APPLICATIONS OF MODEL THEORY TO COMPLEX ANALYSIS

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We use a nonstandard model of analysis to study two main topics in complex analysis.

UNIFORM CONTINUITY AND RATES OF GROWTH OF MEROMORPHIC FUNCTIONS is a unified nonstandard approach to several theories; the Julia-Milloux theorem and Julia exceptional functions, Yosida's class (A), normal meromorphic functions, and Gavrilov's W_p classes. All of these theories are reduced to the study of uniform continuity in an appropriate metric by means of S-continuity in the nonstandard model (which was introduced by A. Robinson).

The connection with the classical Picard theorem is made through a generalization of a result of A. Robinson on S-continuous *-holomorphic functions.

S-continuity offers considerable simplification over the standard sequential approach and permits a new characterization of these growth requirements.

BOUNDED ANALYTIC FUNCTIONS AS THE DUAL OF A BANACH SPACE is a nonstandard approach to the pre-dual Banach space for $H^{\infty}(D)$ which was introduced by Rubel and Shields.

A new characterization of the pre-dual by means of the nonstandard hull of a space of contour integrals infinitesimally near the boundary of an arbitrary region is given.

A new characterization of the strict topology is given in terms of the infinitesimal relation: "h $\stackrel{b}{=}$ k provided $||h-k||$ is finite and $h(z) \approx k(z)$ for $z \in ({}^*D)$".

A new proof of the noncoincidence of the strict and Mackey topologies is given in the case of a smooth finitely connected region. The idea of the proof is that the infinitesimal relation: "h $\stackrel{\gamma}{=}$ k provided $||h-k||$ is finite and $h(z) \approx k(z)$ on nearly all of the boundary", gives rise to a compatible topology finer than the strict topology.

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I. UNIFORM CONTINUITY AND RATES OF GROWTH OF MEROMORPHIC FUNCTIONS

1. Introduction:

The purpose of this section is to show that several theories of the rate of growth of meromorphic functions can be treated in a unified fashion. Introducing appropriate metrics we see that they all amount to a study of uniform continuity (or uniform equicontinuity). In the case of Julia-Milloux's theorem and Gavrilov's W_p classes (5.6) the introduction of the metric is new.

The connection between these theories and the classical Picard theorem is made through Theorem (3.1) via the mechanism of S-continuity. (This generalizes a result of Robinson [19] who introduced S-continuity). In standard terms this relates exceptional values and non-uniformity, but the use of nonstandard analysis and S-continuity results in considerable simplification because non-uniformity and sequences of Milloux circles are reflected in a single S-discontinuity at a remote point. In part, this also extends Robinson's treatment of the holomorphic Julia-Milloux theorem to the meromorphic case.

The nonstandard approach can greatly simplify a number of proofs of known results, particularily in the extensive theory of normal meromorphic functions (4.8). We offer a few simple examples. The original motivation for this work was the study of normal functions, unfortunately a search of the literature revealed that most of our early applications were done during the 1960's by various authors.

Some remarks about our setting are in order. Smooth convex metrics are emphasized because then we have a mean value theorem with which to measure S-continuity. (In this generality even this simple result seems to be new.) A more general discussion (without magnification operators) in terms of uniform continuity or S-continuity is possible.

The domains of functions are viewed as Riemann surfaces for the sake of simplicity. Even in the case of a planar region we wish to emphasize a preferred metric and not be confused with others. This is important since different metrics have different infinitesimals and infinite galaxies. Moreover, some care is needed to apply Theorem (3.1) at remote points, for example, in the plane metric near the boundary of the unit disk the infinitesimals are cut off by the unit circle, whereas Theorem (3.1) applies with respect to the hyperbolic metric. We feel therefore that even in the case of studying boundary behavior of a function defined on the unit disk it is easier to think in terms of a Riemann surface. We hope that this will not cause confusion; the primary examples are the punctured plane, the plane, the unit disk and hyperbolic surfaces (the punctured disk in specific).

We shall use a leisurely elementary style in this section. In effect the point is that some of the known work is more elementary than it might appear because of complicated sequential arguments which our approach avoids.

2. A Mean Value Theorem in Metric Spaces:

A metric space (X, d) is said to be convex if for each $A, B \in X$, there exists $C \in X$, different from A and B, for which $d(A, B) = d(A, C) + d(C, B)$. When a metric space is complete and convex any two points can be connected by a segment, that is, an isometric image of the interval $[0, d(A, B)]$. (The segment need not be unique, of course.) In what follows (X, d) will be a complete convex metric space.

We give an adaptation to metric spaces of a proof of W. A. J. Luxemburg for a mean value theorem in Euclidean n-space. We begin with a lemma following a theorm of P. Levy.

Let (Y, Δ) be an arbitrary metric space. Let $A, B \in X$ and let $[A, B]$ be a segment between A and B.

(2.1) LEMMA:

If $f : (X, d) \to (Y, \Delta)$ is continuous, then for each natural number $n \geq 3$ there exist points $A_n, B_n \in (A, B)$ such that

(1) $d(A, A_n) < d(B_n, A)$

(2) $d(A_n, B_n) = d(A, B)/n$

(3) $$\frac{\Delta(f(A), f(B))}{d(A, B)} \leq \frac{\Delta(f(A_n), f(B_n))}{d(A_n, B_n)}.$$

PROOF:

Let $\psi : [0, d(A, B)] \to [A, B]$ be the isomorphism from the interval in R onto $[A, B]$ in X. Normalize to

$$\varphi(t) = \psi(d(A, B) \cdot t), \qquad t \in [0, 1].$$

Define

$$g(t) = \Delta\left(f(\varphi(t + \tfrac{1}{n})), f(\varphi(t))\right)$$

for fixed but arbitrary $n \geq 3$. Then by the triangle inequality

$$\Delta(f(A), f(B)) \leq \sum_{k=0}^{n-1} g(\frac{k}{n})$$

and

$$\frac{\Delta(f(A), f(B))}{d(A, B)} \leq \frac{1}{n} \sum_{k=0}^{n-1} \frac{g(k/n)}{\left(\frac{d(A, B)}{n}\right)} .$$

Since the terms in the sum are non-negative, either

(1) for every k, $\frac{\Delta(f(A), f(B))}{d(A, B)} = \frac{g(k/n)}{d(A, B)/n}$

or

(2) for some k_0, $\frac{\Delta(f(A), f(B))}{d(A, B)} < \frac{g(k_0/n)}{d(A, B)/n}$.

In case (1) pick $k_0 \neq 0$ or $n-1$ or in case (2) when $k_0 \neq 0$ or $n-1$ we take $A_n = \varphi(k_0/n)$ and $B_n = \varphi(k_0+1/n)$.

In case (2) when $k_0 = 0$ or $n-1$ we do the following. By continuity of $g(t)$ we may move t away from the appropriate endpoint while still maintaining the strict inequality in the expression (2). In this case take $A_n = \varphi(t)$ and $B_n = \varphi(t+\frac{1}{n})$.

This proves the lemma.

The <u>magnification</u> or metric derivative of a function $f: X \to Y$ can be described as follows. Let C be a standard point of $^*X$ whose monad is non-degenerate (i.e., a non-isolated point). Provided that for every pair of points A and B within an infinitesimal of C, $\Delta(f(A), f(C))/d(A, C)$ is finite and infinitesimally close to

$\Delta(f(B), f(C))/d(B, C)$ we say the magnification of f at C is

$$M(\Delta/d)f(C) = Mf(C) = st\left(\frac{\Delta(f(A), f(C))}{d(A, C)}\right)$$

where st denotes the standard part homomorphism. In other words, as long as the standard part exists and is independent of the particular A within an infinitesimal of C, this expression is the magnification.

We describe the operator $M(\Delta/d)$ in the standard model by applying this definition at each standard point. As an operator $M(\Delta/d)$ can be extended to the nonstandard model, though the above description does not apply for internal functions. We leave the limit definition to the reader since we shall not have any need of it.

(2.2) A Mean Value Theorem:

Let (X, d) be a complete convex metric space, let (Y, Δ) be a metric space, and let $f: X \to Y$ have a magnification everywhere on X. Then for every segment $[A, B]$ in X, there exists $C \in (A, B)$ such that

$$\frac{\Delta(f(A), f(B))}{d(A, B)} \le M(\Delta/d)f(C) .$$

PROOF:

f is continuous since $d(x, y) \approx 0$ implies $\Delta(f(x), f(y)) \approx 0$. Thus we may apply the lemma as follows.

Pick $A_3, B_3 \in (A, B)$ satisfying the conditions of the lemma, next pick $A_4, B_4 \in (A_3, B_3)$ and proceed by induction.

Take the nonstandard extension of the sequences $(A_n)_{n \in N}$ and $(B_n)_{n \in N}$. Let ω be an infinite *-natural number and let

$C = st_d(A_\omega) = st_d(B_\omega)$, the standard point C such that $C \overset{d}{=} A_\omega \overset{d}{=} B_\omega$. (A segment is an isometric image of a real compact interval.)

Now we conclude by examining the following inequality.

$$\frac{\Delta(f(A), f(B))}{d(A, B)} \leq \frac{\Delta(f(A_\omega), f(B_\omega))}{d(A_\omega, B_\omega)}$$

$$\leq \frac{d(A_\omega, C)\left[\Delta(f(A_\omega), f(C)) + \delta\right]}{d(A_\omega, B_\omega)d(A_\omega, C)} \quad \frac{d(C, B_\omega)\left[\Delta(f(C), f(B_\omega)) + \epsilon\right]}{d(A_\omega, B_\omega)d(C, B_\omega)}$$

$$\leq M(\Delta/d)f(C) + \xi$$

where δ, ϵ and ξ are infinitesimal.

We use continuity to see that $\Delta(f(A_\omega), f(C)) \approx \Delta(f(A_\omega), f(B_\omega))$, that is, that δ and ϵ are infinitesimal. We use property (1) of the lemma and convexity to combine $d(A_\omega, C) + d(C, B_\omega) = d(A_\omega, B_\omega)$.

This completes the proof of the theorem.

REMARKS:

A non-convex example where the theorem fails is provided by $f(x) = x$ on the interval $[0, 1]$ where the chordal metric of stereographic projection is taken in the domain space and the spherical arc length is taken in the range. Then $M(s/\chi)f(x) = 1$ and $s(f(0), f(1))/\chi(0, 1) = \pi/2 \big/ 2/\sqrt{2} = \pi\sqrt{2}/4 > 1$.

As the following example shows, a minimal growth condition is not possible. $f(x) = |x|$ on $[-1, 1]$ has magnification 1 even at zero whereas $f(\pm 1) = 1$ so nowhere is the magnification below the divided difference.

Applications to meromorphic functions follow.

(2.3) Some examples of complete convex metric spaces are:

1. A Banach space with the metric $||x-y||$.
2. The unit disk with the hyperbolic metric.
3. The Riemann sphere with the great circle arc length metric (or any manifold with global geodesics).
4. The complex plane with zero removed and $|\log \frac{x}{y}|$ (principal value $-\pi < \arg(z) \leq \pi$) as a metric.

The following result is an application of Theorem (2.2). See section 3 for the definition of S-continuity and Robinson [19] for more details.

(2.4) COROLLARY:

If $f:(X, d) \to (Y, \Delta)$ is an internal map whose magnification exists and is finite at each point of the internal complete convex metric space (X, d), then f is uniformily S-continuous on X.

PROOF:

The set of bounds, $B = \{r \in {}^*R^+ : |Mf(x)| < r \text{ for every } x \in X\}$ is an internal set since it is described by an internal statement. Since B contains all infinite positive nonstandard reals it must contain a finite uniform bound for the magnification of f. Now apply (2.2).

While we are on the subject of uniform continuity we give a result (which we use in (6.1) below) that we hope gives the reader who is unfamiliar with nonstandard analysis an idea of the meaning of infinitesimals around a remote point. Sequences without limits in X, but which tend together, play a role analogous to infinitesimals around a remote point.

(2.5) THEOREM:

Let $f:(X,d) \to (Y,\Delta)$ be a continuous standard function. The following are equivalent:

(1) f is uniformly continuous on X.

(2) $^*f$ is S-continuous on the remote points of $^*X$, and hence everywhere on $^*X$.

(3) Every pair of sequences $(x_n : n \in N)$, $(y_n : n \in N)$ which satisfy $d(x_n, y_n) \to 0$ also satisfy $\Delta(f(x_n), f(y_n)) \to 0$. In particular when Y is compact, if f is not uniformly continuous there are sequences $(z_n : n \in N)$, $(w_n : n \in N)$ such that $d(z_n, w_n) \to 0$, $f(z_n) \to a$, and $f(w_n) \to b \neq a$.

PROOF:

The equivalence of (1) and (2) follows from a more general result of Robinson [19], but the fact that (1) implies (2) also follows automatically from interpreting the definition of uniform continuity in the nonstandard model and applying it at a remote point.

That (2) implies (3) follows easily from the nonstandard interpretation of $d(z_n, y_n) \to 0$, namely, that when $\omega \in {}^{\#}N$, $d(x_\omega, y_\omega) \approx 0$. By S—continuity $\Delta(f(x_\omega), f(y_\omega)) \approx 0$ and (3) follows.

We can conclude by showing that if f is not uniformly continuous (3) does not hold. There exists $\varepsilon > 0$ such that for every n there are points x_n and y_n with $d(x_n, y_n) < 1/n$ and $\Delta(f(x_n), f(y_n)) > \varepsilon$, by the negation of uniform continuity.

3. Continuous *-Meromorphic Functions:

In this section we give a basic lemma on S-continuity for internal or *-meromorphic functions. (The reader is referred to Appendix 3 for the notion of a *-transform.) Since the theorem is local in nature and since we wish to consider different metrics even on familiar regions like the punctured plane and the unit disk it seems best to think in terms of coordinate patches on a Riemann surface. If Mer (Ω) denotes the standard space of meromorphic functions on a standard Riemann surface Ω, then *Mer is a function defined on *-Riemann surfaces. No harm seems likely if we extend *Mer to external subsets of a surface by requiring that this means there is a *-region containing the set and a function *-meromorphic in that region.

Let Ω be a *-Riemann surface with a topologically compatible *-metric d. We will say $f \in$ *Mer (Ω) is S-continuous in d provided it is S-continuous in the sense of Robinson [19] as a map from (Ω, d) to $({}^*S, s)$, the *-Riemann sphere with the great circle metric. At $a \in \Omega$ this means that for every standard $\varepsilon > 0$ ($\varepsilon \in \hat{R}^+$) there exists a standard $\delta > 0$ such that $d(z, a) < \delta$ implies $s(f(z), f(a)) < \varepsilon$, or equivalently that $z \overset{d}{=} a$ implies $f(z) \overset{s}{=} f(a)$. Where the notation denotes the respective infinitesimal relations.

Now let (Ω, d) be a standard Riemann surface. Robinson [19] has shown that the near-standard *-meromorphic functions on ${}^*\Omega$ (with respect to uniform convergence on compact subsets) are those which are S-continuous on the near-standard points of ${}^*\Omega$, $ns({}^*\Omega)$. (This can also be shown by writing down the monad of the uniformity of compact convergence in $^*\mathrm{Mer}(\Omega) \times {}^*\mathrm{Mer}(\Omega)$, namely, $(f, g) \in \mu(u_K)$ if and only if $f(z) \overset{s}{=} g(z)$ for all $z \in ns(^*\Omega)$. We use the fact that Ω is locally compact. See section II (4.2) for more on monads of uniformities.) The standard part of a function can be taken pointwise (with respect to s) and $\frac{d}{dz}(st(f)) = st(\frac{df}{dz})$.

Now that we see the basic importance of S-continuity for *-meromorphic functions, we give the following local theorem. We state the result in terms of the plane metric, $p(x, y) = |x-y|$, of a coordinate disk. For this reason some care is necessary in applications at remote points or when several metrics are involved.

Let f be an internal meromorphic function defined on the monad of zero and hence in some finite disk of the complex plane. Continuity refers to the plane metric p as above. Magnification refers to M(s/p).

(3.1) THEOREM:

The following are equivalent:

(1) f is S-continuous at zero.

(2) There exist three values $\alpha, \beta, \gamma \in {}^*S$, finitely separated in the spherical metric, which f does not attain in the infinitesimals, o.

(3) The magnification of f, Mf(z), is finite on the infinitesimals, o.

PROOF:

The equivalence of (1) and (2) follows from a theorem of

Robinson [19, Theorem 6.3.1] and the following lemma:

The near standard transformations of the *-Möbius transformations are those specified by taking three finitely separated points on $^*S$ onto three other finitely separated points on $^*S$.

One way to see this is by first observing that the infinitesimal group is described by taking three finitely separated values to their respective standard parts on S. Direct computation: $\alpha, \beta, \gamma \in {}^*S$ and $\alpha, \gamma \in {}^*C$, then when $a \overset{s}{=} \alpha$, etc.

$$[w, a; b, c] = [z, \alpha; \beta, \gamma] \quad \text{implies}$$

$$\frac{w-a}{w-c} = \varkappa \frac{z-\alpha}{z-\gamma}, \quad \text{where} \quad \varkappa \approx 1 .$$

Then
$$w = [(a-\varkappa c)z + (\varkappa\alpha c - a\gamma)] / [(1-\varkappa)z + (\varkappa\alpha - \gamma)]$$

so that
$$w = \frac{(1+\delta)z+\epsilon}{\eta z+1}, \quad \text{where } \delta,\ \epsilon,\ \eta \text{ are infinitesimal.}$$

A transformation of this last type changes finite values only by an infinitesimal and leaves infinite values infinite so it is within an infinitesimal of the identity. Conversely, an infinitesimal transformation moves each point at most an s-infinitesimal. The pre-images of 0, 1, ∞ will therefore uniquely determine the transformation.

One may now apply Robinson's theorem to the mapping $g = w \circ f$ where w is the *-Möbius transformation taking α, β, γ of (2) into 0, 1, ∞.

Next we show that (1) implies (3). Assume $f(0) = 0$, otherwise work with $w \circ f$ where w is a *-Möbius rotation of the sphere taking f(0) to zero which is justified since $M(s/p)f \equiv M(s/p)w \circ f$. [E.g., $w(z) = z+f(0)/1-f(0)z$.] By continuity and the fact that f is defined on a standard disk around a, we know $|f|$ is smaller than 1 on a disk of standard positive radius, r, about a. Thus, integrating around $|z| = \frac{3}{4} r$,

$$|f'(b)| = |(1/2\pi i) \int (f(z)/(z-b)^2) dz| \le 1/(r/2)^2 , \quad \text{for } b \approx 0 ,$$

and $M(s/p)f(b)$ is finite on $\mathfrak{o}$ since $Mf(b) = |f'(b)|/(1+|f(b)|^2)$. For $z \approx b$,

$$Mf(b) \approx |f(z)-f(b)| \Big/ \left(|z-b| \sqrt{1+|f(z)|^2} \sqrt{1+|f(b)|^2} \right) .$$

The magnification $M(s/p)$ is usually denoted by ρ and is called the spherical derivative. This shows (1) implies (3).

We conclude by remarking that since the segment from 0 to b in $\mathfrak{o}$ is internal, the set

$$\{K : \rho f(z) < K , \quad z \text{ on the segment}\}$$

is internal and contains all infinite numbers, so $\rho f(z)$ is finitely bounded. By Theorem 2.2,

$$s(f(b), f(a)) \le K_0 |b-a| \approx 0 ,$$

so f is S-continuous and (3) implies (1).

We wish to deal with applications of this theorem at remote points of a given metric. In these cases it is necessary that the d-infinitesimals "look" like the monad of zero in $^*C$. The precise reformulation of (3.1) follows.

(3.2) We begin with an example of the difficulties one may encounter (which also shows why we view domains as Riemann surfaces). Take the right half-plane $\{z \in C : Re(z) > 0\}$ with the plane metric $|x-y|$ and consider the function $\exp(1/z)$ in the part of the monad of zero which lies in the right half plane. The function omits the entire unit disk and still fails to be S-continuous.

We take Ω a *-Riemann surface, d a *-metric on Ω and $b \in \Omega$. $^*U$ is the * unit disk, $\{z \in {}^*C : |z| < 1\}$.

(3.3) DEFINITION:

We say b is the center of an S-disk of Ω with respect to d provided there exists an internal conformal (1-1) mapping $\varphi: {}^*U \to \Omega$ satisfying:

(1) $\varphi(0) = b$,

(2) $\varphi(o) = o_d(b)$,

(3) $M(d/p)\varphi(z)$ exists, is finite and non-infinitesimal for z in the monad of zero.

(3.4) COROLLARY:

The conditions of Theorem (3.1) apply at the center $b \in \Omega$ of an S-disk with respect to d where S-continuity in (3.1.1) and the magnification in (3.1.3) are taken with respect to d.

PROOF:

$f(\varphi(z))$ is S-continuous at zero if and only if $f(w)$ is d-S-continuous at b and $M(s/p)f \circ \varphi(z) = M(d/p)\varphi(z) \cdot M(s/d)f(w)$. (Existence of the latter magnification is imposed by (3.3.3).)

4. Invariant Normal Families:

The purpose of this section is to unify several theories of the rate of growth of meromorphic functions whose definitions involve normal families constructed from given functions. These theories are discussed and applications of the basic theorem are given at the end of the section. We feel that for a number of applications S-continuity applied directly is simpler than introducing normal families and this is justified by Corollary (4.3).

We shall discuss a basic setting which is less general than is possible, but which encompasses the three classical cases given in the applications. One generalization is mentioned in (4.5). We feel that a number of results have been unnecessarily obscured in the classical theories and we feel that S-continuity can reveal their true simplicity.

(4.1) Our basic setting is as follows. Ω is a Riemann surface with a topologically compatible convex metric d. We assume that d is asymptotic to the plane metric of a coordinate disk D at its center $a \in D \subseteq \Omega$. In terms of magnification operators this means $M(d/p)id(a) = 1$, where id denotes the identity function and p the plane metric of D. W is a group of conformal d-isometries of Ω onto itself such that for every $b \in \Omega$ there exists a $w \in W$ with $w(a) = b$. As a result of this, d is a smooth metric, $M(d/d)w(z) = 1$, for every $w \in W$ and $z \in \Omega$, and the magnification of a meromorphic function $M(s/d)f(z)$ exists at each point of Ω (since d is asymptotic to p at a).

Let F be a subfamily of $\mathrm{Mer}(\Omega)$. We say F is W-*invariant* provided $F \circ W = \{f \circ w : f \in F \text{ and } w \in W\} = F$. When W consists of all conformal automorphisms of Ω, we say F is *conformally invariant.* If F is an arbitrary family, $F \circ W$ is invariant by the group property of W.

(4.2) THEOREM:

The following are equivalent for a W-*invariant family* F:

(1) F *is a normal family.*

(2) $M(s/d)f(z)$ *is a finite for every* $z \in {}^*\Omega$ *and every* $f \in {}^*F$.

(3) $M(s/d)f(z) < K$ (*a standard constant*) *for every* $z \in \Omega$ *and* $f \in F$.

(4) *Every* $f \in {}^*F$ *is* S-*continuous on all of* ${}^*\Omega$ *and hence uniformly* S-*continuous in the metric* d.

PROOF:

($1 \Longrightarrow 2$). Take $f \in {}^*F$ and $z \in {}^*\Omega$. Let $w(a) = z$, $w \in {}^*W$. $\rho g(a) \equiv M(s/p)g(a) = M(s/d)g(a)$ is finite for every $g \in {}^*F$ by Robinson's characterization of normal families [19, Theorem 6.4.1] and Theorem (3.1). $M(s/d)f(z) = [M(s/d)f(z)] \cdot [M(d/d)w(a)] = M(s/d)\, f \circ w(a)$, so (2) holds.

($2 \Longrightarrow 3$). Since the set of bounds of $Mf(z)$ where $f \in {}^*F$ and $z \in {}^*\Omega$ is internal and contains all the infinite positive numbers there is a standard bound K. Condition (3) holds since its $*$-transform holds (with this K) in the nonstandard model.

($3 \Longrightarrow 4$). Apply Corollary (2.4).

(4⟹1). S-continuity on the near standard points implies that F is a normal family by Robinson [19, Theorem 6.4.1]. (The topological compatibility of d enters as $st(x) = st(y)$ if and only if $d(x, y) \approx 0$, for $x, y \in ns({}^*\Omega)$.)

(4.3) COROLLARY:

When $F = \{f_1, \ldots, f_k\} \circ W$ the conditions of the theorem are equivalent to (standard) uniform continuity of the f_k on all of Ω.

In the setting of (4.8) this is apparently due to Lappan [12].

(4.4) REMARK:

Condition (3) is a generalization of results of Yosida, Noshiro and Lehto-Virtanen.

(4.5) One generalization of this idea is to take W to be conformal mappings defined only on D and such that for every $b \in \Omega$ there is a $w \in W$ with $w(a) = b$. M. F. Behrens has obtained some results in this case where Ω is a disk with holes removed. W consists of dilation of a fixed disk translated within Ω. A function f for which $f \circ W$ is normal is termed regular and our condition (3) states that $\text{dist}(z, b\,\Omega)\,\rho f(z)$ is bounded.

Examples:

(4.6) Julia exceptional functions. Take $\Omega = C \setminus \{0\}$, the punctured plane, $W = \{bz : b \in \Omega\}$, $d(x, y) = \left|\log \frac{x}{y}\right|$ where $-\pi < \arg \leq \pi$ and $a = 1$. Ostrowski [18] discovered that Julia's theorem fails for meromorphic functions, e.g., take $f(z) = \Pi \frac{z-2^n}{z+2^n}$, then $f \circ W$ is normal.

Marty [15] characterized Julia exceptional functions as those for which $|z|\,\rho f|z|$ is bounded. (ρ denotes the spherical derivative, $M(s/p)$, see the proof of (3.1).) This is condition (3) since $M(s/d)f(z) = M(p/d)w(1)\,\rho f(z) = |z|\,\rho f(z)$ where $w(1) = z$. This is because when $x \approx 1$, $M(p/d)w(1) \approx \frac{|z|\;|x-1|}{|\log x|} \approx |z|$, and the far sides are standard numbers, hence equal.

(4.7) Yosida's theory [20]. Take $\Omega = C$, the complex plane, $W = \{z+b : b \in \Omega\}$, $d(x,y) = |x-y|$ and $a = 0$. (Doubly periodic functions arise in this context with Ω as the universal covering surface of the torus.) Yosida obtained condition (3) as necessary and sufficient for normality of $f \circ W$ in the form $\rho f(z) < K$. (We showed in the proof of (3.1) that $M(s/d)f(z) = \rho f(z) = |f'(z)| / (1 + |f(z)|^2)$.)

Yosida also observed that results similar to Julia-Milloux's theorem hold in case normality fails. His results follow from our work in section 5 below.

Yosida also connects this growth requirement with the Nevanlinna characteristic by integrating $\rho f(z)$ in the Ahlfors-Shimizu formula.

(4.8) Normal meromorphic functions: (Noshiro [16], Lehto-Virtanen [14].) Let $\Omega = U = \{z \in C : |z| < 1\}$, the unit disk, $W = \{\exp(i\theta)[(z-a)/(\bar{\alpha}z-1)] : \theta \in R, |\alpha| < 1\}$, all conformal automorphisms of U, $a = 0$ and $d(x,y) = \eta(x,y)$, the hyperbolic metric $(= (1/2)\log[(|x-y| + |\bar{y}x-1|)/(|x-y| - |\bar{y}z-1|)])$. A meromorphic function is called normal if $f \circ W$ is a normal family.

When G is a hyperbolic surface so U is its universal covering surface, a meromorphic function on G is normal provided $\hat{f} = f \circ P$ is normal on U where P is the projection of U onto G. This definition is extrinsic to G but conditions (2), (3) and (4) apply directly by projecting the metric. The S-continuity approach on G could be applied directly.

Noshiro [16] gave condition (3) in the form $(1-z\bar{z})\rho f(z) < K$. We know $M(s/\eta)f(z) = M(s/\eta)f \circ w(0) = [M(p/\eta)w(0)]\rho f(z) = (1-z\bar{z})\rho f(z)$, where $w(0) = z$. This is because $M(p/\eta)w(0) \approx [|((x+z)/(\bar{z}x-1)) - z| / \eta(x,0)] = [(x+z-z(\bar{z}x-1))/(\bar{z}x-1)\eta(x,0)] \approx (1-z\bar{z})(|x|/\eta(x,0)) \approx (1-z\bar{z})$, where $x \approx 0$. Since the far sides are standard and within an infinitesimal they are equal. (The hyperbolic metric is asymptotic to the plane metric at zero, so $(|x|/\eta(x,0)) \approx 1$.)

Applications of a classical nature to the theory of normal functions are possible and the author has given a number of simplified proofs of known results using the nonstandard theory of the metric space (U, η). We give a few examples of this nature which involve a minimum of function theory.

If $\alpha \in {}^*U$, the η-galaxy around α is the set of points a finite distance from α, $G(\alpha) = \{z \in {}^*U : \eta(z,\alpha) \in O\}$.

We begin with an observation of Noshiro [16] which the reader can easily generalize to other settings (see Yosida [20]).

(4.9) If $\varphi(t)$, $0 \le t < 1$, is a continuous curve in U with $|\varphi(t)| \to 1$, and if the normal function f satisfies $f(\varphi(t)) \to b$ as $t \to 1$, then $f \circ W$ has the constant function b as a limit.

Let $a \approx 1$ and $\alpha = \varphi(a)$ be fixed. Take $w(z) = [(z+\alpha)/(\bar{\alpha}z-1)]$. The function $f \circ w$ is S-continuous on $ns(^*U)$ by (4.2), hence $st(f \circ w)$ is meromorphic. Since $f(\varphi(t)) \to b$, whenever $t \approx 1$ we have $f(\varphi(t)) \overset{S}{=} b$, therefore $st(f \circ w) = b$ on the points which map onto the points $\varphi(t)$ which lie in $G(\alpha)$. This means $st(f \circ w)$ is constant on a set with a (non-trivial) adherent point, hence identically constant. Also, $f \circ w \approx b$ on all of $ns(^*U) = G(0)$ and f is near constant on $G(\alpha)$. In fact we see that a necessary and sufficient condition that $f \circ W$ has a constant limit is that f is near constant on a galaxy of $^*U$.

(4.10) Nonstandardizing the work of Hoffman [10], M. F. Behrens [2] has shown that by identifying infinitesimally near-by points of infinite galaxies which contain points of interpolating sequences the galaxies correspond to non-trivial Gleason parts of $H^{\infty}(U)$. Moreover, the pseudo-hyperbolic metric $|(x-y)/(\bar{y}x-1)|$ is infinitesimally close to the parts metric. Now since a normal function is S-continuous on the galaxies, identifying points infinitesimally near-by and taking the standard part of the function gives us a standard function, continuous in the parts metric, defined on the Gleason part. This proves a recent result of Brown-Gauthier [3] that normal functions can be extended to non-trivial parts.

Many other applications could be given, but we refer the reader to the forthcoming monograph of A. J. Lohwater for more on normal functions. The bibliographies of Noshiro [17] and Collingwood-Lohwater [4] contain many other interesting references to normal functions.

5. Milloux Circles and Points of Discontinuity:

Milloux's theory of 'cercles de remplissage' has seen recent interest with generalization in several directions (Lehto [13], Lange [11], Gavrilov [8], [9] and Gauthier [6], [7]). Robinson [19] "nonstandardized" the classical theory obtaining a new lemma for the existence of such circles. Robinson's lemma does not generalize directly the meromorphic functions because the *-sphere has only one S-component. The main idea of replacing sequences of circles with the monad of a discontinuity does extend to a very general setting as we show in this section.

Several of the known results in the various settings reduce to the equivalence of (3.1.2) and (3.1.3) at a discontinuity for an appropriately chosen metric. Our method simplifies the previous approaches to the theory and we hope also shows how nonstandard analysis can be useful when complicated quantification arises. This approach also shows that "Julia-sets" and "Milloux-sets" amount to the same thing since they both reduce to a discontinuity.

In a metric space (X, d) we shall use the notation

$$D_d(B; \epsilon) = \{x \in X : \exists b \in B \quad \text{and} \quad d(x,b) < \epsilon\}$$

where $B \subseteq X$. We also use the *-transform in ${}^*X$. Also

$$o_d(B) = \{x \in {}^*X : \exists b \in B \quad \text{and} \quad d(x,b) \approx 0\} ,$$

shall denote the infinitesimal neighborhood of B.

(5.1) DEFINITIONS:

We say $A \subseteq \Omega$ is a d-*Julia-set for* f if for every (standard) positive ϵ,

$$f(D_d(A;\epsilon)) \supseteq S \setminus \{\alpha, \beta\} ,$$

for two values $\alpha, \beta \in S$, the sphere. (J-set.)

We say A is a d-*Milloux-set for* f if for every positive r, δ, ϵ, there exist $\xi_1, \xi_2 \in S$ and $y \in A$ such that $d(a, y) > r$ and

$$f(D_d(y;\delta)) \supseteq S \setminus D_s(\{\xi_1, \xi_2\};\epsilon) .$$

(M-set.)

The connection between discontinuities and standard J- and M-sets is as follows.

(5.2) THEOREM:

Let b *be the center of an* S-*disk on* Ω *with respect to* d. *If* b *is an* S-*discontinuity in the metric* d *for a standard meromorphic function* f *defined on* Ω *and if* $b \in o_d({}^*A)$, *then* A *is a* J-*set*. *If* b *is in an infinite galaxy, then* A *is an* M-*set*.

PROOF:

We may apply (3.4) at the discontinuity to see that

$$f(o_d(b)) \supseteq {}^*S \setminus o_s(\{\alpha, \beta\})$$

for at most two standard $\alpha, \beta \in S$.

We know that the set of standard points in ${}^*f({}^*[D_d(A;\epsilon)])$ is $f(D_d(A;\epsilon))$. When ϵ is standard $o_d(b) \subseteq {}^*D_d(A;\epsilon) = D_d({}^*A;\epsilon)$ and therefore A is a J-set.

If b is infinite the standard set B given below satisfies $^*B \supseteq O^+ \times (^*R \setminus o) \times (^*R \setminus o)$, that is the first component can be any finite positive real number and the next two any non-infinitesimal. Then $°(^*B) = B = R^+ \times R^+ \times R^+$ and A is an M-set.

$$^*B = \{(r, \delta, \epsilon) : r, \delta, \epsilon > 0 \quad \text{and} \quad \exists\, \xi_1, \xi_2 \in {}^*S \quad \text{and} \quad \exists\, y \in {}^*A$$

$$\text{s.t. } d(y, a) > r \quad \text{and} \quad f(^*D_d(y;\delta)) \supseteq {}^*S \setminus D_s(\xi_1, \xi_2; \epsilon)\}$$

(If r is finite and δ and ϵ non-infinitesimal we take $y \in o_d(b) \cap {}^*A$ and apply the reasoning in the first part of the proof.)

(5.3) REMARKS:

When we begin with a given remote S-discontinuity of a standard function we may obtain a standard sequence which is a J-set or M-set by applying (2.5). (This avoids the somewhat more delicate problem of approximation of a particular point by a standard sequence.) When $\omega \in {}^\#N$, x_ω of the sequence in (2.5.3) is a (remote) S-discontinuity.

If d is finite exactly on $ns(^*\Omega)$, as is the case in (4.6), (4.7) and (4.8) for example, then J-sets and M-sets coincide for standard functions since they are necessarily continuous on $ns(^*\Omega)$.

J-sets and M-sets have infinite discontinuities in their non-standard extensions, so their non-existence in the examples (4.6), (4.7) and (4.8) is equivalent to normality.

Next we extend a result of Marty-Lehto to this setting. If $A \subseteq \Omega$, let

$$\limsup_{z \in A} h(z) = \inf[\sup(h(z) : z \in A \quad \text{and} \quad d(z,a) > r) : r > 0],$$

for real valued functions $h(z)$, where $a \in \Omega$ is fixed.

We shall also assume from now on that if $d(z,a) > r_0$ $(z \in {}^*\Omega)$ for some fixed finite r_0, then z is the center of an S-disk.

(5.4) THEOREM:

If $\limsup_{z \in A} M(s/d)f(z) = \infty$, then ${}^*A$ contains an infinite S-discontinuity of f, or A is an M-set.

PROOF:

The *-transform of lim sup says there is an infinite point $z \in {}^*A$ for which $M(s/d)f(z)$ is infinite since we have assumed $\rho_d(z)$ is an infinitesimal disk we may apply (3.4) which says then that z is a point of discontinuity.

As applications we now consider some of the known results.

(5.5) Julia-Milloux theorem for meromorphic functions: Apply (5.2) to the setting of example (4.6). Observe that since $x \overset{d}{=} y$ in that metric if and only if $|x-y| < \delta|y|$ for some infinitesimal δ, we may substitute the J- and M-set conditions for standard disks $|x-y| < \delta|y|$ which is the classical form. Thus we have the classical result that if $\limsup |z_n| \rho f(z_n) = \infty$ then the sequence (z_n) is an M-set for f.

We have already remarked that $|z| \rho f(z)$ can fail to be infinite in (4.6).

(5.6) Gavrilov's classes W_p $(p \geq 1)$ ([8], [9]): Ω is a punctured disk around ∞. The metric is given locally by the differential form $|dz|/|z|^{2-p}$, $p \geq 1$. $M(s/d)f(z) = |z|^{2-p}\, \rho f(z)$. W_p is the class of uniformly continuous meromorphic functions, which was introduced by the requirement that lim sup $|z|^{2-p}\, \rho f(z) < \infty$. By examining infinitesimals, d-disks may be replaced by $|x-y| < \varepsilon |y|^{2-p}$. The theorem of Gavrilov which follows is immediate.

If $|z_n|^{2-p}\, \rho f(z_n) = \infty$, for a holomorphic function f defined in a neighborhood of ∞, then for every $r > 0$ and $\varepsilon > 0$ there exists a point z_n such that in the disk $|z-z_n| < \varepsilon |z_n|^{2-p}$, f takes on every value in the circle $|w| < r$ with the exception of a set of diameter less than $2/r$.

(5.7) Lehto and Virtanen [14] proved that no meromorphic function can be normal (in the sense of example (4.8)) in the neighborhood of an isolated essential singularity. We put the hyperbolic metric on a disk punctured at ∞ and find $M(s/\eta)f(z) = |z| \log|z|\, \rho f(z)$ must be infinite near ∞. Hence there are the corresponding J- and M-sets in the hyperbolic metric around any sequence on which lim sup $M(s/\eta)f(z) = \infty$. We contrast this with the case of (5.5) where functions can fail to have these sets (see (4.6) above).

(5.8) Functions in the unit disk: The study of M-sets (in the context of (4.8)) for the unit disk originated in Lange [11] and has had many contributors. We give one very simple example (which follows from more refined work of Bagemihl and Seidel [1]; also see Collingwood-

Lohwater's [4] bibliography) to indicate the 'flavor' of the theory.

Let Γ and Λ be boundary paths in U and suppose $f \to \alpha$ along Γ. If $M(s/\eta)f$ is finite on a finite neighborhood of ${}^*\Gamma$, ${}^*D_\eta(\Gamma;\epsilon)$, $(\epsilon \in \hat{R}^+)$, then f is near α on any sub-neighborhood of an infinite point of ${}^*\Gamma$. From this we see that if Λ lies in such a neighborhood of Γ, either $f \to \alpha$ along Λ or ${}^*D_\eta(\Gamma;\epsilon)$ is an M-set. Hence, _if_ Γ _and_ Λ _are finitely separated boundary curves and_ $f \to \alpha$ _on_ Γ, _then either_ $f \to \alpha$ _on_ Λ _or every mutual neighborhood of the curves is an_ M-_set. In particular, if_ f _is normal_, $f \to \alpha$ along Λ.

6. A Note on Two Cluster Set Theorems of Gauthier:

In Gauthier [5] a standard version of the following definition is given. U is the unit disk with η the hyperbolic metric. I is the set of η-infinite points of ${}^*U$, (or the set $\{z : |z| \approx 1\}$). Two standard sets A_1 and $A_2 \subseteq U$ are equivalent if $o_\eta(I \cap {}^*A_1) = o_\eta(I \cap {}^*A_2)$.

We have immediately:

(6.1) THEOREM:

Let $f: U \to S$ _be a continuous function. Then_ f _is uniformly continuous on_ U _if and only if for every pair of equivalent subsets of_ U, $A_1 \sim A_2$, _the cluster sets_ $C(f;A_1)$ _and_ $C(f;A_2)$ _are equal._

The theorem is strictly standard, the proof is nonstandard.

PROOF:

($\Longrightarrow$): The nonstandard characterization of the cluster set is:

$C(f;A_1) = st_s(f(I \cap {}^*A_1)) = st_s(f(o_\eta(I \cap {}^*A_1)) = st_s(f(o_\eta(I \cap {}^*A_2))$ $= C(f;A_2)$. The step $stf(I \cap {}^*A_j) = st(f(o(I \cap {}^*A_j)$ $(j = 1, 2)$ requires uniform continuity.

($\Longleftarrow$): If f is not uniformly continuous there exist sequences $(x_n)_{n \in N}$ and $(y_n)_{n \in N}$ with $\eta(x_n, y_n) \to 0$ and $\eta(0, x_n) \to \infty$, $\eta(0, y_n) \to \infty$ while $f(x_n) \to \alpha \neq \beta \leftarrow f(y_n)$. In this case $\{x_n : n \in N\} \sim \{y_n\}$, but $C(f;\{x_n\}) \neq C(f;\{y_n\})$.

(6.2) A corollary which contains Gauthier's Theorem 2 is: <u>A meromorphic function</u> f <u>is normal on</u> U <u>if and only if</u> $C(f;A_1) = C(f;A_2)$ <u>for every pair of equivalent subsets of</u> U.

In order to establish his results Gauthier introduced the following cluster set in standard terms:

$$\hat{C}(f;A) = st_s\ f(I \cap o_\eta({}^*A)) \quad .$$

Of course, if $A_1 \sim A_2$ we have $\hat{C}(f;A_1) = \hat{C}(f;A_2)$. Now if $f : U \to S$ is S-continuous on ${}^*A$ we have $\hat{C}(f;A) = C(f;A)$ and in light of (5.1) above, when f is meromorphic, either A is an M-set or $\hat{C}(f;A) = C(f;A)$. This sharpens his Corollary 1 and Theorem 1.

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II. BOUNDED ANALYTIC FUNCTIONS AS THE DUAL OF A BANACH SPACE

1. Introduction:

In this section we study a pre-dual for the Banach space of bounded analytic functions on a region which was introduced by Rubel and Shields [7]. Further introductory details are contained in section 2. We also recommend the expository paper of Rubel [5].

In section 3 we give a characterization of the two spaces in terms of the nonstandard hulls of internal spaces of an approximating region. This allows us to approximate arbitrary regions by internally finitely connected ones for the study of the space of analytic functions and leads to a representation of the pre-dual as a quotient space of the internal space L^1 of the boundary.

In section 4 we introduce the topologies of the dual pair and rephrase a number of results in terms of infinitesimal relationships in the nonstandard extension. We also give a new characterization of the strict topology.

In section 5 we give a new proof of a theorem of Rubel and Ryff [6] that the strict and Mackey topologies are noncoincident. We feel that this approach together with the approximations of section 3 may lead to a solution of Rubel's conjecture [5] that these topologies are always noncoincident when a region supports nonconstant bounded analytic functions. This will be taken up in another place.

2. Introduction to H(D) and its Pre-dual P(D):

In what follows D will denote an open connected subset of the Riemann sphere S (i. e., D is a domain or region.) We will assume that D supports non-constant bounded analytic functions. The space $H^{\infty}(D)$ of all bounded holomorphic functions on D with the norm $||h|| = \sup(|h(z)| : z \in D)$ is thus an infinite dimensional Banach space over the complex numbers, C. (For short we denote $H^{\infty}(D)$ by H(D).)

We wish to study the pre-dual of H(D) introduced by Rubel and Shields [7] from the point of view of nonstandard analysis. This leads to a new characterization of the pre-dual when D is infinitely connected though our opening remarks apply to the general case. The basic idea is to replace D by a region bounded by a *-finite number of smooth curves infinitesimally close to the boundary of D. We represent the pre-dual in terms of boundary integrals on the approximating region.

A brief explanation of the *-transform of a concept is given in Appendix 3. We will use this idea freely to extend our vocabulary to a nonstandard model.

A smooth curve is a non-degenerate C^1-curve, meaning that if γ is a parametrization, γ' is a non-zero continuous function from an interval of positive length. A *-smooth curve is the corresponding internal notion.

(2.1) We shall say a *-region G approximates D provided that G is *-finitely connected, has a *-smooth boundary, and satisfies $ns({}^*D) \subseteq G \subseteq {}^*D$. ($ns({}^*D)$ is the set of near-standard points of ${}^*D$).
It is always possible to approximate a standard region in this sense,

since, for example, we may take an infinitely subscripted member of the nonstandard extension of an exhaustion of D. For these preliminaries, further assumptions will not be made about G, though we expect judicious choices to yield more information about the standard spaces.

We denote the internal space of all internal bounded holomorphic functions on G by H(G) and similarily ${}^*(H(D))$ by $H({}^*D)$. This is consistent with the usual notation if we view H as a function defined on regions. Since ${}^*D \supseteq G$, functions in $H({}^*D)$ are mapped into H(G) by restriction and the standard functions (elements of $\widehat{H(D)}$) suffer at most an infinitesimal reduction in norm.

Complex Borel measures with support in D, denoted M(D), are in duality with H(D) by the pairing

$$\langle \mu, h \rangle = \int_D h(z)\, d\mu(z) \quad .$$

The measures which annihilate H(D) are denoted by N(D) = $\{\mu \in M(D) : \langle \mu, h \rangle = 0 \text{ for all } h \in H(D)\}$. M(D) with total variation $|\mu|(D)$ as norm is a Banach space and N(D) is a closed linear subspace. Rubel and Shields [7] have shown that the continuous dual of M(D)/N(D) is H(D). Their proof consists of showing that an arbitrary measure has a representative which is absolutely continuous with respect to two-dimensional Lebesgue measure and of decreased total variation. As a consequence $L^1(D)/N^1(D)$ is also isomorphic to M(D)/N(D) as a pre-dual for H(D), where $L^1(D)$ is the Lebesgue L^1-space and $N^1(D)$ is the set of null functions for H(D),

$\{f \in L^1(D) : \int_D f(z)h(z)dm(z) = 0$ for all $h \in H(D)\}$. This also shows that the pre-dual is separable.

Rubel and Shields [7] also show, in the case of the unit disk, that measures can be swept to the boundary. Their argument is extended to finitely connected regions, G, with smooth boundary, bG, in Appendix 2. In this case we have the Banach-isomorphisms

$$L^1(bG)/N^1(bG) = L^1(G)/N^1(G) = M(G)/N(G) .$$

We shall refer to the pre-dual as P(G) in any of these three roles when G is finitely connected and has smooth boundary, and P(D) for arbitrary regions in the latter two roles. As a function defined on regions P extends to $*$-regions in the nonstandard model via the $*$-transform of its standard characterizations. As a result when G approximates $^*D$ we have the above three internal characterizations of P(G) as either $*$-Borel measures in G, or $*$-L^1-functions in G, or $*$-L^1-functions on bG. The last of these has no standard analog in D, generally speaking, and allows us to apply the Cauchy formula, etc. to these situations. We connect the internal spaces P(G) and H(G) with the standard spaces P(D) and H(D) by means of a nonstandard hull of the internal spaces. The general construction and the fact that the construction is consistent with the duality is contained in Appendix 1. The next section contains some basic results in the specific case.

3. Embedding $\langle P(D), H(D) \rangle$ in the Nonstandard Hulls $\langle P_0(G), H_0(G) \rangle$:

We form the nonstandard hulls of P(G) and H(G), which we denote by $P_0(G)$ and $H_0(G)$, respectively. These are standard Banach spaces and the dual of $P_0(G)$ contains $H_0(G)$ by the general results of Appendix 1.

(3.1) THEOREM:

The standard space H(D) is properly norm embedded in $H_0(G)$.

PROOF:

The standard space H(D) is norm embedded in $H_0(G)$, since a standard function, extended via * to *D, is only infinitesimally reduced in norm by restriction to G.

An example of a function in $H_0(G)$ which does not correspond to any standard function is z^λ, where λ is an infinite natural number and $G = {}^*D = {}^*$-unit disk. The function z^λ has norm one, but pointwise standard part zero.

Since we have chosen G with an internal non-degenerate smooth boundary, every $H_0(G)$ will contain such functions (even if H(D) = C). To see this we only need to consider the Riemann mapping of the inside of one boundary component of G onto *U. Specifically, let $a \in ns({}^*D)$ and $\rho: \tilde{G} \to {}^*U$ be a conformal mapping of the inside of one boundary component of G, $\tilde{G} \supseteq ns({}^*D)$, onto *U such that $\rho(a) = 0$. The function $g(z) = \rho^\lambda(z)$ has all its derivatives of order less than λ equal to zero at a. Since g is S-continuous, being bounded, and since derivative commutes with standard part, $g \approx 0$ on $ns({}^*D)$. (Cf. Robinson [4].) We also know that $|g(\xi)| = 1$ on $b\tilde{G}$, so $||g|| = 1$. This proves (3.1).

Our next result shows that remote elements of finite norm, such as those above, play a role in determining the standard elements of the pre-dual $P_0(G)$. (A function which is non-near-standard in a certain topology is termed remote.)

(3.2) THEOREM:

The standard space $P(D)$ is norm-embedded in the nonstandard hull $P_0(G)$ of the approximating region G as the classes of those norm-finite elements $p \in P(G)$, $||p|| \in O$, which satisfy: $\langle p, h\rangle \approx 0$ for all norm-finite $h \in H(G)$, $||h|| \in O$, such that $st(h) = 0$.

PROOF:

We first show that the natural embedding of $P(D)$ into $P_0(G)$ induced by restriction of standard measures has the desired properties.

Let $\mu \in \widehat{M(D)}$, $(\mu|G)(B) = \mu(B \cap G)$ denotes the restriction which is in $M(G)$. Since D is σ-compact, $|\mu|$ is a regular Borel measure and consequently $|\mu|G|(G) = |\mu|(G) \approx |\mu|(D)$, because $G \supseteq ns(^*D)$ and $ns(^*D)$ is the union monad of the compact subsets of D.

Now let $\epsilon \in \hat{R}^+$ be a positive standard number. There exists K, a standard compact subset of D, for which

$$|\mu|(K) + \frac{\epsilon}{2} > |\mu|(D) \quad .$$

Now

$$\left|\int_G h d\mu\right| \leq ||h|| \int_{G\setminus^*K} d|\mu| + \int_{^*K} |h| d|\mu|$$

$$\leq \frac{\epsilon}{2} + |\mu|(D) \sup(|h(z)| : z \in {}^*K)$$

when $||h||$ is finite. If st (h) $\equiv 0$, the second term is infinitesimal, making $|\langle \mu | G, h \rangle| < \epsilon$ for any standard $\epsilon \in \hat{R}^+$.

The fact that the norm is reduced at most an infinitesimal can be seen as follows using the fact that $\widehat{M(D)}$ is norm-embedded in hull $(M(^*D), 0; \{||\mu - \nu||\})$.

As a measure in $M(^*D)$, $\mu|G$ is norm-infinitesimally-near μ, since $|\mu - (\mu|G)|(^*D) \approx 0$. If ν is equivalent to $\mu|G$ for $H(G)$ as an element of $M(G)$, by extending it to be zero off G we have the equivalence of ν and $\mu|G$ in $M(^*D)$ for $H(^*D)$. Taking nonstandard hulls this simply means $||[\mu|G]||$ in $P_0(G)$ equals $||[\mu]||$ in $P(D)$, because $||\mu + \lambda|| \approx ||\mu|G + \lambda||$ for all $\lambda \in N(^*D)$.

Now we focus our attention on the converse: if $p \in P(G)$ with finite norm and satisfying the property of the theorem, then p corresponds to a standard pre-dual element.

Fix $\epsilon \in \hat{R}^+$, a standard positive real number. Consider the internal family $\mathfrak{F}$ of subsets $F \subseteq M(^*D)$ defined by the internal statement: "$F \in \mathfrak{F}$ provided F is a $*$-finite subset of $M(^*D)$ and if for each $f \in F$ we have $|\langle f, h\rangle| \leq 1$ and $||h|| \leq 1$, then $|\langle p, h\rangle| < \epsilon$." Every $*$-finite set F for which ${}^\circ F \supseteq M(D)$, or $F \supseteq \widehat{M(D)}$, is in $\mathfrak{F}$ by the hypothesis of the theorem. To see this observe that $r\delta_x \in \widehat{M(D)}$ for each $r \in \hat{R}$ and δ_x equal to unit point mass at a standard point $x \in \hat{D}$, whence $|\langle f, h\rangle| \leq 1$ for all $f \in F$ only if h is infinitesimal on $\hat{D}$. Since $||h|| \in O$ implies h is S-continuous on $ns(^*D)$ we have that h is infinitesimal on $ns(^*D)$ or $st(h) \equiv 0$.

A result of Luxemburg [2, Theorem 2.7.11] states that in sufficiently saturated models (see Appendix 3) $\mathfrak{F}$ must contain a standard set whenever $\mathfrak{F}$ is internal and contains all *-finite sets which contain $\widehat{M(D)}$. This means there are finitely many pre-dual elements $p_1, \ldots, p_n \in \widehat{P(D)}$ such that whenever $|\langle p_i, h\rangle| \leq 1$ for $i = 1, 2, \ldots, n$ and $||h|| \leq 1$ then $|\langle p, h\rangle| < \epsilon$. For convenience we assume the p_i are linearly independent.

The remainder of the argument follows Luxemburg [2, p. 84, part (c) $\Rightarrow$ (a)].

On the internal space $K \subseteq H(G)$ given by

$$K = \{k: \langle p_j, k\rangle = 0;\ j = 1, \ldots, n\}$$

we have that the norm of the functional $\langle p, \cdot\rangle$ is less than ϵ. By the *-transform of the Hahn-Banach-Bohnenblust-Sobczyk theorem we may extend this to a functional (in $H(G)'$) $\langle \varphi, \cdot\rangle$ on all of $H(G)$ with norm less than ϵ. For $k \in K$, $\langle p-\varphi, k\rangle = 0$.

A simple induction argument shows that

$$p-\varphi = \sum_{i=1}^{n} c_i p_i \ .$$

Thus,

$$||p-\sum c_i p_i|| = ||\varphi|| < \epsilon \ .$$

Another induction argument shows that c_i are finite since $||p||$ and $||p_i||$ are finite $(i = 1, \ldots, n)$. Let $q = \sum st(c_i)p_i$, then $||p-q|| < \epsilon$ and p is norm-near a standard element of $P(D)$ since we may approximate it to within a standard ϵ by a standard q.

(3.3) COROLLARY:

$f \in L^1(bG)$ <u>with finite norm corresponds to a standard pre-dual element provided that whenever</u> $||h|| \in O$ <u>and</u> $st(h) \equiv 0$, <u>then</u> $\int_{bG} f(z)h(z)|dz| \approx 0$.

There is a connection between this theorem and the completeness theorem of Grothendieck. Also, in case G is standard, the result specializes to a consequence of Luxemburg [2, Theorem 3.17.2] by the characterization of the weak star infinitesimals given in the next section.

4. <u>The Weak-Star, Strict and Mackey Topologies:</u>

In this section we introduce the topologies of interest to us in addition to the norm topology for H(D). The Mackey topology also involves the weak topology on P(D). Much of the work of Rubel and Shields [7] and Rubel and Ryff [6] involves the study of these topologies and their relation to a number of classical problems in function theory. A survey of the standard theory can be found in Rubel [5].

We shall give a nonstandard account of various known results mixed with a few new results. Our point of view is that of a uniform space in the sense of Bourbaki [1]. Luxemburg [2] has given the basic nonstandard treatment. If (X, u) is a uniform space with uniformity u, then the intersection monad of u, $\mu(u) = \cap\hat{u} = \cap({}^*U : U \in u)$, is an external equivalence relation. Conversely, if μ_0 is a monad in ${}^*X \times {}^*X$ and an equivalence relation, it determines a uniformity for X. We shall write $x \overset{u}{=} y$ for x is within a u-infinitesimal of y which means $(x, y) \in \mu(u)$. Also, the set of u-infinitesimals around x, $\mu(u)[x] = \{y \in {}^*X : (x, y) \in \mu(u)\} = o_u(x)$. We caution the reader that $o_u(x)$ is not necessarily a monad, which by definition must be the intersection of the standard sets in a standard family of sets, $\cap \hat{\mathfrak{F}}$.

If X is a vector space, u is a compatible uniformity if and only if $o_u(x) + o_u(y) \subseteq o_u(x+y)$ for every $x, y \in \hat{X}$ and $o(\lambda)\, o_u(x) \subseteq o_u(\lambda x)$ for every $\lambda \in \hat{C}$ and $x \in \hat{X}$. We have in fact: $o_u(x) + o_u(y) = o_u(x+y)$ and $\lambda o_u(x) = o_u(\lambda x)$ for every $x, y \in {}^*X$ and finite λ. The uniformity is Hausdorff if no two standard elements are within an infinitesimal and locally convex if $o_u(x)$ is (externally) convex.

If the pseudo-metrics $(\gamma : \gamma \in \Gamma)$ characterize u in X, then $x \overset{u}{=} y$ if and only if $\gamma(x, y) \approx 0$ for every $\gamma \in \hat{\Gamma}$. The following characterization of weak-star infinitesimals follows from this statement.

(4.1) THEOREM:

<u>In</u> $H({}^*D)$, h <u>is within a weak-star infinitesimal of</u> k, $h \overset{\sigma}{=} k$, <u>if and only if</u> $\langle \mu, h \rangle \approx \langle \mu, k \rangle$ <u>for every standard</u> $\mu \in \widehat{M(D)}$.

Since the point masses are in $\widehat{M(D)}$ for standard points of D, if $h \overset{\sigma}{=} k$, then $h(z) \approx k(z)$ for $z \in \hat{D}$. If in addition $||h-k||$ is finite, then $h(z) \approx k(z)$ for $z \in ns({}^*D)$ by Robinson's result that finitely bounded functions are S-continuous.

(4.2) THEOREM:

<u>There are S-discontinuous</u> σ-<u>infinitesimals</u>.

PROOF:

We construct an S-discontinuous σ-infinitesimal as follows. Let P_1 be a *-finite set which contains all the standard pre-dual elements $\widehat{P(D)} \subseteq P_1 \subseteq P({}^*D)$. Let q be a near standard point for which $q \neq st(q)$. If δ_q denotes the point mass at q we see that $[\delta_q] \notin \widehat{P(D)}$, since there are standard functions which are one-to-one at $st(q)$. Now a function $h \in H({}^*D)$ such that $\langle p_1, h\rangle = 0$ for every $p_1 \in P_1$ and $\langle [\delta_q], h\rangle = 1$ is a σ-infinitesimal, $h \overset{\sigma}{=} 0$, and S-discontinuous at $st(q)$.

Let $\mathfrak{F}$ be a family of complex valued functions defined on a set X. Let Σ be a collection of subsets of X.

(4.3) THEOREM:

The uniformity of uniform convergence on the sets of Σ is characterized by its infinitesimal relation as follows: $f \overset{\Sigma}{=} q$ if and only if $f(s) \approx g(s)$ for all $s \in \nu(\Sigma) = U\hat{\Sigma}$, the union monad of Σ.

The easy proof is left to the reader. (See Bourbaki [1] for the standard version.)

(4.4) COROLLARY:

In the space $H({}^*D)$, the infinitesimal relation for uniform convergence on compact subsets is: $h \overset{K}{=} k$ if and only if $h(z) \approx k(z)$ for every $z \in ns({}^*D)$.

PROOF:

The union monad of the compact subsets is $ns({}^*D)$ by local compactness of D.

Now we can see that σ agrees with compact convergence on finitely bounded sets. This was first observed by Rubel and Shields [7].

(4.5) SUMMARY OF RESULTS ON THE STRICT TOPOLOGY:

The strict topology which is the topology induced on H(D) by the notion of bounded sequential convergence ($h_n \to h$ if $||h_n|| \leq M$ and $h_n \to h$ pointwise) is also the finest topology which agrees with σ on bounded sets. We denote the uniformity by β and then the following are equivalent: (Rubel, Ryff, and Schields [6], [7])

a. $h \overset{\beta}{=} k$.

b. $\langle p, h \rangle \approx \langle p, k \rangle$ for every $p \in \mathrm{cmp}_N(P({}^*D))$, the norm-compact points of $P({}^*D) = \cup[{}^*K : K$ is a standard norm-compact subset of $P(D)]$.

c. $\langle p_n, h \rangle \approx \langle p_n, k \rangle$ for every standard norm-null sequence.

d. $f(z)h(z) \approx f(z)k(z)$ for every $z \in {}^*D$ and every $f \in \widehat{C_0(D)}$ = standard continuous functions vanishing off compact subsets of D.

In particular, if $||h-k||$ is finite and $h(z) \approx k(z)$ for $z \in ns({}^*D)$ then d holds. A partial converse is possible, namely d implies that $h(z) \approx k(z)$ for $z \in ns({}^*D)$ since we may take a standard C_0-function which is one at a given near standard point. This proves:

(4.6) THEOREM:

All β-infinitesimals are S-continuous on $ns({}^*D)$.

We contrast this to the weak-star infinitesimals, Theorem (4.2).

The external equivalence relation "$h \overset{b}{=} k$ provided $||h-k||$ is finite and $h(z) \approx k(z)$ for $z \in ns({}^*D)$" is compatible with the linear structure in the sense that if $h \overset{b}{=} h'$ and $k \overset{b}{=} k'$, then $h+h' \overset{b}{=} k+k'$ and for finite λ, $\lambda h \overset{b}{=} \lambda h'$. Moreover, the equivalence classes are convex. This is not a monadic equivalence relation however and there are norm-infinite β-infinitesimals. The relation $\overset{b}{=}$ is particularly natural and in light of the model-theoretic significance of discrete monads (as the best approximation by standard sets) we feel our next result is not without interest.

(4.7) THEOREM:

The discrete monad of the set of $\overset{b}{=}$-equivalent pairs of elements of $H({}^*D)$ equals the set of β-infinitesimals;

$$\mu(\beta) = \mu_D(\{(h,k) : k \overset{b}{=} k\}) = \cap[{}^*E : E \text{ is standard and } {}^*E \supseteq \overset{b}{=}] .$$

In other words, $\overset{b}{=}$ determines the strict topology.

PROOF:

The above remarks state that $\overset{b}{=} \subseteq \mu(\beta)$, so $\mu_D(\overset{b}{=}) \subseteq \mu(\beta)$ and we need only show that $\mu_D(\overset{b}{=}) \supseteq \mu(\beta)$.

Suppose that ${}^*E \supseteq \overset{b}{=}$ and that $n \in \hat{N}$, a standard natural number. In the nonstandard model the sentence "there exists $f \in C_0({}^*D)$ such that $||h-k|| < n$ and $\left|f(z)\left(h(z)-k(z)\right)\right| < 1$ implies $(h,k) \in {}^*E$," holds since we may take $f(z)$ infinitely large on K a $*$-compact set containing ns($^*D$). The same sentence without $*$ on D and E must hold in the standard model and this means β agrees with the filter of $\mu_D(\overset{b}{=})$ on bounded sets. Since β is the finest topology which agrees with σ on bounded sets, the proof is complete.

Perhaps it is worthwhile to examine the monad of zero. We have that $\mu_D(\{h : h \overset{b}{=} 0\}) = \mu_D(\overset{b}{=})[0] = o_\beta(0)$. This follows from the compatibility of $\overset{b}{=}$ with the linear structure.

(4.8) The final topology for this section is the Mackey topology on H(D) which is the finest whose dual is P(D). We denote it by m(H(D), P(D)) or just m. The Mackey-Arnes theorem states $h \overset{m}{=} k$ if and only if $\langle p, h\rangle \approx \langle p, k\rangle$ for every $p \in cmp_W(P(^*D))$, the weakly compact points of $P(^*D)$.

$m = \beta$ if and only if $cmp_N(P(^*D)) = cmp_W(P(^*D))$, that is, if and only if P(D) has the "Schur property" that weak and norm compactness coincide. Rubel [5] conjectures that $m \neq \beta$ so long as $H(D) \neq C$.

We always have m finer than β, so one only needs to show strict inclusion.

5. Noncoincidence of the Strict and Mackey Topologies:

In this section we give a new approach to a theorem of Rubel and Ryff [6] that states that $\beta(H(G), P(G)) \neq m(H(G), P(G))$ when G is a (standard) finitely connected region with smooth boundary, bG.

We do this by introducing the infinitesimal equlvalence relation that two functions are finitely bounded in difference and infinitesimally close on most of the boundary. This relation gives rise to a standard topology strictly finer than β and coarser than m. We do not know if this new topology equals the Mackey topology and the generalization of these results to the nonstandard hull of H(G) where G approximates $^*D$ remains to be done.

(5.1) DEFINITIONS:

α denotes the uniformity associated with the $L^1(bG)$-norm of the non-tangential boundary values of functions $h \in H(G)$, $|h|_\alpha = \int_{bG} |h(z)|\ |dz|$.

$\xi = \sup(\alpha, \beta)$, the uniformity generated by the entourages $U \cap V$, where $U \in \alpha$ and $V \in \beta$.

Γ is the uniformity associated with the finest topology which agrees with the topology of ξ on bounded sets.

$\stackrel{\gamma}{=}$ is the external equivalence relation on $H(^*G)$ given by $\{(h,k) : \|h-k\| \in O$ and $h(z) \approx k(z)$ on most of $bG\}$, precisely $h \stackrel{\gamma}{=} k$ provided $\|h-k\| \in O$ and there exists T an internally measurable subset of $^*bG$ for which $h(t) \approx k(t)$, $t \in T$ and such that $\int_T |dz| \approx \int_{bG} |dz|$.

The motivation to study α, ξ and Γ actually came from an attempt to understand $\stackrel{\gamma}{=}$ which is a natural refinement of $\stackrel{b}{=}$ (and in fact a strict refinement as is easily seen). An understanding of $\stackrel{\gamma}{=}$

seemed desirable from the beginning since $\overset{\gamma}{=}$-continuous linear functionals satisfy a dominated convergence theorem by a direct nonstandard argument. This hints at an integral representation and compatibility with the dual pair. We were unable to give direct extension procedures for such functionals to integrals, however. We have the following result.

(5.2) THEOREM:

<u>If</u> $||h-k||$ <u>is finite, then</u> $h \overset{\alpha}{=} k$ <u>if and only if</u> $h \overset{\gamma}{=} k$. <u>Moreover,</u> $h \overset{b}{=} k$ <u>and</u> $h \overset{\alpha}{=} k$ <u>if and only if</u> $h \overset{\gamma}{=} k$.

PROOF:

If $h \overset{\gamma}{=} k$ then

$$\int_{{}^*bG} |h(z)-k(z)|\ |dz| \leq \int_{T} |h(z)-k(z)|\ |dz| + ||h-k|| \int_{{}^*bG \setminus T} |dz| \approx 0 .$$

Observe that there is an infinitesimal ε such that for all $t \in T$, $|h(t)-k(t)| < \varepsilon$ since T is internal. This proves $h \overset{\alpha}{=} k$.

Next, if $||h-k|| \in O$ and if $\int |h(z)-k(z)|\ |dz| = \delta \approx 0$, then $|h(t)-k(t)| > r \not\approx 0$ on T with finite positive measure leads to a contradiction, hence $h \overset{\alpha}{=} k$ and $||h-k|| \in O$ implies $h \overset{\gamma}{=} k$.

Next, if $h \overset{b}{=} k$ then $||h-k|| \in O$ and if in addition $h \overset{\alpha}{=} k$ then $h \overset{\gamma}{=} k$ by the first part of the theorem.

Finally, we use Cauchy's formula to show that if $h \overset{\gamma}{=} k$ then $h(z) \approx k(z)$ for $z \in ns({}^*G)$. When $z \in ns({}^*G)$, $1/|z-w|$ is finite for all $w \in {}^*bG$ and

$$|h(z)-k(z)| \leq \int_T \frac{|h(w)-k(w)|}{|z-w|}|dw| + \int_{{}^*bG\setminus T} \frac{\|h-k\|}{|z-w|}|dw| \approx 0 .$$

(5.3) THEOREM:

$\mu_D(\overset{\gamma}{=}) = \mu(\Gamma)$, that is $\overset{\gamma}{=}$ determines Γ, the finest topology which agrees with ξ on bounded sets. Moreover, if $h \overset{\Gamma}{=} k$ and $\|h-k\| \in O$, then $h \overset{\gamma}{=} k$.

PROOF:

The first part is proved the same way that $\mu_D(\overset{b}{=}) = \mu(\beta)$ was proved, namely by showing $\mu_D(\overset{\gamma}{=}) = \inf(F_n)$ where F_n is the filter generated by the sets $W \cap \{(h,k) : \|h-k\| \leq n\}$, $W \in \xi$.

The second statement follows from the last theorem since if $h \overset{\Gamma}{=} k$ then $h \overset{\beta}{=} k$ and since $\|h-k\| \in O$, $h \overset{b}{=} k$.

(5.4) THEOREM:

The strict topology $\beta(H(G), P(G))$ is strictly coarser than ξ which is coarser than Γ which in turn is coarser than the Mackey topology $m(H(G), P(G))$. In particular, $\beta \neq m$.

PROOF:

We prove β is strictly coarser than ξ by showing that α is not coarser than β. Recall the example given above of an infinite power of the Riemann mapping of the inside of one boundary component of G. That function is a β-infinitesimal with modulus one on one whole boundary component, hence not an α-infinitesimal. Thus $(\rho^\lambda, 0) \in \mu(\beta)$ and $(\rho^\lambda, 0) \notin \mu(\alpha) \cap \mu(\beta)$ and ξ is strictly finer than β.

Γ is finer than ξ by definition.

α is coarser than the Mackey topology since any α-continuous functional has a representation as an integral: $\varphi(h) = \int_{bG} h(z)g(z)|dz|$, $g \in L^{\infty}(bG)$ and hence we may view α as a polar topology on a restricted collection of subsets of P(G) making α coarser than m. This means that ξ is coarser than m since both α and β are. Moreover, ξ and β have the same closed convex sets since they are both compatible with the duality $\langle H, P \rangle$.

Finally, we show that Γ is compatible with the duality $\langle H, P \rangle$ and hence coarser than m. Let L be a Γ-closed convex subset of H(G). We shall show that L is β-closed which implies Γ and β have the same continuous linear functionals. By the characterization of β as the topology of bounded sequential convergence it is enough to show that if $(h_n : n \in N) \subseteq L$ and $h_n \xrightarrow{B} h$, then $h \in L$. Since $(h_n : n \in N)$ is bounded the ξ-closed convex hull and the Γ-closed convex hull coincide. Moreover, the Γ-closed convex hull is contained in L. Now since ξ and β have the same closed convex sets, having the same dual spaces, the β-closed convex hull equals the Γ-closed convex hull and $h \in L$. This completes the proof of the theorem.

(5.5) THEOREM:

$h \overset{\alpha}{=} k$ <u>implies</u> $h \overset{K}{=} k$, <u>that is the</u> α-<u>uniformity is finer than uniform convergence on compact subsets of</u> G.

PROOF:

Take $z \in ns({}^{*}G)$ so that $(1/|z-w| \leq M \in O$, for all $w \in {}^{*}bG$. By the Cauchy formula $h(z) \approx k(z)$, and the proof is finished.

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APPENDIX 1

THE NONSTANDARD HULL AND DUALITY

Robinson [4] constructed the completion of a metric space by a nonstandard construction based on the extended metric. Luxemburg [2, Theorem 3.15.1] extended Robinson's construction by showing that it applied to standard uniform spaces and that (in saturated ultra-power models) it gives rise to a larger complete space, the non-standard hull, in which the completion is contained. Also, Machover and Hirschfeld [3] later extended Robinson's construction to standard uniform spaces using entourages.

The first result of this appendix shows that Luxemburg's construction applies to internal or *-uniform spaces. The motivation for this is not abstract generality, but rather to obtain H(D) from hull(H(G)) where G approximates *D as in section II. 3 above. We are sure there are other applications as well. The reader should consult Luxemburg's paper [2] for details not contained in this appendix.

We remark that in [8] the author showed that the nonstandard hull of a standard precompact uniform space is complete for non-standard models which are only enlargements. The use of saturation seems essential in the general case. However, we remind the reader that δ-incomplete ultrapowers (the simplest of all nonstandard models!) are $\aleph_1$-saturated (Luxemberg [2, Theorem 1.6.4]). The outcome of this last remark is that nonstandard hulls of *-metric spaces are complete.

(A1.1) CONSTRUCTION OF THE 'INTERNAL' NONSTANDARD HULL.

Let X be an internal set and $\Lambda = \{d : d \in \Lambda\}$ a (possibly external) set of *-pseudometrics on X, that is, each $d \in \Lambda$ is a map $d: X \times X \to {}^*R^+$ satisfying $d(x,x) = 0$, $d(x,y) = d(y,x)$, and $d(x,z) \leq d(x,y) + d(y,z)$.

Let $O_\Lambda(a)$ be the Λ-galaxy of a, that is, the set of points a finite distance from a, $\{x \in X : \text{for each } d \in \Lambda,\ d(x,a) \in O\}$. Let $o_\Lambda(x) = \{y \in X : d(x,y) \approx 0 \text{ for each } d \in \Lambda\}$. On the set $O_\Lambda(a) / o_\Lambda$, the points of $O_\Lambda(a)$ which have been identified if they are infinitesimally close for each $d \in \Lambda$, the mappings st(d(x, y)) are pseudometrics. We denote the resulting uniform space by

$$\text{hull}(X, a; \Lambda) \quad ,$$

which we refer to as the nonstandard hull of X at a with respect to Λ.

In the case of H(G) in section II, we take a to be the zero function and Λ the single element $||x-y||$.

(A1.2) THEOREM:

<u>Let</u> X <u>be a</u> *-<u>uniform space in a nonstandard</u> $\varkappa$-<u>saturated ultrapower model</u> $^*\mathfrak{X}$. <u>Let</u> $a \in X$ <u>and</u> Λ <u>be a family of</u> *-<u>pseudometrics on</u> X. <u>If</u> $\varkappa > \max(\aleph_0, \text{card}(\Lambda))$, <u>then</u> hull(X, a;$\Lambda$) <u>is a complete uniform space.</u>

The proof in Luxemburg [2, pp. 80 and 81] applies with more work to extend the Cauchy set. We omit the details. The simpler case of the hull of a *-metric space is all we shall use and this is done as follows. Let $(\mathfrak{o}(\alpha_p) : p \in N)$ be a Cauchy sequence in hull$(X, a; \{d\})$. Extend $\alpha : {}^*N \to {}^*\mathcal{X}$ to an internal sequence. Since $\alpha^{-1}(X) \supseteq \hat{N}$ we know $\alpha^{-1}(X)$ contains an initial segment $\{1, \ldots, \omega\}$ with $\omega \in {}^{\#}N$. $\mathfrak{o}(\alpha_\omega)$ is the limit.

Now we shall consider the hulls of a *-Banach space E and its internal continuous dual E'. We assume throughout that we are working in a saturated ultrapower model. Let $E_0 = \text{hull}(E, 0; \{||x-y||\})$ and $(E')_0 = \text{hull}(E', 0; \{||\varphi - \psi||\})$.

We may also consider the external continuous dual of the ordinary Banach space E_0 which we denote by $(E_0)'$.

(A1.3) THEOREM:

$(E')_0$ <u>is a closed subspace of</u> $(E_0)'$ <u>which separates points of</u> E_0.

PROOF:

If $\varphi \in E'$ has finite norm, then $st\varphi(x)$, $x \in E_0$ is well defined, linear, and $||st\varphi|| \approx ||\varphi||$. Now if $||\psi - \varphi|| \approx 0$, then $st(\psi(x)) = st(\varphi(x))$ for $x \in E_0$, so we may view $(E')_0$ as norm embedded in $(E_0)'$. It is closed since it is complete.

If $0 \neq x, y \in E$ with $||x-y|| \not\approx 0$ so that they give rise to distinct elements of E_0, we define φ as follows.

$\varphi(x/||x||) = 1$, $\varphi(y/||y||) = 0$ and $\varphi([a/||x||]x + [b/||y||]y) = a$ and then extend φ internally to all E subject to the condition that $||\varphi|| \leq 1$. Now $st(\varphi)$ separates x and y in E_0.

This last result leads us to ask whether $(E')_0$ is all of $(E_0)'$ and on the basis of the compatibility of nonstandard models with linked algebraic and topological properties, we might expect this to be the case. On the other hand, the construction of nonstandard hulls is external and since E_0 may be quite large it also seems possible that $(E_0)'$ may be strictly larger. We do not know the answer to this question.

APPENDIX 2

SWEEPING MEASURES TO THE BOUNDARY OF SMOOTH REGIONS

In this appendix we extend a result of Rubel and Shields from the unit disk to finitely connected regions with smooth boundary. The *-transform of this result is used in section II. 3 on the approximating region to represent the pre-dual as internal integrals infinitesimally near the boundary of an arbitrary region. Since the techniques of Rubel and Shields [7] are scarcely changed we have put the result in an appendix.

(A2. 1) THEOREM:

<u>Let</u> G <u>be a finitely connected region with smooth boundary</u> bG. <u>Then</u> $M(G)/N(G)$ <u>and</u> $L^1(bG)/N^1(bG)$ <u>are isomorphic pre-dual Banach spaces for</u> $H(G)$.

Specifically, for $\mu \in M(G)$ there is an $f \in L^1(bG)$ such that $\int_G h d\mu = \int_{bG} f(z)h(z)\,|dz|$ for all $h \in H(G)$. Moreover, the norm in the quotient spaces is preserved under this assignment and the assignment is onto.

We begin by stating the lemmas we shall need in order to apply the techniques of Rubel and Shields.

LEMMA 1:

<u>Each</u> $h \in H(G)$ <u>determines a bounded measurable function on</u> bG <u>by the non-tangential limits of</u> h. <u>Moreover, the Cauchy formula holds on</u> bG:

$$h(z) = (1/2\pi i) \int_{bG} (h(w)/(w-z))dw , \qquad z \in G .$$

The smooth simply connected case follows from the fact that the inverse of the Riemann mapping is angle preserving to the boundary and absolutely continuous on the circle. The finitely connected case can be reduced to this by dividing the region with non-overlapping smooth simply connected curves tangent to the successive boundary components. In this way you reduce the region to the study of two simply connected ones which overlap on the interiors of the dividing curves. The boundary integral,

$$\int_{bG} = \int_{C_1} + \int_{C_2} - \sum_{i=0}^{n} \int_{\Gamma_i}$$

and the Cauchy formula follows.

LEMMA 2:

$H^\infty(G)$ viewed as the non-tangential limit functions on bG is a closed subspace of $L^\infty(bG)$, in fact, the L^∞-norm of the non-tangential limit function equals the H^∞-norm of the analytic function.

This follows from the simply connected case by the same procedure as in Lemma 1. The simply connected case follows from the fact that sets of measure zero correspond under the Riemann mapping.

The importance of Lemma 2 is that $L^1(bG)/N^1(bG)$ is a pre-dual for $H^\infty(G)$.

LEMMA 3:

<u>In the weak-star topologies</u> $\sigma(H,M)$ <u>and</u> $\sigma(H,L^1)$ <u>a linear space is closed if and only if it is sequentially closed.</u>

This well-known result appears in Banach's treatise.

LEMMA 4:

<u>A sequence</u> $(h_n) \subseteq H(G)$ <u>converges in</u> $\sigma(H,M)$ <u>to a limit</u> h <u>if and only if</u> (h_n) <u>converges boundedly to</u> h.

This result appears in the paper of Rubel and Shields [7]. (See section II. 4 for the definition of bounded convergence.)

PROOF OF THE THEOREM:

Suppose first that we are given a function $f \in L^1(bG)$. We show that $L(h) = \int_{bG} f(z)h(z)\,|dz|$ is σ-continuous by showing that its kernel is σ-sequentially closed. (Lemma 3.)

Assume $h_n \xrightarrow{\sigma} h$ and $L(h_n) = 0$ for each n. We wish to show $L(h) = 0$. Now $h_n \xrightarrow{B} h$ by Lemma 4 so $||h_n|| \leq B$ and for convenience assume ≤ 1. By Alaoglu's theorem h_n has a $\sigma(H^\infty, L^1/N^1)$-convergent subsequence $k_n \to k$. This means

$$\int k_n(z)f(z)\,|dz| \to \int k(z)f(z)\,|dz|$$

for each $f \in L^1(bG)$. By the Cauchy formula, substituting $\frac{z}{z-w}\frac{dz}{|dz|} = f(z)$, we see that $k(z) = h(z)$. Since L is by definition $\sigma(H^\infty, L^1/N^1)$-continuous $L(k) = 0$ and therefore $L(h) = 0$ which shows L^1/N^1 is contained in M/N.

Conversely, if $K(h) = \int h(z)d\mu(z)$ for $\mu \in M(G)$ we must show that the kernel of K is $\sigma(H^\infty, L^1/N^1)$-sequentially closed. (Lemma 2 and 3.) So we assume $h_n \xrightarrow{\sigma(H^\infty, L^1/N^1)} h$, with $K(h_n) = 0$ for all n. We must show $K(h) = 0$.

By the uniform boundedness principle the h_n are uniformly essentially bounded on bG and hence within G. They converge pointwise by the Cauchy formula. By Lemma 4 therefore $h_n \xrightarrow{\sigma} h$ and $K(h) = 0$.

APPENDIX 3

NONSTANDARD MODELS

In abstract algebra one often studies the properties of a mapping necessary to preserve the structure in question, that is, monomorphisms. The image under such a map is indistinguishable from the original for the sake of the structure in question. (The reals embedded in the complex plane, for example.)

We wish to inject "ordinary mathematics" into a larger theory in a way which preserves its structure, much as the real numbers are embedded in the complex numbers. The reader is no doubt familiar with examples of properties of the real numbers which can be more easily obtained by first embedding R in C. This is analogous to the way one may view applications of nonstandard analysis--we enlarge to simplify.

For our purposes "ordinary mathematics" is the study of the following set $\mathcal{N}$ called the superstructure based on the natural numbers, N. Let $N_0 = N$, $N_k = P\left(\bigcup_{l=0}^{k-1} N_l\right)$, $k = 1, 2, \ldots$, we take $\mathcal{N} = \bigcup_{k=0}^{\infty} N_k$. (P denotes power set, the set of all subsets.)

The structure we are interested in is the membership relation, ϵ, restricted to $\mathcal{N}$. No doubt the reader "believes" that we may base mathematics on set theory--calling $\mathcal{N}$ a (standard) model for "ordinary mathematics" is no more than that "belief". This set is sufficiently large to suit our purposes, for example, the real numbers are in as Dedekind cuts of rationals, which are ordered pairs of integers, which are ordered pairs of natural numbers. (We view (a, b) as

$\{a, \{a, b\}\}$.) Functions are sets of ordered pairs, so $H^{\infty}(D) \in \mathcal{M}$. Riemann surfaces can be embedded in Euclidean space, so they too can be viewed in $\mathcal{M}$. (The reader can convince himself of these statements with a little effort.)

Now we give the essential properties of a nonstandard extension map, $*$, which is defined on $\mathcal{M}$.

$*$ is an injection, it is one-to-one.

$*$ preserves $\in$: if $a \in A \in \mathcal{M}$, then ${}^*a \in {}^*A$.

$*$ preserves equality: ${}^*\{(x,x) : x \in A \in \mathcal{M}\} = \{(y,y) : y \in {}^*A\}$.

$*$ preserves finite sets: if $a_1, a_2, \ldots, a_n \in \mathcal{M}$, then ${}^*\{a_1, \ldots, a_n\} = \{{}^*a_1, \ldots, {}^*a_n\}$.

$*$ preserves basic set operations: ${}^*\phi = \phi$, ${}^*(A \cup B) = {}^*A \cup {}^*B$, ${}^*(A \cap B) = {}^*A \cap {}^*B$, ${}^*(A \setminus B) = {}^*A \setminus {}^*B$, where $A, B \in \mathcal{M}$.

$*$ preserves domains and ranges of n-ary relations and commutes with permutations of n-ary relations. This allows us to extend functions to the extension of their domains and ranges for example.

$*$ preserves atomic standard definitions of sets: ${}^*\{(x,y) : x \in y \in A \in \mathcal{M}\} = \{(z,w) : z \in w \in {}^*A\}$.

$*$ produces a proper extension, that is a NONstandard model: if $A \in \mathcal{M}$ is an infinite set, then $\{{}^*x : x \in A\}$ which is denoted $\hat{A}$, is properly contained in ${}^*A$. This means ${}^*R$ contains infinitesimals, but we remind the reader that it also means that the set of standard subsets of ${}^*R$, $\widehat{P(R)}$ is a proper subset of the internal subsets of ${}^*R$, ${}^*P(R)$.

The image of $*$ is a nonstandard model of ordinary mathematics in the sense that a theorem phrased in terms of sets of $\mathcal{M}$ is true if and only if its *-transform is true, where the *-transform is the same statement with a * on each constant. For example, the nonstandard reals, ${}^*R$, are *-complete and yet externally incomplete. This means every bounded set $A \in {}^*P(R)$ has a least upper bound, nonetheless, the set of infinitesimals $o \in P({}^*R)$ does not have a supremum. In particular, the set of internal subsets of ${}^*R$, ${}^*P(R)$, is properly contained in the set of (all or) external subsets, $P({}^*R)$. Distinguishing between internal and external objects is what allows us to enlarge $\mathcal{M}$ and still preserve all of its properties so far as they can be expressed in terms of the extensions of sets in $\mathcal{M}$.

The *-<u>transform</u> of a statement from ordinary mathematics is the internal notion one obtains in the nonstandard model by first writing the statement in terms of $\in$ and sets of $\mathcal{M}$, being careful to only quantify over elements of $\mathcal{M}$, and then placing a * on each constant in the sentence. A *-continuous function is necessarily internal and satisfies the ϵ-δ definition where ϵ and δ range over ${}^*R$. An S-continuous function may be external and it satisfies the ϵ-δ definition with <u>standard</u> ϵ and δ, that is $\epsilon, \delta \in \hat{R}^+$--being able to work with both internal and external concepts like these makes the theory useful.

It is usually difficult to tell whether a given set is internal, but one case is easy--when the set is described by a sentence which involves only internal constants. We call this the INTERNAL DEFINITION PRINCIPLE and use it extensively above. A special case of this is when the sentence is the *-transform of a sentence from $\mathcal{M}$, then the set is the image under * of the correspondingly defined set in $\mathcal{M}$.

One way to exhibit *-mappings is by means of the ultrapower construction. Nonstandard ultrapower models have extension properties for mappings as well as additional saturation, most important for applications, they will seem quite concrete to many mathematicians while an appeal to the compactness principle may not. We shall not give this construction since it can be found in many places.

In order to work more freely with internal sets we assume our nonstandard model is $\varkappa$-saturated up to some infinite cardinal, usually card $(\mathcal{M})^+$.

Many interesting properties of $\varkappa$-saturated ultrapower models can be found in the paper, A General Theory of Monads by W. A. J. Luxemburg. The above axioms for the *-mapping appear in A Set-Theoretical Characterization of Enlargements by A. Robinson and E. Zakon. Both papers are in the volume Applications of Model Theory to Algebra, Analysis and Probability, edited by W. A. J. Luxemburg.

For our purposes the properties of $\varkappa$-saturated ultrapower needed are Luxemburg's Theorems 2.7.11 and 2.7.12.