STRESSES AND DEFLECTIONS OF UNSWEPT AND

SWEPT THIN-WALLED BEAMS

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SUMMARY

A method is presented for the determination of the stresses and deflections of unswept and swept, thin-walled beams of uniform closed cross section. The cross section, loading distribution and boundary conditions are assumed to be arbitrary. The method is based on the differential equation governing the behavior of orthogonal elastic shells. The differential equation is transformed into a difference equation and the solution obtained by the relaxation technique. A comparison of the theoretical solution and experimental data for a swept back wing with a carry through bay under symmetrical bending showed good agreement.

A tapered wing may be treated by approximating the variation by a series of spanwise steps.

As the difference equations are a system of simultaneous algebraic equations, they may be solved by automatic calculating equipment or by electric analogue computers as well as by the relaxation technique.

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TABLE OF NOTATION

a_{SH} = area of sheet element resisting normal stress - sq. in.

a_{ST} = area of stiffener element - sq. in.

a_j = a_{SH} + a_{ST} = equivalent area of elements resisting normal

stress - sq. in.

A_w = area of monocoque cross section - sq. in.

A_H = horizontal shear resistant area - sq. in.

A_v = vertical shear resistant area - sq. in.

c = normal chord of the outer panel - in.

E = Young's modulus - psi.

E' = $E/(1 - \mu^2)$ - psi.

G = shearing modulus of elasticity - psi.

H = horizontal section shear - lb.

h = height of beam cross section - in.

I_c = central moment of inertia - in.^{4}

r = radius from origin of principal shear axis to a tangent

through the wall element - in.

s = tangential coordinate

t = wall thickness of monocoque section - in.

T = section torque - in. lb.

u = horizontal displacement - in.

v = vertical displacement - in.

V = vertical section shear - lb.

w = axial displacement - in.

$\left.\begin{array}{c} x \\ y \\ z \end{array}\right\}$ = rectangular coordinates

α = angle between tangent to a wall element and the x axis - deg.

γ = shearing strain - radians

ϵ = normal strain - in./in.

μ = Poisson's ratio

φ = angle of twist (rotation) of the cross section - radians

ψ = angle of sweep back - deg.

λ = angle of resolution of normal stress in the outer panel sheet element - deg.

β = included angle between outer panel sheet element and inner longitudinal axis - deg.

ξ = tangential displacement - in.

σ = normal stress - psi.

τ = shearing stress - psi.

SUBSCRIPTS

i = inner panel of swept back wing with a carry through bay

o = outer panel of swept back wing with a carry through bay

j, n = quantities on the shell surface defined by their values at the stiffeners

$k, \mathcal{R}$ = quantities on the shell surface defined by their values over the sheet

s = tangential coordinate

z = rectangular (axial) coordinate

I. INTRODUCTION

Since early in 1947, the structures group at GALCIT has been investigating theoretically and experimentally the effect of sweep upon the stress and deflection distribution in aircraft wings. In 1950, Dr. E. E. Sechler and the author undertook the task of reviewing the existing structural analysis methods, experimental data, and aeroelastic effects for swept wings with solid, thick-walled and thin-walled sections. This review indicated that no satisfactory solution is available for the stresses and deflections in a thin-walled, swept wing structure. The theoretical determination of the stresses that exist in such a swept wing under static load requires the analysis of a swept, redundant, cantilever structure of variable cross section with an elastic support. The available theoretical methods of treatment of this problem are the energy methods, the method of consistent deformations, and methods in which loads are arbitrarily assigned to various elements of the structure.

The energy methods are available to be used in those cases where equilibrium conditions are not sufficient in themselves to specify the stress distribution, and use is made of the fact that the actual distribution corresponds to a minimum of the energy. The energy methods may use either Castigliano's theorem, or a least (virtual) work analysis. Castigliano's theorem proceeds from the expression for the elastic energy U stored in the structure by the application to the points 1, 2, . . . i of the forces P_i, torques T_i, and bending moments M_i, and gives equations for the

displacements λ_i in the direction of the loads P_i, the angles of twist α_i in the direction of the torques T_i, and the angles of rotation θ_i in the direction of the moments M_i. The equations are:

$$\lambda_i = \frac{\partial U}{\partial P_i} \qquad \alpha_i = \frac{\partial U}{\partial T_i} \qquad \theta_i = \frac{\partial U}{\partial M_i}$$

The least work analysis applies the method of minimum complementary energy and minimum potential energy. The principle of minimum complementary energy is a variational condition on the stresses, and the principle of minimum potential energy is a variational condition on the displacements. The former expresses a condition which defines the correct state of stress that satisfies the equilibrium conditions and the boundary conditions on the stresses; whereas the latter determines the correct deformation configuration consistent with the boundary conditions on the deflections.

The principle of the method of consistent deformations assumes that the elements of the structure are considered as free bodies in equilibrium under the action of forces and moments. At the joints, the elements have the same translational and rotational displacements, or in other words, the deformations are consistent. If there are many elements, the method inherently requires the solution of a large number of simultaneous equations. The concept of an elastic axis is not used in the method. The effects of warping of the cross section and shear lag may be considered.

The methods in which loads are arbitrarily assigned to various elements of the structure usually use the elastic axis concept, and consider that one group of elements resists the bending loads, and another group of elements

resists the torsion loads.

However, the knowledge of the structural behavior of swept wings with thin-walled cross sections is not complete. There are no known existing accurate solutions of the problem, and the approximate methods show only fair agreement with experimental data. Therefore, there is a need for a theoretical approach that will be as accurate as possible, and at the same time provide a procedure to obtain an engineering solution.

As a result, a theoretical solution has been developed in which the problem is formulated in terms of the differential equation from the theory of orthogonal elastic shells. The assumption of the preservation of the normal cross section is relaxed in the neighborhood of the oblique support, in order to account for the boundary conditions and the effect of root ribs. The solution of the differential equation is obtained by means of finite difference methods. Although automatic calculating equipment and electric analogue computers may be used to solve the difference equations, the relaxation method is used due to its flexibility for investigational purposes, and its effectiveness as an engineering procedure.

II. FORMULATION OF THE DIFFERENTIAL EQUATION GOVERNING THE ORTHOGONAL SHELL PROBLEM

Contributions to the formulation of the orthogonal shell problem have been made by a large number of authors. An excellent review of these contributions, and additional extensions have been made by Benscoter (Reference 1). The following formulation is taken from Benscoter's work, and modified where necessary to permit the eventual use of numerical calculation.

In this formulation, the following assumptions are made:

(1) The physical beam is replaced by a similar theoretical beam having ribs, flanges, stiffeners, and sheets.

(2) The cross sections of the beam are uniform, and contain closely spaced internal rigid diaphragms which stabilize the shell and preserve the cross-sectional shape. The diaphragms are infinitely stiff in their own planes but are completely free to warp out of their planes.

(3) The flanges and stiffeners are straight and transmit only axial loads.

(4) The sheet is sufficiently thin so that the stresses are uniformly distributed over the thickness of the wall. The sheet transmits axial and shear loads but does not transmit bending loads.

(5) There are no cut outs.

(6) Hooke's Law is assumed.

In the derivation of the differential equation, it is assumed that the beam has a thin-walled, unstiffened cross section, consisting of a single cell without sharp corners or discontinuities in thickness.

The beam and coordinate system are shown in Figure 1, with a point on the beam identified by the coordinate z along axis of the beam, and the coordinate s measured around the peripheral center line of the beam. The idealized differential element of the shell is shown in Figure 2. The element has variable thickness in the s direction and constant thickness in the z direction.

Assuming that Hooke's law is applicable, and neglecting the normal forces due to the diaphragms, the stress strain relationships are

$$\sigma_z - \mu\,\sigma_s = E\,\epsilon_z \qquad \sigma_s - \mu\,\sigma_z = E\,\epsilon_s \qquad \tau_{sz} = G\,\gamma_{sz} \tag{1}$$

where

E = Young's modulus

G = shearing modulus of elasticity

$\epsilon_s,\ \epsilon_z$ = normal strain along the tangential and axial coordinate respectively

$\sigma_s,\ \sigma_z$ = normal stress along the tangential and axial coordinate respectively

μ = Poisson's ratio

τ_{sz} = shearing stress in the sz plane

γ_{sz} = shearing strain in the sz plane

Due to the assumption of stiff diaphragms, $\epsilon_s = 0$. Therefore,

$$\sigma_z = \frac{E}{1-\mu^2} \, \epsilon_z = E' \epsilon_z$$

where   (2)

$$E' = \frac{E}{1-\mu^2}$$

The stress σ_s and strain ϵ_s do not appear in Part II henceforth, and the subscripts of the remaining stresses and strains will be dropped with the following change of notation:

$$\sigma = \sigma_z \qquad \epsilon = \epsilon_z \qquad \tau = \tau_{sz} \qquad \gamma = \gamma_{sz}$$

The strains at a point may be expressed in terms of the axial displacement w and tangential displacement ξ by

$$\epsilon = \frac{\partial w}{\partial z} \qquad \gamma = \frac{\partial w}{\partial s} + \frac{\partial \xi}{\partial z} \qquad (3)$$

The stresses may be expressed in terms of the displacements by

$$\sigma = E' \frac{\partial w}{\partial z} \qquad (4)$$

$$\tau = G \left(\frac{\partial w}{\partial s} + \frac{\partial \xi}{\partial z} \right)$$

where   s, z = tangential and axial coordinate respectively

    ξ, w = tangential and axial displacement respectively.

The forces acting on the differential element are shown in Figure 2. The stress components acting in the circumferential direction on the faces of an element of the thin wall are balanced by the forces from the diaphragms acting on the inner surface of the wall. Hence only the equation of equilibrium of the element in the axial direction need be considered. The equation is

$$\frac{\partial \sigma}{\partial z} \left(t + \frac{1}{2} \frac{dt}{ds} \, ds \right) ds \, dz + \frac{\partial(\tau t)}{\partial s} \, ds \, dz = 0$$

where   t = wall thickness of monocoque section.

Assuming the changes in thickness are small, then

$$t \frac{\partial \sigma}{\partial z} + \frac{\partial (\tau t)}{\partial s} = 0 \qquad (5)$$

In order to obtain an equation for axial displacements, the strain compatibility equation is formed by substituting Equations (4) into Equation (5), giving

$$t \frac{\partial}{\partial z} \left(E' \frac{\partial w}{\partial z} \right) + \frac{\partial}{\partial s} \left[t G \left(\frac{\partial w}{\partial s} + \frac{\partial \xi}{\partial z} \right) \right] = 0$$

or

$$\frac{E' t}{G} \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial s} \left(t \frac{\partial w}{\partial s} \right) + \frac{\partial}{\partial s} \left(t \frac{\partial \xi}{\partial z} \right) = 0$$

It is convenient to use the principal shear axes as the coordinate axes of the cross section. The location of these axes may be computed by the formulas given by Duberg (Reference 2). At any given point of the wall, draw a tangent to the centerline as shown in Figure 3(a). The radius from the origin of the principal shear axes to the tangent is r, while the angle made by the tangent with the positive direction of the x axis is α . The location of the principal shear axes is defined in terms of r and α by the condition that the following line integrals vanish around the cell,

$$\oint t r \cos \alpha \, ds = 0$$
$$\oint t r \sin \alpha \, ds = 0 \qquad (7)$$
$$\oint t \cos \alpha \sin \alpha \, ds = 0$$

The section properties associated with the principal shear axes are defined by:

$$I_c = \oint t\, r^2 ds$$

$$A_H = \oint t\, \cos^2 \alpha\, ds$$

$$A_V = \oint t\, \sin^2 \alpha\, ds \tag{8}$$

where

I_c = central moment of inertia (corresponds to polar moment of inertia)

A_H = horizontal shear resistant area

A_V = vertical shear resistant area

A_W = area of monocoque cross section

$$= \oint t\, ds = A_H + A_V$$

The tangential displacement ξ may be eliminated from Equation (6) by further use of the assumption of rigid bulkheads. The horizontal displacement u and vertical displacement v are parallel to the principal shear axes and regarded as the translations of the origin of the shear axis. The contributions that each of these displacements make to the tangential displacement are illustrated in Figure 3.

If the displacements of the rigid body are small, the rotational and translational contributions may be superimposed to give

$$\xi = r\varphi + u\cos\alpha + v\sin\alpha$$

and

$$\frac{\partial \xi}{\partial z} = r\frac{d\varphi}{dz} + \cos\alpha\frac{du}{dz} + \sin\alpha\frac{dv}{dz} \tag{9}$$

where φ = angle of twist (rotation) of the cross section.

The section torque and external shears are related to the internal shearing stresses by the following equations,

$$T = \oint t \, \tau \, r \, ds$$

$$H = \oint t \, \tau \cos \alpha \, ds$$

$$V = \oint t \, \tau \sin \alpha \, ds$$

where

H = horizontal section shear

T = section torque

V = vertical section shear

If Equation (4) is substituted into Equation (10), then

$$T = G \oint tr \frac{\partial w}{\partial s} \, ds + G \oint tr \frac{\partial \xi}{\partial z} \, ds$$

$$H = G \oint t \cos \alpha \frac{\partial w}{\partial s} \, ds + G \oint t \cos \alpha \frac{\partial \xi}{\partial z} \, ds \qquad (11)$$

$$V = G \oint t \sin \alpha \frac{\partial w}{\partial s} \, ds + G \oint t \sin \alpha \frac{\partial \xi}{\partial z} \, ds$$

Substituting Equation (9), and omitting the integrals which vanish by Equations (7), then

$$T = G \oint tr \frac{\partial w}{\partial s} \, ds + G \frac{d\varphi}{dz} \oint tr^2 ds$$

$$H = G \oint t \cos \alpha \frac{\partial w}{\partial s} \, ds + G \frac{du}{dz} \oint t \cos^2 \alpha \, ds \qquad (12)$$

$$V = G \oint t \sin \alpha \frac{\partial w}{\partial s} \, ds + G \frac{dv}{dz} \oint t \sin^2 \alpha \, ds$$

Introduction of principal shear axes permits the above equations to be solved for the derivatives of φ , u and v independently, rather than as a simultaneous system. Inserting the section constants given in Equation (8) into Equation (12), and solving for the displacement gives

$$\frac{d\varphi}{dz} = \frac{T}{GI_c} - \frac{1}{I_c} \oint tr \frac{\partial w}{\partial s} ds$$

$$\frac{du}{dz} = \frac{H}{GA_H} - \frac{1}{A_H} \oint t \cos\alpha \frac{\partial w}{\partial s} ds$$

$$\frac{dv}{dz} = \frac{V}{GA_v} - \frac{1}{A_v} \oint t \sin\alpha \frac{\partial w}{\partial s} ds$$

The assumption that the cross section is a single cell without sharp corners implies that the functions $t(s)$ and $r(s)$, are continuous. Thus, as it will be seen that $\frac{\partial w}{\partial s}$ is continuous, integration by parts transforms the equations into the following form:

$$\frac{d\varphi}{dz} = \frac{T}{GI_c} + \frac{1}{I_c} \oint w \frac{d(tr)}{ds} ds$$

$$\frac{du}{dz} = \frac{H}{GA_H} + \frac{1}{A_H} \oint w \frac{d(t\cos\alpha)}{ds} ds \qquad (13)$$

$$\frac{dv}{dz} = \frac{V}{GA_v} + \frac{1}{A_v} \oint w \frac{d(t\sin\alpha)}{ds} ds$$

An expression *for* ξ which will permit its elimination from Equation (6) may be obtained from Equation (9) by substituting in Equation (13) to give:

$$\frac{d\xi}{dz} = \frac{Tr}{GI_c} + \frac{H\cos\alpha}{GA_H} + \frac{V\sin\alpha}{GA_v} + \frac{r}{I_c} \oint w \frac{d(tr)}{ds} ds \qquad (14)$$

$$+ \frac{\cos\alpha}{A_H} \oint w \frac{d(t\cos\alpha)}{ds} ds + \frac{\sin\alpha}{A_v} \oint w \frac{d(t\sin\alpha)}{ds} ds$$

This equation may be substituted into Equation (4) to give:

$$t = G \left[\frac{\partial w}{\partial s} + \frac{Tr}{GI_c} + \frac{H\cos\alpha}{GA_H} + \frac{V\sin\alpha}{GA_v} + \right.$$

$$+ \frac{r}{I_c} \oint w \frac{d(tr)}{ds} ds + \frac{\cos\alpha}{A_H} \oint w \frac{d(t\cos\alpha)}{ds} ds \qquad (15)$$

$$\left. + \frac{\sin\alpha}{A_v} \oint w \frac{d(t\sin\alpha)}{ds} ds \right]$$

Equation (14) may be substituted into Equation (6) to obtain the following

expression governing the displacements:

$$\frac{E'}{G} t \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial s}\left(t \frac{\partial w}{\partial s}\right) + \frac{\partial}{\partial s}\left[t\left(\frac{Tr}{GI_c} + \frac{H\cos\alpha}{GA_H} + \frac{V\sin\alpha}{GA_V} + \right.\right.$$

$$\frac{r}{I_c}\oint w \frac{d(tr)}{ds}ds + \frac{\cos\alpha}{A_H}\oint w \frac{d(t\cos\alpha)}{ds}ds +$$

$$\left.\left.\frac{\sin\alpha}{A_V}\oint w \frac{d(t\sin\alpha)}{ds}ds\right)\right] = 0$$

Simplifying,

$$\frac{E't}{G}\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial s}\left(t \frac{\partial w}{\partial s}\right) = -\left[\frac{T}{GI_c}\frac{d(tr)}{ds} + \frac{H}{GA_H}\frac{d(t\cos\alpha)}{ds}\right.$$

$$+ \frac{V}{GA_V}\frac{d(t\sin\alpha)}{ds} + \frac{1}{I_c}\frac{d(tr)}{ds}\oint w \frac{d(tr)}{ds}ds \tag{16}$$

$$\left. + \frac{1}{A_H}\frac{d(t\cos\alpha)}{ds}\oint w \frac{d(t\cos\alpha)}{ds}ds + \frac{1}{A_V}\frac{d(t\sin\alpha)}{ds}\oint w \frac{d(t\sin\alpha)}{ds}ds\right]$$

A similar expression may be developed for the axial stress.

Equation (16) may be simplified if the cross section of the beam and

the corresponding boundary conditions are symmetrical about one or both

principal shear axes. In this event, all or part of the actions of torsion,

horizontal bending, and vertical bending become separated, and permit

the solution of a simple system of independent equations. The development

of like expressions is given in Reference 1.

In the case of a cross section and boundary conditions which are symm-

etrical about both principal shear axes, the equation becomes

$$\frac{E't}{G}\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial s}\left(t \frac{\partial w}{\partial s}\right) = 0 \quad \text{(axial forces)}$$

or,
$$\frac{E't}{G}\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial s}\left(t\frac{\partial w}{\partial s}\right) = -\frac{1}{I_c}\frac{d(tr)}{ds}\left[\frac{T}{G} + \oint w\frac{d(tr)}{ds}ds\right] \tag{17}$$

or,
$$= -\frac{1}{A_H}\frac{d(t\cos\alpha)}{ds}\left[\frac{H}{G} + \oint w\frac{d(t\cos\alpha)}{ds}ds\right]$$

or,
$$= -\frac{1}{A_V}\frac{d(\sin\alpha)}{ds}\left[\frac{V}{G} + \oint w\frac{d(t\sin\alpha)}{ds}ds\right]$$

In the case of a cross section and boundary conditions which are symmetrical about the x axis, the equation becomes

$$\frac{E't}{G}\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial s}\left(t\frac{\partial w}{\partial s}\right) = -\frac{1}{A_H}\frac{d(t\cos\alpha)}{ds}\left[\frac{H}{G} + \oint w\frac{d(t\cos\alpha)}{ds}ds\right] \tag{18}$$

or,
$$= -\frac{1}{I_c}\frac{d(tr)}{ds}\left[\frac{T}{G} + \oint w\frac{d(tr)}{ds}ds\right]$$
$$-\frac{1}{A_V}\frac{d(t\sin\alpha)}{ds}\left[\frac{V}{G} + \oint w\frac{d(t\sin\alpha)}{ds}ds\right]$$

In the case of a cross section and boundary conditions which are symmetrical about the y axis, the equation becomes

$$\frac{E't}{G}\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial s}\left(t\frac{\partial w}{\partial s}\right) = -\frac{1}{A_V}\frac{d(t\sin\alpha)}{ds}\left[\frac{V}{G} + \oint w\frac{d(t\sin\alpha)}{ds}ds\right]$$

or,
$$\tag{19}$$
$$= -\frac{1}{I_c}\frac{d(tr)}{ds}\left[\frac{T}{G} + \oint w\frac{d(tr)}{ds}ds\right]$$
$$-\frac{1}{A_H}\frac{d(t\cos\alpha)}{ds}\left[\frac{H}{G} + \oint w\frac{d(t\cos\alpha)}{ds}ds\right]$$

III. TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS INTO DIFFERENCE EQUATIONS

The solution of linear differential equations with known boundary conditions may be obtained by finite difference methods. The differential equation governing the unknown function is transformed into a difference equation and the solution of the difference equation is obtained at a finite number of points within the boundary. As the set of points at which the desired function is sought may be made arbitrarily dense, the solution of the difference equation approaches the solution of the differential equation to any desired degree of accuracy. The difference problem may be solved by Southwell's relaxation method (Reference 3), which is defined as "a systematic sequence of localized changes of the wanted function that steadily brings the 'residuals' toward their desired value."

The difference equation is obtained from the differential equation by expressing the differentials and integrals in terms of finite differences. The finite differences may be obtained by the procedures given in Reference (4).

Suppose a function $f(u)$ is given for the values a_o, a_1 .... a_n of its argument u. It is required to find the value of the function or its derivatives at some point $x(u)$ in which the intervals of the argument are not constant. This problem may be solved by the application of Newton's formula for unequal values of the argument. (Ref. Chapter II, Reference (4).) In order to derive this formula, consider what are called differences

of tabular values. The quantity

$$f(a_1, a_0) = \frac{f(a_1) - f(a_0)}{a_1 - a_0}$$

is called a divided difference of first order. In the same manner, divided

differences of second and third order are respectively:

$$f(a_2, a_1, a_0) = \frac{f(a_2, a_1) - f(a_1, a_0)}{a_2 - a_0}$$

$$f(a_3, a_2, a_1, a_0) = \frac{f(a_3, a_2, a_1) - f(a_2, a_1, a_0)}{a_3 - a_0}$$

The divided differences of higher orders are formed in the same way.

Let $f(u)$ be a function whose divided differences of (say) order 4

vanish; and suppose that its values for the arguments a_0, a_1, a_2, a_3 are

known. The value of the function for any other argument $x(u)$ may be

obtained in the following way: since the divided differences of order 4

vanish, it follows that the differences of order 3 are constant, or

$$f(u, a_0, a_1, a_2) = f(a_0, a_1, a_2, a_3)$$

By definition of the divided difference of order 3,

$$f(u, a_0, a_1) = f(a_0, a_1, a_2) + (u - a_2) f(u, a_0, a_1, a_2)$$

Again, by definition

$$f(u, a_0) = f(a_0, a_1) + (u - a_1) f(u, a_0, a_1)$$

$$= f(a_0, a_1) + (u - a_1) \quad f(a_0, a_1, a_2) + (u - a_2) f(u, a_0, a_1, a_2)$$

Also, by definition

$$f(u) = f(a_0) + (u - a_0) f(u, a_0)$$

whence

$$f(u) = f(a_0) + (u-a_0) f(a_0, a_1) + (u-a_0)(u-a_1) f(a_0, a_1, a_2)$$

$$+ (u-a_0)(u-a_1)(u-a_2) f(a_0, a_1, a_2, a_3)$$

Now let $(u - a_n) = \alpha_n$, and by induction

$$f(u) = f(a_0) + \alpha_0 f(a_0, a_1) + \alpha_0 \alpha_1 f(a_0, a_1, a_2) + \cdots +$$

$$+ \alpha_0 \alpha_1 \cdots \alpha_{n-1} f(a_0, a_1, \cdots a_n) + Remainder \qquad (20)$$

where Remainder $= \alpha_0 \alpha_1 \cdots \alpha_n f(u, a_0, \cdots a_n)$

The derivatives of a function in terms of its divided differences may be expressed by differentiating both sides of Newton's Equation (20), so that

$$f'(u) = f(a_0, a_1) + (\alpha_0 + \alpha_1) f(a_0, a_1, a_2)$$

$$+ (\alpha_0 \alpha_1 + \alpha_0 \alpha_2 + \alpha_1 \alpha_2) f(a_0, a_1, a_2, a_3) + \cdots \qquad (21)$$

$$\tfrac{1}{2} f''(u) = f(a_0, a_1, a_2) + (\alpha_0 + \alpha_1 + \alpha_2) f(a_0, a_1, a_2, a_3) + \cdots$$

and so on.

In order to obtain any desired accuracy, the Newton Equation (20) for the expansion of a function in terms of its divided differences should include such orders of differences that the remainder term is less than some pre-assigned amount. However, accuracy may also be obtained by a reduction of the length of the intervals. In the following, the divided differences of third and higher orders will be neglected, and the accuracy

of the solution obtained by progressive reduction of the intervals.

There are a number of methods of numerically evaluating a definite integral. Among the most common are the Newton-Cotes formulae, of which the simplest case is

$$\int_a^{a+w} f(u)\, du = \tfrac{1}{2} w \left[f(a) + f(a+w) \right] \tag{22}$$

which is known as the trapezoidal rule. It is exact when $f(u)$ is a function whose first differences are constant over the interval $(a, a+w)$. When $f(u)$ is a function whose first differences are not constant, the difference between the two sides of the above equation may be written

$$\frac{1}{12} w^3 f''(a + \theta w)$$

$$\text{where } 0 < \theta < 1$$

To apply the finite differences procedure to the analysis of a swept wing, it is necessary to form the differences of the value of the dependent variable for unequal intervals of the independent variable. Consider the particular case of the differential Equation (16) for elastic shells. Assume that the solution of the equation is desired in the domain D of the s, z plane bounded by a closed curve Γ as shown in Figure 4. Superimpose on the Domain D an arbitrary orthogonal mesh of lines parallel to the s and z coordinate axes. The lines parallel to the z and s axes are identified by the nomenclature $j - 1$, j, $j + 1$, and $n - 1$, n, $n + 1$, etc. respectively, and the regions of the domain between the lines parallel to the z and s axes are identified by k, $k+1$ and R, $R+1$

etc. respectively. The intervals between the lines parallel to the s axis are constant and are equal to ℓ. The intervals between the lines parallel to the z axis are not constant and are equal to L_K, L_{K+1}, etc. This identifies the mesh nodes and intervals in the neighborhood of a point $(j(s), n(z))$.

In order to transform the differential equation into a difference equation based on the mesh network illustrated in Figure 4, express the terms in Equation (16) as a function of the differences at a point (j, n) as follows. The second derivative of a function is given by Equation (21) to be

$$\frac{1}{2!} f''(u) = f(a_0, a_1, a_2) + \cdots$$

$$f''(u) = \frac{2}{a_0 - a_2} \left[\frac{f(a_0) - f(a_1)}{a_0 - a_1} - \frac{f(a_1) - f(a_2)}{a_1 - a_2} \right] + \cdots$$

Recalling that third order differences are neglected, and that the intervals in the directions parallel to the z and s coordinates are equal and unequal respectively, then the equation becomes

$$\frac{E't}{G} \frac{\partial^2 W}{\partial z^2} = \frac{E't}{G} \left\{ -\frac{2}{2\ell} \left[\frac{W_{n-1} - W_n}{-\ell} - \frac{W_n - W_{n+1}}{-\ell} \right]_j \right\}$$

$$= \frac{E't}{G} \frac{1}{\ell^2} \left(W_{n+1} + W_{n-1} - 2W_n \right)_j$$

and

$$\frac{\partial}{\partial s} \left(t \frac{\partial w}{\partial s} \right) = \frac{2}{-(L_k + L_{k+1})_j} \left\{ \left(\frac{t_k}{-L_k} \right)_j (w_{j-1} - w_j)_n - \left(\frac{t_{k+1}}{-L_{k+1}} \right)_j (w_j - w_{j+1})_n \right\}$$

$$= \frac{2}{(L_k + L_{k+1})_j} \left\{ \left(\frac{t_{k+1}}{L_{k+1}} \right)_j (w_{j+1} - w_j)_n - \left(\frac{t_k}{L_k} \right)_j (w_j - w_{j-1})_n \right\}$$

where, for example the term (W_{n+1})$_j$ refers to the axial displacement

of the point (j, $n+1$) and (L_k)$_j$ refers to the interval in the dir-

ection parallel to the s coordinate in the neighborhood of the point j as

indicated in Figure 4.

The first derivative of a function is given by Equation (21) to be

$$f'(u) = f(a_0, a_1) + (\alpha_0 + \alpha_1) \, f(a_0, a_1, a_2)$$

Consider a function $f\left[t(u), r(u), \alpha(u)\right]$ which is known at $u = a_0, a_1 \ldots$

a_n. It is desired to obtain the first derivative at a point $x = (a_1 + a_2)/2$.

Inserting $u = x$ in Equation (21), then

$$f'(x) = \frac{f(a_0) - f(a_1)}{a_0 - a_1} + \left(\frac{a_1 - a_2}{2} - a_0 + \frac{a_1 + a_2}{2} - a_1\right)\left(\frac{1}{a_0 - a_2}\right) \times$$

$$\left(\frac{f(a_0) - f(a_1)}{a_0 - a_1} - \frac{f(a_1) - f(a_2)}{a_1 - a_2}\right)$$

$$= \frac{f(a_2) - f(a_1)}{a_2 - a_1}$$

In terms of the mesh network, the values of the function are given at

the mid-points of each interval along the lines (n, n + 1, ...) parallel to

the s axis as illustrated by Figure 4. The first derivative is desired at

the node point (j, n) of the mesh. Therefore

$$\frac{d}{ds} \, f\left[t(s), r(s), \alpha(s)\right] = \left[\frac{f(t, r, \alpha)_{k+1} - f(t, r, \alpha)_k}{(L_{k+1} + L_k)/2}\right]_j$$

The line integrals may be evaluated by means of the trapezoidal

rule. The integral terms become

$$\oint w \, \frac{d}{ds} f(t, r, \alpha) \, ds = \sum_j \left[f(t, r, \alpha)_{K+1} - f(t, r, \alpha)_K \right]_j w_j$$

The differential Equation (16) may be transformed into a difference equation by expressing the differentials and integrals in terms of the differences that have been derived. For the point (j, n), the equation becomes

$$\frac{E't}{G} \frac{(w_{n+1} + w_{n-1} - 2w_n)_j}{\ell^2} + \frac{2}{(L_{K+1} + L_K)_j} \left[\left(\frac{t_{K+1}}{L_{K+1}} \right)_j (w_{j+1} - w_j)_n - \left(\frac{t_K}{L_K} \right)_j (w_j - w_{j-1})_n \right]$$

$$= -\frac{2}{I_c} \frac{(t_{K+1} r_{K+1} - t_K r_K)_j}{(L_{K+1} + L_K)_j} \left[\frac{T}{G} + \sum_j (t_{K+1} r_{K+1} - t_K r_K)_j w_j \right]$$

$$-\frac{2}{A_V} \frac{(t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j}{(L_{K+1} + L_K)_j} \left[\frac{V}{G} + \sum_j (t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j w_j \right]$$

$$-\frac{2}{A_H} \frac{(t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j}{(L_{K+1} + L_K)_j} \left[\frac{H}{G} + \sum_j (t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j w_j \right]$$

Multiplying through by $(L_{K+1} + L_K)_j /2$, and letting $a_j = t(L_{K+1} + L_K)_j /2$, the expression becomes:

$$\frac{E' a_j}{G \ell^2} (w_{n+1} + w_{n-1} - 2w_n)_j + \left(\frac{t_{K+1}}{L_{K+1}} \right)_j (w_{j+1} - w_j)_n - \left(\frac{t_K}{L_K} \right)_j (w_j - w_{j-1})_n$$

$$= -\frac{1}{I_c} (t_{K+1} r_{K+1} - t_K r_K)_j \left[\frac{T}{G} + \sum_j (t_{K+1} r_{K+1} - t_K r_K)_j w_j \right]$$

$$-\frac{1}{A_V} (t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j \left[\frac{V}{G} + \sum_j (t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j w_j \right] \quad (23)$$

$$-\frac{1}{A_H} (t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j \left[\frac{H}{G} + \sum_j (t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j w_j \right]$$

where a_j = equivalent area resisting the normal stresses.

In the event that there is symmetry of the cross section and the boundary conditions, the difference equation may be simplified. In the

same manner as in Part II, symmetry about both principal shear

axes permits the equation to be written as

$$\frac{E'a_j}{G\ell^2}(w_{n+1} + w_{n-1} - 2w_n)_j + \left(\frac{t_{K+1}}{L_{K+1}}\right)_j (w_{j+1} + w_j)_n - \left(\frac{t_K}{L_K}\right)_j (w_j - w_{j-1})_n$$

$$= 0$$

(24)

$$\text{or,} \quad = -\frac{1}{I_c}(t_{K+1} r_{K+1} - t_K r_K)_j \left[\frac{T}{G} + \Sigma_j (t_{K+1} r_{K+1} - t_K r_K)_j \, w_j\right]$$

$$\text{or,} \quad = -\frac{1}{A_V}(t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j \left[\frac{V}{G} + \Sigma_j (t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j \, w_j\right]$$

$$\text{or,} \quad = -\frac{1}{A_H}(t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j \left[\frac{H}{G} + \Sigma_j (t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j \, w_j\right]$$

In the case of symmetry about the x axis, the equation becomes

$$\frac{E'a_j}{G\ell^2}(w_{n+1} + w_{n-1} - 2w_n) + \left(\frac{t_{K+1}}{L_{K+1}}\right)(w_{j+1} - w_j)_n - \left(\frac{t_K}{L_K}\right)_j (w_j - w_{j-1})_n$$

$$= -\frac{1}{A_H}(t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j \left[\frac{H}{G} + \Sigma_j (t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j \, w_j\right]$$

(25)

$$\text{or,} \quad = -\frac{1}{I_c}(t_{K+1} r_{K+1} - t_K r_K)_j \left[\frac{T}{G} + \Sigma_j (t_{K+1} r_{K+1} - t_K r_K)_j \, w_j\right]$$

$$- \frac{1}{A_V}(t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j \left[\frac{V}{G} + \Sigma_j (t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j \, w_j\right]$$

In the case of symmetry about the y axis, the equation becomes

$$\frac{E'a_j}{G\ell^2}(w_{n+1} + w_{n-1} - 2w_n)_j + \left(\frac{t_{K+1}}{L_{K+1}}\right)(w_{j+1} - w_j)_n - \left(\frac{t_K}{L_K}\right)_j (w_j - w_{j-1})_n$$

$$= -\frac{1}{A_V}(t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j \left[\frac{V}{G} + \Sigma_j (t_{K+1} \sin\alpha_{K+1} - t_K \sin\alpha_K)_j \, w_j\right]$$

(26)

$$\text{or,} \quad = -\frac{1}{I_c}(t_{K+1} r_{K+1} - t_K r_K)_j \left[\frac{T}{G} + \Sigma_j (t_{K+1} r_{K+1} - t_K r_K)_j \, w_j\right]$$

$$- \frac{1}{A_H}(t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j \left[\frac{H}{G} + \Sigma_j (t_{K+1} \cos\alpha_{K+1} - t_K \cos\alpha_K)_j \, w_j\right]$$

The difference equation is written in terms of differences of the variable at a discrete set of points. In cases where there is a variation of the variable between the points, it is necessary to approximate the variation by some consistent procedure. Consider the wall of the shell on which a mesh network has been superimposed. A curved element of variable thickness in a mesh interval may be approximated by a plane element of constant thickness. This implies that the values of t, r, and α are constant in each mesh interval. In Equation (23) the term a_j was defined as equal to $t(L_{k+1} + L_k)_j/2$, where a_j is the equivalent area resisting the normal stresses. This implies that the area corresponding to the normal stress σ at the point (j, n) is half of the area of the material in each interval neighboring the line j containing the point.

The section properties associated with the principal shear axes are given by Equation (8). The transformation of these equations into difference form gives

$$I_c = \sum_k t_k L_k r_k^2 \tag{27}$$

$$A_H = \sum_k t_k L_k \cos^2 \alpha_k$$

$$A_V = \sum_k t_k L_k \sin^2 \alpha_k$$

The deflections and rotations of the principal shear axis of the shell are given by Equation (13). The transformation of these equations into difference form gives,

$$\frac{d\varphi}{dz} = \frac{1}{I_c}\left[\frac{T}{G} + \Sigma_j\left(t_{k+1}\,r_{k+1} - t_k\,r_k\right)_j w_j\right]$$

$$\frac{du}{dz} = \frac{1}{A_H}\left[\frac{H}{G} + \Sigma_j\left(t_{k+1}\cos\alpha_{k+1} - t_k\cos\alpha_k\right)_j w_j\right]$$

(28)

$$\frac{dv}{dz} = \frac{1}{A_V}\left[\frac{V}{G} + \Sigma_j\left(t_{k+1}\sin\alpha_{k+1} - t_k\sin\alpha_k\right)_j w_j\right]$$

The value of the total deflection or rotation is given by the summation of the terms along the z axis of the shell. The summation is given by the trapezoidal rule. For example the total rotation becomes

$$\varphi = \int_0^z \frac{d\varphi}{d\eta}\,d\eta$$

$$= \Sigma_R \frac{1}{2}\left[\left(\frac{d\varphi}{dz}\right)_{n-1} + \left(\frac{d\varphi}{dz}\right)_n\right]\ell$$

The displacements u_j and v_j of the surface of the shell are functions of the deflections and rotations of the principal shear axis, and may be easily shown to be

$$u_j = u - \varphi y$$

$$v_j = v + \varphi x$$

(29)

The transformation of the differential equation (15) gives the following difference equation for the shear stress,

$$t_k = G\left[\frac{1}{I_k}\left(w_j - w_{j-1}\right) + \frac{T r_k}{G I_c} + \frac{H\cos\alpha_k}{G A_H} + \frac{V\sin\alpha_k}{G A_V}\right.$$

$$+ \frac{r_k}{I_c}\Sigma_j\left(t_{k+1}\,r_{k+1} - t_k\,r_k\right)_j w_j$$

(30)

$$+ \frac{\cos\alpha_k}{A_H}\Sigma_j\left(t_{k+1}\cos\alpha_{k+1} - t_k\cos\alpha_k\right)_j w_j$$

$$+ \frac{\sin\alpha_k}{A_V}\Sigma_j\left(t_{k+1}\sin\alpha_{k+1} - t_k\sin\alpha_k\right)_j w_j$$

The tangential path of integration for determining the shear in the K, Rth element (Figure 4) in Equation (30) may be either along the neighboring (n-1) or (n) th line. If the length of the interval approaches zero, the choice of the path is immaterial. However, for finite intervals the choice of the path affects the result, and by definition, the path chosen is along the (n)th line. In other words, the path is the line neighboring the shear element in the direction of the positive z axis. It is to be noted that the shear stress in each mesh interval is constant.

The normal stress along a line parallel to the z axis is given by Equation (4). By application of Newton's Equation (21) for the derivative, the normal stresses are given by the following difference equations:

$$(\sigma)_{n-1} = \frac{E'}{2\ell} (-3 w_{n-1} + 4 w_n - w_{n+1})_j$$

$$(\sigma)_n = \frac{E'}{2\ell} (w_{n+1} - w_{n-1})_j \qquad (31)$$

$$(\sigma)_{n+1} = \frac{E'}{2\ell} (3 w_{n+1} - 4 w_n + w_{n-1})_j$$

where the subscript notation is in accordance with the nomenclature in Figure 4. It is to be noted, that as the shear stress in each mesh interval is constant, the variation of the normal stress along each line parallel to the z axis is linear.

The formulation of the finite difference problem for unstiffened shells may be applied to stiffened shells provided that two modifications are made to Equation (23). First, the equivalent area of the material resisting the normal stresses (a_j) should include the stiffener area in addition to the wall area. Second, the modulus of elasticity E should be used in place of E'.

In order that the solution of the difference equations in a given domain D be unique, the location of the closed curve Γ bounding the domain and the values of the function or its derivatives must be assigned or determinable. In the cases to be considered, the location of the boundaries are assigned, and intersect nodes of the mesh. The boundary value problems to be considered are "mixed", as the values of the function are given over certain portions of the boundary, and the value of the first derivatives over the remainder. The details of the application of the boundary conditions are discussed in Parts IV and V.

IV. THE RELAXATION PROCEDURE FOR THE SOLUTION OF THE FINITE DIFFERENCE PROBLEM

An excellent introduction on the use of relaxation methods is given by Shaw (Reference 5). This section can only hope to point out the highlights of the procedure, as the serious user will have to master the technique by study of the literature.

The relaxation mesh of the finite difference procedure for the orthogonal shell is a right cylinder (since the displacements are a singly periodic function) with boundary conditions imposed at the two ends. Simplification of the mesh arises when the cross section of the beam and corresponding boundary conditions are symmetrical about one or both principal shear axes. In this case, the cylindrical mesh may be cut along a line corresponding to the intersection of a principal shear axis and the surface of the shell. The boundary condition along the edges of the cut is that the value of either the displacement or its derivative is zero. This is illustrated by Table 1. The cut cylindrical mesh is then unwrapped, and the mesh becomes a rectangle in the cartesian s, z plane. The boundary conditions along the ends of this rectangle are dependent on the particular problem.

The size of the initial mesh is dependent on the rate of change of the wanted function, the ease in "advancing to finer mesh", and the ease of application of "line" and "block" relaxation. The meshes used in

Parts VI and VII are examples of the usable size.

The initial assumptions of the function in the domain D should be made with care in order to avoid excessive labor. For the case of orthogonal shells, much aid may be obtained from simple bending and torsion theory.

In order to briefly illustrate the techniques used, the simple Poisson equation

$$\nabla^2 \phi(x, y) = F \tag{32}$$

will be considered. The procedure for the more complex integro-differential equation for shells is identical. Consider a square domain with square meshes of side h as shown in Figure 5(a) in which the nearby mesh nodes are identified by a letter system. The transformation of the differential Equation (32) gives the following difference equation for the point (0),

$$\phi_a + \phi_b + \phi_c + \phi_d - 4\phi_o - h^2 F = E_o \tag{33}$$

where

ϕ_o = the value of the "wanted" function at point 0

E_o = error of the difference equation at point 0.

Consider that the boundary values are known for the mesh network shown in Figure 5(b), and assume initial values of the function in the domain. By the use of Equation (33), the errors are computed for each point in the set not on the boundary. The purpose of the method is to reduce the error to a value which will approach zero. This is done by the use of relaxation operators.

An example will illustrate the procedure for obtaining a relaxation operator. Assume a change in the value of the function at point (0) of $\varphi = 1$. The question arises, "What are the changes in the errors in the domain?". Inspection of Equation (33) shows that E_0 changes by -4. The equations for the errors at the points a, b, c, d contain φ_0, and thus E_a, E_b, E_c, E_d change by 1. The "point" relaxation operator is diagrammatically shown in Figure 5(c). In the calculations, only the value of the total error should be recorded, whereas usually the increment is recorded in the case of the function φ. Note that the value of the function is written to the left of the vertical line, and the errors to the right of it. At the boundary where the value of the function is specified, the errors are not recorded.

Lines of symmetry may be used with boundary conditions where the derivative of the function is zero. With reference to Figure 5(d), consider that a solution of the problem is desired in the hatched region. If a change in the value of the function is made at a point (o) along a line adjacent to the line of symmetry, then a change is necessary at the corresponding point (o') on the other side of the symmetry line. The difference equations are unchanged except at the line of symmetry. For example at the point c, the difference equation becomes

$$\varphi_e + \varphi_f + 2\varphi_0 - 4\varphi_c - h^2 F = E_c$$

The relaxation operators are unchanged except at the line adjacent to the line of symmetry. The modified operator is diagrammatically illustrated in Figure 5(e).

The procedure for reducing an error is to alter the value of the function at the point in such a manner so as to reduce the error. Usually the points with the largest errors are treated first. The totality of the errors is only reduced by "pushing some of them over the boundary". Any other operation has the effect of distributing them over a larger area. In order to speed up the process, "line" and "block" relaxation operators are used. These are the superposition of a number of adjacent point relaxation operators due to a simultaneous change of the function along a portion of a line or in an area. Usually, they are computed for a unit change of the function and the magnitude adjusted by a multiplier dependent on the totality of the errors. In addition, the practice of "over relaxation" is frequently used with relaxation operators. By these procedures, the errors may be reduced to nearly zero, and the approximate difference equations solved for any given mesh.

In order to obtain increased accuracy, it is necessary to decrease the size of the mesh. In practice, it seems best to commence the problem with a fairly coarse mesh, liquidate the errors everywhere, and then from this first approximation derive a trial solution on a net of finer mesh. As the mesh becomes finer, the resulting successive solutions give an indication of their convergence. The procedure is indicated in Figure 5(f). Let the horizontal and vertical solid lines be the net, of mesh size h, for which an approximate solution has been obtained. Then the net with the next finer mesh will be that at 45°

formed by the broken lines. The mesh size will be $h/\sqrt{2}$. Then for the point O (for Poisson's equation) the difference equation is

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 4\varphi_0 - \frac{h^2}{2} F = E_0$$

Assume the error is zero $(E_0 = 0)$, hence

$$\varphi_0 = \frac{1}{4} \left(- \frac{h^2 F}{2} + \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \right)$$

This gives the trial points for the 45° mesh. This mesh may be liquified, and the procedure repeated to give a new 90° mesh, of mesh size $h/2$, and so on.

V. GENERAL PROCEDURE FOR THE DETERMINATION OF STRESSES AND DEFLECTIONS FOR THIN-WALLED BEAMS

1. Orthogonal Beams

The stresses and deflections of a thin-walled orthogonal beam of uniform closed cross section may be determined by the application of the theory of shells. The procedure may be simplified in the event of symmetry of the cross section and symmetry of the boundary conditions of the beam about one or both of the principal shear axes. The beam is oriented by the coordinate system shown in Figure 1. The relaxation mesh corresponding to the beam is a right cylinder. In the event of symmetry, the simplification of the relaxation mesh and the boundary conditions along the edges of the mesh parallel to the z axis is illustrated by Table 1. The boundary conditions along the edges of the mesh parallel to the s coordinate are dependent upon the particular problem, and in the cases to be considered, are either on the value of the displacement (w) or the first derivative of the displacement. The value of the first derivative of the displacement is expressed in difference form by Equation (31). The length of the intervals of the relaxation mesh and the corresponding idealization of the beam cross section are dependent upon the possibility of simplifying the difference equation, the rate of change of the wanted function, the ease in advancing to finer mesh, and the ease of application of line and block relaxation operators. The nodes and the intervals of the mesh are identified in accordance with

Figure 4. The behavior of the beam is governed by the applicable difference equation for the displacement (w) given in Part III. The point relaxation operators are determined from the difference equation by the methods outlined in Part IV. The initial assumptions of the function in the mesh may be obtained from simple bending and torsion theory. The solution of the difference equations on the displacement are obtained by the procedures in Part IV. The stresses and deflections of the beam are obtained from the equations given in Part III.

The boundary conditions along the edges of the mesh parallel to the s coordinate are illustrated by considering the cases of the simply supported and cantilever beams.

a. Simply Supported Beam

The boundary conditions along the edges of the mesh parallel to the s coordinate are dependent upon the conditions at the ends of the actual beam. For the case of a simply supported beam with transverse loads, the axial stresses at the ends of the beam are zero. In the case of a simply supported beam with end moments, the axial stresses at the ends of the beam are known. In both cases, the stress may be expressed as a derivative of the displacement and transferred into difference form.

b. Cantilever Beam with an Orthogonal Support

The boundary conditions along the edges of the mesh parallel to the s coordinate are dependent upon the conditions at the ends of the

actual beam. For the case of a cantilever beam with transverse

loading, the axial stresses at the free ends of the beam are zero. The

boundary conditions for the fixed end of the beam are dependent upon

the type of support. For a rigid support, the boundary condition is that

the displacement is zero at the root. An example of this case is given

in Part VI. For an elastic support, the boundary conditions require

the compatibility of displacements and equilibrium of forces at the root.

An example of this case is given in Part VII.

2. Cantilever Beam with a Rigid Oblique Support

The stresses and deflections of a swept, cantilever beam of uniform,

thin-walled, closed cross section may be determined by the application

of the theory of shells. Consider the problem of a cantilever beam on a

rigid oblique support (AA'D'D) shown in Figure 6(a). The problem is

equivalent to a cantilever beam with an orthogonal support (AA'E'E)

shown in Figure 6(b), with the condition of no displacement

$(u_j = v_j = w_j = o)$ of the section AA'D'D.

The problem of the equivalent cantilever beam is the same as the

problem of the orthogonal cantilever beam discussed in Part 1 of this

section, with the following exceptions.

The first exception is that the boundary condition applied at the

fixed end becomes a condition applied along a line in the domain adjacent

to the boundary. In other words, it is required to distribute the values

of the function along the boundary adjacent to the line in the domain such

that the desired conditions of the function are achieved along the specified line. The solution of the inverse problem may be obtained by relaxation methods, and is unique. Therefore, the equivalent beam problem may be treated in the same manner as before, with the displacement (w_j) being the unknown function. It remains to discuss the question, in the framework of the orthogonal shell theory, how closely can the conditions along the section AA'D'D be satisfied? In answer to the question, it is obvious that the conditions on the displacement w_j may be exactly satisfied along the section AA'D'D. However, the displacements u_j and v_j are functions of the displacement w_j, the orientation of the orthogonal support AA'E'E, and the beam and loading conditions as shown by Equations (28) and (29). Consider the vertical displacement v_j along the section AA'D'D. The assumption in the formulation of the shell theory that the cross sections normal to the beam are preserved implies that cross sections oblique to the beam are not preserved. It follows, in general, that the condition on the vertical displacement along the section AA'D'D may only be satisfied at 2 points on the contour. In particular, for a wing of conventional box section, the conditions on v_j may only be satisfied at the spars. Consider the lateral displacement u_j along the section AA'D'D. The condition may be satisfied for a point along AA', but only by chance for any other point along AA'D'D. However, the effect of the error in u_j is assumed to be small, and is neglected in the analysis. Therefore, in the frame-

work of the orthogonal shell theory the condition on the displacement w_j is satisfied, and the condition on u_j and v_j is approximately satisfied. The condition on v_j may be satisfied if the assumption on the preservation of the cross section in the root region is relaxed. Consider a finite number of normal forces applied along the lines AD and A'D'. If the magnitudes of the normal forces are such that the deflections (v_j) are zero, then the conditions on v_j are satisfied. However, it turns out that the deflections (v_j) along the cross section AA'D'D are small, and the effects may be neglected.

The second exception is that the deflections of the beam are obtained in the same manner as in Part 1 of this section, with the addition of a solid body translation and rotation to satisfy the conditions on the displacements u_j and v_j at the spars.

3. Beam on Multiple Supports Representing a Sweptback Wing with a Carry Through Bay.

The stresses and deflections of a beam on multiple supports representing a sweptback wing with a carry through bay may be determined by the application of the theory of shells. Consider the problem of the sweptback wing shown in Figure 7(a). The wing is composed of an outer panel cantilevered obliquely to a simply supported inner panel. The intersection of the panels introduces a redundancy that may be removed by making a cut along the intersection. As a result, the problem is equivalent to considering the outer panel as a

cantilevered shell with an orthogonal support (AA'E E) and the inner

panel as a simply supported shell, as shown in Figure 7(b). The condi-

tions of compatibility of displacements and equilibrium of forces of the

shells are required to be satisfied along the section AA'D'D.

The problems of the equivalent wing comprising the equivalent

cantilever beam and the simply supported beam are the same as the

problems discussed in parts 1(a) and 2 of this section, with the exception

of the boundary conditions imposed on the equivalent wing, and the elasti-

city of the oblique support (AA'D'D).

For the moment, consider the boundary conditions on the wing,

assuming rigid wing supports. The boundary conditions on the wing are

dependent upon the load distribution. Any arbitrary load distribution

may be split up into its symmetrical and antisymmetrical parts. In the

symmetrical case, the slope of the wing at the center and the vertical

deflection at the wing supports are zero. For the antisymmetrical case,

the vertical deflections of the wing at the center and at the wing supports

are zero. Thus, for both cases, the vertical deflection of the wing at

the supports is zero and for transverse loading the normal stresses at

the free ends are zero. The displacements (w) and normal stresses

(σ) are zero at the center of the wing for the symmetrical and anti-

symmetrical cases respectively.

Consider the elasticity of the oblique support (AA'D'D). The idealized

support structure is composed of a fuselage rib (AA'D'D), an inboard

orthogonal wing rib (HH'D'D), and the joints of the sheet and stiffeners

between the inner and outer shells as shown in Figure 7(a). In order
to investigate the effects of the ribs the assumption of the preservation
of the cross sections of the shells in the support region is relaxed. It
is assumed that the fuselage rib is infinitely stiff in its own plane and
free to warp out of its plane. In addition, it is assumed that the wing
rib is elastic, pin-ended at HH', and that the shear loads on the top and
bottom rib flanges may be neglected.

.As the equations governing the behavior of the structure are linear,
the method of influence coefficients may be used to determine the
stresses and deflections at any point. The stresses and deflections are
given by

$$\sigma_m = \sum_{n=1}^{r} \sigma_{m,n}^{\sigma,F} F_n$$

$$\delta_m = \sum_{n=1}^{r} \delta_{m,n}^{\delta,F} F_n$$

(34)

where

σ_m , δ_m = stress and deflection at point m

$\sigma_{m,n}^{\sigma,F} , \delta_{m,n}^{\delta,F}$ = stress and deflection influence coefficients for

the behavior at m due to a unit load at point n.

F_n = load at point n

The influence coefficients are determined for the behavior of the equi-
valent wing with no wing rib under the action of external loads, and under
the action of loads imposed by the wing rib. The procedure for determin-
ing the influence coefficients for the external loading and the rib loading

are discussed in the following.

Consider the behavior of the wing with no wing rib under the action of external loads. Orient the equivalent cantilever beam (outer panel) and the simply supported beam (inner panel) as shown in Figure 7(b). The corresponding relaxation meshes divide up the surface of the wing into a series of stiffener and sheet elements. Consider the compatibility of the displacements of a point (j, n). The displacements $(u_j)_o$ and $(w_j)_o$ of the jth stringer of the outer panel along the outer panel coordinate axes $(x)_o$ and $(z)_o$ respectively, are resolved along the inner panel coordinate axes $(x)_i$ and $(z)_i$ as follows:

$$(w_j)_i = (w_j)_o \cos \psi + (u_j)_o \sin \psi$$

$$(u_j)_i = (u_j)_o \cos \psi - (w_j)_o \sin \psi$$

The assumption of a rigid fuselage rib (AA'D'D) implies that $(u_j)_i = 0$, whence

$$(w_j)_i = (w_j)_o / \cos \psi \qquad (35)$$

This establishes the relationship between the displacements (w_j) of the inner and outer panels.

The condition of equilibrium of forces must be satisfied at the joint in order to uniquely determine the displacements. The forces acting at the joints of the stiffener and sheet elements are shown in Figure 7(c). The stresses in the outer panel sheet elements may be resolved into directions normal and parallel to the fuselage rib. The portion of the force normal to the fuselage rib that is contributed by the sheet is pro-

portional to the resolved normal stress σ_{z}' . The resolved normal

stress is given in Timoshenko (Chapter 1, Reference 6) to be

$$\sigma_{z}' = \sigma_{z} \cos^2\lambda + \sigma_s \sin^2\lambda - 2\tau_{sz} \sin\lambda \cos\lambda$$

where λ = angle of resolution of the stress.

By Equation (1)

$$\sigma_s = \mu\,\sigma_z$$

whence

$$\sigma_{z}' = \sigma_{z} (\cos^2\lambda + \mu \sin^2\lambda) - \tau_{sz} \sin 2\lambda \tag{36}$$

For a swept back wing with a carry through bay with a uniform cross

section in each panel, the angle of resolution is

$$\lambda = \psi \,|\cos\alpha|$$

where ψ = angle of sweep

It follows that the portion of the force transmitted across the fuselage

rib by the outer panel sheet element is,

$$\left(\sigma_{z}' \frac{\angle t}{\cos\lambda} \right)_o$$

The force is resolved parallel to the $(z)_i$ axis to give the following

contribution by the sheet element.

$$\left(\sigma_{z}' \frac{\angle t}{\cos\lambda} \right)_o \cos\beta$$

where $\beta = \psi\,|\sin\alpha|$ = included angle between outer sheet

element and $(z)_i$ axis.

The force transmitted across the fuselage rib by the stiffeners is

$(\sigma_{z}\ a_{ST})_j$, where a_{ST} is the area of the stiffener element. The

force is resolved parallel to the $(z)_i$ axis to give the following contribution by the stiffener element.

$$(\sigma_{\bar{z}}\, a_{SH})_0 \cos \psi$$

Since it was assumed that half of the width of the sheet each side of the stiffener acts with the stiffener, the following equation is obtained for the equilibrium of forces at the point (j, n),

$$(\sigma_j\, a_j)_i = \left[(\sigma_{\bar{z}}\, a_{ST})_j\right]_0 \cos \psi + \frac{1}{2}\left[\left(\sigma_{\bar{z}}'\, \frac{Lt}{\cos\lambda}\, \cos\beta\right)_{K+1}\right. \tag{37}$$
$$\left. + \left(\sigma_{\bar{z}}'\, \frac{Lt}{\cos\lambda}\, \cos\beta\right)_K\right]_0$$

where $\sigma_{\bar{z}}'$ is given by Equation (36). The normal and shear stresses may be written in terms of the displacements (w_j), the beam loading, and geometrical characteristics by means of Equations (30) and (31). It follows that by using the boundary conditions on the wing and the equations of compatibility and equilibrium of the joint, the difference equations for the displacements may be solved. The stresses and deflections of the beam are determined, and the influence coefficients given by Equation (34).

Consider the behavior of the wing under the action of loads imposed by the wing rib. To account for the effects of the wing rib, cut the rib at HH', and apply an equal and opposite redundant load P to the rib and to the spar as shown in Figure 7(d). The load P on the rib at HH' applies a moment at DD' to the rear spar of the inner panel equal to

$$M = P\, c \sin \psi$$

where c = normal chord of the outer panel. The load P on the outer

panel is equivalent to a vertical and a torque load applied at the cross

section HH'D'D. The boundary conditions on the displacements and on

the forces at the joint are the same as for the case of the external loads

(Equations (35) and (37)), with the exception of the force equation for the

rear spar flange. For this case, Equation (37) for the equilibrium of

forces includes an additional term in the right hand side to account for

the moment applied by the rib. The expression becomes

$$(\sigma_j \, a_j)_i = \frac{P c \sin \psi}{h} + \left[(\sigma_z \, a_{sr})_j \right]_0 \cos \psi$$
$$+ \frac{1}{2} \left[\left(\sigma_z' \frac{L t}{\cos \lambda} \cos \beta \right)_{k+1} + \left(\sigma_z' \frac{L t}{\cos \lambda} \cos \beta \right)_k \right]_0 \tag{38}$$

where h = depth of beam cross section at root of rear spar. It follows

that the stresses and deflections of the beam may be determined for a

given load P, whence the influence coefficients may be obtained by

Equation (34).

The value of the redundant load P imposed by the wing rib is

determined by considering the compatibility of deflections at the cut.

The deflection of the outer panel front spar must be equal to the de-

flection of the wing rib at the point H. This is given by

$$\delta_H = \delta_{H, f_n (V,H,T)}^{\delta, F} \, F_n (V,H,T) + \delta_{H, f_n (V,H,T)}^{\delta, F} \, F_n (P,M) \tag{39}$$

where f_n(V, H, T), F_n(V, H, T), etc., represent the loading conditions.

The deflection of the wing rib at the point H is a function of the rota-

tion of the rear spar of the inner panel and the elasticity of the rib. The

deflection is given by

$$\delta_H = \left[\theta^{\delta,F}_{D,f_n(V,H,T)} F_n(V,H,T) + \theta^{\delta,F}_{D,M} M \right] c \sin \psi$$

$$+ \frac{1}{3} \frac{Pc^3}{(EI)_{wing\ rib}} \tag{40}$$

where $\quad \theta^{\delta,F}_{m,n} \quad$ = rotation influence coefficient for the behavior at m

due to a unit moment at point n.

The value of P may be determined by equating Equations (39) and (40).

The stresses and deflections of the wing under external load including

the effects of the wing rib are given by Equation (34). An example is

given in Part VII.

VI. COMPARISON OF THE FINITE DIFFERENCE METHOD AND THE

ANALYTICAL SOLUTIONS FOR A CANTILEVER BEAM WITH A

RIGID ORTHOGONAL SUPPORT

The problem of determining the stresses and deflections of a thin-walled beam of uniform thickness subject to variable twist has been formulated by von Karman and Chien (Reference 7). From the formulation, an analytical solution was developed for a beam with a doubly symmetric cross section, and in particular, an example was worked out for a semi-infinite beam of rectangular cross section on a rigid support subject to end twist.

The formulation of the problem given in Part II is in agreement with the formulation by Karman and Chien. Therefore, it is of interest to compare the finite difference and analytical solutions for a representative beam. Consider the tube of rectangular cross section shown in Figure 8(a). The tube has a length of 25.4 in., a cross section 2 in. x 8 in., and a wall thickness of one-eighth inch. The tube is loaded by an end torque of 29,600 inch-pounds. The cross section and boundary conditions are symmetrical about both principal shear axes, and therefore permit maximum simplification of the procedures. The tube is oriented by the coordinate system shown in Figure 8(a) and the simplified relaxation mesh for this case is given by Table 1. As a result the mesh corresponds to the upper left hand quadrant of the cross section. The boundary condition along the edges of the mesh parallel to the z axis is that the dis-

placement (w) is zero. The boundary conditions along the edges of the

mesh parallel to the s coordinate corresponding to the fixed and free ends

of the beam are that the displacement (w) and the derivative of the dis-

placement (σ) are zero respectively. This is illustrated in Figure

8(b). The length of the intervals of the relaxation mesh and idealization

of the cross section may be chosen so as to simplify the difference

Equation (24). It will be seen that it is desirable to let $L_{k+1} = L_k = L$, and

$$\frac{E' a_j L}{G \ell^2 t} = 1 \tag{41}$$

Assuming, μ = .3, and since a_j = Lt, the relationship between the

intervals is given by

$$\ell = 1.690L$$

In this problem, it is relatively easy to advance to finer mesh and

employ line and block relaxation techniques. As a result, the interval

L = 1 in. is chosen for the initial mesh for the entire domain. Subse-

quently, the mesh size is advanced to L = 1/2 inch for the half of the

domain corresponding to the root region, and to L = 1/4 inch for the

domain corresponding to the corner of the cross section in the root

region. The advance to finer mesh employs the procedures outlined in

Part IV, and is illustrated in Figure 8(b). The nodes and intervals of

the mesh are identified in accordance with Figure 4.

The difference equation for a beam under torsion with doubly symme-

trical cross section and boundary condition is

$$\frac{E' a_j}{G \ell^2} \left(w_{n+1} + w_{n-1} - 2 w_n \right)_j + \left(\frac{t_{k+1}}{L_{k+1}} \right) \left(w_{j+1} - w_j \right)_n - \left(\frac{t_k}{L_k} \right)_j \left(w_j - w_{j-1} \right)_n \qquad (24)$$

$$= -\frac{1}{I_c} \left(t_{k+1} r_{k+1} - t_k r_k \right) \left[\frac{T}{G} + \sum_j \left(t_{k+1} r_{k+1} - t_k r_k \right)_j w_j \right]$$

The equation is simplified by the choice of the mesh intervals. Applying Equation (41), the result is

$$\left(w_{n+1} + w_{n-1} \right)_j + \left(w_{j+1} + w_{j-1} \right)_n - 4 w_{j,n} \qquad (42)$$

$$= -\frac{L}{I_c} \left(r_{k+1} - r_k \right)_j \left[\frac{T}{G} + t \sum_j \left(r_{k+1} - r_k \right)_j w_j \right]$$

For every region in the domain except the region corresponding to the corners of the tube, the equation reduces to the Laplace equation,

$$\left(w_{n+1} + w_{n-1} \right)_j + \left(w_{j+1} + w_{j-1} \right)_n - 4 w_{j,n} = 0 \qquad (43)$$

For the region in the domain corresponding to the corner of the tube, Equation (42) becomes

$$\left(w_{n+1} + w_{n-1} \right)_j + \left(w_{j+1} + w_{j-1} \right)_n - 4 w_{j,n} = -L \left(\frac{2.31}{10^3} + 0.45 \, w_{j,n} \right) \qquad (44)$$

where the numerical calculations are given in Appendix A. The relaxation technique is facilitated if decimal values of the variable are avoided, so let

$$w' = w \times 10^6 . \quad \text{Whence Equation (44) becomes}$$

$$\left(w'_{n+1} + w'_{n-1} \right)_j + \left(w'_{j+1} + w'_{j-1} \right)_n - \left(4 - 0.45 \right) w'_{j,n} = -2310 \, L \qquad (45)$$

The difference equations corresponding to the mesh sizes are obtained by inserting the value of the interval (L) into Equation (45). The equation becomes,

$$\left(w'_{n+1} + w'_{n-1} \right)_j + \left(w'_{j+1} + w'_{j-1} \right)_n - K_1 \, w'_{j,n} = -K_2 \qquad (46)$$

where $K_1 = 3.55$, 3.775, 3.8875 and $K_2 = 2310$, 1155, 577 for $L = 1$, $1/2$ and $1/4$ inches respectively.

The point relaxation operators are determined by the procedures outlined in Part IV. Diagrammatically, the relaxation operators are shown in Figure 8(c). By the simple torsion theory, the initial displacements (w) were assumed to be zero for the coarse mesh. The solutions of the difference Equations (43) and (46) in the domain corresponding to the upper left hand quadrant of the cross section with the applicable boundary conditions are obtained by the procedures in Parts IV and V.

The final values of the displacement (w) for the rectangular tube under torsion are shown in Figure 9. The values of the normal stress are calculated from the displacements by Equation (31), and are shown in Figure 9. The values of displacement and stress determined with the fine mesh (L = $1/2$ in.) are plotted as a function of the tangential coordinate in Figures 10 and 11 respectively. The normal stresses along the corner of the tube are plotted as a function of the axial coordinate in Figure 12. The normal stresses at the support for various mesh sizes are compared to the results of the analytical solution by von Karman and Chien by plotting the stresses as a function of the mesh size in Figure 13.

The comparison of the stresses determined analytically and by the finite differences method show good agreement, with a possible exception of the region in the neighborhood of the corner of the tube at the support.

In this region, the large gradients of the normal stress, both in the axial and tangential directions, probably account for the discrepancy. An additional advance to a finer mesh would undoubtedly improve the agreement, but in view of the fact that the mesh interval is already within 5 percent of the width of the tube, it does not appear to be of practical importance to do so. This example illustrates the distribution of the "bending stresses due to torsion" in the support region of the tube, and points out the necessity of an accurate knowledge of the support conditions if precise stress data are desired.

VII. COMPARISON OF THE THEORETICAL SOLUTIONS AND EXPERIMENTAL DATA FOR A SWEPT BACK WING WITH A CARRY THROUGH BAY UNDER SYMMETRICAL BENDING

The National Advisory Committee for Aeronautics has conducted a series of experimental investigations on the behavior of an untapered box beam, representing the main structural component of a full span, two spar, 45° swept wing with a carry through bay. The investigation was reported in references (8) and (9) for the cases of symmetrical and antisymmetrical bending and torsion respectively. The full span model was carefully constructed, and included complete instrumentation to record the normal and shear stresses, in addition to the distortions of the structure. As a result, the NACA experimental data is considered to be unusually reliable, and is compared to the theoretical difference solution for symmetrical bending in order to establish the validity of the theoretical assumptions.

The experimental wing was constructed of aluminum alloy with the swept-back parts consisting of two boxes with their longitudinal axes at right angles, joined by and continuous with a short rectangular carry through bay representing that part of the wing inside the fuselage. The details of the experimental swept-back box beam are shown in Figure 14. The wing was loaded in symmetrical bending with a downward end shear of 2500 pounds acting through the principal shear axes of the tip cross section.

The procedure for theoretically determining the stresses and deflections of the swept wing follow the methods given in Part 3 of Part V. The swept wing is equivalent to a cantilevered shell with an orthogonal support and a shell on a simple support. The beams are oriented by the coordinate system shown in Figure 7(b). The cross section and boundary conditions are symmetrical about the x axis, and therefore permit a simplification of the procedures. The simplified relaxation mesh for this case is given by Table 1. As a result, the mesh corresponds to the upper half of the cross section. Since the purpose of this investigation is to establish a satisfactory minimum idealization of the swept wing in addition to establishing the validity of the theoretical assumptions, the cross section is idealized into four, six, and ten stiffener or flange elements.

The relaxation mesh for the inner and outer panels are shown in Figure 15(a), (b), and (c) for the four, six and ten element beams respectively. The mesh lines (j) parallel to the z axis correspond to the boundaries, the flanges (F_1, F_2) or to the stringers (S). The intervals between the lines corresponding to the flanges and stringers are constant. The mesh lines parallel to the s coordinate are located by virtue of the desired intervals between the mesh lines. For instance, there are three mesh lines (n) of equal interval in the inner panel in order that the difference equations for stresses (31) may be used, and the intervals between the mesh lines (n) in the outer panel are equal to the interval

between the stiffeners and flanges. As it will be seen later, it is desirable to adopt two methods of identifying the modes and intervals of the mesh. For the difference equation, the mesh is identified in accordance with Figure (4) with the addition that the mesh lines parallel to the z axis are annotated F_1, F_2, S, etc. corresponding to the flanges and stringers. For the equations of equilibrium and compatibility, the mesh nodes in the neighborhood of the joint are identified by numbers. The identification of the meshes is illustrated in Figures 15(a), (b) and (c).

The distribution of material for the idealized cross section is governed by the assumptions made in Parts II and III. The total areas of the idealized and actual stiffeners and flanges are equal, with the exception that the idealized flanges contain the equivalent spar web material and the sheet material attached to the flange. The equivalent spar web material is that equivalent area representing the moment carrying capacity of the webs, and for each flange is equal to one sixth of the cross sectional area of the actual web. The areas of the idealized and actual sheet are equal. The concept of effective width has not been introduced, primarily to avoid inconsistencies at the oblique boundary. The numerical calculations for the idealization are carried out in Appendix B. 1, and the idealized cross sections are illustrated in Figure 16.

The idealized support structure discussed in Part 3 of Part V is composed of a fuselage rib, an inboard orthogonal wing rib, and the

joints of the sheet and stiffeners between the inner and outer shells as
shown in Figure 7(a).

It is now desirable to consider the boundary conditions. From
Table 1, the boundary condition along the edges of the mesh parallel to
the z axis is that the displacement (w) is zero. The boundary conditions
along the edges of the mesh parallel to the s coordinate are dependent
upon the boundary conditions on the wing and the conditions at the inter-
section of the inner and outer shells.

The boundary conditions for the wing loaded in symmetrical bending
are that the displacements are zero at the center of the carry through
bay and the stresses are zero at the tip of the wing. This is illustrated
in Figure 15. With regard to the boundary condition applied at the tip
of the wing, there is one important point to be brought up. In thin-walled
beams, the "disturbances" due to the oblique support are nearly
"damped" out in about one (normal) chord length from the support. As
a result, the final convergence of the solution of the difference equations
is greatly speeded up if the boundary conditions are initially applied at a
cross section one chord length from the support. The boundary conditions
are determined by elementary beam theory. The resulting wing is called
a "clipped" wing, and this procedure is used for the six and ten element
beams. The numerical calculations for this boundary condition are
carried out in Appendix B.2.

Boundary conditions are imposed on the compatibility of displacements

and equilibrium of forces at the intersection of the inner and outer shells.
The compatibility of displacements is given by Equation (35). In the
wing with an angle of sweep of 45°, the equation becomes

$$(w_j)_i = 1.414 \, (w_j)_o \qquad (47)$$

As discussed in Part 3 of Part V, the conditions on the equilibrium of
forces, and in addition, the difference equation are dependent upon
whether the behavior of the wing under the action of external loads or
rib loads is being considered. As a result, these equations are obtained
for a general loading, with the treatment of the particular loading condi-
tions discussed as required. Hence, the equation of equilibrium of
forces at the point (j, n) is

$$(\sigma_j \, a_j)_i = \frac{Pc \sin \psi}{h} + \left[(\sigma_z \, a_{ST})_j \right]_o \cos \psi \qquad (38)$$

$$+ \frac{1}{2} \left[(\sigma_z' \frac{Lt}{\cos \lambda} \cos \beta)_{K+1} + (\sigma_z' \frac{Lt}{\cos \lambda} \cos \beta)_K \right]_o$$

where $\quad \sigma_z' = \sigma_z \, (\cos^2 \lambda + \mu \sin^2 \lambda) - \tau_{sz} \sin 2\lambda$

$$\lambda = \psi \, / \cos \alpha \, / \qquad (37)$$

$$\beta = \psi \, / \sin \alpha \, /$$

In the idealized cross section, the spar web does not transmit normal
stress. Thus, only the elements in the idealized cover of the box
contribute to the equilibrium equation. Hence substituting Equation (37)
into Equation (38), the result is

$$(\sigma_j \, a_j)_i = 3P + 0.707 \left[(\sigma_z \, a_{ST})_j \right]_o \qquad (48)$$

$$+ 0.03535 \left\{ \left[(.656 \, \sigma_z - \tau_{sz}) L \right]_{K+1} + \left[(.656 \, \sigma_z - \tau_{sz}) L \right]_K \right\}_o$$

where $\qquad\mu\;=\;.312\qquad$ (Ref. Appendix 3.3)

$$c\;=\;29.7\;\;in.$$

$$h\;=\;7\qquad in.$$

$$t\;=\;0.050\;\;in.\qquad (Ref.\;Figure\;15)$$

$$\lambda\;=\;\psi\;=\;45\;deg.$$

$$\beta\;=\;0$$

By Equation (30), the shear stress (τ_{sz}) for the upper cover sheet of the outer panel becomes

$$\tau_{sz}\;=\;4\times 10^{6}\left[\frac{1}{L_{\kappa}}\,(w_{j}-w_{j-1})+\frac{3.5T}{4\times 10^{6}\times 277.2}\right.$$
$$\left.+\frac{3.5}{277.2}\,(-1.966\,(w_{F_{1}}-w_{F_{2}}))\right]$$

$$E\;=\;10.5\times 10^{6}\;psi$$

where $\qquad G\;=\;4\times 10^{6}\quad psi\qquad$ (Ref. Appendix B.3)

$$I_{c}\;=\;277.2\;\;in.^{4}$$

$$\sum_{j}\,(t_{\kappa+1}\,r_{\kappa+1}-t_{\kappa}\,r_{\kappa})_{j}\,w_{j}\;=\;-1.966\,(w_{F_{1}}-w_{F_{2}})$$

$$r_{\kappa}\;=\;3.5\;\;in.\qquad\qquad (Ref.\;Figure\;15)$$

The relaxation technique is facilitated if decimal values of the variable are avoided, so let $w'\;=\;w\times 10^{5}$, whence

$$\tau_{sz}\;=\;0.01262\,T-\frac{40}{L_{\kappa}}\left[w'_{j-1}-w'_{j}+0.0248\,L_{\kappa}\,(w'_{F_{1}}-w'_{F_{2}})\right] \tag{49}$$

The difference equation for a beam under vertical shear and torsion with the cross section and boundary conditions symmetrical about the

x axis is

$$\frac{E\,a_j}{G\,\ell^2}\,(w_{n+1}+w_{n-1}-2w_n)_j + \left(\frac{t_{k+1}}{L_{k+1}}\right)(w_{j+1}-w_j)_n - \left(\frac{t_k}{L_k}\right)_j(w_j-w_{j-1})_n \qquad (25)$$

$$= -\frac{1}{I_c}\,(t_{k+1}\,r_{k+1}-t_k\,r_k)_j\left[\frac{T}{G}+\sum_j(t_{k+1}\,r_{k+1}-t_k\,r_k)_j\,w_j\right]$$

$$-\frac{1}{A_v}\,(t_{k+1}\sin\alpha_{k+1}-t_k\sin\alpha_k)_j\left[\frac{V}{G}+\sum_j(t_{k+1}\sin\alpha_{k+1}-t_k\sin\alpha_k)_j\,w_j\right]$$

The additional section properties associated with the principal shear axes, and the summation terms are evaluated in Appendix B.3 to give

$$I_c = 528.2 \quad \text{in.}^4 \quad \text{inner panel}$$

$$A_v = 1.092 \text{ in.}^2 \quad \text{inner and outer panels}$$

$$\sum_j(t_{k+1}\,r_{k+1}-t_k\,r_k)_j\,w_j = -2.910\,(w_{F_1}-w_{F_2}) \qquad \text{inner panel}$$

$$\sum_j(t_{k+1}\sin\alpha_{k+1}-t_k\sin\alpha_k)_j\,w_j = -0.156\,(w_{F_1}+w_{F_2}) \qquad \text{inner and outer panels}$$

$$\ell = 7.5 \text{ in.} \quad \text{inner panel}$$

substituting into Equation (25), and recalling that $w' = w \times 10^5$, the expression for the inner panel is

$$.0466\ a_j\,(w'_{n+1}+w'_{n-1}-2w'_n)_j + \left(\frac{t_{k+1}}{L_{k+1}}\right)(w'_{j+1}-w'_j)_n - \left(\frac{t_k}{L_k}\right)_j(w'_j-w'_{j-1})_n$$

$$= -\frac{1}{528.2}\,(t_{k+1}\,r_{k+1}-t_k\,r_k)_j\left[\frac{T}{40}-2.910\,(w'_{F_1}-w'_{F_2})_n\right] \qquad (50)$$

$$-\frac{1}{1.092}\,(t_{k+1}\sin\alpha_{k+1}-t_k\sin\alpha_k)_j\left[\frac{V}{40}-0.156\,(w'_{F_1}+w'_{F_2})_n\right]$$

similarly Equation (25) for the outer panel becomes

$$\frac{2.62\,a_j}{\ell^2}\left(w'_{n+1}+w'_{n-1}-2w'_n\right)_j+\left(\frac{t_{K+1}}{L_{K+1}}\right)_j\left(w'_{j+1}-w'_j\right)_n-\left(\frac{t_K}{L_K}\right)_j\left(w'_j-w'_{j-1}\right)_n$$

$$=-\frac{1}{277.2}\left(t_{K+1}\,r_{K+1}-t_K\,r_K\right)_j\left[\frac{T}{40}-1.966\left(w'_{F_1}-w'_{F_2}\right)_n\right]\qquad(51)$$

$$-\frac{1}{1.092}\left(t_{K+1}\sin\alpha_{K+1}-t_K\sin\alpha_K\right)_j\left[\frac{V}{40}-0.156\left(w'_{F_1}+w'_{F_2}\right)_n\right]$$

Now that the equations of equilibrium of forces and the difference equations for the general loading condition for the swept wing have been derived, the particular loading conditions may be investigated. As indicated in part 3, Part V, the behavior of the wing with an elastic wing rib may be determined by the method of influence coefficients. The influence coefficients are determined for the behavior of the equivalent wing with no wing rib under the action of external loads and under the action of loads imposed by the wing rib. The case of the four-element beam is discussed in detail in this section, and the details for the six and ten-element cases are given in the appendices.

Consider the behavior of the wing with no wing rib under the action of external loads. The compatibility of displacements of the joint are given by Equation (47) to be

$$W_3 = 1.414\ W_5$$
$$W_4 = 1.414\ W_8 \qquad (52)$$

The equilibrium of the joint is governed by Equations (48) and (49). For this case $P = T = 0$ and the section dimensions are given in Figure 15(a). Inserting Equation (49) into (48), the expression becomes

$$2.156 \left(\sigma_j \right)_i = .707(1.111)\left(\sigma_j \right)_o + .03535(29.7)\left\{ .656\left(\sigma_j \right)_o \right.$$

$$\left. + \frac{40}{29.7}\left[w'_{F_1} - w'_{F_2} + .0248\,(29.7)(w'_{F_1} - w'_{F_2}) \right]\right\} \qquad (53)$$

$$\left(\sigma_j \right)_i = .684\left(\sigma_j \right)_o + 1.137\left(w'_{F_1} - w'_{F_2} \right)$$

substitute the expressions for displacements in terms of the stresses

given by Equation (31) into Equation (53), and obtain for flange F_1,

$$\frac{10.5 \times 10^6}{2 \times 7.5 \times 10^5}\left(3w'_3 - 4w'_1 \right) = .684 \times \frac{10.5 \times 10^6}{2 \times 29.7 \times 10^5}\left(-3w'_5 + 4w'_7 - w'_9 \right)$$

$$+ 1.137\left(w'_7 - w'_8 \right)$$

Inserting the compatibility relationship for w'_3 given by Equation (52)

into the above equation gives

$$-w'_5 + .8403w'_1 + .1793w'_7 - .03628w'_9 - .03412\,w'_8 = 0 \qquad (54)$$

Similarly, the expression for the flange F_2 is

$$w'_8 + .9082\,w'_2 + .03921\left(w'_{10} - w'_6 \right) + .03688\,w'_7 = 0 \qquad (55)$$

The difference equations for the inner and outer panels are given by

Equations (50) and (51) respectively. The section dimensions are given

in Figure 15(a). For the inner panel, $V = T = 0$, and the equation for

flange F_1 becomes

$$(.0466)(2.156)(w'_{n+1} + w'_{n-1} - 2w'_n)_j + \left(\frac{.050}{41.8} \right)(w'_{F_2} - w'_{F_1})_n - \left(\frac{.078}{3.5} \right)(w'_{F_1})_n$$

$$= -\frac{1}{528.2}\,(.050 \times 3.5 - .078 \times 20.9)(-2.910)(w'_{F_1} - w'_{F_2})_n$$

$$- \frac{1}{1.092}\,(-.078)(-.156)(w'_{F_1} + w'_{F_2})_n$$

Simplifying, and as $(w_{n-1})_j = 0$ by boundary conditions,

$$-(w'_{F_1})_n + .0210 \, (w'_{F_2})_n + .489 \, (w'_{n+1})_{F_1} = 0 \qquad (56)$$

Similarly for flange F_2,

$$-(w'_{F_2})_n + .0210 \, (w'_{F_1})_n + .489 \, (w'_{n+1})_{F_2} = 0 \qquad (57)$$

For the outer panel, the equation for flange F_1 becomes

$$\frac{2.62 \times 1.853}{(29.7)^2} \, (w'_{n+1} + w'_{n-1} - 2w'_n)_j + \frac{.050}{29.7} \, (w'_{F_2} - w'_{F_1})_n - \frac{.078}{3.5} \, (w'_{F_1})_n$$

$$= -\frac{1}{277.2} \, (.050 \times 3.5 - .078 \times 14.85) \left[\frac{T}{40} - 1.966 \, (w'_{F_1} - w'_{F_2})_n \right]$$

$$- \frac{1}{1.092} \, (-.078) \left[\frac{V}{40} - .156 \, (w'_{F_1} + w'_{F_2})_n \right]$$

Simplifying

$$-(w'_{F_1})_n + .346 \, (w'_{F_2}) + .326 \, (w'_{n+1} + w'_{n-1})_{F_1} - .1058 \, V - .00525 \, T = 0 \qquad (58)$$

Similarly for flange F_2

$$-(w'_{F_2})_n + .346 \, (w'_{F_1})_n + .326 \, (w'_{n+1} + w'_{n-1})_{F_2} - .1058 \, V + .00525 \, T = 0 \qquad (59)$$

The loading condition for the outer panel is $V = 2500$ pounds and $T = 0$.

Inserting the loading conditions, the equations for flange F_1 and F_2 become respectively

$$-(w'_{F_1})_n + .346 \, (w'_{F_2})_n + .326 \, (w'_{n+1} + w'_{n-1})_{F_1} + 265 = 0 \qquad (60)$$

$$-(w'_{F_2})_n + .346 \, (w'_{F_1})_n + .326 \, (w'_{n+1} + w'_{n-1})_{F_2} + 265 = 0$$

Likewise the equations for the six and ten-element beams may be derived. These equations are given in Appendix B.4.

The relaxation operators are determined from the difference equation by the procedures outlined in Part IV. Diagrammatically, the point relaxation operators for the inner and outer panels are shown in Figure 17. The initial displacements (w) for the mesh were obtained by simple bending theory. The solutions of the difference equations in the domain corresponding to the upper half of the cross section with the applicable boundary conditions on the value and derivative of the displacement are obtained by the procedures in Parts IV and V. The final values of displacement for the swept wing with no wing rib under symmetrical bending are shown in Figure 18. The values of the normal stress are calculated from the displacements by Equation (31), and are shown in Figure 18. The deflection and rotation of the beam are calculated by Equation (28), with $u_j = v_j = 0$ at AA' and DD', and are given in Table 2.

Consider the behavior of the wing under loads imposed by the wing rib. With reference to Part 3, Part V, the action of the wing rib is to apply a vertical load P at HH' and a moment (M = Pc sin ψ) at DD'. The vertical load P at HH' is equivalent to a vertical and a torque load applied at the cross section HH'D'D. The loading of the inner panel is V = T = 0. Since the experimental value for P is approximately 1000 pounds, let the loading of the outer panel be P = V = 1000 pounds, T = 14,850 in. pounds for the cross section HH'D'D and inboard, and V = T = 0 outboard of section HH'D'D. The compatibility of displacements

of the joint are unaffected by the loading, and are given by Equation (52).

The equilibrium of the joint is governed by Equations (48) and (49).

Inserting the proper constants, Equation (49) becomes

$$\tau_{sz} = 187 - 2.34(w'_{F_1} - w'_{F_2}) \tag{61}$$

Equation (61) may be substituted into Equation (48) to obtain the equation

of equilibrium of forces

$$2.156 \left(\sigma_j \right)_i = 3P + .707 (1.11)(\sigma_j)_o + .03535 (29.7)$$

$$\left[.656 \left(\sigma_j \right)_o - 187 + 2.34 \left(w'_{F_1} - w'_{F_2} \right) \right] \tag{62}$$

$$(\sigma_j)_i = 1.393P + .684 \left(\sigma_j \right)_o + 1.137 \left(w'_{F_1} - w'_{F_2} \right) - 91$$

The term including P is only applicable to the equilibrium equations for

the rear flange. Substitute the expressions for displacements in terms

of the stresses given by Equation (31) into Equation (62), and obtain for

flange F_1

$$\frac{10.5 \times 10^6}{2 \times 7.5 \times 10^5} (3w'_3 - 4w'_1) = .684 \times \frac{10.5 \times 10^6}{2 \times 29.7 \times 10^5}(-3w'_5 + 4w'_7 - w'_9)$$

$$+ 1.137 \left(w'_7 - w'_8 \right) - 91$$

Inserting the compatibility relationship for w'_3 given by Equation (52)

into the above equation gives

$$-w'_5 + .8403 \, w'_1 + .1793 \, w'_7 - .03628 \, w'_9 - .03412 \, w'_8 - 3 = 0. \tag{63}$$

Similarly, the expression for the flange F_2 is

$$-w'_8 + .9082 \, w'_2 + .03921 \left(w'_{10} - w'_6 \right) + .03688 \, w'_7 + 42 = 0 \tag{64}$$

The difference equations for the inner and outer panels are given by Equations (50) and (51) respectively. For the inner panel, $V = T = 0$, and the equations for flanges F_1 and F_2 are given by Equations (56) and (57). For the outer panel, there are two loading conditions. In the cross section HH'D'D and inboard, the loading is $V = 1000$ pounds and $T = 14,500$ inch-pounds. Inserting the loading conditions into Equations (58) and (59), the equations for flanges F_1 and F_2 become respectively

$$-(W'_{F_1})_n + .346\,(W'_{F_2})_n + .326\,(W'_{n+1} + W'_{n-1})_{F_1} - 184 = 0$$

$$\tag{65}$$

$$-(W'_{F_2})_n + .346\,(W'_{F_1})_n + .326\,(W'_{n+1} + W'_{n-1})_{F_2} - 28 = 0$$

Outboard of section HH'D'D, the loading is $V = T = 0$. Inserting the loading conditions into Equations (58) and (59), the equations for the flanges F_1 and F_2 become respectively

$$-(W'_{F_1})_n + .346\,(W'_{F_2})_n + .326\,(W'_{n+1} + W'_{n-1})_{F_1} = 0 \tag{66}$$

$$-(W'_{F_2})_n + .346\,(W'_{F_1})_n + .326\,(W'_{n+1} + W'_{n-1})_{F_2} = 0$$

Likewise the equations for the six-element beam may be derived. These equations are given in Appendix B.5.

The relaxation operators are unaffected by the loading, and are shown in Figure 17. The solutions of the difference equations in the domain corresponding to the upper half of the cross section with the applicable boundary conditions on the value and the derivative of the displacement are obtained by the procedures in Parts IV and V. The final values of the

displacement for the swept wing under loads imposed by the wing rib are

shown in Figure 19. The values of the normal stress are calculated from

the displacements by Equation (31), and shown in Figure 19.

The deflection and rotation of the beam are calculated by Equation (28),

with $u_j = v_j = 0$ at AA' and DD', and are given in Table 2.

The value of the redundant load P imposed by the wing rib is deter-

mined by considering the compatibility of deflections at the cut. The

deflection of the outer panel front spar must be equal to the deflection of

the wing rib at the point H. The deflection of the outer panel front spar

at the point H is given by Equation (39) to be

$$\delta_H = -.043 + 14.85(-.00295) + \left[.012 + 14.85(00078)\right]\frac{P}{1000} \tag{67}$$

where the incremental deflections are given by Equation (29) and Table 2.

Similarly, the deflection of the wing rib at the point H is given by

Equation (40) to be

$$\delta_H = (29.7)(.707)\left[\frac{-1244}{10^5}\frac{1}{(3.5)} - \frac{30}{10^5}\frac{1}{(3.5)}\frac{P}{1000}\right] - \frac{P(29.7^3)}{3(6.63)(30 \times 10^6)} \tag{68}$$

where the rotations are obtained from the displacements given in

Figures 18 and 19. The value of $P = 167$ pounds is obtained by equating

Equations (67) and (68). Likewise the value of the redundant load P may

be determined for the six-element beam. This is computed in Appendix

B.6.

The stresses and deflections of the wing under external load including

the effects of the wing rib are given by application of Equation (34). The

final values of normal stresses and deformations for the swept wing with a wing rib are shown in Figure 20 and Table 2 respectively.

The axial distribution of the normal stresses for the four, six and ten-element beams for the normal and "clipped" cases is shown in Figure 21(a), (b) and (c). The normal stresses at various cross sections are compared to the experimental data in Figure 22. The deflection and rotation deformations are compared to the experimental data in Figure 23.

The stresses as theoretically determined by the use of an idealized six-element beam show good agreement with experimental data. The stresses determined using a four-element beam do not show the effects of shear lag, and therefore do not have the desired characteristics. However it is to be pointed out that the maximum stresses are closely approximated by this simple idealization. The stresses determined by using a ten-element beam are in close agreement with the stresses given by the six-element beam with the exception of the low stress region in the neighborhood of the front flange joint.

The comparison of the deformations determined by the method of finite differences with the experimental data shows good agreement for the deflections and fair agreement for the rotations. The less accurate solution for the beam rotations is probably due to the assumption of a rigid fuselage rib. A preliminary investigation revealed that the assumption of an elastic fuselage rib improves the agreement. The procedure for taking into account the elasticity of the fuselage rib is

identical to the methods outlined in Part V, with the exception that the

equations of equilibrium of forces and compatibility of displacements in

the transverse direction are not eliminated, and require the consideration

of the elasticity of the fuselage rib. Since the bending loads in the

fuselage rib are high, the problem of approximating the rib loads by a

number of concentrated forces is important.

VIII. CONCLUSION

The method is a rational engineering approach to the problem of determining the stresses and deflections of unswept and swept, thin-walled beams of uniform closed cross section. The cross section, loading distribution, and boundary conditions are assumed to be arbitrary. The method is based on the differential equation governing the behavior of orthogonal elastic shells. The differential equation is transformed into a difference equation, and the solution is obtained by the relaxation technique. The comparison of the theoretical solution and experimental data for a swept back wing with a carry through bay under symmetrical bending showed good agreement.

As a result of the investigation, it is apparent that the swept wing with a carry through bay may be idealized by intersecting orthogonal shells, with elastic fuselage and inboard wing ribs. In the event that there is sufficient interest, it would be desirable to conduct further investigations in order to broaden the field of applicability of this method. The effects of symmetrical and antisymmetrical loading, taper of the cross section, and the influence of rib rigidity on rotational deformations have not been investigated with any completeness. For these conditions, it would be desirable to indicate the relationship between the idealization of the beam and the accuracy of the solution. In addition, there is the

possibility that cut outs and temperature effects may be studied by modification of the theory. In conclusion, it is believed that the method provides a more accurate and rapid means of investigating the behavior of thin-walled beams than was previously available.

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LIST OF APPENDICES

APPENDIX A

NUMERICAL DATA FOR RECTANGULAR TUBE UNDER TORSION

The difference equation governing the behavior of the rectangular

tube is given by Equation (42). The equation is

$$(w_{n+1} + w_{n-1})_j + (w_{j+1} + w_{j-1})_n - 4 w_{j,n} \qquad \text{A-1}$$

$$= -\frac{L}{I_c} (r_{k+1} - r_k)_j \left[\frac{T}{G} + t \Sigma_j (r_{k+1} - r_k)_j w_j \right]$$

The constants in the equation may be evaluated as follows. By

Equation (27)

$$I_c = \Sigma_j t_k r_k^2 L_k \qquad = \frac{2}{8} (8 \times 1 + 2 \times 4^2) = 10 \ \ \text{in.}^4$$

The cross section of the tube and the boundary conditions are doubly

symmetric. Therefore under torsion, the displacements (w_j) at the

corners are doubly antisymmetric. Hence

$$\Sigma_j (r_{k+1} - r_k)_j w_{j,n} = 4(4-1) w_{j,n} = 12 w_{j,n}$$

where w_j is the displacement of the corner of the upper left hand quad-

rant of the tube. The external torque applied to the cross section is

29,600 inch-pounds. Let $E = 10^7$ psi, $\mu = .3$ whence $G = E/2(1+\mu) = 10^7/2.6$.

Inserting the numerical values, Equation (42) becomes

$$(w_{n+1} + w_{n-1})_j + (w_{j+1} + w_{j-1})_n - 4 w_{j,n} \qquad \text{A-2}$$

$$= -\frac{L}{10} (4-1) \left[\frac{(29,600)(2.6)}{10^7} + \frac{1}{8} (12) w_{j,n} \right]$$

$$= -L \left[\frac{2.31}{10^3} + .45 w_{j,n} \right]$$

This corresponds to Equation (44) in Part VI.

APPENDIX B

NUMERICAL DATA FOR SWEPT BOX BEAM UNDER

SYMMETRICAL BENDING

B.1   Idealized Cross Sections of Swept Beam

The details of the NACA swept box beam are shown in Figure 14.

The numerical calculations for idealizing the cross sections of the

swept beam are carried out in the following .

Outer Panel

The idealized outer shell has a length of 118-3/4 inches and a

7 inch by 29.7 inch rectangular cross section. The choice fortuitously

makes the length equal to four times the chord. The cross-sectional

area of the elements are as follows:

Area of each cover sheet (normal to z axis) $= 29.7 \times .050 = 1.485$ sq. in.

Area of each spar web $= 7 \times .078 = 0.546$ sq. in.

Total stiffener area per cover $= 14 (1\text{-}1/2 \times 1/16) = 1.312$ sq. in.

Area of each idealized flange = actual flange area + equivalent spar

web area + area of sheet effectively attached to flange

$$= 2 \tfrac{1}{2} \times \tfrac{1}{8} + .546 \times \tfrac{1}{6} + \left(31 \tfrac{3}{4} - 29.7\right)\left(\tfrac{.050}{2}\right)$$

$$= 0.455 \text{ sq. in.}$$

Also, moment of inertia of section

$$= 2(2 \times .455 + 1.312 + .050 \times 29.7)(3.5)^2$$

$$= 90.8 \quad \text{in.}^4$$

Inner Panel

The idealized inner panel is composed of one half of the carry through bay, and has a length of 15 inches and a 7 inch x 41.8 inch rectangular cross section. (Ref. NACA Langley Field Drawing No. LD-14839. The distance between the outside edges of flange angles = $43 \frac{55}{64}$ inches. Whence $43 \frac{55}{64} - (31 \frac{3}{4} - 29.7) = 41.8$ in. The cross sectional areas of the elements are as follows:

Area of each cover sheet (normal to z axis) = 41.8 x .050 = 2.090 sq. in.

Area of each spar web = 0.546 sq. in.

Total stiffener area per cover = 1.312 sq. in.

Area of each idealized flange = 0.455 sq. in.

The idealized cross section of the beam contains stiffeners and flanges that transmit axial loads, cover sheets that transmit axial and shear loads, and spar webs that transmit only shear loads. The cross-sectional areas may be essentially divided into the areas resisting the axial loads, and the areas resisting the shear loads. The areas resisting the axial loads are made up of equivalent areas of elements (a_j) which are composed of the area of the stiffener element (a_{ST}) and the area of the cover sheet element (a_{SH}) resisting the normal stress. The areas resisting the shear loads are the actual areas of the sheet. As a result, the idealized cross section is made up of equivalent areas of elements resisting the normal stress, and the sheet areas resisting the shear stress. In actuality, the resulting section is the

same as would be obtained by the effective width concept, except that
the effective width concept would be inconsistent at an oblique boundary
where the sheet stresses are resolved as shown in Figure 7(c).

The following example illustrates the idealization into a six-
element beam. Assume that the distribution of the stiffener area
across the cover is uniform. As indicated in Part III, assume that the
area of the cover sheet acting in conjunction with the stiffener area is
half of the area of the sheet material in each interval neighboring the
stiffener. Thus the flange and stiffener areas become for the outer
panel,

$$a_{flange} = (0.455 + \frac{1.312}{4}) + (\frac{1.485}{4})$$

$$= .783 + .371 = 1.154 \text{ sq. in.}$$

$$a_{ST} = \frac{1.312}{2} + \frac{1.485}{2}$$

$$= .656 + .742 = 1.398 \text{ sq. in.}$$

The idealized cross sections are illustrated in Figure 16.

B.2 "Clipped" Wing Boundary Conditions for Six and Ten-Element Beams

The outer panel is clipped a distance of 59.6 in. from the actual
tip as illustrated in Figures 15(b) and (c). The boundary condition on
the stress at the tip of the clipped wing is determined by elementary
beam theory. For a vertical shear of 2500 pounds applied at the tip of

the actual wing, the normal stress at the tip of the clipped wing is

$$\sigma = \frac{My}{I} = \frac{59.6 \times 2500 \times 3.5}{90.8} = 5724 \text{ psi.}$$

Let the mesh line corresponding to the tip of the clipped wing be (n).

Then by Equation (31)

$$\sigma_n = \frac{E}{2\ell}(w_{n+1} - w_{n-1})_j = \frac{105}{2\ell}(w'_{n+1} - w'_{n-1})_j = 5724$$

Thus
$$(w'_{n+1})_j = \frac{5724 \times 2\ell}{105} + (w'_{n-1})_j$$

$$= 109\ell + (w'_{n-1})_j$$

This condition is substituted in the difference equations corresponding

to the tip of the clipped wing. The procedure then follows the methods

outlined in Part IV.

B.3 <u>Evaluation of Section Constants</u>

In order that the comparison between the theory and experiment

may be made on the same basis, let $E = 10.5 \times 10^6$ psi, $G = 4 \times 10^6$ psi,

and $\mu = \frac{E}{2G} - 1 = 0.312$. It is to be noted that this modulus (E) is

different from the modulus used in Appendix A.

The section properties associated with the principal shear axes are

given by Equation (27). Evaluation of the constants for the inner and

outer panels gives,

$$I_c = \sum_k t_k L_k r_k^2$$

$$= 2(.050 \times 41.8 \times 3.5^2 + .078 \times 7 \times 20.9^2) = 528.2 \text{ in.}^4 \text{ (inner panel)}$$

$$= 2(.040 \times 29.7 \times 3.5^2 + .078 \times 7 \times 14.85^2) = 277.2 \text{ in}^4 \text{ (outer panel)}$$

$$A_v = \sum_k t_k L_k \sin^2 \alpha_k$$

$$= 2 \times .078 \times 7 = 1.092 \text{ in.}^2 \text{ (inner and outer panel)}$$

For the NACA swept wing under vertical shear and torsion, the displacements are antisymmetrical about the x axis. Therefore, evaluation of the summation terms in the difference equation for the inner and outer panels gives

$$\sum_j (t_{K+1} r_{K+1} - t_K r_K)_j \, w_j$$

$$= 2(.050 \times 3.5 - .078 \times 20.9)(w_{F_1} - w_{F_2})$$

$$= -2.910(w_{F_1} - w_{F_2}) \text{ (inner panel)}$$

$$= 2(.050 \times 3.5 - .078 \times 14.85)(w_{F_1} - w_{F_2})$$

$$= -1.966(w_{F_1} - w_{F_2} \text{ (outer panel)}$$

$$\sum_j (t_{K+1} \sin \alpha_{K+1} - t_K \sin \alpha_K)_j \, w_j$$

$$= 2(-.078)(w_{F_1} + w_{F_2})$$

$$= -.156(w_{F_1} + w_{F_2}) \text{ (inner and outer panel)}$$

$$\sum_j (t_{K+1} \cos \alpha_{K+1} - t_K \cos \alpha_K)_j \, w_j = 0 \text{ (inner and outer panel)}$$

B.4 Boundary Conditions and Difference Equations for the Behavior of the Wing With No Wing Rib Under External Loads for the Six and Ten-Element Beams

The boundary conditions and difference equations for the behavior of the wing for the six and ten-element beams are obtained by the procedures in Part VII. The notation is given in Figures 15(b) and (c).

For the six-element beam, the compatibility of displacements of the joint are given by Equation (47) to be

$$w'_4 = 1.414 \, w'_7$$

$$w'_5 = 1.414 \, w'_{11}$$

$$w'_6 = 1.414 \, w'_{15}$$

The equations of equilibrium are governed by Equations (48) and (49), and become

$$-w_7' + .7569\, w_1' + .3031\, w_{10}' - .06575\, w_{13}' - .02929\, w_{11}' - .01078\, w_{12}' = 0$$

$$-w_{11}' + .9173\, w_2' + .1057\, w_{14}' - .0785\, w_8' + .03725\,(w_{10}' - w_{15}') + .01002\,(w_{13}' - w_{12}') = 0$$

$$-w_{15}' + .8981\, w_3' + .07801\,(w_{18}' - w_{12}') + .03475\, w_{14}' + .01279\, w_{13}' = 0$$

The difference equations for the inner and outer panels are given by Equations (50) and (51) respectively. For the inner panel, the expressions are

$$-(w_{F_1}')_n + .0246\,(w_{F_2}')_n + .0188\,(w_S')_n + .478\,(w_{n+1}')_{F_1} = 0$$

$$-(w_S')_n + .01464\,(w_{F_1}' + w_{F_2}')_n + .486\,(w_{n+1}')_S = 0$$

$$-(w_{F_2}')_n + .0246\,(w_{F_1}')_n + .0188\,(w_S')_n + .478\,(w_{n+1}')_{F_2} = 0$$

Similarly, for the outer panel

$$-(w_{F_1}')_n + .096\,(w_S')_n + .392\,(w_{n+1}' + w_{n-1}')_{F_1} + .1192\,(w_{F_2}')_n + 128 = 0$$

$$-(w_S')_n + .0842\,(w_{F_1}' + w_{F_2}') + .416\,(w_{n+1}' + w_{n-1}')_S = 0$$

$$-(w_{F_2}')_n + .096\,(w_S')_n + .392\,(w_{n+1}' + w_{n-1}')_{F_2} + .1192\,(w_{F_1}')_n + 128 = 0$$

For the ten-element beam, the compatibility of displacements at the joint are given by Equation (47) to be

$$w_6' = 1.414\ w_{11}'$$
$$w_7' = 1.414\ w_{17}'$$
$$w_8' = 1.414\ w_{23}'$$
$$w_9' = 1.414\ w_{29}'$$
$$w_{10}' = 1.414\ w_{35}'$$

The equations of equilibrium are governed by Equations (48) and (49), and become

$$-w'_{11} + .6310\, w'_1 + .4837\, w'_{16} - .1103\, w'_{21} - .0362\, w'_{17} - .0067\, w'_{20} = 0$$

$$-w'_{17} + .8931\, w'_2 + .2059\, w'_{22} - .1528\, w'_{12} + .0628\, w'_{16} - .0531\, w'_{23}$$
$$+ .0098\, (-w'_{20} + w'_{21} - w'_{25}) = 0$$

$$-w'_{23} + .8931\, w'_3 + .2059\, w'_{28} - .1528\, w'_{18} + .0531\, (w'_{22} - w'_{29})$$
$$+ .0098\, (w'_{21} - w'_{25} + w'_{26} - w'_{30}) = 0$$

$$-w'_{29} + .8931\, w'_4 + .2059\, w'_{34} - .1528\, w'_{24} + .0531\, w'_{28} - .0628\, w'_{35}$$
$$+ .0098\, (w'_{26} - w'_{30} + w'_{31}) = 0$$

$$-w'_{35} + .8863\, w'_5 + .1549\, (w'_{40} - w'_{30}) + .0509\, w'_{34} + .0094\, w'_{31} = 0$$

The difference equations for the inner and outer panels are given by Equations (50) and (51) respectively. For the inner panel, the expressions are

$$-(w'_{F_1})_n + .0348\, (w'_{F_2})_n + .0532\, (w'_{S_1})_n + .456\, (w'_{n+1})_{F_1} = 0$$

$$-(w'_{S_1})_n + .0538\, (w'_{S_2} + w'_{F_1})_n + .446\, (w'_{n+1})_{S_1} = 0$$

$$-(w'_{S_2})_n + .0538\, (w'_{S_3} + w'_{S_1})_n + .446\, (w'_{n+1})_{S_2} = 0$$

$$-(w'_{S_3})_n + .0538\, (w'_{F_2} + w'_{S_2})_n + .446\, (w'_{n+1})_{S_3} = 0$$

$$-(w'_{F_2})_n + .0348\, (w'_{F_1})_n + .0532\, (w'_{S_3})_n + .456\, (w'_{n+1})_{F_2} = 0$$

Similarly, for the outer panel

$$-(w'_{F_1})_n + .0477\, (w'_{F_2})_n + .0769\, (w'_{S_1})_n + .438\, (w'_{n+1} + w'_{n-1})_{F_1} + 51 = 0$$

$$-(w'_{S_1})_n + .0842\, (w'_{S_2} + w'_{F_1})_n + .415\, (w'_{n+1} + w'_{n-1})_{S_1} = 0$$

$$-(w'_{S_2})_n + .0842\, (w'_{S_3} + w'_{S_1})_n + .415\, (w'_{n+1} + w'_{n-1})_{S_2} = 0$$

$$-(w'_{S_3})_n + .0842\, (w'_{F_2} + w'_{S_2})_n + .415\, (w'_{n+1} + w'_{n-1})_{S_3} = 0$$

$$-(w'_{F_2})_n + .0477\, (w'_{F_1})_n + .0769\, (w'_{S_3})_n + .438\, (w'_{n+1} + w'_{n-1})_{F_2} + 51 = 0$$

B. 5 Boundary Conditions and Difference Equations for the Behavior

of the Wing Under Loads Imposed by the Wing Rib for the Six-

Element Beam

The boundary conditions and difference equations for the behavior

of the wing for the six-element beam are obtained by the procedures in

Part VII. The notation is given in Figure 15(b). The compatibility of

displacements of the joint are given by Equation (47) to be

$$w'_4 = 1.414 \, w'_7$$

$$w'_5 = 1.414 \, w'_{11}$$

$$w'_6 = 1.414 \, w'_{15}$$

The equations of equilibrium are governed by Equations (48) and (49),

and become

$$-w'_7 + .7569 \, w'_1 + .3031 \, w'_{10} - .06575 \, w'_{13} - .02929 \, w'_{11} - .01078 \, w'_{12} - 2 = 0$$

$$-w'_{11} + .9173 \, w'_2 + .1057 \, w'_{14} - .0785 \, w'_8 + .03725 \, (w'_{10} - w'_{15}) + .01002 \, (w'_{13} - w'_{12}) - 4 = 0$$

$$-w'_{15} + .8981 \, w'_3 + .07801 \, (w'_{18} - w'_{12}) + .03475 \, w'_{14} + .01279 \, w'_{13} + 72 = 0$$

The difference equations for the inner and outer panels are given by

Equations (50) and (51) respectively. For the inner panel, the

expressions are

$$-(w'_{F_1})_n + .0246 \, (w'_{F_2})_n + .0188 \, (w'_S)_n + .478 \, (w'_{n+1})_{F_1} = 0$$

$$-(w'_S)_n + .01464 \, (w'_{F_1} + w'_{F_2})_n + .486 \, (w'_{n+1})_S = 0$$

$$-(w'_{F_2})_n + .0246 \, (w'_{F_1})_n + .0188 \, (w'_S)_n + .478 \, (w'_{n+1})_{F_2} = 0$$

However, for the outer panel, the loading condition is not constant, and two systems of equations are required. In the cross section HH'D'D and inboard, the equations are

$$-(w'_{F_1})_n + .096\,(w'_S)_n + .392\,(w'_{n+1} + w'_{n-1})_{F_1} + .1192\,(w'_{F_2})_n - 89 = 0$$

$$-(w'_S)_n + .0842\,(w'_F + w'_{F_2})_n + .416\,(w'_{n+1} + w'_{n-1})_S = 0$$

$$-(w'_{F_2})_n + .096\,(w'_S)_n + .392\,(w'_{n+1} + w'_{n-1})_{F_2} + .1192\,(w'_{F_1})_n - 13 = 0$$

Outboard of the cross section HH'D'D, the equations are the same as above with the exception that the constant terms are omitted.

B.6 Determination of Redundant Rib Load (P) for the Six-Element Beam

The value of the redundant load P imposed by the wing rib is determined by considering the compatibility of the deflections of the rib and the spar at point H. The deflection of the outer panel front spar at the point H is given by Equation (39) to be

$$\delta_H = -.0351 + 14.85(-.00237) + \left[.009 + 14.85(\ .00061)\right]\frac{P}{1000} \tag{B.1}$$

where the incremental deflections are given by Equation (29) and Table 2. Similarly the deflection of the wing rib at the point H is given by Equation (40) to be

$$\delta_H = (29.7)\,(.707)\left[\frac{-1328}{10^5\,(3.5)} - \frac{157}{10^5(3.5)}\frac{P}{1000}\right] - \frac{P\,(29.7^3)}{3\,(6.63)\,(30\times10^6)} \tag{B.2}$$

where the rotations are obtained from the displacements given in Figures (18) and (19). The value of P = -135 pounds is obtained by equating Equations (B.1) and (B.2).

LIST OF TABLES

TABLE 1

Morphology of Single Cell Relaxation Meshes

The relaxation mesh for the general orthogonal shell with
arbitrary boundary conditions is a right cylinder. The mesh may
be simplified if the cross section and boundary conditions are
symmetric about one or both of the principal shear axes of the
shell. This is illustrated by the following table. The regions
corresponding to the simplified mesh are cross hatched, and the
boundary conditions for the edges of the mesh parallel to the z axis
are shown.
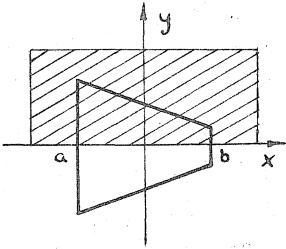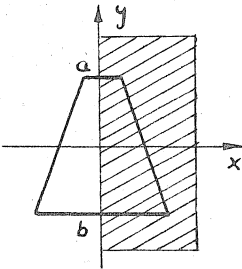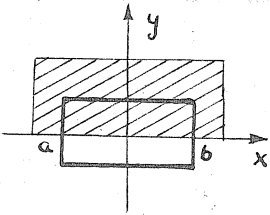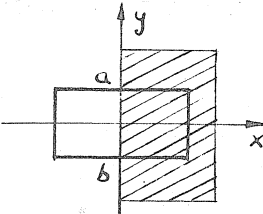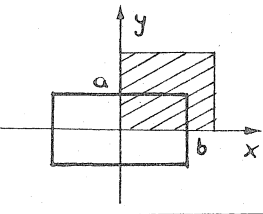
Cross sectional shape	End constraint	Edge	Loading		
			Torsion	Vertical shear	Horizontal shear
Symmetric about x axis	Symmetric about x axis	a, b	$w = 0$	$w = 0$	$\dfrac{\partial w}{\partial s} = 0$
Symmetric about y axis	Symmetric about y axis	a, b	$w = 0$	$\dfrac{\partial w}{\partial s} = 0$	$w = 0$
Doubly Symmetric	Symmetric about x axis	a, b	$w = 0$	$w = 0$	$\dfrac{\partial w}{\partial s} = 0$
	Symmetric about y axis	a, b	$w = 0$	$\dfrac{\partial w}{\partial s} = 0$	$w = 0$
	Doubly symmetric	a	$w = 0$	$\dfrac{\partial w}{\partial s} = 0$	$w = 0$
		b		$w = 0$	$\dfrac{\partial w}{\partial s} = 0$

TABLE 2

DEFLECTION AND ROTATION OF OUTER PANEL OF

THE SWEPT WING

The deflections and rotations of the outer panel of the swept wing are obtained for the four and six element beams by the procedures in Part II, with the boundary conditions $(u_j = v_j = 0$ at AA' and DD'$)$ satisfied at the supports. The results are shown in tabular form in the following.

4 Element Beam

Distance From Tip (in)	External Load No Wing Rib		Rib Load P = 1000		Rib Load P = 167		External Load With Wing Rib	
	Deflection (in)	Rotation (radians)	Deflection (in)	Rotation (radians)	Deflection (in)	Rotation (radians)	Deflection (in)	Rotation (radians)
118.8	-.029	.00196	.008	-.00055	.0013	-.00009	-.028	.00187
89.1	-.043	-.00295	.012	.00078	.0020	.00013	-.041	-.00282
59.4	-.259	-.00422	.025	.00135	.0042	.00023	-.255	-.00399
29.7	-.608	-.00455	.035	.00145	.0058	.00024	-.602	-.00431
0	-1.023	-.00466	.044	.00149	.0073	.00025	-1.016	-.00441

6 Element Beam

Distance From Tip (in)	External Load No Wing Rib		Rib Load P = 1000		Rib Load P = -135 lb.		External Load With Wing Rib	
	Deflection (in)	Rotation (radians)	Deflection (in)	Rotation (radians)	Deflection (in)	Rotation (radians)	Deflection (in)	Rotation (radians)
118.80	-.0446	.00301	.0137	-.00092	-.0018	.00012	-.0464	.00313
103.95	-.0016	-.00081	.0071	.00001	-.0010	.00000	-.0026	-.00081
89.10	-.0351	-.00237	.0090	.00061	-.0012	-.00008	-.0363	-.00245
74.25	-.1275	-.00300	.0117	.00091	-.0016	-.00012	-.1291	-.00312
59.40	-.2658	-.00326	.0124	.00099	-.0017	-.00013	-.2675	-.00339
44.55	-.4396	-.00337	.0129	.00103	-.0017	-.00014	-.4413	-.00351
29.70	-.6398	-.00341	.0133	.00104	-.0018	-.00014	-.6416	-.00355
14.85	-.8568	-.00343	.0136	.00105	-.0018	-.00014	-.8586	-.00357
0	-1.0827	-.00344	.0138	.00105	-.0019	-.00014	-1.0846	-.00358

LIST OF FIGURES

LIST OF FIGURES - Continued

Sign Convention

Positive Forces are in direction of axes
Positive Moments are given by right hand rule
Positive Normal Stresses are tensile
Positive Shear Stresses result from positive torque
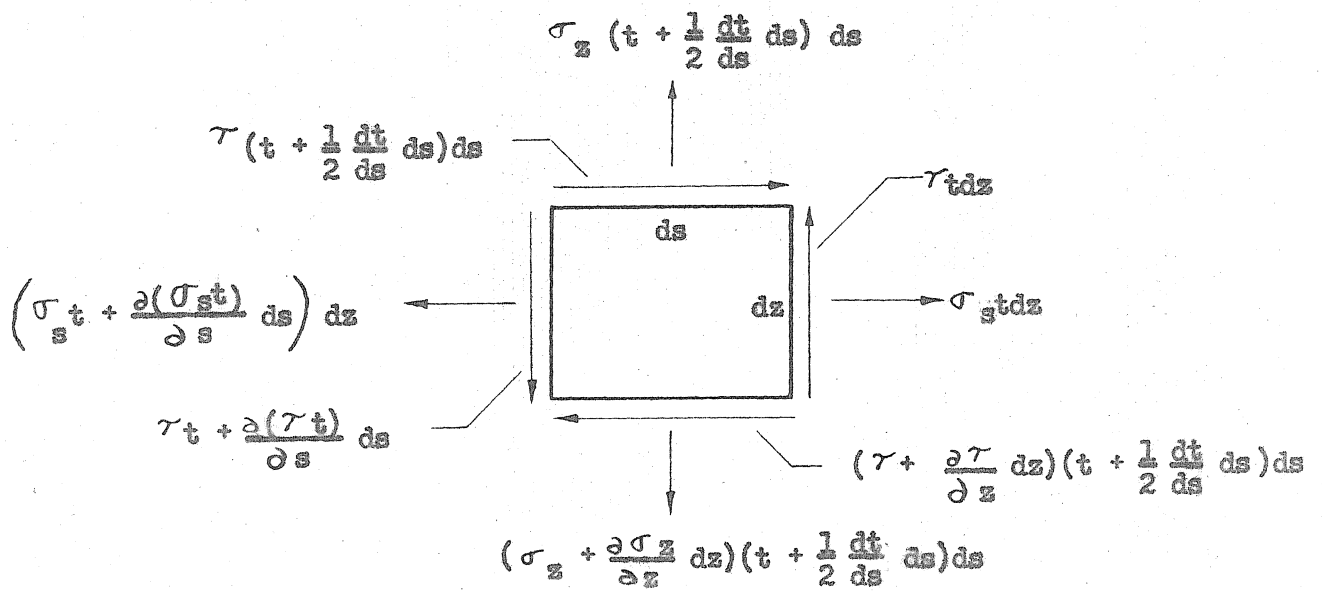
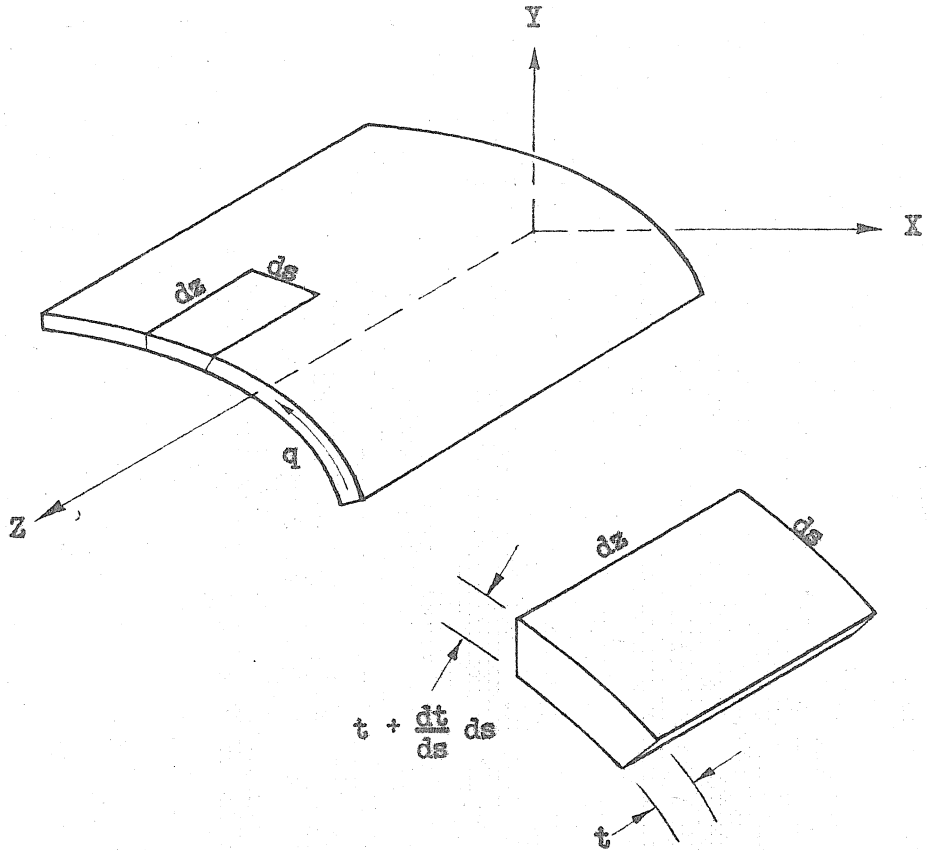FIGURE 1

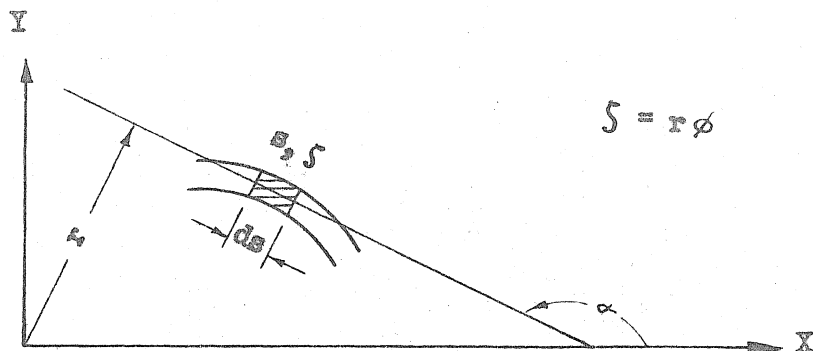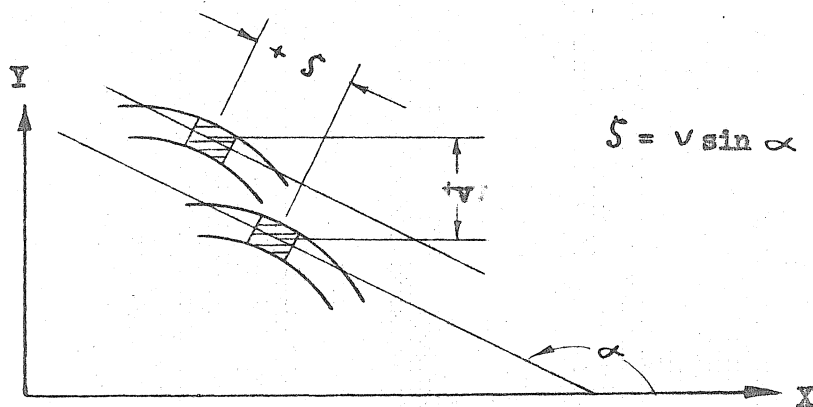COORDINATE SYSTEM FOR ORTHOGONAL SHELL

$$\sigma_z \left(t + \frac{1}{2}\frac{dt}{ds} ds\right) ds$$

$$\tau \left(t + \frac{1}{2}\frac{dt}{ds} ds\right) ds$$

$$\tau_{tdz}$$

$$\left(\sigma_s t + \frac{\partial(\sigma_s t)}{\partial s} ds\right) dz$$

$$\sigma_s tdz$$

$$\tau t + \frac{\partial(\tau t)}{\partial s} ds$$

$$\left(\tau + \frac{\partial \tau}{\partial z} dz\right)\left(t + \frac{1}{2}\frac{dt}{ds} ds\right) ds$$

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z} dz\right)\left(t + \frac{1}{2}\frac{dt}{ds} ds\right) ds$$

ds

dz

FIGURE 2

FORCES ACTING ON A WALL ELEMENT

$S = r \phi$

(a) Due to rotation
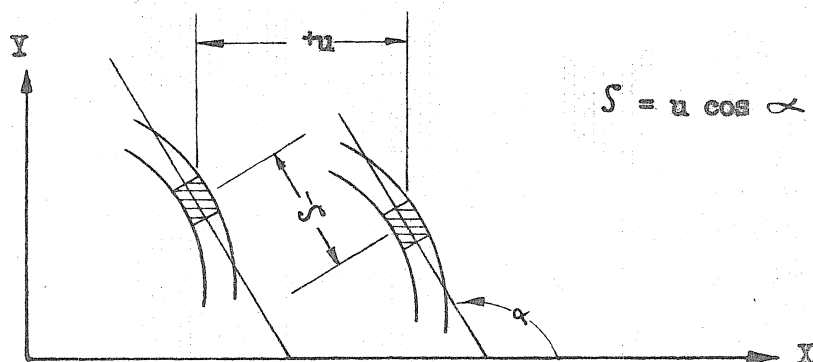
$S = v \sin \alpha$

(b) Due to vertical translation

$S = u \cos \alpha$

(c) Due to horizontal translation
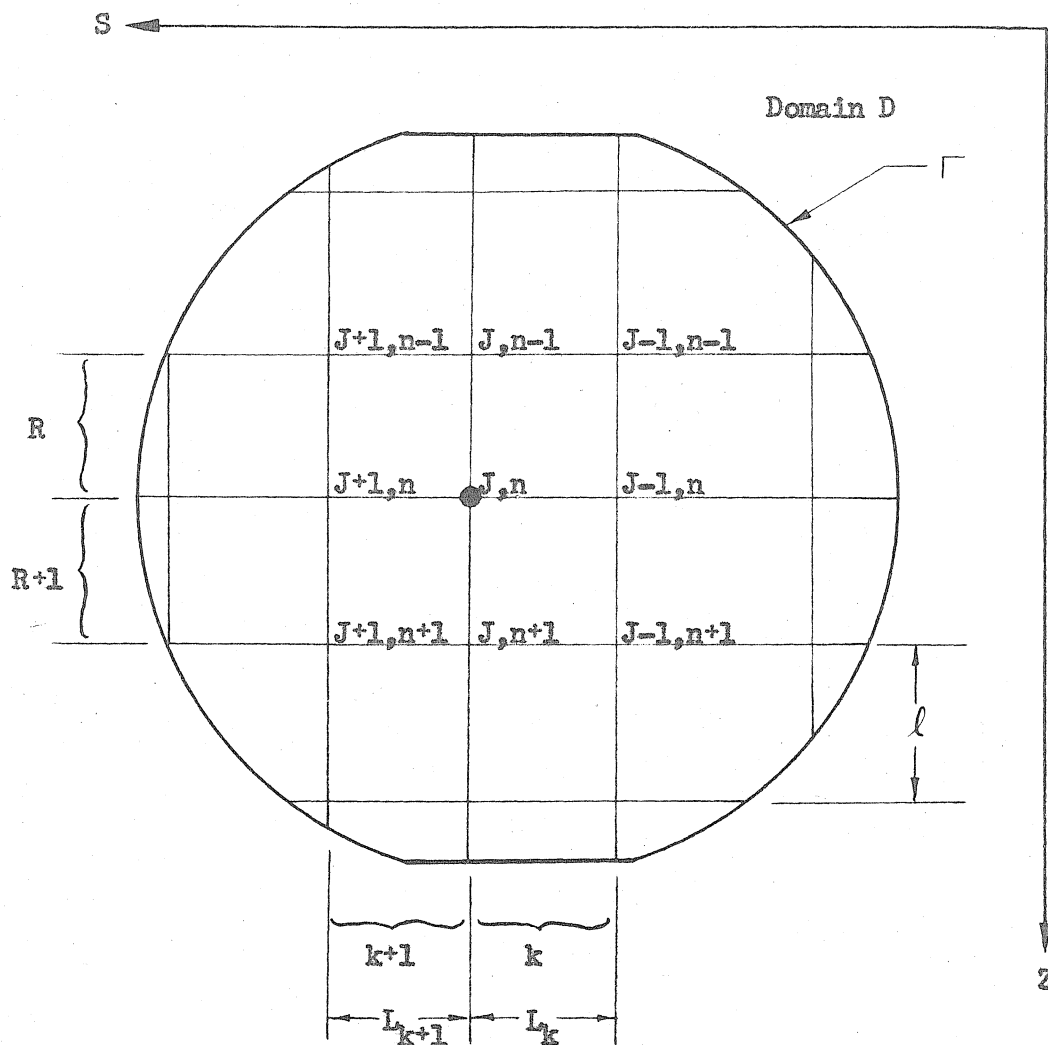
Sign Convention

Positive α is counterclockwise

FIGURE 3

TANGENTIAL DISPLACEMENTS

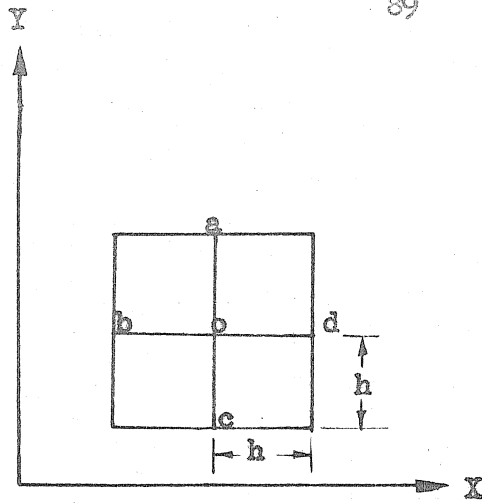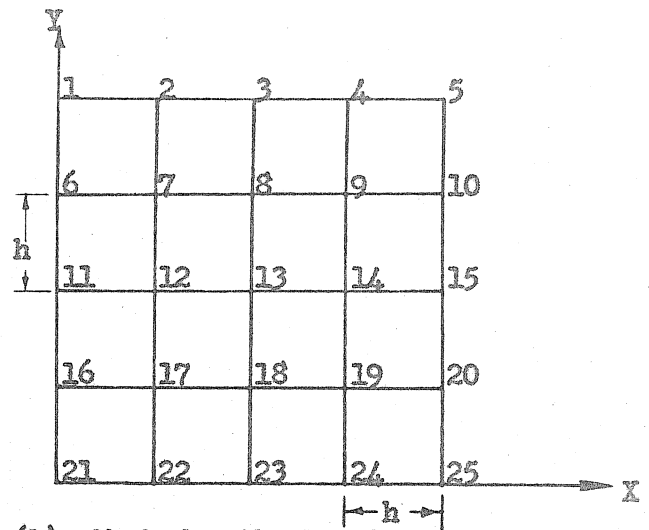FIGURE 4
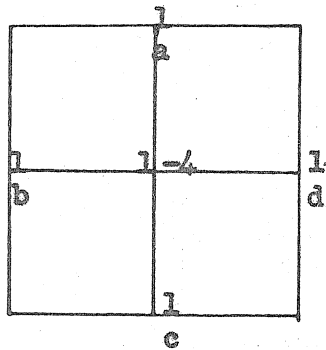
MESH FOR DIFFERENCE EQUATION

(a) Mesh for Difference Equation

(b) Mesh for the Domain

(c) Point Relaxation Operator for Laplace or Poisson Equation
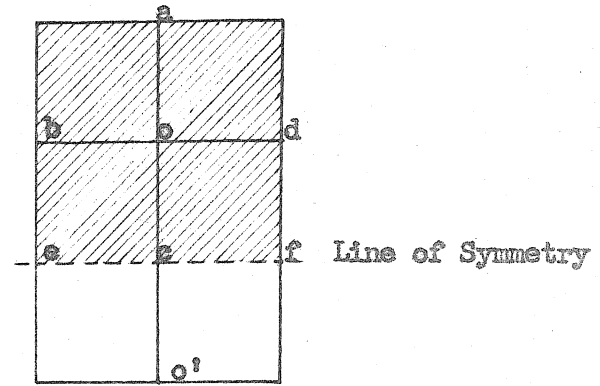
(d) Mesh to Illustrate Line of Symmetry
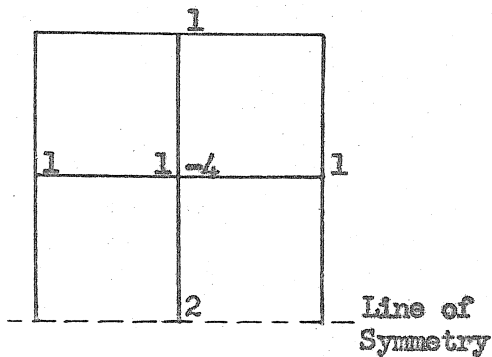
(e) Point Relaxation Operator for Laplace or Poisson Equation for Line Adjacent to Line of Symmetry
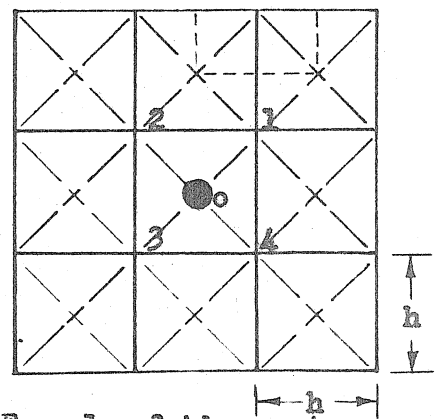
(f) Example of Advance to Finer Mesh

FIGURE 5

RELAXATION TECHNIQUE
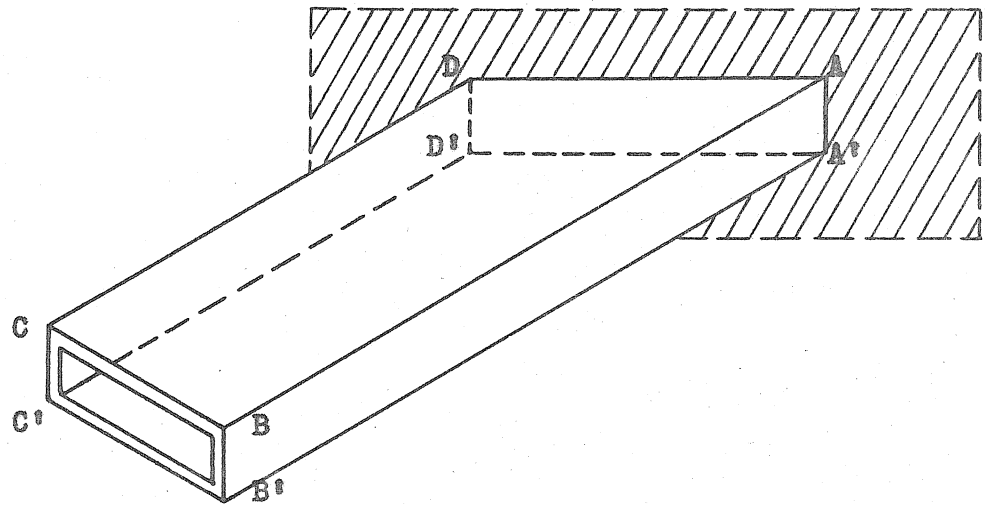
(a) Cantilever Beam on an Oblique Support
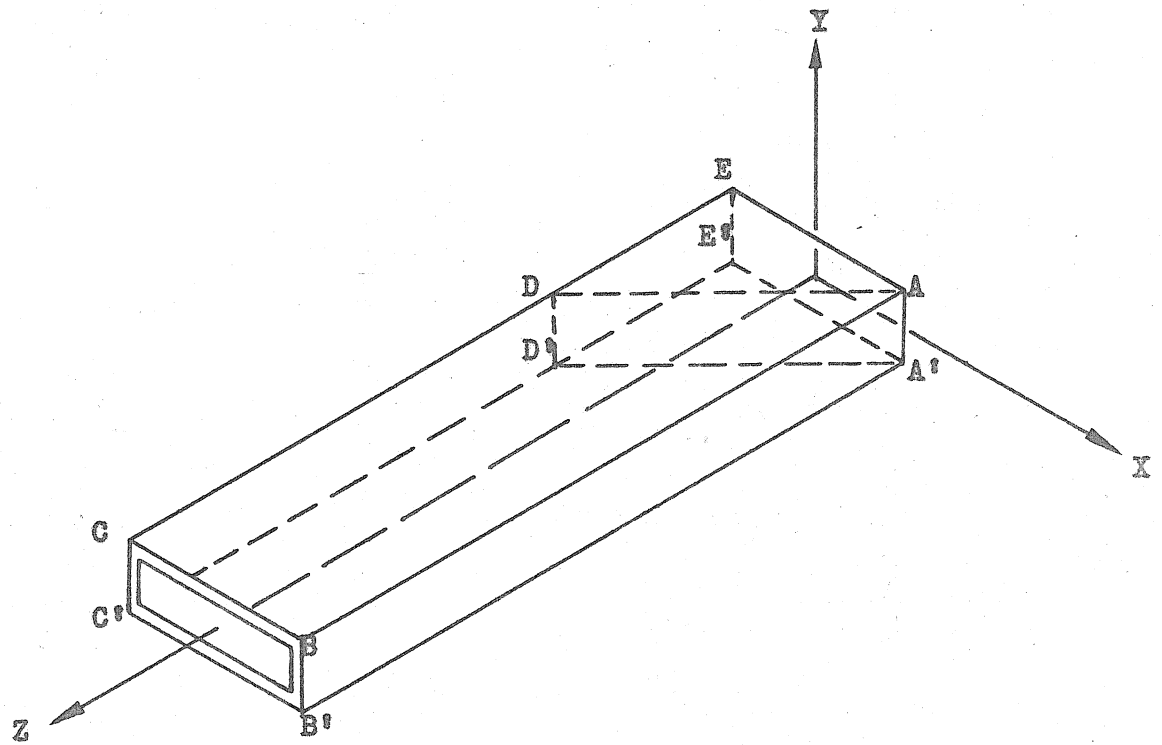
(b) Equivalent Cantilever Beam

FIGURE 6

CANTILEVER BEAM ON AN OBLIQUE SUPPORT

Plane of Symmetry

Inner Panel (Shell)

Fuselage Rib

Wing Rib

Outer Panel (Shell)

Wing Supports
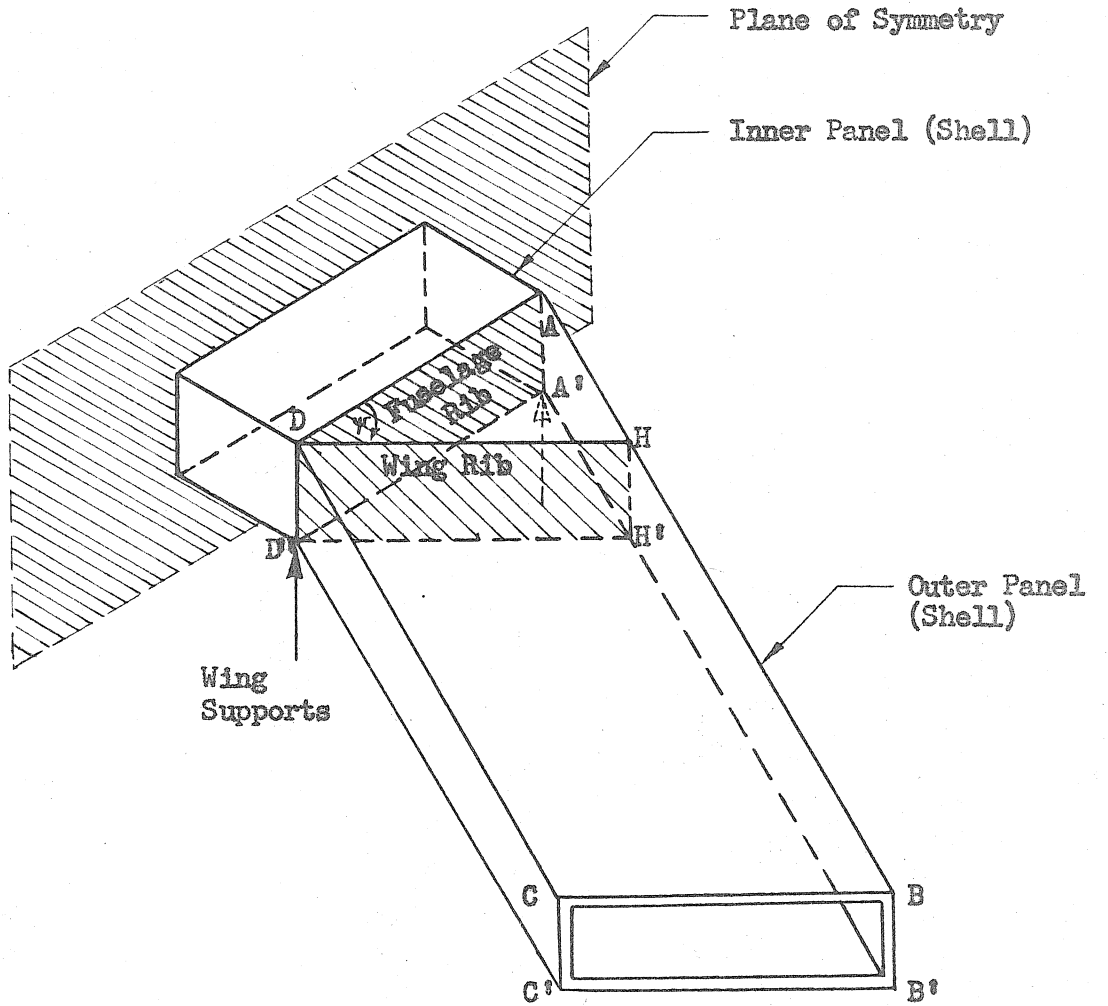
(a) Swept Wing with a Carry Through Bay

FIGURE 7

SWEPT WING WITH A CARRY THROUGH BAY

Plane of Symmetry

F   G $(x)_1$

D   A

$(z)_1$

E
J+1,n-1

J+1,n   J,n-1

J+1,n+1   J,n   J-1,n-1

$(x)_o$

J,n+1   J-1,n

J-1,n+1

C

A

$(z)_o$

$(y)_o$ B

$(x)_o$
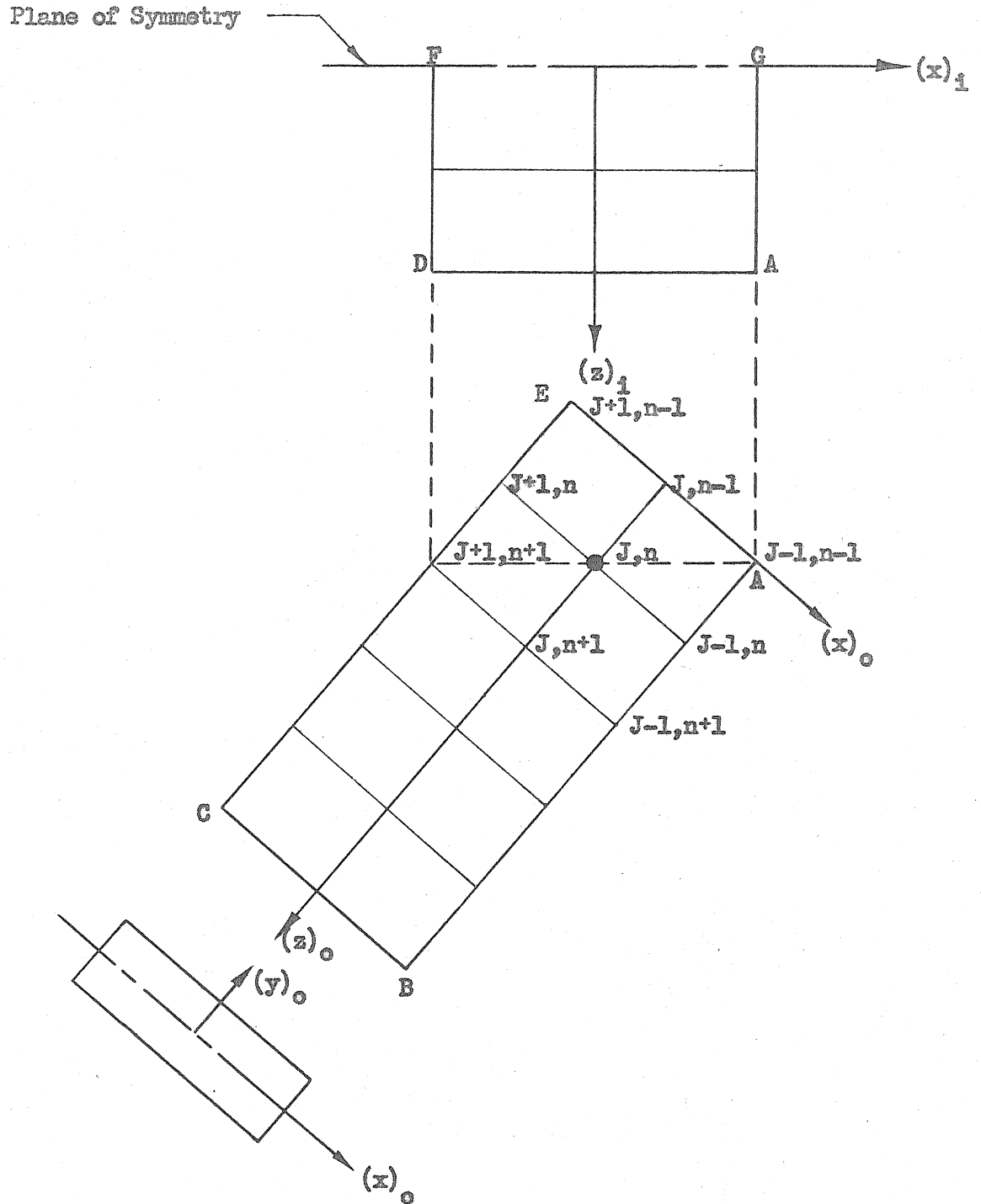
(b)   Equivalent Wing and a Corresponding Relaxation Mesh

FIGURE 7 (CONTINUED)

SWEPT WING WITH A CARRY THROUGH BAY

Inner Shell

Stiffener Element

Sheet Element

Outer Shell

σ_s σ_z

$\sigma_{z'}$

τ_{sz}

σ_z

$\sigma_{z'}$

σ_s
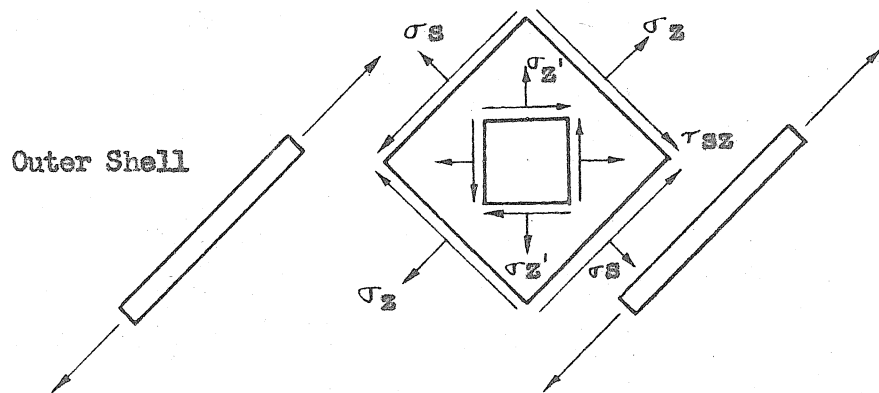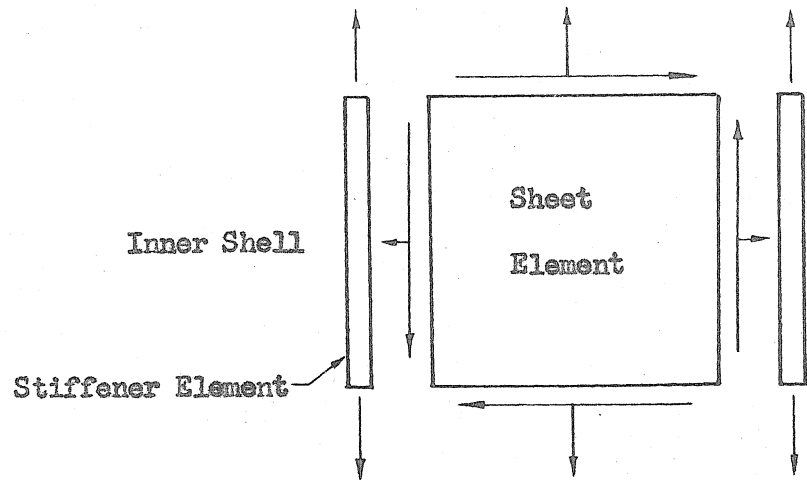
(c) Forces Acting at the Joints of the Stiffener and Sheet Elements

FIGURE 7 (CONTINUED)

SWEPT WING WITH A CARRY THROUGH BAY
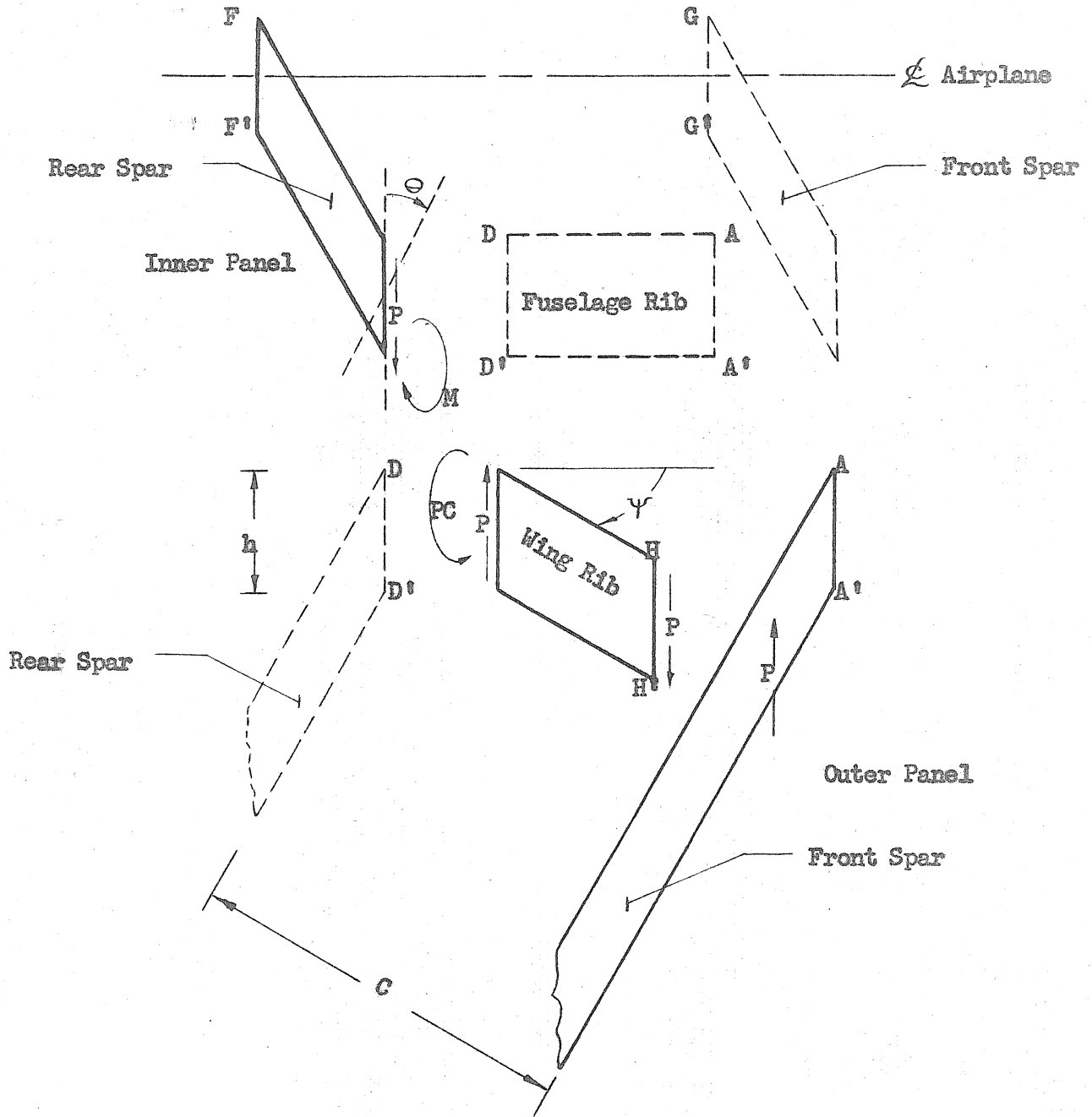
(d)   Idealized Spars and Ribs of Swept Wing

FIGURE 7 (CONTINUED)

SWEPT WING WITH A CARRY THROUGH BAY
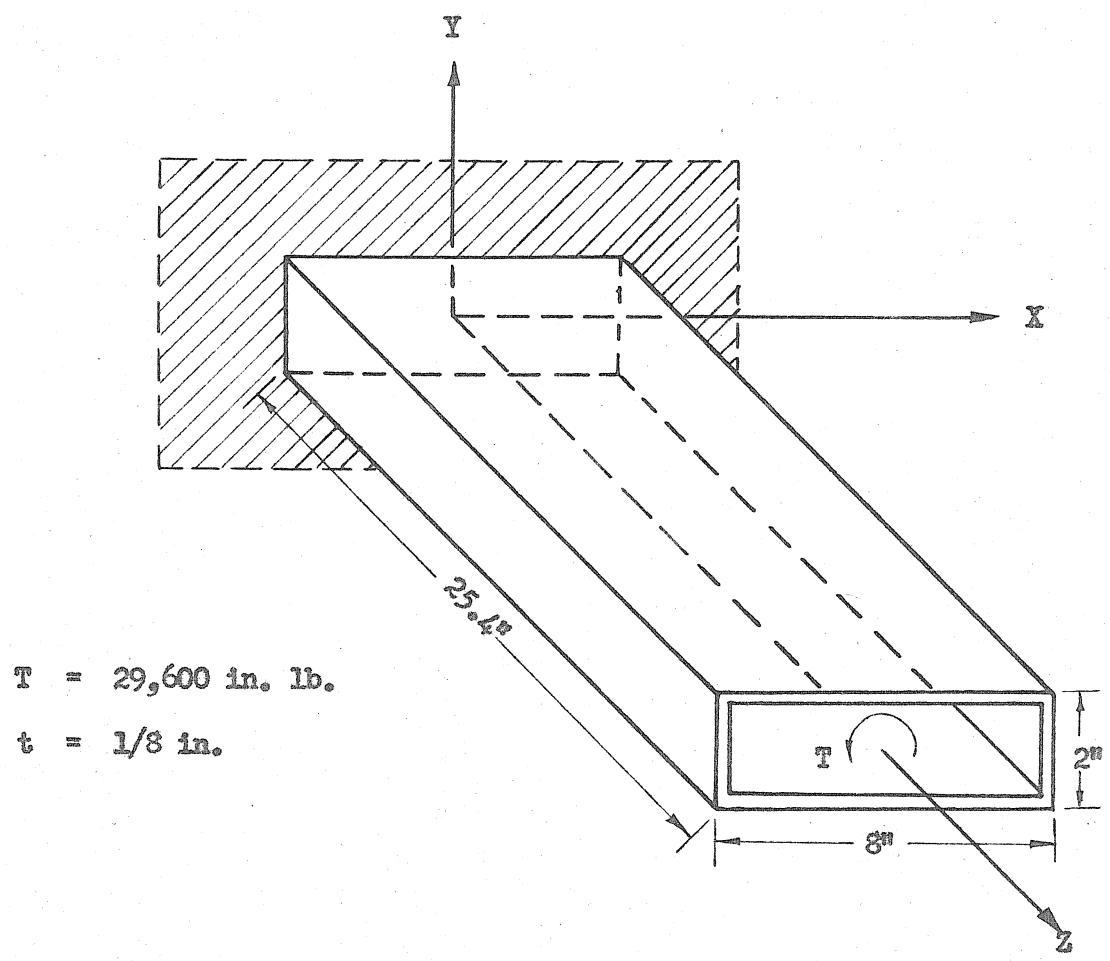
$T = 29,600$ in. lb.

$t = 1/8$ in.

(a) Rectangular Tube

FIGURE 8

RECTANGULAR TUBE UNDER TORSION
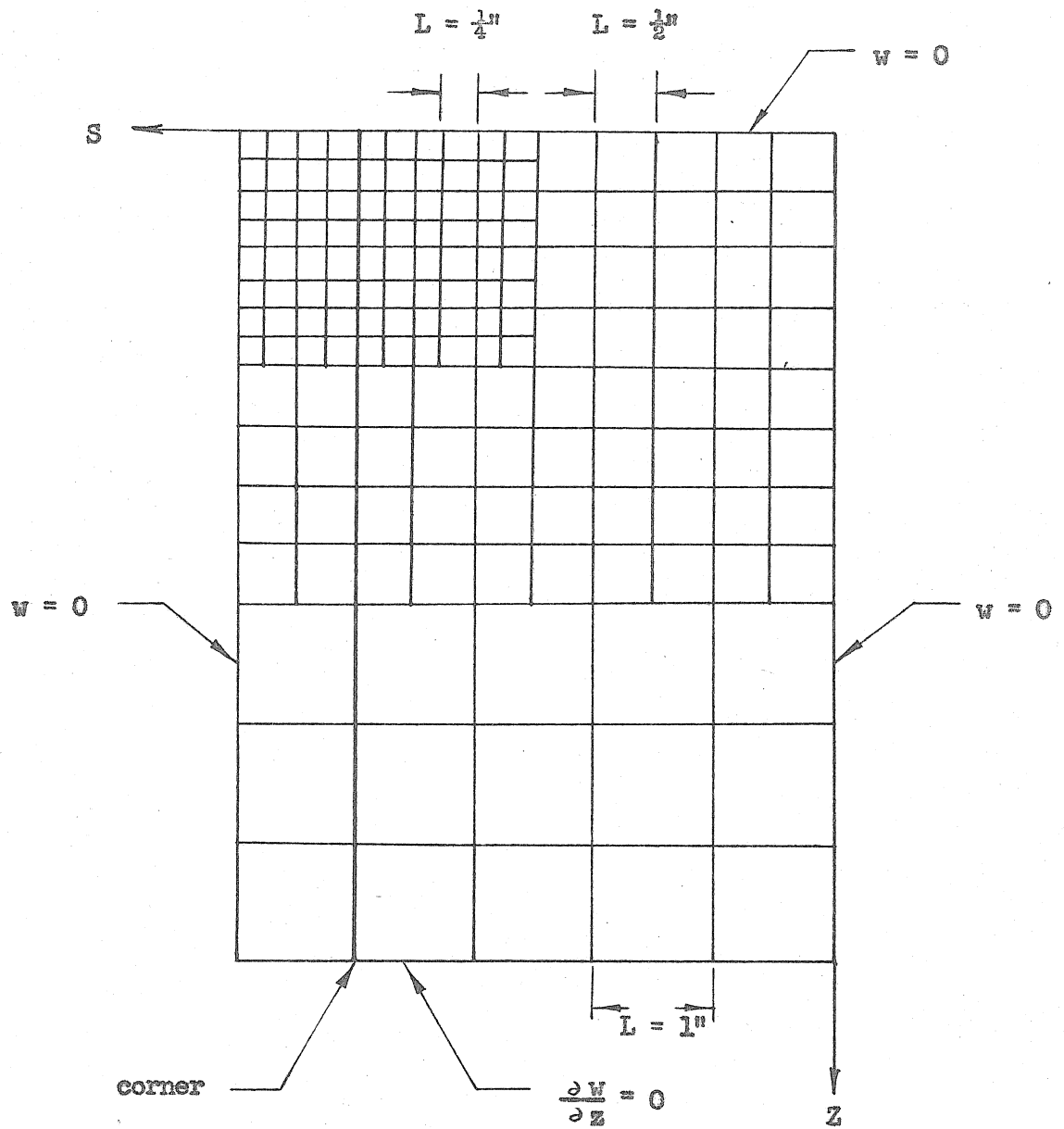
(b)   Relaxation Mesh and Boundary Conditions

FIGURE 8 (CONTINUED)

RECTANGULAR TUBE UNDER TORSION

$$
\begin{array}{ccc}
 & 1 & \\
j+1,n-1 & j,n-1 & j-1,n-1 \\
1 \qquad 1 & -k & 1 \\
j+1,n & j,n & j-1,n \\
 & 1 & \\
j+1,n-1 & j,n+1 & j-1,n+1
\end{array}
$$
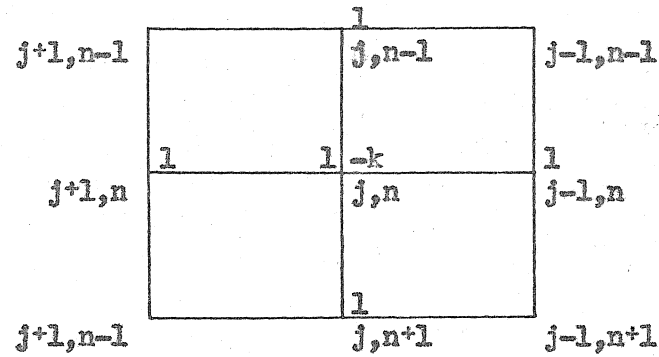
where

k	Region
4	Not Including Corner of Tube
3.55	Corner of Tube (L = 1 in.)
3.775	Corner of Tube (L = $\frac{1}{2}$ in.)
3.8875	Corner of Tube (L = $\frac{1}{4}$ in.)

(c)   Point Relaxation Operator

FIGURE 8 (CONTINUED)

RECTANGULAR TUBE UNDER TORSION

The units of displacement and stress are ins. x 10^-6 and psi, respectively

k	Corner 1		E			c		a
A	0 / 0	12418	0 / 6396	0 / 1426		0 / 1531		0 / 0
C	0 / 0	1497 2 / 7043	839 0 / 4511	471 0 / 2698		216 1 / 1364		0 / 0
E	0 / 0	2167 / 3250	1388 / 2850	830 / 1947		384 -1 / 972		0 / 0
G	0 / 0	2497 -1 / 1635	1716 -3 / 1684	1070 1 / 1271		514 0 / 673		0 / 0
I	0 / 0	2670 0 / 871	1906 -1 / 975	1221 1 / 790		596 -2 / 432		0 / 0
K	0 / 0	2765 -2 / 483	2016 0 / 560	1313 0 / 478		677 -1 / 266		0 / 0
M	0 / 0	2818 0 / 273	2080 -1 / 328	1368 -1 / 283		678 -1 / 159		0 / 0
O	0 / 0	2849 -3 / 156	2117 1 / 188	1400 0 / 166		696 2 / 94		0 / 0
	0 / 0	2866 -1 / 98	2138 1 / 111	1417 0 / 98		707 1 / 58		0 / 0
	0 / 0	2876 -1 / 52	2151 -2 / 65	1430 1 / 59		714 -1 / 36		0 / 0
	0 / 0	2882 -2 / 29	2158 0 / 36	1437 -1 / 36		718 -1 / 20		0 / 0
	0 / 0	2885 -1 / 16	2162 1 / 23	1441 -2 / 20		720 0 / 10		0 / 0
	0 / 0	2887 -1 / 10	2166 1 / 13	1443 0 / 10		721 1 / 7		0 / 0
	0 / 0	2888 0 / 3	2166 -1 / 0	1444 0 / 3		722 0 / 6		0 / 0
	0 / 0	2888 0 / 0	2166 / 0	1444 0 / 0		722 0 / 0		0 / 0
	0 / 0	2888 - / 0	2166 - / 0	1444 - / 0		722 - / 0		0 / 0

(a) Course Mesh (L = 1 in.)

FIGURE 9

FINAL VALUES OF AXIAL DISPLACEMENTS AND

STRESSES FOR RECTANGULAR TUBE UNDER TORSION

Legend (top right):

Displacement | Error
Stress

Figure grid — each cell lists displacement (upper, with error digit) over stress (lower). Units: displacement = in. × 10⁻⁶, stress = psi.

"Corner" is marked above column i.

Row	k	j	i (Corner)	h	g	f	e	d	c	b	a
A	0/0	6669	16218	9718	6539	4641	3348	2334	1482	722	0/0
B	0/0	445 (1) / 4903	1027 (2) / 10486	666 (0) / 7598	468 (0) / 5629	340 (2) / 4199	248 (-1) / 3100	174 (0) / 2190	111 (0) / 1404	54 (0) / 682	0/0
C	0/0	754 (1) / 3341	1611 (1) / 6227	1169 (1) / 5629	866 (0) / 4634	646 (2) / 3660	477 (-1) / 2782	337 (2) / 2008	216 (0) / 1300	105 (2) / 637	0/0
D	0/0	959 (-1) / 2229	1905 (2) / 4050	1532 (0) / 4056	1181 (3) / 3614	903 (1) / 3016	676 (2) / 2379	483 (-1) / 1755	311 (0) / 1150	152 (1) / 572	0/0
E	0/0	1096 (2) / 1508	2236 (-1) / 2769	1793 (1) / 2932	1422 (2) / 2762	1110 (2) / 2412	843 (0) / 1970	607 (2) / 1482	393 (2) / 988	193 (1) / 494	0/0
F	0/0	1190 (2) / 1040	2411 (1) / 1956	1983 (1) / 2145	1606 (0) / 2100	1274 (1) / 1898	979 (-1) / 1586	711 (6) / 1222	463 (1) / 832	228 (2) / 422	0/0
G	0/0	1259 (0) / 760	2537 (-1) / 1410	2123 (-1) / 1580	1745 (1) / 1586	1402 (-1) / 1469	1087 (0) / 1254	795 (1) / 982	521 (-1) / 676	258 (-2) / 344	0/0
H	0/0	1308 (-2) / 546	2628 (1) / 1027	2226 (-1) / 1164	1850 (-1) / 1190	1500 (-2) / 1118	1192 (-2) / 975	862 (0) / 774	567 (-1) / 533	281 (1) / 273	0/0
I	0/0	1343 (0) / 396	2695 (-1) / 754	2302 (-1) / 858	1928 (1) / 890	1574 (-1) / 845	1237 (-2) / 741	914 (0) / 598	603 (0) / 416	300 (-2) / 214	0/0
J	0/0	1369 (-1) / 286	2744 (-2) / 546	2358 (1) / 637	1987 (-1) / 670	1630 (-1) / 637	1286 (0) / 559	954 (-1) / 455	631 (-1) / 318	314 (0) / 162	0/0
K	0/0	1387 (1) / 208	2779 (0) / 396	2400 (-2) / 468	2031 (-2) / 496	1672 (0) / 481	1323 (2) / 429	984 (0) / 344	652 (0) / 240	325 (0) / 124	0/0
L	0/0	1401 (-1) / 156	2805 (1) / 299	2430 (0) / 338	2063 (-1) / 358	1704 (-2) / 358	1352 (-1) / 325	1007 (0) / 260	668 (0) / 182	333 (1) / 91	0/0
M	0/0	1411 (0) / 110	2825 (-2) / 221	2452 (1) / 247	2086 (1) / 260	1727 (-2) / 254	1373 (-1) / 234	1024 (0) / 188	680 (0) / 136	339 (1) / 72	0/0
N	0/0	1418 (-1) / 65	2839 (-2) / 156	2468 (-2) / 156	2103 (2) / 202	1743 (-3) / 156	1388 (0) / 176	1036 (0) / 124	689 (0) / 104	344 (-2) / 46	0/0
O	0/0	1421 / 13	2849 / 104	2476 / 52	2117 / 162	1751 / 52	1400 / 136	1043 / 58	696 / 78	346 / 6	0/0

The units of displacement and stress are ins. × 10⁻⁶ and psi, respectively.

(b) Fine Mesh (L = ½ in.)

FIGURE 9

FINAL VALUES OF AXIAL DISPLACEMENTS AND
STRESSES FOR RECTANGULAR TUBE UNDER TORSION

				Displacement	Error
				Stress	

Corner

	k	j		l	h		g		f		e		d
A	3185	6773	11570	19695	13065	9737	7735	6396	5330	4524	3861	3276	2795	2314
	116 -1	244 0	405 2	648 1	461 -1	356 0	288 1	240 -2	202 2	172 1	147 0	125 1	106 -2	88 -
	2847	5915	9490	14001	10907	8775	7241	6084	5174	4420	3783	3224	2717	2262
B	219	455 0	730 1	1077 2	839 0	675 0	557 1	468 2	398 -1	340 0	291 -1	248 -1	209 1	174 -
	2457	4979	7527	9711	8866	7709	6630	5707	4927	4251	3653	3133	2639	2210
	305 1	627 1	983 -1	1394 0	1143 0	949 1	798 0	679 -1	581 1	499 -1	428 1	366 0	309 1	258 -
	2028	4030	5889	7345	7150	6578	5889	5200	4563	3978	3458	2977	2535	2119
C	375 2	765 1	1183 1	1642 -1	1389 -1	1181 -1	1010 0	868 2	749 -1	646 2	557 -1	477 -1	406 0	337 -
	1651	3237	4654	5772	5798	5538	5135	4641	4147	3666	3224	2782	2392	2002
	432 1	876 1	1342 1	1839 1	1589 1	1375 1	1193 1	1036	900 1	783 1	676 -2	580 2	493 0	412 -
	1339	2613	3744	4654	4758	4667	4420	4095	3718	3341	2951	2587	2236	1898
D	478 2	966 0	1471 0	2000 -1	1755 3	1540 0	1350 2	1183 0	1035 0	903 0	784 0	676 0	576 0	483 -
	1092	2119	3042	3809	3965	3952	3809	3575	3302	3003	2691	2392	2067	1768
	516 0	1039 2	1576 2	2132 2	1894 1	1679 2	1486 -2	1311 2	1154 2	1012 2	883 0	764 -2	652 2	548 -
	897	1742	2509	3159	3328	3354	3276	3120	2925	2691	2444	2171	1898	1612
E	547 1	1100 1	1664 0	2243 0	2011 1	1798 -1	1602 2	1423 2	1260 0	1110 0	972 0	843 1	722 -1	607 -
	741	1456	2093	2652	2808	2860	2821	2730	2587	2405	2197	1963	1729	1482
	573 1	1151 -1	1737 1	2336 0	2110 -1	1899 0	1703 -1	1521 1	1353 1	1197 1	1052 -1	915 -1	785 1	662 -
	624	1209	1742	2223	2366	2431	2431	2379	2275	2132	1963	1768	1560	1352
F	595 -1	1193 1	1798 3	2414 0	2193 1	1985 0	1789 2	1606 1	1435 0	1274 0	1123 -1	979 0	842 0	711 -
	520	1014	1482	1872	2002	2080	2093	2067	2002	1885	1742	1573	1404	1209
	613 1	1229 1	1851 -2	2480 2	2264 1	2059 0	1864 2	1680 2	1507 -1	1342 1	1186 -2	1036 1	893 0	755 -
	442	871	1261	1599	1716	1794	1833	1807	1755	1664	1534	1404	1261	1092
G	629 1	1260 -1	1895 1	2537 0	2325 0	2123 0	1930 -2	1745 1	1570 -2	1402 -1	1241 1	1087 0	939 -2	795 -
	377	741	1066	1378	1482	1547	1586	1573	1521	1456	1352	1248	1118	975
	642 0	1286 -1	1933 1	2586 0	2378 1	2178 1	1986 -1	1801 1	1624 1	1454 0	1290 0	1132 0	979 0	830 -
	312	624	923	1183	1287	1339	1352	1361	1326	1274	1196	1105	988	871
H	653 -	1308 -	1966 -	2628 -	2424 -	2226 -	2034 -	1850 -	1672 -	1500 -	1333 -	1172 -	1015 -	862 -
	260	520	793	1001	1105	1157	1144	1183	1170	1118	1040	975	884	793

The units of displacement and stress are ins. x 10^{-6} and psi, respectively.

(c) Finest Mesh (L = $\frac{1}{4}$ in.)

FIGURE 9

FINAL VALUES OF AXIAL DISPLACEMENTS AND
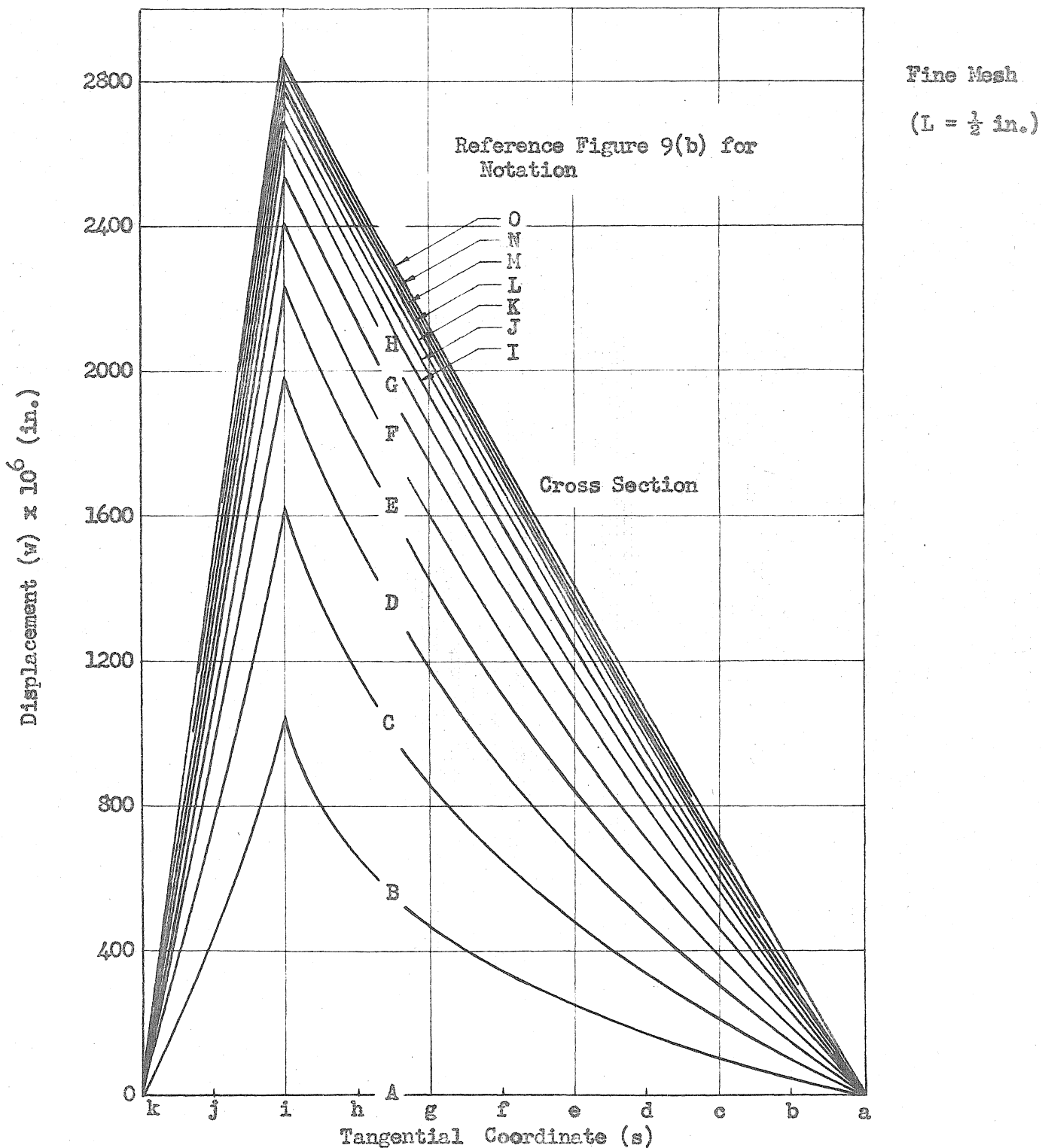
STRESSES FOR RECTANGULAR TUBE UNDER TORSION

FIGURE 10

TANGENTIAL DISTRIBUTION OF AXIAL DISPLACEMENT FOR RECTANGULAR TUBE UNDER TORSION
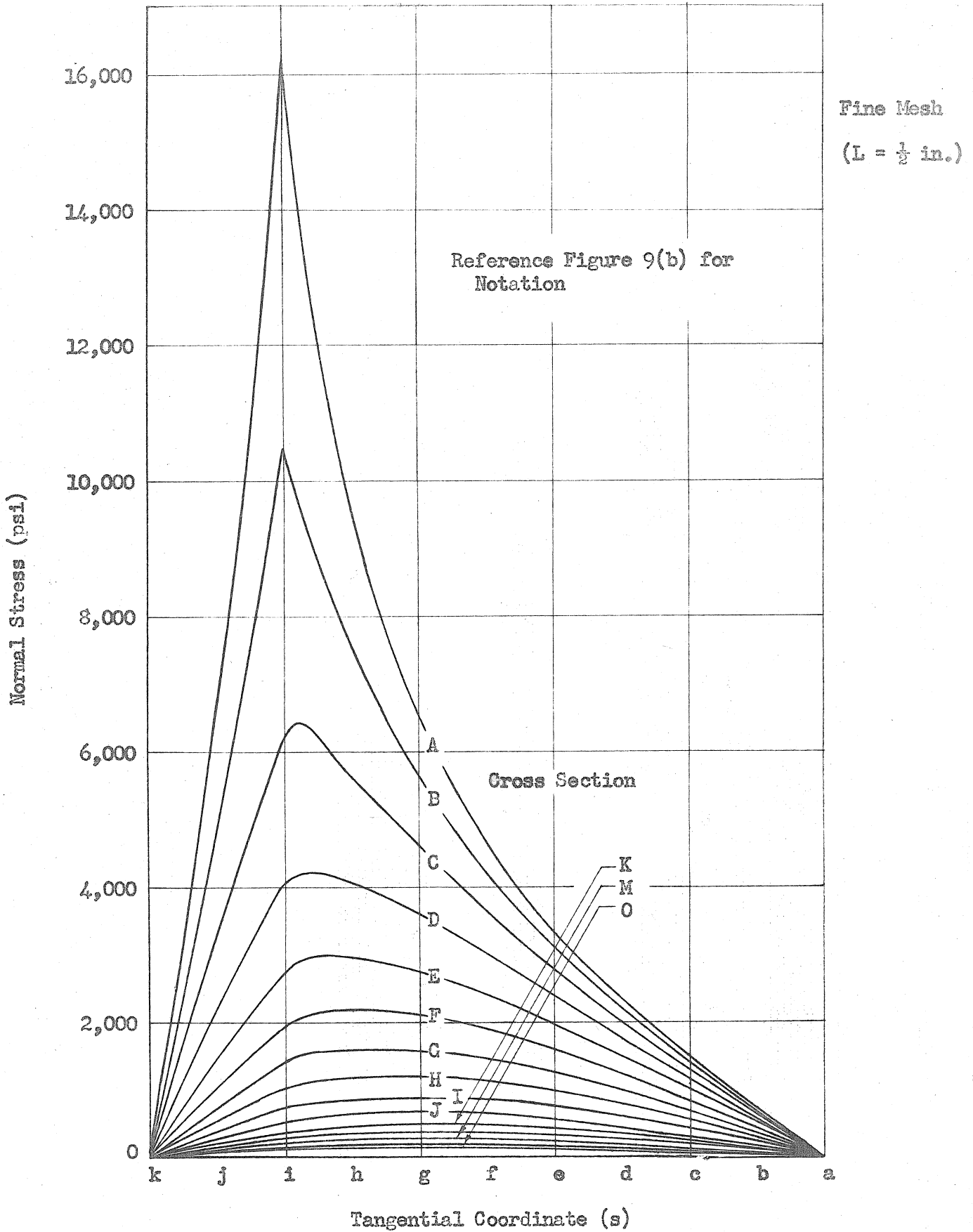
FIGURE 11

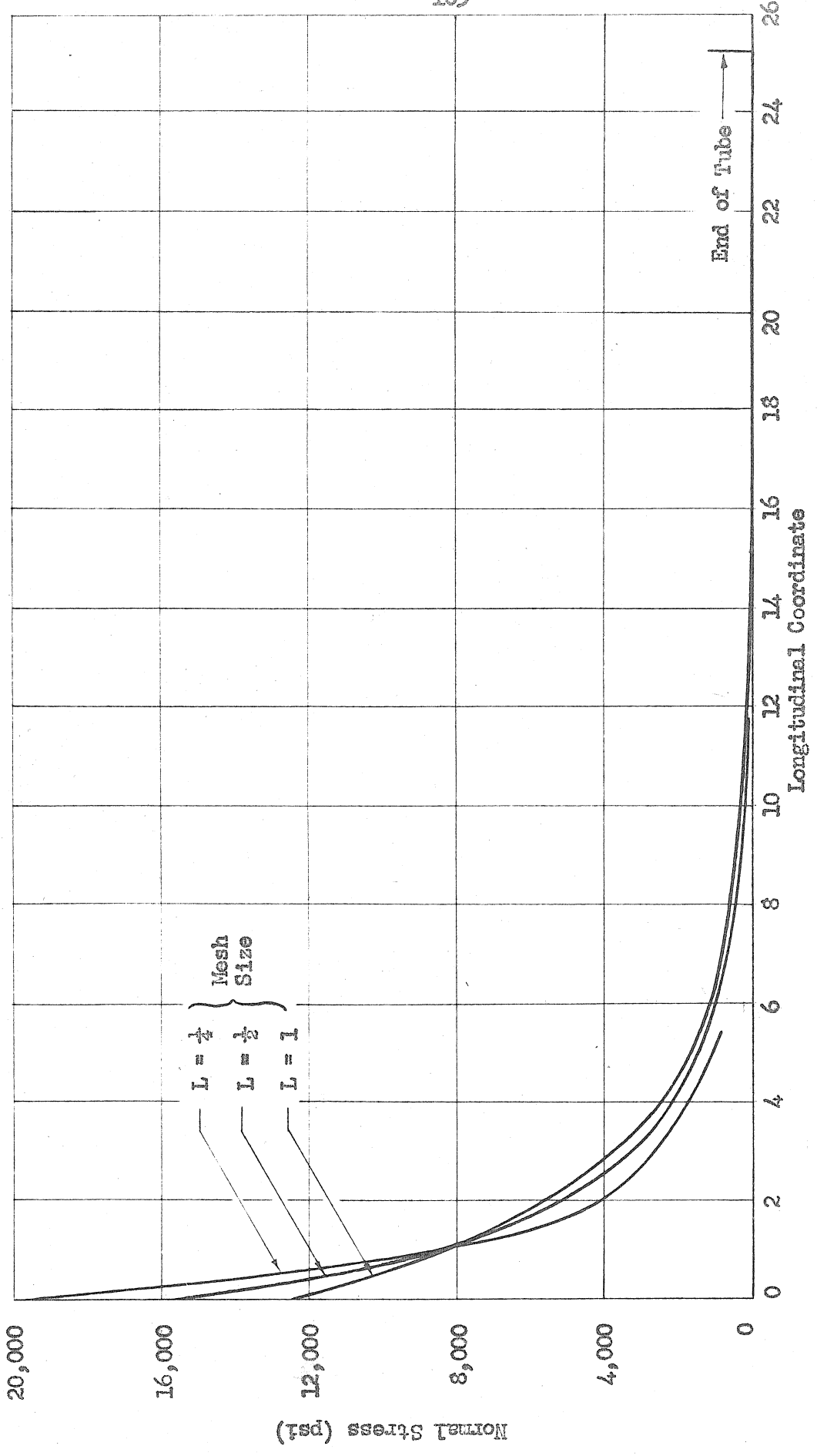TANGENTIAL DISTRIBUTION OF NORMAL STRESS FOR RECTANGULAR TUBE UNDER TORSION

FIGURE 12

AXIAL DISTRIBUTION OF NORMAL STRESS AT CORNER OF RECTANGULAR TUBE UNDER TORSION

Code

△ Analytical Solutions (Kármán-Chien)
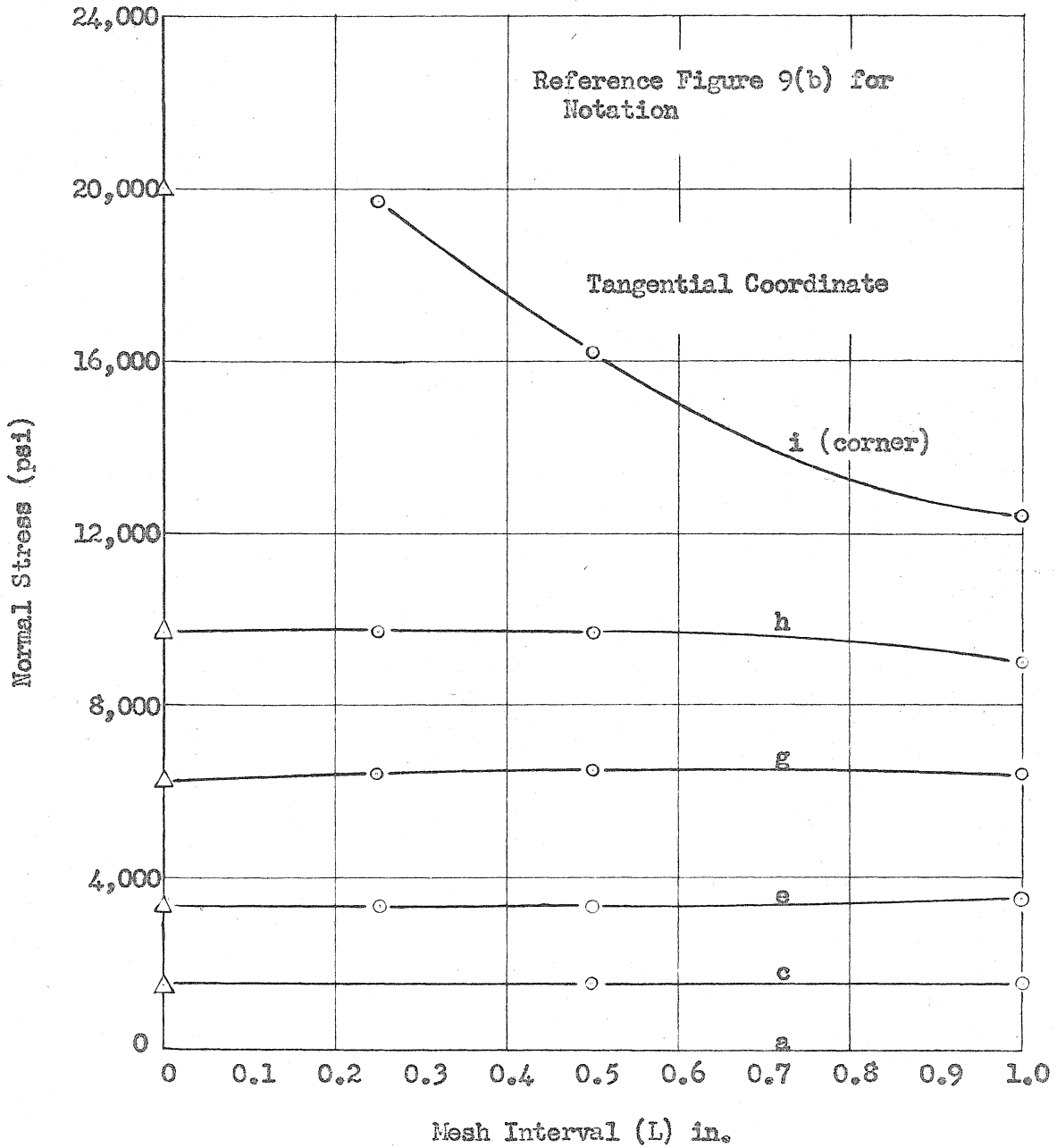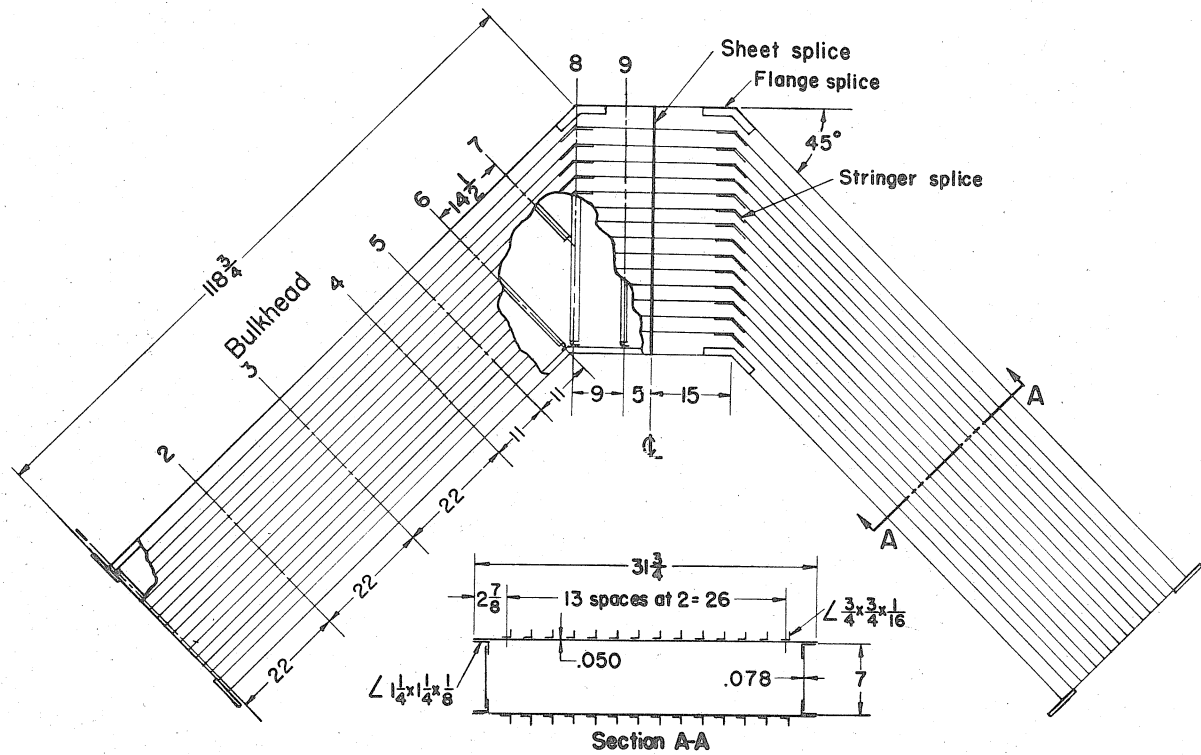
⊙ Finite Difference Solutions

FIGURE 13

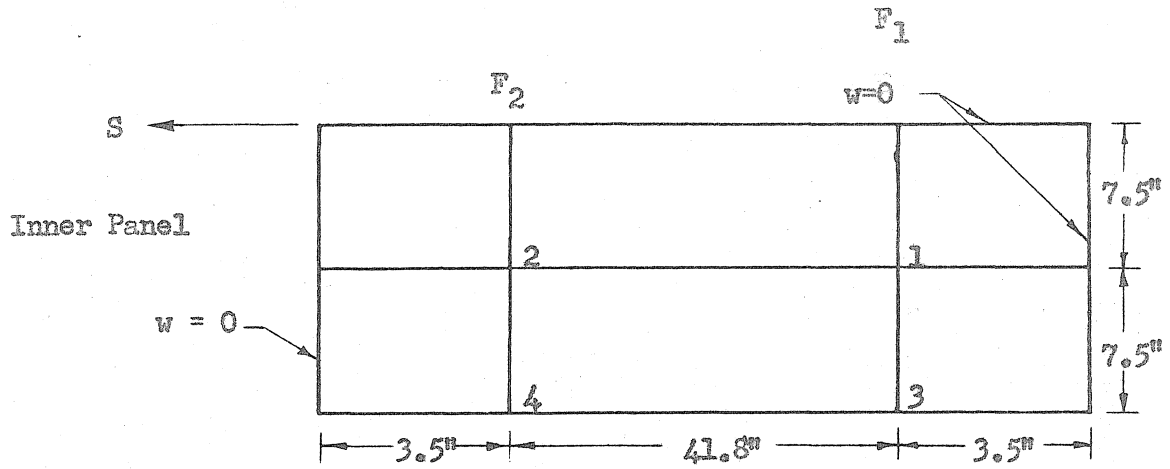NORMAL STRESS AT SUPPORT FOR VARIOUS MESH SIZES
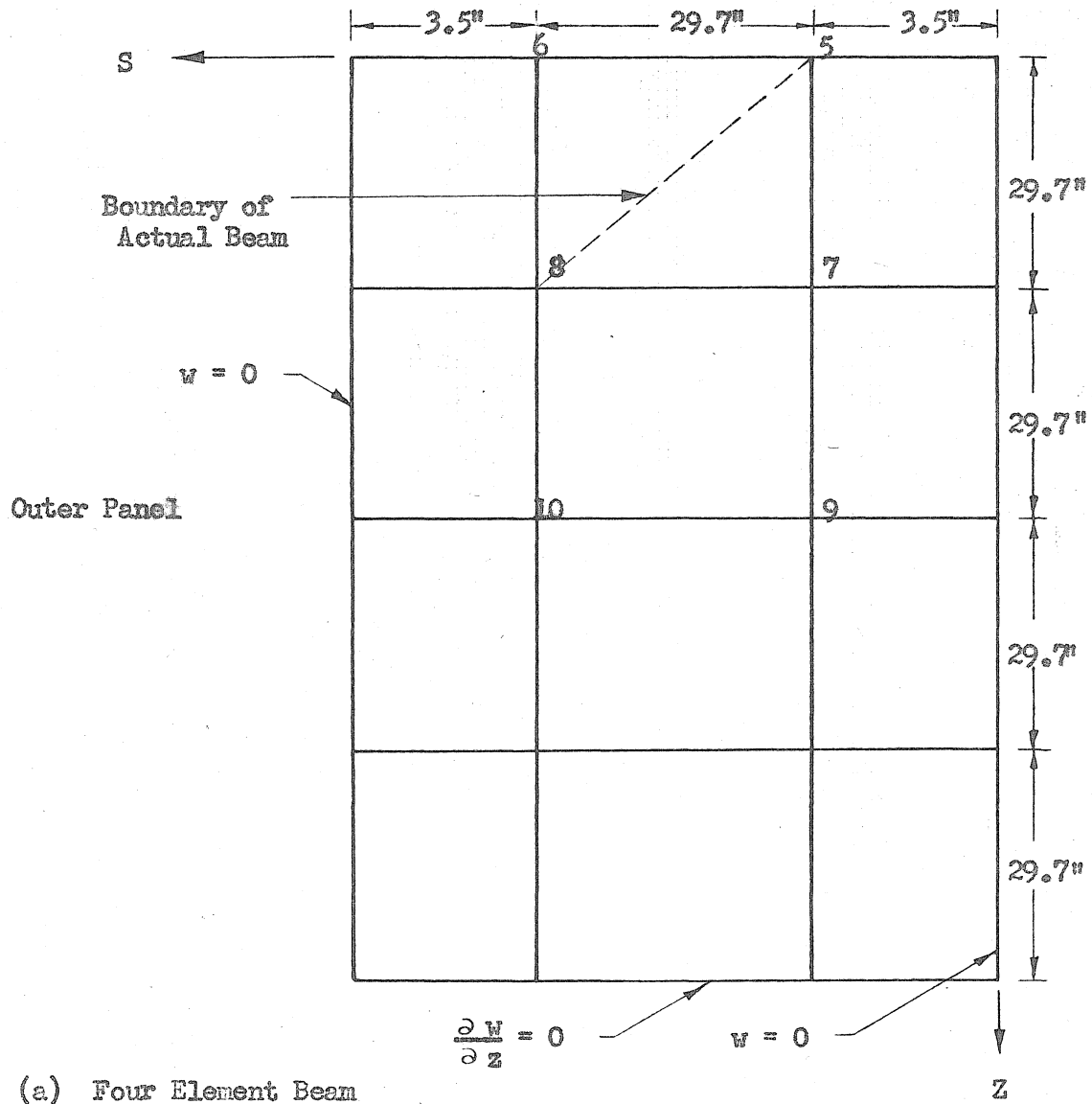
FOR RECTANGULAR TUBE UNDER TORSION

Dimensions in Inches

FIGURE 14

DETAILS OF NACA SWEPT BOX BEAM

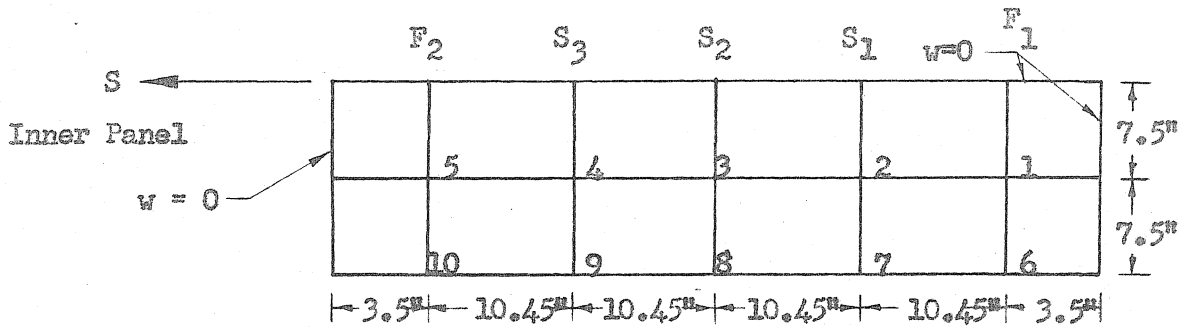(a) Four Element Beam

FIGURE 15

RELAXATION MESH AND BOUNDARY CONDITIONS FOR SWEPT BEAM

Boundary Conditions at Joint Given in Section VII

FIGURE 15

RELAXATION MESH AND BOUNDARY CONDITIONS FOR SWEPT BEAM

FIGURE 15

RELAXATION MESH AND BOUNDARY CONDITIONS FOR SWEPT BEAM

Four Element

Six Element

Ten Element
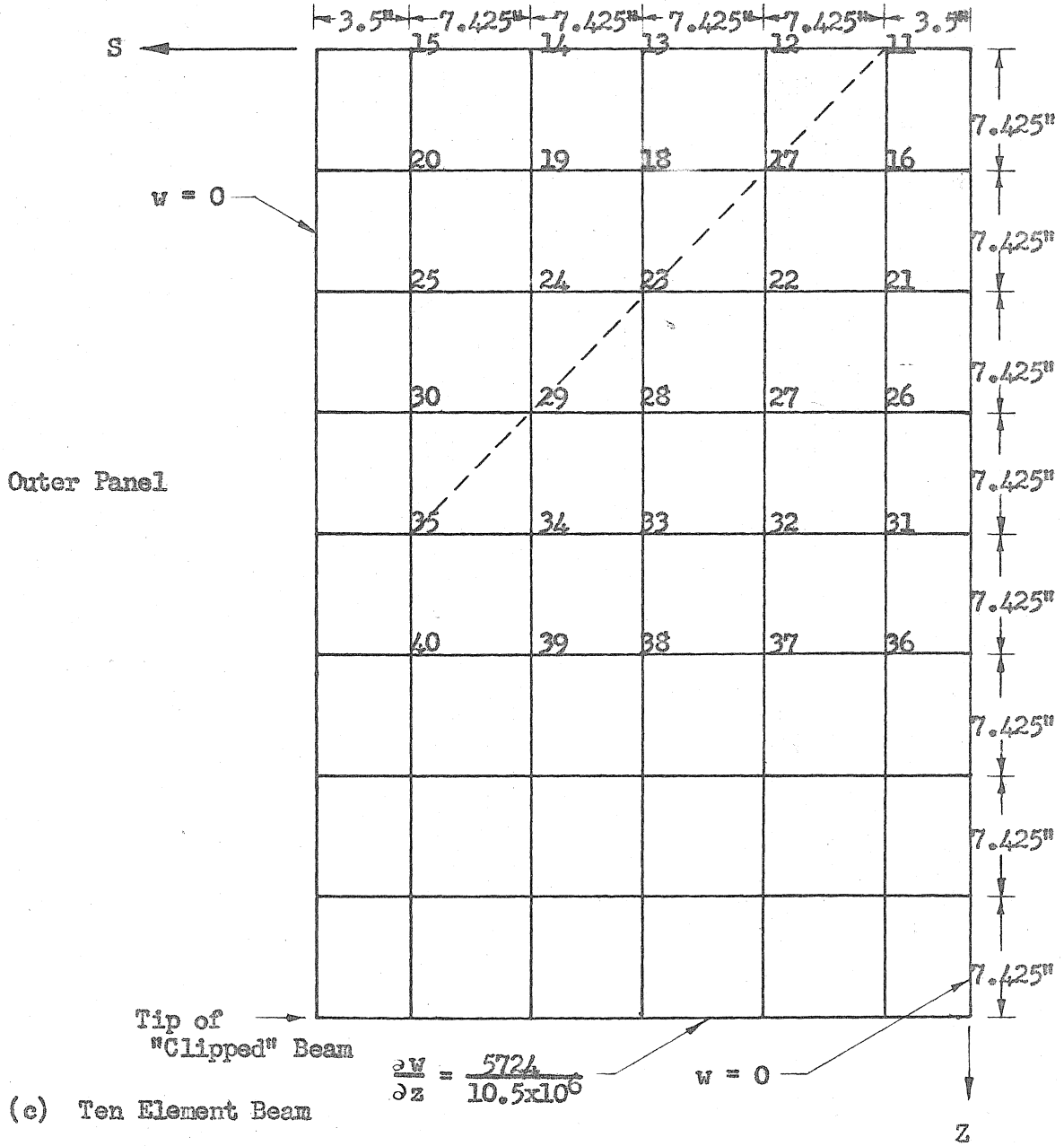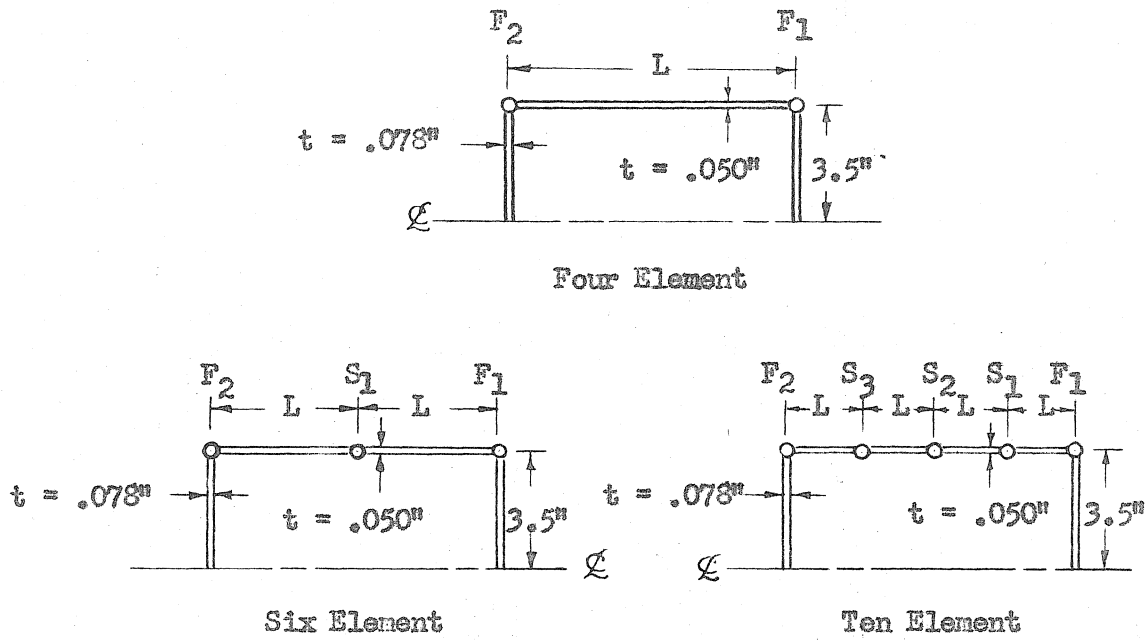
Element	Area (Sq. in.)	Inner Panel			Outer Panel		
		Flange	Stiffener	L(in.)	Flange	Stiffener	L(in.)
4	a_{st} a_{sh} a	1.111 1.045 2.156	– – –	41.8	1.111 .742 1.853	– – –	29.7
6	a_{st} a_{sh} a	.783 .522 1.305	.656 1.045 1.701	20.9	.783 .371 1.154	.656 .742 1.398	14.85
10	a_{st} a_{st} a_{sh}	.619 .261 .880	.328 .522 .850	10.45	.619 .186 .805	.328 .371 .699	7.425

FIGURE 16

IDEALIZED CROSS SECTIONS OF SWEPT BEAM

Four Element Beam

	F₂		F₁	
	.021	1	-1	

Inner Panel

			.326	
		.346	1 -1	
			.326	

(.652)

Outer Panel

Six Element Beam

	F₂	S₁	F₁	
	.0246	.0146	1 -1	

		.0188	1 -1	.0188

Inner Panel

			.392	
	.1192	.0842	1 -1	
			.392	

(.784)

			.416	
	.096	1 -1	.096	
			.416	

(.832)

Outer Panel

Ten Element Beam

	F₂	S₃	S₂	S₁	F₁
	.0348			.0538	1 -1

Inner Panel

				.0538	1 -1	.0532

			.0538	1 -1	.0538	

				.438
	.0477		.0842	1 -1
				.438

(.876)

Outer Panel

				.415	
			.0842	1 -1	.0769
				.415	

(.830)

		.415		
		.0842	1 -1	.0842
		.415		

(.830)

Note: Reference Figure 5e, for line adjacent to tip, use number in parenthesis.

FIGURE 17

POINT RELAXATION OPERATORS FOR SWEPT BEAM

F_2 Inner Panel F_1

Displacement | Error
Stress

Inner Panel grid:

0	—	0	—	0	—	0	—
0		8512		4557		0	
0	—	615	0	319	0	0	—
0		8708		4375		0	
0	—	1244	—	625	—	0	—
0		8904		4193		0	

Outer Panel

0	—	-3264	—	442	—	0	—
0		17878		4556		0	
0	—	880	0	1838	0	0	—
0		11428		5316		0	
0	—	3200	0	3449	0	0	—
0		6337		4763		0	
0	—	4464	1	4532	0	0	—
0		2958		2576		0	
0	—	4873	0	4906	0	0	—
0		0		0		0	

The data at each point of the grid is in accordance with the code shown in the upper right hand corner. The units of displacement and stress are ins. x 10^{-5} and psi, respectively.

(a) Four Element Beam

FIGURE 18

FINAL VALUES OF AXIAL DISPLACEMENTS AND STRESSES FOR THE

SWEPT WING WITH NO WING RIB UNDER SYMMETRICAL BENDING

	F_2		S_1		F_1		Displacement Error Stress

Inner Panel

0	—	0	—	0	—	0	—	0	—
0		8813		5404		3577		0	
0	—	645	0	384	0	243	0	0	—
0		9247		5348		3227		0	
0	—	1321	—	764	—	461	—	0	—
0		9681		5292		2877		0	

Outer Panel

0	—	-4798	—	-77	—	326	—	0	—
0		—		—		2439		0	
0	—	-1232	0	540	0	870	2	0	—
0		—		5394		5253		0	
0	—	934	-1	1449	1	1812	0	0	—
0		12,765		6536		6780		0	
0	—	2379	0	2389	1	2788	-1	0	—
0		8632		6370		6543		0	
0	—	3376	0	3251	2	3663	-1	0	—
0		5724		5724		5724		0	

The units of displacement and stress are ins. $\times 10^{-5}$ and psi, respectively.

(b) Six Element (Clipped) Beam

FIGURE 18

FINAL VALUES OF AXIAL DISPLACEMENTS AND STRESSES FOR THE

SWEPT WING WITH NO WING RIB UNDER SYMMETRICAL BENDING

	F₂		S₁		F₁		Displacement Error	Stress

Inner Panel

	F₂		S₁		F₁		Disp	Err
0 —	0	—	0	—	0	—	0	—
0	8848		5327		3584		0	
0 —	648	0	378	0	243	0	0	—
0	9296		5257		3220		0	
0 —	1328	—	751	—	460	—	0	—
0	9744		5187		2856		0	

Outer Panel

	F₂		S₁		F₁		Disp	Err
0 —	-4813	—	-70	—	325	—	0	—
0	30231		3206		3418		0	
0 —	-1237	0	531	0	864	0	0	—
0	20333		5292		5204		0	
0 —	939	0	1427	0	1797	0	0	—
0	12889		6412		6699		0	
0 —	2409	-1	2345	0	2759	0	0	—
0	8929		6140		6402		0	
0 —	3465	0	3164	0	3608	0	0	—
0	6441		5246		5412		0	
0 —	4231	0	3829	-1	4290	-1	0	—
0	4574		4062		4154		0	
0 —	4759	-1	4313	0	4783	-1	0	—
0	2966		2750		2793		0	
0 —	5070	-1	4607	-1	5080	0	0	—
0	1460		1386		1400		0	
0 —	5172	0	4705	0	5179	0	0	—
0	0		0		0		0	

The units of displacement and stress are ins. x 10^{-5} and psi, respectively.

(c) Six Element Beam

FIGURE 18

FINAL VALUES OF AXIAL DISPLACEMENTS AND STRESSES FOR THE

SWEPT WING WITH NO WING RIB UNDER SYMMETRICAL BENDING

	F_2	S_3	S_2	S_1	F_1	Displacement/Error Stress

Inner Panel

0 –	0 –	0 –	0 –	0 –	0 –	0 –
	8274	6447	5453	4158	2177	0
0 – / 0	619 0	455 0	391 0	303 0	127 0	0 – / 0
	9058	6293	5495	4326	1379	0
0 – / 0	1294 –	899 –	785 –	618 –	197 –	0 – / 0
	9843	6139	5537	4494	581	0

"Clipped" Outer Panel

0 – / 0	–7092 –	696 –	–190 –	212 –	139 –	0 – / 0
	55974	–14107	5621	2546	–445	0
0 – / 0	–3742 0	75 1	195 0	437 0	249 –1	0 – / 0
	38777	–3458	5268	3818	3557	0
0 – / 0	–1608 1	207 –1	555 0	752 0	642 1	0 – / 0
	25406	3967	5324	4914	6343	0
0 – / 0	–149 0	636 0	948 2	1132 0	1146 2	0 – / 0
	17840	6753	5805	5643	7340	0
0 – / 0	915 1	1162 0	1376 1	1550 1	1680 1	0 – / 0
	13343	7509	6159	6053	7481	0
0 – / 0	1738 0	1698 1	1819 2	1988 2	2204 2	0 – / 0
	10493	7410	6279	6215	7227	0
0 – / 0	2399 2	2210 1	2264 1	2429 1	2702 2	0 – / 0
	8528	6972	6215	6173	6795	0
0 – / 0	2944 0	2684 –1	2698 0	2861 0	3165 1	0 – / 0
	7000	6371	6003	5982	6258	0
0 – / 0	3389 1	3111 0	3113 0	3275 0	3587 0	0 – / 0
	5724	5724	5724	5724	5724	0

The units of displacement and stress are ins. x 10^{-5} and psi, respectively.

(d) Ten Element (Clipped) Beam

FIGURE 18

FINAL VALUES OF AXIAL DISPLACEMENTS AND STRESSES FOR THE

SWEPT WING WITH NO WING RIB UNDER SYMMETRICAL BENDING

The units of displacement and stress are ins. $\times 10^{-5}$ and psi, respectively.

(a) Four Element Beam

FIGURE 19

FINAL VALUES OF AXIAL DISPLACEMENTS AND STRESSES FOR

THE SWEPT WING UNDER THE LOADS IMPOSED BY THE WING RIB

	F_2	S_1	F_1	Displacement Stress	Error

Inner Panel

F_2	S_1	F_1	Error
0 / 945	0 / -343	0 / -651	0 / 0
73 / 1099	-26 / -385	-50 / -749	-1 / 0
157 / 1253	-55 / -427	-107 / -847	0 / 0

Outer Panel

F_2	S_1	F_1	Error
842 / -4525	-97 / 604	-76 / -703	0 / 0
339 / -2587	-39 / 216	-151 / -357	0 / 0
110 / -1071	-36 / 21	-177 / 187	0 / 0
36 / -332	-33 / 11	-98 / 350	-1 / 0
16 / 0	-33 / 0	-78 / 0	0 / 0

The units of displacement and stress are ins. x 10^{-5} and psi, respectively.

(b) Six Element (Clipped Beam)

FIGURE 19

FINAL VALUES OF AXIAL DISPLACEMENTS AND STRESSES FOR

THE SWEPT WING UNDER THE LOADS IMPOSED BY THE WING RIB

	F_2	S_1	F_1	Displacement Stress	Error

Inner Panel

	F_2	S_1	F_1	Displacement Stress	Error
0 —	0 —	0 —	0 —	0 —	
	945	−301	−623	0	
0 —	73 0	−23 0	−49 0	0 —	
	1099	−343	−742	0	
0 —	157 —	−49 —	−106 —	0 —	
	1253	−385	−854	0	

Outer Panel

	F_2	S_1	F_1	Displacement Stress	Error
0 —	834 —	−94 —	−75 —	0 —	
	4472	601	−686	0	
0 —	337 0	−35 0	−146 0	0 —	
	−2556	233	−318	0	
0 —	111 0	−28 0	−165 0	0 —	
	−1064	49	247	0	
0 —	36 0	−21 0	−76 −1	0 —	
	−361	46	452	0	
0 —	9 −1	−15 0	−37 −1	0 —	
	−131	35	201	0	
0 —	−1 0	−11 0	−19 −1	0 —	
	−42	25	92	0	
0 —	−3 −1	−8 0	−11 0	0 —	
	−11	18	42	0	
0 —	−4 0	−6 0	−7 −1	0 —	
	−4	11	18	0	
0 —	−4 0	−5 0	−6 0	0 —	
	0	0	0	0	

The units of displacement and stress are ins. x 10^{-5} and psi, respectively.

(c)   Six Element Beam

FIGURE 19

FINAL VALUES OF AXIAL DISPLACEMENTS AND STRESSES FOR

THE SWEPT WING UNDER THE LOADS IMPOSED BY THE WING RIB

F_2 F_1

Inner Panel

0	8538	4391	0
0	8743	4200	0
0	8948	4008	0

Outer Panel

0	17481	4389	0
0	11256	5302	0
0	6301	4803	0
0	2950	2586	0
0	0	0	0

The unit of stress is psi.

(a) Four Element Beam

FIGURE 20

FINAL VALUES OF NORMAL STRESS FOR THE SWEPT WING

WITH A WING RIB UNDER SYMMETRICAL BENDING

	F_2	S_1	F_1	
		Inner Panel		
0	8720	5368	3668	0
0	9148	5303	3320	0
0	9515	5239	2971	0

		Outer Panel		
0	29627	3125	2511	0
0	20678	5261	5247	0
0	13033	6405	6666	0
0	8978	6134	6341	0
0	6459	5241	5385	0
0	4580	4059	4142	0
0	2967	2748	2787	0
0	1461	1385	1398	0
0	0	0	0	0

The unit of stress is psi.

(b) Six Element Beam

FIGURE 20

FINAL VALUES OF NORMAL STRESS FOR THE SWEPT WING

WITH A WING RIB UNDER SYMMETRICAL BENDING
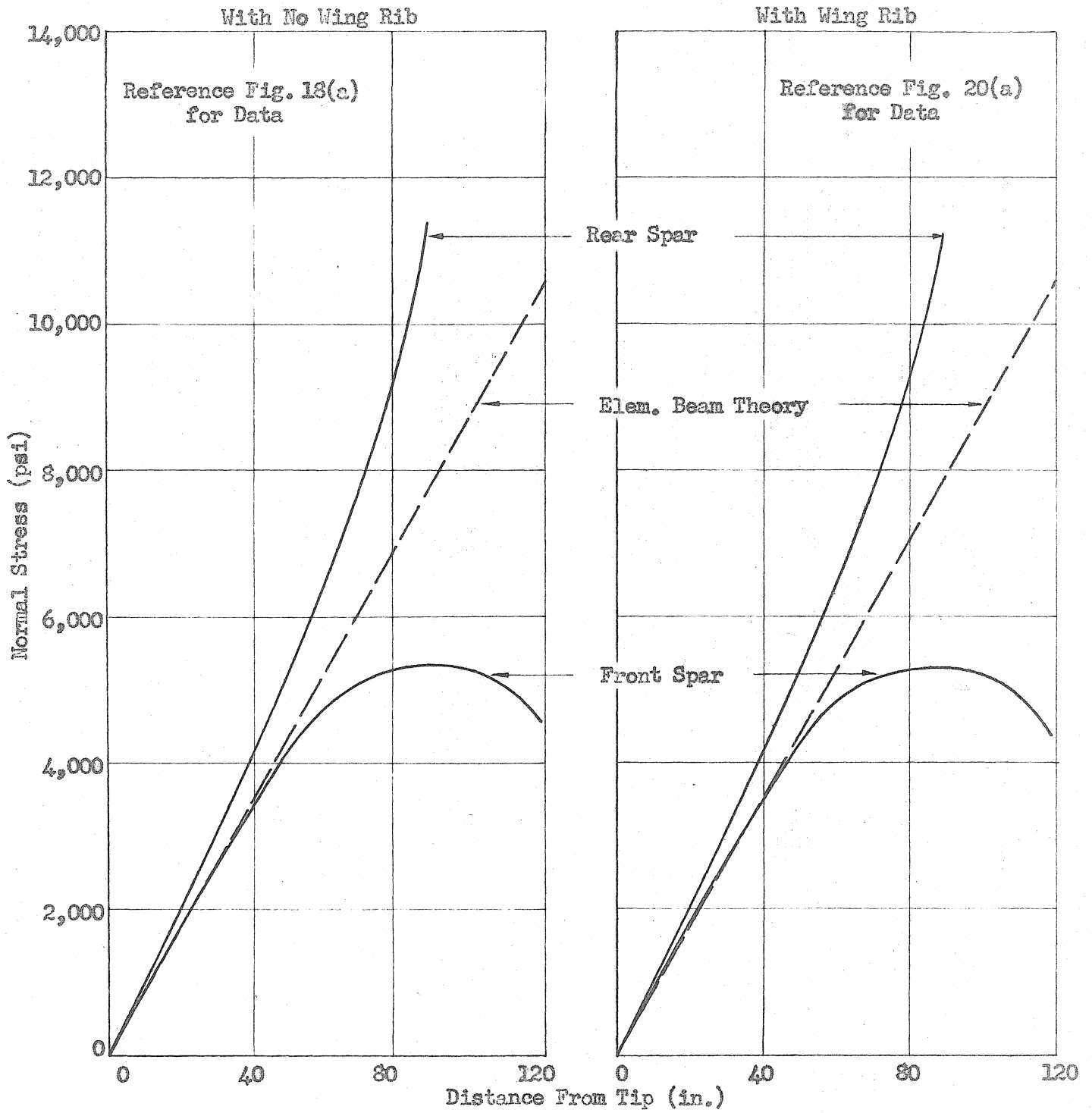
(a) Four Element Beam – Outer Panel

FIGURE 21

STRINGER AND FLANGE STRESSES OF SWEPTBACK BOX BEAM FOR TIP BENDING LOAD

(b) Four Element Beam – Inner Panel

FIGURE 21

STRINGER AND FLANGE STRESSES OF SWEPTBACK BOX BEAM FOR TIP BENDING LOAD

Reference Fig. 15(b) for Notation

Clipped With No Wing Rib

With No Wing Rib

Reference Fig. 18(b) for Data

Reference Fig. 18(c) for Data

Rear Spar

Elem. Beam Theory

Front Spar

S_1

Normal Stress (psi)

Distance From Tip (in.)

(c) Six Element Beam — Outer Panel

FIGURE 21

STRINGER AND FLANGE STRESSES OF SWEPTBACK BOX BEAM FOR TIP BENDING LOAD

Reference Fig. 15(b) for Notation

Clipped With No Wing Rib

With No Wing Rib
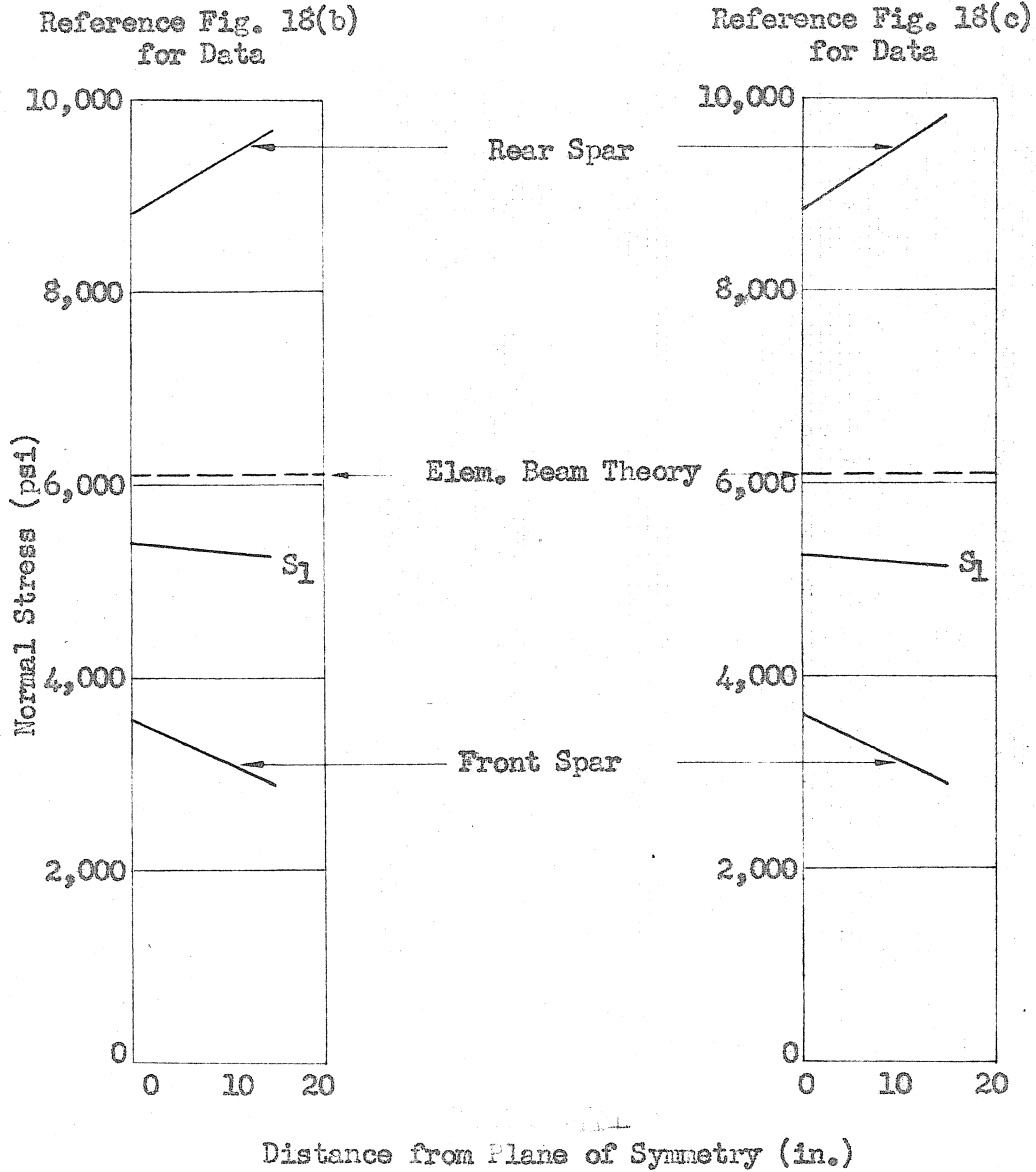
Reference Fig. 18(b)
for Data

Reference Fig. 18(c)
for Data

(d)   Six Element Beam — Inner Panel

FIGURE 21
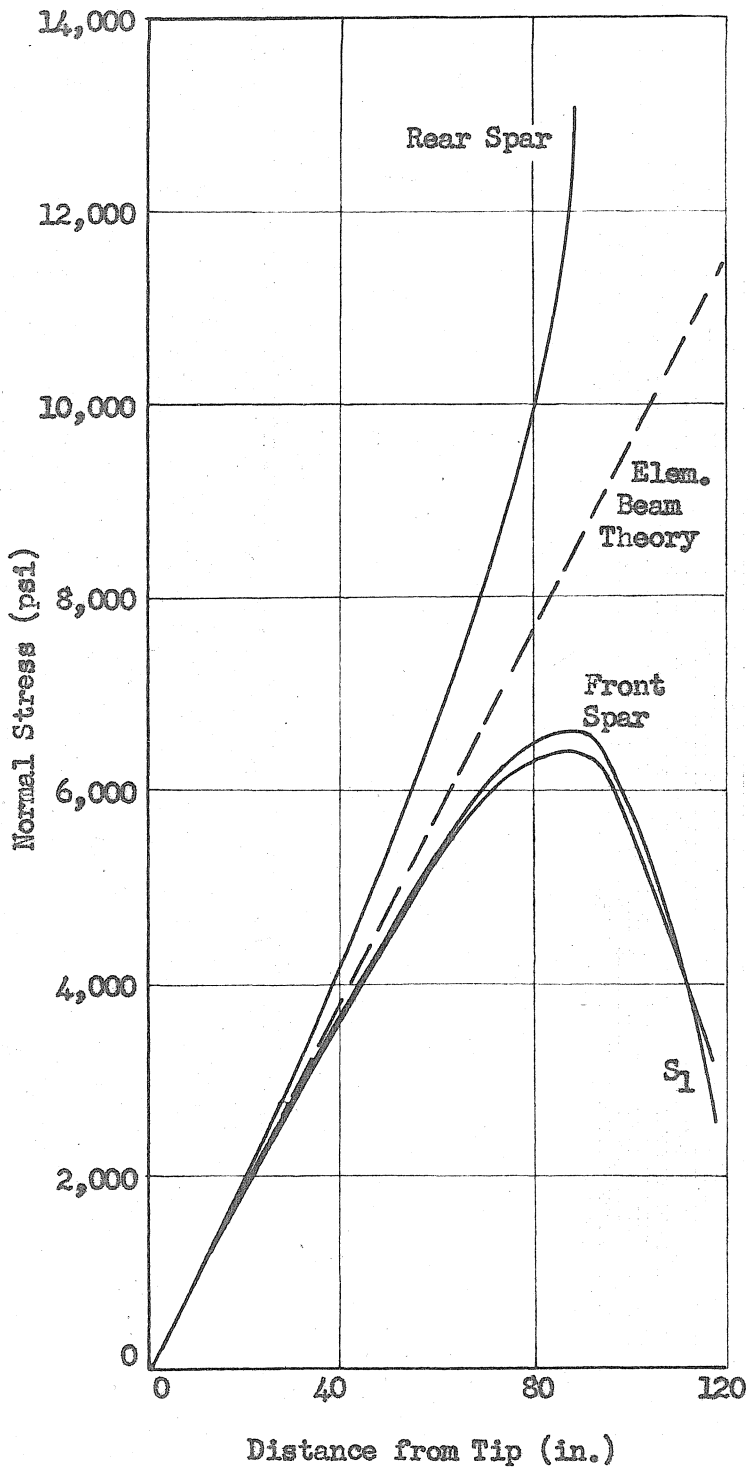
STRINGER AND FLANGE STRESSES OF SWEPTBACK BOX BEAM FOR TIP BENDING LOAD
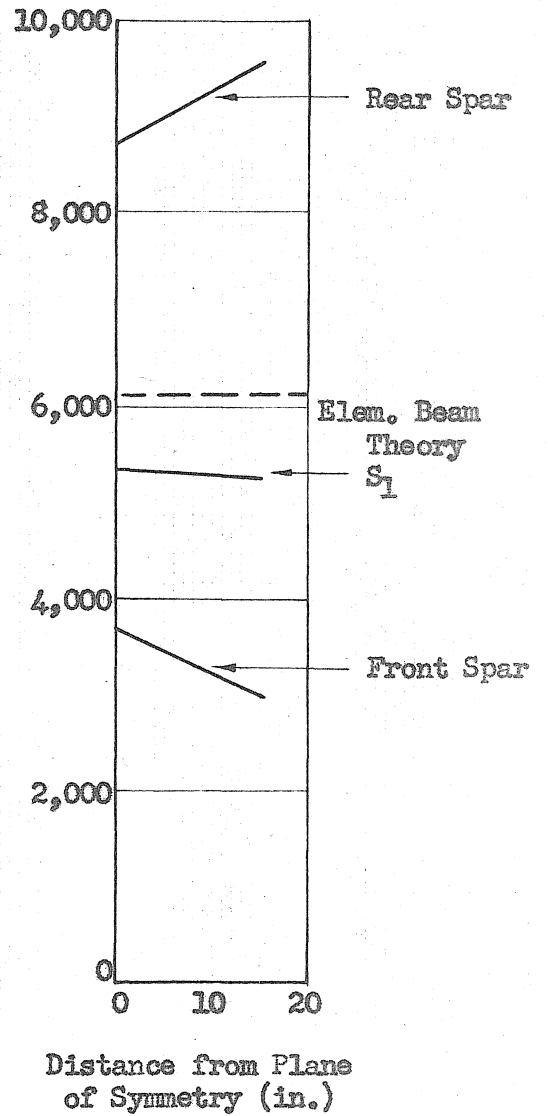
(e) Six Element Beam — With Wing Rib

FIGURE 21

STRINGER AND FLANGE STRESSES OF SWEPTBACK BOX BEAM FOR TIP BENDING LOAD
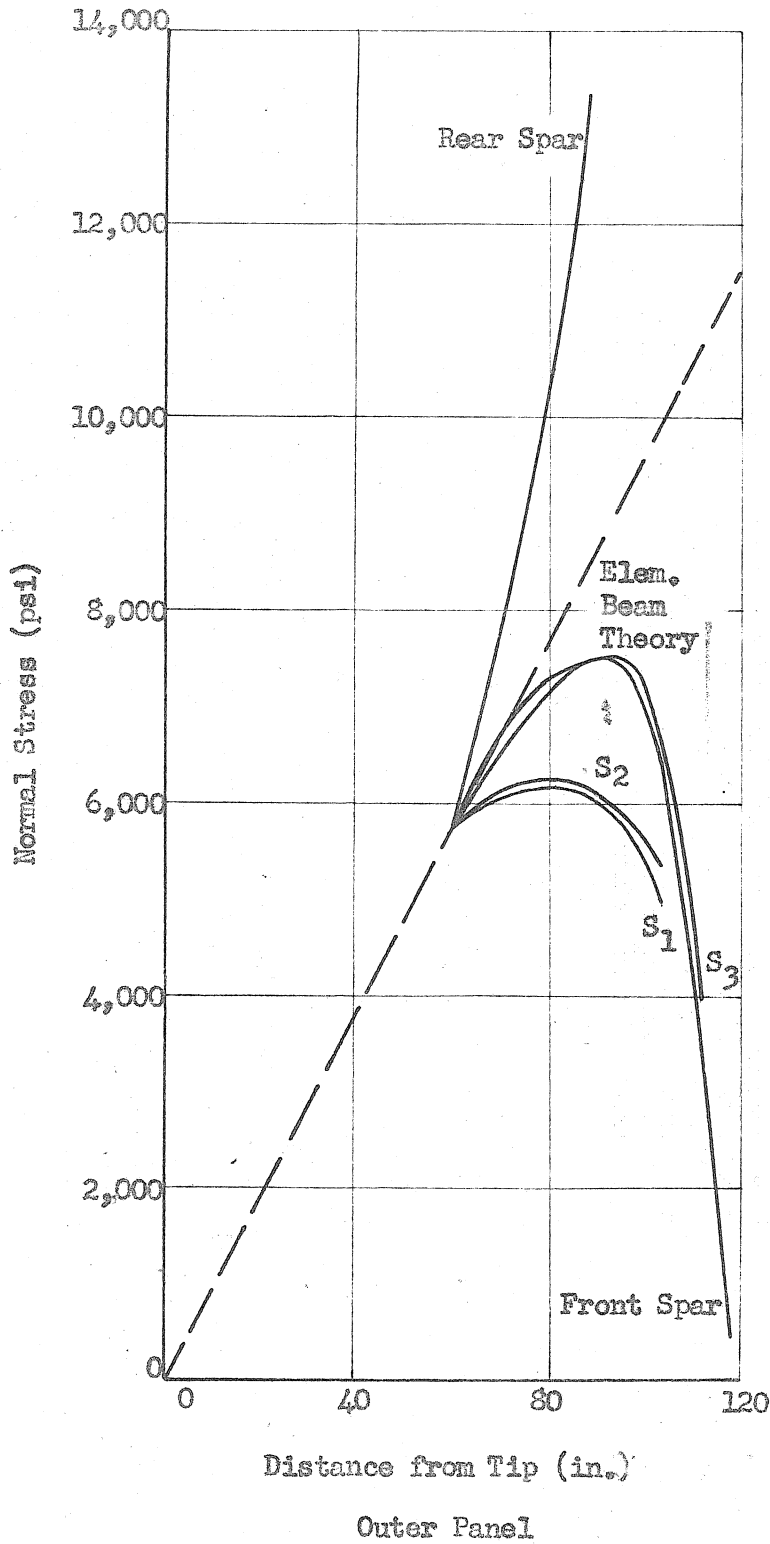
Reference Fig. 18(d)
for Data
Reference Fig. 15(c)
for Notation
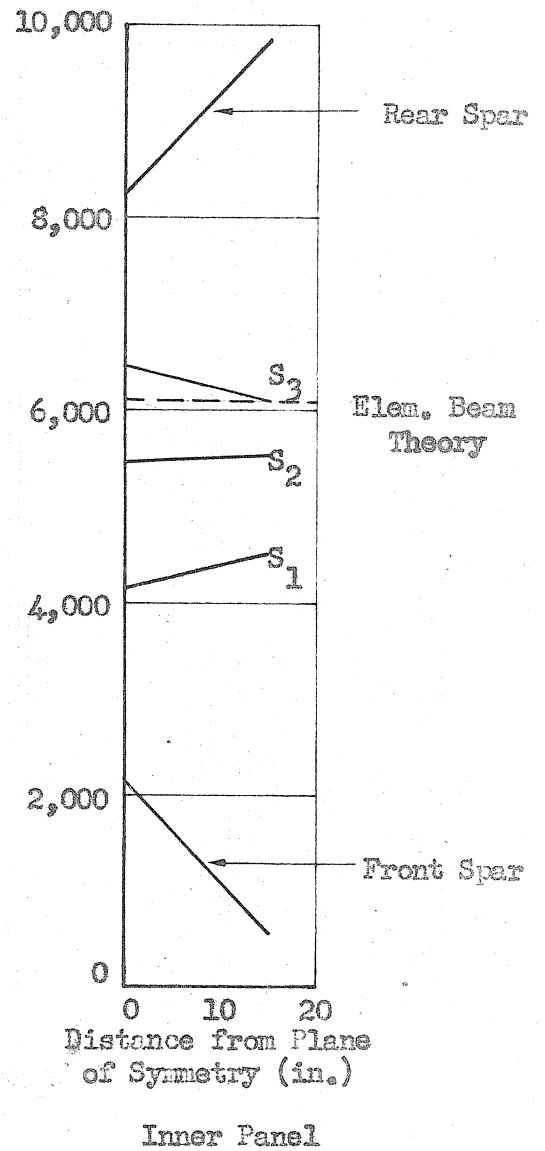
(f) Ten Element Beam — Clipped With No Wing Rib

FIGURE 21

STRINGER AND FLANGE STRESSES OF SWEPTBACK BOX BEAM FOR TIP BENDING LOAD

(a) Cross Section 21 Inches from Tip of Wing

FIGURE 22

TANGENTIAL DISTRIBUTION OF NORMAL STRESS
FOR THE SWEPT WING UNDER SYMMETRICAL BENDING

(b) Cross Section 43 Inches from Tip of Wing

FIGURE 22

TANGENTIAL DISTRIBUTION OF NORMAL STRESS
FOR THE SWEPT WING UNDER SYMMETRICAL BENDING

(c) Cross Section 65 Inches from Tip of Wing

FIGURE 22

TANGENTIAL DISTRIBUTION OF NORMAL STRESS
FOR THE SWEPT WING UNDER SYMMETRICAL BENDING

(d) Cross Section 86 Inches from Tip of Wing

FIGURE 22

TANGENTIAL DISTRIBUTION OF NORMAL STRESS
FOR THE SWEPT WING UNDER SYMMETRICAL BENDING

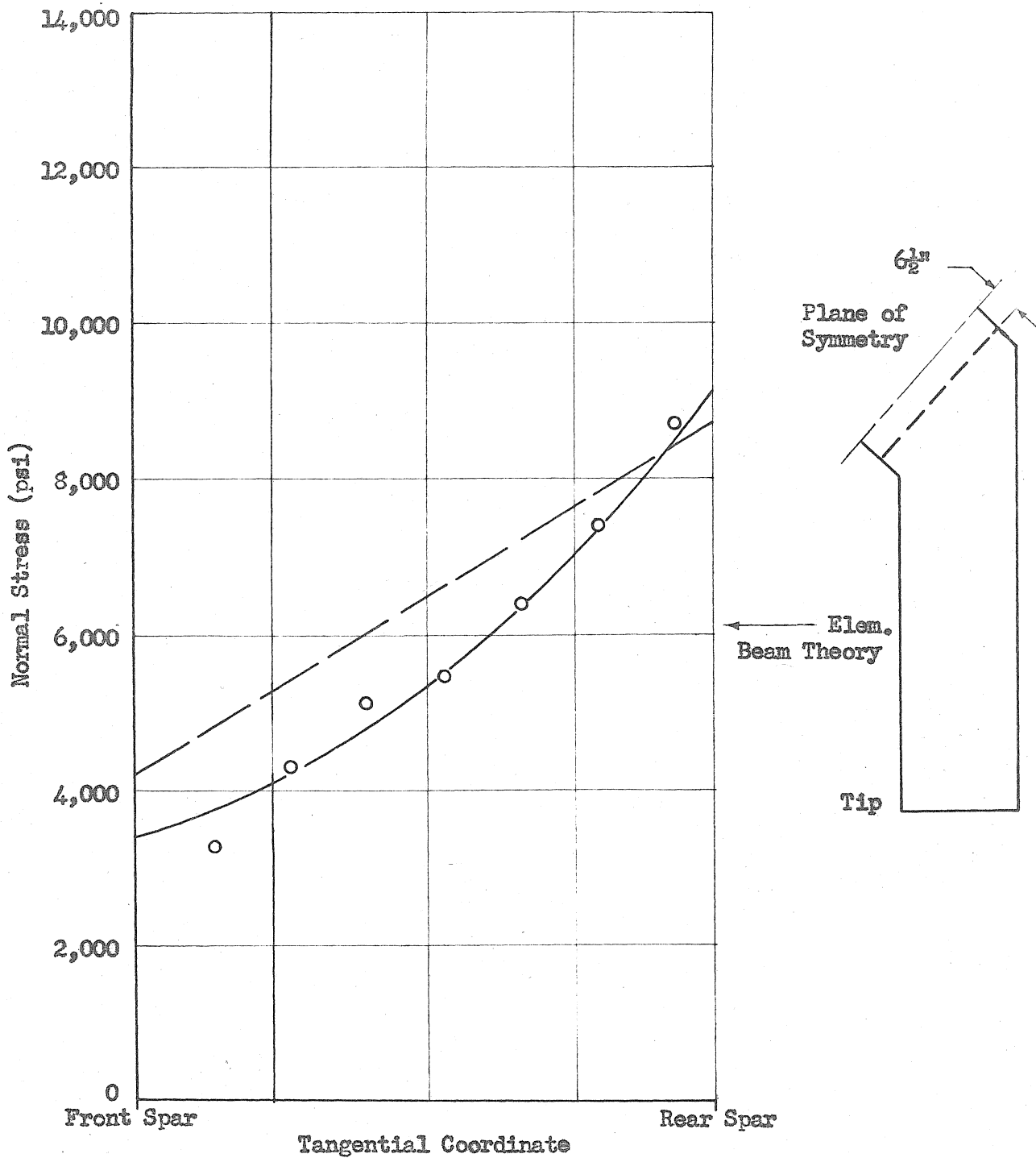Plane of
Symmetry

Elem.
Beam Theory

96"

Tip

Note: Center Point of
Theoretical Curve
Obtained from
Fig. 22(f)

Tangential Coordinate

Front Spar

Rear Spar

Notation

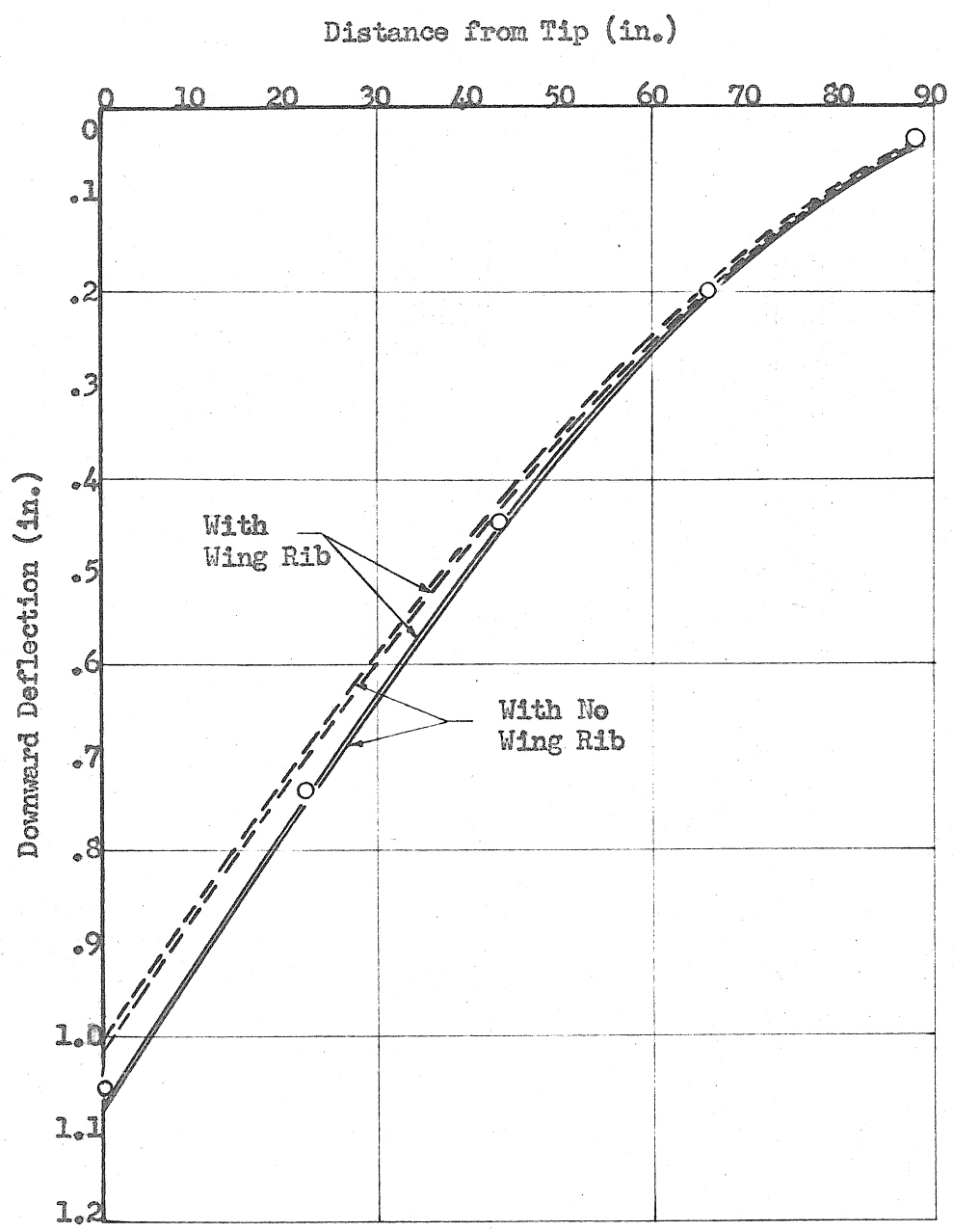O   Experimental Data (Reference N.A.C.A. T.N. 1525)
          Theoretical Curves (Reference Fig. 21)
────   Four Element Beam (With Wing Rib)
────   Six Element Beam (With Wing Rib)

(e)   Cross Section 96 Inches from Tip of Wing

FIGURE 22

TANGENTIAL DISTRIBUTION OF NORMAL STRESS
FOR THE SWEPT WING UNDER SYMMETRICAL BENDING

(f) Cross Section $18\frac{1}{4}$ Inches from Plane of Symmetry

FIGURE 22

TANGENTIAL DISTRIBUTION OF NORMAL STRESS
FOR THE SWEPT WING UNDER SYMMETRICAL BENDING

(g)   Cross Section $6\frac{1}{2}$ Inches from Plane of Symmetry

FIGURE 22

TANGENTIAL DISTRIBUTION OF NORMAL STRESS
FOR THE SWEPT WING UNDER SYMMETRICAL BENDING

○   Experimental Data (Reference N.A.C.A.   T.N. 1525)
          Theoretical Curves (Reference Table 2)

— —   Four Element Beam
——   Six Element Beam

Distance from Tip (in.)

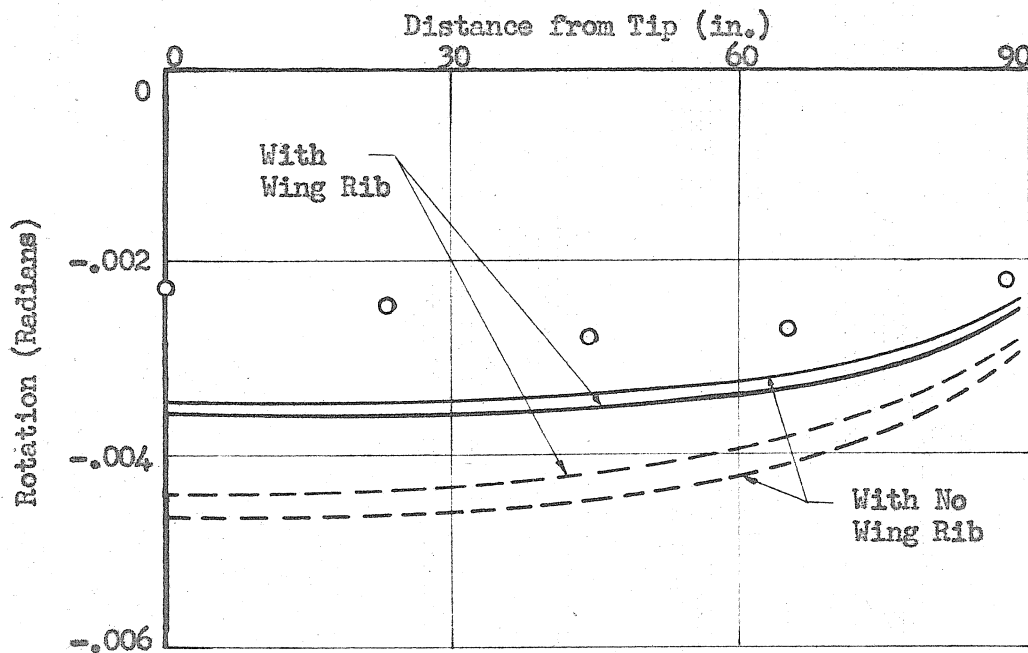(a) Deflections of Centerline of Wing

FIGURE 23

DEFORMATIONS OF SWEPT WING UNDER SYMMETRICAL BENDING

Notation

○ Experimental Data (Reference N.A.C.A. T.N. 1525)
   Theoretical Curves (Reference Table 2)
— —   Four Element Beam
——— Six Element Beam

(b) Rotations of Wing

FIGURE 23

DEFORMATIONS OF SWEPT WING UNDER SYMMETRICAL BENDING