# A DUAL QUARK MODEL WITH SPIN

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Stephen Dean Ellis

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# ABSTRACT

A dual quark model is developed from the usual Veneziano model by explicitly including the Dirac spin of the quarks. Resonances appear without the parity doubling and new ghosts present in previous models with spin. This is accomplished by eliminating the contributions of the negative parity components (MacDowell twins) of the spin $\frac{1}{2}$ quarks through the introduction of fixed J-plane cuts. The resonances belong to an $SU_6$ symmetric spectrum identical, on the leading trajectory, with that of the usual static symmetrical quark model. All resonances couple via $SU_{6_W} \times O_{2_{L_z}}$ symmetric vertices and the model factorizes with essentially the same degeneracy as the usual Veneziano model. As a consequence of requiring these two features the model acquires further new structure which is studied in detail in terms of the asymptotic behavior of the model. This new structure leads to unavoidable "background" contributions to the imaginary parts of the amplitudes not present in previous dual models. This situation is examined and interpreted in the language of Finite Energy Sum Rules.

In order to test the basic features of the model explicit calculations are made for the case of pion-nucleon scattering in the Regge limit. To make the numerical work easier a somewhat simplified version of the model is used. Although the results of the calculations are suggestive of reasonable J-plane structure for the various amplitudes, i.e., the location of Regge pole-fixed cut

interference is reasonable from the standpoint of the data, the overall kinematic behavior of the amplitudes is definitely not compatible with what is measured. However, it is noted that this kinematic behavior depends strongly on those details of the model which were simplified in the present study. If such models are to be unambiguously and successfully tested against data, future studies must treat these details more completely and realistically, including both unitarity and symmetry breaking effects.

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# INTRODUCTION

The research to be described in the following pages involves an effort to combine two of the most interesting concepts to appear in hadron physics in the last eight years. The first is the idea that hadrons can be thought of as composites of spin $\frac{1}{2}$, $SU_3$ triplet objects called quarks.[1] The second is the concept of duality.

In the quark model we assume that mesons are composed of a quark and an anti-quark and that baryons are three quarks. If we then add some internal angular momentum as if, for example, the quarks are bound by a harmonic potential and require that only states totally symmetric in the quarks appear, we can reproduce much of the observed regularity of the particle spectrum.[2] If we further assume that the strong interactions are the same for all the quarks then the strong interactions must be invariant under the operators of the group $SU_6$. Thus we expect resonances to appear in mass degenerate multiplets which are representations of $SU_6$. This is at least approximately true in nature and we shall henceforth work in this limit as a first approximation.

The $SU_6$ symmetry of the strong interactions can, of course, be studied independently of the quark model, but we shall retain the notion of quarks, although not requiring them to exist as free, asymptotic states. This will allow us to keep an "intuitive" picture of what is happening and is suggestive of how certain questions should be answered. In particular, the quark picture leads to quite definite predictions as to how various objects should couple and recent work has stimulated new interest in this field.[3,4] As pointed out in Ref. 4, the simplest sort of coupling that one can imagine for the three $q\bar{q}$ mesons shown in Fig. I-1 is that in the collinear frame (as shown all momenta are along the z axis) quarks a and d are unaffected while b and c annihilate in a $^3P_0$ state to conserve angular momentum and parity. If we further assume that the transverse momenta of b and c are negligible, then the pair bc has $L_z=0$ and thus $S_z=0$. It turns out that a vertex with this structure has a well defined symmetry called $SU_{6W} \times O_{2_{Lz}}$, where the $O_{2_{Lz}}$ reminds us that it is a collinear symmetry invariant under rotations about the z axis. This vertex symmetry has already been well studied and details, which are not required here, can be found in the literature.[5] We only note that the model we are studying does exhibit this symmetry and that many of its predictions are in reasonable agreement with data. There are certain glaring discrepancies,[6] however, which we shall return to later.

The other major concept to understand is duality. This is a somewhat newer and less well understood idea and, although there are many excellent reviews of the subject,[7] we shall discuss much of its basic development and emphasize the points which are of interest here. The basic question which duality seeks to answer can be illustrated by considering the two "Feynman" diagrams in Fig. I-2. The question is, how do we treat these two seemingly different diagrams? In Q.E.D. this might be $e^+e^-$ scattering. Figure I-2a is an s channel bound state, the photon, and Fig. I-2b is a t channel photon exchange.

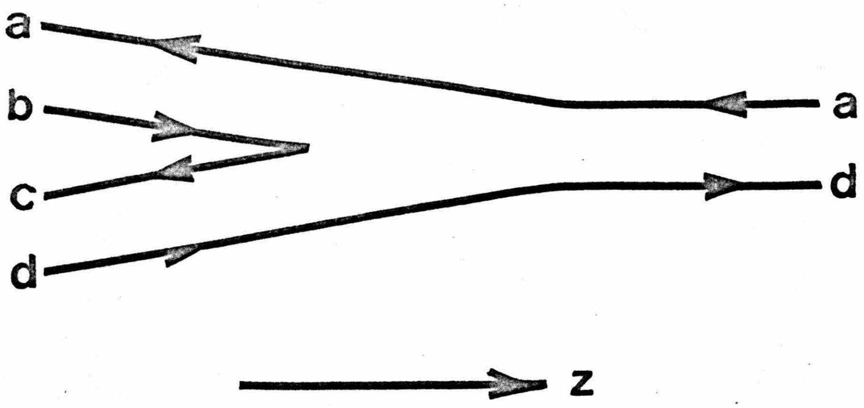



Fig. I-1 Three meson vertex.

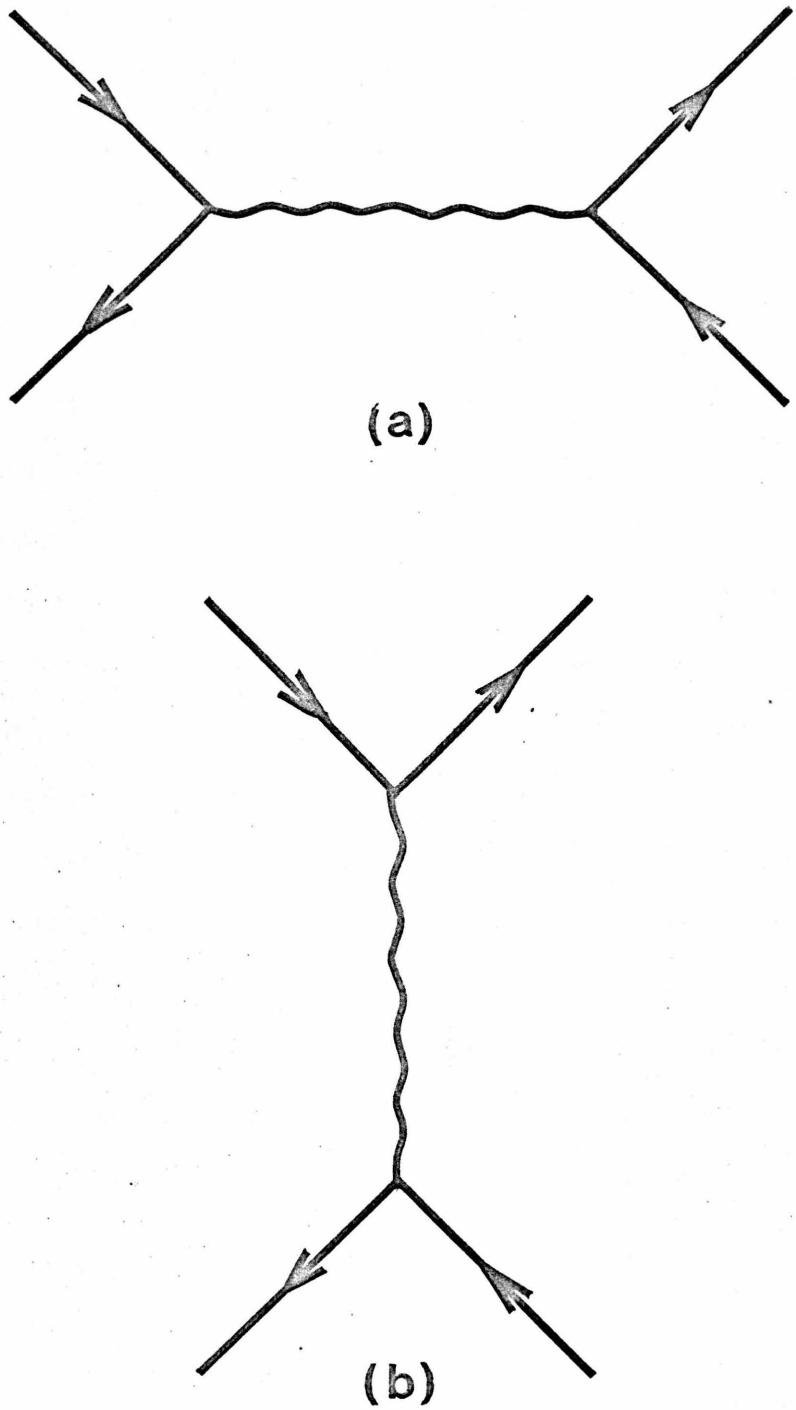



Fig. I-2 Feynman diagrams : (a) direct channel; (b) exchange channel.

In this case, a perturbation approach is appropriate and it is clear that the first order diagrams, a and b, should be added. The next level of complexity is to consider positronium, a real $e^+e^-$ bound state. It no longer seems possible to say that it contributes only to diagram I-2a since it certainly contains some t channel photon exchanges.

Finally, in hadron physics the wavy line in 2a now represents a sum of resonances in the direct channel and that in 2b is a Regge pole exchange (which can also be thought of as a sum of resonances in the exchange channel). Perturbation theory does not seem applicable in this case, and the situation is somewhat less transparent. If one still wants to sum the two diagrams he gets the "interference model." However, the basic idea of duality is that the two diagrams are somehow equivalent and that to add them would lead to double counting. Thus, duality would say that the description of a scattering amplitude by its resonances and the description in terms of its Regge behavior are equivalent, the former being more generally useful at low energies and the latter at high energies. The Regge behavior is, of course, an asymptotic expansion and somehow less "physical" than a resonance description. The duality scheme says that we could ignore questions about Regge poles if only we could completely specify the total resonance content of a given reaction in, say, the s and u channels.

A more concise approach is offered by using Finite Energy Sum Rules (FESR). Consider a scattering amplitude $A(\nu,t)$ which we shall assume to have the following expansion for $\nu$ large, t fixed.

$$A(\nu,t) = \sum_i \beta_i(t) \frac{1 \pm e^{-i\pi\alpha_i(t)}}{\sin\pi\alpha_i(t)} \nu^{\alpha_i(t)} \qquad \text{(I-1)}$$

i.e., it has a Regge expansion. Here the ± sign depends on the crossing properties ($s \leftrightarrow u$, $\nu \sim s-u$) of the amplitude which we take to be well defined. Then if we use the contour shown in Fig. I-3, we find ($A(\nu,t)$ is analytic in $\nu$ except for the 2 cuts)

$$\int_c \nu^n A(\nu,t)\, d\nu = 0 \qquad \text{(I-2)}$$

(n integer) .

If N, the radius of the semi-circle, is large, we can use Eq. I-1 to do the integral on the semi-circles. The result is the standard FESR form

$$\int_0^N \nu^n d\nu \, \text{Im}\, A(\nu,t) = \sum_i \beta_i(t) \frac{N^{\alpha_i(t)+n+1}}{\alpha_i(t)+n+1} \qquad \text{(I-3)}$$

where we have assumed that $A(\nu,t)$ has the crossing behavior $A(\nu,t) = (-)^{n+1} A(-\nu,t)$. The opposite behavior just gives $0 = 0$ (recall imaginary parts have opposite crossing behavior from real parts). To get the FESR we used only analyticity, definite crossing behavior, and Regge asymptotic behavior, i.e., no strong "background integral" contribution.[8] Physics enters the problem when we try to determine 1) how large N must be? 2) which and how many Regge poles are needed on the right-hand side? 3) what should be included on the left-hand side? Note that although Eq. I-3 only involves imaginary parts, the real parts can be found, in principle, from dis-

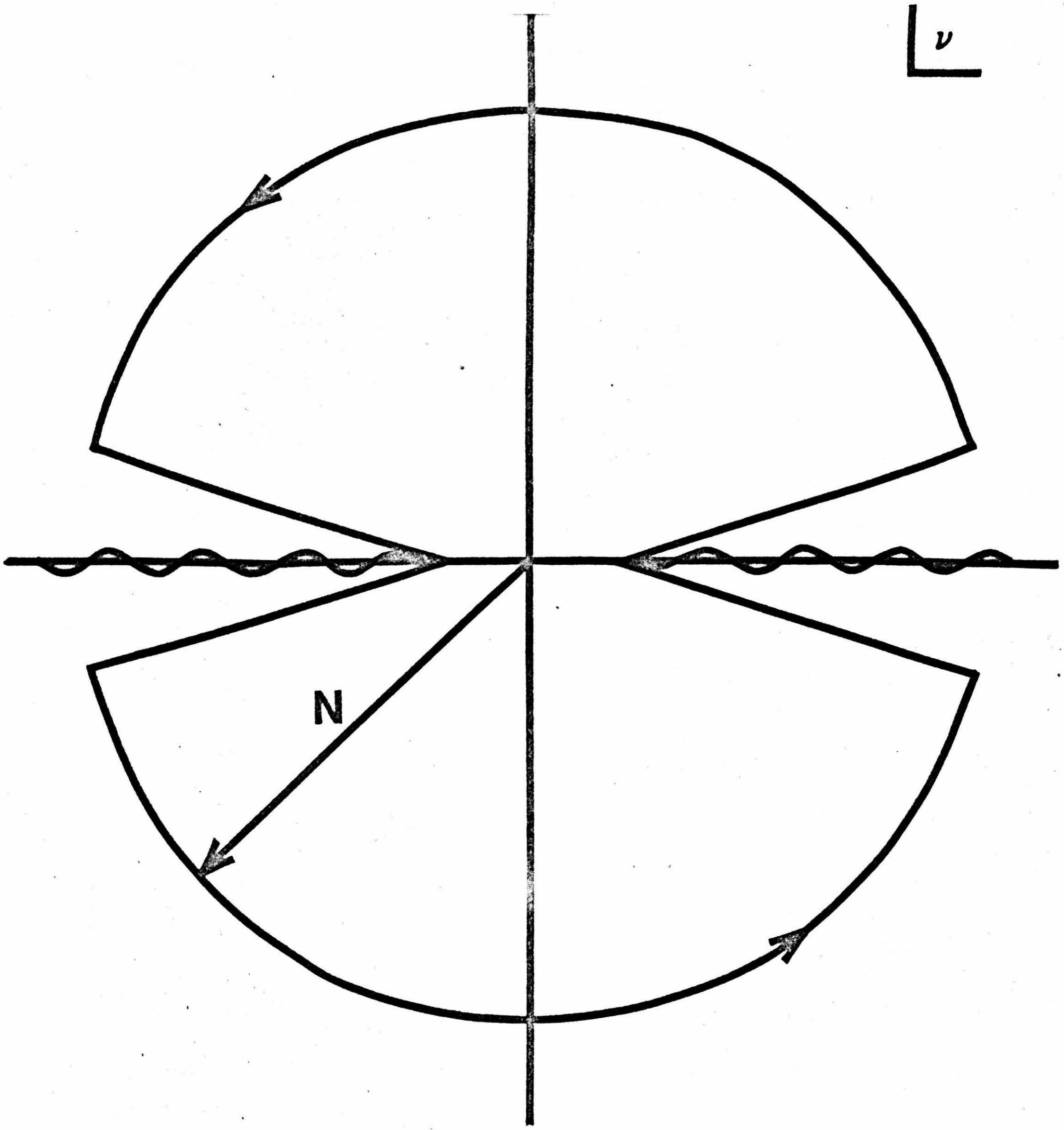



Fig. I-3 Contour for FESR integral.

persion relations.

The first application of the FESR which led to answers to these questions was that of Dolen, Horn and Schmid.[9] They studied $\pi N$ charge exchange where $\rho$ exchange is known to dominate the high energy data and $\alpha_\rho$ and $\beta_\rho$ are well determined. They continued this Regge fit down to intermediate energies where they described the left-hand side of Eq. I-3 with phase shift data (essentially pure resonances). As shown in Fig. I-4, they found that the Regge fit was a good average to the resonance data. This led to the duality suggestion that the Regge description is equivalent to the resonance description, at least on the average. To get the local structure in the low energy resonance region one would have to include a large number of Regge exchanges. In the "interference model" the Regge poles would appear as a background at low energy on top of which would appear the resonances. Thus another way to state duality is that in a FESR which has Regge poles on the right, the left-hand side can be saturated simply by resonances (resonance saturation).

Another striking feature found by Dolen, Horn and Schmid was the t dependence of the FESR. Regge analysis of charge exchange indicates that the $B^{(-)}$ amplitude, which is the I = 1, helicity non-flip amplitude in the t channel center of mass and which dominates $d\sigma/dt$, has a zero near $t \cong -0.6$ as indicated by the dip in $d\sigma/dt$ and predicted by the signature factor of the $\rho$ $(+1 - e^{-i\pi\alpha_\rho(t)})$ which vanishes for $\alpha_\rho = 0$. Similarly the cross over point where

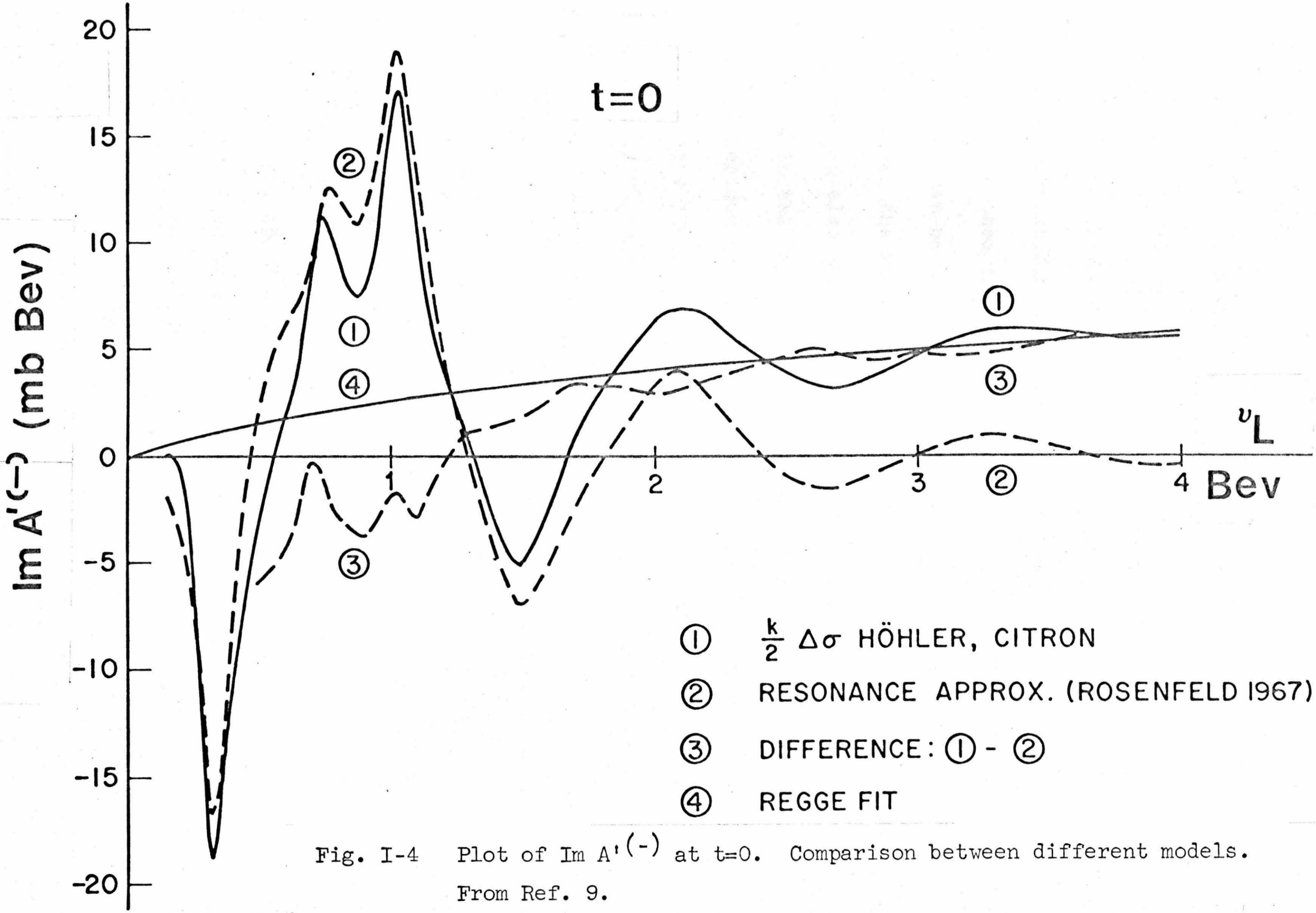



Fig. I-4  Plot of Im $A'^{(-)}$ at t=0. Comparison between different models. From Ref. 9.

$$\frac{d\sigma}{dt}(\pi^-N \text{ elastic}) - \frac{d\sigma}{dt}(\pi^+N \text{ elastic}) = 0$$

suggests that Im $A'^{(-)}$ (the nonflip amplitude) vanishes at $t \approx -0.2$. Dolen, Horn and Schmid found that not only did the resonance saturated left-hand side exhibit these zeroes, but that each of the dominant resonances had them individually as shown in Fig. I-5.

We have seen that in principle, since Regge poles are "dual" to resonances, the Regge parameters can be determined from resonance data. We can also invert the process and discover direct channel resonances in the Regge exchanges. The general idea(10) is that, if the Regge residue $\beta(t)$ has structure in t (e.g. zeroes), when we do a partial wave projection in terms of $P_\ell(\cos\theta_s)$ we will find definite structure in the partial wave amplitude, $a(\ell,s)$. Further, since fixed t structure corresponds to varying $\theta_s$ as we vary s, the partial wave structure will vary with s and can reproduce s channel resonances. Thus the structure of the Regge residue function is "dual" to the direct channel resonances.

This leads us to one of the most interesting and successful predictions of the duality scheme: amplitudes with low energy resonances ($\bar{p}p$, $\pi p$, $K^-p$) should have dips in the angular distribution which persist even at high energies (Regge region), whereas non-resonanting amplitudes (pp, $K^+p$) should have smooth angular distributions. These predictions are quite well satisfied in the data. In an interference model where the Regge behaved part of the amplitude is just a background contribution at low energies, the presence or

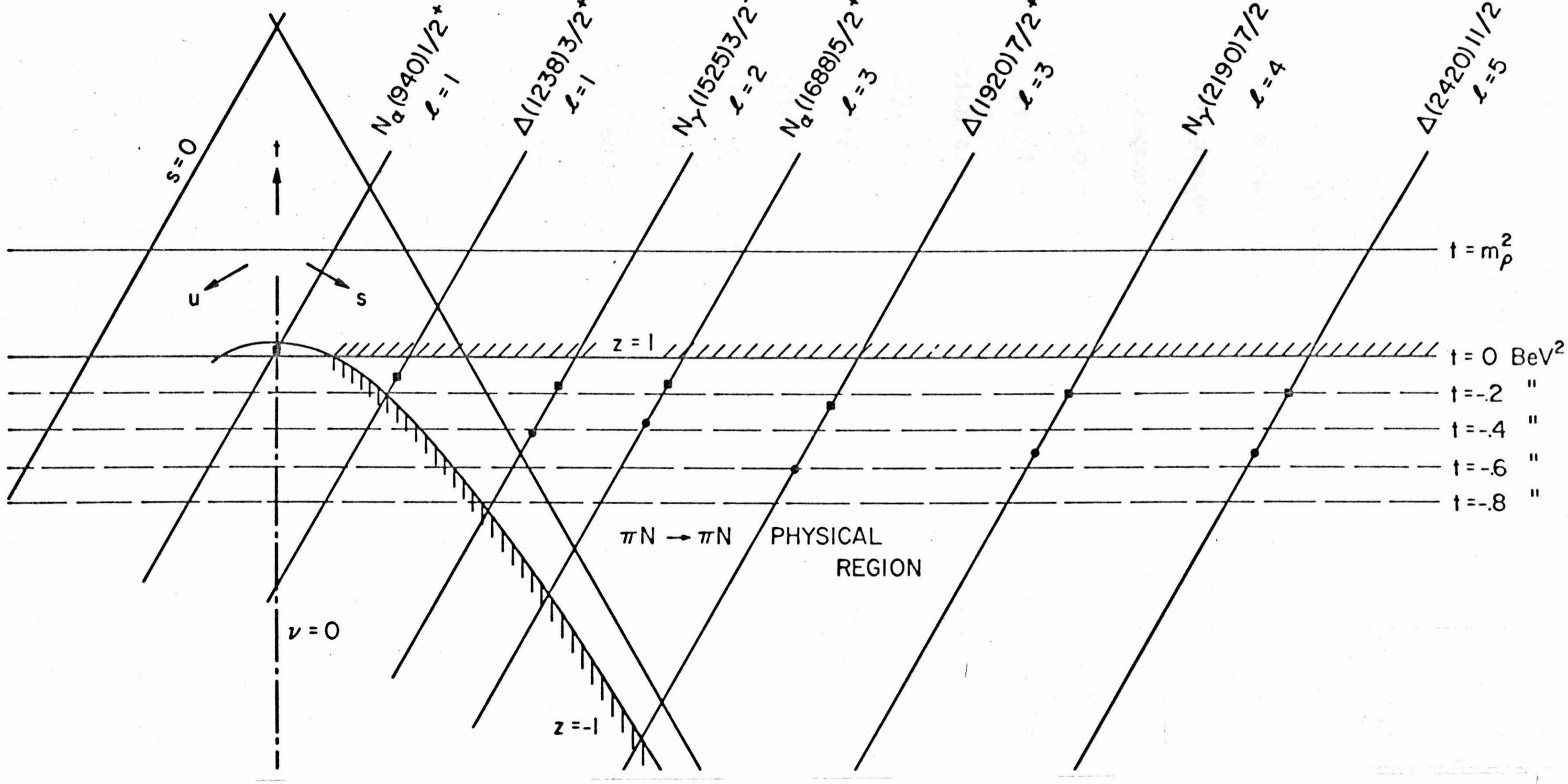



Fig. I-5 First zeroes of the prominent resonances on the Mandelstam plot for πN system. The dots apply to $B^{(-)}$ and the squares to $A'^{(-)}$. From Ref. 9.

absence of resonances makes no statement about the angular structure of the Regge residue. Another point to note is that for fixed t structure the dominant partial wave will have the dependence $\ell_{dom} \approx \sqrt{s}$ as might be expected from a simple impact parameter picture where the hadrons have a definite radius.[11]

Before proceeding we must consider the technical question as to what is a resonance. This is crucial since resonances play a dominant role in our study. Certainly resonances should have definite quantum numbers, they should factorize and they should be observed in both production experiments ($\pi N \rightarrow \pi N^*$) and formation experiments ($\pi N \rightarrow N^* \rightarrow \pi N$). However, the problem of separating resonances and background in the data when the widths become large is very difficult. It is so difficult in fact that all reasonable mathematical realizations of dual amplitudes presently in use (e.g. Veneziano) have had to avoid the problem by working in the narrow resonance limit where resonances are poles on the real s axis. Thus one must give up explicit unitarity. Then diffraction (Pomeron exchange), which is a direct channel unitarity effect, must play a special role in these models. We are led to the "two component" theory of Harari and Freund[12] where the "Pomeron exchange" (vacuum quantum number exchange) is taken to be dual to the direct channel background and the resonances give the ordinary Regge exchanges.

A good example of the problem of resonance identification is the $A_1$. The enhancement in the cross section near the predicted mass of the $A_1$ can also be explained in terms of threshold effects for $\pi\rho$ production plus the strong momentum dependence of the propagator for the exchanged $\pi$ (Deck effect).(13) One would like to say that the two explanations are "dual" to each other and therefore equivalent. However, a problem arises when we attempt to single out an individual resonance. In terms of the FESR, we would want to change the range of integration form being 0 to N to the range $M_{A_1}^2 - \Delta$ to $M_{A_1}^2 + \Delta$ (we could do this by subtracting 2 normal FESR's with the endpoints $N_1 = M_{A_1}^2 - \Delta$ and $N_2 = M_{A_1}^2$ ). The scheme which tries to utilize FESR's modified in this way is called "local duality." Since only the imaginary part of a resonance is localized in s, one should only apply local duality to imaginary parts. Thus it is difficult to say anything about $\pi$ exchange which is primarily real. The whole question of local duality is still poorly understood.

Quarks now enter our duality picture. We wish to postulate as a general rule the observed fact that exotic resonances do not appear in strong interactions. The exotic classification is most easily understood in terms of quarks. Exotic mesons are those which cannot be made from $q\bar{q}$ states whereas exotic baryons are those not contained in q q q states. This leads us into the exciting field of exchange degeneracies,(14) which we shall only discuss peripherally here in order to show the important role played by the quark picture in the application of duality as a predictive tool. The general idea is that if a given channel, say $\pi^+\pi^+$, is not allowed to resonate then the imaginary part of the amplitude given by the resonances must vanish. From the duality principle it follows that the imaginary part of the Regge residue also must vanish. Since the imaginary part in

Eq. I-1 comes from the $e^{-i\pi\alpha(t)}$ term in the signature factor $(1 \pm e^{-i\pi\alpha(t)})$, the absence of an imaginary part results, in general, from the simultaneous exchange of two trajectories with opposite signature and the same $\alpha$ and $\beta$, i.e.,

$$(1 + e^{-i\pi\alpha(t)}) + (1 - e^{-i\pi\alpha(t)}) = 2 = \text{real}.$$

This is just exchange degeneracy. Studies in the meson-meson system lead to exchange degenerate multiplets of mesons, e.g. the $\rho\omega f A_2$ system which is well satisfied in nature. Studies in the meson baryon system give somewhat less satisfactory results and the scheme faces serious difficulties for baryon-baryon scattering. In fact, recent studies in the Veneziano formalism suggest that maybe we should expect exotics to appear in the $\bar{B}$ B channel.[15]

For our purposes the essential point of this discussion is that all the important results (positive and negative) can be summarized in terms of quark-duality diagrams.[16] The rules for writing down such diagrams are simply: 1) All baryons are made out of three quarks, 2) All mesons are made out of a quark and an antiquark (direction of arrow determines quarks and antiquarks), 3) The quark and antiquark from the same particle cannot annihilate ("Zweig's Rule"), 4) Only standard meson or baryon (i.e., $q\,\bar{q}$, $q\,q\,q$ or $\bar{q}\,\bar{q}\,\bar{q}$) channels are allowed to resonate. Such diagrams are illustrated in Fig. I-6 where the usual channel labels are given. For example, Fig. I-6a might represent $\pi^+\pi^-$ scattering in the s channel. It is also $\pi^+\pi^-$ in the t channel. However, in the u channel, this diagram looks as shown in Fig. I-6b, i.e., is exotic $(q\,q\,\bar{q}\,\bar{q})$ in the $\pi^+\pi^+$ channel.

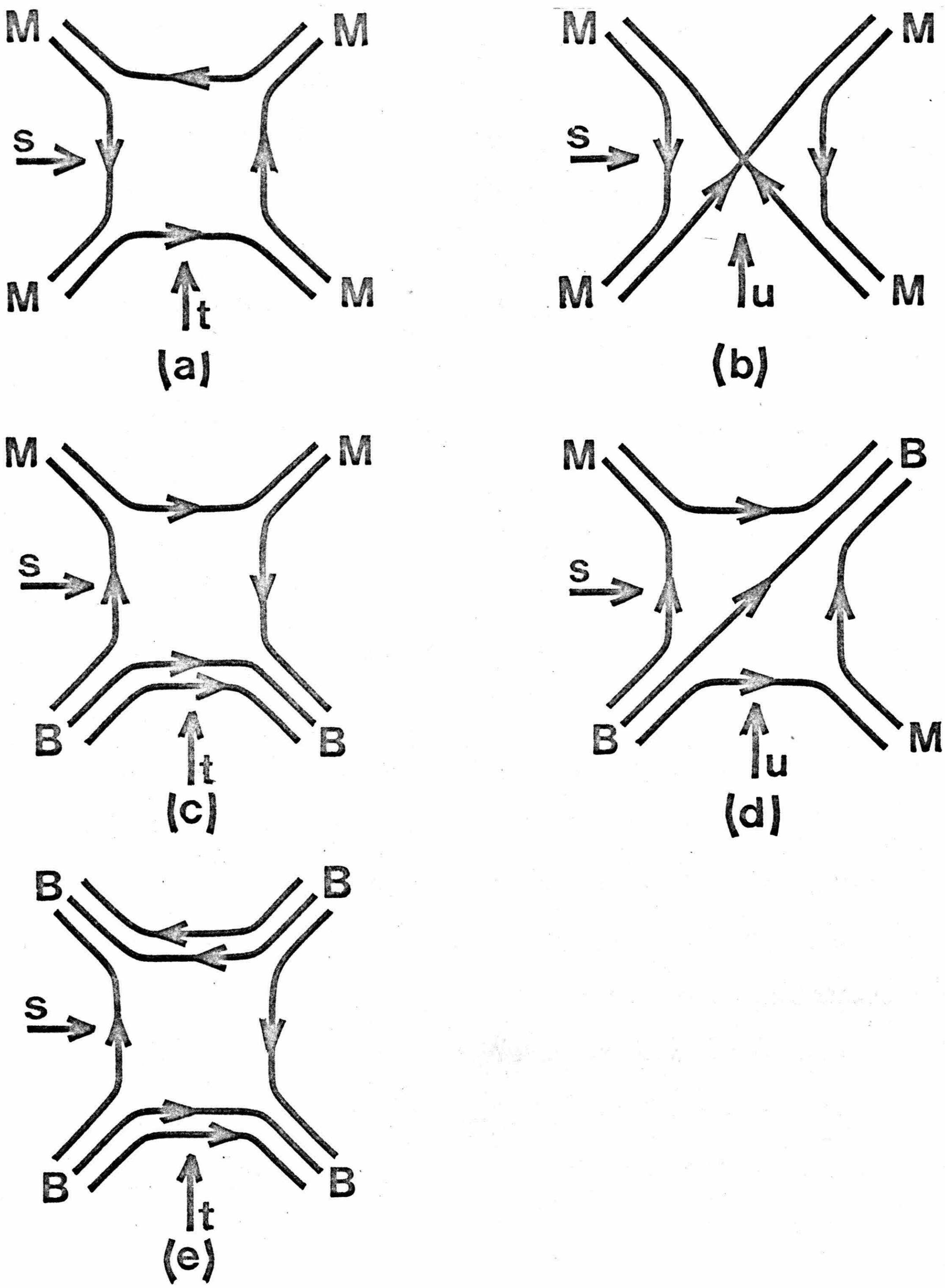



Fig. I-6 Quark diagrams for: (a) forward meson-meson scattering; (b) backward meson-meson scattering; (c) forward meson-baryon scattering; (d) backward meson-baryon scattering; (e) baryon-antibaryon scattering (illegal diagram).

Figures 6c and 6d give the expected MB system diagrams, i.e., in $SU_3$ notation $\underset{\sim}{1}$ and $\underset{\sim}{8}$ in the t channel and $\underset{\sim}{1}, \underset{\sim}{8}$ and $\underset{\sim}{10}$ in the s and u channels. Finally, Fig I-6e shows the illegal diagram, i.e., it contains one channel with $q\ q\ \bar{q}\ \bar{q}$, and another with $q\ q\ q\ \bar{q}\ \bar{q}\ \bar{q}$, which appears in $\bar{B}\ B$ scattering and causes the problem mentioned above.

We note that the question of exchange degeneracies is basically a question of finding a solution of the $SU_3$ crossing matrices. While the duality diagram is certainly a solution to this problem, it is not necessarily the most general. However, it is indeed very suggestive and our task henceforth will be to try to devise a consistent recipe for calculating these diagrams as scattering amplitudes as in a Feynman approach where the diagrams represent a first approximation to hadron physics.

To put this problem in perspective, recall that the dream of S-matrix theory is to derive, guess or buy a function that has the following properties: 1) analyticity in s,t,u, 2) crossing symmetry, 3) unitarity and 4) good asymptotic behavior, e.g. Regge behavior.

Now we wish to add 5) duality, which means that with 1), 2) and 4) giving us FESR's we want the non-diffractive part of the amplitude to be saturated by resonances. By "non-diffractive" we mean an amplitude which involves exchange of non zero quantum numbers and gives rise to falling cross sections in the Regge limit. We find that we can in fact do this, but as suggested above, we must give up unitarity. However, the model does have a natural interpretation which yields a sort of "pseudo unitarity" as will be discussed later. We expect that when unitarity is rigorously included diffractive effects (the Pomeron) will appear in a natural way.

The most widely used mathematical form for the dual amplitude is that first suggested by Veneziano(17) which is essentially just a beta function, e.g.

$$A_V(s,t) = B(1-\alpha(s), 1-\alpha(t)) = \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s) - \alpha(t))} \tag{I-4}$$

The actual amplitude used to describe a specific reaction is usually just a polynomial in s and t times such a function or more generally a sum of such functions where varying integers appear in the different gamma functions. In fact, one can show rather generally that every amplitude which satisfies our initial duality constraints can be written as such a sum.(18)

We shall reserve the mathematical details of the functionry for the next chapter and review just the basic properties of the Veneziano amplitude: 1) narrow resonances in the form of simple poles whenever $\alpha(s)$ or $\alpha(t)$ pass through positive integers except that the denominator eliminates the possibility of double (simultaneous) poles; 2) symmetry in its arguments, a property which can be used to build amplitudes with definite crossing properties, e.g. a function even in $\nu \sim s-u$ is given by $A_V(s,t) + A_V(u,t)$; 3) pole residues which are polynomials in the cross channel variable as long as the trajectory functions are linear, i.e., $\alpha(s) = a + bs$; 4) Regge behavior, e.g.

as $|s| \to \infty$, t fixed $A_V(s,t) \sim \frac{(-s)^{\alpha(t)}}{\Gamma(\alpha(t))\ \sin \pi\alpha(t)}$ as long as $|\arg(s)| > \epsilon$. This last point also requires that the slopes of the trajectory functions in all channels be the same. Whether this is taken as input or a prediction, it is in remarkable agreement with nature. Since the Regge behavior requirement also excludes us from a tiny wedge around the positive real axis ($|\arg(s)| > \epsilon$) where the poles are, we shall take the approach that this wedge somehow represents the elastic unitarity cut (pseudo unitarity). Thus the discontinuity across the wedge is the imaginary part of the amplitude and the interior of the wedge containing the poles becomes the second Riemann sheet of the complex s plane. Further, the distance from the poles to the wedge boundary goes like $(\sin \epsilon)\cdot|s|$ and is taken to be the effective imaginary part of $\alpha(s)$ which is observed in data to grow approximately like the first power of s.

Another interesting feature is that this amplitude has resonances in only two channels as suggested by the simple duality graphs which are always exotic in one channel (out of s, t, u). If there are no exotic channels then more than one quark graph will contribute (as in $\pi p$ scattering) and signature will reappear in a natural way, e.g. if $A\ (s,\ t,\ u) = A\ \ (s,t) + A\ \ (u,t)$ and we consider $|s|,\ |u| \to \infty$, t fixed then

$$A(s,t,u) \sim \Gamma(1-\alpha(t)) \left\{(-s)^{\alpha(t)-1} + (-u)^{\alpha(t)-1}\right\} =$$

$$\Gamma(1-\alpha(t))\ s^{\alpha(t)-1}\ (1-e^{-i\pi\alpha(t)}) \quad . \qquad \text{(I-5)}$$

We also note that the Veneziano amplitude generalizes easily to the case of n external particles.[19]

Although this amplitude does have definite diseases (to be discussed in the next chapter) we shall assume that it is a reasonable way to calculate our quark diagrams in the limit that the quarks are scalars and $SU_3$ singlets.

There remains, then, the task of introducing the $SU_6$ quantum numbers of the quarks. We certainly must include the $SU_3$ factors for the sake of exchange degeneracy studies and as one might expect this is fairly simple to do (as we shall see). The problem of introducing the Dirac spin of the quarks is considerably more troublesome and it is therefore useful to review again why we want to do it. Recall that when we discussed the quark model, it was suggested that nature behaves in many ways as if it were close to an $SU_6$ symmetric limit, e.g. particle spectra which are approximately given by $SU_6$. A further suggestion comes from inelastic electron-proton scattering.[20] It has been found that much of the data can be simply explained in terms of a composite proton, i.e., a proton constructed out of several partons (quarks?). The data, in particular the ratio $\sigma_T/\sigma_S \approx 0$ where $\sigma_S$ is the cross section for scalar photons and $\sigma_T$ is for transverse photons, are strongly in favor of spin $\frac{1}{2}$ partons.

Attempts have already been made to include $SU_6$ quarks in dual models[21] but with the problem that the resonances produced were parity doubled and contained new ghosts (states with negative coupling constant squared).

The new feature of the present work is the appearance of cuts in the complex angular momentum plane which allow us to introduce quark spin without also introducing parity doubling. Following the work of Carlitz and Kislinger,[22] we take the approach that linear trajectories and spin $\frac{1}{2}$ quarks are compatible with non-parity doubled resonances only when there are fixed (Carlitz-Kislinger) cuts in the J-plane. We shall see that the usual factorization, analyticity, and crossing contraints for dual models require the amplitude to exhibit further new J-plane and s-plane structure. Our primary interest during this research will be to study the form and interpretation of all this new structure and to understand how it affects the behavior of the amplitude. In particular, we shall need to restudy our requirements for an acceptable dual amplitude, e.g. that it should have pure Regge pole asymptotic behavior. Also the new forms of measureable quantities such as differential cross sections will be discussed and compared to data. This new complexity is not unexpected and is probably desirable. Simple dual models such as discussed above have been markedly unsuccessful in describing processes involving spin $\frac{1}{2}$ or even integer spin greater than one for the external particles. It has been postulated that the introduction of spin may be incompatible with the original strict duality constraints.[23] It is also worthwhile noting that Regge pole phenomenology shows a strong need for cuts in order to describe the data[24] and that no truly consistent model now exists for generating these cuts.

Having discussed the general direction of the research in this thesis and explained why it is of interest let us proceed to look at the details. In the next chapter we shall discuss the technical details and properties of the simple Veneziano amplitude. Chapter III will bring us to the introduction of quark spin for the simple case of meson-meson scattering where we can study what functionry is required without too much kinematic complexity. In Chapters IV and V we shall study in detail the new structure in the J-plane and s-plane introduced by this model, the behavior of the related functionry, and the meaning of all this in relation to duality. After calculating meson-baryon amplitudes in Chapter VI, we shall attempt comparison with experiment, focusing our attention on those points which result from the inclusion of spin rather than details of the dual model. In the final chapter we shall review the situation and abstract some conclusions about the model.

# II. THE BASIC DUAL AMPLITUDE

In this chapter we shall review some of the basic properties of the Veneziano amplitude.[25] Since we will eventually be interested only in results which are essentially independent of the details of the amplitude, the object here will be to establish a general familiarity with the formalism. Indeed, if improved versions of the basic amplitude are developed (e.g. unitarized or without ghosts) with similar representations to those discussed here, then the present program for introducing the spin of the quark can be easily applied to these new versions.

Recall that the basic Veneziano amplitude is given by:

$$A_V(s,t) = B(-\alpha(s), -\alpha(t)) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{II-1a}$$

$$= \int_0^1 dx\, x^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1} \tag{II-1b}$$

$$= \int_0^1 dx\, x^{-\alpha(t)-1} (1-x)^{-\alpha(s)-1} \tag{II-1c}$$

where the $s \leftrightarrow t$ symmetry is evident in the integral representation because of the $x \leftrightarrow 1-x$ symmetry of the integrand. The beta function is a particularily beautiful amplitude to study because we have not only the integral representation which is very useful for making generalizations but also the explicit gamma function representation whose properties can be looked up in a book.

First let us look at the resonance structure. This is most easily obtained from the integral representation. Expanding the integrand in Eq. II-1b about x = 0 and doing the integral we have

$$A_V(s,t) = \sum_n \frac{\Gamma(n+\alpha(t)+1)}{\Gamma(n+1)\,\Gamma(\alpha(t)+1)} \int_0^1 dx\, x^{-\alpha(s)-1} x^n = \sum_n \frac{\Gamma(n+\alpha(t)+1)}{\Gamma(n+1)\Gamma(\alpha(t)+1)} \frac{1}{n-\alpha(s)}$$

(II-2a)

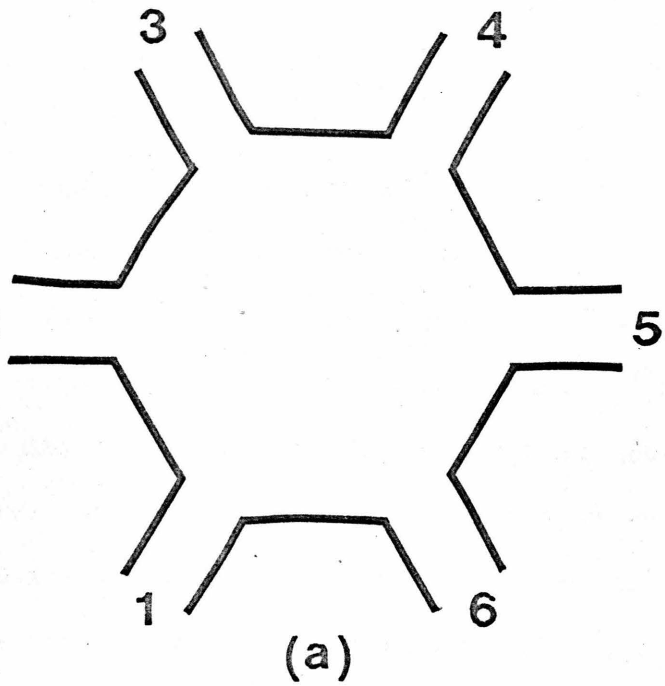
Likewise from Eq. II-1c we get

$$A_V(s,t) = \sum_n \frac{\Gamma(n+\alpha(s)+1)}{\Gamma(n+1)\,\Gamma(\alpha(s)+1)} \frac{1}{n-\alpha(t)} \qquad \text{(II-2b)}$$

These two expansions illustrate several points. First, dual amplitudes can be alternately expressed as sums of resonances in either of the two channels which is an explicit illustration of the duality idea. Second, we see that the residue at the $n^{th}$ pole is a polynomial of $n^{th}$ order in the cross channel trajectory function. As long as we have linear trajectories, it is also a polynomial of the $n^{th}$ order in the cross channel variable. If we partial wave analyze this residue, we will in general find all angular momenta from 0 to n present. This is the well known fact that the Veneziano amplitude contains not only the leading trajectory, but also daughter (lower spin) trajectories. It should be noted that if the trajectory functions were in fact not linear, as happens when an ad hoc term $\sqrt{s_{th}-s}$ is introduced to account for unitarity, then the residue is not a polynomial in cos $\theta$ and contains angular momenta greater than n (ancestors). It is also worthwhile realizing that this daughter structure is not unexpected. We saw during our earlier discussion of the partial wave analysis of Regge poles that fixed t structure in the Regge residue required dominant s channel resonances whose angular momentum increased like $\sqrt{s}$. This is consistent with linear trajectories only if daughter trajectories with dominant coupling are present. This is exactly what happens in the Veneziano amplitude.

The final point to notice about Eqs. II-2a and II-2b is that the two expansions actually converge in different kinematic regions; II-2a for $\alpha(t) < 0$ and II-2b for $\alpha(s) < 0$. The divergence of the sum is what gives the resonances in the cross channel. This explains how a sum of terms, each of which is regular in t, for example, can have poles in t. The integral representation is just the analytic continuation of the function defined by the sums.

Two related and important questions are factorization and degeneracy of the resonances. In the present language these problems are too technical to be discussed, and we shall just present the results of previous work.[26] Recall that our definition of a resonance specified that it should factorize. If we apply this to the original n-point amplitude (n external particles), we require that when we look at a resonance in any channel the n-point amplitude will factorize into an m-point amplitude and an n-m + 2-point amplitude. Further, these lower n amplitudes should be the same as if we had written them down directly with the appropriate external particles. For example, in Fig. II-1 we illustrate the case for n = 6, m = 4. It turns out that in order to satisy these constraints the degeneracy



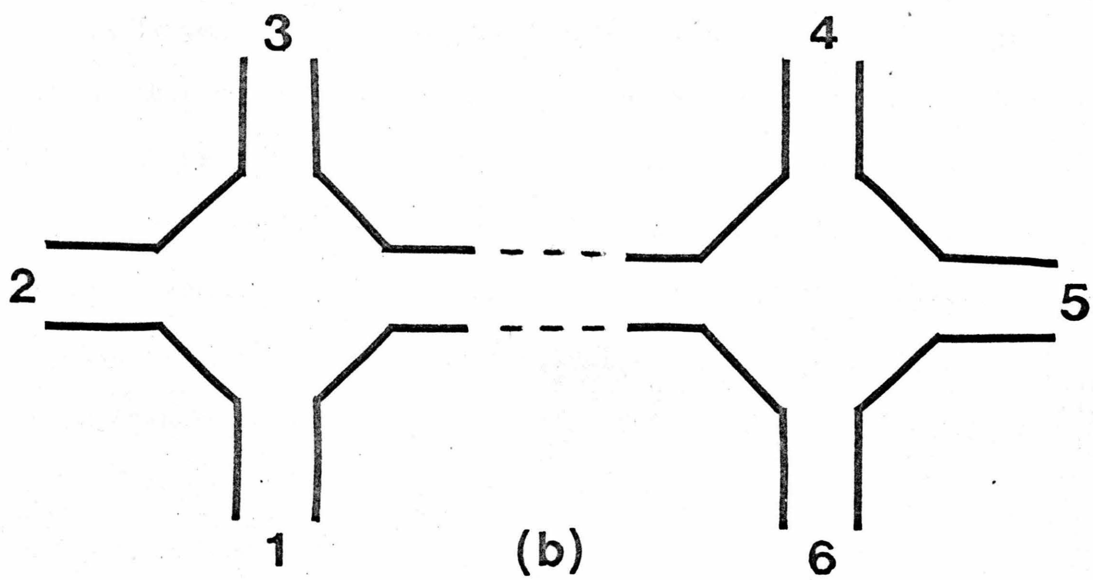




Fig. II-1 Factorization of 6 point diagram: (a) 6 point diagram; (b) two 4 point diagrams.

of the resonances must increase very rapidly with the mass of the resonance. For large values of the mass, m, the number of degenerate states grows like exp(cm) where c is a positive constant. It was also found that some of the states are ghosts, i.e., have negative coupling constant squared. At first sight such a degeneracy may seem untenable. However, similar results occur in statistical models[27] and it is really not surprising in a resonance saturation model where all the channels which are opening up at high energies require resonances to saturate them. A further point which is worth noting is that when the model is made more complicated by adding together several beta functions with arguments which differ by integers (called satellites), in order to include spin for example, an even higher and probably unacceptable degeneracy rising like $\exp(cm^2)$ is found.[28] We shall return to this point later when we discuss how spin is to be introduced in the present model. We will want to construct the model so that the degeneracy is changed from the single Veneziano amplitude only by an overall factor representing the various $SU_6$ states.

The asymptotic behavior of Eq. II-1 is most easily studied in the gamma function representation using the Stirling approximation[29]

$$\Gamma(z) \rightarrow \sqrt{2\pi}\, e^{-z}\, z^{z-\frac{1}{2}} \quad , \qquad |z| \rightarrow \infty, |\arg(z)| < \pi \tag{II-3}$$

and

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{II-4}$$

Using Eq. II-3 with $|s| \to \infty$ , $|\arg(s)|>\epsilon$ so that $|\arg(-s)|< \pi-\epsilon$ and t fixed, we find ( $\alpha(s) = \alpha_{0_s} + \alpha' s$)

$$A_V(s,t) \underset{\substack{s \to \infty \\ t \text{ fixed}}}{\sim} \Gamma(-\alpha(t)) \frac{\sqrt{2\pi}(-\alpha(s))^{-\alpha(s)-\frac{1}{2}} e^{\alpha(s)}}{\sqrt{2\pi}(-\alpha(s))^{-\alpha(s)-\alpha(t)-\frac{1}{2}} e^{\alpha(s)}} \sim \Gamma(-\alpha(t))(-\alpha' s)^{\alpha(t)} \times \left(1 + \mathcal{O}\left(\frac{1}{|s|}\right)\right)$$

(II-5)

To see what are the restrictions on $\epsilon$ we consider the case $|t|$, $|s| \to \infty$ u fixed, i.e., $t \sim -s$. Again using Eqs. II-3 and II-4 with $\alpha(t) = \alpha_{0_t} + \alpha' t$, $\alpha(s) = \alpha_{0_s} + \alpha' s$ and $s + t + u = K$, we find

$$A_V(s,t) = \frac{\Gamma(-\alpha_{0_s} - \alpha' K + \alpha' u + \alpha' t)}{\Gamma(-\alpha_{0_t} - \alpha_{0_s} + \alpha' u - \alpha' K)} \frac{-1}{\Gamma(1+\alpha(t)) \sin \pi\alpha(t)} \sim$$

$$\frac{-\pi(\alpha' t)^{\alpha' u - \alpha_{0_s} - \alpha' K - 1}}{\Gamma(-\alpha_{0_t} - \alpha_{0_s} + \alpha' u - \alpha' K)} \frac{2i}{e^{i\pi\alpha(t)} - e^{-i\pi\alpha(t)}} \qquad \text{(II-6)}$$

which decreases faster than any power of $|s|$, i.e., is smaller than Regge terms, as long as Imt grows faster than $\ell n(t)$, e.g. $\arg(t) = \text{const.} > 0$. Note that as mentioned above this is only true for equal slope trajectories in the two channels. Otherwise a factor $(\alpha' t)^{(\alpha'_s - \alpha'_t)t}$ appears. Since we cannot look at the asymptotic behavior out along the positive real axis we shall consider this "wedge region" to be on the second Riemann sheet as mentioned in Chapter I. To summarize all this, we have as $|s|\to\infty$, $\pi-\epsilon>|\arg(s)|>\epsilon$, t fixed:

$$A_V(s,t) \sim \Gamma(-\alpha(t))(-\alpha' s)^{\alpha(t)} \quad . \qquad \text{(II-7a)}$$

$$A_V(u,t) \sim \Gamma(-\alpha(t))(\alpha' s)^{\alpha(t)} \qquad \text{(II-7b)}$$

$$A_V(s,u) \sim \mathcal{O}(e^{-c|s|}) \quad . \qquad \text{(II-7c)}$$

The scale factor $s_0$ of the usual Regge form appears naturally as $1/\alpha'$ in this model.

In Chapter IV we will see that it is also possible to find the asymptotic behavior from the integral representation which must be used in the general case. However it is very handy to have the gamma function representation results as a check.

For completeness and to be totally honest, this discussion of dual amplitudes must include a short treatment of attempts to unitarize the Veneziano amplitude. The most obvious procedure is to give the trajectory function $\alpha(s)$ an imaginary part.(30) However, as noted above, this leads to nonlinear trajectory functions, nonpolynomial residues, and high spin ancestors. Other programs include treating the Veneziano amplitude as a distribution function and using standard smearing techniques,(31) and treating the Veneziano amplitude as a Born term and applying N/D methods(32) or K matrix methods(33) to it. The first of these procedures is extremely arbitrary and tends to ruin the Regge pole asymptotic behavior. The second and third procedures, while possibly useful in phenomenology, destroy the crossing symmetry of the amplitude.

Finally the most mathematically beautiful approach is that suggested by Kikkawa, Sakita, and Virasoro.[34] Again the Veneziano amplitude is treated as a Born term, but now in a field theoretic sense. Unitarization proceeds by including higher diagrams. The approach is fraught with extreme difficulties such as exponential divergences but has produced some technically beautiful results. In any case, no truly unitarized amplitude is presently available. This, coupled with the conjecture that the Veneziano form is already "partially unitary," i.e., the psuedo-unitarity mentioned above, makes the question of unitarization one of the most pressing problems in the field of dual models.

Before closing this chapter on the "scalar quark" dual amplitude, we shall briefly review the operatorial or "rubber band" formulation of dual amplitudes.[35] Although this approach has not as yet proved particularly useful for the problem of including spin, it is very suggestive of a simple intuitive picture of quark interactions and of relationships between dual models and other quark model work such as Feynman et al.[3] It also gives simple interpretations of certain problems mentioned above and thus may be useful in seeking solutions. The essential assumption of this approach is that the quark and anti-quark of a meson are bound via an infinite number of harmonic oscillators (the modes of the "rubber band"). The quarks themselves do not carry energy, but rather carry the intrinsic quantum numbers, i.e., they specify where on the "rubber band" other hadrons can couple.

We can describe the "rubber band" in terms of a wave function $\Phi_\mu(\xi, \tau)$ which gives the 4-dimensional displacement of the "rubber band" at a point $\xi$ ($2\pi \geq \xi \geq 0$, we are thinking of a circular "rubber band") at a proper time $\tau$, i.e., $\tau$ is the parameter which describes the evolution of $\Phi_\mu$ in time. For mesons in this model the quark and antiquark are at $\xi = 0, \pi$, whereas for baryons the three quarks are at $\xi = 0$, $^{2\pi}/3$ and $^{4\pi}/3$. $\Phi_\mu$ satisfies a Klein-Gordon equation

$$\frac{\partial^2}{\partial^2 \xi} \Phi_\mu(\xi, \tau) - \frac{\partial^2}{\partial^2 \tau} \Phi_\mu(\xi, \tau) = 0 \qquad \text{(II-8)}$$

We can express $\Phi_\mu$ in terms of the creation and annihilation operators for an infinite number of harmonic oscillators as (in a more formal treatment one would have to include the CM co-ordinates)

$$\Phi_\mu(\xi, \tau) = \sum_r \frac{\sqrt{2}}{r} \Big\{ (a_{\mu r}\, e^{ir\tau} + a^+_{\mu r}\, e^{-ir\tau}) \cos r\,\xi$$
$$+\, i(b_{\mu r}\, e^{ir\tau} + b^+_{\mu r}\, e^{-ir\tau}) \sin r\,\xi \Big\} \qquad \text{(II-9)}$$

where

$$[a_{\mu r}, a^+_{\nu s}] = [b_{\mu r}, b^+_{\nu s}] = -g_{\mu\nu}\, \delta_{rs}, \; g_{\mu\nu} = \begin{bmatrix} + & & \\ & - & \\ & & - \end{bmatrix} \qquad \text{(II-10)}$$

and all other commutators vanish. Likewise the total energy is expressed as

$$M^2 = H + m^2 = -\sum_r r\,(a^+_{r\mu}\, a_{r\mu} + b^+_{r\nu}\, b_{r\nu}) + m^2 \qquad \text{(II-11)}$$

where $m^2$ is the ground state mass (effectively this is what specifies the trajectory intercept). Physics enters the problem when we define a vertex which we shall take as

$$\Gamma(k,\xi,\tau) = :e^{ik_\mu \Phi_\mu(\xi\tau)}: \qquad \text{(II-12)}$$

where the : : means normal ordering, i.e., annihilation operators to the right and creation operators to the left in order to avoid an overall factor of infinity. The picture we have in mind is shown in Fig. II-2a. Some external object (the wavy line) couples to the quark at $\xi$. Its only effect is to change the momentum by k, i.e., $e^{ik\Phi}$ is just the displacement operator in momentum space since $\Phi$ is a co-ordinate operator. The weakness in this picture is that we have treated the particle with moment k as being structureless whereas the other two particles are $\bar{q}q$ pairs. This is not unreasonable as long as all the external particles are in the ground state, but the generalization to more complicated cases is unclear.

We are now almost ready to calculate the 4-point amplitude of Fig. II-2b where the Mandelstam variables are defined to be $s=(q_1+q_2)^2=(-q_3-q_4)^2$, $t=(q_2+q_3)^2=(-q_1-q_4)^2 = 2m^2+ 2q_3\cdot q_2$, (all momenta are incoming). There are, in fact, two ways to proceed. The first procedure, which is more formal and more difficult, involves keeping track of the $\tau$ dependence of the vertices which appear and integrating

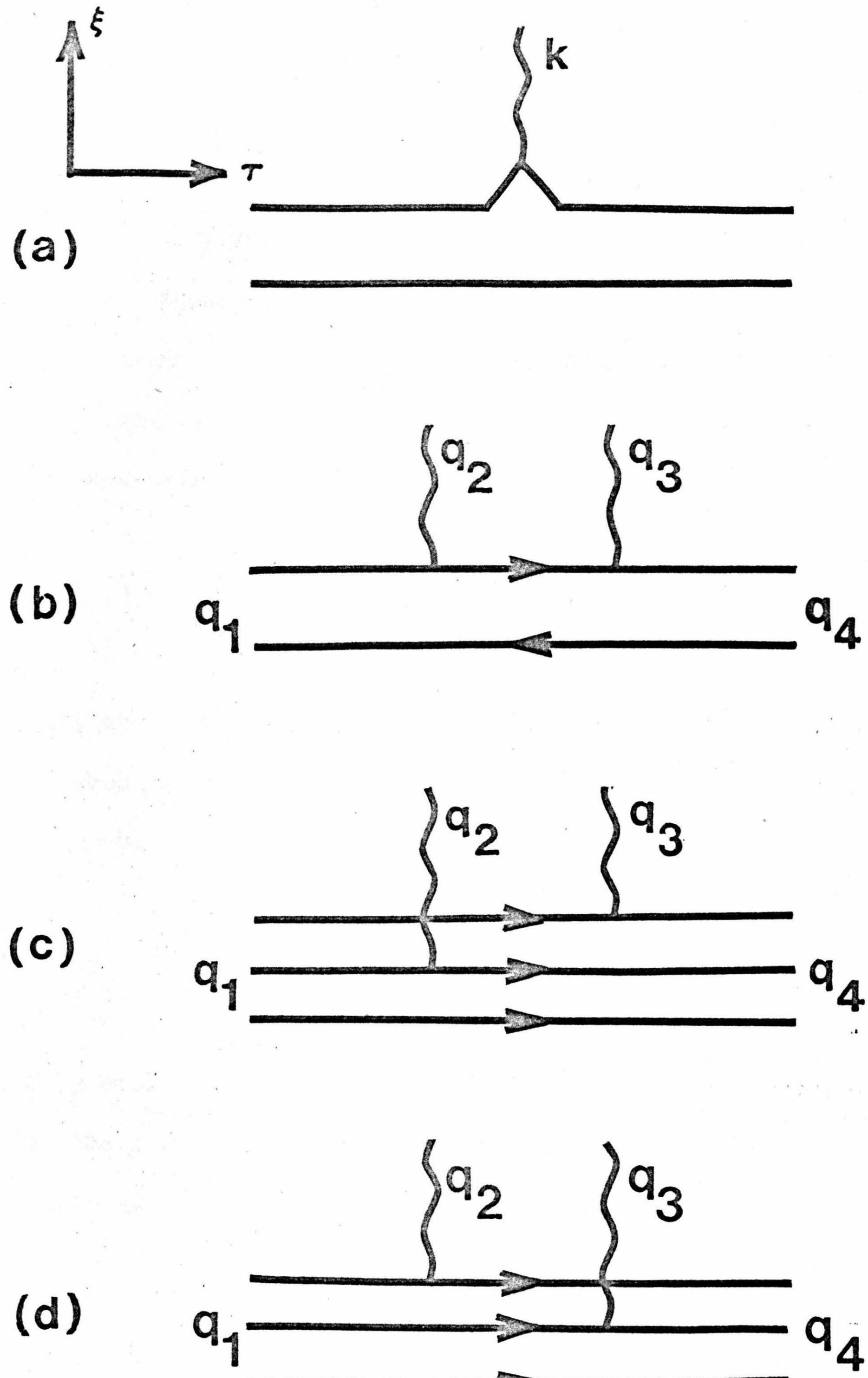



Fig. II-2 Diagrams for calculating Veneziano amplitudes:(a) general vertex;(b) meson-meson process;(c),(d) meson-baryon process.

over the differences of these $\tau$'s (essentially a Heisenberg approach). One must be very careful in this calculation to take account of the CM co-ordinates and momenta which we have not discussed above. The second approach, which is much simpler and more transparent, involves evaluating all vertices at $\tau = 0$ and then putting in an explicit propagator by hand (a Schrodinger approach). We shall use this second procedure with the following propagator:

$$\Delta(s) = \frac{f(\alpha_0, m^2)}{H-\alpha(s)} = \int_0^1 dx\; x^{H-\alpha(s)-1}(1-x)^{-\alpha_0-2m^2-1} \qquad \text{(II-13)}$$

($f(\alpha_0, m^2)$ is, at this level, an ad hoc factor to fix the intercepts). With the external ground state given by $|0>$, the 4-point meson amplitude becomes

$$A_4(s,t) = <0|\, \Gamma(q_3, \xi=0, \tau=0)\, \Delta(s)\, \Gamma(q_2, \xi=0, \tau=0)\, |0> \qquad \text{(II-14)}$$

We can easily evaluate this expression using the coherent state formalism. In the 1 dimensional case (easily generalized) we define a coherent state as

$$|z> \equiv e^{a^+ z}|0> \quad \text{(z a complex number)} \qquad \text{(II-15a)}$$

which has the properties

$$a|z> = z|z> , \qquad e^{a^+ z}|y> = |y+z>,$$

$$(z)^{a^+ a}|y> = |yz> , \qquad <z|y> = e^{z^* y} , \qquad \text{(II-15b)}$$

$$|z> = \sum_n \frac{(z)^n}{\sqrt{n!}} |n> \qquad (|n> \text{ are occupation number states})$$

$$\frac{1}{\pi}\int dz^2\, e^{-|z|^2}\, |z><z| = \mathbb{1}$$

Applying all this to Eq. II-14 gives

$$A_4(s,t) = \int_0^1 dx\, x^{-\alpha(s)-1}(1-x)^{-\alpha_0-2m^2-1} <0|\, e^{\sum_r(\sqrt{\frac{2}{r}})a_r^+\cdot q_3}\, e^{\sum_p(\sqrt{\frac{2}{p}})a_p\cdot q_3}$$

$$x^{-\sum_n n(a_n^+\cdot a_n + b_n^+\cdot b_n)}\, e^{\sum_m\sqrt{\frac{2}{m}}\, a_m^+\cdot q_2}\, e^{\sum_\ell\sqrt{\frac{2}{\ell}}\, a_\ell\cdot q_2}\, |0>$$

$$= \int_0^1 dx\, x^{-\alpha(s)-1}(1-x)^{-\alpha_0-2m^2-1}\prod_{p,m}<\sqrt{\frac{2}{p}}\, q_3|x^m q_2\sqrt{\frac{2}{m}}>$$

$$= \int_0^1 dx\, x^{-\alpha(s)-1}(1-x)^{-\alpha_0-2m^2-1}\prod_m e^{\frac{2}{m}x^m\, q_3\cdot q_2}$$

$$= \int_0^1 dx\, x^{-\alpha(s)-1}\,(1-x)^{-\alpha_0-2m^2-1}\, e^{(-t+2m^2)(-\sum_m\frac{x^m}{m})} \qquad \text{(II-16)}$$

$$= \int_0^1 dx\, x^{-\alpha(s)-1}\,(1-x)^{-\alpha(t)-1} \qquad (\frac{d\alpha(t)}{dt} = 1)$$

where we have used $-\sum\frac{x^m}{m} = \ell n(1-x)$ and the fact that for mesons $\xi = 0$, i.e., the b type modes do not couple. Thus we get directly to the usual Veneziano amplitude. The weak point is the function $f(\alpha_0,m^2)$ which is introduced to fix the intercept in the cross channel. This can be formally handled by introducing a phony 5th dimension, but it still remains a question mark. We note that this problem does not arise for $m^2 = -1$ so that $\alpha_0 = 1$ (recall that $\alpha(m^2) = 0$). This is a well studied case in the lore of dual models because of the many simplifications which occur at the price of having a tachyon ground state. The meaning of this is still unclear.

What can be abstracted from this formalism? First, if we had only a single oscillator as in ref. 3, Eq. II-16 would become the nondual form

$$\bar{A}_4(s,t) = \int_0^1 dx\, x^{-\alpha(s)-1} (1-x)^{-\alpha_0-2m^2-1} e^{x(t-2m^2)} \tag{II-17}$$

which again points out the need for a high degeneracy in order to produce a dual amplitude. Also, since $[a_{0r}, a^+_{0r}] = -1$, the presence of an odd number of time-like excitations is seen as the root of the ghost states mentioned before. These time-like excitations are difficult to interpret physically anyway[3] and we would hope that a gauge condition can eventually be consistently defined so as to decouple them analogously to the case in Q.E.D.

The final important point which we can learn from this formalism and which will be used later is the form of the s,u amplitude in meson-baryon scattering (see Fig. I-6d). From Fig. I-6 we see that in the s,t and u,t quark diagrams the external mesons both couple to the same quark of the baryon, i.e., $\xi = 0$ for both vertices and the calculation yields a beta function as above. However for the s,u diagram two of the baryon quarks are involved, i.e., both $\xi = 0$ and $\xi = \frac{2\pi}{3}$ appear. The calculation yields a different result[36] because the b type modes also appear. This is expected since in the three quark picture of a baryon there are two independent ways of introducing internal angular momentum. A priori we could put two quarks at the same $\xi$ and reproduce the meson results. However, this does not seem as natural as the picture above and does not lead to quark model results. Proceeding to calculate the diagrams of Fig. II-2c we find (no $f(\alpha_o, m^2)$ this time)

$$A_c(s,u) = \int_o^1 dx\, x^{-\alpha(s)-1} \prod_{r,p} {}_a\!\left\langle \sqrt{\tfrac{2}{r}} (\cos \xi_2 r)\, q_3 \right| {}_b\!\left\langle i\sqrt{\tfrac{2}{r}} (\sin \xi_2\, r) q_3 \right|$$

$$x^H \left| \sqrt{\tfrac{2}{p}} (\cos p\, \xi_1) q_2 \right\rangle_a \left| -i\sqrt{\tfrac{2}{p}} (\sin p\, \xi_1) q_2 \right\rangle_b$$

$$= \int_o^1 dx\, x^{-\alpha(s)-1} \exp\left\{ 2q_2 \cdot q_3 \sum_r \frac{x}{r} \cos r\, (\xi_1 - \xi_2) \right\}$$

$$= \int_o^1 dx\, x^{-\alpha(s)-1} \exp\left\{ -2q_2 \cdot q_3 \,\frac{1}{2}\, [(1-xe^{i(\xi_1-\xi_2)})(1-xe^{-i(\xi_1-\xi_2)})] \right\}$$

and with $\xi_1 - \xi_2 = \frac{2\pi}{3}$

$$A_c(s,u) = \int_o^1 dx\, x^{-\alpha(s)-1} \exp\left\{ -(t-2\mu^2)\, \frac{1}{2}\, (1+x+x^2) \right\}$$

$$= \int_o^1 dx\, x^{-\alpha(s)-1} (1+x+x^2)^{-\frac{(t-2\mu^2)}{2}} \tag{II-18a}$$

then use $s + t + u = 2m^2 + 2\mu^2$, $\alpha_{o_u} = -m^2$

$$A_c(s,u) = \int_o^1 \frac{dx}{1+x+x^2} \left( \frac{x}{\sqrt{1+x+x^2}} \right)^{-\alpha(s)-1} \left( \frac{1}{\sqrt{1+x+x^2}} \right)^{-\alpha(u)-1}$$

To calculate Fig. II-2d we need only exchange $q_3 \leftrightarrow q_2$ and thus $s \leftrightarrow u$ to give

$$A_d(s,u) = \int_0^1 \frac{dx}{1+x+x^2}\left(\frac{x}{\sqrt{1+x+x^2}}\right)^{-\alpha(u)-1}\left(\frac{1}{\sqrt{1+x+x^2}}\right)^{-\alpha(s)-1} \tag{II-18b}$$

In Eq. II-17a we use the substitution $z = x/(1+x)$ and in Eq. II-17b $z = 1/(1+x)$. The result of these substitutions is

$$A_{MB}(s,u) = A_c + A_d = \int_0^1 \frac{dz}{1-z(1-z)}\left(\frac{z}{\sqrt{1-z(1-z)}}\right)^{-\alpha(s)-1}\left(\frac{1-z}{\sqrt{1-z(1-z)}}\right)^{-\alpha(u)-1} \tag{II-19}$$

which is similar to but clearly different from the s,t and u,t forms of Eq. II-16. In particular, if we consider the leading trajectory in the s channel, i.e., we expand the integrand of Eq. II-19 about x = 0 and keep only leading order terms in $(\alpha(u)x)$, we find

$$A_{MB}(s,u) = \sum_n \left\{\frac{\left(\frac{\alpha(u)}{2}\right)^n}{n!}\frac{1}{n-\alpha(s)} + \text{lower order in } u\right\} \tag{II-20a}$$

whereas the corresponding s,t term gives

$$A_{MB}(s,t) = \sum_n \left\{\frac{(\alpha(t))^n}{n!}\frac{1}{n-\alpha(s)} + \text{lower order in } t\right\} \tag{II-20b}$$

The remarkable aspect of these two equations is that when one considers the $SU_6$ content of the s channel resonances which result from adding the s,u and s,t quark diagrams, one finds that the leading trajectory looks like the symmetric quark model.[37] The factor $(\frac{1}{2})^n$ is just such that for $n = 0$ $(L = 0)$ there is only the $\underset{\sim}{56}$ multiplet, for $n = 1$ $(L = 1)$ there is only the $\underset{\sim}{70}$, and for higher $n(L)$ there are both $\underset{\sim}{56}$'s and $\underset{\sim}{70}$'s. We shall see this explicitly in Chapter VI when we calculate meson-baryon scattering.

# III. MESON-MESON SCATTERING WITH SPIN

In this chapter we shall discuss the introduction of the quark quantum numbers for the specific case of meson-meson scattering.[38] This example is chosen for its relative simplicity so that various facets of the model can be more easily understood. It is also possible to make direct reference to previous work[21] for this case. In Chapter VI we shall study meson-baryon where we can take up the question of comparing the model to data (bearing in mind the constraint that we are dealing with a totally $SU_6$ symmetric model which can be only approximately like the real world). In the present chapter we shall require of the model only a certain reasonableness and simplicity.

As discussed earlier, we assume that the usual Veneziano amplitude correctly describes the scattering process in the quark diagram of Fig. III-1 for the case of scalar quarks. Now we include the spin of the quarks while still maintaining factorization and Regge asymptotic behavior, and without introducing any new ghosts or parity partners. We assume that all meson resonances can be described as $\bar{q}q$ composites of spin $\frac{1}{2}$, $SU_3$ triplet, positive parity quarks. The resonances appear in mass degenerate $(6,\bar{6};L)$ representations of $SU_6 \times SU_6 \times O_3$.[39] Their total spin is given by the quark spin contribution plus the orbital excitation of the Veneziano kernel. Thus we have a Russell-Saunders sort of description where the L·S coupling vanishes as is observed to be close to the truth, at least for the

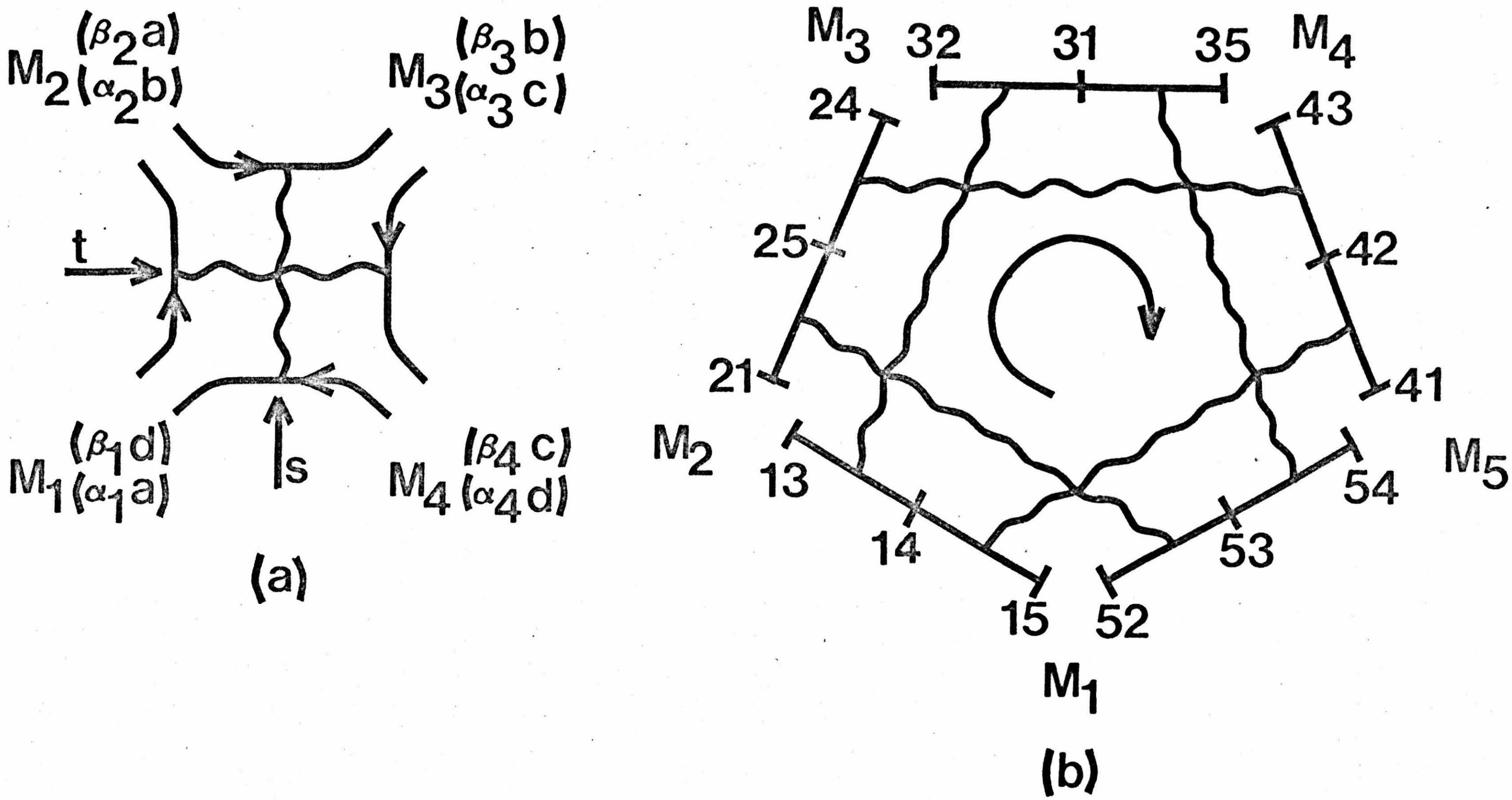



Fig. III-1 Quark diagrams for: (a) 4 point function; (b) 5 point function. Solid lines indicate quarks, wavy lines indicate dual projection operators, and $M_i$ specify meson matrices. The numbers (15), (14), etc., correspond to the indices of the $\alpha$'s in Eq.III-14. Arrows indicate sense of direction of quark lines.

baryons. Coupling of the mesons is assumed to occur via $SU_{6W} x O_{2L_z}$ [40] invariant vertices. The external particles are taken to be any of the nine $0^-$ or nine $1^-$ members of the ground state $(6,\bar{6};0)$ represented by the matrix:[41]

$$M^{(\beta b)}_{(\alpha a)} = \frac{1}{\sqrt{2}}[(1+\frac{\not{p}}{m})\gamma_5]^{\beta}_{\alpha} \; P^b_a + \frac{1}{\sqrt{2}}[(1+\frac{\not{p}}{m})\gamma_\mu]^{\beta}_{\alpha} \, (V_\mu)^b_a \qquad \text{(III-1)}$$

The 3x3 matrices P and V represent the pseudoscalar and vector nonets, p is the meson's momentum, and m is its mass. Note further that M satisfies the Bargman-Wigner equations $(\not{p}-m)^{\gamma}_{\alpha} M^{\beta}_{\gamma} = 0$ and $M^{\gamma}_{\alpha} (\not{p}+m)^{\beta}_{\gamma} = 0$ as befits a matrix whose indices represent a Dirac spinor and anti-Dirac spinor respectively. The M's also have the property that if we couple three of them at a vertex in the fashion $Tr(M_1 M_2 M_3)$, where Tr is a trace in both Dirac and $SU_3$ space, we are guaranteed that the vertex is $SU_{6W} x O_{2L_z}$ invariant in the collinear frame where the symmetry is defined.

Now we shall calculate the amplitude for Fig. III-1a and determine the $SU_6$ quantum numbers of the resonances. In order to do this we use the following identities found, for example, in ref. 21.

In the $SU_3$ world $\quad Tr(AB) = \sum_{i=0}^{8} Tr(A\,\lambda_i)\, Tr(B\,\lambda_i) \qquad$ (III-2)

where the $\lambda_i$ are the usual 3x3 $SU_3$ matrices appropriately normalized.

In the space of Dirac spin

$$\mathrm{Tr}(AB)=\tfrac{1}{4}\Big\{\mathrm{Tr}(A)\mathrm{Tr}(B)+\mathrm{Tr}(A\gamma_5)\mathrm{Tr}(\gamma_5 B)+\mathrm{Tr}(A\gamma_{\mu_A})\mathrm{Tr}(\gamma_{\mu_B}B)$$

$$-\ \mathrm{Tr}(A\frac{\not q_A}{m_A})\mathrm{Tr}(\frac{\not q_B}{m_B}B)\ -\ \mathrm{Tr}(A\gamma_5\gamma_{\mu_A})\ \mathrm{Tr}(\gamma_5\gamma_{\mu_B}\ B)$$

$$+\ \mathrm{Tr}(A\frac{\gamma_5\not q_A}{m_A})\ \mathrm{Tr}(\gamma_5\frac{\not q_B}{m_B})\ -\ \mathrm{Tr}(A\frac{\sigma_{\mu\nu}\ q_{A\gamma}}{m_A})\ \mathrm{Tr}(\frac{\sigma_{\mu\lambda}\ q_{B\lambda}}{m_B}\ B)$$

$$-\ \mathrm{Tr}(A\ \gamma_5\ \frac{\sigma_{\mu\nu}\ q_{A\gamma}}{m_A})\ \mathrm{Tr}\ (\gamma_5\frac{\sigma_{\mu\lambda}\ q_{B\lambda}}{m_A}\ B)\Big\} \qquad \text{(III-3)}$$

where $\gamma_{\mu_A} = \gamma_\mu - q_{\mu_A}\frac{\not q_A}{m_A^2}$, $m_A = m_B = \sqrt{q_A^2}$, $q_A = -q_B$ and $q_A$ and $q_B$ are the total momenta coming into the two vertices A and B. The explicit factorization indicated in Eqs. III-2 and III-3 allows us to identify the $SU_6$ character of the resonances which appear, i.e., written as $\mathrm{Tr}(A\ O_p)\ \mathrm{Tr}(O_p B)$ the resonances have the quantum numbers of $O_p$. If we assume the quark lines of Fig. III-1a just represent $\delta$ functions in $SU_3$ and Dirac space then A is just $M_3 \cdot M_2$ and B is $M_1 \cdot M_4$.

Thus, from Eq. III-2 we learn that our resonances belong to $SU_3$ nonets as desired. By comparing Eq. III-3 with the usual couplings for scalar, pseudoscalar, vector, and axial vectors [21,42]

$$\left\{-i,\ \frac{\not q}{\sqrt{q^2}}\right\},\ \left\{\gamma_5,\ \frac{\gamma_5\not q}{\sqrt{q^2}}\right\},\ \left\{-i(\gamma_\mu - \frac{q_\mu\not q}{q^2}),\ \frac{\sigma_{\mu\nu}q_\nu}{\sqrt{q^2}}\right\}$$

$$\text{and}\ \left\{-i(\gamma_5\gamma_\mu - \frac{q_\mu\gamma_5\not q}{q^2}),\ -i\ \gamma_5\ \frac{\sigma_{\mu\nu}q_\nu}{\sqrt{q^2}}\right\} \qquad \text{(III-4)}$$

and using $-i\ g_{\mu\nu}$ for the spin-1 propagator and $+i$ for the spin 0 propagator, we note that Eq. III-3 implies the exchange of all four types of particles, each with both kinds of couplings. Further, the scalars and axial vectors are ghosts, i.e., have negative coupling constants squared. Now from the quark model we expect only $0^-$ and $1^-$ particles since this is all one would expect from a "real" fermion - antifermion pair. The extra states are effectively the contribution of the MacDowell (parity) partners of the quarks which are present whenever we propagate fermions without being careful. More explicitly, in order to have a Lorentz invariant program we have imbedded the 2 component Pauli spinors in 4 component Dirac spinors and because we have not been careful to eliminate the contribution of the "small" (negative energy) components our resonances actually satisfy a $SU_{12} \times O_3$ - parity doubled symmetry (also called U(6,6) or $\tilde{U}_{12}$) rather than the desired $SU_6 \times O_3$ symmetry.

The point of this model is that the unwanted ghostly, parity partner resonances can easily be eliminated by assuring that the propagators for resonating quarks carry projection operators $(1 \pm \frac{\not{p}}{m})$ just as for ordinary fermions. Here m is the mass of the appropriate $q\,\bar{q}$ resonance. This will insure that resonance wave function will satisfy the Bargman-Wigner equations just as was required of the external wave functions.

Of course in the context of a dual model the appearance of $\frac{1}{m}$ factors requires that new functionry be present to generate them. Also, cuts are introduced in the complex angular momentum plane, a la Carlitz-Kislinger,[22] which influence the asymptotic behavior of the amplitude. In the following we suggest a possible form of the "functionry" and study the corresponding effects on the asymptotic (Regge Region) amplitude and the interpretation of duality.

Thus the 4 point amplitude which describes Fig. III-la may be written as:

$$A(s,t) = g^2 \int_0^1 dz\, z^{-\ell(s)-1} (1-z)^{-\ell(t)-1}$$

$$M_{1(\alpha_1 a)}^{(\beta_1 d)} \; M_{2(\alpha_2 b)}^{(\beta_2 a)} \; M_{3(\alpha_3 c)}^{(\beta_3 b)} \; M_{4(\alpha_4 d)}^{(\beta_4 c)}$$

$$\Delta^{\alpha_1 \alpha_3}_{\beta_2 \beta_4} (p_s, z) \quad \Delta^{\alpha_2 \alpha_4}_{\beta_3 \beta_1} (p_t, 1-z) \qquad \text{(III-5)}$$

Here g is an overall coupling constant, $\ell(s)$ is a linear trajectory function specifying the orbital excitations, and $p_s$ is the s-channel momentum ($p_s^2 = s$). We write the dual projection operator $\Delta^{\alpha\,\gamma}_{\beta\,\delta}$ in the form

$$\Delta^{\alpha\,\gamma}_{\beta\,\delta} (p,z) = \delta^{\alpha}_{\beta}\, \delta^{\gamma}_{\delta} + E_1(p,z)\, (p\cdot\gamma^{\alpha}_{\beta}\, \delta^{\gamma}_{\beta} - \delta^{\alpha}_{\beta}\, p\cdot\gamma^{\gamma}_{\delta}) - E_2(p,z)\, p\cdot\chi^{\alpha}_{\beta}\, p\cdot\gamma^{\gamma}_{\delta} \qquad \text{(III-6)}$$

Note that any function multiplying the $\delta\delta$ term could be absorbed into the basic vertex function which we have assumed here is given by Veneziano.

At each p for which $\ell(p^2)$ is a positive integer we require that $\Delta$ satisfy the conditions

$$\Delta^{\alpha\ \gamma}_{\beta\ \delta}\ (p,o) = (1 + \frac{\not p}{\sqrt{p^2}})^{\alpha}_{\beta}\ (1 - \frac{\not p}{\sqrt{p^2}})^{\gamma}_{\delta} \qquad \text{(III-7a)}$$

$$\frac{\partial^n}{\partial z^n}\ \Delta^{\alpha\ \gamma}_{\beta\ \delta}\ (p,z)\Big|_{z=0} = 0 \qquad n=1,\ 2,\ 3\ \ldots\ \ell(p^2) \qquad \text{(III-7b)}$$

Recall that we must consider z=0 since this is where s-channel poles come from, e.g. in Eq. II-5. Condition III-7b is to insure that the daughter trajectories are also parity-partner free, although this constraint is somewhat questionable since the odd daughters, in general, already contain ghosts.

Further, since the appearance of a $\not p_s$ factor between $M_1$ and $M_2$ when we are looking at a t channel resonance will destroy the $SU_{6W} \times O_{2L_z}$ symmetry of the coupling of $M_1$ and $M_2$ to that resonance, we must also require

$$\Delta^{\alpha\ \gamma}_{\beta\ \delta}\ (p,1) = \delta_{\beta}{}^{\alpha}\ \delta_{\delta}{}^{\gamma} \qquad \text{(III-8a)}$$

$$\frac{\partial^n}{\partial z^n}\ \Delta^{\alpha\ \gamma}_{\beta\ \delta}\ (p,z)\Big|_{z=1} = 0 \qquad n=1,\ 2\ \ldots \qquad \text{(III-8b)}$$

Note that this constraint is also what is required to insure factorization in the sense described in the previous chapter.

Consider the case of the 6 point amplitude illustrated in Fig. II-1. We factor the amplitude by going to a pole in the indicated channel. Then Eq. III-8 guarantees that the quarks of the left-hand 4 point amplitude will not have projection operators which corresponded in the original 6 point amplitude to channels involving also quarks from the right-hand 4 point amplitude. This is just the condition that the two 4 point amplitudes we obtain by factorization are the same as we would write down directly. If this were not the case, this factorization requirement would, in general, lead to a degeneracy greater than in the simple Veneziano amplitude.

We write E in the form $(d\ell/ds = \ell')$

$$E_\sigma(p,z) = \frac{(\ell')^{\sigma/2}}{\Gamma(\sigma/2)} \int_z^1 d\,x\; x^{(\ell' p^2)-1} \left(\ell n\frac{1}{x}\right)^{\sigma/2-1} \Phi(1-{}^z/x) \qquad \text{(III-9)}$$

so that the conditions of Eqs. III-7 and III-8 become

$$\Phi(0) = 0 \qquad \Phi(1) = 1$$

$$\frac{d^n}{dz^n}\,\Phi(z)\Big|_{z=1,0} = 0 \quad n = 1,\ 2\ \ldots \qquad \text{(III-10)}$$

Detailed questions about $\Phi$ will be discussed in Chapter V. For now we just accept the existence of a class of such functions, known as ven der Corput neutralizers.(43) We note in passing that for $\Phi$ equal to 1 and $\sigma = 1$, $E_1 = \mathrm{erf}\,(\sqrt{\ell' p^2 \ln(1/z)})$ as expected from the work of Carlitz and Kislinger.(22)

Now with our projection operator appropriately defined, we return to Eq. III-5 to study the s channel resonances, i.e., $z \approx 0$. For the nth resonance $(\ell(s) \approx n)$

$$A \approx \frac{\Gamma\,(n+\ell(t)+1)}{\Gamma(n+1)\,\Gamma(\ell(t)+1)} \quad \frac{1}{n-\ell(s)} \quad \mathrm{Tr}\left[M_3 M_2\,(1+\frac{\not{p}_s}{m_n})\,M_1 M_4\,(1-\frac{\not{p}_s}{m_n})\right]$$

(III-11)

where $m_n^2 = \frac{n-\ell(0)}{\ell'}$ and Tr denotes a trace on $SU_3$ and Dirac indices. The first factor of Eq. III-11 specifies the orbital excitation; it is just the $n^{th}$ pole of the usual Veneziano amplitude. Now we apply the factorization of Eqs. III-2 and III-3 to the trace of Eq. III-11 where $A = M_3M_2\,(1+\not{p}_s/m_n)$ and $B = M_1M_4\,(1-\not{p}_s/m_n)$. We see that at the pole the two $0^+$ terms cancel as do the $1^+$ terms and so do not resonate. For example, with $q_B = p_s$, $q_A = -p_s$, and $p_s^2 = m_n^2 = m_A^2 = m_B^2$, we have

$$(1+\frac{\not{p}_s}{m_n})\,\frac{\not{q}_A}{m_n} = -\,(1+\frac{\not{p}_s}{m_n}) \quad \text{and} \quad (1-\frac{\not{p}_s}{m_n})\,\frac{\not{q}_B}{m_n} = -\,(1-\frac{\not{p}_s}{m_n}) \; .$$

Thus the first and fourth terms of Eq. III-3 cancel (both $0^+$). Similarly the fifth and eighth ($1^+$) terms also cancel at the pole. The remaining terms can be written as

$$A \approx \frac{\Gamma\,(n+\ell(t)+1)}{\Gamma\,(n+1)\,\Gamma\,(\ell(t)+1)} \quad \frac{1}{n-\ell(s)} \quad \frac{1}{2}\sum_{i=0}^{8}\Big[\mathrm{Tr}\,[M_3M_1\,(\frac{1+\not{p}_s}{m_n})\frac{\gamma_5}{\sqrt{2}}\,\lambda_i\,]$$

$$\times\,\mathrm{Tr}\,[(1-\frac{\not{p}_s}{m_n})\,\frac{\gamma_5}{\sqrt{2}}\,\lambda_i\,M_1M_4\,] + \mathrm{Tr}\,[M_3M_2\,(1+\frac{\not{p}}{m_n})\,\frac{\gamma_\mu}{\sqrt{2}}\,\lambda_i\,](g_{\mu\nu}-\frac{p_{s\mu}\,p_{s\nu}}{m_n^2})$$

$$\times\,\mathrm{Tr}\,[(1-\frac{\not{p}_s}{m_n})\,\frac{\gamma_\nu}{\sqrt{2}}\,\lambda_i\,M_1M_4\,]\Big] \qquad . \qquad \text{(III-12)}$$

As promised our amplitude explicitly contains resonances belonging to $SU_3$ nonets whose total spin corresponds to the orbital excitations of of Veneziano plus a quark spin of either zero or one with the appropriate parity. Note that in general both parities are present but that the projection operators we have introduced allow only a single parity at the poles. Also note that unlike the usual Veneziano model for $\pi\pi$ scattering, where the $\rho$ trajectory could in principle have a particle at $\alpha_\rho(t) = 0$ and $t < 0$, i.e., a tachyon, $\alpha_\rho(t) = 0$ in this model corresponds to $L = -1$ and there is no question about the absence of a particle there.

Before proceeding to discuss details of the J-plane structure of this model a few words should be said about how the model can be extended from the 4-point case to the general n-point case, e.g. the 5-point diagram shown in Fig. III-1b. Although we shall not explicitly study the n-point problem, one of the strengths of the model is its straightforward generalization. We label the channel $(i+1, \ldots., j)$ goes to $(j+1, j+2, \ldots.., i)$ by $(i,j)$ where $i+n = i$. The corresponding momentum is

$$P_{ij} = k_{i+1} + \ldots.. + k_j = -(k_{j+1} + k_{j+2} + \ldots.. + k_i) \qquad \text{(III-13)}$$

where all external momentum are incoming and $k_{i+n} = k_i$. We note that each planar channel $(i,j)$ has to have a "dual projection operator" which is "activated" or "deactivated" as explained above. The general form of the n-point amplitude is

$$A_n = g^{N-2} \int_0^1 \Pi\, dz_{ij}\, z_{ij}^{-\ell(p_{ij}^{\;2})-1} \quad \Pi' \delta(z_{ij} + \prod_{ij} z_{k\ell} - 1)$$

$$\Pi\, \Delta^{\alpha_{i,j+1}\;\alpha_{j,i+1}}_{\alpha_{ij}\;\alpha_{ji}} (p_{ij}, z_{ij}) \prod_{k=1}^{N} M_k \begin{pmatrix} \alpha_{k-1,k+1}, a_k \\ \alpha_{k,k-1}, a_{k+1} \end{pmatrix} \qquad \text{(III-14)}$$

where $\Pi$ means product over all planar channels (of course $(i,j) = (j+1, i+1)$, so that if $(i,j)$ is included, then $(j+1, i-1)$ is to be omitted); $\Pi'$ means product over all channels other than, say, the n-3 channels (1,3), (1,4), ....., (1, n-1); and $\Pi_{ij}$ means product over all channels dual to $(i,j)$, i.e., channels which cannot resonate simultaneously with $(i,j)$. After the two products in the bracket are expanded, each spinor index appears exactly once up and once down and is to be summed over. The indexing of the $\Delta$'s and M's can be understood by referring to the quark diagram, Fig. III-1b, where the solid lines indicate quarks, wavy lines indicate projection operators, and the $M_i$ specify the external meson matrices. Except for the spin factors, Eq. III-14 is just the usual n-point dual resonance amplitude.[19] Just as in the 4-point case, the n-point function factors at all poles.

# IV. ASYMPTOTIC BEHAVIOR AND J-PLANE STRUCTURE

From the preceding chapter we see that there are essentially two features appearing in our model which are new and which make it different from previous models. The first of these is the presence of $1/m_n$ factors which appear because we avoid parity doubling in the manner suggested by Carlitz and Kislinger[22] and which would appear in any Regge exchange model with quarks using an approach similar to ours, independent of duality. The second new feature is the neutralizer function which appears solely because of our duality constraints. It is important to keep this distinction between the two new features in mind.

In this chapter we shall study those facets of the asymptotic behavior and related J-plane structure of our model which are independent of the specific form of the neutralizer. This will prepare us for the next chapter where the neutralizer is discussed in detail and all of the new structure is interpreted in terms of the FESR formalism. That part of the structure which appears also in the ordinary Veneziano amplitude, i.e., s and t channel moving Regge poles and the more subtle point of u channel nonsense wrong signature fixed poles, will not be emphasized as it is well discussed in the literature.[45]

If the amplitude that we wish to study is expressed as a sum of, for example, s channel resonances as in Eq. II– 2, the easiest way to find the large t behavior and the related singularity structure of the s channel partial wave amplitude is to apply a Sommerfeld-Watson transformation. Since we are studying asymptotic behavior in what follows, we are not finding the actual partial wave amplitude, but rather a related amplitude which, to the extent that asymptotic behavior is determined by the singularities in the J-plane, has the same singularity structure as the actual partial wave amplitude. The Sommerfeld-Watson transformation proceeds by the replacement

$$\sum_n \rightarrow \frac{1}{2\pi i} \int d\lambda \, \frac{\pi e^{-i\pi\lambda}}{\sin \pi\lambda}$$

and $n \rightarrow \lambda$ everywhere. The contour in the $\lambda$ plane starts at $+\infty$, comes in along the top of the real axis, goes around the origin, and goes back to $+\infty$ along the bottom of the real axis. The sum $\sum_n$ reappears if we shrink the contour in about the poles of $\frac{1}{\sin \pi\lambda}$. The trick now is to move the contour out, away from the real axis to pick up the contributions from the other $\lambda$ singularities which arise from the form of the amplitude itself. We will be dealing with functions which are well enough behaved that the large circle at infinity does not contribute. These singularities which the integrand has in $\lambda$ are the same as the partial wave amplitude has in the variable $\ell$ as discussed above and we can study the integrand as if it were the partial wave amplitude. If we apply this approach to Eq. II-2a for large t, fixed s, so that $\frac{\Gamma(n+\alpha(t)+1)}{\Gamma(\alpha(t)+1)} \sim (\alpha' t)^n$, we find

$$A_V(s,t) \sim \frac{-1}{2\pi i} \int d\lambda \, \frac{(-\alpha' t)^{\lambda} \, \Gamma(-\lambda)}{\lambda - \alpha(s)} \qquad \text{(IV-1)}$$

where we used $\frac{\pi}{\Gamma(1+\lambda)\sin\pi\lambda} = -\Gamma(-\lambda)$ . Now the integrand has only a simple (Regge) pole singularity and we can evaluate the integral to get the result $\Gamma(-\alpha(s))\,(-\alpha' t)^{\alpha(s)}$ which checks with our work with the gamma functions in Chapter II. If we now consider the factors $1/m_n$ and $1/m_n^2$ which appear in Eq. III-11, we find that singularites of the form

$$\frac{1}{\sqrt{(\lambda-\alpha_o)/\alpha'}} \quad \text{and} \quad \frac{\alpha'}{\lambda-\alpha_o}$$

are introduced. Thus we expect the partial wave amplitude to have fixed cut and fixed pole contributions and the full amplitude to exhibit $(-\alpha' t)^{\alpha_o}$ asymptotic behavior.

We will return to the Sommerfeld-Watson transform in Chapter VI when we numerically evaluate our model. Now it is useful to turn to a more systematic approach which will allow us to inspect the behavior in all channels of an arbitrary amplitude. This formalism is called the modified Mellin transform.(46) This approach will reproduce the Sommerfeld-Watson result but does not require that the amplitude be initially expressed as a sum of partial wave amplitudes. In particular, it is more useful if the amplitude is expressed as an integral. The Mellin transformed amplitude is essentially just the integrand which appears in the final Sommerfeld-Watson contour integral. It is the amplitude with the same singularity structure as the partial wave amplitude and, as we saw above, these singularities determine the asymptotic behavior. For simplicity we shall consider only the behavior of a single term of the projection operator and only in the $\ell$-plane (we get to the J-plane by adding the quark spin).

We define the modified Mellin transform, in the case of large t, fixed s, as

$$\tilde{A}_\sigma \ (\ell(s), \lambda) = \sin (\pi\lambda) \int_1^\infty dt \ t^{-\lambda-1} \int_0^1 dz \ z^{-\ell(s)-1} \ (1-z)^{-\ell(-t)-1} \ E_\sigma(s,z)$$

(IV-2)

where the $\sin(\pi\lambda)$ factor eliminates the extraneous poles at the positive integers. Notice the change in the sign of t to insure convergence. Actually, as will be seen in the next chapter, by changing the path of the z integration one can show that all that is needed for convergence is $|\arg t| > \epsilon$ as expected from Chapter II. Without changing the singularities in $\lambda$ and to allow us to do the integral explicitly we change the t integral lower limit to zero. Setting $\ell' = 1$ for convenience, we have

$$\tilde{A}_\sigma \ (\ell(s),\lambda) \sim \frac{\sin\pi\lambda}{\Gamma(\sigma/2)} \int_0^1 dz \ z^{-\ell(s)-1} \ (1-z)^{-\ell_0-1}$$

$$\int_z^1 dx \ x^{s-1} \ (\ell n \ \frac{1}{x})^{(\sigma/2)-1} \ \Phi \ (1-{}^z/x) \ \int_0^\infty dt \ t^{-\lambda-1} \ (1-z)^t$$

$$= \frac{\sin\pi\lambda \ \Gamma(-\lambda)}{\Gamma(\sigma/2)} \int_0^1 dz \ z^{-\ell(s)-1} \ (1-z)^{-\ell_0-1} \ (\ell n \ \frac{1}{(1-z)})^\lambda$$

$$\int_z^1 dx \ x^{s-1} \ (\ell n \ \frac{1}{x})^{(\sigma/2)-1} \ \Phi \ (1-z/_x)$$

$$= \frac{-\pi}{\Gamma(\sigma/_2) \ \Gamma(1+\lambda)} \int_0^1 dx \ x^{-\ell_0-1} \ (\ell n \ \frac{1}{x})^{(\sigma/2)-1}$$

$$\int_0^1 dy \ y^{-\ell(s)-1} \ (1-xy)^{-\ell_0-1} \ (\ell n \ \frac{1}{1-xy})^\lambda \ \Phi(1-y) \qquad \text{(IV-3)}$$

where the last step involves changing the order of integration and introducing the new variable $y = z/x$. The general approach is to look for the singularities in $\lambda$ coming from the region of integration where the argument of the logarithm goes to unity, i.e., where x and y vanish. Expanding about $x,y = 0$, i.e., $\ell n(1/(-xy)) \sim xy$, $\Phi(1-y) \sim 1$ which corresponds to ignoring the neutralizer and which will require further discussion later, we have

$$\tilde{A}_\sigma\ (\ell(s),\lambda) \sim \frac{-\pi}{\Gamma(1+\lambda)} \int_0^1 \frac{dx\ x^{-\ell_0-1+\lambda}}{\Gamma(\sigma/_2)} \left(\ell n\ \frac{1}{x}\right)^{(\sigma/2)-1} \int_0^1 dy\ y^{-\ell(s)-1+\lambda}$$

$$= \frac{-\pi}{\Gamma(1+\lambda)} \quad \frac{(\lambda-\ell_0)^{-\sigma/2}}{\lambda-\ell(s)} \qquad \text{(IV-4)}$$

(using $\int_0^1 dx\ x^{a-1}\ (\ell n\ \frac{1}{x})^{b-1} = a^{-b}\ \Gamma(b)$ ). Thus we have a moving Regge pole and a multiplicative fixed cut $(\sigma = 1)$ or fixed pole $(\sigma = 2)$. Further terms in the expansion of $(1-xy)^{-\ell_0-1}$ and $(\ell n\ \frac{1}{1-xy})^\lambda$ yield the usual daughter Regge poles and corresponding cuts. The inverse transform is

$$A_\sigma\ (\ell(s),\ t \to \infty) \equiv \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{d\lambda}{\sin\pi\lambda}\ (-t)^\lambda\ .\ \tilde{A}_\sigma\ (\ell(s),\lambda)$$

$$\sim \frac{1}{2\pi i} \int_{\supset} d\lambda\ \frac{(-t)^\lambda\ \Gamma(-\lambda)}{\lambda-\ell(s)}\ (\lambda-\ell_0)^{-\sigma/2} \qquad \text{(IV-5)}$$

which is the Sommerfeld-Watson result as advertised.

Now we can proceed to study the more difficult case of large s, fixed t,

$$\tilde{A}_\sigma(\lambda,\ell(t)) = \sin(\pi\lambda)\int_1^\infty ds\, s^{-\lambda-1}\int_0^1 dz$$

$$\int_z^1 \frac{dx}{x}\left(\frac{z}{x}\right) s_z^{-\ell_0-1}(1-z)^{-\ell(t)-1}\frac{(\ell n\frac{1}{x})^{(\sigma/2)-1}}{\Gamma(\sigma/2)}\Phi(1-z/x)$$

$$\sim \frac{-\pi}{\Gamma(1+\lambda)\Gamma(\sigma/2)}\int_0^1 dx\, x^{-\ell_0-1}(\ell n\frac{1}{x})^{(\sigma/2)-1}$$

$$\int_0^1 dy\, y^{-\ell_0-1}(1-xy)^{-\ell(t)-1}\Phi(1-y)(\ell n\frac{1}{y})\quad (y = z/x). \qquad \text{(IV-6)}$$

This time the $\lambda$ singularities come from $y \sim 1$ which is excluded by the neutralizer, $\Phi$, i.e., $\tilde{A}_\sigma(\lambda,\ell(t))$ is an entire function of $\lambda$. This means that for $s \to \infty$ in some region of the s plane this amplitude will decrease faster than any power of s. The details depend on $\Phi$ and will be studied in the next chapter. For completeness we shall see what happens if $\Phi = 1$, i.e., no neutralizer. For this case it is convenient to rewrite the above equation as: (with $\Phi = 1$)

$$\tilde{A}_\sigma(\lambda,\ell(t)) \sim \sin(\pi\lambda) \int_0^\infty ds\, s^{-\lambda-1} \int_0^1 dz\, z^{-\ell(t)-1} (1-z)^{-\ell_0-1} \int_{1-z}^1 \frac{dx}{x} \left(\frac{1-z}{x}\right) \frac{\left(\ell n \frac{1}{x}\right)^{(\sigma/2)-1}}{(\sigma/2)}$$

$$= \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)} \int_0^1 dz\, z^{-\ell(t)-1} (1-z)^{-\ell_0-1} \int_0^z \frac{dr}{1-r} \left(\ell n \frac{1}{1-r}\right)^{(\sigma/2)-1} \left(\ell n \frac{1-r}{1-z}\right)^{\lambda} \qquad (r = 1-x)$$

$$= \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)} \int_0^1 dz\, z^{-\ell(t)} (1-z)^{-\ell_0-1} \int_0^1 \frac{dy}{1-yz} \left(\ell n \frac{1}{1-yz}\right)^{(\sigma/2)-1} \left(\ell n \frac{1-yz}{1-z}\right)^{\lambda} \qquad (r = yz) \qquad \text{(IV-7)}$$

Evidently the $\lambda$ singularities come from the two boundaries $y = 1$, $z = 0$. Expanding in the region where these boundaries intersect we find, to first order in z and $v = 1 - y$

$$\tilde{A}_\sigma(\lambda,\ell(t)) \sim \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)} \int_0^1 dz\, z^{\lambda-\ell(t)+(\sigma/2)-1} \int_0^1 dv\, v^{\lambda}$$

$$= \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)} \frac{1}{\lambda-\ell(t)+(\sigma/2)} \frac{1}{\lambda+1} \quad (+\text{ daughters}) \qquad \text{(IV-8)}$$

Without the neutralizer we find two problems. The amplitude has both fixed poles at the negative integers $(1/(1+\lambda))$ and moving poles which for $\sigma = 1$ are shifted by $\frac{1}{2}$ from the usual positions (called cousins for the simple reason that the terms ancestors and daughters are already in use). This is another indication that some sort of neutralizer is required. By repeating the above analysis (Eqs. IV-6 to IV-8) with a neutralizer of the form $\Phi(1-z)$ or $\Phi(1-x)$ one can see that neither form is satisfactory since one of the above diseases will still persist in either case. We can exhibit the effects of both of the above terms on the asymptotic behavior if we consider a simple Carlitz-Kislinger[22] type amplitude with a cut in the s channel, for large s. It is of the form

$$A^{ck}_{\sigma=1}(s\to\infty,\ell(t)) = \lim_{s\to\infty}\int_0^1 dx\, x^{-\ell(s)-1}(1-x)^{-\ell(t)-1}\,\frac{\operatorname{erf}\left(\sqrt{s\ell n\frac{1}{x}}\right)}{\sqrt{s}}$$

$$\sim \int dx\, x^{-\ell(s)-1}(1-x)^{-\ell(t)-1}\left(\frac{1}{\sqrt{s}} - \frac{x^s}{\sqrt{\pi\,\ell n\frac{1}{x}}\;s}\right)$$

$$\sim \frac{s^{\ell(t)}}{\sqrt{s}}\int dy\, e^{-y}(1-y)^{-\ell(t)-1} \quad (x=e^{-y/s}) \quad - \frac{1}{s}\int dx\, x^{-\ell_o-1}\frac{(1-x)^{-\ell(t)-1}}{\sqrt{\pi\ell n\frac{1}{x}}}$$

$$\sim f(t)\, s^{\ell(t)-\frac{1}{2}} - \frac{g(t)}{s} \qquad \text{(IV-9)}$$

We come now to the question of fixed u, large s,t behavior, i.e., u channel partial wave amplitudes. This is a much more complex and subtle problem and we shall not carry the discussion into any great detail. First we define the variables $U=\frac{1}{2}(s+t)$ and $V=\frac{1}{2}(s-t)$ so that in the limit we are interested in U is fixed and V becomes large. Using the usual manipulations we have

$$A_\sigma\,(\ell(s),\,\ell(t)) = \frac{1}{\Gamma(\sigma/2)} \int_0^1 dx\, x^{-\ell_o-1} \left(\ell n\,\frac{1}{x}\right)^{(\sigma/2)-1}$$

$$\int_0^1 dy\,(y\,(1-xy))^{-\ell_o-U\,-1} \left(\frac{1-xy}{y}\right)^{-V} \Phi(1-y) \qquad \text{(IV-10)}$$

To apply the Mellin transform we must separate the two cases $1\text{-}xy/y > 1$ $(y < \frac{1}{1+x})$ and $1\text{-}xy/y < 1$ $(y > \frac{1}{1+x})$ and treat them separately to insure convergence of the transform, i.e., $A(U,V)$ for case I and $A(U,-V)$ for II.

$$\tilde{A}_{\sigma_I}\,(\ell(U),\lambda) \sim \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)}$$

$$\int_0^1 dx\, x^{-\ell_o-1} \left(\ell n\,\frac{1}{x}\right)^{(\sigma/2)-1} \int_0^{\frac{1}{1+x}} dy\,(y(1-xy))^{-\ell(U)\,-1}$$

$$\times \quad \Phi(1-y)\,\left(\ell n\,\left(\frac{1-xy}{y}\right)\right)^{\lambda} \qquad \text{(IV-11a)}$$

$$\tilde{A}_{\sigma_{II}}(\ell(U),\lambda) \sim \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)}$$

$$\int_0^1 dx\, x^{-\ell_o\,-1} \left(\ell n\,\frac{1}{x}\right)^{(\sigma/2)-1} \int_{\frac{1}{1+x}}^1 dy\,(y(1-xy))^{-\ell(U)\,-1}$$

$$\times \quad \Phi(1-y)\,\left(\ell n\,\left(\frac{1}{1-xy}\right)\right)^{\lambda} \qquad \text{(IV-11b)}$$

Changing variables to $y = (1-r)/(1+x)$ in Eq. IV-11a and $y=(1+r)/(1+x)$ in Eq. IV-11b and keeping only first order terms in r, i.e., the $\lambda$ singularities come from $y \sim 1/1+x$, we get

$$\tilde{A}_{\sigma_I}(\ell(U),\lambda) \sim \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)}$$

$$\int_0^1 dx\, x^{-\ell_0-1} \left(\ell n \frac{1}{x}\right)^{(\sigma/2)-1} \Phi\left(\frac{1}{1+x}\right)(1+x)^{\lambda+2\ell(U)+1} \int_0^1 dr\, r^{\lambda}$$

$$= \frac{f(U,\lambda)}{\lambda+1} \qquad \text{(IV-12a)}$$

$$\tilde{A}_{\sigma_{II}}(\ell(U),\lambda) \sim \frac{-\pi}{\Gamma(\sigma/2)\,\Gamma(1+\lambda)}$$

$$\int_0^1 dx\, x^{-\ell_0-1} \left(\ell n \frac{1}{x}\right)^{(\sigma/2)-1} \Phi\left(\frac{x}{1+x}\right)(1+x)^{\lambda+2\ell(U)+1} \int_0^x dr\, r^{\lambda}$$

$$= \frac{1}{1+\lambda} \cdot \frac{-\pi}{\Gamma(\sigma/2)\Gamma(1+\lambda)}$$

$$\int_0^1 dx\, x^{\lambda-\ell_0} \left(\ell n \frac{1}{x}\right)^{(\sigma/2)-1} \left(\frac{x}{1+x}\right)(1+x)^{\lambda+2\ell(U)+1} \qquad \text{(IV-12b)}$$

We notice two points: both amplitudes have fixed poles at minus one and if it weren't for the fact that $\Phi(\frac{x}{1+x})$ vanishes at x = 0, $\tilde{A}_{\sigma_{II}}$ would also have a singularity at $\lambda = \ell_0 - 1$. To the extent that the neutralizer causes most of the integral to come from x ~ 1, the two amplitudes are equal. When we invert this transform we recall that for $A_{\sigma_I}$ we used $+V$ and that for $A_{\sigma_{II}}$ we used $-V$. Thus for $A_{\sigma_I}$ the inversion integral is $\int d\lambda\, V^\lambda$ and the fixed poles gives $\frac{f(U)}{V}$. For $A_{\sigma_{II}}$ we have $\int d\lambda\, (-V)^\lambda$ with the result $\frac{f(U)}{-V}$. So fixed poles at odd negative integers (nonsense wrong signature fixed poles) do not contribute to the total amplitude if the residues at the poles are the same in $A_I$ and $A_{II}$. For the ordinary Veneziano amplitude $(A_{\sigma=0})$ only odd integer poles appear and the amplitude decreases exponentially (for $|\arg(s)| > \epsilon$) as we saw in Chapter II. In general, this is not true of our $A_\sigma$'s but one could, in principle, use the requirement of only odd integer fixed poles to determine the form of $\Phi$. However, since what is required of u channel behavior is unclear, and since we are primarily interested in those facets of the model which are independent of the detailed behavior of $\Phi$, we shall not pursue this point further. In the following chapters the assumption will be made that nothing exciting happens in the u channel.

To summarize, we have found that $A_\sigma(s,t)$, which corresponds to the $(1/m_n)^\sigma$ term in the s channel projection operator, has partial wave amplitudes in the t and u channels which are essentially entire functions. Thus, for fixed t or u and large s, $A_\sigma(s,t)$ decreases faster than any power of s with the details depending on $\Phi$. In the s channel the partial wave amplitude contains both a moving Regge pole and a fixed cut or fixed pole and $A_\sigma(s,t)$ contributes to the Regge behavior. We shall find that again the details are affected by the neutralizer. Similar behavior holds for the other parts of the total amplitude with the corresponding structure in the other channels.

# V. THE NEUTRALIZER FUNCTION AND THE FESR

Although, as emphasized previously, we are primarily interested in results which are independent of the exact form of the neutralizer, it is still of value to study the general form of such functions[43,47] and to determine what freedom exists in our model for their definition. These considerations are important in order to be able to give a meaningful interpretation to the presence of the neutralizer and for making comparisons with data as in the next chapter.

Recall that the neutralizer first appeared in Chapter III where it had to satisfy the following constraints:

$$\Phi(0) = 0, \ \Phi(1) = 1, \ \frac{d^n}{dz^n} \Phi(z) \Big|_{z = 0,1} = 0, \quad n = 1, 2, 3 \ldots \tag{III-10}$$

In this chapter we study the further restrictions placed on $\Phi$ in order to guarantee the asymptotic behavior we anticipate from our work in the previous chapter. For example, Eq. III-10 is satisfied by $\Phi(z) = \theta(z - \frac{1}{\gamma})$, $1 < \gamma < \infty$ where $\theta(z)$ is the usual step function: $\theta(z) = 0$ $\mathrm{Re}\,z < 0$, $\theta(z) = 1$, $\mathrm{Re}\,z > 0$. However, it will turn out that this function is too singular (not very surprisingly) to yield the required behavior in the entire s plane. Instead we shall need a smoother function such as that suggested by Suzuki[43]

$$\Phi(z) = \frac{1}{c} \int_0^z dx \exp \left\{ -(\ell n\, \ell n \frac{1}{x}) \ell n \frac{1}{x} - (\ell n\, \ell n \frac{1}{1-x}) \ell n \frac{1}{1-x} \right\}$$

$$c = \int_0^1 dx \exp \left\{ -(\ell n\, \ell n \frac{1}{x}) \ell n \frac{1}{x} - (\ell n\, \ell n \frac{1}{1-x}) \ell n \frac{1}{1-x} \right\} \qquad \text{(V-1)}$$

However, the step function does have the advantage of calculational simplicity and will be used as an approximation to $\Phi$ in the next chapter. We should also keep in mind the possibility of relaxing some of the constraints of Eq. III-9, for example those at $z = 1$, since this corresponds to allowing the daughter trajectories to have ghostly parity partners and, as mentioned in Chapter II, the daughters in the Veneziano amplitude already contain some ghosts.

As a warm up let us consider why it is, in terms of its integral representation, that the ordinary Veneziano amplitude,

$$A_V(s,t) = \int_0^1 dz\, z^{-\ell(s)-1} (1-z)^{-\ell(t)-1} \qquad \text{(V-2)}$$

has Regge behavior for $|s| \to \infty$ , $|\arg s| > \epsilon$, t fixed. First change variables to $z = e^{-r}$ to get

$$A_V(s,t) = \int_0^\infty dr\, e^{+r\ell(s)} (1-e^{-r})^{-\ell(t)-1} \quad . \qquad \text{(V-3)}$$

It is immediately obvious that, for $\pi > |\arg s| > \pi/2$ , the integral is at least well defined since $\mathrm{Re}(rs) < 0$. This is in fact the case studied earlier with the Mellin transform. The contributions to the integral from all finite r, i.e., $r > \frac{1}{|s|}$ , will vanish exponentially as $|s| \to \infty$ . The interesting contribution comes from small r (again let $\ell' = 1$).

$$A_V(s,t) \underset{|s| \to \infty}{\sim} \int_0 dr\, e^{rs} \quad r^{-\ell(t)-1} \qquad \text{(set } y = -rs)$$

$$\sim \int_0^{\infty} dy\, e^{-y} \; y^{-\ell(t)-1} \, (-s)^{\ell(t)} = \Gamma(-\ell(t))(-s)^{\ell(t)} \; . \qquad \text{(V-4)}$$

To find just the basic power behavior we can use the method of steepest descent.[48]

$$A_V(s \to \infty,\, t) \sim \int dr\, e^{|s|(-r + \frac{(\ell(t)+1)}{|s|} \;\ell n \frac{1}{r})} \qquad . \qquad \text{(V-5)}$$

The maximum of the function in the exponent is for $r \sim \frac{-(\ell(t)+1)}{|s|}$ (we are assuming $\ell(t)+1 < 0$), and we find

$$A_V(s \to -\infty, t) \sim \sqrt{\frac{2\pi(\ell(t)+1)}{|s|^2}} \quad e^{\ell(t)+1} \left( - \frac{|s|}{(\ell(t)+1)} \right)^{\ell(t)+1} \qquad \text{(V-6)}$$

which has the $(-s)^{\ell(t)}$ correctly, but the t dependence, i.e., the $\Gamma(-\ell(t))$ , is somewhat hard to pick out. However, since we are interested only in the gross s dependence we shall continue to use this method (which gives us the first term in an asymptotic expansion in 1/s).

Now, how do we define the integral for $|\arg s| \leq \frac{\pi}{2}$ ? We know that it is well defined from our study of the gamma function form in Chapter II. This is done by rotating the ray in the r plane along which we do the integral so as to maintain $\mathrm{Re}(sr) < 0$. Defining $r = \rho e^{i\Psi}$ and $s = \eta e^{i\Theta}$, we require $\cos(\Theta + \Psi) < 0$, i.e., $|\Theta + \Psi| > \frac{\pi}{2}$. We can make this rotation because the integrand vanishes exponentially for $\rho \to \infty$, $|\Psi| < \frac{\pi}{2}$, at least when $|\Theta| > \frac{\pi}{2}$, and is entire in the right-hand r plane so that all rays are equivalent. Once the rotation is made we can easily define the analytic continuation of the integral for $0 < |\Theta| \leq \frac{\pi}{2}$. The real s axis, $\Theta = 0$, is excluded because the second factor in the integrand is poorly behaved for $|\Psi| > \frac{\pi}{2}$ and for $|\Psi| = \frac{\pi}{2}$ the whole integral is poorly defined. This is to be expected because the positive real axis contains all our narrow resonances. The results obtained this way agree with those of Chapter II.

As we learned in the previous chapter, the behavior of the s channel projection operator for $|s| \to \infty$ should depend on the explicit form of the neutralizer and that is the case we shall study here. Recall that this term should decrease faster than any power of s. We define again

$$A_\sigma(s,t) \equiv \int_0^1 dz\, z^{-\ell(s)-1} (1-z)^{-\ell(t)-1} \int_z^1 \frac{dx\, x^{s-1}}{\Gamma(\sigma/2)} \left(\ell n \frac{1}{x}\right)^{(\sigma/2)-1} \Phi(1-y) \tag{V-7}$$

where we use the following general form

$$\Phi(1-y) = \int_y^1 dv\ f(v) = \int_0^{\ell n \frac{1}{y}} dr\ e^{-r}\ e^{g(r)} \quad . \qquad \text{(V-8)}$$

To simplify matters we do all but the r integral in the usual way to get

$$A_\sigma\ (s,t) = \sum_n \frac{\Gamma(n+\ell(t)+1)}{\Gamma(n+1)\ \Gamma(\ell(t)+1)}\ \frac{1}{n-\ell(s)}\ (n-\ell_o)^{-\sigma/2}$$

$$\times \int_0^\infty dr\ e^{-r(n-\ell(s)+1)}\ e^{g(r)} \quad . \qquad \text{(V-9)}$$

For large $|s|$ we recall from the previous chapter that with $\Phi$ equal one the sum without the integral has only power behavior in s for $\pi > |\arg s| > \frac{\pi}{2}$ and the continuation to $\epsilon < |\arg s| < \frac{\pi}{2}$ goes through as above. Thus we can study the properties of just

$$F(s) = \int_0^\infty dr\ e^{rs}\ e^{g(r)} \qquad (F(-1) = \Phi(1) = 1) \quad . \qquad \text{(V-10)}$$

If it decreases faster than any power of s as $|s| \to \infty$, $|\arg s| > \epsilon$ then the whole sum decreases faster than any power, i.e., does not contribute to Regge behavior (the term n in the exponent only helps the convergence). This is consistent with our constraint that the part of the amplitude represented by Eq. V-9, which is multiplied by $\not{p}_s$ in the s channel projection operator, should not contribute to t channel resonances. Recall that t channel resonances come from a "controlled" divergence of the sum of s channel resonances. If the factor $F(\ell(s) -n -1)$ in Eq. V-9 decreases faster than any power of n as $n \to \infty$, as the above statement about F(s) implies, then the sum will not diverge for any finite $\ell(t)$. We have, so to speak, pushed all the t channel poles out to infinity.

For convenience we shall consider breaking $g(r)$ into two terms, $g(r) = g_1(r) + g_2(r)$, such that for $r \to 0$, $g_1(r)$ is well behaved (where well behaved means has a well defined, noninfinite limit) and $g_2(r)$ goes to $-\infty$, whereas for $|r| \to \infty$, $g_1(r)$ goes to $-\infty$ and $g_2(r)$ is well behaved. In the variable v we have $f(v) = f_1(v)\, f_2(v)$ and, in Eq. V-1, $f_1(v) = f_2(1-v)$. We see that such properties are exactly what are required for $\Phi$ to satisfy Eq. III-10. Furthermore, $g_1$ and $g_2$ serve different purposes and in this formulation we can study them separately. Since $g_1(r)$ $(f_1(v))$ governs the behavior for $|r| \to \infty$ $(v \to 0)$, it tells us about the s channel, i.e., nonparity doubled daughters. Thus it is related to the asymptotic behavior in s for s positive real. Likewise, $g_2(r)$ $(f_2(v))$ governs the $r \to 0$ $(v \to 1)$ region and determines whether there will be any t channel resonances. So it gives the asymptotic s behavior, $|\arg s| > \epsilon$. Also, the restrictions on $f_1$ and $f_2$ (and thus $g_1$ and $g_2$) are in general different. Since a path in the r plane along a ray of finite angle $< \frac{\pi}{2}$ maps into a path in the v plane which always approaches 1 from below but spirals in to the origin, $f_2$ and its derivatives need only vanish for v approaching 1 from below, whereas $f_1$ must vanish uniformly for $v \to 0$.

As with the Veneziano amplitude the integral in Eq. V-10 has the property that contributions from finite r will vanish exponentially as $|s| \to \infty$ for $|\arg s| > \frac{\pi}{2}$. If $g_1(f_1)$ satisfies the constraint mentioned above (in fact, we only need that $\mathrm{Re}\ g_1(r) < -\mathrm{Re}\ rs$ as $|r| \to \infty$, $|\arg r| \leq \frac{\pi}{2}$) then we can continue F(s) over the entire s plane excluding some region about the positive real axis as before and all the interesting contribution comes from r near 0. Then we have $(s = \eta e^{i\Theta}, r = \rho e^{i\Psi})$

$$F(s) \underset{n \to \infty,\ \Theta \neq 0}{\sim} \int_0^\infty dr\ e^{\eta(re^{i\Theta} + \frac{g_2(r)}{\eta})} \quad . \tag{V-11}$$

Applying the method of steepest descent as before yields

$$F(s) \underset{n \to \infty,\ \Theta \neq 0}{\sim} \frac{1}{\sqrt{g_2''(r_o)}}\ e^{r_o s + g_2(r_o)} \tag{V-12}$$

where $r_o$ is the solution of $g_2'(r_o) = -s$. For example, if $g_2(r) = \ell n(1-e^{-n})\ \ell n\ \ell n(\frac{1}{1-e^{-r}})$ as suggested by Eq. V-1 so that for small r we have $g_2(r) \sim \ell n\ r\ \ell n\ \ell n\ \frac{1}{r}$, then $\rho_o \sim \frac{1}{\eta \ell n\ \ell n \eta}$ so that $(g_2''(r_o) \sim -\eta^2\ \ell n\ \ell n\ \eta)$

$$|F(s)| \underset{\eta \to \infty,\ \Theta \neq 0}{\sim} (\frac{1}{\eta})^{\ell n\ \ell n\ \eta + 1} \tag{V-13}$$

where we have assumed $\cos(\Theta + \Psi) \sim -1$. This falls faster than any power of $\eta$ and since it appears in a sum (Eq. V-9) which already decreases at a rate comparable to Regge terms, i.e., $\eta^{\ell(t)}$, $\eta$ need not be too large before this term is negligible. Another example is $g_2(r) = -\frac{1}{r}$ in which case $\rho_o \sim \frac{1}{\sqrt{\eta}}$ and we have

$$|F(s)| \sim \eta^{-3/4} e^{-2|\sin\theta/2|\sqrt{\eta}}$$
$$\eta \to \infty, \ \theta \neq 0 \qquad \text{(V-14)}$$

which is convergent for any $|\theta| > 0$. Actually, this is a limiting case for if we take $g_2 = -\frac{1}{r^n}$ for large n we find

$$|F(s)| \sim (n)^{\frac{n+2}{2(n+1)}-1} \eta^{-\frac{n+2}{2(n+1)}} e^{(\cos(\frac{\pi+n\theta}{n+1}) n^{\frac{1}{n+1}} \eta^{\frac{n}{n+1}})}$$
$$n \to \infty, \ \theta \neq 0 \qquad \text{(V-15)}$$

Thus for n very large we have $F \sim e^{-(\ )\eta}$ but only for $|\theta| > \frac{\pi}{n}(\frac{n+1}{2}-1) \sim \frac{\pi}{2}$, i.e., we cannot have an exponential decrease in the entire s plane. What is happening is that $g_2(r)$ no longer goes to $-\infty$ as $r \to 0$ except for a very narrow range in $\Psi$ so that we cannot rotate the r contour freely. An explicit example is the step function mentioned earlier. For $\Phi(1-y) = \theta(\frac{1}{\gamma}-y)$ we have $f(v) = \delta(\frac{1}{\gamma}-v)$ and $F(s) = \gamma^s$ which is exponentially decaying for $\pi > |\theta| > \frac{\pi}{2}$ but divergent for $|\theta| < \frac{\pi}{2}$. In summary we have seen that we can certainly define a $g_2(r)$ so that F(s) decreases faster than any power as $|s| \to \infty$, $|\arg s| > \epsilon$.

The next point to consider is the role of $g_1(r)$. We see that if $g_1(r) \to -\infty$ as $|r| \to \infty$ fast enough the r integral is will defined even for $\theta = 0$, $\mathrm{Re}(sr) \geq 0$. In this case, the dominant contribution is for large r and we can apply the method of steepest descent to

$$F(s) \underset{\eta \to \infty, \theta = 0}{\sim} \int dr\, e^{rs + g_1(r)} \sim \frac{1}{\sqrt{g_1''(r_o)}}\, e^{r_o s + g_1(r_o)} \qquad \text{(V-16)}$$

where $g_1'(r_o) = -s$. Again taking Eq. 1 as an example $g_1(r) = -r\,\ell n\, r$, $r_o = e^{s-1}$ and

$$F(s) \underset{\eta \to \infty,\ \theta = 0}{\sim} e^{\frac{s-1}{2}}\, e^{e^{s}} \qquad \text{(V-17)}$$

This is true as long as $Re(e^s) > 0$, i.e., $\cos(\eta \sin\theta) > 0$ or $\sin\theta < \frac{\pi}{2\eta}$, so that the angular region where this is true becomes vanishing small as $\eta \to \infty$. Outside of this region the large r contribution is again decreasing exponentially and the dominant contribution comes from small r as discussed earlier. Note that in this example $Re\, g_1(r) \to -\infty$ as $\rho \to \infty$ for all $|\Psi| < \frac{\pi}{2}$ so that it is consistent that the large r contribution goes away exponentially in $\eta$ everywhere except right on the positive real s axis. If we take $g_1 = -r^2$ then $|F(s)| \sim e^{\eta^2 \cos 2\theta}$ as long as $|\theta| \leq \frac{\pi}{4}$, i.e., F is very large in a finite wedge about the positive real s axis. This is simply because $g_1(r) \to +\infty$ if $|\Psi| > \frac{\pi}{4}$ so that we cannot rotate in the r plane throught more than $\pm 45^{o}$. This corresponds to the earlier statement that $f_1(v)$ must vanish uniformly as $v \to 0$ (which it doesn't here) in order to be able to rotate the contour in the r plane freely. Finally, if we take $g_1(r) = 0$

$(f(v) = 1)$ $F(s)$ has poles along the positive real axis in the s plane.

Summarizing again, we have that $g_2(r)$ determines the asymptotic behavior of $F(s)$ except in a region about the positive real s axis. The size of this region and the behavior of $F(s)$ within it is determined by $g_1(r)$. This whole question can be placed in a more mathematical framework by consulting a theorem of Phragmén and Lindelöf.[49] For our purposes this theorem states that if the function $F(s)$ $(s = \eta\, e^{i\theta})$ is regular in the region between and on the two rays $\theta = \pm\frac{\pi}{2\alpha}$, is bounded by M on the two rays, and is of order $e^{\eta^\beta}$, $\beta < \alpha$, as $\eta \to \infty$ uniformly in the region between the two rays, then $F(s)$ is bounded by M everywhere in the region. Thus, if we pick $g_2(r)$ so that $M = 0$, and pick $g_1$ and $g_2$ so that we may take $\alpha$ arbitrarily large, then, for $\theta = 0, F(\eta)$ must be larger than $e^{\eta^\beta}$, $\beta$ arbitrarily large but fixed, as $\eta \to \infty$. This is just what we saw, either 1) $F(\eta) \sim e^{e^\eta} > e^{\eta^\beta}$; 2) $F(s)$ was not regular for s real (had poles); or 3) we were not allowed to make $\alpha$ arbitrarily large.

It is also worthwhile to point out how the neutralizer affects the large t, fixed s behavior. Recall that when we found the Mellin transform for this case in the previous chapter (Eq. IV-4), we effectively ignored the neutralizer. Using Eqs. V-8 and V-10 we can redo that calculation to find

$$\tilde{A}_\sigma\ (\alpha(s),\lambda) \sim \frac{-\pi}{\Gamma(1+\lambda)}\ \frac{(\lambda - \ell_0)}{\lambda - \alpha(s)}^{-\sigma/2}\ F(\ell(s) - \lambda - 1) \qquad \text{(V-18)}$$

We see that the behavior of this amplitude in the $\lambda$ plane is affected by the neutralizer and so the asymptotic behavior of the total amplitude is also affected as we shall see explicitly in the next chapter.

Now let us consider what all this means in terms of the usual duality-FESR picture. We have shown that, for an amplitude involving s and t channel resonances, we have Regge asymptotic behavior, with cuts as well as poles, for $|s| \to \infty$, fixed t provided $|\arg s| > \epsilon$ and $|s| \to \infty$, fixed u provided $\pi - \epsilon > |\arg s| > \epsilon$ similarly to the usual Veneziano amplitude. Consider the contour integral shown in Fig. V-1 from which FESR's are derived

$$\int_{\mathcal{C}} ds' \;\; A(s',t) \;=\; 0 \qquad \text{(V-19)}$$

where $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$. In the limit $R \to \infty$, where R is the radius of the ring $\mathcal{C}_1$, $\int_{\mathcal{C}_1} ds' \, A(s',t)$ has an asymptotic expansion in terms of t channel poles and cuts in the usual way. The integral over $\mathcal{C}_2$, i.e., the left-hand side of the usual FESR (see Eq. I-3), is equal to the integral over $\mathcal{C}_3$ plus the contribution of the poles contained inside the contour $\mathcal{C}_2 - \mathcal{C}_3$. In the ordinary Veneziano model the contribution of $\mathcal{C}_3$ can be made negligibly small by making $\epsilon$ small and choosing the path of $\mathcal{C}_3$ appropriately. This is possible because the beta function is well behaved in between the poles on the real axis. In fact, it vanishes at one point on the real s axis between each pair of poles because of the gamma function in the denominator. Under these conditions the imaginary part of the amplitude, i.e., the left-hand side of the FESR, is saturated by the resonances. However,

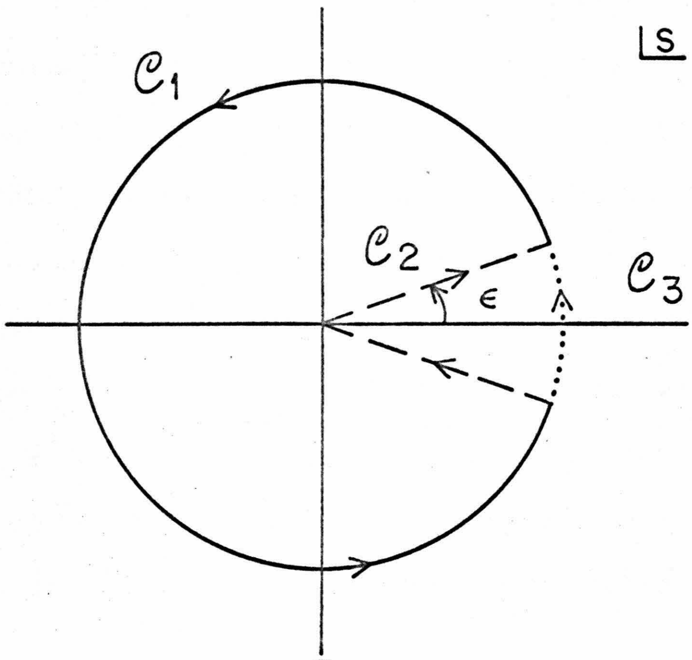



Fig. V-1 Integration contour for A(s,t) in Eq.V-19.

in the present model, including spin, the contribution of $\mathcal{C}_3$ cannot be neglected. Hence the integral of $A(s,t)$ over $\mathcal{C}_2$ is no longer simply the sum of the poles inside the wedge but contains also an additional "background" contribution, i.e., the integral over $\mathcal{C}_3$ .

Since the presence of this background term is an essential feature of this model and is important for an understanding of what "duality" means when spin is included, at least as in this model, let us review why the background term is there. We have seen that, in order to avoid parity doubling and new ghosts at the resonances when we include spin, we have introduced projection operators. Additionally we have required that all vertices exhibit $SU_{6_W} \times O_{2_{Lz}}$ symmetric couplings and that factorization proceed as in the usual Veneziano amplitude. This means that, for example, terms in the t channel projection operator which involve $\not{p}_t$ (see Eq. III-5) cannot contribute to s channel resonances. So these terms, which yield the fixed cut, fixed pole and singular $(\frac{1}{\sqrt{t}}$ or $\frac{1}{t})$ part of the t channel Regge pole, appear as a background contribution in the s channel and account for part of the integral over $\mathcal{C}_3$ . Recall this is the part of the amplitude which has had all its s channel poles "pushed" out to infinity. We have also seen that those parts of the s channel projection operator which involve $\not{p}_s$ cannot contribute to t channel resonances (for the same reasons as above) and hence cannot contribute to the Regge asymptotic behavior. Recall that outside of $C_2$ these $\not{p}_s$ terms, which contain an s channel neutralizer, decrease faster than any power of s as $|s| \to \infty$ . Inside of the wedge they are growing very rapidly. Thus there is a contribution to the $\mathcal{C}_3$ integral which, except for a remainder which decreases faster than any power of R as $R \to \infty$, just cancels the contribution to the pole residues that arises from the $\not{p}_s$ terms.

To interpret all of this we first notice that the "bare" resonances in the various channels, i.e., those parts of the pole residues which were present in the ordinary Veneziano without spin, are still dual to each other in the usual sense. To avoid parity doubling when spin is added fixed cuts, fixed poles and Regge poles with singular residues appear in the complex angular momentum plane. From the arguments above we identify these terms as being dual to background. As mentioned in the Introduction, it is not unexpected that such new features should accompany the inclusion of spin. In order to give some interpretation to the neutralizer we shall have to become somewhat speculative. Recall that we chose to interpret the wedge, $|\arg s| > \epsilon$ ($\mathcal{C}_2$ in Fig. V-1) forced on us in the usual Veneziano model, as being essentially the elastic unitarity cut such that the imaginary part of the amplitude is given by the discontinuity across the wedge which is saturated by the resonances.

$$\int ds' \;\; \mathrm{Im}\, A_V = \int_{\mathcal{C}_2} ds' \; A(s',t) \;\; = \;\; \Sigma \text{ poles} \qquad . \qquad \text{(V-20)}$$

Now we interpret the new wedge requirements arising from the s channel neutralizer as also representing a cut. In this case the head of the cut, i.e., the point where the argument of $F(\lambda(-s) - \lambda-1)$ passes through -1 (see Eq. V-18), depends on both $\lambda(\ell_s)$ and s. Thus it is a moving Regge cut, i.e., cut in both $\lambda$ and s. If we write a part of the amplitude which contains a neutralizer in terms of a sum over s channel resonances as in Eqs. V-9 and V-10, we get

$$A(s,t) = \sum_n \frac{\Gamma(n+\ell(t)+1)}{\Gamma(n+1)\,\Gamma(\ell(t)+1)} \cdot \frac{(n-\ell_0)^{-\sigma/2}}{n-\ell(s)} \; F(\ell(s)-n-1) \; . \qquad \text{(V-21)}$$

Then the above interpretation of the function F leads "naturally" to the idea that the imaginary part of such an amplitude comes not only from the usual "elastic unitarity" cut (wedge) required by the poles on the real axis $(1/(n-\ell(s))$, but also includes contributions from a series of cuts $(F(\ell(s)-n-1))$ which start near each of the poles. This is suggestive of inelastic effects, i.e., the cuts "represent" the opening of the inelastic channels. Recall that the contribution of the neutralizer terms to the $\mathcal{C}_3$ integral was such as to subtract from the resonance contribution to the imaginary part of the amplitude, again reminiscent of inelastic effects. So now we must write

$$\int ds' \; \mathrm{Im}\, A \; = \int_{\mathcal{C}_2} A \, ds' \; = \; \Sigma \text{ poles} + \int_{\mathcal{C}_3} A \, ds' \qquad \text{(V-22)}$$

where the first term on the right is "related" to elastic effects (i.e., gives the elastic width) and the second term is "related" to inelastic effects (i.e., gives the difference between the total width of a resonance and its elastic width).

This picture of the neutralizer as giving the inelastic unitarity cuts is admittedly speculative. However, it is important to recognize that we have not given a dynamical reason for why any of the new structure, i.e., fixed cut, fixed pole, or moving cut, should be present. We have only postulated that it must be present in order to avoid parity doubling and ghosts. So maybe it is not surprising that the structure that appears is suggestive of unitarity corrections which is the usual way that Regge cuts appear in models of hadrons. It is also reasonable that such cuts should require that we change our basic concept of dual amplitudes.[50] We can no longer require that the imaginary part in the FESR should be given purely by resonances. However, for a specific reaction where Regge cuts make only a small contribution and in an energy range where inelasticity effects are small, we can still expect that the simple duality of Dolen, Horn and Schmid will be approximately true. As we shall see in the next chapter the A' amplitude in $\pi N$ charge exchange studied by Dolen, Horn and Schmid, has in fact no Regge cut contribution at $t = 0$. Further, since the resonances contribute with varying sign, neutralizer effects can be expected to cancel, especially at the low energies involved, and the present model may still be consistent with the data. In any case, the question of separating resonance effects from neutralizer effects in anything other than our narrow resonance model (e.g. the data) would seem to be very difficult to answer.

To pursue these ideas further we will need a model exhibiting some degree of rigorous unitarity. Since such a model does not at present exist we shall stop here while suggesting that this is an interesting topic for future study. In the next chapter we shall calculate $\pi N$ charge exchange with the model in its present form.

# VI. MESON-BARYON SCATTERING

Now that we have a dual model of strong interactions which includes spin it is both feasible and useful to attempt comparison with the large amounts of data available for meson-baryon scattering. It is also important to decide beforehand which features of the data we can reasonably expect the model to exhibit. By construction the model should exhibit the correct resonance spectrum to the extent that nature is like a symmetric quark model. Detailed questions of breaking this degeneracy must be left for future studies. Similar considerations apply to the study of the coupling at resonances which are $SU_{6_W}$ symmetric by construction. We shall limit ourselves to a consideration of the general structure of angular distributions in the Regge limit. In particular, we want to study $\frac{d\sigma}{dt}$ for $\pi N$ charge exchange. There is a good deal of structure in this cross section and the related amplitudes, and as a result it has been one of the most critical tests of various Regge models.[24] We might expect that our dual quark model will give reasonable results for this reaction since the exchange degeneracy of the vector and tensor mesons, which our model exhibits by construction, is well established as discussed in Chapter I. Also the $\pi$, which is the $SU_6$ partner of the $\rho$ and is clearly not degenerate in mass, cannot contribute to the t channel resonances because it cannot couple to two other $\pi$'s. We shall also look briefly at the backward direction, i.e., baryon exchange, in order to determine the general form. However, numerical comparisons are not appropriate here since the predicted exchange degeneracies are observed to be badly broken. Comparisons with data for baryon exchange will have to await a more complete model.

Before discussing the most characteristic features of the forward scattering data let us review the structure of the $\pi N$ charge exchange scattering amplitude.[51] We define the usual kinematic variables

$$s = (p+q)^2 = (q'+p')^2, \quad p_s = p+q = p'+q' ,$$

$$t = (p-p')^2 = (q'-q)^2, \quad p_t = p-p' = q'-q ,$$

and $$u = (p-q')^2 = (q-p')^2 , \quad p_u = p-q' = p'-q ,$$

where p and q are the momenta of the incoming nucleon and pion respectively, and p' and q' refer to the outgoing states (see Fig. VI-1 where the appropriate quark graphs are shown). The actual scattering amplitude has the form

$$T(s,t) = \bar{u}(p') \left\{ A^{(-)}(s,t) + \frac{(q+q')\cdot\gamma}{2} B^{(-)}(s,t) \right\} u(p) . \qquad \text{(VI-1)}$$

Here the superscript (-) specifies the crossing symmetry $s \leftrightarrow u$ which is odd for the case of charge exchange. Also, the amplitude corresponds to pure isospin one in the t channel. The A' amplitude, defined by

$$A'(s,t) = A(s,t) + \frac{m(s-u)}{4m^2 - t} B \qquad \text{(VI-2)}$$

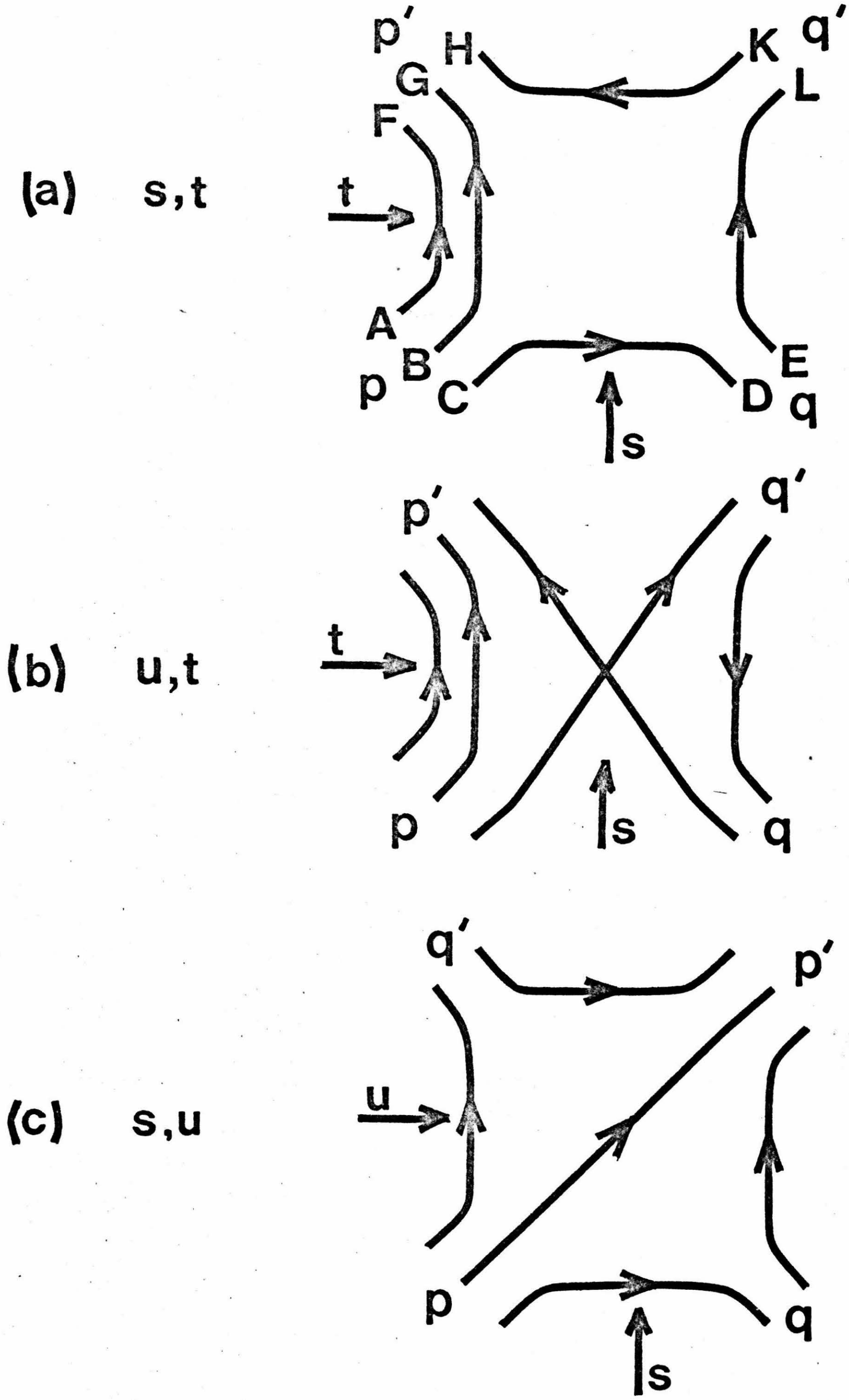



Fig. VI-1 Quark diagrams for meson-baryon scattering. A,B,C, etc., are indices as they appear in Eq.VI-8.

where m is the nucleon mass ($\mu = \pi$ mass), is the helicity nonflip amplitude in the t channel center of mass. Similarly, the t channel helicity flip amplitude is given by B(s,t) except for kinematic factors. The differential cross section is given by

$$\frac{d\sigma}{dt} = \frac{1}{\pi s}\left(\frac{m}{4k}\right)^2 \left[\left(1-\frac{t}{4m^2}\right)|A'|^2 + \frac{t}{4m^2}\left\{s - \frac{(m+w)^2}{1-\frac{t}{4m^2}}\right\}|B|^2\right] \qquad \text{(VI-3)}$$

where $w = \frac{(s - m^2 - \mu^2)}{2m}$ is the laboratory energy of the pion and $k^2 = \frac{(s - \mu^2 - m^2)^2 - 4m^2\mu^2}{4s}$. The polarization of the recoil nucleon is given by

$$P(t)\frac{d\sigma}{dt} = -\frac{\sin\Theta}{16\pi\sqrt{s}} \; \text{Im}\,(A'B^*) \qquad \text{(VI-4)}$$

where $\Theta$ is the scattering angle in the s channel center of mass.

The simplest features of the data which we might hope to explain in terms of the present model are: 1) a slight dip in the forward direction ($t = 0$) indicating that the $B^{(-)}$ (helicity flip) amplitude is dominant; 2) a dip in $\frac{d\sigma}{dt}$ at $t \approx -0.6$ which correlates with the zero in $\text{Im}B^{(-)}$ at $t = -0.5$ found in the FESR's discussed in Chapter I, and with the zero in the signature factor of the $\rho$ trajectory which is thought to dominate in this reaction; 3) a zero in $\text{Im}A'^{(-)}$ at $t \approx -0.15$ also found in FESR's and correlated with the zero in $\frac{d\sigma}{dt}(\pi^- p \text{ elastic}) - \frac{d\sigma}{dt}(\pi^+ p \text{ elastic})$ (cross over zero) at about the same t; 4) a polarization of approximately 15% or larger for t in the range -0.2 to -0.4. It is interesting to note that the last two features do not readily appear in simple Regge or Veneziano models.

To begin the calculation for the present model, we define the $SU_6$ wave function of the nucleon as[52]

$$\Psi(p)\begin{Bmatrix} ABC \\ abc \end{Bmatrix} = \frac{1}{\sqrt{6}}\left\{[(1+\frac{\not p}{m})\gamma_5\, C]_{ab}\ \epsilon_{ABD}\, N_{c,C}^{\ D} + \text{cyclic permutations}\right\} \tag{VI-5}$$

where m is the 56 multiplet mass, a b c are Dirac indices, A B C are $SU_3$ indices, and $N_{c,C}^{\ D}$ is a Dirac spinor times the standard $SU_3$ matrix describing the nucleons:

$$N_{c,C}^{\ D} = u_c(p)\, B_C^D = u_c(p)\begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & \Sigma^+ & p \\ \Sigma^- & \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & n \\ -\Xi^- & \Xi^0 & \frac{2}{\sqrt{6}}\Lambda^0 \end{pmatrix}_C^D \tag{VI-6}$$

C is a 4x4 matrix with the properties $C^T = -C = C^+ = C^{-1}$, $C^{-1}\gamma_u C = -(\gamma_u)^T$, $C^{-1}\gamma_5 C = \gamma_5^{\ T}$ and $\overline{C} = C$. $\epsilon_{ABC}$ is antisymmetric in all its indices.

To study t channel Regge exchanges we need to calculate

Fig. VI-1a, i.e., the s,t diagram, in the large s, fixed t limit. Recall this means that the projection operator for the s channel resonances becomes just the unit operator, i.e., the terms like $\frac{1}{m_s}$, $\frac{1}{m_s^2}$ are decreasing faster than any power of s. Also, the u,t diagram will give a similar contribution but with s replaced by -s(u = - s + const.). In fact, this is just what gives us signature. Finally, as described above the s, u diagram does not contribute to the large s and fixed t Regge behavior. Note that what we are really calculating is just the contribution of the s,t diagram to the t channel resonances and we find the Regge behavior in the s channel by the usual Sommerfeld-Watson techniques. Thus the simplest way to think about the calculation is that the scattering amplitude T(s,t) is given by

$$\underset{\substack{s \to \infty \\ t \text{ fixed}}}{T(s,t)} \sim g^2 \sum_n \frac{(\ell' s)^n}{\Gamma(n+1)(n-\ell(t))} \Big\{ G(n,s,t) \text{ (the s,t contribution)} + e^{-i\pi n} G(n,-s,t) \text{ (the u,t contribution)} \Big\} \qquad \text{(VI-7)}$$

where $\ell(m_\rho^2) = 0$ so that $\ell(t) = \alpha_\rho(t) - 1$. Now we calculate

$G(n,s,t)$ from $\left(m_n = \sqrt{\frac{n - \ell_0}{\ell'}}\right)$

$$G(n,s,t) = \bar{\Psi}\begin{Bmatrix}FGH\\ fgh\end{Bmatrix} \delta^K_H \left(1 - \frac{\not{p}_t F}{m_n}\right)^k_h \left\{\left[\gamma_5\left(1 + \frac{\not{q}'}{\mu}\right)\right]^\ell_k \bar{P}^L_K\right\}$$

$$(1)^e_\ell \ \delta^E_L \left\{\left[\left(1 + \frac{\not{q}}{\mu}\right)\gamma_5\right]^d_e P^D_E\right\} (1)^a_f \ \delta^A_F \ (1)^b_g \ \delta^B_G$$

$$\left(1 + \frac{\not{p}_t F}{m_n}\right)^c_d \delta^C_D \ \Psi\begin{Bmatrix}ABC\\ abc\end{Bmatrix} \tag{VI-8}$$

where we have used the pseudoscalar wave functions from Chapter III. Recall P is the $SU_3$ wave function for the pseudoscalar mesons.

$$P^A_B = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta^0 \end{pmatrix}^A_B \tag{VI-9}$$

The evaluation of Eq. VI-8 is straightforward but extremely tedious. Details can be found in the appendix and we shall only present the results here. In the following equation all extraneous overall constants are set equal to unity since they get absorbed into the unknown overall coupling constant $g^2$ anyway. Also, the traces over $SU_3$ indices which occur are explicitly exhibited using the notation

$< \; > \; = \; \mathrm{Tr}$ .

$$G(n,s,t) = \bar{u} \Bigg[ s \Bigg\{ < \overline{B}\overline{P}PB > \Bigg[ \left(1 + \frac{tF}{m_n^2}\right) (40\, m^2 + 16\, m\mu - 6\, t)$$

$$+ \frac{F}{m_n} (160\, m^2 \mu + 16\, mt - 24\, \mu t) \Bigg] + ( < \overline{B}B\overline{P}P > - < \overline{B}B > < \overline{P}P > ) .$$

$$\times \Bigg[ \left(1 + \frac{tF}{m_n^2}\right) (8\, m^2 + 32\, m\mu + 6\, t) + \frac{F}{m_n} (32\, m^2 \mu + 32\, mt + 24\, \mu t\,) \Bigg] \Bigg\}$$

$$- 4 \left(\frac{\not{\Delta} + \not{\Delta}'}{2}\right) (4\, m^2 - t) \Bigg[ (m + \mu)\left(1 + \frac{tF}{m_n^2}\right) + \frac{F}{m_n} (4\, m\mu + t) \Bigg] (5 < \overline{B}\overline{P}PB >$$

$$+ < \overline{B}B\overline{P}P > - < \overline{B}B > < \overline{P}P > ) \Bigg] u \qquad . \qquad \text{(VI-10)}$$

To evaluate the u, t contribution we need only change s to -s and P to $\overline{P}$ ($\overline{P}$ to P) in the above expression. F is the remnant of the neutralizer function discussed in Chapter V, i.e,

$$F = F(\ell(t) - n - 1) = \int_0^\infty dr\, e^{-r(n+1 - \ell(t))}\, e^{g(r)} = \int_0^1 dv\, v^{n-\ell(t)}\, f(v)$$

(VI-11)

where $F(-1) = 1$, and $F(z)$ decreases faster than any power of $|z|$ as $|z| \to \infty$, $|\arg z| > \epsilon$ . Note that in our simple treatment of the neutralizer we have essentially taken $F^2 = F$, i.e., the neutralizer with the $\frac{1}{m_n}$ term is the same as that with the $\frac{1}{m_n^2}$ term.

From Eq. VI-10 we can pick off the f/d ratio for the coupling of the vector meson trajectory to nucleon-antinucleon. If the coupling has the form

$$a < \overline{B}\overline{P}PB > + b < \overline{B}B\overline{P}P > = \frac{a-b}{2} < \overline{B}[\overline{P}P,B] > + \frac{a+b}{2} < \overline{B}\left\{\overline{P}P,B\right\} > \tag{VI-12a}$$

$$\text{then} \quad f/d = \frac{a-b}{a+b} \tag{VI-12b}$$

Thus for the B amplitude (the coefficient of $\frac{\not{q} + \not{q}'}{2}$) we have $\left(f/d\right)_B = \frac{5-1}{6} = 2/3$ which is the usual $SU_{6_W}$ value. The value of $f/d$ for the A amplitude involves $m_n$ and $F$ and thus is strongly model dependent. Using the calculations which will be discussed below, we find $f/d|_{A_{t=0}} = 12/21$. Note that the usual $SU_{6_W}$ result is[53] $f/d|_A = 2/3 - \frac{\sqrt{t}}{2m}$, whereas the experimental observation that the f meson seems to decouple from the A amplitude suggests $f/d|_A = 1/3$ (the value such that the $SU_3$ coupling for the f meson is zero). The usual $SU_{6_W}$ result gives approximately this value at $t = m_\rho^2$ but, of course, not at other values of t. This problem of decoupling the f meson would be interesting to pursue in the present model, but it has not yet been studied.

The A' amplitude can be calculated by using Eq. VI-2 and we find

$$A' = \frac{6g^2}{\ell'} \sum_n \frac{(\ell' s)^{n+1}}{\Gamma(n+1)\,(n-\ell(t))} \left\{ [(1+\frac{tF}{m_n^2})\,(4\,m\mu + t) + \frac{4tF}{m_n}\,(m+\mu)] \right.$$

$$\left. \times\ [- < \overline{B}\overline{P}PB > + < \overline{B}B\overline{P}P > - < \overline{B}B > < \overline{P}P >] + (s \rightarrow -s,\ P \leftrightarrow \overline{P}) \right\} \tag{VI-13}$$

so that $f/d = \infty$ as expected from the $SU_{6_W}$ symmetry. In order to put the above amplitude in a form easy to understand and evaluate, we perform a Sommerfeld-Watson transform to get, for the case of charge exchange, (let $\ell' = 1$)

$$A'^{(-)} \simeq \frac{6 g_s^2}{\sqrt{2}\, 2\pi i} \int d\lambda \frac{[(-s)^\lambda + (s)^\lambda]\, \Gamma(-\lambda)}{\lambda - \ell(t)} \left[ \left(1 + \frac{tF(\ell(t) - \lambda - 1)}{\lambda - \ell_o}\right)(4\, m\mu + t) + \frac{4t}{\sqrt{\lambda - \ell_o}}\, F\,(\ell(t) - \lambda - 1)\,(m + \mu) \right] \quad . \qquad \text{(VI-14)}$$

The contour defining this integral includes the moving pole $\left(\frac{1}{\lambda - \ell(t)}\right)$ and the fixed singularities given by $\frac{1}{\sqrt{\lambda - \ell_o}}$ and $\frac{1}{\lambda - \ell_o}$ . For the B amplitude we get

$$B^{(-)} \simeq \frac{20\, g^2}{\sqrt{2}\, 2\pi i} (4m^2 - t) \int d\lambda \frac{[(-s)^\lambda + s^\lambda]\, \Gamma(-\lambda)}{\lambda - \ell(t)} \left[ (m + \mu)\left(1 + \frac{tF\,(\ell(t) - \lambda - 1)}{\lambda - \ell_o}\right) + \frac{F(\ell(t) - \lambda - 1)}{\sqrt{\lambda - \ell_o}} (4m\mu + t) \right] \quad .$$

(VI-15)

By studying these formulae, several points can be learned even before we attempt to numerically evaluate the formulae. First we see that just because of the numerical coefficients we can expect that the flip amplitude, B, will dominate for $t \neq 0$ in qualitative agreement with the data.

Next note that there are in general three contributions to each amplitude: 1) the Regge pole part coming from the $\frac{1}{\lambda - \ell(t)}$ factor whose s dependence is $s^{\ell(t)}$ (recall we set $s_o = \frac{1}{\ell'} = 1$ everywhere) and whose phase is given by $e^{-i\,\pi/2\,\ell(t)}$ ($\ell_o = -\frac{1}{2}$ for the $\rho$ trajectory); 2) the fixed cut contribution $(\frac{1}{\sqrt{\lambda - \ell_o}})$ which is dominated by the singularity at the head of the cut and has s dependence essentially $\frac{s^{\ell o}}{\sqrt{\ell n\ s}}$ and phase varying rather slowly but in the same direction as the Regge term; 3) the fixed pole part $(\frac{1}{\lambda - \ell_o})$ which goes like $s^{\ell_o}$ with phase given by $e^{-i \frac{\pi}{2} \ell_o}$ .

Since the fixed pole and fixed cut terms resulted from terms containing $\not{p}_t$ in the t channel projection operator, we can expect that they are multiplied by coefficients which vanish as $\not{p}_t \to 0$, i.e., $t \to 0$. For the B (flip) amplitude this is not a constraint since an explicit t factor appears in front of $|B|^2$ in $\frac{d\sigma}{dt}$ (see Eq. VI-3) just from kinematics. However, for the A' (nonflip) amplitude we know that a t factor must multiply both the fixed pole and fixed cut as is seen explicitly in Eq. VI-14. Thus, for $t < 0$ both of these contributions will be approximately 180° out of phase with the Regge pole term and we can expect cancellations, i.e., zeroes in both the real and imaginary parts of A' for fairly small t. We will see this explicitly shortly. For the B amplitude (Eq. VI-15), it turns out that the fixed pole is still multiplied by t whereas the fixed cut has no factor t. Then for some range of t we can expect approximate cancellation between the two fixed contributions. In summary, we can expect a strong fixed cut contribution for the non flip amplitude A' and possible cancellation with the Regge term at small t; whereas we can expect cancellation between the fixed terms in the flip amplitude B and thus possible domination by the Regge term. This is just the sort of behavior we need if we expect to fit the qualitative aspects of the data, i.e., the zero in Im A' at small t and the zero in B which occurs naturally in the Regge term due to the nonsense wrong signature zero at $\ell(t) = -1$ . This last fact is just a result of the $\cos(\frac{\pi}{2}\ell(t))$ factor which appears in the Regge residue from $(1 + e^{-i\pi\ell(t)}) = 2e^{-i\frac{\pi}{2}\ell(t)}\cos(\frac{\pi}{2}\ell(t))$. We also see in Eq. VI-14 explicit verification of the statement made at the end of the last chapter that the fixed singularities do not contribute to A' at $t = 0$.

Note that the structure of the two amplitudes is actually more systematic than discussed above since only the factors $(1 + \frac{tF}{\lambda - \ell_o})$ and $\frac{F}{\sqrt{\lambda - \ell_o}}$ appear. Also, the coefficients are effectively either $(4m\mu + t)$ or $(m + \mu)$ and the two just switch in going from A' to B.

The expressions in Eqs. VI-14 and VI-15 have actually been evaluated and used to calculate $\frac{d\sigma}{dt}$ and P(t). For consistency we have used approximate $SU_6$ values for the masses, i.e, $\mu^2 = 0.6$ GeV and $m^2 = 1.0$ GeV. We have also used $\ell_o = -0.5$ . The Regge pole and fixed pole terms can be done analytically while the fixed cut requires numerical evaluation. The actual numerical work was done by Geoffrey C. Fox. The neutralizer form $F(\ell(t) - \lambda - 1)$ was approximated as

$$F\ (\ell(t)\ -\lambda\ -1)\ =\ \gamma^{\ell(t)\ -\lambda} \qquad \text{(VI-16)}$$

which corresponds to $\Phi(1-y)$ being a step function with the step at $\frac{1}{\gamma}$ as discussed in Chapter V. We have already seen that this expression is not too good an approximation. In particular, it ignores our interpretation of F as a Regge cut. However, it does make the calculation simple which is all we desire for our present qualitative study. Note that this will not affect the Regge pole term which comes from $\lambda = \ell(t)$ but does introduce a factor $\gamma^t$ in the fixed terms which come mostly from $\lambda \approx \ell_o$. In Fig. VI-2 we see the result of calculating $\frac{d\sigma}{dt}$ with $\gamma = 1$, i.e., no neutralizer. Also shown are idealized data points which show the general structure of the actual experimental data (from ref. 24). It is evident that our amplitudes need some strongly t dependent factor, e.g. $\gamma^t$ for large $\gamma$, in order for $\frac{d\sigma}{dt}$ to have a t dependence similar to the data. Thus it is unfortunately the case that any attempts to fit the data must depend crucially on the form of the neutralizer which is, as yet, poorly defined.

In Figures VI-3 and VI-4 we see the results of calculations with $\gamma = 10$. There are two important points to notice. 1) The structure of the B amplitude is reasonable, i.e., zeroes in both the real and imaginary parts near $t = -0.6$ which result from the signature zero in the Regge pole term plus a cancellation of the fixed pole and fixed cut terms in the same t region. The t region

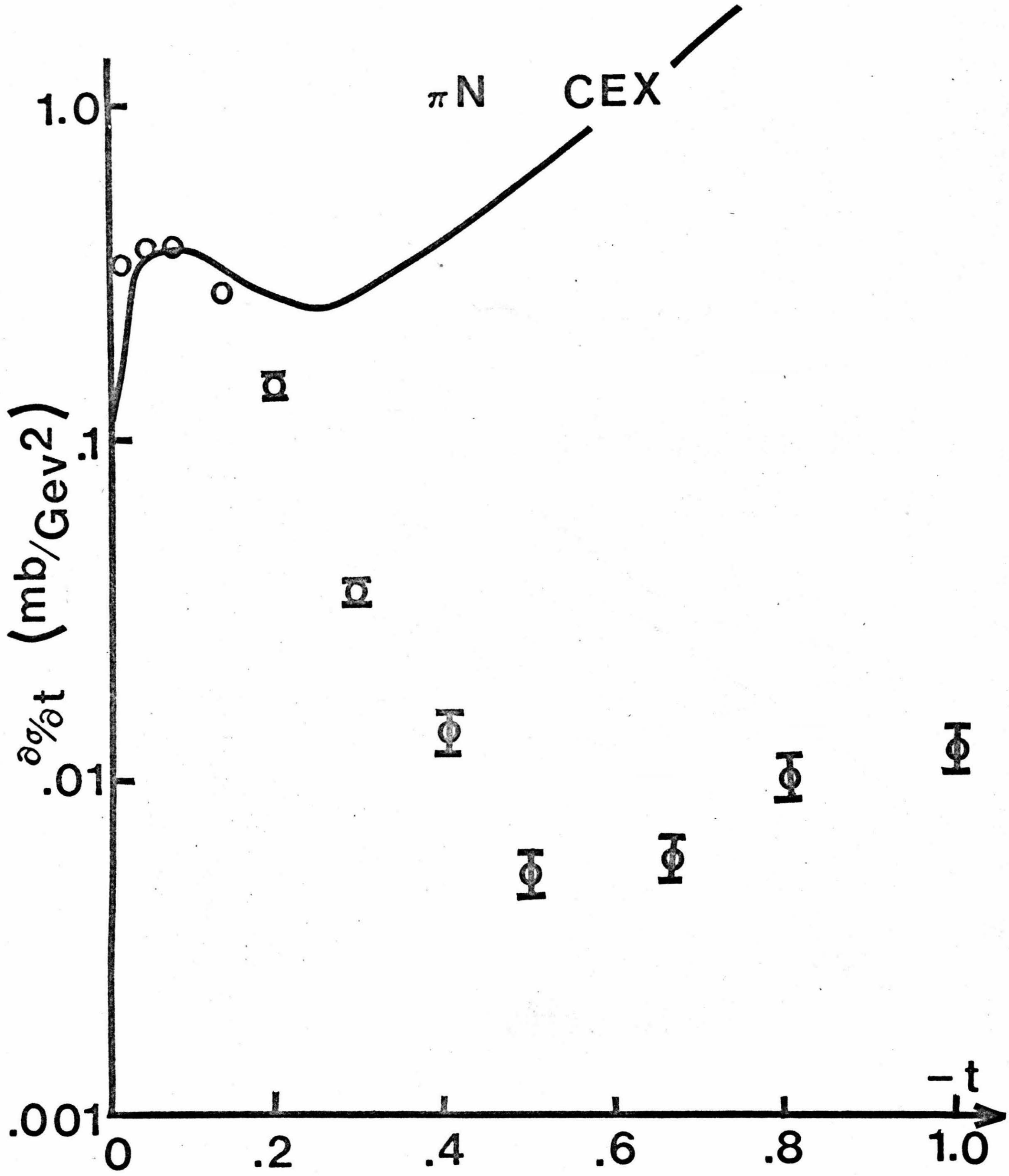



Fig. VI-2 Calculated differential cross section with no neutralizer, i.e., $\gamma=1$. $P_L$ is 5.9 Gev/c. Some data points are shown for comparison.

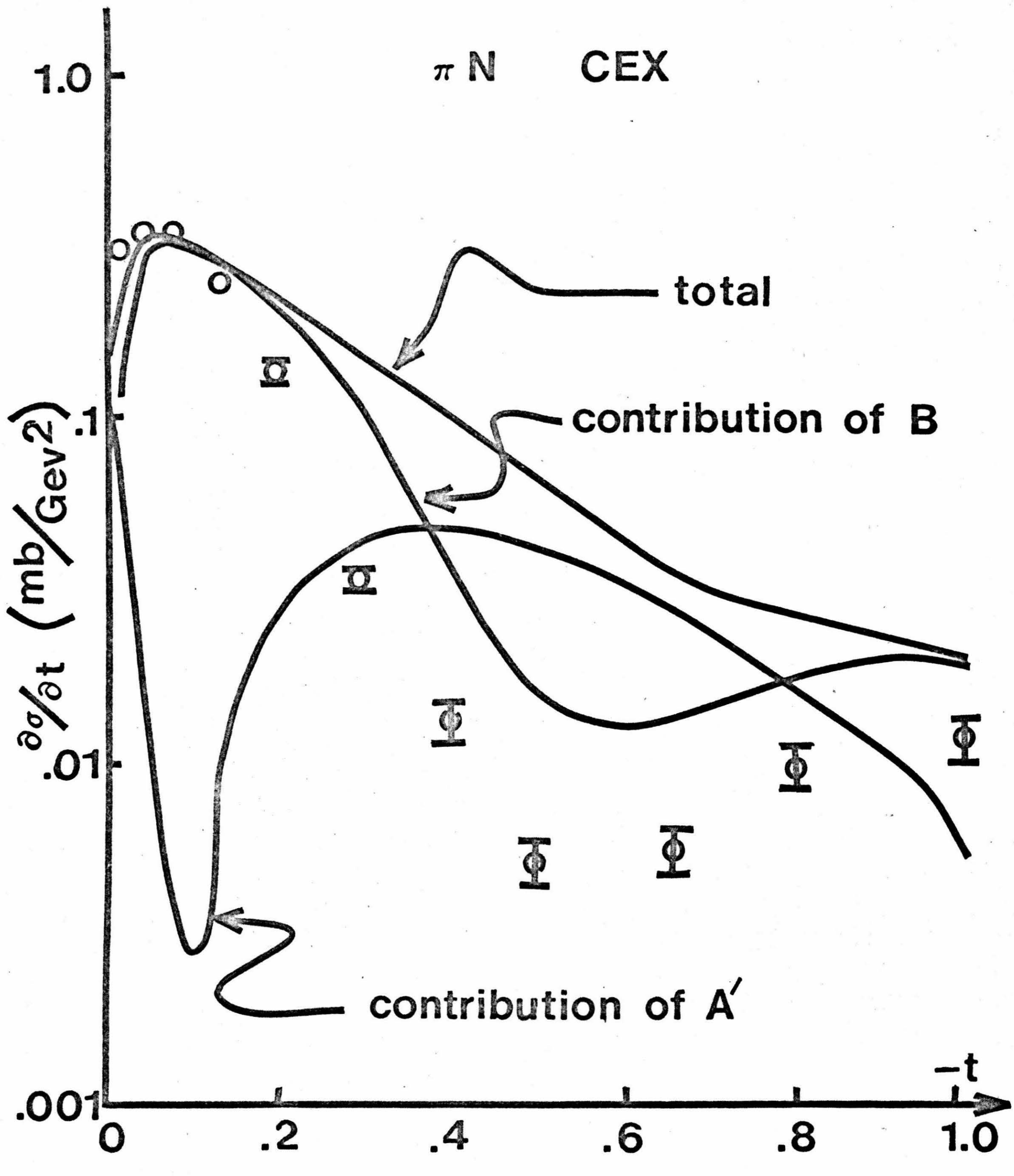



Fig. VI-3 Calculated differential cross section for $\gamma$=10, $P_L$=5.9 Gev/c, $\mu^2$=0.6 $Gev^2$, $m^2$=1.0 $Gev^2$, and $\ell_o$=-0.5. Some data are shown for comparison. Contributions of individual amplitudes are also shown.

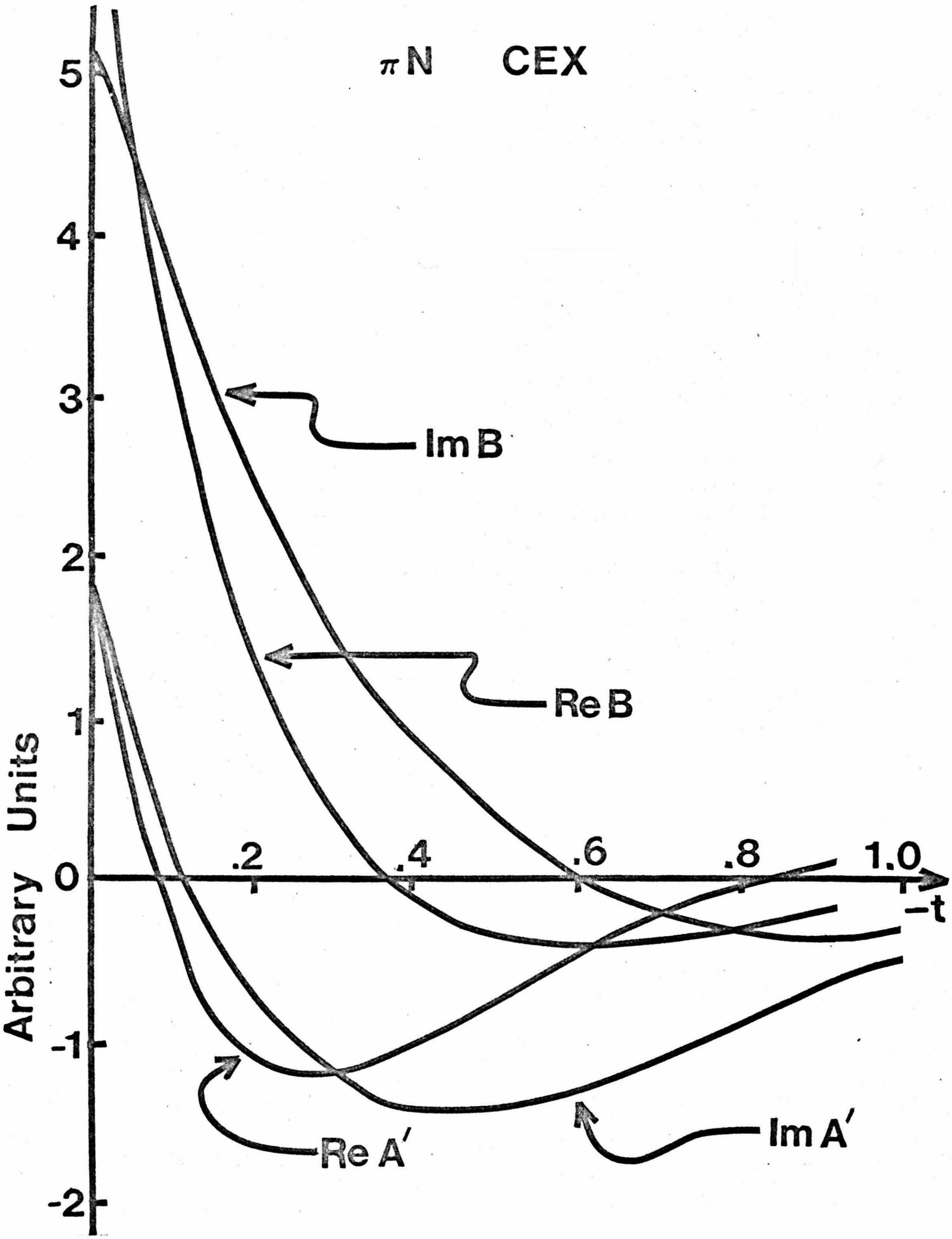



Fig. VI-4 Calculated values of amplitudes A' and B for $\gamma$=10, $P_L$=5.9 Gev/c, $\mu^2$=0.6 Gev$^2$, $m^2$=1.0 Gev$^2$, and $\ell_o$=-0.5.

of this cancellation depends on the values of the masses used, e.g. for a smaller value of $\mu$ the cancellation occurs at smaller t, but it depends only slightly on the value of $\gamma$. 2) The structure of the A' amplitude is also reasonable in that it has a zero in its imaginary part at $t \approx -0.1$ . However, $|A'|^2$ is much too large for $t < -0.3$ and in fact fills in the dip in $|B|^2$ at $t = -0.6$ to make $\frac{d\sigma}{dt}$ quite smooth. Also, the data indicate that the zero in Im A' should occur at $t \simeq -0.2$ , not $-0.1$ . Both of these problems could be remedied by stronger neutralizer effects, e.g. larger $\gamma$ , but at this point it is clear that the simplified fashion in which we have treated the neutralizer is not sufficient. If we just increase $\gamma$ the $\lambda$ integrals soon become poorly defined and, in any case, we saw that the step function form of the neutralizer was not consistent with Regge behavior. Without doing further calculations with a more complicated form of the neutralizer which would, for example, treat it as a Regge cut, we shall take the attitude that the model is basically consistent with the zero structure of the data.

However, there is one more piece of data which is much more restrictive. This is the fact that for t in the range -.15 to -.4 there is a fairly sizable POSITIVE polarization of 15 to 40%. The present model shows this fairly large polarization but with NEGATIVE sign. Recall that if $\Phi_{A'}$ is the phase of the A' amplitude and $\Phi_B$ is the phase of the B amplitude, then Eq. VI-4 tells us that the sign of the polarization is given by the sign of $\sin(\Phi_B - \Phi_{A'})$, which is positive if $\pi > \Phi_B - \Phi_{A'} > 0$ . From Fig. VI-4 we see that

$\Phi_{A'}$ is approximately $-\frac{3\pi}{4}$ for the t range of interest while $\Phi_B$ is close to $\frac{\pi}{2}$ so that $\Phi_B - \Phi_{A'} \sim \frac{5\pi}{4}$ and the polarization is negative. However, changes in the neutralizer may strongly affect the result since, if $\Phi_B$ were slightly smaller and $\Phi_{A'}$ slightly less negative, $\Phi_B - \Phi_{A'}$ could be less than $\pi$ rather than greater than $\pi$. This question can only be answered by a more detailed numerical study, but the polarization data are certainly an important question and at present the model is inconsistent with the data.

We can also calculate backward[(54)] scattering in the Regge region in the present model. In this case we are looking at the resonance structure in the u channel with u fixed and $|s| \to \infty$. The interesting amplitudes are those corresponding to $I = 3/2$ and $I = \frac{1}{2}$ in the u channel which are easily separated by considering $\pi^- p$ elastic scattering (pure $I_u = 3/2$) and $\pi^+ p$ elastic scattering (a mixture of $I_u = \frac{1}{2}$ and $I_u = 3/2$). The question of comparing the model to the data is clouded by the absence of much structure in $\frac{d\sigma}{du}$, and the fact that the predicted exchange degeneracies are poorly satisfied for nonstrange baryons, as mentioned above. Note that, since the present model has degenerate $\underset{\sim}{56}$'s and $\underset{\sim}{70}$'s at all L except $L = 0$ (pure $\underset{\sim}{56}$) and $L = 1$ (pure $\underset{\sim}{70}$), it cannot have any signature and thus cannot have wrong signature zeroes. Another pressing problem is that our model will consider the quark spin 3/2 particles as the leading trajectory, whereas in nature $SU_6$ is badly enough broken that the actual quark spin $\frac{1}{2}$ trajectories, which are allowed to contribute in general (unlike the $\pi$ for meson exchange above), lie almost as high on a Chew-Frautschi plot as the quark spin 3/2 trajectories. Thus we should not a priori expect to explain the one clear feature in $\frac{d\sigma}{du}$ which is a dip at $u = -0.2$ in $\pi^+ p$ backward scattering. This dip is often attributed to a nonsense wrong signature zero in the quark spin $\frac{1}{2}$, $I = \frac{1}{2}$ trajectory (ordinary nucleon) which appears because the $I = \frac{1}{2}$ trajectories are not in fact exchange degenerate as we would predict from our model (recall exchange degenerate trajectories yield exchanges with no signature and thus no zeroes). This trajectory is also not the leading contribution in our model so its affect is ignored to leading order in s.

However, we did notice structure for meson exchange which had nothing to do with signature zeroes, i.e., the Regge pole - fixed cut and pole interference in the A' amplitude at small t. We can take the attitude that some structure will result just because of the spin content of a certain amplitude, e.g. because of the relative sign of the Regge term and the fixed cut-fixed pole terms. This is similar to an optical picture where dips in differential cross-sections occur at the zeroes of $J_n\left(\sqrt{-\binom{t}{u}}\,b\right)$ where n is the amount of spin flip and $b \approx 1$ fermi. For example, the spin nonflip amplitude is given by a $J_0$ and has a dip near $\begin{matrix} t \\ u \end{matrix} = -0.2\ \mathrm{GeV}^2$. Thus any dip in $\frac{d\sigma}{du}$ in our model would result from a pole-cut interference in the dominant amplitude rather than a signature zero.

The amplitudes which we shall calculate are $A + m B$, which is proportional, in the Regge region, to the s channel helicity flip amplitude, and B, which is proportional to the s channel helicity nonflip amplitude. For u near zero ($\theta_s \approx 180^\circ$) these amplitudes are also approximately the spin nonflip and spin flip amplitudes respectively. We shall present the results as an integral in the $\lambda$ plane as in the meson exchange example above. Recall that there are two contributions, one from the u, t diagram which goes like $s^\lambda$ and one from the s, u diagram which goes like $(\frac{-s}{2})^\lambda$ as we discussed in Chapter II (see Eq. II-19a). For $I_u = 3/2$ we have ( $\sigma = m + \mu$, $\ell' = 1$)

$$(A + mB)^{I_u \cong 3/2}_{\substack{hel\\flip}} = \frac{g^2 s}{2\pi i}$$

$$\int \frac{d\lambda\ \Gamma(-\lambda)}{\lambda - \ell(u)} \left\{ s^\lambda \left[ \left(\frac{3 + 5u\ F(\ell(u) - \lambda - 1)}{\lambda - \ell_o}\right) (u + \sigma^2) \right.\right.$$

$$\left. + 2\,\sigma\, u\, (7 + \frac{u}{\lambda - \ell_o}) \ \frac{F(\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}} \right]$$

$$+ (\frac{-s}{2})^\lambda \left[ (5 + \frac{11u\ F\ (\ell(u) - \lambda - 1)}{\lambda - \ell_o}) \ (u + \sigma^2) \right.$$

$$\left.\left. + 2\,\sigma\, u\ (13 + \frac{3u}{\lambda - \ell_o}) \ \frac{F\ (\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}} \right] \right\} \qquad \text{(VI-17a)}$$

$$B_{\substack{hel\\ nonflip}}^{I_u = 3/2} \sim \frac{-g^2 s}{2\pi i}$$

$$\int d\lambda \; \frac{\Gamma(-\lambda)}{\lambda - \ell(u)} \left\{ s^\lambda \left[ \left(1 + \frac{7\, u\, F\,(\ell(u) - \lambda - 1)}{\lambda - \ell_o}\right) 2\,\sigma \right.\right.$$

$$+ (u + \sigma^2)\,\left(5 + \frac{3u}{\lambda - \ell_o}\right)\, \frac{F\,(\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}}$$

$$+ \left(\frac{-s}{2}\right)^\lambda \quad \left(3 + \frac{13\, u\, F\,(\ell(u) - \lambda - 1)}{\lambda - \ell_o}\right)\, 2\,\sigma$$

$$\left.\left. + (u + \sigma^2)\,\left(11 + \frac{5u}{\lambda - \ell_o}\right)\, \frac{F\,(\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}} \right]\right\} \qquad \text{(VI-17b)}$$

$$(A + mB)_{\substack{hel\\ flip}}^{I_u = 3/2} \cong \frac{g^2}{2\pi i}\, \left(\frac{s}{2}\right)$$

$$\int \frac{d\lambda\, \Gamma(-\lambda)}{\lambda - \ell(u)} \left\{ s^\lambda \left[ \left(15 - \frac{11\, u\, F\,(\ell(u) - \lambda - 1)}{\lambda - \ell_o}\right)\,(u + \sigma^2) \right.\right.$$

$$+ 2\,\sigma\, u\,\left(17 - \frac{13u}{\lambda - \ell_o}\right)\, \frac{F\,(\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}}$$

$$+ \left(\frac{-s}{2}\right)^\lambda \quad \left(10 - \frac{14\, u\, F\,(\ell(u) - \lambda - 1)}{\lambda - \ell_o}\right)\,(u + \sigma^2)$$

$$\left.\left. + 2\,\sigma\, u\, \left(8 - \frac{12u}{\lambda - \ell_o}\right)\, \frac{F\,(\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}} \right]\right\} \qquad \text{(VI-18a)}$$

$$B_{\substack{hel \\ nonflip}}^{I_u = 1/2} = \frac{g^2}{2\pi i} \left(\frac{s}{2}\right)$$

$$\int \frac{d\lambda\, \Gamma(-\lambda)}{\lambda - \ell(u)} \left\{ s^\lambda \left[ \left(13 - \frac{17\, u\, F\,(\ell(u) - \lambda - 1)}{\lambda - \ell_o}\right) 2\,\sigma \right.\right.$$

$$\left. + \; (u + \sigma^2) \left(11 - \frac{15\, u}{\lambda - \ell_o}\right) \frac{F\,(\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}} \right]$$

$$+ \; \left(\frac{-s}{2}\right)^\lambda \left[ \left(12 - \frac{8\, u\, F\,(\ell(u) - \lambda - 1)}{\lambda - \ell_o}\right) 2\,\sigma \right.$$

$$\left.\left. + \; (u + \sigma^2) \left(14 - \frac{10\, u}{\lambda - \ell_o}\right) \frac{F\,(\ell(u) - \lambda - 1)}{\sqrt{\lambda - \ell_o}} \right] \right\} \quad . \qquad \text{(VI-18b)}$$

One of the first things to notice about these equations is the explicit realization of a point made earlier about the symmetric quark model resonance spectrum. The u-channel resonances in the above equations come from $\ell(u) = \lambda = 0, 1, 2 \ldots$ . For $\lambda = 0$, i.e., internal orbital angular momentum zero, we expect only a $\underset{\sim}{56}$ and no $I = \frac{1}{2}$ , quark spin 3/2, nonstrange particle. We see that in Eqs. VI-18a and b the part of the integrand in brackets vanishes at $\lambda = \ell(u) = 0$ as required. Likewise for $\lambda = 1$, orbital angular momentum equal to one, we expect only a $\underset{\sim}{70}$ and thus no $I = 3/2$, quark spin 3/2, resonances, i.e., in Eqs. VI-17a and b the quantity in brackets vanishes for $\lambda = \ell(u) = 1$ . For all higher $\lambda$ ($L_u$) we expect both $\underset{\sim}{56}$'s and $\underset{\sim}{70}$'s, i.e., resonances in both I spin channels as is observed in Eqs. VI-17 and 18.

As in the meson exchange case, we observe that these baryon exchange amplitudes exhibit a certain degree of regularity in their structure. Specifically, the fixed pole term always appears multiplied by u and with a coefficient similar to the pure Regge term (the term with no F factor). Also, the two terms with $\frac{1}{\sqrt{\lambda - \ell_o}}$ appear together in similar fashion. Aside from the various integers, only two coefficients appear, i.e., $u + \sigma^2$ and $2\sigma$, and again they change roles in going from the nonflip to the flip amplitude. Since all the terms involving $\lambda - \ell_o$ were multiplied by $\not{p}_u$ in the original projector operator we expect them to be multiplied by a coefficient which vanishes as $\not{p}_u \to 0$, $u \to 0$ (of course $p_u \equiv 0$ also requires $m = \mu$). This is not a constraint for the helicity nonflip amplitude which is also spin flip near $u = 0$ and thus automatically has a kinematic factor which vanishes at $\theta = 180^o$ to conserve angular momentum, i.e., $u = 0$ to first order in s. However, for the helicity flip amplitude $(A + mB)$, which is also spin nonflip near $u = 0$, the fixed cut terms pick up an extra u factor and we can expect strong interference with the Regge term. Such interference effects are actually observed when Eqs. VI-17 and VI-18 are evaluated by the same methods as we used in the meson exchange case. The helicity flip amplitude $(A + mB)$ does exhibit zeroes in both the real and imaginary parts in the range $-0.35 < u < -0.05$, whereas the helicity nonflip amplitude (B) in general does not show such structure. However, the location of the zeroes depends strongly on the intercepts chosen for the various trajectories which, of course, determine the phase of the fixed $\lambda$ singularities. For example, to obtain the zeroes mentioned above, intercepts suggested by the data were used, i.e., $\ell_o(I_u = 3/2) = -1.3$, $\ell_o(I_u = 1/2) = -0.8$, which already break the $SU_6$ symmetry. Also, in order to obtain even qualitative agreement with the data on $\frac{d\sigma}{du}$ the helicity flip amplitude $(A + mB)$ would have to dominate the $I_u = 1/2$ exchange and not the $I_u = 3/2$ (to obtain a dip in $\pi^+p$ and not $\pi^-p$). In the present model the helicity nonflip amplitude (B) is always large and $\frac{d\sigma}{du}$ shows little structure. As in the meson exchange case the cross section does not fall fast enough as $|u|$ increases indicating a need for stronger neutralizer effects so that a detailed comparison with data is not meaningful at present.

We can try to abstract from all this certain systematics of quark spin amplitudes which result essentially from the $\not{p}$ structure of the projection operators. In amplitudes which are not required by kinematics to vanish at $\theta = 0$ or $\pi$, i.e., amplitudes which are essentially spin nonflip amplitudes in these kinematic regions, we expect strong cancellation between the Regge pole term and the fixed cut term and between the fixed pole $(\frac{1}{\lambda - \ell_o})$ and the $(\frac{1}{\lambda - \ell_o})^{3/2}$ terms if the latter is present. For amplitudes which vanish automatically we can expect the fixed cut to have the same sign as the Regge pole and cancel instead with the other fixed $\lambda$ singularities. As we have seen these systematics are suggestive of structure which is in qualitative agreement with the data. However, when it comes to actual comparison with the data including the question of overall $t$ or $u$ dependence, i.e., decreasing fast enough as $|t|$, $|u|$ increases, the question of exact relative phases as exhibited in polarizations, and the question of relative sizes of flip and nonflip amplitudes, the model fares quite poorly. We can explain these difficulties in terms of requiring a better treatment of the neutralizer and of $SU_6$ breaking, but our original hope of checking the model against data independently of such details has certaintly not been realized. The model, as it stands, is only suggestive of reasonable structure for the various amplitudes and is of little value for actually fitting the data.

For the sake of completeness it should be pointed out that the detailed structure of the amplitudes obtained in the present model is considerably more complex than that suggested by the original work of Carlitz and Kislinger,[22] although the general conclusions, about where strong cut effects can be expected, are quite similar. This difference arises not only from the fact that we are now studying a dual model, but also from a basic difference in the way in which the two models are constructed. In the second paper of Carlitz and Kislinger, where meson exchange is discussed, the primary emphasis is on constructing nonsingular, $SU_{6_W}$ symmetric vertices. The total amplitude is obtained by using these vertices to couple the external particles to the exchanged resonances and then summing the resonances a la Van Hove. Due to the explicit factorization of the amplitude, even away from the resonances, much of the initial complexity of the vertices can be and is absorbed into the overall coupling constant leaving a fairly simple result. In the present model the primary interest is in constructing a dual 4-point function involving explicit quarks and their propagators. We have seen that in order to avoid contributions to the resonances from what are effectively the MacDowell (parity) partners of the quarks we have introduced projection operators. This guarantees that at the poles we have the same structure as Carlitz and Kislinger, but away from the poles the MacDowell partners still contribute to the amplitude, i.e., although only the $J^P = 1^-$ and $0^-$ channels are allowed to resonate there is still a nonzero $1^+$ and $0^+$ amplitude present. Even in the resonant channels the coupling away from the poles is more complicated than in Carlitz and Kislinger. As a result, our total amplitude does not factorize except at the poles and the simplifications of Carlitz and Kislinger are not possible. This problem is related to the usual ambiguity about how to continue an amplitude away from its poles. The model of Carlitz and Kislinger simply suggests a different program for this continuation than in the present model.

# VII. SUMMARY

Let us briefly review what it is that we have accomplished in the work discussed above. We have succeeded in writing down a "dual quark model" which explicitly exhibits the Dirac spin of the quarks. The model has the spectrum of the static symmetric quark model and has vertices which are symmetric under $SU_{6_W} \times O_{2_{L_Z}}$. In order to avoid parity doubling of the resonances we have introduced fixed J-plane cuts and poles. To maintain the symmetry of the vertices and to insure the usual factorization properties we have introduced further new functionry, i.e., the neutralizer, which we have suggested should be interpreted as a moving Regge cut possibly related to inelastic effects. This new structure has led to unavoidable background contributions to the FESR's and hence to a modified interpretation of duality. This new complexity was, however, not unexpected.

Finally, we have attempted comparison with data for $\pi N$ scattering using a simplified version of the neutralizer function with the hope of testing the general form of the model independent of the details of the neutralizer. The amplitudes calculated are in fact suggestive of systematic structure which is in qualitative agreement with nature, i.e., that spin flip amplitudes should be dominated by pure Regge poles, whereas spin nonflip amplitudes should exhibit strong interference between poles and cuts. However, the actual calculational results are in poor agreement with the data and the form of the disagreement suggests that the details of the neutralizer cannot be ignored if the model is to succeed at all. In its present simplified form the model is definitely unsatisfactory for parameterizing the data.

Before considering discarding this model there are several areas which deserve further study. First is an effort to make a numerical study using a more satisfactory form for the neutralizer, including the constraints of full Regge behavior as discussed in Chapter V, and possibly trying to relate the form to some model for the cut structure arising from unitarization. It would also be interesting to use this model as a vehicle to study schemes for breaking both the $SU_{6_W}$ coupling symmetry and the $SU_6$ resonance symmetry. We can expect that when these two problems are studied within the framework of a dual model there will exist contraints which relate the two effects. It is interesting to note that the $\not{p}_t$ terms, for example, which we eliminated from s channel resonance couplings via the neutralizer in order to maintain $SU_{6_W}$ symmetry are quite similar to the $SU_{6_W}$ breaking terms used by Colglazier and Rosner(6) to study the decays of the $A_1$ and B mesons. This suggests that in a more complete model such terms should contribute to at least some of the resonance couplings and could give the observed symmetry breaking. Another interesting suggestion, which has not yet been thoroughly or consistently studied, is that of Humble, Vaughn and Zia.(55) They want to physically interpret the trajectories which appear, in the absence of the neutralizer, at a position in the Chew-Frautschi plot shifted by $\frac{1}{2}$ from the input trajectories. These trajectories were mentioned in Chapter IV, where they were called cousins, and are suppressed in our model. Although such trajectories are difficult to interpret in a model with quarks (not a part of the model of Humble, Vaughn and Zia), the $\frac{1}{2}$ shift is very suggestive of the observed size of $SU_6$ breaking. Since both of these studies bear on the structure of the neutralizer, their successful completion might lead to a model which realizes the good features suggested by the present model while not exhibiting its difficulties.

# APPENDIX

In this appendix we wish to give an illustrative example of how the $SU_6$ calculations were carried out for the quark diagrams for meson-baryon scattering. In particular, we shall look at the s,t diagram (Fig. VI-1a) in the limit of large s and fixed t so that only the t channel projection operator contributes factors other than the unity operator. What follows then is an explicit evaluation of Eq. VI-8.

$$\begin{aligned} G(n,s,t,) = \bar{\Psi} & \begin{Bmatrix} FGH \\ fgh \end{Bmatrix} \delta^K_H \left(1 - \frac{\not{p}_t F}{m_n}\right)^k_h \\ & \left\{ \left[\gamma_5 \left(1 + \frac{\not{q}'}{\mu}\right)\right]^\ell_k \bar{P}^L_K \right\} (1)^e_\ell \, \delta^E_L \\ & \left\{ \left[\left(1 + \frac{\not{q}}{\mu}\right)\gamma_5\right]^d_e P^D_E \right\} (1)^a_f \\ & (1)^b_g \, \delta^B_G \left(1 + \frac{\not{p}_t F}{m_n}\right)^c_d \delta^C_D \, \Psi \begin{Bmatrix} ABC \\ abc \end{Bmatrix} \end{aligned} \qquad \text{(VI-8)}$$

Recall that A,B,C are $SU_3$ indices and a, b, c are Dirac indices. Repeated indices ($\searrow$) imply summation. The nucleon is defined by[52] (from Eq. VI-5)

$$\Psi(p) \begin{Bmatrix} ABC \\ abc \end{Bmatrix} = \frac{1}{\sqrt{6}} \left\{ \left[ (1 + \frac{\not{p}}{m}) \gamma_5 C \right]^b_a \epsilon_{ABJ} \; u_c(p) \; B^J_C \right.$$

$$+ \left[ (1 + \frac{\not{p}}{m}) \gamma_5 C \right]^c_b \epsilon_{BCJ} \; u_a(p) \; B^J_A$$

$$\left. + \left[ (1 + \frac{\not{p}}{m}) \gamma_5 C \right]^a_c \epsilon_{CAJ} \; u_b(p) \; B^J_B \right\} \tag{A-1}$$

The outgoing state, momentum $\not{p}'$ , is defined as

$$\overline{\Psi}(p') \begin{Bmatrix} FGH \\ fgh \end{Bmatrix} = - \frac{1}{\sqrt{6}} \left\{ \left[ C \gamma_5 (1 + \frac{\not{p}'}{m}) \right]^g_f \epsilon^{FGR} \; \bar{u}^h(p') \; \overline{B}^H_R \right.$$

$$+ \left[ C \gamma_5 (1 + \frac{\not{p}}{m}) \right]^h_g \epsilon^{GHR} \; \bar{u}^f(p') \; \overline{B}^F_R$$

$$\left. + \left[ C \gamma_5 (1 + \frac{\not{p}'}{m}) \right]^f_h \epsilon^{HFR} \; \bar{u}^g(p') \; \overline{B}^G_R \right\} \tag{A-2}$$

The properties of the matrix $C$ are $C^T = -C = C^+ = C^{-1}$, $C^{-1} \gamma_\mu C = -(\gamma_\mu)^T$, $C^{-1} \gamma_5 C = \gamma_5{}^T$, and $\overline{C} = C$. The tensor $\epsilon^{ABC}$ is antisymmetric in all of its indices. As mentioned in the text, we henceforth drop all of the overall constants since there is some unknown coupling constant anyway.

When we substitute Eqs. A-1 and A-2 into Eq. VI-8, there will be nine distinct terms coming from the three incoming nucleon terms times the three outgoing nucleon terms. For simplicity we shall work them out one at a time. The first term has the form

$$G_1 = \epsilon^{FGR}\, \bar{B}^{H}_{R}\, \delta^{K}_{H}\, \bar{P}^{L}_{K}\, \delta^{E}_{L}\, P^{D}_{E}\, \delta^{C}_{D}\, \delta^{A}_{F}\, \delta^{B}_{G}\, \epsilon_{ABJ}\, B^{J}_{C}$$

$$\times \left[C\,\gamma_5 (1 + \frac{\not{p}'}{m})\right]^{g}_{f} \bar{u}^{h}(p')\, (1 - \frac{\not{p}_t F}{m_n})^{k}_{h} \left[\gamma_5\, (1 + \frac{\not{q}'}{\mu})\right]^{\ell}_{k} (1)^{e}_{\ell}$$

$$\left[(1 + \frac{\not{q}'}{\mu})\,\gamma_5\right]^{d}_{e} (1)^{a}_{f}\, (1)^{b}_{g}\, (1 + \frac{\not{p}_t}{m_n} F)^{c}_{d} \left[(1 + \frac{\not{p}}{m})\,\gamma_5\, C\right]^{b}_{a} u_c(p)\,. \qquad \text{(A-3)}$$

To work out the $SU_3$ part of $G_1$ and the terms given below, the following identities are useful.

$$\epsilon^{ABC}\,\epsilon_{ABD} = 2\,\delta^{C}_{D} \qquad \text{(A-4a)}$$

$$\epsilon^{ABC}\,\epsilon_{ADF} = \delta^{B}_{D}\,\delta^{C}_{F} - \delta^{B}_{F}\,\delta^{C}_{D} \qquad \text{(A-4b)}$$

$$\epsilon^{ABC}\,\epsilon_{DFG} = \delta^{A}_{D}\,\delta^{B}_{F}\,\delta^{C}_{G} + \delta^{A}_{F}\,\delta^{B}_{G}\,\delta^{C}_{D} + \delta^{A}_{G}\,\delta^{B}_{D}\,\delta^{C}_{F} - \delta^{A}_{F}\,\delta^{B}_{D}\,\delta^{C}_{G}$$
$$- \delta^{A}_{D}\,\delta^{B}_{G}\,\delta^{C}_{F} - \delta^{A}_{G}\,\delta^{B}_{F}\,\delta^{C}_{D}\,. \qquad \text{(A-4c)}$$

Thus the $SU_3$ factor in $G_1$ becomes

$$\bar{B}^{H}_{R}\,\bar{P}^{L}_{H}\,P^{D}_{L}\,B^{J}_{D}\,\epsilon^{ABR}\,\epsilon_{ABJ} = 2\,\bar{B}^{H}_{R}\,\bar{P}^{L}_{H}\,P^{D}_{L}\,B^{J}_{D}\,\delta^{R}_{J} = 2 < \overline{B}\overline{P}PB > \qquad \text{(A-5)}$$

with the notation $< > = \mathrm{Tr}$. In order to write the Dirac spin part of $G_1$ in the usual matrix multiplication form we must take the transpose of one of the nucleon factors to get the indices in the correct position. Working on the incoming nucleon we have

$$\left[(1 + \frac{\not p}{m}) \gamma_5 C\right]_a^b = \left(\left[(1 + \frac{\not p}{m}) \gamma_5 C\right]^T\right)_b^a = \left(\left[C^T \gamma_5{}^T (1 + \frac{\not p^T}{m})\right]\right)_b^a$$

$$= \left[-C C^{-1} \gamma_5 C (1 - \frac{C^{-1} \not p C}{m})\right]_b^a = -\left[(1 + \frac{\not p}{m}) \gamma_5 C\right]_b^a \qquad \text{(A-6)}$$

where we have used the properties of C given above. Now we can write $G_1$ as (using $(1)_f^a = (1)_a^f$ )

$$G_1 = 2 < \overline{B}PPB > \bar{u}(p') \left[(1 - \frac{\not p_t}{m_n} F) \gamma_5 (1 + \frac{\not q'}{\mu})(1 + \frac{\not q}{\mu}) \gamma_5 (1 + \frac{\not p_t}{m_n} F)\right] u(p)$$

$$\times \ \mathrm{Tr} \left[C \gamma_5 (1 + \frac{\not p'}{m})(1 + \frac{\not p}{m}) \gamma_5 C (-)\right]$$

$$= 2 < \overline{B}PPB > \bar{u}(p') \left[(1 - \frac{\not p_t}{m_n} F)(1 - \frac{\not q'}{\mu})(1 - \frac{\not q}{\mu})(1 + \frac{\not p_t}{m_n} F)\right] u(p)$$

$$\times \ \mathrm{Tr} \left[(1 + \frac{\not p'}{m}) (1 + \frac{\not p}{m})\right] \qquad . \qquad \text{(A-7)}$$

Let us proceed by evaluating the trace factor which we shall label 2K(t).

$$2\,K(t) \equiv \mathrm{Tr}\left[(1 + \frac{\not{p}'}{m})(1 + \frac{\not{p}}{m})\right] = \mathrm{Tr}\left[1 + \frac{\not{p} + \not{p}'}{m} + \frac{\not{p}\,\not{p}'}{m^2}\right]$$

$$= 4\,(1 + \frac{p \cdot p'}{m^2}) = 4\,(1 + \frac{2m^2 - t}{2m^2}) = 2\,(4 - \frac{t}{m^2}) \quad . \tag{A-8}$$

To simplify the rest of $G_1$ we need to work out the form of $\bar{u}(p')\ \Theta_p\ u(p)$ for the various operators $\Theta_p$ which appear. We expect on general principles that the result can always be expressed as $\bar{u}(p')\ \Theta_p\ u(p) = \bar{u}\ (a + \not{Q}\,b)\ u$ where $\not{Q} = \frac{1}{2}(\not{q} + \not{q}')$ . In the following table we present the general results which can easily be verified by the reader (see e.g. ref. 51). Recall that $s = (p + q)^2 = (p' + q')^2$ , $t = (p - p')^2 = (q' - q)^2$, $p_t = (p - p') = (q' - q)$, $u = (p - q')^2 = (p' - q)^2$ .

TABLE I

| $\Theta_p$ | a | b | $\Theta_p$ | a | b | $\Theta_p$ | a | b |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | $\not{q}'\not{q}$ | $s-m^2$ | $-2m$ | $\not{q}'\not{q}\not{p}_t$ | 0 | $-t$ |
| $\not{q}$ | 0 | 1 | $\not{q}\not{p}_t$ | $u-m^2-\mu^2$ | $2m$ | $\not{p}+\not{q}'\not{q}\not{p}_t$ | $t(u-m^2)$ | $2mt$ |
| $\not{q}'$ | 0 | 1 | $\not{q}'\not{p}_t$ | $-s+m^2+\mu^2$ | $2m$ | $\not{p}\not{q}$ | $\frac{1}{2}(s-u)$ | $-m$ |
| $\not{p}$ | m | 0 | $\not{p}_t\not{p}_t$ | t | 0 | $\not{q}\not{p}'$ | $\frac{1}{2}(s-u)$ | $-m$ |
| $\not{p}'$ | m | 0 | $\not{p}_t\not{q}'\not{q}$ | 0 | t | $\not{p}\not{p}'$ | $m^2-t$ | 0 |
| $\not{p}_t$ | 0 | 0 | $\not{p}_t\not{q}'\not{p}_t$ | 0 | $-t$ | $\not{p}\not{q}\not{p}'$ | $m(s-u)$ | $t-3m^2$ |
| $\not{p}_t\not{q}$ | $s-m^2-\mu^2$ | $-2m$ | $\not{p}_t\not{q}\not{p}_t$ | 0 | $-t$ | | | |
| $\not{p}_t\not{q}$ | $-u+m^2+\mu^2$ | $-2m$ | | | | | | |

Using the table we find

$$\bar{u}\,\left(1 - \frac{\not{p}_t}{m_n}F\right)\left(1 - \frac{\not{q}}{\mu}\right)\left(1 - \frac{\not{q}}{\mu}\right)\left(1 + \frac{\not{p}_t}{m_n}F\right)u$$

$$= \bar{u}\left[1 - \frac{\not{p}_t}{m_n}F + \frac{\not{p}_t}{m_n}F - \frac{\not{q}'}{\mu} - \frac{\not{q}}{\mu} + \frac{\not{p}_t \not{q}'}{m_n\mu}F + \frac{\not{p}_t \not{q}}{m_n\mu}F + \frac{\not{p}_t\not{p}_t}{m_n^2}F + \frac{\not{q}'\not{q}}{\mu^2} - \frac{\not{q}'\not{p}_t}{\mu m_n}F\right.$$

$$\left. - \frac{\not{q}\,\not{p}_t}{\mu m_n}F - \frac{\not{p}_t\not{q}'\not{q}}{\mu^2 m_n}F + \frac{\not{p}_t\not{q}'\not{p}_t}{\mu m_n^2}F + \frac{\not{p}_t\not{q}\,\not{p}_t}{m_n^2\mu}F + \frac{\not{q}'\not{q}\,\not{p}_t}{m_n\mu^2}F - \frac{\not{p}_t\,\not{q}'\not{q}\,\not{p}_t}{\mu^2 m_n^2}F\right]u$$

$$= \bar{u}\left[1 + \frac{-u + m^2 + \mu^2}{m_n\,\mu}F + \frac{s - m^2 - \mu^2}{m_n\,\mu}F + \frac{t}{m_n\mu}F + \frac{s - m^2}{\mu^2} + \frac{s - m^2 - \mu^2}{\mu\, m_n}F\right.$$

$$- \frac{u - m^2 - \mu^2}{\mu\, m_n}F - \frac{t\,(u - m^2)}{\mu^2\, m_n^2}F$$

$$\left. + \not{q}\left\{-\frac{2}{\mu} - \frac{4m}{m_n\mu}F - \frac{2m}{\mu^2} - \frac{4m}{m_n\mu}F - \frac{t}{m_n\mu^2}F - \frac{2t}{m_n^2\,\mu}F - \frac{t}{\mu^2\, m_n}F - \frac{2\,m\,t}{\mu^2\, m_n^2}F\right\}\right]u$$

$$= \bar{u}\left[\frac{s - m^2 - \mu^2}{\mu^2} + \frac{2\,F}{m_n\,\mu}(s - u) - \frac{t\,F}{\mu^2\, m_n^2}(u - m^2)\right.$$

$$\left. - \not{q}\left\{\frac{2\,(m + \mu)}{\mu^2} + \frac{2\,F}{m_n\,\mu^2}(4\,m\,\mu + t) + \frac{2t\,F}{\mu^2\, m_n^2}(\mu + m)\right\}\right]u \qquad \text{(A-9)}$$

Note that in our simplified treatment of the neutralizer $F^2 = F$. Since we are interested in the large $|s|$ , $|u|$ , fixed t limit ( $u \sim -s$), we can further simplify $G_1$ by defining the functions

$$g\ (s,t) \equiv \frac{s}{\mu^2}\left[1 + \frac{4\,\mu\, F}{m_n} + \frac{t\, F}{m_n^2}\right] \qquad \text{(A-10a)}$$

$$h\ (s,t) \equiv \frac{-2}{\mu^2}\left[(m+\mu)\left(1 + \frac{t\,F}{m_n^2}\right) + \frac{F}{m_n}\ (4\, m\, \mu + t)\right] \qquad \text{(A-10b)}$$

So, using Eqs. A-7, A-8, and A-10 we have

$$G_1\ (n,s,t) \cong 4 < \overline{B}\overline{P}PB > K(t)\ \bar{u}\ (g + \not{q}\ h)\ \ u \qquad \text{(A-11)}$$

With the above calculation as practice, we can proceed to evaluate the other eight terms in fairly short order. The ordering used in the following is to successively look at each term in $\overline{\Psi}$ for a fixed term in $\Psi$.

$$G_2 = \epsilon^{HFR}\ \overline{B}^{G}_{R}\ \delta^{K}_{H}\ \overline{P}^{L}_{K}\ \delta^{E}_{L}\ P^{D}_{E}\ \delta^{C}_{D}\ \delta^{A}_{F}\ \delta^{B}_{G}\ \epsilon_{ABJ}\ B^{J}_{C}$$

$$\times \left[C\,\gamma_5\ \left(1 + \frac{\not{p}'}{m}\right)\right]^{h}_{g}\ \bar{u}^{f}(p')\left(1 - \frac{\not{p}_t}{m_n}F\right)^{k}_{h}\left[\gamma_5\left(1 + \frac{\not{q}'}{\mu}\right)\right]^{\ell}_{k}\ (1)^{e}_{\ell}\left[\left(1+\frac{\not{q}}{\mu}\right)\gamma_5\right]^{d}_{e}$$

$$\times\ (1)^{a}_{f}\ (1)^{b}_{g}\ \left(1 + \frac{\not{p}_t}{m_n}F\right)^{c}_{d}\left[\left(1 + \frac{\not{p}}{m}\right)\gamma_5\ C\right]^{b}_{a}\ \ u_c(p)$$

$$= \left[< \overline{B} > < \overline{P}PB > - < \overline{B}\overline{P}PB >\right]\ \bar{u}\left\{\left(1 + \frac{\not{p}}{m}\right)\gamma_5\ C\ C\ \gamma_5\right.$$

$$\left. \times\ \left(1 + \frac{\not{p}'}{m}\right)\left(1 - \frac{\not{p}_t}{m_n}F\right)\gamma_5\ \left(1 + \frac{\not{q}'}{\mu}\right)\left(1 + \frac{\not{q}}{\mu}\right)\gamma_5\ \left(1 + \frac{\not{p}_t}{m_n}F\right)\right\}\ u$$

$$= \left[< \overline{B}\overline{P}PB > - < \overline{B} > < \overline{P}PB >\right]\bar{u}\ \left\{\left(1 + \frac{\not{p}}{m}\right)\ \left(1 + \frac{\not{p}'}{m}\right)\ \left(1 - \frac{\not{p}_t}{m_n}F\right)\right.$$

$$\left. \times\ \left(1 - \frac{\not{q}'}{\mu}\right)\cdot\left(1 - \frac{\not{q}}{\mu}\right)\ \left(1 + \frac{\not{p}_t}{m_n}F\right)\right\}\ u \qquad \text{(A-12)}$$

(Recall that C C = -1)

To evaluate this expression we note that

$$\bar{u}(p')\left(1+\frac{\not p}{m}\right)\left(1+\frac{\not p'}{m}\right) = \bar{u}(p')\left(1+\frac{\not p}{m}+\frac{\not p'}{m}+\frac{\not p \not p}{m^2}\right)$$

$$= \bar{u}\left(2+\frac{\not p}{m}-\frac{\not p}{m}+\frac{2\, p\cdot p'}{m}\right) = K(t)\,\bar{u}(p') \qquad \text{(A-13)}$$

where we have used Eq. A-8. The remainder of the expression between $\bar{u}(p')$ and $u(p)$ is just what we had in Eq. A-9, so that

$$G_2(n,s,t) = \left[<\overline{B}\overline{P}PB> - <\overline{B}><\overline{P}PB>\right] K(t)\,\bar{u}\,(g+\not q h)\,u \qquad \text{(A-14)}$$

The next term is

$$G_3 = \epsilon^{HFR}\,\overline{B}^{G}_{R}\,\delta^{K}_{H}\,\overline{P}^{L}_{K}\,\delta^{E}_{L}\,P^{D}_{E}\,\delta^{C}_{D}\,\delta^{A}_{F}\,\delta^{B}_{G}\,\epsilon_{ABJ}\,B^{J}_{C}$$

$$\times \left[C\gamma_5\left(1+\frac{\not p}{m}\right)\right]^{f}_{h} \bar{u}^{g}(p')\left(1-\frac{\not p_t}{m_n}F\right)^{k}_{h}\left[\gamma_5\left(1+\frac{\not q'}{\mu}\right)\right]^{\ell}_{k}(1)^{e}_{\ell}$$

$$\times\left[\left(1+\frac{\not q}{\mu}\right)\gamma_5\right]^{d}_{e}$$

$$\times (1)^{a}_{f}\,(1)^{b}_{g}\left(1+\frac{\not p_t}{m_n}F\right)^{c}_{d}\left[\left(1+\frac{\not p}{m}\right)\gamma_5\, C\right]^{b}_{a} u_c(p)$$

$$= \left[<\overline{B}><\overline{P}PB> - <\overline{B}\overline{P}PB>\right]\bar{u}\left\{(-1)\left(1+\frac{\not p}{m}\right)\gamma_5\, C\,(-1)\,C\,\gamma_5\right.$$

$$\left.\times\left(1+\frac{\not p'}{m}\right)\left(1-\frac{\not p_t}{m_n}F\right)\gamma_5\left(1+\frac{\not q'}{\mu}\right)\left(1+\frac{\not q}{\mu}\right)\gamma_5\left(1+\frac{\not p_t}{m_n}F\right)\right\}u$$

$$= \left[<\overline{B}\overline{P}PB> - <\overline{B}><\overline{P}PB>\right]\bar{u}\left\{\left(1+\frac{\not p}{m}\right)\left(1+\frac{\not p'}{m}\right)\left(1-\frac{\not p_t}{m_n}F\right)\right.$$

$$\left.\times\left(1-\frac{\not q'}{\mu}\right)\left(1-\frac{\not q}{\mu}\right)\left(1+\frac{\not p_t}{m_n}F\right)\right\}u \qquad \text{(A-15)}$$

where we have used Eq. A-6 and the corresponding form for $\left[C\,\gamma_5\,(1+\frac{\not p}{m})\right]^f_h$. Inspection shows that Eq. A-15 is identical to Eq. A-12.

$$G_3 = G_2 = \,<\overline{B}\overline{P}PB> - <\overline{B}><\overline{P}PB>\; K(t)\;\; \bar{u}\,(g + \not q h)\, u \qquad \text{(A-16)}$$

Next we have

$$G_4 = \epsilon^{FGR}\,\overline{B}^H_R\;\delta^K_H\;\overline{P}^L_K\;\delta^E_L\;P^D_E\;\delta^C_D\;\delta^A_F\;\delta^B_G\;\epsilon_{BCJ}\;B^J_A$$

$$\left[C\,\gamma_5\,(1+\frac{\not p'}{m})\right]^g_f\;\bar{u}^h(p')\,(1-\frac{\not p_t}{m_n}F)^k_h\;\left[\gamma_5\,(1+\frac{\not q'}{\mu})\right]^\ell_e\,(1)^e_\ell$$

$$\times\;\left[(1+\frac{\not q}{\mu})\,\gamma_5\right]^d_e\,(1)^a_f\,(1)^b_g\,(1+\frac{\not p_t}{m_n}F)^c_d\;\left[(1+\frac{\not p}{m})\,\gamma_5\,C\right]^c_b\;u_a(p)$$

$$= \left[<\overline{B}\overline{P}P><\overline{B}> - <\overline{B}\overline{P}PB>\right]\;\bar{u}\left\{(1-\frac{\not p_t}{m_n}F)\,\gamma_5\,(1+\frac{\not q'}{\mu})\,(1+\frac{\not q}{\mu})\,\gamma_5\right.$$

$$\left.\times\;(1+\frac{\not p_t}{m_n}F)\,(-1)\,(1+\frac{\not p}{m})\,\gamma_5\,C\,(-1)\,C\,\gamma_5\,(1+\frac{\not p'}{m})\right\}u$$

$$= \left[<\overline{B}\overline{P}PB> - <\overline{B}\overline{P}P><B>\right]\;\bar{u}\left\{(1-\frac{\not p_t}{m_n}F)\,(1-\frac{\not q'}{\mu})\,(1-\frac{\not q}{\mu})\right.$$

$$\left.\times\;(1+\frac{\not p_t}{m_n}F)\,(1+\frac{\not p}{m})\,(1+\frac{\not p'}{m})\right\}u \qquad . \qquad \text{(A-17)}$$

Similarly to Eq. A-13 we find

$$(1+\frac{\not p}{m})\,(1+\frac{\not p'}{m})\,u(p) = (1+\frac{\not p}{m}+\frac{\not p'}{m}+\frac{\not p\,\not p'}{m^2})\,u(p) = K(t)\,u(p). \qquad \text{(A-18)}$$

So again we have just Eq. A-9 and we find

$$G_4 = \left[ < \overline{B}\overline{P}PB > - < \overline{B}\overline{P}P > < B > \right] K(t) \ \bar{u} \ (g + \not{Q}h) \ u \qquad \text{(A-19)}$$

The fifth term has the form

$$G_5 = \epsilon^{GHR} \ \overline{B}^{F}_{R} \ \delta^{K}_{H} \ \overline{P}^{L}_{K} \ \delta^{E}_{L} \ P^{D}_{E} \ \delta^{C}_{D} \ \delta^{A}_{F} \ \delta^{B}_{G} \ \epsilon_{BCJ} \ B^{J}_{A}$$

$$\times \left[ C \gamma_5 (1 + \frac{\not{p}'}{m}) \right]^h_g \bar{u}^f (p') (1 - \frac{\not{p}_t}{m_n} F)^k_h \left[ \gamma_5 (1 + \frac{\not{q}'}{\mu}) \right]^\ell_k (1)^e_\ell$$

$$\times \left[ (1 + \frac{\not{q}}{\mu}) \gamma_5 \right]^d_e (1)^a_f (1)^b_g (1 + \frac{\not{p}_t}{m_n} F)^c_d \left[ (1 + \frac{\not{p}}{m}) \gamma_5 C \right]^c_b u_a(p)$$

$$= \left[ < \overline{B}B > < \overline{P}P > - < \overline{B}B\overline{P}P > \right] \bar{u} \ u \ \mathrm{Tr} \left[ C \gamma_5 (1 + \frac{\not{p}'}{m}) (1 - \frac{\not{p}_t}{m_n} F) \gamma_5 \right.$$

$$\left. \times (1 + \frac{\not{q}'}{\mu}) (1 + \frac{\not{q}}{\mu}) \gamma_5 (1 + \frac{\not{p}_t}{m_n} F) (-1) (1 + \frac{\not{p}}{m}) \gamma_5 C \right]$$

$$= \left[ < \overline{B}B > < \overline{P}P > - < \overline{B}B\overline{P}P > \right] \bar{u} \ u \ \mathrm{Tr} \left[ (1 + \frac{\not{p}'}{m}) (1 - \frac{\not{p}_t}{m_n} F) \right.$$

$$\left. \times (1 - \frac{\not{q}'}{\mu}) (1 - \frac{\not{q}}{\mu}) (1 + \frac{\not{p}_t}{m_n} F) (1 + \frac{\not{p}}{m}) \right] \qquad \text{(A-20)}$$

Using the fact that $(1 + \frac{\not{p}'}{m})$ acting to the right and $(1 + \frac{\not{p}}{m})$ acting to the left are, for our purposes, just like $\bar{u}(p')$ $u(p)$, i.e, $(1 + \frac{\not{p}'}{m})\not{p}' = (1 + \frac{\not{p}'}{m})m$ and $\not{p}(1 + \frac{\not{p}}{m}) = m(1 + \frac{\not{p}}{m})$, the trace factor is easily simplified to give $\mathrm{Tr}\left[(1 + \frac{\not{p}'}{m})(g + Q/h)(1 + \frac{\not{p}}{m})\right]$. This is straightforward to evaluate using $p \cdot Q = p' \cdot Q = \frac{s-u}{4}$.

To leading order in s we obtain

$$G_5 = \left[ < \bar{B}B > < \bar{P}P > - < \bar{B}B\bar{P}P > \right] \bar{u}\, u \left[ 2\, g\, K(t) + \frac{4\, s}{m}\, h \right] \tag{A-21}$$

The last term which we shall calculate explicitly is

$$G_6 = \epsilon^{HFR}\ \bar{B}^{G}_{R}\ \delta^{K}_{H}\ \bar{P}^{L}_{K}\ \delta^{E}_{L}\ P^{D}_{E}\ \delta^{C}_{D}\ \delta^{A}_{F}\ \delta^{B}_{G}\ \epsilon_{BCJ}\ B^{J}_{A}$$

$$\times \left[ C\, \gamma_5 \left(1 + \frac{\not{p}}{m}\right) \right]^{f}_{h} \bar{u}^{g}(p')\left(1 - \frac{\not{p}_t}{m_n} F\right)^{k}_{h} \left[ \gamma_5 \left(1 + \frac{\not{q}'}{\mu}\right) \right]^{\ell}_{k} (1)^{e}_{\ell}$$

$$\times \left[ \left(1 + \frac{\not{q}}{\mu}\right) \gamma_5 \right]^{d}_{e} (1)^{a}_{f}\, (1)^{b}_{g} \left(1 + \frac{\not{p}_t}{m_n} F\right)^{c}_{d} \left[ \left(1 + \frac{\not{p}}{m}\right) \gamma_5\, C \right]^{c}_{b} u_a(p)$$

$$= \Big[ < \bar{B}\bar{P}PB > + < \bar{P}P > < \bar{B} > < B > + < \bar{B}B\bar{P}P > - < \bar{P}P > < \bar{B}B >$$

$$- < B\bar{P}P > < \bar{B} > - < \overline{BPP} > < B > \Big]$$

$$\bar{u} \left\{ \left(1 + \frac{\not{p}}{m}\right) \gamma_5\, C \left(1 - \frac{C^{-1}\, \not{p}_t\, C}{m_n} F\right) C^{-1} \gamma_5\, C \left(1 - \frac{C^{-1}\, \not{q}\, C}{\mu}\right) \right.$$

$$\left. \left(1 - \frac{C^{-1}\, \not{q}\, C}{\mu}\right) C^{-1} \gamma_5\, C \left(1 + \frac{C^{-1}\, \not{p}_t\, C}{m_n} F\right) C\, \gamma_5 \left(1 + \frac{\not{p}'}{m}\right) \right\} u$$

$$= \Big[ < \bar{P}P > < \bar{B}B > + < B\bar{P}P > < \bar{B} > + < \overline{BPP} > < B > - < \overline{BPPB} >$$

$$- < \bar{P}P > < B > < \bar{B} > - < \bar{B}B\bar{P}P > \Big]$$

$$\cdot\ \bar{u} \left\{ \left(1 + \frac{\not{p}}{m}\right) \left(1 + \frac{\not{p}_t}{m_n} F\right) \left(1 - \frac{\not{q}}{\mu}\right) \left(1 - \frac{\not{q}'}{\mu}\right) \left(1 - \frac{\not{p}_t}{m_n} F\right) \left(1 + \frac{\not{p}'}{m}\right) \right\} u\ .$$

(A-22)

We can simplify the Dirac portion of this expression by noticing that the quantity in the brackets is just like the expression appearing in the trace of Eq. A-20 except that we must interchange $q \leftrightarrow q'$, $p \leftrightarrow p'$, and $p_t \leftrightarrow -p_t$. Of course these changes do not affect the variables s, t, and Q so that $g + \not{Q}\, h$ appears as before and we obtain

$$G_6 = \Big[ <\bar{P}P><BB> + <B\bar{P}P><\bar{B}> + <\overline{BP}P><B> - <\overline{BP}PB> - <\bar{P}P><B><\bar{B}> - <\bar{B}B\bar{P}P> \Big] \times \bar{u} \left\{ \left(1 + \frac{\not{p}}{m}\right)(g + \not{Q}\, h)\left(1 + \frac{\not{p}'}{m}\right) \right\} u$$

$$= \Big[ <\bar{P}P><\bar{B}B> + <B\bar{P}P><\bar{B}> + <\overline{BP}P><B> - <\overline{BP}PB> - <\bar{P}P><B><\bar{B}> - <\bar{B}B\bar{P}P> \Big] \times \bar{u} \left\{ g\left(4 - \frac{t}{m^2}\right) + \frac{2}{m}(s-u)\, h - \not{Q}\, h \left(4 - \frac{t}{m^2}\right) \right\} u \quad . \qquad \text{(A-23)}$$

Finally, to leading order in s, we have

$$G_6 \cong \Big[ <\bar{P}P><\bar{B}B> + <B\bar{P}P><\bar{B}> + <\overline{BP}P><B> - <\overline{BP}PB> - <\bar{P}P><B><\bar{B}> - <\bar{B}B\bar{P}P> \Big] \times \bar{u} \left\{ g\, K(t) - \not{Q}\, h\, K(t) + \frac{4\, s}{m} h \right\} u \qquad \text{(A-24)}$$

To complete these calculations, the reader can easily verify that the following equations are true.

$$G_7 = G_4 \qquad \text{(A-25a)}$$

$$G_8 = G_6 \qquad \text{(A-25b)}$$

$$G_9 = G_5 \qquad \text{(A-25c)}$$

Adding up all of these contributions and noting that $< B > = < \bar{B} > = 0$ (see Eq. VI-6), we arrive at the result shown in Eq. VI-10 (to get to that result we must also remove the inessential overall factor $\frac{1}{\mu^2 m^2}$).

$$G(n,s,t) = \mu^2 m^2 \sum_{i=1}^{q} = (\mu^2 m^2)\, \bar{u} \Big\{ < \overline{B}\overline{P}PB > \Big[ 6\, g\, K(t) - 8 \frac{s}{m} h + 10\, \not{q}\, K(t) h \Big]$$
$$+ \big( < \bar{B}B > < \bar{P}P > - < \bar{B}B\bar{P}P > \big) \Big[ 6\, g\, K(t) + 8 \frac{s}{m} h - 2\, \not{q}\, K(t) h \Big] \Big\} u$$
$$= \bar{u} \Big\{ s \Big[ < \overline{B}\overline{P}PB > \Big\{ \big(1 + \frac{t\, F}{m_n^2}\big)(40\, m^2 + 16\, m\mu - 6\, t) + \frac{F}{m_n}$$
$$(160\, m^2 \mu + 16\, mt - 24\, \mu t) \Big\} + \big( < \bar{B}B\bar{P}P > - < \bar{B}B > < \bar{P}P > \big)$$
$$\times \Big\{ \big(1 + \frac{t\, F}{m_n^2}\big)(8\, m^2 + 32\, m\mu + 6\, t) + \frac{F}{m_n}(32\, m^2 \mu + 32 mt + 24 \mu t) \Big\} \Big]$$
$$- 4\, \not{q}\, (4\, m^2 - t) \Big[ (m + \mu)\big(1 + \frac{t\, F}{m_n^2}\big) + \frac{F}{m_n}\, (4\, m\mu + t) \Big]$$
$$\times \big( 5 < \overline{B}\overline{P}PB > + < \bar{B}B\bar{P}P > - < \bar{B}B > < \bar{P}P > \big) \Big\}\, u \,. \qquad \text{(VI-10)}$$

The other quark diagrams of Fig. VI-1 and the other kinematic limits are all calculated in similar fashion.

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