SOME TOPICS IN GRAND UNIFIED MODELS
AND
THE COSMOLOGICAL BARYON ASYMMETRY

Thesis by

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Abstract

In part I of this thesis some of the parameters relevant to the production of a cosmological baryon number asymmetry are considered in the context of grand unified models. General expressions for the average baryon number generated in the free decays of bosons are derived. The CP violation necessary for the generation of a baryon excess is discussed for a variety of $SU(5)$ models. The kinematics of baryon number production in an illustrative $SO(10)$ model is discussed in detail. In part II a viable $SO(10)$ model is constructed which reproduces the phenomenological fermion mass and mixing angle values. A detailed discussion of the beta function for this model is presented. This analysis includes the effects of scalars.

Contents

References

1) Introduction

In the beginning, the idea of grand unification [1] was introduced as an economizing gesture to reduce the number of possible Yang-Mills [2] couplings. An immediate consequence of this was (under some circumstances) to reduce the plethora of parameters that appear in the Yukawa couplings of a model. The Higgs self-couplings, however, do not fare so well under this treatment for two reasons: the number of unconstrained parameters generally increases and the vial containing the noxious problem of hierarchies [3,8] is uncorked. There is also another problem endemic to grand unification that is shared by both fermions and scalars (especially when one considers models larger than the minimal $SU(5)$ scheme). This is the proliferation of degrees of freedom. One is forced to consider the possibility of some (presently) unobserved fermions and many unobserved scalars. In the end, the program of grand unification, although its original aim was to father simplicity, has given rise to a rather large amount of complexity. Nonetheless, in spite of or, perhaps, because of this complexity, grand unified models possess a number of interesting features.

One of the first things grand unification forced one to consider was the possibility of the decay of the proton [4,5]. Vector induced proton decay did not exist in a theory based on the product of a flavor group with $SU(3)$ of color: $G_{fl} \otimes SU(3)_c$. However, scalar induced proton decay certainly could have been put into a $G_{fl} \otimes SU(3)_c$ theory ad hoc through the inclusion of scalar representations with appropriate quantum numbers. The context of grand unification is a natural one in which to consider vector-induced proton decay.

A similar situation exists for other esoteric processes. Those arising from the presence of the extra fermions (either charged or neutral) could certainly have been considered in a $G_{fl} \otimes SU(3)_c$ theory and recently such ideas have been examined, prompted by their appearance in grand unified models [6]. Grand unification beyond $SU(5)$ requires one to consider massive neutral fermions and the associated neutrino oscillation and lepton number violation phenomena [7].

The presence of a large number of degrees of freedom in larger grand unified models necessitates the examination of their effect on the renormalization of the parameters in the model (notably the gauge couplings and, hence, the Weinberg angle) [8,9,10].

In grand unified models the global symmetry structure can generally be very rich, allowing one to experiment with a large number of naturalness conditions in an attempt to reproduce phenomenological mass and mixing angle (and CP violation parameter) values. The possible presence of of zeroth order mass relations and of the soft breaking of symmetries each allows one to consider exactly calculable quantities, permitting, perhaps, the construction of a model in which the electron family's parameters are strictly perturbative.

To explore the ideas mentioned above, it is by no means necessary to introduce grand unification, but it does act as a natural matrix in which to consider them together. Just as one was able to consider the $SU(2)_L \otimes U(1)_Y$ model as "a framework for organizing huge quantities of experimental data," [11] so too grand unified models can be considered as a framework for considering a large number of theoretical possibilities.

One may wonder about the possible significance of grand unified models on a level somewhat deeper than the pragmatic and the organizational. Unification without gravity is only partial unification. So too is unification without a criterion for choosing which of many possible models is the most correct one. Grand unified models may be criticized on both of these accounts. The following operational philosophy is certainly a reasonable one to adopt. Grand unified models are worth exploring both from the point of view of being a laboratory for theoretical ideas and from the hope that one such model will turn out to be a limit of a fundamental theory yet to be discovered.

A few such theoretical ideas are considered in this thesis. In the first part we discuss some aspects of the cosmological baryon asymmetry in the context of grand unified models. There are two perspectives that one may take in considering baryon number violating processes in the very early universe. First is the grand unified modeler's perspective. From this point of view one notes that there are only two laboratories in which the effects of the "intermediate vector baseballs* [53]" are manifest. One is caverns in salt mines where, it is hoped by some, the decay of the proton may be observed. The other is at the superhigh temperatures that were possibly present in the very early universe: $T \gtrsim 10^{15}$ GeV. At such temperatures the rates for baryon number violating interactions are competitive with those that conserve baryon number. The second perspective is that of the cosmologist who poses the so-called initial condition question: which observational cosmological facts must be taken as initial conditions (isotropy? homogeneity? thermal equilibrium?...) and

*Thank you Sid.

which ones may or must be derived? In this context grand unified models act as a self-consistent setting in which to discuss how the cosmological baryon number asymmetry does not *need* to be imposed as an initial condition on the evolution of the universe. In the body of part I a number of topics relevant to the calculation of the baryon asymmetry are discussed. To set the stage for this we review, in chapter 2, the method of calculating the magnitude of the cosmological asymmetry in an arbitrary grand unified model*.

In the second part of this thesis we discuss the construction of a grand unified model based on the simple Lie group $SO(10)$ which acts as a natural generalization of $SU(5)$. In this model we are able to reproduce the phenomenological fermion mass and mixing angle values. It is a general feature of models based on gauge groups larger than $SU(5)$ that there may be more than one level of symmetry breaking; $SO(10)$ has this feature. The various predicitons of a grand unified model depend upon the complexity of the symmetry breaking. Notable in this regard is the effect of multiple symmetry breaking scales on the running of the gauge couplings; hence we present an analysis of the beta function in this model.

In summary then, this thesis asks again the age old question, "What is the cat's last name?" This question cannot be answered as is pointed out in the poem which follows, nonetheless it is great fun to try. Eliot has put it [54]:

*The text of chapter 2 is essentially that of a paper by J. A. Harvey, E. W. Kolb, D. B. Reiss and S. Wolfram recently submitted to Physical Review Letters.

The naming of cats is a difficult matter,
        It isn't just one of your holiday games;
You may think at first that I'm as mad as a hatter
When I tell you, a cat must have THREE DIFFERENT NAMES.
First of all, there's the name that the family use daily,
        Such as Peter, Agustus, Alonzo or James,
Such as Victor or Jonathan, George or Bill Bailey -
        All of them sensible everyday names.
There are fancier names if you think they sound sweeter,
        Some for the gentlemen, some for the dames:
Such as Plato, Ademetus, Electra, Demeter -
        But all of them sensible everyday names.
But I tell you, a cat needs a name that's particular,
        A name that's peculiar, and more dignified,
Else how can he keep up his tail perpendicular,
        Or spread out his whiskers, or cherish his pride?
Of names of this kind, I can give you a quorum,
        Such as Munkustrap, Quaxo, or Coricopat,
Such as Bombalurina, or else Jellyorum -
        Names that never belong to more than one cat.
But above and beyond there's still one name left over,
        And that is the name that you never will guess;
The name that no human research can discover -
        But THE CAT HIMSELF KNOWS, and will never confess.
When you notice a cat in profound meditation,
        The reason, I tell you, is always the same:
His mind is engaged in a rapt contemplation
        Of the thought, of the thought, of the thought of his name:
                His ineffable effable
                Effanineffable
Deep and inscrutable singular Name.

PART I

2) Cosmological Baryon Generation in Grand Unified Models

Cosmology is potentially an important source of information on the baryon number (B) violating interactions expected in most grand unified gauge models. Any net B imposed as an initial condition on the universe should have been rapidly destroyed by any B-violating interactions. To account for the observed baryon number density to photon number density ratio, $n_B/n_\gamma \simeq 10^{-9}$, a net baryon number must subsequently have been generated. This requires, in addition to B violation, the violation of C and CP (and hence T) invariance, along with departures from thermal equilibrium [12,13]. This chapter outlines the complete calculation of n_B/n_γ generation in specific grand unified models in the context of the standard hot big bang model of the early universe. The method we present allows for the exact treatment of an arbitrary number of superheavy bosons as well as the presence of nonthermalizing modes [14]. We summarize results for several realistic $SU(5)$ models. Many details and extensions are discussed in ref. [15].

We denote heavy bosons generically by χ and light fermions by a, b,.... The number density n_i of a particle i and that of its antiparticle $n_{\bar{i}}$ are parametrized by $i_+ \equiv (n_i + n_{\bar{i}})/n_\gamma$ and $i_- \equiv (n_i - n_{\bar{i}})/n_\gamma$. The time development of these quantities is described by a set of coupled Boltzmann transport equations. For the heavy bosons these are [13,15]

$$\dot{\chi}_+ = -\sum_{a,b} \langle\Gamma(\chi \to ab)\rangle \, (\chi_+ - \chi_+^{eq}) \tag{2.1a}$$

$$\dot{\chi}_- = -\sum_{a,b} \langle\Gamma(\chi \to ab)\rangle \, (\chi_- - (a_- + b_-)\, \chi_+^{eq}) \tag{2.1b}$$

where dots denote time derivatives and the expansion of the universe is accounted for through division by n_γ in the definitions of $i_\pm$. The first terms on the right side of eqns (2.1a) and (2.1b) correspond to free decays of χ and $\bar{\chi}$ with partial rates $<\Gamma(\chi\to ab)>$ averaged over the decaying χ energy spectrum. The second terms account for back reactions in which the χ decay products interact to produce χ. The equilibrium number density χ_+^{eq} is obtained by integrating the $\exp[-E_\chi/T]$ equilibrium Maxwell-Boltzmann phase space density. In equilibrium, $\chi_+=\chi_+^{eq}$ and $\dot{\chi}_+=0$; the expansion of the universe produces deviations from equilibrium at temperatures $T\sim m_\chi$.

The densities of fermion species develop according to

$$\dot{f}_- = \sum_{a,b,\chi} <\Gamma(\chi\to ab)> (N_f)_{ab}\,\{(\chi_+ - \chi_+^{eq})\,R(\chi\to ab) + 2\chi_- - (a_- + b_-)\,\chi_+^{eq}\}$$

$$+\sum_{a,b,c,d,\chi} n_a\,[(N_f)_{ab} - (N_f)_{cd}]\,\{a_- + b_- - c_- - d_-\}\,<|v|\,\sigma'_\chi(ab\to cd)> \quad , \qquad (2.2)$$

where $(N_f)_{ab}$ denotes the number of particles of type f in the state ab. $R(\chi\to ab)$ denotes the difference in branching ratios between the CP conjugate decays $\chi\to a\,b$ and $\bar{\chi}\to\bar{a}\,\bar{b}$ divided by the full rate for χ decay; it vanishes if CP is conserved. The first part of the first term on the right side of eqn (2.2) therefore represents the production of an asymmetry in fermion number densities as a result of CP-violating decays of a symmetrical χ, $\bar{\chi}$ mixture. The second part causes asymmetries , χ_- , between χ and $\bar{\chi}$ to be transferred to the fermions when the χ ($\bar{\chi}$) decays. The third part gives a correction to the rate for inverse decays resulting from the deviation of the fermion number densities from their equilibrium value. The second term in eqn (2.2) represents the production and destruction of fermions by two-to-two scattering processes. σ'_χ is the cross-section for this scattering mediated by χ exchange, but with the term corresponding to a real intermediate χ removed (since this is already accounted for by χ

decay and inverse decay processes).

The number of independent particle densities to be treated in eqns (2.1) and (2.2) may be reduced by using unbroken symmetries (gauge and global). For non-Abelian groups, any asymmetries are shared symmetrically among members of each irreducible representation. If only a subset of the interactions that may potentially contribute to eqn (2.2) is included, there may be additional symmetries leading to further conserved combinations of fermion number densities (e.g., Π conservation in the absence of Higgs-fermion couplings for the models discussed below).

Let f^i_- $(i=1, \cdots, N_f)$ be the independent fermion asymmetries and χ^α_- $(\alpha=1, \ldots, N_\chi)$ the independent supermassive boson asymmetries. It is convenient to form a set $\vec{Q}$ which consists of independent quantum number densities B, L, etc... related to $\vec{F}=\{f^i_-, \chi^\alpha_-\}$ by a unitary transformation, $\vec{Q}=H\,\vec{F}$, $\vec{F}=H^{-1}\,\vec{Q}$.

The thermalization of a quantum number Q_i through reactions of a particular boson χ is given from eqn (2.2) by $\dot{Q}_i=\sum_\chi \chi_+^{eq}\, M^\chi_{ij}\, Q_j$, where $M^\chi_{ij}=\sum_{k,l} \Delta Q_i(\chi\to f^k f^l)\langle\Gamma(\chi\to f^k f^l)\rangle(H^{-1}_{kj}+H^{-1}_{lj})$ and $\Delta Q_i(\chi\to f^k f^l)$ represents the change in the value of Q_i through the reaction $\chi\to f^k f^l$. Boltzmann's H theorem requires that the eigenvalues of M^χ are all real and nonpositive. Any zero eigenvalues reveal additional symmetries; the corresponding eigenvector of number densities is then conserved in χ reactions (e.g., Π in vector boson exchanges in $SU(5)$).

We consider two grand unified models based on $SU(5)$. In each case a family of fermions transforms as a reducible representation $(\bar{5}\oplus 10)_i$, labeled by the family index i. The following Higgs representations are taken to couple to fermions: in model I (minimal $SU(5)$), a single 5 of

Higgs, H_5; in model II, H_5 and an additional 5 of Higgs, $H_{5'}$. The Yukawa couplings in these models have the schematic form $(\bar{5}_i\ (D_\alpha)_{ij}\ 10_j)\ H_\alpha + (10_i (U_\alpha)_{ij}\ 10_j)\ \bar{H}_\alpha$.

It is shown in chapter 4 that a CP-violating nonzero $R(\chi\to ab)$ enters through an imaginary part of the product of the couplings in diagrams in which one boson is exchanged between the ab produced in the χ decay. The sum over a and b in eqn (2.2) runs over all types and families of fermions; thus, for fixed fermion types, $R(\chi\to ab)$ is proportional to a family space trace of Yukawa coupling matrices. In model I the first diagram exhibiting CP violation involves only Higgs bosons and is of eighth order in the Yukawa couplings [16,17,15]. This is discussed in chapter 6. It is proportional to the imaginary part of the family space trace, $Tr[UU^\dagger UD^2U^\dagger D^2]$, suggesting the rough estimate $R\sim\alpha^3\,(m_F/m_W)^6\,\varepsilon/(128\pi^3)$ $=4\times10^{-9}\,(m_F/m_W)^6\,\varepsilon$, with $|\varepsilon|\lesssim1$, where m_F is the mass of the heaviest fermion. (Stability of the effective potential requires that $m_F\lesssim\sqrt{3}m_W$ [18] and hence $R\lesssim10^{-8}\,\varepsilon$, making the production of an adequate baryon asymmetry implausible in this model.)

In model II (discussed in chapter 7), both H_5 and $H_{5'}$ have only the single B-violating component*, (3, 1, -1/3); since 5 is a complex representation one may form complex linear combinations so that the (3, 1, -1/3) in both 5 and 5' is separately a mass eigenstate. This suffices to show that no CP violation may occur for gauge boson decay with Higgs scalar exchange (or vice versa). CP violation may occur at $O(\alpha(m_F/m_W)^2)$ through 5 decay with 5' exchange (and vice versa) [19].

* In this notation the first entry is the $SU(3)$ multiplicity, the second is the $SU(2)_L$ multiplicity and the last the value of the weak hypercharge Y normalized so that the charge operator is given by $Q=T_{3L}-Y$.

$SU(3) \otimes SU(2)_L \otimes U(1)_Y$ symmetry allows the 15 independent fermion fields in a family of an $SU(5)$ model to be reduced to the set U_L, $(U^c)_L$, $(D^c)_L$, E_L and $(E^c)_L$ (the subscript L denotes the left-handed helicity state and c denotes charge conjugation). The model contains a (3, 2, 5/6) of B-violating vector bosons X (with number densities parametrized by X_- and X_+). We consider the case where there are n_s (=1 or 2) scalars, S_1, S_2,..., S_{n_S}, transforming as (3, 1, -1/3) (with number densities parametrized by S_{i-} and S_{i+}). These models possess a locally conserved weak hypercharge whose initial value we assume to be zero. The models exhibit two further zero eigenmodes. The first is $B-L$ which has zero eigenvalue (is conserved) in all boson interactions. A second zero eigenmode, $\Pi=-3(D^c)_{L-}-2E_{L-}$, is present if scalar-fermion interactions are removed [14]. Π (termed "fiveness") corresponds to the net number density of the fermion species appearing in the 5 representation. A density Π_0 generated through Higgs decays would be distributed as $B = -\Pi_0/10$, $\nu_-=-\Pi_0/5$ through Π-conserving X interactions. Π_0 may be destroyed through exchanges of light Higgs bosons. A convenient choice of independent combinations of fermion densities is $n_B/n_\gamma \equiv B = 2D_{L-}-(U^c)_{L-}-(D^c)_{L-}$, Π and $\nu_-=E_{L-}$.

For model I, according to the estimate for $R(S \to ab)$ given above, an adequate baryon number asymmetry will be generated only if very heavy fermions exist ($m_F \sim m_W$)*. Fig. 2.1a shows the baryon asymmetry (taking m_X=5×10^{14} GeV and α=1/40) as a function of m_S/m_X for m_F/m_W=1 and m_F/m_W=3 obtained by numerically integrating the Boltzmann transport equations (2.1) and (2.2). When $m_S/m_X \gg 1$, X exchanges thermalize the B

* Similar conclusions have recently been reached in ref. [20].

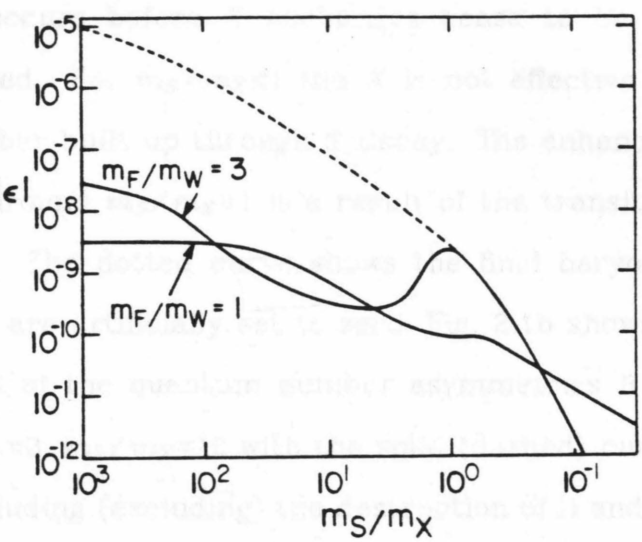

Fig. 2.1a: Baryon number density as a function of the Higgs boson (S) mass generated in the minimal $SU(5)$ model in which the heaviest fermion has mass m_F. Results are for α=1/40, m_X=5×10^{14}GeV. The CP violation parameter ε is unknown but less than 1.

produced in S decay to the value $-\Pi/10$; meanwhile, Π is reduced by light Higgs interactions. The final B attained is determined by the reduction in Π that occurs before X exchanges cease to be important and B becomes fixed. For $m_S/m_X<1$ the X is not effective in destroying the baryon number built up through S decay. The enhancement in the final value of B around $m_S/m_X=1$ is a result of the transition between these two regions. The dotted curve shows the final baryon number if all X interactions are artificially set to zero. Fig. 2.1b shows the temperature development of the quantum number asymmetries B, Π and ν_- for the case $m_F/m_\varphi=3$, $m_S/m_X=10$ with the solid (dashed) curves indicating the effect of including (excluding) the destruction of Π and ν_- by the interactions of the light Higgs doublet.

For model II the final baryon number density as a function of m_{S_1}/m_X is shown in fig. 2.2 for different choices of m_{S_2}/m_X. Note that, when $m_1=m_2$, we have (assuming $(\Gamma_{S_1})_{total}=(\Gamma_{S_2})_{total}$ in the Born approximation) $R(S_1\rightarrow ab)=-R(S_2\rightarrow ab)$ and hence no B is generated. For $m_{S_i}>m_X$ the additional decay mode $S_i\rightarrow X+\varphi$ (where φ is a light Higgs boson) decreases the effective CP violation, $R(S_i\rightarrow ab)$, in S_i decay. For $m_{S_2}>m_X$ and $m_{S_1}>m_X$, the final B is negative and determined by vector thermalization of the positive Π produced in S_2 decay. For $m_{S_2}>m_X$ but $m_{S_1}<0.1\,m_X$, the final baryon number is positive and determined mainly by inverse decays into S_1. The dominant term governing the time evolution of B for $T\gtrsim m_{S_1}$ is $\dot{B}\propto S_{1+}^{eq}\langle\Gamma_{S_1}\rangle(14\nu_- - 12B+7\Pi)$ with similar equations for $\dot{\nu}_-$ and $\dot{\Pi}_-$. Since $\Pi>0$, $\Pi>\nu_-$ and $\Pi>B$, this term tends to drive B positive. In general there are three linear combinations of B, ν_- and Π which decrease as exponentials until cut off at temperatures below m_{S_1}. The final value of B thus

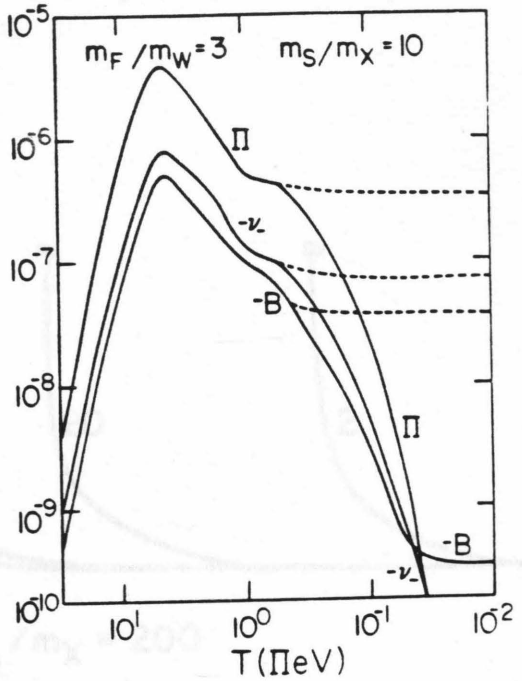

Fig. 2.1b. Evolution of independent quantum number densities as a function of temperature in the minimal $SU(5)$ model. B denotes the net baryon number, ν_- the asymmetry between $\bar{\nu}$ and ν densities and Π the total asymmetry between fermions in the 5 and $\bar{5}$ representations of SU(5); 1ΠeV = 10^{24} eV. In these graphs the parameter ε has been scaled out. The dashed curves are results obtained by neglecting light Higgs boson exchange processes.

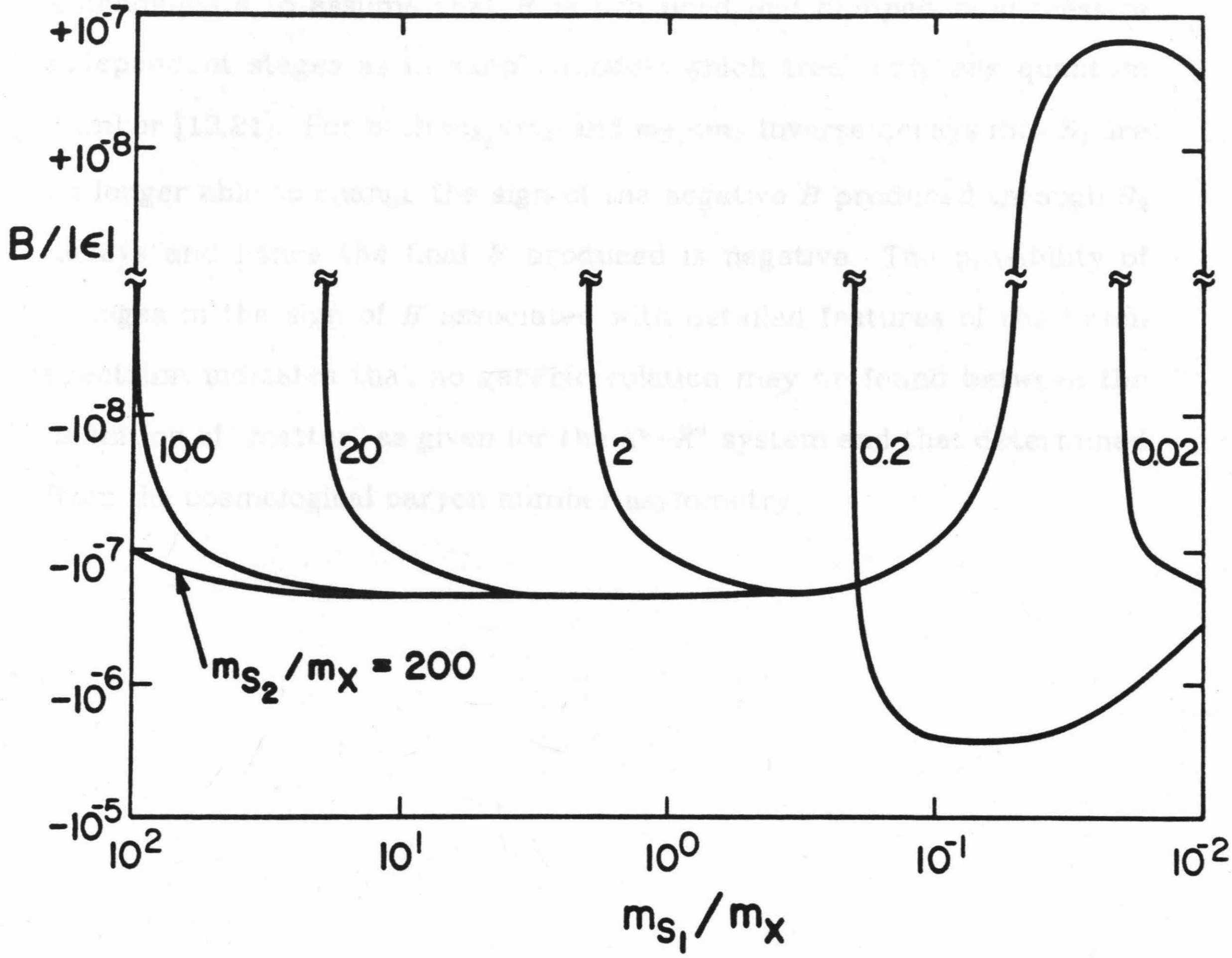

Fig. 2.2: Baryon number density for an $SU(5)$ model with two baryon number violating Higgs bosons (S_1, S_2) as a function of the S_1 mass for different choices of the S_2 mass. The results are for $\alpha=1/40$ and $m_X=5\times10^{14}$ GeV. The CP violation parameter ε is unknown but less than 1.

depends sensitively on the initial values of Π, ν_- and B. For this reason, it is inadequate to assume that B is produced and damped in successive independent stages as in simple models which treat only one quantum number [13,21]. For both $m_{S_2}<m_X$ and $m_{S_1}<m_X$ inverse decays into S_1 are no longer able to change the sign of the negative B produced through S_2 decays and hence the final B produced is negative. The possibility of changes in the sign of B associated with detailed features of the boson spectrum indicates that no generic relation may be found between the definition of "matter" as given for the $K^0-\bar{K}^0$ system and that determined from the cosmological baryon number asymmetry.

3) B and B-L Violation in Models With SU(5) Singlet Fermions

At temperatures at which baryon number production is thought to occur, $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ symmetry will be unbroken. With this assumption we may analyze the possible baryon number violating vector and scalar bosons which may occur in a renormalizable theory. This analysis has been done for fermions with the quantum numbers of the $\bar{5}\oplus 10$ representation of $SU(5)$ [19]. For this case the baryon violating vector bosons come in two varieties, X and X', with $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ transformation properties (3, 2, 5/6) and (3, 2, -1/6) respectively. The possible baryon violating scalars are $S\sim(3, 1, 1/3)$, $S_1\sim(3, 1, 4/3)$ and $S_2\sim(3, 3, 1/3)$. Fermi statistics require that S_1 and S_2 couple to fermions antisymmetrically in family space in order to violate baryon number (hence, they cannot give a tree-level contribution to the proton decay rate). With the conventional assignments of baryon number (B) and lepton number (L), it is found that all of these baryon number violating bosons preserve $B-L$.

With the $SO(10)$ model in mind we extend this analysis to include an $SU(5)$ singlet fermion $N_L\sim(1, 1, 0)$. We assume here that the N_L has a Majorana mass; consequently it may not carry any quantum numbers. The fermion fields considered in our analysis are listed in table 3.1. We assume that this pattern of fermions is repeated for the heavier families. Lorentz invariance requires that renormalizable vector couplings have the form $\Psi_a^\dagger\sigma^\mu\Psi_b V_\mu$ while renormalizable scalar couplings have the form $\Psi_a^T\sigma_2\Psi_b S$ with V_μ and S vector and scalar fields respectively. By taking the relevant products of fermion fields appearing in Table 3.1 we obtain the possible $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ transformation properties of the vector and scalar fields along with their values of B and $B-L$. These values of B and

Field	$SU(3)\otimes SU(2)_L\otimes U(1)_Y$
$l=\begin{pmatrix}\nu\\ e\end{pmatrix}$	(1, 2, 1/2)
$q=\begin{pmatrix}u\\ d\end{pmatrix}$	(3, 2, -1/6)
e^c	(1, 1, -1)
u^c	($\bar{3}$, 1, 2/3)
d^c	($\bar{3}$, 1, -1/3)
N	(1, 1, 0)

Table 3.1

Fermion fields and their associated $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ representation content.

$B-L$ are listed in tables 3.2 and 3.3 respectively. Bosons which may violate B or $B-L$ are indicated by a dash. We find no new B-violating vector or scalar bosons. However, the X' vectors and S scalars are now capable of violating $B-L$ due to their interactions with N_L. The additional $B-L$ violating vectors transform as (3, 1, −2/3) and (1, 1, 1) and are gauge fields for the $SU(4)$ and $SU(2)_R$ subgroups of $SO(10)$ respectively. The additional $B-L$ violating scalars transform as (1, 2, 1/2) (the ordinary Higgs doublet of $SU(2)_L \otimes U(1)_Y$), (3, 2, -1/6) and (1, 1, 1). These scalars are found in the following $SO(10)$ representations which may couple to fermions:

$$(1,2,1/2) \subset 10,\ 120,\ 126$$

$$(3,2,-1/6) \subset 126 \tag{3.1}$$

$$(1,1,1) \subset 120,\ 126$$

If the effective symmetry is $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ or $SU(4) \otimes SU(2)_L \otimes U(1)_R$, then a Majorana mass for the N_L is forbidden by the $SU(2)_R$ or $U(1)_R$ symmetry and the N_L must be treated similarly to the other fermions. In particular, the gauged $B-L$ symmetry present in SO(10) will be unbroken and we must assign a value $B-L = 1$ to the N_L. If the effective symmetry is $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, then, as will be discussed in chapter 8, the presence of an unbroken charge conjugation symmetry forbids the production of a baryon asymmetry [22].

For the tables which follow, in (**i**, **j**, **k**), **i** denotes the $SU(3)$ representation, **j** denotes the $SU(2)$ representation and **k** denotes the $U(1)_Y$ charge Y. Y is given here corresponding to a definition for the electric charge $Q = T_3 - Y$, where T_3 is the diagonal generator of $SU(2)$ normalized to $Tr[T_3^2]=1/2$ in the 5 representation of $SU(5)$.

	B	$B-L$
(8, 3, 0)	0	0
(8, 1, 0)	0	0
(8, 1, -1)	0	0
(6, 2, -5/6)	2/3	2/3
(6, 2, 1/6)	2/3	2/3
(3, 3, -2/3)	1/3	4/3
(3, 2, -1/6)	–	–
(3, 2, 5/6)	–	-2/3
(3, 1, -2/3)	1/3	–
(3, 1, -5/3)	1/3	4/3
(3, 1, 1/3)	1/3	1/3
(1, 3, 0)	0	0
(1, 2, -3/2)	0	2
(1, 2, -1/2)	0	1
(1, 1, 0)	0	0
(1, 1, 1)	0	–

Table 3.2

Vectors that may couple to the fermions of table 3.1. Their $SU(3) \otimes SU(2)_L \otimes U(1)_Y$ representation content is given along with the associated values of B and $B-L$. Vectors that may have more than one value for these quantities are indicated by a dash.

	B	$B-L$
(8, 2, 1/2)	0	0
(6, 3, -1/3)	2/3	2/3
(6, 1, -1/3)	2/3	2/3
(6, 1, -4/3)	2/3	2/3
(6, 1, 2/3)	2/3	2/3
(3, 3, 1/3)	–	-2/3
(3, 2, -7/6)	1/3	4/3
(3, 2, -1/6)	1/3	–
(3, 1, 1/3)	–	–
(3, 1, 4/3)	–	-2/3
(3, 1, -2/3)	1/3	1/3
(1, 3, 1)	0	-2
(1, 2, 1/2)	0	–
(1, 1, 1)	0	–
(1, 1, -2)	0	2
(1, 1, 0)	0	0

Table 3.3

Scalars that may couple to the fermions of table 3.1. Their $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ representation content is given along with the associated values of B and $B-L$. Scalars that may have more than one value for these quantities are indicated by a dash.

4) Baryon Number Generation in Free Decays

In this chapter we describe the calculation of the average baryon number produced in the free decays of an equal mixture of particles χ and their antiparticles (CP conjugates) $\bar{\chi}$. This asymmetry is parametrized by the quantity

$$R_\chi \equiv \sum_f B_f \left\{ \frac{\Gamma(\chi \to f)}{\Gamma_\chi} - \frac{\Gamma(\bar{\chi} \to \bar{f})}{\Gamma_{\bar{\chi}}} \right\} \tag{4.1}$$

where $\Gamma(\chi \to f)$ denotes the partial width for decay of χ to the final state f, Γ_χ is the total χ decay width and B_f is the baryon number of the state f (so that $B_f = -B_{\bar{f}}$).

In treating the statistical mechanics of baryon number production it is convenient to choose a basis so that the χ are mass eigenstates. For (4.1) to be nonzero, CP must be violated in the decays of χ and $\bar{\chi}$. As discussed below (and proved in general in the first reference of [19] and in [13]), this requires interference between the Born amplitude for the decay and a one-loop correction with an absorptive part. In addition, the couplings of the particles participating in the decay must be relatively complex.

We consider first the simplest nontrivial case: two massive bosons, X and Y, coupled to four fermion species i_1, i_2, i_3 and i_4, through the vertices of fig. 4.1 and their CP conjugates*. In the Born approximation, these vertices lead to the decay processes $X \to \bar{i}_1 i_2$, $X \to \bar{i}_3 i_4$, $Y \to \bar{i}_3 i_1$, $Y \to \bar{i}_4 i_2$ and the corresponding CP-conjugate processes. We denote the coupling

*These vertices may be represented schematically by the interaction Lagrangian

$$L \sim i_2^\dagger X i_1 + i_4^\dagger X i_3 + i_1^\dagger Y i_3 + i_2^\dagger Y i_4 + \text{h.c.}$$

where all Lorentz structure has been suppressed.

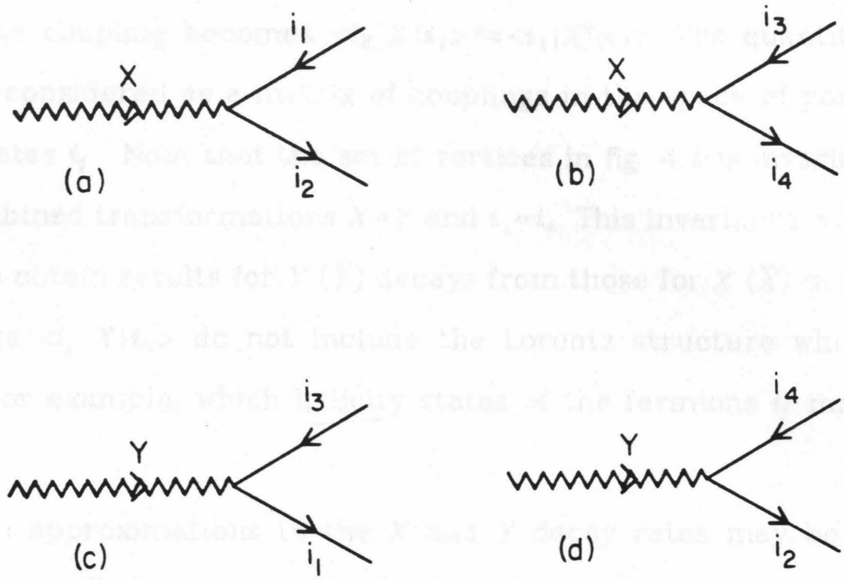

Fig. 4.1: Boson-fermion-fermion vertices. The jagged lines indicate generic bosons.

in, for example, the vertex in fig. 4.1a by $<i_2|X|i_1>$ so that the CP-conjugate coupling becomes $<i_2|X|i_1>^* = <i_1|X^\dagger|i_2>$. The quantity X here may be considered as a matrix of couplings in the space of possible fermion states i_j. Note that the set of vertices in fig. 4.1 is invariant under the combined transformations $X \leftrightarrow Y$ and $i_1 \leftrightarrow i_4$. This invariance will be used below to obtain results for Y ($\bar{Y}$) decays from those for X ($\bar{X}$) decays. The couplings $<i_j|X|i_k>$ do not include the Lorentz structure which determines, for example, which helicity states of the fermions i_j may contribute.

Born approximations to the X and Y decay rates may be obtained directly from the vertices of fig. 4.1. For example

$$\Gamma(X \to i_2 \bar{i}_1)_{Born} = I_X^{1\,2} \; |<i_2|X|i_1>|^2$$

$$\equiv I_X^{1\,2} <i_2|X|i_1> <i_1|X^\dagger|i_2>. \qquad (4.2)$$

Here $I_X^{1\,2}$ accounts for the kinematic structure of the process $X \to i_2 \bar{i}_1$; it gives the complete result if all couplings are set to one. Expressions for I_X for the cases where X is a scalar and a vector are given in appendix C. From eqn (4.2) it is evident that $\Gamma(X \to i_2 \bar{i}_1)_{Born} = \Gamma(\bar{X} \to \bar{i}_2 i_1)_{Born}$, and hence R_X vanishes in this approximation. To obtain a nonzero result for R_X, one must include corrections arising from interference of the one-loop contributions shown in fig. 4.2 with the Born amplitudes of fig. 4.1. Consider, for example, the interference of the diagrams of fig. 4.1a and fig. 4.2b. The resulting term in the squared amplitude is shown as fig. 4.3a. There the dotted line is a "unitarity cut"; each cut line represents a physical on-mass-shell particle. The amplitude for the diagram fig. 4.3a is then given by

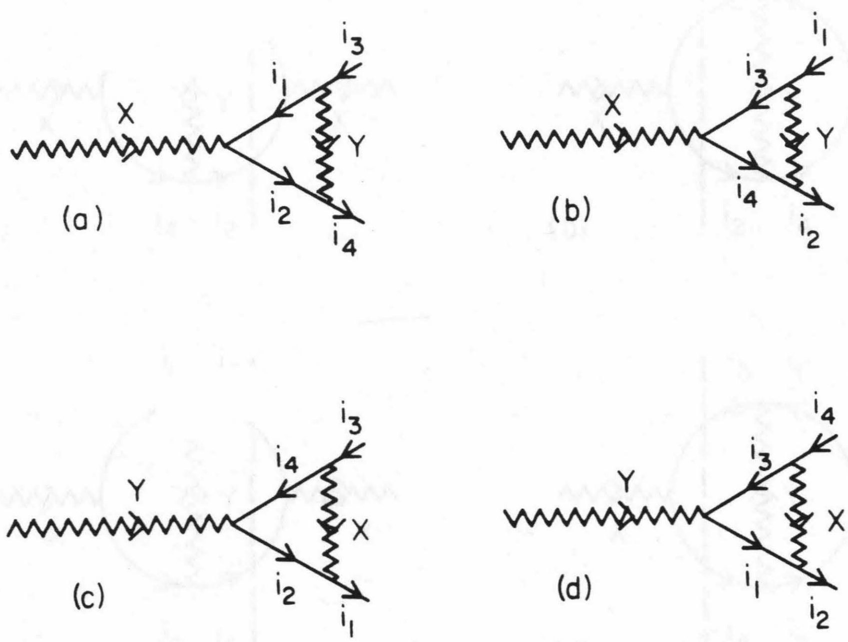

Fig. 4.2: One particle exchange corrections to the diagrams of fig. 4.1. The jagged lines indicate generic bosons.

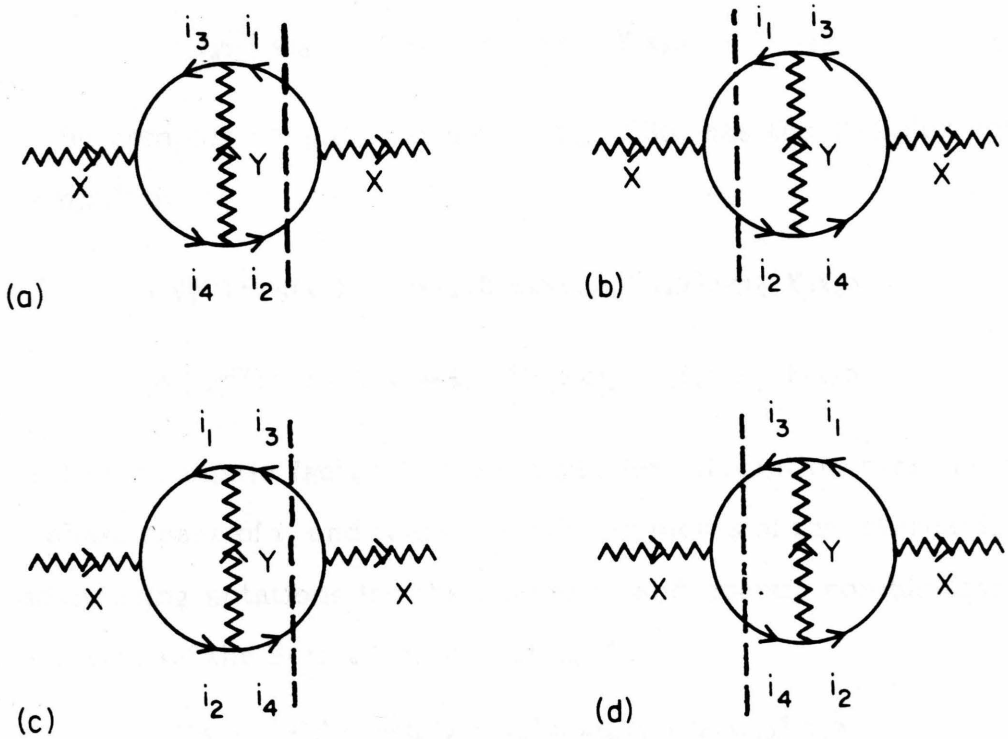

Fig. 4.3: Lowest order (non-Born) contributions to the decay rate of the generic boson X. The dashed line is a unitarity cut.

$$I_{XY}^{1234}\,[<i_3|Y^\dagger|i_1><i_4|X|i_3><i_2|Y|i_4>][<i_2|X|i_1>]^*$$

$$= I_{XY}^{1234}\,<i_3|Y^\dagger|i_1><i_4|X|i_3><i_2|Y|i_4><i_1|X^\dagger|i_2>, \tag{4.3}$$

while the complex conjugate diagram, fig. 4.3b, has the complex conjugate amplitude

$$(I_{XY}^{1234})^*\,[<i_3|Y^\dagger|i_1><i_4|X|i_3><i_2|Y|i_4>]^*<i_2|X|i_1>$$

$$= (I_{XY}^{1234})^*\,<i_2|X|i_1><i_4|Y^\dagger|i_2><i_3|X^\dagger|i_4><i_1|Y|i_3> \tag{4.4}$$

where the kinematic factor I_{XY}^{1234} accounts for integration over the final state phase space of i_2 and $\bar{i}_1$ and over the momenta of the internal i_4 and $\bar{i}_3$. Introducing notations for the quadratic and quartic combinations of the couplings of the Born terms and of fig. 4.3

$$\begin{aligned} \Xi_{jk}^{\chi} &= (\Xi_{jk}^{\chi})^\dagger \equiv |<i_k|\chi|i_j>|^2 = <i_k|\chi|i_j><i_j|\chi^\dagger|i_k> \\ \Omega_{1234} &= <i_3|Y^\dagger|i_1><i_1|X^\dagger|i_2><i_2|Y|i_4><i_4|X|i_3> \end{aligned} \tag{4.5}$$

one may write the one-loop approximation to the $X\rightarrow i_2\bar{i}_1$ decay rate which is obtained by adding the results (4.2), (4.3) and (4.4) as

$$\Gamma(X\rightarrow i_2\bar{i}_1) = I_X^{12}\Xi_{12}^X + I_{XY}^{1234}\Omega_{1234} + (I_{XY}^{1234}\Omega_{1234})^*. \tag{4.6}$$

The kinematic factors, I_X, of the Born approximation are always real. However, the kinematic factors I_{XY} for loop diagrams may have an imaginary part whenever the internal fermion lines have sufficiently small masses that they may propagate on their mass shells in the intermediate state. In the one-loop diagrams of fig. 4.3, this occurs when the threshold conditions

$$m_X \geq m_3 + m_4 \tag{4.7}$$

and

$$m_X \geq m_1 + m_2. \tag{4.8}$$

are satisfied. With light intermediate fermions therefore, I_{XY} always exhibits an imaginary part. Results for $\mathrm{Im} I_{XY}^{ijkl}$ in a variety of cases are given in appendix B.

We now consider the CP-conjugate decay $\overline{X} \to \overline{i}_2 i_1$. To obtain the CP-conjugate amplitude all couplings must be complex conjugated. The kinematic factors , however, are unaffected by the CP-conjugation (this is manifest in the fact that reversal of the direction of fermion lines in a closed loop does not affect the associated Dirac trace). Thus, to one-loop order, the complete result for $\Gamma(\overline{X} \to \overline{i}_2 i_1)$ becomes

$$\Gamma(\overline{X} \to \overline{i}_2 i_1) = I_X^{12} \Xi_{12}^X + I_{XY}^{1234} \Omega_{1234}{}^* + (I_{XY}^{1234})^* \Omega_{1234} \,. \tag{4.9}$$

The diagrams for the decays $X \to i_4 \overline{i}_3$ and $\overline{X} \to \overline{i}_4 i_3$ are shown as figs. 4.3c and 4.3d. The loop diagrams differ from those for the decays $X \to i_2 \overline{i}_1$ and $\overline{X} \to \overline{i}_2 i_1$ only in that the unitary cut is taken through the i_3 and i_4 rather than the i_1 and i_2 lines. In analogy with eqns (4.8) and (4.9) we obtain

$$\Gamma(X \to i_4 \overline{i}_3) = I_X^{34} \Xi_{34} + I_{XY}^{3412} \Omega_{1234}{}^* + (I_{XY}^{3412})^* \Omega_{1234} \,, \tag{4.10}$$

and

$$\Gamma(\overline{X} \to \overline{i}_4 i_3) = I_X^{34} \Xi_{34} + I_{XY}^{3412} \Omega_{1234} + (I_{XY}^{3412} \Omega_{1234})^* \,. \tag{4.11}$$

Using the results of eqns (4.7) through (4.11) together with eqn (4.1) we can compute the average baryon number produced in the free decays of an equal number of X's and $\overline{X}$'s. The one-loop contribution to this asymmetry from the $i_1\bar{i}_2$ and $\bar{i}_1 i_2$ final states is given by

$$R_X^{12} = (B_{i_2} - B_{i_1}) \frac{[\Gamma(X \to i_2\bar{i}_1) - \Gamma(\bar{X} \to \bar{i}_2 i_1)]}{[\Gamma(X \to i_2\bar{i}_1) + \Gamma(X \to i_4\bar{i}_3)]}$$

$$= (B_{i_2} - B_{i_1}) \frac{[I_{XY}^{1234}\Omega_{1234} + (I_{XY}^{1234}\Omega_{1234})^* - I_{XY}^{1234}\Omega_{1234}{}^* - (I_{XY}^{1234})^*\Omega_{1234}]}{[I_X^{12}\Xi_{12}^X + I_X^{34}\Xi_{34}^X]}$$

$$= -4\,\frac{(B_{i_2} - B_{i_1})}{\Gamma_X}\mathrm{Im}[I_{XY}^{1234}]\mathrm{Im}[\Omega_{1234}]\,. \tag{4.12}$$

The analogous result for the 34 final state is

$$R_X^{34} = -4\,\frac{(B_{i_4} - B_{i_3})}{\Gamma_X}\mathrm{Im}[I_{XY}^{3412}]\mathrm{Im}[\Omega_{1234}{}^*]$$

$$= 4\,\frac{(B_{i_4} - B_{i_3})}{\Gamma_X}\mathrm{Im}[I_{XY}^{3412}]\mathrm{Im}[\Omega_{1234}]\,. \tag{4.13}$$

The kinematic factors $\mathrm{Im}[I_{XY}^{1234}]$ and $\mathrm{Im}[I_{XY}^{3412}]$ are obtained from diagrams involving two unitarity cuts (as in fig. 4.4): one through the i_1 and i_2 lines and the other through the i_3 and i_4 lines. The resulting quantities are invariant under the combined interchanges $i_1 \leftrightarrow i_3$ and $i_2 \leftrightarrow i_4$ and consequently are equal:

$$\mathrm{Im}[I_{XY}^{1234}] = \mathrm{Im}[I_{XY}^{3412}]. \tag{4.14}$$

Hence $R_X^{12}/R_X^{34} = (B_{i_1} - B_{i_2})/(B_{i_4} - B_{i_3})$, as expected. Notice that, if all intermediate fermions have zero mass, then the I_{XY}^{1234} are completely independent of their upper indices; corrections from small fermion masses are of order* $(m_f/m_X)^2$.

*Corrections of order m_f/m_X vanish due to the helicity structure of the relevant diagrams.

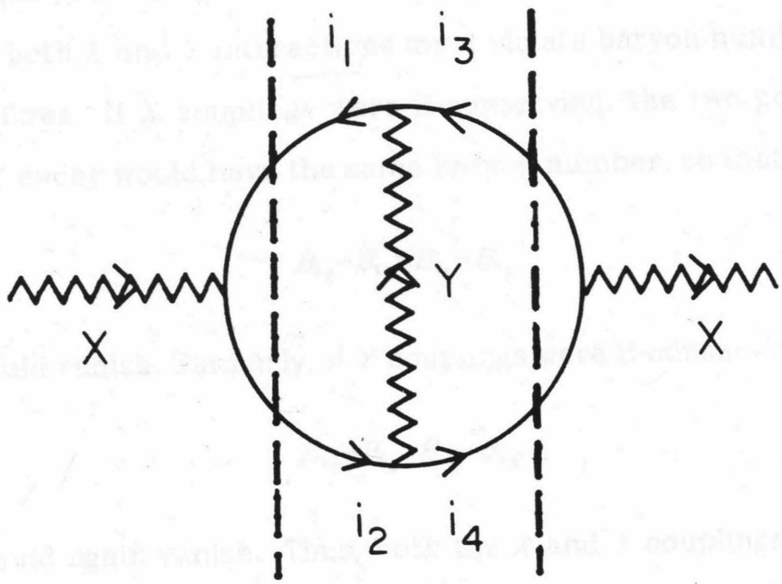

Fig. 4.4: The double cut diagram that represents the contribution of the generic boson X to the baryon asymmetry in its free decays.

Upon adding the contributions (4.12) and (4.13) we obtain the final result:

$$R_X = \frac{4}{\Gamma_X}\text{Im}[I_{XY}^{1234}]\text{Im}[\Omega_{1234}][B_{i_4}-B_{i_3}-(B_{i_2}-B_{i_1})]\,. \qquad (4.15)$$

The conditions for the kinematic factor $\text{Im}[I_{XY}^{1234}]$ to be nonvanishing were given in eqns (4.5) and (4.6). A further condition for R_X to be nonvanishing is that both X and Y interactions must violate baryon number. This is seen as follows. If X couplings were B-conserving, the two possible final states in X decay would have the same baryon number, so that

$$B_{i_2}-B_{i_1}=B_{i_4}-B_{i_3} \qquad (4.16)$$

and R_X would vanish. Similarly, if Y couplings were B-conserving,

$$B_{i_2}-B_{i_4}=B_{i_1}-B_{i_3} \qquad (4.17)$$

and R_X would again vanish. Thus, both the X and Y couplings must be B-violating to obtain a nonvanishing R_X. Furthermore, even if X and Y are baryon violating, graphs which do not exhibit this quality do not contribute to R_X. Thus, although it is not necessary that i_1, i_2, i_3 and i_4 all be distinct (it is, of course, necessary that at least one be different from the others), they must be such that $B_{i_4}-B_{i_3}-(B_{i_2}-B_{i_4})$ is nonzero if R_X is to be so. This is as implied by the general theorem given in the first reference of [19] and in [13], that there is no contribution to R_X from graphs of lowest order in baryon number violating and arbitrary order in baryon number conserving interactions.

The asymmetry R_Y produced in Y and $\bar{Y}$ decays may be obtained from (4.15) by the transformation $X \leftrightarrow Y$ and $i_3 \leftrightarrow i_4$, yielding

$$R_Y = \frac{4}{\Gamma_Y}\mathrm{Im}[\Omega_{1234}{}^*\mathrm{Im}[I_{YX}^{3142}][B_{i_4}-B_{i_3}-(B_{i_2}-B_{i_1})]$$

$$= -\frac{4}{\Gamma_Y}\mathrm{Im}[\Omega_{1234}]\mathrm{Im}[I_{YX}^{3142}][B_{i_4}-B_{i_3}-(B_{i_2}-B_{i_1})] \qquad (4.18)$$

and so

$$R_X/R_Y = -\mathrm{Im}(I_{XY}^{1234})/\mathrm{Im}(I_{YX}^{3142}). \qquad (4.19)$$

It follows that the average baryon number produced in the free decay of an equal number of X, $\bar{X}$, Y and $\bar{Y}$ is

$$R_{X+Y} = 4\left\{\frac{\mathrm{Im}[I_{XY}^{1234}]}{\Gamma_X} - \frac{\mathrm{Im}[I_{YX}^{3142}]}{\Gamma_Y}\right\} \times \mathrm{Im}[\Omega_{1234}][B_{i_4}-B_{i_3}-(B_{i_2}-B_{i_1})]. \qquad (4.20)$$

Even if the R_X and R_Y are nonvanishing on their own, for the total to be nonzero, the terms in the brace must not cancel. This requires that the particles X and Y be distinct either in mass or in the Lorentz structure of their couplings (e.g., one vector and one scalar) and that $\Gamma_X \neq \Gamma_Y$. The brace typically vanishes if X and Y are in the same irreducible representation of an unbroken symmetry group.

If more than the minimal set of four fermion species is present, the result (4.20) must be summed over all possible contributing $\{i_j\}$. It must also be summed over all possible (X,Y) pairs. Whenever the fermions have equal masses on the scale of m_X, the corresponding kinematic factors may be factored out of the summation as follows from the comment immediately following eqn (4.14).

Eqn (4.20) is also valid, with slight modification, when the intermediate particles in fig. 4.4 are bosons rather than fermions, as illustrated in fig. 4.5. If the intermediate bosons Z_1 and Z_2 themselves have B-violating decays, their weight B_f in eqn (4.1) is the average baryon number

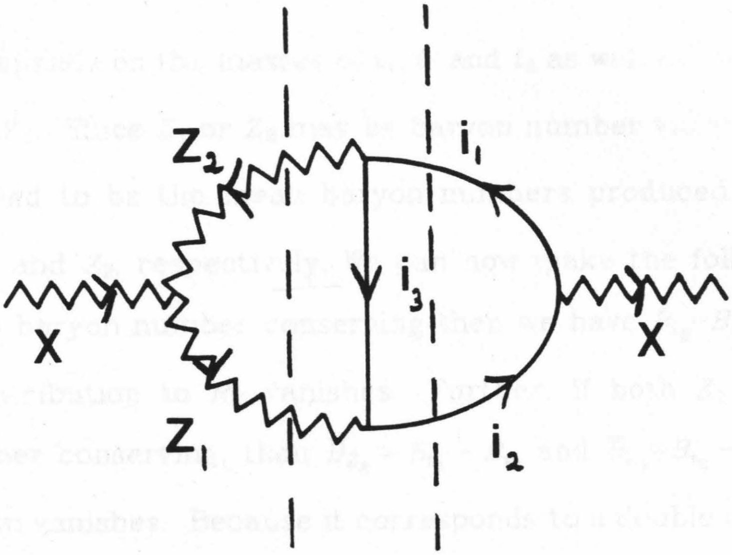

Fig. 4.5: A digram like that shown in fig. 4.4 but involving a three-boson vertex.

produced in their decays. It gives the contribution to R_X

$$R_X = \frac{4}{\Gamma_X} \mathrm{Im}(\tilde{\Omega}) \mathrm{Im}(I^{123}_{XZ_1Z_2})[B_{i_2}-B_{i_1}+\overline{B}_{Z_1}-\overline{B}_{Z_2}]. \tag{4.21}$$

Here $I^{123}_{XZ_1Z_2}$ depends on the masses of i_1, i_2 and i_3 as well as on the masses of X, Z_1 and Z_2. Since Z_1 or Z_2 may be baryon number violating, $\overline{B}_{Z_1}$ and $\overline{B}_{Z_2}$ are defined to be the mean baryon numbers produced in the free decays of Z_1 and Z_2, respectively. We can now make the following argument: if X is baryon number conserving then we have $B_{i_2}-B_{i_1} = \overline{B}_{Z_2} - \overline{B}_{Z_1}$ and this contribution to R_X vanishes. Further, if both Z_1 and Z_2 are baryon number conserving, then $\overline{B}_{Z_2} = B_{i_1} - B_{i_3}$ and $\overline{B}_{Z_1}=B_{i_2} - B_{i_3}$ and the diagram again vanishes. Because it corresponds to a double cut diagram, the expression for $\mathrm{Im}(I^{123}_{XZ_1Z_2})$ has the threshold conditions

$$m_X > m_{Z_1} + m_{Z_2} \qquad \text{and} \qquad m_X > m_1 + m_2 . \tag{4.22}$$

The expression for $\tilde{\Omega}$ is (analogous to eqn (4.5))

$$\tilde{\Omega} = \langle Z_2 | X | Z_1 \rangle \langle i_2 | Z_2 | i_3 \rangle \langle i_3 | Z_1^\dagger | i_1 \rangle \langle i_1 | X^\dagger | i_2 \rangle . \tag{4.23}$$

The *individual* baryon asymmetry parameters R_X for X decays enter the complete Boltzmann transport equations discussed in chapter 2 and in [13]. These parameters determine the final baryon asymmetry by themselves only if back reactions (inverse decays) and 2→2 scatterings are ignored [13]. The total contribution to the baryon asymmetry from decays of two species of bosons, X and Y, thus is not generally a simple sum of their corresponding parameters R_X and R_Y: if X and Y have different masses, the importance of back reactions and 2→2 scatterings may be different in the two cases.

The discussion above concerns the one-loop contributions to baryon asymmetry. In the generic case, an asymmetry occurs at this order if it is to occur at any order. However, in some simple models (such as the minimal $SU(5)$ model considered in chapter 6) the one-loop contribution vanishes, but there are higher loop contributions which are finite. In such cases the detailed analysis given above must be suitably generalized by summing over all possible unitarity cuts through the multiloop diagram.

We now discuss the value of the CP-violating coupling parameter $\mathrm{Im}[\Omega]$ defined in eqn (4.12) (a general discussion of its structure is presented in appendix D). We assume here that the i_j are all fermions with masses much smaller than m_X and m_Y.

In a grand unified model based upon a gauge group G a family of fermions will transform either as a reducible or an irreducible representation. These models are conveniently cast in terms of left-handed fermion fields. The two simplest examples to keep in mind are the $SU(5)$ model where a family of fermions transforms as the reducible representation $\bar{5}\oplus 10$, and an $SO(10)$ model where a family can transform irreducibly as a 16.

In writing down eqns (4.15) and (4.18) we have assumed that the fermion mass eigenstates are states of definite baryon number. This is guaranteed by the unbroken $SU(3)$ symmetry if no exotic assignments of baryon number are made to the weak eigenstates (i.e., all $SU(3)$ singlet fermions have B=0 and all $SU(3)$ triplet fermions have B=0); no Majorana mass terms for quarks may appear in the Lagrangian. We further assume (though this does not affect our discussion very much) that all fermions are $SU(3)$ triplets or singlets. Thus, fermions may mix within and between families so long as the mixings respect the quantum numbers of the unbroken local and global symmetries of the model. In the boson sector of the model there may be mixings (when allowed by the other quantum numbers of the model) among the baryon number conserving bosons and mixings among the baryon number violating bosons, but no mixing may occur between these two classes of bosons.

As discussed in appendix B, the coupling of gauge vector bosons to massless fermions may always be taken as real. Hence if both X and Y are gauge vector bosons, the CP violation parameter $\mathrm{Im}[\Omega]$ will always vanish in this case: contributions from processes which only involve vectors come from the fermion mass matrix.

We now consider the case in which X is a gauge vector boson V and Y is a Higgs scalar boson S, as illustrated in fig. 4.6. (Interchange of the identifications of X and Y is irrelevant for this discussion since this merely complex conjugates Ω; however, this interchange does effect the Born rate.) The diagonal nature of the gauge couplings requires that the fermions i_1 and i_2 lie in the same irreducible representation $\mathbf{f}_1$ of the gauge group (and similarly i_3 and i_4 lie in the same irreducible representation $\mathbf{f}_2$). Scalar bosons contributing to fig. 4.6 must lie in irreducible representations $\mathbf{s}_i$ such that

$$\mathbf{f}_1 \otimes \bar{\mathbf{f}}_2 \supset \mathbf{s}_\alpha \ . \tag{4.24}$$

The exchanged mass eigenstate scalar boson S is in general a linear combination of components which have the same transformation properties under some subgroup of the gauge group (e.g., for $SU(5)$ the relevant subgroup is $SU(3) \otimes SU(2)_L \otimes U(1)_Y$):

$$S = \alpha_1 S_1 + \alpha_2 S_2 + \cdots \tag{4.25}$$

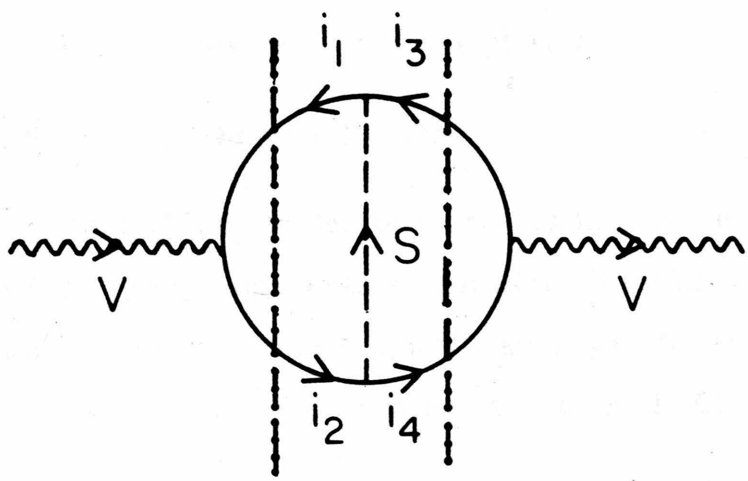

Fig. 4.6: Vector decay with scalar exchange.

Note that this linear combination may include both mixing between the irreducible representations s_a and within a given irreducible representation (an example of both such mixings is discussed in the context of an illustrative $SO(10)$ model in chapter 9). We shall assume for now that at most two components are present; the generalization to an arbitrary number is immediate. In this case,

$$\mathrm{Im}[\Omega^{1234}]=\mathrm{Im}[\,Tr\,[<i_3|S^\dagger|i_1><i_2|S|i_4>]]$$

$$=\mathrm{Im}[\,Tr\,[(\alpha_1{}^*<i_3|S_1^\dagger|i_1>+\alpha_2{}^*<i_3|S_2^\dagger|i_1>)$$

$$\times(\alpha_1<i_2|S_1|i_4>+\alpha_2<i_2|S_2|i_4>)]] \qquad (4.26)$$

where we have dropped the real factor corresponding to the gauge boson couplings, and the trace represents a sum over all fermion representations (usually "families"). Since $i_1,i_2\subset f_1$ and $i_3,i_4\subset f_2$, the couplings $<i_2|S_a|i_4>$ and $<i_1|S_a|i_3>$ are related by a real Clebsch-Gordan coefficient:

$$<i_2|S_a|i_4>=C_a<i_1|S_a|i_3> \ . \qquad (4.27)$$

Hence

$$\mathrm{Im}[\Omega]=\mathrm{Im}[\,Tr[(\alpha_1{}^*<i_3|S_1^\dagger|i_1>\alpha_2{}^*<i_3|S_2^\dagger|i_1>(C_1\alpha_1<i_1|S_1|i_3>+C_2\alpha_2<i_1|S_2|i_3>)]]$$

$$=\mathrm{Im}[\,Tr[(C_2\alpha_1{}^*\alpha_2<i_3|S_1^\dagger|i_1><i_1|S_2|i_3>+C_1\alpha_1\alpha_2{}^*<i_3|S_2^\dagger|i_1><i_1|S_1|i_3>)]]$$

$$=(C_2-C_1)\mathrm{Im}[\,Tr[\alpha_1{}^*\alpha_2<i_1|S_1|i_3><i_3|S_2^\dagger|i_1>]] \ . \qquad (4.28)$$

Thus, if $C_1=C_2$, then this contribution vanishes. This is inevitable if all relevant Higgs bosons lie in replications of the same irreducible representation of the gauge group, and if this representation contains only one B-violating component. Examples of cases in which $C_1 \neq C_2$ are the $SU(5)$ model with a 5_H and a 45_H (case B in chapter 7) and an $SO(10)$ model with a 10_H and a 120_H or a 126_H. In these models, CP violation may occur at the one-loop level from scalar boson exchange in vector boson decay. Notice that since in the absence of spontaneous symmetry breakdown only one of the α_j is nonzero, the result (4.28) yields no CP violation in this case.

The case of vector boson exchange in scalar boson decay (illustrated in fig. 4.7) is exactly analogous to the case of scalar exchange in vector decay discussed above. When fig. 4.7 contributes, it is often important by virtue of the large value of the vector couplings relative to the scalar ones.

We now consider CP violation arising from scalar boson (S') exchange in scalar (S) boson decay, as illustrated in fig. 4.8. If only one B-violating Higgs boson is present, then the decaying and exchanged boson must be identical, and the results discussed above show that fig. 4.8 can give no CP violation. This is the case for the minimal $SU(5)$ model. (However, as described in chapter 6, CP violation may occur in higher-order diagrams.) We consider for now the case in which all fermions are effectively massless. Then, in analogy with (4.24), the contributing scalar bosons must appear in representations $\mathbf{s}_\alpha$ such that

$$\mathbf{f}_1 \otimes \bar{\mathbf{f}}_2 \subset \bar{\mathbf{s}}_\alpha \tag{4.29}$$

$$\mathbf{f}_4 \otimes \bar{\mathbf{f}}_3 \subset \mathbf{s}_\alpha$$

$$\mathbf{f}_2 \otimes \bar{\mathbf{f}}_4 \subset \mathbf{s}'_\alpha$$

$$\mathbf{f}_3 \otimes \bar{\mathbf{f}}_1 \subset \bar{\mathbf{s}}'_\alpha$$

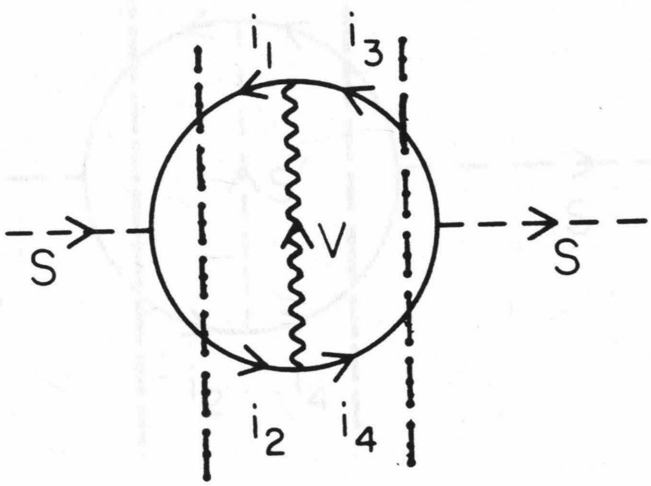

Fig. 4.7: Scalar decay with vector exchange.

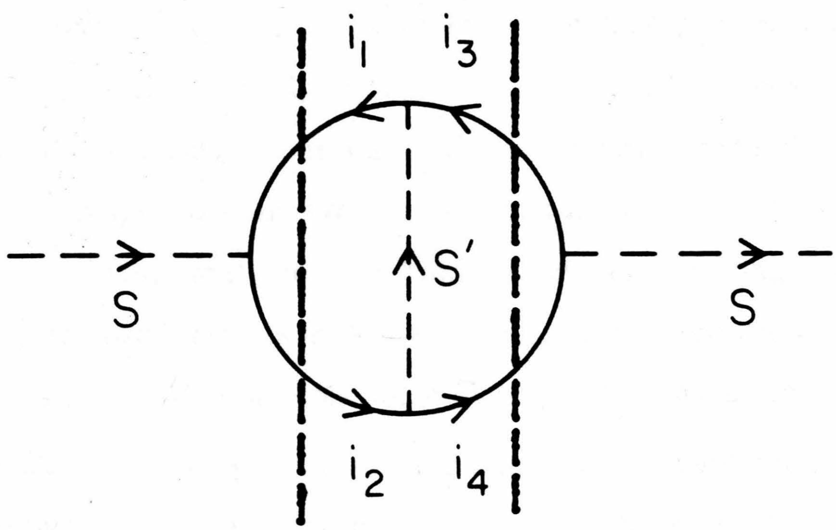

Fig. 4.8: Scalar decay with scalar exchange.

If all the left-handed fermions lie in the same complex irreducible representation, **f** (or sequence of such identical representations), then $\mathbf{f}_1 = \bar{\mathbf{f}}_2 = \bar{\mathbf{f}}_3 = \mathbf{f}_4$ and these constraints become

$$\mathbf{f}\otimes\mathbf{f} \supset \mathbf{s}_\alpha,\ \mathbf{s}'_\alpha,\ \bar{\mathbf{s}}_\alpha,\ \bar{\mathbf{s}}'_\alpha \tag{4.30}$$

For low-dimensionality representations, this requires $\mathbf{s}_\alpha$ and $\mathbf{s}'_\alpha$ to be real representations. Hence in $SO(10)$ models where all fermions lie in the 16 representation, only 10_H or 120_H may contribute to fig. 4.8; the 126_H which appears in $16_f\otimes 16_f$ is complex. (For high-dimensional fermion representations, some complex Higgs representations may satisfy (4.30): an example is the 126_H occurring in the symmetric product $144_f\otimes 144_f$ of $SO(10)$.) After spontaneous symmetry breakdown, mixing between scalar bosons may occur, and the constraints (4.29) are no longer applicable. Thus, in both $SU(5)$ models with several Higgs representations coupling to fermions, and in $SO(10)$ models, fig. 4.8 can yield CP violation.

The discussion above has assumed that all relevant fermion species are effectively massless. With gauge groups such as $SO(10)$ or $E(6)$, it is common for fermions with $SU(2)$ singlet and thus potentially large mass terms to exist. The effect of such fermions in intermediate states of figs. 4.6 through 4.8 and in vector decay through vector exchange diagrams is always suppressed by $O(m_f^2/m_X^2)$. If only a single massive fermion exists, then it can introduce no CP-violating effects into vector decay through vector exchange; a single massive fermion is, however, sufficient to generate CP violation in figs. 4.6 through 4.8 even when (4.28) vanishes.

It is certainly worth noting that, though the analysis of this chapter has focused on baryon number, the expressions that we have derived are by no means restricted to that quantum number. The expressions are valid to describe the generation of any quantum number in the free decays of X, $\bar{X}$, Y or $\bar{Y}$. Thus, for example, to describe lepton number generation we need to replace the B_i's by the relevant lepton number assignments. Furthermore, although our analysis focused on the diagrams of fig. 4.4 as the first nonvanishing contributions to R_X, it may be that those diagrams give a vanishing contribution to R_X for a particular model (an example is the minimal $SU(5)$ model discussed in chapter 6). In that event the discussion given here goes through with very little change: the quantity $I\Omega$ then arises from the lowest order contributing diagrams.

5) SU(5) Models

$SU(5)$ is the simplest group (and the only one of rank 4) which contains the group which so successfully describes the existing low energy phenomenology* [1], $SU(3)\otimes SU(2)_L\otimes U(1)$. The vector bosons transform according to the adjoint representation, 24. The symmetry breaking $SU(5)\to SU(3)\otimes SU(2)\otimes U(1)$ is typically effected by means of a 24_H of Higgs scalars which is postulated to attain a superlarge vacuum expectation value so that the phenomenological constraints on the decay rate of the proton [23] may be maintained. A family of fermions, which consists of 15 left-handed fields, falls into the reducible representation $\bar{5}\oplus 10$. Such a family has the (generic) particle content

$$\bar{5}_f = \{\vec{D}_L^c;\ \nu_L,\ E_L\} \tag{5.1a}$$

$$10_f = \{\vec{U}_L,\vec{D}_L;\vec{U}_L^c;E_L^c\}\ , \tag{5.1b}$$

where the superscript c stands for charge conjugation and the vector sign indicates transformation as an $SU(3)$ triplet. Thus $\vec{U}_L^c$ transforms as a $\bar{3}$.

Scalar fields which couple to fermions must transform according to representations that appear in the decompositions of $\bar{5}\otimes\bar{5}$, $\bar{5}\otimes 10$ or $10\otimes 10$:

$$\bar{5}\otimes\bar{5} = \overline{10}_A+\overline{15}_S \tag{5.2a}$$

$$\bar{5}\otimes 10 = 5+45 \tag{5.2b}$$

$$10\otimes 10 = (\bar{5}+50)_S + (\overline{45})_A\ . \tag{5.2c}$$

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*Sometimes refered to as the "Holy Trinity."

With the assignments (5.1), of these representations the only ones which have a neutral (zero electric charge) component (and hence can have a vacuum expectation value and contribute to the fermion mass matrix) are 5, 15 and 45. The inclusion of a 15 of Higgs would allow the left-handed neutrino to have a Majorana mass (and would thereby violate the $B-L$ symmetry usually present in the broken $SU(5)$ theory with only 5's or 45's of Higgs [24]). It appears difficult to make such a Majorana mass naturally small in an $SU(5)$ model with the fermions (5.1); the 15 is usually excluded on these grounds.

The simplest viable set of Higgs is a single 5_H (in addition, of course, to the 24_H). A model with a single 45_H, though theoretically sound, has phenomenological problems because it gives the mass relation $m_b / m_\tau = 1/3$ at the unification scale and this is very difficult to reconcile with experiment. Alternatives to this so-called minimal $SU(5)$ model that effect the charged fermion mass matrix while leaving the neutral mass matrix unaltered are to have more than one 5_H or to have a 5_H and a 45_H or to have some arbitrary number of each. A discussion of the CP violation necessary for cosmological baryon number production in such models is discussed in chapter 7. Another possibility is to add a 50_H of Higgs (this, of course, has no effect on the fermion mass matrix at tree level); this case is relevant for the "primordial" $SO(10)$ model discussed in chapter 9.

The reducibility of the fermion representation implies that, even with a single Higgs representation (5_H or 45_H), there are two independent Yukawa coupling matrices. One couples to the product $\bar{5}_i \otimes 10_j$ (where i and j index fermion families) and the other couples to the product $10_i \otimes 10_j$. The former yields (after the Higgs have obtained their vacuum expectation values) the D and E mass matrices and the latter the U mass matrix.

The $SU(5)$ representations introduced above may be decomposed according to the embedding $SU(5) \supset SU(3) \otimes SU(2)_L \otimes U(1)_Y$ as

$$5 = (3, 1, 1/3) + (1, 2, -1/2) \tag{5.3}$$

$$10 = (3, 2, -1/6) + (\bar{3}, 1, 2/3) + (1, 1, -1) , \tag{5.4}$$

$$15 = (6, 1, 2/3) + (3, 2, -1/6) + (1, 3, -1) \tag{5.5}$$

$$24 = (8, 1, 0) + (\bar{3}, 2, -5/6) + (3, 2, 5/6) + (1, 1, 0) + (1, 3, 0) \tag{5.6}$$

$$45 = (1, 2, -1/2) + (8, 2, -1/2) + (\bar{6}, 1, 1/3) + (\bar{3}, 1, -4/3)$$
$$+ (3, 1, 1/3) + (3, 3, 1/3) + (\bar{3}, 2, 7/6) \tag{5.7}$$

$$50 = (6, 3, -1/3) + (8, 2, 1/2) + (3, 2, -7/6) + (\bar{6}, 1, +4/3)$$
$$+ (\bar{3}, 1, -1/3) + (1, 1, -2) . \tag{5.8}$$

6) CP Violation in the Minimal SU(5) Model

In this chapter we discuss the result that, for the minimal $SU(5)$ model, the first possibly nonzero contribution to $\mathrm{Im}(\Omega)$ (in the notation of chapter 8) occurs at eighth order in the couplings [16].

In minimal $SU(5)$ one puts each family of fermions in the reducible representation $\overline{5}_i \oplus 10_i$, where the index i is a family index. As we discussed above, the Higgs multiplets in the minimal model are taken to be a 5_H and a 24_H ; the 24_H cannot couple directly to fermions. The coupling of the fermions to the 5_H may be written schematically as

$$5_H \cdot (10_i (h_U)_{ij} 10_j) + \overline{5}_H \cdot (\overline{5}_i (h_D)_{ij} 10_j) \tag{6.1}$$

where summation over repeated indices is implied and where h_D and h_U are the Yukawa coupling matrices in family space. In eqn (6.1) there should also appear group coupling coefficients to make the respective terms to transform as $SU(5)$ singlets. Such coefficients may be taken to be real as discussed in appendix B. Furthermore, there is only a single representation of Higgs that couples to fermions and that representation contains only one baryon number violating scalar (up to $SU(3)$ degeneracy); thus, there is no mixing among the relevant baryon number violating scalars. Consequently, any group coupling coefficients that appear in Ω can be factored out of $\mathrm{Im}(\Omega)$; thus, for the present discussion, they cause us no consternation and we will therefore suppress them. This will also be the case for an extended $SU(5)$ model with two 5_H representations; however, for an extended $SU(5)$ model with a 5_H and a 45_H this will not be valid since the group coupling coefficients for coupling to the 5_H differ from those for coupling to the 45_H. The details of these cases are discussed in chapter 7. Thus, all matrices that we consider in this chapter are matrices in family space and all traces are over family indices. Off-diagonal elements in these matrices represent transitions between different families.

The coupling of the gauge vector bosons to the fermion fields is diagonal in family space. We write those couplings schematically as

$$\frac{g}{\sqrt{2}}[24_V \cdot (\overline{10}_i \cdot 10_i) + 24_V \cdot (\overline{5}_i \cdot 5_i)]. \qquad (6.2)$$

In the broken theory the errors introduced by neglecting the mixing of fermions induced by the off-diagonal elements in h_D and h_U are of $0(m_f^2/M^2)$, where m_f is a typical fermion mass and M is the mass of an internal boson line (see appendix C for explicit calculations in the two-loop case). In the unbroken theory (i.e., at high temperatures) the fermions propagate as massless particles. Either way we are justified in neglecting fermion mixing.

The types of vertices that may appear in a diagram fall into two classes: the couplings of fermions to bosons and the couplings of bosons to one another. For the minimal $SU(5)$ model, the latter class has all real (non-CP violating) couplings. Thus, all factors arising from such vertices may be factored out of Im(Ω). The reasons for this are as follows. In the minimal $SU(5)$ model there is no spontaneous CP violation (the 24_H is a real representation and hence its vacuum expectation value is real; any phase appearing in the vacuum expectation value of the 5_H can be rotated away in exactly the same fashion as in the $SU(2)_L \otimes U(1)_Y$ electroweak theory with one Higgs doublet.) Furthermore, there can be no intrinsic CP violation in the Higgs potential (which CP violation would be necessary for the three- or four-boson couplings to be CP non-invariant) because it is Hermitian. For example,

$$\lambda \bar{5}_H \cdot 5_H \cdot 24_H + \lambda^* (\bar{5}_H \cdot 5_H \cdot 24_H)^\dagger$$

$$= \lambda \bar{5}_H \cdot 5_H \cdot 24_H + \lambda^* (\bar{5}_H)^\dagger \cdot (5_H)^\dagger \cdot 24_H$$

$$= (\lambda + \lambda^*) \bar{5}_H \cdot 5_H \cdot 24_H , \tag{6.3}$$

which is CP invariant. In treating CP violation in the minimal $SU(5)$ model to lowest order, we therefore consider diagrams with no three or four-boson vertices. This leaves us with diagrams consisting of one or more closed fermion loops and with only fermion-fermion-boson vertices. It therefore suffices to determine what the lowest order is for which the family-space trace corresponding to a single fermion loop bubble diagram with no multiple boson vertices has an imaginary part.

The fermion-boson coupling vertices in the minimal $SU(5)$ model are shown in fig. 6.1. The lowestorder corrections to B-violating decays in this model are given in fig. 6.2. Each diagram is proportional to a trace in family space over the products of coupling matrices occurring around the closed fermion loop. The trace for fig. 6.2(a1) and 6.2(a2) is trivial, hence there can be no CP violation from vector boson exchanges in vector boson decays. For fig. 6.2(a1), the relevant trace is $Tr[(h_D)^\dagger (h_D)]$, which is real. Similarly, fig. 6.2(b2) involves $Tr[(h_U)^\dagger (h_U)]$ which is again real. Figs. 6.2c yield the same traces and are thus also CP-conserving. Finally, fig. 6.2d gives $Tr[(h_D)^\dagger (h_D)(h_U)^\dagger (h_U)]$, which is manifestly real. Thus, none of the diagrams in fig. 6.2 can give rise to CP violation.

A systematic investigation of possible three-loop diagrams reveals that none can have CP violation. For example, fig. 6.3 yields the trace

$$Tr[h_D^\dagger h_U h_D^\dagger h_D h_U^\dagger h_D] = [Tr[h_D^\dagger h_U h_D^\dagger h_D h_U^\dagger h_D]]^* \ . \tag{6.4}$$

The first diagrams for which the corresponding traces are not necessarily

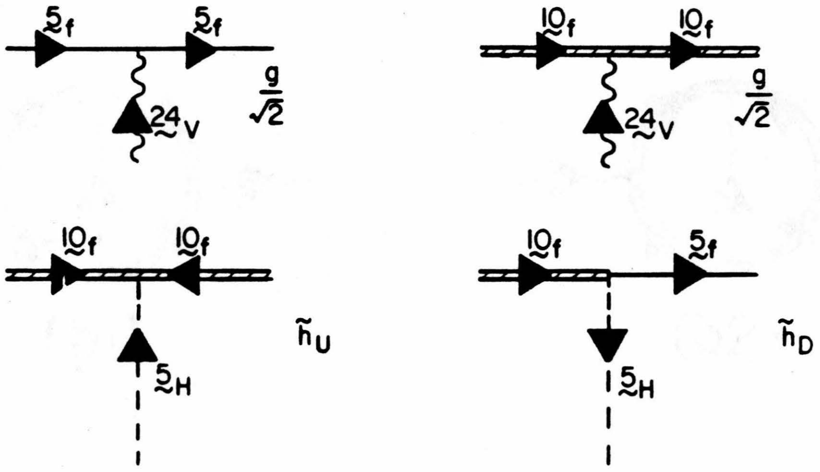

Fig. 6.1: Boson-fermion-fermion vertices in the minimal $SU(5)$ model.

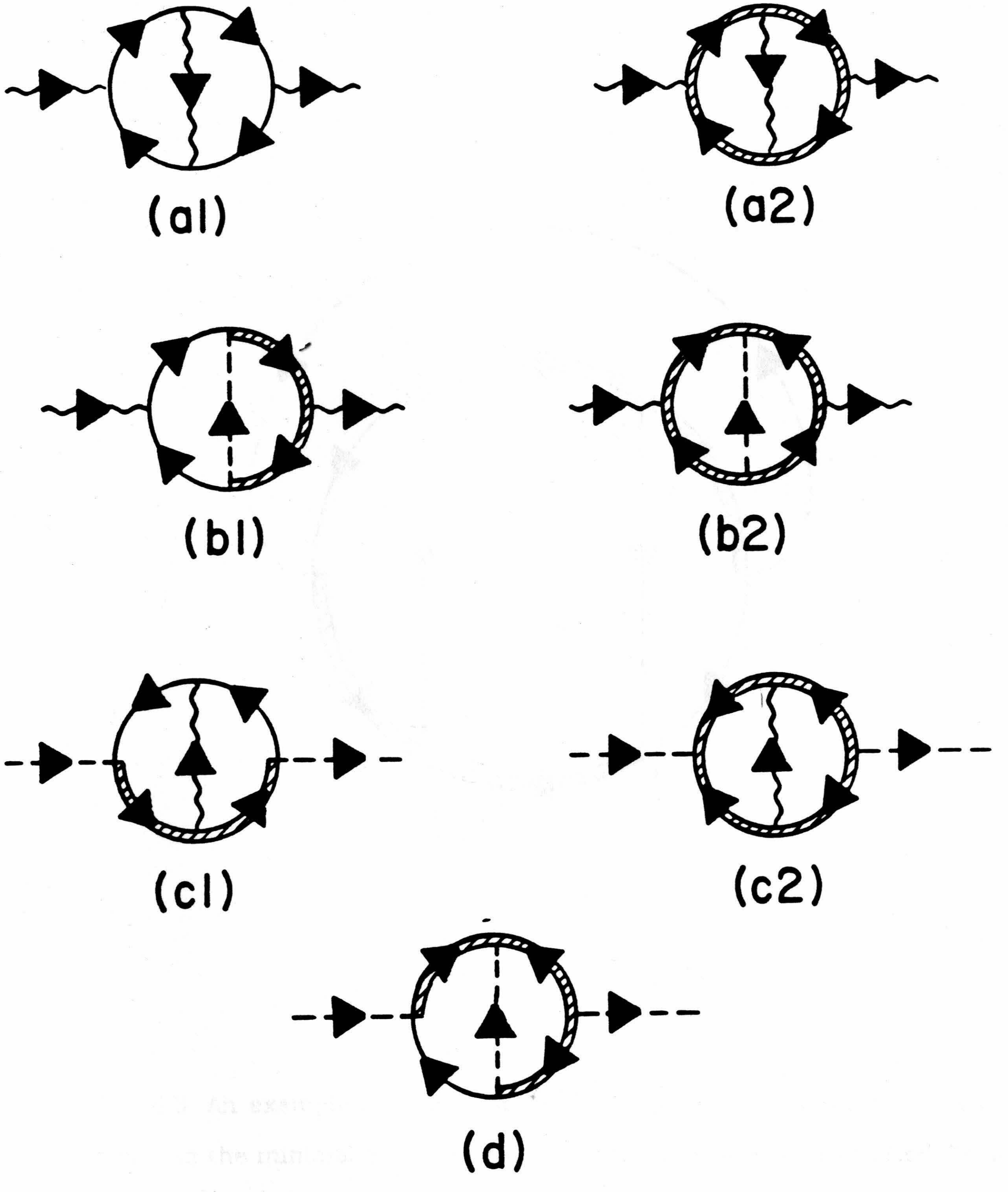

Fig. 6.2: Lowest order (non-Born) diagrams for the decay of bosons in the minimal $SU(5)$ model. Unitarity cuts are not exhibited. These diagrams do not give CP violation.

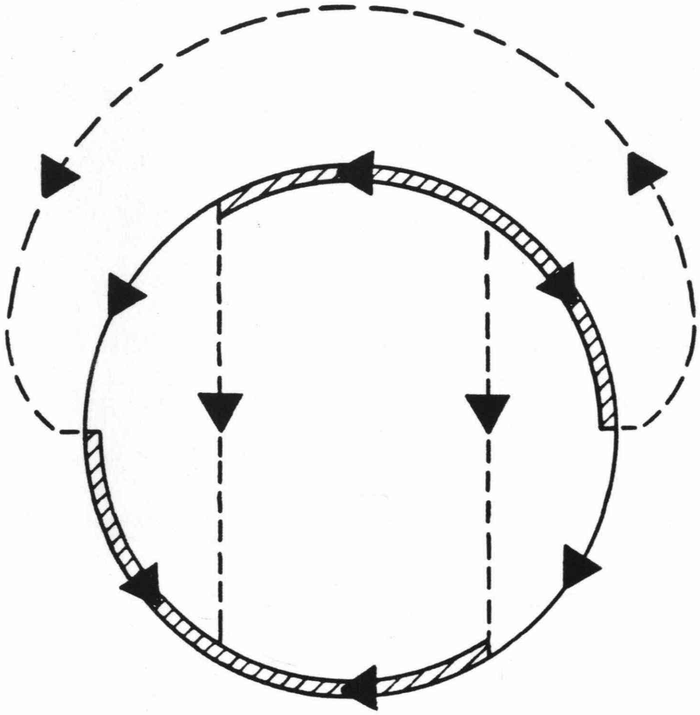

Fig. 6.3: An example of a sixth order diagram for the decay of a scalar boson in the minimal $SU(5)$ model. Unitarity cuts are not exhibited. This diagram does not give CP violation.

(PAGE 53 IS BLANK, DUE TO ERROR IN PAGINATION) real appear in the next order [16]. The relevant diagrams are shown in fig. 6.4, and are proportional to the trace*

$$Tr[(h_U)(h_U)^\dagger(h_U)(h_D)^2(h_U)^\dagger(h_D)^2]. \qquad (6.5)$$

At this point it is worth noting that there is an additional freedom in this model which allows us to take either h_D or h_U to be real and diagonal. In earlier work [16] it was believed that this freedom was necessary to show that all of the diagrams of less than eighth order do not have a CP-violating imaginary trace; however, as we have shown above, it is not necessary to use this freedom. We review the discussion of this symmetry here because it is interesting to see how, by using the available freedom (under the assumption of massless fermions), the CP violation may be isolated.

With the neglect of fermion mixing we can perform a unitary redefinition of fields in family space

$$\left.\begin{aligned} 5_i &= V_{ij}\tilde{5}_j \\ 10_j &= U_{jk}\,\tilde{10}_k \end{aligned}\right\} \qquad (6.6)$$

$$V^*_{ji} = (V^{-1})_{ij}; \quad U^*_{kj} = (U^{-1})_{jk}. \qquad (6.7)$$

Upon applying these redefinitions and suppressing explicit family indices, the couplings (6.1) become

$$(\tilde{\bar{5}}\cdot V^\dagger\cdot(h_D)\cdot U\cdot\tilde{10})\cdot\bar{5}_H+(\tilde{10}^T\cdot U^T\cdot(h_U)\cdot U\cdot\tilde{10})\cdot 5_H$$

$$= (\tilde{\bar{5}}\cdot(\tilde{h}_D)\cdot\tilde{10})\cdot\bar{5}_H + (\tilde{10}^T\cdot(\tilde{h}_U)\cdot\tilde{10})\cdot 5_H \qquad (6.8)$$

where

$$\left.\begin{aligned} (\tilde{h}_D) &= V^\dagger\cdot(h_D)\cdot U \\ (\tilde{h}_U) &= U^T\cdot(h_U)\cdot U \end{aligned}\right\}. \qquad (6.9)$$

***In this expression h_D is taken to be real and diagonal (this may always be done, as is discussed in what follows). For such an h_D all of the diagrams in fig. 6.4 have equal weight.**

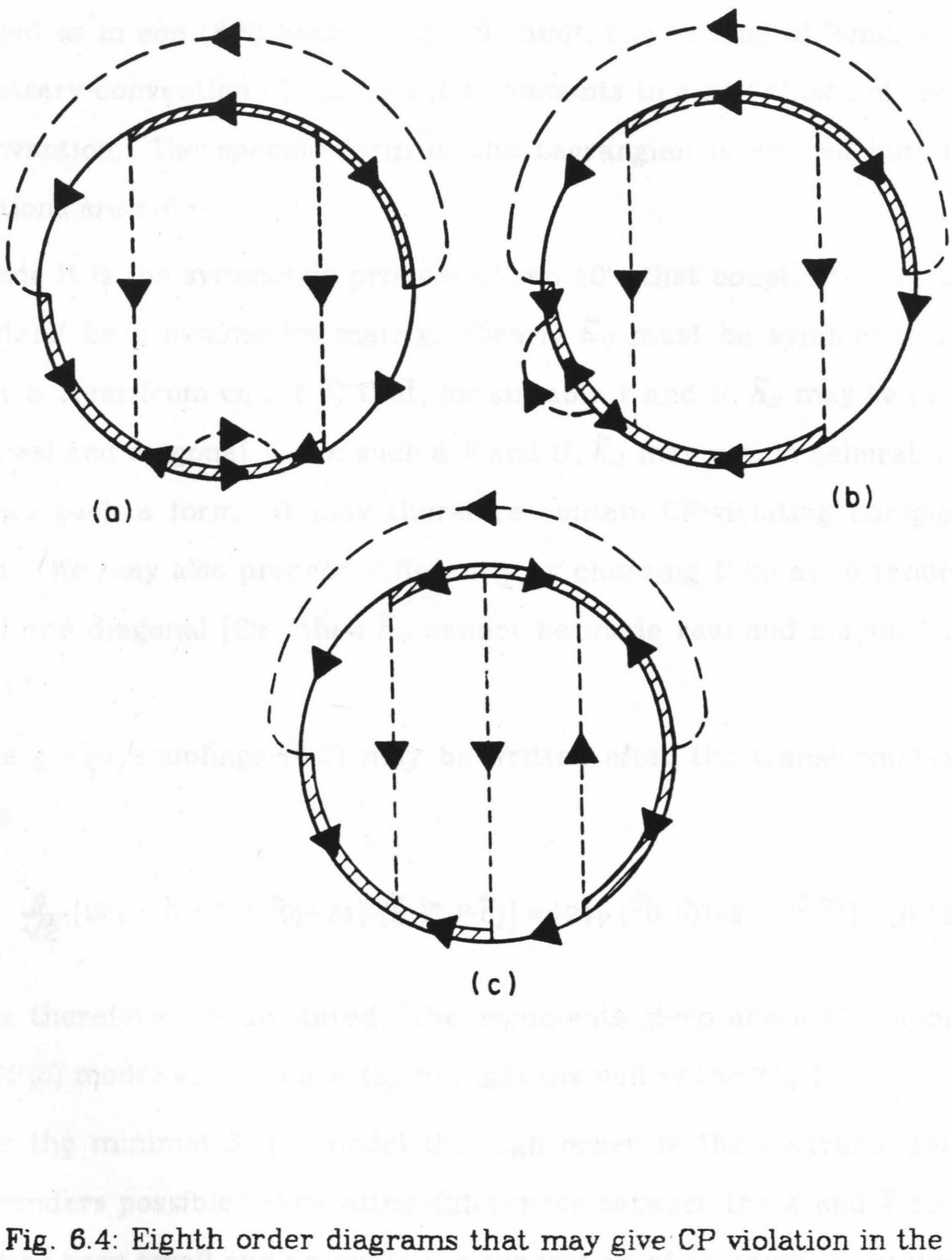

Fig. 6.4: Eighth order diagrams that may give CP violation in the Minimal $SU(5)$ model. Unitarity cuts are not exhibited.

The 5's of fermions and the 10's of fermions can be independently redefined as in eqn (6.6) because, in this limit, the naming of families is an arbitrary convention. Thus, eqn (6.6) amounts to a new choice of family convention. The specific form of the Lagrangian is altered but its predictions are not.

Since it is the symmetric product of two 10's that couples to 5_H , h_U must itself be a symmetric matrix. Clearly $\tilde{h}_U$ must be symmetric as well. It is clear from eqn (6.6) that, for suitable V and U, $\tilde{h}_D$ may be rendered real and diagonal. Given such a V and U, $\tilde{h}_U$ may not, in general, be cast into such a form. It may therefore contain CP-violating complex entries. (We may also proceed differently by choosing U so as to render $\tilde{h}_U$ real and diagonal [25], then $\tilde{h}_D$ cannot be made real and diagonal in general.)

The gauge couplings (6.2) may be written after the transformation (6.6) as

$$\frac{g}{\sqrt{2}}[24_V\cdot(\overline{\tilde{10}}\cdot U^\dagger\cdot U\cdot\tilde{10})+24_V\cdot(\overline{\tilde{5}}\cdot V^\dagger\cdot V\cdot\tilde{5})] = [24_V\cdot(\overline{\tilde{10}}\cdot\tilde{10})+24_V\cdot(\overline{\tilde{5}}\cdot\tilde{5})] \quad (6.10)$$

and are therefore left unaltered. The arguments given above also apply in an $SU(5)$ model with a single 45_H of Higgs (as well as the 24_H).

For the minimal $SU(5)$ model the high order of the diagrams generally renders possible CP-violating differences between the X and $\bar{X}$ partial widths very small and prevents the generation of an adequate baryon asymmetry. A rough estimate for the magnitude of the parameter R of eqn (4.1) arising from these diagrams is

$$R \simeq \frac{4}{7}\mathrm{Im}[I](g^2/2)^3\left[\sum_i(\tilde{h}_{U_i}+\tilde{h}_{D_i})\right]^6\varepsilon, \quad (6.11)$$

where ε is a CP violation parameter ($\varepsilon=\sin\delta$ where δ is a CP violating phase angle)$|\varepsilon|\leq 1$. The Yukawa couplings are dominated by the heaviest fermion F. The momentum integration factor $\mathrm{Im}[I]$ is given very roughly simply by the volume of available phase space for each loop integration: $\mathrm{Im}[I] \sim (1/8\pi^2)^3$. Thus one may estimate

$$R \sim \frac{\alpha^3}{128\pi^3}\left[\frac{m_F}{m_W}\right]^6 \varepsilon \simeq 4\times 10^{-9}\left[\frac{m_F}{m_W}\right]^6 \varepsilon. \qquad (6.12)$$

This is completely inadequate unless very heavy fermions exist in a family transformaing as $\bar{5}\oplus 10$. With the usual symmetry breaking mechanism, $m_F \lesssim \sqrt{3}\, m_W$, so that $R \lesssim \times 10^{-8}\varepsilon$. In principle, one may make unitarity cuts through the diagrams of fig. 6.4 to obtain either two-body or three-body final states. However, the fact that the exchanged bosons have the same mass as the decaying bosons renders all but two-body final states energetically forbidden.

Above we have considered only decays to fermion final states. CP violation can enter in the minimal model only through intrinsic complex mixings between fermion families: fermion intermediate states are therefore necessary for CP violation. Decays such as $S \to X\varphi$ or $X \to S\varphi$ to boson final states (where φ is an $SU(2)_L$ doublet scalar) therefore exhibit CP violation only through internal fermion loops and at very high order in perturbation theory (always at an order higher than if one has only fermion final states).

7) CP Violation in Alternative SU(5) Models

The minimal $SU(5)$ model is economical in its choice of Higgs representations, but that choice is by no means necessary. From the point of view of generating an acceptable baryon number asymmetry, there are two simple modifications of the $SU(5)$ Higgs structure that may be made: adding a further 5_H (case A) or the addition of a 45_H (case B). Of course, more complicated Higgs structures may be chosen.

We first discuss case A, in which two 5_H's, denoted 5_{H1} and 5_{H2}, appear. The coupling of these Higgs to fermions may then be written in the form

$$(\overline{5}_f \cdot (h_{D1}) \cdot 10_f) \cdot \overline{5}_{H1} + (\overline{5}_f \cdot (h_{D2}) \cdot 10_f) \cdot \overline{5}_{H2}$$

$$+ (10_f^T \cdot (h_{U1}) \cdot 10_f) \cdot 5_{H1} + (10_f^T \cdot (h_{U2}) \cdot 10_f) \cdot 5_{H2}. \tag{7.1}$$

The 5_{H1} and 5_{H2} here may be chosen to be mass eigenstates. This is possible because any linear combination of 5_H's is also a 5_H (complex linear combinations are permitted since 5_H is a complex representation) and furthermore there is only a single baryon number violating scalar in each 5_H up to $SU(3)$ degeneracy.

The diagrams for corrections to gauge boson decay through Higgs exchange are as in fig. (6.2), except that either of the two 5_H's may be exchanged. In each case, to lowest order, the CP-violating part is proportional to $\mathrm{Im}[Tr(h^a h^{a\dagger})] = 0$. Similarly, no CP violation is generated by gauge boson exchange corrections to the decays of 5_{H1} or 5_{H2} Higgs bosons. Exchanges of $5_{H\alpha}$ in $5_{H\alpha}$ decay may be treated just as in the minimal model which contains only a single 5_H discussed in chapter 6: CP violation in such cases was shown to vanish until eighth order. However, exchange of 5_{H1} in 5_{H2} decay (or vice versa) may lead to

CP violation at fourth order. The coupling factor associated with this diagram is $\Omega = Tr[(h_{D1})(h_{D2})^\dagger(h_{U1})(h_{U2})^\dagger]$. One may apply the unitary transformations (6.3) to render real and diagonal either h_{D1} or h_{D2}, but not, in general, both. Hence here, in general, $\mathrm{Im}[\Omega] \neq 0$, so that CP violation may occur in 5_{H2} decays through 5_{H1} exchange (or vice versa) at the one-loop ($O(\alpha)$) order. This is shown in fig. 7.1.

We now discuss case B defined above, involving a 5_H and a 45_H. Their coupling to fermions may be written in the form

$$(\bar{5}_f \cdot (h_D) \cdot 10_f) \cdot \bar{5}_H + (\bar{5}_f \cdot (H_D) \cdot 10_f) \cdot \overline{45}_H$$

$$+ (10_f \cdot (h_U) \cdot 10_f) \cdot 5_H + (10_f^T \cdot (H_U) \cdot 10_f) \cdot 45_H. \qquad (7.2)$$

In this case, Higgs with definite $SU(5)$ transformation properties will not, in general, contain baryon number violating scalars that are mass eigenstates. A (3, 1, 1/3) (B-violating) component exists in both 5_H and 45_H; the mass eigenstates will be linear combinations of these components. The presence of a $\bar{5}_H \cdot 45_H \cdot 24_H \cdot 24_H$ term in the Higgs potential enforces such a mixing between the (3, 1, 1/3) in the 5_H and that in the 45_H . This term cannot be removed by the imposition of a discrete symmetry without affecting the Yukawa terms. A cubic term such as $\bar{5}_H \cdot 45_H \cdot 24_H$ can be excluded by the symmetry $24_H \to -24_H$. We denote the (3, 1, 1/3) mass eigenstates (assumed mixtures of that in the 5_H and that in the 45_H) by S_1 and S_2. The couplings of fermions to these mass eigenstates are linear combinations of the h_D, H_D and h_U, H_U appearing in (7.2). However, in general, the linear combinations will be different for different fermions within a single family by virtue of the different Clebsch-Gordan coefficients in coupling fermions to the (3, 1, 1/3) in 5_H and to the (3, 1, 1/3) in the 45_H. If we call the (3, 1, 1/3) in the 5_H, S_a (where a is the

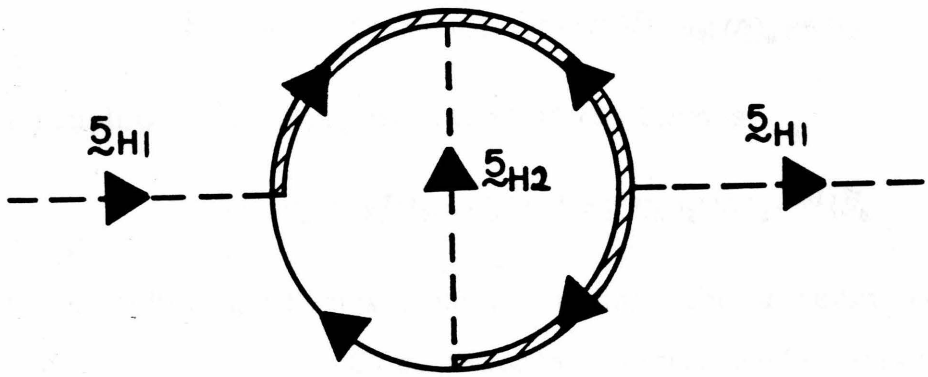

Fig. 7.1: Scalar decay with scalar exchange in an extended $SU(5)$ model with two 5's of Higgs. This diagram may give CP vioation.

$SU(3)$ index) and call the (3, 1, 1/3) in the 45_H, $\tilde{S}_a$, then we may write their couplings to fermions as follows. The coupling of S_a coming from the $\bar{5}\cdot 10\cdot \bar{5}_H$ term is

$$\{E_L^T h_D \sigma_2 (U_L)^a + \nu_L^T h_D \sigma_2 (D_L)^a + (D_L^c)_b^T h_D \sigma_2 (U_L^c)_d \varepsilon^{abd}\} S_a \quad . \tag{7.3}$$

The coupling of $\tilde{S}_a$ coming from the $\bar{5}\cdot 10\cdot 45_H$ term is

$$\{E_L^T H_D \sigma_2 (U_L)^a + \nu_L^T H_D \sigma_2 (D_L)^a - (D_L^c)_b^T H_D \sigma_2 (U_L)_d \varepsilon^{abd}\} \tilde{S}_a \quad . \tag{7.4}$$

Aside from having differing Yukawa couplings, the preceding two expressions also differ in their Clebsch-Gordan coefficients (notably the crucial minus sign in (7.4) as compared to (7.3)). Eqn (4.28) then shows that we may now have a contribution to the baryon asymmetry due to the CP violation in vector exchange in scalar decay (and vice versa). Similarly, gauge boson exchanges in Higgs boson decays may also yield CP violation. The structure of CP violation for Higgs boson exchanges in Higgs boson decay is analogous to the model A discussed above.

It is worth noting that there are two more baryon number violating scalars in Model B: a ($\bar{3}$, 1, −4/3) and a (3, 3, 1/3), both contained in the 45_H. We call them S_3 and S_4 respectively. Diagrams involving only an S_3 or those involving only an S_4 show no possible CP violation until eighth order in the couplings if there is no intrinsic CP violation in the Higgs potential. Furthermore, in the decays of gauge vector bosons there is no CP violation through the exchange of a single S_3 or S_4 as follows from eqn (4.28) since S_3 and S_4 are necessarily mass eigenstates. Also, since each couples to fermions identically (within a factor of a real Clebsch-Gordan coefficient), it follows that a diagram that only involves S_3 and S_4 will show no CP violation until eighth order, although a diagram for, say, S_1

decay through S_3 exchange may have CP violation at fourth order.

8) Introduction to SO(10)

Grand unified models based on $SU(5)$ are the most economical as well as being the simplest for actual calculations. However, the assignment of a (left-handed) family to the reducible $\bar{5}\oplus 10$ representation has a number of ugly features. Some of the particles belong to different irreducible representations than their antiparticles and, although the anomalies cancel between the $\bar{5}$ and 10 representations of fermions [1], this cancellation appears rather artificial from the standpoint of $SU(5)$. In addition, many $SU(5)$ models contain a global quantum number corresponding to baryon number minus lepton number, $B-L$ [24]. These features may be removed by embedding the $SU(5)$ theory in an $SO(10)$ model with the fermions assigned to the lowest dimensional spinor representation [26].

The defining representation of $SO(10)$, 10, is real and has the following $SU(5)$ decomposition:

$$10 = 5 + \bar{5}. \tag{8.1}$$

The lowest dimensional spinor representations of $SO(10)$ are* 16 and its conjugate, $\overline{16}$. All representations of $SO(10)$ can be built out of products of 16 and $\overline{16}$ among themselves. The $SU(5)$ decomposition of 16 is

$$16 = 1 + \bar{5} + 10\ . \tag{8.2}$$

We see that a single family of fermions can be accommodated in the 16 of $SO(10)$. The anomaly cancellation that occurs in $SU(5)$ between the $\bar{5}$ and the 10 of fermions has a natural explanation by choosing the 16 as the

*It is here that the (often confused) distinction between $O(10)$ and $SO(10)$ occurs. The lowest dimensional spinor representation of $O(10)$ is 32 and is self-conjugate. Under $O(10)\supset SO(10)$, $32 = 16 + \overline{16}$. 32 is irreducible in $O(10)$, but reducible in $SO(10)$.

fermion representation in $SO(10)$. The 16 is anomaly free as are all representations of $SO(n)$ for $n>6$. This is equivalent to the statement that the symmetric product of the adjoint with itself does not contain the adjoint and hence the d coefficients vanish.

$SO(10)$ is of rank 5 and contains $SU(5)$ as a subgroup with the maximal embedding

$$SO(10) \supset SU(5) \otimes U(1). \tag{8.3}$$

The $SU(5)$ model may be considered as being embedded in an $SO(10)$ model in a sense similar to the way the $SU(3) \otimes SU(2)_L \otimes U(1)_Y$ model is embedded in $SU(5)$. This point of view is expressed by the following discussion. In the simplest $SU(5)$ models (with only 5's and 45's of Higgs) there is a global $U(1)$ symmetry in addition to the gauged $SU(5)$ symmetry. When $SU(5)$ undergoes spontaneous symmetry breakdown, this global $U(1)$ is broken, as is the combination of generators through which the Z^0 boson couples. A linear combination of these generators, however, survives as a global symmetry* and corresponds to $B-L$ [27]. If we demand that there be no ungauged continuous symmetries**, then we

*We will refer to this process by which a global symmetry survives spontaneous symmetry breaking as the 't Hooft mechanism [28]. This mechanism can be used for both discrete as well as continuous symmetries. In the latter case, however, it provides one with a method for avoiding Goldstone bosons. In the case of discrete symmetries it allows for what may possibly be a very simple symmetry to transmute into a much richer symmetry. If one takes a somewhat proletarian attitude towards model building (and, for example, postulating various messy discrete symmetries to force particular results from a given model), this fact gives one hope that embedding such a model in a larger model may give rise to a *simpler* natural symmetry structure. An example of this is given in the context of an illustrative $SO(10)$ model in chapter 13 and appendix E.

**This is, of course, a bias which is not necessitated at present by any well defined theoretical principles just as long as any global symmetries that are spontaneously broken are rendered harmless by the 't Hooft mechanism. Other alternatives are to explicitly break the symmetry with terms of dimension 2 or 3 or terms of dimension 4 in the Lagrangian. This cannot always be done as is evidenced by the $SU(5)$ example where one has to use the 't Hooft mechanism of necessity. It was this *feeling* that continuous global symmetries are in some sense incompatible with the ideas of local quantum field theory that originally led Yang and Mills to must consider $SU(5)\otimes U(1)$ as the gauge group. $SO(10)$ is then the smallest simple group containing $SU(5)\otimes U(1)$. The gauge vector boson corresponding to the $U(1)$ factor can then mediate $B-L$ violating processes.

From eqn (8.2) we see that the price to pay for this $SO(10)$ unification is an $SU(5)$ singlet fermion. Since the electric charge operator is entirely contained in $SU(5)$, this fermion is neutral. We denote this extra field by N_L. It provides a charge conjugate partner for the left-handed neutrino and thus allows a $\Delta I_w=1/2$ Dirac mass term for the neutrino. The potential disaster of neutrino masses of the order of the $\Delta I_w=1/2$ breaking may be avoided if the N_L acquires a very large, $\Delta I_w=0$, Majorana mass, M_N [29, 27]. The neutral lepton mass matrix will then have the form*

$$\begin{pmatrix} 0 & m_q \\ m_q^T & M_N \end{pmatrix} \tag{8.4}$$

with m_q a matrix with entries of the order of the observed quark masses and M_N the Majorana mass matrix for the N_L. For $M_N \gg m_q$ the eigenvalues of this matrix are given approximately by the eigenvalues of M_N, which are the masses of the N_L, and the eigenvalues of the matrix $m_q^T M_N^{-1} m_q$, which are the light neutrino masses. As a result of this mechanism, $SO(10)$ models naturally predict the existence of neutrino masses and hence neutrino oscillations. The N_L can be given a large Majorana mass either directly through a 126 of Higgs that obtains a large vacuum

the concept of the gauge field [2].

*The zero entry in this matrix is not necessary. All that is required is a condition such as being $< O(m_q)$. Such a Majorana mass for ν_L would have $\Delta I_w = 1$ and must be relatively small so as not to disturb $m_W/(m_Z \cos\theta_W) \simeq 1$. A Majorana mass for ν_L can be accommodated in $SU(5)$ by including a 15 of Higgs which obtains a vacuum expectation value or by means of a 10 of Higgs. In the latter case, the left-handed neutrino's Majorana mass arises as a calculable correction to the zeroth order mass relation which requires it to vanish [31].

expectation value along its $SU(5)$ singlet direction or through radiative corrections [30].

The vector bosons transform as the 45 dimensional (adjoint) representation of $SO(10)$ which has the chiral decomposition ($SO(10) \supset SU(4) \otimes SU(2)_L \otimes SU(2)_R$),

$$45 = (6,2,2) + (15,1,1) + (1,3,1) + (1,1,3) \quad . \tag{8.5}$$

The last three representations correspond to the gauge bosons of $SU(4)$, $SU(2)_L$ and $SU(2)_R$ respectively. The (6, 2, 2) contains the usual leptoquark-diquarks (X, Y) of $SU(5)$ transforming as (3, 2, 5/6) under $SU(3) \otimes SU(2)_L \otimes U(1)_Y$, their antiparticles, an additional doublet of leptoquark-diquarks, (X', Y'), transforming as (3, 2, -1/6) and their anti-particles. The gauge bosons of $SU(4)$ contain the gluons of $SU(3)$ and an additional color triplet field transforming as (3, 1, 2/3), which we denote by V, its antiparticle and a color singlet field. The gauge bosons of $SU(2)_R$ transform as (1,1,-1), (1,1,0), and (1,1,1) which we denote by W_R^+, W_R^0, and W_R^- respectively.

The scalar fields which couple directly to fermions must transform according to representations which appear in the Clebsch-Gordan decomposition of $16 \otimes 16$:

$$(16 \otimes 16) = (10 + 126)_S + (120)_A \quad , \tag{8.6}$$

where S and A denote the symmetric and the antisymmetric parts. The $SU(5)$ decompositions of some $SO(10)$ representations are given in table 8.1; their $SU(3) \otimes SU(2)_L \otimes U(1)_Y$ decompositions can be obtained easily from this table and eqns (5.3) through (5.8). The 10 and 120 are real representations; the 126 is complex. One may also include other Higgs

$10=5+\bar{5}$
$16=1+\bar{5}+10$
$45=1+10+\overline{10}+24$
$54=15+\overline{15}+24$
$120=5+\bar{5}+10+\overline{10}+45+\overline{45}$
$126=1+\bar{5}+10+\overline{15}+45+50$

Table 8.1

$SU(5)$ decompositions of some $SO(10)$ representations.

multiplets which do not have direct couplings to fermions. Typically an adjoint, 45, or a 54, is chosen. The minimal set of Higgs necessary to break $SO(10)$ down to $SU(3)\otimes U(1)_{EM}$ and give masses to all fermions is 10_H, 16_H and 45_H. With only these Higgs one finds the tree level mass relations

$$m_\nu = m_u = m_d = m_e \tag{8.7}$$

at the unification scale, with the same relations holding for the heavier families. Thus we see that for the minimal set of Higgs the mass of the everyday neutrino is predicted to be far too large.

The generation of a net baryon number from symmetrical initial conditions requires the presence of both C and CP violation [12,13]. In $SU(2)_L \otimes U(1)_Y$ weak interaction models and $SU(5)$ grand unified models no C operator may be defined since there is no left-handed antineutrino to act as the charge conjugate partner of the left-handed neutrino. In some larger models, such as $SO(10)$ or $E(6)$, each fermion has a potential charge conjugate partner or is an eigenstate of C hence a C operation may be defined which is a symmetry of the unbroken theory [32]. The production of a C-odd quantum number (such as B or L) in these models therefore depends on the interplay between the sources of C violation and the processes which violate the quantum number under consideration.

The lack of B production in a C-symmetric theory may be seen by considering the decays of B-violating bosons χ and their antiparticles $\bar{\chi}$ as well as the decays of their charge conjugate partners χ^c and $\bar{\chi}^c$. The B produced by the decays of an equal mixture of χ and $\bar{\chi}$ into the specific final state $\bar{i}_1 i_2$ and the charge conjugate decays of χ^c and $\bar{\chi}^c$ into the state $\bar{i}_1^c i_2^c$ is proportional to the quantity (see eqn (4.12))

$$R_X^{12} + (R_X^{12})^c = \mathrm{Im}I \ \mathrm{Im}\Omega \, (B_{i_2} - B_{i_1}) + \mathrm{Im}I^c \ \ \mathrm{Im}\Omega^c \, (B_{i_2^c} - B_{i_1^c}) \tag{8.8}$$

where I represents the integral over the intermediate momenta and final state phase space for the decay and Ω is a product of the relevant couplings. The lowest order contributions to I and Ω were discussed in chapter 4. I^c and Ω^c are the corresponding quantities for the charge conjugate reaction. In a C-symmetric theory, $I = I^c$ and $\Omega = \Omega^c$, while, since B is C-odd, $B_{i_2} = -B_{i_2^c}$ and $B_{i_1} = -B_{i_1^c}$ causing $R_X^{12} + (R_X^{12})^c$ to vanish.

We now restrict our attention to $SO(10)$ grand unified models. The presence of a charge conjugate partner for the neutrino, N_L, allows the definition of a C operation for all fermion fields appearing in the theory. In terms of the $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ subgroup of $SO(10)$, C interchanges the two $SU(2)$'s, as well as conjugating them, and also conjugates the representations of $SU(4)$ [22]. It may be shown that all C violation in the fermion mass matrix must lie in the part of the 126 representation of $SO(10)$ which gives a Majorana mass to N_L. This C-violating mass term allows for the production of a nonzero B since $\mathrm{Im}I$ is no longer equal to $\mathrm{Im}I^c$. Expanding I and I^c in powers of M_N / M_X gives

$$R_X^{12} + (R_X^{12})^c = O(M_N^2 / M_X^2) \tag{8.9}$$

where M_X is the mass of the decaying boson.

If all asymmetries can be expressed in terms of C odd quantum numbers then, (8.9) constrains the possible values of M_N / M_X if we demand that the theory be able to produce the observed baryon asymmetry. However, in the general case, asymmetries which have no definite behavior under C must be considered. Large asymmetries in such quantum numbers may be produced even if the theory is in a C-conserving phase (e.g., $SO(10)$ broken to $SU(4) \otimes SU(2)_L \otimes SU(2)_R$) [33]. These asymmetries may later be converted into a baryon asymmetry by B-violating reactions which occur in a C-violating phase of the theory. These reactions will be able to produce a sufficient baryon asymmetry only if there exist B-violating bosons with masses less than the transition temperature between the C-conserving and C-violating phases of the theory. For $SO(10) \to SU(4) \otimes SU(2)_L \otimes SU(2)_R$ the $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ symmetry must not persist to temperatures below $\sim 10^{12}$ GeV if an adequate B is to be produced. A detailed discussion of some of these ideas is presented in the context of an illustrative $SO(10)$ model in the following chapter.

9) Analysis of an Illustrative SO(10) Model

We saw in chapter 8 that N_L can obtain a large Majorana mass if we have the fermions couple to a 126_H of Higgs which obtains a large vacuum expectation value along its $SU(5)$ singlet direction. Vacuum expectation values along other components of the 126_H break $SU(2)\otimes U(1)$ (see table 8.1 and eqns (5.3) through (5.8)) and hence must be small relative to the singlet vacuum expectation value. If $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ is unbroken* the vacuum expectation value of 126_H is purely along its $SU(5)$ singlet, $<126_H>\sim 1$. To effect a complete symmetry breakdown of $SO(10)$ to $SU(3)\otimes SU(2)\otimes U(1)$ we need another Higgs representation**. Either a 54_H or a 45_H is usually chosen so as to conform with typical $SU(5)$ models since both contain a 24 of $SU(5)$. The size of the 54_H (or 45_H) vacuum expectation value along the $SU(5)$ 24 direction is O(10^{15} GeV) so as to conform with the bound imposed by the nonobservation of proton decay [23]. Depending on the relative sizes of $<54_H>$ (or $<45_H>$) and $<126_H>$ one will have different symmetry breaking patterns for $SO(10)$.

To consider the production of a cosmological baryon number asymmetry in an $SO(10)$ model with $<126>\sim 1$ we will consider the simplified case where there is only one family of fermions. We will discuss the contributions to the asymmetry due to the free decays of baryon number violating bosons. The free decays of the N_L do not contribute appreciably to the baryon asymmetry because the baryon number violating decays of the N_L are generally into a three-body fermion final state, whereas the main contribution to the Born rate is through the two-body final state

*This is generally the case at the high temperatures present in the standard model of the early universe.
**With $<126>\sim 1$ alone, $SO(10)$ breaks down only to $SU(5)$.

$N_L \to \nu_L \varphi$ where φ is an $SU(2)_L$ doublet scalar field [34]. The contribution of the decay of the N_L is thereby greatly supressed.

We assume that there is no intrinsic CP violation in this model; however, it has been shown [35,36] that there can be a source of high temperature spontaneous CP violation with a calculable phase. The Higgs potential contains the following quartic terms involving only the 126_H:

$$\lambda_1(126_H \cdot \overline{126}_H)^2 + \lambda_2\{(126_H)^4 + (\overline{126}_H)^4\} \tag{9.1}$$

where λ_1 and λ_2 are real. If we write $<126_H> = \rho e^{i\varphi}$ then (9.1) gives

$$\{\lambda_1 + 2\lambda_2 \cos(4\varphi)\}\rho^4 \,. \tag{9.2}$$

If we choose λ_1, $\lambda_2 > 0$ then* the potential has its minimum when $\varphi = \pm\pi/4$ or $\pm 3\pi/4$. These two cases are not independent in the following since the quantity that always enters into the calculations is $<126>^2$. Such a CP-violating phase enters the theory at a scale $O(|<126_H>|)$ and is the only CP violation present between that scale and the scale at which the next level of symmetry breaking occurs. Thus, if the vacuum expectation value of the 54_H is greater than that of the 126_H (this is the most interesting case since the chain of symmetry breaking is $SO(10) \to SU(4) \otimes SU(2)_L \otimes SU(2)_R \to SU(3) \otimes SU(2)_L \otimes U(1)_R$), then this CP violation will be absent in the $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ phase; any decays that occur in this phase will conserve baryon number by virtue of the absence of CP violation. Even if there had been CP violation in this phase (for example

*To insure the stability of the potential we must also have $2\lambda_2 < \lambda_1$. The sign of φ must be so as to generate the correct sign of the baryon number asymmetry consistent with our conventions for particles and antiparticles. Also, $(126 \cdot \overline{126})^2$ may couple to form a singlet in four ways [33,36]; however, only the term constructed from having $126 \cdot \overline{126}$ transform as a singlet has a φ_1^4 term in its expansion (where φ_1 is the $SU(5)$ singlet in the 126_H) and thus the remaining quartic invariants do not contribute to eqn (9.2).

due to the presence of intrinsic CP violation in the Lagrangian), a baryon asymmetry could not be generated directly by the decays of baryon number violating bosons because of the presence of an unbroken charge conjugation operation as discussed in chapter 8.

The Yukawa term that we currently have in this model, $(16 \cdot 16) \cdot \overline{126}_H$, has a global $U_X(1)$ symmetry with a charge X for which (by convention) $X=1$ for the 16 and $X=2$ for the 126_H ($X=-2$ for $\overline{126}_H$). (This global $U_X(1)$ is broken explicitly by the $(126_H)^4$ term in the Higgs potential as well as by other terms to be discussed below.) When 126_H gets its vacuum expectation value, $U_X(1)$ is broken as is the local $U(1)_R$ appearing in $SU(5) \otimes U(1)_R$ (we call the charge corresponding to the local $U(1)_R$, R). If the vacuum expectation value of 126_H is along any one direction in its $SU(5)$ decomposition (as is true in our case), then the 't Hooft mechanism is operable, yielding a global $U(1)_Z$ after symmetry breaking with a corresponding charge (for $<126_H> \sim 1$),

$$Z=(X-\frac{R}{5}) \ . \tag{9.3}$$

The analysis of this symmetry is discussed in appendix E.

This symmetry allows us to classify the possible scalar mass terms in the model, since they all must be Z (as well as $SU(3) \otimes SU(2)_L \otimes U(1)_Y$) invariant. To this end we list all of the values of Z for 54_H and the 126_H. All components of the 54_H have $Z=0$. For 126_H the 1, $\bar{5}$, 10, $\overline{15}$, 45 and 50 have Z values respectively,

$$0, \frac{8}{5}, \frac{4}{5}, \frac{16}{5}, \frac{12}{5} \text{ and } \frac{8}{5} . \tag{9.4}$$

In this model CP violation may occur through the scalar mass matrix and hence, in the scalar's mass eigenstate basis, through the scalar rotation matrix; in particular this occurs when complex entries occur in the mass matrix of the baryon number violating scalars that couple directly to fermions.

In addition to the terms (9.1) there are other quartic terms appearing in the Higgs potential*:

$$126_H \cdot \overline{126}_H \cdot (54_H)^2 \ ; \ (54_H)^4 \ . \tag{9.5}$$

If we assume that the Higgs potential consists only of quartic terms** we can immediately state, within factors of quartic coupling constants and Clebsch-Gordan coefficients, what the scalar mass terms are. We write the fields appearing in the 126_H as φ_1, $\varphi_5^\dagger$, φ_{10}, $\varphi_{15}^\dagger$ φ_{45} and $\varphi_{\overline{50}}^\dagger$; and we write the fields appearing in 54_H as ψ_{15}, $\psi_{\overline{15}}$ and ψ_{24}. Note that since 54_H is self-conjugate, $\psi_{15}^\dagger = \psi_{\overline{15}}$.

From $(126_H)^4$ we can only get

$$\varphi_1 \varphi_1 < \varphi_1 >^2 \ . \tag{9.6}$$

and from $(\overline{126}_H)^4$ we get the Hermitian conjugate of this,

---

*At this point the term $(126_H)^2(54_H)^2$ could appear, but is excluded on the basis of the discrete residue of the $U(1)_X$ symmetry. Later we shall need to include these terms when we break the $U(1)_X$ symmetry further by the inclusion of a real 10 of Higgs. Furthermore, as mentioned above, there are, for example, four ways to couple 126, 126, $\overline{126}$ and $\overline{126}$ to form a singlet.

**This is the case if the symmetry breaking is generated by the Coleman-Weinberg mechanism [37] which one would wish to occur so as to exclude dimensionful parameters from the bare Lagrangian. Dimensional transmutation may then occur. Cubic terms can always be excluded from the Higgs potential by means of a discrete symmetry (in fact the discrete residue of the X symmetry will suffice to do this). The presence of possible quadratic terms does not affect the following analysis very much since they yield only a diagonal contribution to the baryon number violating scalar boson mass matrix.

$$\varphi_1^\dagger \varphi_1^\dagger \langle \varphi_1^\dagger \rangle^2 \,. \tag{9.7}$$

From $(126_H \cdot \overline{126}_H)^2$, if a 126_H and a $\overline{126}_H$ get a vacuum expectation value we get

$$\varphi_1^\dagger \varphi_1 \,,\; \varphi_5^\dagger \varphi_5 \,,\; \varphi_{10}^\dagger \varphi_{10} \,,\; \varphi_{15}^\dagger \varphi_{15} \,,\; \varphi_{45}^\dagger \varphi_{45} \,,\; \varphi_{\overline{50}}^\dagger \varphi_{\overline{50}} \,, \tag{9.8}$$

each multiplied by $|\langle \varphi_1 \rangle|^2$. If 126_H and 126_H get their vacuum expectation values then we get

$$\varphi_1 \varphi_1 \langle \varphi_1^\dagger \rangle^2 \,, \tag{9.9}$$

and if $\overline{126}_H$ and 126_H are given their vacuum expectation values the term

$$\varphi_1^\dagger \varphi_1^\dagger \langle \varphi_1 \rangle^2 \tag{9.10}$$

arises.

From $(126_H \cdot \overline{126}_H) \cdot (54_H)^2$, if both 54_H's get their vacuum expectation values, then we have the same terms as in eqn (9.8) but multiplied by $\langle \psi_{24} \rangle^2$. Other possible terms such as $\varphi_5 \varphi_{\overline{50}}$ are excluded, at this point, by the Z symmetry; we shall have occasion to include them later. Finally, if 126_H and $\overline{126}_H$ get their vacuum expectation values ,we get the terms

$$\psi_{15}^\dagger \psi_{15} |\langle \varphi_1 \rangle|^2 \,,\; \psi_{24} \psi_{24} |\langle \varphi_1 \rangle|^2 \,. \tag{9.11}$$

Since all the terms above which involve baryon number violating scalars have real coefficients, it is clear that one cannot have a contribution to the baryon asymmetry in the present form of this model because CP is not violated.

There is a simple extension of the model, however, which yields nontrivial results. If we introduce a Higgs representation transforming as a real 10, 10_H, then it can couple to the fermions through

$$(16 \cdot 16) \cdot 10_H \ . \tag{9.12}$$

(The 10 of $SO(10)$ has the $SU(5)$ decomposition $10=\bar{5}+5$, where, for a single scalar representation, the $\bar{5}$ is conjugate to the 5; thus, if we write $\tilde{\varphi}_{\bar{5}}$ for the $\bar{5}$ in the 10_H and $\tilde{\varphi}_5$ for the 5, we have $\tilde{\varphi}_{\bar{5}}=\tilde{\varphi}_5^\dagger$.) The X symmetry is now explicitly broken down to the discrete symmetry generated by the transformations

$$\left.\begin{array}{c} 16_f \to \pm i\, 16_f \\ 126_H \to -126_H \\ 10_H \to -10_H \end{array}\right\} \ . \tag{9.13}$$

All quartic terms that are $SO(10)$ singlets and which can made from 126_H, 54_H and 10_H respect this discrete symmetry. These terms are listed in Table 9.1. Before the X symmetry is broken down to the discrete symmetry (9.13) we have the Z symmetry for which

$$\left.\begin{array}{r} Z=\dfrac{8}{5} \text{ for the } \tilde{5} \\ \text{and } Z=\dfrac{12}{5} \text{ for the } \tilde{\bar{5}} \end{array}\right\} . \tag{9.14}$$

After the X symmetry is broken down to the discrete symmetry (9.14) the Z symmetry breaks down as well in the following fashion.

If, under $U(1)_X$, a given field (say φ) transforms as

$$\varphi \to e^{iX\alpha}\varphi \ , \tag{9.15}$$

then, after the 126_H gets a vacuum expectation value along its $SU(5)$ singlet direction, the Lagrangian continues to be globally invariant under the action of $U(1)_Z$:

$$\varphi \rightarrow e^{iZ\alpha}\varphi = e^{i\left[X-\frac{R}{5}\right]\alpha}\varphi . \qquad (9.16)$$

When the $U(1)_X$ symmetry is explicitly broken down to the discrete symmetry (9.13), then, upon comparing (9.13) with (9.15), we see that this is equivalent to the constraint $\alpha=n\pi/2$ (with n an arbitrary integer) in (9.15) and, hence, in (9.16). With this restriction a field with $Z=12/5$ and one with $Z=-8/5$ have identical transformations under $U(1)_Z$ since

$$e^{i(12/5)n\pi/2}=e^{-i(8/5)n\pi/2} , \qquad (9.17)$$

as must certainly be the case since $\tilde{\varphi}_5=\tilde{\varphi}_5^\dagger$. Thus, the $U(1)_Z$ symmetry allows* the (3, 1, 1/3) in the $\tilde{5}$ to mix with those in the 5, 45 and $\overline{50}$ that come from the 126_H.

If $SU(5)$ is unbroken, then certainly the (3, 1, 1/3) in the $\overline{50}$, for example, cannot mix with those in the 45, 5 or $\tilde{5}$ since the mass terms from which these mixings arise must themselves be $SU(5)$ invariant. The breaking of $SU(5)$ comes, in this model, from the 54_H obtaining a vacuum expectation value. Thus, any term of the form $\varphi_5^\dagger\varphi_{\overline{50}}$ or $\tilde{\varphi}_5^\dagger\varphi_{\overline{50}}$ must be proportional** to $<\psi_{24}>$. The term $\varphi_5^\dagger\varphi_{\overline{50}}$ could therefore come from $(126\cdot\overline{126})\cdot 54^2$, $(126\cdot\overline{126})\cdot 54\cdot 126$ or $(126\cdot\overline{126})\cdot 54\cdot\overline{126}$; however, one cannot form an $SO(10)$ singlet with the latter two possibilities, nor, for that matter, are they permitted by the Z symmetry. The term $\tilde{\varphi}_5^\dagger\varphi_{\overline{50}}$ could come from $(10\cdot 126)\cdot 54^2$, $(10\cdot 126)\cdot 54\cdot 126$ or $(10\cdot 126)\cdot 54\cdot\overline{126}$; however, none of these can

*It also turns out that the 15 and the 10 in the 126 now have the same transformation properties as one another under the remaining Z symmetry; however, neither the 10 or the 15 contain any baryon number violating scalars as can be seen from eqns (5.3) through (5.8) and table 3.3.

**The full mass term must, at the $SU(5)$ level, be an invariant before the $24\subset 54$ obtains its vacuum expectation value; thus, any candidate mass term must be checked to show that one can form a singlet from the relevant product of $SU(5)$ representations.

combine to form singlets of $SO(10)$; thus, the term $\tilde{\varphi}_5^\dagger \varphi_{\overline{50}}$ cannot occur. Mixing between φ_5 and $\varphi_{\overline{50}}$ will occur through the term $126\overline{126}\cdot 54^2$. A similar analysis shows that φ_{45} mixes with the φ_5 and the $\varphi_{\overline{50}}$, but not with the $\tilde{\varphi}_5$, and the mixing occurs again through the term $126^2\cdot 54^2$.

Mixing between the $\tilde{5}$ in the 10_H and the 5 in the 126_H is permitted by the Z symmetry and does indeed occur through the term

$$10_H\cdot\overline{126}_H\cdot(126_H)^2+\text{h.c.} \tag{9.18}$$

Further quartic terms consistent with the symmetries of the previous terms are

$$\begin{gathered}(10_H)^4\,,\,(10_H)^2\cdot(126_H)^2\,,\\ (10_H)^2\cdot(126_H\cdot\overline{126}_H)\,,\,(10_H)^2\cdot(54_H)^2\end{gathered} \tag{9.19}$$

and their Hermitian conjugates. The quartic terms appearing in the Higgs potential are summarized in table 9.1 [33,36]. The entries in the baryon number violating scalar boson mass matrix that may come from these terms are summarized* in table 9.2. The suppressed coefficients of these mass terms are typically the product of a quartic coupling constant, a combinatoric factor and a Clebsch-Gordan coefficient. In light of our ignorance of these factors (especially the quartic coupling constants) we will take all the coefficients to have the common value λ. We choose $\langle\varphi_1\rangle^2/|\langle\varphi_1\rangle|^2=+i$ (the case $\langle\varphi_1\rangle^2/|\langle\varphi_1\rangle|^2=-i$ can be obtained easily from the following). Defining $\varepsilon=\langle\psi_{24}\rangle^2/|\langle\varphi_1\rangle|^2$ we get the following scalar mass matrix in the $(\varphi_{\overline{50}}, \varphi_{45}, \varphi_5, \tilde{\varphi}_5)$ basis:

$$\lambda|\langle\varphi_1\rangle|^2\begin{bmatrix}1+\varepsilon & \varepsilon & \varepsilon & 0\\ \varepsilon & 1+\varepsilon & \varepsilon & 0\\ \varepsilon & \varepsilon & 1+\varepsilon & 1-i\\ 0 & 0 & 1+i & 1+\varepsilon\end{bmatrix}\,. \tag{9.20}$$

*We only write those involving the (3, 1, 1/3) in the φ_5, $\tilde{\varphi}_5$ φ_{45} and $\varphi_{\overline{50}}$ since all others lead to (at most) baryon violating interactions with no CP violation.

10^4
54^4
$(126\cdot\overline{126})^2$ 126^4
$10^2\cdot 54^2$
$126\cdot\overline{126}\cdot 54^2$ $126^2\cdot 54^2$
$10^2\cdot 126\cdot\overline{126}$ $10^2\cdot 126^2$
$10\cdot\overline{126}\cdot(126\cdot\overline{126})$

Table 9.1

All quartic terms in a Higgs potential involving a 54, a 126 and a real 10. The number of possible terms of each type is not indicated.

$(126\cdot\overline{126})^2$	$\varphi_5\varphi_5^\dagger\,|\langle\varphi_1\rangle|^2$ $\varphi_{\overline{50}}\varphi_{\overline{50}}^\dagger\,|\langle\varphi_1\rangle|^2$ $\varphi_{45}\varphi_{45}^\dagger\,|\langle\varphi_1\rangle|^2$
$10^2\cdot 54^2$	$\tilde{\varphi}_5\tilde{\varphi}_5^\dagger\langle\psi_{24}\rangle^2$
$126\cdot\overline{126}\cdot 54^2$	$\varphi_5\varphi_5^\dagger\langle\psi_{24}\rangle^2$ $\varphi_{\overline{50}}\varphi_{\overline{50}}^\dagger\langle\psi_{24}\rangle^2$ $\varphi_{45}\varphi_{45}^\dagger\langle\psi_{24}\rangle^2$ $\varphi_5\varphi_{\overline{50}}^\dagger\langle\psi_{24}\rangle^2$
$126^2\cdot 54^2$	$\varphi_{45}\varphi_5^\dagger$ $\varphi_{45}\varphi_{\overline{50}}^\dagger$
$10^2\cdot 126\cdot\overline{126}$	$\tilde{\varphi}_5\tilde{\varphi}_5^\dagger\,|\langle\varphi_1\rangle|^2$
$10\cdot 126\cdot(126\cdot\overline{126})$	$\tilde{\varphi}_5\varphi_5^\dagger\,|\langle\varphi_1\rangle|^2$ $\varphi_5\tilde{\varphi}_5^\dagger\langle\varphi_1\rangle^2$

Table 9.2

Terms in the Higgs potential that give rise to entries in the mass matrix of (3, 1, 1/3) baryon number violating scalar bosons along with the associated mass terms.

If we write the eigenvalues of this matrix as μ^2 and write

$$x \equiv 1+\varepsilon-\frac{\mu^2}{\lambda|<\varphi_1>|^2}\ , \tag{9.21}$$

then the characteristic equation is

$$x^4-(3\varepsilon^2+2)x^2+2x\varepsilon^2+2\varepsilon^2=0\ . \tag{9.22}$$

We can write down one solution of this exactly, $x=\varepsilon$, giving

$$\mu_1^2=\lambda|<\varphi_1>|^2\ . \tag{9.23}$$

The remaining solutions can be given approximately for large ε as

$$\mu_2^2=\lambda|<\varphi_1>|^2\left\{3\varepsilon+1+\frac{1}{3\varepsilon}+\cdots\right\} \tag{9.24}$$

$$\mu_3^2=\lambda|<\varphi_1>|^2\left\{\varepsilon+1+\frac{1}{\varepsilon}+..\right\} \tag{9.25}$$

$$\mu_4^2=\lambda|<\varphi_1>|^2\left\{1-\frac{4}{3\varepsilon}+\cdots\right\}. \tag{9.26}$$

Thus, for large ε, μ_2 and μ_3 decouple from equilibrium at a temperature at which the CP violation that we are considering has not yet turned on, and thus a baryon asymmetry cannot be generated through their decays. To get a nonzero baryon number asymmetry we need to have a scalar that decouples from equilibrium at a temperature that is less than the $SU(4)\otimes SU(2)_L\otimes SU(2)_R \rightarrow SU(3)\otimes SU(2)_L\otimes U(1)_R$ transition temperature*; for

*We neglect here the question of the supercooling and possible associated entropy generation that may occur when a phase transition occurs via tha Coleman-Weinberg mechanism [38]. This may be avoided by having small negative quadratic terms in the Higgs potential; as mentioned above, such terms would not substantially affect our analysis.

$\lambda<1$ this may occur. Thus, the decays of the lighter baryon number violating scalars may give rise to a baryon number asymmetry if they have CP violation in their decays.

The eigenvectors associated with the above eigenvalues are respectively

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \tag{9.27a}$$

$$\begin{pmatrix} \frac{1}{\sqrt{3}} - \frac{7}{36\sqrt{3}}\frac{1}{\varepsilon^2} \\ \frac{1}{\sqrt{3}} - \frac{7}{36\sqrt{3}}\frac{1}{\varepsilon^2} \\ \frac{1}{\sqrt{3}} + \frac{5}{36\sqrt{3}}\frac{1}{\varepsilon^2} \\ e^{i\pi/4}\left(\frac{1}{\sqrt{6}\varepsilon}\right) \end{pmatrix}, \tag{9.27b}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}\varepsilon} \\ \frac{1}{\sqrt{2}\varepsilon} \\ -\frac{1}{\sqrt{2}\varepsilon} \\ e^{i\pi/4}\left(-1+\frac{3}{4\varepsilon^2}\right) \end{pmatrix} \tag{9.27c}$$

and

$$\begin{pmatrix} \frac{1}{\sqrt{6}} - \frac{5\sqrt{2}}{9\sqrt{3}}\frac{1}{\varepsilon^2} \\ \frac{1}{\sqrt{6}} - \frac{5\sqrt{2}}{9\sqrt{3}}\frac{1}{\varepsilon^2} \\ -\frac{\sqrt{2}}{\sqrt{3}} + \frac{4\sqrt{2}}{9\sqrt{3}}\frac{1}{\varepsilon^2} \\ e^{i\pi/4}\left(\frac{2}{\sqrt{3}\varepsilon}\right) \end{pmatrix}. \tag{9.27d}$$

Of the two light scalars, μ_1 and μ_4 ($\mu_4<\mu_1$), only the eigenvector of the lighter one is associated with a complex eigenvector. Thus, its decay, through the exchange of a vector, may exhibit CP violation. The other light scalar, μ_1, may not violate CP in such a decay, although it may do so through the exchange of another scalar. Generally, the decay of a scalar through the exchange of a vector produces a baryon asymmetry greater than that produced through the exchange of another scalar by a factor g^2/Y^2 where g is a typical gauge coupling constant and Y is a typical Yukawa coupling constant. Thus, unless the Yukawa couplings are very large, an estimation of the baryon asymmetry produced through the free decays will be dominated by the vector exchange diagrams. For illustrative purposes we only consider these diagrams. This model allows for scalar exchange in vector decay since the double-cut diagram for scalar exchange in scalar decay is related to that of vector exchange in scalar decay simply by complex conjugation; however, the relevant vectors have masses that are obtained in the $SO(10)\to SU(4)\otimes SU(2)_L\otimes SU(2)_R$ symmetry breaking and decouple from equlibrium at a temperature generally greater than that at which our high temperature CP violation has turned on (certainly this is true if $\varepsilon\gg1$ as is the case that we are investigating). Also, the Born rates for the two processes are generally different, making the latter process larger than the former by the factor g^2/Y^2. If we write the eigenvectors, eqns (9.27), as

$$\begin{pmatrix}\alpha_j\\ \beta_j\\ \gamma_j\\ \delta_j\end{pmatrix} \quad ; \quad j=1,2,3,4 \tag{9.28}$$

then the mass eigenstates, χ_j, may be written as

$$\chi_j=\alpha_j\varphi_{\overline{50}}+\beta_j\varphi_{45}+\gamma_j\varphi_5+\delta_j\tilde{\varphi}_5 \tag{9.29}$$

and this may be inverted to give

$$\left.\begin{array}{l}\varphi_{\overline{50}}=\alpha_j\chi_j\\ \varphi_{45}=\beta_j{}^*\chi_j\\ \varphi_5=\gamma_j{}^*\chi_j\\ \tilde{\varphi}_5=\delta_j{}^*\chi_j\end{array}\right\} \tag{9.30}$$

where we have taken α_j to be real by convention (this convention has already been imposed in the expressions given above for the eigenvectors). If we write the $SO(10)$ Yukawa couplings as

$$A(16\cdot 16)\cdot\overline{126}+B(16\cdot 16)\cdot 10+\text{h.c.} \tag{9.31}$$

(note that A and B may be taken to be real since we are considering the case of no intrinsic CP violation), we find the neutral fermion mass matrix to be

$$\begin{pmatrix}0 & 0\\ 0 & e^{i\pi/4}A\,|\langle\varphi_1\rangle|\end{pmatrix}. \tag{9.32}$$

This is rendered real by working with the field N'_L related to N_L by

$$N'_L=e^{i\pi/8}N_L\ . \tag{9.33}$$

Using this we find the effective $SU(5)$ Yukawa couplings to be

$$e^{i\pi/8}A(\bar{5}_f\cdot 1'_f)\cdot 5+A(10_f\cdot 10_f)\cdot 5+A(10_f\cdot 10_f)\cdot\overline{50}+A(\bar{5}_f\cdot 10_f)\cdot\overline{45}$$

$$+e^{i\pi/8}B(\bar{5}_f\cdot 1'_f)\cdot\tilde{5}+B(\bar{5}_f\cdot 10_f)\cdot\tilde{\bar{5}}+B(10_f\cdot 10_f)\cdot\tilde{5}\ . \tag{9.34}$$

We note here that the Clebsch-Gordan coefficients for the coupling of the (3, 1, 1/3) in a $\overline{50}$ to $10_f\cdot 10_f$ are different from those for the coupling of the (3, 1, 1/3) in a 5 to the same quantity. The former case has the coupling

$$\{(e_L^c)^T\sigma_2(u_L^c)^a+\frac{1}{2}(u_L)_b^T\sigma_2(d_L)_d\,\varepsilon^{abd}\}S_a \tag{9.35}$$

whereas the latter case has the coupling

$$[(e_L^c)^T\sigma_2(u_L^c)^a+(u_L)_b^T\sigma_2(d_L)_d\,\varepsilon^{abd}]S_a \ . \tag{9.36}$$

The couplings of the χ_j to $\bar{5}_f\cdot 10_f$ do not appear in the diagrams of a χ_j decay through X' or Y' exchange and therefore are not relevant to our calculation. Thus, in writing down the relevant Yukawa terms for these processes, we will ignore these couplings. In terms of the χ_j's the Yukawa couplings therefore read (referring back to our eigenvectors, eqns (9.27), we write $\alpha_j=a_j$, $\beta_j=b_j$, $\gamma_j=c_j$ and $\delta_j=e^{i\pi/4}d_j$ where a_j, b_j, c_j and d_j are all real):

$$[e^{i\pi/8}[(d_L^c)^a]^T\sigma_2N'_L(Ac_j+e^{-i\pi/4}Bd_j)+(e_L^c)^T\sigma_2(u_L^c)^a\{A(a_j+c_j)+e^{-i\pi/4}Bd_j\}$$

$$+(u_L)_b(d_L)_c\,\varepsilon^{abc}\{A(\frac{1}{2}a_j+c_j)+e^{-i\pi/4}Bd_j\}]\,\chi_{aj} \ . \tag{9.37}$$

The couplings of the X' and Y' vector bosons (the baryon number violating vector bosons that are in $SO(10)$ but not in $SU(5)$) are [33]

$$-\frac{g}{\sqrt{2}}X'^{\mu}_a\{e_L^\dagger\sigma_\mu(u_L^c)^a-e^{i\pi/8}d_{La}^\dagger\sigma_\mu N'_L+\varepsilon^{abd}[(d_L^c)^b]^\dagger\sigma_\mu u_{Ld}\}+\text{h.c.} \tag{9.38}$$

and

$$\frac{g}{\sqrt{2}}Y'^{\mu}_a\{-\nu_L^\dagger\sigma_\mu(u_L^c)^a+e^{i\pi/8}\,u_{La}^\dagger\sigma_\mu N'_L+\varepsilon^{abd}[(d_L^c)^b]^\dagger\sigma_\mu u_{Ld}\}+\text{h.c.} \tag{9.39}$$

Note that the sum of eqns (9.38) and (9.39) is invariant under $SU(2)_L$.

As discussed in chaper 8, in the limit of vanishing N_L mass the baryon asymmety generated in free decays will vanish in this $SO(10)$ model since, in this limit, there is an unbroken charge conjugation operator. We need therefore consider only those diagrams which involve an N_L. There are two such diagrams as shown in fig. 9.1. Note that, since the X' and the Y' constitute an $SU(2)_L$ doublet, the diagram of fig. 9.1a can be obtained from that of fig. 9.1b by an $SU(2)_L$ rotation; therefore, the contribution of the two diagrams are identical (this is also verifiable by explicit computation). The weight of each diagram is

$$\frac{1}{2}\Omega=3g^2(Ac_j+e^{-i\pi/4}Bd_j)(A\{\frac{1}{2}a_j+c_j\}+e^{-i\pi/4}Bd_j)^* \tag{9.40}$$

and its imaginary part is

$$\mathrm{Im}\Omega=-3g^2\frac{AB}{\sqrt{2}}a_jd_j \quad . \tag{9.41}$$

For the scalar, μ_4, we have

$$\mathrm{Im}\Omega=-3g^2AB/\varepsilon \quad . \tag{9.42}$$

To compute the average baryon number generated in the decays of the light scalar we must divide the above results by the full (Born) rate for its decay as noted in eqn (4.1). The total rate for the free decay of the χ_j is proportional to (here, $u=m_N/m_j$)

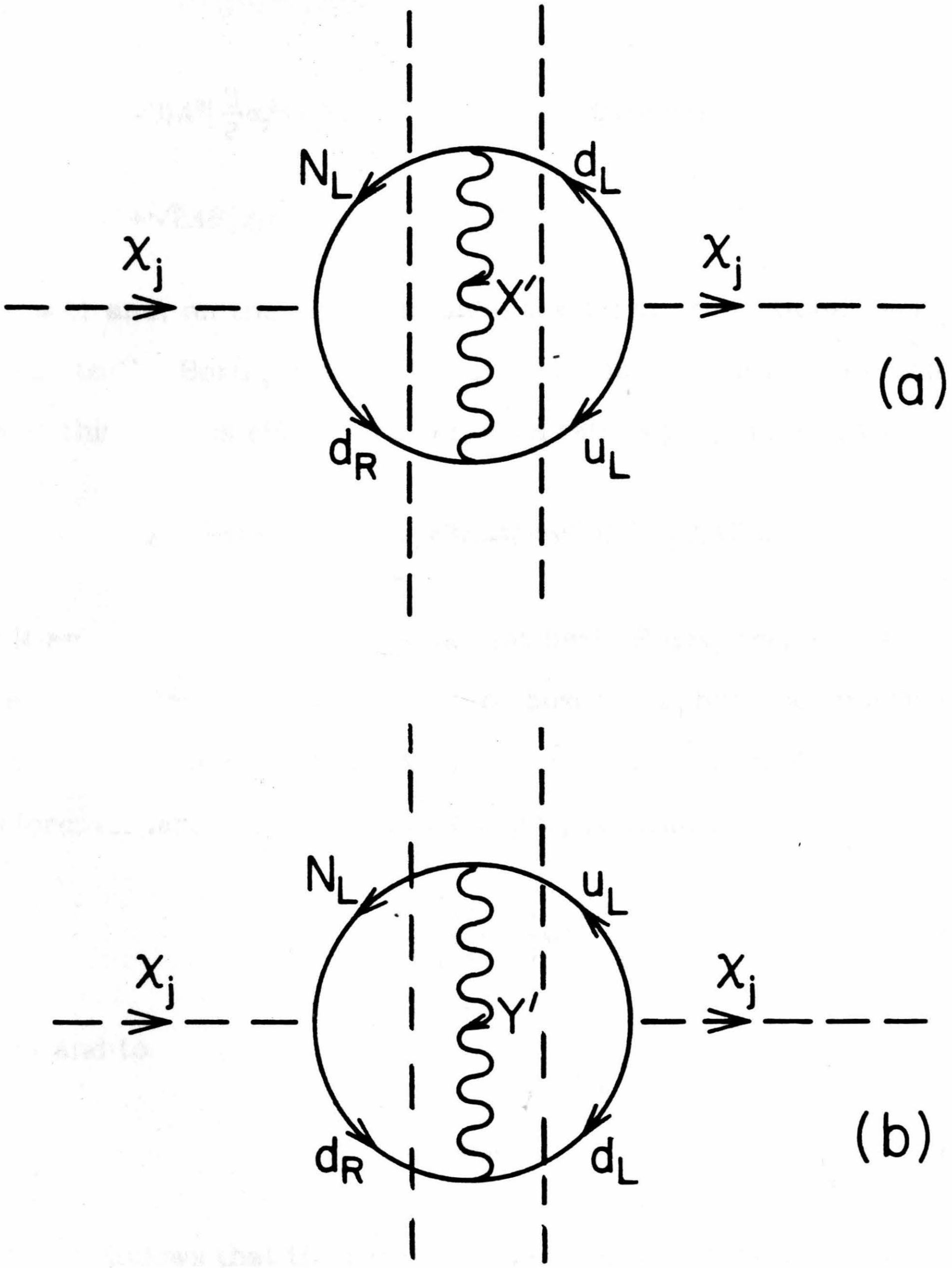

Fig. 9.1: Scalar decay with vector exchange diagrams in the "primordial" $SO(10)$ model that can give rise to a baryon asymmetry (for $j=4$). Only diagrams that involve N_L may contribute.

$$3\{|Ac_j+Be^{-i\pi/4}d_j|^2(1-u^2)+|A(a_j+c_j)+Be^{-i\pi/4}d_j|^2$$

$$+2|A(\frac{1}{2}a_j+c_j)+Be^{-i\pi/4}d_j|^2\}$$

$$=3\{A^2[\frac{3}{2}a_j^2+c_j^2(4-u^2)+4a_jc_j]+B_2d_j^2(4-u^2)$$

$$+\sqrt{2}AB[c_jd_j(4-u^2)+2a_jd_j]\} \qquad (9.43)$$

where $u<1$ and, on the left hand side, the term proportional to $\{1-u^2\}$ is that due to the Born graph involving N_L in the final state (see eqn C-15). For $u>1$ this term is absent and the Born rate is proportional to

$$3\{A^2[\frac{3}{2}a_j^2+3c_j^2+4a_jc_j]+3B_2d_j^2+\sqrt{2}AB[3c_jd_j+2a_jd_j]\} \ . \qquad (9.44)$$

Both A and B are presumably small (at best, $B\sim gm_q/m_W$ and $A\sim gm_N/m_\Phi$ where m_Φ is a typical mass of a vector boson that becomes massive when the transition $SU(4)\otimes SU(2)_L\otimes SU(2)_R \rightarrow SU(3)\otimes SU(2)_L\otimes U(1)_R$ occurs); therefore, for large ε, the Born rate is proportional to

$$3A^2\left\{\frac{19}{12}-\frac{2}{3}u^2\right\} \qquad (9.45a)$$

for $u<1$ and to

$$A^2\frac{11}{4} \qquad (9.45b)$$

for $u>1$. It follows that the ratio of ImΩ to the Born rate is

$$-g^2\frac{1}{\varepsilon}\frac{B}{A}\frac{1}{\left\{\frac{19}{12}-\frac{2}{3}u^2\right\}} \qquad u<1 \qquad (9.46a)$$

$$-\frac{g^2}{\varepsilon}\frac{B}{A}\frac{12}{11} \qquad u>1\ . \qquad (9.46b)$$

In computing the average baryon number produced in the free decays of these scalars we must multiply this result by the baryon number factor -1 and by the difference, $\mathrm{Im}[I_{SV}(v,u)]-\mathrm{Im}[I_{SV}(v,0)]$, where the momentum space weight, $\mathrm{Im}[I_{SV}(v,u)]$ is given in eqn C-13. This gives

$$\Delta B = g^2 \frac{1}{\varepsilon}\frac{B}{A}\frac{1}{8\pi}\frac{1}{\left(\frac{19}{12}-\frac{2}{3}u^2\right)}\left\{\ln\left[\frac{v^2}{1+v^2}\right]-(1-u^2)\ln\left[\frac{v^2}{1+v^2-u^2}\right]\right\} \tag{9.47a}$$

for $u<1$ (where $v=m_X/m_S$), and

$$\frac{1}{8\pi}\frac{g^2}{\varepsilon}\frac{12}{11}\frac{B}{A}\ln\left[\frac{v^2}{1+v^2}\right] \tag{9.47b}$$

for $u>1$. We also know that $M_X \sim g<\psi_{24}>$; thus $v^2 \sim \frac{g^2}{\lambda}\varepsilon$. If ε is large enough so that v is also large, then we can make a Taylor expansion of ΔB in powers of $1/\varepsilon$. Keeping only the lowest order term we have

$$\Delta B \simeq \frac{1}{8\pi}\frac{1}{\varepsilon^2}\frac{B}{A}\frac{\lambda}{\left(\frac{19}{12}-\frac{2}{3}u^2\right)}[1-(1-u^2)^2]$$

$$=\frac{1}{8\pi}\frac{1}{\varepsilon^2}B\sqrt{\lambda}\left(\frac{u^3-2u}{\frac{19}{12}-\frac{2}{3}u^2}\right) \tag{9.48a}$$

for $u<1$ (where, in the last equality, we have used the fact that $u=A/\sqrt{\lambda}$), and

$$\Delta B \simeq \frac{3}{8\pi}B\frac{\sqrt{\lambda}}{\varepsilon^2}\frac{1}{u} \tag{9.48b}$$

for $u>1$. We can make a rough estimate of the maximum ΔB one can expect in this model as follows. Stability of the effective potential requires that, if $\lambda \lesssim g^2$ then $(A, B) \lesssim g$ and, if $\lambda \gtrsim g^2$, then $(A, B) \lesssim \lambda$ [18].

Although A and B are bounded above in these two cases respectively by g and λ, they are not bounded below. Thus, for fixed λ, we can vary u from 0 to $\sim g/\sqrt{\lambda}$ in the first case, and from 0 to $\sqrt{\lambda}$ in the second case. In the first case since, for fixed λ, ΔB is monotonically increasing as a function of u in the region $0<u<1$ and is monotonically decreasing for $u>1$ and, since the maximum permitted value of u is $g/\sqrt{\lambda}>1$, it is clear that ΔB is maximized by choosing $u\sim 1$. The maximum choice for λ under these assumptions is $\lambda \sim g^2$ giving a maximum ΔB of $\Delta B \sim 3\alpha/2\varepsilon^2$. In the second case, for fixed λ, the maximum value of u is $\sqrt{\lambda}$; thus, for $\lambda<1$, the maximum of ΔB is obtained at $u\sim\sqrt{\lambda}$, while for $\lambda>1$ the maximum is obtained at $u\sim 1$. However, for $\lambda>1$ no baryon asymmetry will be generated since then the mass of the μ_4 will be greater than the temperature at which the $SU(4)\otimes SU(2)_L\otimes SU(2)_R \rightarrow SU(3)\otimes SU(2)_L\otimes U(1)_R$ phase transition occurs and so there will be no CP violation in its decays (this is independent of the statement that perturbation theory may not now be valid in the scalar sector and the arguments that we are using here will then probably not be valid). Thus we want $\lambda<1$ (we are of course assuming that $g^2<1$). If we saturate this bound we find that $\Delta B<3/8\pi\varepsilon^2$. If g is very small there can be a substantial difference between the results of these two cases. In practice, however, we have $g^2/4\pi\sim 1/40$ and hence $g^2\sim 1/3$.

The value of ΔB is an upper bound for the value of the baryon number to photon number ratio that can be produced in the context of a grand unified model. In fact, if we ignore the dynamics contained in the Boltzmann transport equations (2.1) and (2.2), then the value of n_B/n_γ is related to ΔB by a statistical factor that is generally $O(N_X/N)$ where N_X is the number of bosons participating in the free decay process and N is the total number of particles with mass less than that of the relevent decaying bosons [12,19]. In a grand unified model this factor has the potential of being quite small. It follows that (for $n_B/n_\gamma \sim 10^{-9}$) we can have a maximum $\varepsilon \sim 10^4$. This in turn gives a minimum value $\langle 126_H \rangle \sim 1^{11}$ if we assume the minimum value $\langle 54_H \rangle \sim 10^{15}$ which is required to keep the proton sufficiently stable.

Thus we have seen, in a rather detailed example, how the breaking of a charge conjugation symmetry may lead to the generation of a baryon excess. The magnitude of this excess depends upon the scale at which the breaking of the charge conjugation symmetry occurs. In the present example the magnitude of this scale is proportional to the mass of the N_L. In the limit $m_N \to 0$, ΔB vanishes.

PART II

10) Mass Matrices in SU(5)

The phenomenological mass relations [39]

$$\frac{m_e}{m_\mu} \simeq \frac{1}{10}\frac{m_d}{m_s} \tag{10.1}$$

$$\frac{m_b}{m_\tau} \simeq 1 \tag{10.2}$$

must be reproduced in any viable grand unified model. Since both sides of (10.1) are renormalized in the same way, it is a relation valid at all scales and, in particular, at the grand unification scale; (10.2) is a relation valid at the grand unification scale. A somewhat weaker set of relations (weaker in the sense that use of the renormalization group may not be a valid procedure) may be obtained from current algebra as

$$\frac{m_e}{m_d} \simeq \frac{1}{3}; \quad \frac{m_\mu}{m_s} \simeq 3; \quad \frac{m_\tau}{m_b} \simeq 1 \tag{10.3}$$

at the grand unification scale. It is the first of the relations in (10.3) that one is most insecure about; nonetheless, these relations are consistent with (10.1) and (10.2).

In an $SU(5)$ model it is rather easy to institute the relations $m_\mu/m_s \simeq 3$ and $m_\tau/m_b \simeq 1$ in a natural way by having the μ family obtain its masses solely through coupling to a 45 of Higgs and having the τ family get its masses through a 5 of Higgs. However, to incorporate the relation (10.1) in a natural way is a trickier business. Nonetheless, it has almost been done with the following choice of Yukawa terms [39] (the Higgs representations are three 5's, 5_H, $5'_H$, $5''_H$ and a 45_H):

$$\{A(\bar{5}_2 \cdot 10_1) + A'(\bar{5}_1 \cdot 10_2) + B(\bar{5}_3 \cdot 10_3)\} \cdot \bar{5}_H + C(\bar{5}_2 \cdot 10_2) \cdot 45_H$$

$$+ \{D(10_1 \cdot 10_2) + E(10_3 \cdot 10_3)\} \cdot 5'_H + F(10_2 \cdot 10_3) \cdot 5''_H \,. \tag{10.4}$$

The naturalness of these terms is maintained by several $U(1)$'s which must be broken softly (i.e., by terms of dimension ≤ 3) in the Higgs potential. The latter fact then allows for there to be calculable corrections to the Yukawa terms (10.4) and hence the possible inclusion of terms that do not have the required form. The predicted mass relations may thereby be altered.

The mass matrices obtained from (10.4) are schematically

$$M_{-1/3}=\begin{pmatrix} 0 & A' & 0 \\ A & C & 0 \\ 0 & 0 & B \end{pmatrix} \tag{10.5}$$

$$M_{lepton}=\begin{pmatrix} 0 & A' & 0 \\ A & -3C & 0 \\ 0 & 0 & B \end{pmatrix} \tag{10.6}$$

$$M_{2/3}=\begin{pmatrix} 0 & D & 0 \\ D & 0 & F \\ 0 & F & E \end{pmatrix} \tag{10.7}$$

where, for brevity, we have absorbed vacuum expectation values into the definitions of the couplings and we have omitted CP-violating phases. These mass matrices yield the desired mass relations under the unnatural assumption

$$A \simeq A' \tag{10.8}$$

and the "fermion-mass-hierarchy" assumption

$$B \gg C >> A \,. \tag{10.9}$$

One may also obtain a prediction for the magnitude of the Cabibbo angle:

$$\tan(\vartheta_c) \simeq \sqrt{m_d / m_s} \, . \tag{10.10}$$

The details of such calculations are discussed below in the context of $SO(10)$.

11) Mass Matrices in A Viable SO(10) Model

To construct a set of $SO(10)$ Yukawa terms that behave like those in (10.4) when restricted to the $SU(5)$ level we proceed as follows [35,36]. To reproduce the $(\bar{5}_2 \cdot 10_2) \cdot 45_H$ term in (10.4) we must use a 126_2 coupling to $(16_2 \cdot 16_2)_S$. The terms $[A\bar{5}_2 \cdot 10_1 + A'\bar{5}_1 \cdot 10_2 + B\bar{5}_3 \cdot 10_3] \cdot \bar{5}_H + [D10_1 \cdot 10_2 + E10_3 \cdot 10_3] \cdot 5'_H$ can be easily obtained by coupling $16_1 \cdot 16_2$ and $16_3 \cdot 16_3$ to the same complex 10_H $(=10_1 + i10_2)$ of Higgs. The relevant $SU(5)$ decompositions are

$$10_H = \bar{5} + \tilde{5} \tag{11.1}$$

$$126_H = 1 + \bar{5} + 10 + \overline{15} + 45 + 50 \tag{11.2}$$

and

$$16 = 1 + \bar{5} + 10 \ . \tag{11.3}$$

(Note that we have emphasized that we have given the decomposition for a *complex* 10_H in (11.1) by having $\tilde{5} \neq 5$.) To assure the coupling $(10_2 \cdot 10_3) \cdot 5''$ as in (2.4) without a $(\bar{5}_2 \cdot 10_3) \cdot \bar{5}_H$ or a $(\bar{5}_3 \cdot 10_2) \cdot \bar{5}_H$ term, we couple $16_2 \cdot 16_3$ to a 126_3:

$$d(16_2 \cdot 16_3) \cdot \overline{126}_3 \ . \tag{11.4}$$

The Yukawa terms of this model are, thus far,

$$(a16_1 \cdot 16_2 + b16_3 \cdot 16_3) \cdot 10_H + c(16_2 \cdot 16_2)\overline{126}_2 + d(16_2 \cdot 16_3) \cdot \overline{126}_3 \ . \tag{11.5}$$

If we now assume that, for a range of parameters in the Higgs potential, the 126_2 has its vacuum expectation value purely along its $SU(5)$ 45 direction and that the 126_3 has its vacuum expectation value along its $SU(5)$ $\bar{5}$ direction,

$$\left.\begin{matrix} <126_2> \sim 45_{H2} \\ <126_3> \sim \bar{5}_{H3} \end{matrix}\right\}, \tag{11.6}$$

then the predictions of (10.4) for charged fermions are reproduced. This would not have been possible without choosing 126_3 different from 126_2.

When the Yukawa terms (11.5) are expanded with the decompositions (11.1), (11.2) and (11.3) we obtain the following terms which contribute to the neutral fermion mass matrix when the Higgs fields get vacuum expectation values:

$$[a(1_1 \cdot \bar{5}_2 + 1_2 \cdot \bar{5}_1) + b\, 1_3 \cdot \bar{5}_3] \cdot \tilde{5}_H \ . \tag{11.7}$$

If we view the CP conjugate of the $SU(5)$ singlet fermion as a right handed partner for the neutrino, then we see that the terms (11.7) give Dirac masses to the neutrinos of the same order of magnitude as those of the charge 2/3 quarks. If, however, in addition to the Dirac mass, m, mentioned above, the neutral singlet fermion has a Majorana mass, M, then the neutrino mass matrix has the following form in the (ν_L, N_L) basis* [29,27]:

$$\begin{bmatrix} 0 & m \\ m & M \end{bmatrix} . \tag{11.8}$$

If $M \gg m$, then the eigenvalues of this matrix are approximately M and $m(m/M)$. Thus, one eigenvalue is naturally very large relative to the charge 2/3 quark mass and the other is very small. It is the latter eigenstate which we identify with the garden variety neutrino. Such an eigenstate would be primarily ν_L with a small amplitude (proportional to m^2/M) for helicity flip into ν_R $(=(N_L)^{CP})$.

*Also see chapter 8.

The mechanism described above is effected in this $SO(10)$ model by coupling the relevant combinations of fermion multiplets (in the case of eqn (11.5), the combination $e\,16_1 \cdot 16_2 + f\,16_3 \cdot 16_3$) to a 126_1 which obtains a superlarge vacuum expectation value along its $SU(5)$ singlet direction. Further, to preserve the predictions of eqn (11.5) along with the assumption (11.6), we assume that the 126_1 has vacuum expectation values only along its $SU(5)$ singlet and $\bar{5}$ directions*:

$$<126_1> \sim 1_{H1} + \bar{5}_{H1} \,. \tag{11.9}$$

The complete Yukawa couplings are now

$$(a\,16_1 \cdot 16_2 + b\,16_3 \cdot 16_3) \cdot 10_H + c\,(16_2 \cdot 16_2) \cdot \overline{126}_2$$
$$+ d\,(16_2 \cdot 16_3) \cdot \overline{126}_3 + (e\,16_1 \cdot 16_2 + f\,16_3 \cdot 16_3) \cdot \overline{126}_1 \,. \tag{11.10}$$

These couplings are natural, the naturalness being maintained by two global $U(1)$ symmetries which will be explicitly broken down to discrete symmetries in the Higgs potential. We call the charges associated with these $U(1)$ symmetries X and Y. Their values for the representations present in this model are summarized in the following table:

	16_1	16_2	16_3	10	126_1	126_2	126_3
X	-3/2	1/2	-1/2	1	-1	1	0
Y	1	-1	0	0	0	-2	-1

*If $<\bar{5}_{H1}>=0$, a t-quark mass relation follows that is phenomenologically unacceptable. This vacuum expectation value is also required for naturalness reasons [33,36].

When the Higgs acquire vacuum expectation values, these phase symmetries will be spontaneously broken. Since they are not gauged, massless Goldstone-Nambu bosons will result. To avoid their presence we have to break X and Y in such a way as to preserve the naturalness of the Yukawa couplings. Remarkably enough this can be done through the following property of the 126 representation: the fourfold fully symmetrized product $(126^4)_S$ contains one $SO(10)$ singlet. Thus, we require that the Higgs potential contain terms like

$$\lambda_1(126_1)^4+\lambda_2(126_2)^4+\ldots+\text{h.c.} \tag{11.11}$$

The first term breaks X to a discrete symmetry mod 4; the second one breaks Y to another discrete symmetry mod 8. These two discrete symmetries suffice to maintain the naturalness of the Yukawa couplings while avoiding the problem of massless bosons. The remaining Higgs self-couplings are selected so as to honor the remaining discrete symmetries.

The terms, at the $SU(5)$ level, in eqn (11.10) which are relevant for the computation of the fermion mass matrices are:

$$(a\,10_1\cdot 10_2+b\,10_3\cdot 10_3)\cdot\tilde{5}_H+[e\,10_1\cdot 10_2+f\,10_3\cdot 10_3]\cdot 5_{H1}$$

$$+[\mathbf{a}(10_1\cdot\bar{5}_2+10_2\cdot\bar{5}_1)+\mathbf{b}\,10_3\cdot\bar{5}_3]\cdot\bar{5}_H+\mathbf{d}(10_2\cdot 10_3)\cdot 5_{H3}+c\,(\bar{5}_2\cdot 10_2)\cdot 45_{H2} \tag{11.12}$$

for charged fermions and

$$[\mathbf{a}(1_1\cdot\bar{5}_2+1_2\cdot\bar{5}_1)+\mathbf{b}\,1_3\cdot\bar{5}_3]\cdot\tilde{5}_H+(e\,1_1\cdot 1_2+\mathbf{f}\,1_3\cdot 1_3)\cdot 1_{H1} \tag{11.13}$$

for neutral fermions.

We will discuss here only the charged fermion mass matrices. (The neutral fermion mass matrices are discussed in [33] and [36].) The first thing worth noting is that the unnatural relation (10.8) is natural in this $SO(10)$ scheme. This is obviated by comparison of (10.4) with (11.12). If we now write

$$\left.\begin{array}{l} <5_H>=re^{i\vartheta} \\ <\tilde{5}_H>=pe^{i\kappa} \\ <5_{H1}>=te^{i\eta} \\ <5_{H3}>=qe^{i\mu} \\ <45_{H2}>=se^{i\chi} \end{array}\right\} \tag{11.14}$$

and define

$$ar \equiv R,\ br \equiv T,$$

$$dq \equiv Q,\ cs \equiv S,$$

$$ape^{i\kappa}+ete^{i\eta} \equiv Pe^{i\delta},$$

$$bpe^{i\kappa}+fte^{i\eta} \equiv Ve^{i\zeta}$$

then the charged fermion mass matricies are:

charged $-1/3$ quarks:

$$M_{-1/3}=\begin{bmatrix} 0 & Re^{i\vartheta} & 0 \\ Re^{i\vartheta} & Se^{i\chi} & 0 \\ 0 & 0 & Te^{i\vartheta} \end{bmatrix} \tag{11.15}$$

charged $2/3$ quarks:

$$M_{2/3}=\begin{bmatrix} 0 & Pe^{i\delta} & 0 \\ Pe^{i\delta} & 0 & Qe^{i\mu} \\ 0 & Qe^{i\mu} & Ve^{i\zeta} \end{bmatrix} \tag{11.16}$$

charged leptons:

$$M_l=\begin{bmatrix} 0 & Re^{i\vartheta} & 0 \\ Re^{i\vartheta} & -3Se^{i\chi} & 0 \\ 0 & 0 & Te^{i\vartheta} \end{bmatrix}. \tag{11.17}$$

By a suitable redefinition of the fermion fields (discussed below) the phases can be removed from these matrices. (The phases certainly reappear, for example, in the charge current sector of the theory.) For the mass matrices with the phases removed we write respectively $\widetilde{M}_{-1/3}$, $\widetilde{M}_{2/3}$ and $\widetilde{M}_l$. The eigenvalues of both $\widetilde{M}_{-1/3}$ and $\widetilde{M}_l$ are easy to compute since they are both block diagonal. The eigenvalues of $\widetilde{M}_{2/3}$ are easily computed in the limit $V \gg Q \gg P$, while assuming one eigenvalue to be $O(V)$ and the remaining two to be $\ll V$. In this limit and in the limit $T \gg S \gg R$ for $\widetilde{M}_{-1/3}$ and $\widetilde{M}_l$ we may identify the fermion masses as

$$\left.\begin{array}{r} m_d \simeq \dfrac{R^2}{S} \\ m_s \simeq S \\ m_b \simeq T \end{array}\right\} , \tag{11.18}$$

$$\left.\begin{array}{r} m_e \simeq \dfrac{R^2}{3S} \\ m_\mu \simeq 3S \\ m_\tau \simeq T \end{array}\right\} \tag{11.19}$$

and

$$\left.\begin{array}{r} m_u \simeq \dfrac{1}{2}\left\{\left[\dfrac{Q^4}{V^2}+4P^2\right]^{1/2}-\dfrac{Q^2}{V}\right\} \\ m_c \simeq \dfrac{1}{2}\left\{\left[\dfrac{Q^4}{V^2}+4P^2\right]^{1/2}+\dfrac{Q^2}{V}\right\} \\ m_t \simeq V \end{array}\right\} . \tag{11.20}$$

The mass relations (10.1), (10.2) and (10.3) follow from (11.18), (11.19) and (11.20). These expressions can be used to solve for the parameters that appear in $\widetilde{M}_{-1/3}$, $\widetilde{M}_{2/3}$ and $\widetilde{M}_l$ in terms of the fermion masses.

$\widetilde{M}_{-1/3}$, $\widetilde{M}_{2/3}$ and $\widetilde{M}_l$ are real and symmetric and are diagonalized by orthogonal matrices which we denote respectively by $U_{-1/3}$, $U_{2/3}$ and U_l:

$$U_{-1/3}\widetilde{M}_{-1/3}U^T_{-1/3} = \begin{pmatrix} -m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \tag{11.21}$$

$$U_{2/3}\widetilde{M}_{2/3}U^T_{2/3} = \begin{pmatrix} -m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \tag{11.22}$$

$$U_l\widetilde{M}_l U_l^T = \begin{pmatrix} m_e & 0 & 0 \\ 0 & -m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{11.23}$$

$U_{-1/3}$ is easily computed to be

$$U_{-1/3} \simeq \begin{pmatrix} 1 & \left(\frac{m_d}{m_s}\right)^{1/2} & 0 \\ -\left(\frac{m_d}{m_s}\right) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11.24}$$

Similarly for U_l:

$$U_l \simeq \begin{pmatrix} 1 & -\frac{1}{3}\left(\frac{m_d}{m_s}\right)^{1/2} & 0 \\ \frac{1}{3}\left(\frac{m_d}{m_s}\right)^{1/2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11.25}$$

$U_{2/3}$ is somewhat messier but nonetheless straightforward:

$$U_{2/3} \simeq \begin{pmatrix} 1 & -\left(\frac{m_u}{m_c}\right)^{1/2} & \frac{m_c}{m_t}\left(\frac{m_u}{m_t}\right)^{1/2} \\ \left(\frac{m_u}{m_c}\right)^{1/2} & 1 & \left(\frac{m_c}{m_t}\right)^{1/2} \\ -\left(\frac{m_u}{m_t}\right)^{1/2} & -\left(\frac{m_c}{m_t}\right)^{1/2} & 1 \end{pmatrix} . \tag{11.26}$$

If we write the column vector, in family (weak eigenstate) space, of charge −1/3 left-handed quarks as $L_{-1/3}$ and that for the right-handed components as $R_{-1/3}$, the mass term in the Lagrangian is

$$L^{\dagger}_{-1/3} M_{-1/3} R_{-1/3} \tag{11.27}$$

before any phase redefinitions of the fields have been made. We can redefine each of the fermion fields by an arbitrary phase

$$\left. \begin{aligned} L_{-1/3} &= \mathbf{L}_{-1/3} \widetilde{L}_{-1/3} \\ R_{-1/3} &= \mathbf{R}_{-1/3} \widetilde{R}_{-1/3} \end{aligned} \right\} \tag{11.28}$$

where $\mathbf{L}_{-1/3}$ and $\mathbf{R}_{-1/3}$ are diagonal matrices of the form

$$\mathbf{L}_{-1/3} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix} \tag{11.29}$$

and

$$\mathbf{R}_{-1/3} = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} . \tag{11.30}$$

We want

$$\begin{aligned} L^{\dagger}_{-1/3} M_{-1/3} R_{-1/3} &= \widetilde{L}^{\dagger}_{-1/3} \mathbf{L}^{*}_{-1/3} M_{-1/3} \mathbf{R}_{-1/3} \widetilde{R}_{-1/3} \\ &= \widetilde{L}^{\dagger}_{-1/3} \widetilde{M}_{-1/3} \widetilde{R}_{-1/3} , \end{aligned} \tag{11.31}$$

or

$$\widetilde{M}_{-1/3}=\mathbf{L}^{*}_{-1/3}M_{-1/3}\mathbf{R}_{-1/3}\,. \tag{11.32}$$

Note that if this is true then it follows that

$$\widetilde{M}_{l}=\mathbf{L}^{*}_{-1/3}M_{l}\mathbf{R}_{-1/3} \tag{11.33}$$

although one might wish to choose $\mathbf{L}_l$ and $\mathbf{R}_l$ different from $\mathbf{L}_{-1/3}$ and $\mathbf{R}_{-1/3}$ for reasons of convenience. Eqn (11.32) results in the following:

$$\left.\begin{array}{l}\vartheta+\beta_1-\alpha_2=0\\ \vartheta+\beta_2-\alpha_1=0\\ \chi+\beta_2-\alpha_2=0\\ \vartheta+\beta_3-\alpha_3=0\end{array}\right\}. \tag{11.34}$$

Similarly, if we write

$$\mathbf{L}_{2/3}=\begin{pmatrix} e^{i\gamma_1} & 0 & 0\\ 0 & e^{i\gamma_2} & 0\\ 0 & 0 & e^{i\gamma_3}\end{pmatrix} \tag{11.35}$$

and

$$\mathbf{R}_{2/3}=\begin{pmatrix} e^{i\varphi_1} & 0 & 0\\ 0 & e^{i\varphi_2} & 0\\ 0 & 0 & e^{i\varphi_3}\end{pmatrix}, \tag{11.36}$$

then from

$$\widetilde{M}_{2/3}=\mathbf{L}^{*}_{2/3}M_{2/3}\mathbf{R}_{2/3} \tag{11.37}$$

we get

$$\left.\begin{array}{l}\delta+\varphi_1-\gamma_2=0\\ \delta+\varphi_2-\gamma_1=0\\ \mu+\varphi_2-\gamma_3=0\\ \mu+\varphi_3-\gamma_2=0\\ \zeta+\varphi_3-\gamma_3=0\end{array}\right\}. \tag{11.38}$$

Of the phases α_1, α_2, α_3, β_1, β_2 and β_3 only five are independent insofar as effecting changes on $M_{-1/3}$ is concerned (in particular the transformation $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=0$ is sterile). Thus, we choose $\alpha_1=0$; and similarly for charge 2/3 quarks we choose $\gamma_1=0$. Thus, from eqns (11.34) we are left with one degree of freedom and from eqns (11.38) we are left with no degrees of freedom.

The left-handed charged current coupling to the usual $(SU(2)_L)$ W_L^+ boson is

$$j_\mu^- = L_{2/3}^\dagger \sigma_\mu L_{-1/3} \tag{11.39}$$

or, from eqn (11.27) and its analogue for charge 2/3 quarks,

$$j_\mu^- = \tilde{L}_{2/3}^\dagger \sigma_\mu \mathbf{L}_{2/3}^* \mathbf{L}_{-1/3} L_{-1/3}\,. \tag{11.40}$$

We can write $\tilde{L}_{-1/3}$ and $\tilde{L}_{2/3}$ in terms of the mass eigenstates:

$$\tilde{L}_{-1/3} = U_{-1/3} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix},$$

$$\tag{11.41}$$

$$\tilde{L}_{2/3} = U_{2/3} \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix}.$$

Thus, in the mass eigenstate basis, we have

$$j_\mu^- = (u_L^\dagger\ c_L^\dagger\ t_L^\dagger)\sigma_\mu U_{2/3}^T \mathbf{L}_{2/3}^* \mathbf{L}_{-1/3} U_{-1/3} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}. \tag{11.42}$$

The matrix that appears in this expression to be acting on the left-handed charge -1/3 mass eigenstates,

$$U_{2/3}^{T}\mathbf{L}_{2/3}^{*}\mathbf{L}_{-1/3}U_{-1/3}\,, \tag{11.43}$$

is explicitly

$$\begin{pmatrix} 1-e^{ia}\left(\frac{m_u m_d}{m_s m_c}\right)^{1/2} & \left(\frac{m_d}{m_s}\right)^{1/2}+e^{ia}\left(\frac{m_u}{m_c}\right)^{1/2} & -e^{ib}\left(\frac{m_u}{m_t}\right)^{1/2} \\ -\left[\left(\frac{m_u}{m_c}\right)^{1/2}+e^{ia}\left(\frac{m_d}{m_s}\right)^{1/2}\right] & -\left(\frac{m_u m_d}{m_c m_s}\right)^{1/2}+e^{ia} & -e^{ib}\left(\frac{m_c}{m_t}\right)^{1/2} \\ \frac{m_c}{m_t}\left(\frac{m_d}{m_t}\right)^{1/2}-e^{ia}\left(\frac{m_d m_c}{m_s m_t}\right)^{1/2} & \frac{m_c}{m_t}\left(\frac{m_d m_u}{m_t m_s}\right)^{1/2}+e^{ia}\left(\frac{m_c}{m_t}\right)^{1/2} & e^{ib} \end{pmatrix}, \tag{11.44}$$

where

$$a=\alpha_2-\gamma_2$$

and (11.45)

$$b=\alpha_3-\gamma_3\,.$$

The quantity a is completely determined by eqns (11.34) and eqns (11.38) to be

$$a=\chi-\vartheta-2\mu+\delta+\zeta\,. \tag{11.46}$$

The remaining unconstrained quantity in eqns (11.34) (i.e., β_3) can be used to set

$$b=0\,. \tag{11.47}$$

Thus the matrix (11.44) has a CP-violating phase. Eqn (11.44) is not in standard Kobayashi-Maskawa form [40]; however, it can be rendered so, if necessary, by a phase redefinition of the left-handed mass eigenstate fields.

12) The Beta Function in This SO(10) Model

In general, the symmetry breaking in a grand unified model based upon a gauge group G will proceed in several steps before arriving at the low energy model $G_{fl}\otimes SU(3)$ where G_{fl} is the electroweak gauge group and is at least as large as $SU(2)_L\otimes U(1)$. After the ith step there will be a remaining unbroken gauge group G_i which is a valid symmetry up to a scale $O(2M)$ where M is the mass of a gauge vector boson which is in G_{i-1} and not in G_i (note that we are writing the sequence of symmetry breaking as $G\to G_1\to G_2\to\cdots\to G_i\to\cdots\to[G_{fl}\otimes SU(3)]$).

In the $SU(5)$ model there is only one possible pattern of symmetry breaking compatible with low energy phenomenology: $SU(5)\to SU(3)\otimes SU(2)_L\otimes U(1)_Y$. However, as soon as one considers larger groups, the possible patterns of symmetry breaking compatible with the world as we know it become more numerous. Such models offer some hope of partially filling the "desert" region between 300 GeV and 10^{15} GeV which is present in the simplest grand unified models based on $SU(5)$.

In a generic $SO(10)$ model there are a number of possible symmetry breaking patterns as illustrated in fig. 12.1 [38,41]. Of these we consider those with the intermediate scale gauge group $SU(4)\otimes SU(2)_L\otimes SU(2)_R$ to be of particular interest because of the presence of $SU(4)$ as a generalized (Pati-Salam) color group with lepton number as the fourth color [42]. In this chapter we discuss the running of the various Yang-Mills couplings for the symmetry breaking scheme

$$SO(10) \underset{54_H}{\to} SU(4)\otimes SU(2)_L\otimes SU(2)_R \underset{126_1}{\to} SU(3)\otimes SU(2)_L\otimes U(1)_Y$$

of the model described in the preceding chapter. We compute the renormalization of the $m_{-1/3}/m_l$ mass relations down from the unification

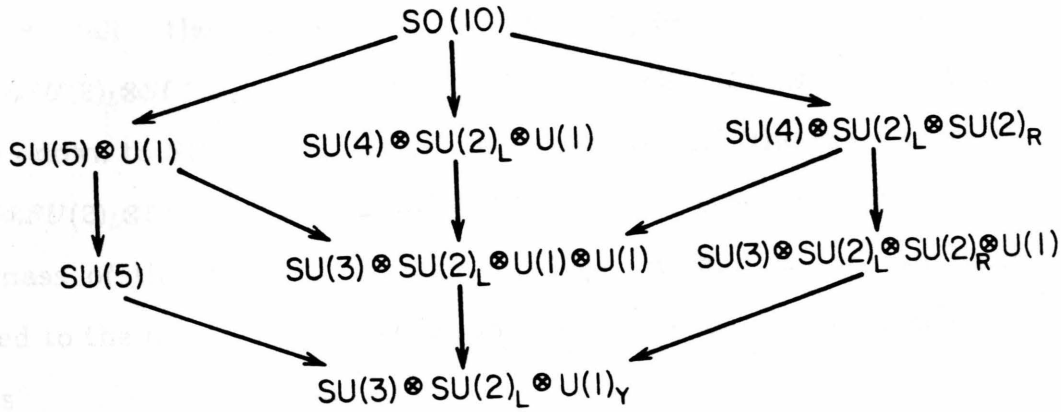

Fig. 12.1: Possible paths of symmetry breaking for the group $SO(10)$. By suitable choice of Higgs representations any given sequence of symmetry breaking may occur that is consistent with the flow of the lines in this figure.

scale. We consider the effects of running the couplings on the value of $\sin^2\vartheta_W$ and on the lifetime of the proton. Furthermore, since this model is only temporarily free, we compute the position of the Landau "singularity".

We call the scale at which $SO(10)$ breaks down to $SU(4)\otimes SU(2)_L\otimes SU(2)_R$, m_2; the scale at which $SU(4)\otimes SU(2)_L\otimes SU(2)_R$ breaks down to $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ is called m_1; and, the scale at which $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ breaks to $SU(3)\otimes U(1)_{EM}$ is called m_0. m_0 is related to the mass of the W boson through $m_0 \simeq 2\,M_W$. m_1 and m_2 are similarly related to the masses of the vector bosons that become massive at those scales.

With forethought we choose to normalize the generators of $SO(10)$ to 2 in the 16 representation:

$$Tr[T(16)^2] = 2 \ . \tag{12.1}$$

This will give us the expression for the electric charge operator,

$$Q \doteq T_{3L}^{(10)} + \left(\frac{5}{3}\right)^{1/2} Y \ , \tag{12.2}$$

where $T_{3L}^{(10)}$ is the diagonal $SU(2)_L$ generator (the superscript indicates that it is normalized as embedded in $SO(10)$) and Y is the $U(1)_Y$ generator as embedded in $SO(10)$, so as to conform with the traditional $SU(5)$ expression. This latter point is assured if we normalize the generators of any $SU(n)$ subgroup of $SO(10)$ to 1/2 in the n representation:

$$Tr[T^{(n)}(n)^2] = \frac{1}{2} \ . \tag{12.3}$$

By tracking the three low energy couplings up to the scale m_2 at which $SO(10)$ first breaks, we will get three expressions that depend on the parameters m_1, m_2 and $\alpha_{10}(m_2)$ (α_{10} is the $SO(10)$ coupling squared divided by 4π). These can be determined by using as inputs the values of α_3, $\sin^2\vartheta_W$ and α_{EM} at m_υ. The qualitative behavior of the couplings for a simple assumption about scalar thresholds is shown in fig. 12.2. We work with beta functions to lowest order in g and treat all mass thresholds in the theta function approximation.

We write g_{10} for the $SO(10)$ coupling ($\alpha_{10}=g_{10}^2/4\pi$); similarly we write g_4 for $SU(4)$, g_{2L} for $SU(2)_L$, g_{2R} for $SU(2)_R$, g_Y for $U(1)_Y$ and e for $U(1)_{EM}$. We use a similar notation for the quantities b that appear in the respective beta functions.

In general a Yang-Mills coupling g runs according to ($\mu<m$) [43]

$$g(\mu)^{-2}=g(m)^{-2}+2b\ln\left(\frac{m}{\mu}\right), \tag{12.4}$$

assuming that there are no mass thresholds between μ and m and that the coupling remains perturbative in that region. The only difficulty in running, say, what starts as the $SU(3)$ coupling up to the unification scale is in determining the boundary conditions applicable as one goes from one reigon to an adjacent one. With the normalizations in eqns (12.1) and (12.3) we have

$$g_3(m_1)=g_4(m_1)\ ;\ g_4(m_2)=g_{10}(m_2)\ ;$$

$$g_{2L}(m_2)=g_{10}(m_2)\ ;\ g_{2R}(m_2)=g_{10}(m_2)\ ;$$

$$g_Y(m_1)^{-2}=\sin^2\varphi\ g_R(m_1)^{-2}+\cos^2\varphi\ g_4(m_1)^{-2}\ ; \tag{12.5}$$

$$e(m_\upsilon)^{-2}=f^2[\sin^2\eta\ g_Y(m_\upsilon)^{-2}+\cos^2\eta\ g_{2L}(m_\upsilon)^{-2}]\ ;$$

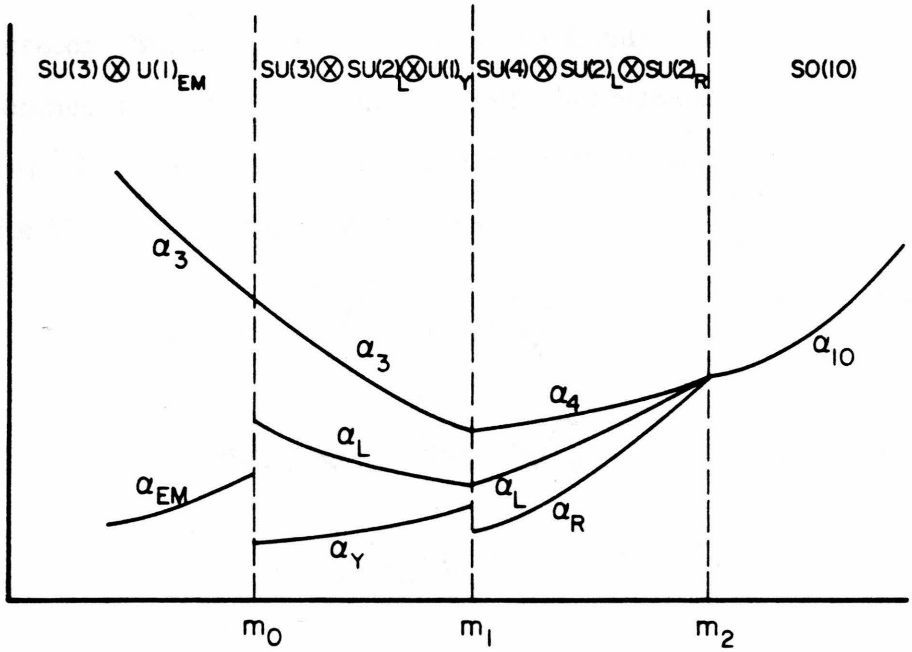

Fig 12.2: The scaling behavior of the couplings in this $SO(10)$ model.

and g_{2L} is continuous across the threshold at m_1. In eqn (12.5), φ is a mixing angle which specifies which linear combination of the $SU(2)_R$ diagonal generator T_{3R} and the diagonal generator T_{15} in $SU(4)$ becomes the $U(1)_Y$ generator. Similarly, η specifies which linear combination of T_{3L} and Y becomes Q, and f is a constant which normalizes the electron charge to -1. The conditions that Y=0 for the $SU(5)$ singlet fermion and that Q=0 for the $SU(2)_L$ doublet neutrino give

$$\sin\varphi = \sqrt{\frac{3}{5}};\ \cos\varphi = \sqrt{\frac{2}{5}};$$

$$\sin\eta = \sqrt{\frac{5}{8}};\ \cos\eta = \sqrt{\frac{3}{8}}; \qquad (12.6)$$

$$f = \sqrt{\frac{8}{3}}\ .$$

The expression for, say, $g_{\bar{Y}}^{-2}$ is obtained from the following argument (which can easily be generalized to cases more complicated than the linear combination of $U(1)$'s that we review here [9]). Say that we have vector fields A and B coupling respectively with generators X and Y and coupling constants g and g' :

$$g\ AX + g'BY\ .$$

Then, if symmetry breaking leaves the combination $aX+bY$ unbroken (where $a^2+b^2=1$), we have

$$gAX+g'BY=\tilde{g}C(aX+bY)+\mathbf{g}D(cX+dY)$$

where D is the vector which couples through the broken generator $cX+dY$ (this generator is not necessarily orthogonal to $aX+bY$), $\tilde{g}$ is the coupling constant for the vector field C and $\mathbf{g}$ is that for the vector D. This gives

$$gA=\tilde{g}aC+gcD$$

and

$$g'B=\tilde{g}bC+gdD$$

The orthonormality of A and B then gives

$$g^2=\tilde{g}^2a^2+g^2c^2,$$

$$g'^2=\tilde{g}^2b^2+g^2d^2$$

and

$$0=\tilde{g}^2ab+g^2cd,$$

from which follows

$$\tilde{g}^{-2}=a^2g^{-2}+b^2g'^{-2}.$$

It is from this that the last two boundary conditions in eqn (12.5) were obtained. (If the normalization conditions (12.1) or (12.3) are changed the conditions in eqn (12.5) change accordingly. For example, if we choose $Tr[T(16)^2]=1/2$ then we get $g_4(m_2)^2=g_{10}(m_2)^2/4$.)

Using eqns (12.4), (12.5) and the definition of the Weinberg angle [8],

$$W\equiv\frac{1}{\sin^2\vartheta_W}=g_{2L}(m_o)^2\left\{\frac{C^2}{g_{2L}(m_o)^2}+\frac{1}{g_Y(m_o)^2}\right\}, \qquad (12.7)$$

where $C^2=5/3$ (again, the specific form of this equation depends upon the normalization conditions), we get the following three relations:

$$8\pi\{\tau+b_3y+(b_4-b_3)z\} = \alpha_3(m_o)^{-1}, \quad (12.8)$$

$$8\pi f^2\{\tau+(b_{2L}\cos^2\eta+b_Y\sin^2\eta)y$$
$$+[(b'_{2L}-b_{2L})\cos^2\eta+(b_{2R}\sin^2\varphi+b_4\cos^2\varphi-b_Y)\sin^2\eta]z\} = \alpha_{EM}(m_o)^{-1}, \quad (12.9)$$

and

$$(1+C^2-W)\tau+[b_{2L}(1-W)+C^2b_Y]y$$
$$+[(b'_{2L}-b_{2L})(1-W)+C^2(b_2R\sin^2\varphi+b_4\cos^2\varphi-b_Y)]z = 0, \quad (12.10)$$

where

$$\tau \equiv [8\pi\alpha_{10}(m_2)]^{-1}$$

$$y \equiv \ln\left[\frac{m_2}{m_o}\right] \quad (12.11)$$

and

$$z \equiv \ln\left[\frac{m_2}{m_1}\right].$$

b_{2L} is the $SU(2)_L$ value for b in (12.4) between m_o and m_1 and b'_{2L} is the value between m_1 and m_2.

In calculating the various b's we use the expression obtained from the one-loop beta function [43],

$$b = \frac{-1}{16\pi^2}\frac{1}{r}\{\frac{11}{3}I_2(\mathrm{Adj})-\frac{2}{3}\sum_f I_2(f)-\frac{1}{6}\sum_s I_2(s)\}, \quad (12.12)$$

where r = rank of the group, and the second index of the representation r, $I_2(r)$ is given by

$$\frac{I_2(r)}{r} = Tr[T(r)^2]. \quad (12.13)$$

(Tables of I_2 for the simple compact Lie groups are given in [44].) The summations in eqn (12.12) are over all relevant left-handed fermion representations f and real scalar representations s; complex scalar representations are counted twice since each complex scalar field contains two degrees of freedom; this comment also applies to doubled real or pseudoreal representations. Adj stands for the (vector) adjoint representation.

To consider the scalar and fermion thresholds it is convenient to consider the $SU(4)\otimes SU(2)_L\otimes SU(2)_R$ decomposition of the 10, 16, 54 and 126:

$$10=(6,1,1)+(1,2,2) \tag{12.14}$$

$$16=(4,2,1)+(\bar{4},1,2) \tag{12.14a}$$

$$54=(1,1,1)+(20,1,1)+(6,2,2)+(1,3,3) \tag{12.15}$$

$$126=(6,1,1)+(10,3,1)+(\overline{10},1,3)+(15,2,2)\ . \tag{12.16}$$

In order to decide at which of the scales m_0, m_1 or m_2 a given Higgs obtains a mass, we adopt the following ansatz: a scalar multiplet gets the largest possible mass consistent with its vacuum expectation value and the symmetry present at that scale. Thus, a multiplet which gets no vacuum expectation value (such as the (6,1,1) in the complex 10) will have a mass of $O(m_2)$. This assignment assures that all baryon number violating scalars will have a mass of $O(m_2)$. Certainly other assignments may be made since the only necessary constraints are that the $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ theory be consistent with phenomenology at accessible energies and that the proton not decay too fast. Our choice is the one with the fewest low mass scalars.

As mentioned above, in the complex 10 of Higgs the (6,1,1) has a mass $\sim m_2$; the (1,2,2) will have a mass $\sim m_0$. Since, by assumption, <54>$\sim m_2$, we put all of the masses of the (real) 54 $\sim m_2$. The representations 126_2 and 126_3 have vacuum expectation values $O(m_0)$ (in the 126_2 case it is along the $SU(5)$ $\bar{5}$ direction and for the 126_3 it is along the 45 direction). In each case the vacuum expectation value is contained in the (15, 2, 2). The $SU(3)$ decomposition of the 15 of $SU(4)$, $15=1^c+3^c+\bar{3}^c+8^c$, then shows that we will have a mass for the (1^c, 2, 2) of $O(m_0)$ and masses for the remaining particles in (15, 2, 2) $O(m_1)$. The other particles in 126_2 and 126_3 may all be given masses $O(m_2)$. For the 126_1 we have the addition of a vacuum expectation value along the $SU(5)$ singlet direction $O(m_1)$. The $SU(5)$ singlet is contained in the ($\overline{10}$, 1, 3) which will therefore have a mass $\sim m_1$. These results are summarized in table 12.1.

With this information we can now compute b_Y, b_{2R}, b_{2L}, b_3 and b_4 (and b_{10}). With the normalizations (12.1) and (12.3) we get, for F families of fermions,

$$16\pi^2 b_Y=\frac{4}{3}F+\frac{4}{5};\ 16\pi^2 b_{2L}=\frac{4}{3}F-\frac{20}{3};$$

$$16\pi^2 b_{2R}=\frac{4}{3}F+\frac{44}{3};\ 16\pi^2 b'_{2L}=\frac{4}{3}F+8 \tag{12.17}$$

$$16\pi^2 b_3=\frac{4}{3}F-11;\ 16\pi^2 b_4=\frac{4}{3}F+\frac{13}{3};$$

$$16\pi^2 b_{10}=\frac{4}{3}F+8.$$

For F=3 these give

scale	scalars	multiplicity
m_0	(1, 2, 2)	2
	$(1^c, 2, 2)$ {$\subset$ (15, 2, 2)}	6
m_1	(1, 2, 2)	2
	$(\overline{10}, 1, 3)$	1
	(15, 2, 2)	6
m_2	10	2
	54	1
	126	3

Table 12.1

Scalars with masses less than or equal to the respective scales m_0, m_1 and m_2. The superscript c indicates that the $SU(3)$ representation is being specified. In the remaining cases (except for m_2) the $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ transformation properties are specified. For the case of m_2, the $SO(10)$ representation is given.

$$16\pi^2 b_Y=24/5\,;\ 16\pi^2 b_{2L}=-8/3\,;$$

$$16\pi^2 b_{2R}=56/3\,;\ 16\pi^2 b'_{2L}=12\,; \tag{12.18}$$

$$16\pi^2 b_3=-7\,;\ 16\pi^2 b_4=25/3\,;$$

$$16\pi^2 b_{10}=12.$$

Thus, in this scenario, the only couplings which diminish with increasing Q^2 are the $SU(3)$ and $SU(2)_L$ couplings in the region from m_o to m_1.

The predictions for the ratio of the charged lepton to that of the charged -1/3 quark are valid down to the scale m_1, where $SU(4)$ is broken. To determine $m_{-1/3}/m_l$ at m_o we must renormalize through [4,45,46]

$$\frac{m_{-1/3}(m_o)}{m_l(m_o)}=\frac{m_{-1/3}(m_1)}{m_l(m_1)}\left\{\frac{\alpha_Y(m_o)}{\alpha_Y(m_1)}\right\}^{3/4F}\left\{\frac{\alpha_3(m_o)}{\alpha_3(m_1)}\right\}^{12/(33-4F)} \tag{12.19}$$

where F is the number of families. From eqns (12.5) and (12.7) we find that

$$\alpha_Y(m_o) = f^2\{\cos^2\eta(W-C^2)^{-1}+\sin^2\eta\}\alpha_{EM}(m_o). \tag{12.20}$$

If we write $m_{-1/3}(m_o)/m_l(m_o)=R(F)\,m_{-1/3}(m_1)/m_l(m_1)$, then, using eqns (12.4), (12.6) and the two preceding equations, we find

$$R(F)=[1-8\pi b_3\alpha_3(m_o)(y-z)]^{12/(33-4F)}$$

$$\times[1-8\pi b_Y\alpha_{EM}(m_o)\{\frac{3}{8}(W-\frac{5}{3})^{-1}+\frac{5}{3}\}(y-z)]^{3/4F}. \tag{12.21}$$

Given $\alpha_{EM}(m_o)$ and $\alpha_3(m_o)$, this expression depends on F through b_Y and b_3 and through the exponents. Neither y nor z depends upon F. Aside from the explicit dependence upon W, R depends upon W implicitly through y and z. In this expression for R we are neglecting scalar thresholds.

To estimate the proton lifetime in this scheme we use the result from BEGN [4] for the $SU(5)$ model,

$$\Gamma(p \to lepton + any) = \frac{1}{\tau_p}$$

$$\simeq \frac{9\pi}{16}|\psi(0)|^2 \frac{m_q^2}{M_X^4}(\alpha_{gum})^{10/7}[\alpha_3(m_{proton})]^{4/7}\left\{\frac{614}{3}\right\} \quad (12.22)$$

for three families. For rough comparison we would wish to replace α_{gum} by $\alpha_{10}(m_2)$ and replace M_X by m_2. Thus the ratio of the BEGN estimate to the present one is

$$\frac{1}{4}\left[\frac{\alpha_{gum}}{\alpha_{10}(m_2)}\right]^{10/7}\left[\frac{m_2}{M_X}\right]^4 . \quad (12.23)$$

Using the $SU(5)$ result $M_X \simeq 4\times10^{14}$ Gev and $\alpha_{gum} \simeq 1/40$, the requirement that the proton not decay too fast implies that (12.23) be greater than one:

$$\frac{\tau_p(SO(10))}{\tau_p(SU(5))} \simeq \frac{(m_2)^4[\alpha_{10}(m_2)]^{-10/7}}{2\times10^{61}} > 1 . \quad (12.24)$$

It is unclear whether the relevant parameter for a perturbation expansion is α or $\alpha Tr(T^2)$ (where T is the generator in the representation relevant to a vertex in the diagram being considered) or any of a large number of other possibilities. Thus, to ask where, for a non-asymptotically free theory, the expansion parameter becomes non-perturbative is ambiguous. However, a measure of where perturbation theory fails that is independent of these questions is the position of the so-called Landau singularity. This is the value of m in eqn (12.4) for which $g(m)^{-2}$ vanishes. In the present model it is given by

$$m_L = m_2 \exp[1/\{8\pi b_{10}\alpha_{10}(m_2)\}]$$

$$= m_2 \exp[\pi/\{6\alpha_{10}(m_2)\}] \,. \tag{12.25}$$

It is a matter of personal opinion whether one wants m_L to be greater than, less than or of the order of the Planck mass, m_P; "aesthetic" arguments can be made for each point of view. The resolution of this question must certainly wait for a theory that includes the effects of gravitation at the quantum level. What seems clear nonetheless is that asymptotic freedom is by no means a necessary condition to impose on a grand unified model.

The results of these calculations are given in figs. 12.3 through 12.6. (For these calculations, in computing m_W we have taken into account its variation with $\sin^2(\vartheta_W)$ given by

$$m_W^2 = \frac{\pi\alpha_{EM}}{\sqrt{2}G_f \sin^2\vartheta_W} \,. \tag{12.26}$$

This has a negligible effect on m_L, but does affect $\tau_p(SO(10))/\tau_p(SU(5))$ somewhat. In all calculations we have used $\alpha_{EM}(2m_W)\simeq 1/128$.) In figs. 12.3 we have ploted m_2 and m_L versus $\sin^2\vartheta_W$. What is notable about these graphs is that the presence of the Landau "singularity" gives an upper bound for the Weinberg angle in each case; this upper bound is determined by the condition $m_2=m_L$. However, it is necessary to note that it is not clear that the Landau "singularity" is a true physical singularity as opposed to, for example, being just a relic of perturbation theory. Thus, it is not clear whether this upper bound is physically significant or not.

In figs. 12.4 we have plotted m_2 and m_1 versus $\sin^2\vartheta_W$. What should be noted from these graphs is that with our symmetry breaking scheme we have a lower bound for $\sin^2\vartheta_W$: the value for which $m_1=m_2$. The inclusion of scalars in the calculation generally raises this value (in the absence of scalars the value is ~0.20 [48]). Independent of this is the fact that the value of $\sin^2\vartheta_W$ becomes larger as the difference between the scales (at which the symmetry breakings occur) grows.

In figs. 12.5 we show the values of $R(3)$ and $R(4)$ as a function of $\sin^2\vartheta_W$. These values are somewhat larger than those in the absence of scalars [48].

In fig. 12.6 is plotted the values of $\tau_P(SO(10))/\tau_P(SU(5))$. It is clear that considering a model that is more complicated than the minimal $SU(5)$ scheme can increase the proton's lifetime by many orders of magnitude. It is worth noting in this context that the actual value of the proton lifetime is very sensitive to the values of the various input parameters. Thus, a calculation that uses only the lowest order beta function cannot be used to obtain precise results since higher order corrections can effect the results significantly. Furthermore, uncertainties in the location of scalar thresholds can also effect the results* [47].

The results of this chapter show that one must be quite careful in making catagorical statements about the specific numerical predictions of grand unified models**.

*This is independent of the possibility that an ansatz different from the one that we have chosen will effect the lowest order beta function significantly by having many more low mass scalars.
**Calculations similar to those presented here have recently been given in [49].

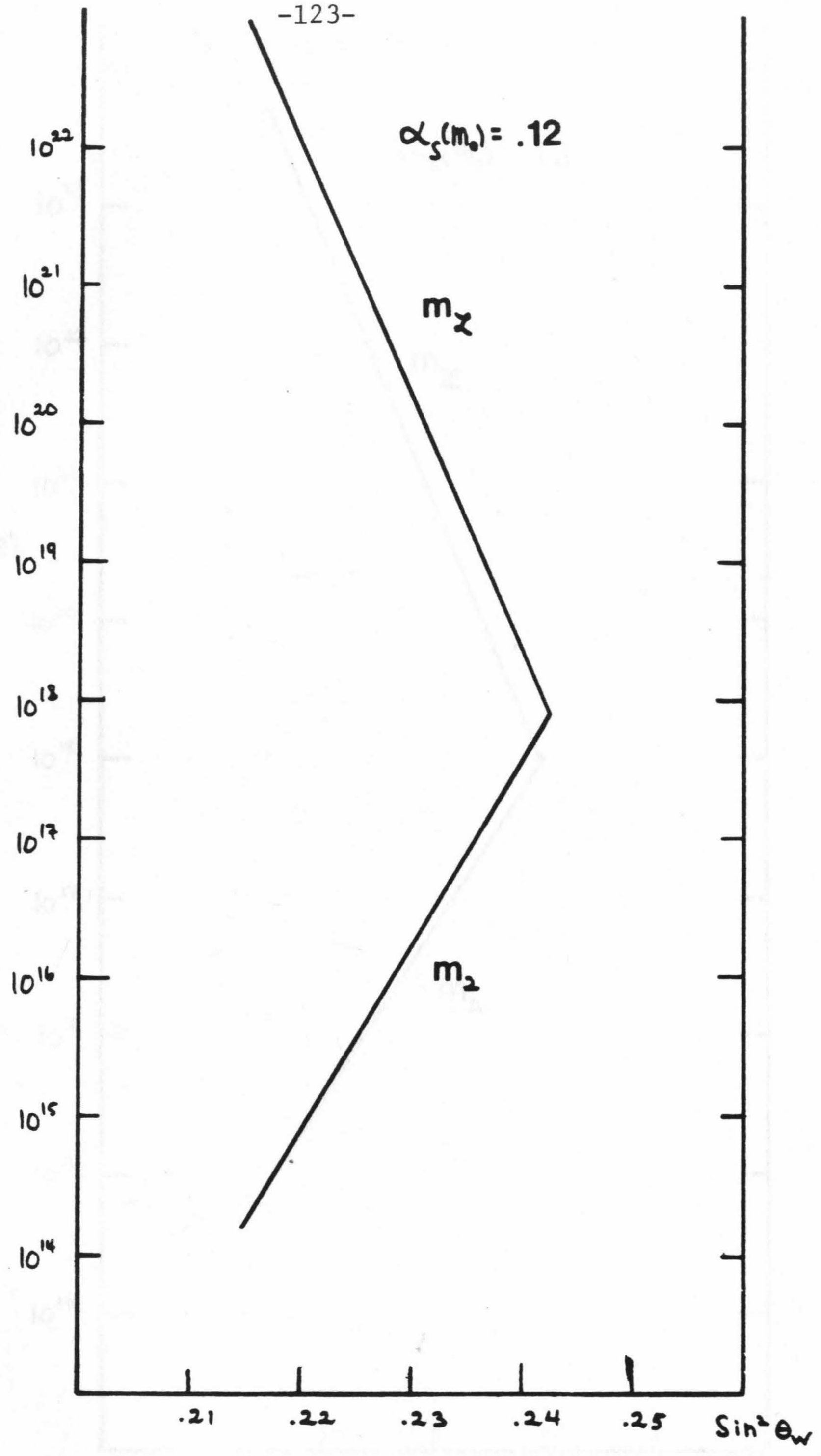

Fig. 12.3a: m_2 and m_L as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.12.

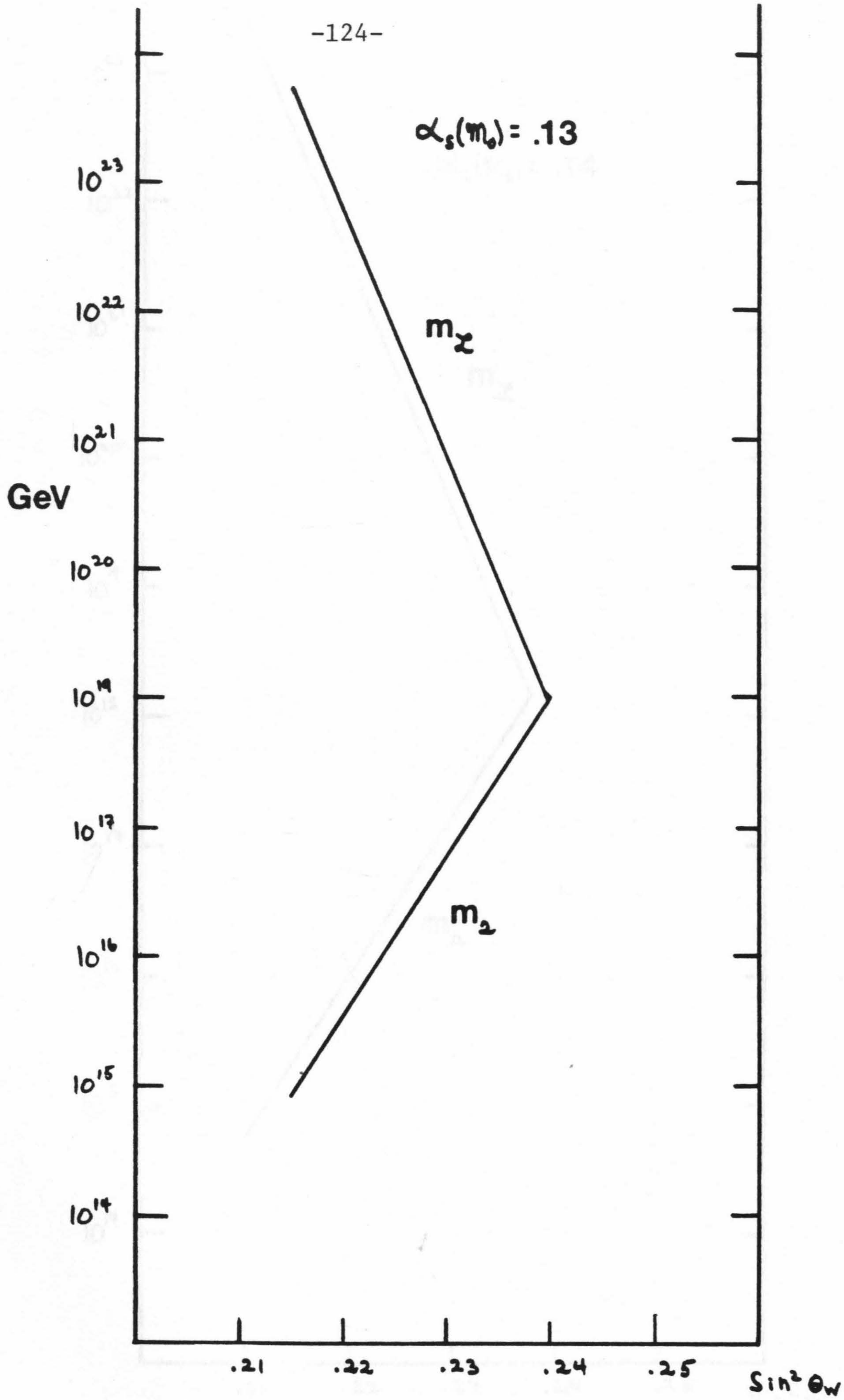

Fig. 12.3b: m_2 and m_L as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_o)$=0.13.

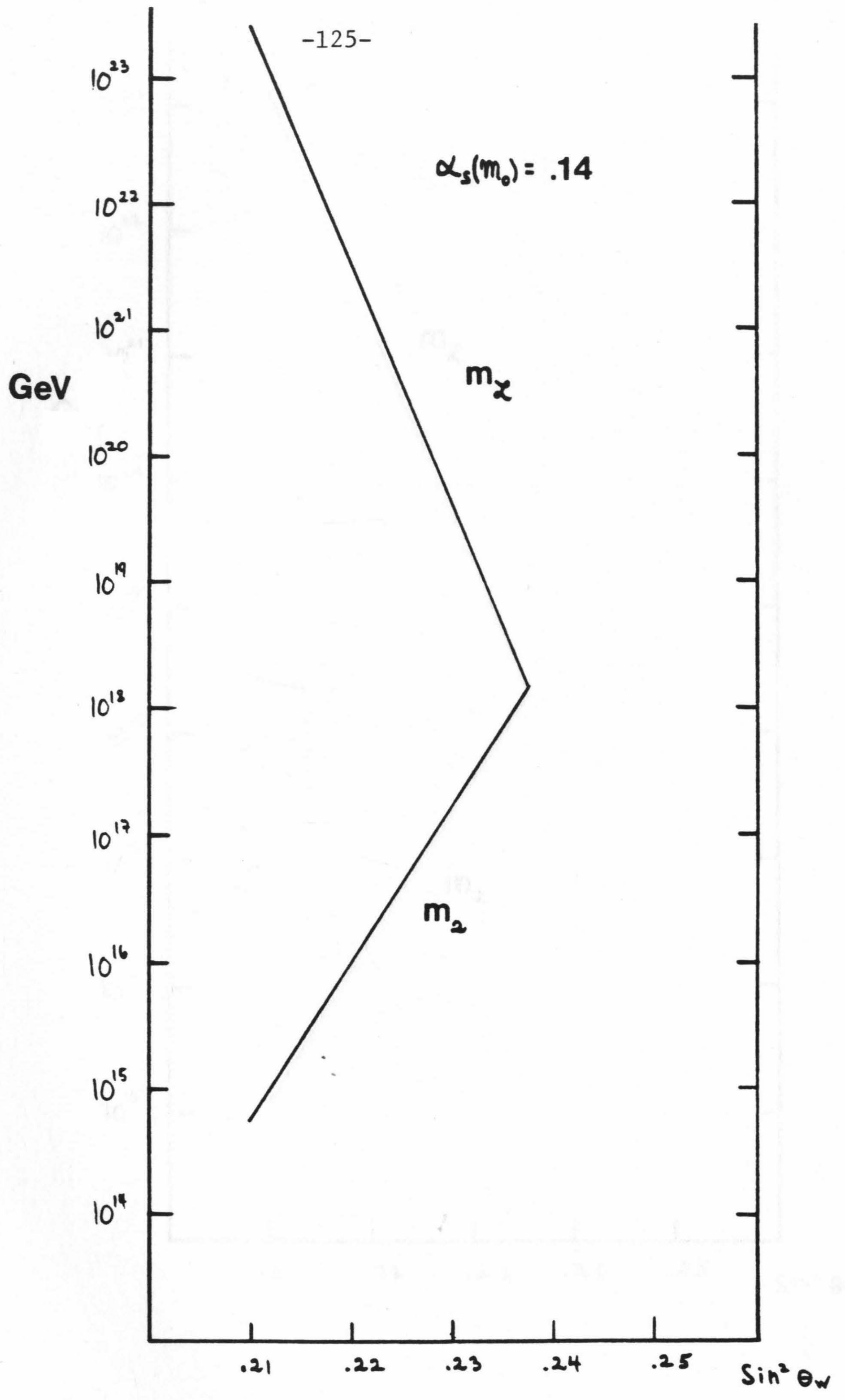

Fig. 12.3c: m_2 and m_L as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.14.

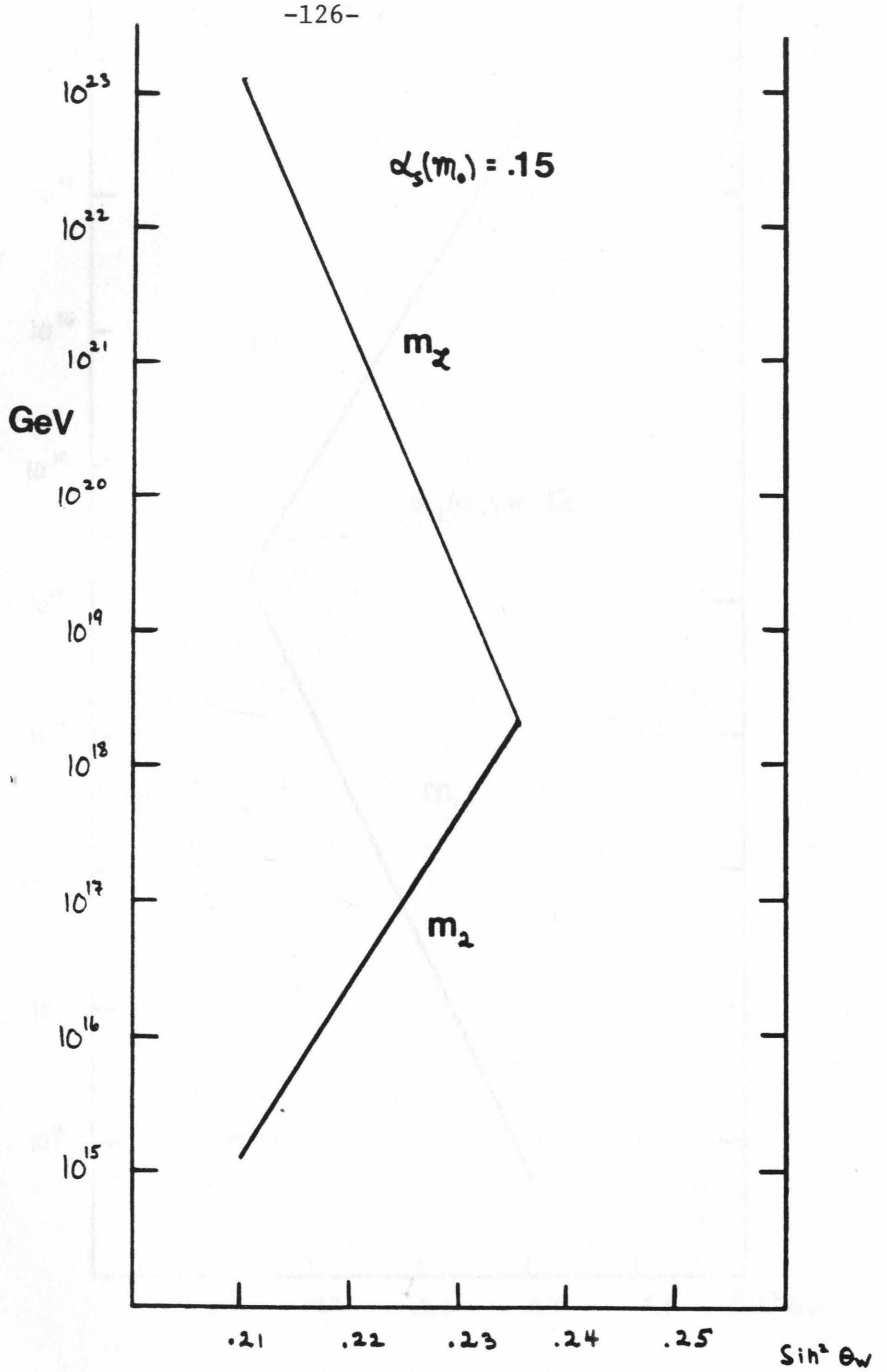

Fig. 12.3d: m_2 and m_L as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.15.

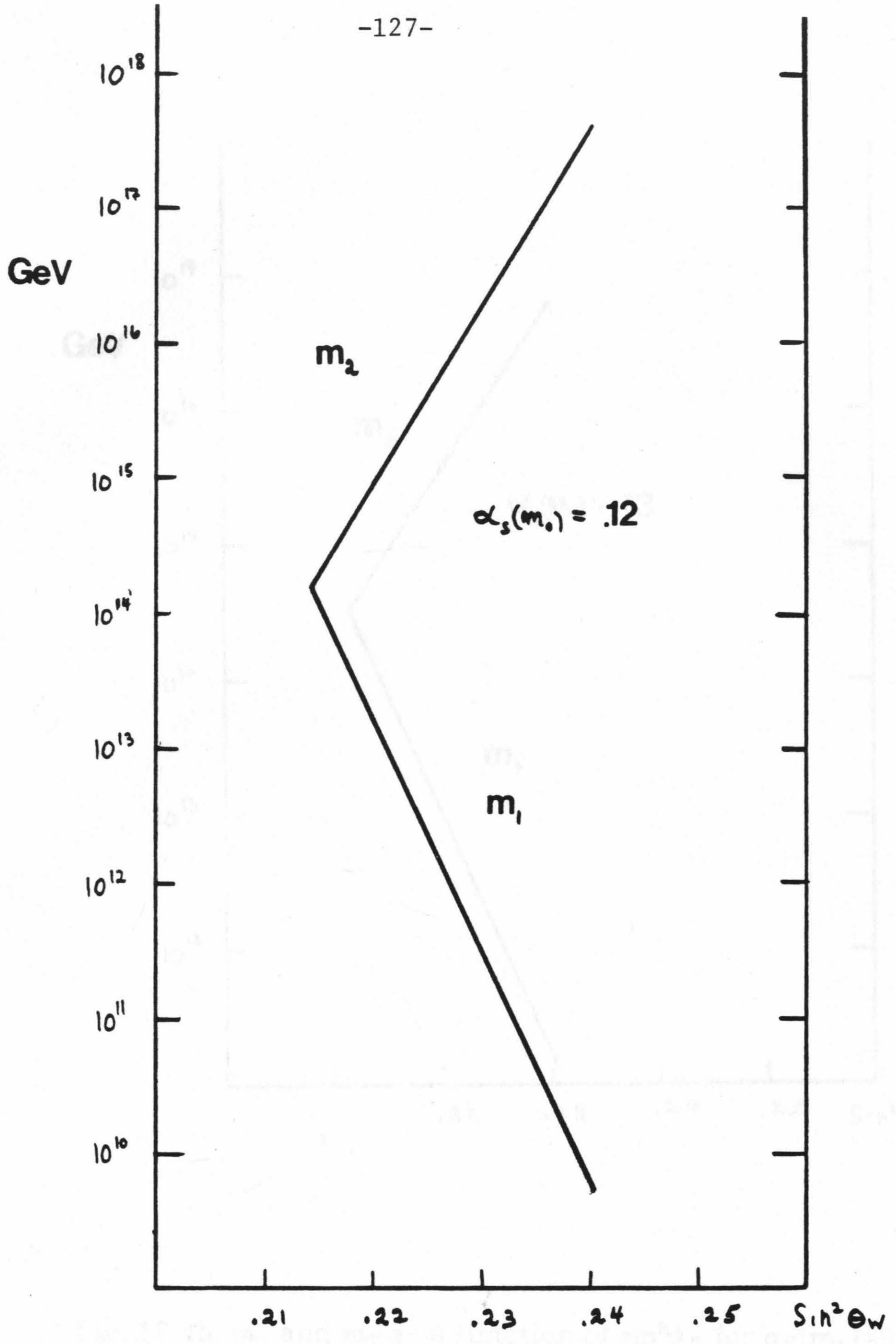

Fig. 12.4a: m_1 and m_2 as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.12.

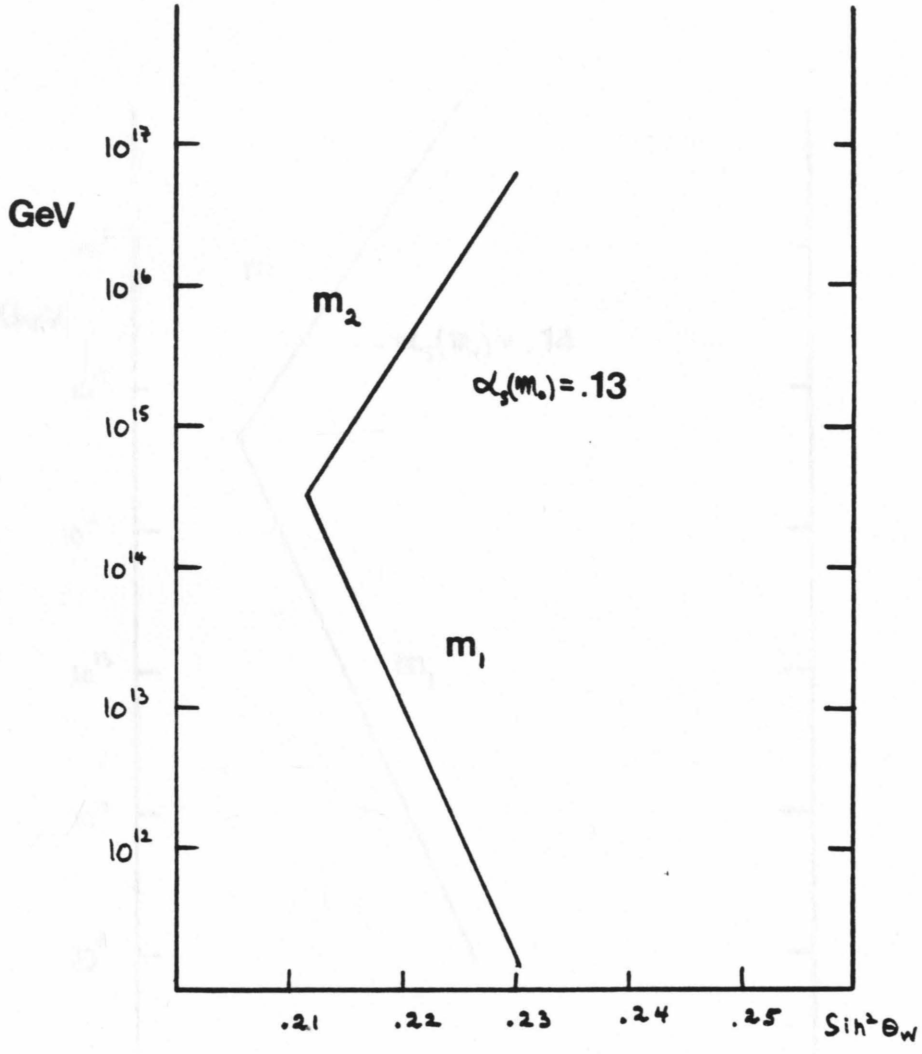

Fig. 12.4b: m_1 and m_2 as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.13.

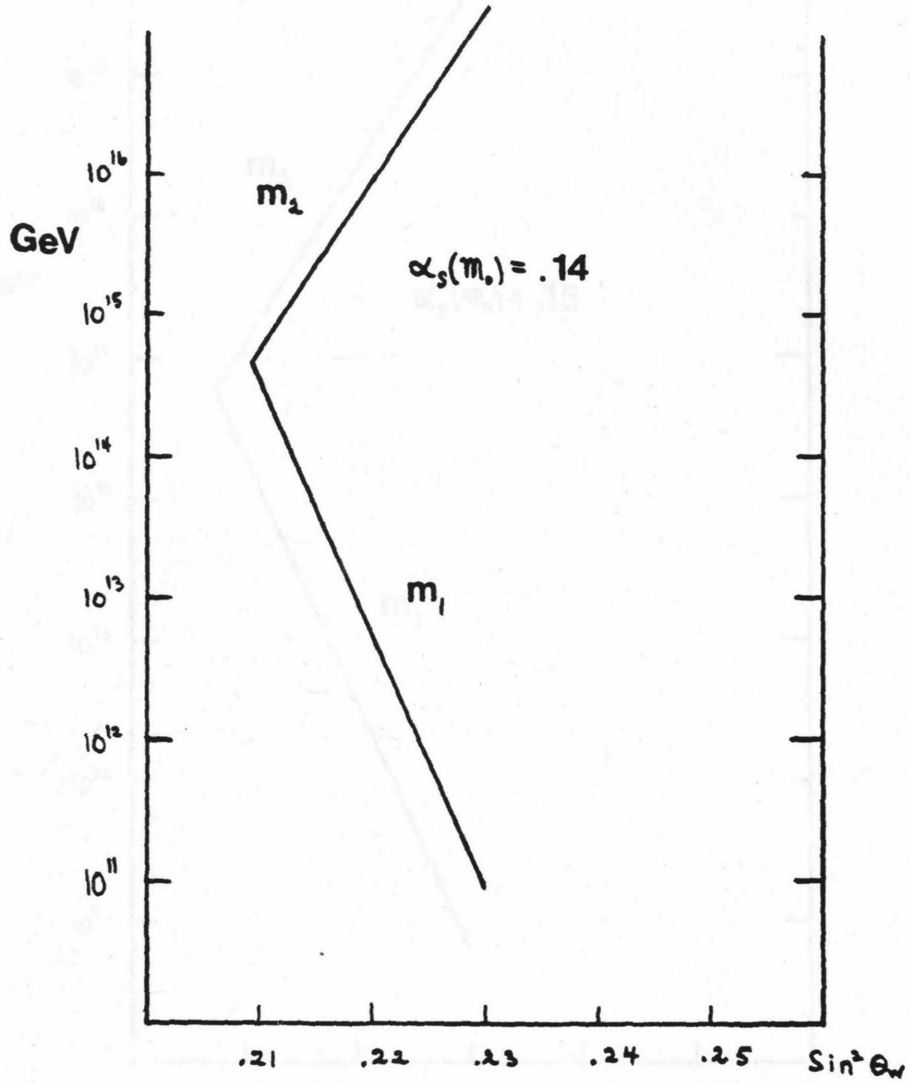

Fig. 12.4c: m_1 and m_2 as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.14.

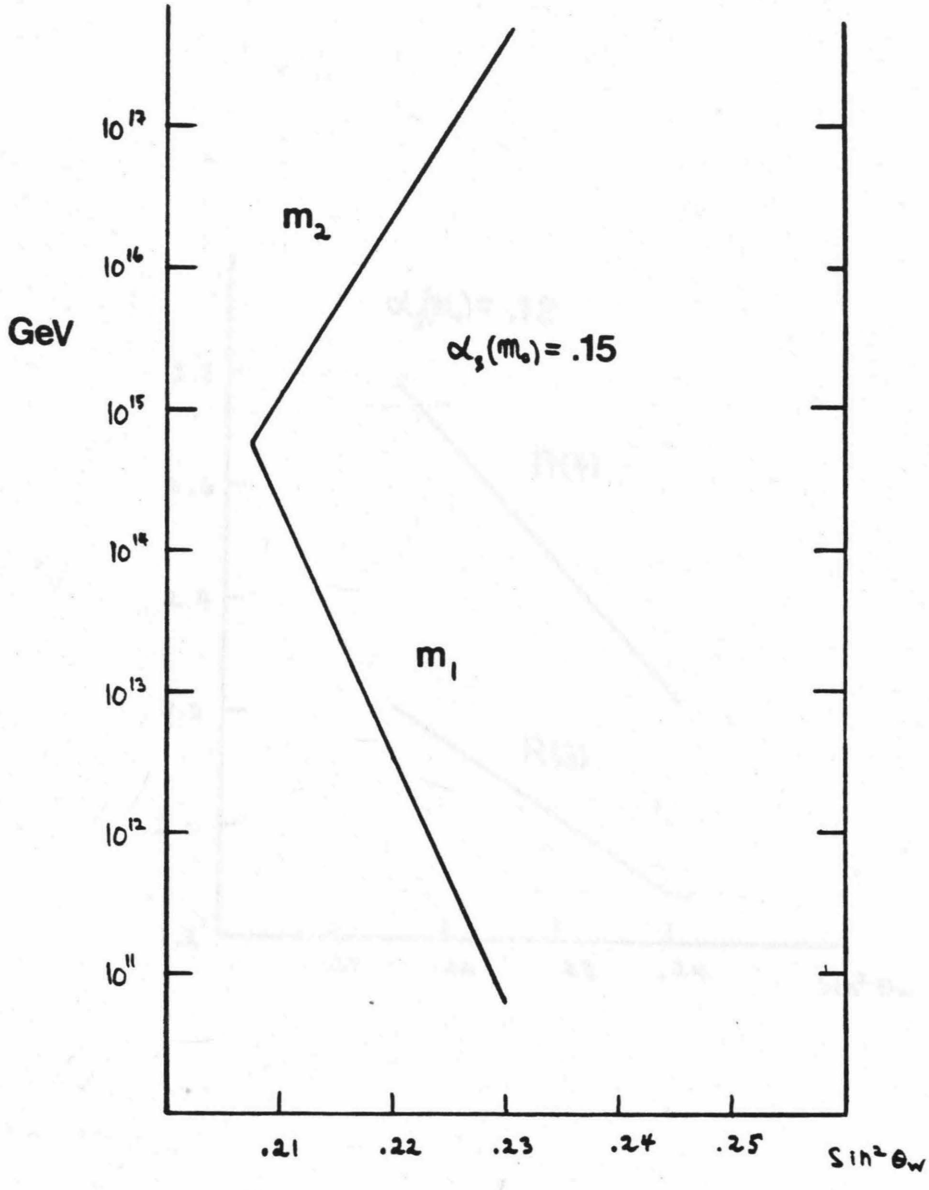

Fig. 12.4d: m_1 and m_2 as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_o)$=0.15.

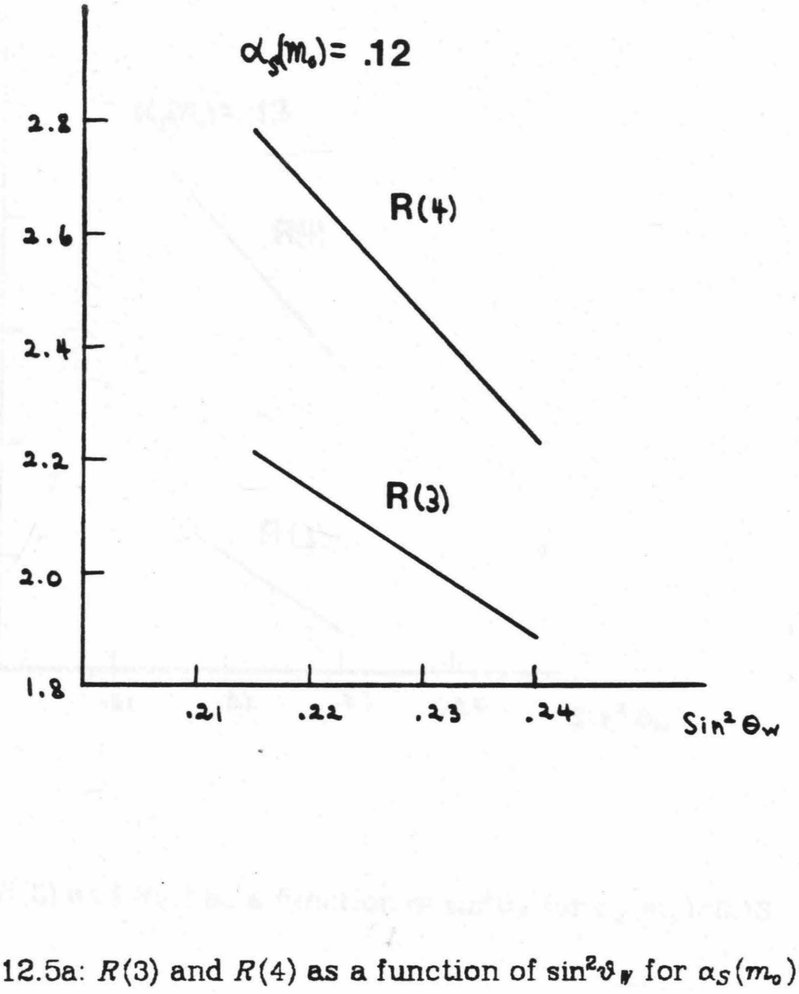

Fig. 12.5a: $R(3)$ and $R(4)$ as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_o)$=0.12.

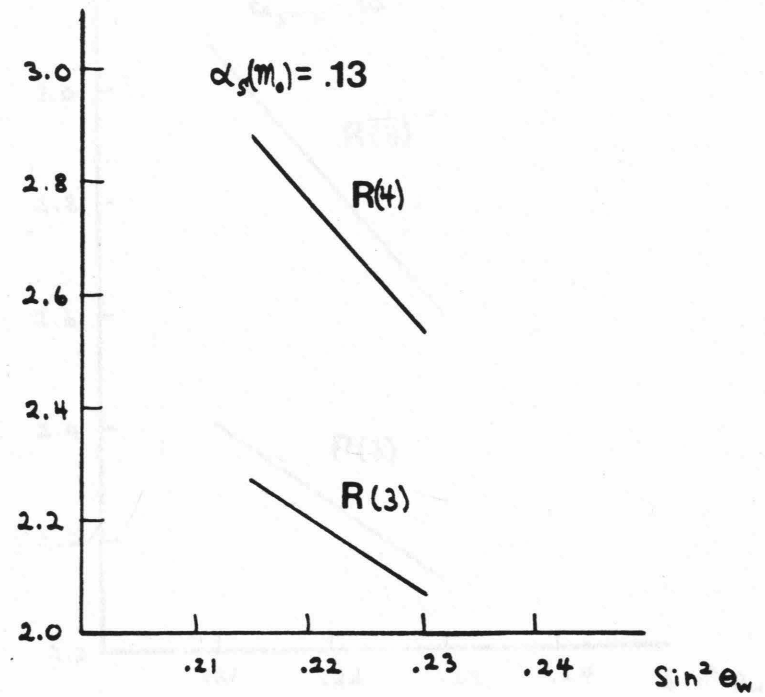

Fig. 12.5b: $R(3)$ and $R(4)$ as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_o)$=0.13.

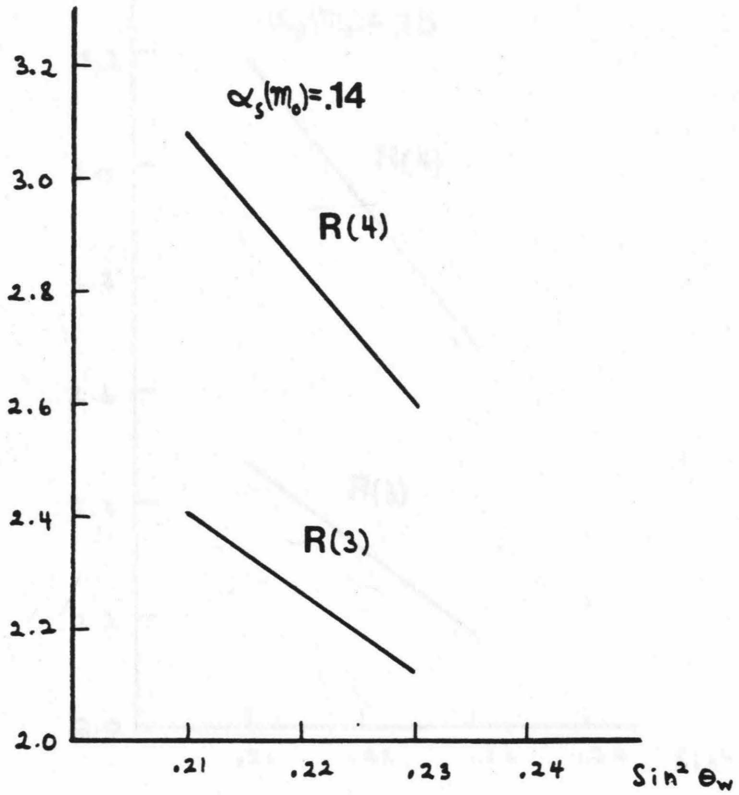

Fig. 12.5c: $R(3)$ and $R(4)$ as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.14.

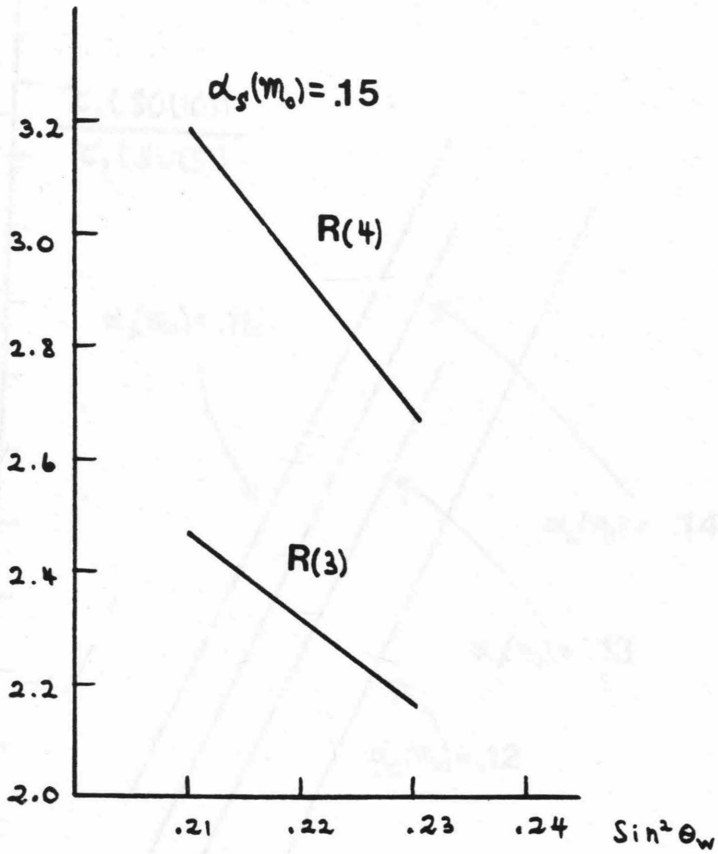

Fig. 12.5d: $R(3)$ and $R(4)$ as a function of $\sin^2\vartheta_W$ for $\alpha_S(m_0)$=0.15.

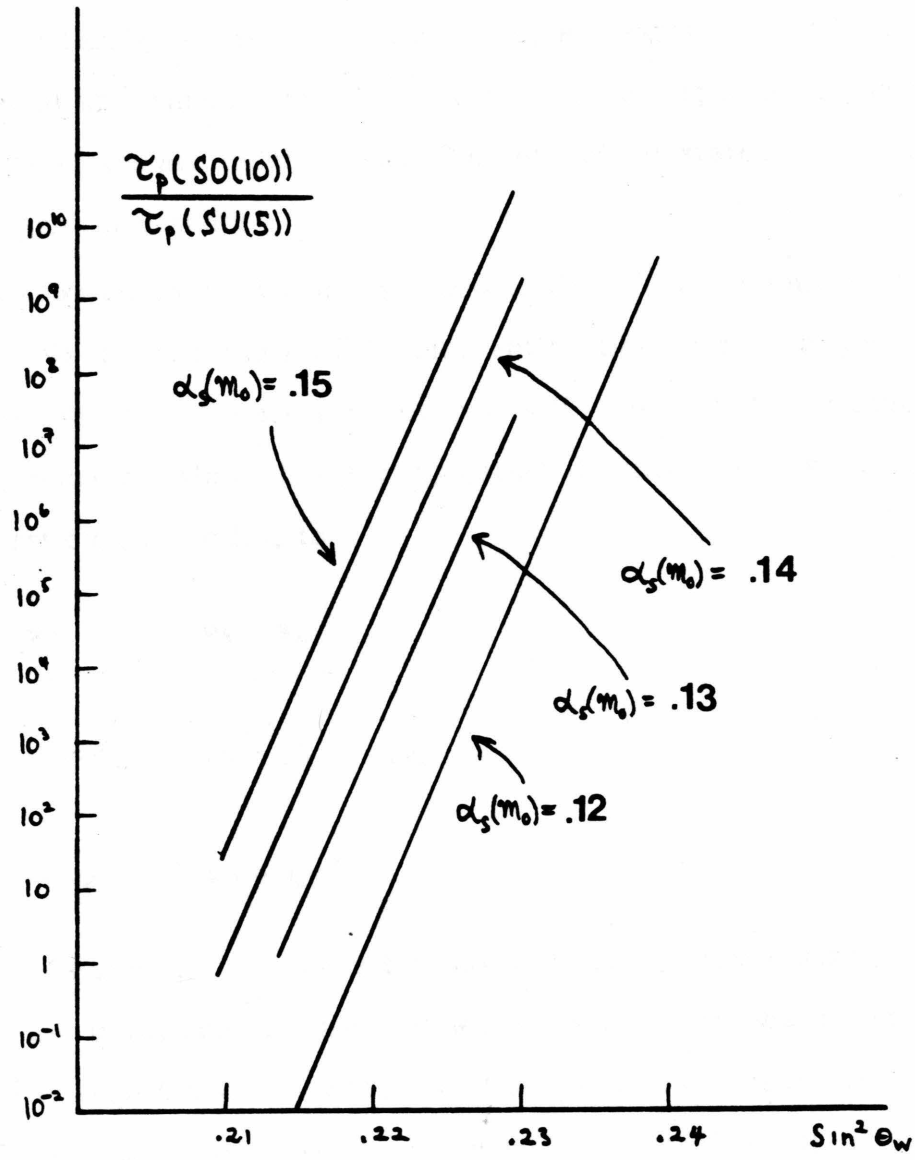

Fig. 12.6: $\tau_P(SO(10))/\tau_P(SU(5))$ as a function of $\sin^2\vartheta_W$ for the various values of $\alpha_S(m_o)$.

Appendix A) Notation for Fermion Fields

We describe spin-1/2 fermions by two-component fields of definite chirality: left-handed fields are denoted by ψ_L and right-handed fields by ψ_R. For massless fermions, chirality and helicity are equivalent and the two chirality states are independent. Only one of the states need therefore be present in a model.

For the two-component fields, ψ_L^c denotes the left-handed antiparticle of ψ_R, while ψ_R^c denotes the right-handed antiparticle of ψ_L. For fields in which both helicity states are present, parity (P) serves to interchange L and R components, while charge conjugation (C) interchanges particles with antiparticles, according to:

P: $\psi_L \to \psi_R$ $\qquad$ $\psi_R \to \psi_L$

C: $\psi_L \to \psi_L^c = \sigma_2 \psi_R^*$ $\qquad$ $\psi_R \to \psi_R^c = -\sigma_2 \psi_L^*$

CP: $\psi_L \to -\sigma_2 \psi_L^*$ $\qquad$ $\psi_R \to \sigma_2 \psi_R^*$

where σ_2 is a Pauli matrix. These transformations are summarized in fig. A.1. Note the important feature that while the separate operations of C and P interchange L and R components, the combined CP transformation does not modify the helicity state. Hence while the definition of individual C and P transformation properties generally requires the presence of both L and R states, CP transformation properties may always be defined for massless particles with only a single helicity state.

The two-component fermion fields may be collected into a four-component Dirac spinor describing a fermion of arbitrary helicity: $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$.

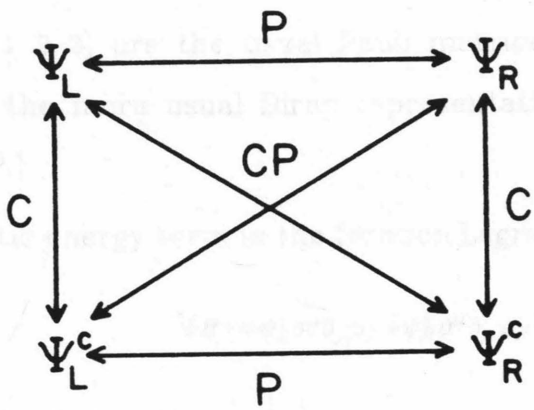

Fig. A.1: Illustration of the action of the operators C, P and CP on the two helicity components of a four-component spinor or, equivalently, two independent two-component Weyl spinors.

It is convenient to take the Dirac gamma matrices which act on this spinor in the Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} ,$$

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where σ^i $(i=1, 2, 3)$ are the usual Pauli matrices. (This representation differs from the more usual Dirac representation simply by the interchange, $\gamma^0 \leftrightarrow \gamma^5$.)

The kinetic energy term in the fermion Lagrangian is given by

$$\bar{\Psi} \partial\!\!\!/ \Psi = \psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + \psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R$$

with $\sigma^\mu = (1, \sigma^i)$, $\bar{\sigma}^\mu = (1, -\sigma^i)$.

Fermion fields for which both helicity states are present may give a Dirac mass term:

$$m\bar{\Psi}\Psi = m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R) .$$

If only one helicity is present, say ψ_L, no Dirac mass term may be constructed, but a Majorana mass term is still possible:

$$m\overline{\Psi^c}\frac{(1+\gamma_5)}{2}\Psi = m\psi_L^T \sigma_2 \psi_L .$$

Here the charge-conjugate four-component spinor Ψ^c is given by

$$\Psi^c = \begin{pmatrix} \psi_L^c \\ \psi_R^c \end{pmatrix} = \begin{pmatrix} -\sigma_2 \psi_R{}^* \\ \sigma_2 \psi_L{}^* \end{pmatrix} .$$

For a fermion field with only a single helicity state, it is sometimes convenient to define a four-component Majorana spinor,

$$\Psi_M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix} ,$$

in terms of which the Majorana mass term becomes $\frac{m}{2}\bar{\Psi}_M \Psi_M$.

Note that fields with Majorana mass terms may not carry any $U(1)_Q$ charges since the mass term is not invariant under the global gauge transformation $\psi_L \to e^{i\alpha Q}\psi_L$.

Appendix B) The CP Operation in Grand Unified Models

The generation of a baryon excess from an initially symmetrical state requires CP violating interactions. In this appendix we discuss some of the properties of the CP operator in the context of grand unified models constructed from a standard Yang-Mills action*.

We consider first a complex <u>scalar</u> field $\varphi(x,t)$. It is necessary to distinguish the field operator $\hat{\varphi}$ from the "fields" φ obtained as the expectation values of this operator in particular states. It is the q-number field operator which appears in the canonical quantization procedure; the c-number field appears in the path integral formalism; we generally work with the latter.

The actions of parity (P), charge conjugation (C) and time reversal (T) on a complex scalar field are given, up to arbitrary phases, by:

P: $\varphi(x,t) \to \varphi(-x,t)$

C: $\varphi(x,t) \to \varphi^*(x,t)$
$\hat{\varphi}(x,t) \to \hat{\varphi}^\dagger(x,t)$

T: $\varphi(x,t) \to \varphi^*(x,-t)$
$\hat{\varphi}(x,t) \to \hat{\varphi}(x,-t)$

The transformations P and C are represented by unitary operators, which act on $\hat{\varphi}$ just as on φ. T is an antiunitary operator, which reverses the order of factors in products of field operators. It thus interchanges the bra and ket states in an expectation value and complex conjugates the

*This presumes that we are talking only about compact Lie groups. For noncompact Lie groups the discussion presented here must be extended.

field φ. The combined operation of CPT on $\varphi(x,t)$ yields $\varphi(-x,-t)$ and is equivalent, as usual, to a generalized Lorentz transformation (total 4-inversion).

The P, C and T transformations above are modified for particles with spin. Their action on spin 1/2 fermions is outlined in appendix A. Note that separate P and C transformations interchange chirality states, while the combined CP or T transformations do not. Thus, massless particles with only one chirality or helicity state have definite behavior under CP, but the action of C or P may not be defined. For spin 1 fields, P and T transformations reverse respectively the space and time components of the polarization vector; they may therefore be considered to "raise" or "lower" the Lorentz vector index on a vector potential A_μ.

If a simple scalar charge is associated with the field φ above, then the C operation serves to reverse this charge. When a field carries a non-Abelian charge the action of the C transformation on this charge is more complicated, and again may not be defined.

We shall consider a field transforming under a unitary representation r of some simple compact Lie group G that acts as the gauge group for a grand unified model. The analysis presented below can be easily extended to take into account any additional global symmetries of the model. The group G will, for now, be assumed unbroken. A column vector of fields η transforms under the action of a finite element of G according to

$$\eta \to e^{iT^a\omega^a}\,\eta \tag{B.1}$$

where the ω^a are the group parameters of the transformation and the T^a are the generators of G in the representation r. If the ω^a are chosen to be real, then the T^a are Hermitian.

"Charges" or "quantum numbers" are usually associated with the elements of the Cartan subalgebra or center of G. The elements of this algebra will be denoted H_l, where l runs from 1 to the rank of the group. This algebra consists of the maximal set of commuting generators of G and thus generates the maximal Abelian subgroup of G (it therefore may be written as a product of $U(1)$ factors). The H_l may be rendered simultaneously diagonal by a unitary transformation on the representation space (which does not affect the Hermiticity of the T^a). We denote the vector η of fields in this basis by η^D, and the H_l by H_l^D. In this basis, the CP transformation is defined by

$$\mathrm{CP}[\eta_j^D] = e^{i\alpha_j}\Lambda(\eta_j^D)^* \tag{B.2}$$

where Λ is a matrix acting on Lorentz indices, and no sum is taken over j. (For the case of scalar fields Λ=1, for fermion fields $\Lambda=\pm\sigma_2$ depending on whether η_D is left- or right-handed respectively (see appendix A) and for vector fields $\Lambda=g_{\mu\nu}$ or $g^{\mu\nu}$; i.e., Λ lowers or raises an index on a vector field.) The phase α_j is, for the moment, arbitrary, but we will show later that it may be taken to vanish. With this definition, the transformation

$$\eta^D \to e^{iH_D^l\omega^l}\,\eta^D \,, \tag{B.3}$$

becomes for the CP conjugate fields:

$$\mathrm{CP}[\eta^D] \to e^{-iH_D^l\omega^l}\,\mathrm{CP}[\eta^D] \,. \tag{B.4}$$

Hence the CP transformation reverses signs of all the charges associated with H^l. If a set of fields transforms according to a representation r, the CP conjugate fields transform according to the conjugate representation $\bar{r}$.

If the representation r is irreducible, then the phase factors $e^{i\alpha_j}$ appearing in eqn (B.2) must all be equal, so that $\alpha_j=\alpha$ for all j, since one may perform an arbitrary group transformation on η^D before applying CP. The common phase, α, may then be removed by an overall phase redefinition of all fields. We shall usually assume below that r is irreducible; reducible representations may be treated by considering separately each of their irreducible components.

We have defined CP transformations above in terms of the "diagonal" basis η^D; below we shall consider other bases η obtained by unitary transformations:

$$\eta=\mathbf{U}\eta^D. \tag{B.4}$$

The action of CP on fields in this basis is given from (B.2) and (B.4) by

$$\mathrm{CP}[\eta]=\mathbf{U}\mathbf{U}^T\Lambda\eta^* \tag{B.5}$$

since $\mathrm{CP}(\mathbf{U}^\dagger\eta)=\mathbf{U}^\dagger\mathrm{CP}(\eta)$.

The action of CP on the vector $\Lambda(\eta^D)^*$ transforming as the conjugate representation $\bar{r}$ is

$$\mathrm{CP}(\Lambda(\eta^D)^*)=e^{i\beta}\Lambda'\Lambda^*\eta^D \tag{B.6}$$

where, if $\Lambda=\sigma_2$ $(\Lambda=g_{\mu\nu})$ then $\Lambda'=-\sigma_2$ $(\Lambda'=g^{\mu\nu})$, so

$$\Lambda'\Lambda=1\,. \tag{B.7}$$

The phase β must be such that the kinetic term for η^D be CP invariant. The analogue of eqn (B.5) is

$$\mathrm{CP}(\Lambda\eta^*)=e^{i\beta}U^*U^\dagger\eta\,. \tag{B.8}$$

The form of CP transformations for fields in a representation r depends on the relationship of r to the conjugate representation $\bar{r}$. Representations with three basic characteristics may be distinguished [50]:

Complex — $\bar{r}$ and r are completely inequivalent (e.g. fundamental representation of $SU(n)$ for $n>2$); for complex representations the singlet does not appear in the decomposition of $r\otimes r$.

Real — $\bar{r}$ is equivalent to r and there is a basis in which the representation matrices are purely real (in which basis $\bar{r}$ is equal to r) (e.g. all adjoint representations of compact simple Lie groups); for real representations the singlet appears in the decomposition of the symmetric part of $r\otimes r$.

Pseudoreal — $\bar{r}$ is equivalent to r, but there is no basis in which r is equal to $\bar{r}$ (e.g. fundamental representations of symplectic groups); for pseudoreal representations the singlet appears in the decomposition of the antisymmetric part of $r\otimes r$. All pseudoreal representations have even dimension.

For our purposes the basis of greatest interest for a complex representation is that in which the Cartan subalgebra is diagonal; the CP transformation properties of a set of fields transforming according to r are simplest in this basis. For any representation, under the action of the group, η transforms as

$$\eta \to g\eta \; . \tag{B.9}$$

where g is an element of the (unitary) representation r in the η basis. It follows that

$$\eta^\dagger \to \eta^\dagger g^\dagger \; . \tag{B.10}$$

If r is a complex representation and if η are scalar fields then $\partial_\mu \eta^\dagger \partial^\mu \eta$ is a group (and Lorentz) invariant (we have not yet gauged G; gauging replaces all derivatives by gauge covariant derivatives). If the η are (left-handed) fermi fields then $i(\eta^\dagger \sigma_\mu \partial^\mu \eta - (\partial^\mu \eta^\dagger)\sigma_\mu \eta)$ is a group (and Lorentz) invariant. In both cases the requirement of CP invariance gives $\beta=0$. Thus $(\mathrm{CP})^2=1$ when acting on a complex representation.

Real and pseudoreal representations both have the property that, for an arbitrary basis there is a unitary matrix $\mathbf{V}$ such that all of the representation matrices satisfy $g^*=\mathbf{V}g\mathbf{V}^\dagger$. If we apply this relation twice we find $g=\mathbf{V}^*\mathbf{V}g\mathbf{V}^\dagger\mathbf{V}^T$. Thus the (unitary) matrix $\mathbf{V}^*\mathbf{V}$ commutes with the representation and thus, by Shur's lemma, must be proportional to the identity: $\mathbf{V}^*\mathbf{V}=\alpha\mathbf{I}$. Since $\mathbf{V}$ is unitary it follows that $\mathbf{V}^T=\alpha\mathbf{V}$. Applying this relation twice then gives $\alpha^2=1$; thus, the matrix $\mathbf{V}$ is either symmetric or antisymmetric. These two alternatives correspond to the real and the pseudoreal cases respectively. We now discuss these cases in turn.

If r is a real representation, then one may choose a basis η^R in which all the representation matrices g are real (orthogonal). The action of the group cannot mix the real with the imaginary parts of η^R. Thus, for scalar fields we may choose η^R to be real; this gives us the minimal set of fields needed to construct the representation. For fermi fields we are obligated to take an appropriate linear combination of fields transforming as r to assure definite transformation properties under the Lorentz group; i.e., the representation must be "doubled". (There exist an equivalence class of representation η^R connected by orthogonal similarity transformations.) This basis is obtained by the unitary transformation $\mathbf{U}$ such that

$$g^R \equiv \mathbf{U} g \mathbf{U}^\dagger = (g^R)^* \tag{B.11}$$

is real; hence

$$g^* = \mathbf{U}^T \mathbf{U} g (\mathbf{U}^T \mathbf{U})^\dagger \tag{B.12}$$

so that the matrix which effects the equivalence between r and $\bar{r}$ is symmetric (this is related to the fact that the singlet is in the symmetric part of $r \otimes r$). The η^R is distinct from the basis η^D in which the Cartan subalgebra is diagonal, but may be obtained from it by a unitary transformation $\mathbf{U}_R$. In the η^R basis, the action of the CP operator is obtained from eqn (B.5) as

$$\mathrm{CP}[\eta^R] = \mathbf{U}_R \mathbf{U}_R^T \Lambda (\eta^R)^* \ . \tag{B.13}$$

(For scalars the complex conjugation on the right-hand side of this equation is superfluous.) It is clear that we may choose $\mathbf{U}_R$ so that each of its elements is either purely real or purely imaginary. It follows that $\mathbf{U}_R \mathbf{U}_R^T$ is a real (orthogonal) matrix and that $(\mathbf{U}_R \mathbf{U}_R^T)^2 = 1$. By an appropriate orthogonal matrix we may diagonalize $\mathbf{U}_R \mathbf{U}_R^T$; its eigenvalues are 1's and -1's. Consequently we may work with a basis in which each η_j^R is an eigenstate of CP (it is a member of the equivalence class mentioned above). If, choosing η^R to be real, we act on it a second time with CP we get $(\mathrm{CP})^2(\eta^R) = (\mathbf{U}_R \mathbf{U}_R^T)^2 \Lambda \Lambda' \eta^R = \eta^R$. Thus $(\mathrm{CP})^2 = 1$ when acting on a real representation.

A pseudoreal representation, r, is one for which r is equivalent to $\bar{r}$, but there is no basis in which the representation matrices are real. In any basis there is a unitary matrix C such that

$$g^{*}=\mathbf{C}g\,\mathbf{C}^{\dagger} \tag{B.14}$$

and $\mathbf{C}$ is antisymmetric (this is connected with the fact that, for a pseudo-real representation, the singlet is in the antisymmetric part of $r\otimes r$):

$$\mathbf{C}^{T}=-\mathbf{C}\,. \tag{B.15}$$

Although there is no basis in which the representation matrices are real there is a basis (with representation matrices g^{Z}) with the property

$$\mathbf{Z}g^{Z}=(g^{Z})^{*}\mathbf{Z}\,, \tag{B.16}$$

where $\mathbf{Z}$ is the antisymmetric orthogonal matrix [50]

$$\mathbf{Z}=\begin{pmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}=\mathrm{diag}(i\tau_2,\, i\tau_2,\, \ldots)\,. \tag{B.17}$$

Note that since $\mathbf{Z}$ is related to $\mathbf{C}$ by a unitary similarity transformation, and since $\mathbf{Z}^2=-1$, it follows that $\mathbf{C}^2=-1$.

If η transforms as a pseudoreal representation, then a conjugation matrix C may be defined so that the constraint $C\eta=\eta^{*}$ may be imposed as long as this constraint does not violate any other symmetries of the theory (in particular, Lorentz invariance; this constraint cannot be imposed on the fermi fields of definite chirality with which we work). This amounts to the statement that the minimal number of degrees of freedom for a set of scalar fields transforming as a pseudoreal representation may be chosen to be equal to the dimension of the representation, as is certainly true for a real representation, but definitely not true for a complex representation. The obvious choice for this conjugation matrix is $C=\mathbf{C}$. Thus, if η is a set of scalar fields with this constraint imposed, the usual kinetic term vanishes identically; $\partial_\mu\eta^\dagger\partial^\mu\eta=\partial_\mu\eta^T\mathbf{C}^T\partial^\mu\eta=0$ since $\mathbf{C}$ is antisymmetric [32,51,52]. A normal kinetic term can be formed for scalar fields transforming as a pseudoreal representation only if the representation is "doubled" by taking an additional set of scalars χ satisfying $\chi^*=\mathbf{C}\chi$ and forming the combination $\rho\equiv\eta+i\chi$; then the kinetic term for ρ becomes $\partial_\mu\rho^\dagger\partial^\mu\rho=2i\partial_\mu\chi^T\mathbf{C}\partial^\mu\eta$, which is nonvanishing. We can act with CP twice on the fields ρ as follows: $(\mathrm{CP})^2\rho=\mathrm{CP}(\eta^*+i\chi^*)=\mathbf{C}\mathrm{CP}(\eta+i\chi)=\mathbf{C}^2\rho=-\rho$, since $\mathbf{C}^2=-1$ [32]. We can build a fermi field (of definite chirality) as follows. If ψ and ξ satisfy $\psi^*=\mathbf{C}\psi$ and $\xi^*=\mathbf{C}\xi$ and transform, under CP as $\psi\to\sigma_2\psi^*=\mathbf{C}\sigma_2\psi$ and $\xi\to-\sigma_2\xi^*=-\mathbf{C}\sigma_2\xi$, then the combination $\Psi=\psi+i\xi$ transforms as a left-handed field. Under CP $\Psi\to\sigma_2\Psi^*=\mathbf{C}\sigma_2(\psi-i\xi)$ and, under the action of CP once again, $(\mathrm{CP})^2$: $\Psi\to\mathbf{C}^2\sigma_2(\psi+i\xi)=-\Psi$, which is the same result as for the scalar case [32,51].

Given two vectors η and χ transforming as the pseudoreal representation $\boldsymbol{r}$, there are two ways to form invariants under G (for simplicity we write in the $\mathbf{Z}$ basis):

$$[\mathbf{Z}\Lambda(\eta^Z)^*]^\dagger\chi^Z=(\eta^Z)^T\Lambda\mathbf{Z}\chi^Z \tag{B.18}$$

and

$$(\eta^Z)^\dagger\chi^Z\,. \tag{B.19}$$

Whether one of these expressions is useful depends upon whether it can be made to have useful transformation properties under the Lorentz group. If η^Z are left-handed fermi fields (and therefore we are taking a "doubled" pseudoreal representation) and $\chi^Z = i\sigma_\mu \partial^\mu \eta^Z$, then it is eqn (B.19) that is the Lorentz singlet; if $\chi^Z = \eta^Z$, then eqn (B.18) is the Lorentz singlet [32,51].

We now consider the action of CP on gauge vector fields A^a_μ. Under an infinitesimal gauge transformation parametrized by ω^a, the gauge potentials behave according to

$$A^b_\mu \to A^b_\mu + iC^b_{c\,a}\,\omega^c A^a_\mu + \partial_\mu \omega^b \quad , \tag{B.20}$$

where the C^a_{bc} are the structure constants for G defined in terms of the infinitesimal generators (the Lie algebra) T^a by

$$[T^b, T^c] = C^a_{bc}\, T^a \quad . \tag{B.21}$$

To discuss CP we need only consider global gauge transformations, for which the last term in (B.20) is absent. The gauge vectors are real if the generators T^a are chosen to be Hermitian (and hence if the group parameters are chosen to be real).

By an appropriate choice of basis on the group manifold one can choose $C^a_{b\,c} = if_{abc}$ where f_{abc} is completely antisymmetric and real. This is the choice that is usually made. By virtue of the fact that the structure constants satisfy the Jacobi relation one may choose them to be the elements of the generator matrices in the adjoint representation. In this case the choice $C^a_{b\,c} = if_{abc}$ renders the adjoint representation matrices real. It is not possible to choose the generators in the adjoint representation to be the structure constants, to have Hermitian generators and to have the Cartan subalgebra diagonal all at the same time. If a basis is chosen so that the T^a are all Hermitian, then the fields A^a may all be taken as real. This basis makes the reality of the adjoint representation manifest, and, as discussed above, gives

$$\mathrm{CP}[A^a_\mu] = \varepsilon^a A^{a\mu} \tag{B.22}$$

where $\varepsilon^a = \pm 1$ (depending on the value of a).

Using the above results we shall now discuss the conditions that the matrices of the group generators and the Clebsch-Gordan coefficients must satisfy so that the action of CP on the fields transforms a given term in the Lagrangian into its Hermitian conjugate (when all couplings are set to unity).

If ψ_i is a column vector of (left-handed) fermions transforming as some representation r, then in its gauge invariant kinetic term we have the term

$$i\psi_i^\dagger \sigma^\mu \psi_j \, T^a_{ij} A^a_\mu \,. \tag{B.23}$$

If we work in the basis of r in which the Cartan subalgebra is diagonal then, upon acting with CP on the fields, we get (note that $\sigma_2 \sigma^\mu \sigma_2 = (\sigma_\mu)^T$)

$$-i\psi_j^\dagger \sigma_\mu \psi_i \, T^a_{ij} \varepsilon^a A^{a\mu} \tag{B.24}$$

where we have used the anticommutativity of the fermi fields and the thrice-repeated index a is summed over as are all other repeated indices. For eqn (B.24) to be the Hermitian conjugate of eqn (B.23) we must have

$$T^a_{ij} = \varepsilon^a \, T^a_{ji} \qquad (\text{no sum on } a) \,. \tag{B.25}$$

Thus we must have $\varepsilon^a = +1$ for the elements of the diagonal Cartan subalgebra. The remaining generators (in the basis in which the Cartan subalgebra is diagonal) are purely real or purely imaginary accordingly as

$\varepsilon^a = \pm 1$. These conditions also suffice to render the gauge invariant scalar kinetic term CP invariant.

We may now digress for a moment to consider the gauge vector boson mass matrix and its eigenstates at tree level. This matrix is real and therefore cannot itself violate CP. Consequently its eigenstates, if chosen to be Hermitian fields, must also be eigenstates of CP. Generally, those eigenstates will not be states with definite $U(1)$ quantum numbers (unless those quantum numbers all vanish). In that case there will be at least a twofold degeneracy in the mass matrix where the mass eigenstates have opposite CP eigenvalues. If we call such a pair of eigenstates A_μ^+ and A_μ^- (where $\pm$ indicates the CP eigenvalue) then the combinations $(A_\mu^+ \pm A_\mu^-)/\sqrt{2}$ are states of definite $U(1)$ quantum numbers and

$$\mathrm{CP}[\frac{1}{\sqrt{2}}(A_\mu^+ \pm A_\mu^-)] = \frac{1}{\sqrt{2}}(A^{\mu +} \mp A^{\mu -}) . \tag{B.26}$$

If $T^\pm$ is the linear combination of generators that $A_\mu^\pm$ couples through, then $(A_\mu^+ \pm iA_\mu^-)/\sqrt{2}$ couples through $(T^+ \pm iT^-)/\sqrt{2}$, which is purely real.

To discuss the Yukawa terms we consider (left-handed) fermi fields ψ_i and χ_m transforming as irreducible representations r_ψ and r_χ respectively. We also consider scalar fields φ_a transforming as some irreducible representation r_φ appearing in the Clebsch-Gordan decomposition of $r_\psi \otimes r_\chi$. A group invariant Yukawa term is then of the form

$$i\psi_i^T \sigma_2 \psi_m \varphi_a R_{ima} \tag{B.27}$$

where R_{ima} are group-coupling coefficients to couple r_ψ, r_χ and r_φ to make a singlet. We assume that the bases for the representations r_ψ and r_χ are chosen so that the Cartan subalgebra is diagonal. If we also choose the basis for r_φ so that the Cartan subalgebra is diagonal, then the action of

CP changes (B.22) into

$$i\psi_i^\dagger \sigma_2 \chi_m^\dagger \varphi_\alpha^\dagger R_{im\alpha} \,. \tag{B.28}$$

For this to be the Hermitian conjugate of (B.27), $R_{im\alpha}$ must be real. If r_φ is a real representation in the real basis (we call the group-coupling coefficients $R^R_{im\alpha}$ in this basis) then $CP(\varphi_\alpha)=\varepsilon_\alpha\varphi_\alpha$ (no sum on α) where $\varepsilon_\alpha=\pm1$ as discussed above. The action of CP then changes (B.27) into

$$i\psi_i^\dagger \sigma_2 \chi_m^\dagger \varphi_\alpha \varepsilon_\alpha R^R_{im\alpha} \,. \tag{B.29}$$

We must then have

$$R^{R*}_{im\alpha} = R^R_{im\alpha}\,\varepsilon_\alpha \qquad \text{(no sum on } \alpha) \,. \tag{B.30}$$

Thus, $R^R_{im\alpha}$ is purely real or purely imaginary accordingly as $\varepsilon_\alpha=\pm1$.

The group-coupling coefficients for the coupling of scalars to scalars can be treated along the same lines as above.

Appendix C) Momentum Space Weights: Double-cut Diagrams

In this appendix we evaluate the momentum space weight of the "double-cut" diagram of fig. 4.4. We present the results normalized by the Born diagram with massless fermions.

We take the fermions i_1, i_2, i_3 and i_4 in fig. 4.4 to have masses m_1, m_2, m_3 and m_4 respectively. X and Y are bosons with masses m_X and m_Y. The couplings to be used depend upon whether X (Y) is a scalar or a vector boson.

We have, in the center of mass system of the X,

$$\mathrm{Im}[I] = -\frac{1}{4\pi^2 Tr[\not{q}h(\not{q}-\not{p})h]} \int d^4k \frac{\delta(k^2-m_2^2)\delta((k-p)^2-m_1^2)}{(k-q)^2-m_Y^2}$$

$$\times Tr[(\not{q}+m_4)e(\not{k}+m_2)f(\not{k}-\not{p}+m_1)g(\not{q}-\not{p}+m_3)h] \qquad \text{(C.1)}$$

$$=-\frac{\lambda(m_X, m_1, m_2)}{32\pi} \frac{\vartheta(m_X-(m_1+m_2))\,\vartheta(m_X-(m_3+m_4))}{Tr[\not{q}h(\not{q}-\not{p})h]} \int_{-1}^{1} dx \frac{1}{(k-q)^2-m_Y^2}$$

$$\times Tr[(\not{q}+m_{44})e(\not{k}+m_2)f(\not{k}-\not{p}+m_1)g(\not{q}-\not{p}+m_3)h] \qquad \text{(C.2)}$$

where

$$\left.\begin{aligned} x &= \frac{\vec{k}\cdot\vec{q}}{|\vec{k}\,||\vec{q}\,|} \\ \lambda(m_X, m_1, m_2) &= \left[1+\frac{(m_1^2-m_2^2)^2}{m_X^4}-\frac{2(m_1^2+m_2^2)}{m_X^2}\right]^{1/2} \end{aligned}\right\} \qquad \text{(C.3)}$$

and c, d, e, f and g are various products of gamma matrices and chiral projection operators which depend upon the specific diagram being evaluated.

Some further kinematical facts are:

$$p\cdot q=m_X q_0=\frac{m_X}{2}(m_X^2-m_3^2+m_4^2) \tag{C.4}$$

$$k\cdot p=m_X k_0=m_X\left\{\frac{m_X^2}{16}\lambda^2(m_X,\,m_1,\,m_2)+m_2^2\right\}^{1/2} \tag{C.5}$$

$$\left.\begin{array}{c} k\cdot q=k_0 q_0-|\vec{k}\,|\,|\vec{q}\,|x \\ k^2=m_2^2\,;\; q^2=m_4^2 \end{array}\right\}. \tag{C.6}$$

As an example, take the case of vector exchange in vector decay with all of the fermion masses set equal to zero. We obtain

$$\mathrm{Im}[I_{VV}]=\frac{-1}{32\pi Tr\,[\not{q}\gamma_\mu \mathrm{P}(\not{q}-\not{p})\gamma^\mu \mathrm{P}]}\int_{-1}^{1}dx\,\frac{1}{(k-q)^2-m_Y^2}$$

$$\times Tr\,[\not{q}\gamma_\mu \mathrm{P}\not{k}\gamma_\nu \mathrm{P}(\not{k}-\not{p})\gamma^\mu \mathrm{P}(\not{q}-\not{p}))\gamma^\nu \mathrm{P}]$$

$$=\frac{1}{16\pi}\left\{2(1+v^2)^2\ln\left[\frac{v^2}{1+v^2}\right]+2v^2+3\right\} \tag{C.7}$$

where $\mathrm{P}=(1\pm\gamma_5)/2$ and $v\equiv m_Y/m_X$.

The results for the remaining three cases are

$$\mathrm{Im}[I_{VS}(v)]=-\frac{1}{16\pi}\left\{v^4\ln\left[\frac{v^2}{1+v^2}\right]+v^2-\frac{1}{2}\right\}\ . \tag{C.8}$$

$$\mathrm{Im}[I_{SV}(v)]=\frac{1}{8\pi}\ln\left[\frac{v^2}{1+v^2}\right] \tag{C.9}$$

and

$$\mathrm{Im}[I_{SS}(v)]=-\frac{1}{16\pi}\left\{v^2\ln\left[\frac{v^2}{1+v^2}\right]+1\right\}\ . \tag{C.10}$$

Plots of these functions are given in fig. C.1.

For the case $m_2=m_3=m_4=0$ we define $u \equiv m_1/m_X$. We then get the following exact results and Taylor expansions about $u=0$:

$$\mathrm{Im}[I_W(v,u)]=\frac{1}{16\pi}\left\{2(1+v^2)(1+v^2-u^2)\ln\left[\frac{v^2}{1+v^2-u^2}\right]+(1-u^2)(2v^2-u^2+3)\right\}$$

$$=\frac{1}{16\pi}\left\{2(1+v^2)\ln\left[\frac{v^2}{1+v^2}\right]+2v^2+3\right.$$

$$\left.-\left[2+(1-v^2)\ln\left[\frac{v^2}{1+v^2}\right]\right]u^2-\frac{u^6}{3(1+v^2)}+\cdots\right\} \tag{C.11}$$

$$\mathrm{Im}[I_{VS}(v,u)]=-\frac{1}{16\pi}\left\{v^2(v^2-u^2)\ln\left[\frac{v^2}{1+v^2-u^2}\right]+\frac{1}{2}(1-u^2)(2v^2-1)-\frac{1}{2}u^2(1-u^2)\right\}$$

$$=-\frac{1}{16\pi}\left\{v^4\ln\left[\frac{v^2}{1+v^2}\right]+v^2-\frac{1}{2}-\frac{u^2v^2}{1+v^2}\left((1+2v^2)\ln\left[\frac{v^2}{1+v^2}\right]+1\right)\right.$$

$$\left.+\frac{u^4}{2(1+v^2)}-\frac{u^6(3+v^2)v^2}{6(1+v^2)^3}+\cdots\right\} \tag{C.12}$$

$$\mathrm{Im}[I_{SV}(v,u)]=\frac{1}{8\pi}(1-u^2)\ln\left[\frac{v^2}{1+v^2-u^2}\right]$$

$$=\frac{1}{8\pi}\left\{\ln\left[\frac{v^2}{1+v^2}\right]-u^2\left\{\ln\left[\frac{v^2}{1+v^2}\right]-\left(\frac{1}{1+v^2}\right)\right\}\right.$$

$$\left.-\frac{u^4(1+2v^2)}{2(1+v^2)^2}-\frac{u^6(1+3v^2)}{3(1+v^2)^3}\cdots\right\} \tag{C.13}$$

and

$$\mathrm{Im}[I_{SS}(v,u)]=-\frac{1}{16\pi}\left\{1-u^2+v^2\ln\left[\frac{v^2}{1+v^2-u^2}\right]\right\}$$

$$=-\frac{1}{16\pi}\left\{1+v^2\ln\left[\frac{v^2}{1+v^2}\right]-\frac{u^2}{1+v^2}+\frac{u^4v^2}{2(1+v^2)^2}+\frac{u^6v^2}{3(1+v^2)^3}+\cdots\right\} \tag{C.14}$$

We note here the expression for the Born rate (when all couplings are set to unity) of the vector and scalar decay diagrams (normalized by the rate with massless fermions):

$$1-\left(\frac{m_1}{m_X}\right)^2-\left(\frac{m_2}{m_X}\right)^2 \quad . \tag{C.15}$$

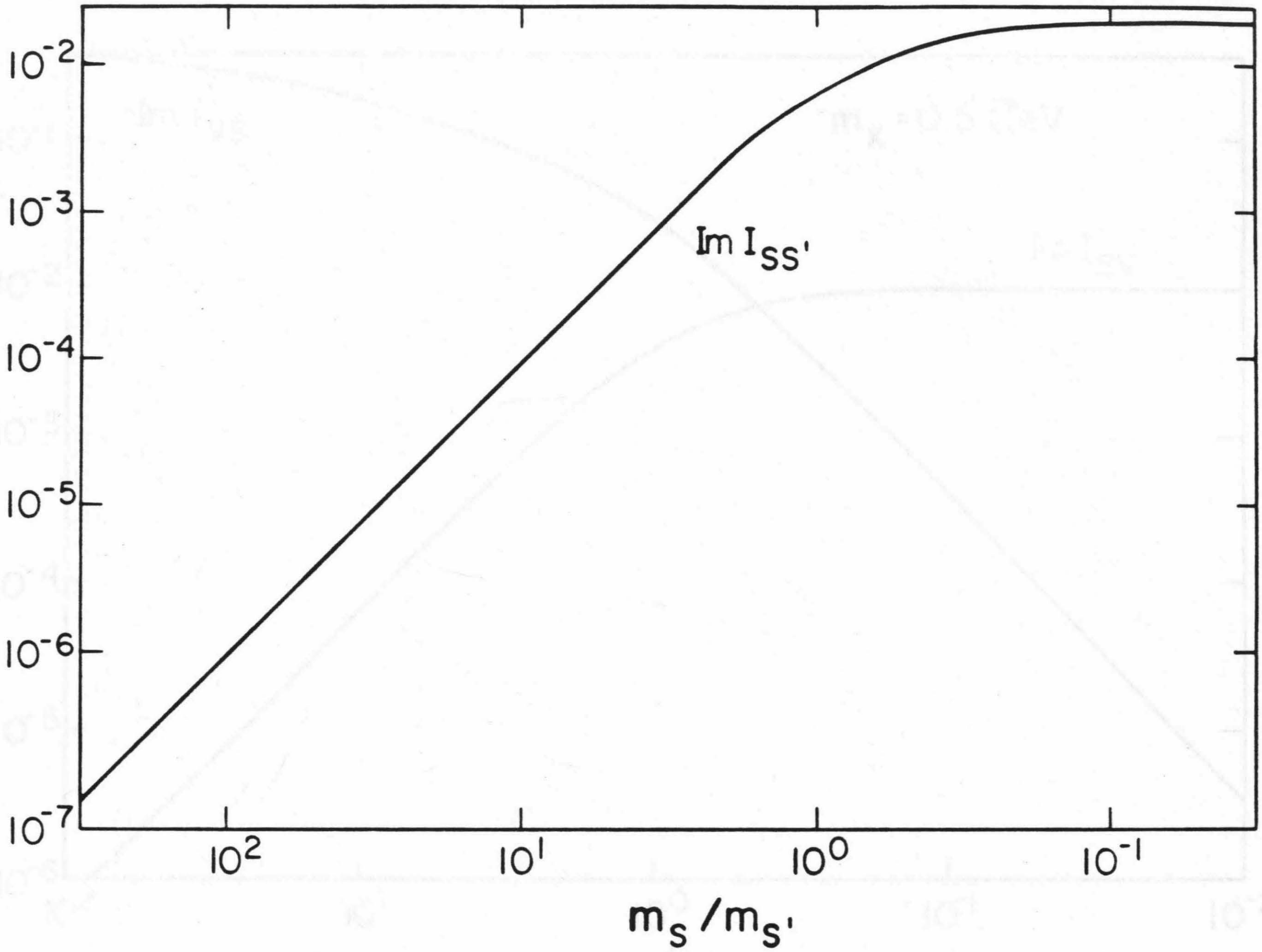

Fig. C.1a: Momentum space weight for scalar decay with scalar exchange for massless fermions.

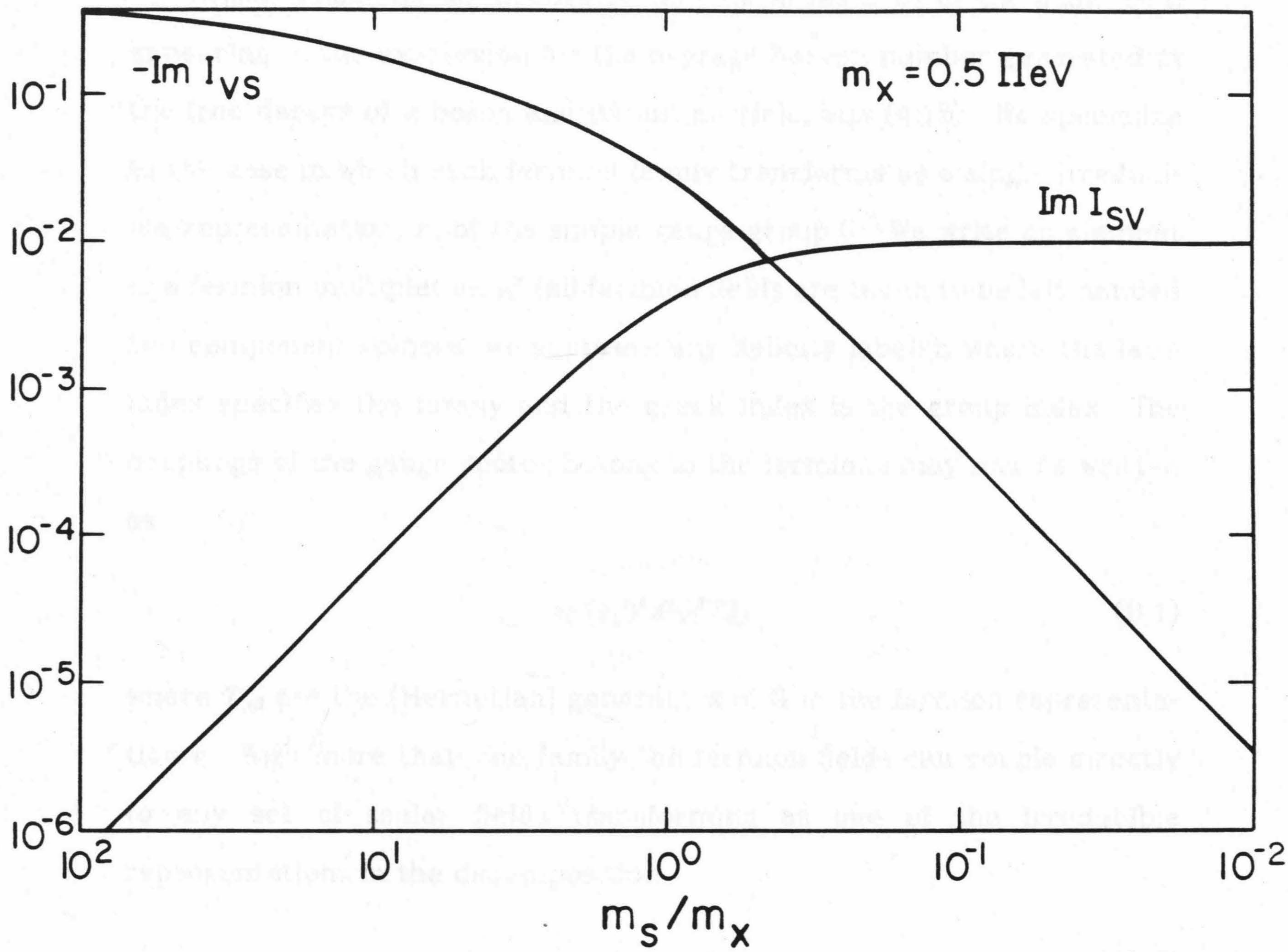

Fig. C.1b: Momentum space weights for scalar decay with vector exchange (SV) and for vector decay with scalar exchange (VS) for massless fermions. The mass of the vector is taken to be 0.5 ΠeV.

Appendix D) Im(Ω)

In this appendix we discuss some general features of the quantity Ω appearing in the expression for the average baryon number generated in the free decays of a boson and its antiparticle, eqn (4.15). We specialize to the case in which each fermion family transforms as a single irreducible representation, r, of the simple gauge group G. We write an element of a fermion multiplet as ψ_i^α (all fermion fields are taken to be left-handed two component spinors; we suppress any helicity labels), where the latin index specifies the family and the greek index is the group index. The couplings of the gauge vector bosons to the fermions may now be written as

$$ig\,(\psi_i^\alpha)^\dagger A\!\!\!/^{\gamma} \psi_i^\beta T^\gamma_{\alpha\beta} \tag{D.1}$$

where $T_{\alpha\beta}$ are the (Hermitian) generators of G in the fermion representation r. With more than one family the fermion fields can couple directly to any set of scalar fields transforming as one of the irreducible representations in the decomposition

$$r \otimes r = r_1 \oplus r_2 \oplus \ldots \oplus r_a \oplus \ldots \tag{D.2}$$

(if all fermions appear in a single irreducible representation, they can only couple to scalar representatins appearing in the symmetric part of (D.2)). In general there may be several scalar multiplets which transform according to a given irreducible representation (whether or not it appears in (D.2)); that is, the scalar sector may have its own family structure. We write the lth scalar family which transforms according to r_a as $(\varphi^a)_l^\gamma$. The Yukawa couplings can then be written as

$$(\psi_i^\alpha)^T \sigma_2 \psi_j^\beta (R^a)^{\alpha\beta\gamma} \{(h^a)_{ijk} (\varphi^a)_k^\gamma + (\tilde{h}^a)_{ijk} (\Delta^a)^{\gamma\rho} [(\varphi^a)_k^\rho]^\dagger\} + \text{h.c.}\,, \tag{D.3}$$

where all repeated indices are summed over (even those repeated thrice). The $(h^a)_{ijk}$ are the Yukawa coupling constants and the $(R^a)^{\alpha\beta\gamma}$ are the Clebsch-Gordan coefficients coupling $\bar{r}_a$ to $r \otimes r$ to make a singlet. The $(R^a)^{\alpha\beta\gamma}$ may be taken as real when all fermion and scalar representations are taken in the basis in which their Cartan sublagebra is diagonal (as discussed in appendix B); also, for a given r_a, $(R^a)^{\alpha\beta\gamma}$ is either symmetric or antisymmetric under the interchange of α and β. Δ^a is a unitary matrix defined to vanish when r_b is a complex representation. If r_a is a real representation, then $(\Delta^a)^{\gamma\rho}$ is symmetric, whereas if r_a is pseudoreal, $(\Delta^a)^{\gamma\rho}$ is antisymmetric. Furthermore, if r_a is a real representation and a basis is chosen in which the representation matrices are real, then $(\Delta^a)^{\gamma\rho}=\delta^{\gamma\rho}$. The second term is an independent term in the Lagrangian only if r is a *doubled* real or pseudoreal representation.

The various fields appearing in (D.1) and (D.3) may be rewritten in terms of mass eigenstate fields by the unitary transformations

$$\psi_i^a = U_i^a{}_m \psi_m \tag{D.4a}$$

$$(\varphi^a)_k^\gamma = (V^a)_{kP}^\gamma \varphi_P + (\tilde{V}^a)_{kP}^\gamma \varphi_P^\dagger \tag{D.4b}$$

$$A^\gamma = W^{\gamma S} A^S + (W^{\gamma S})^* A^{S\dagger} . \tag{D.4c}$$

In eqn (D.4b) if, for a given P, the field φ_P is a Hermitian field, then

$$(V^a)_{kP}^\gamma = (\tilde{V}^a)_{kP}^\gamma . \tag{D.5}$$

Furthermore, if the representation r_a is a (undoubled) real representation and the basis of the representation space is chosen so that the fields $(\varphi^a)_k^\gamma$ are Hermitian (real), it follows that

$$(V^a)_{kP}^\gamma = [(\tilde{V}^a)_{kP}^\gamma]^* \tag{D.6}$$

(this condition was used in writing eqn (D.5c); the vector mass matrix is discussed in appendix B). These transformations satisfy the unitary relations

$$(U_{i\ m}^{a})^{*}U_{j\ m}^{\beta}=\delta_{\alpha\beta}\delta_{ij}\ , \qquad \text{(D.7a)}$$

$$[(V^{a})_{kP}^{\gamma}]^{*}(V^{b})_{lP}^{\delta}+[(\tilde{V}^{a})_{kP}^{\gamma}]^{*}(\tilde{V}^{b})_{lP}^{\delta}=\delta_{\gamma\delta}\delta_{kl}\delta_{ab} \quad \text{summed on } P\ , \qquad \text{(D.7b)}$$

$$\left.\begin{array}{r} W^{\gamma S}(W^{\varepsilon S})^{*}=\delta_{\gamma\varepsilon} \\ W^{\gamma S}(W^{\gamma T})^{*}=\delta_{ST} \end{array}\right\} . \qquad \text{(D.7c)}$$

In eqn (D.7b) the indices a and b refer only to representations that appear in the decomposition of $r\otimes r$, i.e., representations that can couple to the fermion-fermion operator. There may be other scalar representations in the model which do not couple directly to fermions (indeed, in general, with fermions transforming according to the fundamental representation of the gauge group, there must be such scalar multiplets except in the case of $E(6)$ * [27]). Thus, to invert the expression (D.7b), one must allow the indices a and b to run over a set of values which includes these multiplets as well. However, the inverses of the expressions (D.4) are not needed in the following.

In terms of the mass eigenstate fields eqns (D.1) and (D.3) become

$$ig\,(U_{i\ m}^{a})^{*}U_{i\ n}^{\beta}\,W^{\gamma S}\,T_{\alpha\beta}^{\gamma}\,\psi_{m}^{\dagger}A^{S}\psi_{n} \qquad \text{(D.8)}$$

*In an $E(6)$ model a fermion family (or, in some incarnations of the model, more than one fermion family) can be put into the fundamental 27 dimensional representation. To break $E(6)$ down to $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ and then to $SU(3)\otimes U(1)_{EM}$ we need only use scalar multiplets that appear in the decomposition of $27\otimes 27$:

$$27\otimes 27=(\overline{27}+351')_S+351_A\ .$$

$E(6)$ is the only compact simple Lie group with this property.

and

$$U_{i}^{\alpha}{}_{m}U_{j}^{\beta}{}_{n}(\psi_m)^{CP\dagger}\psi_n(R^{a})^{\alpha\beta\gamma}\{[(h^{a})_{ijk}(V^{a})^{\gamma}_{kP}+(\tilde{h}^{a})_{ijk}[(\tilde{V}^{a})^{\rho}_{kP}]^*(\Delta^{a})^{\gamma\rho}]\varphi_P$$

$$+[(h^{a})_{ijk}(\tilde{V}^{a})^{\gamma}_{kP}+(\tilde{h}^{a})_{ijk}[(V^{a})^{\rho}_{kP}]^*(\Delta^{a})^{\gamma\rho}]\varphi^*_P\}+\text{h.c.}$$

$$=U_{i}^{\alpha}{}_{m}U_{j}^{\beta}{}_{n}(\psi_m)^{CP\dagger}\psi_n\{(\Gamma^{a}_{P})^{\gamma}_{ij}\varphi_P+(\tilde{\Gamma}^{a}_{P})^{\gamma}_{ij}\varphi^*_P\}(R^{a})^{\alpha\beta\gamma}+\text{h.c.}\,, \tag{D.9}$$

where

$$(\Gamma^{a}_{P})^{\gamma}_{ij}\equiv\{(h^{a})_{ijk}(V^{a})^{\gamma}_{kP}+(\tilde{h}^{a})_{ijk}[(\tilde{V}^{a})^{\rho}_{kP}]^*(\Delta^{a})^{\gamma\rho}\}\,, \tag{D.10}$$

and

$$(\tilde{\Gamma}^{a}_{P})^{\gamma}_{ij}\equiv\{(h^{a})_{ijk}(\tilde{V}^{a})^{\gamma}_{kP}+(\tilde{h}^{a})_{ijk}[(V^{a})^{\rho}_{kP}]^*(\Delta^{a})^{\gamma\rho}\} \tag{D.11}$$

(there is no sum on a in the two preceding equations). We have the following symmetry as a consequence of the Pauli principle:

$$(R^{a})^{\alpha\beta\gamma}(\Gamma^{a}_{P})^{\gamma}_{ij}=(R^{a})^{\beta\alpha\gamma}(\Gamma^{a}_{P})^{\gamma}_{ji} \tag{D.12a}$$

and

$$(R^{a})^{\alpha\beta\gamma}(\tilde{\Gamma}^{a}_{P})^{\gamma}_{ij}=(R^{a})^{\beta\alpha\gamma}(\tilde{\Gamma}^{a}_{P})^{\gamma}_{ji}\,. \tag{D.12b}$$

We can now consider the quantity

$$\mathbf{A}\equiv\Omega(B_{i_2}-B_{i_1}+B_{i_3}-B_{i_4})\,, \tag{D.13}$$

which appears in eqn (4.15), in the limit where the fermion masses are much smaller than boson masses. In this case the imaginary part of the loop integrals, $\text{Im}[I_{XY}]$ are (as discussed in chapter 4) independent of their indicies i_1, i_2, i_3 and i_4 and we may therefore sum over the relevant indices (the fermion mass eigenstate indices) in the expression for **A**. In the resulting expression the contributions of fermion mixing drop out due to the condition (D.7a) and the absence of mixing at this order between fermions of different baryon number. This last fact allows us to implement eqn (D.7a) in spite of the presence of the factor $B_{i_2}-B_{i_1}+B_{i_3}-B_{i_4}$ in the summed version of eqn (D.15).

For the case of vector exchange in vector decay (fig. D.1) we find

$$\mathbf{A}_{VV}=T^P_{\mu\nu}T^P_{\varphi\eta}T^S_{\alpha\beta}T^S_{\rho\varepsilon}U_i{}^\beta{}_q(U_i{}^\alpha{}_p)^*U_l{}^\eta{}_p(U_l{}^\varphi{}_n)^*$$

$$\times U_k{}^\varepsilon{}_n(U_k{}^\rho{}_m)^*U_j{}^\nu{}_m(U_j{}^\mu{}_q)^*[B_m+B_p-(B_q+B_n)]$$

$$=N_f\, T^P_{\mu\nu}T^P_{\varphi\alpha}T^S_{\alpha\mu}T^S_{\nu\varphi}[B_\nu+B_\alpha-(B_\mu+B_\varphi)]\,, \qquad \text{(D.14)}$$

where N_f is the number of fermion families and

$$T^P_{\alpha\beta}\equiv W^{\gamma P}T^\gamma_{\alpha\beta} \qquad \text{(D.15)}$$

is real as discussed in appendix B. There is no summation on P or S in (D.14) because the bosons are specific mass eigenstates (neither the decaying nor the exchanged boson species is summed over). As a consequence of the reality of T^P and T^S it follows that

$$\mathrm{Im}(\mathbf{A}_{VV})=0\,. \qquad \text{(D.16)}$$

The expressions for $\mathbf{A}_{SV}$, $\mathbf{A}_{VS}$ and $\mathbf{A}_{SS}$ are not, in general, real*.

For $\mathbf{A}_{SV}$ there are two types of diagram that may contribute as shown in fig. D.2. Diagrams in which the directions of the fermion arrows are reversed are equivalent to one of the diagrams already listed. Fig. D.2a gives the contribution

*They are unreal.

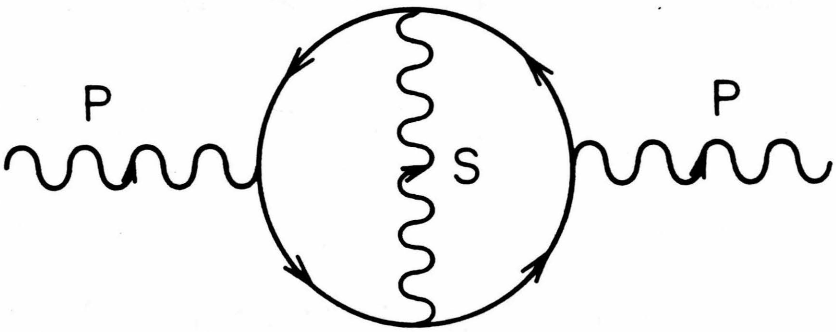

Fig. D.1: Vector decay with vector exchange. Arrows on the fermion lines indicate the flow of the left-handed helicity.

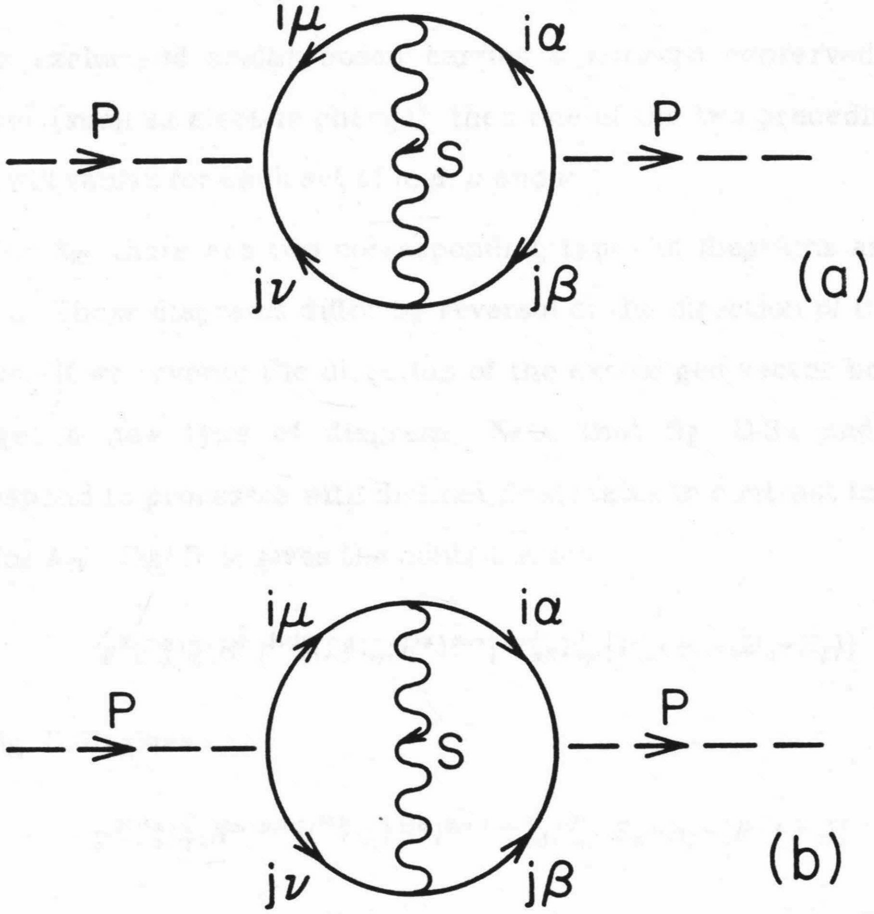

Fig. D.2: Scalar decay with vector exchange. Arrows on the fermion lines indicate the flow of the left-handed helicity.

$$g^2 T^P_{\nu\mu} T^P_{\alpha\beta} (\Gamma^a_S)^\gamma_{ij} (R^a)^{\alpha\mu\gamma} [(\Gamma^b_S)^\delta_{ij} (R^b)^{\beta\nu\delta}]^* \{B_\alpha - B_\beta - (B_\nu - B_\mu)\} \,. \qquad \text{(D.17)}$$

and fig. D.2b gives

$$g^2 T^P_{\nu\mu} T^P_{\alpha\beta} (\tilde{\Gamma}^a_S)^\gamma_{ij} (R^a)^{\alpha\mu\gamma} [(\tilde{\Gamma}^b_S)^\delta_{ij} (R^b)^{\beta\nu\delta}]^* [B_\alpha - B_\beta - (B_\nu - B_\mu)] \,. \qquad \text{(D.18)}$$

If the exchanged scalar boson carries a nonzero conserved quantum number (such as electric charge), then one of the two preceding expressions will vanish for each set of α, β, μ and ν.

For $\mathbf{A}_{VS}$ there are two corresponding types of diagrams as shown in fig. D.3. These diagrams differ by reversal of the direction of the fermion arrows. If we reverse the direction of the exchanged vector boson we do not get a new type of diagram. Note that fig. D.3a and fig. D.3b correspond to processes with distinct final states in contrast to the situation for $\mathbf{A}_{SV}$. Fig. D.3a gives the contribution

$$g^2 (\Gamma^b_S)^\delta_{ij} (R^b)^{\beta\nu\delta} [(\Gamma^a_S)^\gamma_{ij} (R^a)^{\alpha\mu\gamma}]^* T^P_{\alpha\beta} T^P_{\nu\mu} [B_\mu + B_\nu - (B_\alpha + B_\beta)] \qquad \text{(D.19)}$$

and fig. D.3b gives

$$g^2 (\tilde{\Gamma}^a_S)^\gamma_{ij} (R^a)^{\alpha\mu\gamma} [(\tilde{\Gamma}^b_S)^\delta_{ij} (R^b)^{\beta\nu\delta}]^* T^P_{\alpha\beta} T^P_{\nu\mu} [B_\alpha + B_\beta - (B_\mu + B_\nu)] \,. \qquad \text{(D.20)}$$

If S or P carry a conserved quantum number, then one of the two preceding expressions vanishes for each choice of α, β, μ and ν. Apart from the baryon number factors, eqns (D.19) and (D.20) are the complex conjugates of eqns (D.17) and (D.18) respectively.

For $\mathbf{A}_{SS}$ we again have two types of diagrams as shown in fig. D.4. The diagrams corespond to processes with distinct final states. Reversal of the direction of the exchanged scalar boson yields no new type of diagram. Fig. D.4a gives the contribution

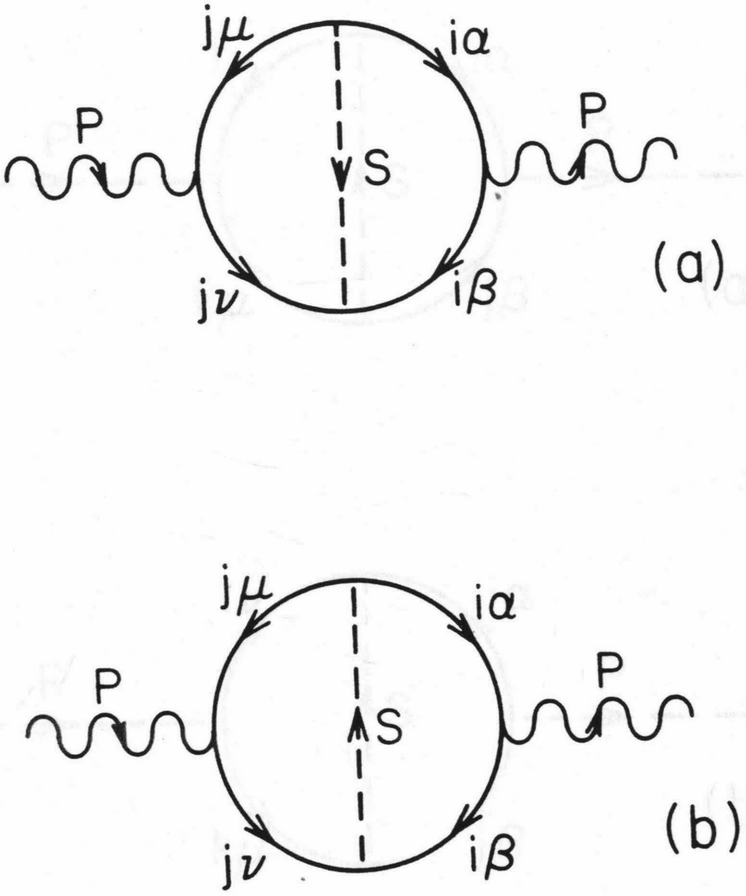

Fig. D.3: Vector decay with scalar exchange. Arrows on the fermion lines indicate the flow of the left-handed helicity.

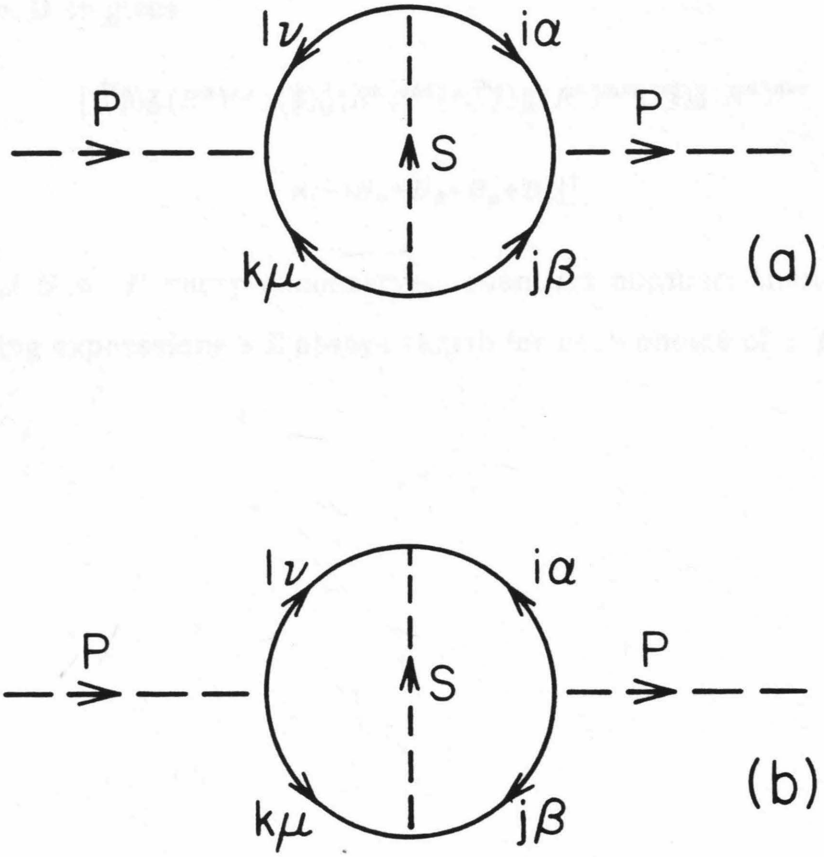

Fig. D.4: Scalar decay with scalar exchange. Arrows on the fermion lines indicate the flow of the left-handed helicity.

$$(\Gamma_P^a)_{lk}^{\gamma}(R^a)^{\nu\mu\gamma}(\tilde{\Gamma}_P^b)_{ij}^{\delta}(R^b)^{\alpha\beta\delta}[(\Gamma_S^c)_{jk}^{\eta}(R^c)^{\beta\mu\eta}(\tilde{\Gamma}_S^d)_{il}^{\varepsilon}(R^d)^{\alpha\nu\varepsilon}]^* \tag{D.21}$$

$$\times[B_\alpha+B_\beta+B_\mu+B_\nu]\,,$$

while fig. D.4b gives

$$[(\tilde{\Gamma}_P^a)_{lk}^{\gamma}(R^a)^{\nu\mu\gamma}(\Gamma_P^b)_{ij}^{\delta}(R^b)^{\alpha\beta\delta}]^*(\tilde{\Gamma}_S^c)_{jk}^{\eta}(R^c)^{\beta\mu\eta}(\Gamma_S^d)_{il}^{\varepsilon}(R^d)^{\alpha\nu\varepsilon} \tag{D.22}$$

$$\times[-\{B_\alpha+B_\beta+B_\mu+B_\nu\}]\,.$$

Again, if S or P carry a conserved quantum number, then one of the preceding expressions will always vanish for each choice of α, β, μ and ν.

Appendix E) Symmetries of the Illustrative SO(10) Model

In the model of chapter 9 we have a single family of fermions, 16_f, the obligatory adjoint of vectors, 45_V and, for the sake of the present discussion, we consider the following set of scalar representations: a 126_H, a complex 10_H ($=10_1+i10_2$ where 10_1 and 10_2 are real representations) and a 54_H. Later we may replace the complex 10 by a real 10 and thereby specialize to the discrete symmetry, 9.14, of the model of chapter 9. We choose the Yukawa terms in this model to be

$$A(16_f \cdot 16_f) \cdot \overline{126}_H + B(16_f \cdot 16_f) \cdot 10_H^\dagger + \text{h.c.} \tag{E.1}$$

Ignoring for the moment the terms in the Higgs potential, this model possesses a global $U(1)_X$ symmetry. If we call the generator of $U(1)_X$, X, then we have $X=1$ for 16_f, $X=2$ for 126_H and $X=2$ for 10_H; all other values for X are zero. These values corespond to the transformations

$$\left.\begin{array}{c} 16_f \to e^{i\alpha} 16_f \\ 126_H \to e^{i2\alpha} 126_H \\ 10_H \to e^{i2\alpha} 10_H \end{array}\right\}. \tag{E.2}$$

$U(1)_X$ may be broken down to a discrete symmetry by various terms in the Higgs potential: as long as this discrete symmetry is large enough the Yukawa terms (E.1) are natural. However, for the particular case where the discrete symmetry is that generated by the transformations

$$\left.\begin{array}{c} 16_f \to \pm i\, 16_f \\ 126_H \to -126_H \\ 10_H \to -10_H \end{array}\right\}. \tag{E.3}$$

the Yukawa terms (E.1) must either be augmented by the term

$$C(16_f \cdot 16_f) \cdot 10_H \tag{E.4}$$

or the theory may be modified so that the 10_H is real. The discrete symmetry (E.3) may be considered as a special case of (E.2) for $\alpha=\pm\pi/2$. We will therefore treat the theory as if it had the full $U(1)$ symmetry, specializing later to the case $\alpha=\pm\pi/2$ (and thus adding in the parameter C if the 10_H is complex). The symmetry (E.3) is the symmetry of the most general Yukawa coupling in this model. We will see below that although this symmetry is very simple it leads to a richer symmetry after $SO(10)$ has been broken via the $SU(5)$ singlet vacuum expectation value of the 126_H.

The global symmetry $U(1)_X$ is spontaneously broken when 126_H obtains a vacuum expectation value, as is the local group $U(1)_R$ appearing in $SO(10)\supset SU(5)\otimes U(1)_R$. $U(1)_R$ distinguishes among the terms in the $SU(5)$ decomposition of a given $SO(10)$ representation. We call the generator of $U(1)_R$, R. At temperatures sufficiently large so that $SU(3)\otimes SU(2)_L\otimes U(1)_Y$ is unbroken, 126_H will have a vacuum expectation value only along its $SU(5)$ singlet direction. At such a temperature a linear combination of X and R that vanishes along that direction will still generate a global $U(1)_Z$ symmetry even though $U(1)_R$ and $U(1)_X$ are separately spontaneously broken*. To determine the relevant linear combination we must compute the values of R for the components of 126_H (in particular the $SU(5)$ singlet component).

We call the values of R for the components in the decomposition $126=1+\bar{5}+10+\overline{15}+45+50$ respectively a, b, c, d, e and f. Similarly for $16=1+\bar{5}+10$ we use β, γ and δ; and for $10=\bar{5}+5$ (where 10 is real) we use ρ and $-\rho$ (where only in the last case have we implemented the

*At this temperature therefore no Goldstone bosons would appear even if $U(1)_X$ was respected by the Higgs potential. However, at lower temperatures this new global $U(1)_Z$ will generally be broken in a way that produces a Goldstone boson. Thus, we must break $U(1)_X$ down to a discrete symmetry in the Higgs potential.

tracelessness of X).

Two conditions on β, γ and δ can be obtained by considering the antisymmetric product of two 16's

$$(16\otimes 16)_A = 120 \,. \tag{E.5}$$

In the $SU(5)$ decomposition of the (real) 120,

$$120 = 5+\bar{5}+10+\overline{10}+45+\overline{45} \,, \tag{E.6}$$

the 5 and the $\bar{5}$ are conjugates of one another: similarly for the 10 and $\overline{10}$ and for the 45 and $\overline{45}$. Thus the values of R for each of these pairs are equal and opposite. The 5 in (E.6) arises in the combination

$$(\bar{5}\otimes 10 + 10\otimes\bar{5})_A \tag{E.7}$$

appearing in (E.5). Similarly the $\bar{5}$ comes from

$$(\bar{5}\otimes 1 + 1\otimes\bar{5})_A \,. \tag{E.8}$$

Thus it follows that

$$\gamma+\delta = -(\beta+\gamma) \,. \tag{E.9}$$

Similarly considering the 45 and $\overline{45}$ gives us

$$\gamma+\delta = -2\delta \,. \tag{E.10}$$

Thus

$$\gamma = -3\delta = -3; \quad \beta = 5\delta = 5 \,. \tag{E.11}$$

(We choose to normalize so that $\delta=1$.) If we now consider the symmetric product of two 16's,

$$(16\otimes 16)_S = 10 + 126 \tag{E.12}$$

and demand that the trace of R be zero, we find that

$$a=10,\ b=2,\ c=6,\ d=-6,\ e=-2,\ f=2 \text{ and } p=2\ . \tag{E.13}$$

The linear combination of X and R that we seek, Z, is then

$$Z = X-2\frac{R}{a}$$

$$= X-\frac{R}{5}. \tag{E.14}$$

The values of Z for the components of 16_f, 10_H, 54_H and 126_H are summarized in table E.1. As noted in chapter 9, when the X symmetry is broken to the discrete symmetry (E.3), the Z values 8/5 and 12/5 are equivalent as are the values 4/5 and 16/5.

$16_f = 1(0)+\bar{5}(\frac{8}{5})+10(\frac{4}{5})$
$10_H = \bar{5}(\frac{8}{5})+5(\frac{12}{5})$
$54_H = =15(0)+\overline{15}(0)+24(0)$
$126_H = 1(0)+\bar{5}(\frac{8}{5})+10(\frac{4}{5})+\overline{15}(\frac{16}{5})+45(\frac{12}{5})+50(\frac{8}{5})$

Table E. 1

The $SU(5)$ decompositions of some $SO(10)$ representations and their associated values of Z.

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