TWO STUDIES OF DYNAMICAL CALCULATIONS IN TWO MESON SYSTEMS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

May 1965

Acknowledgements

The author wishes to thank his advisor Dr. S.C. Frautschi, for many stimulating discussions, and for critical assistance in preparing this thesis.

The author also gratefully acknowledges useful discussions with the following people:- Drs. Ernest Abers, L. Balazs, R. Dashen and R. Gerbracht.

Thanks are due to Dr. M. Gell-Mann for sponsoring IBM 7094 computer calculations employed to verify features of the calculations of part I of this thesis.

Finally, the author thanks the Schlumberger Foundation, the Rand Corporation, the Woodrow Wilson Foundation, and the California Institute of Technology for financial support during his graduate student life.

ABSTRACT

This thesis describes dynamical calculations in dispersion theory relating to meson states. The first part discusses the bound state pseudoscalar mesons, utilizing a pole model representation of the binding forces.

In the one-channel pseudoscalar vector calculation, and in a calculation coupling this to the baryon-antibaryon channel, fairly modest forces suffice to produce a tightly bound state. However, the output coupling constants are much larger than those fed into the calculation, so that bootstrap self consistency is not attained. It is also interesting that the baryon-antibaryon channel seems to be a very significant contributor to the dynamical generation of bound state pseudoscalar mesons.

The second part of this work considers electromagnetic mass splittings of mesons. In a dynamical scheme for generating the mesons, mass splittings of the derived multiplets arise both because of mass splittings of the particles participating in the relevant scattering channels, and, because of photon exchanges. A crude inelastic bootstrap calculation of the ρ vector meson is developed and then subject to electromagnetic perturbations. As a corresponding pseudoscalar meson bootstrap is not available, we can only relate pseudoscalar and vector mass differences here.

A rough estimate which results for the ρ meson is $M_{\rho 0} - M_{\rho +} \gtrsim$ 11 MeV. A yet cruder estimate of the $K^*(885)$ mass difference is $MK_0^* - MK_+^* \sim 5$ MeV.

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Part I. Dynamical Calculations of Pseudoscalar Meson Bound States

Introduction

In recent years one approach to the problem of theoretically predicting the observed spectrum of bound and resonant states of "fundamental" particles has been that of dynamical calculations based on utilization of unitarity and analyticity of the S matrix[1]. One can employ knowledge of these principles to obtain dispersion relations for partial wave amplitudes, and one can employ a prescription usually based on field theory, to guess at a plausible input force for the calculation. Many calculations have been performed with various approximations[2]. These calculations have been moderately successful in obtaining a self-consistent solution for angular momentum one, negative parity, amplitudes possessing resonant states closely corresponding to the observed vector mesons. It is of interest therefore, to consider similar calculations of pseudoscalar mesons.

We shall discuss the dynamical calculation of a pseudoscalar (PS) bound state in a pseudoscalar-vector (PS-V) channel and also study the effects of including a baryon-antibaryon (B-$\bar{B}$) channel. A pole approximation to single particle exchange forces will be employed.

One finds that plausible forces easily suffice to give adequate binding for the low energy bound states in these channels. However, self-consistency in the coupling constants is not achieved; the output coupling constant from a calculation turns out to be considerably larger than the input coupling constant required to obtain the desired bound state. Another interesting result is that in spite of its high threshold mass, the $B\text{-}\bar{B}$ state contributes significantly to the origin of the 0^- bound state.

In Section 1 we define relevant amplitudes. Section 2 contains reviews of the N/D dispersion theory and develops the pole-model approximation. In Section 3 we consider in detail the PS-V channel and obtain (numerically) the results referred to above. In Section 4 we present several comments on aspects of the previous calculation. Finally, in Section 5, we describe a coupled channel (PS-V and $B\text{-}\bar{B}$) calculation.

1. Definitions

It is first necessary to define the amplitudes discussed in subsequent sections. The conventional S matrix(3) is related to a T matrix as in Equation (1):

$$S = 1 + \frac{(2\pi)^4 \delta^4(\sum_f P_f - \sum_i P_i)\ i\ T}{\prod_{i,f} \sqrt{2E_i\ 2E_f}} \ . \tag{1}$$

For two-particle elastic scattering one then finds that the differential scattering cross-section in the center of momentum frame (c-m) is given by

$$\frac{d\sigma}{d\Omega} = \left| f \right|^2 \times \frac{q_{final}}{q_{initial}} \tag{2}$$

where

$$f = \frac{T}{8\pi\omega} \ .$$

Here P, p, q, k etc., will refer to 4-momenta, E_i to a single particle energy, ω to the total c.m. energy, $s = \omega^2$.

With the above convention, a positive T corresponds to an attractive force. A state of angular momentum J, parity π, will be denoted by J^π. The 0^- partial wave amplitude is defined by (3)

$$t_{J=0}^{\lambda'\lambda} = 1/2 \int_{-1}^{1} d(\cos\ \theta)\ \frac{T^{\lambda'\lambda}(s,\cos\ \theta)}{16\pi} \tag{3}$$

for helicities λ', λ.

It satisfies the (elastic) unitarity condition (4)

$$\mathrm{Im}\ t_{J=0}^{\lambda'\lambda} = \frac{2p}{\omega} \left| t_{J=0}^{\lambda'\lambda} \right|^2 \tag{4}$$

where p is the c.m. 3-momentum.

The 0^- state in a PS-V channel occurs in only one amplitude (only one "sense" amplitude, in the notation of M. Gell-Mann et al.[4]), namely the helicity[6] zero, orbital P-wave amplitude. It can also occur in the S wave, spin singlet B-$\bar{B}$ channel (as well as in V-V channels, etc.).

Before returning to specific details of calculating the relevant amplitudes, it is appropriate to discuss the equations to be used in dynamical calculations, which are the subject of the next section.

2. N/D Equations

In this section we present a brief discussion of dispersion relation equations for partial wave amplitudes. The discussion will briefly mention questions relating to the convergence of the integrals occurring in dispersion relations, and then will present the approximations to be employed in subsequent calculations.

To begin, then, let us consider some analytic properties of a single channel partial wave amplitude which we call t when normalized as in Section 1. Because of the unitarity of the scattering matrix, t will have a cut along the positive real axis of the complex S plane, running from some branch point S_1 to ∞. Also, t possesses cuts on the left hand axis and off the real axis, which are due to various exchanges of particles.

With the normalization of I, t corresponds to a unitary elastic scattering amplitude of the form (5)

$$t = \frac{\omega}{2p} e^{i\delta} \sin \delta \ . \tag{5}$$

At threshold, t(s) vanishes like $p^{2\ell}$ (5). One now can write $t = \zeta N/D$ where ζ is a factor chosen to ensure the proper threshold behaviour of t, D has the unitary cuts of t, and N has the force cuts ("left-hand cuts") of t. Let $\rho = 2p/\omega$, and let B be a function having the exact left-hand cuts of t; as a first approximation I take B to be the sum of Born single-particle-exchange force diagrams as calculated from standard perturbation theory. Let $\bar{B} = B/\zeta$.

One can simply obtain the following dispersion relations for N and D (7).

$$N = \bar{B} + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\rho\zeta N}{s'-s}\left[\bar{B}(s') - \frac{s-s_t}{s'-s_t}\bar{B}(s)\right] ds' \qquad (6)$$

$$D = 1 - \frac{s-s_t}{\pi}\int_{s_1}^{\infty} \frac{\rho\zeta N}{(s'-s)(s'-s_t)} ,$$

or alternately,

$$N = \bar{B}D + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\rho\zeta NB}{s'-s}\, ds' .$$

Because solutions of the integral N/D equations are independent of subtraction parameters(8), one can normalize D to 1 at arbitrary s_t.

Bound states correspond to zeros of D, i.e. to poles of t. A bound state pole may be overlapped by a force cut, so that the so-called determinantal approximation, $N = \bar{B}$, could result in a complex residue at the pole. However, when employing Eq. (6) we see that Im N = 0 at D = 0; we thus prefer to employ Eq. (6) for bound state calculations.

We now consider the convergence of the integrals in Eq. (6), with certain "model" assumptions about B, which give a B resembling a reasonably modified field theory amplitude. The question of convergence of the integrals depends on both the factors ζ and B. For S-waves, a natural threshold factor ζ will not diverge at high energies, so that a divergence can arise only if B grows too rapidly. For higher partial waves, a typical threshold factor ζ will diverge as some power of s, for $s \to \infty$, and therefore, convergence will be controlled more by the nature of the threshold factor.

We are, in fact, interested in a higher partial wave, the P-wave, and we shall now show that B vanishing monotonically as $s \to \infty$ does not ensure convergence. This will have the consequence that if one wishes to choose a simple, monotonic B, then one must choose ζ appropriately so as to obtain convergent integrals. As an example of the typical high-energy behaviour to be expected of partial wave exchange amplitudes let us first consider the consequences of assumed Regge behaviour on the asymptotic behaviour of t_J. One now would have $T \backsim b(t)\, s^{\alpha(t)}$ for large S, where $\alpha(t)$ is the dominant Regge trajectory, and $t = -2q^2 (1-\cos\theta)$ for elastic scattering. Therefore,

$$t_J \;\alpha \int_{-1}^{+1} dx\, P_J(x)\, b(t)\, s^{\alpha(t)} \qquad (x = \cos\theta)$$

$$= \left(\int_{-1}^{+1} s^{\alpha(t)} dx \right) P_J(x)\, b(t) \Bigg|_{x=-1}^{x=+1} - \int \left(\int s^{\alpha(t)} dx \right) \frac{d}{dx} \left(b(t) P_J(x) \right) \qquad (7)$$

As $s \to \infty$, $x = \pm 1$ corresponds to $t = 0, -\infty$. If we take $\alpha(t)$ approximately constant as $t \to -\infty$, and linear as $t \to 0$, one then obtains from the first term of Eq. (7):

$$t_J(s) \backsim \left(\frac{s^{\alpha(0)} \text{x const}}{2q^2 \ln s \left(\frac{dx}{dt}\right)_0} + s^{\alpha(-\infty)} \right) \left(1 + 0\left(\frac{1}{\ln s} \dots\right) \right). \qquad (8)$$

For a more specific example, consider

$$\alpha \simeq \alpha(-\infty) \left[1 + \frac{\alpha(0)-\alpha(-\infty)}{t+\alpha(-\infty)} \right] . \qquad (9)$$

Assuming that $\alpha(0) - \alpha(-\infty) > 0$, and $b(t)$ = constant, one may project out a partial wave from $s^{\alpha(t)}$. One finds

$$t_{J=0} \propto 2s^{\alpha(-\infty)} \left[1 + 0 \, \frac{(\ln s)^2}{2q^2} \right] \qquad (10)$$

$$+ \frac{\alpha(-\infty)}{2q^2} s^{\alpha(0)} + \text{terms vanishing more rapidly as } s \to \infty.$$

For "PS exchange"; $\alpha(0) < 0$, so that the partial wave amplitudes should vanish at least as fast as $1/\ln s$ for $s \to \infty$. It is important to note that $t_{J=0}(s)$ decreases smoothly at large S. Thus, one cannot invoke high-energy oscillations of t_J (when Regge behaviour is dominant) as a mechanism ensuring convergence of the N/D equations. This has also been pointed out recently in a more precise manner by Omnes[10]. Furthermore, application of an arbitrarily sharp monotonic cutoff to B will not ensure convergence. For example, if $\bar{B} \sim 1/S^p$, then the kernel of the integral equation for N becomes (taking $s_t = 0$)

$$\sum_{m=0}^{p=2} \frac{s'^m \, s^{p-2-m}}{s'^{(p-1)} \, s^{p-1}} \frac{\rho_\ell}{s'} . \qquad (11)$$

Because one now has (for integer p) a sum of separable kernels, one can legitimately infer from the above expression that $N \underset{s \to \infty}{\to} O(1/s)$. (One can establish this even if $\bar{B} \sim \exp(-as)$.) One now finds that the integrand of the N equation behaves asymptotically like

$$\zeta \; (s') \times (N \sim 1/s') \times (1/s'^2). \qquad (12)$$

This only gives a convergent integral if $\zeta(s') < O(s'^2)$.

Another example may be enlightening. If B has the form $\sum_{p=2}^{3} Cp/s^p$, then one can indeed choose Cp so that $N \underset{s \to \infty}{\sim} 1/s^2$, but $\bar{B}$ now will change sign at some intermediate energy.

From such crude illustrations it becomes evident that the convergence of the integrals encountered is not merely dependent on asymptotic behaviour of B, but on delicate cancellations; B must be exactly some non-monotonic function or the integrals diverge when $\zeta > s^2$. In this sense, the N/D subtracted equations are "unstable", as has been shown by Gatland et al.(11) who consider a form of the N/D equations formulated in terms of integrals over the left hand cuts. Because of these convergence difficulties, ζ will here be chosen to have the asymptotic behaviour $\zeta \leq s$.

Now let us consider useful approximations. One often finds that the Born approximation to $\bar{B}$ may well be approximated by a single pole form for s > threshold. Suppose that $\bar{B} = A/s-a$ for $s \geqslant s_1$, we may employ the fact of independence of subtraction parameters to choose $s_t = a$. One now obtains, for $\zeta = s-s_1$, (in a P wave problem)

$$N = \frac{A}{s-a} \quad s \geqslant s_1 \tag{13a}$$

$$D = 1 - \frac{s-a}{\pi} A \int_{s_1}^{\infty} \frac{\rho(s'-s_1}{(s'-a)^2(s'-s)} \tag{13b}$$

$$D' = - \frac{A}{\pi} \int_{s_1}^{\infty} \frac{\rho(s'-s_1)}{(s'-s)^2(s'-a)} \tag{13c}$$

$$N_{D=0} = \frac{A^2}{\pi} \int \frac{\rho(s'-s_1)}{(s'-a)^2(s'-s)} = A/s-a \Big)_{D=0} \quad . \tag{13d}$$

One can evaluate these integrals exactly (see Appendix A1), but the results are long, messy expressions. If one is interested in bound states, i.e. $s < s_1$, not near s_1, it is a reasonable approximation to take $\rho = \bar{\rho}$, constant (e.g. $\bar{\rho} \sim \pi/4$ say). The difference between ρ and $\bar{\rho}$ is only significant near threshold, but the factor $s'-s_1$ reduces the contribution of this range to the integral.

With this further approximation one obtains

$$D = 1 - \frac{A}{\pi}\bar{\rho} \left\{ 1 + \frac{s-s_1}{s-a} \ln \left(\frac{s_1-a}{|s-s_1|} \right) \right\} \tag{14a}$$

$$D' = \frac{A}{\pi}\bar{\rho}\frac{1}{s-a} \left\{ 1 - \frac{s_1-a}{s-a} \ln \left(\frac{s_1-a}{|s-s_1|} \right) \right\} . \tag{14b}$$

One can now obtain

$$N/D' \Big]_{D=0} = \frac{(\pi/\bar{\rho})s-s_1}{(s-a)-\frac{(\pi/\bar{\rho})}{A}(s_1-a)} \Bigg]_{D=0} . \tag{15}$$

In the next section we turn to specific details of the PS-V channel.

3. The PS-V channel

In this section we mainly discuss the calculation of a bound state pseudoscalar pole in a PS-V channel. This will be performed using a single pole model approximation to one-particle exchange forces, and will turn out to give a bound state of appropriate mass, but with too large a coupling constant obtained as output.

A brief discussion is given concerning the situation encountered for various pseudoscalar masses. The possibility of a single channel dynamical origin for a recently discovered PS meson at 960 MeV is also considered.

To begin the discussion then, we first note that the relevant PS-V amplitude having $J^{\pi} = 0^{-}$ is the helicity-zero amplitude. We approximate the forces in this amplitude by PS and V exchange (in the U channel) as in Figure 1. We satisfy the requirements of crossing symmetry of the amplitude to the extent that in a self-consistent calculation we demand that the output coupling constant of a bound state should equal the input coupling constant for the same state (contributing to the exchange channel).

It should be noted that if one were considering the scattering of mass-degenerate octets of mesons, one would again be performing a dynamical calculation for a single channel. Thus the present study can be considered to represent either scattering of two specific particles, or of degenerate multiplets of particles.

In the present calculation, we adopt the policy of ignoring possible V-V-V couplings. One motivation for this is the desire for simplicity, and to avoid introducing additional parameters. It has been shown(9) that the V^3 exchange force in the PS-V 0^- channel for SU_3 octets of particles is attractive; this therefore causes a smaller V-PS-PS coupling to be required. Therefore, the difficulty in obtaining $g^2_{out} = g^2_{in}$ would actually be increased by such a force.

Let M, μ be the 1^- and 0^- masses, respectively. One has then the following expression for the 3-momentum (cm):

$$p^2 = (s-(M+\mu)^2)\ (s-(M-\mu)^2)/4s. \qquad (16)$$

The couplings employed are given below, in momentum space notation; the 4 vector notation adopted here is $a \cdot b = a_0 b_0 - \vec{a} \cdot \vec{b}$. At the PS-PS-V vertex, p', p are final, initial PS momenta; ϵ is the vecton polarization (see Fig. 2); one has a factor

$$g\ \epsilon \cdot (p' + p)\ . \qquad (17a)$$

At the V-V-PS vertex, q',q are final, initial V momenta, with respective polarizations ϵ', ϵ, one has a factor

$$\left(\frac{\bar{g}}{\mu}\right)\ \epsilon_{\mu\nu\lambda\sigma}\, q_\mu\, q'_\nu\, \epsilon_\lambda\, \epsilon'_\sigma\ . \qquad (17b)$$

Both coupling constants must be multiplied by appropriate isospin factors. It is useful to note that for helicity zero,

$$\epsilon^\mu_{(0)} \equiv \frac{p_0}{Mp}\ p^\mu - \delta_{\mu 0}\,\frac{M}{p}\ . \qquad (18)$$

Because of the antisymmetry of the V-V-PS coupling, when the external vector helicity is zero, only the $\delta_{\mu,0}$ term contributes at the vertex.

One now obtains the following Born matrix elements[24]:

$$t_{J=0^-}\Big)_{\text{PS exchange}} = C_1 \frac{g^2}{4\pi}\frac{1}{2M^2}\left[A + C \ln\left(1 + \frac{4p^2}{\beta+m^2-M^2}\right)\right] \qquad (19a)$$

$$t_{J=0^-}\Big)_{\text{V exchange}} = C_2 \frac{\bar{g}^2}{4\pi}\frac{M^2}{8\mu^2}\left[\bar{A} + \bar{C} \ln\left(1 + \frac{4p^2}{\beta+M_{ex}^2-M^2}\right)\right] \qquad (19b)$$

where m^2, M_{ex}^2 are the exchanged PS, V masses.

The expressions for A, C, etc., are given below, in terms of p^2, s, so that one can easily obtain the threshold behaviour.

$$\beta = M^2 - (M^2-\mu^2)^2/s$$

$$A = p^2 + \frac{\beta+2\mu^2-M^2m^2}{2} - \frac{M^2}{2p^2}\ (\beta + m^2 - M^2)$$

$$2C = M^2 - \beta + (\beta + m^2 - M^2)\left[M^2 - \frac{\beta+\mu^2}{2} + \frac{1}{4}(\beta + m^2 - M^2\right]/p^2$$
$$+ M^2 (\beta + m^2 - M^2)^2/4p^4 \qquad (20)$$

$$\bar{A} = 1 + \beta/2p^2 + (M_{ex}^2 - M^2)/2p^2$$

$$\bar{C} = -(\beta + M_{ex}^2 - M^2)\left(2 + \frac{\beta+M_{ex}^2-M^2}{2p^2}\right)/4p^2 .$$

If one wishes to have a conserved vector current theory then one must include an extra "seagull" diagram (see Fig. 3) which is done here. This simply increases A by another p^2.

The factors C_1, C_2 are crossing matrix elements, either for SU_2 or SU_3 or whatever group is under consideration. For example, if all states involved are SU_3 octets, then $C_1 = -C_2 = +1/2$. If PS,V states are π and ρ, then it turns out that again

$C_1 = - C_2 = + 1/2$. In these cases, not only V exchange[11], but also PS exchange gives an attractive force. For the purpose of relating the residue of $t_{J=0^-}$ at a pole to the coupling constant, one wants to know the Born amplitude for the S-channel intermediate state. This pole diagram (Fig. 4) contributes an amplitude

$$t_{J=0^-} = - \left(\frac{g^2}{4\pi}\right) \frac{sp^2}{M^2(s-\mu^2)} . \tag{21}$$

When considering degenerate SU_3 octet scattering, one would expect a PS-mass of perhaps 500 MeV compared to a V-mass of about 800 MeV. From the decay rates of the ω and ρ mesons[12] one can estimate $\bar{g}^2$, g^2, and one then finds that PS exchange dominates[13]. Subsequently, only the PS exchange force will be used to determine approximate parameters in the pole model for the exchange force.

Finally, it should be noted that the one-particle exchange terms have left hand cuts lying only on the axis; off axis cuts arise from typical t-channel diagrams, which are neglected here (because V-V-V couplings are omitted for simplicity).

Now we shall investigate some annoying aspects of an unequal mass elastic channel. If one chooses the threshold factor $\zeta = p^2$, then $t = \zeta N/D$ will, in practice, have a spurious pole at $s = 0$, since, in practice, the calculated N/D will not vanish at $s = 0$ without a further subtraction being made in the N dispersion relation. Using Equation (21), one has the condition for self-consistency of the dynamical calculation:

$$\left(\frac{g_{in}^2}{4\pi}\right) = \left(\frac{g_{out}^2}{4\pi}\right) \equiv - \frac{M^2}{s_B}\frac{N}{D'}\Bigg]_{D=0} \tag{22}$$

Since all dispersion relations should presumably be continuous functions of external masses, p^2 would appear a reasonable threshold factor. However, Eq. (22) shows that, in practice, N will not have a zero at s = 0, and so for small s_B, g_{out}^2 will tend to be large, making self-consistency difficult to obtain.

One might consider using $\zeta = s - (M + \mu)^2 \equiv s - s_1$. Now one would have

$$\left(\frac{g^2}{4\pi}\right)_{out} = \frac{-4M^2}{s_B - (M-\mu)^2}\ \frac{N}{D'} ; \tag{23}$$

this has the disadvantage of not having a definite sign as M/μ varies.

Still another compromise, giving an amplitude with appropriate zeros, and no pole at s = 0, would be $\zeta = sp^2$. This would give divergence problems, however, in any practical calculation (see the previous section).

It is thus not evident which choice of ζ offers the least disadvantage. The policy to be adopted in the present approximate considerations is as follows. When μ is appreciably greater than $M-\mu$, p^2 is approximately equal to $(s-s_1)/4$ (e.g. when μ = 500 MeV, M = 800 MeV) at the bound state energy. Therefore, for this range of masses, we choose $\zeta = s-s_1$. It will be found subsequently that the present approximation can give a low bound state mass but too large an output coupling constant: these features also were obtained for a considerable range of masses and for either $\zeta = p^2$ or $s-s_1$ in computer calculations which the author has carried out. In subsequent discussion of numerical detail, we will illustrate these features for the case $\mu^2 = 2$, $M^2 = 2.56$, in units of 500 MeV. This choice corresponds to a heavy PS meson, and is employed here merely to show that the features of the calculation mentioned above are not peculiar to very low mass states such as the pion.

The D integral would converge if one retained the full Born amplitude, but the author feels that the Born amplitude overestimates the force grossly at high energies and therefore will employ a high energy cutoff. Thus the Born amplitude, behaving like s as $s \to \infty$, will be multiplied by a factor $\left[1 + (s - s_1/z)^2\right]^{-1}$. For the above masses, we choose Z to be 40, i.e. $W_z = 4$ BeV. We cannot dispute whether or not this is too high a cutoff without having at our disposal information about high-energy behaviour of amplitudes. However, moving the cutoff closer to threshold is equivalent to moving the pole used to approximate the force towards s = 0 (on the left hand real axis); this is not found to appreciably alter the above-mentioned features of the results. One can also argue that this cutoff is not an unreasonably low energy, as follows. Single particle U-channel exchange involves $-s < U < 0$ as $s \to \infty$.

Now, with a typical Regge trajectory slope, if $\alpha(700 \text{ MeV}) = 0$, then $\alpha(0)$ is appreciably less than zero, so that Regge behaviour modifies the effective spin of the exchanged particles, damping the exchange amplitudes even at fairly low energies..

We are finally ready to complete our calculation; fitting the force roughly by a pole, one finds that a fairly good approximation to $t_{J=0^-}/C_1$ is $(G \equiv g^2/4\pi)$, for $s > s_1$,

$$t_{0^-}/C_1 \backsim \frac{3.1\ G}{s+8} \quad . \tag{24}$$

Using Eqs. (14) and (15), with $C_1 = 1/2$, one finds that requiring the output pole to occur at $s = 2$ leads to the following results:

i) $G_{in} \backsim 7$ compared to 3 from experiment[14],

ii) $G_{out}/G_{in} \backsim 6$.

The second result is not sensitive to the location of the "pseudo-pole" position in expression (24) or to the location of the desired bound state when it is well below threshold.

For very light masses, however, a new feature develops. When $\mu < 1/2\ M$, the vector meson is unstable. If one still wishes to discuss a two particle (PS-V) scattering channel as a reasonable approximation to actual three particle scattering, one encounters singularities in the physical region, due to the contributions of non-virtual PS exchange[15]. Some S-matrix theorists[16] propose that in such a case the object appropriate to describing the scattering is essentially the residue of a three-particle scattering amplitude at the pole in the relevant two-particle variable (or in the second sheet with respect to that variable). As shown by Hwa[17] this amplitude possesses a complex unitarity cut, with a short portion of force cut occurring below the former cut; the separation of these two cuts is related to the unstable particle width. Figure 5a roughly indicates these properties: Figure 5b suggests a crude approximation one might employ for the singularities of Fig. 5a. A further approximation might be to neglect ImN on the unitary cut due to the off axis short cut. This effectively gives the usual N/D equations, except that the singularities of the short cut are damped because of its displacement from the real axis. One might expect such an effect from an appropriate resonance approximation to a three-particle amplitude[18]. At any rate, from the point of view of these suggested approximations, one finds (as is verified by computer calculations) that the same qualitative features of a dynamical calculation are obtained as before.

In addition to peculiarities arising from unstable vectors, one also encounters inelastic behaviour of the amplitude in real life. One can consider the effect of inelasticity on a bound state far from inelastic threshold in the following manner. For example, let the ratio of total to elastic cross-sections for two particle scattering be constant, say = R. This has the chief effect of increasing the strength of the input force and therefore decreases the input G needed for fixed bound state energy. However, G_{out}/G_{in} is not substantially affected. This will be confirmed in a later section. If Regge behaviour <u>drastically</u> reduces the strength of the force, then one would require rather larger G_{in} to bind, but again, one would expect G_{out}/G_{in} to be more or less as before.

In summary then, one finds that the forces available are quite adequate to give rise to low energy bound states, but one cannot achieve consistency in the coupling constant, G_{out}/G_{in} being $\gg 1$. This situation has been encountered recently by other authors seeking bound states in B-$\bar{B}$ scattering(19), and in scalar-scalar scattering. The large output coupling constant obtained seems to be due to several factors:

1. The (field theory) pole amplitude has a residue proportional to $G \times 4sp^2$, so that G_{out} is proportional to $\left[\zeta N/D'\right] /4sp^2$. For equal external masses, $4p^2$ is equal to $s-s_1$. When $\mu \gg (M-\mu)^2$ as in the numerical example cited earlier, $4p^2$ is again approximated well by $s-s_1$. Thus, for when s is less than s threshold, $\zeta\ (4sp^2)^{-1}$ is considerably larger than for a heavy resonance, giving a large G_{out}. This situation also occurs in the V-V and B-$\bar{B}$ 0^- channels (which are approximately "equal mass-scattering" channels).

2. At $D = 0$, the present model gives $N = G/s-a$ (as a consequence of solving the integral equations) which is much larger for bound states than for resonances further from the force "pseudopole". This does not apply to the "N^*" considered as an "$N^*\pi$" bound state because "N^*" is quite close to the "$N^*\pi$" threshold.

In case the reader feels suspicious about the present approximations, several comments are next offered.

It might be noted that application of this pole approximation to elastic π-π scattering gives a self-consistent resonance close to threshold, in agreement with the results of many authors[21], so that the present approximations succeed where other more detailed calculations also succeed. Also an approximate B-$\bar{B}$ calculation carried out was in very close agreement with the findings of the Reggeistic calculation of reference (19).

Of course, one should be aware that if scalar mesons (s) exist, the PS-S channel would be relevant. In this channel the pole residue is not proportional to s, so that some of the difficulties referred to above might not occur.

When considering bound states of fairly high mass (say, near threshold), the present difficulties seem to be much alleviated. We discuss the question of higher mass pseudoscalars in Appendix B1.

Before turning to a multichannel calculation, a section will next be devoted to some other interesting aspects of single channel calculations.

4. Heuristic Arguments

In this section we present a crude non-relativistic argument to indicate that a rather moderate meson-baryon interaction may suffice to bind a low mass 0^- state in a heavy B-$\bar{B}$ channel. We also give a possible mechanism whereby a self-consistent 0^- dynamical generation could occur, namely, the occurrence of a high mass 0^- meson (or a higher mass SU_3 0^- multiplet).

It is amusing to obtain a feeling for the strength of force needed to bind even a B-$\bar{B}$, to give a PS bound state, in terms of some familiar concepts. We note that the Born approximation for the scattering amplitude from a square well of depth V and radius a, for a particle of (effective) mass μ, is

$$f(k) = \frac{2\mu V}{k^3} \left[\sin ka - ka \cos ka \right] , \tag{25}$$

where k is the momentum transfer. Therefore, at threshold,

$$f(0) \equiv f_{s \text{ wave}} \equiv \frac{2}{3} \mu V a^3 . \tag{26}$$

One can obtain the s wave $J = 0^-$ B-$\bar{B}$ scattering amplitude due to vector exchange, with a crossing coefficient of 1/2, at threshold in the Born approximation; it turns out to be

$$\left(\frac{g^2_{VN\bar{N}}}{4\pi} \right) \frac{M_N}{{M_V}^2} . \tag{27}$$

We now adopt the notation of Schiff[21].

Let $\xi = a \sqrt{M_N} \sqrt{V-2M_N}$

$\eta = a \sqrt{M_N} \sqrt{2M_N}$.

We take the binding energy to equal $2M_N$, and the range $= 2/M_N$ (this of course makes the non-relativistic approximation bad). Then the condition that the appropriate bound state occurs is:

$$\cot y = - \frac{2\sqrt{2}}{y} , \tag{28}$$

where $y = 2 \sqrt{\frac{V-2M_N}{M_N}}$.

This has the solution V = 3.25 M, which can be used in Eqs. (26) and (27) to obtain

$$\frac{g^2_{VN\bar{N}}}{4\pi} \approx 5.7 \ . \tag{29}$$

Thus, a not unreasonably large coupling strength gives very strong binding.

One might also question the effect of the existence of another state in the 0^- channel. If it occurred for an energy higher than the low-lying bound state of interest here, then D' has to change sign as s varies from one zero to another of D, and simultaneously, N would have to change sign in order that both states have real couplings. If the zero of N were to occur near the lower eigenenergy, then the output coupling constant could be reasonably small. In practice, simple approximations for a one-channel calculation (as in the next section) do not exhibit such behaviour. One can, however, force the amplitude to possess a second pole by treating the second state as though it were elementary, and thereby including the pole diagram for the second state with the input forces. We therefore want to solve the N/D integral equations for a two pole model of the force, which will now be performed.

Let the pole diagram for the second state (energy $= \sqrt{b}$) be $\beta^{\zeta}/s-b$, β a constant. One then obtains the following results (in the approximation $\rho \sim \bar{\rho}$):

$$\bar{B} = \frac{A}{s-a} \text{ as before } + \frac{\beta}{s-b} \tag{30a}$$

$$N = \frac{n_1}{s-a} + \frac{n_2}{s-b}$$

Subtracting D at a, one finds that $n_1 = A$ and

$$n_2 = \frac{\beta - \frac{A\beta\bar{\rho}}{\pi}\left[1 + \frac{b-s_1}{b-a}\ln\left(\frac{s_1-a}{|s_1-b|}\right)\right]}{1 - \frac{\beta}{\pi}\bar{\rho}\left[1 + \frac{a-s_1}{a-b}\ln\left(\frac{s_1-b}{s_1-a}\right)\right]} \tag{30c}$$

$$D = 1 - \frac{A}{\pi}\bar{\rho}\left[1 + \frac{s-s_1}{s-a}\ln\left(\frac{s_1-a}{s_1-s}\right)\right]$$

$$- \frac{n_2(s-a)}{\pi}\bar{\rho}\left[\frac{s_1-s}{(s-a)(s-b)}\ln(s_1-s) + \right. \tag{31a}$$

$$\left. + \frac{s_1-a}{(s-a)(b-a)}\ln(s_1-a) + \frac{s_1-b}{(s-b)(a-b)}\ln(s_1-b)\right]$$

$$D' = \frac{A\bar{\rho}}{\pi}\frac{1}{s-a}\left[1 - \frac{s_1-a}{s-a}\ln\frac{s_1-a}{s_1-s}\right]$$

$$+ \frac{n_2}{\pi}\bar{\rho}\frac{1}{s-b}\left[1 + \frac{s_1-b}{s-b}\ln\frac{s_1-s}{s_1-b}\right] \tag{31b}$$

$$N_{D=0} = \frac{A^2\bar{\rho}}{\pi(s-a)} \left[1 + \frac{s-s_1}{s-a} \ln\left(\frac{s_1-a}{s_1-s}\right)\right] \left(\frac{-\beta}{n_2}\right) + \frac{A}{s-a} \left(1 + \frac{\beta}{n_2}\right)$$
$$+ \frac{\beta n_2}{\pi} \frac{\bar{\rho}}{s-b} \left[1 + \frac{s-s_1}{s-b} \ln \left(\frac{s_1-b}{s_1-s}\right)\right]. \qquad (31c)$$

Adopting the basic force $(A/s-a)$ to be that used earlier for PS exchange in a PS-V channel, one finds that self consistency will be realizable when the second state is bound near threshold; it need only be bound with a reasonable coupling constant. The coupling constant for the derived state is also reasonably small, and one indeed finds that N has a zero near the derived bound state energy. Under these circumstances, we find that $\bar{B} < 0$ near threshold, but > 0 for higher energies, while $N > 0$ for $s > s_1$ (these observations are based on inserting numbers into Eqs. (30) and (31)). This still gives a strong force but also gives a small N at the bound state because of cancellation in the expression for $N)_{D=0} \equiv 1/\pi \int B\rho N/s'-s \, ds$).

In general, if one desires to have $N_{D=0}$ small, then a sign change of BN at some energy is helpful to achieve this. If B were of constant sign, but N changed sign, then there would be cancellation in the D integral. The effective force would be rather small, so it is preferable that B change sign, as happens in the above situation.

To summarize this two-state discussion, it is found that the existence of a second higher energy state (0^-) could imply a more favorable situation for a self consistent calculation of a low-lying bound state.

5. Two-Channel Calculation

In this section a crude approximation to a dynamical calculation will be performed for coupled PS-V and B-$\bar{B}$ channels. First though, the material of section 2 will be extended to a multi-channel case.

Let t_{fi} be a partial wave amplitude from state i to state f. (states of definite parity, total isospin, strangeness, etc.); part of the present approximation consists of restricting considerations to two-particle states. In the present case, the B-$\bar{B}$ threshold is considerably higher than some three-particle scattering thresholds; the present discussion will only be intended to give some feeling for the contribution of a second, albeit high-mass, channel.

Unitarity imposes the condition

$$\mathrm{Im}\, t_{fi} = \sum_k t^*_{fk}\, t_{ki}\; \theta(s-s_{k\ \text{threshold}})\; \rho_k \tag{32}$$

where

$$\rho_k = \frac{2p_k}{W}.$$

To ensure appropriate threshold behaviour of t_{fi}, we define a threshold factor ζ_{fi} possessing the desired threshold behaviour, and define h_{fi} by

$$t_{fi} = \zeta_{fi} h_{fi}. \tag{33}$$

Because the threshold factor is factorizable, one has

$$\zeta_{fi} = \eta_f\, \eta_i\ ,$$

and therefore, Eq. (32) becomes

$$\text{Im } h_{fi} = \sum_k h^*_{fk} h_{ki} \theta_k \rho_k \zeta_{kk} . \tag{34}$$

One similarly defines $\bar{B}_{ij} = B_{ij}/\zeta_{ij}$, where B_{ij} is a function possessing the exact left hand cuts of t_{ij}. From now on it is more convenient to avoid writing indices by employing matrix notation.

$$\text{Let } \chi \equiv \chi_{ik} = \theta(s-s_{k \text{ threshold}}) \rho_k \zeta_{kk} \delta_{ik} \tag{35a}$$

$$h \equiv h_{ik} = (N/D^{-1})_{ik} \tag{35b}$$

$$\bar{B} \equiv \bar{B}_{ik} . \tag{35c}$$

If one writes a once subtracted (at s_t) dispersion relation for D, one obtains

$$\begin{aligned} N &= \bar{B}\, D + \frac{1}{\pi} \int \frac{B\chi N}{s'-s} \\ &= \bar{B} + \frac{1}{\pi} \int \frac{ds'}{s'-s} \left(\bar{B}(s') - \frac{s-s_t}{s'-s_t} \bar{B}(s) \right) \chi N \end{aligned} \tag{36a}$$

$$D = 1 - \frac{s-s_t}{\pi} \int \frac{\chi N}{(s'-s)(s-s_t)} . \tag{36b}$$

As a model of multichannel effects we consider a two-channel single pole approximation. In order to connect this model of the forces to field theory Born terms, the pole coefficients in the various channels will be chosen to give forces at low energies which are roughly the same strength as the Born forces due to one-particle exchange. To simplify this procedure, let us consider equal mass P and V states; this at least suffices to give a fairly representative illustration of the effect of the second channel. In the one-pole model

$$N = \frac{1}{s+a} \left\{ \begin{array}{cc} & \begin{array}{cc} PS\ 1 & B\bar{B}\ 2 \end{array} \\ \begin{array}{c} PS\ \ 1 \\ \\ \bar{B}B\ \ 2 \end{array} & \begin{pmatrix} C & E \\ & \\ E & A \end{pmatrix} \end{array} \right\} . \tag{37}$$

To ensure proper threshold behaviour, define

$$\zeta_{11} = p_1^{\;2} \; (= \frac{s}{4} - 1) \text{ and } \zeta_{22} = 1 \; , \tag{38}$$

where units are employed such that $M_V^{\;2} = M_{PS}^2 = 1$. This reflects the fact that the 0^- state occurs in the PS-V P-wave amplitude and the B-$\bar{B}$ S-wave amplitude. The approximation will be made that $2q_{PV}/W$ and $2q_{N\bar{N}}/W$ can be approximated by a constant in all integrals, say $\pi/4$. This is a very good approximation for the PV channel (as discussed earlier) but possibly slightly overestimates the B$\bar{B}$ contribution.

One now obtains the equivalent of Eqs. (14) and (15) (subtracting at a)

$$D_{11} = 1 - \frac{C}{16} \left[1 + \frac{s-4}{s+a} \ln \frac{4+a}{s-4} \right] \equiv 1 - \frac{C}{16} \; I_1$$

$$D_{22} = 1 - \frac{A}{4} \; \left[\frac{-1}{4M_N^{\;2}+a} + \frac{1}{s+a} \ln \left(\frac{4M_N^{\;2}+a}{4M_N^{\;2}-s} \right) \right] = 1 - \frac{A}{4} I_2$$

$$D_{12} = - \frac{E}{16} I_1 \quad \text{and} \quad D_{21} = - \frac{E}{4} I_2 \; . \tag{39}$$

Analogously to the single channel case, a bound state here corresponds to a zero of det $(D) = D_{11}D_{22} - D_{12}D_{21}$. At this energy, the residue is given by (in this pole model, $N=\bar{B}$)

$$\text{Residue } h = \frac{1}{\pi}\int \frac{\bar{B}\,\chi\,N}{(s'-s)}\,\frac{\bar{D}}{(\det D)'}\,, \tag{40}$$

where $\bar{D}$ is defined by $D^{-1} = \bar{D}/\det D$. For example, in the one-pole model, at $\det D = 0$, one can show that

$$\text{Residue } h_{11} = \left[CD_{22} - ED_{21}D_{22} + A\,D_{21}^{\;2}\right] / \left[(s+a)(\det D)'\right]. \tag{41}$$

Equations (39), (40) and (41) are the basis of the two-channel calculation to be described. First though one must obtain estimates for the coefficients A,C,E.

A reasonable low energy numerical approximation to the (equal mass) PS-V channel PS exchange (19a) is found to be

$$h_{11} = G_{PPV}\,\gamma\,\frac{15}{s+12}\,, \tag{42}$$

where γ is a group theoretic (e.g. isospin) crossing factor, and $G \equiv g^2/4\pi$.

The forces from single particle exchange in the $B\bar{B}$ channel are, where α', α'' are crossing coefficients,

$$B^{0-}_{\text{V exchange}} = \alpha'\,G_{VBB}\,\frac{s-2M_N^{\;2}}{s-4M_N^{\;2}}\;Q_0\left(1+\frac{2M_V^{\;2}}{s-4M_N^{\;2}}\right) \tag{43a}$$

$$B^{0^-}_{\text{PS exchange}} = \alpha''\,G_{BB\bar{B}}\,\frac{1}{4}\left[1-\frac{2}{s-4M_N^{\;2}}\;Q_0\left(1+\frac{2}{s-4M_N^{\;2}}\right)\right]. \tag{43b}$$

With the strong interactions SU_3 symmetric, one finds $\alpha' > 0$, $\alpha'' > 0$. With typical strength of the forces the net force will be attractive and somewhat stronger than $B_V^{0^-}$. (In the subsequent example, matching A to (43a)), we take $G_{VNN} = 2$, $G_{PNN} \backsim 10$, $M_N = 2$.) The V exchange force looks approximately as in Fig. 6, so that a pole approximation is a poor representation of this force, but probably suffices to represent the strength of the force. In any case, matching a pole to B_V at $s/4M_N^2 = 2$, one obtains

$$A \backsim (9 + 6M_N^2) \ln \left(1 + \frac{4M_N^2}{M_V^2}\right) \alpha G_{VN\bar{N}} , \qquad (44)$$

where α is some effective crossing coefficient.

The Born force h_{12} is due to the diagrams exhibited in Fig. 7, and is given below

$$h_{12} \equiv \frac{t_{12}}{P_{PV}} = \left[\frac{s-2M_V^2}{P_{PV}^2 M_V} + \frac{4s\ P_{PV}^2 - (s-2M_V^2)^2}{4M_V\ P_{PV}^2\ P_{B\bar{B}}} \quad Q_0\left(\frac{s-2M_V^2}{4P_{PV}\ P_{B\bar{B}}}\right)\right] \qquad (45)$$

$$\times\ \frac{\beta}{4} \sqrt{G_{VN\bar{N}}\ G_{PNN}} \ .$$

Again β is a crossing coefficient. Note that $1/P_{B\bar{B}}\ Q_0$ ( ) is a real function even for $s < 4M_N^2$, so that this off-diagonal force possesses no peculiar kinematic cuts as sometimes occurs for off-diagonal amplitudes. One can also verify that the above expression for t_{12} indeed vanishes as P_{PV} at $s = 4$. This force, at low energies, tends to look like a pole force except for a narrow spike at P-V threshold (for t_{12}/P_{PV}, not t_{12}). It is not obvious where one should match pole force and field theory Born term. If the matching is done at $s = 4M_N M_V$ (the geometric mean of the two thresholds), then one finds that E is proportional to $M_N \sqrt{G_{VNN} G_{PNN}}$, which is annoying for $M \to \infty$. This question of behaviour as $M \to \infty$ for a coupled channel calculation in discussed in Appendix A; for now we shall match at $s = 4M_N M_V$.

To obtain a specification via dynamics of all the parameters entering the above expressions for the forces would require simultaneous bootstrapping of several channels (more than 2). This is not the intention here; this study, rather, aims to assess the importance of the $B\bar{B}$ channel, which is now done by fixing all but G_{PPV} at reasonable values" and then considering the result of numerical calculation. Accordingly, let us adopt the following parameters: $G_{VNN} = 2$, $G_{PNN} \backsim 10$, $M_N = 2$, $\alpha = \beta = \gamma = 1/2$ (this is a typical crossing coefficient). Employing Eqs. (39) and (41), one now finds the following results:

i) G_{PPV}, one-channel calculation with equal meson masses, $= 3.5$

G_{PPV}, coupled PS-V, $B\bar{B}$ $= 1.4$

ii) G_{out}/G_{in}, coupled PS-V, $B\bar{B}$ $= 6.4$.

The qualitative feature of the second result is unchanged even under appreciable variation of the strength of the inter-channel coupling (although the magnitude of G_{PPV} naturally varies inversely as the magnitude of $G_{VN\bar{N}}$ when the latter provides attractive forces in the $B\bar{B}$ channel).

The author considers the relevance of the above calculation to be that, in spite of the high threshold mass of the B-$\bar{B}$, the channel is probably essential to a dynamical understanding of (at least) the pseudoscalar mesons.

Conclusion

In the previous sections, crudely approximate dynamical calculations have been performed to obtain a bound pseudoscalar state, in both PS-V and B-$\bar{B}$ channels. The author wishes to emphasize the following inferences to be obtained from these calculations:

1. The plausibility of the idea that even low lying bound states are composite, demonstrated by the fact that only relatively modest forces are required to give rise to bound states in model calculations.

2. There seems to be difficulty in obtaining a dynamical calculation which is self-consistent in the coupling strengths for low lying states.

3. High mass channels, contrary to commonly expressed opinion, are quite likely to be important contributors to the dynamical generation of the observed particle spectrum. The importance of the B-$\bar{B}$ channel in the present calculations is but a (re)-confirmation of the relevance of the Fermi-Yang model for the mesons[23].

Appendix A1. Coupled channels with high threshold mass

This appendix discusses coupled channels in the approximation and notation of Section 5. First, a few comments on the pole model approximation are necessary. It should be realized that the inelastic amplitude PS-V$\leftrightarrow$B-$\bar{B}$ will proceed via fermion exchange and therefore the higher the fermion mass, the further off the mass shell this fermion will be. Consequently, one might expect that there will be appreciable damping of the amplitude below the Born value. Therefore, as we let $M_N \rightarrow \infty$ ($M_V = 1$), we shall consider E to be non-divergent, rather than proportional to some power of M_N as results from matching the Born term and a pole amplitude.

We now consider the exact pole model expression for

$$D_{21} = -\frac{s+a}{\pi} \; E \int_b^{\infty} \frac{2q_{BB}}{W} \frac{ds'}{(s'-s)(s'+a)^2}$$

$$= -\frac{s+a}{\pi} \; E \int_b^{\infty} ds' \, \frac{(s'-b)}{X} \; \frac{1}{(s'-s)(s'+a)^2} \qquad \text{(A1.1)}$$

where

$$X = \left[s'(s'-b) \right]^{1/2} .$$

This, of course, is for the case where channel 2 is an S-wave channel for which one customarily takes $\zeta = 1$.

This can be integrated exactly, giving the expression below:

$$D_{21}^{(s)} = -\frac{E}{\pi}\left[-\sqrt{\frac{b-s}{s}}\,\frac{1}{s+a}\left\{\frac{\pi}{2} - \sin^{-1}\left(1 - \frac{2s}{b}\right)\right\}\right.$$
$$+\sqrt{\frac{a+b}{a}}\left\{\frac{1}{s+a} + \frac{b}{2a(a+b)}\right\}\ln\left(1 + \frac{2a}{b} + \frac{2\sqrt{a}\,\sqrt{a+b}}{b}\right) \qquad \text{(A1.2)}$$
$$\left. - \frac{1}{a}\right] .$$

As $b \to \infty$, this expression approaches

$$D_{21} = -\frac{E}{\pi}\left(0\left(\frac{1}{b}\right)^{2}\right) \sim E \times 0\left(\frac{1}{M_N^{\;2}}\right)^{2}. \qquad \text{(A1.3)}$$

Thus with E less divergent than M_N^2, one sees that $D_{21} \times D_{12} \to 0$ as $M_N \to \infty$ and so the high mass S-wave channel does indeed decouple from a low threshold mass channel. It isn't necessary to employ the exact pole model expressions here because A2) vanishes the same (for $M_N \to \infty$) as the approximate expression in Eq. (39).

If a P-wave channel of high mass were coupled to the first P-wave channel, then one would be dealing with the following integral in D_{21} (taking ζ = s-b for a threshold P-wave factor):

$$I = (s+a)\int_b^\infty \frac{(s'-b)^2}{\sqrt{s'(s'-b)}}\;\frac{1}{(s'-s)(s'+a)^2} \qquad \text{(A1.4)}$$

$$= \frac{(s-b)^2}{(s+a)\sqrt{s(b-s)}}\left[\frac{\pi}{2} - \sin^{-1}\left(1 - \frac{2s}{b}\right)\right] + \frac{a+b}{a}$$
$$\qquad \text{(A1.5)}$$
$$+\sqrt{\frac{a+b}{a}}\left[\frac{2s+a-b}{s+a} - \frac{b+2a}{2a}\right]\ln\left(1 + \frac{2a}{b} + \frac{2\sqrt{a(a+b)}}{b}\right).$$

As $b \to \infty$ one easily obtains from (A1.4) the result that I is of the order $1/b$. The channel coupling term $D_{12}\, D_{21}$ is proportional to E^2/b in our pole model. Consequently, if $E(b)$ increases slower than $\sqrt{b}$, we still have channel decoupling, i.e., $D_{12}\, D_{21} \to D$. Note, however, that with our monotonic force and the usual threshold factor, the high mass P-wave channel decouples less rapidly than a high mass S-wave channel.

Appendix B1

In this appendix we discuss the possibility of a heavy ($\geq$ 1 BeV) PS meson arising in a single channel calculation.

In the present approximation it appears that one could obtain a bound PS state near threshold with moderate input forces and with a small output coupling constant (which is due to larger D' and S_B) if the relevant forces were attractive. In fact, it has recently been established that another PS meson exists at 960 MeV, and with quantum numbers $I\ (J^{\pi G})\ C = 0\ (0^{-+}) +$ ( ) . It is therefore interesting to consider how this state might arise dynamically in a degenerate SU_3 symmetric situation.

A common assumption currently is that the X is an SU_3 singlet. In this case, the requirement of charge conjugation invariance forbids it to couple to the SU_3 channels P_8-V_8 or P_1-V_1. (Recall that P_8-V_8-P_1 coupling would necessarily include an interaction of the form $\rho^0\pi^0$ X which violates C invariance.)

Next, let us consider the assumption that this state (call it the X) is a member of an SU_3 octet P_8'. It has the same quantum numbers as the $\eta(550)$, so P_8' has all the same quantum numbers as P_8 (the original PS_8). One therefore expects to find it as a bound state in the P_8-V_8 system, which has considerably lower threshold then the P_8'-V_8 channels. Indeed, from the above discussion, no particular difficulty in achieving a bound state with small (output) coupling is anticipated. Admittedly, this qualitative discussion does not consider whether self consistency for both octets could be achieved. However, one cannot pass judgement yet on this question since one certainly does not see how to obtain two bound states in one channel with any hitherto examined model of the relevant forces, (it is quite possible that when considering a more realistic multi-channel situation one might obtain 2 bound states, but no evidence for this is available). This difficulty is to be expected whenever one adopts a model with a monotonic force, which is unlikely to give several zeros in D.

Another possibility is that a unitary singlet X could occur in the V_8-V_8 channel. However, the V-V channel Born terms from PS exchange (ignoring possible V-V-V couplings), are much too small to produce a singlet bound state (if one employs a "reasonable" cutoff in the calculation) with a reasonable input coupling constant.

It is of incidental interest to consider the C = - unitary singlet PS channel. One finds that the octet exchange forces in the P_8-V_8 channel are repulsive (and the singlet exchange force is very small because of a small SU_3 crossing coefficient (1/8)), thus no (bound) C = - singlet is expected unless the higher threshold channel consisting of V_1-P_1 (C = +) can bootstrap a P_1(C = -). One would thus have to consider the possibility of P_1(C = -) and P_1(C = +) being bootstrapped in the V_1(-) - P_1(+) and V_1(-) - P_1(-) channels.

Finally, returning to the dynamics of the P_1(C=+) one finds that it can indeed occur in a Baryon-antibaryon channel. Even in the present approximation the output coupling constant would be reasonable for $M \sim 1$ BeV. Again, since the forces would come mainly from the P_8, V_8 exchanges, one cannot make any assertions about self consistency from only single channel considerations.

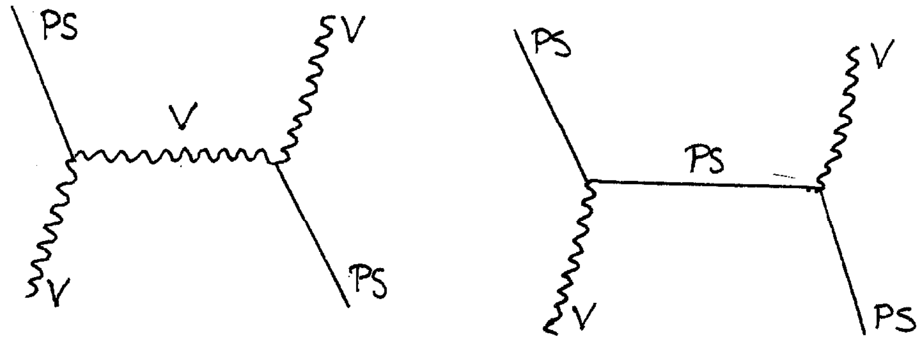


Figure 1. Born Exchange Diagrams

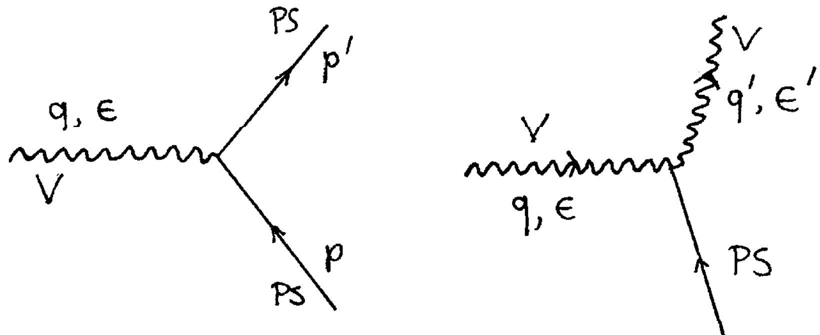


Figure 2. Vertex Kinematics

Figure 3.
"Seagull Diagram"

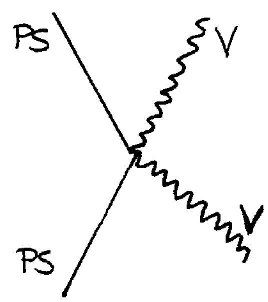
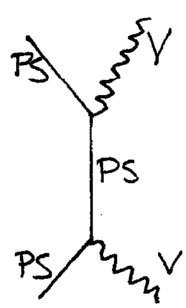


Figure 4.
"Pole" Diagram



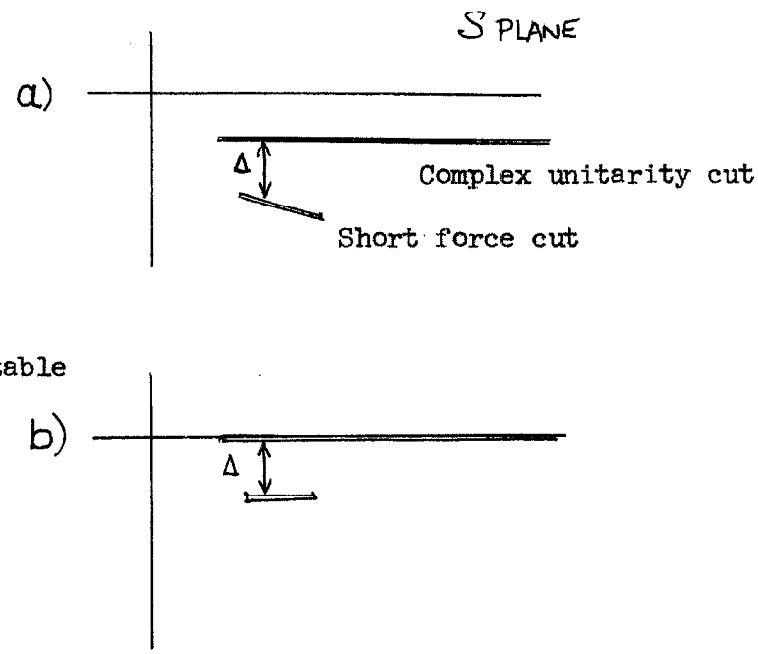


Fig. 5a) and b) .

Cuts of t for unstable vecton

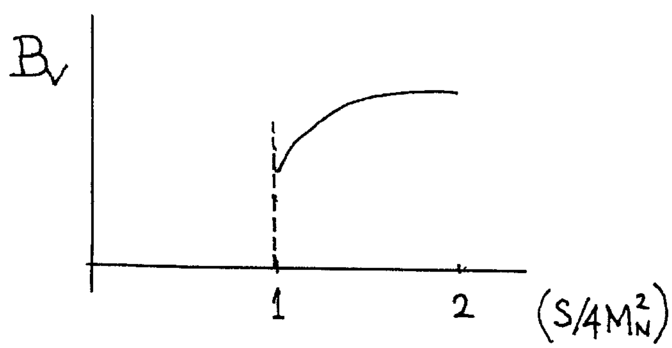


Fig. 6

B-B̄ force V exchange

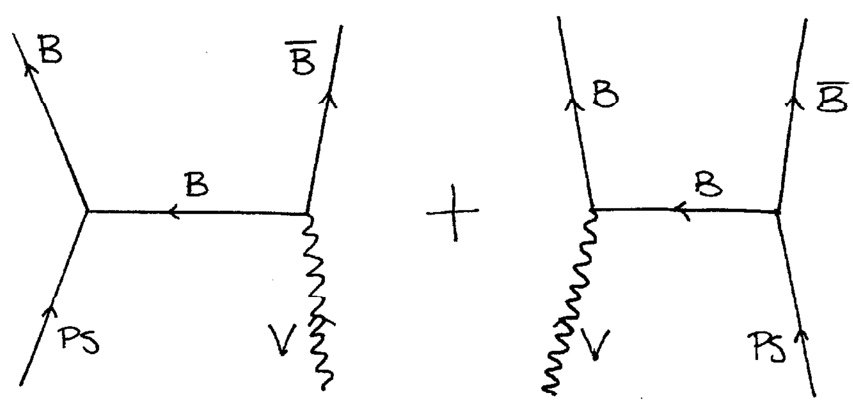


Fig. 7

B-B̄ ↔ PS-V
Born exchange diagrams

Part II. On Meson Electromagnetic Mass Splitting Calculations

Introduction

In this study we investigate the calculation of electromagnetic mass differences of mesons. The mesons are to be considered as composite states in the sense that these particle states are in a one to one correspondence with S matrix poles having a dynamical origin. Electromagnetic effects are thus to be considered as perturbations upon a system of strongly interacting particles, and will be studied by examining perturbations of a strong interaction dynamical calculation.

In the present approach we want to study two effects causing mass splittings among the members of a strong interaction isospin multiplet of particles. The first effect is that of the electromagnetic mass splittings of the particle multiplets which participate as external and exchanged particles in a scattering channel. The second effect is that of exchanged photons modifying the basic strong interaction forces, and is understood physically from the following example. Consider a neutral composite particle occurring mainly in a single channel comprising one positive and one negative strongly interacting particle (SIP). Photon exchange is here expected to provide an extra attractive force supplementing the strong interactions.

Since the present investigation requires a strong interaction dynamical calculation to perturb, it is appropriate to comment on the strong interaction calculations which have appeared to date in the literature. Many authors[1] have performed calculations in the channels possessing the quantum numbers of the known vector mesons. Approximate methods of "dispersion" were generally employed. The binding forces were either represented in terms of single particle exchange forces due to vector exchange, or a model was constructed without explicit vector exchange but utilizing crossing symmetry of the scattering amplitudes. Because the latter case is difficult to perturb in an algebraically simple manner, the one particle exchange model of two-particle scattering will be adopted here.

The calculations found in the literature demonstrated the existence of self-consistent bootstrapped vector resonant states, although multichannel calculations were often essential. However, it is not clear that a self-consistent vector bootstrap is achieved when one employs more realistic forces and the full integral equations determining a partial wave amplitude[2]. Rather, in Part I of this work it was seen that no simple approximation is likely to lead to a self-consistent bootstrap of the pseudoscalar mesons[3,4]. The policy adopted here is that the vector mesons presently come closest to having been dynamically generated in a consistent calculation. Consequently, this study will be restricted to a specific example of the electromagnetic mass splittings of a vector multiplet.

A brief outline of the succeeding sections is now presented. In Section 1 and 2 we will discuss, in general, techniques for studying perturbations of a dynamical calculation. In Sections 3 and 4 we specialize these techniques to a "pole-model" approximation. In Section 5 we then numerically apply all derived methods to the example of the ρ meson. In Sections 7, 8, and 9 we consider in general and specifically for the ρ case, the effects of inelasticity. In Section 10 we consider the ρ and K^* in an SU_3 degenerate model.

The results are summarized briefly by two statements for the specific example of the ρ in a π-π channel.

1. The ρ mass splittings are primarily determined by the mass splittings of the participating particles in the π-π channel. The effects of exchanged photons appear to be less important. One estimates from a one-channel calculation that $M_{\rho^0} - M_{\rho^+} \gtrsim$ 11 MeV.

2. The above mentioned results are not qualitatively changed when one includes inelastic effects. In fact, the mass splitting probably becomes even larger.

1. General Considerations on Mass Shifts

In general, a composite model for a particle is characterized by a function whose argument depends on energy and on the various masses and couplings of those particles involved in the forces contributing to a given channel. Some condition on this function specifies the location of the desired composite state; the energy satisfying this appropriate condition is thus dependent on various masses, etc., of relevant particles. The first decision one faces is that of choosing the function appropriately, from the point of view of calculational simplicity.

In several recent papers[5] an S matrix dispersion theory method for calculating mass shifts within particle multiplets has been proposed. This method involves dispersion relations for the change in a scattering amplitude due to a perturbing force. The relevant equations in turn involve integrals whose integrand depends on the square of the conventional D function of scattering theory (for a specified angular momentum, parity, etc.). When it was applied to baryon meson interactions with a static model approximation to the exchange forces, this method has had considerable success[5]. In such a case, however, the nature of the approximation made it possible to obtain simple D functions which did not greatly complicate the necessary integrations. One reason for this was that in such static model calculations one only needed knowledge of D over a relatively small region of the real axis, and therefore, could employ linear approximations to D.

In meson-meson channels, the kinetic energies of the mesons are much greater than in the static model meson baryon case, i.e. the kinematics is fully relativistic. Consequently, a somewhat different approach would be desirable. We now proceed to outline such an approach. Let us first consider a one-channel calculation of the amplitudes for elastic scattering of particles A,B (in appropriate charge states) with the forces dominated by exchange of particle C, and with electromagnetic "driving forces" due to γ exchange, etc. We let the scattering amplitude in a channel of given J, and parity, with appropriate threshold factor divided out, be written in the form $\bar{t} = N/D$ where

$$D = D(S, M_A^2, M_B^2, M_C^2, "\gamma", G_j) . \qquad (1)$$

Here, S is the square of the centre of mass total energy and "γ" symbolically represents the γ exchange forces, etc. G_j is the square of the conventional strong coupling constant divided by 4π. For simplicity let us assume that the desired composite state is C itself (otherwise one need only sum over various exchange forces). For different charge states of the composite state C, D will depend on different charge states of the external and exchanged masses. The condition that a composite state exists as a consequence of the forces present is that

$$D(s = M_C^2, \ldots\ldots) = 0 \qquad (2)$$

(for a resonant state we require that Re $D(s = M_C^2, \ldots) = 0$)

One also requires that the amplitude satisfy crossing symmetry or, at least, that the output residue at the composite state pole (or at Re D = 0) should be equal to the input coupling constant, to within kinematic factors.

This gives an equation of the form

$$G_{input} = \left(\frac{N}{D'}\right) \frac{1}{K} \quad , \qquad (3)$$

where $G = g^2/4\pi$, g a dimensionless coupling constant, $D' = \partial D/\partial s$ and K is a kinematical factor.

Now, suppose that a particle labelled by i belongs to an isospin multiplet. For simplicity, let us here assume that the mass splittings within the multiplet can be characterized by only one number, which we shall call $\delta\, m_i^2$. This will be the case for an $I = 1$ meson multiplet, for instance, where charge conjugation invariance requires that the two charged states have equal mass. In a one-particle exchange model of a force, there are four external particles and one exchanged particle labelled by $i = 1 \ldots 5$. The i^{th} type of particle participating in the scattering in a channel called C here, will be in a charge state appropriate to the over-all charge state of C. The masses of the i^{th} particles participating in scattering for two different C charge states may differ; the difference can be written as $C_i\ \delta m_i$: C_i will be either 0 or ± 1 (depending on whether the same, or differently charged i states contribute).

We require condition (2) for two different charge states of C; this gives the equation

$$\sum_{i=1}^{4} C_i\ \delta m_i^2 \frac{\partial D}{\partial m_i^2} + C_{ex}\ \delta m_c^2 \frac{\partial D}{\partial m^2_{exchange \equiv C}}$$
$$+ \sum_i \frac{\partial D}{\partial G_i}\, \delta G_i + \frac{\partial D}{\partial S}\, C_s\ \delta m_c^2 + \frac{\partial D}{\partial \gamma}\, \text{"}\delta\gamma\text{"} = 0\ . \tag{4}$$

In addition to the above conditions, one can simplify calculations by explicitly requiring that the bootstrap equations be scale invariant. Scale invariance is the statement that if all input $(\text{masses})^2$ are increased by a common factor β and all G's are unchanged, then a bootstrap solution originally giving mass m_c becomes a consistent solution for $(\text{mass})^2 = \beta\, m_c^{\;2}$ with the same G. This yields the equation, in the case of only C exchanged,

$$\sum_{i=1}^{4} m_i^{\;2} \frac{\partial D}{\partial m_i^{\;2}} + m_c^{\;2} \left(\frac{\partial D}{\partial m_{ex}^2} + \frac{\partial D}{\partial s} \right) = 0 \;. \qquad (5)$$

Similarly, if in Eq. (3), we write $h = N/D'K = G_{in}$, then one obtains two equations for the changes in h analogous to Eqs. (4) and (5): namely

$$\delta G = \sum_i C_i \;\delta m_i^{\;2} \frac{\partial h}{\partial m_i^{\;2}} + \left(C_{ex} \frac{\partial h}{\partial m_c^{\;2}} + C_s \frac{\partial h}{\partial s} \right) \delta m_c^{\;2} \qquad (6)$$

$$+ \frac{\partial h}{\partial \gamma} \;\text{"}\delta\gamma\text{"} + \delta G_i \text{ terms}$$

and

$$0 = \sum_i m_i^{\;2} \frac{\partial h}{\partial m_i^{\;2}} + m_c^{\;2} \left(\frac{\partial h}{\partial m_c^{\;2}} + \frac{\partial h}{\partial s} \right). \qquad (7)$$

Equations (4) - (7) are the analogues of the equations of reference 5.

In a multichannel problem, one writes $\underset{\sim}{\bar{t}} = \underset{\sim}{N}\, \underset{\sim}{D}^{-1}$ where "$\sim$" denotes a matrix. The occurrence of a composite state is related to $|\underset{\sim}{D}| = 0$ (determinant $\underset{\sim}{D} = 0$).

One now has equations exactly like (4) and (5) with D replaced by $|\underset{\sim}{D}|$ and with more indices required to describe the additional channels.

If we write $\underset{\sim}{D}^{-1} = \bar{\underset{\sim}{D}}/|\underset{\sim}{D}|$

then

$$\bar{\underset{\sim}{t}} = \underset{\sim}{N}\,\bar{\underset{\sim}{D}}/|\underset{\sim}{D}| \, . \tag{8a}$$

The analogue of equation (2) is

$$K_{ij}\,G_{ij} \equiv K_{ij}\sqrt{G_i G_j} = \frac{(\underset{\sim}{N}\,\bar{\underset{\sim}{D}})_{ij}}{\mathrm{Tr}\left(\frac{\partial \underset{\sim}{D}}{\partial s}\,\bar{\underset{\sim}{D}}\right)} \quad . \tag{8b}$$

The analogue of equation (6) is (α now labels particles external and exchanged); no ij summation is implied

$$\begin{aligned} + \frac{\delta G_{ij}}{G_{ij}} \equiv & \sum_\alpha \left(\frac{\partial \underset{\sim}{N}}{\partial m_\alpha^2}\,\bar{\underset{\sim}{D}} + \underset{\sim}{N}\,\frac{\partial \bar{\underset{\sim}{D}}}{\partial m_\alpha^2} \right)_{ij} \left[\frac{C_\alpha^{ij} \cdot \delta m_\alpha^2}{(\underset{\sim}{N}\,\bar{\underset{\sim}{D}})_{ij}} + \delta G_{ij}\ \text{terms} \right] \\ & - \frac{\mathrm{Tr}\left\{ \sum_\alpha \left(\frac{\partial^2}{\partial m_\alpha^2 \partial s}\,\underset{\sim}{D} \right) \bar{\underset{\sim}{D}} + \frac{\partial \underset{\sim}{D}}{\partial s}\,\frac{\partial \bar{\underset{\sim}{D}}}{\partial m_\alpha^2} \right\}_{ij} C_\alpha^{ij}\,\delta m^2{}_\alpha}{\mathrm{Tr}\left(\frac{\partial \underset{\sim}{D}}{\partial s}\,\bar{\underset{\sim}{D}} \right)} \\ & - \sum_\alpha \frac{\partial K_{ij}}{\partial m_\alpha^2}\,\frac{C_\alpha^{ij}\,\delta m_\alpha^2}{K_{ij}} \quad . \end{aligned} \tag{9}$$

As in reference (5) one can employ not only scale invariance, but group theory to simplify the complexity of the coefficients of the δm_α^2 in these equations.

2. Photon Driving Forces

In order to illustrate the photon forces which can arise, we employ a specific example here. The π-π channel with ρ exchange is therefore illustrated in Fig. 1, which presents the various perturbation theory photon corrections which occur in the different possible total charge states of the isospin one channel.

If one adopts the input force for a dynamical calculation to be one photon exchange (Fig. 1a), and if one solves the usual N/D equations for the J=1, negative parity, J=1 partial wave amplitude, then one's final amplitude should include the effects of γ-ρ "ladders"(7). Figure 1g is such a γ-ρ "ladder" in the U channel. This type of diagram, however, (with charged vecton exchange) will have a non-renormalizable ultraviolet divergence.

In Dashen's(8) calculation of the n-p mass difference, static model kinematics and the re-normalizability of the U channel γ-N exchanges permitted him to argue that such exchanges were unimportant. These situations do not pertain here, though, and the author cannot present comparable arguments. It should be pointed out, however, that the t-channel γ-ρ exchanges will turn out to be small, so one is hopefully not omitting any large effects if one omits U channel γ-ρ exchanges. Consequently, motivated mainly by the desire for a simple model of photon exchange forces, the author <u>will</u> omit U channel exchanges. (This is stated in the language of the Mandelstam representation as omitting the 3rd or u-t double spectral function.)

Photons modifying a vertex (Figs. 1e and 1f) can be considered to modify the effective coupling constant of the strong $\rho\pi\pi$ interaction (although the modification constitutes a somewhat energy-dependent form factor), these modifications will therefore be considered to be included in the coupling constant perturbations δG. The photon intermediate state (Fig. h) is unlikely to be a dynamically bound π-π state, and therefore the photon pole Born term (Fig. 1h) must be added separately to the input Born perturbing terms. However, one can check that its effect is small compared to photon exchange effects.

Because the photon has zero mass, there will also be problems arising from infrared divergences. In particular, a one-photon exchange amplitude will be proportional (in the ℓ channel) to $Q_\ell\,(1 + \lambda/2q^2)$ which diverges as the fictitious photon mass λ goes to zero[8]. Unless one accounts for brehmsstrahlung properly, the S matrix acquires an infinite phase proportional to the above factor on the unitary cut (extending from $4m_\pi^2$ to ∞). The D function consequently also acquires an infinite phase. If one calculates N and D with the above one γ exchange amplitude as input force, one obtains finite terms (as λ=0) + terms proportional to $Q_\ell\,(1 + \lambda/2q^2)$. All such divergent terms can be interpreted as coming from an expansion of the divergent S matrix phase factor and can then be discarded. To be more precise, in this calculation, whenever a term diverges like $\ln \lambda$, the part of the expression diverging as $Q_1\,(1 + \lambda/2q^2)$ was dropped. This is equivalent to dropping the part of any expression diverging like $\ln\,(\sqrt{\lambda e}/2q)$, $e = 2.718\ldots$, which is the same prescription as adopted in reference (8). This infinite phase factor occurring in the S matrix does not affect physical scattering predictions and in that sense is of no physical significance except that it is the momentum space equivalent of the infinite phase shift proportional to ln (2kr) appearing in outgoing Coulomb scattered waves. To see that the infinite phase will diverge as $Q_1(1 + \lambda/2q^2)$ we calculate the divergent Born approximation to the Coulomb partial wave phase shift. For P-wave scattering, we observe that the relevant scattering amplitude from photon exchange is proportional to $(t - \lambda)^{-1}$. This corresponds to a co-ordinate space potential $V(r) \sim e^{-\sqrt{\lambda}\, r}/r$. Now we have (in NR theory)

$$\delta_\ell \text{ proportional to } \int j_\ell^2\,(qr)V(r)\, r^2\, dr \tag{10a}$$

(e.g. see reference 21 of part I,this thesis)

For P waves, this gives δ_ℓ proportional to

$$\int_0^\infty J_{3/2}^2\,(qr)\; e^{-\sqrt{\lambda}\, r}\, dr \ , \tag{10b}$$

which in turn is proportional to[14)]

$$Q_1\,\left(1 + \frac{\lambda}{2q^2}\right) . \tag{10c}$$

The lowest order estimate of the contribution to the non-Coulomb divergent part of D from one γ exchange would be obtained from a determinantal approximation $\delta_\ell N = B_\gamma$. If we employ subscript f to denote the "finite part" of a Coulomb divergent quantity, we have for a once subtracted D function

$$\delta_{\ell} D)_f = - \left[\left(\frac{s-s_0}{\pi} \right) P \int_{s_1}^{\infty} \frac{ds' \ \rho(s') \ B_\gamma^{\ell}(s')}{(s'-s)(s'-s_0)} \right]_f \tag{11a}$$

ρ is a phase space factor.

In this "determinantal" approximation one also has in Eq. (6)

$$\frac{\partial h}{\partial \gamma} \quad "\delta\gamma" = \frac{\delta_{\ell} G}{G} = \left[\frac{\delta_{\ell} N)_f}{N} - \frac{\delta_{\ell} D')_f}{D'} \right] \tag{11b}$$

Here however, $N = B_\gamma^{\ell}$ and so $(\delta_\ell N)_f = 0$. If one solved the full integral equations for N and D, $\delta_\ell N_f$ would no longer be zero. We shall return to this point later.

3. Pole Model P Wave Single Channel Bootstrap

The next problem is to obtain reasonably simple approximations for N and D which in some way exhibit a dependence of external and exchanged masses. A reasonable model consists of approximating the forces involved by poles. We first must define precisely what amplitude we wish to calculate. In the present case of equal external masses (to order ℓ^2) let us write a dispersion relation for the amplitude $\bar{t}$ given by

$$\bar{t} = \frac{w}{2q} \frac{1}{4q^2} e^{i\delta} \sin \delta \ , \tag{12}$$

(Note that $4q^2 = s - 4\mu^2$ to order ℓ^2).

We write $\bar{t} = N/D$, with N and D, respectively, possessing the left hand and right hand cuts of $\bar{t}$. Henceforth let threshold be denoted by s_1 ($\approx 4\mu^2$) and let $\rho = 2q/\sqrt{s}$.

We shall now adopt as input force the ρ exchange Born diagram B. The Born amplitude, near threshold, is approximately constant after removal of a "threshold factor" p^2; since it needs to be "cut off" to have the correct high-energy behaviour, a reasonable approximation to $B/4p^2$ should be a pole term $\mathcal{Q}G/s-a$, with "a" quite far out on the negative real s axis, i.e., this amplitude corresponds to

$$B = \frac{4p^2\mathcal{Q}G}{s-a} , \qquad (12)$$

which no longer has the unphysical logarithm divergence, at high energies, of the perturbation theory Born term.

In solving the full integral equations for N and D we note that the solution is independent of subtraction point, and so subtract at s = a. This then gives the solutions below, equations (13) to (15), which are also set forth in the first part of this thesis,

$$N = \frac{\mathcal{Q}G}{s-a} , \; s > s_1 \qquad (13a)$$

$$D = 1 - \frac{s-a}{\pi}\mathcal{Q}G \; P\int_{s_1}^{\infty} \frac{\rho(s'-s_1)}{(s'-a)^2(s'-s)} \quad . \qquad (13b)$$

It is a very good approximation to take $\rho = 1$ here. When ρ ($= 2q/\sqrt{s}$) tends towards to zero as $\sqrt{s'-s_1}$, the integrand is already vanishing as $(s'-s_1)$, so little error is introduced. It is easy enough to explicitly evaluate the above integral, but the expressions obtained are much simpler when one makes the approximation $\rho = 1$.

One then obtains, from (13b),

$$\text{Re } D = 1 - \frac{\alpha G}{\pi}\left\{1 + \frac{s-s_1}{s-a} \ln\left(\frac{s_1-a}{s-s_1}\right)\right\}. \tag{14}$$

At Re D = 0, one also has

$$\frac{N}{D'} = \frac{\pi(s-s_1)}{(s-a) - \frac{\pi}{\alpha G}(s_1-a)} \Rightarrow \frac{1}{X} G_{out} \tag{15}$$

In this approximation, one has ($\bar{B} \equiv B/4p^2$),

$$\delta N = \delta\bar{B} + \frac{1}{\pi}\int\left[\delta\bar{B}(s') - \frac{s-a}{s'-a}\,\delta\bar{B}(s)\right]\frac{N(s')}{s'-s}\ . \tag{16}$$

If one adopts a determinantal approach for calculating photon driving forces (i.e. if one takes N = Born term B) then Eq. (10) describes the change in D. If, however, one wishes to solve the integral equation for N, one finds that for a pole model strong interaction force, one obtains ($A \equiv \alpha G$),

$$\delta_e N)_f = -\frac{A}{s-a} \times \left[\delta D)_f \text{ of equation (10)}\right]\ . \tag{17}$$

Unfortunately, to obtain $\delta D)_f$ one now has to calculate the "finite part" of a dispersion integral whose integrand includes $\delta_e N \equiv \delta_e N)_f + \delta_e N)_{divergent}$. In order to select the "finite part" of these integrals one must be able to evaluate the integrals analytically which is not feasible except in the determinantal approximation. Therefore, only determinantal calculations are carried out numerically here. The errors involved in this approximation are discussed later.

In Eqs. (8) one considers the parameters of $D(s_1, \mathcal{a}, a)$ as functions of all the masses involved. One can estimate the quantities $\partial\mathcal{a}/\partial M_{ex}^2$, $\partial a/\partial M_{ex}^2$ by matching the mass derivatives of Eq. (12a) with those of the actual Born term, at two energies. Of course, the mass derivatives of the actual forces vary with energy, but one may hope that if the important contribution to the dispersion integrals comes from a finite energy range, then the approximation of constant mass derivatives of the pole model parameters should not give misleading results. The only alternative to a simple pole model is a numerical integration procedure for integrals of the mass derivatives of Born "force" amplitudes. Since the Born amplitudes need to be damped at high energies to be realistic, one is then faced with the problem of mass dependent parametrization of the damping factor. That procedure is even less appealing than a pole model approximation to this author. We now have decided to adopt a pole model with constant mass derivatives of the pole model parameters. Based on this idea we proceed to utilize equations (14) and (15), the bootstrap equations.

If one writes

$$\frac{\partial}{\partial M_{ex}^2} \begin{pmatrix} \mathcal{a} \\ a \end{pmatrix} \equiv \begin{pmatrix} \mathcal{a}_M \\ a_M \end{pmatrix} ,$$

then from all the above equations one can establish the following, after some tedious algebra (Appendix B),

$$\frac{\delta G}{G} + \delta_e D_{\rho 0} = \left(\delta M^2 + \frac{M^2}{2\mu^2} \delta\mu^2 \right) x \tag{18a}$$

$$\left[\left(1 - \frac{a_G}{\pi} \frac{M^2-a}{s_1-a}\right) \left(- \frac{a_M}{M^2-a} - \frac{s_1-a}{(M^2-a)(M^2-s_1)}\right) - \frac{a_M}{a} \right]$$

$$\frac{\delta G}{G} - \frac{X_\pi (s_1-M^2)}{G(M^2-a)} \left[\frac{\delta_e D'_{\rho 0}}{D'_{\rho 0}} - \frac{\delta_e N_{\rho 0}}{N} \right]$$

$$= \left(\delta M^2 + \frac{M^2}{2\mu^2} \delta\mu^2 \right) \quad x \tag{18b}$$

$$\left[\frac{1}{G(M^2-a)} \left[(G-\pi a) a_M - (X_\pi + G) \right] - \frac{\pi(s_1-a)}{a_G (M^2-a)} \frac{a_M}{a} \right]$$

Equations (18a) and (18b) comes respectively from (8a) and (8b).

4. The Born Amplitude and Photon Force

In the previous sections we presented general discussion and set up the general formalism necessary for a calculation of electromagnetic mass differences. We proceed in this section with calculation of the Born amplitude and the δD due to photon exchange in the strong interaction pole model. Numerical calculation is carried out in the next section. Our first task is to write down the Born amplitudes for the exchange forces in the $\ell = 1$ channel. As we shall later perform numerical calculation only for the I = 1 π-π channel with ρ (and γ) exchange forces, the ensuing Born expressions are for this case. Taking account of appropriate boson symmetrization of states, one obtains for the $\pi\pi$ channel, ρ exchange force,

$$\bar{t}_{\rho\ \text{exchange}} = \frac{G_{\rho\pi\pi}}{32q^2} \left(4 + \frac{s+M^2}{q^2}\right) Q_1 \left(1 + \frac{M^2}{2q^2}\right). \qquad (19)$$

Here $G_{\rho\pi\pi} \equiv g^2{}_{\rho\pi\pi}/4\pi = 4(\gamma^2{}_{\rho\pi\pi}/4\pi$ of reference 1a).

q = cm 3-momentum

Q_1 is the Legendre function of the second kind of order one.

The pole diagram (ρ intermediate state in the S channel) is

$$\frac{-G_{\rho\pi\pi}}{12(s-M_\rho^2)}\ . \qquad (20)$$

The γ exchange contribution may be obtained easily from Eq. (19). For the ρ^0 channel, the photon exchange diagram contributes just the expression of Eq. (18) with M^2 replaced by λ, the square of the fictitious photon mass, and with $g^2{}_{\rho\pi\pi}/4\pi$ replaced by $e^2/4\pi = 1/137$. The contribution of the photon exchange force to δD, is most easily calculated from the total Born term for photon exchange, rather than from a partial wave projection of the Born photon term. The procedure employed follows:

We have in the ρ^0 channel (with phase space factor = 1)

$$B_\gamma = \left(\frac{e^2}{16\pi} \left[-1 + \frac{2s+\lambda^2-4\mu^2}{-(t-\lambda)} \right] + \text{U channel contribution} \right) \times 1/2 \quad . \qquad (21)$$

In the determinantal approximation this contributes to $D_{\rho^0}^{\ell=1}$

$$= -1/2 \int_{-1}^{1} P_\ell(x) \frac{e^2}{16\pi} \left(\frac{s-s_0}{\pi}\right) \int_{s_1}^{\infty} \frac{ds'}{(s'-s)(s'-s_0)} \left[\frac{(2s'-s_1)}{(s'-\alpha)}\right] \qquad (22)$$

where

$$\alpha = s_1 - \frac{2\lambda}{1-x}$$

$$= -\left(\frac{e^2}{8\pi^2}\right) 1/2 \int_{-1}^{1} \frac{P_\ell(x)}{(1-x)} \left[\ln(s_1-\alpha) \left[\frac{2s-s_1}{s-\alpha} - \frac{2s_0-s_1}{s_0-\alpha}\right] - \frac{2s-s_1}{s-\alpha} \ln(s_1-s_1) + \frac{2s_0-s_1}{s_0-\alpha} \ln(s_1-s_0) \right],$$

(23)

Now, we must consider certain integrals: First

$$1/2 \int_{-1}^{1} \frac{P_\ell}{1-x} \frac{1}{s-\alpha} = \frac{1}{s-s_1} \times Q_\ell\left(1 + \frac{\lambda}{2q^2}\right),$$

which is to be discarded, as it is just the Coulomb divergent phase referred to earlier. Also, one has

$$1/2 \int_{-1}^{1} \frac{P_\ell(x)}{1-x} \frac{\ln(s_1-\alpha)}{s-\alpha}$$

(24)

$$= \frac{\ln(2\lambda)}{s-s_1} Q_\ell\left(1 + \frac{\lambda}{2q^2}\right) - 1/2 \int_{1}^{1} \frac{P_\ell(x) \ln(1-x)}{\left[\left(1 + \frac{\lambda}{2q^2}\right) - x\right] (s-s_1)}$$

The 2nd term is equal to

$$- \frac{1}{8q^2} \int_0^2 \frac{du \ln u\,(1-u)}{\gamma_q + u} \; , \; \gamma_q = \frac{\lambda}{2q^2}$$

$$= - \frac{1}{8q^2} \int_0^2 du \ln u \left(\frac{1+\gamma_q}{\gamma_q + u} - 1 \right) \qquad (25)$$

$$= \frac{1}{4q^2} (\ln 2 - 2) - \frac{1+\gamma_q}{8q^2} \int_0^2 \frac{du \ln u}{\gamma_q + u} \; .$$

Similarly, one will have an integral like that of the second term in Eq. (25), but with $\gamma_q = 2\lambda/s_1 + s$, replaced by $\gamma_k = 2\lambda/s_0 - s_1$. These two integrals are evaluated in Appendix 1. Finally, removing all terms divergent in λ, in particular, removing a term $\mathcal{Q}_\ell(1 + \lambda/2q^2)$ wherever ln λ divergence occurs, one obtains the "finite part" of $\delta_e D$ as given in Eq. (26), where $4k^2 = - s_1 + s_0$,

$$\delta D_{\rho 0})_f = - \frac{e^2}{8\pi^2} \left[\frac{(2s_0 - s_1) \ln (s_1 - s_0)}{8k^2} \ln \left| \frac{k^2}{q^2} \right| + \frac{(2s - s_1)\pi^2}{48q^2} - (2s_0 - s_1)\, f_1\,(q^2, k^2) \right] \qquad (26)$$

$$f_1(q^2, k^1) = \frac{1}{4k^2} \left[- \frac{5\pi^2}{24} + \ln \left| \frac{k^2}{q^2} \right| \left(1 - \frac{\ln 2}{2} + \frac{\ln \left| k^2/q^2 \right|}{4} + 1/2\,(\ln 8q^2 - 2) \right) \right] \qquad (26a)$$

From Eq. (26) one also obtains

$$\delta D'_{\rho 0}\Big|_f = -\frac{e^2}{8\pi^2}\left[\frac{\pi^2}{24q^2}\left(1 - \frac{2s-s_1}{8q^2}\right) + \frac{s_1-2a}{32k^2q^2}\ \ln\left(\frac{s_1-a}{4q^2}\right)\right] . \quad (27)$$

Actually, in Eq. (21) B_γ should be multiplied by $F^2(t)$, where F is the pion (isovector) electromagnetic form factor. This is discussed in Appendix D.

Before proceeding any further, it should be stated that an elastic π-π calculation is only able to bootstrap a ρ which lies very close to $\pi\pi$ threshold and which requires a very large G, and a pole close to s = 0[9]. This is so unrealistic that it is pointless to carry out the full mass shift calculation for this bootstrap. Instead, the pole parameters will be adjusted so as to give a ρ at 750 MeV (the width will be far too large as calculated from Eq. (2)). Certain salient features which the author feels will be present in any "real bootstrap"[10] will be discussed, employing Eq. (18). Equation (18b) comes from the condition ($G_{out} \propto N/D' = G_{in}$) which cannot be satisfied in a simple bootstrap here, but for reasonable forces one can get Re D = 0 at 750 MeV, and so one expects Eq. (18a) to retain some significance. Thus, the following calculation of an elastic $\pi\pi$ channel is not a proper bootstrap. However it will indeed be found that when inelasticity is added, a bootstrap is possible, and the features of the electromagnetic mass splitting are little different from those to be obtained in the elastic case

5. Elastic $\rho\pi\pi$ Calculation

In this section we consider the dynamically generated ρ in a $\pi\pi$

channel . The section is organized in the following manner First we consider some general aspects of the π-π channel and rederive our expression for the ρ mass shift, ignoring photon driving forces. Then we consider the effect of the photon exchange force by itself, pretending that the scattering and exchanged particles possess no electromagnetic mass splittings. Finally, we consider both types of perturbation together.

The $\rho+$ occurs in $\pi^+\pi^0$ channel in which $\rho+$ exchange provides the binding force. The ρ_0 occurs in the $\pi^+\pi^-$ channel with ρ_0 exchange providing the binding force. This is illustrated in Fig. 4.

Evaluating all expressions in Eqs. (4) - (7) at $s = M^2$, we obtain Eqs. (4a) - (7a). Here $\delta\mu^2 = \mu_{\pi^+}^2 - \mu_{\pi 0}^2$, $\delta M^2 = M_{\rho+}^2 - M_{\rho 0}^2$, and $\delta G = G_{\rho 0\pi^+\pi^-} - G_{\rho^+\pi^+\pi^0}$. Again we symbolically write the perturbation due to photon forces as "$\delta\gamma$" $\partial/\partial\gamma$.

$$\frac{\partial D}{\partial \gamma}\,\text{"}\delta\gamma\text{"} - 2\delta\mu^2 \frac{\partial D}{\partial \mu^2} + \delta M^2 \left[\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^2}\right] - \frac{\delta G}{G} = 0 \tag{4a}$$

$$4\mu^2 \frac{\partial D}{\partial \mu^2} + M^2\left[\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^2}\right] = 0 \tag{5a}$$

$$\delta G = - 2\delta\mu^2 \frac{\partial h}{\partial \mu^2} + \delta M^2\left[\frac{\partial h}{\partial s} + \frac{\partial h}{\partial M_{ex}^2}\right] + \frac{\partial h}{\partial \gamma}\,\text{"}\delta\gamma\text{"} \tag{6a}$$

$$4\mu^2 \frac{\partial h}{\partial \mu^2} + M^2 \left[\frac{\partial h}{\partial s} + \frac{\partial h}{\partial M_{ex}^2}\right] = 0 \ . \tag{7a}$$

These equations have the solutions

$$\frac{\delta G}{G} - \frac{\partial D}{\partial \gamma} \text{"}\delta\gamma\text{"} = \left(\delta M^2 + \frac{M^2}{2\mu^2} \delta\mu^2\right) \left[\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^2}\right] \tag{8a}$$

$$\delta G - \frac{\partial h}{\partial \gamma} \text{"}\delta\gamma\text{"} = \left(\delta M^2 + \frac{M^2}{2\mu^2} \delta\mu^2\right) \left[\frac{\partial h}{\partial s} + \frac{\partial h}{\partial M_{ex}^2}\right] . \tag{8b}$$

In general, the square bracket factors of Eqs (8a) and (8b) are not equal. Consequently one has, in the absence of photon driving forces, the solution found in reference (5),

$$\begin{aligned} &\delta G = 0 \\ &\delta M^2 + \frac{M^2}{2\mu^2} \delta\mu^2 = 0 . \end{aligned} \tag{28}$$

We shall now begin numerical calculations. First, we observe that in order to produce a resonance at $s = 29$ (in units of μ^2) with a reasonable coupling constant, one needs to represent the force by a pole quite far out on the negative real S axis.

With the approximation

$$\bar{t} = \frac{.34\, G}{s+200} , \tag{29}$$

which gives the correct strength of the Born force near threshold, one requires $G = 7.5$. This "pseudopole" is much further out on the negative real axis than was the case for the PS-V 0^- calculation of part I of this thesis. In the latter case, however, the Born amplitude exhibited a marked decrease as the energy increased above threshold (that is, Born/p^2 decreased). In the π-π-ρ case, however, the Born amplitude shows no indication of decreasing above threshold, and so corresponds to a force pseudopole much further from threshold.

One now finds that (s_0 = - 200)

$$\delta_e D_{\rho 0})_f = - \left(\frac{e^2}{4\pi}\right) x \ (1.3). \tag{30a}$$

In addition, one obtains for this case,

$$\partial D/\partial s = - .003 \ . \tag{30b}$$

From these numbers, one finds that, neglecting all effects except the photon exchange force

$$\frac{\delta M^2}{M^2} = \frac{\delta D_{\rho 0})_\delta}{(\partial d/\partial s)} = \frac{3.1}{M^2} \ . \tag{30c}$$

This corresponds to about a 40 MeV mass shift which is unreasonably large. However, sound arguments will now be presented which considerably lower this number.

One might hope to approximate γ exchange by a pole on the negative real S axis, but closer to threshold than the shorter range ρ exchange pole. (The shorter the range of a force, the more its contribution persists up to high energies. This is equivalent to saying that a pseudopole representing the force should be at more negative energy for a shorter range force since then the pole force drops off slower at high energies.)

If one examines a two-pole model of the force with the second pole representing γ exchange being closer to s = 0 and with a residue of order e^2, one can easily solve the integral equation for N. One finds typically that for the second pole located at - 25 or - 50 ($x\mu^2$) the strength of $\delta_e N$ is only about half the strength of the input 2nd pole. This indicates that the determinantal approximation overestimates the photon exchange contribution to δD by perhaps a factor of 2.

In addition, taking into account $\gamma\pi\pi$ form factors, one finds another reduction by a factor of 2, bringing the purely photon exchange mass splitting down to about 10 MeV.

In view of these comments, it is not unreasonable to take instead of (30a),

$$\delta D_{\rho 0} \sim -.6 \left(\frac{e^2}{4\pi}\right) . \tag{31}$$

Taking into account the γ-π-π form factor would further reduce the size of the RHS of Eq. (31).

From Eq. (31) and the known value of $\partial D/\partial s$ in our model, we would estimate $\delta M \lesssim 18$ MeV from this Coulomb binding. Form factors at the γ-π-π vertices would substantially reduce this, say to 12 MeV. That this is reasonable can be argued via semi-classical considerations. A priori, we expect that if ρ exchange is the dominant force, the range of forces binding the π's is less than $1/M_\rho$, or about 1/5 fermi. One estimates now that the Coulomb binding energy should be $\sim e^2/(\text{range})$, or about 7 MeV. One should realize that the elastic π-π-ρ bootstrap fails to give a heavy ρ; $\partial D/\partial s$ is too small, at large energies, to give a small output $g^2/4\pi$. This tends to give too large an δM.

Now that we have examined the effect of photon exchange, let us proceed to the effects of the mass splittings of the participating particles. One first wants to obtain estimates for the mass derivatives of the pole model parameters. One can do so by the method referred to in Section 3. Using

$$\frac{\partial \bar{t}}{\partial M_{ex}^{\;2}} = \frac{1}{32p^2} \left[\left(4 + \frac{s+M^2}{p^2}\right) Q_1' \left(1 + \frac{M^2}{2p^2}\right) \frac{1}{2p^2} + \frac{1}{p^2} \; Q_1 \left(1 + \frac{M^2}{2p^2}\right) \right] \tag{32}$$

$$\left(Q_1'(x) \equiv \frac{\partial Q(x)}{\partial x} \right) , \tag{33}$$

one can estimate that $\alpha_M \approx -.06$

$a_M \approx 28.5$.

This is an average of what one obtains by matching derivatives of $\bar{t}$, and $\bar{t}$ pole at (a) $s = 8$ and 16, and

(b) at $s = 16$ and 24.

Employing these numbers in Eq. (18a) one is struck by the fact that $\partial D/\partial M_{ex}^{\;2} \gg \partial D/\partial s$. (In the language of reference (5), the A matrix elements are $\gg 1$.) This is quite different from the situation in static model meson baryon calculations[5]. The reason for this is that here the forces are fairly short range; changing the exchange mass corresponds to changing the sampling of the P wave centrifugal repulsive barrier between the two pions. At short ranges, the centrifugal barrier varies rapidly with radius, and hence the force is very sensitive to the exchange mass.

The result of putting the above numbers into Eq. (18a) is

$$4.5\left(\frac{\delta M^2}{M^2} + 1/2\,\frac{\delta\mu^2}{\mu^2}\right) = -\ .6\left(\frac{e^2}{4\pi}\right) + \frac{\delta G}{G}\ , \tag{34}$$

or, using $\delta\mu^2/\mu^2 = +\ .06$

$$\frac{\delta M^2}{M^2} = -\ .03 + \frac{\frac{\delta G}{G} - .6(e^2/4\pi)}{4.5}\ . \tag{35}$$

Thus, with any reasonably small $\delta G/G$ (e.g. $\lesssim 4\%$) one sees that $\delta M^2/M^2 \simeq -\ .02$ to $-\ .03$, or

$$\underline{\underline{M^0 - M^+ \sim +7.5 \text{ to } \sim +11 \text{ MeV}}}. \tag{36}$$

The important features are that 1) the exchange mass dependence is very important; if the derivative $\partial D/\partial s$ were somewhat larger (as could arise from inelasticity) the number 4.5 might be decreased, but only slightly. 2) The electromagnetic driving term does not appear to have a large effect again because of the dominant effect of changing the exchange mass compared to changing the pole position. These features seem likely to remain when improved dynamical calculations are undertaken. The magnitude of $\delta G/G$ cannot be estimated without actually having a bootstrap. However, as long as $\delta G/G$ is of the same size as $\delta M^2/M^2$, $\delta\mu^2/\mu^2$, one expects from the above discussion, that δM will certainly remain negative and of order 8 or more MeV.

6. Inelasticity

The coupling of additional channels to a given channel can seriously alter the dynamics of a resonance. One can examine this situation partially without even performing a multichannel calculation (which for the present type of mass shift calculation would be very messy) by introducing inelasticity semi-phenomenologically into a single channel calculation. Denoting the ratio of total cross-section to elastic cross-section by R, we have

$$R = \frac{\sigma_{TOT}}{\sigma_{el}} = 1 + \sum_{n \neq i} \frac{q_n}{q_i} \frac{|t_{ni}|^2}{|t_{ii}|^2} \qquad (37)$$

where t_{ij} corresponds to $\omega/2(q_i x q_j)^{1/2}\, e^{i\delta} \sin \delta$.

A simple model for R is expressed by

$$R = \bar{R}\ \theta(s-c)\ , \qquad \bar{R}\ \text{constant}\ . \qquad (38)$$

For a P wave pole model, one has corresponding to equations (14) and (15), the following:

$$N = \frac{A}{s-a}\ ,\ s > s_1,\ A = \alpha G\ , \qquad (39)$$

$$\mathrm{Re}\ D = 1 - \frac{A}{\pi}\left[1 + \frac{s-s_1}{s-a}\ln\frac{s_1-a}{s-s_1} - (\bar{R}-1)\left\{\frac{s-s_1}{s-a}\ln\frac{c-s}{c-a} - \frac{s_1-a}{c-a}\right\}\right] \tag{40}$$

$$D' = \frac{A}{\pi(s-a)}\left[1 - \frac{s_1-a}{s-a}\ln\left(\frac{s_1-a}{s-s_1}\right) - (\bar{R}-1)\left\{\frac{s-s_1}{c-s} - \frac{s_1-a}{s-a}\ln\left(\frac{c-s}{c-a}\right)\right\}\right].$$

Near (below) the inelastic threshold, there is a strong attractive force. Also in this region, as S approaches C, one finds that $G_{out}/G_{in} \rightarrow 0$. Thus there is a tendency to have a resonance below an inelastic threshold[12]. The main problem with present bootstraps, that of G_{out}/G_{in} being $\gg 1$, is also alleviated. Of course, in the present case, non-S wave inelasticity does not really commence abruptly at an inelastic threshold. Nevertheless, for a steeply rising R, the above features should still be manifested.

The inelasticity also depends on external and exchanged masses and hence contributes to Eq. (4a) an extra term, which for constant $\bar{R}$, can be written as

$$\frac{\partial D}{\partial R}\left(\sum_\alpha C_\alpha\ \delta M_\alpha^2\ \frac{\partial \bar{R}}{\partial M_\alpha^2} \equiv \delta\bar{R}\right). \tag{41}$$

Similarly, scale invariance implies

$$0 = \sum_i M_i^{\ 2}\frac{\partial D}{\partial M_i^2} + M_c^{\ 2}\left(\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^{\ 2}}\right) + \text{driving force term}$$

$$+ \frac{\partial D}{\partial R}\left(\sum_\alpha M_\alpha^{\ 2}\frac{\partial \bar{R}}{\partial \alpha}\right). \tag{42}$$

However, scale invariance is really nothing more than a statement that a function is dimensionless. R is dimensionless and independently scale invarient, so that the last term of (42) is independently zero. Thus, in the present π-π example, including inelasticity, Eq. (8a) is replaced by

$$\frac{\delta G}{G} - \underset{\text{forces}}{\text{driving}} = \left(\delta M^2 + \frac{M^2}{2\mu^2} \quad \delta\mu^2\right)\left(\frac{\partial D}{\partial s} + \frac{\partial D}{\partial M_{ex}^{\ 2}}\right) + \frac{\partial D}{\partial \bar{R}} \quad \delta\bar{R}. \quad (43)$$

Of course, the S and M derivatives depend on $\bar{R}$ now.

In the π-π problem, Balasz has considered inelasticity in a Reggeistic calculation, and found it to be quite large ($R \geqslant 4$) for s well above threshold. From a survey of works on π-π scattering calculations one obtains the impression that most authors come close to bootstrapping the ρ, or actually succeed, when sufficient inelasticity is taken into account. Most approaches characterize R in a manner which is not susceptible, however, to providing the mass derivatives of R which one requires for a mass shift calculation.

Those who have performed multichannel calculations have found that the π-ω channel is 2nd most important in determining the ρ position, after the π-π channel. One could obtain some feeling for the mass derivatives of R by examination of the off-diagonal single particle exchange force diagrams connecting the π-π and π-ω channels, but this might not at all adequately represent the inelastic amplitudes dependence on the particle masses. The details are left to Appendix C.

The $\pi\omega$ inelastic threshold is, in units of μ^2, at $s \sim 45$; since the $\pi\omega$ channel is a P wave channel for $J = 1^-$, the inelasticity rises as $p^4_{\pi\omega}$.

A step function R therefore overestimates the tendency for inelasticity to give a consistent resonance nearby.
For $C = 45$, $\bar{R} = 4$, $a = 200$, one finds $G_{out}/G_{in} = 1.5$
For $C = 40$, $\bar{R} = 4$, $a = 200$, $G_{out}/G_{in} \sim 1.25$.

This latter case further exaggerates the actual effect of inelasticity by moving the inelastic threshold and the resonance closer together.

For $C = 45$, $\bar{R} = 4$, $a = 100$, one finds $G_{out}/G_{in} \simeq 1$, so we shall adopt this as a model for our mass shift calculation. The simplest approach would be to omit the mass dependences of R, and see what happens.

The next section deals in more detail with this calculation.

7) Inelastic Bootstrap and Mass Shifts

In the above section we have obtained a crude inelastic model giving a consistent bootstrapped ρ. Let us now initially neglect the mass dependences of R, and exam ine a model where $\bar{R}$ is independent of energy. One can then easily establish Eq. (44)

$$\frac{\partial D}{\partial M_{ex}^{\;2}} = - \frac{\mathcal{A}_M}{\mathcal{A}} - \frac{\mathcal{A}_G}{\pi}\,\mathcal{A}_M\, f_2(s) \;, \tag{44}$$

$$\begin{aligned} f_2(s) = \frac{s-s_1}{s-a}\Bigg[&\frac{\ln(s_1-a/s-s_1)}{s-a} - \frac{1}{s-a} \\ &- (\bar{R}-1)\left\{\frac{\ln(c-s/c-a)}{s-a} + \frac{1}{c-a}\right\}\Bigg] \\ &- (\bar{R}-1)\left(\frac{1}{c-a} - \frac{s_1-a}{(c-a)^2}\right) . \end{aligned} \tag{44a}$$

Also, one has

$$G = -\,12\,\frac{N}{D'}\bigg|_{s=M_\rho^{\;2}}$$

in which the factor $\mathcal{A}$ cancels out. Using (40) to write G in the form $-\,12\pi/f_3(s)$, one has

$$\begin{aligned} \frac{\delta G}{G} &= \left(-\,\frac{\delta f_3(s)}{f_3(s)}\right) \\ &= \left(\delta M^2 + \frac{M^2}{2\mu^2}\,\delta\mu^2\right) f_4(s)\Big/\left(\frac{\pi(s-a)}{A}\,D'\right) + h_D \end{aligned} \tag{45}$$

$$\begin{aligned} f_4(s) = \frac{a_M}{s-a}\Bigg(&-1 - \frac{s-s_1}{s-a}\ln\frac{s_1-a}{s-s_1} + (\bar{R}-1)\left[\frac{s-s_1}{s-a}\ln\frac{c-s}{c-a} - \frac{s_1-a}{c-a}\right]\Bigg) \\ &- \frac{s_1-a}{(s-a)^2}\ln\left(\frac{s_1-a}{s-s_1}\right) - \frac{s_1-a}{(s-a)(s-s_1)} \\ &+ (\bar{R}-1)\left[\frac{1}{c-s}\left(1 + \frac{s_1-a}{s-a}\right) + \frac{s-s_1}{(c-s)^2} + \frac{s_1-a}{(s-a)^2}\ln\left(\frac{c-s}{c-a}\right)\right]. \end{aligned} \tag{46a}$$

and

$$h_D = \left(\frac{\delta N}{N} - \frac{\delta D'}{D'}\right)_{\text{Electromagnetic}}$$

$$= - \left(\frac{\delta N}{N} - \frac{\delta D'}{D'}\right)_{\rho^0,\ \text{Electromagnetic}} \quad (46c)$$

With the model referred to above, $a = -100$, $\mathcal{a} = .18$, $\bar{R} = 4$, $G = 3.7$; one finds by matching mass derivatives with the field theory Born term at $s = 8, 16$, that

$$\mathcal{a}_M = -\ .036 \text{ and } a_M = +\ 17.0\ . \quad (47)$$

Using Eqs. (44) and (40) one finds now that

$$\frac{\partial D}{\partial M_{ex}^2} = +\ .197$$

$$\frac{\partial D}{\partial s} = -\ .016\ . \quad (48)$$

Note that, as asserted earlier, the calculation is very sensitive to the exchanged mass, compared to the position of the resonance.

One also finds that

$$\left.\delta D_{\rho^0}\right)_f = \left(-\ 1.1\ \frac{e^2}{4\pi}\right) \quad (49)$$

plus an inelastic term. This would lead to a Coulomb binding energy given by $\delta M^2\ \ \partial D/\partial s \lesseqqgtr -\ 1.1\ e^2/4\pi$, or

$$\delta M = +\ 6.4 \text{ MeV} \quad (50)$$

+ an inelastic contribution

in the determinantal approximation. One might estimate that the inelastic contribution (coming from $\int_0^\infty \bar{R}$ etc.) would increase this estimate to 20 MeV.

However, one finds that the form factor contributions to the force reduce the elastic contributions to δD from $- 1.1\ e^2/4\pi$ to only $- .6\ e^2/4\pi$, using the formulae of Appendix D. Thus, one might estimate that including the inelastic contributions, one would have $\delta D \leqslant - 2.0\ e^2/4\pi$, corresponding to $\delta M = 11.5$ MeV. Correcting for determinantal overestimation of the forces would further reduce δM (due to only photon exchange) to a very reasonable value.

This is exactly what one expects from arguments given earlier; a non-determinantal calculation would give a somewhat smaller result.

Using the above equations, one finds from Eqs. (8a) and (8b)

$$\frac{\delta G}{G} + \delta D_{\rho=0} = 5.2 \left(\frac{\delta M^2}{M^2} + \frac{\delta \mu^2}{2\mu^2} \right) \tag{51a}$$

$$\frac{\delta G}{G} + \left(\frac{\delta_e N}{N} - \frac{\delta_e D'}{D'} \right)_{\rho 0} = + .2 \left(\frac{\delta M^2}{M^2} + \frac{\delta \mu^2}{2\mu^2} \right) \quad . \tag{51b}$$

Thus

$$\frac{\delta D'}{D'} - \frac{\delta_e N}{N} + \delta\ D_{\rho=0} = 5 \left(\frac{\delta M^2}{M^2} + \frac{\delta \mu^2}{2\mu^2} \right) \tag{52a}$$

and

$$\frac{\delta G}{G} \approx \left(\frac{\delta_e D'}{D'} - \frac{\delta N}{N} \right)_{\rho 0} . \tag{52b}$$

If one solves the integral equation for N and D one finds now that

$$\left.\frac{\delta N_f}{N}\right)_{\rho 0} = \left.\delta D_{\rho 0}\right) \text{ determinantal}$$

$$= + .04.$$

Also, in the determinantal approximation, $\delta D'/D')_{\rho 0}$ is only about + .001 and will be less in the exact solution. Thus one obtains $\delta G/G < + 1\%$.

The left hand side of Eq. (52a) is similarly estimated to be of order - .07, and one therefore obtains

$$\frac{\delta M^2}{M^2} = - \frac{\delta \mu^2}{2\mu^2} - .004 .$$

One sees that the electromagnetic driving forces do not appear very important and that the resultant answer is essentially group theoretical. Finally, let us write down this estimate for the mass shift:

$$M_{\rho 0} - M_{\rho +} = + 11 \text{ MeV} . \tag{53}$$

When taking into account the mass dependence of R, in the R = constant model, we have to employ Eq. (43) and the equation corresponding to (8b). Here

$$\frac{\partial D}{\partial R} = + \frac{A}{\pi} \left\{ \frac{s-s_1}{s-a} \ln \frac{c-s}{s-a} - \frac{s_1-a}{c-a} \right\} \tag{54}$$

which in the present case

$$\cong - .24 .$$

From Appendix C, discussing δR, we therefore see that there will be added on to the right hand side of Eq. (51c) a term of the form $(\delta M^2)\,(<0) + (\delta\mu^2)(>0)$. This tends to produce a further negative mass splitting of the $\rho+$ and $\rho 0$.

8. Two Channel Approach to Inelasticity

In this section we discuss briefly a genuine two channel approach to the ρ dynamical generation and its relevance to the ρ mass splittings. The discussion will concentrate on the mass splittings of the participating particles and will not consider the photon exchange forces. The general trend of expected results (if such a calculation were executed in detail) will be found to agree with the results of Section 8 for non-diagonal forces.

Let us consider the coupled channels $\pi\pi$ (called "1") and $\pi\omega$ (called "2"). The relevant matrix expression of the coupled channel partial wave amplitudes is described in Eqs. (8). All the amplitudes will possess resonances due to their common factor $\left|\det \underset{\sim}{D}\right|^{-1}$. This determinant is simply $D_{11}D_{22} - D_{12}D_{21}$ here, we assume that our multichannel model of the forces is such that N_{ij} and D_{ij} depend only on the i j channel forces. The π-ω channel is attractive for $J = 1^-$, $I = 1$, but D_{22} will be greater than + 0 above threshold. We also assume that purely elastic forces in the π-π channel are insufficient to bind the ρ, this was the case, too, in the previous inelastic model for the ρ. Thus $D_{11}(M_{\rho^2})$

is positive (and less than 1).

Next, we carry out the type of analysis employed previously, utilizing the idea of scale invariance of the individual D_{ij} when this is useful. (This is possible because the D_{ij} are each separately dimensionless.) As before, we obtain

$$\delta D \equiv \delta \left[\det D\right] = \delta D_{11} D_{22} + D_{11} \delta D_{22} - \delta D_{12} D_{21} - D_{12} \delta D_{21}, \quad (55)$$

and

$$\delta D_{11} \equiv \left(\delta M^2 + \frac{M^2}{2\mu^2} \quad \delta \mu^2 \right) \left(\frac{\partial D_{11}}{\partial s} + \frac{\partial D_{11}}{\partial M_{ex}^2} \right) . \quad (56)$$

In the $\pi\omega$ channel, both external π and exchanged ρ have the same charge as the over-all charge of the channel. This gives as a consequence (where $\delta\left[\;\right] = \left[\;\right]$+ state - $\left[\;\right]$ 0 state),

$$\delta D_{22} = 2 \frac{\partial D_{22}}{\partial \mu^2} \delta \mu^2 + \left(\frac{\partial D_{22}}{\partial s} + \frac{\partial D_{22}}{\partial M_{ex}^2} \right) \delta M^2 \quad . \quad (57a)$$

Scale invariance gives the relation

$$2\mu^2 \frac{\partial D_{22}}{2\mu^2} + 2M_{\omega}^2 \frac{\partial D_{22}}{\partial M_{\omega}^2} + \left(\frac{\partial D_{22}}{\partial s} + \frac{\partial D_{22}}{2M_{ex}^2} \right) M^2 = 0 \; . \quad (57b)$$

Equations (57a) and (57b) then imply equation (58) below:

$$\delta D_{22} = \left(\delta M^2 - \delta \mu^2 \frac{M^2}{\mu^2} \right) \left(\frac{\partial D_{22}}{\partial s} + \frac{\partial D_{22}}{\partial M_{ex}^2} \right) - \delta \mu^2 \frac{M_{\omega}^2}{\mu^2} \frac{\partial D_{22}}{\partial M_{\omega}^2} . \quad (58)$$

We can write symbolically

$$D_{21}\ \delta D_{12} = D_{21}{}^* \left[\frac{\partial D_{12}}{\partial M_{ex}{}^2} \left(-\frac{1}{2}\ \delta M^2 \right) \ + \text{small } \delta\mu^2 \text{ term} \right]. \qquad (59)$$

As usual, $D_{21}\ \partial D_{12}/\partial M_{ex}{}^2 < 0$ and therefore Eq. (59) has the form

$$\delta M^2 \times (> 0) + \text{small } \delta\mu^2 \text{ term}. \qquad (59a)$$

In Eq. (56a) we observe that for $M_\omega < M_\rho + M_\pi$, the 2-2 force is attractive. As earlier, let us assume that $\partial D_{ij}/\partial M_{ex}{}^2$ are the largest derivatives. We then have

$$\delta D_{22} \sim (\delta M^2) \times (> 0) - \delta\mu^2 \times (> 0)\ . \qquad (58a)$$

Combining these equations, we have:

$$\delta\,|D| = (D_{22}, > 0) \times \left(\delta M^2 + \frac{M^2}{2\mu^2}\,\delta\mu^2\right) \times (> 0)$$

$$+ (D_{11}, > 0) \times (\delta M^2 \times (> 0) - \delta\mu^2(> 0)) \qquad (60)$$

$$- \left[\delta M^2 \times (> 0) + \text{small } \delta\mu^2 \text{ term}\right] \text{ from } D_{12},\ D_{21}.$$

These two opposing changes in the δM^2 coefficient, and the $\delta\mu^2$ term from D_{12} are difficult to ascertain, therefore, any over-all change is obscure. However, the off-diagonal term decreases the ratio of the coefficients of δM^2 and $\delta\mu^2$, in agreement with previous statements.

It is amusing to also consider what can be learned about the ρ mass splitting by considering the dynamical generation of the π in a $\pi\rho$ channel, although no simple calculation seems likely to succeed in this case. If one takes the dominant exchange force to be due to π exchange, then the π^0 occurs in the amplitude connecting a $\rho + \pi^-$ state to a $\rho^-\pi^+$ state, with π^0 exchange. The π^+ occurs in the amplitude which=1/2 the sum of the $\pi^+ \rho^0 \to \pi^+ \rho^0$ and $\pi^0 \rho^+ \to \pi^0 \rho^+$ amplitudes, with π^+ exchange. This is illustrated in Fig. 5. Assuming that one has a model of the forces in which D is just proportional to an integral over the sum of Born exchange terms, one obtains:

$$\delta D \equiv D^{(+)} - D^{(0)} \equiv 0$$

as usual, which implies

$$\left(\frac{\partial D}{\partial \mu_{ex}^2} + \frac{\partial D}{\partial s} \right) \delta\mu^2 - \delta M^2 \frac{\partial D}{\partial M_{ex}^2} - \delta\mu^2 \frac{\partial D}{\partial \mu^2} + 1/2 \frac{\delta G}{G} = 0. \quad (61)$$

Scale invariance imposes the condition

$$\left(\frac{\partial D}{\partial \mu_{ex}^2} + \frac{\partial D}{\partial s} \right) \mu^2 + 2M^2 \frac{\partial D}{\partial M_\rho^2} + 2\mu^2 \frac{\partial D}{\partial \mu^2} = 0. \quad (61a)$$

From this , one obtains finally, (neglecting γ exchange effects)

$$1/2 \frac{\delta G}{G} = \delta\mu^2 \left[\frac{3\partial D}{\partial \mu^2} + \frac{2M^2}{\mu^2} \frac{\partial D}{\partial M^2} \right] + \delta M^2 \frac{\partial D}{\partial M^2} . \quad (62)$$

From an examination of the field theory Born exchange amplitudes and their mass derivatives, it is very plausible that

$$- 3 \frac{\partial D}{\partial \mu^2} \ll \frac{M^2}{\mu^2} \frac{\partial D}{\partial M^2} . \qquad (62a)$$

Thus, one obtains,

$$1/2 \frac{\delta G}{G} + \frac{\partial D}{\partial M^2} (- \delta M^2) \leqslant \frac{2M^2}{\mu^2} \delta\mu^2 \frac{\partial D}{\partial M^2} . \qquad (63)$$

This predicts (for $\partial G/2G$ small) a negative mass difference, as was obtained too, from consideration of the ρ bootstrap. It is likely that additional channels would reduce $\left| \delta M^2/M^2 \right|$ from the above value of $\backsim +2\ \delta\mu^2/\mu^2$ in the π^0 calculation. At the same time we have seen that an additional channel in the ρ bootstrap will increase $\left| \delta M^2/M^2 \right|$ from the π-π calculation value of $\delta\mu^2/2\mu^2$. All indications thus point to $M_{\rho^0} - M_{\rho^+} \geqslant \backsim 10$ MeV.

9. SU_3 Degenerate PS_8 - PS_8 Model for Vectons

In previous section we have considered two approaches to an inelastic dynamical description of the ρ. In the present section we consider yet another model and its implications for electromagnetic mass splittings. The assumptions of the model now to be considered are:

1. The relevant strong interactions are SU_3 invariant. This implies that the PS_8 are mass degenerate, with mass μ, and the V_8 are mass degenerate, with mass M. This degeneracy evidently violates the facts of life about PS strong mass splittings. It is assumed here simply to get a feeling for electromagetic mass shifts without being forced to carry out a perturbation on a non-degenerate

(2-) coupled channel problem; such a calculation would be discouragingly messy;

2. The V_8 are dynamically generated solely in the PS_8-PS_8 scattering channels.

3. The forces in these channels come entirely from V_8 exchange. We are thus ignoring the complications of the existence of a V_1 in order to retain simplicity.

The idea of only the PS_8 - PS_8 channels sufficing to dynamically predict V_8 resonances was successfully employed by other authors[14, 15] who took into account the observed physical PS_8 masses. Therefore, the present degenerate model may not be too remote from current calculations in the literature when used to give electromagnetic mass differences.

We observe that there are two different ways to approach perturbations of a degenerate coupled channel problem. The first approach is to consider the degenerate problem in terms of a single channel calculation resulting from diagonalization of a multichannel matrix calculation. A vector state β can be expressed in terms of two particle (PS-PS) states which we label by a single index i,j (where C_i are appropriate coefficients, specified by SU_3 here)

$$\left| \beta \right\rangle = \sum_i \lambda_i \left| i \right\rangle \tag{64}$$

One then is dealing with a single channel calculation (for each member β of the V octet) in which the force coming from one-V exchange is given by the "t matrix":

$$t_\beta = \sum_i \lambda_i \lambda_j \overline{g_{ij}} \, t_{ij}^{J\beta} . \tag{65}$$

In Eq. (65), t_{ij}^J is the t- matrix partial wave amplitude connecting two-particle states i and j, and $\overline{g_{ij}}$ is an appropriate product of coupling constants for a one-particle exchange amplitude. We now have

$$D_\beta = 1 - d_\beta = 1 - \text{integral over } t_\beta . \tag{66}$$

Therefore, one can consider electromagnetic perturbations as below:

$$\begin{aligned} \delta D_\beta &\equiv D_{\beta +}(M_{\beta +}^2) - D_{\beta 0}(M_{\beta 0}^2), \text{ the difference between} \\ &\quad \text{D for two charge states (+) and (0),} \\ &\equiv \sum_i \lambda_i \lambda_j \overline{g_{ij}} \, \delta t_{ij}^{J,\beta} + \delta g_i \text{ terms.} \end{aligned} \tag{67}$$

This is just the same as Eq. (5), with detailed attention being paid to the fact that a sum over various two-particle amplitudes is involved.

If we define

$$\begin{aligned} \frac{\partial f}{\partial m}_{\text{external}} &= f{,}1 \\ \frac{\partial f}{\partial m}_{\text{exchanged}} &= f{,}2 \\ \frac{\partial f}{\partial s} &= f{,}s \end{aligned} \tag{68}$$

for any function f, then Eq. (67) gives the equation below:

$$\delta D_\beta = \sum_i \lambda_i \lambda_j \overline{g_{ij}} \; {}^{\beta}\Delta_{ij}^{\alpha} \left[d_{\beta,\alpha} \right] \delta M^{2\alpha} \qquad (69)$$

$$+ \; \delta g \text{ terms}$$

where $\alpha = 1,2,s$ and

${}^{\beta}\Delta_{ij}^{\alpha}$ is a set of coefficients analogous to the C_i of Eq. (4).

d_β is *the degenerate mass d function.*

Because we are dealing with a diagonalized one channel calculation, we also have the corresponding equations for the perturbations of output pole residues:

$$\delta \left[\sum_{ij} \lambda_i \lambda_j \, g_i g_j \times \begin{pmatrix}\text{possible kinematic factor} \\ \text{conveniently set} = 1\end{pmatrix} \right]_{\text{channel } \beta}$$

$$\equiv \delta \left(\frac{N}{D'} \right)_\beta$$

$$= \sum_i \lambda_i \lambda_j \overline{g_{ij}} \; {}^{\beta}\Delta_{ij}^{\alpha} \left[\frac{t_{\beta,\alpha}}{D'_\beta} - \frac{t_\beta}{(D'_\beta)^2} \, d'_{\beta,\alpha} \right] \delta M^2 \qquad (70)$$

In both Eqs. (69) and (70), one can employ the scale invariance of appropriate quantities to relate derivatives with respect to external mass to derivatives with repsect to exchanged mass and energy. Furthermore, it might be the case that the "exchange-mass derivatives" of the relevant quantities would be the dominant terms in (69) and (70) as a consequence of dynamical peculiarities. This was seen to occur earlier in the π-π-ρ exchange calculation. In this case, because the same coefficients "Δ" appear in equations (69) and (70), one finds the same dependence of both equations on the $\delta M^{2\alpha}$ (to within an over-all proportionality factor). Although the "δg" terms will not generally be the same in (69) and (70), one might suppose them to be relatively small compared to the $\delta M^{2\alpha}$ terms, as occurred in the π-π-ρ calculation of preceding sections. One then might obtain a crude estimate of the solution by setting the $\delta M^{2\alpha}$ terms separately equal to zero.

All of this discussion has really been to provide partial justification for the following plan of procedure: we will examine only the $\delta M^{2\alpha}$ terms, ignoring the hopefully small, and certainly complicated, δg_i terms, in the perturbation equations. Actually, we will now use a simple two- channel approach to the ρ and K^* calculation since we can then simply refer the reader to the current literature for well-known results about the Born amplitude t-matrices.

The ρ occurs in the $\pi\pi$ and $K\bar{K}$ channels, while the K^* occurs in the coupled $K\pi$ and $K\eta$ channels. Neglecting electromagnetic mass splittings, let f be the PS-PS, V exchange, $J^\pi = 1^-$ amplitude multiplied by an appropriate coupling constant squared. Then one obtains the following t-matrix, where the SU_3 values of various couplings have been employed (see reference 17):

$$t_\rho^{BORN} = \begin{matrix} \\ \text{"}\pi\pi\text{"} \\ \\ \end{matrix} \begin{pmatrix} \text{"}\pi\pi\text{"} & \\ 4_\rho & \sqrt{8}_{K^*} \\ \sqrt{8}_{K^*} & 3\omega-1_\rho \\ & \text{"}K\bar{K}\text{"} \end{pmatrix}_{\text{"}K\bar{K}\text{"}} \times f \qquad (71a)$$

Here the subscripts refer to the exchanged particles providing the forces. One also has:

$$t_{K^*}^{BORN} = \begin{array}{c|cc|c} & \text{"K}\pi\text{"} & & \\ \text{"K}\pi\text{"} & 4\rho - 1K^* & 3K^* & \\ & 3K^* & 3K^* & \text{"K}\eta\text{"} \\ & & \text{"K}\eta\text{"} & \end{array} \times f \qquad (71b)$$

One needs to know details of the charge states contributing to these forces. In order to obtain this information the following assorted facts are relevant:

$$\left| K\pi \right\rangle^{+}_{I=1/2} = \frac{K^+\pi^0 - \sqrt{2}K^0\pi^+}{\sqrt{3}} \qquad (72a)$$

$$\left| K\pi \right\rangle^{0}_{I=1/2} = \frac{K^0\pi^0 + \sqrt{2}K^+\pi^-}{\sqrt{3}} \qquad (72b)$$

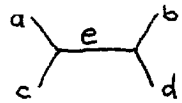
$$\left| K\bar{K} \right\rangle^{0}_{I=1/2} = 1/\sqrt{2}\ \left(|K^+K^-\rangle - |K^0\bar{K}^0\rangle \right) \qquad (72c)$$

$2\left|g_{\rho^0 K^+ K^+}\right| = \left|g_{\rho^0 \pi^+ \pi^-}\right| = \left|g_{\rho^+ \pi^+ \pi^0}\right| = 1$ in appropriate units (in an SU_3 scheme)

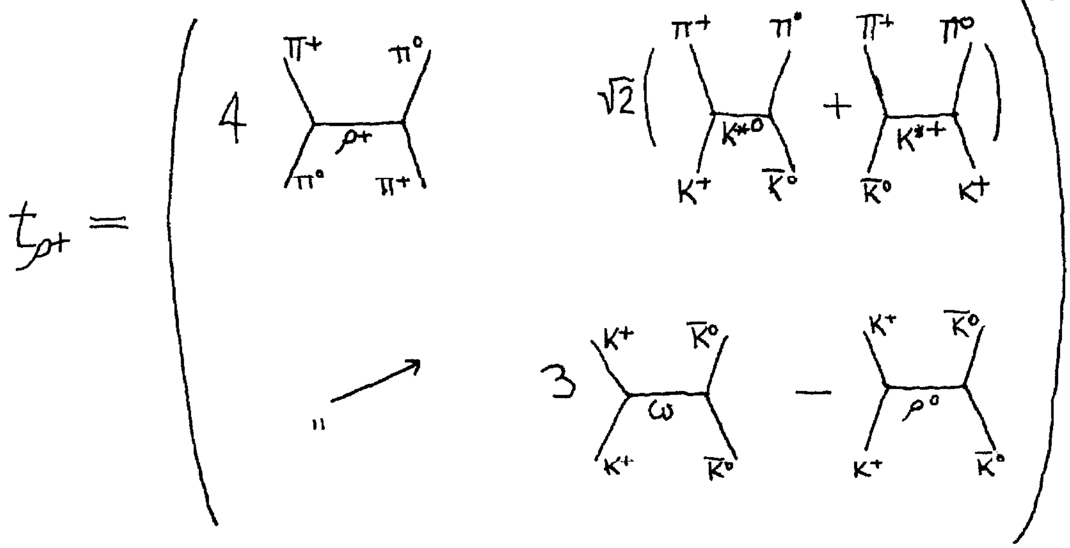
$$\left|g_{\rho^- K^0 K^+}\right| = \left|g_{K^{*+} K^0 \pi^-}\right| = \sqrt{2}$$

$$\left|g_{K^* K \eta}\right| = \left|g_{\omega K \bar{K}}\right| = \sqrt{3} \qquad (72d)$$

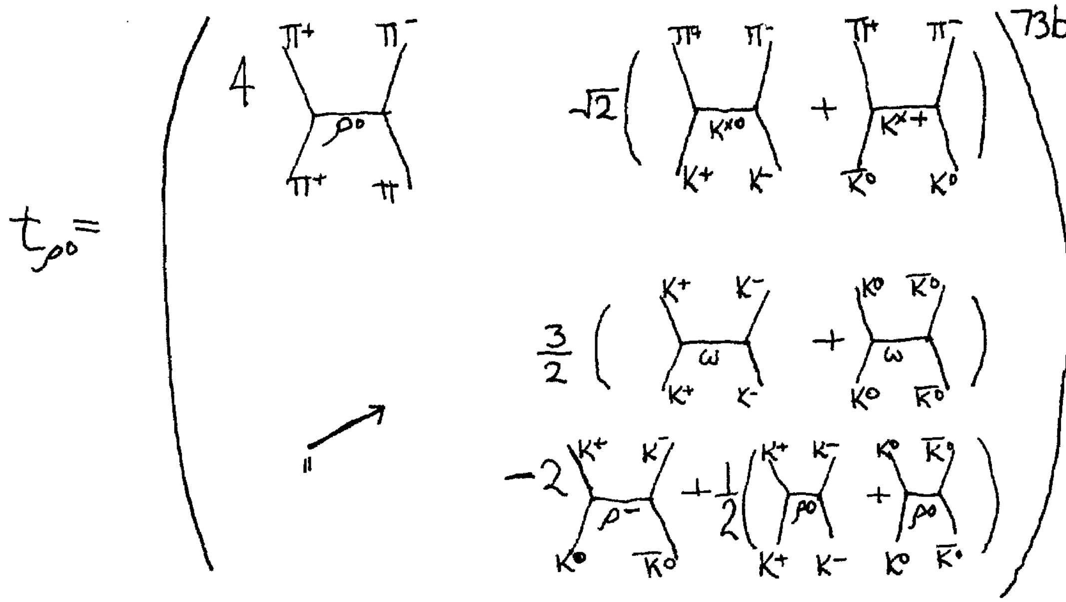
$$\left|g_{K^{*0} K^0 \pi^0}\right| = 1.$$

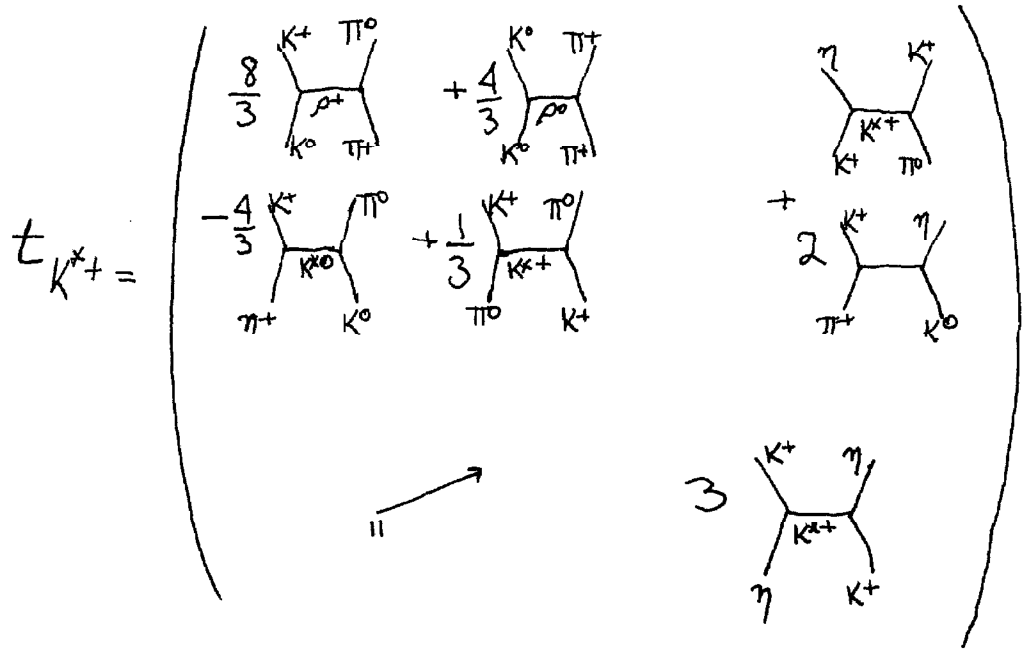
We now introduce a pictorial representation of the matrix elements The symbol  represents the amplitude without coupling constants , or signs, for particles a,b,c,d,e in their obvious roles. One now finds the following charge state decompositions of the t matrices

73a)

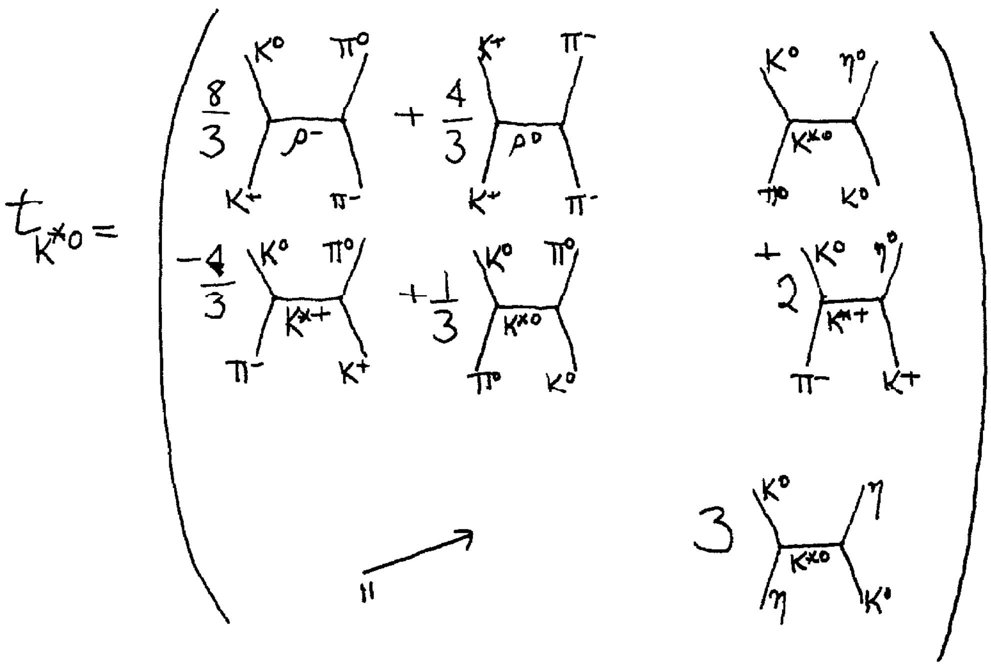


73b)




$t_{K^{*+}} =$
8/3
4/3
-4/3
1/3
2
3
K+
π0
K0
π+
ρ+
ρ0
K*0
K*+
η

73c)


$t_{K^{*0}} =$
8/3
4/3
-4/3
1/3
2
3
K0
π0
K+
π-
ρ-
ρ0
K*+
K*0
η

73d)

We now use the fact that the matrix D is related to t by

$$D = 1 - \text{integral over } t \tag{74}$$

in a determinantal approximation (or in a one-pole approximation, etc.) This permits us to write $D = 1 - d$, where d has the form of the above t's. Finally, to find the difference between (determinant D) for different charge states we adopt the following convenient artifice. We find the difference between $D^{(+)}$ (and $D^{(0)}$) and a hypothetical D_0 for which all particles have the neutral charged state mass as a simple "counting" procedure, and then easily obtain $\delta|\det|D$: thus, we have, using the previously established notation for derivatives of d:

$$D_0^\rho \left(\overset{\text{masses}}{\text{degenerate}}\right) = 1 - \begin{pmatrix} 4 & \sqrt{8} \\ \sqrt{8} & 2 \end{pmatrix} d \tag{74}$$

$$|D| = 1 - 6d.$$

$$D^{\rho+}(s=M^2_{\rho+}) \equiv D_0 - \begin{pmatrix} 4\left[2\delta^\pi d_1 + \delta^\rho(d_2+d_s)\right] & \frac{\sqrt{8}}{2}\left[\begin{matrix} 2d_1(\delta^\pi+\delta^K) \\ + d_2\delta^{K*}+2d_s\delta^\rho \end{matrix}\right] \\ {}'' \nearrow & \begin{matrix} 3\left[2d_1\delta^K+d_s\delta^\rho\right] \\ - 2\delta^K d_1 - d_s\delta^\rho \end{matrix} \end{pmatrix} \tag{75a}$$

where $\delta^\pi = \mu^2{}_{\pi^+} - \mu^2{}_{\pi^0}$, $\delta^K = \mu^\delta{}_{K^+} - \mu^2{}_{K^0}$ and similarly for δ^K, δ^ρ.

$$D^{\rho 0}(s=M^2_{\rho 0}) \equiv D_0 - \begin{pmatrix} 16\ d_1\delta^\pi & \frac{\sqrt{8}}{2}\left[(4\ \delta^\pi+2\delta^K)d_1 + d_2\ \delta K^*\right] \\ \text{"} \nearrow & \frac{3}{2} \times 4\delta^K\ d_1 - 2(2\delta^K d_1 + \delta^\rho d_2) + \frac{1}{2} \times 4d_1\ \delta^K. \end{pmatrix} \tag{75b}$$

$$\delta D \equiv + \begin{pmatrix} 8d_1\delta^\pi - 4\delta^\rho(d_2+d_s) & \sqrt{8}\ (d_1\delta^\pi - \delta^\rho\ d_s) \\ \sqrt{8}\ (d_1\delta^\pi - \delta^\rho\ d_s) & -\delta^\rho\ (2d_2 + 2d_s) \end{pmatrix}. \tag{75c}$$

From Eqs. (75), we obtain

$$\begin{aligned} \delta\left|\det\right| D &\equiv \text{Trace (co-factor matrix of } D \times \delta D) \\ \therefore \quad 0 &= 8\ d_1\ \delta^\pi - \delta^\rho\ (6d_s + 10/3d_2) \end{aligned} \tag{76}$$

when calculated at $s = \bar{M}^2_\rho$, where $d = 1/6$.
Using the scale invariance of d at $s = M^2$,

$$4\mu^2 d_1 + M^2\ (d_2 + d_s) = 0 \tag{77}$$

and so, finally, one obtains, neglecting " δg " terms

$$\delta^\rho\left[\frac{5}{3}\ d,_2 + 3d,_s\right] + \delta^\pi\ \frac{M^2}{\mu^2}\left[d,_2 + d,_s\right] = 0 \ . \tag{78}$$

Now, let us consider the K^* channels. By the same technique employed for the ρ channel, we find that

$$D_{K^{*+}}(M^2_{K^+}) - D_{K^{*0}}(M^2_{K^0}) = \delta D_{K^*}$$

$$= \begin{pmatrix} \begin{pmatrix} -2d_1\delta^K + \frac{5}{3} d_2\delta^{K^*} \\ + 3d_s\delta^{K^*} \end{pmatrix} & \begin{pmatrix} 2d_1\delta^{K^*} \\ + (3d_s - d_2)\delta^{K^*} \end{pmatrix} \\ \\ \begin{matrix} 2d_1\delta^{K^*} \\ + (3d_s - d_2)\delta^{K^*} \end{matrix} & 6d_1\delta^K + 3(d_2+d_s)\delta^{K^*} \end{pmatrix} \tag{79a}$$

whereas

$$D_0^{K^*} \overset{\text{masses}}{(\text{degenerate})} = 1 - \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} d \ . \tag{79b}$$

$$|D| = 1 - 6d$$

Equations (79a,b) now provide us with

$$\delta \left| \det D^{K^*} \right| \text{ as in Equation (76) .}$$

Therefore, setting this equal to zero, we obtain

$$0 = 2d_1\delta^K + \delta^{K^*}(3d_s + \frac{2}{3} d_2) \tag{80}$$

Again using the scale invariance of d, we obtain from (80)

$$\delta^{K^*}\left[\frac{2}{3} d_2 + 3d_s\right] - \delta^K \frac{M^2}{2\mu^2} \left[d_2 + d_s\right] = 0. \tag{81}$$

We note now that from Eqs. (78) and (81)

$$\frac{2\delta K^*}{\delta\rho} = - \frac{\delta^K}{\delta\pi} \left(\frac{3d_s + 5/3\ d_2}{3d_s + 2/3\ d_2} \right) . \tag{82}$$

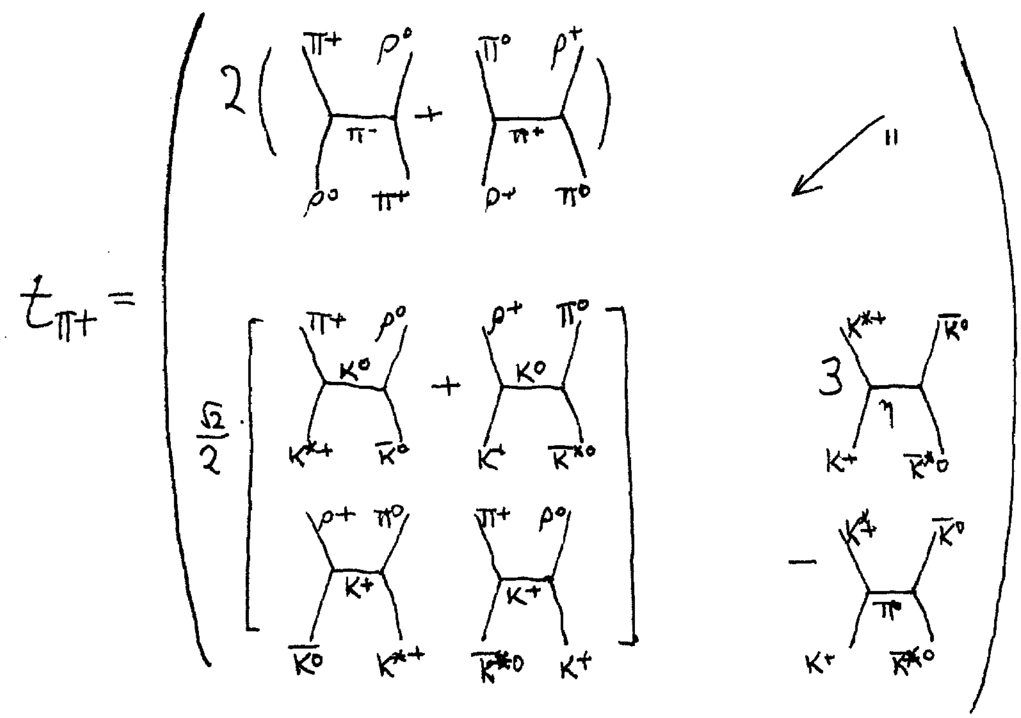
In our model, we dropped the "δg_1" terms and photon force in this study of electromagnetic perturbations, the reader should recall.

If we make a further dynamical assumption, which at least appeared warranted in the earlier π-π-ρ calculations, that $d_2 \gg d_s$, we obtain, finally,

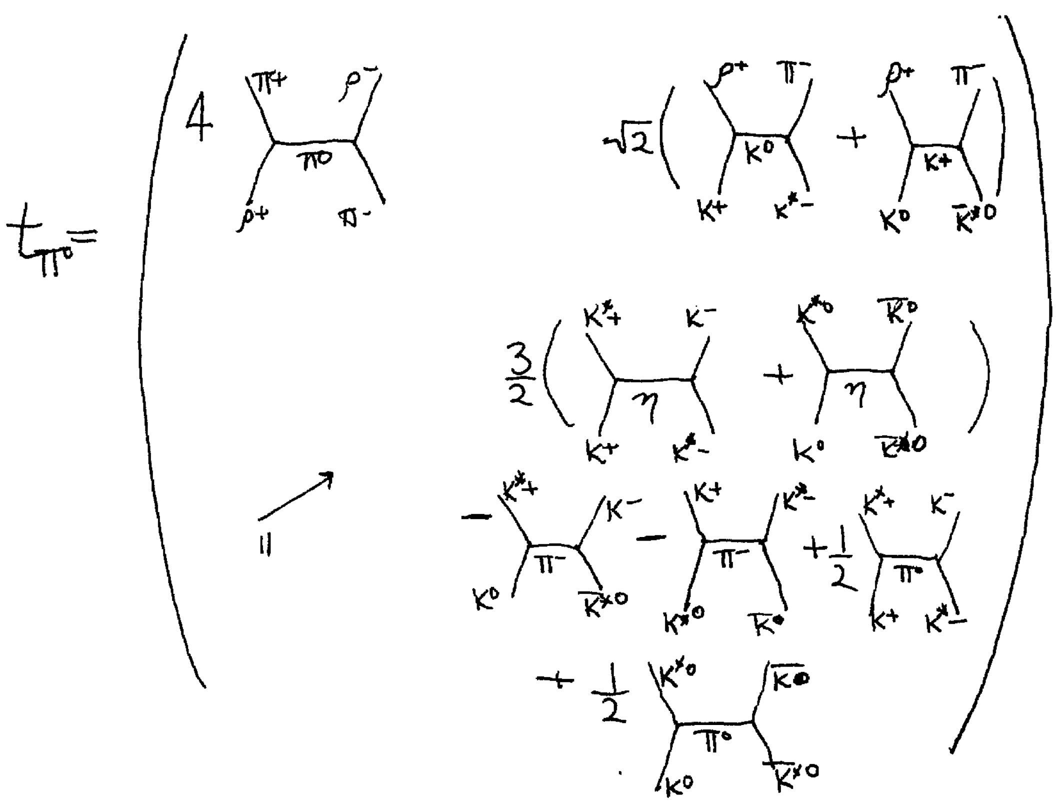
$$\delta K^* = + \frac{3}{4} \frac{M^2}{\mu^2} \delta^K$$
$$\delta^\rho = - \frac{3}{5} \frac{M^2}{\mu^2} \delta^\pi \quad . \tag{83}$$

Although part 1 of this thesis indicates that a PS_8-V_8 model of the PS_8 states probably cannot give a self-consistent bootstrap, it is still amusing to consider the consequences of such an assumed model on predicted mass splittings. Without a strong interaction calculation available, one cannot actually obtain the PS and V_8 electromagnetic mass differences. However, one can investigate the mass dependence of the perturbation equations in this model to see if it is roughly compatible with the above results of the PS_8-$P\delta_8$ model of the vector mesons. It will indeed be found that the expected signs of the vecton mass differences are unchanged from the signs in Eqs. (83), while the magnitude of the splittings are a little different.

First, we present the charge state decomposed t matrices in Eq. (84):


$t_{\pi^+} =$
2
π^+ ρ^0 π^- ρ^0 π^+
π^0 ρ^+ π^+ ρ^+ π^0
$\frac{\sqrt{2}}{2}$
K^0
K^+
3
η
π^0

(84)


$t_{\pi^0} =$
4
π^+ ρ^- π^0 ρ^+ π^-
$\sqrt{2}$
K^0
K^+
$\frac{3}{2}$
η
π^-
$\frac{1}{2}$
π^0

These two t matrices enable us to calculate δD_π: However, we first need an additional definition: let $d_1 = \partial d/\partial\mu^2_{FS\ \text{external}}$ and let $\bar{d}_1 = \partial d/\partial\mu^2_{V\ \text{external}}$. As before, let $d_2 = \partial d/\partial\mu^2_{\text{exchanged}}$.

Then we obtain

$$\delta D_\pi = \begin{pmatrix} 4\left[\delta^\pi(d_2+d_s) - \delta^\pi d_1 - \delta^\rho \bar{d}_1\right] & '' \\ \frac{\sqrt{8}}{2}\left(-\delta^\pi d_1 - \delta^\rho \bar{d}_1 + 2\delta^\pi d_s\right) & 2\delta^\pi(d_2 + d_s) \end{pmatrix} \tag{85}$$

As before, the "adjoint" matrix of D is

$$\frac{1}{3}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} . \tag{86}$$

One now obtains

$$0 = \delta\left|\det D_\pi\right| = \left(-4\left[\delta^\pi d_1 + \delta^\rho \bar{d}_1\right] + 6\delta^\pi (d_2 + d_s)\right) - \frac{8}{3}\delta^\pi d_2 . \tag{87}$$

Scale invariance implies that at $s = \mu^2$,

$$(d_2 + d_s)\mu^2 + 2(M^2\bar{d}_1 + \mu^2 d_1) = 0. \tag{88}$$

Therefore, again considering only δM^2 terms and ignoring δg_i terms in the order e^2 perturbation, we obtain:

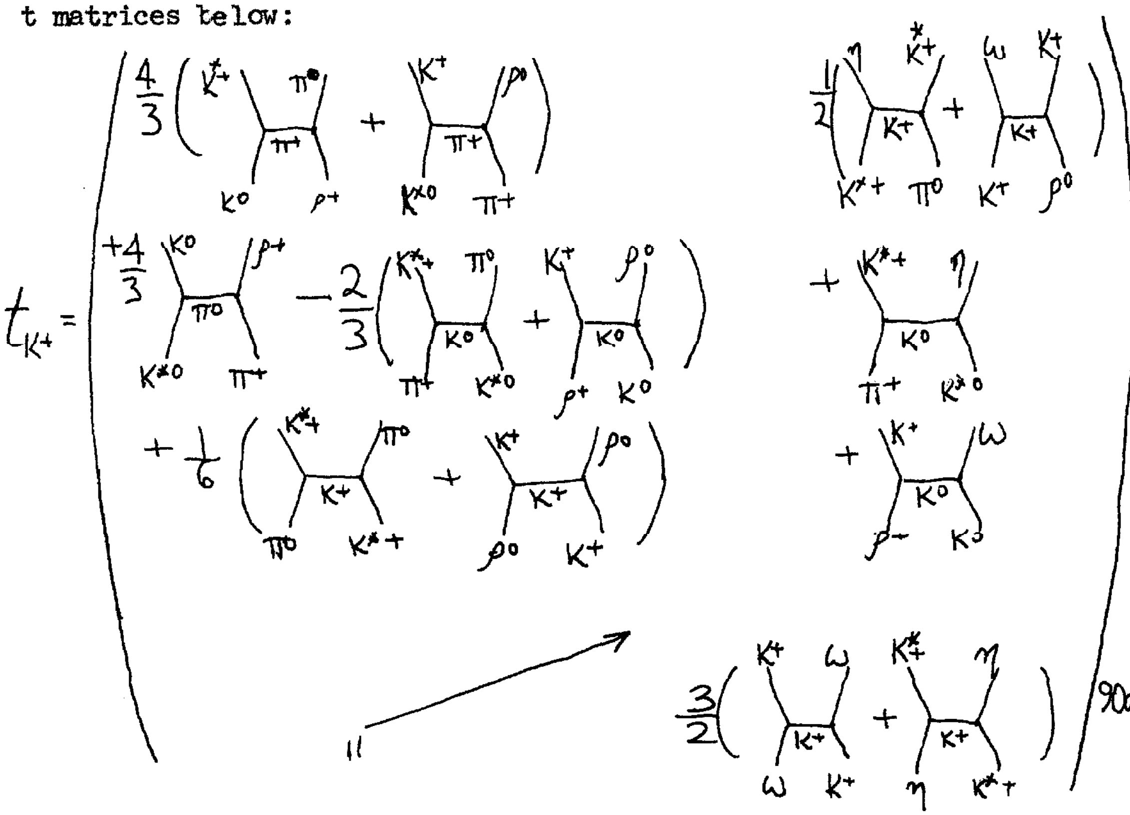
$$\delta^\rho \bar{d}_1 = -\delta^\pi\left[\frac{5}{3}\left(\frac{M^2}{\mu^2}\right)\bar{d}_1 + \frac{8}{3}(d_1 + d_s)\right] . \tag{89}$$

If, as was the case in previous π-ρ calculations, $\bar{d}_1 > 0$, while $d_1, d_s < 0$, then we obtain the estimate

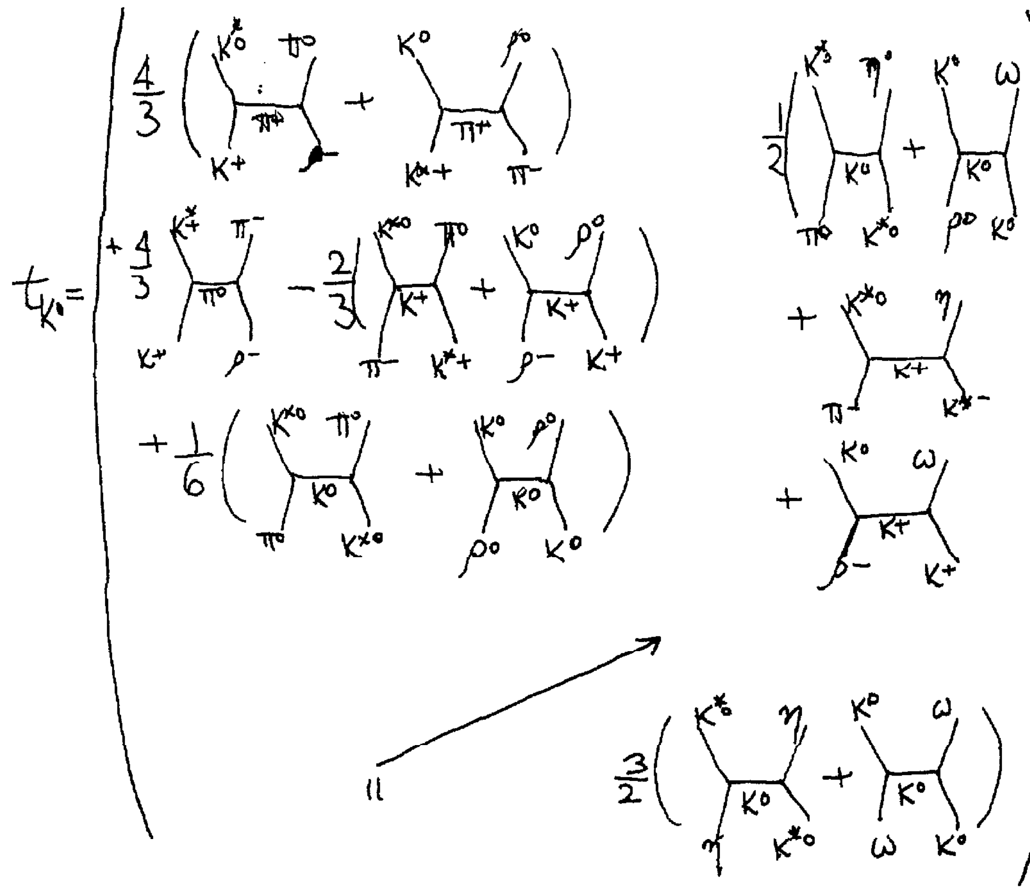
$$\delta^{\rho} \geqslant -\delta^{\pi} \times \frac{5}{3}\left(\frac{M^2}{\mu^2}\right) \qquad (89a)$$

Notice that again this implies $\delta^{\rho} < 0$, and as in the discussion of Section 8, this PS-V model of the PS octet gives a larger mass splitting than the PS-PS model of the V octet. The latter (V octet) estimate is probably closer to the truth, because the dynamical calculation, self consistently, of the PS_8 seems less likely in our PS-V model than the bootstrap of the V_8 in the PS-PS model.

For the sake of completeness, we include the calculation of δ^{K^*} from the PS-V model for PS_8. In the case of the K we now have the t matrices below:



(90a)



(90b)

One obtains δ det D_K from the above equations:

$$0 = \delta \det D_K = \delta^{K}(d_1 + 3d_s + \frac{2}{3} d_2) + \delta^{K^*} \bar{d}_1 . \qquad (91)$$

Using the condition of scale invariance of d referred to previously, this condition becomes:

$$\delta^{K^*}\bar{d}_1 \simeq \delta^{K} \left[\frac{4}{3} \frac{M^2}{\mu^2} \bar{d}_1 + \left(\frac{1}{3} d_1 - \frac{4}{3} d_s\right)\right] . \qquad (92)$$

From this we estimate that

$$\delta^{K^*} \sim + \frac{4}{3} \frac{M^2}{\mu^2} \delta^{K} . \qquad (92a)$$

This estimate is again slightly larger than that obtained from the PS_8-PS_8 model of V_8.

We have presented only a crude approximation to the calculations required in an SU_3 degenerate octet model, neglecting the effects of coupling constant shifts and photon forces. However, we consider it significant that both V_8 and less reliable PS_8 models make reasonable similar predictions about the vecton mass differences. Presumably, if one had proper models for these J^{π} channels and if one included the actual physical masses of particles as input, then the present scheme could be extended to provide the absolute scale of mass splittings as well as relations of the above type. Such a programme unfortunately appears to be very complicated, and beyond the scope of the present study.

Summary

We have adopted a pole model characterization of the strong interaction force in the π-π channel, and have examined order e^2 perturbations of the resultant dynamical equations which describe the composite ρ vector meson. For both elastic and inelastic calculations, the exchange mass dependence of the perturbations is quite sensitive, and more important than the photon exchange perturbations.

From the inelastic calculation, in particular, one obtains the estimate that $M_{\rho^0} - M_{\rho^+} \geqslant 11$ MeV. This general feature is expected, as was shown from a genuine two channel ρ calculation, and also, roughly, from a π calculation.

A calculation based on a SU_3 degenerate mass PS_8-PS_8 model of the vector mesons also gave a rough estimate for $\delta M^{2\rho}/M^2{}_\rho$ which agrees well with the above estimates. The estimate for δ^{K^*} is $K^{*+} - K^{*0} = - 5.5$ MeV.

Appendix A

Here we will examine two integrals occurring in the calculation of $\delta_e D)_f$: First

$$\int_0^2 du \frac{\ln u}{u + \gamma_q} \equiv I_q$$

$$\equiv \int_{\gamma_q}^{2+\gamma_q} \frac{\ln(x-\gamma_q}{x} dx, \ \gamma_q > 0 \qquad \text{(A1)}$$

This can be found to be(11)

$$\frac{(\ln 2)^2}{2} - \frac{(\ln \gamma_q)^2}{2} - \frac{\pi^2}{6}$$

$$+ 0(\gamma_q) \ , \qquad \text{(A2)}$$

taking into account that $\gamma_q \rightarrow 0$ as $\lambda \rightarrow 0$.

Next, we consider

$$\int_0^2 du \frac{\ln u}{u-|\gamma_k|} \equiv I_k \qquad \text{(A3)}$$

where the principle value of the integral is understood

$$= \int_0^{2|\gamma_k|} du \frac{\ln u}{u-|\gamma_k|} + \int_{2|\gamma_k|}^2 du \frac{\ln u}{u-|\gamma_k|} \ . \qquad \text{(A4)}$$

The first term of (A4)

$$= \mathrm{P}\int_0^{2|\gamma_k|} \frac{1}{u-|\gamma_k|} \left[\ln|\gamma_k| + \frac{u-|\gamma_k|}{|\gamma_k|} - \frac{1}{2} \frac{(u-|\gamma_k|)^2}{|\gamma_k|^2} + \right]$$

$$= 2 \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{4} \ . \qquad \text{(A5)}$$

Also

$$\int_{2|\gamma_k|}^{2} \frac{\ln u}{u-|\gamma_k|} = \int_{|\gamma_k|}^{2|\gamma_k|} \frac{\ln(x+|\gamma_k|)}{x}\, dx$$

$$\text{as } \lambda \to 0 = \frac{(\ln 2)^2}{2} - \frac{(\ln|\gamma_k|)^2}{2} + \frac{\pi^2}{6} \tag{A6}$$

$$+ 0(\gamma_k).$$

$\therefore$ as $\lambda \to 0$

$$P\!\!\int_{0}^{2} \frac{\ln u}{u-|\gamma_k|} = \frac{5}{12}\ \pi^2 - \frac{(\ln|\gamma_k|)^2}{2}$$

$$+ \frac{(\ln(2))^2}{2} \quad .$$

Appendix B.

In this appendix we sketch the derivation of equations (18a).

Using the condition that at $s = M_\rho^2$, Re D = 0, one easily obtains

$$\frac{\partial D}{\partial s} = \frac{1}{M - s_1}\left[\frac{\mathcal{a}G}{\pi} - \frac{s_1 - a}{M^2 - a}\right]$$

$$\frac{\partial D}{\partial M_{ex}^2} = -\frac{\mathcal{a}_M}{\mathcal{a}} + a_M\left[-\frac{1}{M^2 - a} + \frac{\mathcal{a}G}{\pi(s_1 - a)}\right], \qquad \text{(B1)}$$

which substituted into (8a) gives (18a). From Eq. (15) one has

$$G = +\frac{X\pi(s_1 - M^2) + \pi/\mathcal{a}(s_1 - a)}{M^2 - a} + X\frac{\pi(s_1 - M^2)}{M^2 - a}\,\Delta\,. \qquad \text{(B2)}$$

where Δ contains the effect of the electromagnetic driving force, i.e.,

$$\frac{N}{D'} = \left(\frac{N}{D'}\right)_{strong}(1 + \Delta_{em})$$

and $\therefore$

$$\Delta = \left(\frac{\delta_e N_f}{N} - \frac{\delta D'_f}{D'}\right), \qquad \text{(B3)}$$

To zeroth order in the electromagnetic interactions

$$\frac{\partial G}{\partial M_{ex}^2} = \frac{\partial G}{\partial \alpha} \alpha_M + \frac{\partial G}{\partial a} a_M$$

$$= \left(G - \frac{\pi}{\alpha}\right) \frac{a_M}{M^2 - a} - \frac{\pi(s_1 - a)}{\alpha(M^2 - a)} \frac{\alpha_M}{\alpha} \tag{B4a}$$

$$\frac{\partial G}{\partial M^2} = - \frac{X\pi}{M^2 - a} - \frac{G}{M^2 - a} \quad . \tag{B4b}$$

Substituting these results into Eq. (8b) gives Eq. (18b).

Appendix C

In this appendix we consider the diagram π-$\pi \rightarrow \pi\ \omega$ via ρ exchange in order to get some feeling for the inelasticity mass dependence.

The amplitude $t^{J=1^-}$ corresponding to $e^{i\delta}\sin\delta/q$ is given by

$$t = \frac{\sqrt{2}}{3M\pi} \sqrt{\frac{\gamma^2_{\rho\pi\omega}}{4\pi} \frac{\gamma^2_{\rho\pi\pi}}{4\pi}} \left[Q_2(\theta) - Q_0(\theta) \right] < 0\ , \qquad \text{(C1)}$$

where

$$\theta = \frac{s^2 - s(\mu_1{}^2 + \mu_3{}^2 + \mu_4{}^2 + M_\omega{}^2) + (\mu_3{}^2 - \mu_4{}^2)(\mu_1{}^2 - M_\omega{}^2) + 2M_\rho^2\, s}{4s\ p_\pi\ p_\omega} \qquad \text{(C1a)}$$

The labelling of particles is given in Fig. 2.

The amplitude (C1) is the I = 1 amplitude and comes from the sums of graphs with different $\pi \cdot \rho$ charge states. This is indicated in Fig. 3.

The M_ρ dependence is already clear from this: the charged channel has an ρ^+ exchange, i.e. is "lighter" by amount δM^2; therefore, $|t|^2$ charged will be "bigger" by amount proportional to δM^2, i.e., expect that

$$\frac{\partial R}{\partial M^2} \propto \delta M^2\ x\ (> 0)\ .$$

If we write $Q = Q_2(\theta) - Q_0(\theta)$, then

$$\frac{\partial t}{\partial M_\rho^{\ 2}} = \frac{\sqrt{2}}{3M\pi} \sqrt{\left(\frac{\gamma^2_{\rho\pi\omega}}{4\pi}\right)\left(\frac{\gamma^2_{\rho\pi\pi}}{4\pi}\right)} \frac{\partial Q/\partial\theta}{4p_i pf} \ (\ -\ \delta M_\rho^{\ 2}) \cdot \qquad (C2)$$

Now

$$R = \frac{q_{\pi\omega}}{q_{\pi\pi}} \frac{|t|^2_{\pi\pi\ \pi\omega}}{|t|^2_{\pi\pi\ \pi\pi}} + 1$$

$$\therefore + R_{\rho^+} - R_{\rho^0} \equiv \delta R = 2(R-1)\left[\frac{\delta|t_{inel}|}{|t_{inel}|} - \frac{\delta|t_{el}|}{|t_{el}|} + \frac{\delta(q_{\pi\omega}/q_{\pi\pi})}{(q_{\pi\omega}/q_{\pi\pi})}\right] . \qquad (C3)$$

From (C2), using fact $t < 0$, we have

$$\delta|t| = -\ \delta t$$

$$\equiv -\ \frac{\partial t}{\partial M^2_\rho}\ \delta M^2_\rho \propto (\delta M^2_\rho)\ x\ (> 0)\ ,$$

as asserted above.

Also note that in elastic π-π scattering via ρ echange

$$\delta|t_{el}| \propto (\ -\ \delta M_\rho 2)\ x\ (> 0)\ ,$$

since the charged channel has the heavier exchange mass and therefore contributes weaker force. One concludes then that

$$\delta R \propto (\delta M^2)\ x\ (> 0)\ . \qquad (C4)$$

After tedious algebra, one can evaluate the effects of all the mass shifts in the diagrams of Fig. 3. One then obtains

$$
\begin{aligned}
\delta R &\equiv R(+) - R(-) \\
&= \sqrt{2}(R-1) \left[\frac{\partial q/\partial \sigma}{4p_\pi p_\omega Q}\right] \left[\frac{1}{2sp_\omega^2} \begin{pmatrix} s(M_\rho^2 + M_\omega^2 - \mu^2) \\ - (M_\omega^2 - \mu^2)(M_\omega^2 + \mu^2 - M_\rho^2) \end{pmatrix} \delta\mu^2 \right. \\
&\qquad \left. - \frac{1}{4p_\pi^2} (2M_\rho^2 - M_\omega^2 + \mu^2)\,\delta\mu^2 - \delta M_\rho^2 \right] \\
&\quad + \delta G \text{ terms} \\
&\quad + \text{terms from } (\pi\text{-}\pi) \qquad (C5) \\
&= \delta\mu^2 \times (< 0) + \delta M_\rho^2 \times (> 0) .
\end{aligned}
$$

One can understand this result (for $\delta\mu^2$) as follows. The π-ω momentum is always less than the π-π momentum and is therefore more sensitive to mass changes. In the charged channel, there is a heavier π in $\pi\omega$ channel, this decreases the π-ω momentum and hence decreases the size of the amplitude (which must be $\propto p_{\pi\omega}^2$ near threshold). Similarly, in the π-π channel, the charged channel has the lighter π's and hence contributes a larger amplitude, which contributes a term $\sim (\delta\mu^2) \times (< 0)$ to δR. Similarly, $\delta(q_{\pi\omega}/q_{\pi\pi})$ is proportional to $\delta\mu^2 \times (< 0)$. So finally, from examining the mass dependences of the ρ exchange diagrams, we guess that

$$\delta R \propto (\delta\mu^2)(< 0) + (\delta M^2)(> 0). \qquad (C6)$$

Appendix D

This section discusses the effect of the pion electromagnetic form factor on the determinantal estimate for δD_e. A reasonable approximation to the form factor is (when $t < 0$)

$$F_\pi(t) = \frac{-M_\rho^2}{t-M_\rho^2} \quad . \tag{D1}$$

With this, the force term equivalent to the square bracket of Eq. (21) becomes

$$\frac{-M^4}{(t-M^2)^2}\left[1 + \frac{2s-4\mu^2}{M^2}\right] + \frac{2s-\Sigma}{t-M^2} - \frac{2s-\Sigma}{t-\lambda} \quad . \tag{D2}$$

First, consider the $1/t-M^2$ term. As in Section 5, it is easiest to calculate the contribution to D by first integrating w.r.t s' and then projecting out the relevant partial wave (here $\ell = 1$). No new problem arises for the $1/t-M^2$ term, and one obtains its contribution to D to be (I_q k are defined in Appendix 1) as in Eq. (D3), where

$$a_q = 1 + M^2/2q^2, \quad a_k = 1 + M^2/2k^2$$

$$= \quad + \frac{e^2}{8\pi^2}\left[\frac{2s-s_1}{4q^2} \ln\left(\frac{2M^2}{2s-s_1}\right) \mathcal{Q}_1(a_q) - \frac{2s_0-s_1}{4k^2} \ln\left(\frac{2M^2}{s_1-s_0}\right) \mathcal{Q}_1(a_k)\right]$$

$$
-\frac{e^2}{8\pi^2}\left[\frac{2s-s_1}{4q^2}\left(1-\ln 2+\frac{a_q}{2}\,I_q\right) - \frac{2s_0-s_1}{4k^2}\left(1-\ln 2+\frac{a_k}{2}\,I_k\right)\right] \tag{D3}
$$

Proceeding similarly with the $1/(t-M^2)^2$ terms one obtains

$$
\begin{aligned}
&-\frac{e^2}{8\pi^2}\left[\frac{2s+M^2-s_1}{4q^2}\,\ell_1\left(1+\frac{M^2}{2q^2}\right) - \frac{2s_0+M^2-s_1}{4k^2}\,\ell_1\left(1+\frac{M^2}{2k^2}\right)\right] \\
&-\frac{e^2M^2}{4\pi^2}\left[\left(\frac{a_q}{a_q^2-1} - \ell_0\left(1+\frac{M^2}{2q^2}\right)\right)\left(\frac{2s+M^2-s_1}{16q^4}\right)\ln\left(\frac{s-s_1}{2M^2}\right)\right. \\
&\qquad\left. -\left(\frac{a_k}{a_k^2-1} - \ell_0\left(1+\frac{M^2}{2k^2}\right)\right)\left(\frac{2s_0+M^2-s_1}{16k^4}\right)\ln\left(\frac{s_1-s_0}{2M^2}\right)\right] \\
&-\frac{e^2M^2}{8\pi^2}\left[\frac{2s_0+M^2-s_1}{16k^4}\times\left(I_k + a_k\frac{d}{d\gamma_k}I_k\right) - \frac{2s+M^2-s_1}{16q^4}\times\left(I_q + a_q\frac{d}{d\gamma_q}I_q\right)\right].
\end{aligned} \tag{D4}
$$

In obtaining this, one employs the fact that

$$
\int\frac{\ln u\,du}{(u+\gamma)^2} \equiv -\frac{d}{d\gamma}\int\frac{\ln u\,du}{(u+\gamma)} .
$$

FIGURE II-1 : ELECTROMAGNETIC CORRECTIONS to π – π SCATTERING

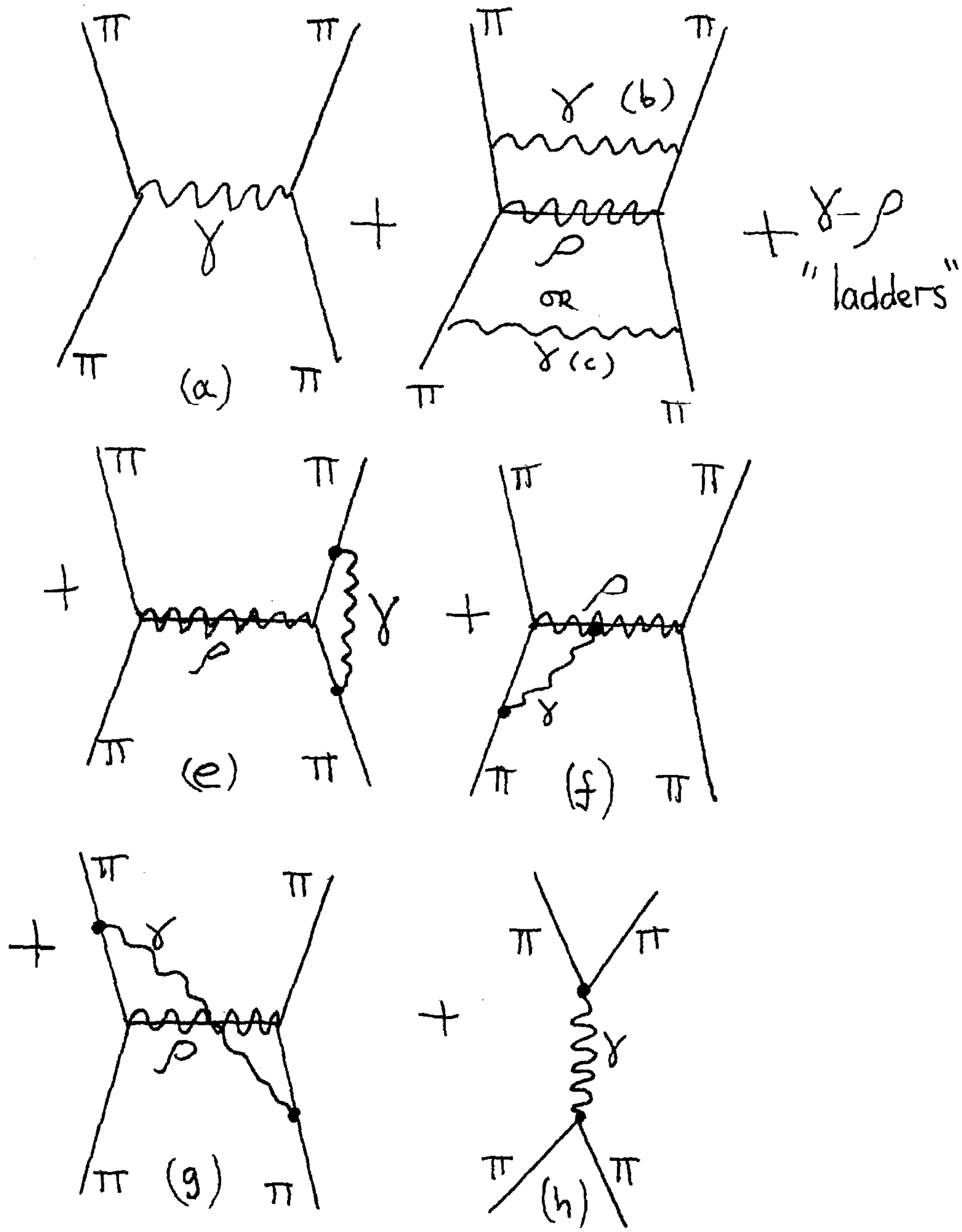


FIGURE II-2: INELASTIC BORN TERM

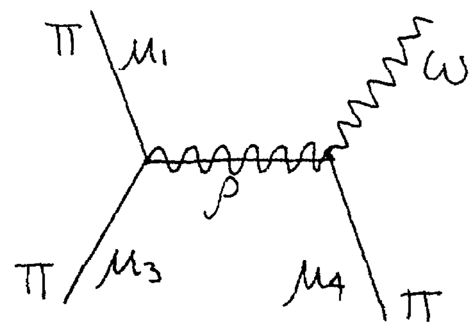


FIGURE II-3
I=1 ππ-πω charge states:

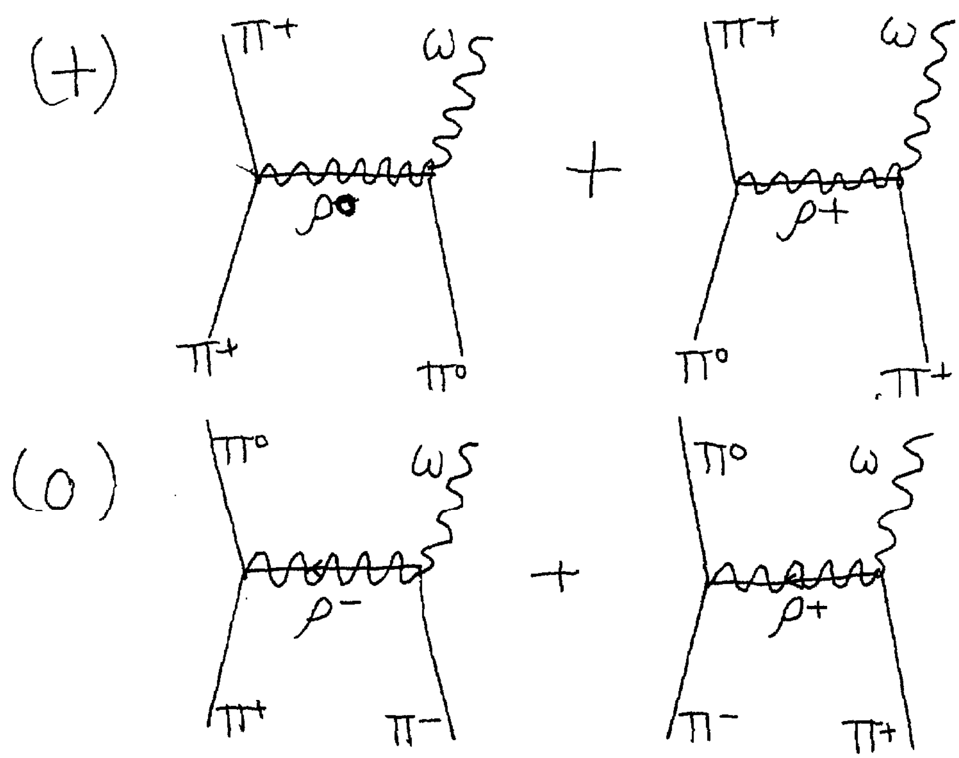


FIGURE II-4

I=1 $\pi\pi - \pi\pi$ charge states

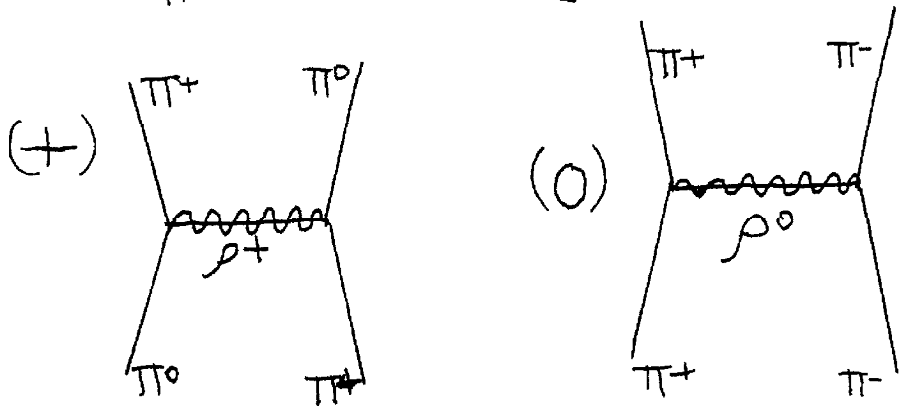
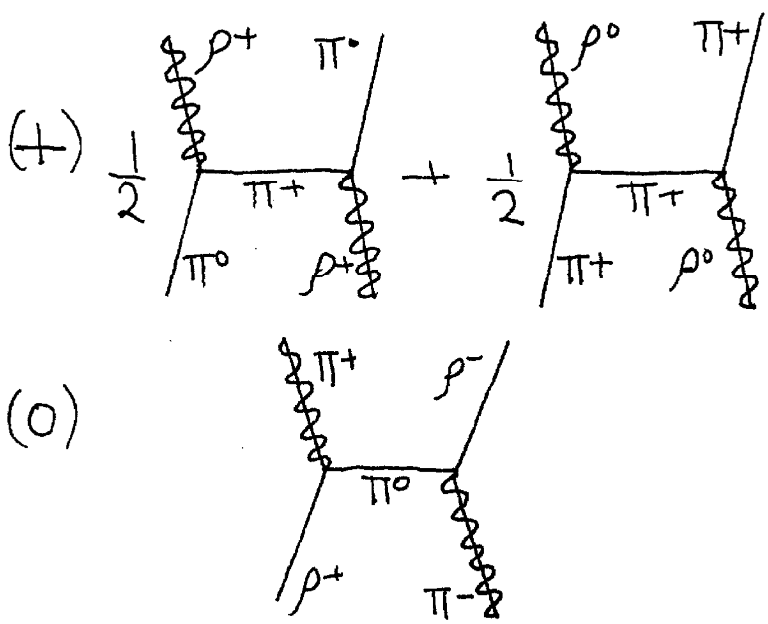


FIGURE II-5

I=1 $\pi\rho - \pi\rho$ charge states



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