and in many senses simpler) way from the Coulomb index of the 4d SCFT $T[\Sigma, G]$! This is precisely the approach taken in Chapter 3 and [16] for tamely ramified Σ. We now proceed to study the Coulomb branch index of the general Argyres-Douglas theories to uncover the wild Hitchin characters.

We end this section with two remarks. The first is about the large-k limit of the Hitchin character. In this limit, it is related to another interesting invariant of $\mathcal{M}_H$ called the "equivariant volume" studied in [127]

$$\mathrm{Vol}_\beta(\mathcal{M}_H) = \int_{\mathcal{M}_H} \exp(k\widetilde{\omega}_I) = \sum_{F_d} e^{-\beta\mu(F_d)} \int_{F_d} \frac{e^{\omega_I}}{\mathrm{eu}_\beta(F_d)} \quad (4.46)$$

where $\mathrm{eu}_\beta(F_d)$ is the equivariant Euler class of the normal bundle of F_d,

$$\mathrm{eu}_\beta(F_d) = \prod_{i=1}^{\mathrm{codim}_{\mathbb{C}} F_d} (x_i + \beta n_i). \quad (4.47)$$

The second remark is about the quantization of the monodromy parameter α (and also the α_1 and α_2). In the definition of the moduli space $\mathcal{M}_H$, this parameter can take arbitrary values inside the Weyl alcove $\mathrm{Lie}(\mathbb{T})/W_{\mathrm{aff}}$ subject to no restrictions. However, only for discrete values of the monodromy parameter, $\mathcal{M}_H$ is quantizable. The allowed values are given by the characters of G modulo W_{aff} action (or equivalently integrable representations of G at level k.)

$$k\alpha \in \Lambda_{\mathrm{char}}(G)/W_{\mathrm{aff}} = \mathrm{Hom}(G, U(1))/W_{\mathrm{aff}}, \quad (4.48)$$

which ensures the prequantum line bundle $\mathcal{L}^{\otimes k}$ has integral periods over $\mathcal{M}_H$ (see Chapter 3 for completely parallel discussion of this phenomenon in the tame case.) For $G = SU(2)$, we often use the integral parameter

$$\lambda = 2k\alpha \in \{0, 1, \ldots, k\}. \tag{4.49}$$

The discretization of α can also be understood from the SCFT side. For a quantum field theory with flavor symmetry ${}^L G$ on $M_3 \times \mathbb{R}$, one can deform the system — and also its Coulomb branch — by turning on a flavor holonomy in $\mathrm{Hom}(\pi_1 M_3, {}^L G)/{}^L G$. When $M_3 = L(k, 1)$, the homomorphism $\pi_1 = \mathbb{Z}_k \to^L G$ up to conjugation is precisely classified by elements in

$$\Lambda_{\text{cochar}}({}^L G) = \Lambda_{\text{char}}(G) \tag{4.50}$$

modulo affine Weyl symmetry.[14]

4.3 The Coulomb branch index of AD theories from $\mathcal{N} = 1$ Lagrangian

Now our task is to compute the Coulomb branch index of Argyres-Douglas theories on the lens space $L(k, 1)$. This is, however, a rather nontrivial problem, since these theories are generically strongly-interacting, non-Lagrangian SCFTs. Their original construction using singular loci of the Coulomb branch of $\mathcal{N} = 2$ super Yang-Mills theory is not of much use: the IR R-symmetries are emergent, the Seiberg-Witten curves are derived from a subtle scaling limits (see $e.g.$ [128] for discussion of this issue), and the Higgs branches are intrinsic to the superconformal point itself [123]. Also, no known dualities can relate them to Lagrangian theories. For example, in Chapter 3 the generalized Argyres-Seiberg duality is very powerful for study of Coulomb index of class $\mathcal{S}$ theories, but its analogue for Argyres-Douglas theories is not good enough to enable the computation of superconformal indices, since the two S-duality frames in general both consist of non-Lagrangian theories [129–132].

Recently, the author of [109–111] discovered that a certain class of four-dimensional $\mathcal{N} = 1$ *Lagrangian* theories exhibit supersymmetry enhancement under RG flow. In particular, some of them flow to $\mathcal{N} = 2$ Argyres-Douglas theories. The $\mathcal{N} = 1$

[14] In Chapter 3, the importance of distinguishing between G and ${}^L G$ was emphasized. However, for the wild Hitchin moduli space that we study, the difference is not as prominent, because Σ is now restricted to be $\mathbb{C}\mathbf{P}^1$, making the Hitchin character insensitive to global structure of the gauge group. In fact, the wild Hitchin characters we will consider are complete determined by the Lie algebra $\mathfrak{g}$, provided that we analytically continuate $k\alpha$ to be a weight of $\mathfrak{g}$. Because of this, we will use the simply-connected group — $SU(2)$ in the rank-2 case — for both the gauge group of the SCFT and the moduli space.

description allows one to track down the flow of R-charges and identify the flavor symmetry from the UV, making the computation of the full superconformal index possible.

In this section we will use their prescription to calculate the Coulomb branch index of Argyres-Douglas theories on $S^1 \times L(k,1)$. Investigation of their properties, which is somewhat independent of the main subject of the chapter, is presented in Appendix C, which consists of two subsections. The TQFT properties of the Coulomb branch indices make up Appendix C.1. When there is only tame ramifications, the lens space Coulomb branch index of $T[\Sigma]$ gives rise to a very interesting 2D TQFT on Σ [16]. In the presence of irregular singularities, the geometry of Σ is highly constrained, and only a remnant of the TQFT cutting-and-gluing rules is present, which tells us how to close the regular puncture on the south pole to go from the (A_1, D_{K+1}) theory to (A_1, A_{K-2}).

In Appendix C.2, we consider the dimensional reduction of Argyres-Douglas theories, which will be relevant later when we discuss the large-k behavior of the Hitchin character. The main motivation is to resolve an apparent puzzle: any fractional $U(1)_r$ charges in four dimensions should disappear upon dimensional reduction, since it is impossible to have fractional R-charges in the resulting three-dimensional $\mathcal{N}=4$ theory, whose R-symmetry is enhanced to $SU(2)_C \times SU(2)_H$. The solution lies in the mixing between the topological symmetry and the R-symmetry, similar to what was first discussed in [112] using Schur index. Here we shall confirm the statement from Coulomb branch point of view directly.

In the following we begin with a brief review of the construction [109–111] and present an integral formula for the Coulomb branch index on lens spaces.

The construction

In the flavor-current multiplet of a 4d $\mathcal{N}=2$ SCFT, the lowest component is known as the "moment map operator", which we will denote as $\widehat{\mu}$. It is valued in $\mathfrak{f}^*$, the dual of the Lie algebra of the flavor symmetry F, and transforms in the $\mathbf{3}_0$ of the $SU(2)_R \times U(1)_r$ R-symmetry. In other words, if the Cartan generators of $SU(2)_R$ and $U(1)_r$ is I_3 and r, then

$$I_3(\widehat{\mu}) = 1, \quad \text{and} \quad r(\widehat{\mu}) = 0. \tag{4.51}$$

The idea of [109–111] is to couple the moment map operator $\widehat{\mu}$ with an additional $\mathcal{N}=1$ "meson" chiral multiplet M in the adjoint representation $\mathfrak{f}$ of F via the superpotential

$$W = \langle \widehat{\mu}, M \rangle \tag{4.52}$$

and give M a nilpotent vev $\langle M \rangle$. If the $\mathcal{N} = 2$ theory we start with has a Lagrangian description (the case that we will be mainly interested in below), such deformation will give mass to some components of quarks, which would be integrated out during the RG flow.

The Jacobson-Morozov theorem states that a nilpotent vev $\langle M \rangle \in \mathfrak{f}^+$ specifies a Lie algebra homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{f}$. The commutant of the image of ρ is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{f}$. This subalgebra $\mathfrak{h}$ is the Lie algebra of the residual flavor symmetry H. In the presence of the nilpotent vev, $\mathfrak{f}$ (and similarly $\mathfrak{f}^*$) can be decomposed into representations of $\mathfrak{su}(2) \times \mathfrak{h}$ as

$$\mathfrak{f} = \sum_j V_j \otimes R_j, \tag{4.53}$$

where the summation runs over all possible spin-j representations V_j of $\mathfrak{su}(2)$, and R_j carries a representation of $\mathfrak{h}$. Both M and $\hat{\mu}$ can be similarly decomposed

$$M = \sum_{j,j_3} \widetilde{M}_{j,j_3}, \quad \widehat{\mu} = \sum_{j,j_3} \widehat{\mu}_{j,j_3}, \tag{4.54}$$

where M_{j,j_3} also carries the R_j representation of $\mathfrak{h}$ that we omitted. Here (j, j_3) is the quantum number for the $\mathfrak{su}(2)$ representation V_j. Among them, $M_{1,1}$ will acquire a vev v, and we re-define M to the fluctuation $M - \langle M \rangle$. Then, the superpotential (4.52) decomposes as

$$W = v\mu_{1,-1} + \sum_j \langle M_{j,-j}, \widehat{\mu}_{j,j} \rangle. \tag{4.55}$$

Note that only the $-j$ component of the spin-j representation of $\mathfrak{su}(2)$ for the M's remains coupled in the theory, as the other components giving rise to irrelevant deformations [110].

Next, we examine the R-charge of the deformed theory. In the original theory, we denote $(J_+, J_-) = (2I_3, 2r)$ and a combination of them will be the genuine $U(1)_R$ charge of the $\mathcal{N} = 1$ theory, leaving the other as the flavor symmetry $\mathcal{F} = (J_+ - J_-)/2$. Upon RG flow to the infrared SCFT, the flavor symmetry would generally mix with the naive assignment of $U(1)_R$ charge:

$$R = \frac{1}{2}(J_+ + J_-) + \frac{\epsilon}{2}(J_+ - J_-). \tag{4.56}$$

matter	$Sp(N)$	(J_+, J_-)
q	$\square$	$(1, 0)$
q'	$\square$	$(1, -4N - 2)$
ϕ	adj	$(0, 2)$
$M_j,\ j = 1, 3, \ldots 4N + 1$	$\mathbf{1}$	$(0, 2j + 2)$
M'_{2N+1}	$\mathbf{1}$	$(0, 4N + 4)$

Table 4.2: The $\mathcal{N} = 1$ matter content for the $Sp(N)$ gauge theory that flows to (A_1, A_{2N}) Argyres-Douglas theory. ρ is given by the principal embedding, and j takes values in the exponents of $\mathfrak{f}$. For $\mathfrak{f} = \mathfrak{so}(4N + 4)$, the exponents are $\{2N + 1; 1, 3, \ldots, 4N + 1\}$.

The exact value of the mixing parameter ϵ can be determined via a-maximization [133] and its modification to accommodate decoupled free fields along the RG flow [134]. In the following, we summarize the $\mathcal{N} = 1$ Lagrangian theory and the embedding ρ found in [110, 111] that are conjectured to give rise to Argyres-Douglas theories relevant for this chapter.

Lagrangian for (A_1, A_{2N}) theory. The $\mathcal{N} = 1$ Lagrangian is obtained by starting with $\mathcal{N} = 2$ SQCD with $Sp(N)$ gauge group[15] plus $2N+2$ flavors of hypermultiplets. The initial flavor symmetry is $F = SO(4N+4)$ and we pick the principal embedding, given by the partition $[4N + 3, 1]$. The resulting $\mathcal{N} = 1$ matter contents are listed in Table 4.2. Under the RG flow the Casimir operators $\mathrm{Tr}\,\phi^{2i}$ with $i = 1, 2, \ldots, N$ and M_j with $j = 1, 3, \ldots, 2N + 1$ and M'_{2N+1} decouple. The mixing parameter in (4.56) is

$$\epsilon = \frac{7 + 6N}{9 + 6N}. \tag{4.57}$$

Lagrangian for (A_1, A_{2N-1}) theory. Similarly one starts with $\mathcal{N} = 2$ SQCD with $SU(N)$ gauge group and $2N$ fundamental hypermultiplets with $SU(2N) \times U(1)_B$ flavor symmetry. We again take the principal embedding. The matter content is summarized in Table 4.3. Using a-maximization we see that M_j with $j = 1, 2, \ldots, N$, along with all Casimir operators, become free and decoupled. The mixing parameter in (4.56) is

$$\epsilon = \frac{3N + 1}{3N + 3}. \tag{4.58}$$

[15] We adopt the convention that $Sp(1) \simeq SU(2)$.

matter	$SU(N)$	$U(1)_B$	(J_+, J_-)
q	$\square$	1	$(1, -2N+1)$
$\tilde{q}$	$\overline{\square}$	-1	$(1, -2N+1)$
ϕ	adj	0	$(0, 2)$
$M_j,\ j = 1, 2, \dots 2N-1$	$\mathbf{1}$	0	$(0, 2j+2)$

Table 4.3: The $\mathcal{N} = 1$ matter content for the $SU(N)$ gauge theory that flows to (A_1, A_{2N-1}) Argyres-Douglas theory. ρ is again the principal embedding, and j ranges over the exponents of $\mathfrak{su}(2N)$.

It is worthwhile to emphasize that the extra $U(1)_B$ symmetry would become the flavor symmetry of the Argyres-Douglas theory. In particular, when $N = 2$, it is enhanced to $SU(2)_B$. This $U(1)_B$ symmetry is the physical origin of the gauge monodromy α in Section 4.2.

Lagrangian for (A_1, D_{2N+1}) theory. Just as the (A_1, A_{2N}) theories, the starting point is the $\mathcal{N} = 2$ SCFT with $Sp(N)$ gauge group and $2N+2$ fundamental hypermultiplets. However, the nilpotent embedding ρ is no longer the principal one; rather it is now given by the partition $[4N+1, 1^3]$, whose commutant subgroup is $SO(3)$ [111]. The Lagrangian of the theory is given in Table 4.4. Among mesons and Casimir operators $\mathrm{Tr}\ \phi^i$, only M_j for $j = 2N+1, 2N+3, \dots, 4N-1$ remain interacting. The mixing parameter in (4.56) is found to be

$$\epsilon = \frac{6N+1}{6N+3}. \tag{4.59}$$

In this case, the UV $SO(3)$ residual flavor symmetry group is identified as the IR $SU(2)$ flavor symmetry coming from the simple puncture.

Lagrangian for (A_1, D_{2N}) theory. Similar to the (A_1, A_{2N-1}) case, we start with the $SU(N)$ gauge theory with $2N$ fundamental hypermultiplets, but choose ρ to be the embedding given by the partition $[2N-1, 1]$. This leaves a $U(1)_a \times U(1)_b$ residual flavor symmetry, the first of which is the baryonic symmetry that we started with. The Lagrangian is summarized in Table 4.5. Under RG flow, the decoupled gauge invariant operators are Casimir operators $\mathrm{Tr}\,\phi^i, i = 2, 3, \dots, N$, M_j with $j = 0, 1, \dots, N-1$ and $(M, \tilde{M})$. The a-maximization gives the mixing parameter

$$\epsilon = 1 - \frac{2}{3N}. \tag{4.60}$$

matter	$Sp(N)$	$SO(3)$	(J_+, J_-)
q	$\square$	$\mathbf{3}$	$(1, 0)$
q'	$\square$	$\mathbf{1}$	$(1, -4N)$
ϕ	adj	$\mathbf{1}$	$(0, 2)$
$M_j,\ j = 1, 3, \ldots 4N - 1$	$\mathbf{1}$	$\mathbf{1}$	$(0, 2j + 2)$
M'_{2N}	$\mathbf{1}$	$\mathbf{3}$	$(0, 4N + 2)$
M'_0	$\mathbf{1}$	$\mathbf{3}$	$(0, 2)$

Table 4.4: The $\mathcal{N} = 1$ matter content for the $Sp(N)$ gauge theory that flows to (A_1, D_{2N+1}) Argyres-Douglas theory.

matter	$SU(N)$	$U(1)_a$	$U(1)_b$	(J_+, J_-)
q	$\square$	1	$2N - 1$	$(1, 0)$
$\tilde{q}$	$\overline{\square}$	-1	$-2N + 1$	$(1, 0)$
q'	$\square$	1	-1	$(1, 2 - 2N)$
$\tilde{q}'$	$\overline{\square}$	-1	$+1$	$(1, 2 - 2N)$
ϕ	adj	0	0	$(0, 2)$
$M_j,\ j = 0, 1, \ldots 2N - 2$	$\mathbf{1}$	0	0	$(0, 2j + 2)$
M	$\mathbf{1}$	0	$2N$	$(0, 2N)$
$\widetilde{M}$	$\mathbf{1}$	0	$-2N$	$(0, 2N)$

Table 4.5: The $\mathcal{N} = 1$ matter content for the $SU(N)$ gauge theory that flows to (A_1, D_{2N}) Argyres-Douglas theory.

In the IR, one combination of $U(1)_a$ and $U(1)_b$ would become the Cartan of the enhanced $SU(2)$ flavor symmetry.

Coulomb branch index on lens spaces

The $\mathcal{N} = 1$ constructions of the generalized Argyres-Douglas theories enable one to compute their $\mathcal{N} = 2$ superconformal index by identifying the additional R-symmetry with a flavor symmetry of the $\mathcal{N} = 1$ theory. As the ordinary superconformal index on $S^1 \times S^3$, the $\mathcal{N} = 1$ lens space index can be defined in terms of the trace over Hilbert space on $L(k,1)$ [79, 135]

$$\mathcal{I}_{\mathcal{N}=1}(p,q) = \mathrm{Tr}\,(-1)^F p^{j_1+j_2+R/2} q^{j_2-j_1+R/2} \xi^{\mathcal{F}} \prod_i a_i^{f_i} \exp(-\beta'\delta'), \tag{4.61}$$

where $j_{1,2}$ are the Cartans of the $SO(4)_E \simeq SU(2)_1 \times SU(2)_2$ rotation group, R counts the superconformal $U(1)_R$ charge of the states. We also introduce the flavor fugacity ξ for the symmetry $\mathcal{F} = (J_+ - J_-)/2$ inherited from the $\mathcal{N} = 2$ R-symmetry. Finally, δ' is the commutator of a particular supercharge Q chosen in defining the index. It is given by

$$\delta' = \{Q, Q^\dagger\} = E - 2j_1 + \frac{3R}{2}, \tag{4.62}$$

where E the conformal dimension. Supersymmetry ensures that only states annihilated by Q contribute in (4.61); hence the results are independent of β' and one can restrict the trace to be taken over the space of BPS states.

One advantage of the lens space index comes from the non-trivial fundamental group of $L(k,1)$, making it sensitive to the global structure of the gauge group [79]. Also, the gauge theory living on $L(k,1)$ has degenerate vacua labelled by holonomies around the Hopf fiber, so the Hilbert space will be decomposed into different holonomy sectors. All of these make the lens index a richer invariant than the ordinary superconformal index.

For a theory with a Lagrangian, the lens space index can be computed by first multiplying contributions from free matter multiplets after $\mathbb{Z}_k$-projection, then integrating over the (unbroken) gauge group determined by a given holonomy sector, and finally summing over all inequivalent sectors. We introduce the elliptic Gamma function

$$\Gamma(z;p,q) = \prod_{j,k=0}^{+\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}. \tag{4.63}$$

Then, for a chiral superfield with gauge or flavor fugacity/holonomy (b,m) we have

$$I_\chi(m,b) = I_0^\chi(m,b) \cdot \Gamma\Big((pq)^{\frac{R}{2}} q^{k-m} b; q^k, pq\Big) \Gamma\Big((pq)^{\frac{R}{2}} p^m b; p^k, pq\Big) \tag{4.64}$$

with the prefactor related to the Casimir energy

$$I_0^\chi(m,b) = \left((pq)^{\frac{1-R}{2}} b^{-1}\right)^{\frac{m(k-m)}{2k}} \left(\frac{p}{q}\right)^{\frac{m(k-m)(k-2m)}{12k}}. \tag{4.65}$$

For a vector multiplet the contribution is

$$I_V(m,b) = \frac{I_0^V(m,b)}{\Gamma\big(q^m b^{-1}; q^k, pq\big)\Gamma\big(p^{k-m} b^{-1}; p^k, pq\big)} \tag{4.66}$$

with

$$I_0^V(m,b) = \left((pq)^{\frac{1}{2}} b^{-1}\right)^{-\frac{m(k-m)}{2k}} \left(\frac{q}{p}\right)^{\frac{m(k-m)(k-2m)}{12k}}. \tag{4.67}$$

Notice that we will not turn on flavor holonomy for the $U(1)$ flavor symmetry $\mathcal{F}$ along the Hopf fiber. This is because it is part of the $\mathcal{N} = 2$ R-symmetry; turning on background holonomy for it will break the $\mathcal{N} = 2$ supersymmetry.

To connect (4.61) with $\mathcal{N} = 2$ lens space index, recall the definition of the latter is [60, 78]

$$\mathcal{I}_{\mathcal{N}=2}(p,q,t) = \mathrm{Tr}\,(-1)^F p^{j_1+j_2+r} q^{j_2-j_1+r} t^{R-r} \prod_i a_i^{f_i} \exp(-\beta'' \delta''), \tag{4.68}$$

where the index counts states with $SU(2)_R \times U(1)_r$ charge (R, r) that are BPS with respect to $\delta'' = E - 2j_2 - 2R - r$. To recover the above $\mathcal{N} = 2$ index from (4.61), we make the substitution

$$\xi \to \left(t(pq)^{-\frac{2}{3}}\right)^{\gamma} \tag{4.69}$$

for some constant γ depending on how $U(1)_{\mathcal{F}}$ is embedded inside $SU(2)_R \times U(1)_r$.

Finally, we take the "Coulomb branch limit" of the $\mathcal{N} = 2$ lens space index,

$$p, q, t \to 0, \quad \frac{pq}{t} = \mathrm{t} \ \text{ fixed.} \tag{4.70}$$

The trace formula (4.68) then reduces to

$$\mathcal{I}^C_{\mathcal{N}=2} = \mathrm{Tr}_C(-1)^F \mathrm{t}^{r-R} \prod_i a_i^{f_i}, \tag{4.71}$$

where the trace is taken over BPS states annihilated by both $\widetilde{Q}_{1\dot{-}}$ and $\widetilde{Q}_{2\dot{+}}$ (*i.e.*, satisfying $E - 2j_2 - 2R + r = E + 2j_2 + 2R + r = 0$.) Notice that, in our convention, $L(k,1)$ is a quotient of S^3 by $\mathbb{Z}_k \subset U(1)_{\text{Hopf}} \subset SU(2)_1$. Since both $\widetilde{Q}_{1\dot{-}}$ and $\widetilde{Q}_{2\dot{+}}$ transform trivially under $SU(2)_1$, they are preserved after the $\mathbb{Z}_k$ quotient. Hence the trace formula (4.71) is well-defined.

For all known examples the Coulomb branch operators have $R = 0$, so the above limit effectively counts $U(1)_r$ charge. For a Lagrangian theory, when $k = 1$ this limit counts the short multiplet $\overline{\mathcal{E}}_{r,(0,0)}$ [58], whose lowest component parametrizes the Coulomb branch vacua of the SCFT.

Below we will list the integral formulae for the Coulomb branch indices of Argyres-Douglas theories that we are interested in throughout this chapter. In computing the lens space index we have removed contributions from the decoupled fields.

(A_1, A_{2N}) **theories.** We have

$$\mathcal{I}_{(A_1,A_{2N})} = \prod_{i=1}^{N} \frac{1}{1-\mathrm{t}^{\frac{2(N+i+1)}{2N+3}}} \prod_{i=1}^{N} \frac{1-\mathrm{t}^{\frac{2i}{2N+3}}}{1-\mathrm{t}^{\frac{1}{2N+3}}}$$

$$\times \sum_{m_i} \prod_{\alpha>0} \left(\mathrm{t}^{\frac{2}{2N+3}}\right)^{-\frac{1}{2}(\llbracket\alpha(m)\rrbracket-\frac{1}{k}\llbracket\alpha(m)\rrbracket^2)} \prod_{i=1}^{N} \left(\mathrm{t}^{\frac{4(N+1)}{2N+3}}\right)^{\frac{1}{2}(\llbracket m_i\rrbracket-\frac{1}{k}\llbracket m_i\rrbracket^2)} \tag{4.72}$$

$$\times \frac{1}{|\mathcal{W}_m|} \oint [d\mathbf{z}] \prod_{\llbracket\alpha(m)\rrbracket=0} \frac{1-\mathbf{z}^\alpha}{1-\mathrm{t}^{\frac{1}{2N+3}}\mathbf{z}^\alpha}\,,$$

where the integral is taken over the unbroken subgroup of $Sp(N)$ with respect to a given set of holonomies $\{m_i\}$. Here, $|\mathcal{W}_m|$ is the order of Weyl group for the residual gauge symmetry. The constant γ (4.69) is $\gamma = 1/(2N+3)$. We use the notation $\llbracket x\rrbracket$ to denote the remainder of x modulo k.

(A_1, A_{2N-1}) **theories.** After taking $\gamma = 1/(N+1)$ and the Coulomb branch limit, we have

$$\mathcal{I}_{(A_1,A_{2N-1})} = \prod_{i=1}^{N-1} \frac{1}{1-\mathrm{t}^{\frac{2N+1-i}{N+1}}} \prod_{i=1}^{N-1} \frac{1-\mathrm{t}^{\frac{i+1}{N+1}}}{1-\mathrm{t}^{\frac{1}{N+1}}}$$

$$\times \sum_{m_i} \prod_{\alpha>0} \left(\mathrm{t}^{\frac{2}{N+1}}\right)^{-\frac{1}{2}(\llbracket\alpha(m)\rrbracket-\frac{1}{k}\llbracket\alpha(m)\rrbracket^2)} \prod_{i=1}^{N} \left(\mathrm{t}^{\frac{2N}{N+1}}\right)^{\frac{1}{2}(\llbracket m_i+n\rrbracket-\frac{1}{k}\llbracket m_i+n\rrbracket^2)} \tag{4.73}$$

$$\times \frac{1}{|\mathcal{W}_m|} \oint [d\mathbf{z}] \prod_{\llbracket\alpha(m)\rrbracket=0} \frac{1-z_i/z_j}{1-\mathrm{t}^{\frac{1}{N+1}}z_i/z_j}\,,$$

where we have introduced $U(1)$ flavor holonomy n and the integral is taken over the (unbroken subgroup of) $SU(N)$. Specifically, suppose the gauge holonomy breaks the gauge group $SU(N)$ as

$$SU(N) \to SU(N_1)\times SU(N_2)\times \dots SU(N_l)\times U(1)^r\,, \tag{4.74}$$

where $N-1 = (N_1-1)+(N_2-1)+\cdots+(N_l-1)+r$ then we have

$$\frac{1}{|\mathcal{W}_m|} \oint [d\mathbf{z}] \prod_{\llbracket\alpha(m)\rrbracket=0} \frac{1-z_i/z_j}{1-\mathrm{t}^{\frac{1}{N+1}}z_i/z_j} = \prod_{i=1}^{l}\prod_{j=1}^{N_l-1} \frac{1-\mathrm{t}^{\frac{1}{N+1}}}{1-\mathrm{t}^{\frac{j+1}{N+1}}}. \tag{4.75}$$

To derive the general formula, we assume the $U(1)$ flavor holonomy n is an integer. In fact, we will see in Section 4.4 that n is allowed to take value in $\mathbb{Z}/N$. In fact, n is the quantization of the monodromy around irregular puncture. Its allowed values differ from λ in (4.49) since they are identified respectively in the UV and IR. Their relation is $\lambda = [\![Nn]\!] = 2k\alpha$. The index takes the following form:

$$\mathrm{t}^{\frac{1}{N+1}\left([\![Nn]\!]-\frac{1}{k}[\![Nn]\!]^2\right)}(1+\ldots), \tag{4.76}$$

where the ellipsis stands for terms with only *positive* powers of t.

(A_1, D_{2N+1}) **theories.** We have

$$\begin{aligned}\mathcal{I}_{(A_1,D_{2N+1})} =& \prod_{j=1}^{N} \frac{1}{1-\mathrm{t}^{\frac{4N+2-2j}{2N+1}}} \prod_{j=1}^{N} \frac{1-\mathrm{t}^{\frac{2j}{2N+1}}}{1-\mathrm{t}^{\frac{1}{2N+1}}} \\ &\times \sum_{m_i} \prod_{\alpha>0} \left(\mathrm{t}^{\frac{2}{2N+1}}\right)^{-\frac{[\![\alpha(m)]\!](k-[\![\alpha(m)]\!])}{2k}} \prod_i \left(\mathrm{t}^2\right)^{\frac{[\![m_i]\!](k-[\![m_i]\!])}{2k}} \left(\mathrm{t}^{\frac{1}{2N+1}}\right)^{\frac{[\![m_i\pm 2n]\!](k-[\![m_i\pm 2n]\!])}{2k}} \\ &\times \frac{1}{|\mathcal{W}_m|} \oint [d\mathbf{z}] \prod_{[\![\alpha(m)]\!]=0} \frac{1-\mathbf{z}^\alpha}{1-\mathrm{t}^{\frac{1}{2N+1}}\mathbf{z}^\alpha} ,\end{aligned} \tag{4.77}$$

where n is regarded as the holonomy for $SU(2)$ symmetry in the IR,[16] which is related to the quantized monodromy around the regular puncture at the south pole by $\lambda = [\![2n]\!] = 2k\alpha$. The constant γ here is $1/(2N+1)$. As in (A_1, A_{2N}) case, the integral is taken over the unbroken subgroup of $Sp(N)$. Note that here we allow n to a half-integer. This fact also plays an important role when we discuss TQFT structure in Appendix C.1. As before, the closed expression of the index contains a normalization factor

$$\mathrm{t}^{\frac{N}{2N+1}\left([\![2n]\!]-\frac{1}{k}[\![2n]\!]^2\right)}. \tag{4.78}$$

[16] The factor of 2 in front of n is due to the fact that the quarks q in the UV transform in the triplet **3** of $SU(2)$.

(A_1, D_{2N}) **theories.** Similarly, the index formula is

$$\begin{aligned}\mathcal{I}_{(A_1,D_{2N})} =& \prod_{j=N+1}^{2N-1} \frac{1}{1-\mathrm{t}^{\frac{j}{N}}} \prod_{j=1}^{N-1} \frac{1-\mathrm{t}^{\frac{j+1}{N}}}{1-\mathrm{t}^{\frac{1}{N}}} \\ &\times \sum_{m_i} \prod_{\alpha>0} \left(\mathrm{t}^{\frac{2}{N}}\right)^{-\frac{[\![\alpha(m)]\!](k-[\![\alpha(m)]\!])}{2k}} \prod_i \left(\mathrm{t}^{\frac{1}{N}}\right)^{\frac{[\![m_i+n_1+(2N-1)n_2]\!](k-[\![m_i+n_1+(2N-1)n_2]\!])}{2k}} \\ &\times \prod_i \left(\mathrm{t}^{\frac{2N-1}{N}}\right)^{\frac{[\![m_i+n_1-n_2]\!](k-[\![m_i+n_1-n_2]\!])}{2k}} \\ &\times \frac{1}{|\mathcal{W}_m|} \oint [d\mathbf{z}] \prod_{[\![\alpha(m)]\!]=0} \frac{1-z_i/z_j}{1-\mathrm{t}^{\frac{1}{N}} z_i/z_j} ,\end{aligned} \tag{4.79}$$

where we have introduced (n_1, n_2) to represent the $(U(1)_a, U(1)_b)$ flavor holonomy repsectively. The constant $\gamma = 1/N$, and the integral is over the (unbroken subgroup of) $SU(N)$. Its precise value is given in (4.75) by substituting $\mathrm{t}^{1/(N+1)}$ with $\mathrm{t}^{1/N}$. In (4.79) the computation was done assuming $n_{1,2} \in \mathbb{Z}$ so that the gauge holonomies m_i are all integers. However, the allowed set of values are in fact larger. We will return to this issue in Section 4.4. The relations to monodromies around wild and simple punctures are given by, respectively,

$$\lambda_1 = [\![Nn_1]\!] = 2k\alpha_1, \quad \lambda_2 = [\![2Nn_2]\!] = 2k\alpha_2. \tag{4.80}$$

Again, the evaluation of (4.79) gives a normalization factor

$$(\mathrm{t})^{\frac{N-1}{2N}\left([\![2Nn_2]\!]-\frac{1}{k}[\![2Nn_2]\!]^2\right)+\frac{1}{2N}\left([\![Nn_1+Nn_2]\!]-\frac{1}{k}[\![Nn_1+Nn_2]\!]^2\right)+\frac{1}{2N}\left([\![Nn_1-Nn_2]\!]-\frac{1}{k}[\![Nn_1-Nn_2]\!]^2\right)}. \tag{4.81}$$

4.4 Wild Hitchin characters

Now that we have the integral expressions for the Coulomb branch indices of Argyres-Douglas theories (4.72), (4.73), (4.77) and (4.79), we will evaluate them explicitly in this section.

Before presenting the results, we remark that the Coulomb indices have several highly non-trivial properties. Anticipating the equality between the index and wild Hitchin characters, we can often understand these properties from geometry.

1. **Positivity.** The Coulomb branch index as a series in t always has positive coefficients. This phenomenon is not obvious from the integral expression.

From the geometric side, this is a simple corollary of the "vanishing theorem" for the wild Hitchin moduli space

$$H^i(\mathcal{M}_H, \mathcal{L}^{\otimes k}) = 0 \quad \text{for } i > 0. \tag{4.82}$$

This further implies that, on the physics side, all Coulomb BPS states on $L(k,1)$ are bosonic. This positivity phenomenon is the analogue of those observed in [136] and [137] with wild ramifications.

2. **Splitting.** The indices always turn out to be rational functions. Further, they split as a sum over fixed points — a form predicted by the Atiyah-Bott localization formula from the geometry side (4.40). This will allow us to extract geometric data for moduli spaces directly. However, the interpretation of this decomposition is not clear at the level of the BPS Hilbert spaces $\mathcal{H}_{\text{Coulomb}}$. It is not even clear that the $\mathcal{H}_{\text{Coulomb}}$ can be decomposed in similar ways, as the individual contributions from some fixed points do not have positivity.

3. **Fractional dimensions.** One notable feature of Argyres-Douglas theories is the fractional scaling dimensions of their Coulomb branch operators. From the point of view of the Hitchin action, this comes from the fractional action on the z coordinate. For example, the $U(1)$ action on $\mathcal{M}_{2,2N+1}$ involves a rotation of the base curve $\mathbb{CP}^1$ with coordinate z by

$$\rho_\theta: \quad z \mapsto e^{-i\frac{2}{2N+3}\theta} z. \tag{4.83}$$

Therefore only the $(2N+3)$-fold cover of the $U(1)$ defines a (genuine non-projective) group action, and the Hitchin character will be a power series in $\mathsf{t}^{\frac{1}{2N+3}}$. In all four families of moduli spaces ($\mathcal{M}_{2,K}$ versus $\widetilde{\mathcal{M}}_{2,K}$; K either even or odd) $K+2$ is always the number of Stokes rays centered at the irregular singularity, and the Hitchin character will be a power series in $\mathsf{t}^{\frac{1}{K+2}}$. When K is even, one can check that the $(K+2)/2$-fold cover of the $U(1)$ given by ρ_θ defines a group action, and the Hitchin character will contain integral powers of $\mathsf{t}^{\frac{2}{K+2}}$ as a consequence.

We will start this section by giving formulae for the wild Hitchin characters in Section 4.4. In Section 4.4, the large-k limit of the wild Hitchin character is discussed. This limit effectively reduces the theory to three dimensions; by taking the mirror symmetry $\mathcal{M}_H$ is realized as the Higgs branch of a 3d $\mathcal{N}=4$ quiver gauge theory. This is in accordance with the mathematical work [138]. By comparing 3d index and 4d index, we will see how good this approximation is on the nilpotent cone. As a byproduct, we give a physical interpretation of the fixed points from the 3d mirror point of view.

In Appendix E, we will present mathematical calculations that directly confirm the physical prediction: the Coulomb branch index of Argyres-Douglas theory indeed computes the wild Hitchin character for $\mathcal{M}_H(\Sigma, PSL(2,\mathbb{C})) := {}^L\mathcal{M}_H$.

As we have explained — and we will soon offer another explanation from the physics perspective — the Hitchin character is not sensitive to the difference between $\mathcal{M}_H(\Sigma, SL(2,\mathbb{C}))$ and $\mathcal{M}_H(\Sigma, PSL(2,\mathbb{C}))$ when Σ is a sphere with at most two punctures. In fact, one can directly check that the fixed points are exactly the same with identical ambient geometry. As a consequence, the Hitchin character for $\mathcal{M}_H(\Sigma, SL(2,\mathbb{C}))$ can be obtained via "analytic continuation" of λ, λ_1 and λ_2 by allowing them to take odd values. So we will not emphasize the difference between $\mathcal{M}_H$ and ${}^L\mathcal{M}_H$ in this section, unless specified.

The wild Hitchin character as a fixed-point sum

The moduli space $\mathcal{M}_{2,2N+1}$. A nice illustrative example to start is the (A_1, A_2) theory with no flavor symmetry at all. The Coulomb branch index is

$$\mathcal{I}_{(A_1,A_2)} = \frac{1}{(1-\mathrm{t}^{\frac{2}{5}})(1-\mathrm{t}^{\frac{3}{5}})} + \frac{\mathrm{t}^{\frac{k}{5}}}{(1-\mathrm{t}^{\frac{6}{5}})(1-\mathrm{t}^{-\frac{1}{5}})}. \tag{4.84}$$

On the other hand, the moduli space $\mathcal{M}_{2,3}$ has two complex dimensions, and we have the fixed points and the associated eigenvalues of the circle action on normal bundles obtained in Appendix E:

$$\varphi_0^* = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} dz, \quad \varphi_1^* = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix} dz, \tag{4.85}$$

with moment maps $\mu = 1/40$ and $9/40$ respectively. After shifting the two moment maps simultaneously by $1/40$,[17] we get $\mu_1 = 0$ and $\mu_2 = 1/5$. These are precisely the power entering the numerator of each term in (4.84)! Furthermore, from the denominator of each term, we are able to read off the weights of the circle action

[17] We normalize the Hitchin character such that the $\mathrm{t} = 0$ limit gives 1. The ambiguity of multiplying a monomial $\mathrm{t}^{\Delta\mu}$ to the Hitchin character corresponds to redefining the $U(1)$ action such that it rotates the fiber of the line bundle $\mathcal{L}$ as well.

on the two-dimensional normal bundle of each fixed points — they are respectively $(2/5, 3/5)$ and $(6/5, -1/5)$. This is directly checked in Appendix E from geometry, providing strong evidence for our proposal (4.1). Also, notice the ubiquity of number "5" — the number of Stokes rays associated with the irregular singularity.

The formula (4.84) encodes various interesting information about the geometry and topology of the moduli space. As in the tame case, the moment map (which agrees with (4.32) at fixed points) is expected to be a perfect Morse function on $\mathcal{M}_H$. The fixed points are critical points of μ, and the positive- (negative-)eigenvalue subspaces of the normal bundle correspond to the upward (downward) Morse flows. In particular, we know that the top fixed point in $\mathcal{M}_{2,3}$ has Morse index 2 and the downward flow from it coincides with the nilpotent cone — the singular fiber of the Hitchin fibration with Kodaira type II [139]. Then the Poincaré polynomial of $\mathcal{M}_{2,3}$ is

$$\mathcal{P}(\mathcal{M}_{2,3}) = 1 + \mathrm{r}^2. \tag{4.86}$$

Another important quantity is the equivariant volume of $\mathcal{M}_{2,3}$ as given in (4.46)

$$\mathrm{Vol}_\beta(\mathcal{M}_{2,3}) = \frac{25}{6\beta^2}(1 - e^{-\frac{1}{5}\beta}). \tag{4.87}$$

Note that as $\beta \to +\infty$, the volume scale as β^{-2}, with the negative power of β being the complex dimension of $\mathcal{M}_{2,3}$. This is unlike the tame situation, where β scales according to half the dimension of $\mathcal{M}_H$. Intuitively, this is because, while Higgs field is responsible for half of the dimensions of $\mathcal{M}_H$ in tame case, they are responsible for all dimensions in the wild Hitchin moduli space, as a G-bundle has no moduli over Σ in the cases that we consider.

We now give a general formula of the wild Hitchin character for $\mathcal{M}_{2,2N+1}$, predicted by the Coulomb index and proved in Appendix E. There are $N+1$ fixed points in the moduli space $P_0, P_1, \ldots, P_N$. They have moment maps given by

$$\mu_i = \frac{i(i+1)}{2(2N+3)}, \quad i = 0, 1, 2, \ldots, N, \tag{4.88}$$

where we have already shifted a universal constant so that P_0 as moment map 0. The weights are given in (E.22), and the wild Hitchin character reads

$$\mathcal{I}(\mathcal{M}_{2,2N+1}) = \sum_{i=0}^{N} \frac{\mathfrak{t}^{\frac{i(i+1)}{2(2N+3)}k}}{\prod_{l=1}^{i}\left(1 - \mathfrak{t}^{\frac{2(N+l+1)}{2N+3}}\right)\left(1 - \mathfrak{t}^{-\frac{2l-1}{2N+3}}\right)\prod_{l=i+1}^{N}\left(1 - \mathfrak{t}^{\frac{2l+1}{2N+3}}\right)\left(1 - \mathfrak{t}^{\frac{2(N-l+1)}{2N+3}}\right)}. \tag{4.89}$$

The Morse index of P_i is $2i$, so the Poincaré polynomial of $\mathcal{M}_{2,2N+1}$ is

$$\mathcal{P}(\mathcal{M}_{2,2N+1}) = 1 + \mathfrak{r}^2 + \mathfrak{r}^4 + \cdots + \mathfrak{r}^{2N} = \frac{1-\mathfrak{r}^{2N+2}}{1-\mathfrak{r}^2}. \tag{4.90}$$

The moduli space $\widetilde{\mathcal{M}}_{2,2N-1}$. A closely related moduli space is $\widetilde{\mathcal{M}}_{2,2N-1}$, which has regular puncture at the south pole of Σ in addition to the irregular puncture $I_{2,2N-1}$ at the north pole. Then the gauge connection has monodromy $A \sim \alpha d\theta$ around the regular puncture, and $\lambda = 2k\alpha = \{0, 1, \ldots, k\}$ is quantized and are integrable weights of $\widehat{\mathfrak{su}}(2)_k$.[18] Again we will absorb the normalization constant (4.78) appearing in the superconformal index so that the index as a series in $\mathfrak{t}$ will start with 1.

Next we present the wild Hitchin character for the moduli space $\widetilde{\mathcal{M}}_{2,2N-1}$. We begin with the example $\widetilde{\mathcal{M}}_{2,1}$, or Argyres-Douglas theory of type (A_1, D_3). Denote $\lambda := 2k\alpha = 2n$ valued in $\{0, 1, \ldots, k\}$. Then, we have

$$\mathcal{I}_{(A_1,D_3)} = \frac{1}{(1-\mathfrak{t}^{\frac{1}{3}})(1-\mathfrak{t}^{\frac{2}{3}})} + \frac{\mathfrak{t}^{\frac{\lambda}{3}} + \mathfrak{t}^{\frac{k-\lambda}{3}}}{(1-\mathfrak{t}^{-\frac{1}{3}})(1-\mathfrak{t}^{\frac{4}{3}})}. \tag{4.91}$$

This formula tells us that $\widetilde{\mathcal{M}}_{2,1}$ has three fixed points under the Hitchin action. One of them has the lowest moment map 0 with weights on the normal bundle $(1/2, 2/3)$, while the other two have moment maps $\mu_1^{(1)} = 2\alpha/3$ and $\mu_1^{(2)} = (1-2\alpha)/3$. These results are also confirmed by mathematical calculations in Appendix E. Using Morse theory, we get the Poincaré polynomial of $\widetilde{\mathcal{M}}_{2,1}$

$$\mathcal{P}(\widetilde{\mathcal{M}}_{2,1}) = 1 + 2\mathfrak{r}^2. \tag{4.92}$$

---

[18] λ starts life as a weight of $SO(3)$, since the physical set-up computes the Hitchin character of ${}^L\widetilde{\mathcal{M}}_{SU(2)} = \widetilde{\mathcal{M}}_{SO(3)}$ according to (4.1). As we have explained, from the geometric side, the difference between $\widetilde{\mathcal{M}}_{SU(2)}$ and $\widetilde{\mathcal{M}}_{SO(3)}$ is almost negligible for the purpose of studying wild Hitchin characters — one only needs to analytically continuate λ to go from one moduli space to another. This phenomenon has a counterpart in the index computation as well. Being an $SU(2)$ flavor holonomy, a natural set of values for λ without violating charge quantization condition is $0, 2, \ldots, 2\lfloor k/2 \rfloor$ [16]. However, in the expression (4.77), there is no problem with simply allowing $\lambda = 2n$ to take odd values. This can be understood from the perspective of the $\mathcal{N} = 1$ Lagrangian theories listed in Table 4.4. There all the matter contents are assembled either in the trivial or the vector representation of the global $SO(3)$ symmetry, and these two representations cannot distinguish $SU(2)$ from $SO(3)$; as a consequence if we expand the full superconformal index and look at the BPS spectrum of Argyres-Douglas theory, only representations for $SO(3)$ will appear. This means odd λ does not violate the charge quantization condition, and can be allowed. Furthermore, since the superconformal index of (A_1, A_{2N}) can be obtained from (A_1, D_{2N+3}) by closing the regular puncture through (C.8), one immediately concludes that the Hitchin characters for $\mathcal{M}_{2,2N+1}$ and for the Langlands dual ${}^L\mathcal{M}_{2,2N+1}$ are exactly the same.

And the equivariant volume is given by

$$\mathrm{Vol}_\beta(\widetilde{\mathcal{M}}_{2,1}) = \frac{9}{4\beta^2}(2 - e^{-\frac{2\alpha}{3}\beta} - e^{-\frac{1-2\alpha}{3}\beta}). \tag{4.93}$$

As $\widetilde{\mathcal{M}}_{2,1}$ has hyper-Kähler dimension one, it is an elliptic surface in complex structure I. The only singular fiber is the nilpotent cone with Kodaira type III [139] ($i.e.$ labeled by the affine A_1 Dynkin diagram, see Figure 4.1). It consists of two $\mathbb{C}\mathbf{P}^1$ with the intersection matrix given by

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}. \tag{4.94}$$

The null vector of the intersection matrix should be identified with the homology class of the Hitchin fiber,

$$[\mathbf{F}] = 2\,[D_1] + 2\,[D_2]\,. \tag{4.95}$$

This relation translates into (see [69] and Chapter 3 [16] for review of this relation

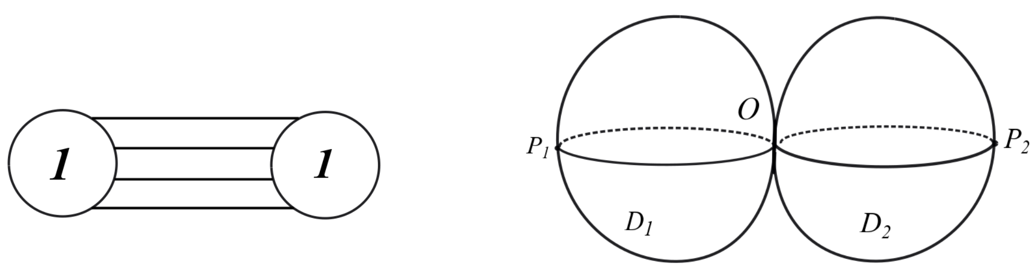


Figure 4.1: Left: the affine A_1 Dynkin diagram. Right: the nilpotent cone of Hitchin fibration for $\widetilde{\mathcal{M}}_{2,1}$, consisting of two $\mathbb{C}\mathbf{P}^1$ intersecting at O with intersection number 2. Together with P_1, P_2, they comprise the three fixed points of the Hitchin moduli space $\widetilde{\mathcal{M}}_{2,1}$.

as well as examples with tame ramifications)

$$\mathrm{Vol}(\mathbf{F}) = 2\mathrm{Vol}(D_1) + 2\mathrm{Vol}(D_2), \tag{4.96}$$

which is indeed visible from the Hitchin character. It is not hard to see that for each $\mathbb{C}\mathbf{P}^1$, the volumes are $\mathrm{Vol}(D_1) = 3\mu_1^{(1)} = 2\alpha$ and $\mathrm{Vol}(D_2) = 3\mu_1^{(2)} = 1 - 2\alpha$

respectively. (The factor "3" is due to the weights $-1/3$ that corresponds to the downward Morse flow.) Consequently, we see (4.96) is exactly true, with $\mathrm{Vol}(\mathbf{F}) = 2$ in our normalization.

We now give a general statement for the wild moduli space $\widetilde{\mathcal{M}}_{2,2N-1}$. There are $2N+1$ fixed points, divided into $N+1$ groups. We label them as $P_i^{(1,2)}$, $i = 0, 1, \ldots, N$. The i-th group contains two fixed points for $i > 0$ and one fixed points for $i = 0$. The $U(1)$ weights on the $2N$-dimensional normal bundle to P_i is given by

$$\begin{aligned} \epsilon_l &= -\frac{2l-1}{2N+1}, \quad \tilde{\epsilon}_l = \frac{2N+2l}{2N+1}, \quad l = 1, 2, \ldots, i \\ \epsilon_l &= \frac{2l-1}{2N+1}, \quad \tilde{\epsilon}_l = \frac{2N+2-2l}{2N+1}, \quad l = i+1, i+2, \ldots N. \end{aligned} \tag{4.97}$$

The normal bundle can be decomposed into the tangent space to the nilpotent cone plus its orthogonal complement, and ϵ_l and $\tilde{\epsilon}_l$ correspond respectively to the former and the latter.

For the 0-th fixed point the moment map is 0, while for the i-th group with $i > 0$, the two moment map values are

$$\mu_i^{(1)} = \frac{i(i+1)}{2(2N+1)} - \frac{i}{2N+1}(2\alpha), \quad \mu_i^{(2)} = \frac{(i-1)i}{2(2N+1)} + \frac{i}{2N+1}(2\alpha) \tag{4.98}$$

where α is again the monodromy around the simple puncture. Then the wild Hitchin character is

$$\begin{aligned} \mathcal{I}\left(\widetilde{\mathcal{M}}_{2,2N-1}\right) &= \frac{1}{\prod_{l=1}^{N}\left(1-\mathrm{t}^{\frac{2l-1}{2N+1}}\right)\left(1-\mathrm{t}^{\frac{2N+2-2l}{2N+1}}\right)} \\ &+ \sum_{i=1}^{N} \frac{\mathrm{t}^{k\mu_i^{(1)}} + \mathrm{t}^{k\mu_i^{(2)}}}{\prod_{l=1}^{i}\left(1-\mathrm{t}^{\frac{2N+2l}{2N+1}}\right)\left(1-\mathrm{t}^{-\frac{2l-1}{2N+1}}\right)\prod_{l=i+1}^{N}\left(1-\mathrm{t}^{\frac{2l-1}{2N+1}}\right)\left(1-\mathrm{t}^{\frac{2N+2-2l}{2N+1}}\right)}, \end{aligned} \tag{4.99}$$

which precisely agrees with the mathematical calculation in Appendix E. The Morse index of P_i is again $2i$, giving the Poincaré polynomial of the moduli space

$$\mathcal{P}(\widetilde{\mathcal{M}}_{2,2N-1}) = 1 + 2\mathrm{r}^2 + 2\mathrm{r}^4 + \ldots 2\mathrm{r}^{2N}. \tag{4.100}$$

The moduli space $\mathcal{M}_{2,2N}$. Compared to its cousin $\mathcal{M}_{2,2N+1}$, the moduli space $\mathcal{M}_{2,2N}$ depends on an additional parameter α giving the formal monodromy of the gauge field around the irregular singularity, again subject to the quantization condition $2k\alpha = 0, 1, \ldots, k$. On the physics side, it is identified with the holonomy of the $U(1)_B$ flavor symmetry of the (A_1, A_{2N-1}) theory.

From this point forward, the level of difficulty in finding fixed points via geometry increases significantly; on the contrary, the physical computation is still tractable, yielding many predictions for the moduli space.

When $N = 1$ the physical theory is a single hypermultiplet, and the index is just a multiplicative factor (4.76). When $N = 2$ the moduli space is isomorphic to $\widetilde{\mathcal{M}}_{2,1}$; and two Argyes-Douglas theories (A_1, A_3) and (A_1, D_3) are identical [18]. Hence in this section we begin with the next simplest example $\mathcal{M}_{2,6}$. After absorbing the normalization constant (4.76) similar to previous examples, we arrive at the expression

$$\mathcal{I}_{(A_1,A_5)} = \frac{\mathrm{t}^{\frac{k-\lambda}{2}} + \mathrm{t}^{\frac{\lambda}{2}} + \mathrm{t}^{\frac{k}{2}}}{(1-\mathrm{t}^{\frac{6}{4}})(1-\mathrm{t}^{\frac{5}{4}})(1-\mathrm{t}^{-\frac{2}{4}})(1-\mathrm{t}^{-\frac{1}{4}})} + \frac{\mathrm{t}^{\frac{k-\lambda}{4}} + \mathrm{t}^{\frac{\lambda}{4}}}{(1-\mathrm{t}^{\frac{3}{4}})(1-\mathrm{t}^{\frac{5}{4}})(1-\mathrm{t}^{\frac{1}{4}})(1-\mathrm{t}^{-\frac{1}{4}})}$$
$$+ \frac{1}{(1-\mathrm{t}^{\frac{3}{4}})(1-\mathrm{t}^{\frac{2}{4}})(1-\mathrm{t}^{\frac{2}{4}})(1-\mathrm{t}^{\frac{1}{4}})}. \tag{4.101}$$

The index formula predicts that there are six fixed points under the Hitchin action, with their weights on the normal bundle manifest in the denominators. The Poincaré polynomial is then

$$\mathcal{P}(\mathcal{M}_{2,6}) = 1 + 2\mathrm{r}^2 + 3\mathrm{r}^4. \tag{4.102}$$

And the equivariant volume is

$$\mathrm{Vol}_\beta(\mathcal{M}_{2,6}) = \frac{64}{15\beta^4}\Big(e^{-\frac{1-2\alpha}{2}\beta} + e^{-\frac{2\alpha}{2}\beta} + e^{-\frac{1}{2}\beta} - 4e^{-\frac{1-2\alpha}{4}\beta} - 4e^{-\frac{2\alpha}{4}\beta} + 5\Big). \tag{4.103}$$

We now write down the general formula for the Hitchin character of $\mathcal{M}_{2,2N}$. The moduli space has N groups of fixed points. We label the group by $i = 0, 1, \ldots, N-1$ with increasing Morse indices. The i-th group contains $i+1$ isolated fixed points $P_i^{(j)}$ with $j = 0, 1, \ldots, i$. The weights on the normal bundle for each group are as follows:

$$\begin{aligned} \epsilon_l &= \frac{N+1+l}{N+1}, \quad \tilde{\epsilon}_l = -\frac{l}{N+1}, \quad l = 1, 2, \ldots, i \\ \epsilon_l &= \frac{N-l}{N+1}, \quad \tilde{\epsilon}_l = \frac{l+1}{N+1}, \quad l = i+1, i+2, \ldots, N-1. \end{aligned} \tag{4.104}$$

Within the group the moment maps are organized in a specific pattern:

$$\begin{aligned} j \text{ odd:} \quad \mu_i^{(j)} &= \frac{(2i-j+1)(j+1)}{4(N+1)} - \frac{i-j+1}{N+1}(2\alpha) \\ j \text{ even:} \quad \mu_i^{(j)} &= \frac{(2i-j+2)j}{4(N+1)} + \frac{i-j}{N+1}(2\alpha). \end{aligned} \tag{4.105}$$

Then the wild Hitchin character is

$$\mathcal{I}(\mathcal{M}_{2,2N}) = \sum_{i=0}^{N-1} \frac{\sum_{j=0}^{i} \mathrm{t}^{k\mu_i^{(j)}}}{\prod_{l=1}^{i}\left(1-\mathrm{t}^{\frac{N+1+l}{N+1}}\right)\left(1-\mathrm{t}^{-\frac{l}{N+1}}\right)\prod_{l=i+1}^{N-1}\left(1-\mathrm{t}^{\frac{N-l}{N+1}}\right)\left(1-\mathrm{t}^{\frac{l+1}{N+1}}\right)} \tag{4.106}$$

and from it we can write down immediately the Poincaré polynomial

$$\mathcal{P}(\mathcal{M}_{2,2N}) = 1 + 2\mathrm{r}^2 + 3\mathrm{r}^4 + 4\mathrm{r}^6 + \cdots + N\mathrm{r}^{2(N-1)}. \tag{4.107}$$

In the large-k limit, some of the moment maps $\mu_i^{(j)}$ in the numerator of (4.106) will stay at $O(1)$ and become large after multiplied by k, even when $\lambda = 2k\alpha$ is fixed, and the contribution from the corresponding fixed points will be exponentially suppressed. We see that for each group in (4.105) only one fixed point survives, namely the one with $j = 0$. These fixed points are the only ones visible in the three-dimensional reduction of Argyres-Douglas theories. We will revisit this problem in Section 4.4.

The moduli space $\widetilde{\mathcal{M}}_{2,2N-2}$. We now turn to the last of the four families of wild Hitchin moduli spaces, $\widetilde{\mathcal{M}}_{2,2N-2}$, which is arguably also the most complicated. It is the moduli space associated with Riemann sphere with one irregular singularity $I_{2,2N-2}$ and one regular singularity, with monodromy parameters α_1 and α_2. The corresponding Argyres-Douglas theory (A_1, D_{2N}) generically has $U(1) \times SU(2)$ flavor symmetry, and $\lambda_1 = 2k\alpha_1$ and $\lambda_2 = 2k\alpha_2$ in (4.80) label their holonomies along the Hopf fiber of $L(k, 1)$.

Let us again start from the simplest example: $\widetilde{\mathcal{M}}_{2,2}$ or (A_1, D_4) Argyres-Douglas theory. The hyper-Kähler dimension of this moduli space is again one; we thus expect to understand the geometric picture more concretely. Modulo the normalization constant, (4.81), we have

$$\mathcal{I}_{(A_1,D_4)} = \frac{\mathrm{t}^{k\mu_1^{(0)}} + \mathrm{t}^{k\mu_1^{(1)}} + \mathrm{t}^{k\mu_1^{(2)}}}{(1-\mathrm{t}^{\frac{3}{2}})(1-\mathrm{t}^{-\frac{1}{2}})} + \frac{1}{(1-\mathrm{t}^{\frac{1}{2}})(1-\mathrm{t}^{\frac{1}{2}})}. \tag{4.108}$$

The moment map values are

$$\begin{aligned}
\mu_1^{(0)} &= \frac{1}{2} - \frac{1}{2k} \max\left([\![\lambda_1 + \frac{\lambda_2}{2}]\!], \lambda_2 \right) \\
\mu_1^{(1)} &= \frac{1}{2k} \min\left([\![\lambda_1 + \frac{\lambda_2}{2}]\!], \lambda_2 \right) \\
\mu_1^{(2)} &= \frac{1}{2k} \max\left([\![\lambda_1 + \frac{\lambda_2}{2}]\!], \lambda_2 \right) - \frac{1}{2k} \min\left([\![\lambda_1 + \frac{\lambda_2}{2}]\!], \lambda_2 \right).
\end{aligned} \tag{4.109}$$

Here, when $(\lambda_1 + \lambda_2/2) \notin \mathbb{Z}$, the character formula (4.108) shall be set to zero.

From the wild Hitchin character (4.108), we know the Poincaré polynomial is

$$\mathcal{P}(\widetilde{\mathcal{M}}_{2,2}) = 1 + 3\mathrm{r}^2. \tag{4.110}$$

$\widetilde{\mathcal{M}}_{2,2}$ is another elliptic surface, and the nilpotent cone is of Kodaira type IV [139], labeled by the affine A_2 Dynkin diagram. It contains three $\mathbb{C}\mathbf{P}^1$'s, which we denote as $D_{1,2,3}$, and the intersection matrix is given by

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \tag{4.111}$$

$D_{1,2,3}$ each contains one of the three fixed points with Morse index 2, see Figure 4.2 for illustration. The null vector of the intersection matrix gives the homology class of the Hitchin fiber,

$$[\mathbf{F}] = 2\,[D_1] + 2\,[D_2] + 2\,[D_3], \tag{4.112}$$

which can be translated into a relation about the volumes

$$\mathrm{Vol}(\mathbf{F}) = 2\mathrm{Vol}(D_1) + 2\mathrm{Vol}(D_2) + 2\mathrm{Vol}(D_3). \tag{4.113}$$

Indeed, the three moment map values (4.109) satisfy

$$4\mu_1^{(0)} + 4\mu_1^{(1)} + 4\mu_1^{(2)} = 2 = \mathrm{Vol}(\mathbf{F}). \tag{4.114}$$

We now write down the general wild Hitchin character for the moduli space $\widetilde{\mathcal{M}}_{2,2N-2}$. There are N groups of fixed points, we label them as $i = 0, 1, \ldots, N-1$. The i-th group contains $2i + 1$ fixed points with Morse index i. The expression looks like

$$\mathcal{I}(\widetilde{\mathcal{M}}_{2,2N-2}) = \sum_{i=0}^{N-1} \frac{\sum_{j=0}^{2i} \mathrm{t}^{k\mu_i^{(j)}}}{\prod_{l=1}^{i}\left(1 - \mathrm{t}^{\frac{l+N}{N}}\right)\left(1 - \mathrm{t}^{-\frac{l}{N}}\right) \prod_{l=i+1}^{N-1}\left(1 - \mathrm{t}^{\frac{l}{N}}\right)\left(1 - \mathrm{t}^{\frac{N-l}{N}}\right)}. \tag{4.115}$$

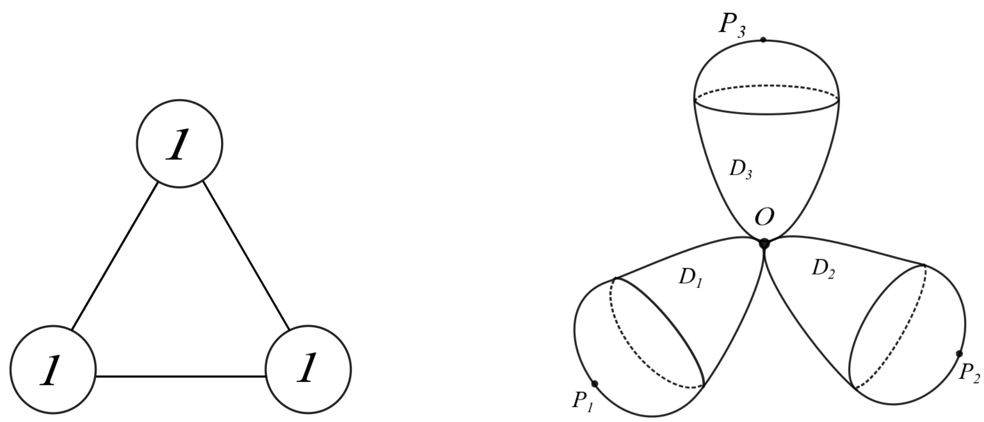


Figure 4.2: Left: the affine A_2 Dynkin diagram, with Dynkin label indicated at each node. Right: the nilpotent cone of singular fibration, consisting of three $\mathbb{C}\mathbf{P}^1$ intersecting at O. The spheres are distorted a little to accommodate the common intersection. Together with P_1, P_2 and P_3, they comprise the four fixed points of the Hitchin moduli space $\widetilde{\mathcal{M}}_{2,2}$.

Explicit formulae for the moment map $\mu_i^{(j)}$ when λ_1 and λ_2 are zero are given after (4.156). In general, they are functions of $[\![\lambda_1 + \lambda_2/2]\!]$ and λ_2, with the quantization condition of $(\lambda_1 + \lambda_2/2)$ being an integer. Moreover, for the i-th group of fixed points, the sum of the moment map values,

$$\sum_{j=0}^{2i} \mu_i^{(j)} = \frac{1}{6N} i(i+1)(2i+1), \tag{4.116}$$

is independent of the monodromy parameters.

We can similarly obtain the Poincaré polynomial for this moduli space,

$$\mathcal{P}(\widetilde{\mathcal{M}}_{2,2N-2}) = 1 + 3\mathfrak{r}^2 + 5\mathfrak{r}^4 + \cdots + (2N-1)\mathfrak{r}^{2N-2}. \tag{4.117}$$

Fixed points from the three-dimensional mirror theory

One interesting limit of the superconformal index on $S^1 \times L(k,1)$ is the large-k limit, where the Hopf fiber shrinks and the spacetime geometry effectively becomes $S^1 \times S^2$. In this limit, the 4d $\mathcal{N}=2$ theory becomes a three-dimensional $\mathcal{N}=4$ theory $T_{3\mathrm{d}}[\Sigma, G]$. Its 3d mirror $T_{3\mathrm{d}}^{\mathrm{mir.}}[\Sigma, G]$ sometimes admits a Lagrangian description [140, 141]. The original Coulomb branch vacua of $T_{3\mathrm{d}}[\Sigma, G]$ becomes the Higgs branch vacua in the mirror frame. What is the relation between the Hitchin moduli space $\mathcal{M}_H$ and the Coulomb branch $\mathcal{M}^*$ of $T_{3\mathrm{d}}[\Sigma, G]$? Intuitively, we expect that the latter is an "approximation" of the former because some degrees of freedom become massive and integrated out. More precisely, under the RG flow to the IR, we zoom in onto a small neighborhood of the origin of the Coulomb branch. As a consequence, the Coulomb branch $\mathcal{M}^*$ of $T_{3\mathrm{d}}[\Sigma]$ is a linearized version of $\mathcal{M}_H$, given by a finite-dimensional hyper-Kähler quotient of vector spaces — in other word, $\mathcal{M}^*$ is a quiver variety consisting of holomorphically trivial $G_{\mathbb{C}}$-bundle over Σ.

This precisely agrees with the discovery of [138]: there it was proved mathematically that the wild Hitchin moduli space $\mathcal{M}_H$ contains the quiver variety $\mathcal{M}^*$ as an open dense subset, parametrizing irregular connections on a trivial bundle on $\mathbb{C}\mathbf{P}^1$. Furthermore, $\mathcal{M}^*$ contains a subset of the $U(1)$ fixed points in $\mathcal{M}_H$. These fixed points can be identified with massive vacua of $T_{3\mathrm{d}}^{\mathrm{mir.}}[\Sigma, G]$ on the Higgs branch, giving much easier access to them compared with the rest.[19] To recap, we have the following relations:

$$\boxed{\begin{array}{rl} \text{Hitchin moduli space } \mathcal{M}_H \rightsquigarrow & \text{quiver variety } \mathcal{M}^* \\[1em] \text{Coulomb branch of } T[\Sigma] \text{ on } S^1 \rightsquigarrow & \text{Higgs branch of } T_{3\mathrm{d}}^{\mathrm{mir.}}[\Sigma] \\[1em] \text{"lowest" fixed points on } \mathcal{M}_H \rightsquigarrow & \text{massive Higgs branch vacua} \end{array}}\,. \qquad (4.118)$$

These relations also suggest that there is a relation between the Hitchin character and the Higgs branch index of $T_{3\mathrm{d}}^{\mathrm{mir.}}[\Sigma]$, as we will show below. Recall that the 3d $\mathcal{N}=4$ index is given by [142]

$$\mathcal{I}_{\mathcal{N}=4}^{3\mathrm{d}} = \mathrm{Tr}_{\mathcal{H}}(-1)^F \mathfrak{q}^{j_2+\frac{1}{2}(R_H+R_C)} \mathfrak{v}^{R_H-R_C} e^{-2\beta(\widetilde{E}-R_H-R_C-j_2)}, \qquad (4.119)$$

where j_2 is the angular momentum with respect to the Cartan of the $SO(3)$ Lorentz group and $R_{C,H}$ are respectively the Cartans of $SU(2)_C \times SU(2)_H$ R-symmetry. There are two interesting limits:

$$\begin{aligned} &\text{Coulomb limit}: \quad \mathfrak{q}, \mathfrak{v} \to 0, \quad \frac{\mathfrak{q}^{\frac{1}{2}}}{\mathfrak{v}} = \mathfrak{t} \text{ fixed}, \\ &\text{Higgs limit}: \quad \mathfrak{q}, \mathfrak{v}^{-1} \to 0, \quad \mathfrak{q}^{\frac{1}{2}}\mathfrak{v} = \mathfrak{t}' \text{ fixed}. \end{aligned} \qquad (4.120)$$

As we will work with $T_{3\mathrm{d}}^{\mathrm{mir.}}[\Sigma]$ in the mirror frame, the Higgs branch limit is that one that interests us.

[19] Note that no analogue exists in four dimensions, simply because Coulomb branch cannot be lifted without breaking supersymmetry.

3d mirror of (A_1, A_{2N-1}) **theory.** To begin with, let us first turn to (A_1, A_{2N-1}) theory whose three-dimensional mirror is $\mathcal{N} = 4$ SQED with N fundamental hypermutiplets. The Higgs branch has an $SU(N)$ flavor symmetry while the Coulomb branch has $U(1)_J$ topological symmetry that can be identified with the flavor symmetry of the initial (A_1, A_{2N-1}) theory. Let (z_i, m_i) be the fugacities and monopole numbers for the $SU(N)$ flavor symmetry and let (b, n) be the fugacity and monopole number for the $U(1)_J$ topological symmetry. The fugacities z_i are subject to the constraint $\prod_i z_i = 1$, while m_i will all be zero. The Higgs branch index is given by

$$\begin{aligned}\mathcal{I}_H^{3\mathrm{d}} &= (1-\mathrm{t}')\prod_{i=1}^{N}\delta_{m_i,0}\oint\frac{dw}{2\pi i w}w^{Nn}\prod_{i=1}^{N}\frac{1}{(1-\mathrm{t}'^{\frac{1}{2}}wz_i)(1-\mathrm{t}'^{\frac{1}{2}}w^{-1}z_i^{-1})}\\ &= \left(\prod_{i=1}^{N}\delta_{m_i,0}\right)\sum_{i=1}^{N}\mathrm{t}'^{\frac{|Nn|}{2}}z_i^{-|Nn|}\prod_{j\neq i}\frac{1}{1-\mathrm{t}'z_j/z_i}\frac{1}{1-z_i/z_j}.\end{aligned} \tag{4.121}$$

To recover the $k \to +\infty$ limit of the (A_1, A_{2N-1}) Coulomb branch index (4.106), we make the following substitution:

$$z_i \to \mathrm{t}'^{(N+1-2i)/(2N+2)}, \quad i = 1, 2, \ldots, N. \tag{4.122}$$

This substitution (4.122) can be interpreted as the mixing between topological symmetry and $SU(2)_C$ symmetry on the Coulomb branch of $T_{3\mathrm{d}}[\Sigma]$, which is further examined in Appendix C.2. After the substitution, the index can be written as

$$\mathcal{I}_H^{3\mathrm{d}} = \mathrm{t}'^{\frac{1}{N+1}|Nn|}\sum_{i=1}^{N}\frac{\mathrm{t}'^{\frac{i-1}{N+1}|Nn|}}{\prod_{j\neq i}\left(1-\mathrm{t}'^{\frac{N+1+i-j}{N+1}}\right)\left(1-\mathrm{t}'^{\frac{j-i}{N+1}}\right)}, \tag{4.123}$$

where each term in the summation is the residue at a massive vacuum. Comparing to the Hitchin character (4.106), one finds that only a subset of fixed points in $\mathcal{M}_H$ contribute to $\mathcal{I}_H^{3\mathrm{d}}$. Namely, these are fixed points that live in $\mathcal{M}^* \subset \mathcal{M}_H$.

For pedagogy, we describe these massive supersymmetric vacua explicitly. Our description is again in the mirror frame and one can easily interpret them in the original frame. First we turn on the real FI parameter $t_{\mathbb{R}}$, and the Higgs branch (which is a hyper-Kähler cone) gets resolved to be $T^*\mathbb{C}\mathbf{P}^{N-1}$. The $SU(N)$ flavor symmetry and $SU(2)_H$ acts on $T^*\mathbb{C}\mathbf{P}^{N-1}$, and the $U(1)$ Hitchin action is embedded into the Cartan of $SU(N)\times SU(2)_H$, with the embedding given by (4.122). Then, one can study the fixed points under this $U(1)$ subgroup. It turns out that there are N of them, computed in Appendix D. As the equivariant parameters of the $SU(N)$ flavor symmetry are the masses of hypermultiplets, these fixed points can be interpreted as massive vacua of the theory when mass parameters are turned on according to the mixing (4.122).

On the other hand, from the perspective of $\mathcal{M}_H$, the contributing fixed points are also straightforward to identify: they are precisely the ones whose moment map values multiplied by k remain finite in the large-k limit, and there are precisely N of them. Summing up their contributions gives back (4.123).

3d mirror of (A_1, D_{2N}) **theory.** Now we turn to Argyres-Douglas theories of type (A_1, D_{2N}), which are also known to have three-dimensional mirrors with Lagrangian descriptions [18]. The mirror theory of (A_1, D_{2N}) is given by a quiver $U(1) \times U(1)$ gauge theory, with $N-1$ charged hypermultiplets between two gauge nodes. These hypermultiplets enjoy an $SU(N-1)$ flavor symmetry. Moreover, there is one hypermultiplet only charged under the first $U(1)$ gauge group while another hypermultiplet is charged only under the second $U(1)$ gauge group. There is also an additional $U(1)$ flavor symmetry that rotates $N+1$ hypermultiplets together with charge $1/2$. See the quiver diagram in Figure 4.3.

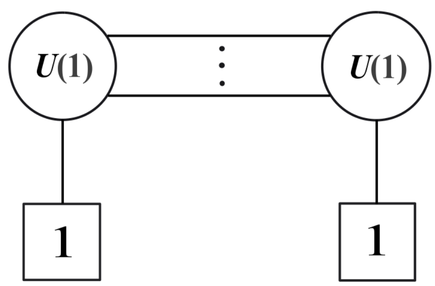


Figure 4.3: The 3d mirror of (A_1, D_{2N}) theories. There are $N-1$ hypermultiplet between two $U(1)$ gauge nodes, and there are additional one hypermultiplet charged under each node.

The index computation is similar. We will use $N(n_2 - n_1)$ and $N(n_1 + n_2)$ to denote monopole numbers for the $U(1) \times U(1)$ topological symmetry on the Coulomb branch. They come from the combination of flavor holonomies of the parent Argyres-Douglas theory. Besides the fugacity z for $U(1)$ flavor symmetry, we also include a_i, $i = 1, \dots, N-1$ as the fugacities for the extra $SU(N-1)$ flavor symmetry, subject to the constraint $\prod a_i = 1$. The associated background flavor monopole numbers all vanish, similar to the previous case. Then we have the index formula:

$$\mathcal{I}_H^{3\mathrm{d},D_{2N}} = (1-\mathrm{t}')^2 \oint \frac{dw_1}{2\pi i w_1}\frac{dw_2}{2\pi i w_2} w_1^{N(n_2-n_1)} w_2^{N(n_1+n_2)} \times \frac{1}{1-\mathrm{t}'^{\frac{1}{2}}(w_1 z^{\frac{1}{2}})^{\pm}} \frac{1}{1-\mathrm{t}'^{\frac{1}{2}}(w_2 z^{\frac{1}{2}})^{\pm}} \prod_{i=1}^{N-1} \frac{1}{1-\mathrm{t}'^{\frac{1}{2}}(w_1 w_2^{-1} a_i z^{\frac{1}{2}})^{\pm}}. \tag{4.124}$$

In the computation we have set $z = 1$ as it will not mix with the R-symmetry (see Appendix C.2 for more details). To evaluate the integral, we can assume without loss of generality that $n_2 > n_1 > 0$. Then summing over residues gives

$$\begin{aligned}\mathcal{I}_H^{3\mathrm{d},D_{2N}} &= \mathrm{t}'^{Nn_2} \prod_{i=1}^{N-1} \frac{1}{(1-\mathrm{t}'^{\frac{1}{2}}a_i)(1-\mathrm{t}'^{\frac{1}{2}}a_i^{-1})} \\ &+ \sum_{j=1}^{N-1} \frac{(\mathrm{t}'a_j)^{N(n_1+n_2)}\mathrm{t}'^{\frac{N}{2}(n_2-n_1)}}{(1-\mathrm{t}'^{\frac{3}{2}}a_j)(1-\mathrm{t}'^{-\frac{1}{2}}a_j^{-1})} \prod_{i\neq j} \frac{1}{(1-\mathrm{t}'a_j/a_i)(1-a_i/a_j)} \\ &+ \sum_{j=1}^{N-1} \frac{\mathrm{t}'^{Nn_2}\left(\mathrm{t}'^{\frac{1}{2}}a_j^{-1}\right)^{N(n_2-n_1)}}{(1-\mathrm{t}'^{\frac{3}{2}}a_j^{-1})(1-\mathrm{t}'^{-\frac{1}{2}}a_j)} \prod_{i\neq j} \frac{1}{(1-\mathrm{t}'a_i/a_j)(1-a_j/a_i)}.\end{aligned} \tag{4.125}$$

It is not hard to see that the following substitution would recover the parent Hitchin character (4.115) at $k \longrightarrow +\infty$:

$$a_j \longrightarrow \mathrm{t}'^{\frac{j}{N}-\frac{1}{2}}. \tag{4.126}$$

Similarly, the residue sums in (4.125) are in one to one correspondence with massive vacua of the 3d mirror theory, which are also identified with the fixed points under the $U(1) \subset SU(N-1) \times SU(2)_H$ action on the Higgs branch. Explicit calculations done in Appendix D show that there are precisely $2N+1$ fixed points, which, from Hitchin moduli space point of view, are exactly those with vanishing moment map in the large-k limit.

In summary, considering the three-dimensional mirror theory gives physical interpretation to the fixed points in $\mathcal{M}^*$ as discrete vacua of the mass-deformed theory. The fixed-point sum can be thought of as a sum of residues in the Higgs branch localization [143].

4.5 Vertex operator algebras

In previous sections, we have given a very strong test of the proposed isomorphism (4.1) for Argyres-Douglas theories. In this section, we enrich this correspondence to the triangle (4.3) by introducing another player into the story — vertex operator algebras (VOAs).

VOA from geometric Langlands correspondence

One motivation for incorporating VOAs is the celebrated geometric Langlands correspondence (see [144] and [145] for pedagogical reviews on this subject), which conjectures the equivalence of two derived categories,

$$\boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_{G_{\mathbb{C}}}} \quad = \quad \boxed{\text{coherent sheaves on } \mathrm{Loc}_{{}^L G_{\mathbb{C}}}} \,. \tag{4.127}$$

The gauge theory approach to the geometric Langlands program, started by [96], suggests that the above relation naturally fits inside a triangle,

$$\begin{array}{ccc} \text{A-branes in } (\mathcal{M}_H, \omega_K) & \overset{\textcircled{1}}{\longleftrightarrow} & \text{B-branes in } ({}^L\mathcal{M}_H, J) \\ \textcircled{2}\nwarrow\!\!\searrow & & \swarrow\!\!\nearrow\textcircled{3} \\ & \mathcal{D}\text{-modules on } \mathrm{Bun}_{G_{\mathbb{C}}} & \end{array} \,. \tag{4.128}$$

The geometric Langlands correspondence (4.127) now becomes the arrow $\textcircled{3}$ on the bottom-right of (4.128), as the B-brane category of ${}^L\mathcal{M}_H$ is closely related to the derived category of coherent sheaves on $\mathrm{Loc}_{{}^L G_{\mathbb{C}}}$. The arrow $\textcircled{1}$ on the top is the homological mirror symmetry (or S-duality from the 4d guage theory viewpoint). The arrow $\textcircled{2}$, a new relation, was proposed in Section 11 of [96] and is related to the "brane quantization" of $\mathrm{Bun}_{G_{\mathbb{C}}}$ [33] (see also [146] for more examples and [147, 148] for an alternative way to establish the equivalence).

Now let us return to the diagram

$$\begin{array}{ccc} \text{Coulomb index of } \mathcal{T} & \longleftrightarrow & \text{quantization of } {}^L\mathcal{M}_{\mathcal{T}} \\ \nwarrow\!\!\searrow & & \swarrow\!\!\nearrow \\ & \text{VOA } \chi_{\mathcal{T}} & \end{array} \,. \tag{4.129}$$

The top arrow for class $\mathcal{S}$ theories explained in Chapter 3 is in fact the result of ① in (4.128) as we review below. Then one expects there is a VOA that fits into the diagram, giving rise to $\mathcal{D}$-modules via the conformal block construction (see *e.g.* part III of [144]).

To understand the top arrow from homological mirror symmetry, one first rewrites the Coulomb BPS states on $L(k,1)$, view as T^2 fibered over an interval,[20] in the categorical language

$$\mathcal{H}_{\text{Coulomb}} = \text{Hom}_{C_A}(A_0, ST^kS \cdot A_0). \tag{4.130}$$

Here C_A is the category of boundary conditions on T^2 (or "A-branes" in $\mathcal{M}_H$) of the Argyres-Douglas theory, and $A_0 \in C_A$ is the boundary condition given by the solid torus $D^2 \times S^1$, and ST^kS is an element of $SL(2,\mathbb{Z})$ that acts on C_A via the modular group action on T^2. Suppressing one S^1 circle and the time direction, the geometry near the endpoint of the interval is given by the tip of a cigar, and the brane A_0 associated with this geometry is conjectured to be the "oper brane." The generator $S \in SL(2,\mathbb{Z})$ acts as homological mirror symmetry, transforming C_A into C_B — the category of B-branes in ${}^L\mathcal{M}_H$, and the mirror of A_0 is expected to be $S \cdot A_0 = B_0 = \mathcal{O}$, the structure sheaf of ${}^L\mathcal{M}_H$. Then acting on (4.130) by S gives

$$\text{Hom}_{C_A}(A_0, ST^kS \cdot A_0) = \text{Hom}_{C_B}(B_0, T^k \cdot B_0). \tag{4.131}$$

As $T \in SL(2,\mathbb{Z})$ acts on objects in C_B by tensoring with the determinant line bundle $\mathcal{L}$, the right-hand side is precisely the geometric quantization of ${}^L\mathcal{M}_H$,

$$\mathcal{H}(\Sigma, {}^LG, k) = H^\bullet\left({}^L\mathcal{M}_H, \mathcal{L}^{\otimes k}\right) = \text{Hom}_{C_B}(B_0, T^k \cdot B_0). \tag{4.132}$$

If a VOA fits into the triangle (4.129) via the correspondence between A-branes and $\mathcal{D}$-modules, there should be a modular tensor category C_χ of representations of the VOA, and there is a similar vector space

$$\text{Hom}_{C_\chi}(\chi_0, ST^kS \cdot \chi_0). \tag{4.133}$$

The module χ_0 corresponding to the oper brane A_0 is expected to be the vacuum module, and ST^kS acts by modular transform. The "geometric Langlands triangle"

[20] As observed in [16] and [15], the Coulomb index is the same as a topologically twisted partition function. This enables us to treat the physical theory as if it is a TQFT and freely deform the metric on $L(k,1)$.

(4.128) states that all the above three vector spaces are isomorphic, which implies, at the level of dimensions,

$$\dim \mathcal{H}(\mathcal{M}_H) = \mathcal{I}_{\text{Coulomb}} = (ST^k S)_{0,0}. \tag{4.134}$$

As the first two quantities can be refined by $\mathfrak{t}$, one expects the S- and T-matrices for the VOA should also be refined. However, for the VOAs that will appear (such as Virasoro minimal models), the refinement is not known, and we will only check the relation (4.134) at a root of unity $\mathfrak{t} = e^{2\pi i}$.[21]

With flavor holonomy. Moreover, with flavor symmetry G from the singularities of the Riemann surface, we also consider the Coulomb index on $L(k,1)$ in the presence of a flavor holonomy along the Hopf fiber labeled by $\lambda \in \Lambda_{\text{cochar}}(G)/k\Lambda_{\text{cochar}}(G)$. This is equivalent to inserting a surface defect at the core of a solid torus in the decomposition of $L(k,1)$, carrying a monodromy determined by λ. It will change (4.130) into

$$\mathcal{H}_{\text{Coulomb}}(\lambda) = \text{Hom}_{C_A}(A_0, ST^k S \cdot A_\lambda), \tag{4.135}$$

where

$$A_\lambda = L_\lambda A_0 \tag{4.136}$$

with L_λ representing the action of the surface defect on boundary conditions. These defects are analogous to the 't Hooft line operators — in fact, they are constantly referred to as "'t Hooft-like operators" in [65] — and change the parabolic weights at the singularities on Σ. Then, the relation between A-branes and $\mathcal{D}$-modules predicts that there exists a corresponding operator (which we again denote as L_λ) in the category C_χ. Now, the VOA has $\widehat{\mathfrak{g}}$ affine Kac-Moody symmetry, whose modules are labeled by the weights λ of $\widehat{\mathfrak{g}}$, and one expects the action of L_λ on the vacuum module is given by

$$L_\lambda \cdot \chi_0 = \chi_{-\lambda}. \tag{4.137}$$

Then, in the presence of flavor holonomies, one expects the following relation:

$$\dim \mathcal{H}(\mathcal{M}_H, \lambda) = \mathcal{I}_{\text{Coulomb},\lambda} = (ST^k S)_{0,-\lambda}. \tag{4.138}$$

At this stage we do not know *a priori* what is the right VOA when $\mathcal{M}_H$ is a wild Hitchin moduli space, but we conjecture that it is given by the VOA under the

[21] As the wild Hitchin character involves fractional powers of $\mathfrak{t}$, such limit is different from $\mathfrak{t} \to 1$ and is in fact associated with a non-trivial root of unity. Also, the ambiguity of normalizing the Hitchin character by a monomial in $\mathfrak{t}$ now becomes the ambiguity of a phase factor in matching (4.134).

"SCFT/VOA correspondence" discovered in [19, 97–99, 149, 150]. Indeed, for theories of class $\mathcal{S}$, this correspondence gives, for each maximal tame puncture, an affine Kac-Moody symmetry at the critical level — the one that gives rise to a specific type of $\mathcal{D}$-modules central to the geometric Langlands program known as Hecke eigensheaves. In the rest of this section, we will review this correspondence and check that the above relations (4.134) and (4.138) hold for wild Hitchin moduli spaces. It will be an interesting problem to explain why this construction gives the correct $\mathcal{D}$-modules relevant for this particular problem.

Moreover, as shown in [151], general characters of certain $2d$ VOAs can be expressed by the Schur indices with line operator insertion of corresponding $4d$ theory. Our results can be interpreted as a relation between the Coulomb branch indices and the modular transformation of Schur indices with line operator insertion of AD theories. The modular properties of indices without any operator insertion of $4d$ theories are studied in [152, 153] and their modular properties are related to the 't Hooft anomalies of the theory. It is interesting to further study the $4d$ interpretation of modular S transformations on indices with line operator insertion and their relation with Coulomb branch indices.

2d VOAs from 4d SCFTs

As was first discovered in [19], every four-dimensional $\mathcal{N} = 2$ superconformal theory contains a protected subsector of BPS operators, given by the cohomology of certain nilpotent supercharge $\mathbb{Q}$, when these operators lie on a complex plane inside $\mathbb{R}^4$. These BPS operators are precisely the ones that enter into the Schur limit of the 4d $\mathcal{N} = 2$ superconformal index [58]. Moreover, the operator product expansion (OPE) of these operators are meromorphic, and they can be assembled into a two-dimensional *vertex operator algebra*. The central charges of the 4d SCFT and the 2d VOA are related by

$$c_{2d} = -12 c_{4d}, \tag{4.139}$$

which implies that all VOAs obtained in this way are necessarily non-unitary. If the parent four-dimensional theory enjoys a global symmetry given by a Lie group, then it will be enhanced to an affine Lie symmetry on the VOA side. The relation between the flavor central charge and the level for the affine symmetry is given by

$$k_{2d} = -\frac{1}{2} k_{4d}. \tag{4.140}$$

AD theory	VOA
(A_1, A_{2N})	$(2, 2N+3)$ minimal model
(A_1, A_{2N-1})	$\mathcal{B}_{N+1}$ algebra
(A_1, D_{2N+1})	$\widehat{\mathfrak{sl}}(2)_k$ at level $k = -\frac{4N}{2N+1}$
(A_1, D_{2N})	$\mathcal{W}_N$ algebra

Table 4.6: Examples of Argyres-Douglas theories and corresponding VOAs. To be more precise, in the (A_1, A_{2N-1}) case, it is the subregular quantum Hamiltonian reduction of $\widehat{\mathfrak{sl}}(N)_k$ at level $k = -N^2/(N+1)$ [154, 155]. In the (A_1, D_{2N}) case, it is the non-regular quantum Hamiltonian reduction of $\widehat{\mathfrak{sl}}(N+1)_k$ with $k = -(N-1)^2/N$ [154]. For details about quantum Hamiltonian reduction, see [156].

Examples of these VOAs are identified on a case-by-case basis [97–99, 149, 150]. We listed some examples of Argyres-Douglas theories in Table 4.6. For the case of (A_1, A_{2N-1}) and (A_1, D_{2N}), the VOAs are identified very recently in [154].

As was mentioned, the VOA has a very close relationship with the Schur operators. In particular, the Schur limit of the superconformal index is equal to the vacuum character of the VOA.[22] In contrast, Coulomb branch operators do not enter into the $\mathbb{Q}$-cohomology and are not counted by the Schur index. However, it turns out that the Coulomb branch index is related to the VOA in a quite surprising manner — the *modular transformation* property of the latter is captured by the Coulomb branch index, as we have motivated using the geometric Langlands correspondence in (4.134) and (4.138).

To check these relations explicitly, we need to identify the relevant representation categories C_χ of the VOAs listed in Table 4.6 that are closed under modular transforms. For the (A_1, A_{2N}) series, the answer is clear — the $(2, 2N+3)$ minimal model specifies a category of highest-weight modules of the Virasoro algebra. For the rest, we will also give the relevant category later in this section. But what about a more general theory $\mathcal{T}$? Once we obtain the VOA $\chi_\mathcal{T}$, how is the category $C_{\chi_\mathcal{T}}$ that is relevant for the Coulomb index of $\mathcal{T}$ constructed?

An obvious candidate would be the category of all representations of $\chi_\mathcal{T}$, but it cannot be the right answer as it is too large and there are many non-highest-weight modules whose conformal dimensions are not bounded from below nor above.

[22] On the other hand, the Schur index that incorporates line defects maybe used to probe non-vacuum modules, see [151].

Nonetheless, there is a natural procedure, called "semi-simplification" [157], that gives precisely the category we are interested in. Specifically, one forms a new quotient category, denoted as $\mathcal{O}^{\mathrm{s}}_{\chi\mathcal{T}}$, by modding out the negligible morphisms [158, 159] and keeping only simple objects with non-zero categorical dimensions. This category is believed to be a modular tensor category [157], and in each class of modules there is at least one module with bounded conformal dimensions (the "highest-weight" module). And we conjecture that

$$\mathcal{O}^{\mathrm{s}}_{\chi\mathcal{T}} = \mathcal{C}_{\chi\mathcal{T}} \tag{4.141}$$

is the category fitting in the triangle (4.3).

This conjecture will be verified in the four series of Argyres-Douglas theories that we study in this chapter. In the following we show that the wild Hitchin character (or Coulomb branch index) at $\mathfrak{t} \to e^{2\pi i}$ is indeed given by a matrix element of the modular transformation ST^kS in $\mathcal{C}_\chi$. In fact, in order for the relation (4.134) to be correct for all k, it is necessary to have a one-to-one correspondence between fixed points in $\mathcal{M}_H$ and modules in the category $\mathcal{C}_\chi$.

VOAs of Argyres-Douglas theories

(A_1, A_{2N}) **theories and Virasoro minimal models.** The observation of [99], by comparing the central charge (4.139), indicates that the associated VOA for (A_1, A_{2N}) Argyres-Douglas theory is the $(2, 2N+3)$ Virasoro minimal model. (Recall that $2N+3$ is also the number of Stokes rays centered at the irregular singularity.) The minimal model contains a finite number of highest-weight representations labeled by the conformal dimension $h_{r,s}$, where $s = 0$ and $1 \leq r+1 \leq 2N+2$.[23] Among these representations, there are $N+1$ independent ones given by $r = 0, 1, \ldots, N$ — exactly the same as the number of fixed points in the wild Hitchin moduli space $\mathcal{M}_{2,2N+1}$!

In [124], the one-to-one correspondence between the fixed points in $\mathcal{M}_{2,2N+1}$ and representations in the Virasoro minimal model is spelled out. Namely, if one defines the effective central charge

$$c_{\text{eff}} = c - 24h_{r,s}, \tag{4.142}$$

then there is a simple relation between c_{eff} and the moment map μ

$$\boxed{\mu = \frac{1}{24}(1 - c_{\text{eff}})}\,. \tag{4.143}$$

[23] Unlike the usual convention in the literature here we shift r and s by 1 so that the vacuum corresponds to $(r, s) = (0, 0)$.

Here the moment map values are calculated around (E.5), *without* the further shift we made in the last section. Later, we extend this observation to all the other types of wild rank-two Hitchin moduli spaces, with emphasis on the perspective of modular transformations, where this correspondence finds its natural home.

To see the relation between the wild Hitchin character (4.89) of $\mathcal{M}_{2,2N+1}$ and the modular transformation of $(2, 2N+3)$ minimal model, recall that characters of these $N+1$ modules form an $N+1$-dimensional representation of $SL(2,\mathbb{Z})$, with the S- and T-matrices given by

$$\begin{aligned} \mathcal{S}_{r,\rho} &= \frac{2}{\sqrt{2N+3}}(-1)^{N+r+\rho}\sin\left(\frac{2\pi(r+1)(\rho+1)}{2N+3}\right), \\ \mathcal{T}_{r,\rho} &= \delta_{r\rho}e^{2\pi i(h_{r,\rho}-c/24)}, \end{aligned} \tag{4.144}$$

where r and ρ run from 0 to N. With the help of (4.144) one can show that

$$\mathcal{I}(\mathcal{M}_{2,2N+1}) = \mathfrak{t}^{\frac{k}{8(2N+3)}}\mathcal{I}_{(A_1,A_{2N})}|_{\mathfrak{t}\to e^{2\pi i}} = e^{\frac{\pi i k}{12}}\left(\mathcal{S}\mathcal{T}^k\mathcal{S}\right)_{0,0}. \tag{4.145}$$

(A_1, D_{2N+1}) **theories and Kac-Moody algebras.** It was conjectured in [99, 150] that the corresponding VOA is the affine Kac-Moody algebra $\widehat{\mathfrak{su}}(2)_{k_F}$ for which

$$k_F = -2 + \frac{2}{2N+1}. \tag{4.146}$$

which is a boundary admissible level [160]. Notice that -2 is the critical level for $\widehat{\mathfrak{su}}(2)$, while $2N+1$ is again the number of Stokes rays on Σ. There is a notion of "admissible representations" for the Kac-Moody algebra, which is the analogue of integrable representations for Kac-Moody algebra at positive integer level (see *e.g.* [161, Sec. 18]). These representations are highest-weight modules, and are objects in the quotient category $\mathcal{O}^{\mathrm{s}}_{\chi}$. Their fusion rules and representation theory remained controversial for years, and were completely solved and understood (in the case of $N = 1$ for instance) recently in [162, 163] (see also the reference therein).

Let $\widehat{\omega}_0$ and $\widehat{\omega}_1$ be the fundamental weights of $\widehat{\mathfrak{su}}(2)$. A highest-weight representation for $\widehat{\mathfrak{su}}(2)_\kappa$ is called *admissible*, if the highest weight $\widehat{\lambda} = [\lambda_0, \lambda_1] := \lambda_0\widehat{\omega}_0 + \lambda_1\widehat{\omega}_1$, can be decomposed as

$$\widehat{\lambda} = \widehat{\lambda}^I - (\kappa+2)\widehat{\lambda}^F. \tag{4.147}$$

Here, if we write $\kappa = t/u$ with $t \in \mathbb{Z}\backslash\{0\}$, then $u \in \mathbb{Z}^+$ and $(t,u) = 1$. In our case $t = -4N$ and $u = 2N+1$. $\widehat{\lambda}^I$ and $\widehat{\lambda}^F$ are integrable representations for $\widehat{\mathfrak{su}}(2)$ at level $k^I = u(\kappa+2) - 2$ and $k^F = u - 1$, respectively. Specializing to our case, we see that $\widehat{\lambda}^I = 0$ and $\hat{\lambda}^F = 2N$, so the admissible representations are in one-to-one correspondence with the $2N+1$ integrable representations of $\widehat{\mathfrak{su}}(2)_{2N}$. This is again the same number as the fixed points of the moduli space $\widetilde{\mathcal{M}}_{2,2N-1}$! Let us see if there is a similar relation for the moment maps.

For each admissible module, the conformal dimension is given by

$$h_{\widehat{\lambda}} = \frac{\lambda_1(\lambda_1+2)}{4(\kappa+2)}. \tag{4.148}$$

If we denote the highest weight of the i-th integrable representation of $\widehat{\mathfrak{su}}(2)_{2N}$ as $[2N-i, i]$ for $i = 0, 1, \ldots, 2N$, then we have

$$\lambda_1^i = -\frac{2i}{2N+1}, \quad h^i_{\widehat{\lambda}} = -\frac{i(2N+1-i)}{2(2N+1)}. \tag{4.149}$$

In order to see the relation between (4.149) with the values of the moment map in (4.98), we relabel the indices. Additionally, to get rid of overall phase factors, we shift the moment map

$$\mu \to \mu + \frac{1}{8(2N+1)} + \frac{2N}{2N+1}\alpha. \tag{4.150}$$

Such a shift is not as *ad hoc* as it appears — the second term is the minimal moment map value computed in (E.28) for $\alpha = 0$, while the third term comes from the linear piece of the normalization factor (4.78). Then, we have the following correspondence:

$$\boxed{\mu = \left(h_{\widehat{\lambda}} - \frac{c}{24} + \frac{1}{8}\right) - \lambda_1\alpha}\,. \tag{4.151}$$

Hence the moment maps in (4.99) are in one-to-one correspondence with admissible representations of the Kac-Moody algebra $\widehat{\mathfrak{su}}(2)_{k_F}$. This also explains why the fixed points are assembled into groups — the two fixed points in each group are precisely the ones that are related by an outer-automorphism of the Kac-Moody algebra (recall that the outer-automorphism group is $\mathbb{Z}_2$, the same as the center of $SU(2)$).

The characters of admissible modules of $\widehat{\mathfrak{su}}(2)_\kappa$ also form a representation of the modular group and the S- and T-matrices are

$$\begin{aligned}\mathcal{S}_{\widehat{\lambda},\widehat{\mu}} = \sqrt{\frac{2}{u^2(\kappa+2)}}\,(-1)^{\mu_1^F(\lambda_1^I+1)+\lambda_1^F(\mu_1^I+1)} \\ \times e^{-i\pi\mu_1^F\lambda_1^F(\kappa+2)} \sin\left[\frac{\pi(\lambda_1^I+1)(\mu_1^I+1)}{\kappa+2}\right], \end{aligned} \tag{4.152}$$

$$\mathcal{T}_{\widehat{\lambda},\widehat{\mu}} = \delta_{\widehat{\lambda}\widehat{\mu}}\, e^{2\pi i(h_{\widehat{\lambda}}-c/24)},$$

with $\kappa = t/u$ being the level of the affine $\widehat{\mathfrak{su}}(2)$. Using (4.152) we have

$$\mathcal{I}(\widetilde{\mathcal{M}}_{2,2N-1})(\mathrm{t} \to e^{2\pi i}, \lambda = 0) = e^{\frac{k\pi i}{4}} \left(\mathcal{S}\mathcal{T}^k\mathcal{S}\right)_{0,0} . \tag{4.153}$$

When the monodromy is non-zero, the moment map changes accordingly. In fact we have

$$\mathcal{I}(\widetilde{\mathcal{M}}_{2,2N-1})(\mathrm{t} \to e^{2\pi i}, \lambda) = e^{\frac{k\pi i}{4}} \left(\mathcal{S}\mathcal{T}^k\mathcal{S}\right)_{0,(2N+1-\lambda)} . \tag{4.154}$$

(A_1, D_{2N}) theories and $\mathcal{W}_N$ algebra. As we have seen in Table 4.6, the VOA in this case is given by the $\mathcal{W}_N$ algebra, which is a non-regular quantum Hamiltonian reduction of affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(N+1)_k$ at level $k = -(N-1)^2/N$. The set of modules generated by spectral flow are considered in [154]. For a given VOA χ, in general there are two types of modules: the "local" modules and the "twisted" modules. A local module [164] in the braided category $\mathcal{O}^{\mathrm{s}}_{\chi}$ (*cf.* Section 4.5) is a module M of χ with no non-trivial monodromy. A twisted module is attached to the automorphism of χ [165], similar to the twisted sectors in string theory on orbifolds. For our $\mathcal{W}_N$ algebra, the precise details of the modules depend on whether N is even or odd. For simplicity, we will focus in this section on the even case where all local modules are closed under modular transformations [154]. They are parametrized by the set

$$(s, s') \in \{-N \le s \le N-1, 0 \le s' \le N-1, s + s' \in 2\mathbb{Z}\}. \tag{4.155}$$

It is not hard to see that the number of local modules is N^2 — exactly the same as the number of fixed points on Hitchin moduli space $\widetilde{\mathcal{M}}_{2,2N-2}$.

By picking suitable representatives of local modules, their conformal dimensions are bounded from below and given by

$$h_{(s,s')} = \frac{s^2 - s'^2}{4N} - \frac{|s|}{2} + \begin{cases} 0, & \text{for } |s+s'| \le N \text{ and } |s-s'| \le N, \\ (s+s')/2 - N/2, & \text{for } s+s' > N, \\ (s'-s)/2 - N/2, & \text{for } s-s' < -N. \end{cases} \tag{4.156}$$

Then, we find that for vanishing flavor holonomies, there is the relation

$$\boxed{\mu(\lambda_1 = \lambda_2 = 0) = h - \frac{c}{24} + \frac{1}{6}}, \tag{4.157}$$

where the central charge of $\mathcal{W}_N$ algebra is given by $c = 4 - 6N$. We also have the modular transformation data among those N^2 modules [154],

$$\begin{aligned}\mathcal{T}_{(\ell,\ell'),(s,s')} &= \delta_{\ell,s}\delta_{\ell',s'}\exp\left[2\pi i\Big(h_{(s,s')} - \frac{c}{24}\Big)\right],\\ \mathcal{S}_{(\ell,\ell'),(s,s')} &= \frac{1}{N}\exp\left[-\frac{\pi i}{N}(s\ell - s'\ell')\right].\end{aligned} \tag{4.158}$$

It can be verified that

$$\mathcal{I}(\widetilde{\mathcal{M}}_{2,2N-2})(\mathrm{t} \to e^{2\pi i}, \lambda_1 = \lambda_2 = 0) = e^{\frac{k\pi i}{3}}\Big(\mathcal{S}\mathcal{T}^k\mathcal{S}\Big)_{00}. \tag{4.159}$$

We note that the above matching becomes subtle when N is odd, where modular transformation turns local modules into twisted modules. Moreover, the vacuum module (which is local), is half-integer graded and thus have "wrong statistics" [166]. On the contrary, our index formula for the Hitchin moduli space $\widetilde{\mathcal{M}}_{2,2N-2}$ does not exhibit drastic difference between odd and even N. It will be interesting to understand the precise relation here.

(A_1, A_{2N-3}) theories and $\mathcal{B}_N$ algebra. Finally, we remark on the last type of Argyres-Douglas theory. We will be very brief here. As the (A_1, A_{2N-3}) theory is related to the (A_1, D_{2N}) theory via Higgsing, the VOA $\mathcal{B}_N$ in Table 4.6 can be similarly constructed via quantum Hamiltonian reduction of the $\mathcal{W}_N$ algebra introduced above. As in previous case, the representation theory of the VOA again depends on the parity of N. For N odd, local modules are preserved under modular transformation [154, 167]. By carefully picking a set of basis, it is clear that the modules are in one-to-one correspondence with fixed points (the total number is $N(N-1)/2$), and the moment map values match with effective central charges. When N is even, much less is known about the relevant categorical property. It will be interesting to understand this situation further.

Other examples

In fact, the correspondence between fixed manifolds on the Hitchin moduli space under the circle action and modules in $\mathcal{O}^{\mathrm{s}}_{\chi}$ of VOAs is much more general. To supplement our previous discussion focused on Argyres-Douglas theories, here we list such correspondence for other $T[\Sigma]$'s where the VOAs are known. For a tame puncture decorated by a parabolic subgroup of $G_{\mathbb{C}}$ (usually in the A_{N-1} series), we will use $[s_1, s_2, \ldots, s_l]$ to denote the associated Young tableau with each column of heights $s_1, \ldots, s_l$. If for a given Young tableau there are n_s columns with height

s, then the flavor symmetry associated with the puncture is $S(\prod_s U(n_s))$. In this notation the maximal puncture is $[1, 1, \ldots, 1]$.

- (A_1, D_4) Argyres-Douglas theory. The VOA is $\widehat{\mathfrak{su}}(3)_{-\frac{3}{2}}$. The Hitchin moduli space $\widetilde{\mathcal{M}}_{2,2}$ has four fixed points, corresponding to the four admissible modules of the affine Kac-Moody algebra. The relation between effective central charge and moment maps are checked with the help of (4.156), but one can also check directly using results from the Kac-Moody algebra. One again sees that $\mu(\lambda_1 = \lambda_2 = 0) = -c_{\text{eff}}/24 + 1/6$.

- $SU(2)$ gauge theory with four hypermultiplets. The Hitchin moduli space has $SU(2)$ gauge group, defined on S^2 with four tame punctures. There are five fixed manifolds — one $\mathbb{C}\mathbf{P}^1$ plus four points, and they all lie on the nilpotent cone of Kodaira type I_0^*. When the holonomies are set to zero, the moment map values are $\{0, 0, 0, 0, 1\}$. The VOA is $\widehat{\mathfrak{so}}(8)_{-2}$. There are five highest-weight modules belonging to the category O_χ^s, which for Kac-Moody algebras always coincide with Bernstein-Gelfand-Gelfand's category O [168]. The corresponding highest weights are $\{-2\omega_1, -2\omega_3, -2\omega_4, -\omega_2, 0\}$ with conformal dimensions $\{-1, -1, -1, -1, 0\}$ [169]. Then we see that $\mu(\lambda_{1,2,3,4} = 0) = -c_{\text{eff}}/24 + 5/12$.

- T_3 theory [73]. The Hitchin moduli space is associated with S^2 with three maximal tame punctures, with gauge group $SU(3)$. The moduli space has seven fixed manifolds: one $\mathbb{C}\mathbf{P}^1$ plus six fixed points lying on the nilpotent cone of Kodaira type IV^* [16]. The associated VOA is the affine Kac-Moody algebra $\widehat{\mathfrak{e}_6}$ at level -3 [97, 98]. There are exactly seven highest-weight modules in the category O [168]. The highest weights are, respectively, $\{0, -\omega_4, -2\omega_2 + \omega_3 - \omega_4, \omega_2 - 2\omega_3, -2\omega_1 + \omega_2 - 2\omega_3 + \omega_4, -2\omega_5 + \omega_6, -3\omega_6\}$ with conformal dimension $\{0, -2, -2, -2, -2, -2, -2\}$. It is not hard to check from the results of [16] that the relation between moment maps and effective central charges with zero holonomy is given by $\mu = -c_{\text{eff}}/24 + 11/12$.

- E_7 SCFT [170]. The associated Hitchin system has $G = SU(4)$, and Σ is a sphere with three tame punctures. Two of them are maximal punctures, while the third one is a next-to-minimal puncture $[2, 2]$ [91]. By comparing the central charges, it is not hard to see that the VOA should be the affine Kac-Moody algebra $\widehat{\mathfrak{e}_7}$ at level -4. Although [16] did not present the calculation of Hitchin character in this case, the steps of calculation were outlined using generalized Argyres-Seiberg duality. The fixed manifolds consist of one $\mathbb{C}\mathbf{P}^1$ plus seven points, all of which stay on the nilpotent cone of Kodaira type III*. Again there are in total eight highest-weight modules of the VOA [168].

- E_8 SCFT [170]. Now $G = SU(6)$ and Σ is a three-punctured sphere, with one maximal puncture, one $[2, 2, 2]$ puncture and one $[3, 3]$ puncture. The moduli space contains nine fixed manifolds — one $\mathbb{C}\mathbf{P}^1$ and eight fixed points all lying on the nilpotent cone of Kodaira type II*. One finds the VOA is the affine Kac-Moody algebra $\widehat{\mathfrak{e}_8}$ at level -6, which has exactly nine highest-weight modules in the category O [168].

It is also quite curious to note that in all cases, the vacuum module corresponds to the *top* fixed point with largest moment map. This is in line with the relation between the vacuum module and the oper brane — the support of the latter is on the Hitchin section, which intersects the nilpotent cone at the top.

Based on the above observations, we formulate the general conjecture that relates the Coulomb branch vacua and the representation of VOA as follows.

Conjecture. Given a four-dimensional $\mathcal{N} = 2$ SCFT $\mathcal{T}$, the fixed points on the Coulomb branch $\mathcal{M}_{\mathcal{T}}$ on $S^1 \times \mathbb{R}^3$ under the $U(1)_r$ action are in one-to-one correspondence with the highest-weight modules of the VOA $\chi_{\mathcal{T}}$ associated with $\mathcal{T}$, in the modular tensor category $O^{\mathrm{s}}_{\chi_{\mathcal{T}}}$ obtained from semi-simplification,

$$\boxed{U(1)_r \text{ fixed points in } \mathcal{M}_{\mathcal{T}}} \quad \longleftrightarrow \quad \boxed{\text{objects in } O^{\mathrm{s}}_{\chi_{\mathcal{T}}}} . \tag{4.160}$$

One may also wish to formulate the correspondence on the categorical level, not just on the level of objects. For this one needs to find the replacement on the left-hand side, and a natural candidate is the following. Consider the theory $\mathcal{T}$ on $\mathbb{R}_{\text{time}} \times D^2 \times S^1$, then weakly gauging $U(1)_{r-R}$ (a subgroup of the R-symmetry group $SU(2)_R \times U(1)_r$ generated by $j_r - j_{3,R}$) will break half of the supersymmetries. The resulting theory $\mathcal{T}'$ will have vacua given by connected components of $U(1)$ fixed points in $\mathcal{M}_{\mathcal{T}}$. Then we have the category of boundary conditions at the spacial infinity $\partial(D^2 \times S^1) = T^2$, which we denote as $\mathcal{T}'(T^2)$. This is a modular tensor category, on which the modular group acts via the mapping class group action of the spacial boundary T^2. Then the above conjecture may be formulated as the equivalence between two modular tensor categories — the "categorical SCFT/VOA correspondence" — as

$$\mathcal{T}'\left(T^2\right) = O^{\mathrm{s}}_{\chi_{\mathcal{T}}}. \tag{4.161}$$

Chapter 5

CLASSIFICATION OF ARGYRES-DOUGLAS THEORIES AND S-DUALITY

In previous chapters, we have used supersymmetric quantum field theory to understand the geometry, such as Chern-Simons invariants and Hitchin moduli spaces. Conversely, we may use the geometry to understand the field theory side and this is precisely what we do in this chapter.

5.1 S-duality for Argyres-Douglas theories

Given a four dimensional $\mathcal{N} = 2$ superconformal field theory (SCFT) with marginal deformations, it is interesting to write down its weakly coupled gauge theory descriptions. In such descriptions, gauge couplings take the role of the coordinate on the conformal manifold and the gauge theory is interpreted as conformal gauging of various strongly coupled isolated SCFTs [76]. It is quite common to find more than one weakly coupled descriptions, and they are S-dual to each other as the gauge couplings are often related by, *e.g.*, $\tau \propto -\frac{1}{\tau}$. Finding all weakly coupled gauge theory descriptions is often very difficult for a generic strongly coupled $\mathcal{N} = 2$ SCFT.

The above questions are solved for class $\mathcal{S}$ theory where the Coulomb branch spectrum has integral scaling dimensions: one represents our theory by a Riemann surface Σ with *regular* singularity so that S-duality is interpreted as different degeneration limits of Σ into three punctured sphere [2]; once a degeneration is given, the remaining task is to identify the theory corresponding to a three punctured sphere, as well as the gauge group associated to the cylinder connecting those three punctured spheres. In class $\mathcal{S}$ theory framework, Σ appears naturally as the manifold on which we compactify 6d $(2,0)$ theory. Certain $\mathcal{N} = 2$ SCFTs and their S-duality can be studied via geometric engineering, see [171].

There is a different type of $\mathcal{N} = 2$ SCFT called Argyres-Douglas (AD) theories [18, 100]. The Coulomb branch spectrum of these theories has fractional scaling dimension and they also admit marginal deformations. Again, one can engineer such AD theories by using $(2,0)$ theory on Riemann spheres $\Sigma_{g=0}$ with *irregular* singularity[1]. Since we can not interpret the exact marginal deformations as the geometric moduli of Σ, there is no clue how weakly coupled gauge theory descriptions can be written down in general, besides some simple cases where one can analyze the Seiberg-Witten curve directly [129].

It came as quite a surprise that one can still interpret S-duality of A_{N-1}-type AD theory in terms of an auxiliary punctured Riemann surface [132]. The main idea of [132] is giving a map from Σ with irregular singularities to a punctured Riemann sphere $\Sigma^{'}$, and then find weakly coupled gauge theory as the degeneration limit of $\Sigma^{'}$ into three punctured sphere.

The main purpose of this chapter is to generalize the idea of [132] to AD theories engineered using general 6d (2, 0) theory of type $\mathfrak{g}$. The major results of this chapter are

- We revisit the classification of irregular singularity of class (k, b) in [18, 104]:

$$\Phi \sim \frac{T_k}{z^{2+\frac{k}{b}}} + \sum_{-b \leq l < k} \frac{T_l}{z^{2+\frac{l}{b}}} \tag{5.1}$$

  and find new irregular singularity which gives SCFT in four dimensions. Briefly, they are the configuration for which

  (i) T_k is regular-semisimple, whose classification was studied in [104].
  (ii) The new cases are that T_k is semisimple.
  (iii) Fix a pair (k, b) and type T_k, we can consider the degeneration of T_k and the crucial constraint is that the corresponding Levi subalgebra has to be the same for T_l, $l > -b$.

- We successfully represent our theory by an auxiliary punctured sphere from the data defining our theory from 6d (2,0) SCFT framework, and we then find weakly coupled gauge theory descriptions by studying degeneration limit of new punctured sphere.

For instance, we find that for $\mathfrak{g} = D_N$, $b = 1$ and large k and all coefficient matrices regular semisimple, one typical duality frame looks like

$$\mathcal{T}_1 - SO(4) - \mathcal{T}_2 - SO(6) - \mathcal{T}_3 - \cdots\cdots - \mathcal{T}_{N-2} - SO(2N-2) - \mathcal{T}_{N-1},$$

[1] We will henceforth drop the subscript $g = 0$ in what follows to denote the Riemann sphere.

where $\mathcal{T}_i$ is given by D_{i+1} theory $\left(III_{k,1}^{[1;2i]^{\times(k+1)},[1^{i+1};0]},[1^{2i+2}]\right)$. The notation we use to label the AD theories is

$$\left(III_{k,b}^{\{\mathfrak{l}_i\}},\quad Q\right), \tag{5.2}$$

where III means type-III singularity in the sense of [18], and $\{\mathfrak{l}_i\}$ are Levi subalgebra for each coefficient matrix T_i and Q is the label for regular puncture. Each notation will be explained in the main text.

The same theory has a second duality frame, given by

$$\widehat{\mathcal{T}_1} \;\frac{\quad}{}\; SU(2) \;\frac{\quad}{}\; \widehat{\mathcal{T}_2} \;\frac{\quad}{}\; SU(3) \;\frac{\quad}{}\; \widehat{\mathcal{T}_3} \;\frac{\quad}{}\; \cdots\cdots \;\frac{\quad}{}\; \widehat{\mathcal{T}}_{N-1} \;\frac{\quad}{}\; SU(N) \;\frac{\quad}{}\; \widehat{\mathcal{T}}'_N,$$

where $\widehat{\mathcal{T}_i}$, $1\leq i\leq N-1$ is given by $\left(III_{k,1}^{[i,1]^{\times(k+1)},[1^{i+1}]},[1^{i+1}]\right)$, and $\widehat{\mathcal{T}}'_N$ is given by $\left(III_{k,1}^{[N;0]^{\times(k+1)},[1^{2N};0]},Q\right)$. An unexpected corollary is that the quiver with $SO(2n)$ gauge groups are dual to quivers with $SU(n)$ gauge groups, and each intermediate matter content does not have to be engineered from the same $\mathfrak{g}$-type in 6d. Similar feature appears when $\mathfrak{g}=E_{6,7,8}$, as will be demonstrated in this work.

The chapter is organized as follows. In section 5.2 we briefly review regular punctures and their associated local data, and then proceed to classify (untwisted) irregular punctures for $\mathfrak{g}=D_N$ and $\mathfrak{g}=E_{6,7,8}$ theories. We give relevant Coulomb branch spectrum. The map from Σ to Σ' is described in Section 5.3. Section 5.4 is devoted to study the duality frames for D_N theories. We consider both untwisted and twisted theories. Finally, we study S-duality frame for $E_{6,7,8}$ theories in section 5.5.

5.2 SCFTs from M5 branes

M5 brane compactifications on Riemann surface Σ provide a large class of $\mathcal{N}=2$ superconformal theories in four dimensions. To characterize the theory, one needs to specify a Lie algebra $\mathfrak{g}$ of ADE type, the genus g of the Riemann surface, and the punctures on Σ. Regular punctures are the loci where the Higgs field Φ has at most simple poles; while irregular punctures are those with Φ having higher order poles. The class $\mathcal{S}$ theories developed in [2] are SCFTs with Σ of arbitrary genus and arbitrary number of regular punctures, but no irregular puncture. Later, it was realized that one may construct much larger class of theories by utilizing irregular punctures [18, 103, 172]. However, in this case the Riemann surface is highly constrained. One may use either

- A Riemann sphere with only one irregular puncture at the north pole;
- A Riemann sphere with one irregular puncture at the north pole and one regular puncture at the south pole,

where the genus $g = 0$ condition is to ensure the $\mathbb{C}^*$ action on the Hitchin system, which guarantees $U(1)_r$ R-symmetry and superconformality. This reduces classification of theories into classification of punctures. In this section we revisit the classification and find new irregular singularity which will produce new SCFTs.

Classification of punctures

Regular punctures. Near the regular puncture, the Higgs field takes the form

$$\Phi \sim \frac{\Lambda}{z} + M, \tag{5.3}$$

and classification of regular puncture is essentially classification of nilpotent orbits. The puncture itself is associated with the *Nahm label*, while Λ is given by the *Hitchin label*. They are related by the *Spaltenstein map*. We now briefly review the classification.

Lie algebra $\mathfrak{g} = A_{N-1}$. The nilpotent orbit is classified by the partition $Y = \left[n_1^{h_1}, \ldots, n_r^{h_r}\right]$, where n_i are column heights, and the flavor symmetry is [2, 91]

$$G_{\text{flavor}} = S\left(\prod_{i=1}^{r} U(h_i)\right). \tag{5.4}$$

The spectral curve is

$$\det(x - \Phi(z)) = 0 \longrightarrow x^N + \sum_{i=2}^{N} \phi_i(z) x^{N-i} = 0. \tag{5.5}$$

Each ϕ_i is the meromorphic differentials on the Riemann surface, living in the space $H^0(\Sigma, K^{\otimes i})$. The order of pole p_i of the regular puncture at ϕ_i determines the local dimension of Coulomb branch spectrum with scaling dimension $\Delta = i$. It is given by $p_i = i - s_i$, where s_i is the height of i-th box of the Young Tableaux Y; here the labeling is row by row starting from bottom left corner.

Lie algebra $\mathfrak{g} = D_N$. We now review classification of regular punctures of D_N algebra. For a more elaborated study, the readers may consult [113, 173].

A regular puncture of type $\mathfrak{g} = D_N$ is labelled by a partition of $2N$, but not every partition is valid. It is a requirement that the even integers appear even times, which we will call a *D-partition*. Moreover, if all the entries of the partition are even, we call it *very even* D-partition. The very even partition corresponds to two nilpotent orbit, which we will label as $O^{I}_{[\cdot]}$ and $O^{II}_{[\cdot]}$. We again use a Young tableau with decreasing column heights to represent such a partition, and we call it a *Nahm partition*. Given a Nahm partition, the residual flavor symmetry is given by

$$G_{\text{flavor}} = \prod_{h \text{ odd}} \text{Spin}(n^h) \times \prod_{h \text{ even}} \text{Sp}\Big(n^h\Big). \tag{5.6}$$

We are interested in the contribution to the Coulomb branch dimension from each puncture. When $\mathfrak{g} = A_{N-1}$ case we simply take transpose and obtain a *Hitchin partition* [91]. However, for $\mathfrak{g} = D_N$ the transpose does not guarantee a valid Young tableaux. Instead it must be followed by what is called *D-collapse*, denoted as $(\cdot)_D$, which is described as follows:

(i) Given a partition of $2N$, take the longest even entry n, which occurs with odd multiplicity (if the multiplicity is greater than 1, take the last entry of that value), then picking the largest integer m which is smaller than $n-1$ and then change the two entries to be $(n, m) \rightarrow (n-1, m+1)$.

(ii) Repeat the process for the next longest even integer with odd multiplicity.

The *Spaltenstein map* $\mathfrak{S}$ of a given partition d is given by $(d^{\mathsf{T}})_D$ and we obtain the resulting *Hitchin partition* or *Hitchin diagram*[2].

The Spaltenstein map is neither one-to-one nor onto; it is not an involution as the ordinary transpose either. The set of Young diagram where $\mathfrak{S}$ is an involution is called *special*. More generally, we have $\mathfrak{S}^3 = \mathfrak{S}$.

Given a regular puncture data, one wishes to calculate its local contribution to the Coulomb branch. We begin with the special diagram.

Using the convention in [173], we can construct the local singularity of Higgs field in the Hitchin system as (5.3) where Λ is an $\mathfrak{so}(2N)$ nilpotent matrix associated to the Hitchin diagram and M is a generic $\mathfrak{so}(2N)$ matrix. Then, the spectral curve is

[2]Unlike [173], here we define the Hitchin diagram to be the one after transpose, so that when reading Young diagram one always reads column heights.

identified as the SW curve of the theory, which takes the form

$$\det(x - \Phi(z)) = x^{2N} + \sum_{i=1}^{N-1} x^{2(N-i)}\phi_{2i}(z) + \tilde{\phi}(z)^2. \tag{5.7}$$

We call $\tilde{\phi}$ the Pfaffian. This also determines the order of poles for each coefficient ϕ_{2i} and $\tilde{\phi}$. We will use p^{α}_{2i} to label the order of poles for the former, and $\tilde{p}^{\alpha}$ to label the order of poles for the latter. The superscript α denotes the α-th puncture.

The coefficient for the leading order singularity for those ϕ's and $\tilde{\phi}$ are not independent, but satisfy complicated relations [173, 174]. Note that the Coulomb branch dimensions of D_N class $\mathcal{S}$ theory are not just the degrees for the differentials; in fact the Coulomb branch is the subvariety of

$$V_C = \bigoplus_{k=1}^{N-1} H^0(\Sigma, K^{2k}) \oplus \bigoplus_{k=3}^{N-1} W_k \oplus H^0(\Sigma, K^N), \tag{5.8}$$

where W_k's are vector spaces of degree k. If we take $c^{(k)}_l$ to be the coefficients for the l-th order pole of ϕ_k, then the relation will be either polynomial relations in $c^{(k)}_l$ or involving both $c^{(k)}_l$ and $a^{(k)}$, where $a^{(k)}$ is a basis for W_k. For most of the punctures, the constraints are of the form

$$c^{(k)}_l = \dots, \tag{5.9}$$

while for certain very even punctures, as $\tilde{\phi}$ and ϕ_N may share the same order of poles, the constraints would become

$$c^{(N)}_l \pm 2\tilde{c}_l = \dots. \tag{5.10}$$

For examples of these constraints, see [173].

When the Nahm partition d is non-special, one needs to be more careful. The pole structure of such a puncture is precisely the same as taking $d_s = \mathfrak{S}^2(d)$, but some of the constraints imposed on d_s should be relaxed. In order to distinguish two Nahm partitions with the same Hitchin partition, one associates with the latter a discrete group, and the map

$$d_{\text{Nahm}} \longrightarrow (\mathfrak{S}(d_{\text{Nahm}}), C(d_{\text{Nahm}})) \tag{5.11}$$

makes the Spaltenstein dual one-to-one. This is studied by Sommers and Achar [175–177] and introduced in the physical context in [113].

Now we proceed to compute the number of dimension k operators on the Coulomb branch, denoted as d_k. We have

$$d_{2k} = (1-4k)(1-g) + \sum_{\alpha}(p^{\alpha}_{2k} - s^{\alpha}_{2k} + t^{\alpha}_{2k}), \tag{5.12}$$

where g is the genus of Riemann surface, s^{α}_{2k} is the number of constraints of homogeneous degree $2k$, and t^{α}_{2k} is the number of $a^{(2k)}$ parameters that give the constraints $c^{(4k)}_l = \left(a^{(2k)}\right)^2$. For d_{2k+1}, since there are no odd degree differentials, the numbers are

$$d_{2k+1} = \sum_{\alpha} t^{\alpha}_{2k+1}, \tag{5.13}$$

which is independent of genus. Finally, we take special care for d_N. When N is even, it receives contributions from both ϕ_N and the Pfaffian $\tilde{\phi}$. We have

$$d_N = 2(1-2N)(1-g) + \sum_{\alpha}(p^{\alpha}_N - s^{\alpha}_N) + \sum_{\alpha}\tilde{p}^{\alpha}. \tag{5.14}$$

When N is odd, it only receives contribution from the Pfaffian:

$$d_N = (1-2N)(1-g) + \sum_{\alpha}\tilde{p}^{\alpha}. \tag{5.15}$$

Lie algebra $\mathfrak{g} = E_{6,7,8}$. Unlike classical algebras, Young tableau are no longer suitable for labelling those elements in exceptional algebras. So we need to introduce some more mathematical notions. Let $\mathfrak{l}$ be a Levi subalgebra, and $O^{\mathfrak{l}}_e$ is the distinguished nilpotent orbit in $\mathfrak{l}$. We have

Theorem [178]. There is one-to-one correspondence between nilpotent orbits of $\mathfrak{g}$ and conjugacy classes of pairs $(\mathfrak{l}, O^{\mathfrak{l}}_e)$ under adjoint action of G.

The theorem provides a way to label nilpotent orbits. For a given pair $(\mathfrak{l}, O^{\mathfrak{l}}_e)$, let X_N denote the Cartan type of semi-simple part of $\mathfrak{l}$. $O^{\mathfrak{l}}_e$ in $\mathfrak{l}$ gives a weighted Dynkin diagram, in which there are i zero labels. Then the nilpotent orbit is labelled as $X_N(a_i)$. In case there are two orbits with same X_N and i, we will denote one as $X_N(a_i)$ and the other as $X_N(b_i)$. Furthermore if $\mathfrak{g}$ has two root lengths and one simple component of $\mathfrak{l}$ involves short roots, then we put a tilde over it. An exception of above is E_7, where it has one root length, but it turns out to have three pairs of nonconjugate isomorphic Levi-subalgebras. We will use a prime for one in a given pair, but a double prime for the other one. Such labels are *Bala-Carter labels*.

The complete list of nilpotent orbits for E_6 and E_7 theory is given in [179, 180]. We will examine them in more detail later in this section and in section 5.5.

Irregular puncture

Grading of the Lie algebra. We now classify irregular punctures of type $\mathfrak{g}$. We adopt the Lie-algebraic techniques reviewed in the following. Recall that for an irregular puncture at $z \sim 0$, the asymptotic solution for the Higgs field Φ looks like [18, 103, 104, 172]

$$\Phi \sim \frac{T_k}{z^{2+\frac{k}{b}}} + \sum_{-b \leq l < k} \frac{T_l}{z^{2+\frac{l}{b}}}, \tag{5.16}$$

where all T_l's are semisimple elements in Lie algebra $\mathfrak{g}$, and we also require that (k, b) are coprime. The Higgs field shall be singled valued when z circles around complex plane, $z \longrightarrow ze^{2\pi i}$, which means the resulting scalar multiplication of T_l comes from gauge transformation:

$$T_l \longrightarrow e^{\frac{2\pi i l}{b}}\, T_l = \sigma\, T_l\, \sigma^{-1} \tag{5.17}$$

for σ a G-gauge transformation. This condition can be satisfied provided that there is a finite order automorphism (torsion automorphism) that gives grading to the Lie algebra:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}_b} \mathfrak{g}^j. \tag{5.18}$$

All such torsion automorphisms are classified in [181–183], and they admit a convenient graphical representation called *Kac diagrams*. A Kac diagram $\mathbb{D}$ for $\mathfrak{g}$ is an extended Dynkin diagram of $\mathfrak{g}$ with labels $(s_0, s_1, \ldots s_r)$ on each nodes, called *Kac coordinates*, where r is the rank of $\mathfrak{g}$. Here s_0 is always set to be 2. Let $(\alpha_1, \ldots, \alpha_r)$ be simple roots, together with the highest root $-\alpha_0 = \sum_{i=1}^{r} a_i \alpha_i$ where $(a_1, \ldots, a_r)$ are the mark. We also define the zeroth mark a_0 to be 1. Then the torsion automorphism associated with $\mathbb{D}$ has order $m = \sum_{i=0}^{r} a_i s_i$ and acts on an element associated with simple root α_i as

$$\sigma: \quad \mathfrak{g}_{\alpha_i} \longrightarrow \epsilon^{s_i} \mathfrak{g}_{\alpha_i}, \qquad i = 1, 2, \ldots r, \tag{5.19}$$

and extend to the whole algebra $\mathfrak{g}$ via multiplication. Here ϵ is the mth primitive root of unity. It is a mathematical theorem [184] that all s_i can only be $0, 1$ or 2. We call $\mathbb{D}$ *even* if all its Kac coordinates are even, otherwise $\mathbb{D}$ is called *odd*. For even diagrams, we may divide the coordinate and the order m by 2 since the odd grading never shows up in (5.18). We will adopt this convention in what follows implicitly[3].

[3] This convention would not cause any confusion because if even diagrams are encountered, the label s_0 would be reduced to 1; for odd diagrams this label remains to be 2, so no confusion would arise.

There are two quantities in the grading of special physical importance. The *rank* of the G^0 module $\mathfrak{g}^j$, denoted as $\text{rank}(G^0|\mathfrak{g}^j)$, is defined as the dimension of a maximal abelian subspace of $\mathfrak{g}^j$, consisting of semisimple elements [185]. We are interested in the case where $\mathfrak{g}^1$ has positive rank: $r = \text{rank}(G^0|\mathfrak{g}^1) > 0$. Another quantity is the intersection of centralizer of semi-simple part of $\mathfrak{g}^1$ with $\mathfrak{g}^0$, and this will give the maximal possible flavor symmetry.

As we get matrix T_j out of $\mathfrak{g}^j$, we are interested in the case where $\mathfrak{g}^j$ generically contains regular semisimple element. We call such grading *regular semisimple*. A natural way to generate regular semisimple grading is to use nilpotent orbits. For $\mathfrak{g} = A_{N-1}$ it is given in [132]. We give the details of D_N and $E_{6,7,8}$ in Appendix G. Note when coefficient matrices are all regular semisimple, the AD theory with only irregular singularity can be mapped to type IIB string probing three-fold compound Du Val (cDV) singularities [186], which we review in Appendix F. We list the final results in table 5.1. This is a refinement and generalization of the classification done in [104, 132]. We emphasize here that the grading when $\mathfrak{g}^j$ generically contain semisimple elements are also crucial for obtaining SCFTs; here b may be more arbitrary. Such grading will be called *semisimple*.

In classical Lie algebra, semisimple element T_i can be represented by the matrices. In order for the spectral curve $\det(x - \Phi(z))$ to have integral power for monomials, the matrices for leading coefficient T_k is highly constrained. In particular, when $\mathfrak{g} = A_{N-1}$, we have

$$T = \begin{pmatrix} a_1\Xi & & & \\ & \ddots & & \\ & & a_r\Xi & \\ & & & 0_{(N-rb)} \end{pmatrix}. \tag{5.20}$$

Here Ξ is a $b \times b$ diagonal matrix with entries $\{1, \omega, \omega^2, \ldots, \omega^{b-1}\}$ for ω a b-th root of unity $\exp(2\pi i/b)$. For $\mathfrak{g} = D_N$, things are more subtle and T depends on whether b is even or odd. A representative of Cartan subalgebra is

$$\begin{pmatrix} Z & 0 \\ 0 & -Z^{\mathsf{T}} \end{pmatrix}, \tag{5.21}$$

$\mathfrak{g}$	Singularity	b
A_{N-1}	$x_1^2 + x_2^2 + x_3^N + z^{kN/b} = 0$	$b\|N$
	$x_1^2 + x_2^2 + x_3^N + x_3 z^{k(N-1)/b} = 0$	$b\|(N-1)$
D_N	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^{k(2N-2)/b} = 0$	$b\|(2N-2)$
	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^{kN/b} x_3 = 0$	$b\|N$
E_6	$x_1^2 + x_2^3 + x_3^4 + z^{12k/b} = 0$	$b\|12$
	$x_1^2 + x_2^3 + x_3^4 + z^{9k/b} x_3 = 0$	$b\|9$
	$x_1^2 + x_2^3 + x_3^4 + z^{8k/b} x_2 = 0$	$b\|8$
E_7	$x_1^2 + x_2^3 + x_2 x_3^3 + z^{18k/b} = 0$	$b\|18$
	$x_1^2 + x_2^3 + x_2 x_3^3 + z^{14k/b} x_3 = 0$	$b\|14$
E_8	$x_1^2 + x_2^3 + x_3^5 + z^{30k/b} = 0$	$b\|30$
	$x_1^2 + x_2^3 + x_3^5 + z^{24k/b} x_3 = 0$	$b\|24$
	$x_1^2 + x_2^3 + x_3^5 + z^{20k/b} x_2 = 0$	$b\|20$

Table 5.1: Classification of irregular singularities with regular semisimple coefficient matrices and the 3-fold singularities corresponding to them. In the table, $b|N$ means that b is a divisor of N.

where $Z \in \mathrm{Mat}_{N\times N}(\mathbb{C})$. When b is odd, we have

$$Z = \begin{pmatrix} 0_{N-br} & & & \\ & a_1\Xi & & \\ & & \ddots & \\ & & & a_r\Xi \end{pmatrix}. \tag{5.22}$$

When b is even, we define $\Xi' = \{1, \omega^2, \omega^4, \ldots, \omega^{b-2}\}$, then $\Xi = \Xi' \cup (-\Xi')$. Then

order of singularity b	mass parameter	exact marginal deformations
$b\|N$	$N/b-1$	$N/b-1$
$b\|(N-1)$	$(N-1)/b$	$(N-1)/b-1$

Table 5.2: Summary of mass parameters and number of exact marginal deformations in A_{N-1}.

the coefficient matrix takes the form

$$Z = \begin{pmatrix} 0_{N-rb/2} & & & \\ & a_1\Xi' & & \\ & & \ddots & \\ & & & a_r\Xi' \end{pmatrix}. \tag{5.23}$$

Counting of physical parameters in two cases are different, as we will see momentarily. In particular, the allowed mass parameters are different for these two situations.

From irregular puncture to parameters in SCFT. We have classified the allowed order of poles for Higgs field in (5.16), and write down in classical algebras the coefficient matrix T_i. The free parameters in T_i encode exact marginal deformations and number of mass parameters.

Based on the discussion above and the coefficient matrix, we conclude that the number of mass parameters is equal to $\mathrm{rank}(\mathfrak{g}^0)$ and the number of exact marginal deformation is given by $\mathrm{rank}(G^0|\mathfrak{g}^k)-1$ if the leading matrix is in $\mathfrak{g}^k$. We may list the maximal number of exact marginal deformations and number of mass parameters in tables 5.2 - 5.6. We focus here only in the case when T's are regular semisimple, while for semisimple situation the counting is similar.

• **Argyres-Douglas matter**. We call the AD theory without any marginal deformations the *Argyres-Douglas matter*. They are isolated SCFTs and thus are the fundamental building blocks in S-duality. In the weakly coupled description, we should be able to decompose the theory into Argyres-Douglas matter connected by gauge groups.

Degeneration and graded Coulomb branch dimension. Our previous discussion focused on the case where we choose a generic regular semisimple element for a

order of singularity b	mass parameter	exact marginal deformations
odd, $b\|N$	N/b	$N/b - 1$
even, $b\|N$	0	$2N/b - 1$
odd, $b\|(2N-2)$	$(N-1)/b + 1$	$(N-1)/b - 1$
even, $b\|(2N-2)$	1 or 0	$(2N-2)/b - 1$

Table 5.3: Summary of mass parameters and number of exact marginal deformations in D_N. Note when b is even divisor of $2N-2$ but not a divisor of $N-1$, the number of mass parameter is zero, otherwise it is one.

order of singularity b	mass parameter	exact marginal deformations
12	0	0
9	0	0
8	1	0
6	0	1
4	2	1
3	0	2
2	2	3

Table 5.4: Summary of mass parameters and number of exact marginal deformations in E_6.

order of singularity b	mass parameter	exact marginal deformations
18	0	0
14	0	0
9	1	0
7	1	0
6	0	2
3	1	2
2	0	6

Table 5.5: Summary of mass parameters and number of exact marginal deformations in E_7.

order of singularity b	mass parameter	exact marginal deformations
30	0	0
24	0	0
20	0	0
15	0	0
12	0	1
10	0	1
8	0	1
6	0	3
5	0	1
4	0	3
3	0	3
2	0	7

Table 5.6: Summary of mass parameters and number of exact marginal deformations in E_8.

given positive rank grading. More generally, we may consider T_k semisimple. We first examine the singularity where $b = 1$:

$$\Phi \sim \frac{T_\ell}{z^\ell} + \frac{T_{\ell-1}}{z^{\ell-1}} + \cdots + \frac{T_1}{z^1}, \tag{5.24}$$

with $T_\ell \subset \cdots \subset T_2 \subset T_1$ [105]. For this type of singularity, the *local* contribution to the dimension of Coulomb branch is

$$\dim_{\mathbb{C}}^{\rho} \text{Coulomb} = \frac{1}{2} \sum_{i=1}^{\ell} \dim(\mathcal{O}_{T_i}). \tag{5.25}$$

This formula indicates that the Coulomb branch dimensions are summation of each semisimple orbit in the irregular singularity. It is reminiscent of the regular puncture case, where the local contribution to Coulomb branch of each puncture is given by half-dimension of the nilpotent orbits, $\dim_{\mathbb{C}}^{\rho} \text{Coulomb} = \frac{1}{2} \dim \mathfrak{S}(\mathcal{O}_\rho)$ [113].

To label the degenerate irregular puncture, one may specify the centralizer for each T_ℓ. Given a semisimple element $x \in \mathfrak{g}$, the centralizer $\mathfrak{g}^x$ is called a *Levi subalgebra*, denoted as $\mathfrak{l}$. In general, it may be expressed by

$$\mathfrak{l} = \mathfrak{h} \oplus \sum_{\Delta' \subset \Delta} \mathfrak{g}_\alpha, \tag{5.26}$$

where $\mathfrak{h}$ is a Cartan subalgebra and Δ' is a subset of the simple root Δ of $\mathfrak{g}$. We care about its semisimple part, which is the commutator $[\mathfrak{l}, \mathfrak{l}]$.

The classification of the Levi subalgebra is known. For $\mathfrak{g}$ of ADE type, we have

- $\mathfrak{g} = A_{N-1}$: $\mathfrak{l} = A_{i_1} \oplus A_{i_2} \oplus \ldots A_{i_k}$, with $(i_1 + 1) + \cdots + (i_k + 1) = N$.
- $\mathfrak{g} = D_N$: $\mathfrak{l} = A_{i_1} \oplus A_{i_2} \oplus \ldots A_{i_k} \oplus D_j$, with $(i_1 + 1) + \cdots + (i_k + 1) + j = N$.
- $\mathfrak{g} = E_6$: $\mathfrak{l} = E_6,\ D_5,\ A_5,\ A_4 + A_1,\ 2A_2 + A_1,\ D_4,\ A_4,\ A_3 + A_1,\ 2A_2,\ A_2 + 2A_1,\ A_3,\ A_2 + A_1,\ 3A_1,\ A_2,\ 2A_1,\ A_1,\ 0$.
- $\mathfrak{g} = E_7$: $E_7,\ E_6,\ D_6,\ D_5 + A_1,\ A_6,\ A_5 + A_1,\ A_4 + A_2,\ A_3 + A_2 + A_1,\ D_5,\ D_4 + A_1,\ A_5',\ A_5'',\ A_4 + A_1,\ A_3 + A_2,\ A_3 + 2A_1,\ 2A_2 + A_1,\ A_2 + 3A_1,\ D_4,\ A_4,\ (A_3 + A_1)',\ (A_3 + A_1)'',\ 2A_2,\ A_2 + 2A_1,\ 4A_1,\ A_3,\ A_2 + A_1,\ (3A_1)',\ (3A_1)'',\ A_2,\ 2A_1,\ A_1,\ 0$.
- $\mathfrak{g} = E_8$: $E_8,\ E_7,\ E_6 + A_1,\ D_7,\ D_5 + A_2,\ A_7,\ A_6 + A_1,\ A_4 + A_3,\ A_4 + A_2 + A_1,\ E_6,\ D_6,\ D_5 + A_1,\ D_4 + A_2,\ A_6,\ A_5 + A_1,\ A_4 + A_2,\ A_4 + 2A_1,\ 2A_3,\ A_3 + A_2 + A_1,\ 2A_2 + 2A_1,\ D_5,\ D_4 + A_1,\ A_5,\ A_4 + A_1,\ A_3 + A_2,\ A_3 + 2A_1,\ 2A_2 + A_1,\ A_2 + A_1,\ D_4,\ A_4,\ A_3 + A_1,\ 2A_2,\ A_2 + 2A_1,\ 4A_1,\ A_3,\ A_2 + A_1,\ 3A_1,\ A_2,\ 2A_1,\ A_1,\ 0$.

Fixing the Levi subalgebra for T_i, the corresponding dimension for the semisimple orbit is given by

$$\dim(O_{T_i}) = \dim G - \dim L_i. \tag{5.27}$$

We emphasize here that Levi subalgebra itself completely specify the irregular puncture. However, they may share the semisimple part $[\mathfrak{l}, \mathfrak{l}]$. The SCFTs defined by them can be very different. Motivated by the similarity between (5.25) and that of regular punctures, we wish to use nilpotent orbit to label the semisimple orbit O_{T_i}, so that one can calculate the graded Coulomb branch spectrum.

The correspondence lies in the theorem we introduced before: there is a one-to-one correspondence between the nilpotent orbit $O_\rho^{\mathfrak{g}}$ and the pair $(\mathfrak{l}, O_e^{\mathfrak{l}})$. Moreover, we only consider those nilpotent orbit with principal $O_e^{\mathfrak{l}}$. For $\mathfrak{g} = A_{N-1}$, principal orbit is labelled by partition $[N]$, while for D_N, it is the partition $[2N-1, 1]$. Then, given a Nahm label whose $O_e^{\mathfrak{l}}$ is principal, we take the Levi subalgebra piece $\mathfrak{l}$ out of the pair $(\mathfrak{l}, \mathcal{O}_e^{\mathfrak{l}})$; we use the Nahm label ρ as the tag such T_i. We conjecture that this fully characterize the coefficients T_i.

To check the validity, we recall orbit induction [187, 188]. Let $\mathcal{O}_{\bar{e}}^{\mathfrak{l}}$ be an arbitrary nilpotent orbit in $\mathfrak{l}$. Take a generic element m in the center $\mathfrak{z}$ of $\mathfrak{l}$. We define

$$\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\bar{e}}^{\mathfrak{l}} := \lim_{m \to 0} \mathcal{O}_{m+\bar{e}}, \tag{5.28}$$

which is a nilpotent orbit in $\mathfrak{g}$. It is a theorem that the induction preserves codimension:

$$\dim G - \dim_{\mathbb{C}} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\bar{e}}^{\mathfrak{l}} = \dim L - \dim_{\mathbb{C}} \mathcal{O}_{\bar{e}}^{\mathfrak{l}}. \tag{5.29}$$

In particular, when $\mathcal{O}_{\bar{e}}^{\mathfrak{l}}$ is zero orbit in $\mathfrak{l}$, from (5.29) we immediately conclude that

$$\dim \mathcal{O}_T = \dim G - \dim L = \dim_{\mathbb{C}} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_0^{\mathfrak{l}}, \tag{5.30}$$

for T the semisimple orbit fixed by L. The Bala-Carter theory is related to orbit induction via [178]

$$\dim \mathfrak{S}(\mathcal{O}_\rho) = \dim_{\mathbb{C}} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathfrak{S}(\mathcal{O}_{\text{principal}}^{\mathfrak{l}}) = \dim_{\mathbb{C}} \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_0^{\mathfrak{l}} = \dim \mathcal{O}_T. \tag{5.31}$$

Therefore, treating each semisimple orbit $\mathcal{O}_T$ as a nilpotent orbit $\mathcal{O}_\rho$, their local contribution to Coulomb branch is exactly the same.

In the A_{N-1} case, Levi subalgebra contains only A_i pieces; the distinguished nilpotent orbit in it is unique, which is $[i+1]$. Therefore, we have a one-to-one correspondence between Nahm partitions and Levi subalgebra. More specifically, a semisimple element of the form

$$x = \mathrm{diag}(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots, a_k), \tag{5.32}$$

where a_i appears r_i times, has Levi subgroup

$$L = S[U(r_1) \times U(r_2) \times \cdots \times U(r_k)], \tag{5.33}$$

whose Nahm label is precisely $[r_1, r_2, \ldots, r_k]$.

For D_N case, if the semisimple element we take looks like

$$x = \mathrm{diag}(a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k, -a_1, \ldots, -a_1, \ldots, -a_k, \ldots, -a_k, 0, \ldots, 0), \tag{5.34}$$

where a_i appears r_i times and 0 appears $\tilde{r}$ times with $\sum 2r_i + \tilde{r} = 0$, the Levi subgroup is given by

$$L = \prod_i U(r_i) \times SO(\tilde{r}). \tag{5.35}$$

We call L of type $[r_1,\ldots,r_k;\tilde{r}]$. Here we see clearly the ambiguity in labelling the coefficient T_i using Levi subalgebra. For instance, when $\mathfrak{g}=D_4$, we have $[1;6]$ and $[4;0]$ having the same Levi subalgebra, but clearly they are different type of matrices and the SCFT associated with them have distinct symmetries and spectrum. We will examine them in more detail in section 5.4.

With Nahm labels for each T_i, we are now able to compute the graded Coulomb branch spectrum. For each Nahm label, we have a collection of the pole structure $\{p^{\alpha}_{i_1},\ldots,p^{\alpha}_{i_r}\}$ for i_k the degrees of differentials. There are also constraints that reduce or modifies the moduli. Then we conjecture that, at differential of degree k the number of graded moduli is given by

$$d_k=\sum_{\alpha}\left(p^{\alpha}_k-s^{\alpha}_k+t^{\alpha}_k\right)-2k+1. \tag{5.36}$$

They come from the term u_i in $(u_0+u_1z+\ldots+u_{d_k-1}z^{d_k-1})x^{h^\vee-k}$, with $h^\vee$ the dual Coxeter number.

However, it might happen that there are constraints of the form $c^{(2k)}=\left(a^{(k)}\right)^2$ in which k is not a degree for the differentials. In this case, t_k should be added to the some $k'>k$ such that $d^{\text{local}}_{k'}<k'-1$.

When a regular puncture with some Nahm label is added to the south pole, one may use the same procedure to determine the contributions of each differential to the Coulomb branch moduli. We denote them as $\{d^{(\text{reg})}_k\}$. Then, we simply extend the power of $z_\beta x^{2(N-k)}$ to $-d^{(\text{reg})}_k<\beta<d_k$.

• *Example*: let us consider an E_6 irregular puncture of class $(k,1)$ where k is very large. Take $T_\ell=\cdots=T_2$ with Levi subalgebra D_5, and T_1 with Levi subalgebra 0. We associate to T_i with $i\geq 2$ Nahm label D_5. As a regular puncture, it has pole structure $\{1,2,3,4,4,6\}$ with complicated relations [179]:

$$\begin{aligned}
&c^{(6)}_3=\frac{3}{2}c^{(2)}_1a^{(4)}_2, \qquad c^{(8)}_4=3\left(a^{(4)}_2\right)^2,\\
&c^{(9)}_4=-\frac{1}{4}c^{(5)}_2a^{(4)}_2, \qquad c^{(12)}_6=\frac{3}{2}\left(a^{(4)}_2\right)^3,\\
&c^{(12)}_5=\frac{3}{4}c^{(8)}_3a^{(4)}_2.
\end{aligned} \tag{5.37}$$

After subtracting it we have pole structure $\{1,2,2,3,3,4\}$. There is one new moduli $a^{(4)}$, and we add it to ϕ_5. The Nahm label 0 has pole structure $\{1,4,5,7,8,11\}$. Then we have the Coulomb branch spectrum from such irregular puncture as

$$\begin{aligned} &\phi_2 : \frac{2k}{k+1}, \ldots, \frac{k+2}{k+1}, \quad \phi_5 : \frac{5k}{k+1}, \ldots, \frac{2k+3}{k+1}, \\ &\phi_6 : \frac{6k}{k+1}, \ldots, \frac{4k+5}{k+1}, \quad \phi_8 : \frac{8k}{k+1}, \ldots, \frac{5k+6}{k+1}, \\ &\phi_9 : \frac{9k}{k+1}, \ldots, \frac{6k+7}{k+1}, \quad \phi_{12} : \frac{12k}{k+1}, \ldots, \frac{8k+9}{k+1}. \end{aligned} \tag{5.38}$$

One can carry out similar analysis for general irregular singularity of class (k, b). The idea is to define a cover coordinate ω and reduce the problem to integral order of pole. Consider an irregular singularity defined by the following data $\Phi = T/z^{2+\frac{k}{b}} + \ldots$; we define a cover coordinate $z = \omega^b$ and the Higgs field is reduced to

$$\Phi = \frac{T'}{\omega^{k+b+1}} + \ldots \tag{5.39}$$

Here T' is another semisimple element deduced from T, see examples in section 5.4. Once we go to this cover coordinate, we can use above study of degeneration of irregular singularity with integral order of pole. We emphasize here that not all degeneration are allowed due to the specific form of T.

Constraint from conformal invariance. As we mentioned, not all choices of semisimple coefficient T_i define SCFTs. Consider the case $b = 1$, and the irregular singularity is captured by by a sequence of Levi subgroup $\mathfrak{l}_\ell \supset \mathfrak{l}_{\ell-1} \supset \ldots \supset \mathfrak{l}_1$. The necessary condition is that the number of parameters in the leading order matrix T_k should be no less than the number of exact marginal deformations. As will be shown later, it turns out that this condition imposes the constraint that

$$\mathfrak{l}_\ell = \mathfrak{l}_{\ell-1} \ldots = \mathfrak{l}_2 = \mathfrak{l}, \tag{5.40}$$

with $\mathfrak{l}_1$ arbitrary. Then we have following simple counting rule of our SCFT:

- The maximal number of exact marginal deformation is equal to $r - r_{\mathfrak{l}} - 1$, where r the rank of $\mathfrak{g}$ and $r_{\mathfrak{l}}$ the rank of semi-simple part of $\mathfrak{l}$. The extra minus one comes from scaling of coordinates.
- The maximal flavor symmetry is $G_{\mathfrak{l}} \times U(1)^{r-r_{\mathfrak{l}}}$, and here $G_{\mathfrak{l}}$ is the semi-simple part of $\mathfrak{l}$.

Similarly, for $b \neq 1$, the conformal invariance implies that all the coefficients except T_1 should have the same Levi subalgebra. This is automatic when the grading is regular semisimple, but it is an extra restriction on general semi-simple grading. For example, consider A_{N-1} type (2,0) theory with following irregular singularity whose leading order matrix takes the form:

$$T = \begin{pmatrix} a_1\Xi & & & \\ & \ddots & & \\ & & a_r\Xi & \\ & & & 0_{(N-rb)} \end{pmatrix}. \tag{5.41}$$

When the subleading term in (5.16) has integral order, the corresponding matrix can take the following general form:

$$T^{'} = \begin{pmatrix} a_1'\mathbb{I}_b & & & \\ & \ddots & & \\ & & a_r'\mathbb{I}_b & \\ & & & \mathbb{K}_{(N-rb)} \end{pmatrix}. \tag{5.42}$$

Here $\mathbb{I}_b$ is the identify matrix with size b, and $\mathbb{K}_{N-rb}$ is a generic diagonal matrix. However, due to the constraints, only for $\mathbb{K}_{N-rb} = \kappa\,\mathbb{I}_{N-rb}$, $T^{'}$ has the same Levi-subalgebra as T. This situation is missed in previous studies [132].

SW curve and Newton polygon

Recall that the SW curve is identified as the spectral curve $\det(x - \Phi(z))$ in the Hitchin system. For regular semisimple coefficient T_i without regular puncture, we may map the curve to the mini-versal deformation of three fold singularity in type IIB construction. For given Lie algebra $\mathfrak{g}$, we have the deformed singularity:

$$\begin{aligned}
&A_{N-1}:\ x_1^2 + x_2^2 + x_3^N + \phi_2(z)x_3^{N-2} + \ldots + \phi_{N-1}(z)x_3 + \phi_N(z) = 0,\\
&D_N:\ x_1^2 + x_2^{N-1} + x_2x_3^2 + \phi_2(z)x_2^{N-2} + \ldots + \phi_{2N-4}(z)x_2 + \phi_{2N-2}(z) + \tilde{\phi}_N(z)x_3 = 0,\\
&E_6:\ x_1^2 + x_2^3 + x_3^4 + \phi_2(z)x_2x_3^2 + \phi_5(z)x_2x_3 + \phi_6(z)x_3^2 + \phi_8(z)x_2 + \phi_9(z)x_3 + \phi_{12}(z) = 0,\\
&E_7:\ x_1^2 + x_2^3 + x_2x_3^3 + \phi_2(z)x_2^2x_3 + \phi_6(z)x_2^2 + \phi_8(z)x_2x_3 + \phi_{10}(z)x_3^2\\
&\qquad + \phi_{12}(z)x_2 + \phi_{14}(z)x_3 + \phi_{18}(z) = 0,\\
&E_8:\ x_1^2 + x_2^3 + x_3^5 + \phi_2(z)x_2x_3^3 + \phi_8(z)x_2x_3^2 + \phi_{12}(z)x_3^3 +\\
&\qquad \phi_{14}(z)x_2x_3 + \phi_{18}(z)x_3^2 + \phi_{20}(z)x_2 + \phi_{24}(z)x_3 + \phi_{30}(z) = 0,
\end{aligned} \tag{5.43}$$

and ϕ_i is the degree i differential on Riemann surface.

A useful diagrammatic approach to represent SW curve is to use Newton polygon. When irregular singularity degenerates, the spectrum is a subset of that in regular semisimple T_i's, so understanding Newton polygon in regular semisimple case is enough.

The rules for drawing and reading off scaling dimensions for Coulomb branch spectrum is explained in [18, 104]. In particular, the curve at the conformal point determines the scaling dimension for x and z, by requiring that the SW differential $\lambda = xdz$ has scaling dimension 1.

• $\mathfrak{g} = A_{N-1}$. The Newton polygon for regular semisimple coefficient matrices is already given in [18] and we do not repeat here. Here we draw the polygon when T is semisimple for some semisimple grading, in the form (5.41). We give one example; see Figure 5.1.

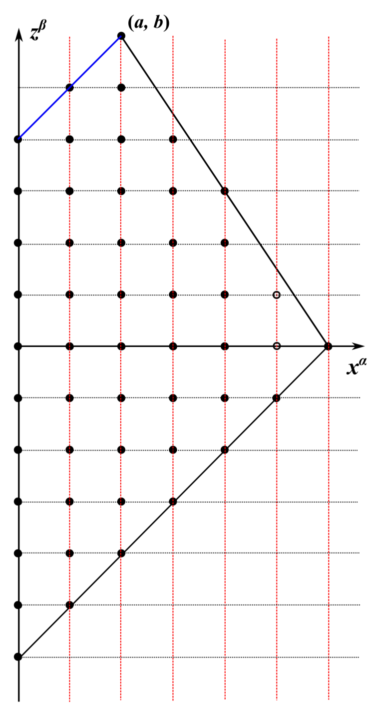


Figure 5.1: An example of Newton polygon for A_5 theory with semisimple grading. Each black dot represents a monomial in SW curve. The white dots mean that the monomials are omitted. The letters have scaling dimension $[x] = 3/5$, $[z] = 2/5$. In general, if the vertex at the top has coordinate (a, b), then we have the relation $(N - a)[x] = b[z]$ and $[x] + [z] = 1$.

• $\mathfrak{g} = D_N$. There are two types of Newton polygon, associated with Higgs field

$$\Phi \sim \frac{T}{z^{2+\frac{k}{N}}}, \qquad \Phi \sim \frac{T}{z^{2+\frac{k}{2N-2}}}, \tag{5.44}$$

We denote two types and their SW curves at conformal point as

$$\begin{aligned} D_N^{(N)}[k]: &\quad x^{2N} + z^{2k} = 0, \\ D_N^{(2N-2)}[k]: &\quad x^{2N} + x^2 z^k = 0. \end{aligned} \tag{5.45}$$

The full curve away from conformal point, and with various couplings turned on, is given by (5.7). In Figure 5.2, we list examples of such a Newton polygon.

• $\mathfrak{g} = E_{6,7,8}$. We can consider Newton polygon from the 3-fold singularities. In this way we may draw the independent differentials unambiguously. We give the case for E_6 with $b = 8, 9, 12$ in Figure 5.3. The other two exceptional algebras are similar.

5.3 Mapping to a punctured Riemann surface

As we mentioned in section 5.1, to generate S-duality we construct an auxiliary Riemann sphere Σ' from the initial Riemann sphere Σ with irregular punctures. We now describe the rules. The motivation for such construction comes from 3d mirror in class $\mathcal{S}$ theory [130, 140, 141]. To recapitulate the idea, from 3d mirror perspective we may interpret the Gaiotto duality as splitting out the quiver theories with three quiver legs. Each quiver leg carries a corresponding flavor symmetry on the Coulomb branch and can be gauged. The 3d mirror of A_{N-1} type Argyres-Douglas theories are know and they are also constructed out of quiver legs. We then regard each quiver leg as a "marked point" on the Riemann sphere Σ'. Unlike the class $\mathcal{S}$ counterpart, now there will be more types of marked points with different rank.

Recall our setup is that the initial Riemann sphere Σ is given by one irregular singularity of class (k, b), with coefficient satisfying

$$T_\ell = T_{\ell-1} = \cdots = T_3 = T_2, \qquad T_1 \text{ arbitrary}, \qquad \ell = k + b + 1, \tag{5.46}$$

possibly with a regular puncture Q. We denote it as $\left(III_{k,b}^{\{\mathfrak{l}_i\}_{i=1}^{\ell}}, \; Q\right)$, where $\mathfrak{l}_i$ is the Levi subalgebra for the semisimple element T_i. We now describe the construction of Σ'.

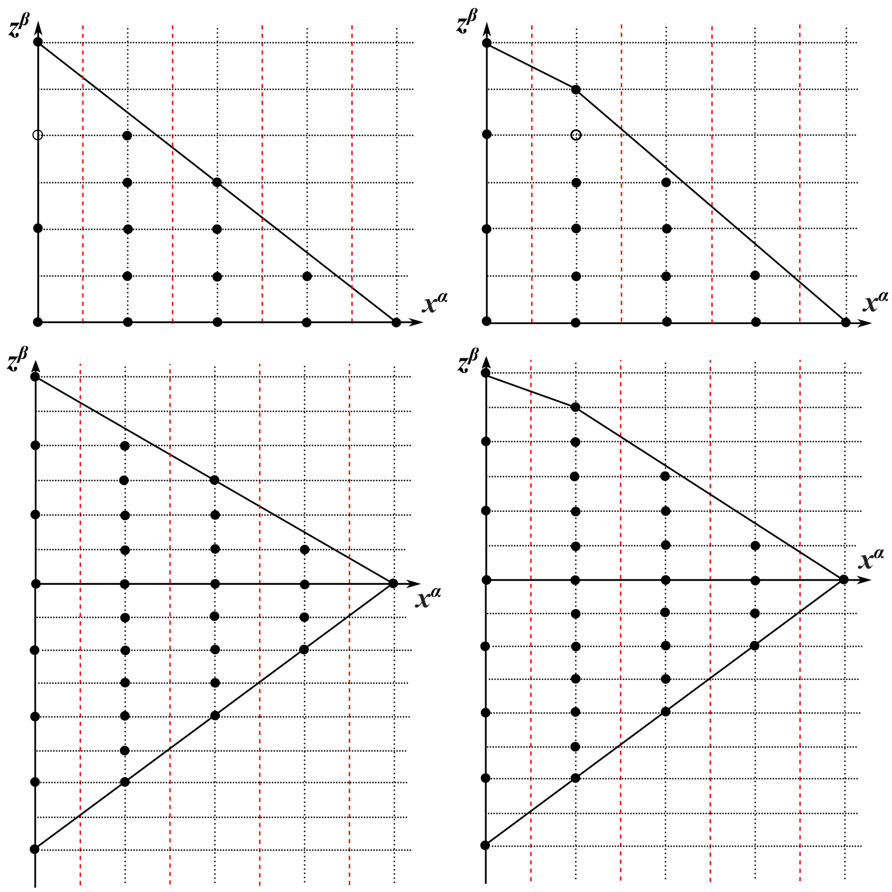


Figure 5.2: A collection of Newton polygon for examples of SCFT with $\mathfrak{g} = D_N$. Each black dot represents a monomial in SW curve in the form of $x^\alpha z^\beta$; except that for the x^0 axis, each term represents the Pfaffian $\tilde{\phi}$, so we shall read it as $\sqrt{z^\beta}$. The white dots mean that the monomials are omitted. The upper left diagram gives $D_4^{(4)}[3]$ theory, while the upper right diagram gives $D_4^{(6)}[5]$. The two lower diagrams represent the same irregular puncture, but with an additional regular puncture (*e.g.* maximal) at the south pole. We denote them as $(D_4^{(4)}[3], F)$ and $(D_4^{(6)}[5], F)$ theory, respectively.

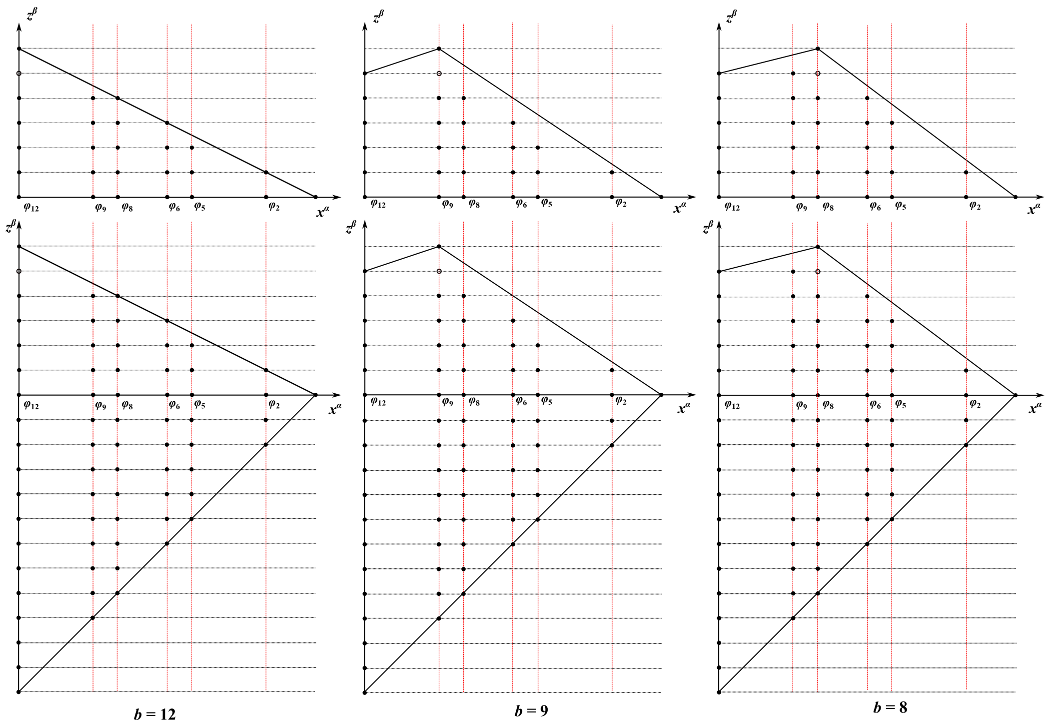


Figure 5.3: A collection of Newton polygons for examples of SCFT with $\mathfrak{g} = E_6$. Each black dot represents a monomial in SW curve in the three fold form. The white dots mean that the monomials are omitted. The upper left diagram gives $b = 12$, $k = 6$ theory, while the upper middle diagram gives $b = 9$, $k = 6$ theory and the upper right gives $b = 8$, $k = 6$ theory. The three lower diagrams represent the same irregular puncture, but with an additional regular puncture (*e.g.* maximal) at south pole.

• **Lie algebra** $\mathfrak{g} = A_{N-1}$. A generic matrix looks like

$$T_i = \mathrm{diag}\left(\underbrace{a_1\Xi_b, \dots, a_1\Xi_b}_{r_1}, \dots, \underbrace{a_s\Xi_b, \dots, a_s\Xi_b}_{r_s}, \underbrace{0, \dots, 0}_{N-(\sum r_j)b}\right), \qquad 2 \le i \le \ell, \tag{5.47}$$

The theory is represented by a sphere with one red marked point (denoted as a cross $\times$) representing regular singularity; one blue marked point (denoted as a square ■) representing 0's in T_i, which is further associated with a Young tableaux with size $N - (\sum r_j)b$ to specify its partition in T_1. There are s black marked points (denoted as black dots •) with size r_j, $j = 1, \dots, s$ and each marked point carrying a Young tableaux of size r_j. Notice that there are $s-1$ exact marginal deformations which are the same as the dimensions of the complex structure moduli of punctured sphere.

There are two exceptions: if $b = 1$, the blue marked point is just the same as the black marked point. If $k = 1, b = 1$, the red marked point is the same as the black marked point as well [132].

• **Lie algebra** $\mathfrak{g} = D_N$. We have the representative of Cartan subalgebra as (5.21) and when b is odd,

$$Z = \mathrm{diag}(\underbrace{a_1\Xi_b, \ldots, a_1\Xi_b}_{r_1}, \ldots, \underbrace{a_s\Xi_b, \ldots, a_s\Xi_b}_{r_s}, \underbrace{0, \ldots, 0}_{N-(\sum r_j)b}), \tag{5.48}$$

while when b is even,

$$Z = \mathrm{diag}(\underbrace{a_1\Xi'_{b/2}, \ldots, a_1\Xi'_{b/2}}_{r_1}, \ldots, \underbrace{a_s\Xi'_{b/2}, \ldots, a_s\Xi'_{b/2}}_{r_s}, \underbrace{0, \ldots, 0}_{N-(\sum r_j)b/2}). \tag{5.49}$$

The theory is represented by a Riemann sphere with one red cross representing regular singularity, one blue puncture representing 0's in T_i; we also have a D-partition of $2\big[N - (\sum r_j)b\big]$ to specify further partition in T_1. Moreover, there are s black marked point with size r_j, $j = 1, \ldots, s$ and each marked point carrying a Young tableaux of size r_j (no requirement on the parity of entries).

• **Lie algebra** $\mathfrak{g} = E_{6,7,8}$: Let us start with the case $b = 1$, and the irregular puncture is labelled by Levi-subalgebra $L_l = \ldots = L_2 = \mathfrak{l}$ and a trivial Levi-subalgebra L_1. We note that there is at most one non-A type Lie algebra for $\mathfrak{l}$: $\mathfrak{l} = A_{i_1} + \ldots + A_{i_k} + \mathfrak{h}$; Let's define $a = \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{h}) - \sum_{j=1}^{k}(i_j + 1)$, we have the following situations:

- $a \geq 0$: we have k black punctures with flavor symmetry $U(i_j+1)$, $j = 1, \ldots, k$, and a more black marked point with $U(1)$ flavor symmetry; we have a blue puncture with H favor symmetry ($\mathfrak{h} = \mathrm{Lie}(H)$), and finally a red puncture representing the regular singularity.
- $a < 0$: When there is a $2A_1$ factor in $\mathfrak{l}$, we regard it as D_2 group and use a blue puncture for it; when the rank of $\mathfrak{l}$ is $\mathrm{rank}(\mathfrak{g}) - 1$, we put all A-type factor of $\mathfrak{l}$ in a single black marked point.

The $b \neq 1$ case can be worked out similarly.

AD matter and S-duality

We now discuss in more detail about the AD matter for $b = 1$. Recall that the number of exact marginal deformations is equal to $r - r_{\mathfrak{l}} - 1$, where $r = \mathrm{rank}(\mathfrak{g})$, and

Lie algebra $\mathfrak{g}$	Levi subalgebra associated to AD matter
A_{N-1}	$A_n + A_m,\ \ (n+1)+(m+1)=N$
D_N	$A_n + D_m,\ \ n+1+m=N$
E_6	$D_5,\ A_5,\ A_4+A_1,\ 2A_2+A_1$
E_7	$E_6,\ D_6,\ D_5+A_1,\ A_6,\ A_5+A_1,\ A_4+A_2,\ A_3+A_2+A_1$
E_8	$E_7,\ E_6+A_1,\ D_7,\ D_5+A_2,\ A_7,\ A_6+A_1,\ A_4+A_3,\ A_4+A_2+A_1$

Table 5.7: Possible Levi subalgebra for T_ℓ that corresponds to AD matter without exact marginal deformations.

$r_\mathfrak{l} = \mathrm{rank}(\mathfrak{l})$. The AD matter is then given by the Levi subalgebra with rank $r-1$. We can list all the possible Levi subalgebra for AD matters in table 5.7.

S-duality frames. With the auxiliary Riemann sphere Σ', we conjecture that the S-duality frame is given by different degeneration limit of Σ'; the quiver theory is given by gauge groups connecting Argyres-Douglas matter without exact marginal deformations. For AD theories of type $\mathfrak{g}$, the AD matter is given by three punctured sphere Σ': one red cross, one blue square, and one black dot. The rank of black dot plus the rank of blue square should equal to the rank of the red cross. See figure 5.4 for an illustration. Each marked points carry a flavor symmetry. Their flavor central charge is given by [132, 189]

$$k_G^{\mathrm{red}} = h^\vee - \frac{b}{k+b}, \quad k_G^{\mathrm{black/blue}} = h^\vee + \frac{b}{k+b}, \tag{5.50}$$

where $h^\vee$ is the dual Coxeter number of G. This constraints the configuration such that one can only connect black dot and red cross, or blue square with red cross to cancel one-loop beta function.

Central charges

The central charges a and c can be computed as follows [189, 190]:

$$2a - c = \frac{1}{4}\sum(2[u_i]-1), \quad a - c = -\frac{1}{24}\dim_{\mathbb{H}}\mathrm{Higgs}. \tag{5.51}$$

This formula is valid for the theory admits a Lagrangian 3d mirror. We know how to compute the Coulomb branch spectrum, and so the only remaining piece is the dimension of the Higgs branch, which can be read from the mirror.

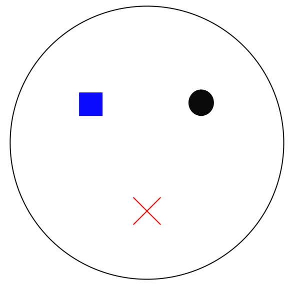


Figure 5.4: An example of Argyres-Douglas matter of type $\mathfrak{g}$. The theory has no exact marginal deformations, and in the meantime the punctured Riemann sphere Σ' has no complex structure moduli.

For theories with $b = 1$, the local contribution to the Higgs branch dimension with flavor symmetry G for red marked point is

$$\dim_{\mathbb{H}}^{\text{red}} \text{Higgs} = \frac{1}{2}(\dim G - \text{rank}(G)), \tag{5.52}$$

while for blue and black marked point, we have

$$\dim_{\mathbb{H}}^{\text{blue/black}} \text{Higgs} = \frac{1}{2}(\dim G + \text{rank}(G)). \tag{5.53}$$

The total contribution to the Higgs branch is the summation of them, except that for A_{N-1}, we need to subtract one.

5.4 S-duality for D_N theory

Class $(k, 1)$

In this section we first consider $\mathfrak{g} = D_N$, and the irregular singularity we take to be

$$\Phi = \frac{T_\ell}{z^\ell} + \frac{T_{\ell-1}}{z^{\ell-1}} + \cdots + \frac{T_1}{z} + T_{\text{reg}}, \tag{5.54}$$

where T_{reg} is the regular terms. This amounts to take $k = \ell - 2$, $b = 1$[4]. We settle the questions raised in previous sections: (i) we show which choices of T_i's give legitimate deformation for SCFT, (ii) we illustrate how to count graded Coulomb branch spectrum, and (iii) how to obtain its S-dual theory. In dealing with these questions, we first utilize the case $D_3 \simeq A_3$, where we already know the results [132].

[4]Careful readers may wonder whether $n_1 = 1$ comes from $D_N^{(N)}[k']$ or $D_N^{(2N-2)}[k']$, as their relevant coefficient matrices are different. However, in the case $n_1 = 1$, leaving two diagonal entries to be zero has the same Levi subgroup ($SO(2)$) as that of leaving it to be diag$(a, -a)$, which is $U(1)$. So two cases actually coincide.

Coulomb branch spectrum. Recall that when irregular puncture degenerates, one maps each semisimple orbit O_{T_i} to a nilpotent orbit with the same dimension. We may use the recipe of regular punctures to calculate the Coulomb branch spectrum. Let us see how this works.

Example 1: non-degenerating D_4 theory of class $(1,1)$. As we have $\ell = 3$, there are three regular punctures whose labels are $\left[1^8\right]$. For such a maximal puncture, the pole structure for the differential is $\{p_2, p_4, p_6; \tilde{p}\} = \{1,3,5;3\}$ and there are no relations. Then, the total contributions to the moduli are $\{d_2, d_4, d_6; \tilde{d_4}\} = \{0,2,4;2\}$. This is consistent with the Newton polygon of $D_4^{(4)}[4]$.

Example 2: degenerating D_4 theory of class $(1,1)$. In this example we take T_3 and T_2 to be labelled by Levi subalgebra of type $[1,1,1;2]$, while T_1 is still of type $[1,1,1,1;0]$. For the former, we see that it is the same as the Levi subalgebra $[1,1,1,1;0]$. Then we are back to the previous example. This is indeed the same spectrum as indicated by Newton polygon of $D_4^{(6)}[6]$.

Example 3: degenerating D_3 theory of class $(1,1)$. We take T_3 and T_2 to have Levi subalgebra of type $[2,1;0]$, giving a regular puncture labelled by Nahm partition $[2,2,1,1]$. In terms of Nahm partition for A_3, they are equivalent to $[2,1,1]$. We also take T_1 to be maximal. From A_3, the algorithm in [18] determines the set of Coulomb branch operators to be $\{3/2\}$. In the language of D_3, the partition $[2,2,1,1]$ gives the pole structure $\{1,2;2\}$, while the maximal puncture has pole structure $\{1,3;2\}$; both of them have no constraints. Then, $\{d_2, d_4; \tilde{d_3}\} = \{0,0;1\}$, giving a Coulomb branch moduli with dimension $3/2$. So we see two approaches agree.

Constraints on coefficient matrices. As we mentioned before, not every choice of $\{T_\ell, T_{\ell-1}, \ldots, T_1\}$ is allowed for the SCFT to exist. Those which are allowed must have $T_\ell = \cdots = T_2$, and T_1 is a further partition of them. In this section we show why this is so.

The idea of our approach is that the total number of exact marginal deformations shall not exceed the maximum determined by the leading matrix T_ℓ. We examine it on a case by case basis.

D_3. In this case we may directly use the results of [132]. Our claim holds.

D_4. First of all we list the correspondence between the Nahm label of the regular

Levi subalgebra	matrix Z	regular puncture	pole structure	constraints	flavor symmetry
$[1,1,1,1;0]$	$\mathrm{diag}(a,b,c,d)$	$[1^8]$	$\{1,3,5;3\}$	–	–
$[2,1,1;0]$	$\mathrm{diag}(a,a,b,c)$	$[2^2,1^4]$	$\{1,3,4;3\}$	–	$SU(2)$
$[1,1;4]$	$\mathrm{diag}(0,0,b,c)$	$[3,1^5]$	$\{1,3,4;2\}$	–	$SO(4)$
$[2,2;0]$	$\mathrm{diag}(a,a,b,\pm b)$	$[2^4]^{I,II}$	$\{1,3,4;3\}$	$c_3^{(4)} \pm 2\tilde{c}_3 = 0$	$SU(2)\times SU(2)$
$[3,1;0]$	$\mathrm{diag}(a,a,a,b)$	$[3,3,1,1]$	$\{1,2,4;2\}$	$c_4^{(6)} = (a_3)^2$	$SU(3)$
$[2;4]$	$\mathrm{diag}(a,a,0,0)$	$[3,2,2,1]^*$	$\{1,2,4;2\}$	–	$SU(2)\times SO(4)$
$[1;6]$	$\mathrm{diag}(0,0,0,a)$	$[5,1,1,1]$	$\{1,2,2;1\}$	–	$SO(6)$
$[4;0]$	$\mathrm{diag}(a,a,a,\pm a)$	$[4,4]^{I,II}$	$\{1,2,3;2\}$	$c_2^{(4)} \pm 2\tilde{c}_2 = (c_1^{(2)})^2/4$, $c_3^{(6)} = \mp\tilde{c}_2 c_1^{(2)}$	$SU(4)$

Table 5.8: Association of a nilpotent orbit to a Levi subalgebra for D_4. Here Z follows the convention in (5.21). The partition $[3,2,2,1]$ is non-special, and we use the * to mark it. In the last column we list the semisimple part of maximal possible flavor symmetry. The partition $[5,3]$ and $[7,1]$ are excluded; the first one is non-principal in $\mathfrak{so}(8)$ while the second gives trivial zero matrix.

puncture and the Levi subalgebra in table 5.8. The regular puncture data are taken from [173]. There are several remarks. For very even partitions, we have two matrix representation for two nilpotent orbits; they cannot be related by Weyl group actions[5]. Moreover, we also see that there are multiple coefficient matrices sharing the same Levi subalgebra; *e.g.* $[4;0]$ and $[1;6]$. Therefore, we do need regular puncture and Nahm label to distinguish them. Finally, we need to exclude orbit which is itself distinguished in D_4, as their Levi subalgebra is maximal, meaning we have zero matrix.

Now consider $\ell = 3$, and T_3 has the Levi subalgebra $[1,1;4]$, with one exact marginal deformation. One can further partition it into the orbit with Levi subalgebra $[2,1,1;0]$ and $[1,1,1,1;0]$. If we pick T_2 to be $[2,1,1;0]$, then no matter what we choose for T_1, there will be two dimension-2 operators; this is a contradiction. So T_2 must be equal to T_3.

The second example has $\ell = 3$, but T_3 now is associated with $[3,3,1,1]$. This puncture has a relation $c_4^{(6)} = (a^{(3)})^2$, so we remove one moduli from ϕ_6, and add one moduli to ϕ_4. The possible subpartitions are $[2^2,1^4]$, $[1^8]$. If $T_2 \neq T_3$ then there

[5] The Weyl group acts on entries of $Z = \mathrm{diag}(a_1, a_2, \ldots, a_N)$ by permuting them or simultaneously flip signs of even number of elements.

will be two exact marginal deformations from ϕ_4 and $\tilde{\phi}$. This is a contradiction, so we must have $T_2 = T_3$.

As a third example, we may take $\ell = 4$ and T_4 corresponding to the regular punctures $[2^4]$, whose pole structure is $\{1, 3, 4; 3\}$, with one constraints $c_3^{(4)} \pm 2\tilde{c}_3 = 0$. Then each of the local contribution to Coulomb moduli is $\{d_2, d_4, d_6; d_3\} = \{1, 2, 4; 3\}$. From the matrix representation we know there is one exact marginal coupling. If we pick T_3 to be $[2^2, 1^4]$, then by simple calculation we see that there are two dimension 2 operators. So we have to pick $T_3 = T_4$. Similarly, we have to pick $T_2 = T_3 = T_4$. Therefore, we again conclude that we must have $T_4 = T_3 = T_2$, while T_1 can be arbitrary.

D_5. We now check the constraints for the Lie algebra D_5. To begin with, we list the type of Levi-subgroup and its associated regular puncture in table 5.9. Now we examine the constraints on coefficient matrices. We first take $\ell = 3$, and pick T_3 to be of the type $[3, 2; 0]$ whose associated regular puncture is $[3, 3, 2, 2]$. There is a constraint $c_6^{(8)} = \left(c_3^{(4)}\right)^2/4$, so the local contribution to Coulomb branch is $\{d_2, d_4, d_6, d_8; d_5\} = \{1, 3, 4, 5; 3\}$. If we take T_2 to be *e.g.*, $[2^4, 1^2]$, then the moduli from $\tilde{\phi}$ contribute one more exact marginal deformations other than ϕ_4, which is a contradiction. Therefore, we again conclude that we must have $T_3 = T_2$, with arbitrary subpartition T_1.

Based on the above examples and analogous test for other examples, we are now ready to make a conjecture about the classification of SCFT for degenerating irregular singularities:

• **Conjecture.** In order for the maximal irregular singularity (5.54) of type D to define a viable SCFT in four dimensions, we must have $T_\ell = T_{\ell-1} = \cdots = T_2$ $(\ell \geq 3)$, while T_1 can be arbitrary subpartition of T_i.

We emphasize at last that when $\ell = 2$, the scaling for x in SW curve is zero. Therefore, we may have arbitrary partition T_2 and T_1, so that $\mathcal{O}_{T_2} \subset \mathcal{O}_{T_1}$.

Generating S-duality frame. With the above ingredients in hand, we are now ready to present an algorithm that generates S-duality for various Argyres-Douglas theories of D type. This may subject to various consistency checks. For example, the collection of the Coulomb branch spectrum should match on both sides; the conformal anomaly coefficients (central charges) (a, c) should be identical. The latter may be computed from (5.51).

Levi subalgebra	matrix Z	regular puncture	pole structure	constraints	flavor symmetry
[1, 1, 1, 1, 1; 0]	diag(a, b, c, d, e)	$[1^{10}]$	$\{1, 3, 5, 7; 4\}$	–	–
[2, 1, 1, 1; 0]	diag(a, a, b, c, d)	$[2^2, 1^6]$	$\{1, 3, 5, 6; 4\}$	–	$SU(2)$
[1, 1, 1; 4]	diag$(0, 0, a, b, c)$	$[3, 1^7]$	$\{1, 3, 5, 6; 3\}$	–	$SO(4)$
[2, 2, 1; 0]	diag(a, a, b, b, c)	$[2^4, 1^2]$	$\{1, 3, 4, 6; 4\}$	–	$SU(2) \times SU(2)$
[3, 1, 1; 0]	diag(a, a, a, b, c)	$[3^2, 1^4]$	$\{1, 3, 4, 6; 3\}$	$c_6^{(8)} = \left(a^{(4)}\right)^2$	$SU(3)$
[2, 1; 4]	diag$(a, a, b, 0, 0)$	$[3, 2^2, 1^3]^*$	$\{1, 3, 4, 6; 3\}$	–	$SU(2) \times SO(4)$
[3, 2; 0]	diag(a, a, a, b, b)	[3, 3, 2, 2]	$\{1, 3, 4, 6; 3\}$	$c_6^{(8)} = \left(c_3^{(4)}\right)^2/4$	$SU(3) \times SU(2)$
[3; 4]	diag$(0, 0, a, a, a)$	[3, 3, 3, 1]	$\{1, 2, 4, 5; 3\}$	–	$SU(3) \times SO(4)$
[1, 1; 6]	diag$(0, 0, 0, a, b)$	$[5, 1^5]$	$\{1, 3, 4, 4; 2\}$	–	$SO(6)$
[4, 1; 0]	diag(a, a, a, a, b)	[4, 4, 1, 1]	$\{1, 2, 4, 5; 3\}$	$c_4^{(6)} = (a^{(3)})^2$, $c_5^{(8)} = 2a^{(3)}\tilde{c}_3$	$SU(4)$
[2; 6]	diag$(0, 0, 0, a, a)$	$[5, 2, 2, 1]^*$	$\{1, 2, 4, 4; 2\}$	–	$SU(2) \times SO(6)$
[5; 0]	diag(a, a, a, a, a)	[5, 5]	$\{1, 2, 3, 4; 2\}$	$c'^{(4)}_2 \equiv c_2^{(4)} - (c_1^{(2)})^2/4$, $c_3^{(6)} = c_1^{(2)} c'^{(4)}_2/2$, $c_4^{(8)} = \left(c'^{(4)}_2\right)^2$	$SU(5)$
[1; 8]	diag$(0, 0, 0, 0, a)$	[7, 1, 1, 1]	$\{1, 2, 2, 2; 1\}$	–	$SO(8)$

Table 5.9: Association of a nilpotent orbit to a Levi subalgebra for D_5. Z is the convention taken in (5.21). the Nahm partition [5, 3, 1, 1], [7, 3] and [9, 1] are excluded.

Duality at large k. For such theories with $\ell = k + 2$, if we take the Levi subalgebra of $T_\ell = \cdots = T_2$ to be of type $[r_1, \ldots r_n; \tilde{r}]$, then there are $n - 1$ exact marginal couplings. For each r_i, $1 \leq i \leq n$ as well as $\tilde{r}$ there is further partition of it in T_1:

$$[r_i; 0] \to \left[m_1^{(i)}, \ldots, m_{s_i}^{(i)}\right], \quad \sum_{j=1}^{s_i} m_j^{(i)} = r_i,$$
$$[0; \widetilde{r}] \to [\widetilde{m}_1, \ldots, \widetilde{m}_s; \widetilde{r}'], \quad 2\sum_{j=1}^{s} \widetilde{m}_j + \widetilde{r}' = \widetilde{r}. \tag{5.55}$$

The Argyres-Douglas matter is given by Z in (5.21) of the leading coefficient matrix

T_ℓ:

$$Z_1 = \begin{pmatrix} a & & & & & \\ & \ddots & & & & \\ & & a & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}. \tag{5.56}$$

They are given by a three-punctured sphere with one black dot of type $[r_1, \ldots, r_m]$ with $\sum r_i = n$ for n being the number of a's, one blue square which is degeneration of $[0; 2N - 2n]$ and one red cross. However, we note the exception when $N = 2$: in this case, since the theory is in fact given by two copies of $SU(2)$ group, so the Argyres-Douglas matter is represented differently. We will see this momentarily.

Example 1: $D_3 \simeq A_3$. This case can be analyzed from either Lie algebra perspective. Let us take T_ℓ to be regular semisimple. We also add a regular puncture labelled by a red cross. One duality frame is given in the first line of figure 5.5.

We can perform various checks for this duality. First of all, (A_1, D_{2k+2}) theory has Coulomb branch spectrum

$$\Delta(O_i) = 2 - \frac{i}{k+1}, \quad i = 1, 2, \ldots, k. \tag{5.57}$$

For the middle theory, for simplicity we focus on the case where the regular puncture is maximal, but replacing it with any regular puncture does not affect the result. The Coulomb branch spectrum for this theory is

$$\begin{aligned} \Delta(O) = & \frac{2k+3}{k+1}, \frac{2k+4}{k+1}, \ldots, \frac{4k+4}{k+1}, \\ & \frac{2k+3}{k+1}, \frac{2k+4}{k+1}, \ldots, \frac{3k+3}{k+1}, \\ & \frac{k+2}{k+1}, \frac{k+3}{k+1}, \ldots, \frac{2k+2}{k+1}. \end{aligned} \tag{5.58}$$

We see that along with two $SU(2)$ gauge groups, the combined Coulomb branch spectrum nicely reproduces all the operators of the initial theory. Secondly, we may calculate the central charge. We know the central charges for (A_1, D_{2k+2}) theory are

$$a = \frac{k}{2} + \frac{1}{12}, \quad c = \frac{k}{2} + \frac{1}{6}. \tag{5.59}$$

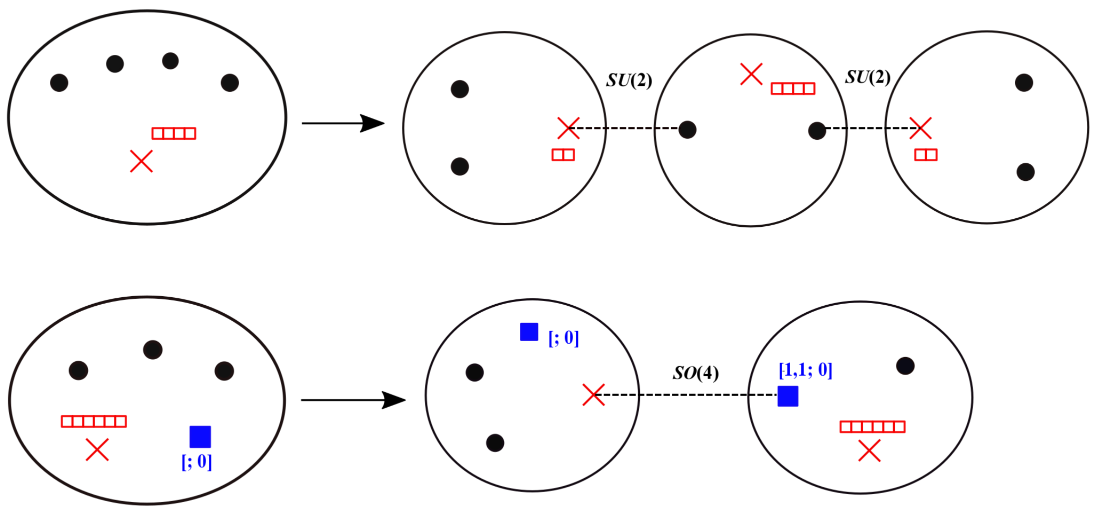


Figure 5.5: Comparison of S-duality from A_3 (upper half) and D_3 (lower half) perspective. From the A_3 point of view, each black dot is given by [1], and the new red marked point after degeneration is given by $SU(2)$ puncture $[1,1]$. The two theories on the left and right sides are (A_1, D_{2k+2}) theory, which is given by irregular puncture whose $T_{k+2}, \ldots, T_1 = [1,1]$, and one regular puncture. The theory in the middle is $(III_{k,1}^{[2,2]^{\times(k+1)},[1,1,1,1]}, F)$ theory. Here F denotes maximal puncture. From the D_3 point of view, two (A_1, D_{2k+2}) theories combine together and form a D_2 type theory. The theory on the right is $(III_{k,1}^{[1;4]^{\times(k+1)},[1^3;0]}, F)$.

The central charges for the initial theory are, with the help of (5.51) and three dimensional mirror,

$$a = 5k + \frac{55}{8}, \quad c = 5k + \frac{58}{8}. \tag{5.60}$$

The central charges for the middle theory are obtained similarly:

$$a = 4k + \frac{131}{24}, \quad c = 4k + \frac{142}{24}. \tag{5.61}$$

We find that

$$\begin{aligned} a_{(I_{4,4k},F)} &= 2a^V_{SU(2)} + 2a_{(A_1,D_{2k+2})} + a_{(III_{k,1}^{[2,2]^{\times(k+1)},[1,1,1,1]},F)}, \\ c_{(I_{4,4k},F)} &= 2c^V_{SU(2)} + 2c_{(A_1,D_{2k+2})} + c_{(III_{k,1}^{[2,2]^{\times(k+1)},[1,1,1,1]},F)}. \end{aligned} \tag{5.62}$$

Here a^V and c^V denote the contribution from vector multiplet. Finally, we may check the flavor central charge and beta functions for the gauge group. The flavor central charge for $SU(2)$ symmetry of (A_1, D_{2k+2}) theory is $(2k+1)/(k+1)$. The middle theory has flavor symmetry $SU(2)^2 \times SU(4)$. Each $SU(2)$ factor has flavor central charge $2 + 1/(k+1)$, so we have a total of 4, which exactly cancels with the beta function of the gauge group.

Now we use D_3 perspective to analyze the S-duality. See the second line of figure 5.5 for illustration. It is not hard to figure out the correct puncture after degeneration of the Riemann sphere. To compare the Coulomb branch spectrum, we assume maximal regular puncture. For the theory on the left hand side, using Newton polygon we have

$$\begin{aligned}\Delta(O) =& \frac{k+2}{k+1}, \frac{k+3}{k+1}, \dots, \frac{2k+1}{k+1}, \\ & \frac{k+2}{k+1}, \frac{k+3}{k+1}, \dots, \frac{2k+1}{k+1}.\end{aligned} \tag{5.63}$$

We see it is nothing but the two copy of (A_1, D_{2k+2}) theories. For the theory on the right hand side, the spectrum is exactly the same as the A_3 theory $(III_{k,1}^{[2,2]^{\times(k+1)},[1,1,1,1]}, F)$. We thus conjecture that

$$a_{(III_{k,1}^{[1;4]^{\times(k+1)},[1^3;0]},F)} = 4k + \frac{131}{24}, \quad c_{(III_{k,1}^{[1;4]^{\times(k+1)},[1^3;0]},F)} = 4k + \frac{142}{24}. \tag{5.64}$$

This is the same as computed by the recipe in section 5.3.

There is another duality frame described in figure 5.6. From D_3 perspective, we get another type of Argyres-Douglas matter and the flavor symmetry is now carried by a black dot, which is in fact $SU(3)$. It connects to the left to an A_2 theory with all T_i's regular semisimple. This theory can further degenerate according to the rules of A_{N-1} theories, and we do not picture it. We conjecture that the central charges for the theory $\left(III_{k,1}^{[3;0]^{\times(k+1)},[1,1,1;0]}, F\right)$ are

$$a_{\left(III_{k,1}^{[3;0]^{\times(k+1)},[1,1,1;0]},F\right)} = 3k + \frac{17}{4}, \quad c_{\left(III_{k,1}^{[3;0]^{\times(k+1)},[1,1,1;0]},F\right)} = 3k + \frac{19}{4}. \tag{5.65}$$

Example 2: D_4. Now we consider a more complicated example. Let us take a generic large $\ell > 3$ and all the coefficient matrices to be regular semisimple, $T_\ell = \cdots = T_1 = [1^4; 0]$. There are several ways to get weakly coupled duality frame, which is described in figure 5.7. The regular puncture can be arbitrary. We have checked that their Coulomb branch spectrum matches with the initial theory, as well as the fact that all gauge couplings are conformal.

For (a) in figure 5.7, we can compute the central charges for the theory $\left(III_{k,1}^{[1;6]^{\times(k+1)},[1^4;0]}, Q\right)$ when Q is a trivial regular puncture. Recall that the initial theory may be mapped

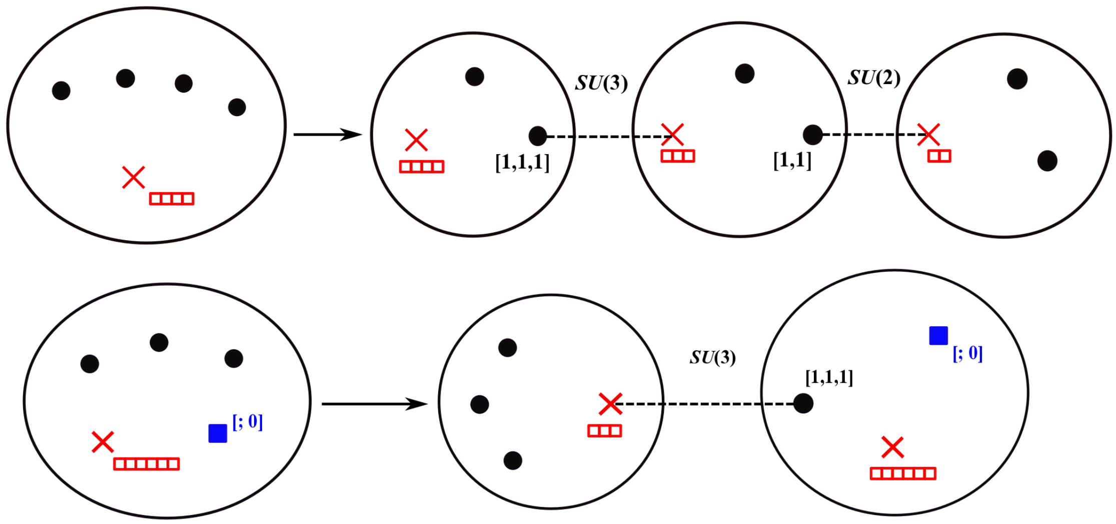


Figure 5.6: Another S-duality frame. The upper one is from A_3 perspective. Here in the weakly coupled description, the rightmost theory is still (A_1, D_{2k+2}), the middle theory is given by $\left(III_{k,1}^{[2,1]^{\times(k+1)},[1,1,1]}, F\right)$, and the leftmost theory is given by $\left(III_{k,1}^{[3,1]^{\times(k+1)},[1,1,1,1]}, F\right)$. The lower one is from the D_3 perspective. The left theory without blue marked points should be understood as A_2 theory. The right hand theory is given by $\left(III_{k,1}^{[3;0]^{\times(k+1)},[1,1,1;0]}, F\right)$. All the computation can be done similarly by replacing full puncture F to be other arbitrary regular puncture Q.

to hypersurface singularity in type IIB construction:

$$a_{\left(III_{k,1}^{[1^4;0]^{\times(k+2)}}, S\right)} = \frac{84k^2 - 5k - 5}{6(k+1)}, \quad c_{\left(III_{k,1}^{[1^4;0]^{\times(k+2)}}, S\right)} = \frac{42k^2 - 2k - 2}{3(k+1)}, \tag{5.66}$$

while we already know the central charges for (A_1, D_{2k+2}) and $\left(III_{k,1}^{[1;4]^{\times(k+1)},[1^3;0]}, F\right)$ theory in (5.64). Therefore we have

$$a_{\left(III_{k,1}^{[1;6]^{\times(k+1)},[1^4;0]}, S\right)} = \frac{54k^2 - 95k - 65}{6(k+1)}, \quad c_{\left(III_{k,1}^{[1;6]^{\times(k+1)},[1^4;0]}, S\right)} = \frac{108k^2 - 185k - 125}{12(k+1)}. \tag{5.67}$$

This is the same as computed from (5.51).

Notice that in (a) of figure 5.7, the leftmost and middle theory may combine together, which is nothing but the theory $\left(III_{k,1}^{[1^3;0]^{\times(k+2)}}, F\right)$. We can obtain another duality frame by using an $SU(3)$ gauge group. See (b) of figure 5.7.

We can try to split another kind of Argyres-Douglas matter, and use the black dot to carry flavor symmetry. The duality frames are depicted in (c) and (d) in figure

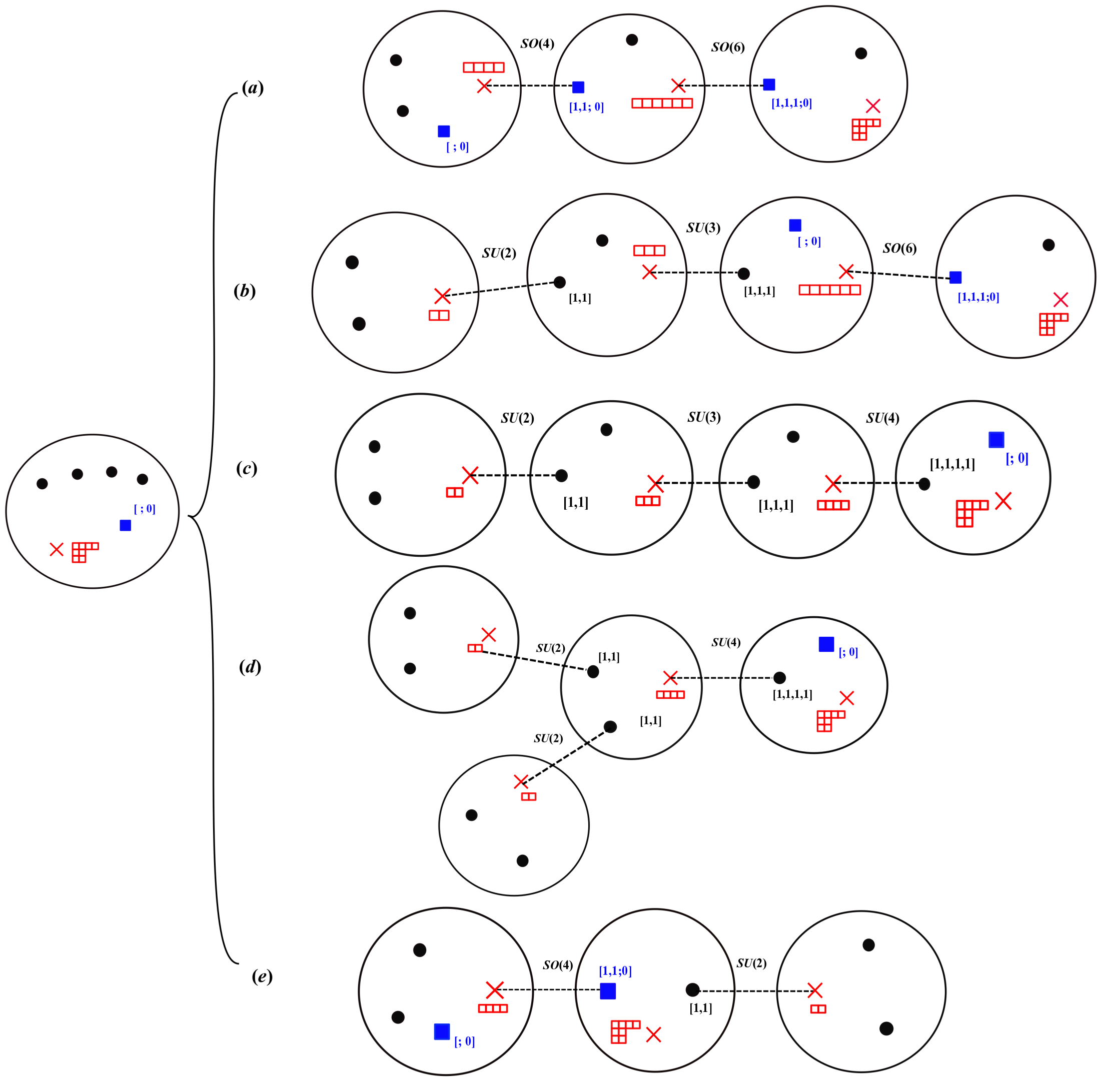


Figure 5.7: The weakly coupled duality frame for D_4 theory of class $(k, 1)$. For (a), the leftmost theory is two copies of (A_1, D_{2k+2}), the middle theory is given by $\left(III^{[1;4]^{\times(k+1)},[1^3;0]}_{k,1}, F\right)$, and the rightmost theory is given by $\left(III^{[1;6]^{\times(k+1)},[1^4;0]}_{k,1}, Q\right)$ where Q is any D_4 regular puncture.
For (b), the leftmost theory is (A_1, D_{2k+2}), followed by the theory $\left(III^{[2,1]^{\times(k+1)},[1^3]}_{k,1}, F\right)$. This is then followed by $\left(III^{[3,1]^{\times(k+1)},[1^4;0]}_{k,1}, F\right)$, and the rightmost theory is still $\left(III^{[1;6]^{\times(k+1)},[1^4;0]}_{k,1}, Q\right)$.
For (c) and (d), the rightmost theory is given by $\left(III^{[4;0]^{\times(k+1)},[1^4;0]}_{k,1}, Q\right)$. Then there are two different ways the theory $\left(III^{[1^4]^{\times(k+2)}}_{k,1}, F\right)$ can be further degenerated.
Finally for (e), the leftmost theory is again two copies of (A_1, D_{2k+2}) theory. The middle theory is D_4 theory $\left(III^{[2;4]^{\times(k+1)},[1^4]}_{k,1}, F\right)$, and the rightmost theory is given by (A_1, D_{2k+2}).

5.7. Again, we can compute the central charges for the Argyres-Douglas matter $\left(III_{k,1}^{[4;0]^{\times(k+1)},[1^4;0]},S\right)$:

$$a_{\left(III_{k,1}^{[4;0]^{\times(k+1)},[1^4;0]},S\right)} = \frac{108k^2-145k-85}{12(k+1)}, \quad c_{\left(III_{k,1}^{[4;0]^{\times(k+1)},[1^4;0]},S\right)} = \frac{27k^2-35k-20}{3(k+1)}, \tag{5.68}$$

same as computed from (5.51).

By comparing the duality frames, we see a surprising fact in four dimensional quiver gauge theory. In particular, (a) in figure 5.7 has $SO(2n)$ gauge groups while (c) in figure 5.7 has $SU(n)$ gauge groups. The Argyres-Douglas matter they couple to are completely different, and our prescription says they are the same theory!

General D_N. Based on the above two examples, we may conjecture the S-duality for D_N theories of class $(k,1)$ for large. The weakly coupled description can be obtained recursively, by splitting Argyres-Douglas matter one by one. See figure 5.8 for illustration of two examples of such splitting. In the first way we get the Argyres-Douglas matter $\left(III_{k,1}^{[1;2N-2]^{\times(k+1)},[1^N;0]},Q\right)$, with remaining theory $\left(III_{k,1}^{[1^{N-1};0]^{\times(k+2)}},F\right)$. The gauge group in between is $SO(2N-2)$. In the second way, we get the Argyres-Douglas matter $\left(III_{k,1}^{[N;0]^{\times(k+1)},[1^N;0]},Q\right)$, with remaining theory $\left(III_{k,1}^{[1^N]^{\times(k+1)}},F\right)$. The gauge group is $SU(N)$. The central charges (a,c) for special cases of regular puncture can be computed similarly.

Duality at small k. We see previously that when k is large enough, new punctures appearing in the degeneration limit are all full punctures. We argue here that when k is small, this does not have to be so. In this section, we focus on D_5 theory, with coefficient matrices $T_\ell = \cdots = T_1 = [1,\ldots,1;0]$ and one trivial regular puncture. The auxiliary Riemann sphere is given by five black dots of type [1], one trivial blue square and one trivial red cross. We will focus on the linear quiver only.

D_5 theory. The linear quivers we consider are depicted in figure 5.9.

After some lengthy calculations, we find that, for the first quiver (where red crosses are all connected with blue squares), when $k=1$, the quiver theory is

$$\left(III_{1,1}^{[1^2;0]^{\times 3}},[1^4]\right) \overset{SO(4)}{—} \left(III_{1,1}^{[1;4]^{\times 2},[1^3;0]},[1^6]\right) \overset{SO(5)}{—} \left(III_{1,1}^{[1;6]^{\times 2},[1^4;0]},[5,1^3]\right) \overset{SO(3)}{—} \left(III_{1,1}^{[1;8]^{\times 2},[1^2;6]},[9,1]\right).$$

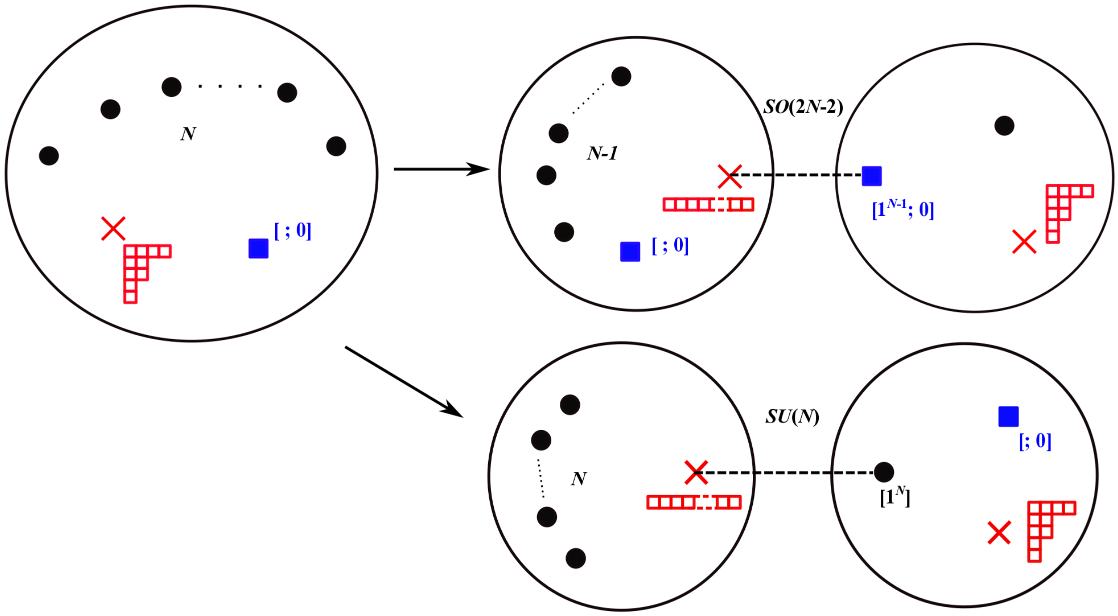


Figure 5.8: The weakly coupled duality frame for D_N theory of class $(k, 1)$. One starts with maximal irregular puncture and a regular puncture, and recursively degenerate a sequence of Argyres-Douglas matter. The first line gives Argyres-Douglas matter $\left(III_{k,1}^{[1;2N-2]^{\times(k+1)},[1^N;0]}, Q\right)$ and the second line gives $\left(III_{k,1}^{[N;0]^{\times(k+1)},[1^N;0]}, Q\right)$. We get in general a quiver with SU and SO gauge groups.

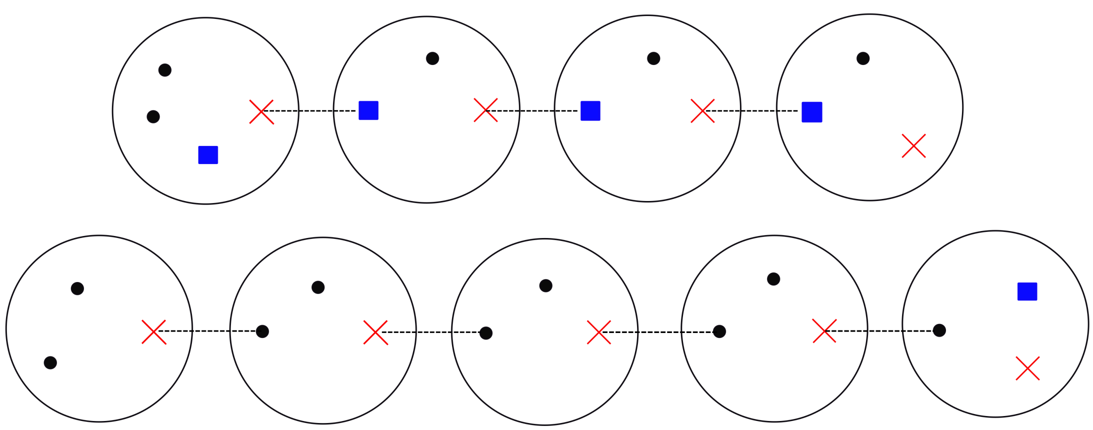


Figure 5.9: The linear quiver that we will examine for k small, when $\mathfrak{g} = D_5$.

In particular, we have checked the central charge and confirm that the middle gauge group is indeed $SO(5)$. Moreover, its left regular puncture is superficially $[1^6]$ but only $SO(5)$ symmetry remains, similar to the right blue marked points $[;6]$[6].

For $k = 2$, we have the quiver

$$
\begin{array}{ccccccc}
 & SO(4) & & SO(6) & & SO(5) & \\
 \left(III^{[1^2;0]^{\times 4}}_{2,1},[1^4]\right) & & \left(III^{[1;4]^{\times 3},[1^3;0]}_{2,1},[1^6]\right) & & \left(III^{[1;6]^{\times 3},[1^4;0]}_{2,1},[3,1^5]\right) & & \left(III^{[1;8]^{\times 3},[1^3;4]}_{2,1},[9,1]\right).
\end{array}
$$

For $k = 3$, we have the quiver

$$
\begin{array}{ccccccc}
 & SO(4) & & SO(6) & & SO(8) & \\
 \left(III^{[1^2;0]^{\times 5}}_{3,1},[1^4]\right) & & \left(III^{[1;4]^{\times 4},[1^3;0]}_{3,1},[1^6]\right) & & \left(III^{[1;6]^{\times 4},[1^4;0]}_{3,1},[1^8]\right) & & \left(III^{[1;8]^{\times 4},[1^5;0]}_{3,1},[9,1]\right).
\end{array}
$$

Finally, for $k > 3$ we reduce to the case in previous section. It is curious to see that some of the gauge group becomes smaller and smaller when k decreases, due to appearance of next-to-maximal puncture. Moreover, there are theories (*i.e.* $\left(III^{[1;8]^{\times 2},[1^2;6]}_{1,1},[9,1]\right)$) whose Coulomb branch spectrum is empty. When this happens, the theory is in fact a collection of free hypermultiplets.

The same situation happens for the second type of D_5 quiver. When k starts decreasing, the sizes of some gauge groups for the quiver theory decrease. When $k = 1$ we get:

$$
\begin{array}{ccccccccc}
 & SU(2) & & SU(3) & & SU(4) & & SU(2) & \\
 \left(III^{[1,1]^{\times 3}}_{1,1},[1^2]\right) & & \left(III^{[2,1]^{\times 2},[1^3]}_{1,1},[1^3]\right) & & \left(III^{[3,1]^{\times 2},[1^4]}_{1,1},[1^4]\right) & & \left(III^{[4,1]^{\times 2},[1^5]}_{1,1},[2,2,1]\right) & & \left(III^{[5;0]^{\times 2},[2,2,1;0]}_{1,1},[9,1]\right).
\end{array}
$$

When $k = 2$, we have the quiver

$$
\begin{array}{ccccccccc}
 & SU(2) & & SU(3) & & SU(4) & & SU(5) & \\
 \left(III^{[1,1]^{\times 4}}_{2,1},[1^2]\right) & & \left(III^{[2,1]^{\times 3},[1^3]}_{2,1},[1^3]\right) & & \left(III^{[3,1]^{\times 3},[1^4]}_{2,1},[1^4]\right) & & \left(III^{[4,1]^{\times 3},[1^5]}_{2,1},[1^5]\right) & & \left(III^{[5;0]^{\times 3},[1^5;0]}_{2,1},[9,1]\right).
\end{array}
$$

[6] We could imagine a similar situation of three hypermultiplets with $SO(6)$ symmetry for six half-hypermultiplets. We then only gauge five of them with $SO(5)$ gauge group. In this way, one mass parameter is frozen, so we get a total of two mass parameters.

Finally when $k > 2$, all the gauge groups do not change anymore and stay as those in previous section.

We can carry out similar analysis for all D_N theory when k is small. This indicates that as we vary the external data, the new punctures appearing in the degeneration limit vary as well.

Class (k, b)

For general $b > 1$ and (k, b) coprime, we need to classify which irregular punctures engineer superconformal theories, and study its duality as before. One subtlety that appears here is that, unlike $b = 1$ case in previous section, here we need to carefully distinguish between whether b is an odd/even divisor of $N/2N - 2$, as their numbers of exact marginal deformations are different.

Coulomb branch spectrum and degenerating coefficient matrices. We elaborate here the procedure how to count graded Coulomb branch dimension for general $b > 1$.

(i) *b is an odd divisor of N.* We may label the degenerating matrices similar to labelling the Levi subgroup: $[r_1, \ldots, r_n; \tilde{r}]$, where $\sum 2br_i + \tilde{r} = 2N$, and there are $n - 1$ exact marginal deformations. To calculate the Coulomb branch spectrum, we first introduce a covering coordinate $z = w^b$, such that the pole structure becomes

$$\frac{T_\ell}{z^{2+\frac{k}{b}}} \to \frac{T'_\ell}{w^{k+b+1}}, \tag{5.69}$$

and T'_ℓ is given by Levi subgroup of type $[r_1, \ldots, r_1, \ldots, r_n, \ldots r_n; \tilde{r}]$, where r_i is repeated b times. Then we are back to the case $b = 1$. This would give the maximal degree d_{2i} in the monomial $w^{d_{2i}} x^{2N-2i}$ that gives Coulomb branch moduli. The monomial corresponds to the degree $2i$ differential ϕ_{2i}, and after converting back to coordinate z, we have the degree of z in $z^{d'_{2i}} x^{2N-2i}$ as

$$d'_{2i} \leq \left\lfloor \frac{d_{2i} - 2i(b-1)}{b} \right\rfloor, \tag{5.70}$$

and similar for the Pfaffian $\tilde{\phi}$.

(ii) *b is an even divisor of N.* We can label the matrix T_ℓ as $[r_1, r_2, \ldots, r_n; \tilde{r}]$ such that $\sum br_i + \tilde{r} = 2N$. Then, we take the change of variables $z = w^b$, and T'_ℓ is given by repeating each r_i $(b/2)$ times, while $\tilde{r}$ is the same. This reduces to the class $(k, 1)$ theories.

(iii) *b is an odd divisor of* $2N-2$. We use $[r_1,\ldots,r_n;\tilde{r}]$ to label the Levi subgroup, which satisfies $2b\sum r_i+\tilde{r}=2N-2$. To get the Coulomb branch spectrum, we again change the coordinates $z=w^b$, and the new coefficient matrix T'_ℓ is now given by Levi subgroup of type $[r_1,\ldots,r_1,\ldots,r_n,\ldots,r_n;\tilde{r}]$, where each r_i appears b times. This again reduces to the class $(k,1)$ theories.

(iv) *b is an even divisor of* $2N-2$. This case is similar once we know the procedure in cases (ii) and (iii). We omit the details.

The above prescription also indicates the constraints on coefficient matrices in order for the resulting 4d theory is a SCFT. We conclude that T_i should satisfy $T_\ell=\cdots=T_2$, T_1 is arbitrary.

To see our prescription is the right one, we can check the case D_4. As an example, we can consider the Higgs field

$$\Phi\sim\frac{T_\ell}{z^{2+\frac{1}{4}}}+\ldots,\quad \ell=6, \tag{5.71}$$

and all T_i to be $[1,1;0]$. Using the above procedure, we know that at ϕ_6 there is a nontrivial moduli whose scaling dimension is $6/5$. This is exactly the same as that given by hypersurface singularity in type IIB construction. Similarly, we may take D_5 theory:

$$\Phi\sim\frac{T_\ell}{z^{2+\frac{1}{4}}}+\ldots,\quad \ell=6, \tag{5.72}$$

and all T_i's given by $[1,1;2]$. After changing variables we have T'_i given by $[1,1,1,1;2]$, which is the same as $[1^5;0]$. Then we have two Coulomb branch moduli with scaling dimension $\{6/5,8/5\}$, same as predicted by type IIB construction.

Duality frames. Now we study the S-duality for these theories. As one example, we may consider D_4 theory of class $(k,b)=(3,2)$, and T_ℓ is given by $[1,1,1,1;0]$. We put an extra trivial regular puncture at the south pole. This theory has Coulomb branch spectrum

$$\Delta(O)=\left\{\left(\frac{6}{5}\right)^{\times 4},\left(\frac{8}{5}\right)^{\times 3},(2)^{\times 3},\left(\frac{12}{5}\right)^{\times 3},\frac{14}{5},\frac{16}{5},\frac{18}{5}\right\}. \tag{5.73}$$

In the degeneration limit, we get three theories, described in figure 5.10. The middle theory $\left(III^{[1;4]^{\times 5},[1,1,1;0]}_{3,2},[3,1,1,1]\right)$ gets further twisted in the sense mentioned in the next subsection, and has Coulomb branch spectrum $\{6/5,8/5,12/5,12/5,14/5,16/5,18/5\}$.

Besides it, the far left theory is two copies of (A_1, D_5) theory with Coulomb branch spectrum $\{8/5, 6/5\}$ each. The far right theory is an untwisted theory, given by $\left(III_{3,2}^{[1;6]^{\times 5},[1,1;4]}, S\right)$, giving spectrum $\{12/5, 6/5\}$. Along with the $SO(4)$ and $SU(2)$ gauge group, we see that the Coulomb branch spectrum nicely matches together. We conjecture that this is the weakly coupled description for the initial Argyres-Douglas theory.

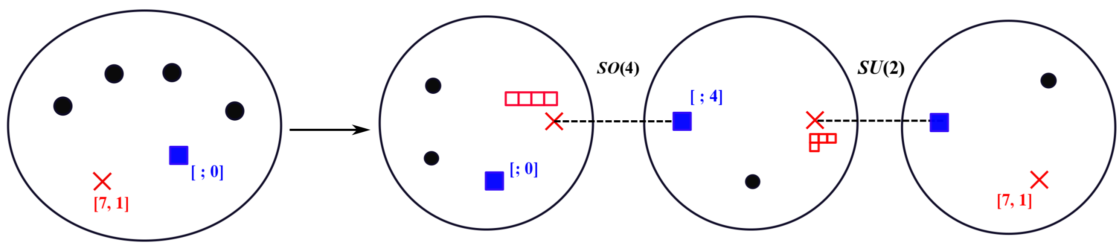


Figure 5.10: S-duality for D_4 theory of class $(3, 2)$. Here we pick the coefficient matrices to be of type $[1, 1, 1, 1; 0]$, with a trivial regular puncture (this setup can be relaxed to general D_4 regular punctures). In the degeneration limit, we get $SO(4) \times SU(2)$ gauge group plus three Argyes-Douglas matter. The leftmost theory is in fact two copies of (A_1, D_5) theory, while the middle theory is given by twisted D_3 theory, given by twisting the theory $\left(III_{3,2}^{[1;4]^{\times 5},[1,1,1;0]}, [3, 1, 1, 1]\right)$. The rightmost theory is $\left(III_{3,2}^{[1;6]^{\times 5},[1,1;4]}, S\right)$ theory.

In this example, each gauge coupling is exactly conformal as well.

As a second example, we consider D_3 theory of class $(3, 2)$. The coefficient matrices are given by $T_6 = \cdots = T_2 = T_1 = [1, 1; 2]$. We put a trivial regular puncture at the south pole. This theory has Coulomb branch spectrum

$$\Delta(O) = \left\{\frac{6}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2, \frac{12}{5}\right\}, \tag{5.74}$$

and is represented by an auxiliary Riemann sphere with two black dots of type [1], one blue square of size 2 and one trivial red cross. See figure 5.11. After degeneration, we get two theories. We compute that the first theory is a twisting of $\left(III^{[1;2]^{\times 5},[1,1;0]}, [1^4]\right)$, having spectrum $\{6/5, 7/5, 8/5, 9/5\}$. The second theory $\left(III_{3,2}^{[1;4]^{\times 5},[1,1,1;0]}, S\right)$ has spectrum $\{12/5, 6/5\}$. The middle gauge group is $SO(3)$, although the two sides superficially have $SO(4)$ symmetry.

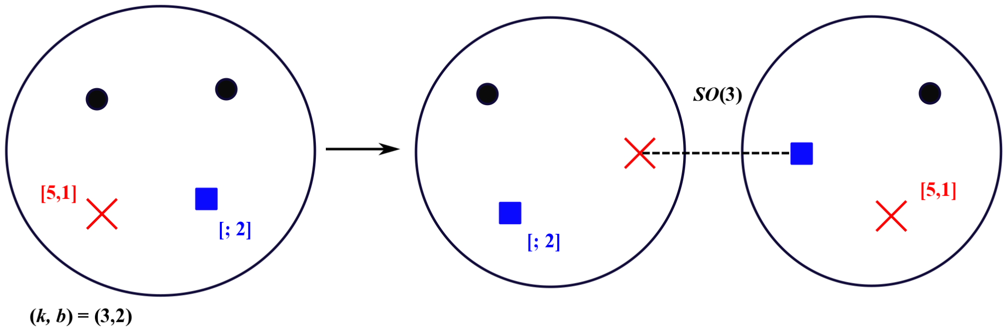


Figure 5.11: S-duality for D_3 theory of class $(3,2)$. Here we pick the coefficient matrices to be of type $[1,1;2]$, with a trivial regular puncture (this setup can be relaxed to general D_3 regular punctures).

$\mathbb{Z}_2$-twisted theory

If the Lie algebra $\mathfrak{g}$ has a nontrivial automorphism group $\mathrm{Out}(\mathfrak{g})$, then one may consider *twisted* punctures. This means that as one goes around the puncture, the Higgs field undergoes an action of nontrivial element $o \in \mathrm{Out}(\mathfrak{g})$:

$$\Phi(e^{2\pi i}z) = h[o(\Phi(z))]h^{-1}, \tag{5.75}$$

where $h \in \mathfrak{g}/\mathfrak{j}^\vee$ with $\mathfrak{j}^\vee$ the invariant subalgebra under $\mathrm{Out}(\mathfrak{g})$. Let us denote $\mathfrak{j}$ the Langlands dual of $\mathfrak{j}^\vee$.

In this section we solely consider D_N theory with automorphism group $\mathbb{Z}_2$. It has invariant subalgebra $\mathfrak{j}^\vee = B_{N-1}$ whose Langlands dual is $\mathfrak{j} = C_{N-1}$. For more details of other Lie algebra $\mathfrak{g}$, see [113, 191–195]. We review some background for twisted regular punctures as in [192], and then proceed to understand twisted irregular punctures and their S-duality. For previous study of S-duality for twisted theory, see [196, 197].

Twisted regular punctures. Following [192], a regular twisted D_N punctures are labelled by nilpotent orbit of C_{N-1}, or a C-*partition* d of $2N-2$, where all odd parts appear with even multiplicity. To fix the local Higgs field, note that $\mathbb{Z}_2$ automorphism group split the Lie algebra $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{j}_1 \oplus \mathfrak{j}_{-1}$, with eigenvalue ± 1 respectively. Apparently, $\mathfrak{j}_1 = B_{N-1}$. The Higgs field behaves as

$$\Phi \sim \frac{\Lambda}{z} + \frac{\Lambda'}{z^{1/2}} + M, \tag{5.76}$$

where Λ' is a generic element of $\mathfrak{j}_{-1}$ and M is a generic element of $\mathfrak{j}_1$. Λ is an element residing in the nilpotent orbit of B_{N-1}, which is given by a *B-partition* of $2N-1$, where all even parts appear with even multiplicity. It is again related to the C-partition d via the Spaltenstein map $\mathfrak{S}$. To be more specific, we have $\mathfrak{S}(d) = \left(d^{+\mathsf{T}}\right)_B$:

- First, "+" means one add an entry 1 to the C-partition d;
- Then, perform transpose of d^+, corresponding to the superscript T;
- Finally, $(\cdot)_B$ denotes the *B-collapse*. The procedure is the same as D-collapse in section 5.2.

For later use we will also introduce the action $\mathfrak{S}$ on a B-partition d'. This should give a C-partition. Concretely, we have $\mathfrak{S}(d') = (d'^{\mathsf{T}-})_C$:

- First, "T" means one take transpose of d';
- Then, perform reduction of d'^{T}, corresponding to subtract the last entry of d'^{T} by 1;
- Finally, $(\cdot)_C$ denotes the *C-collapse*. The procedure is the same as B- and D-collapse except that it now operates on the odd part which appears even multiplicity.

Given a regular puncture with a C-partition, we may read off its residual flavor symmetry as

$$G_{\text{flavor}} = \prod_{h \text{ even}} SO(n^h) \times \prod_{h \text{ odd}} Sp\left(n^h\right). \tag{5.77}$$

We may also calculate the pole structure of each differential ϕ_{2i} and the Pfaffian $\tilde{\phi}$ in the Seiberg-Witten curve (5.7). We denote them as $\{p_2, p_4, \ldots, p_{2N-2}; \tilde{p}\}$; in the twisted case, the pole order of the Pfaffian $\tilde{\phi}$ is always half-integer.

As in the untwisted case, the coefficient for the leading singularity of each differential may not be independent from each other. There are constraints for $c_l^{(2k)}$, which we adopt the same notation as for the untwisted regular puncture. The constraints of the form

$$c_l^{(2k)} = \left(a_{l/2}^{(k)}\right)^2 \tag{5.78}$$

effectively remove one Coulomb branch moduli at degree $2k$ and increase one Coulomb branch moduli at degree k, while the constraints of the form

$$c_l^{(2k)} = \ldots \tag{5.79}$$

only remove one moduli at degree $2k$. For the algorithm of counting constraints for each differentials and complete list for the pole structures, see reference [192]. After knowing all the pole structures and constraints on their coefficients, we can now compute the graded Coulomb branch dimensions exactly as those done in section 5.2. We can also express the local contribution to the Coulomb branch moduli as

$$\dim_{\mathbb{C}}^{\rho} \text{Coulomb} = \frac{1}{2}\left[\dim_{\mathbb{C}} \mathfrak{S}(O_\rho) + \dim \mathfrak{g}/\mathfrak{j}^{\vee}\right], \tag{5.80}$$

here O_ρ is a nilpotent orbit in C_{N-1} and $\mathfrak{S}(O_\rho)$ is a nilpotent orbit in B_{N-1}.

Twisted irregular puncture. Now we turn to twisted irregular puncture. We only consider the "maximal twisted irregular singularities". The form of the Higgs field is, in our $\mathbb{Z}_2$ twisting,

$$\Phi \sim \frac{T_\ell}{z^\ell} + \frac{U_\ell}{z^{\ell-1/2}} + \frac{T_{\ell-1}}{z^{\ell-1}} + \frac{U_{\ell-1}}{z^{\ell-3/2}} + \cdots + \frac{T_1}{z} + \ldots . \tag{5.81}$$

Here all the T_i's are in the invariant subalgebra $\mathfrak{so}(2N-1)$ and all U_i's are in its complement $\mathfrak{j}_{-1}$. To get the Coulomb branch dimension, note that the nontrivial element $o \in \text{Out}(\mathfrak{g})$ acts on the differentials in the SW curve as

$$\begin{aligned} o: \quad & \phi_{2i} \longrightarrow \phi_{2i} \ \text{ for } 1 \le i \le N-1, \\ & \tilde{\phi}_N \longrightarrow -\tilde{\phi}_N. \end{aligned} \tag{5.82}$$

Then, the Coulomb branch dimension coming from the twisted irregular singularities can be written as [104]:

$$\dim_{\mathbb{C}}^{\rho} \text{Coulomb} = \frac{1}{2}\left[\sum_{i=1}^{\ell} \dim T_i + \sum_{i=2}^{\ell}(\dim \mathfrak{g}/\mathfrak{j}^{\vee} - 1) + \dim \mathfrak{g}/\mathfrak{j}^{\vee}\right]. \tag{5.83}$$

In the above formula, the -1 term in the middle summand comes from treating $U_i, 2 \le i \le \ell$ as parameter instead of moduli of the theory. It corresponds to the Pfaffian $\tilde{\phi}_N$ which switches sign under $o \in \text{Out}(\mathfrak{g})$.

As in the untwisted case, we are also interested in the degeneration of T_i and the graded Coulomb branch dimension. First of all, we know that as an $\mathfrak{so}(2N-1)$ matrix, T_i can be written down as

$$\begin{pmatrix} 0 & u & v \\ -v^{\mathsf{T}} & Z_1 & Z_2 \\ -u^{\mathsf{T}} & Z_3 & -Z_1 \end{pmatrix}, \tag{5.84}$$

with $Z_{1,2,3}$ $(N-1)\times(N-1)$ matrices, and $Z_{2,3}$ are skew symmetric; while u, v are row vectors of size $N-1$. After appropriate diagonalization, only Z_1 is nonvanishing. So a Levi subalgebra can be labelled by $[r_1,\ldots,r_n;\tilde{r}+1]$, with $\tilde{r}+1$ always an odd number. The associated Levi subgroup is

$$L=\prod_i U(r_i)\times SO(\tilde{r}+1). \tag{5.85}$$

Now we state our proposal for whether a given twisted irregular puncture defines a SCFT in four dimensions. Similar to untwisted case, we require that $T_\ell = T_{\ell-1} = \cdots = T_2$ and T_1 can be further arbitrary partition of $T_{i\geq 2}$. When all the T_i's are regular semisimple, we can draw Newton polygon for these theories. They are the same as untwisted case, except that the monomials living in the Pfaffian $\tilde{\phi}_N$ get shift down one half unit [104].

Example: D_4 maximal twisted irregular puncture with $\ell = 3$. We consider all T_i to be regular semisimple $\mathfrak{so}(7)$ element $[1,1,1;1]$, plus a trivial twisted regular puncture. From the Newton polygon, we know the spectrum for this theory is $\{2,3/2,3,5/2,2,3/2,7/4,5/4\}$.

S-duality for twisted D_N theory of class $(k,1)$. Having all the necessary techniques at hand, we are now ready to apply the algorithm previously developed and generate S-duality frame. We state our rules as follows for theory of class $(k,1)$ with $k=\ell-2$.

- Given coefficient matrices $T_\ell=\cdots=T_2=[r_1,\ldots,r_n;\tilde{r}+1]$, and T_1 being further partition of T_i, we represent the theory on an auxiliary Riemann sphere with n black dots with size r_i, $1\leq i\leq n$, a blue square with size $\tilde{r}$, and a red cross representing the regular puncture, labelled by a C-partition of $2N-2$.
- Different S-duality frames are given by different degeneration limit of the auxiliary Riemann sphere.
- Finally, one needs to figure out the newly appeared punctures. The gauge group can only connect a red cross and a blue square (Sp gauge group). This is different from the untwisted case we considered before.

Let us proceed to examine the examples. We first give a comprehensive discussion of D_4 theory.

Duality at large k. We have initially three black dots of type [1], a trivial blue square and an arbitrary red cross representing a regular puncture. This theory has a part of the Coulomb branch spectrum coming from irregular puncture:

$$\begin{aligned}\Delta(O) &= \frac{k+2}{k+1}, \dots, \frac{2k}{k+1}, \\ &= \frac{k+2}{k+1}, \dots, \frac{4k}{k+1}, \\ &= \frac{k+2}{k+1}, \dots, \frac{6k}{k+1}, \\ &= \frac{k+3/2}{k+1}, \dots, \frac{4k-1/2}{k+1}.\end{aligned} \tag{5.86}$$

The S-duality frame for this theory is given in figure 5.12.

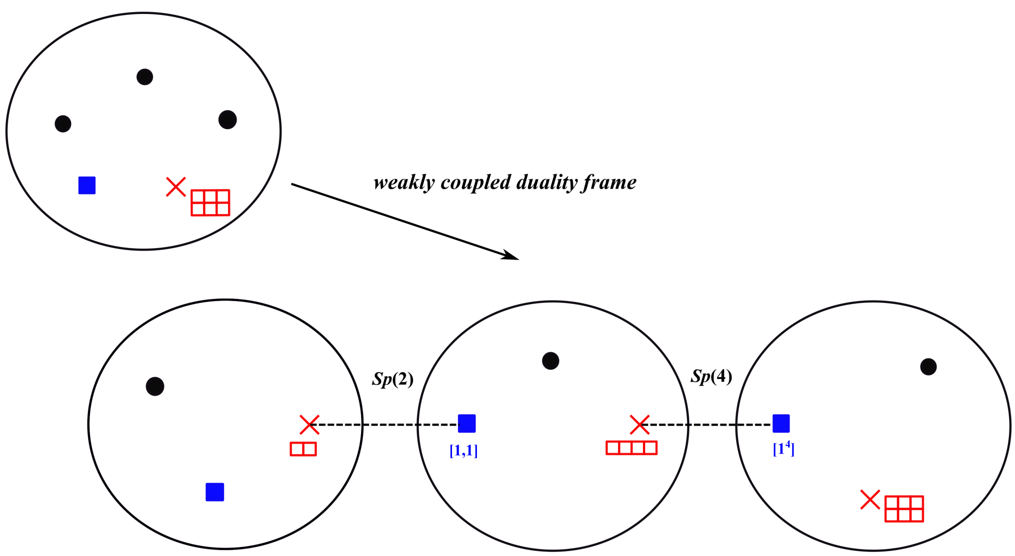


Figure 5.12: S-duality for twisted D_4 theory of class $(k, 1)$ with large k. Each Argyres-Douglas matter is connected with Sp gauge group. Assembling the black dot and the blue square we can read off the data for the irregular puncture and thus identify the theory.

The duality frame in figure 5.12 tells us the Coulomb branch spectrum of each piece. The leftmost theory $\left(III_{k,1}^{[1;1]^{\times(k+2)}}, F\right)$ has the spectrum

$$\begin{aligned}\Delta_1(O) &= \frac{k+2}{k+1}, \dots, \frac{2k+1}{k+1}, \\ &= \frac{k+3/2}{k+1}, \dots, \frac{2k+3/2}{k+1}.\end{aligned} \tag{5.87}$$

The rightmost theory is given by $\left(III_{k,1}^{[1;5]^{\times(k+1)},[1,1,1;1]}, Q\right)$ whose spectrum comes from the irregular part is

$$\begin{aligned}\Delta_2(O) &= \frac{k+2}{k+1}, \ldots, \frac{2k}{k+1}, \\ &= \frac{2k+3}{k+1}, \ldots, \frac{4k}{k+1}, \\ &= \frac{4k+5}{k+1}, \ldots, \frac{6k}{k+1}, \\ &= \frac{3k+7/2}{k+1}, \ldots, \frac{4k-1/2}{k+1}.\end{aligned} \tag{5.88}$$

Finally, the middle theory is $\left(III_{k,1}^{[1;3]^{\times(k+1)},[1,1;1]}, F\right)$. It contributes to the Coulomb branch spectrum coming from the irregular puncture

$$\begin{aligned}\Delta_3(O) &= \frac{k+2}{k+1}, \ldots, \frac{2k+1}{k+1}, \\ &= \frac{2k+3}{k+1}, \ldots, \frac{4k+3}{k+1}, \\ &= \frac{2k+5/2}{k+1}, \ldots, \frac{3k+5/2}{k+1}.\end{aligned} \tag{5.89}$$

These three pieces nicely assemble together and form the total spectrum of original theory. We thus have $Sp(2) \times Sp(4)$ gauge groups.

Duality at small k. Similar to the untwisted case, we expect that some of the gauge group would be smaller. We now focus on a trivial twisted regular puncture in figure 5.12. Analysis for other twisted regular punctures are analogous.

We find that for $k = 1$,

$$\left(III_{1,1}^{[1;1]^{\times 3}}, [1,1]\right) \overset{Sp(2)}{\text{———}} \left(III_{1,1}^{[1;3]^{\times 2},[1,1;1]}, [2,1,1]\right) \overset{Sp(2)}{\text{———}} \left(III_{1,1}^{[1;5]^{\times 2},[1,1;3]}, [6]\right).$$

When $k \geq 2$, the second $Sp(2)$ gauge group becomes $Sp(4)$ and we reduce to the large k calculations.

S-duality of D_N theory. When k is large, the intermediate gauge group in the degeneration limit does not depend on which twisted regular puncture one puts, and they are all full punctures. To obtain the duality frames, we can again follow the recursive procedure by splitting the Argyres-Douglas matter one by one. See the example of such splitting in figure 5.13. Again, due to twisting things become more constraining, and all matter should have a blue square on its auxiliary Riemann sphere.

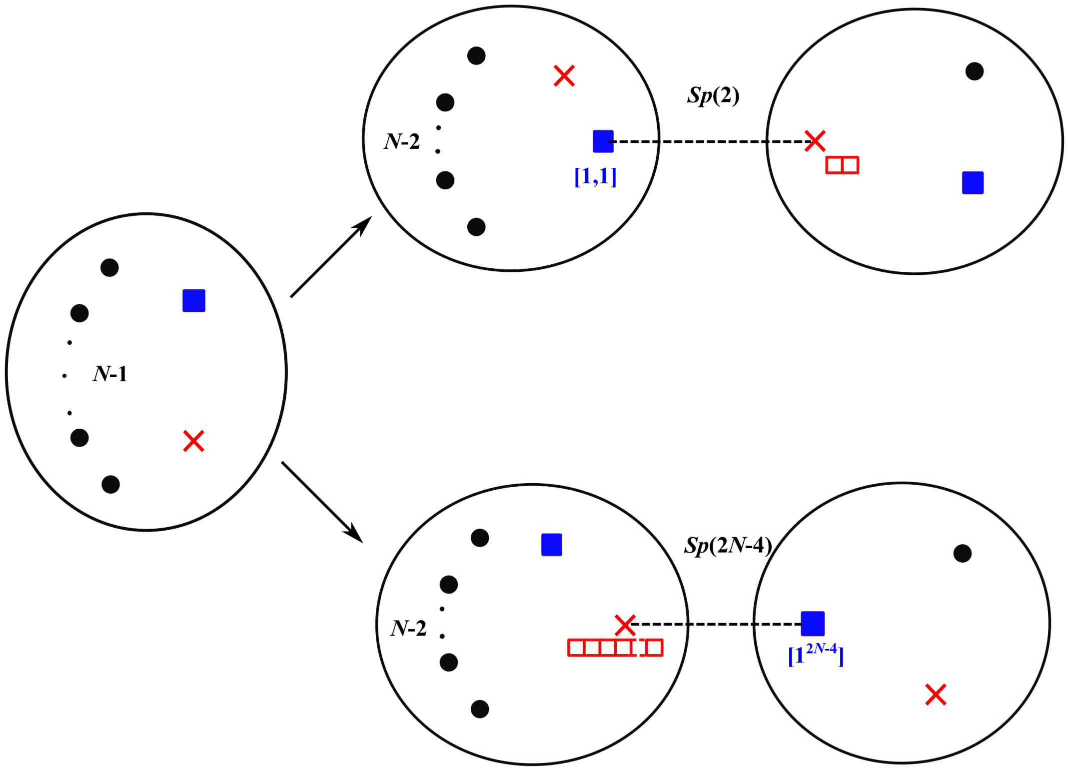


Figure 5.13: S-duality for twisted D_N theory of class $(k, 1)$ with large k. Here we present the duality frame recursively by splitting the Argyres-Douglas matter. In the first line we split a theory $\left(III_{k,1}^{[1;1]^{\times(k+2)}}, F\right)$ with F a full D_2 twisted puncture; in the second line we split a theory $\left(III_{k,1}^{[1;2N-3]^{\times(k+1)},[1^{N-1};1]}, Q\right)$ with original regular puncture Q.

When k is small, some of the intermediate puncture would be smaller. One needs to figure out those punctures carefully. We leave the details to interested readers.

5.5 Comments on S-duality for E-type theories

Finally, we turn to the duality frames for $\mathfrak{g} = \mathfrak{e}_{6,7,8}$. We focus on the Lie algebra $\mathfrak{e}_6$ while state our conjecture for $\mathfrak{e}_7$ and $\mathfrak{e}_8$ case.

A complete list of all the relevant data for regular punctures can be found in [179, 180, 194]. We will use some of their results here for studying irregular puncture.

Levi subalgebra $\mathfrak{l}$	Nahm Bala-Carter label
0	0
A_1	A_1
$2A_1$	$2A_1$
$3A_1$	$(3A_1)^*$
A_2	A_2
$A_2 + A_1$	$A_2 + A_1$
$2A_2$	$2A_2$
A_3	A_3
$2A_2 + A_1$	$(2A_2 + A_1)^*$
$A_2 + 2A_1$	$A_2 + 2A_1$
$A_3 + A_1$	$(A_3 + A_1)^*$
D_4	D_4
A_4	A_4
$A_4 + A_1$	$A_4 + A_1$
A_5	$(A_5)^*$
D_5	D_5

Table 5.10: The correspondence between Nahm label and the Levi subalgebra. The Levi subalgebra E_6 is omitted as it does not give any irregular puncture. We use $*$ to denote the non-special nilpotent orbit. The pole structure and constraints can be found in [179]. Again, we exclude those with non-principal orbit in the Levi subalgebra.

Irregular puncture and S-duality for E_6 theory

We focus on the irregular singularity (5.54). The first task is to characterize the degeneration of coefficient matrices. Those matrices T_i, $1 \leq i \leq \ell$ shall be represented by a Levi subalgebra $\mathfrak{l}$. See section 5.2 for the list of conjugacy classes. For each Levi subalgebra $\mathfrak{l}$, we associate a nilpotent orbit with Nahm label. Since we are already using Bala-Carter's notation, we can directly read of $\mathfrak{l}$. See table 5.10. Here we exclude Bala-Carter label of the form $E_6(\cdot)$, as it gives maximal Levi subalgebra so the irregular puncture is trivial.

We are now ready to count the Coulomb branch spectrum for a given E_6 irregular puncture of class $(k, 1)$, were $\ell = k + 2$. We use the SW curve from type IIB construction, whose isolated singularity has the form[7]

$$x_1^2 + x_2^3 + x_3^4 + z^{12k} = 0, \tag{5.90}$$

whose deformation looks like

$$x_1^2 + x_2^3 + x_3^4 + \phi_2(z)x_2x_3^2 + \phi_5(z)x_2x_3 + \phi_6(z)x_3^2 + \phi_8(z)x_2 + \phi_9(z)x_3 + \phi_{12}(z) = 0, \tag{5.91}$$

where at the singularity $\phi_{12} = z^{12k}$. The Coulomb branch spectrum is encoded in these Casimirs. For example, when $k = 1$ and regular semisimple coefficients, we know the scaling dimensions for each letter are

$$[x_1] = 3, \;\; [x_2] = 2, \;\; [x_3] = \frac{3}{2}, \;\; [z] = \frac{1}{2}. \tag{5.92}$$

By enumerating the quotient algebra generator of this hypersurface singularity we know that the number of moduli for each differential is $\{d_2, d_5, d_6, d_8, d_9, d_{12}\} = \{0, 3, 4, 6, 7, 10\}$. This is consistent with adding pole structures and subtract global contribution of three maximal E_6 regular punctures.

S-duality for E_6 theory. We now study the S-duality for E_6 theory of class $(k, 1)$, with coefficient all regular semisimple. From the D_N S-duality, we know that the Levi subalgebra directly relates to the flavor symmetry. If we take the coefficient matrix to be regular semisimple, then our initial theory is given by a sphere with six black dots, one trivial blue square and one red cross (which is an arbitrary E_6 regular puncture.

We only consider large k situation. In type IIB construction (5.91), the scaling dimensions for each letter are

$$[x_1] = \frac{6k}{k+1}, \;\; [x_2] = \frac{4k}{k+1}, \;\; [x_3] = \frac{3k}{k+1}, \;\; [z] = \frac{1}{k+1}. \tag{5.93}$$

So we have the spectrum of initial theory coming from irregular puncture as

$$\begin{aligned} &\phi_2 : \frac{2k}{k+1}, \ldots, \frac{k+2}{k+1}, \qquad \phi_5 : \frac{5k}{k+1}, \ldots, \frac{k+2}{k+1}, \\ &\phi_6 : \frac{6k}{k+1}, \ldots, \frac{k+2}{k+1}, \qquad \phi_8 : \frac{8k}{k+1}, \ldots, \frac{k+2}{k+1}, \\ &\phi_9 : \frac{9k}{k+1}, \ldots, \frac{k+2}{k+1}, \qquad \phi_{12} : \frac{12k}{k+1}, \ldots, \frac{k+2}{k+1}. \end{aligned} \tag{5.94}$$

[7] As we consider $(k, 1)$ theory, there is no distinction between whether it comes from $b = 8, 9$ or 12. We can simply pick anyone of them.

There are several ways to split Argyres-Douglas matter. For example, we may pop out two black dots and one trivial blue square. We get the duality frame

$$\begin{array}{ccc} & SO(4) & \\ & \diagup \quad \diagdown & \\ \left(III_{k,1}^{(2A_1)^{\times(k+1)},0}, Q_{E_6}\right) & & \left(III_{k,1}^{[1,1;0]^{\times k+2}}, [1^4]\right), \end{array}$$

and here the right hand side theory is two copies of (A_1, D_{2k+2}) theory. This duality frame persists to $k = 1$. We have checked that the central charge matches.

The second way is to pop out a trivial black dot and the E_6 regular puncture. This results in D_5 gauge group:

$$\begin{array}{ccc} & SO(10) & \\ & \diagup \quad \diagdown & \\ \left(III_{k,1}^{[1^5;0]^{\times(k+2)}}, [1^{10}]\right) & & \left(III_{k,1}^{(D_5)^{\times(k+1)},0}, Q\right), \end{array}$$

where the theory $\left(III_{k,1}^{[1^5;0]^{\times(k+2)}}, [1^{10}]\right)$ can be further degenerate according to D_N type rules. The spectrum counting is explained in the example in section 5.2. We see it correctly reproduces $SO(10)$ flavor symmetry. We have also checked that the central charge matches.

Another way is to give $SU(6)$ gauge group in the degeneration limit, by poping out a trivial blue puncture and red cross.

$$\begin{array}{ccc} & SU(6) & \\ & \diagup \quad \diagdown & \\ \left(III_{k,1}^{[1^6]^{\times(k+2)}}, [1^6]\right) & & \left(III_{k,1}^{(A_5)^{\times(k+1)},0}, Q_{E_6}\right). \end{array}$$

We find that the central charges match as well.

E_7 and E_8 theory

Finally, we turn to E_7 and E_8 Argyres-Douglas theories. Tinkertoys for E_7 theories have been worked out in [180]. Similar ideas go through and we will outline the steps here. The key ingredient is to use type IIB construction to count the moduli. For E_7 theory, the deformed singularity has the form

$$\begin{aligned} x_1^2 + x_2^3 + x_2 x_3^3 + \phi_2(z) x_2^2 x_3 + \phi_6(z) x_2^2 + \phi_8(z) x_2 x_3 \\ + \phi_{10}(z) x_3^2 + \phi_{12}(z) x_2 + \phi_{14}(z) x_3 + \phi_{18}(z) = 0, \end{aligned} \tag{5.95}$$

where $\{\phi_2, \phi_6, \phi_8, \phi_{10}, \phi_{12}, \phi_{14}, \phi_{18}\}$ are independent differentials. For E_8 theory, the deformed hypersurface singularity has the form:

$$\begin{aligned} &x_1^2 + x_2^3 + x_3^5 + \phi_2(z)x_2x_3^3 + \phi_8(z)x_2x_3^2 + \phi_{12}(z)x_3^3 \\ &\quad + \phi_{14}(z)x_2x_3 + \phi_{18}(z)x_3^2 + \phi_{20}(z)x_2 + \phi_{24}(z)x_3 + \phi_{30}(z) = 0, \end{aligned} \tag{5.96}$$

where $\{\phi_2, \phi_8, \phi_{12}, \phi_{14}, \phi_{18}, \phi_{20}, \phi_{24}, \phi_{30}\}$ are independent differentials.

The regular puncture for these two exceptional algebras are again given the Bala-Carter label. One can read off the Levi subalgebra similar as before. This then provides the way of counting Coulomb branch spectrum. The duality frame can then be inferred by comparing the spectrum in the degeneration limit, and checked with central charge computation (5.51).

For example, we have in $\mathfrak{e}_7$ theory one duality frame which looks like

$$\begin{array}{ccccc} & & E_6 & & \\ & \diagup & & \diagdown & \\ \left(III_{k,1}^{(0)^{\times(k+2)}}, F_{\mathfrak{e}_6}\right) & & & & \left(III_{k,1}^{(E_6)^{\times(k+1)},0}, Q\right), \end{array}$$

where $F_{\mathfrak{e}_6}$ is the full E_6 regular puncture. Another duality frame is

$$\begin{array}{ccccc} & & SU(7) & & \\ & \diagup & & \diagdown & \\ \left(III_{k,1}^{[1^7]^{\times(k+2)}}, [1^7]\right) & & & & \left(III_{k,1}^{(A_6)^{\times(k+1)},0}, Q\right). \end{array}$$

For $\mathfrak{e}_8$ theory, we have the duality frames

$$\begin{array}{ccccc} & & E_7 & & \\ & \diagup & & \diagdown & \\ \left(III_{k,1}^{(0)^{\times(k+2)}}, F_{\mathfrak{e}_7}\right) & & & & \left(III_{k,1}^{(E_7)^{\times(k+1)},0}, Q\right), \end{array}$$

and

$$\begin{array}{ccccc} & & SU(8) & & \\ & \diagup & & \diagdown & \\ \left(III_{k,1}^{(0)^{\times(k+2)}}, [1^8]\right) & & & & \left(III_{k,1}^{(A_7)^{\times(k+1)},0}, Q\right). \end{array}$$

We have checked that the central charges and the Coulomb branch spectrum matches. The left hand theory of each duality frames can be further degenerated according to known rules for lower rank ADE Lie algebras, and we do not picture them anymore. Here we see the interesting duality appears again: the quivers with E_N type gauge group are dual to quivers with A_{N-1} type quivers.

Chapter 6

QUANTUM CHIRAL RINGS IN FOUR DIMENSIONAL $\mathcal{N} = 1$ ADJOINT SQCD

6.1 Overview of $\mathcal{N} = 1$ theories

In this chapter, we study theories with less supersymmetry in four dimensions. The realm of $\mathcal{N} = 1$ theories in four dimensions exhibits various interesting phenomena, among which electric magnetic dualities play an important role. The pioneering work of Seiberg [22] demonstrated the IR equivalence of two seemingly distinct gauge theories, in which he showed several exact matchings between operators, moduli space of vacua, and massless excitations near singularities. This provides many insights into the non-abelian gauge dynamics of $\mathcal{N} = 1$ theories.

Soon it was realized that such dualities are generic for $\mathcal{N} = 1$ theories [198, 199]. In [200, 201], an attempt was made by Kutasov to analyze the dynamics of $\mathcal{N} = 1$ SQCD with fundamental matter plus one adjoint chiral multiplet (ASQCD)[1]. He showed that by properly adding a superpotential term for adjoint chiral multiplet that truncates the chiral ring of the theory, a generalized version of Seiberg duality also exists. This duality undergoes various semi-classical consistency checks [202], and it also sheds light on the quantum chiral ring relations in the original electric theory: a quantum chiral ring relation for Coulomb operators are in fact classical combinatoric constraints in the dual theory. The duality was further explored by [203, 204] to understand the spectra of the confining theory; the corresponding effective superpotential was written down. It was shown there that the effective superpotential is generated by multi-instanton effects in the dual theory.

Meanwhile, another important progress was achieved by the seminal work of Dijkgraaf and Vafa [205] in probing $\mathcal{N} = 1$ dynamics. They conjectured that the effective superpotential of a wide class of $\mathcal{N} = 1$ supersymmetric gauge theories can actually be calculated perturbatively in a closely related matrix model, whose potential is just the classical superpotential of the gauge theory. A striking conclusion was that only *planar* diagrams in the matrix model suffice. Later, Cachazo *et al* [206] provided a purely field-theoretic proof of the correspondence proposed by Dijkgraaf

[1] In the rest of the chapter, we will call the ASQCD with tree level superpotential considered in [201] for adjoint superfield "Kutasov model".

and Vafa, by analyzing Konishi anomalies and chiral rings of $U(N)$ gauge theory with one adjoint chiral multiplet. The powerful conjecture of [205] makes many exact computation in $\mathcal{N} = 1$ theories (with or without adjoint superfield) accessible; to name several but not all of them, see for instance, [207–217].

However, even with the proposal of duality and the tools from matrix model, there are many other peculiar phenomena in ASQCD that escape precise understanding. For instance, with the aid of a-maximization [133, 218, 219], one discovers that for Kutasov model at large N, some chiral operators decouple and become free under RG flow, introducing in the IR so-called "accidential symmetry"[134]. Moreover, in [220] the author found that in such class of theories there are UV irrelevant operators whose scaling dimensions cross marginality under the flow, and hence are "dangerously irrelevant" [221]. The appearance of such operators are quite counter-intuitive in the sense that in the Morse theory interpretation, RG flow is usually triggered by relevant operators; in other words, the relevant operators are "consumed" along the RG trajectory, and its number should thus decrease along the flow. This "marginality crossing" behavior is in fact special only to $\mathcal{N} = 1$ theories in four dimensions; indeed, as shown in [139], $\mathcal{N} = 2$ theories do not admit dangerously irrelevant operators.

Resolving these peculiarities in $\mathcal{N} = 1$ ASQCD often requires a more precise understanding of vacuum structure, and it is our main motivation of this chapter. We will focus on chiral rings of Kutasov model as well as its mass deformed counterpart. The chiral ring probes the vacua of the theory, and tells us about the quantitative behavior at low energies: $e.g.$, chiral symmetry breaking, confinement, and electric-magnetic duality. The complete chiral ring relation for $U(N)$ theory with one adjoint chiral superfield is obtained in [222], and our work is a generalization of that.

We remark that Kutasov model falls into an ADE classification of SQCD with adjoints [23]. This series are revisited recently in [223], where some puzzles are found. We hope that the full analysis of the quantum chiral ring would resolve these puzzles and eventually help to understand the entire ADE series[2] or ASQCD without superpotentials.

[2] See, for instance, [224] on some related work.

Background and summary

In this chapter we analyze the chiral ring of four-dimensional $\mathcal{N} = 1$ supersymmetric $U(N_c)$ gauge theory with one chiral multiplet Φ in the adjoint representation of $U(N_c)$, and N_f fundamental as well as antifundamental chiral multiplets $\widetilde{Q}_{\tilde{f}}$ and Q^f where $f, \tilde{f} = 1, 2, \ldots, N_f$. We consider asymptotic free theories, namely $2N_c > N_f$. The Lagrangian of the theory is

$$\mathcal{L} = \frac{1}{g^2}\left[\int d^4\theta\, Q_i^\dagger e^V Q^i + \widetilde{Q}_{\tilde{i}} e^{-V} \widetilde{Q}^{\dagger\tilde{i}} + \Phi^\dagger e^{[V,\cdot]}\Phi\right] + \frac{1}{4g^2}\left(\int d^2\theta\, W^\alpha W_\alpha + c.c.\right), \tag{6.1}$$

where for simplicity we do not distinguish between the $U(1)$ couplings in $U(N_c)$ and $SU(N_c)$ couplings, unlike that of [225]. We also think of $U(N_c)$ Kutasov model as coming from $SU(N_c)$ model by gauging the $U(1)$ baryon symmetry. Kutasov model also requires a superpotential of Φ labelled by a positive integer k,

$$W(\Phi) = \frac{h}{k+1}\operatorname{Tr}\Phi^{k+1}, \tag{6.2}$$

and the UV theory enjoys an $SU(N_f)_L \times SU(N_f)_R \times U(1)_r$ symmetry. In this chapter, we mostly focus on $k = 2$.

For $kN_f < N_c$, the theory does not have a quantum vacua; for $kN_f = N_c$ the vacua is modified quantum mechanically; for $kN_f = N_c + 1$ the theory is s-confining, and the effective potential is given by a set of composite degrees of freedom with an irrelevant potential. For $kN_f > N_c$ the theory admits a dual magnetic description with gauge group $U(kN_f - N_c)$.

Kutasov model in general has nontrivial moduli spaces, to understand its quantum chiral ring/quantum vacua, one adds proper deformations to the tree level potential (6.2) to collapse the flat directions. The most general single trace deformation we can add is [208, 209]

$$W_{\text{tree}} = \operatorname{Tr}\widetilde{W}(\Phi) + \widetilde{Q}_{\tilde{f}} m_f^{\tilde{f}}(\Phi) Q^f, \tag{6.3}$$

where

$$\widetilde{W}(z) = \sum_{n=0}^{k} \frac{1}{n+1} g_n z^{n+1}, \tag{6.4a}$$

$$m_f^{\tilde{f}}(z) = \sum_{n=1}^{l+1} m_{f,n}^{\tilde{f}} z^{n-1}. \tag{6.4b}$$

Also we define $L = lN_f$.

We will call such theory with deformed superpotential (6.3) the "mass deformed" version or "deformed cousin" of Kutasov model. In the bulk of the chapter we will be frequently comparing massive and massless theories.

The chapter is organized as follows. In section 6.2 we review some well-known facts about the chiral ring for $U(N_c)$ ASQCD. We classify chiral operators and describe their relations, with special emphasis on two equivalent descriptions: the algebraic description in terms of generators and relations, as well as the geometric description in terms of expectation values for various composite fields.

In section 6.3 we calculate the the classical chiral ring and describe different branches of the moduli space.

After that, section 6.4 is devoted to understand the quantum corrections to the chiral ring. We will list the complete Konishi anomaly equations that give the perturbative chiral ring. The nonperturbative corrections come from certain resolvent operators, whose periods over one cycles of some auxilliary Riemann surface should be integer [209]. It has also been known how to solve the off-shell vacuum expectation values for mass deformed theory [209]; and in this chapter we solve them *on-shell*. In the mass deformed theory, the classical vacua are shifted by quantum effects and there are nonvanishing gaugino condensations. With the inclusion of a new Konishi anomaly equation, we are able to prove that the solutions of the chiral ring are in one-to-one correspondence of the supersymmetric vacua. Then, we focus on massless Kutasov model itself. The difficulty of understanding the flat direction of the moduli, unlike that of SQCD, is that the theory has more possible deformations. We will examine a special massless limit and its implications.

Finally, section 6.5 applies the established framework to some examples of massless model. We will see the existence of quantum corrections directly.

6.2 Chiral rings in $\mathcal{N} = 1$ theories

Following the notation of [206, 208] we review some basics of chiral rings of four dimensional $\mathcal{N} = 1$ theories, with fundamental plus adjoint matter. An operator O is *chiral* if it is annihilated by a pair of supercharges of the same chirality: $[\overline{Q}_{\dot{\alpha}}, O\} = 0$. One readily checks that a product of two chiral operators is again a chiral operator, and therefore chiral operators form a ring.

In the chiral ring, one defines an equivalence relation, namely two chiral operators are equivalent if they differ by a $\overline{Q}_{\dot{\alpha}}$-exact term. Modulo this equivalence relation, a chiral operator is independent of the position since

$$\frac{\partial}{\partial x^\mu}O(x) = [P^\mu, O(x)] = \{\overline{Q}^{\dot\alpha}, [Q^\alpha, O(x)]\}. \tag{6.5}$$

Therefore, the correlation function of the form $\langle O^{(1)}(x_1)O^{(2)}(x_2)\dots O^{(n)}(x_n)\rangle$ is independent of each coordinate $x_1, x_2, \dots x_n$. It is then possible to move each operator insertion to be mutually far away, such that the expectation value factorizes:

$$\langle O^{(1)}(x_1)O^{(2)}(x_2)\dots O^{(n)}(x_n)\rangle = \langle O^{(1)}\rangle\langle O^{(2)}\rangle\dots\langle O^{(n)}\rangle. \tag{6.6}$$

For ASQCD, we need to classify all the possible chiral operators modulo $\overline{Q}_{\dot\alpha}$-exact terms. A crucial fact used in [206, 226] is that, for an adjoint valued chiral superfield O,

$$\left[\overline{Q}_{\dot\alpha}, D_{\alpha\dot\alpha}O\right\} = [W_\alpha, O\}, \tag{6.7}$$

which implies the adjoint superfield Φ commutes with vector superfield W_α while W_α anti-commutes with W_β. For fundamentals, $W_\alpha Q^f$ as well as $\tilde{Q}_{\tilde f}W_\alpha$ is not in the chiral ring [208]. Therefore the possible candidates for the ring are

$$u_k = \mathrm{Tr}\,\Phi^k, \tag{6.8a}$$

$$w_{\alpha,k} = \frac{1}{4\pi}\mathrm{Tr}\,\Phi^k W_\alpha, \tag{6.8b}$$

$$r_k = -\frac{1}{32\pi}\mathrm{Tr}\,\Phi^k W_\alpha W^\alpha, \tag{6.8c}$$

$$v^f_{\tilde f,k} = \widetilde{Q}_{\tilde f}\Phi^k Q^f. \tag{6.8d}$$

We name u_k the Casimir operators, r_k the generalized glueballs, $w_{\alpha,k}$ the generalized photinos, and v_k the generalized mesons[3]. Their form suggests to define resolvent operators as the generating function of these chiral operators

$$T(z) = \mathrm{Tr}\,\frac{1}{z-\Phi}, \tag{6.9a}$$

$$w_\alpha(z) = \frac{1}{4\pi}\mathrm{Tr}\,\frac{W_\alpha}{z-\Phi}, \tag{6.9b}$$

$$R(z) = -\frac{1}{32\pi^2}\mathrm{Tr}\,\frac{W_\alpha W^\alpha}{z-\Phi}, \tag{6.9c}$$

$$M^f_{\tilde f}(z) = \widetilde{Q}_{\tilde f}\frac{1}{z-\Phi}Q^f. \tag{6.9d}$$

[3]There is a slight notation difference between here and what people usually call "generalized mesons" in the literature. What we mean by v_k is often denoted as M_{k+1}.

We will be mostly interested in the resolvent $T(z)$, $R(z)$ and $M(z)$. For supersymmetric vacua, the chiral operators $w_{\alpha,k}$ are zero [222]. Although there are nontrivial ring relations among $w_{\alpha,k}$, for solving the vacua we can temporarily neglect them, see section 6.4. For $U(N_c)$ theories, the single baryon $B^{[i_1,\ldots,i_k][i_{k+1},\ldots,i_{N_c}]}$ formed by dressed quark is not gauge invariant; but the composite $\widetilde{B}B$ is. However, such operators are not in the chiral ring since they can be expressed in terms of generalized mesons, and thus are not independent.

In general, whether at classical or quantum level, the chiral ring of a theory $\mathcal{T}$ is a quotient of polynomial ring by some ideal, $\mathcal{S}$:

$$\mathcal{R}(\mathcal{T}) = \mathbb{C}[u_k, w_{\alpha,k}, r_k, v^f_{\tilde{f},k}]/\mathcal{S}. \tag{6.10}$$

We call the ideal $\mathcal{S}$ the chiral ring relation. Such notation provides two interpretations of the chiral rings. First, the solution satisfying the relation given by $\mathcal{S}$ parametrize the supersymmetric vacua. Hence one thinks of the moduli space of vacua as an algebraic variety defined by ideal $\mathcal{S}$ in the polynomial ring. Second, the chiral ring is the coordinate ring defined on the variety. These two interpretations establish an algebraic and geometric connections between chiral rings and vacua of the theory, similar to the stories in classical algebraic geometry.

Specifically, let $\mathsf{V}(\cdot)$ denote the operation of taking algebraic varieties of an ideal, $\mathsf{I}(\cdot)$ the operation of taking polynomials vanishes on the algebraic variety, then by Hilbert's Nullstellensatz,

$$\mathsf{I}(\mathsf{V}(\mathcal{S})) = \sqrt{\mathcal{S}}, \tag{6.11}$$

with $\sqrt{\mathcal{S}}$ the radical ideal. In modern language of schemes, we have $\mathsf{V}(\mathcal{S}) := \operatorname{Spec}\mathcal{R}$.

A remark is in order. Unlike (twisted) chiral ring in two dimensions, in four dimensions the $\mathcal{N} = 1$ chiral ring cannot be formulated in term of cohomology [227]. The intuitive reason for that is the supercharges (of the same chirality) as part of the definition in the cochain complex carries Lorentz indices, which are rotated into each other under $SO(4)$ Lorentz group. Since one may construct an example that cohomological description fails for a particular supercharge $\overline{Q}_{\dot{1}}$, one sees that it fails for all linear combination of two supercharges $a^{\dot{\alpha}}\overline{Q}_{\dot{\alpha}}$.

In what follows, we denote $\widehat{\mathcal{S}}$ as the quantum relations of Kutasov model, and correspondingly $\widehat{\mathcal{R}}$ for quantum chiral rings.

6.3 Classical chiral rings of Kutasov model

Generalities

In this subsection we mainly focus on the massless model with superpotential (6.2). We will briefly comment on its relation to the mass deformed counterpart at the end.

From the Lagrangian of the theory we know that the corresponding D-term equation reads

$$[\Phi^\dagger, \Phi] + (Q^i Q_i^\dagger - \widetilde{Q}^{\dagger\tilde{i}} \widetilde{Q}_{\tilde{i}}) = 0, \tag{6.12}$$

while the F-term constraint is

$$\Phi^k = 0, \tag{6.13}$$

so Φ is nilpotent[4] with degree k. The nilpotent matrix always has degree no bigger than its order, so for simplicity we only discuss $k \leq N_c$ in this paper[5]. The only nilpotent matrix which is diagonalizable is zero matrix; others can only be put into Jordan normal form:

$$\Phi = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix}, \tag{6.14}$$

where the Jordan block J_i is

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}. \tag{6.15}$$

The nilpotency implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. A Jordan block J_i is uniquely determined by its order N_i. Thus a nilpotent matrix can be labelled by a partition of N_c, $[N_1, N_2, \ldots, N_n]$, characterizing the size of Jordan blocks : $N_1 + N_2 + \cdots + N_n = N_c$ with $k \geq N_1 \geq N_2 \geq \cdots \geq N_n$. We use the symbol Y as a Young tableau with i-th

[4] This is not true for $SU(N_c)$ theories, where a traceless condition should be imposed. This additional constraint makes Φ either diagonalizable or nilpotent. See [203, 204] for more details.

[5] Strictly speaking, $k = N_c$ case is in fact a double trace superpotential, as $\operatorname{Tr} X^{N_c+1}$ is not independent.

row of length N_i. It is a Young tableau for partition of N_c into integers no larger than k.

For nilpotent matrix, we always have

$$\operatorname{Tr}\Phi^j = 0, \quad j > 0, \tag{6.16}$$

which means that classically the vevs of Casimir operators u_j in (6.8a) are always zero. Note this does not mean $u_j = 0$ *in the chiral ring*[6]. In the meantime, the vevs of generalized glueballs r_j are in general proportional to the strong coupling scale $\Lambda^{2N_c - N_f}$, and are constrained by fermionic statistics. Since they can only be formulated using adjoint Φ and vector superfield W_α as in (6.8), the constraints are exactly the same as that in [222] and we will not include them in current analysis. Therefore, modulo generalized glueballs and photinos, the classical chiral ring of $U(N_c)$ Kutasov model is a quotient ring of the polynomial ring generated by generalized mesons and Casimir operators:

$$\mathcal{R}_{N_c,N_f,k} = \mathbb{C}\left[u_1, u_2, \ldots, u_{k-1}, v^f_{0,\tilde{f}}, v^f_{0,\tilde{f}}, \ldots, v^f_{k-1,\tilde{f}}\right] / \mathcal{S}(u_1, u_2, \ldots, u_{k-1}, v_0, v_1, \ldots, v_{k-1}). \tag{6.17}$$

The constraint $\mathcal{S}(u_1, u_2, \ldots, u_{k-1}, v_0, v_1, \ldots, v_{k-1})$ is hard to compute in general. A powerful tool that helps is from computational algebraic geometry. To be more specific, classically we can form a quotient ring using microscopic fields:

$$\mathcal{R}_{\text{micro}} = \mathbb{C}\left[\widetilde{Q}^\alpha_{\tilde{f}}, Q^f_\alpha, \Phi^\alpha_\beta\right] / \mathcal{S}_F, \tag{6.18}$$

where $\mathcal{S}_F$ comes from F-term equations of the superpotential. We do not have to consider the D-term once we complexify the gauge group [228]. The vacuum is parameterized by gauge invariant data, *c.f.* equation (6.10). The natural map arising from composing microscopic field into gauge invariant ones extends to a map between rings:

$$\psi : \mathbb{C}\left[u_k, v^f_{k,\tilde{f}}\right] \to \mathcal{R}_{\text{micro}}. \tag{6.19}$$

Then by definition

$$\mathcal{S} = \ker\psi. \tag{6.20}$$

Computation of this kernel is standard in the theory of Gröbner basis [229, 230]. This method has already been adopted in understanding the vacua and computing

[6] In mathematical language, the two coordinate ring may define the same classical algebraic varieties, but they do not define the same scheme.

Hilbert series of the vacuum moduli, see *e.g.* [231, 232]. In section 6.3, we will explicitly see how this works.

The above algebraic construction is quite abstract. We now turn to concrete description in terms of the moduli space of vacua. As we already know, the Coulomb branch vev $\langle\Phi\rangle$ is parametrized by Young tableau $[N_1, N_2, \ldots, N_n]$. There are two cases to consider:

(1) When all $N_i = 1$. The D-term equation becomes that of SQCD with fundamental matter, and there is nontrivial Higgs branch. For $kN_f > N_c + 1$, at the root of the Higgs branch the theory is conjectured to be in non-abelian Coulomb phase [202].

(2) $N_i > 1$ for some i. Since nontrivial Jordan block does not commute with its conjugate, in general the vevs of quark superfields $\langle Q\rangle$ and $\langle\tilde{Q}\rangle$ are not zero. We will call it the mixed branch.

In (6.16) we see the vevs of gauge invariant Casimir operators are always zero. However the above two cases reveal there are distinct branches in the vacuum moduli. Then the natural question is how can one distinguish between them. Classically, we might tell which branch we are in by looking at the flat directions of generalized mesons. In the branch $[1, 1, \ldots, 1]$ only v_0 is nontrivial, but for other branches more non-trivial generalized mesons appear. However, we will not use such descriptions because such flat directions receive quantum corrections.

Alternatively one can try to study the branch when the deformation (6.4a) is turned on. Moreover we require the deformation is sufficiently generic and $g_0 \neq 0$ in (6.4a). It is not hard to see that now Φ must be diagonalizable, with entries the roots of polynomial

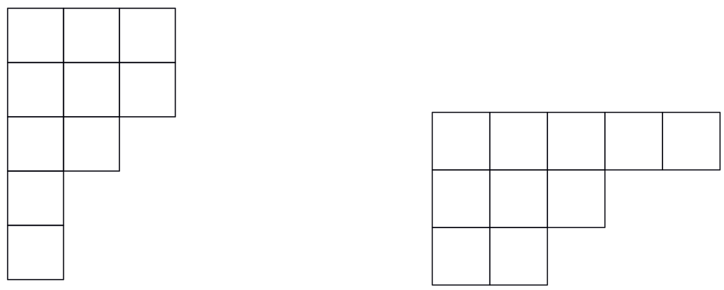
$$\widetilde{W}'(z) = \sum_{n=0}^{k} g_n z^n = \prod_{j=1}^{k}(z - a_j). \tag{6.21}$$

Then the Coulomb branch vev $\langle\Phi\rangle$ is labelled by integers $s_1 \geq s_2 \geq \cdots \geq s_k$, the number of each root of (6.21). Therefore we can label this in in terms of another Young diagram Y': $[\underline{s_1}, \underline{s_2}, \ldots, \underline{s_k}]$, the partition of N_c into no more than k integers[7]. It is a standard fact that

$$Y' = Y^D, \tag{6.22}$$

[7] We use an underline to remind the reader that they are Young tableau for mass deformed theory.

where Y^D is the dual Young diagram of Y. This is also frequently used in the literature as the mapping between nilpotent element and semisimple element. Careful readers now may worry that the mapping is not one-to-one; one can permute the roots $\{a_i\}$ corresponding to the integer $\{s_i\}$. However, there is a natural way to make this mapping one-to-one, due to the fact that their semi-classical unbroken gauge group for a given set of s_i are uniquely fixed regardless of permutation of roots: $U(s_1) \times U(s_2) \times \cdots U(s_k)$. Therefore we may define our map from a nilpotent $\langle\Phi\rangle$ to the image taking the rank of unbroken subgroup of $U(N_c)$. In figure 6.1 we give an example of the correspondence of the Young diagrams.



(a) (b)

Figure 6.1: The deformation of nilpotent matrix in the group $U(10)_{\mathbb{C}} \simeq GL(10)$. In (a) the nilpotent matrix is labelled by $Y = [3,3,2,1,1]$, while the deformed matrix is given by $Y' = [5,3,2]$, with low energy gauge group $U(5) \times U(3) \times U(2)$.

This identification is more robust than the previous one in the sense that patterns of unbroken gauge group are rigid against quantum corrections. We will see that it is indeed the case in section 6.4.

As we have seen that the deformation (6.4a) is important to distinguish between different branches, it is illustrative to summarize what the vacua look like if the full deformation (6.3) is turned on [209]. In this case, the vacua consist of Coulomb branch (pseudo-confining branch) and Higgs branch. For Coulomb branch, we have

$$\langle\Phi\rangle = \mathrm{diag}(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots a_k), \quad \langle \widetilde{Q}_{\tilde{f}} \rangle = \langle Q^f \rangle = 0. \tag{6.23}$$

For Higgs branch we have

$$\langle\Phi\rangle = \mathrm{diag}(b, a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots a_k), \tag{6.24a}$$

$$\langle\widetilde{Q}^{\beta}_{\tilde{f}}\rangle = \langle Q^{f}_{\beta}\rangle = 0, \quad \beta = 2, 3, \ldots N_c, \tag{6.24b}$$

$$Q^f_1\left(\sum_{n=1}^{l+1}(n-1)b^{n-2}m^{\tilde{f}}_{f,n}\right)\widetilde{Q}^1_{\tilde{f}} + \widetilde{W}'(b) = 0, \tag{6.24c}$$

where b is the root of $B(z) = \det\left[m^{\tilde{f}}_{f}(z)\right] = 0$. Similar reasoning to that of [209] reveals that root b can only appear in $\langle\Phi\rangle$ once. The solution can also be elegantly packaged as

$$M(z) = -\sum_{I=1}^{lN_f}\frac{r_I\widetilde{W}'(b_I)}{z-b_I}\frac{1}{2\pi i}\oint_{b_I}\frac{1}{m(x)}dx, \tag{6.25}$$

where $r_I = 0, 1$ is the number of b_I in the diagonal of $\langle\Phi\rangle$. This solution of classical Higgs branch will be important in section 6.4.

Example: $U(2)$ **theory with** $k = 2$

Having discussed generalities, it is time to get refreshed by a couple of examples. In this subsection we will be illustrating the case $N_c = 2$, $k = 2$ with $N_f = 1, 2$. We have two choices of Young tableau for $\langle\Phi\rangle$: $[1, 1]$ or $[2]$. Upon deformations by (6.4a), $[1, 1]$ corresponds to the dual vacua $[\underline{2}]$ where the gauge group remains unbroken as $U(2)$, but $[2]$ corresponds to the dual vacua $[\underline{1}, \underline{1}]$ where gauge group is broken down to $U(1)^2$. For $[1, 1]$ branch, $v_1 = 0$ but it is nonzero for $[2]$. Since Φ^2 vanishes, one concludes that $v_j = 0$ for $j \geq 2$. Therefore we know classically,

$$\mathcal{R}_{2,N_f,2} = \mathbb{C}\left[u_1, v_0, v_1\right]/\mathcal{S}(u_1, v_0, v_1). \tag{6.26}$$

Next we turn to the classical relation $\mathcal{S}$. A nice computer program that produces the kernel of the map ψ in (6.19) is `Macaulay 2` [233, 234]. In the following we list the relations $\mathcal{S}(u_1, v_0, v_1)$ for $N_f = 1, 2$:

- $N_f = 1$.

$$\mathcal{R}_{2,1,2} = \mathbb{C}\left[u_1, v_0, v_1\right]/\langle u_1^3, u_1^2 v_1, u_1 v_1^2, u_1^2 v_0 - 2u_1 v_1\rangle. \tag{6.27}$$

Notice that u_1 is nilpotent in the chiral ring; the classical relation implies that $u_1 = 0$ as an algebraic variety, and the rest constraints in the relations are trivially satisfied. So v_0, v_1 take arbitrary complex values.

• $N_f = 2$. It turns out that the relation can be compactly cast as

$$\mathcal{S}_{2,2,2} = \langle u_1^3, u_1^2 v_1, v_1 \det v_1, u_1 \det v_0 - \det(v_0 + v_1) + \det v_0 + \det v_1, \\ u_1 v^j_{1,i} v^l_{1,k}, u_1^2 v_0 - 2u_1 v_1, u_1(v^j_{0,i} v^l_{1,k} - v^l_{0,k} v^j_{1,i}), v_1 \det(v_0 + v_1) - v_1 \det v_0 - v_0 \det v_1 \rangle. \tag{6.28}$$

In solving the chiral ring, we see again that the nilpotent element $u_1 = 0$. What remains are $\det v_1 = 0$, following from the fact $\langle \Phi \rangle$ has rank 1, and $\det(v_0 + v_1) - \det v_0 = 0$.

Examples: $U(3)$ **theory with** $k = 2$

Our next example is $U(3)$ theory with $k = 2$. Here we only analyze $N_f = 1$. For large numbers of flavors, the relations quickly become very complicated. The adjoint chiral multiplet has two choices of vevs:

$$\langle \Phi \rangle_{[1,1,1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \langle \Phi \rangle_{[2,1]} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.29}$$

For $N_f = 1$, we again see the chiral ring is generated by u_1, v_0 and v_1 as:

$$\mathcal{R}_{3,1,2} = \mathbb{C}\left[u_1, v_0, v_1\right] / \langle u_1^4, u_1^3 v_1, u_1^2 v_1^2, u_1^3 v_0 - 3u_1^2 v_1 \rangle. \tag{6.30}$$

The Casimir operator u_1 is again nilpotent in the chiral ring.

General $U(N_c)$ **theory with** $k = 2$

Motivated by our study of $U(2)$ and $U(3)$ theories with $k = 2$, we conjecture the classical constraints for general $U(N_c)$ theory with N_f fundamental flavors, with $k = 2$ as follows. The superpotential (6.2) forces the nilpotent matrix $\langle \Phi \rangle$ to be

$$Y_{N_c, N_f, 2} = [2, 2, \ldots, 2, 1, 1, \ldots 1], \tag{6.31}$$

where we denote n_2 as number of order 2 Jordan block, then the trivial Jordan block has number $N_c - 2n_2$. Apparently, the chiral ring relation should not depend on the choice of n_2. For $N_f = 1$, we can write down the complete relations $\mathcal{S}$, but for other numbers of flavors, we only write down relations in $\sqrt{\mathcal{S}}$. They may not necessarily be the true chiral ring relation, as the chiral ring contains nilpotent elements.

- $N_f = 1$:

$$\mathcal{R}_{N_c,1,2} = \mathbb{C}[u_1, v_0, v_1] / \langle u_1^{N_c+1}, u_1^{N_c} v_1, u_1^{N_c-1} v_1^2, u_1^{N_c} v_0 - N_c u_1^{N_c-1} v_1 \rangle. \tag{6.32}$$

- $N_f \le \lfloor N_c/2 \rfloor$. The solutions to $\mathcal{S}$ do not constrain v_0 and v_1, they can take arbitrary complex values. This may be confirmed in the $N_f = 1$ case above when taking $u_1 = 0$. We thus have $\sqrt{\mathcal{S}} = \langle u_1 \rangle$.

- $\lfloor N_c/2 \rfloor < N_f < N_c$. Since the adjoint chiral superfield is built from rank $\le \lfloor N_c/2 \rfloor$ data, the second generalized meson becomes degenerate. The solution for v_1 satisfies:

$$v_{1,j_1}^{[i_1} v_{1,j_2}^{i_2} \cdots v_{1,j_{\lfloor N_c/2 \rfloor+1}}^{i_{\lfloor N_c/2 \rfloor+1}]} = 0, \tag{6.33}$$

  and there are no additional constraints on v_0.

- $N_c \le N_f$. We define $\tilde{v} = v_0 + v_1$. In addition to (6.33), we have

$$\tilde{v}_{j_1}^{[i_1} \tilde{v}_{j_2}^{i_2} \cdots \tilde{v}_{j_{N_c}}^{i_{N_c}]} - v_{0,j_1}^{[i_1} v_{0,j_2}^{i_2} \cdots v_{0,j_{N_c}}^{i_{N_c}]} = 0. \tag{6.34}$$

  When $N_c < N_f$ we have yet another relation coming from the degeneration of first generalized meson v_0:

$$v_{0,j_1}^{[i_1} v_{0,j_2}^{i_2} \cdots v_{0,j_{N_c+1}}^{i_{N_c+1}]} = 0. \tag{6.35}$$

6.4 Quantum chiral rings

In this section we analyze quantum chiral rings. When dealing with the quantum vacua with nontrivial flat directions, the usual way is to deform the theory, endowing all the matter with a mass and then taking appropriate limit [235]. We thus introduce the deformation (6.3) first and study the resulting vacuum expectation values of gauge invariant chiral operators; by taking the limit one ends up with some particular point on the vacuum moduli.

We emphasize that such a way recovers vacua as an algebraic variety (or the radical ideal), but not the true chiral ring, by Hilbert's Nullstellensatz (6.11).

Perturbative corrections

The F-term constraint from the superpotential is obtained via chiral rotations $X \to X + \delta X$ where X is some chiral superfield in the Lagrangian. It can also be viewed as conservation law of the current

$$J = \operatorname{Tr} \bar{X} e^V \delta X \tag{6.36}$$

with a source term, which is subjected to Konishi anomaly [236, 237] and its generalized versions [206, 238]. If we pick $\delta X = f(\widetilde{Q}, Q, \Phi, W_\alpha)$, where f is a holomorphic function of its arguments, the conservation equation can be written as

$$\overline{D}^2 J = \mathrm{Tr}\, f(\widetilde{Q}, Q, \Phi, W_\alpha)\frac{\partial W_{\mathrm{tree}}}{\partial X} + \text{ anomaly } + \overline{D}.(\dots)\,. \tag{6.37}$$

Here we may drop the $\bar{D}(\dots)$ term and set to zero the left hand side of (6.37) since it is a $\overline{Q}_{\dot{\alpha}}$ commutator, therefore zero in the chiral ring. In the Dijkgraaf-Vafa conjecture, the Konishi anomaly equations are identified as the loop equations of the matrix model [239].

The one-loop anomaly can be computed as that in [206]. For instance, given an adjoint superfield X and its variation as above, we have

$$\text{anomaly } = \sum_{ijkl} A_{ij,kl}\frac{\partial f(\widetilde{Q}, Q, \Phi, W_\alpha)_{ji}}{\partial \Phi_{kl}}, \tag{6.38}$$

where the coefficient $A_{ij,kl}$ is

$$A_{ij,kl} = \frac{1}{32\pi^2}\left[(W_\alpha W^\alpha)_{il}\delta_{jk} + (W_\alpha W^\alpha)_{jk}\delta_{il} - 2(W_\alpha)_{il}(W^\alpha)_{jk}\right]. \tag{6.39}$$

For the mass-deformed ASQCD, the five independent Konishi anomaly equations are [208, 209]:

$$\mathrm{Tr}\,\frac{\widetilde{W}'(\Phi)}{z-\Phi} + \widetilde{Q}_{\tilde{f}}\frac{m'^{\tilde{f}}_{f}(\Phi)}{z-\Phi}Q^f = 2R(z)T(z) + w_\alpha(z)w^\alpha(z), \tag{6.40a}$$

$$\frac{1}{4\pi}\mathrm{Tr}\,\frac{\widetilde{W}'(\Phi)W_\alpha}{z-\Phi} = 2R(z)w_\alpha(z), \tag{6.40b}$$

$$-\frac{1}{32\pi^2}\mathrm{Tr}\,\frac{\widetilde{W}'(\Phi)W_\alpha W^\alpha}{z-\Phi} = R(z)^2, \tag{6.40c}$$

$$\lambda^f_{f'}\widetilde{Q}_{\tilde{f}}\frac{m^{\tilde{f}}_{f}(\Phi)}{z-\Phi}Q^{f'} = \lambda^f_f R(z), \tag{6.40d}$$

$$\tilde{\lambda}^{\tilde{f}'}_{\tilde{f}}\widetilde{Q}_{\tilde{f}'}\frac{m^{\tilde{f}}_{f}(\Phi)}{z-\Phi}Q^{f} = \tilde{\lambda}^{\tilde{f}}_{\tilde{f}} R(z). \tag{6.40e}$$

The right hand side of equation (6.40a) - (6.40e) is the anomaly at one loop; Setting them to zero reduces to classical F-term equations. Expanding both sides of (6.40) around $z \to +\infty$, and comparing coefficients with the same power of z give perturbative corrections to the chiral ring of the massive theory.

There is one more Konishi anomaly. For an arbitrary matrix $h^{\tilde{g}}_{g}$, we take our chiral rotation to be

$$\delta\Phi = \frac{1}{z-\Phi}\widetilde{Q}_{\tilde{g}}h^{\tilde{g}}_{g}Q^g\frac{1}{z-\Phi}. \tag{6.41}$$

Then we can write down the sixth Konishi anomaly equation:

$$\widetilde{Q}_{\tilde{g}} \frac{W'(\Phi)}{(z-\Phi)^2} Q^g + \sum_{n=1}^{l+1} \sum_{m=0}^{n-2} \widetilde{Q}_{\tilde{f}} \frac{\Phi^m}{z-\Phi} Q^g m^{\tilde{f}}_{f,n} \widetilde{Q}_{\tilde{g}} \frac{\Phi^{n-2-m}}{z-\Phi} Q^f = 2R(z) \widetilde{Q}_{\tilde{g}} \frac{1}{(z-\Phi)^2} Q^g, \tag{6.42}$$

where we have removed $h^{\tilde{g}}_g$ on both sides. We have also dropped terms that contain $W_\alpha Q^f$ or $\widetilde{Q}_{\tilde{f}} W_\alpha$ since they are not in the chiral ring.

The off-shell quantum Coulomb branch vacua have been solved by Cachazo, Seiberg and Witten as [209] using the anomaly equations (6.40):

$$2R(z) = \widetilde{W}'(z) - \sqrt{\widetilde{W}'(z)^2 + f(z)}, \tag{6.43a}$$

$$M(z) = -\sum_{i=1}^{n} \frac{1}{2\pi i} \oint_{A_i} \frac{R(x)}{x-z} \frac{1}{m(x)} dx, \tag{6.43b}$$

$$T(z) = \frac{B'(z)}{2B(z)} - \sum_{I=1}^{L} \frac{y(q_I)}{2y(z)(z-z_I)} + \frac{g(z)}{y(z)}, \tag{6.43c}$$

where $m(x)$ is the abbreviation for $m^{\tilde{f}}_f(z)$ in (6.4b) and $y(z)^2 = \widetilde{W}'(z)^2 + f(z)$. Because of $y(z)$, the solution is defined on a genus $k-1$ Riemann surface Σ. A_i are the cycles that surrounds the i-th cut, smearing of the classical Coulomb branch singularity, and q_I's are the point corresponding to Higgs branch in the first sheet of Σ as a double cover of complex plane. Finally,

$$f(z) = \frac{1}{8\pi^2} \mathrm{Tr} \frac{(\widetilde{W}'(z) - \widetilde{W}'(\Phi)) W_\alpha W^\alpha}{z-\Phi} \tag{6.44a}$$

$$g(z) = \left\langle \mathrm{Tr} \frac{\widetilde{W}'(z) - \widetilde{W}'(\Phi)}{z-\Phi} \right\rangle - \frac{1}{2} \sum_{I=1}^{L} \frac{\widetilde{W}'(z) - \widetilde{W}'(z_I)}{z-z_I}. \tag{6.44b}$$

In solving these equations, it is required that when z approaches to q_I, the residue of $T(z)$ should be at most one [209]. We conjecture that this condition is encoded in the sixth anomaly equation (6.42), which will be clear in section 6.4 and 6.4. Note the above solutions are off-shell, with $f(z)$ being some generic degree $k-1$ polynomial. We will solve these equation on-shell later.

Exactness of Konishi anomaly. A natural question to ask is if the Konishi anomaly receives further quantum corrections. Consider first the perturbative higher loop corrections. The UV coupling τ_{UV} is replaced by dynamical scale Λ. To use holomorphy we write down the symmetry when all the couplings as well as scale Λ

	Δ	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_R$	$U(1)_\Phi$	$U(1)_\theta$
Φ	1	**1**	**1**	0	$\frac{2}{3}$	1	0
Q	1	$\square$	**1**	1	$\frac{2}{3}$	0	$\frac{2}{3}$
$\widetilde{Q}$	1	**1**	$\overline{\square}$	1	$\frac{2}{3}$	0	$\frac{2}{3}$
g_n	$2-n$	**1**	**1**	0	$\frac{2}{3}(2-n)$	$-n-1$	2
$m^{\tilde{f}}_{f,n}$	$2-n$	$\overline{\square}$	$\square$	-2	$\frac{2}{3}(2-n)$	$1-n$	$\frac{2}{3}$
W_α	$\frac{3}{2}$	**1**	**1**	0	1	0	1
Λ^{2N-N_f}	$2N-N_f$	**1**	**1**	$2N_f$	$\frac{2}{3}(2N-N_f)$	$2N$	$-\frac{2}{3}N_f$

Table 6.1: Summary of charge assignments for operators and couplings. Note these charges are chosen so that there are no quantum anomalies.

are treated as background superfields. Following [206], the combination $U(1)_\theta = -2U(1)_\Phi/3 + U(1)_R$ is defined for convenience.

Consider first $f = \delta\Phi \propto \Phi$. This variation is considered in [206] and is the coefficient of z^{-2} in the expansion of (6.40a). The difference between our case and [206] is we need to worry about the appearance of $m^{\tilde{f}}_{f,n}$. The right hand side in the expansion (6.40a) has terms proportional to W_α^2, so it is charged $(0,0,2)$ under $U(1)_A \times U(1)_\Phi \times U(1)_\theta$. Acceptable corrections should not depend on the negative power of couplings since they should vanish if couplings are zero. The only possible terms are $g_n\Phi^{n+1}$, W_α^2 and $m^{\tilde{f}}_{f,n}\widetilde{Q}_{\tilde{f}}\Phi^{n-1}Q^f$, but they are already present in one loop.

The general case when $\delta\Phi \propto \Phi^m$ is similar, where the charge under $U(1)_A \times U(1)_\Phi \times U(1)_\theta$ becomes $(0, m-1, 2)$. The terms already presented in the one-loop expression are $g_n\Phi^{n+m}$, $m^{\tilde{f}}_{f,n}\widetilde{Q}_{\tilde{f}}\Phi^{n+m-2}Q^f$ and $\sum_{l=0}^{m-1} \mathrm{Tr}\, W_\alpha^2\Phi^{m-l-1}\mathrm{Tr}\,\Phi^l$, all of which have the right charge.

Likewise we can consider $\delta Q^f \propto \Phi^m Q^f$ which is the z^{-m-1} coefficient in the expansion of (6.40d). As a result similar to previous argument, we see no higher loop correction is possible which is in accordance with symmetry and holomorphy.

Finally, we can consider $\delta\Phi \propto \Phi^m Q^g h_g^{\tilde{g}}\widetilde{Q}_{\tilde{g}}\Phi^n$ in (6.42). For simplicity we illustrate $m = n = 0$ only. This is the z^{-2} coefficient in the expansion. It is charged $(2, -1, 10/3)$ under $U(1)_A \times U(1)_\Phi \times U(1)_\theta$. Once again, the allowable term $\widetilde{Q}_{\tilde{f}}\Phi^m Q^g m^{\tilde{f}}_{f,n}\widetilde{Q}_{\tilde{g}}\Phi^{n-2-m}Q^f$ is already there at one-loop.

Nonperturbatively we should study the algebra of chiral rotations and the Wess-Zumino consistency condition on the anomaly [240], following the line of [241]. We define the generators of the algebra as

$$L_n = \Phi^{n+1}\frac{\delta}{\delta\Phi}, \tag{6.45a}$$

$$Q_{n,\alpha} = \frac{1}{4\pi}W_\alpha\Phi^{n+1}\frac{\delta}{\delta\Phi}, \tag{6.45b}$$

$$R_n = -\frac{1}{32\pi^2}W_\alpha W^\alpha\Phi^{n+1}\frac{\delta}{\delta\Phi}, \tag{6.45c}$$

$$M^f_{f',n} = \Phi^n Q^f\frac{\delta}{\delta Q^{f'}}, \tag{6.45d}$$

$$\widetilde{M}^{\tilde{f}'}_{\tilde{f},n} = \widetilde{Q}_{\tilde{f}}\Phi^n\frac{\delta}{\delta\widetilde{Q}_{\tilde{f}'}}. \tag{6.45e}$$

Classically they satisfy commutation relations which are an extension of Virasoro algebra:

$$\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n}, \quad [L_m, Q_{n,\alpha}] = (n-m)Q_{n+m,\alpha},\\
&[L_m, R_n] = (n-m)R_{m+n}, \quad \{Q_{m,\alpha}, Q_{n,\alpha}\} = -\epsilon_{\alpha\beta}(n-m)R_{n+m},\\
&[Q_{m,\alpha}, R_n] = 0, \qquad [R_m, R_n] = 0,\\
&[M^f_{f',n}, M^g_{g',m}] = \delta^g_{f'}M^f_{g',n+m} - \delta^f_{g'}M^g_{f',n+m},\\
&[\widetilde{M}^{\tilde{f}'}_{\tilde{f},n}, \widetilde{M}^{\tilde{g}'}_{\tilde{g},m}] = \delta^{\tilde{f}'}_{\tilde{g}}\widetilde{M}^{\tilde{g}'}_{\tilde{f},n+m} - \delta^{\tilde{g}'}_{\tilde{f}}\widetilde{M}^{\tilde{f}'}_{\tilde{g},n+m},\\
&[M^f_{f',n}, \widetilde{M}^{\tilde{g}'}_{\tilde{g},m}] = 0,\\
&[L_n, M^f_{f',m}] = mM^f_{f',n+m}, \qquad [L_n, \tilde{M}^{\tilde{f}'}_{\tilde{f},m}] = m\widetilde{M}^{\tilde{f}'}_{\tilde{f},m+n},\\
&[Q_{n,\alpha}, M^f_{f',m}] = 0, \qquad [Q_{n,\alpha}, \widetilde{M}^{\tilde{f}'}_{\tilde{f},m}] = 0,\\
&[R_n, M^f_{f',m}] = 0, \qquad [R_n, \widetilde{M}^{\tilde{f}'}_{\tilde{f},m}] = 0.
\end{aligned} \tag{6.46}$$

One can in principle include the generator

$$K_{s,t} = \Phi^s\widetilde{Q}_{\tilde{g}}h^{\tilde{g}}_g Q^g\Phi^t\frac{\delta}{\delta\Phi}; \tag{6.47}$$

here we do not consider the algebra involving $K_{s,t}$, since when acting on generalized mesons the transformation is not linear anymore. Note due to the presence of fundamentals, there is no $U(1)$ shift symmetry, unlike the case with adjoint only. In terms of these operators, the Konishi anomaly can be expressed as a representation of the algebra:

$$
\begin{aligned}
L_n W_{\text{eff}} &= \mathcal{L}_n, & M^{f}_{f',n} W_{\text{eff}} &= \mathcal{M}^{f}_{f',n}, \\
Q_{n,\alpha} W_{\text{eff}} &= \mathcal{Q}_{n,\alpha}, & \widetilde{M}^{\tilde{f}'}_{\tilde{f},n} W_{\text{eff}} &= \widetilde{\mathcal{M}}^{\tilde{f}'}_{\tilde{f},n}, \\
R_n W_{\text{eff}} &= \mathcal{R}_n.
\end{aligned}
\tag{6.48}
$$

It is not hard to check that these perturbative anomalies $\mathcal{L}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{M}$, and $\widetilde{\mathcal{M}}$ satisfy the Wess-Zumino consistency conditions and thus form a representation of the chiral rotation algebra.

Now we are ready to check the nonperturbative corrections both to the algebra and the Konishi anomalies. Our theory has an axial $U(1)_A$ symmetry. The generators L, Q, R, M and $\widetilde{M}$ all have charge 0 under the $U(1)_A$. Then the correction to the commutation relations should not carry $U(1)_A$ charge as well. But the scale Λ^{2N-N_f} has charge $2N_f$. The only way to cancel it is to use powers of $m^{\tilde{f}}_{f,k}$. To extract singlet from the flavor symmetry, we have to antisymmetrize the indices:

$$
\epsilon_{\tilde{i}_1\tilde{i}_2\ldots\tilde{i}_{N_f}} \epsilon^{i_1 i_2 \ldots i_{N_f}} m^{\tilde{i}_1}_{i_1,n_1} m^{\tilde{i}_2}_{i_2,n_2} \ldots m^{\tilde{i}_{N_f}}_{i_{N_f},n_{N_f}}. \tag{6.49}
$$

When those m's are finite, one expects that all the non-perturbative corrections can be absorbed into redefinition of the elements in the algebra [242, 243]. We leave the detailed proof to the future work.

Nonperturbative corrections

There are other relations in the chiral ring of nonperturbative origin, and typically involving strong coupling scale. Recall that our gauge group is of finite rank, the Casimir operators $\{u_i = \operatorname{Tr}\Phi^i\}_{i=0}^{+\infty}$ are not all independent. The constraint comes from the characteristic polynomial of matrix Φ:

$$
u_{N_c+p} = \mathcal{F}(u_1, u_2, \ldots, u_{N_c-1}, u_{N_c}), \quad p \in \mathbb{Z}^+. \tag{6.50}
$$

Classically, if we denote $P(z) = \det(z - \Phi) = z^{N_c} + p_1 z^{N_c-1} + \cdots + p_{N_c-1} z + p_{N_c}$ as the characteristic polynomial, then the above relation can be packaged as

$$
\frac{P'(z)}{P(z)} = T(z). \tag{6.51}
$$

The left hand side of (6.51) depends on finite number of parameters $p_1, \ldots, p_{N_c}$ while the right hand side of (6.51) contains all the Casimir operators. This implies the classical constraint (6.50).

Quantum mechanically (6.50) gets modified by instanton effects, turning into

$$u_{N_c+p} = \widehat{\mathcal{F}}(u_1, u_2, \ldots, u_{N_c-1}, u_{N_c}; \Lambda^{2N_c-N_f}), \quad p \in \mathbb{Z}^+. \tag{6.52}$$

This can be deduced based on the fact that the resolvent $T(z)$ has quantized periods. Indeed, if we focus on the classical Coulomb branch solution (6.23), then $T(z)$ has a pole when z approaches to one of the root a_i with residue equal to number of entries of a_i. Integrate around small cycle around a_i we have

$$\frac{1}{2\pi i}\oint_{a_i} T(z)dz = N_i \in \mathbb{Z}. \tag{6.53}$$

Quantum mechanically the poles a_i are smeared into cuts A_i, and the complex plane becomes a Riemann surfaces $\Sigma : y(z)^2 = W'(z)^2 + f(z)$ (6.43), but the quantization condition is the same [209]:

$$\frac{1}{2\pi i}\oint_{A_i} T(z)dz = N_i \in \mathbb{Z}, \tag{6.54}$$

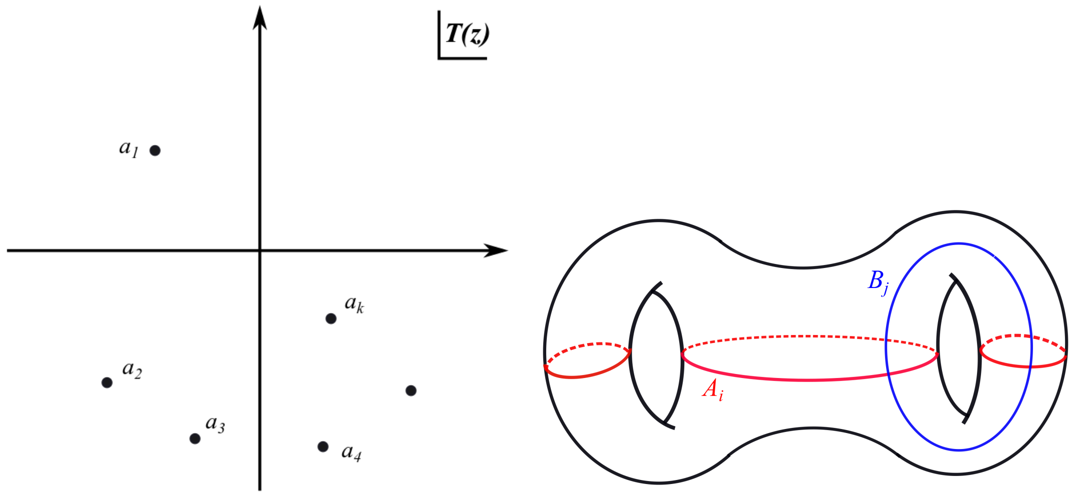
still giving the rank of unbroken gauge group. Hence the rank is robust against quantum corrections, in accordance with what we mentioned in section 6.3. See figure 6.2 for illustration.

Moreover there are other quantization conditions. Pick the compact cycle B_i of the Riemann surface whose intersection number with A_i is δ_{ij}. The field equation of $T(z)$ implies that

$$\frac{1}{2\pi i}\oint_{B_i} T(z)dz = -N_i' \in \mathbb{Z}. \tag{6.55}$$

This is proved by computing the effective superpotential and studying its field equations; so this relation is on shell [209]. Quantization condition of the resolvent $T(z)$ over cycles of Σ implies that $T(z) = d\log\xi(z)$ for some function $\xi(z)$ on Riemann surface Σ.

Another way of understanding the quantization condition for $T(z)$ is as follows. Once we expand the anomaly equations (6.40) and impose (6.52), the set of equations are overdetermined; there are more equations than variables. In order for the recursion relation to admit solutions, it is necessary and sufficient that the periods of $T(z)$ are



(a) (b)

Figure 6.2: The classical (a) and quantum (b) picture of describing resolvent $T(z)$. Classically, $T(z)$ takes value on a complex plane, with poles located at the root of (6.21). Quantum mechanically, the complex plane becomes a Riemann surface described by $y(z)^2 = \widetilde{W}'(z)^2 + f(z)$; the poles a_i becomes cuts A_i. We also choose B_i that intersects only A_i. The quantization condition is around the cycle A_i and B_j.

quantized. This statement is proved by Ferrari and collaborators [243, 244]. If one defines $T(z) = F'(z)/F(z)$ then [244] concludes that

$$F(z) + \frac{\gamma B(z)}{F(z)} = P(z) \tag{6.56}$$

with degree N polynomial $P(z)$. Then

$$F(z) = \frac{1}{2}\left(P(z) + \sqrt{P^2(z) - 4\gamma B(z)}\right), \tag{6.57}$$

and

$$T(z) = \frac{d}{dz}\log\left(P(z) + \sqrt{P^2(z) - 4\gamma B(z)}\right). \tag{6.58}$$

The factor γ can be chosen so that when $m(z) = (M + z)\delta_f^{\tilde{f}}$, in the square root of (6.58) $P(z)^2 - 4\gamma B(z)$ should reduce to standard Seiberg-Witten curve; when $m(z) = m_f^{\tilde{f}}$ it should reduce to that of [206]. Therefore it is natural that $\gamma = \Lambda^{2N-N_f}$. This is consistent with [209]. By setting $\Lambda = 0$ one can get back to the classical results:

$$T(z) = \frac{F'(z)}{F(z)} = \frac{P'(z)}{P(z)} \tag{6.59}$$

so the degree N polynomial $P(z)$ can actually be identified as $\det(z-\Phi)$, that is why we used the same symbol as that of (6.51). Note that the expression in the square root of (6.58) is precisely what is conjectured by Kapustin [245] to be the $\mathcal{N}=1$ analogue of Seiberg-Witten curve.

The quantum corrected formula (6.58) *is* a chiral ring relation, since (6.58) is satisfied on all supersymmetric vacua of the theory.

The photino w_α will be corrected as well. (6.56) holds for arbitrary Φ, so it holds for $\Phi + \epsilon M$ for arbitrary small ϵ and any matrix M. Taking derivative with respect to ϵ in $T(z) = F'(z)/F(z)$, we have

$$\operatorname{Tr} \frac{M}{(z-\Phi)^2} = -\partial_z \left(\frac{F^{(\epsilon)}(z)}{F(z)} \right), \tag{6.60}$$

where we have introduced

$$F^{(\epsilon)} = -\partial_\epsilon F(z;\epsilon) = -\partial_\epsilon \left[\frac{1}{2} \left(P(z;\epsilon) + \sqrt{P^2(z;\epsilon) - 4\gamma B(z)} \right) \right] \tag{6.61}$$

with $P(z;\epsilon) = \det(z - \Phi - \epsilon M)$. Take $M = W_\alpha$ and integrate over (6.60), we get

$$w_\alpha = \frac{1}{4\pi} \frac{-\partial_\epsilon P(z;\epsilon)}{\sqrt{P^2(z) - 4\gamma B(z)}} \Bigg|_{\epsilon \to 0}. \tag{6.62}$$

This is a new relation. However, as w_α has trivial expectation value for supersymmetric vacua, we will not need this relation in the future.

Comparison with perturbative chiral ring. After nonperturbative analysis, let us take a quick look at how perturbative ring looks like. By perturbative chiral ring we mean the strong coupling scale $\Lambda \to 0$, and the chiral ring relation is governed by one-loop Konishi anomaly alone.

First we show that perturbatively there is no gaugino condensations. Recall our theory is governed by Riemann surfaces parametrized by $y(z)^2 = \widetilde{W}'(z)^2 + f(z)$. The nonperturbative formula (6.58) gives another parametrization of the Riemann surface Σ: $P^2(z) - 4\Lambda^{2N-N_f} B(z)$. Requiring consistency of the theory means the Riemann surfaces must factorize properly [209]:

$$\begin{aligned} P^2(z) - 4\Lambda^{2N-N_f} B(z) &= H^2(z) C(z), \\ \widetilde{W}'(z)^2 + f(z) &= G(z)^2 C(z), \end{aligned} \tag{6.63}$$

where $G(z)$ and $H(z)$ are some polynomials. Perturbatively $\Lambda = 0$, so we see $\widetilde{W}'(z)^2 + f(z)$ is a perfect square. However since $\widetilde{W}'(z)$ has degree k while $f(z)$ has degree $k-1$, this is impossible unless $f(z)=0$. Plug into (6.43a), we see we must have $R(z)=0$. Plug into (6.40d), we go back to the classical F-term for the Higgs branch. Therefore, perturbation theory does not alter the classical Higgs branch vacua.

Examples of chiral ring solution

We have introduced the gadgets to compute the quantum chiral ring of the massive theory in previous subsections, *c.f.* equations (6.40) and (6.58). In this section we explicitly see how chiral ring solutions give supersymmetric quantum vacua, in a one-to-one manner.

Let us consider a massive $U(2)$ theory with one flavor, and $k=2$. This model is considered in section 6.3; here we assume the tree level superpotential to be

$$W_{\text{tree}} = \frac{1}{3}\operatorname{Tr}\Phi^3 - \frac{1}{2}\operatorname{Tr}\Phi^2 + \widetilde{Q}(1+\Phi)Q, \tag{6.64}$$

where we pick all the coupling to be ± 1 for simplicity. Let us focus first on classical chiral ring. The expectation value of Φ can have either pseudo-confining vacua or Higgs vacua (modulo Weyl equivalence):

$$\langle\Phi\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.65}$$

This can be computed using entirely the chiral ring. Our strategy is to solve (6.40) and then rule out certain solution using (6.42). Classically there is no gaugino condensation so $R(z)=0$. Expanding with respect to large z we have

$$\begin{aligned} u_{n+2} - u_{n+1} + v_n &= 0, \\ v_{n+1} + v_n &= 0. \end{aligned} \tag{6.66}$$

These equations give $u_1 = u_3 = u_5 = \dots$, and $u_2 = u_4 = u_6 = \dots$. There are also chiral ring relations for the adjoints. We know from (6.51):

$$P(z) = \det(z-\Phi) = \sum_{i=0}^{N} p_i z^{N-i}. \tag{6.67}$$

The coefficients p_i of $P(z)$ are related to u_j by Newton's identity

$$p_n = -\frac{1}{n}\sum_{i=1}^{n} u_i p_{n-i}, \tag{6.68}$$

so we obtain two equations on the generators u_1 and u_2:

$$\begin{aligned} u_2 - u_1^2 - \frac{u_2^2}{2} + \frac{u_1^2 u_2}{2} &= 0, \\ u_1 + \frac{u_1^3}{2} - \frac{3}{2} u_1 u_2 &= 0. \end{aligned} \tag{6.69}$$

These two equations actually contain six solutions, which are

$$(u_1, u_2) = (0,0),\ (1,1),\ (2,2),\ (-1,1),\ (0,2),\ (-2,2). \tag{6.70}$$

Here the first five solutions are exactly listed in (6.65), including both Coulomb and Higgs vacua; the last one is not a physical solution, which corresponds to putting two -1 (the root of $1+z$) in the diagonal of $\langle\Phi\rangle$.

Remember that we still have one extra Konishi anomaly equation (6.42), which imposes additional constraint on generalized mesons. The recursion relation reads:

$$(n+1)(v_{n+2} - v_{n+1}) + \sum_{i=0}^{n} v_i v_{n-i} = 0. \tag{6.71}$$

Notice that this equation is satisfied for all Coulomb branch vacua; the recurrence is also satisfied for the vacua $(u_1, u_2) = (-1, 1)$ and $(0, 2)$. However $(-2, 2)$ is ruled out. Therefore, our classical chiral ring relation gives a complete solution which is identical to solving the F-term equations.

In [209] the first five Konishi anomaly equations are used. There the way to make the solution physically sensible is to impose by hand that the residue of the resolvent $T(z)$ at the Higgs branch singularity should be at most 1; this extra condition is valid both at classical and quantum level. We conjecture that this residue condition is equivalent to imposing another Konishi anomaly (6.42). We prove it in section 6.4.

Next we would like to analyze the quantum chiral ring of the model (6.64). Quantum mechanically the anomaly equations read:

$$\begin{aligned} u_{n+2} - u_{n+1} + v_n &= 2\sum_{i=0}^{n-1} r_i u_{n-i-1}, \\ v_{n+1} + v_n &= r_n, \\ r_{n+2} - r_{n+1} &= \sum_{i=0}^{n-1} r_i r_{n-i-1}, \\ (n+1)(v_{n+2} - v_{n+1}) + \sum_{i=0}^{n} v_i v_{n-i} &= 2\sum_{i=0}^{n-1} (n-i) r_i v_{n-i-1}. \end{aligned} \tag{6.72}$$

Likewise we read off the constraints of Casimir operators by expanding

$$T(z) = \frac{P'(z)}{\sqrt{P^2(z) - 4\Lambda^3(1+z)}} - \frac{2\Lambda^3}{\sqrt{P^2(z) - 4\Lambda^3(1+z)}} \frac{1}{P(z) + \sqrt{P^2(z) - 4\Lambda^3(1+z)}} \tag{6.73}$$

with $P(z) = p_0 z^2 + p_1 z + p_2$. Then we obtain the following relations on u_i:

$$\begin{aligned}
u_3 &= 3\Lambda^3 - \frac{1}{2}u_1^3 + \frac{3}{2}u_1 u_2, \\
u_4 &= 4\Lambda^3(1 + 2u_1) - \frac{1}{2}u_1^4 + u_1^2 u_2 + \frac{1}{2}u_2^2, \\
u_5 &= 10\Lambda^3\left(u_1^2 + u_1 + \frac{1}{2}u_2\right) - \frac{1}{4}u_1^5 + \frac{5}{4}u_1 u_2^2, \\
u_6 &= 9\Lambda^6 - \frac{3}{4}u_1^4 u_2 + 18\Lambda^3\left(\frac{1}{3}u_1^3 + \frac{2}{3}u_1^2 + u_1 u_2 + \frac{1}{3}u_2\right) + \frac{3}{2}u_1^2 u_2^2 + \frac{1}{4}u_2^3, \\
&\dots
\end{aligned} \tag{6.74}$$

To the order of u_6 we can completely determine the expectation value of u_1 and u_2 and get rid of any unphysical solutions. One can use the elimination theory to get the final equation for u_1:

$$(u_1 - 1)\left(u_1^3(u_1+1)(u_1-2)^2 - 9u_1(8u_1^2 + 9u_1 + 4)\Lambda^3 - 27\Lambda^6\right) = 0. \tag{6.75}$$

Note that the vacua are corrected by instantons. When setting $\Lambda \to 0$ we get back to the classical solutions. In particular we recognize one vacuum in the solution with eigenvalue diag(0, 1) for $\langle\Phi\rangle$. When $u_1 = 1$, we can solve that $u_2 = 1$, thus determining the characteristic polynomial $P(z) = z^2 - z$. For generalized glueballs we have $2r_0 = r_1 = 2\Lambda^3$. Therefore we can package it as

$$\begin{aligned}
T(z) &= \frac{d}{dz}\log\left[z^2 - z + \sqrt{(z^2-z)^2 - 4\Lambda^3(1+z)}\right], \\
R(z) &= \frac{1}{2}\left(z^2 - z - \sqrt{(z^2-z)^2 - 4\Lambda^3(z-1) - 8\Lambda^3}\right), \\
M(z) &= \frac{R(z)}{1+z}.
\end{aligned} \tag{6.76}$$

For this solution, the two Riemann surfaces defined by $y(z)^2 = \widetilde{W}'(z)^2 + f(z)$ and $\tilde{y}(z)^2 = P(z)^2 - 4\Lambda^3 B(z)$ match exactly. The reason that $u_1 = 1$ is not quantum corrected by instantons is that this vacua corresponds to residual $U(1)\times U(1)$ gauge symmetry; Coulomb branch vevs leave both monopoles massive, so in the low energy there are still two independent photons. Moreover from the expression of $T(z)$ we know in this case instanton corrections begin to enter only for superpotential with $k \geq 3$.

Isomorphism of Coulomb branch vacua. In writing down the quantum chiral ring associated to (6.3), we see that the only quantity that enters into the formula is $B(z) = \det\left[m_f^{\tilde{f}}(z)\right]$, which is a degree lN_f polynomial. This means for various choices of l and N_f, one can pick distinct l and N_f such that $B(z)$ is identical. It is natural to conjecture that for these choices the Coulomb branch vevs are exactly the same. This is confirmed by explicit examples (for one example, see appendix J), thus prove the claim of [245].

Solution of the chiral ring and supersymmetric vacua

We now turn to the proof that solutions of the chiral ring in the mass deformed theory are in one to one correspondence with supersymmetric vacua. We also show that the extra anomaly equation (6.42) implies residue constraint on the Higgs branch, proposed by [209].

We begin by proving that the one-to-one correspondence holds for Coulomb branch vacua. Classically, it is obvious that those vacua are exactly contained in the chiral ring by setting $\langle Q\rangle = \langle\widetilde{Q}\rangle = 0$ and $R(z) = 0$ in the Konishi anomaly (6.40):

$$\mathrm{Tr}\,\frac{\widetilde{W}'(\Phi)}{z-\Phi} = 0, \tag{6.77}$$

since this is just a gauge invariant way of writing F-term equations.

Conversely, we show the solution of Konishi anomaly is contained in F-term solution. For Coulomb branch vacua, the proof is very similar to that of [222]. One can write

$$0 = \mathrm{Tr}\,\frac{\widetilde{W}'(\Phi) - \widetilde{W}'(z) + \widetilde{W}'(z)}{z-\Phi} = -\zeta(z) + \widetilde{W}'(z)T(z), \tag{6.78}$$

where $\zeta(z)$ is a degree $k-1$ polynomial. Therefore we have an equality:

$$T(z) = \frac{P'(z)}{P(z)} = \frac{\zeta(z)}{\widetilde{W}'(z)}, \tag{6.79}$$

or in the product form $P'(z)\widetilde{W}'(z) = \zeta(z)P(z)$. Over complex field $\mathbb{C}$ the polynomials can be factorized, so the general solution is of the form

$$\begin{aligned} \zeta(z) &= E(z)\tilde{\zeta}(z), \quad P(z) = F(z)H(z), \\ \widetilde{W}'(z) &= E(z)F(z), \quad P'(z) = \tilde{\zeta}(z)H(z). \end{aligned} \tag{6.80}$$

Then we have $T(z) = \tilde{\zeta}(z)/F(z)$. But the root of $F(z) = \prod_{i=1}^{n}(z - \lambda_i)$ is the subset of root of $\widetilde{W}'(z)$, and since $\tilde{\zeta}(z)$ is of degree $n - 1$, so we obtain:

$$T(z) = \sum_{i=1}^{n} \frac{\nu_i}{z - \lambda_i}. \tag{6.81}$$

By definition of $T(z)$ one concludes that all ν_i are integers, labelling the number of entries of λ_i in the diagonal of $\langle\Phi\rangle$. So this solution can be obtained by solving F-term.

Next we turn to the classical Higgs branch. This part of the proof is new. Again it is obvious that the F-term equations admit solutions that are all solutions of chiral ring relations. Conversely, suppose the fractional decomposition of resolvent $T(z)$ is

$$T(z) = \sum_{I} \frac{r_I}{z - b_I} + \ldots, \tag{6.82}$$

where the dots represent the terms coming from roots of $\widetilde{W}'(z)$ as in (6.81). Moreover we also claim [209] the solution of $M(z)$ classically is given by (6.25). Plug into (6.40d) we examine the singular part in z while ignoring the regular part and obtain:

$$m_f^{\tilde{f}}(z) M_{\tilde{f}}^{f}(z) = 0. \tag{6.83}$$

We integrate this formula around b_I and notice the singularity comes from $M(z)$ while $m(z)$ is a polynomial, and we conclude that $m_f^{\tilde{f}}(b_I)$ is a degenerate matrix, namely

$$B(b_I) = \det m_f^{\tilde{f}}(b_I) = 0, \tag{6.84}$$

so b must be a root of $B(z)$. However, straightforward computation shows that the Konishi anomaly equations (6.40a) - (6.40e) even admits solution of $T(z)$ and $M(z)$ with $r_I > 1$. This is exactly what happens in section 6.4. We now show that the sixth anomaly equation (6.42) imposes the condition $r_I = 0$ or 1.

For simplicity and avoiding clutter of notation, we assume the superpotential to be $W_Q = m_1 \widetilde{Q} Q + m_2 \widetilde{Q} \Phi Q$ but we keep $\widetilde{W}_\Phi$ generic. Moreover, to linearize (6.42) we restrict our chiral rotation to be

$$\delta\Phi = \frac{1}{z - \Phi} \widetilde{Q}_{\tilde{g}} h_g^{\tilde{g}} Q^g, \tag{6.85}$$

then it is not hard to see that the singular part of (6.42) becomes

$$W'(z) M(z)_{\tilde{g}}^{g} + v_{0,\tilde{g}}^{f} m_{2,f}^{\tilde{f}} M(z)_{\tilde{f}}^{g} = 0 \tag{6.86}$$

with $M(z)$ being substituted with explicit expression we arrive at $-r_I + r_I^2 = 0$, namely it can only take value 0 and 1. In proving this we use the following fact:

$$\frac{1}{(2\pi i)^2}\oint_{b_I}\oint_{b_J}\left(\frac{1}{m(x)}\right)^f_{\tilde{g}} m_{2,f}^{\tilde{f}}\left(\frac{1}{m(y)}\right)^g_{\tilde{f}} dxdy = \frac{\delta_{IJ}}{2\pi i}\oint_{b_I}\left(\frac{1}{m(x)}\right)^g_{\tilde{g}} dx. \tag{6.87}$$

The conclusion with $r_I = 0, 1$ is exactly the same as the residue condition proposed in [209]. Therefore we conclude that the solution of chiral ring is in one to one correspondence with the supersymmetric vacua at the classical level.

We now comment on the correspondence at the quantum level. We again divide our vacua into Coulomb branch and Higgs branch. Note first that the residue condition $r_I = 0, 1$ cannot be modified at the quantum level. Otherwise if one turns off the strong coupling scale Λ and perturbative anomaly, then the residue condition at classical level is violated. Put another way, an integral constraint is robust against quantum corrections.

On the Coulomb branch, the low energy behavior is determined by factorization of the matrix model curve $y(z)^2 = \widetilde{W}'(z)^2 + f(z)$. If there are $k-n$ massless monopoles, then we have

$$\begin{aligned} \widetilde{W}'(z)^2 + f(z) &= H_{k-n}^2(z)F(z), \\ P(z)^2 - 4\Lambda^{2N_c-N_f}B(z) &= Q_{N-n}^2(z)F(z), \end{aligned} \tag{6.88}$$

so that $F(z)$ is a degree $2n$ polynomial, giving a genus $n-1$ Riemann surface. The number of independent photinos is n. The period of the resolvent $T(z)$ around cycles of Riemann surface give the unbroken rank of the gauge group. These vacua degenerates in a one-to-one manner to the classical supersymmetric vacua.

Massless limit and Kutasov model

We have seen how to calculate the classical and quantum chiral ring of the mass deformed theory by means of solving the recursion relations. In this subsection we will approach the massless limit by setting

$$g_n \to 0, \quad (n < k) \qquad \text{and} \qquad m_f^{\tilde{f}}(z) \to 0, \tag{6.89}$$

and obtain the moduli space of vacua for massless Kutasov model. Again, we emphasize that in this way we only recover the radical of the ring relations as an ideal.

How many parameters are enough? Unlike ordinary SQCD [235] where $\mathrm{Tr}\, mM$ is the only choice of single trace operator deformation, for Kutasov model there are many more deformation parameters. Just as is written in (6.3), we may add

(1) Casimir deformations: $g_n \mathrm{Tr}\,\Phi^{n+1}$ for $n < k$;

(2) Generalized meson deformations: $\mathrm{Tr}\, m_n v_n$ for $nN_f < 2N_c$ [209, 245][8].

Generally, it is required to add all deformations and take various allowed limits. Unfortunately, it would be a cumbersome task. We would like to examine their physical significance and whether their number could be reduced.

Let us begin by Casimir deformations, (6.4a). For $k > 1$, these deformations are used to resolve the nilpotent matrix Φ into a semisimple matrix, $c.f.$ section 6.3. Let there be s_1 of a_1 in the diagonal of $\langle\Phi\rangle$. The low energy gauge group contains a factor $U(s_1)$ and some W-bosons become massive, with mass

$$M_W = |a_1 - a_i|, \quad i \neq 1, \tag{6.90}$$

and Φ acquires mass which is a function of a_i's as well. So tuning g_i's is essentially tuning physical mass parameters. Therefore we have to at least keep the mass generic and distinct; hence the most general (6.4a) is required.

Next we turn to generalized meson deformation (6.4b) with $lN_f < 2N_c$. The claim is that *if one takes generic limit*[9]*, only the first meson deformation, Tr* mv_0 *is sufficient*. We expect such limit probes a subset of true quantum moduli space.

To understand this, we compare the most general deformation (6.4b) and deformation using only $\mathrm{Tr}\, mv_0 = m_f^{\tilde f}\widetilde{Q}_{\tilde f}Q^f$. It is quite obvious that two cases share identical Coulomb branch vacua. For the latter, there is no Higgs branch vacua classically; while for the former case, it is given by (6.24).

Now we take the generic limit. From (6.24c) we learn that the second term in left hand side approaches to a finite quantity while the terms in the bracket goes to zero as $m_{f,n}^{\tilde f} \to 0$. To have solutions we must require at least one of $\langle Q_1^f\rangle$ and $\langle \widetilde{Q}_{\tilde f}^1\rangle$ goes to infinity, which is a run-away vacua. Therefore, we conclude that the extra Higgs branch vacua are absent; the two kinds of deformation are equivalent.

Quantum mechanically, the solution of $M(z)$ for arbitrary vacuum is given by [209]:

$$M(z) = R(z)\frac{1}{m(z)} - \sum_{I=1}^{L} \frac{r_I \widetilde{W}'(z_I) + (1-2r_I)R(q_I)}{z - z_I}\frac{1}{2\pi i}\oint_{z_I}\frac{1}{m(x)}dx, \tag{6.91}$$

[8]The reason for this requirement is that (1) the generalized meson deformations are all relevant; (2) the metric of the Coulomb branch is positive definite; (3) the $\mathcal{N} = 2$ theory whose curve is isomorphic to that in the square root of (6.58) is asymptotically free.

[9]By generic limit we mean that the roots of $\widetilde{W}'(z)$ and $B(z)$ are kept distinct.

where z_I for $I = 1, \dots L = lN_f$ is the roots of $B(z)$. When $r_I = 1$, poles of $T(z)$ around z_I is on the first sheet, while $r_I = 0$ the second sheet. When all $r_I = 0$, we return to the Coulomb branch vacua, (6.43b). A fact that we will prove in the Appendix I is $R(z) = 0$ in the final limit, so if there exists some $r_J = 1$ we see that the second term of $M(z)$ is infinite, assuming no accidental cancellation appears.

However, in the classical expression (6.24c), we see a flat direction opens up if b happens to be the root of $\widetilde{W}'(z)$. These would recover some missing Higgs branches. Therefore, to completely reproduce the flat directions in the quantum vacua, $B(z) = \det\left[m_f^{\tilde{f}}(z)\right]$ should have at least many roots as $\widetilde{W}'(z)$. Therefore, we conjecture that the sufficient number of meson deformations should satisfy:

$$k - 1 \le lN_f < 2N_c. \tag{6.92}$$

Here we write $k - 1$ instead of k, as an overall $U(1)$ factor in the gauge group does not affect the result.

Even for $l = 1$, the computation of chiral ring is quite challenging. Relegating detailed study for future, here we only focus on the potential with $l = 0$:

$$W_{\text{tree}} = \sum_{n=0}^{k} \frac{g_n}{n+1} \operatorname{Tr} \Phi^{n+1} + m_f^{\tilde{f}} \widetilde{Q}_{\tilde{f}} Q^f \tag{6.93}$$

to probe a subset of vacuum structure. We will see in certain cases it already has very nontrivial consequences. For convenience, we will take $g_k = 1$ in later examples. Note that $k = 2$ is special. We know the most general deformation is $\tilde{W}'(z) = z^2 + \theta z + \nu$. No matter which root one picks, we always get the mass

$$\left|\widetilde{W}''(z_{1,2})\right| = \Delta_2 = \sqrt{\theta^2 - 4\nu} \tag{6.94}$$

so what matters is the discriminant. We can thus set $\theta = 0$ for a further simplification.

With the deformation $\operatorname{Tr} m v_0$ only, the six Konishi anomaly equations are no longer mutually independent. In fact, the anomaly (6.42) can be deduced from (6.40c) and (6.40d). We have seen that this is true classically in section 6.4. Quantum mechanically we can expand (6.42) in terms of $z \to \infty$:

$$(n+1) \sum_{i=0}^{k} g_i v_{n+i} = 2 \sum_{i=0}^{n-1} (n-i) r_i v_{n-i-1}, \tag{6.95}$$

where we omitted the flavor indices. Now multiplying both sides by mass matrix $m_g^{\tilde{g}}$, using (6.40d), and massage the dummy indices a little we get

$$2(n+1) \sum_{i=0}^{k} g_i r_{n+i} = 2(n+1) \sum_{i=0}^{n-1} r_i r_{n-i-1}. \tag{6.96}$$

We see this is exactly the recursion relation given by (6.40c). Therefore in the following computation we will ignore (6.42) unless stated.

In Appendix I, we examine some general properties of the vacuum expectation values in the massless limit, from the recursion relations.

6.5 Examples of quantum chiral rings

In this section we study examples of massless $\widehat{S}$. These various examples also give further confirmation on the statements we made previously in section 6.4.

$k = 1$: the vacua of $U(N_c)$ SQCD

We begin with $k = 1$, the superpotential (6.2) is essentially a mass term. When the scale Λ of the theory is smaller than the mass scale of the adjoint, Φ can be integrated out in the IR and the theory is effectively given by $U(N_c)$ SQCD. This RG flow has been analyzed in [246, 247], while $U(N_c)$ SQCD was studied in [248, 249]. Since Φ is invisible in the IR, there is no need to add Casimir deformation (6.4a), in consistent with (6.92).

$U(N_c)$ SQCD with N_f fundamental flavors can be thought of as gauging the baryon symmetry of $SU(N_c)$ theory, under which the quark and anti-quark have charge ± 1 respectively. When $N_c \geq N_f$ the classical chiral ring is generated by mesons $\widetilde{Q}_{\tilde{f}} Q^f$ freely; while for $N_c < N_f$ there are nontrivial relations among mesons [248]:

$$M^{[i_1}_{j_1} M^{i_2}_{j_2} \dots M^{i_{N_c+1}]}_{j_{N_c+1}} = 0. \tag{6.97}$$

This relation arises since mesons of order N_f are built from rank N_c data; and in particular for $N_f = N_c + 1$ the relation becomes $\det M = 0$.

In the following we scale the mass g_1 in (6.2) to be 1, and its dependence can be easily recovered. The superpotential we use is

$$W_{\text{tree}} = \frac{1}{2} \operatorname{Tr} \Phi^2 + m^{\tilde{f}}_{f} \widetilde{Q}_{\tilde{f}} Q^f . \tag{6.98}$$

A short cut to analyze the quantum vacua is to directly apply (6.43c). However we will try a more elaborated way by solving the recursion relation directly. This will be helpful later.

First, the recursion relation for generalized glueball in (6.40c) can be solved explicitly:

$$r_{n+1} = \sum_{i=0}^{n-1} r_i r_{n-i-1}, \tag{6.99}$$

which is actually a recursion relation for binomial coefficients in $(1+x)^{1/2}$. By induction,

$$r_{2j} = \frac{2^j(2j-1)!!}{(j+1)!} r_0^{j+1}, \quad r_{2j+1} = 0. \tag{6.100}$$

Next we focus on the recursion relation (6.40a):

$$u_{n+1} = 2\sum_{i=0}^{n-1} r_i u_{n-i-1}. \tag{6.101}$$

Similar induction tells us that

$$u_{2j} = \frac{2^j(2j-1)!!}{j!} r_0^j u_0, \quad u_{2j+1} = 0, \tag{6.102}$$

with initial condition $u_0 = N_c$. We can plug them into the series of $T(z)$ and get

$$\begin{aligned} T(z) &= \sum_{n=0}^{+\infty} \frac{u_{2n}}{z^{2n+1}} \\ &= \frac{u_0}{z}\sum_{n=0}^{+\infty} \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} \left(-\frac{4r_0}{z^2}\right)^n \\ &= \frac{N_c}{z}\left(1-\frac{4r_0}{z^2}\right)^{-\frac{1}{2}} \end{aligned} \tag{6.103}$$

which is exactly the same as given by (6.43c).

We define $\widetilde{\Lambda}^{2N_c} = (\det m)\Lambda^{2N_c-N_f}$ by scale matching condition. In the meanwhile there is a degree N_c-polynomial $P(z)$ with leading coefficient 1 such that

$$T(z) = \frac{N_c}{z}\left(1-\frac{4r_0}{z^2}\right)^{-\frac{1}{2}} = \frac{P'(z)}{\sqrt{P(z)^2 - 4\widetilde{\Lambda}^{2N_c}}}. \tag{6.104}$$

Integrate both sides and note that the only way that $P(z)$ is a polynomial with leading coefficient 1 is that

$$r_0 \sim \tilde{\Lambda}^2 = (\det m)^{\frac{1}{N_c}} \Lambda^{\frac{2N_c-N_f}{N_c}}. \tag{6.105}$$

Hence,

$$\langle \widetilde{Q}Q\rangle = v_0 = \left(\frac{1}{m}\right) r_0 = \Lambda^{\frac{2N_c-N_f}{N_c}} (\det m)^{\frac{1}{N_c}} \left(\frac{1}{m}\right). \tag{6.106}$$

However, we should remember the scale Λ appeared here is not the scale Λ_L of low energy effective SQCD. They are related by scale matching condition

$$\Lambda_L^{3N_c-N_f} = \Lambda^{2N_c-N_f}. \tag{6.107}$$

Substituting into (6.105) and (6.106) we see the results for vevs of mesons and gaugino condensation is exactly given by that of [22, 235].

Now we can list the quantum chiral ring for above cases.

(1) $N_c > N_f$. There is no supersymmetric ground state; which means the ideal $\widehat{\mathcal{S}}$ contains unit, so $\widehat{\mathcal{R}}$ is empty;

(2) $N_c = N_f$. It is easy to see $\det v_0 = \Lambda^{N_c}$. Therefore the quantum moduli space is smoothed out.

(3) $N_c < N_f$. The quantum moduli space is the same as the classical one, thus

$$\widehat{\mathcal{R}}_{N_c,N_f,1} = \mathcal{R}_{N_c,N_f,1}. \tag{6.108}$$

$U(2)$ **theory with** $k = 2$ **revisited**

In this section we analyze the quantum chiral ring of the examples given in 6.3. As mentioned before we will use the superpotential (6.93) to deform the Kutasov model:

$$W_{\text{tree}} = \frac{1}{3}\mathrm{Tr}\,\Phi^3 - \tau^2 \mathrm{Tr}\,\Phi + m_f^{\tilde{f}} \widetilde{Q}_{\tilde{f}} Q^f, \tag{6.109}$$

where we define $\tau^2 = -g_0$. We can use (6.40a) and (6.40c) to solve for the Casimir u_j and generalized glueball r_j first. There are two types of solution:

• 1^{st} Solution[10]:

$$\begin{aligned}
u_1 &= -\left[4\tau^2 - 8(\det m)^{\frac{1}{2}}\Lambda^{\frac{4-N_f}{2}}\right]^{\frac{1}{2}}, \\
r_0 &= -(\det m)^{\frac{1}{2}}\,\Lambda^{\frac{4-N_f}{2}}\left[4\tau^2 - 8(\det m)^{\frac{1}{2}}\Lambda^{\frac{4-N_f}{2}}\right]^{\frac{1}{2}}, \\
r_1 &= (\det m)^{\frac{1}{2}}\,\Lambda^{\frac{4-N_f}{2}}\left[2\tau^2 - 3(\det m)^{\frac{1}{2}}\Lambda^{\frac{4-N_f}{2}}\right].
\end{aligned} \tag{6.110}$$

• 2^{nd} Solution:

$$u_1 = 0, \quad u_2 = 2\tau^2, \quad r_0 = 0, \quad r_1 = (\det m)\Lambda^{4-N_f}. \tag{6.111}$$

[10] In writing a solution like this, we assume the convention $(x)^{\frac{1}{2}} = \pm\sqrt{x}$, namely one can flip *simultaneously* the sign for the square root. So the above solution has in fact four independent solutions. We do not apply this rule to the strong coupling scale $x = \Lambda$.

Higher order operators are zero in the limit. These two solutions are in fact the quantum deformed version of the classical vacua [1, 1] and [2] in section 6.3. Indeed, the classical Coulomb vacua for the massive theory is either diag(τ, τ) or diag$(\tau, -\tau)$. The corresponding Young tableau is $[\underline{2}]$ and $[\underline{1}, \underline{1}]$, which is dual to the Young tableau of nilpotent matrix [1, 1] and [2].

The vevs of generalized meson is related to glueballs by (6.40d) as $v_j = r_j m^{-1}$. The resulting quotient is an indeterminate, whose value depend on how τ and $m_f^{\tilde{f}}$ approach to zero.

• $N_f = 1$. For the vacuum [2], we see in the massless limit:

$$u_1 = 0, \quad u_2 = 0, \quad v_0 = 0, \quad v_1 = \Lambda^3. \tag{6.112}$$

This vacuum is quantum mechanically modified, as we have $kN_f = N_c$. Going back to table 6.1, we see immediately that the charge of Λ^3 is exactly the same as the charge of v_1. This is consistent with holomorphy. However, we fail to produce flat direction for v_0 in this particular limit.

For the vacua [1, 1] we see that

$$\begin{aligned} v_0 &= -2m^{-\frac{1}{2}}\Lambda^{\frac{3}{2}}\left[\tau^2 - 2m^{\frac{1}{2}}\Lambda^{\frac{3}{2}}\right]^{\frac{1}{2}}, \\ v_1 &= m^{-\frac{1}{2}}\Lambda^{\frac{3}{2}}\left[2\tau^2 - 3m^{\frac{1}{2}}\Lambda^{\frac{3}{2}}\right]. \end{aligned} \tag{6.113}$$

Here we have the freedom to tune parameters τ and m simultaneously. Consider

$$\tau^2 - 2m^{\frac{1}{2}}\Lambda^{\frac{3}{2}} \approx \eta^2 m^{\alpha}\Lambda. \tag{6.114}$$

For (6.113) not to diverge in the limit, we must have $\alpha \geq 1$. To the leading order we may pick $\alpha = 1$. Plug this in, and we see

$$v_0 = -\eta\Lambda^2 \in \mathbb{C}, \quad v_1 = \Lambda^3, \tag{6.115}$$

so v_1 is again corrected by one-instanton effect, although it has zero classical moduli. We conjecture $v_1 = \Lambda^3$ holds for the entire vacua from all possible limit. Note because of this that the Higgs branch of Kutasov model is smoothed out, so there are no singularities on the moduli space. This is the $k = 2$ analogue of smooth moduli space for $N_c = N_f$ in SQCD.

Here we see a qualitative difference between Kutasov model and its deformed cousin. If we keep the deformation parameter τ finite, then taking $m \to 0$ gives divergent

v_0 and v_1. This is in accordance with [201, 202]; the finite τ endows adjoint chiral multiplet a mass, so the low energy effective theory is just $U(2)$ SQCD with one flavor. It is a well-known fact that ADS superpotential lift the vacuum and the theory does not have a ground state [22, 235, 250]. But this will not happen in Kutasov model where we have seen that simultaneous parameter-tuning still preserves the flat direction.

• $N_f = 2$. This is the simplest case when the theory is in conformal window [134, 220]. Now $m_f^{\tilde{f}}$ is a 2×2 matrix. For simplicity we will take it to be diagonal, $m = \text{diag}(\mu_1, \mu_2)$.

Consider vacuum [2] first. Everything remains the same except there is no instanton correction anymore: $v_1 = 0$. For vacuum [1, 1], the expressions are similar:

$$\begin{aligned} v_0 &= -2(\det m)^{\frac{1}{2}}\,\Lambda\Big[\tau^2 - 2(\det m)^{\frac{1}{2}}\Lambda\Big]^{\frac{1}{2}}\left(\frac{1}{m}\right), \\ v_1 &= (\det m)^{\frac{1}{2}}\,\Lambda\,\Big[2\tau^2 - 3(\det m)^{\frac{1}{2}}\Lambda\Big]\left(\frac{1}{m}\right). \end{aligned} \tag{6.116}$$

We see no matter how one tunes the parameter, v_1 is always zero in the limit[11]. We conclude that generic massless limit could not recover flat directions for v_1. However, it is possible to give flat direction to v_0.

The origin of Higgs branch $v_0 = 0$ remains. This means that at the singularity, the $SU(2)_L \times SU(2)_R$ chiral symmetry is unbroken, and the theory is in non-abelian Coulomb phase. The IR behavior exhibits Kutasov duality.

Here we can also see the difference between Kutasov model and its deformed cousin. When τ is finite, we have $\det v_0 = 4\tau^2\Lambda^2$. Since the adjoint superfield Φ is massive with mass 2τ, we see $4\tau^2\Lambda^2$ is nothing but the low energy scale Λ_L^4 of SQCD. This is precisely the quantum modified moduli space of SQCD.

[11]For instance, we can consider the tuning

$$\tau^2 - 2(\mu_1\mu_2)^{\frac{1}{2}}\Lambda \approx \eta\mu_1^{\alpha}\mu_2^{\beta}\Lambda^2, \quad 0 < \alpha, \beta < 1, \tag{6.117}$$

where we choose $\alpha, \beta < 1$ for the reason that v_0 does not diverge. One sees that

$$v_1 \propto \mu_1^{\frac{\alpha}{2}}\mu_2^{\frac{\beta}{2}} v_0 \tag{6.118}$$

after dropping factors which is zero in the limit. Since $\mu_{1,2} \to 0$ and α, β are positive, we see $v_1 \to 0$ in the limit.

$U(3)$ theory with $k = 2$ revisited

Next we turn to the $U(3)$ theory whose classical chiral ring is analyzed in section 6.3. The superpotential deformation used is again (6.109).

• 1^{st} Solution. This is the one corresponding to $[2,1]$ vacuum:

$$\begin{aligned} u_1 &= -\tau, \quad u_2 = 3\tau^2, \\ r_0 &= -(\det m)^{\frac{1}{2}}\Lambda^{\frac{6-N_f}{2}}, \\ r_1 &= (\det m)^{\frac{1}{2}}\Lambda^{\frac{6-N_f}{2}}\tau. \end{aligned} \tag{6.119}$$

• 2^{nd} Solution. This is the one corresponding to $[1,1,1]$ vacuum:

$$\begin{aligned} u_1 &= -3\sqrt{\tau^2 - 2(\Lambda^{6-N_f}\det m)^{\frac{1}{3}}}, \\ u_2 &= 3\tau^2, \\ r_0 &= -2(\Lambda^{6-N_f}\det m)^{\frac{1}{3}}\sqrt{\tau^2 - 2(\Lambda^{6-N_f}\det m)^{\frac{1}{3}}}, \\ r_1 &= 2(\Lambda^{6-N_f}\det m)^{\frac{1}{3}}\left[\tau^2 - \frac{3}{2}(\Lambda^{6-N_f}\det m)^{\frac{1}{3}}\right]. \end{aligned} \tag{6.120}$$

To get the vevs of generalized mesons v_0 and v_1 we again divide r_0 and r_1 by mass matrix m.

We mainly focus on $N_f = 1$ and this is the region for $kN_f < N_c$. We immediately see $[2,1]$ vacua is non-existent. For $[1,1,1]$ vacuum, we have to be more careful since there is a possibility of tuning parameters. However, to make v_0 finite we need to set:

$$\tau^2 - 2(\Lambda^{6-N_f}\det m)^{\frac{1}{3}} \propto m^{\frac{4}{3}} + \text{higher order terms}. \tag{6.121}$$

But this makes v_1 divergent. Therefore, the vacua is quantum mechanically erased, and the chiral ring is empty:

$$\widehat{R}_{3,1,2} = \varnothing. \tag{6.122}$$

This is consistent with the semi-classical analysis of [200, 202].

Chiral ring relation from magnetic dual

In [202], Kutasov, Schwimmer and Seiberg conjectured a quantum chiral ring relation for the Casimir operators $\mathrm{Tr}\,\Phi^n$. Classically these operators are constrained by the superpotential terms as well as the characteristic polynomial of Φ; however, quantum mechanically the characteristic polynomial coming from the adjoint Ψ in the magnetic theory should also be added to the electric theory, via duality maps that send $\mathrm{Tr}\,\Psi^n$ to the combination of $\mathrm{Tr}\,\Phi^m$. In this way the quantum Coulomb vacua on both sides match.

Here we would like to check this statement explicitly. We consider $U(4)$ theory with $N_f = 3$ and $k = 2$ with mass deformation only for adjoint field Φ:

$$W_\Phi = \frac{1}{3}\mathrm{Tr}\,\Phi^3 - \frac{1}{2}\mathrm{Tr}\,\Phi^2. \tag{6.123}$$

Classically, the theory has five vacua that are labelled by diagonal entries of $\langle\Phi\rangle =$ diag$(0,0,0,0)$, diag$(0,0,0,1)$, diag$(0,0,1,1)$, diag$(0,1,1,1)$, diag$(1,1,1,1)$. This can be packaged into two equations obtained from characteristic polynomial as follows. From Konishi anomaly equation (6.40), we set the right hand side of (6.40a) to zero and get the recursion relation:

$$u_{n+2} - u_{n+1} = 0. \tag{6.124}$$

Moreover, the fact that $T(z) = P'(z)/P(z)$ where $P(z)$ is a degree 4 polynomial implies that u_i for $i > 4$ can be expressed by $u_{1,2,3,4}$. Using above recursion relation we can easily obtain:

$$u_2\left(u_2^4 - 10u_2^3 + 35u_2^2 - 50u_2 + 24\right) = 0, \tag{6.125}$$

which is the classical relation coming from "electric" characteristic polynomial[12].

Let us now see what happens quantum mechanically. To compute quantum corrections we endow all quarks with mass by deforming the superpotential as[13]

$$W = \frac{1}{3}\mathrm{Tr}\,\Phi^3 - \frac{1}{2}\mathrm{Tr}\,\Phi^2 + m_f^{\tilde f}\widetilde{Q}_{\tilde f}Q^f \tag{6.126}$$

and we expect some of the vacua would be erased when $m_f^{\tilde f} \to 0$. Indeed such vacuum has two types of solutions. For the first one, it is a deformation of $\langle\Phi\rangle =$ diag$(0,0,0,0)$:

$$\begin{aligned} u_1 &= 2 - 2\left[1 - 8\left(\det m\Lambda^5\right)^{\frac{1}{4}}\right]^{\frac{1}{2}}, \\ v_0 &= -(\det m\Lambda^5)^{\frac{1}{4}}\left(\frac{1}{m}\right). \end{aligned} \tag{6.127}$$

[12] Our results are slightly different from that of [202] in the sense that there are more vacua because the gauge group is unitary. For special unitary gauge group the traceless condition reduces the number of allowed vacua by about one half. Therefore, we would have $N_c/2$ when N_c is even as in [202].

[13] Because Φ is massive now, deforming by $\mathrm{Tr}\,mv_0$ is enough.

Moreover, since $N_f = 3$ we learned that $\det v_0$ is infinite. Therefore this vacuum is not present at quantum level. Similar reasoning shows that the vacuum which is the deformation of $\langle\Phi\rangle = \mathrm{diag}(1, 1, 1, 1)$ is also absent. The total number is reduced from 5 to 3, corresponding to $u_1 = 1, 2, 3$.

Physically, these two run-away vacua precisely correspond to the parameter regime where ADS superpotential is generated at low energies ($N_f < N_c$) after Φ is integrated out. The idea of [202] is that such elimination is equivalent to including the characteristic polynomial from magnetic dual via operator mapping. We now demonstrate that this is true.

First of all, it is straightforward to check that as $m_f^{\tilde{f}} \to 0$ the vevs of Casimir operators are not quantum shifted. Following [202] we define

$$\widehat{\Phi} = \Phi - \frac{1}{2}\mathbb{I}, \tag{6.128}$$

where $\mathbb{I}$ is the unit matrix. Then the superpotential becomes:

$$W_\Phi = \frac{1}{3}\mathrm{Tr}\,\widehat{\Phi}^3 - \frac{1}{4}\mathrm{Tr}\,\widehat{\Phi} - \frac{1}{3}. \tag{6.129}$$

Kutasov duality proposes that the magnetic dual is a $U(2)$ gauge theory with $N_f = 3$ flavors of quarks and generalized mesons, plus an adjoint field Ψ with superpotential

$$\widehat{W} = \widehat{W}_\Psi + \widehat{W}_q = \sum_{i=0}^{2} \frac{\hat{g}_i}{i+1}\mathrm{Tr}\,\Psi^i + \sum_{j=0}^{1} v_j \tilde{q}\,\Psi^{1-j} q. \tag{6.130}$$

When focusing on Coulomb branch, we can perform a similar trick and turn the superpotential of Ψ part into

$$\widehat{W}_\Psi = \frac{\hat{t}_0}{3}\mathrm{Tr}\,\widehat{\Psi}^3 + \hat{t}_2\mathrm{Tr}\,\widehat{\Psi} + \hat{\alpha}, \tag{6.131}$$

where $\hat{\alpha}$ is some constant. The coupling and operator mappings given in [202] tell us that

$$\hat{t}_0 = 1, \quad \hat{t}_2 = -\frac{1}{4}. \tag{6.132}$$

Then it is not hard to see that for dual theory, the Coulomb branch has three allowed choices:

$$\langle\widehat{\Psi}\rangle = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \tag{6.133}$$

and one can deduce the magnetic characteristic polynomial following the same procedure as before:

$$\hat{u}_1^3 - \hat{u}_1 = 0, \quad \hat{u}_2 = \frac{1}{2} \tag{6.134}$$

with $\hat{u}_i = \mathrm{Tr}\,\widehat{\Psi}^i$. Applying the operator mapping derived in [202] we have

$$\mathrm{Tr}\,\widehat{\Psi} = -\mathrm{Tr}\,\widehat{\Phi} = -\mathrm{Tr}\,\Phi + 2, \tag{6.135}$$

so we need to add to the electric theory one more constraint, which is

$$\begin{aligned} 0 &= (-u_1 + 2)^3 - (-u_1 + 2) \\ &= -u_1^3 + 6u_1^2 - 11u_1 + 6, \end{aligned} \tag{6.136}$$

the solution of which is restricted to $u_1 = 1, 2, 3$, exactly as that computed directly from chiral rings of electric theory.

Chapter 7

EPILOGUE

In this dissertation we have discussed topics regarding physical and mathematical aspects of supersymmetric quantum field theory in various dimensions. In particular, we calculated generating functions of BPS spectrum and showed that they are in fact equivalent to various geometric invariants. For three-dimensional $\mathcal{N} = 2$ theories, the 3d-3d correspondence allows one to reproduce $G_{\mathbb{C}}$ Chern-Simons partition functions; for four dimensional $\mathcal{N} = 2$ theories, the index/TQFT correspondence realizes geometric quantization of Hitchin moduli space. Geometric setup of M5 brane compactification connects 3d and 4d partition functions, and nicely illustrates how partition functions for distinct theories and invariants are closely related.

On the other hand, M5 brane configuration makes it possible to understand physical theories and their dynamics directly from geometry. We classified the Argyres-Douglas theory of D_N and $E_{6,7,8}$ type based on classification of irregular punctures in the Hitchin system. We also developed a systematic way of counting graded dimension. Generalizing the construction in [132], we obtained duality frames for these AD theories, and found a novel duality between quivers with SO/E_N gauge groups and quivers with SU gauge groups.

There are many further questions that are potentially interesting based on the results in this dissertation. The first question is whether we can understand more general Hitchin moduli space (for instance, without $U(1)$ Hitchin action in wild ramification case) from physics. An arbitrary Riemann surface with arbitrary irregular punctures usually engineers asymptotic free theories; one may ask if properties of the theory on the Coulomb branch can be related to wild Hitchin moduli space; this may help understand how to geometrically quantize $\mathcal{M}_H$ when there is no Hamiltonian $U(1)$ action. A more involved question is to understand why fixed points on $\mathcal{M}_H$ are mapped to representations of VOAs: Are there string theory interpretations? what do they imply on the mathematical side?

In Chapter 4 we saw that the Coulomb index at $\mathfrak{t} \to e^{2\pi i}$ produces modular transformations. It is further observed in [251] that the modular matrices admit one parameter deformation that gives the full Coulomb branch index. Recall that the character of a given highest weight representation of VOA is calculated by Schur index, and the Macdonald index is the one parameter deformation of it as well. One then wonders what are the precise relation between Macdonald index and the Coulomb branch index. Are modular transformations of the former given by the latter?

As for the Argyres-Douglas theories themselves, there are also further open questions. For instance, it will be nice to provide further interpretation of the auxiliary Riemann sphere. In other words, can one engineer weakly coupled quiver theories in string theory, and the duality is interpreted as operations on the geometry side? On the other hand, calculation of superconformal index for D_N and $E_{6,7,8}$ type theories would be interesting as well, as it probes more exotic type of Hitchin moduli space and four manifold invariants [252], as well as characters of VOAs.

The reason and mathematical rigor behind all the above connections between physics and geometry are hitherto unknown. This is largely due to the fact that no satisfying definition of quantum field theory exists, or more specifically, no rigorous formulation of path integrals. When there is supersymmetry, the tool of localization reduces the infinite dimensional integral to a finite one, and this is where mathematicians begin their work. Nonetheless, an establishment of the framework would perhaps reveal insights even more profound than one could naively expect. In this framework, (supersymmetric) quantum field theory serves as a bridge spanning across distinct mathematical branches, and becomes itself as a novel object to study. Moreover, such framework would inevitably benefit the physics side of quantum field theory, where our daily observations of elementary particles and universe are relied upon.

We also studied the vacuum structure of certain $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions. As the number of supercharges is lowered, we saw that the observables are less protected by supersymmetry, and the calculation becomes harder and harder while the quantum corrections become more and more complicated. For general $\mathcal{N} = 1$ theories, both Coulomb branch and Higgs branch receive corrections. However, many exact calculations are still available, and there is a long history in exploring electric-magnetic dualities. One could ask whether a generalization of the calculation in Chapter 6 is possible to study general adjoint SQCD of ADE type.

For non-supersymmetric theories, there are only limited techniques such as perturbative expansion and anomalies. Progress has been made on using anomalies to understand the non-perturbative phase diagrams [253–257], as well as dualities [258–261]. Note that the phases and dualities put forward in the literature are still conjectural in nature, and direct analytical confirmation is lacking. However, many of those ideas are either inspired, or can be verified by the techniques in supersymmetry, for example see [262, 263].

Although supersymmetry is not realized in Nature, from my own perspective, I tend to view it not only as a potential mathematical framework, powerful but not yet full-fledged, but as a playground for developing formal methodologies and deepening structural understandings of quantum field theories in general. It is a symphony that shall never end.

Appendix A

COMPLEX CHERN-SIMONS THEORY ON LENS SPACES

Lens space $L(p,q)$ can be obtained by gluing two solid tori $S^1 \times D^2$ along their boundary T^2's using an element in $\mathrm{MCG}(T^2) = SL(2,\mathbb{Z})$:

$$\begin{pmatrix} -q & * \\ p & * \end{pmatrix} \begin{pmatrix} m \\ l \end{pmatrix} = \begin{pmatrix} m' \\ l' \end{pmatrix}. \tag{A.1}$$

Here (m,l) and (m',l') are meridian and longitude circles of the two copies of $T^2 = \partial(S^1 \times D^2)$. So the meridian m' of one torus is mapped to $-qm + pl$ of the other torus. As for l, we do not need to track what it is mapped into as the choice only affects the framing of $L(p,q)$. A canonical choice of an $SL(2,\mathbb{Z})$ element in (A.1) is given by

$$ST^{c_1}ST^{c_2}S\ldots T^{c_n}S, \tag{A.2}$$

where $(c_1, c_2, \ldots, c_n)$ are coefficients in continued fraction expansion of p/q. For $q = 1$, the element that gives $L(p,1)$ is

$$ST^pS. \tag{A.3}$$

As $SL(2,\mathbb{Z})$ naturally acts on the Hilbert space $\mathcal{H}^{\mathrm{CS}}(T^2;G)$ of the Chern-Simons theory on the two-torus, one has

$$Z_{\mathrm{CS}}(L(p,q);G) = \langle 0|\mathcal{S}\mathcal{T}^{c_1}\mathcal{S}\mathcal{T}^{c_2}\mathcal{S}\ldots\mathcal{T}^{c_n}\mathcal{S}|0\rangle. \tag{A.4}$$

Here $|0\rangle \in \mathcal{H}$ is the state associated to the solid torus while $\mathcal{S}$ and $\mathcal{T}$ give the action of $S,T \in SL(2,\mathbb{Z})$ on $\mathcal{H}$. When G is compact, $\mathcal{S}$ and $\mathcal{T}$ are known from the study of the 2D WZW model and affine Lie algebra [264] and can be directly used to evaluate (A.4). Partition functions of Chern-Simons theory on lens spaces were first obtained precisely in this manner in [265] for $SU(2)$ and in [266, 267] for higher rank gauge groups. Define $\hat{k} = k + \check{h}$, and then the partition function of the

G Chern-Simons theory on $L(p,q)$ is given by

$$\begin{aligned} Z(L(p,q),\hat{k}) =& \frac{1}{(\hat{k}|p|)^{N/2}} \exp\left(\frac{i\pi}{\hat{k}} s(q,p)|\rho|^2\right) \\ &\times \sum_{w\in W} \det(w) \exp\left(-\frac{2\pi i}{p\hat{k}}\langle \rho, w(\rho)\rangle\right) \\ &\times \sum_{m\in Y^\vee/pY^\vee} \exp\left(i\pi \frac{q}{p}\hat{k}|m|^2\right) \exp\left(2\pi i \frac{1}{p}\langle m, q\rho - w(\rho)\rangle\right). \end{aligned} \tag{A.5}$$

Here $s(q,p)$ is the Dedekind sum:

$$s(q,p) = \frac{1}{4p}\sum_{n=1}^{p-1} \cot\left(\frac{\pi n}{p}\right)\cot\left(\frac{\pi q n}{p}\right), \tag{A.6}$$

ρ the Weyl vector of the Lie algebra $\mathfrak{g}$, W the Weyl group, $Y^\vee$ the coroot lattice, N the rank of the gauge group, and the inner product, $\langle\cdot,\cdot\rangle$, is taken with respect to the standard Killing form of $\mathfrak{g}$.

Now we start computing the partition function of complex Chern-Simons theory using (2.50) for $G_{\mathbb{C}} = GL(N,\mathbb{C})$. The first step is to separate (A.5) into contributions from different flat connections. As discussed in section 2.3, the moduli space $\mathcal{M}_{\text{flat}}$ of $U(N)$ flat connections of $L(p,q)$ — whose foundamental group is $\mathbb{Z}_p$ — consists of discrete points. Each point can be labelled by $(a_1, a_2, \ldots, a_N)$, where the a_j's are the p-th roots of unity. For convenience we use a different set of labels, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathfrak{g}^*$, with the α_j's being integers between 0 and $p-1$ that satisfy

$$e^{2\pi i \alpha_j/p} = a_j. \tag{A.7}$$

Then (A.5) can be rewritten as [268]:

$$\begin{aligned} Z(L(p,q),\hat{k}) &= \frac{1}{N!}\sum_{\alpha} Z_\alpha(L(p,q),\hat{k}), \\ Z_\alpha(L(p,q),\hat{k}) &= \frac{1}{(\hat{k}|p|)^{l/2}} \exp\left(\frac{i\pi}{\hat{k}} N(N^2-1)s(q,p)\right) \exp\left(i\pi\frac{q}{p}\hat{k}|\alpha|^2\right) \\ &\sum_{w,\tilde{w}\in S_N} \det(w)\exp\left(-\frac{2\pi i}{p\hat{k}}\langle\rho, w(\rho)\rangle\right)\exp\left(2\pi i\frac{1}{p}\langle \tilde{w}(\alpha), q\rho - w(\rho)\rangle\right). \end{aligned} \tag{A.8}$$

The set $\{\alpha\}$ is redundant for labelling flat connections in $\mathcal{M}_{\text{flat}}$ because the Weyl group $\mathcal{W} = S_N \subset U(N)$ acts on $\{\alpha\}$ by permuting the α_j's. We will use $\tilde{\alpha}$ to denote equivalence classes of α under Weyl group action and each $\tilde{\alpha}$ corresponds to one flat connection modulo gauge transformations. A canonical representative of $\widetilde{\alpha}$ is given by $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N$. Using $\tilde{\alpha}$, (A.5) can be written as

$$Z(L(p,q),\hat{k}) = \sum_{\tilde{\alpha}} \frac{1}{|\mathcal{W}_{\tilde{\alpha}}|} Z_{\tilde{\alpha}}(L(p,q),\hat{k}), \tag{A.9}$$

where $\mathcal{W}_{\tilde{\alpha}} \subset \mathcal{W}$ is the stabilizer subgroup of $\tilde{\alpha} \in \mathfrak{g}^*$.

Using the naive way (2.49) of computing the partition function of complex Chern-Simons theory when $\mathcal{M}_{\text{flat}}$ is zero-dimensional, one has

$$Z(G_{\mathbb{C}};\tau,\overline{\tau}) = \frac{1}{N!} \sum_{\alpha} Z_\alpha\left(G; \frac{\tau}{2} - \check{h}\right) Z_\alpha\left(G; \frac{\overline{\tau}}{2} - \check{h}\right). \tag{A.10}$$

Notice that using $\tilde{\alpha}$ labels, this is

$$Z(G_{\mathbb{C}};\tau,\overline{\tau}) = \sum_{\widetilde{\alpha}} \frac{1}{|\mathcal{W}_{\widetilde{\alpha}}|} Z_{\widetilde{\alpha}}\left(G; \frac{\tau}{2} - \check{h}\right) Z_{\widetilde{\alpha}}\left(G; \frac{\overline{\tau}}{2} - \check{h}\right), \tag{A.11}$$

and the $\frac{1}{|\mathcal{W}_{\tilde{\alpha}}|}$ factor should not be squared. This is because $G_{\mathbb{C}}$ and G have the same Weyl group $\mathcal{W}$ and in complex Chern-Simons theory $\mathcal{W}$ acts simultaneously on $\mathcal{A}$ and $\overline{\mathcal{A}}$.

(A.11), together with (A.8), is the equation we use to compute the partition function of the complex Chern-Simons theory. In the making of the table 2.1, we have dropped a universal factor

$$\left(\frac{4}{\tau\overline{\tau}}\right)^{N/2} \propto (\ln q)^N. \tag{A.12}$$

This matches the factor that is also omitted on the supersymmetric index side.

Appendix B

ANALYTIC FORMULA OF $\widehat{SU}(3)_K$ FUSION COEFFICIENTS

The notation of this section is from [84]. Specifically, we define the following quantities:

$$\begin{aligned} k_0^{min} &= \max(\lambda_1+\lambda_2, \mu_1+\mu_2, \nu_1+\nu_2, a-\min(\lambda_1,\mu_1,\nu_1), b-\min(\lambda_2,\mu_2,\nu_2)), \\ k_0^{max} &= \min(a,b), \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} a &= \frac{1}{3}(2(\lambda_1+\mu_1+\nu_1)+\lambda_2+\mu_2+\nu_2), \\ b &= \frac{1}{3}(\lambda_1+\mu_1+\nu_1+2(\lambda_2+\mu_2+\nu_2)). \end{aligned} \tag{B.2}$$

Moreover we introduce

$$\delta = \begin{cases} 1 & \text{if } k_0^{max} \geq k_0^{min} \text{ and } a,b > 0, a,b \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{B.3}$$

With these definition we can compactly write our ordinary $su(3)$ representation ring and its fusion coefficient as

$$N_{\lambda\mu\nu} = (k_0^{max} - k_0^{min} + 1)\delta, \tag{B.4}$$

and we also define a list of $N_{\lambda\mu\nu}$ integers:

$$k_0^i = \{k_0^{min}, k_0^{min}+1, \ldots, k_0^{max}\}. \tag{B.5}$$

Then the $\widehat{su}(3)_k$ fusion coefficients can be written as

$$f_{\lambda\mu\nu}(\mathrm{t}=0) \equiv N_{\lambda\mu\nu}^{(k)} = \begin{cases} \max(i) \quad \text{such that } k > k_0^i \text{ and } N_{\lambda\mu\nu} \neq 0, \\ 0 \qquad \text{if } N_{\lambda\mu\nu} = 0 \text{ or } k < k_0^1. \end{cases} \tag{B.6}$$

Appendix C

PROPERTIES OF THE COULOMB BRANCH INDEX

C.1 TQFT structure

As the $\mathcal{N} = 2$ superconformal index of the class $\mathcal{S}$ theories $T[\Sigma_{g,s}; G]$ does not depend on complex moduli of Σ, it has a TQFT structure [58]. This further implies that the index can be computed by cutting and gluing the Riemann surface. As all Riemann surfaces can be reduced to cylinders and pairs of pants, one should be able to recast the superconformal index into the form

$$\mathcal{I}(T[\Sigma_{g,s}; G]; \mathbf{a}_1, \ldots, \mathbf{a}_s) = \sum_{\alpha} (C_{\alpha\alpha\alpha})^{2g-2+s} \prod_{i=1}^{s} \psi^{\alpha}(\mathbf{a}_i) \tag{C.1}$$

by choosing a basis in the TQFT Hilbert space to make the "fusion coefficients" $C_{\alpha\beta\gamma}$ associated with a pair of pants diagonal, and the "metric" $\eta_{\alpha\beta}$ associated with a cylinder proportional to the identity matrix $\delta_{\alpha\beta}$. Here $C_{\alpha\alpha\alpha}$ is also known as the "structure constant," $\psi^{\alpha}(\mathbf{a}_i)$ is called the "wave function" with flavor fugacity $\mathbf{a}_i$ at the puncture.[1]

Now let us specialize to the Coulomb branch index for class $\mathcal{S}$ theories on $S^1 \times L(k, 1)$ and recall the TQFT structure studied in [16]. Unlike the usual lens space index where the holonomies take integral values, in [16] the authors defined the "full index" by summing over 't Hooft fluxes, allowing fractional holonomies as long as charge quantization condition is satisfied. In the case of theories of type $\mathfrak{g} = \mathfrak{su}(2)$, this means that the holonomy m_i at each puncture takes value in $\{0, 1/2, 1, \ldots, k/2\}$. These holonomies form the Hilbert space of the TQFT, and are essentially the set of integrable representations of $\widehat{\mathfrak{su}}(2)_k$. After appropriate normalization of the states, (C.1) has the following form [15, 16]:

$$\mathcal{I}(T[\Sigma_{g,s}; \widehat{su}(2)]; m_1, \ldots, m_s) = \sum_{l=0}^{k} C_l^{2g-2+s} \prod_{i=1}^{s} \psi^l(m_i), \tag{C.2}$$

where

$$C_l = \frac{L_l^{-1}}{\sqrt{1-\mathfrak{t}} \sin\theta_l |1 - \mathfrak{t}\, e^{2i\theta_l}|^2} \tag{C.3}$$

[1] The diagonalizability of the TQFT structure constant is not a guaranteed property when the TQFT Hilbert space is infinite-dimensional (*e.g.*, for Schur limit of lens space index, it seems that one could not simultaneously diagonalize flavor fugacity variable and flavor holonomy variable [59]). But the cutting and gluing rules still apply.

and

$$\psi^l(m) = \sqrt{1-\mathrm{t}}\; L_l \times \begin{cases} (1+\mathrm{t})\sin\theta_l, \quad m=0, \\ \sin 2\theta_l, \quad m = 1/2, \\ \sin 3\theta_l - \mathrm{t}\sin\theta_l, \quad m=1, \\ \sin 4\theta_l - \mathrm{t}\sin 2\theta_l, \quad m = 3/2, \\ \vdots \\ \sin k\theta_l - \mathrm{t}\sin(k-2)\theta_l, \quad m = (k-1)/2, \\ \sin(k+1)\theta_l - \mathrm{t}\sin(k-1)\theta_l, \quad m = k/2. \end{cases} \tag{C.4}$$

Here the normalization constant is

$$L_l^{-2} = \frac{k+2}{2}|1-\mathrm{t}\,e^{2i\theta_l}|^2 + 2\mathrm{t}\cos 2\theta_l - 2\mathrm{t}^2 \tag{C.5}$$

and those θ_l's are the $k+1$ solutions in $(0,\pi)$ to the Bethe ansatz equation,

$$e^{2ik\theta}\left(\frac{e^{i\theta} - \mathrm{t}\,e^{-i\theta}}{\mathrm{t}\,e^{i\theta} - e^{-i\theta}}\right)^2 = 1. \tag{C.6}$$

Moreover the metric in this basis is given by $\eta^{\lambda\lambda} = (1-\mathrm{t}^2, 1-\mathrm{t}, \ldots, 1-\mathrm{t}, 1-\mathrm{t}^2)$.

What happens when irregular punctures are present? It may not even make sense to talk about TQFT structure, because for a Riemann surface $\Sigma_{g,\ell,\{n_\alpha\}}$ with arbitrary genus g plus ℓ regular punctures and an arbitrary number of irregular ones labeled by $\{n_\alpha\}$, the $U(1)_r$ symmetry is broken and the resulting theory is generically asymptotically free [103, 172] instead of superconformal. For instance, consider gauging the diagonal $SU(2)$ group of (A_1, D_K) and (A_1, D_M) theory by an $SU(2)$ vector multiplet. Each side has a flavor central charge $k_{SU(2)} = 4(K-1)/K$ and $k'_{SU(2)} = 4(M-1)/M$; the gauging would contribute to the one-loop running of gauge coupling as

$$b_0 = 2\left(\frac{1}{K} + \frac{1}{M}\right) > 0. \tag{C.7}$$

If one tries to extend the superconformal index of Argyres-Douglas theory to an arbitrary Riemann surface $\Sigma_{g,\ell,\{n_\alpha\}}$ by cutting and gluing, the interpretation of the "index" obtained at the end it is not obvious. In the case of the Schur index and the Macdonald index, it turns out that the cutting-and-gluing procedure computes the index of the UV fixed point, consisting of free multiplets with canonical choice of scaling dimensions [269].

Let us now examine the Coulomb branch limit. In order to define a viable TQFT structure as (C.1), a necessary condition is that one has to be able to consistently close the regular puncture. This means we should be able to reduce (A_1, D_{K+1}) to (A_1, A_{K-2}) theory since the Riemann sphere associated with the two theories differ only by an extra regular puncture. On the field theory side, one observes the Coulomb branch scaling dimensions of (A_1, D_{K+1}) and (A_1, A_{K-2}) theories are very similar, giving further evidence that these two theories are related.

In the language of TQFT, there is a natural "cap state" that tells us how to close a regular puncture. Let us begin with (A_1, D_{2N+1}) and (A_1, A_{2N-2}) theories. Recall that the lens space index (4.77) of (A_1, D_{2N+1}) contains a normalization factor (4.78) which can be absorbed in the redefinition of the states (labeled by the holonomy n) inserted in the regular puncture. Then it is not hard to check that if we define

$$\langle\phi'| = \langle 0'| - \mathfrak{t}^{\frac{2N}{2N+1}}\langle 1'| \tag{C.8}$$

then this is precisely the cap that reduces the index of (A_1, D_{2N+1}) theories into (A_1, A_{2N-2}) theories. Recall that in the equivariant Verlinde TQFT, the cap state is decomposed as

$$\langle\phi| = \langle 0| - \mathfrak{t}\langle 1|. \tag{C.9}$$

The only difference is the $\mathfrak{t}$ here is replaced with $\mathfrak{t}^{\frac{2N}{2N+1}}$ in (C.8). This is due to the fact that, in the presence of an irregular singularity, the $U(1)$ Hitchin action will also rotate the Σ, and the neighborhood of south pole (at $z = 0$) is also rotated,

$$\rho_\theta : \quad z \mapsto e^{-i\frac{2}{2N+1}\theta} z. \tag{C.10}$$

So the state $\langle\phi'|$ is no longer associated with the ordinary cap, but with the "rotating cap", and similarly for $\langle 0'|$ and $\langle 1'|$.

From the cap states (C.8), it is not hard to argue that the structure constants and wavefunctions associated with regular puncture cannot remain simultaneously the same as those in (C.3) and (C.4). This is simply because the cap state is given by $\sum_l C_l^{-1}\eta_{nn}\psi^l(n)$ which should depend on N.

Let us now turn to the (A_1, D_{2N+2}) and (A_1, A_{2N-1}) case. Unlike the previous situation, the latter theory contains an additional $U(1)$ flavor symmetry so that the existence of the cap state $\langle\phi'|$ is more non-trivial. Similarly, there is a normalization constant for each theory that needs to be absorbed. For the (A_1, A_{2N-1}) theory, the normalization constant is (4.76) which shall be absorbed in the definition of irregular puncture wavefunction $\hat{\psi}^l_{2N}$; while for (A_1, D_{2N}) theory, the quantity is (4.81). Note that there is "entanglement" between the two factors of the $U(1) \times SU(2)$ flavor symmetry, and one cannot split it into a product of two functions that depend on n_1 and n_2 separately.

In order to go from (A_1, D_{2N+2}) to (A_1, A_{2N-1}), we should properly identify the residual $U(1)$ symmetry and which combination of n_1 and n_2 is enhanced to $SU(2)$ in the IR. In fact, [111] shows that the mixing to $SU(2)$ is given by $(1/2N+2)U(1)_b$. Therefore, we identify $(N+1)n_2$ as the $SU(2)$ holonomy, while the residual symmetry is identified as

$$n \sim \frac{N+1}{N} n_1. \tag{C.11}$$

Then it is a straightforward computation to see that the cap state for the regular puncture of (A_1, D_{2N+2}) can be defined as

$$\langle \phi' | = \langle 0' | - \left\langle \left(\frac{1}{N+1} \right)' \right| \times \begin{cases} \mathrm{t}, & \text{for } n_1 = 0 \\ \mathrm{t}^{\frac{N}{N+1}}, & \text{for } n_1 > 0 \end{cases} \tag{C.12}$$

Here, the value inside the bra is for n_2. Note the following peculiar behavior: when n_1 (the holonomy for $U(1)$ symmetry carried by the irregular puncture) is zero, then the cap state becomes the ordinary one in the tame case [15, 16], while for non-zero n_1 the irregular puncture starts to affect in a non-local way the regular puncture on the other side. Similar to the previous case, one can argue that the structure constants and the wave function for the regular puncture cannot be made identical to the tame case (C.3) and (C.4) simultaneously.

We do not yet know what this quantity computes for arbitrary $\Sigma_{g,\ell,\{n_\alpha\}}$ wild quiver gauge theories via cutting and gluing. What we have found above is a consistent way to define the TQFT structure (C.1) solely for Argyres-Douglas theories. A clear picture may be achieved once the irregular states in TQFT are better understood, as was studied in CFT [270–272].

C.2 Symmetry mixing on the Coulomb branch

In Section 4.4, we mentioned that (4.122) and (4.126) can be interpreted as the mixing between $U(1)_r$ symmetry and topological symmetry on the Coulomb branch. We now explain why this is so. We focus on the $T_{3\mathrm{d}}[\Sigma]$ side instead of its mirror

$T_{\rm 3d}^{\rm mir.}[\Sigma]$, and the fugacities assigned on the Higgs branch of $T_{\rm 3d}^{\rm mir.}[\Sigma]$ become those for the topological symmetry on the Coulomb branch of $T_{\rm 3d}[\Sigma]$. The trace formula (4.119) in the Coulomb limit becomes

$$\mathcal{I}_C^{\rm 3d} = {\rm Tr}_{\mathcal{H}_C} {\rm t}^{R_C - R_H} \mathbf{z}^{\mathbf{f}_J} \tag{C.13}$$

with the BPS Hilbert space $\mathcal{H}_C$ containing those states satisfying $\widetilde{E} = R_C$ and $R_H = -j_2$. Here $\mathbf{f}_J$ is the charge under topological symmetry. To further simplify (C.13), we claim $R_H = 0$. To see this, let us go back to 4d $\mathcal{N} = 2$ index and ask what type of short multiplets are counted by Coulomb branch limit. In general, two types will enter [58]: they are of type $\overline{\mathcal{E}}_{r,(j_1,0)}$ and $\overline{\mathcal{D}}_{0,(j_1,0)}$. It was shown in [273] that for Argyres-Douglas theories considered in this chapter, no short multiplet of above two types with $j_1 > 0$ occur. Since $\overline{\mathcal{D}}_{0,(0,0)}$ is a subclass of $\overline{\mathcal{E}}_{r,(0,0)}$ it suffices to say that the Coulomb branch index only counts the $\overline{\mathcal{E}}_{r,(0,0)}$ multiplet for Argyres-Douglas theories. After dimensional reduction, it becomes clear that $R_H = 0$ in (C.13) since $\overline{\mathcal{E}}_{r,(0,0)}$ carries the trivial representation of $SU(2)_R$.

Therefore, the substitution we have made in (4.122) and (4.126) only mixes topological symmetry with $SU(2)_C$ symmetry. Under mirror symmetry, $SU(2)_C$ and $SU(2)_H$ are exchanged, and the topological symmetry becomes the flavor symmetry in the mirror frame. To see explicitly the operator mapping, consider (A_1, A_{2N-1}) theories with a rank-$(N-1)$ Coulomb branch, for which the mixing is given by (4.122) and (4.123). After comparing with (4.106), we see that the 4d $\mathcal{N} = 2$ Coulomb branch operators come from the ${\rm t}' z_j/z_i$ term with $i = N$ and $j = 1, 2, \ldots, N-1$. They are precisely the Higgs branch operators $X^j Y_1$, where (X^i, Y_i) are two $\mathcal{N} = 2$ chiral fields in the i-th hypermultiplet.[2]

We now turn to the (A_1, D_{2N}) Argyres-Douglas theory, whose three-dimensional mirror is given in Figure 4.3 [18]. The Higgs branch index is given by (4.125) and the substitution made there is (4.126). Note that we set the $U(1)$ fugacity to be 1, implying that this symmetry does not mix with the R-symmetry. In particular, when $N = 2$, the non-abelian part of the topological symmetry is trivial, so we have no mixing at all! This is actually quite reasonable, because the $U(1)_r$ charge (1/2) of the Coulomb branch operator of (A_1, D_4) theory automatically satisfies the $SU(2)_C$ quantization condition.

[2]The results here differ slightly from that of [112] due to a different choice of matrix representations of Cartan element. The two conventions can be mapped to each other. We thank Matthew Buican for discussion and clarification.

For general (A_1, D_{2N}) theories with $N > 2$ the Coulomb branch operators no longer have half-integral scaling dimensions, so the symmetry mixing (4.126) should be non-trivial. It is not hard to single out the term in the denominator of (4.125) that gives rise to those Coulomb branch operators.

Unfortunately, it is not known in the current literature what is the three-dimensional mirror of (A_1, A_{2N}) and (A_1, D_{2N+1}) Argyres-Douglas theories. The absence of Higgs branch in the (A_1, A_{2N}) theories indicates that their 3d mirror cannot be given by quiver theory. The computation of Coulomb branch index and $k \to +\infty$ limit shows that the $T_{3d}[\Sigma]$ must have topological symmetry.

Appendix D

MASSIVE VACUA OF THREE-DIMENSIONAL QUIVER THEORY

In this appendix we give explicit steps in solving the massive vacua for certain three-dimensional $\mathcal{N} = 4$ quiver gauge theories. These are the mirrors of three-dimensional reduction of Argyres-Douglas theories. As mentioned in Section 4.4, the problem of finding the $U(1)$ fixed points is equivalent to the problem of finding the massive vacua with masses turned on according to the embedding $U(1) \subset G_{\text{R-sym}} \times G_{\text{flavor}}$. More precisely, this embedding will specify a one-dimensional subspace of the Lie algebra of $\mathfrak{g}_{\text{R-sym}} \oplus \mathfrak{g}_{\text{flavor}}$ and its dual, where mass parameters lives.[1] However, as the number of massive vacua are the same for a generic embedding and $U(1)_{\text{Hitchin}}$ is generic (in the sense that fixed points are isolated), we will work with a generic choice of mass parameters to simplify the notation, which will still lead to the right number of vacua.

D.1 (A_1, A_{2N-1}) Argyres-Douglas theory

The three dimensional mirror is $\mathcal{N} = 4$ SQED with N flavors of hypermultiplets. Let us denote (X_i, Y_i) where $i = 1, 2, \ldots, N$ as the chiral component for the N hypermultiplets, and Φ (σ) as the complex (real) scalar in the $U(1)$ vector multiplet. We turn on complex masses $m^i_{\mathbb{C}}$ and real FI parameter $t_{\mathbb{R}} < 0$, and denote the induced action $(\mathbb{C}^*)_m$. The BPS equations are

$$\begin{aligned} &X \cdot Y = 0, \qquad |X|^2 - |Y|^2 + t_{\mathbb{R}} = 0, \\ &(\Phi + m_{\mathbb{C}}) \cdot X = 0, \quad \sigma \cdot X = 0, \\ &(\Phi + m_{\mathbb{C}}) \cdot Y = 0, \quad \sigma \cdot Y = 0. \end{aligned} \tag{D.1}$$

The solution is easy to describe, given by

$$\sigma = 0, \quad \Phi = -m^i_{\mathbb{C}}, \quad Y = 0, \quad X = (0, \ldots, 0, \sqrt{-t_{\mathbb{R}}}, 0, \ldots, 0), \tag{D.2}$$

for $i = 1, 2, \ldots, N$. So there are N fixed points under $(\mathbb{C}^*)_m$ action.

[1] Turning on mass parameters associated with R-symmetry will in general break supersymmetry. For us, it will break 3d $\mathcal{N} = 4$ to 3d $\mathcal{N} = 2$.

D.2 (A_1, D_{2N}) Argyres-Douglas theory

The three dimensional mirror is a $U(1) \times U(1)$ quiver gauge theory with $N-1$ hypermultiplets (X_i, Y_i) stretching between two gauge nodes, one single hypermultiplet (A_1, B_1) only charged under the first $U(1)$, and another single hypermultiplet only charged under the second $U(1)$. The superpotential of the theory is

$$W = \sum_{i=1}^{N-1} (\Phi_1 - \Phi_2 + m_{\mathbb{C}}^i) X_i Y_i + (\Phi_1 + M_1) A_1 B_1 + (\Phi_2 + M_2) A_2 B_2, \tag{D.3}$$

where $m_{\mathbb{C}}^i, M_{1,2}$ are the complex masses. We have the following constraints on the space of allowed vacua:

$$\begin{aligned} &(\Phi_1 - \Phi_2 + m_{\mathbb{C}}^i) X_i = 0, \quad (\Phi_1 - \Phi_2 + m_{\mathbb{C}}^i) Y_i = 0, \\ &(\Phi_1 + M_1) A_1 = 0, \quad (\Phi_1 + M_1) B_1 = 0, \\ &(\Phi_2 + M_2) A_2 = 0, \quad (\Phi_2 + M_2) B_2 = 0, \\ &\sum_{i=1}^{N-1} X_i Y_i + A_1 B_1 = 0, \quad -\sum_{i=1}^{N-1} X_i Y_i + A_2 B_2 = 0, \end{aligned} \tag{D.4}$$

where $\Phi_{1,2}$ are the complex scalar in the gauge group. Since we have set the real mass to be zero, the vevs of real scalars $\sigma_{1,2}$ in the vector multiplet will automatically be zero. We also must impose the D-term equation,

$$\begin{aligned} &\sum_{i=1}^{N-1} (|X_i|^2 - |Y_i|^2) + |A_1|^2 - |B_1|^2 = t_{\mathbb{R}}^1, \\ &\sum_{i=1}^{N-1} (|X_i|^2 - |Y_i|^2) + |A_2|^2 - |B_2|^2 = t_{\mathbb{R}}^2. \end{aligned} \tag{D.5}$$

For simplicity and without loss of generality, we will assume that the real FI parameters $t_{\mathbb{R}}^{1,2} > 0$. Let us try to solve the above equations.

(a) Suppose $\Phi_1 - \Phi_2 + m_{\mathbb{C}}^i \neq 0$ for all i.

This means that $X_i = Y_i = 0$ for all i. Then we get $A_1 B_1 = A_2 B_2 = 0$. But they cannot be simultaneously zero, otherwise the D-term condition would be violated. Therefore we see that only $B_1 = B_2 = 0$, and $|A_1| = \sqrt{t_{\mathbb{R}}^1}$, $|A_2| = \sqrt{t_{\mathbb{R}}^2}$. This fixes $\Phi_1 = -M_1$ and $\Phi_2 = -M_2$. This gives one solution.

(b) There exists one i such that $\Phi_1 - \Phi_2 + m^i_{\mathbb{C}} = 0$.

This implies that $X_j = Y_j = 0$ whenever $j \neq i$ since the $m^i_{\mathbb{C}}$'s are kept generic. Now if we assume neither $\Phi_1 + M_1$ and $\Phi_2 + M_2$ is zero, then we should have $A_1 = A_2 = B_1 = B_2 = 0$. Then we see that $|X_i|^2 - |Y_i|^2$ equals both to $t^1_{\mathbb{R}}$ and $t^2_{\mathbb{R}}$, which is impossible since the real FI parameters are also generic.

We conclude that $\Phi_1 = -M_1$ or $\Phi_2 = -M_2$ (they cannot simultaneously hold). If the former is true, then $A_2 = B_2 = 0$, and $X_i Y_i = A_1 B_1 = 0$. We then see that $Y_i = 0$ and $|X_i| = \sqrt{t^2_{\mathbb{R}}}$, and $|A_1|^2 - |B_1|^2 = t^1_{\mathbb{R}} - t^2_{\mathbb{R}}$. Depending on whether $t^1_{\mathbb{R}} > t^2_{\mathbb{R}}$ or $t^1_{\mathbb{R}} < t^2_{\mathbb{R}}$ we can solve for A_1 and B_1. In this way we get $N - 1$ solutions.

Similarly, if the latter is true, we also get $N - 1$ solutions. So in total, we have $2N - 1$ solutions, which is exactly what we want.

Appendix E

FIXED POINTS UNDER $U(1)$ HITCHIN ACTION

In this appendix we give the explicit form of fixed points by solving the Hitchin equations. We only consider moduli spaces $\mathcal{M}_{2,2N+1}$ and $\widetilde{\mathcal{M}}_{2,2N-1}$. In the case $\mathcal{M}_{2,2N+1}$, the fixed points and corresponding values of μ are described in [124]. We check in detail the weights on the normal bundle for each fixed point and argue that they agree precisely with physical interpretations. In the case $\widetilde{\mathcal{M}}_{2,2N-1}$, we generalize the methods in [124] to describe the fixed points, and then check the weights. Throughout this section, we adopt the convention specified around (4.9).

E.1 Fixed points on $\mathcal{M}_{2,2N+1}$

For given N, the $U(1)$ fixed points are labeled by an integer $\ell = 0, 1, \ldots, N$ up to gauge equivalence. In terms of the triple $(\bar{\partial}_E, h, \varphi)$, they are given by

$$\begin{aligned} &\bar{\partial}_E = \bar{\partial}, \\ &\varphi^*_\ell = \begin{pmatrix} 0 & z^{N-\ell} \\ z^{N+1+\ell} & 0 \end{pmatrix} dz, \\ &h = \begin{pmatrix} |z|^{\frac{1+2\ell}{2}} e^U & \\ & |z|^{-\frac{1+2\ell}{2}} e^{-U} \end{pmatrix}, \end{aligned} \tag{E.1}$$

where $U = U(|z|)$ is the unique solution of the ordinary differential equation [274]

$$\left(\frac{d^2}{d|z|^2} + \frac{1}{|z|} \frac{d}{d|z|} \right) U = 8|z|^{2N+1} \sinh(2U) \tag{E.2}$$

satisfying the following boundary conditions:

$$\begin{aligned} &U(|z|) \sim -\frac{1+2\ell}{2} \ln |z| + \ldots \quad |z| \to 0, \\ &U(|z|) \sim 0, \quad |z| \to \infty. \end{aligned} \tag{E.3}$$

The boundary condition at $|z| = 0$ guarantees that the Hermitian metric h is smooth there; therefore the Chern connection $D = \partial + \bar{\partial} + h^{-1}\partial h$ has trivial monodromy.

The gauge transformation g_θ which undoes the $U(1)$ action (4.30) on (E.1) is

$$g_\theta = \begin{pmatrix} e^{\frac{1+2\ell}{2(2N+3)}i\theta} & \\ & e^{-\frac{1+2\ell}{2(2N+3)}i\theta} \end{pmatrix}. \tag{E.4}$$

The moment map (4.32) can be interpreted as a regularized L^2-norm of the Higgs field. Consequently, at the $U(1)$ fixed point labeled by the integer ℓ, we have from (4.32):

$$\begin{aligned} \mu_\ell &= \frac{i}{\pi}\int |z|^{2N+1}(\cosh 2U - 1)dz \wedge d\bar{z} \\ &= \frac{(1+2\ell)^2}{8(2N+3)}. \end{aligned} \tag{E.5}$$

The $U(1)$ action also acts on the tangent space $T_{(\bar{\partial},\varphi,h)}\mathcal{M}_{2,2N+1}$ to each fixed point. Let $\dot{\varphi} \in \Omega^{(1,0)}(\mathbb{C}\mathbf{P}^1; \mathrm{End}E)$ be the variation of the Higgs field. We say that the $U(1)$ action acts on $\dot{\varphi}$ with weight ϖ if

$$e^{i\theta}\rho_\theta^*\dot{\varphi} = e^{i\varpi\theta}g_\theta^{-1}\dot{\varphi}\, g_\theta, \tag{E.6}$$

where g_θ is given in (E.4).

As in [12, 86], one can define the complex symplectic form on the tangent space $(\dot{A}, \dot{\varphi})$ as

$$\omega'((\dot{A}_1, \dot{\varphi}_1), (\dot{A}_2, \dot{\varphi}_2)) = \int \mathrm{Tr}\,(\dot{\varphi}_2 \wedge \Psi_1 - \dot{\varphi}_1 \wedge \Psi_2), \tag{E.7}$$

where Ψ is the image of the identification from $\Omega^1(\mathbb{C}\mathbf{P}^1, \mathrm{ad}(P))$ to $\Omega^{(0,1)}(\mathbb{C}\mathbf{P}^1, \mathrm{ad}(P)\otimes\mathbb{C})$. Then it is immediate that the complex symplectic form ω' has charge 1 under the circle action. The existence of such form implies that the weights are paired on the tangent space: if there is a weight ϖ on the tangent space, there is also a weight $1-\varpi$. This statement will be confirmed in examples shortly.

Our strategy in determining these weights relies heavily on permissible deformations of Higgs field and (E.6). By the word "permissible" we mean that, (i) its spectral curve must be that of (4.18) with $K = 2N+1$ with vanishing coupling constants; (ii) it does not originate from infinitesimal meromorphic gauge transformation $\dot{\varphi} = [\varphi, \varkappa]$ for $\varkappa \in \mathfrak{sl}(2,\mathbb{C})$, and (iii) it does not introduce extra singularities; (iv) it does not alter leading nilpotent coefficient matrix. The goal is then to enumerate these inequivalent permissible deformations. Moreover, it suffices to consider the deformation to the linear order and ignore all higher order terms.

Let us begin with the case $\mathcal{M}_{2,3}$, pick a small parameter υ and focus on the first fixed points

$$\varphi_1^* = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix} dz. \tag{E.8}$$

To preserve the spectral curve (4.18), there are two simple linear deformations one could write down:

$$\dot{\varphi}_1 = \begin{pmatrix} 0 & 0 \\ \upsilon & 0 \end{pmatrix} dz, \quad \begin{pmatrix} \upsilon & 0 \\ 0 & -\upsilon \end{pmatrix} dz. \tag{E.9}$$

However, the second deformation is a gauge artifact, while the first one is legitimate with the weight being $6/5$. We then conclude that the other paired weight must be $-1/5$. Indeed one could find the corresponding deformation as

$$\dot{\varphi}_1 = \begin{pmatrix} \upsilon z^2 & 0 \\ 0 & -\upsilon z^2 \end{pmatrix} dz + o(\upsilon). \tag{E.10}$$

The determinant of $\varphi_1^* + \dot{\varphi}_1$ equals to $-z^3 dz^2$ up to quadratic terms in υ, so such deformation stays on the nilpotent cone.

On the other hand, we have another fixed point

$$\varphi_0^* = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} dz. \tag{E.11}$$

We see that the diagonal deformation is allowed at this time, since gauge transformation with essential singularity is forbidden. This deformation has weight $3/5$, whose paired weight is $2/5$. The associated deformation for the latter weight is then

$$\dot{\varphi}_0 = \begin{pmatrix} 0 & -\upsilon \\ \upsilon z & 0 \end{pmatrix} dz + o(\upsilon). \tag{E.12}$$

Now we generalize the above procedure to wild Hitchin moduli space $\mathcal{M}_{2,2N+1}$ with $N > 1$. Let us consider the ℓ-th fixed point in (E.1). For $j = 0, \cdots, \ell - 1$, the following family of deformations come from infinitesimal deformations $\dot{\varphi}$ of the lower-left entry of the Higgs field:

$$\dot{\varphi}_\ell^{(j)} = \begin{pmatrix} 0 & 0 \\ \upsilon z^j & 0 \end{pmatrix} dz. \tag{E.13}$$

The associated determinant that enters spectral curve is

$$-\det(\varphi_\ell^{(j)}) = (z^{2N+1} + \upsilon z^{N-\ell+j})dz^2. \tag{E.14}$$

So (E.13) is a permissible deformation. The associated series of weights are

$$\varpi_j = \frac{2(N+\ell+1-j)}{2N+3} > 1, \qquad j = 0, \ldots, \ell-1. \tag{E.15}$$

The moment map is largest at the fixed point $\ell = N$. There are N such deformations, and this family of deformations at $\ell = N$ should be thought of as (the analogue of) the Hitchin section.

Because of the complex symplectic form ω' in (E.7), there are weights that are paired with those in (E.15):

$$\varpi_j = \frac{-1-2j}{2N+3} < 0, \qquad j = 0, \ldots, \ell-1, \tag{E.16}$$

where we have relabeled the indices. They are downward Morse flows, so must stay on the nilpotent cone. In other words, the corresponding family of deformations $\varphi_\ell^{(j)}$ preserves the spectral curve $-\det(\varphi_\ell^{(j)}) = z^{2N+1}dz^2$:

$$\dot{\varphi}_\ell^{(j)} = \begin{pmatrix} \upsilon z^{N+j+1} & 0 \\ 0 & -\upsilon z^{N+j+1} \end{pmatrix} dz + o(\upsilon). \tag{E.17}$$

This particular type of deformation, (E.17) also appears in [275].

The remaining $2(N-\ell)$ weights are between 0 and 1. Let us consider one family of deformations labeled by $j = 0, \ldots, N-\ell-1$, which is the diagonal deformation:

$$\dot{\varphi}_\ell^{(j)} = \begin{pmatrix} \upsilon z^j & 0 \\ 0 & -\upsilon z^j \end{pmatrix} dz, \tag{E.18}$$

and the determinant is $-\det(\varphi_\ell^{(j)}) = z^{2N+1}dz^2$, meaning such deformation stays on the nilpotent cone. The associated series of weights are

$$\varpi_j = \frac{2N+1-2j}{2N+3}, \qquad j = 0, \ldots, N-\ell-1. \tag{E.19}$$

The rest weights correspond to deformations $\dot{\varphi}$ which involve both the upper-right and lower-left entries. They can be written as

$$\dot{\varphi}_\ell^{(j)} = \begin{pmatrix} 0 & -\upsilon z^j \\ \upsilon z^{1+2\ell+j} & 0 \end{pmatrix} dz + o(\upsilon), \tag{E.20}$$

whose determinant can be verified to lie in the Hitchin base $\mathcal{B}$. The associated weights are

$$\varpi_j = \frac{2(N-\ell-j)}{2N+3}, \qquad j = 0, \ldots, N-\ell-1. \tag{E.21}$$

These weights, after a reordering of indices, pair with the weights in (E.19). In summary, we have the following weights for the ℓ-th fixed points on the tangent space:

$$\varpi_j = \frac{2(N+1+j)}{2N+3}, \quad j = 1, 2, \ldots, \ell, \tag{E.22a}$$

$$\varpi_j = -\frac{2j-1}{2N+3}, \qquad j = 1, 2, \ldots, \ell, \tag{E.22b}$$

$$\varpi_j = \frac{2j+1}{2N+3}, \qquad j = \ell+1, \ell+2, \ldots, N, \tag{E.22c}$$

$$\varpi_j = \frac{2(N-j+1)}{2N+3}, \quad j = \ell+1, \ell+2, \ldots, N. \tag{E.22d}$$

These weights are precisely matched with the wild Hitchin character for $\mathcal{M}_{2,2N+1}$ in Section 4.4.

E.2 Fixed points on $\widetilde{\mathcal{M}}_{2,2N-1}$

The fixed points on $\widetilde{\mathcal{M}}_{2,2N-1}$ are quite straightforward to obtain: one merely allows a regular singularity at $z = 0$, whose monodromy for gauge connection is denoted as α. Expressed in terms of a triple $(\bar{\partial}_E, h, \varphi)$ these fixed points are

$$\begin{aligned} \bar{\partial}_E &= \bar{\partial}, \\ \varphi &= \begin{pmatrix} 0 & z^{\ell} \\ z^{2N-1-\ell} & 0 \end{pmatrix} dz, \\ h &= \begin{pmatrix} |z|^{\frac{2N-1-2\ell}{2}} e^{U} & 0 \\ 0 & |z|^{-\frac{2N-1-2\ell}{2}} e^{-U} \end{pmatrix}, \end{aligned} \tag{E.23}$$

where the index ℓ is an integer such that $-1 < \ell + 2\alpha < 2N$ [274]. The function $U(|z|)$ is the unique solution of

$$\left(\frac{d^2}{d|z|^2} + \frac{1}{|z|}\frac{d}{d|z|}\right) U = 8|z|^{2N-1} \sinh(2U) \tag{E.24}$$

satisfying the following boundary conditions:

$$\begin{aligned} &U(|z|) \sim \left(-\frac{2N-1-2\ell}{2} + 2\alpha\right) \ln|z| + \ldots \quad |z| \to 0, \\ &U(|z|) \sim 0, \quad |z| \to \infty. \end{aligned} \tag{E.25}$$

The asymptotics of $U(|z|)$ guarantees that near $z \sim 0$, the harmonic metrics all satisfy

$$h \sim \begin{pmatrix} |z|^{2\alpha} & 0 \\ 0 & |z|^{-2\alpha} \end{pmatrix} \tag{E.26}$$

so that the gauge connection indeed has monodromy $A \sim \alpha d\theta$. Computing the regularized value of the moment map (4.32) at each of these $U(1)$ fixed points, we get

$$\mu'(\ell) = \frac{1}{2(2N+1)} \left(-\frac{2N-1-2\ell}{2} + 2\alpha\right)^2 . \tag{E.27}$$

In our case, $2\alpha \in (0,1)$, these $2N+1$ fixed points are unique up to gauge transformation and are labeled by $\ell = -1, \cdots, 2N-1$. As in previous case, to match the physical predication we usually need to subtract the lowest moment map value. The minimal value, μ'_{min} occurs at $\ell = N-1$:

$$\mu'_{\text{min}} = \frac{1}{2(2N+1)} \left(-\frac{1}{2} + 2\alpha\right)^2 . \tag{E.28}$$

Letting

$$\mu = \mu' - \mu'_{\text{min}}, \tag{E.29}$$

the values of μ are

$$\mu = \frac{i(i+1)}{2(2N+1)} - \frac{i}{2N+1}(2\alpha), \qquad i = N, N-1, \ldots, -N+1, -N, \tag{E.30}$$

where we have relabeled the indices by setting $i = N - \ell - 1$. Note that these are precisely the values of the moment map appearing in (4.98).

Now we turn to the weights on the normal bundle of these fixed points. Notice that we do not have to compute everything from scratch, because the fixed points in (E.23), except $\ell = -1$, are automatically fixed points for the moduli space $\mathcal{M}_{2,2N-1}$, *cf.* (E.1). However, we are missing two weights since

$$\dim_{\mathbb{C}} \widetilde{\mathcal{M}}_{2,2N-1} = \dim_{\mathbb{C}} \mathcal{M}_{2,2N-1} + 2. \tag{E.31}$$

These two additional weights are very easy to obtain, since the associated deformations of the Higgs fields involve z^{-1}. We then have:

$$\epsilon_N = \frac{2N-1}{2N+1}, \qquad \tilde{\epsilon}_N = \frac{2}{2N+1}. \tag{E.32}$$

The weights for $\ell = -1$ are new, but they are computed in a similar way and we omit the details.

Appendix F

TYPE IIB CONSTRUCTION FOR AD THEORIES

Consider type IIB string theory on isolated hypersurface singularity in $\mathbb{C}^4$:

$$W(x_1, x_2, x_3, x_4) = 0, \qquad W(\lambda^{q_i} x_i) = \lambda W(x_i), \tag{F.1}$$

where the condition of isolation at $x_i = 0$ means $dW = 0$ if and only if $x_i = 0$. The quasi-homogeneity in above formula plus the constraint $\sum q_i > 1$ guarantees that the theory has $U(1)_r$ symmetry, $i.e$ it is superconformal.

The Coulomb branch of resulting four dimensional $\mathcal{N} = 2$ SCFT is encoded in the mini-versal deformation of the singularity:

$$F(x_i, \lambda_a) = W(x_i) + \sum_{a=1}^{\mu} \lambda_a \phi_a, \tag{F.2}$$

where $\{\phi_a\}$ are a monomial basis of the quotient algebra

$$\mathcal{A}_W = \mathbb{C}[x_1, x_2, x_3, x_4] \Big/ \Big/ \left\langle \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \frac{\partial W}{\partial x_3}, \frac{\partial W}{\partial x_4} \right\rangle . \tag{F.3}$$

The dimension μ of the algebra as a vector space is the *Minor number*, given by

$$\mu = \prod_{i=1}^{4} \left(\frac{1}{q^i} - 1 \right). \tag{F.4}$$

The mini-versal deformation can be identified with the SW curve of the theory.

BPS particles in the SCFT can be thought of as D3 brane wrapping special Lagrangian cycles in the deformed geometry. The integration of the holomorphic three form,

$$\Omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{dF} \tag{F.5}$$

on the three cycles gives the BPS mass of the theory. Thus, we require that Ω should have mass dimension 1. This determines the scaling dimension of the parameter λ_a:

$$[\lambda_a] = \alpha(1 - [\phi_a]), \tag{F.6}$$

where $\alpha = 1/(\sum q_i - 1)$.

The central charges of the theory is given by [190]:

$$a = \frac{R(A)}{4} + \frac{R(B)}{6} + \frac{5r}{24} + \frac{h}{24}, \quad c = \frac{R(B)}{3} + \frac{r}{6} + \frac{h}{12}. \tag{F.7}$$

Here $R(A)$ is given by summation of Coulomb branch spectrum:

$$R(A) = \sum_{[u_i]>1} ([u_i] - 1), \tag{F.8}$$

and r, h are number of free vector multiplets and hypermultiplets of the theory at generic point of the Coulomb branch. In our cases, r equals the rank of Coulomb branch and h is zero. Finally, we have [276]

$$R(B) = \frac{\mu\alpha}{4}. \tag{F.9}$$

Appendix G

GRADING OF LIE ALGEBRA FROM NILPOTENT ORBIT

A natural way of generating torsion automorphism is to use nilpotent orbit in $\mathfrak{g}$. Let e be a nilpotent element, which may be included in an $\mathfrak{sl}_2$ triple $\{e, h, f\}$ such that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. With respect to the adjoint action $\operatorname{ad} h$, $\mathfrak{g}$ decompose into eigenspaces:

$$\mathfrak{g} = \bigoplus_{i=-d}^{d} \mathfrak{g}_i, \tag{G.1}$$

where d is called the depth. Proper re-assembling of $\mathfrak{g}_i$ gives (5.18), hence fixes a torsion automorphism σ_e of order m. We call the nilpotent element e even (odd) if the corresponding Kac diagram $\mathbb{D}_e$ is even (odd). In fact $\mathbb{D}_e$ is identical to the weighted Dynkin diagram $\widehat{\mathbb{D}}_e$ [178]. Moreover, we have the relation $m = d + 2$ and $\mathfrak{g}^2 = \mathfrak{g}_2 + \mathfrak{g}_{-d}$.

A *cyclic* element of the semisimple Lie algebra $\mathfrak{g}$ associated with nilpotent element e is the one of the form $e + F$, for $F \in \mathfrak{g}_{-d}$. We say e is of *nilpotent* (resp. *semisimple* or *regular semisimple*) *type* if any cyclic element associated with e is nilpotent (resp. any *generic* cyclic element associated with e is semisimple or regular semisimple). Otherwise, e is called mixed type [185]. A theorem of [185] is that e is of nilpotent type if and only if the depth d is odd. We see that T_2 precisely corresponds to the cyclic element. In order to get regular semisimple coefficient matrices, it is clear that one needs e of regular semisimple type. In fact, except for $\mathfrak{g} = A_{N-1}$ case, all nilpotent elements of regular semi-simple type generate even Kac diagram $\mathbb{D}_e$[1].

However, nilpotents e of regular semisimple type do not exhaust all the torsion automorphism we are interested in. To complete the list, we examine the problem from another point of view. When a cyclic element $e + F$ is regular semisimple, its centralizer $\mathfrak{h}'$ is a Cartan subalgebra. σ_e leaves $\mathfrak{h}'$ invariant, thus induces a regular element w_e in the Weyl group. When e gives even $\mathbb{D}_e$, w_e and σ_e have the same order, called the *regular number* of w_e. The regular element and its regular number are classified in [277], and nilpotents of regular semisimple type do not cover all of them.

[1] By this we mean that the nilpotents with partition $[n, n, \ldots, n, 1]$ for $\mathfrak{g} = A_{N-1}$, though of regular semisimple type, are not even.

The remaining regular numbers, fortunately, are all divisors of those of σ_e. Hence, we can obtain the Kac diagrams from taking appropriate power of some σ_e. Their Kac coordinates are determined from the following algorithm [182, 183]. Suppose we start with automorphism σ_e of order m and Kac coordinates $(s_0, s_1, \ldots, s_r)$ and we wish to construct automorphism of order $n < m$ by taking $\sigma^{m/n}$. We first replace the label s_0 by

$$s_0 \longrightarrow n - \sum_{i=1}^{N} a_i s_i. \tag{G.2}$$

Now s_0 will be necessarily negative. After that, we pick one negative label s_j at each time for $j = 0, 1, \ldots, N$, and change the label into $(s_0', s_1', \ldots, s_r')$ such that

$$s_i' = s_i - \langle \alpha_i, \alpha_j^\vee \rangle s_j, \quad i = 0, 1, \ldots, r, \tag{G.3}$$

where $\alpha^\vee$ is the coroot. One repeats the procedure until finally all $(s_0, \ldots, s_r)$ are positive. This gives the Kac diagram that corresponds to the automorphism with order n. The Kac diagram obtained is unambiguous, independent of which element e we start with.

We now use nilpotent elements to obtain the grading. For $\mathfrak{g} = A_{N-1}$, this is done in [132]. We mainly examine the classification when $\mathfrak{g} = D_N$ and $E_{6,7,8}$.

• **The Lie algebra** $\mathfrak{g} = D_N$. Nilpotent element e is of semi-simple type if and only if

(i) The embedding is $[n_1, \ldots, n_1, 1, \ldots, 1]$ where n_1 has even multiplicity;

(ii) $[2m+1, 2m-1, 1, \ldots, 1]$ with $m \geq 1$;

(iii) $[n_1, 1, \ldots, 1]$ for $n_1 \geq 5$.

In particular, e is of regular semi-simple type if and only if in (i) n_1 is odd and 1 occurs at most twice; in (ii) $p \leq 4$; in (iii) $p \leq 2$. In each case we can compute $b = d + 2$ where d is the depth. They are (i) $d = 2n_1 - 2$; (ii) $d = 2n_1 - 4 = 4m - 2$; (iii) $d = 2n_1 - 4$ [185]. As is known, these nilpotent elements are all even. Next we examine each case of regular semi-simple type in more detail.

Nilpotent embedding of case (i). When the partition is $[n_1, n_1, \ldots, n_1]$, we see n_1 must be a divisor of N. Therefore we have the Higgs field

$$\Phi \sim \frac{T}{z^{2+\frac{k}{n_1}}} \tag{G.4}$$

with $(k, n_1) = 1$. Note that when N is even, the partition $[N, N]$ is not allowed. This case will be recovered in case (ii).

When the partition is $[n_1, \ldots, n_1, 1]$, then we know n_1 divides $2N - 1$. But n_1 must have even multiplicity, so this case is excluded.

When the partition is $[n_1, \ldots, n_1, 1, 1]$, then n_1, being an odd number, must divide $N - 1$. Then we get (G.4) as well (but the matrix T is different).

Nilpotent embedding of case (ii). There can only be no 1 or two 1's in the Young tableaux. For the former, we have $4m = 2N$. So this case exists only when N is even number. The Higgs field is

$$\Phi \sim \frac{T}{z^{2+\frac{k}{N}}} \tag{G.5}$$

with $(k, N) = 1$. For the latter, we have $4m = 2N - 2$ (which means $N - 1$ must be even), and the Higgs field is

$$\Phi \sim \frac{T}{z^{2+\frac{k}{N-1}}} \tag{G.6}$$

for $(k, N - 1) = 1$.

Nilpotent embedding of case (iii). When $p = 1$, we have the partition $[2N]$. This violates the rule for D-partition.

When $p = 2$ we have $n_1 = 2N - 1$, so the order of ϵ is $4N - 4$. We get the Higgs field

$$\Phi \sim \frac{T}{z^{2+\frac{k}{2N-2}}}. \tag{G.7}$$

In summary, with classification of nilpotent orbit of regular semi-simple type, for N odd, we have recovered $b = N$ and all its divisors $b = n_1$ (no even divisors). For N even, we can recover $b = N$ as well and all its odd divisor. But we *could not recover its even divisors* using the above technique. Similarly, we have recovered $b = 2N - 2$ and $b = N - 1$ as well as all odd divisors of $N - 1$, but we missed all the even divisors of $2N - 2$ except $N - 1$ itself.

The recovery of the missing cases can be achieved with the prescription introduced around (G.2) and (G.3). We give some examples in appendix H. Here we only

nilpotent orbit	depth	order	Higgs field
$D_4(a_1)$	6	4	$\Phi \sim T/z^{2+\frac{k}{4}}$
$E_6(a_3)$	10	6	$\Phi \sim T/z^{2+\frac{k}{6}}$
D_5	14	8	$\Phi \sim T/z^{2+\frac{k}{8}}$
$E_6(a_1)$	16	9	$\Phi \sim T/z^{2+\frac{k}{9}}$
E_6	22	12	$\Phi \sim T/z^{2+\frac{k}{12}}$

Table G.1: Summary of nilpotent elements of regular semi-simple type in E_6.

nilpotent orbit	depth	order	Higgs field
$E_7(a_5)$	10	6	$\Phi \sim T/z^{2+\frac{k}{6}}$
A_6	12	7	$\Phi \sim T/z^{2+\frac{k}{7}}$
$E_6(a_1)$	16	9	$\Phi \sim T/z^{2+\frac{k}{9}}$
$E_7(a_1)$	26	14	$\Phi \sim T/z^{2+\frac{k}{14}}$
E_7	34	18	$\Phi \sim T/z^{2+\frac{k}{18}}$

Table G.2: Summary of nilpotent elements of regular semi-simple type in E_7.

mention that such procedure is unambiguous, $i.e.$ the resulting Kac diagram is the same regardless of which parent torsion automorphism we use[2].

• **The Lie algebra** $\mathfrak{g} = E_{6,7,8}$. As in the previous case, we would like to first find all nilpotent elements of regular semi-simple type. They are listed in table G.1 - table G.3, along with their order and the singular Higgs field behavior. One can also use the pole data to read off the 3-fold singularity.

Again, the above classification does not exhaust the possibility of the order of poles. We expect that we should be able to get all divisors for the denominator. We still can use the same algorithm to generate them, and they are unambiguous. We recover the missing Kac diagram in appendix H.

[2]More specifically, they should descend from the *same* "parent". For instance, fix D_N, if n_1 and n_2 are *both* divisors of N and $n_1|n_2$, then the torsion automorphism of σ_1 of order n_1 is the same whether we start with $\sigma_{[2m+1,2m-1]}$ by taking N/n_1-th power, or with σ_2 of order n_2 by taking n_2/n_1-th power. See appendix H for more detail.

nilpotent orbit	depth	order	Higgs field
$E_8(a_7)$	10	6	$\Phi \sim T/z^{2+\frac{k}{6}}$
$E_8(a_6)$	18	10	$\Phi \sim T/z^{2+\frac{k}{10}}$
$E_8(a_5)$	22	12	$\Phi \sim T/z^{2+\frac{k}{12}}$
$E_8(a_4)$	28	15	$\Phi \sim T/z^{2+\frac{k}{15}}$
$E_8(a_2)$	38	20	$\Phi \sim T/z^{2+\frac{k}{20}}$
$E_8(a_1)$	46	24	$\Phi \sim T/z^{2+\frac{k}{24}}$
E_8	58	30	$\Phi \sim T/z^{2+\frac{k}{30}}$

Table G.3: Summary of nilpotent elements of regular semi-simple type in E_8.

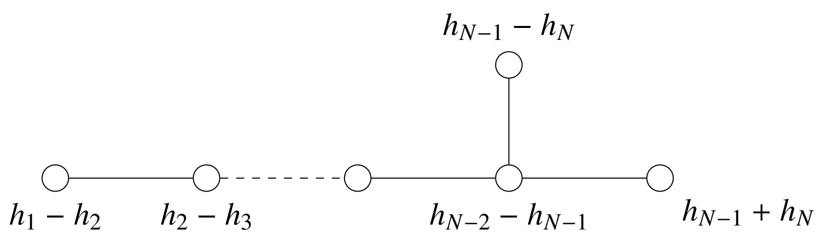
Appendix H

RECOVER MISSING KAC DIAGRAMS

Here we shall give examples of how to generate those Kac diagrams of torsion automorphisms that are missing from considering nilpotent embedding, as in appendix G. To begin with, we first explain in $\mathfrak{g} = D_N$ case how to write down the weighted Dynkin diagrams for automorphisms of the form σ_e. For a thorough mathematical treatment, the readers may consult [178].

Assume that e is represented by a Young tableau $Y = \left[n_1, n_2, \ldots, n_p\right]$, and $n_1 + \cdots + n_p = 2N$. Moreover we assume Y is not very even[1], which is what we concern. For each n_i we get a sequence $\{n_i - 1, n_i - 3, \ldots, -n_i + 3, -n_i + 1\}$. Combining the sequences for all i, we may arrange them in a decreasing order and the first N elements are apparently non-negative, and we denote them as $\{h_1, h_2, \ldots, h_N\}$. Now the Kac coordinate on the Dynkin diagram of D_N is given as follows:

σ_Y :



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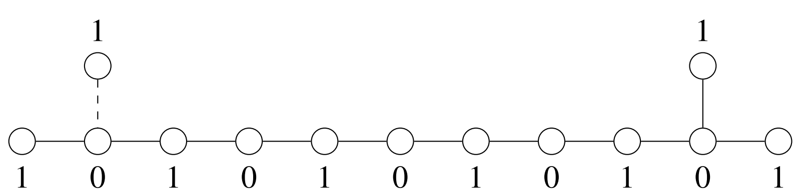
Then, we add the highest root α_0 and make it an extended Dynkin diagram, and put the label $s_0 = 2$ for it. If in addition the Kac diagram is even, by our convention we divide each label by 2.

Now we present examples showing the unambiguity of generating Kac diagrams. We take $N = 12$. The order 12 torsion automorphism is obtained by the nilpotent element with partition [13, 11], so its affine weighted Dynkin diagram is

$\sigma_{[13,11]}$:



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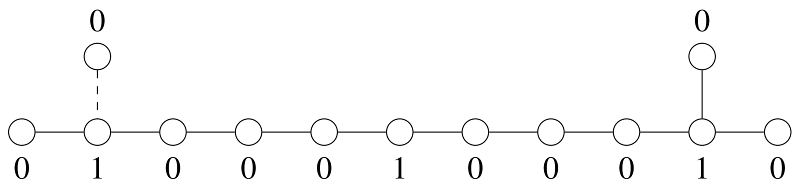
[1]For weighted Dynkin diagrams of very even element, see [178].

where we used dashed line to indicate the affine root. We may use the algorithm from (G.2) and (G.3) to generate an order 6 torsion automorphism. It is given by:

$\sigma_{\{6\}}$:



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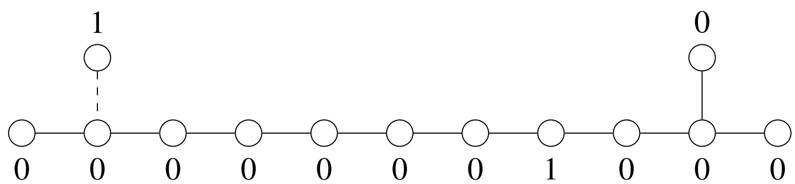
Since this diagram does not come from any nilpotent element e, we just use a subscript $\{6\}$ to indicate its order. With this diagram, we can further generate an order 3 nilpotent element by taking a twice power of $\sigma_{\{6\}}$. The same algorithm gives a Kac diagram:

$\sigma_{\{3\}}$:



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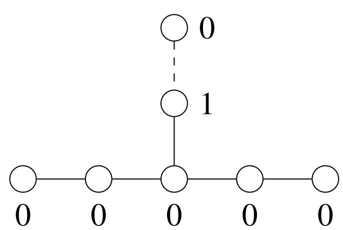
This Kac diagram is precisely the same as the affine weighted Dynkin diagram of the nilpotent element $\left[3^8\right]$. So we see there is no ambiguity.

As a second example, we take $N = 9$. The same argument as above shows that the Kac diagram for order 8 torsion automorphism constructed from nilpotent element of partition [9, 7, 1, 1], is exactly identical to the one obtained by square of the torsion automorphism from the element [17, 1].

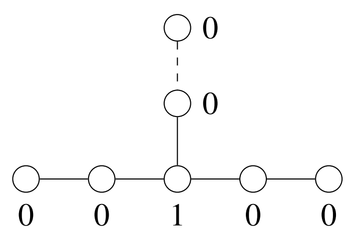
For $\mathfrak{g} = E_{6,7,8}$ case, the Kac diagrams for nilpotent elements of regular semisimple type are given in [185]. With the same procedure, we can recover missing Kac diagrams as follows.

For $\mathfrak{g} = E_6$, we missed order 2 and order 3 element, their Kac diagrams are, respectively:

$\sigma^{E_6}_{\{2\}}$:



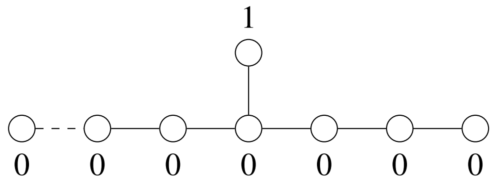
, $\sigma^{E_6}_{\{3\}}$:



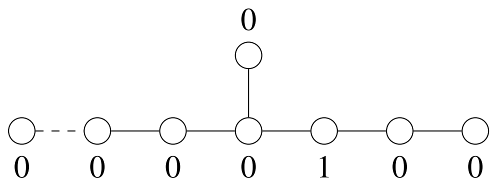
.

For $\mathfrak{g} = E_7$ we also missed the order 2 and order 3 torsion automorphisms. The Kac diagram can also be obtained:

$\sigma^{E_7}_{\{2\}}$:



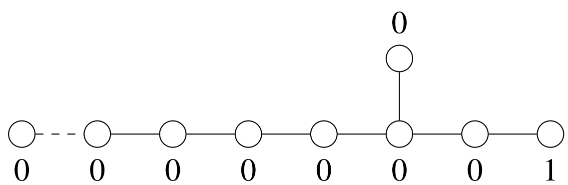
$\sigma^{E_7}_{\{3\}}$:



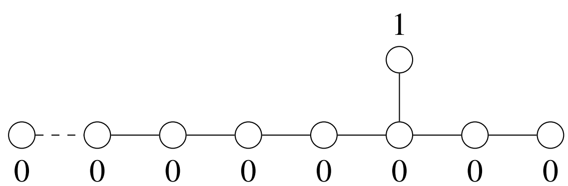
.

Finally, for $\mathfrak{g} = E_8$, we have missed the torsion automorphisms of order $2, 3, 4, 5, 8$. They can be recovered by weighted Dynkin diagrams of nilpotent elements of regular semi-simple type. We list them as follows:

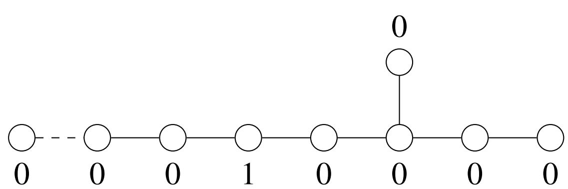
$\sigma^{E_8}_{\{2\}}$:



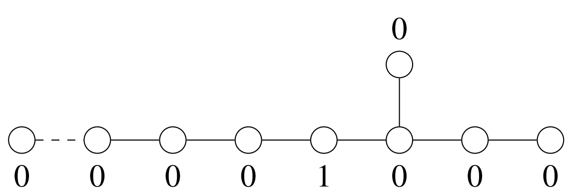
$\sigma^{E_8}_{\{3\}}$:



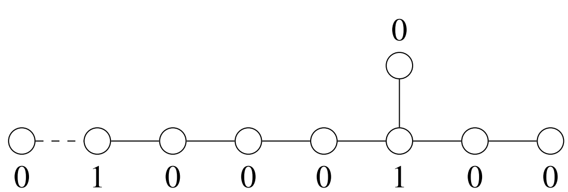
$\sigma^{E_8}_{\{4\}}$:



$\sigma^{E_8}_{\{5\}}$:



$\sigma^{E_8}_{\{8\}}$:



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Appendix I

GENERAL PROPERTIES OF THE RECURSION RELATIONS

In this appendix we wish to extract some universal properties of the vacua for all N_c, N_f, and k with deformation (6.93), and the massless limit.

Before diving into technical proof, we may imagine how the vacuum looks like by physical argument. First, we know Φ is classically nilpotent, labelled by a set of discrete integers. In other words, Φ is already "quantized" at the classical level, and quantum corrections cannot modify it. So we expect $u_j = 0$ quantum mechanically as well. Moreover, the superpotential (6.2) truncates the chiral ring, and we expect this is also true at quantum level. Specifically, we expect there exists an integer k_0 such that for $j \geq k_0$ all $v_j = 0$. Classically $k_0 = k$.

We prove the following claims. Some claims can be proven even for most general deformations (6.3). We will use a * notation to indicate this situation.

Claim 1. All generalized glueball has trivial vevs $r_j = 0$, implying $R(z) = 0$. Thus $U(N_c)$ Kutasov model does not have non-trivial gaugino condensations.*

Proof. From Konishi anomaly (6.40d), we can expand around $z \to +\infty$ and look at coefficients of z^{-n-1}. It reads

$$\sum_{j=1}^{l+1} m_{f,j}^{\tilde{f}} v_{\tilde{f},n+j-1}^{f'} = \delta_f^{f'} r_n. \tag{I.1}$$

A physically sensible solution of the quantum chiral ring should have all the elements u_n, r_n and v_n as functions of parameters $\{g_i, m_{f,l}^{\tilde{f}}\}$, and they must be finite when the parameters approach zero. Therefore taking the limit of both sides of above equations, and picking $f = f'$, we immediately see

$$r_n = 0. \tag{I.2}$$

In particular, $r_0 \propto \mathrm{Tr}\, W_\alpha W^\alpha \sim \langle \lambda\lambda \rangle = 0$. □

Claim 2. There exists k_0 such that for all $j \geq k_0$, $v_j = 0$ in the chiral ring.

Proof. Here we assume superpotential (6.93). Then Konishi anomaly (6.40d) and (6.40e) tell us that

$$[m.v_n]_f^{f'} = \delta_f^{f'} r_n, \quad [v_n.m]_{\tilde{f}'}^{\tilde{f}} = \delta_{\tilde{f}'}^{\tilde{f}} r_n, \tag{I.3}$$

which means m and v_n commute and the product is a diagonal matrix, proportional to r_n times the identity. Then from the Konishi anomaly (6.40c)

$$\sum_{i=0}^{k} g_i r_{n+i} = \sum_{i=0}^{n-1} r_i r_{n-i-1}. \tag{I.4}$$

One can think of it as a matrix equation, and substitute each r_n by $m.v_n$ and multiply m^{-1} on both sides. Taking limit on both sides we see $v_{k+n} = 0$ for all $n \geq 0$. Thus the truncation is at least as far as in classical case. □

Claim 3.* $u_{k+n} = 0$ *for all* $n \geq 0$.

Proof. This time we use Konishi anomaly (6.40a). One obtains

$$\sum_{i=k}^{0} g_i u_{n+i} + \sum_{j=1}^{l+1} (j-1) m_{f,j}^{\tilde{f}} v_{\tilde{f},n+j-2}^{f} = 2 \sum_{i=0}^{n-1} r_i u_{n-i-1}. \tag{I.5}$$

Again taking the limit on both sides and use the condition that $r_n = 0$ of claim 1, and all parameters except g_k is infinitesimally small, we see that $u_{k+n} = 0$ for any non-negative integer n. □

Claim 4. $u_1 = u_2 = \cdots = u_{k-1} = 0$.

Proof. We will use induction. Notice first that

$$T(z)^2 \Big(P(z)^2 - \widetilde{\Lambda}^{2N} \Big) = P'(z)^2, \tag{I.6}$$

where $\widetilde{\Lambda}^{2N} = (\det m) \Lambda^{2N-N_f}$ and $P(z) = p_N + p_{N-1} z + \ldots p_1 z^{N-1} + z^N$. It is now safe to take massless limits on both side[1], and because of claim 3, we obtain an equality:

$$\begin{aligned} \left(\frac{u_{k-1}}{z^k} + \cdots + \frac{N}{z} \right) & \Big(p_N + p_{N-1} z + \cdots + p_1 z^{N-1} + z^N \Big) \\ &= p_{N-1} + 2 p_{N-2} z + \cdots + (N-1) p_1 z^{N-2} + N z^{N-1}. \end{aligned} \tag{I.7}$$

Now suppose $k = 2$. The comparing coefficients on both sides tells us $u_1 p_N = 0$. Then we must have $p_N = 0$, otherwise we are done. Then by iterating the procedure we see $p_1 = p_2 = \cdots = p_N = 0$; then $u_1 = 0$ so the claim is valid for $k = 2$. Suppose

[1] Here one should first show that p_i are all finite in the limit. Indeed, with deformation (6.93) p_i can be expressed by polynomial of $u_1, \ldots u_N$ and no instanton factor would enter. In other words the expressions are the same as classical case.

the claim is true for $k-1$, now we proceed to the case of k. Again by comparing the coefficients of (I.7), under the condition $u_{k-1} \neq 0$ (otherwise we are done by assumption), we see all p_i's vanish. Therefore, u_{k-1} must vanish as well. So the proof is complete. □

Although expectation values of Casimir operators and generalized mesons are zero, they may not be trivial in the chiral ring. We conclude that quantum mechanically, in general the chiral ring of Kutasov model can still be written as

$$\widehat{\mathcal{R}}_{N_c,N_f,k} = \mathbb{C}[u_1, u_2, \ldots, u_{k-1}, v_0, v_1, \ldots, v_{k-1}]/\widehat{\mathcal{S}}(u_1, u_2, \ldots, u_{k-1}, v_0, v_1, \ldots, v_{k-1}), \tag{I.8}$$

where we have omitted the generalized glueball and photinos $w_{\alpha,k}$.

Appendix J

ISOMORPHISM OF COULOMB BRANCH VACUA

In this appendix, we consider two examples that the quantum Coulomb branch receive exactly the same corrections. We take the gauge group to be $U(2)$.

J.1 $N_f = 1, l = 2$

We pick the superpotential to be

$$W = \frac{1}{3}\mathrm{Tr}\,\Phi^3 - \frac{1}{2}\mathrm{Tr}\,\Phi^2 + \widetilde{Q}(2 + 3\Phi + \Phi^2)Q. \tag{J.1}$$

The recursion relation becomes

$$\begin{aligned} u_{n+2} - u_{n+1} + 2v_{n+1} + 3v_n &= 2\sum_{i=0}^{n-1} r_i u_{n-i-1}, \\ r_{n+2} - r_{n+1} &= \sum_{i=0}^{n-1} r_i r_{n-i-1}, \\ v_{n+2} + 3v_{n+1} + 2v_n &= r_n, \\ (n+1)(v_{n+2} - v_{n+1}) + 3\sum_{i=0}^{n} v_i v_{n-i} + 2\sum_{i=0}^{n} v_i v_{n-i+1} &= 2\sum_{i=0}^{n-1}(n-i) r_i v_{n-i-1}. \end{aligned} \tag{J.2}$$

Classical vacua. At classical level one can set the right hand side of above recurrence formulae to be zero and only consider the first, third and fourth equations. Then one can first solve the generalized mesons:

$$v_n = (-2)^n C_1 + (-1)^n C_2, \tag{J.3}$$

where $C_{1,2}$ are two parameters that determine the initial condition. Then we can further plug the expression in the first equation of (J.2) and eliminate additional variables. So the classical chiral ring relation for u_1 is

$$(u_1 - 2)(u_1 - 1)u_1(u_1 + 1)(u_1 + 2)(u_1 + 3) = 0. \tag{J.4}$$

This precisely corresponds to 3 Coulomb branch vacua and 5 Higgs branch vacua.

Quantum vacua. The quantum recursion relation can be solved leaving single generator u_1 as usual. We expect that the quantum moduli space is a deformation of the classical one in the sense that if we take the strong coupling scale $\Lambda \to 0$, we should recover classical chiral ring, possibly with increased multiplicities of the roots. Indeed in this case we have

$$\begin{aligned}(u_1-1)(u_1+3)\Big(&u_1^8-(7+52\Lambda^3)u_1^6-(2+376\Lambda^3)u_1^5+(12-926\Lambda^3-204\Lambda^6)u_1^4\\ &+(8-1000\Lambda^3-976\Lambda^6)u_1^3-(498\Lambda^3+1552\Lambda^6+160\Lambda^9)u_1^2\\ &-(100\Lambda^3+1120\Lambda^6+448\Lambda^9)u_1-275\Lambda^6-160\Lambda^9+64\Lambda^{12}\Big)=0.\end{aligned} \tag{J.5}$$

J.2 $N_f = 2, l = 1$

We take the superpotential to be

$$W=\frac{1}{3}\mathrm{Tr}\,\Phi^3-\frac{1}{2}\mathrm{Tr}\,\Phi^2+m_{1,f}^{\tilde f}\widetilde{Q}_{\tilde f}Q^f+m_{2,f}^{\tilde f}\tilde{Q}_{\tilde f}\Phi Q^f, \tag{J.6}$$

and we use the chiral symmetry to cast m_1 into diagonal form and assume it to be

$$m_{1,f}^{\tilde f}=\begin{pmatrix}1&0\\0&2\end{pmatrix}, \tag{J.7}$$

while in principle m_2 does not have to be diagonal, but we require it to be invertible. To make things simple we set

$$m_{2,f}^{\tilde f}=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{J.8}$$

Classical vacua. The recursion relation is

$$\begin{aligned}&u_{n+2}-u_{n+1}+\mathrm{Tr}\;m_2.v_n=0,\\ &m_{1,f}^{\tilde f}v_{n,\tilde f}^{f'}+m_{2,f}^{\tilde f}v_{n+1,\tilde f}^{f'}=0,\\ &(n+1)(v_{n+2}-v_{n+1})_{\tilde g}^{g}+\sum_{i=0}^{n}v_{i,\tilde g}^{f}m_{2,f}^{\tilde f}v_{n-i,\tilde f}^{g}=0.\end{aligned} \tag{J.9}$$

From the second equation we see $v_n=-m_1.v_{n-1}=(-m_1)^n v_0=v_0(-m_1)^n$. This fact means v_0 must be a diagonal matrix, and so are all generalized mesons. Then one can again eliminate variables and obtain the relation for the generator u_1, so that we arrive at

$$(u_1-2)(u_1-1)u_1(u_1+1)(u_1+2)(u_1+3)=0, \tag{J.10}$$

and also the recursion relation could uniquely determine the vevs of generalized mesons.

Quantum vacua. The right hand side of recursion relations should be supplemented by the anomalies. Since one also has $v_n.m_1 = m_1.v_n$ so generalized mesons are still diagonal. The nonperturbative corrections to Casimir operators:

$$T(z) = \frac{d}{dz}\log\left(P(z)^2 + \sqrt{P(z)^2 - 4(1+z)(2+z)\Lambda^2}\right) \tag{J.11}$$

is in fact the same as $N_f = 1, l = 2$ case, except the substitution $\Lambda^3 \to \Lambda^2$. After some lengthy calculation we obtain the relation for the generator u_1:

$$\begin{aligned}(u_1-1)(u_1+3)\Big(&u_1^8 - (7+52\Lambda^2)u_1^6 - (2+376\Lambda^2)u_1^5 + (12-926\Lambda^2-204\Lambda^4)u_1^4 \\ &+(8-1000\Lambda^2-976\Lambda^4)u_1^3 - (498\Lambda^2+1552\Lambda^4+160\Lambda^6)u_1^2 \\ &-(100\Lambda^2+1120\Lambda^4+448\Lambda^6)u_1 - 275\Lambda^4 - 160\Lambda^6 + 64\Lambda^8\Big) = 0.\end{aligned} \tag{J.12}$$

It is not surprising to see that the expression is isomorphic to (J.5), and the quantum shift to the chiral ring generator u_1 is exactly the same. This isomorphism can be attribute to the fact that the curve $P(z)^2 - 4\Lambda^{2N_c-N_f}B(z)$ is isomorphic on the Coulomb branch.

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