A SECOND ORDER SOLUTION FOR AN OSCILLATING, TWO-DIMENSIONAL, SUPERSONIC AIRFOIL

Thesis by

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SUMMARY

In this paper a second-order solution, for the forces and moments produced by an oscillating two-dimensional airfoil of arbitrary cross section, has been determined. This solution was obtained by means of an iteration procedure. In the iteration procedure it was necessary to have a linearized solution of simple, closed form which was valid throughout the whole x, y plane. Existing solutions did not satisfy these requirements, thus, it was first necessary to develop a new linearized or first-order velocity potential. This potential was developed as a power series approximation, in frequency, to the exact linearized solution. Six terms of this series were developed and this sixth-order solution shown to be within a few percent of the exact linearized solution for reduced frequencies ($\bar{\omega} = \frac{\omega \bar{C}}{\beta^2 V_o}$) less than 1.3.

The first two terms of the series approximation were then used in the iteration process to produce the second-order solution in thickness. This solution which is valid to second-order in thickness and frequency has been determined for an oscillating airfoil of general cross section.

The second-order terms were found to have a relatively strong influence on the final solution, particularly for the pitching moment. It will be seen in Section V that in many cases the second-order terms are larger in magnitude than the corresponding first order-terms and thus reverse the tendencies indicated by first-order theory. In particular, it was shown that the theoretical instability predicted by linearized theory for an airfoil of zero thickness is completely eliminated for an airfoil having a thickness ratio as small as three percent.

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SYMBOLS

A, B, D	functions of M_o and γ (See Fig. 8).
c, c_o	the speed of sound; the subscript zero refers to free stream conditions.
$\overline{C}$	chord of airfoil.
C_L	lift coefficient.
C_M	pitching moment coefficient.
C_p	pressure coefficient; subscripts u and ℓ refer to upper and lower surfaces.
C_{p_T}	pressure coefficient differential between upper and lower surfaces $C_{p_T} = C_{p_\ell} - C_{p_u}$.
f	an arbitrary function of $(x - y\beta)$ which is defined as $f(\xi) = f'(\xi) = 0$ for $\xi \leq 0$; $f'(\xi) = \frac{\partial f(\xi)}{\partial \xi}$, and f and f' are continuous.
g	an arbitrary function of $(x - y\beta)$ which is defined as $g(\xi) = g'(\xi) = 0$ for $\xi \leq \overline{C}$, and g and g' are continuous.
H	a function of the first order (in α) velocity potential which forms the non-homogeneous portion of the wave equation to be satisfied by the second order velocity potential.
M_o	free stream Mach number.
p, p_o	pressure; the subscript zero refers to free stream conditions.
q	perturbation velocity $= \sqrt{u^2 + v^2}$.
t	time.
T	thickness of airfoil.
u, v	perturbation velocities in the x, y directions.
V_o	free stream velocity.

x, y	coordinates, fixed in the leading edge of the airfoil with the positive x axis in the direction of the free stream
x_o	non-dimensional position of the point of rotation.
α	angle of attack = $\alpha_M \cos \omega t$.
α_M	maximum angle of attack; $\alpha_M << 1$.
$\dot{\alpha}$	partial derivative of α with respect to time.
β	$\sqrt{M_o^2 - 1}$.
γ	adiabatic exponent.
δ	constant; $\delta << 1$.
ε	constant; $\varepsilon << 1$; used as half thickness of wedge.
ρ	density.
$\bar{\bar{\Phi}}$	complete velocity potential.
$\bar{\Phi}$	complete perturbation velocity potential
ϕ and σ	portion of velocity potential used for discussion of flat plate and oscillating wedge.

Φ_n^m	perturbation velocity potential correct to m^{th} order in angle of attack α and n^{th} order in frequency ω.
ϕ_n^m	one term in series expansion of Φ_n^m, it includes the m^{th} term in α and the n^{th} term in frequency.
ϕ	the portion of ϕ_n^2 which satisfies the non-homogeneous wave equation; the ''particular solution''.
σ	the portion of ϕ_n^2 which satisfies the boundary conditions on the surface; the ''correction potential''.

ψ and θ	are used in place of ϕ and σ for the discussion of more complex airfoils.
ω	angular frequency in radians/sec.
$\bar{\omega}$	the reduced frequency = $\dfrac{\omega \bar{C}}{\beta^2 V_o}$

SECTION I. INTRODUCTION

The linearized theory for the supersonic, irrotational, non-viscous, flow of a perfect gas past an oscillating wing is at present in a reasonably complete state of development. Progressing from the first investigations of non-stationary supersonic flow, by Possio[1] and von Borbély[2], a number of authors including Garrick and Rubinow[3], [4], Temple and Jahn[5], and Miles[6] have considered the two-dimensional oscillating airfoil problem. In the last few years a number of solutions have been determined for non-stationary three-dimensional wings, for example; Stewart and Li[7] (rectangular wing), Miles[8], [9] (rectangular and delta wings), Hipsch[10] (delta wing with subsonic leading edge), Froelich[11] (delta wing with supersonic leading edge), and Parkinson[12] (delta wing with supersonic leading edge). Thus, it seems probable that within a few years the linearized solution, for almost any planform will be available.

From previous experience with stationary flows it is reasonable to expect that these solutions should give forces and moments which will be in good agreement with the actual forces and moments for some combinations of Mach number and thickness. However, for other combinations of Mach number and thickness the linearized results will probably be quite inaccurate and solutions which allow a larger range of validity will have to be obtained. Now the previously mentioned linearized solutions indicate the effect of Mach number but take no account of thickness. The purpose of this paper is to produce a solution which takes account of the effects of thickness. Or stated in another way, the reason for this study is to produce a second-order solution in angle of attack (and thickness) so that the effects of thickness on the forces and moments produced by an oscillating airfoil may be evaluated.

The prototype of the higher order solution in supersonic flow is Busemann's[13] series for the surface pressure on a two dimensional body in steady flow. More recently Van Dyke[14] has solved the problem of second-order steady supersonic flow past a body of revolution. In both of these cases the second-order solution gave a pressure coefficient which was sufficiently accurate for almost any purpose.

This study involves the use of the same method of attack as was used by Van Dyke[14] - - namely, an iteration process in which each step of the iteration is dependent on the previous solution. Hence, to determine the second-order velocity potential, the first-order potential must be available. However, to utilize the first-order potential in the iteration process it had to be defined throughout the whole x, y plane and also be of a comparatively simple closed form. Since none of the available solutions satisfied these requirements, a satisfactory first-order velocity potential had to be determined. This complete potential was calculated in the form of a series approximation to the exact linearized velocity potential.

In 1946 Guderley[15] produced the first two terms of the series approximation to the linearized velocity potential. This solution was complete (in the sense that it was valid throughout the whole x, y plane) and it was in a very simple form. It is this expression for the linearized potential which will be utilized in the determination of the second-order solution. The use of this form of the potential leads to a final solution which is accurate only when the frequency is less than an upper bound which is dependent on the number of terms of the expansion which are retained. Of course, n terms of the series could be retained and the second-order solutions made accurate to any desired degree, however, as might be expected the labor involved is more than directly proportional to the number of terms retained. The compromise in this study consisted of taking the first two terms of the series expansion. Thus, the final result is a solution valid to second-order in angle of attack and frequency [1]. This solution will be found accurate for most non-stationary problems that do not include flutter, and does indicate that the second order solution can probably not be neglected in the case of flutter.

In Section II the background material for this study is presented; it will include: A discussion of the assumptions and the iteration process; the differential equations which must be satisfied by the first- and second-order velocity potentials; the boundary conditions for the first- and second-order potentials; and the expression for the pressure coefficient as a function of the velocity potential.

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[1] There is a tendency for the expression ''order'' to become confusing in this paper since order can refer to the maximum angle of attack α_M, the thickness parameter ε and the frequency ω. The maximum angle of attack and the thickness determine the order of the solution in the usual sense of the word, i.e., the first-order (in α_M or ε) is the linearized solution. The maximum angle of attack and the thickness parameter are taken to be of similar magnitude, α_M is of $0(\varepsilon)$, and for brevity are referred to as order in angle of attack. Any order solution in angle of attack may then be expanded into a series solution in powers of the frequency. Terms of this expansion are referred to as order in frequency.

In Section III the series expansion of the linearized velocity potential is considered. The first two terms [2] of the series expansion are developed. This is the result as presented by Guderley(15). Guderley's paper did not indicate the method of development but presumably it was similar to that given here. Later in this Section the same method is applied to produce the first six terms of the series [3]. The forces produced by the series expansion solution are then compared with the exact solution to find out in what range the series solution is valid.

In Section IV the second-order potential, in angle of attack, is determined for the case of an oscillating airfoil of zero thickness. The expressions for the lift and moment are then determined.

Section V extends the results of Section IV, first to the case of a wedge, secondly, to the case of a double wedge and finally, to an airfoil of general shape. The numerical results for a modified double wedge section are calculated and compared with the first-order solution. It is shown that the instability of an oscillating airfoil of zero thickness as indicated by linearized theory is actually non-existent for any feasible airfoil section when second-order terms are considered.

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[2] In this paper the expression first n terms will mean the solution valid to the n^{th} order.

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[3] Although only the first two terms of the series are to be used in determining the second-order solutions in angle of attack the series expansion is, by itself, of value. First it is in a simple closed form while the exact solution may only be solved by means of tables, and thus, the series solution has immediate application in a Fourier solution of other non-oscillatory motions. Secondly, it is valid away from the airfoil (i.e., $y \neq 0$) and hence, may be of interest for interaction or interference problems.

SECTION II. THE ITERATION PROCEDURE

In this section the general results which are necessary for the application of the iteration procedure to a specific problem are presented. The basic assumptions are outlined. Under these assumptions an exact perturbation potential is determined. Then, by means of the iteration process the linear differential equations, which must be satisfied by the first- and second-order potentials, are calculated. The boundary conditions which these potentials must satisfy are then determined, these boundary conditions together with the differential equations completely specify the unknown potentials. Finally, the second-order expression for the pressure is calculated as a function of the velocity potential.

Basic Assumptions

The problem to be considered is the study of the surface pressures produced by the uniform flow of a perfect gas past an oscillating airfoil of general shape (See Fig. 1).

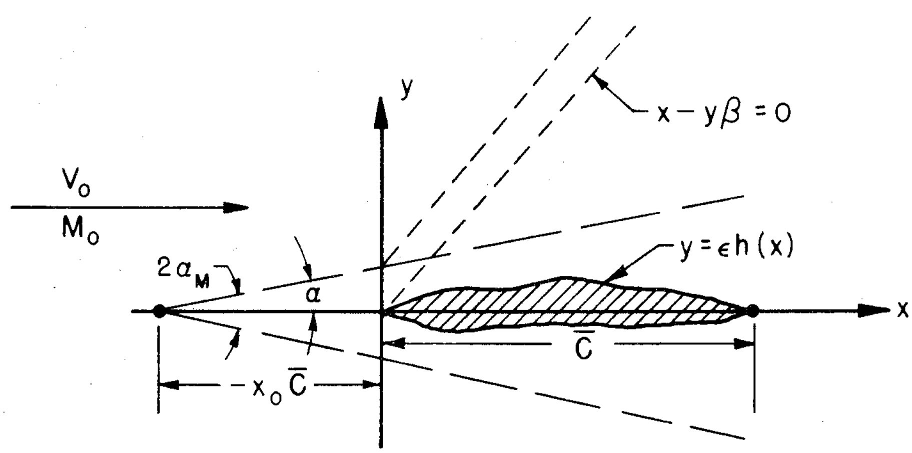

Fig. 1

The airfoil is required to be slender and the maximum angle of attack α_M is required to be small. The slenderness is measured by the symbol ε since the thickness is given by the equation $y = \varepsilon\, h(x)$. Thus, by requiring that α_M and ε be much less than 1, and that the slope of the surface at any point be small (of $O(\varepsilon)$), the perturbation velocities at any point on the surface are required to be small as compared to the free stream velocity. The coordinate system is chosen so that the center of coordinates is fixed in the leading edge of the airfoil when $\alpha = 0$. The x axis is in the direction of the free stream velocity. The airfoil rotates about a point ($x = x_o \bar{C}$) located a non-dimensional distance x_o downstream from the leading edge. The angle of attack is defined by the symbol $\alpha = \alpha_M \cos \omega t$, where ω is the angular velocity of the oscillation. The angular velocity ω is limited by requiring that the reduced frequency $\bar{\omega}$ ($\bar{\omega} = \dfrac{\omega \bar{C}}{\beta^2 V_o}$) be small. The flow is required to be uniform upstream from the leading edge and to be irrotational, isentropic and non-viscous. The assumption that the flow is irrotational and isentropic is valid in this problem since the effects of rotationality and changes in entropy are of third and higher order in angle of attack.

The Exact Potential Equation

Under the limitations of the previous assumptions there are, in two dimensions the following well known results;

Euler's Equations,

$$\begin{aligned} \bar{u}_t + \bar{u}\,\bar{u}_x + \bar{v}\,\bar{u}_y &= -1/\rho\, p_x , \\ \bar{v}_t + \bar{u}\,\bar{v}_x + \bar{v}\,\bar{v}_y &= -1/\rho\, p_y , \end{aligned} \tag{1}$$

here $\bar{u}$ and $\bar{v}$ are the velocity components in the x and y directions, ρ is the density and p the pressure, The Continuity Equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \bar{u}) + \frac{\partial}{\partial y}(\rho \bar{v}) = 0 \,, \tag{2}$$

and

The Condition of Irrotationality,

$$\bar{v}_x = \bar{u}_y \,. \tag{3}$$

Now since the flow is irrotational and isentropic there exists a velocity potential $\bar{\bar{\Phi}}$ such that $\bar{u} = \bar{\bar{\Phi}}_x$ and $\bar{v} = \bar{\bar{\Phi}}_y$, and the speed of sound c is given by the relationship,

$$c^2 = \frac{\gamma p}{\rho} \,. \tag{4}$$

Replacing the velocity components $\bar{u}$ and $\bar{v}$ by $\bar{\bar{\Phi}}_x$ and $\bar{\bar{\Phi}}_y$ and combining Eqs. (1), (3) and (4), results in the non-stationary Bernoulli Equation,

$$\bar{\bar{\Phi}}_t + \frac{1}{2} q^2 + \frac{c^2}{\gamma - 1} = \frac{c_0^2}{\gamma - 1} \,. \tag{5}$$

Here q is the total velocity ($q^2 = \bar{u}^2 + \bar{v}^2 = \bar{\bar{\Phi}}_x{}^2 + \bar{\bar{\Phi}}_y{}^2$), and the subscript zero refers to free stream conditions. Combining Eqs. (2) and (5) the potential equation is found to be

$$\bar{\bar{\Phi}}_{xx}(c^2 - \bar{\bar{\Phi}}_x^2) + \bar{\bar{\Phi}}_{yy}(c^2 - \bar{\bar{\Phi}}_y^2) - \bar{\bar{\Phi}}_{tt} - 2\bar{\bar{\Phi}}_x \bar{\bar{\Phi}}_y \bar{\bar{\Phi}}_{xy} - 2\bar{\bar{\Phi}}_x \bar{\bar{\Phi}}_{xt} - 2\bar{\bar{\Phi}}_y \bar{\bar{\Phi}}_{yt} = 0 \,. \tag{6}$$

These last two equations are now rewritten in tensor notation since the iteration procedure is considerably shortened by use of this form of the potential.

$$\frac{\partial \bar{\bar{\Phi}}}{\partial t} + \frac{1}{2}\frac{\partial \bar{\bar{\Phi}}}{\partial X_i}\frac{\partial \bar{\bar{\Phi}}}{\partial X_i} + \frac{c^2}{\gamma - 1} = \frac{c_0^2}{\gamma - 1} \,, \tag{5a}$$

and

$$\frac{\partial^2 \bar{\bar{\Phi}}}{\partial t^2} + 2\frac{\partial \bar{\bar{\Phi}}}{\partial X_i}\frac{\partial^2 \bar{\bar{\Phi}}}{\partial X_i \partial t} + \frac{\partial \bar{\bar{\Phi}}}{\partial X_i}\frac{\partial \bar{\bar{\Phi}}}{\partial X_j}\frac{\partial^2 \bar{\bar{\Phi}}}{\partial X_i \partial X_j} = c^2 \frac{\partial^2 \bar{\bar{\Phi}}}{\partial X_i \partial X_i} \,. \tag{6a}$$

Solution by Iteration

The complete potential equations Eq. (6) or (6a) is a non-linear equation and hence, may not be solved directly. The approach to be used is that first suggested by Prandtl[16]. It was later applied to planar subsonic flow by Görtler[17], Hantzsche and Wendt[18], [19], Imai and Oyama[20], [21] and Kaplan[22], [23]. Schmieden and Kawalki[24] applied the same method to subsonic flow past an ellipsoid of revolution. Van Dyke[14] applied it to supersonic flow past a body of revolution. In this method of solution it is assumed that the velocity potential $\bar{\Phi}$ may be approximated by the expression,

$$\bar{\Phi} = V_0 x + \alpha \phi^1 + \alpha^2 \phi^2 + \alpha^3 \phi^3 + \cdots + \alpha^n \phi^n. \tag{7}$$

Since the term $V_o x$ has been included in Eq. (7) the ϕ's represent the perturbation potential, ϕ^n representing the n^{th} order potential. Since in this study only the second-order potential is desired, the terms ϕ^n for $n > 2$ may be neglected. The derivatives of $\bar{\Phi}$ are given by the expressions,

$$\bar{\Phi}_{x_i} = V_0\, \delta_i^x + \alpha\, \phi^1_{x_i} + \alpha^2 \phi^2_{x_i} + \cdots ,$$

$$\bar{\Phi}_t = \alpha\, \phi^1_t + \alpha^2 \phi^2_t + \cdots .$$

δ_i^x is the Kronecker delta symbol so that $\delta_i^x = \begin{cases} 0 \text{ for } i \neq x \\ 1 \text{ for } i = x \end{cases}$

Putting the series expansion Eq. (7) for $\bar{\Phi}$ into Eq. (6a), and replacing the speed of sound c in Eq. (6a) by its corresponding function of $\bar{\Phi}$ from Eq. (5a), the potential equation becomes:

(terms containing α)

$$\Box \phi^1 = \beta^2 \phi^1_{xx} - \phi^1_{yy} + \frac{2M_0^2}{V_0} \phi^1_{xt} + \frac{M_0^2}{V_0^2} \phi^1_{tt} = 0 , \tag{8}$$

here $\frac{V_o}{c_o}$ has been replaced by the Mach number M_o and $\beta = \sqrt{M_o^2 - 1}$,

(terms containing α^2)

$$\begin{aligned} \Box \phi^2 &= H(\phi^1) \\ &= -\frac{M_0^2}{V_0^2} \Big\{ (\gamma-1)(\phi^1_{xx} + \phi^1_{yy})(\phi^1_x + \tfrac{1}{V_0} \phi^1_t) \\ &\quad + 2\phi^1_x (\phi^1_{xx} + \tfrac{1}{V_0} \phi^1_{xt}) + 2\phi^1_y (\phi^1_{xy} + \tfrac{1}{V_0} \phi^1_{yt}) \Big\} . \end{aligned} \tag{9}$$

These are the differential equations which must be satisfied by the first- and second-order potentials ϕ^1 and ϕ^2. Both of these equations are linear and may be solved by the usual methods applicable to linear differential equations. Although only the first- and second-order equations are derived here the higher order equations may easily be calculated. The higher order equations would be of the form,

$$\Box\phi^3 = I(\phi^1, \phi^2),$$

$$\Box\phi^4 = J(\phi^1, \phi^2, \phi^3),$$

etc.

The method of solution is now apparent. First one obtains the first-order solution ϕ^1 which satisfies Eq. (8), the homogeneous wave equation, and the proper first-order boundary conditions. Then the function $H(\phi^1)$ is determined. Next a second-order potential ϕ^2, which satisfies Eq. (9) (the non-homogeneous wave equation) and the proper second-order boundary conditions, is calculated.

The expression for the speed of sound c will be needed later. From Eq. (5a)

$$c^2 = c_0^2 - \left(\frac{\gamma-1}{2}\right)\left(\bar{\bar{\Phi}}_{x_i}\bar{\bar{\Phi}}_{x_i} - V_0^2 + 2\bar{\bar{\Phi}}_t\right),$$

or

$$c^2 = c_0^2 - \left(\frac{\gamma-1}{2}\right)\Big\{\alpha\left[\phi_t^1 + V_0\phi_x^1\right] + \frac{\alpha^2}{2}\left[2V_0\phi_x^2 + (\phi_y^2)^2 + (\phi_x^2)^2 + 2\phi_t^2\right]\Big\}. \qquad (10)$$

An interesting and important point in the application of the iteration process has been investigated in detail by Van Dyke. The next two and a half paragraphs are a shortened version of Van Dyke's discussion, for a more complete survey of the problem see Ref. 14. It should be kept in mind that this particular discussion concerns only stationary flow.

It will be noted that this iteration process has a superficial resemblance to the Picard process for hyperbolic equations in two independent variables (Reference: Courant and Hilbert, Vol. II, p.317). There is however, an essential difference. In the Picard process the characteristic lines of the differential equations are known at the outset since the functions F_n do not depend on the highest order derivative. In the iteration process however, the characteristic lines are unknown and it might seem that the characteristics would have to be revised after each step of the iteration process. Thus, each step except the first, would involve equations with non-constant coefficients. The subsonic counterpart of such a procedure is known to converge under proper conditions (Reference: Courant and Hilbert, Vol. II, pp. 288-289).

The iteration process as chosen makes no provision for such a revision and at each stage of the iteration process the equation has the original characteristics of the first-order potential (i.e., $\frac{dy}{dx} = \frac{1}{\beta}$, here only the downstream running characteristics are considered). These are the Mach lines of the undisturbed flow and are also characteristics in the mathematical sense (Reference: Courant and Hilbert, Vol. II, Chapter 5). Now the second-order characteristics in the physical sense, that is lines along which discontinuities in velocity are propagated, depend on the second-order potential and are different from the mathematical second-order characteristics which are still given by $\frac{dy}{dx} = \frac{1}{\beta}$. However, when the physical characteristics are calculated from the second-order potential they are found to be coincident with the expected revised characteristics. Hence, if discontinuities do not occur, the mathematical characteristics (those of the free stream) behave physically as if they had been revised.

This connection between the original and revised characteristics can be interpreted physically. The right hand side, of the non-homogeneous wave equation which is to be satisfied by the second-order potential, may be regarded as representing the effect of a known distribution of supersonic sources throughout the flow field. The influence of this source distribution spreads downstream along the original characteristics. The resulting velocity changes are just such that the second-order velocities become constant along the revised rather than the original characteristics. As was previously mentioned this discussion was valid for the stationary case or in the limiting case $\omega \to 0$. For the general two-dimensional non-stationary problem the characteristics surfaces are too complicated for a discussion of this type, however, the method is mathematically the same for the non-stationary as it is in the stationary case/thus it is apparent that the mathematical and physical results must again be compatible. The discussion of the failure of the iteration procedure to give a valid solution in the neighborhood of a discontinuity in the flow will be taken up later.

Boundary Conditions

The flow is required to satisfy two boundary conditions:

(1) The resultant velocity must be tangent to the surface of the body.

(2) All perturbation velocities must vanish upstream from the plane $x = 0$.

From the theory of hyperbolic differential equations (Reference: Courant and Hilbert, Vol. II, p. 172) it is known that these two requirements are sufficient to determine the solution.

Consider the Tangency Condition

If the surface of the plate is given as y = h (x, t), the tangency condition at the surface of the body is

$$\left(\Phi^2_y\right)_{y=h(x,t)} - \frac{\partial h}{\partial t} = \left(\frac{\partial h}{\partial x}\right)\left(V_0 + \Phi^2_x\right)_{y=h(x,t)} . \tag{11}$$

Here the terms on the left represent the difference in vertical velocity between the flow and the plate while the term $V_o + \Phi^2_x$ is the resultant velocity in the direction of flow. If Φ^2 is replaced by $\phi^1 + \phi^2$, where ϕ^1, $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial t}$ are of the order α while ϕ^2 is of order α^2, the boundary condition at the surface to first-order in α is:

$$\left(\phi^1_y\right)_{y=0} = V_0 \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} . \tag{12}$$

The exact boundary condition is

$$\left[\frac{\Phi^2_y - \frac{\partial h}{\partial t}}{V_0 + \Phi^2_x}\right]_{y=h} = \frac{\partial h}{\partial x} ,$$

or since $\Phi^2 = \phi^1 + \phi^2$ it follows that

$$\left(\phi^1_y + \phi^2_y\right)_{y=h} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\left(V_0 + \phi^1_x + \phi^2_x\right)_{y=h} .$$

Then since

$$\frac{\partial h}{\partial x}\left(V_0 + \phi^1_x + \phi^2_x\right)_{y=h} = \frac{\partial h}{\partial x}\left(V_0 + \phi^1_x\right) + O(\alpha^3) ,$$

and

$$\left(\phi^1_y + \phi^2_y\right)_{y=h} = \left(\phi^1_y\right)_{y=h} + \left(\phi^2_y\right)_{y=0} + O(\alpha^3) ,$$

the second-order boundary condition may be expressed as,

$$\left(\phi^2_y\right)_{y=0} = \left(-\phi^1_y\right)_{y=h} + \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\left(V_0 + \phi^1_x\right)_{y=0} .$$

Now eliminating $V_o \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}$ by means of Eq. 12 the second-order boundary condition is seen to be,

$$\left(\phi^2_y\right)_{y=0} = \left(\phi^1_x\right)_{y=0} \frac{\partial h}{\partial x} + \left[\left(\phi^1_y\right)_{y=0} - \left(\phi^1_y\right)_{y=h}\right] . \tag{13}$$

Consider the condition that the perturbation potential vanish for $x \lessgtr 0$:

From the theory of hyperbolic differential equations (Reference: Courant and Hilbert, Vol. II) this requirement may be stated and satisfied in two physically different and yet mathematically similar ways. Since both definitions are used later the two definitions will be presented.

The first approach requires that two conditions be imposed on the so-called time-like surface x = 0 and one condition along the space-like surface of the body. The previously mentioned tangency condition is space-like hence, we have to impose two conditions on the plane x = 0. These are,

$$\bar{\Phi}(0, y, t) = 0 \, , \quad \text{and} \quad \bar{\Phi}_x(0, y, t) = 0 \, . \tag{14}$$

These conditions along with the differential equation and the tangency condition completely specify the problem.

The second approach specifies the potential upstream by requiring that the potential be zero on the Mach line from the leading edge of the airfoil.

$$\phi'(x, \frac{x}{\beta}, t) = 0 \tag{15}$$

As expressed by Eq. (15) the potential goes to zero on the Mach wave through the origin and not on the oscillating Mach line from the leading edge. However, since the movement of the airfoil is considered to be small, the movement of the Mach wave only produces effects of second-order in angle of attack. Therefore, since this criterion is only used in the determination of the first-order solution the second-order effects may be neglected. Physically this requirement corresponds to the known fact that the perturbation potential is zero upstream from the disturbance produced by the leading edge, and since the potential is continuous it must in first-order theory be zero on the Mach line from the leading edge. Mathematically this approach corresponds to specifying one space-like condition (the tangency) and one condition along a characteristic.

The Pressure Coefficient:

The flow is isentropic, hence,

$$p/p_o = \left(c^2/c_o^2\right)^{\gamma/\gamma-1}.$$

Thus, the pressure coefficient $C_p = \dfrac{p - p_o}{\frac{1}{2}\rho_o V_o^2}$ is given by the expression,

$$C_p = \frac{2}{\rho_o V_o^2}\left[\left(\frac{c^2}{c_o^2}\right)^{\frac{\gamma}{\gamma-1}} - 1\right].$$

After substituting $\dfrac{c^2}{c_o^2}$ from Eq. (10),

$$C_p = \frac{2}{\rho_o V_o^2}\left\{\left[1 - M_o^2(\gamma+1)\left[\frac{1}{V_o}\Phi_x + \frac{1}{V_o^2}\Phi_t + \frac{1}{2V_o^2}\left(\Phi_x^2 + \Phi_y^2\right)\right]\right]^{\frac{\gamma}{\gamma-1}} - 1\right\},$$

and since the right-hand portion of the function in the bracket is less than one the bracket may be expanded to give

$$C_p = -2\left\{\frac{1}{V_o}\Phi_x + \frac{1}{V_o^2}\Phi_t - \frac{1}{2V_o^2}\left[\beta^2(\Phi_x)^2 - (\Phi_y)^2 + \frac{2M_o}{V_o}\Phi_x\Phi_t + \frac{M_o^2}{V_o^2}(\Phi_t)^2\right]\right\}. \quad (16)$$

The third and higher order terms have of course been neglected.

SECTION III. FIRST-ORDER SOLUTION IN ANGLE OF ATTACK

It was shown in Section II that the first-order velocity potential ϕ^1 must satisfy the following conditions,

$$\Box\phi'=0 \qquad \text{(See Eq. 8)}, \qquad (17A)$$

$$(\phi'_y)_{y=0} = V_0 \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \qquad \text{(See Eq. 12)}, \qquad (17B)$$

$$\phi(x=y\beta) = 0 \qquad \text{(See Eq. 15)}. \qquad (17C)$$

These conditions exactly specify the linearized velocity potential and, as was previously pointed out, numerous solutions which satisfy Eqs. (17) are available. However, these solutions cannot (See Section II) be used in the iteration procedure. In this Section a first-order potential ϕ^1 (x, y, t) will be developed which is both in closed form and valid throughout the whole flow field. However, the solution will no longer be an exact first-order solution but will consist of the first 6 terms of the series expansion in frequency (of the exact linearized solution).

This Section is divided into three parts. In the first part the expression determined by Guderley(15), that is the second-order solution in frequency, is derived in detail. The second part briefly utilizes the same method to extend the solution to sixth-order. In part three the expressions for the lift and pitching moment coefficients of an airfoil of zero thickness are calculated and compared with the exact solutions.

Guderley's Second Order Solution in Frequency

In 1946 Guderley(15) wrote a paper entitled ''The Pressure Distribution on a Flat Plate Oscillating with a Small Frequency''. In this unpublished paper he produced without any development the first two terms of the series expansion in frequency (of the linearized velocity potential) for the flow over an oscillating flat plate. This same solution will now be developed. The method of attack is interesting because it allows an algebraic determination of any desired number of the terms of the series expansion. The method is not restricted to oscillatory motions, but actually appears to be useful in the solution of any continuous non-stationary, supersonic airfoil problem. To be more specific it is believed that this method could lead to first- and second-order solutions to the problems of an airfoil experiencing a constant rate of change of angle of attack, an airfoil experiencing a constant acceleration in forward velocity or other continuous type motions.

The location of the surface of the airfoil (See Fig. 1) is given by the relation

$$y = \alpha\,(x - x_o\,\bar{C}), \qquad (\text{for } 0 \leq x \leq \bar{C})$$

where $\alpha = \alpha_M \cos \omega t$. Hence, the slope $(\frac{\partial y}{\partial x})$ and the vertical velocity $(\frac{\partial y}{\partial t})$ of the plate are,

$$\frac{\partial y}{\partial x} = \alpha$$

and [4]

$$\frac{\partial y}{\partial t} = \dot{\alpha}\,(x - x_o\,\bar{C}).$$

The tangency condition, Eq. (17B) is thus,

$$(\phi'_y)_{y=0} = V_0\,\alpha + \dot{\alpha}(x - x_o\bar{C}) \; . \tag{17D}$$

[4] Although it will not be specifically mentioned in the remainder of this paper, the equations giving the location of the surface of the airfoil, the vertical velocity of the airfoil, etc., are of course restricted to the region 0 - x - chord. Similarly the velocity potential is restricted to an appropriate strip of the x, y plane.

Now consider the series expansion of ϕ^1 in powers of a non-dimensionalized frequency $\bar{\omega}$. The so-called reduced frequency $\bar{\omega}$ ($\bar{\omega} = \frac{\omega \bar{C}}{\beta^2 V_o}$) is initially required to be small ($\bar{\omega} \ll 1$) in order that the higher order terms in frequency may be neglected. Later it will be seen that the sixth-order solution gives a good approximation to the exact solution for $\bar{\omega} < 1.4$. The series approximation to ϕ^1 is

$$\phi' = \phi_1' + \bar{\omega}\phi_2' + \bar{\omega}^2\phi_3' + \cdots + \bar{\omega}^n\phi_{n+1}' \quad . \tag{18}$$

Where ϕ_n^1 is the nth order term in the series. It should be noted that while the so-called first-order term in frequency does not contain $\bar{\omega}$, this term does contain the parameter $\alpha = \alpha_M \cos \omega t$. Thus, this term is due to the instantaneous position of the airfoil and is the first-order term in the frequency expansion.

From Eq. (17D) it was seen that ϕ_y^1 at y = 0 was of the form,

$$(\phi_y')_{y=0} = V_o \alpha + \dot{\alpha}(X - X_o\bar{C}) \quad .$$

Hence, ϕ_2^1 must be,

$$\phi_2' = \int_0^y (\phi_y')_{y=0} dy + \alpha\{AX + DY^2 + FX^2 + HXY + JX^3 + \cdots\} + \dot{\alpha}\{BX + EY^2 + GX^2 + IXY + KX^3 + \cdots\} \tag{19}$$

where D, E $\cdots$ K are functions of Mach number and velocity. The higher terms in frequency (i.e., $\ddot{\alpha}$, $\dddot{\alpha}$, etc.) are not included since only a second-order solution in frequency is desired. ϕ_2^1 may now be determined algebraically by substituting Eq. (19) into Eqs. (17A) and (17C). When this is done it will be seen that a set of n algebraic equations are derived which are exactly sufficient to determine the n unknown coefficients D, E $\cdots$ n. To simplify the algebra the unknown coefficients J $\cdots$ n will be taken to be zero immediately, although the same result would be obtained if they were allowed to remain through the derivation.

Integrating Eq. (19) it is found that ϕ_2^1 is of the form,

$$\begin{aligned}\phi_2' &= yV_0\alpha + yx\dot{\alpha} - y\dot{\alpha}X_0\bar{c} \\ &+ \alpha\{Ax + Dy^2 + Fx^2 + Hxy\} \\ &+ \dot{\alpha}\{Bx + Ey^2 + Gx^2 + Ixy\} \quad .\end{aligned} \tag{20}$$

Now substituting Eq. (20) into Eqs. (17A) and (17C) it follows that:

$$\begin{aligned}\Box\phi_2' = 0 = \beta^2(2F\alpha + 2G\dot{\alpha}) - 2(D\alpha + E\dot{\alpha}) \\ + \frac{2M_0^2}{V_0}(A\dot{\alpha} + 2xF\dot{\alpha} + 2Hy\dot{\alpha}) + O(\ddot{\alpha}),\end{aligned}$$

and

$$\begin{aligned}\phi(y = \tfrac{x}{\beta}) = 0 = \frac{V_0\alpha x}{\beta} + \frac{x^2\dot{\alpha}}{\beta} - \frac{xX_0\bar{c}\dot{\alpha}}{\beta} + A\alpha x + B\dot{\alpha}x + \frac{D\alpha x^2}{\beta^2} \\ + \frac{E\dot{\alpha}x^2}{\beta} + F\alpha x^2 + G\dot{\alpha}x^2 + \frac{H\alpha x^2}{\beta} + \frac{I\dot{\alpha}x^2}{\beta} \quad .\end{aligned}$$

Rearranging these two equations according to the variables α, $\dot{\alpha}$, x and y,

$$\begin{aligned}&\alpha(2F\beta^2 - 2D) = 0 \quad , \\ &\dot{\alpha}(2G\beta^2 - 2E + \frac{2M_0^2}{V_0}A) = 0 \, , \\ &x\dot{\alpha}(\frac{4M_0^2}{V_0}F) = 0 \quad , \\ &y\dot{\alpha}(\frac{4M_0^2}{V_0}H) = 0 \quad , \\ &x\alpha(\frac{V_0}{\beta} + A) = 0 \quad , \\ &x^2\alpha(\frac{D}{\beta^2} + F + \frac{H}{\beta}) = 0 \quad , \\ &x\dot{\alpha}(B - \frac{X_0\bar{c}}{\beta}) = 0 \quad , \\ &x^2\dot{\alpha}(\frac{1}{\beta} + \frac{E}{\beta^2} + G + \frac{I}{\beta}) = 0 \quad .\end{aligned}$$

a set of eight simultaneous algebraic equations with eight unknowns, is determined. Again note that if twenty unknown coefficients had been used there would have been twenty simultaneous equations and the last twelve coefficients would be given as zero. Solving these eight simultaneous equations the coefficients are found to be

$$A = -\frac{V_o}{\beta}, \qquad F = 0,$$
$$B = x_o\bar{c}/\beta, \qquad G = 1/2\beta^3,$$
$$D = 0, \qquad H = 0,$$
$$E = \frac{1-2M_o^2}{2\beta}, \qquad I = 0.$$

Hence, to second-order in frequency the linearized velocity potential is given as

$$\phi' = y^2\left(\frac{\dot{\alpha}}{2\beta}\right)\left(1-2M_o^2\right) + y\left[\dot{\alpha}(x-x_o\bar{c}) + \alpha V_o\right] + \frac{x^2}{\beta^2}\left(\frac{\dot{\alpha}}{2\beta}\right) + \frac{x}{\beta}\left[x_o\bar{c}\,\dot{\alpha} - \alpha V_o\right]. \tag{21}$$

It is apparent that some unnecessary work was done in this derivation by including extra terms in Eq. (19). For example D, F, H and I could have been immediately eliminated by realizing that as $\omega \to 0$ the expression for the potential must approach the solution for the flow over a stationary surface at an angle of attack $\alpha = \alpha_M \cos \omega t$. Since the solution for this problem is known to be

$$\phi = V_o\, \alpha \left(y - \frac{x}{\beta}\right)$$

D, F and H could then have been eliminated. Eq. (21) is the solution given by Guderley for the potential of a flat plate oscillating such that $\bar{\omega} << 1$. As will be seen later in this Section this solution approximates the exact linearized solution within a few percent for $\bar{\omega} \leq .3$ (this would include such motion as control movements, stability, etc.).

The Sixth-Order Solution in Frequency

In this part the previously determined second-order solution will be extended to include the linearized solution valid to sixth-order in frequency. As will be seen later the sixth-order solution is within

1 percent of the exact for $\overline{\omega} \leq .9$, while the previously obtained solution is only valid to 1 percent for $\overline{\omega} \leq .15$. Hence, for many flutter problems the simple final form for the forces and moments as given by the sixth-order solution will give satisfactory results. A linearized potential of the form,

$$\phi' = \phi_1' + \overline{\omega}\,\phi_2' + \overline{\omega}^2\phi_3' + \cdots + \overline{\omega}^5\phi_6' ,$$

must now be determined such that ϕ^1 satisfies the conditions,

$$(\phi_y')_{y=0} = V_0\alpha + \dot{\alpha}(x - x_0\bar{c}),$$

$$\phi'(y = x/\beta) = 0, \qquad \text{See Eq. (17)}.$$

$$\Box\phi' = 0 + O(\overline{\omega}^6).$$

The method of solution is exactly the same as before and is made especially easy if two simple rules are followed,

a) No term $Ax^n y$ may appear since $\phi_y \neq 0$ and the tangency conditions would be changed.

b) The symmetry shown in the first two terms of the series expansion is continued throughout the higher terms and thus

$$\phi_n^1 = F\,(x^m y^p) \text{ where } \left(\begin{array}{c} m + p = n \\ 0 \leq m \text{ and } p \leq n \end{array}\right).$$

The final sixth-order solution in frequency that satisfies the requirements of Eqs. (17) to $O(\overline{\omega}^6)$ is presented below.

$$\Phi_6' = V_0\alpha\left\{y - \frac{x}{\beta}\right\} + \dot{\alpha}\left\{y^2\left(\frac{1-2M_0^2}{2\beta}\right) + xy + x^2\left(\frac{1}{2\beta^3}\right) - x_0\bar{c}\left(y - \frac{x}{\beta}\right)\right\}$$

$$- \frac{\ddot{\alpha}M_0^2}{V_0}\left\{x^3\left(\frac{1}{4\beta^5}\right) - y^3(1/2) + xy^2\left(\frac{2\beta^2-1}{4\beta^3}\right) - \frac{x_0\bar{c}}{2\beta}\left(\frac{x^2}{\beta^2} - y^2\right)\right\}$$

$$+ \frac{\dddot{\alpha}M_0^2}{V_0^2}\left\{x^4\left(\frac{1+4M_0^2}{48\beta^7}\right) - y^4\left(\frac{3-12M_0^2+8M_0^4}{48\beta^3}\right) + \frac{xy^3}{6} - \frac{x^2y^2}{8\beta^5}\right.$$

$$\left. - x_0\bar{c}\left(-\frac{x^3}{12\beta^5}(2M_0^2+1) + \frac{xy^2}{4\beta^3} + \frac{y^3}{6}\right)\right\} - \frac{\overset{IV}{\alpha}M_0^4}{192V_0^3}\left\{x^5\left(\frac{3+4M_0^2}{\beta^9}\right) - 8y^5\right.$$

$$- \frac{10x^3y^2}{\beta^7} + \frac{xy^4}{\beta^5}(8M_0^4 - 20M_0^2 + 15) - 4x_0\bar{c}\left(\frac{x^4}{\beta^7}(2M_0^2+3) - \frac{6x^2y^2}{\beta^5}\right.$$

$$+\frac{y^4}{\beta^3}(3-2M_0^2)\Big)\Big\} + \frac{\overset{V}{\alpha} M_0^4}{1920 V_0^4}\Big\{X^6\Big(\frac{8M_0^4+9M_0^2+1}{\beta^{11}}\Big) - \frac{5X^4y^2}{\beta^9}(15M_0^2+1)$$
$$+\frac{4X^5X_0\bar{c}}{7\beta^9}(28M_0^4+109M_0^2+8) + \frac{40X_0\bar{c}X^3y^2}{7\beta^7}(11M_0^2-6) + \frac{15X^2y^4}{\beta^7}(1-3M_0^2) \qquad (22)$$
$$+16Xy^5 - 16X_0\bar{c}y^5 + \frac{y^6}{3}(15-99M_0^2+121M_0^4-48M_0^6) + \frac{20X_0\bar{c}Xy^4}{7\beta^5}(25M_0^2+8)\Big\}.$$

In the calculation of the linearized pressure on the oscillating plate only the potential on the plane y = 0 is required. This potential is presented in Eq. (23).

$$(\phi_0')_{y=0} = -\frac{V_0\alpha X}{\beta} + \dot{\alpha}\Big\{\frac{X^2}{2\beta^3} + \frac{X_0\bar{c}X}{\beta}\Big\} - \frac{\ddot{\alpha} M_0^2}{V_0}\Big\{\frac{X^3}{4\beta^5} - \frac{X_0\bar{c}X^2}{2\beta^3}\Big\} + \frac{\dddot{\alpha} M_0^2}{V_0^2}\Big\{X^4\Big(\frac{1+4M_0^2}{48\beta^7}\Big)$$
$$+\frac{X_0\bar{c}X^3}{12\beta^5}(2M_0^2+1)\Big\} - \frac{\overset{IV}{\alpha} M_0^4}{192 V_0^3}\Big\{X^5\Big(\frac{3+4M_0^2}{\beta^9}\Big) - \frac{4X_0\bar{c}X^4}{\beta^7}(2M_0^2+3)\Big\} \qquad (23)$$
$$+\frac{\overset{V}{\alpha} M_0^4}{1920 V_0^4}\Big\{X^6\Big(\frac{8M_0^4+9M_0^2+1}{\beta^{11}}\Big) + \frac{4X_0\bar{c}X^5}{7\beta^9}(28M_0^4+109M_0^2+8)\Big\}.$$

The fourth-order solution included in Eq. (23) has recently been obtained by Watkins[25] who expanded the first-order solution in powers of $\bar{\omega}$ and then integrated to obtain the fourth-order solution in frequency.

The Linearized Lift and Pitching Moment Coefficients

In this part the pressure, lift and pitching moment will be calculated from the series expansion and a comparison will be made between the series and the exact solutions. The unstable tendency (i.e., negative damping) of an oscillating airfoil of zero thickness, which was first pointed out by Garrick and Rubinow[17], is also discussed.

The linearized pressure coefficient is given by Eq. (16) if all terms containing α^2 are dropped, i.e.,

$$C_p = -2\left(\frac{1}{V_0}\phi_x^1 + \frac{1}{V_0^2}\phi_t^1\right). \tag{24}$$

Now substituting the potential ϕ_6^1 (Eq. (22)) for ϕ^1 in Eq. (24) the pressure coefficient is seen to be,

$$C_p = \frac{2\alpha_n}{\beta}\Bigg\{\cos\omega t\Bigg[1 - \frac{\omega^2}{V_0^2}\left(X^2\left[\frac{M_0^2+2}{4\beta^4}\right] + \frac{X_0\bar{c}X}{\beta^2}\right) + \frac{\omega^4}{V_0^4}\Bigg(\left[\frac{4M_0^6+27M_0^4+4M_0^2}{192\beta^8}\right]X^4 + X_0\bar{c}X^3\left[\frac{4M_0^4+M_0^2}{12\beta^6}\right]\Bigg)\Bigg] - \frac{\omega}{V_0}\sin\omega t\Bigg[X\left(\frac{M_0^2-2}{\beta^2}\right) - X_0\bar{c} + \frac{\omega^2}{V_0^2}\left(\frac{X^3(M_0^4+4M_0^2)}{12\beta^6} + \frac{3X_0\bar{c}X^2M_0^2}{4\beta^4}\right) - \frac{\omega^4}{V_0^4}\left(X^5\left[\frac{2M_0^8+16M_0^6+9M_0^4}{480\beta^{10}}\right] + X_0\bar{c}X^4\left[\frac{5(19M_0^6+10M_0^4)}{672\beta^8}\right]\right)\Bigg]\Bigg\}. \tag{25}$$

Thus, the section lift coefficient

$$C_L = \frac{2}{\bar{c}}\int_0^{\bar{c}} C_p\, dX$$

is,

$$C_L = \frac{4\alpha_n}{\beta}\Bigg\{\cos\omega t\Bigg[1 - \frac{\bar{\omega}^2}{12}\left(M_0^2+2+6X_0\beta^2\right) + \frac{\bar{\omega}^4}{960}\Big(4M_0^6+27M_0^4+4M_0^2 + 20X_0\beta^2(4M_0^4+M_0^2)\Big)\Bigg] - \frac{\bar{\omega}}{2}\sin\omega t\Bigg[\left(M_0^2-2-2X_0\beta^2\right) + \frac{\bar{\omega}^2}{24}\left(M_0^4+4M_0^2+12X_0\beta^2M_0^2\right) - \frac{\bar{\omega}^4}{20{,}160}\Big(7(2M_0^8+16M_0^6+9M_0^4) + 30X_0\beta^2(19M_0^6+10M_0^4)\Big)\Bigg]\Bigg\}. \tag{26}$$

The pitching moment coefficient as calculated about the point of rotation x_0 is given by the equation,

$$C_m = -\frac{2}{\bar{c}^2}\int_0^{\bar{c}} (X - X_0\bar{c})\,C_p\, dX$$

or,

$$C_m = -\frac{4\alpha_m}{\beta}\Bigg\{\cos\omega t\Bigg[\left(\frac{1}{2}-X_0\right) - \frac{\bar{\omega}^2}{16}\left(M_0^2+2+X_0(4M_0^2-8) - 8X_0^2\beta^2\right) + \frac{\bar{\omega}^4}{5670}\Big(20M_0^6 + 135M_0^4+20M_0^2\Big) + X_0\left(360M_0^6-396M_0^4-120M_0^2\right) - 120X_0^2\beta^2(4M_0^4+M_0^2)\Big)\Bigg] - \frac{\bar{\omega}}{6}\sin\omega t\Bigg[\left(2M_0^2-4-3X_0(2\beta^2-1)+6X_0^2\beta^2\right) + \frac{\bar{\omega}^2M_0^2}{40}\Big((4M_0^2+16) + X_0(40M_0^2-65) - 60X_0^2\beta^2\Big) - \frac{\bar{\omega}^4M_0^4}{6720}\Big(24M_0^4+192M_0^2+108 + X_0(936M_0^4-552M_0^2-563) - 30X_0^2\beta^2(19M_0^2+10)\Big)\Bigg]\Bigg\}. \tag{27}$$

The solution for the lift, Eq. (26) will now be compared with the lift as determined by the exact linearized solution. The exact solution was computed from the tabular data presented by von Borbély[12]. The comparisons will be made at three Mach numbers ($M_o = \frac{10}{9}, \frac{10}{7}$ and $\frac{10}{5}$) and plotted versus $\beta^2\ \bar{\omega}$. The reason for plotting versus $\beta^2\ \bar{\omega}$ is that it tends to separate the results which otherwise lie very close to each other. As may be seen from Eq. (26) the lift is given in the form

$$C_L = \frac{4\ \alpha_M}{\beta} (P \cos \omega\ t + Q \sin \omega\ t).$$

Where P and Q are functions of ($\bar{\omega}$, M_o and x_o). Figure 2 shows the function P as calculated from the second-order, fourth-order, sixth-order and exact solutions. Figure 3 shows the function Q under the same conditions. Both Figures 2 and 3 are presented for the condition $x_o = 0$, that is the case when the airfoil is rotating about its leading edge. It will be noted that for relatively low speed motions such as are found in stability calculations or control deflections (where $\bar{\omega}$ may be expected to be less than .4) the second-order solution in frequency is within a few percent of the exact solution. For high-speed motions such as flutter the sixth-order solution will probably give a sufficiently accurate result.

Now consider the unstable tendency of the oscillating flat plate. In 1946 Garrick and Rubinow[17] noted that as the frequency ω approached zero the damping moment (represented by the odd ''Q'' terms) would be negative for certain combinations of Mach number and location of the point of rotation. Their result is exactly that given by the present second-order solution in frequency. The region in which the motion will be unstable is indicated on Fig. 4. As a matter of interest this unstable region is also presented for frequencies of $\beta^2 \bar{\omega} = .25$ and $\beta^2 \bar{\omega} = .5$ (as calculated from the fourth-order solution). The region of instability is seen to be reduced somewhat by the higher order terms in frequency.

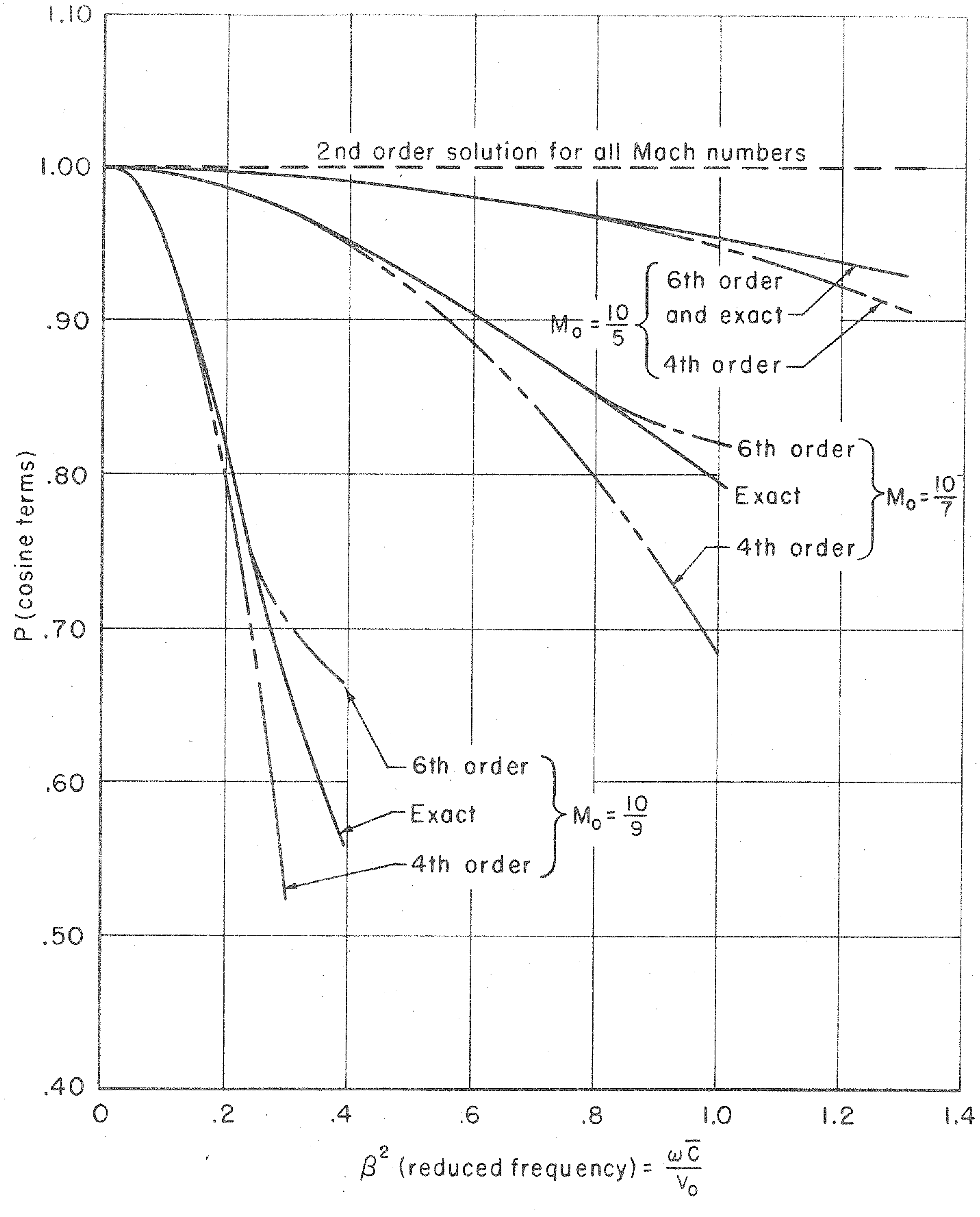

Fig. 2 — Comparison of the even terms in the series solution and the exact solution for the linearized lift of an oscillating airfoil

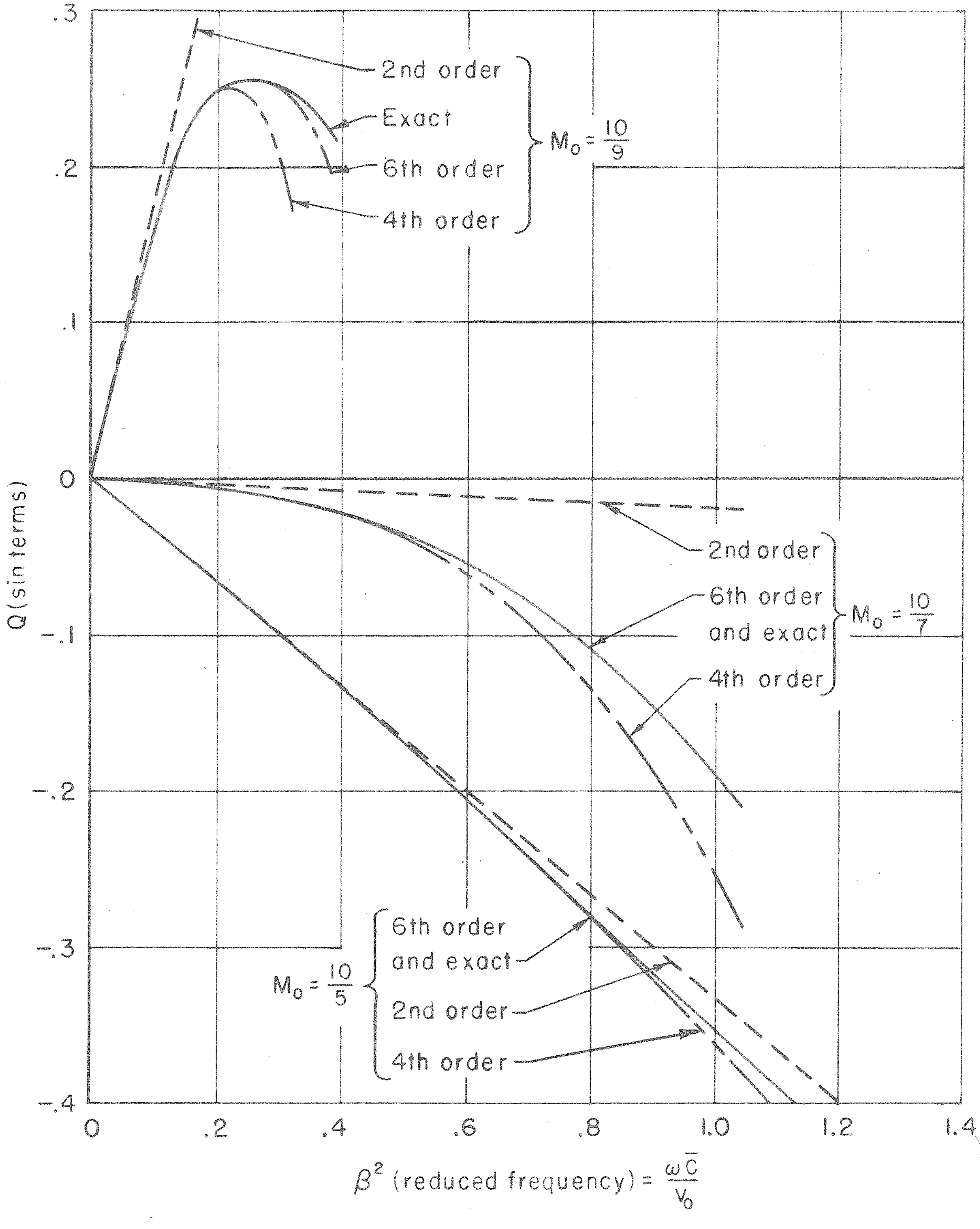

Fig. 3 — Comparison of the odd terms in the series solution and the exact solution for the linearized lift of an oscillating airfoil

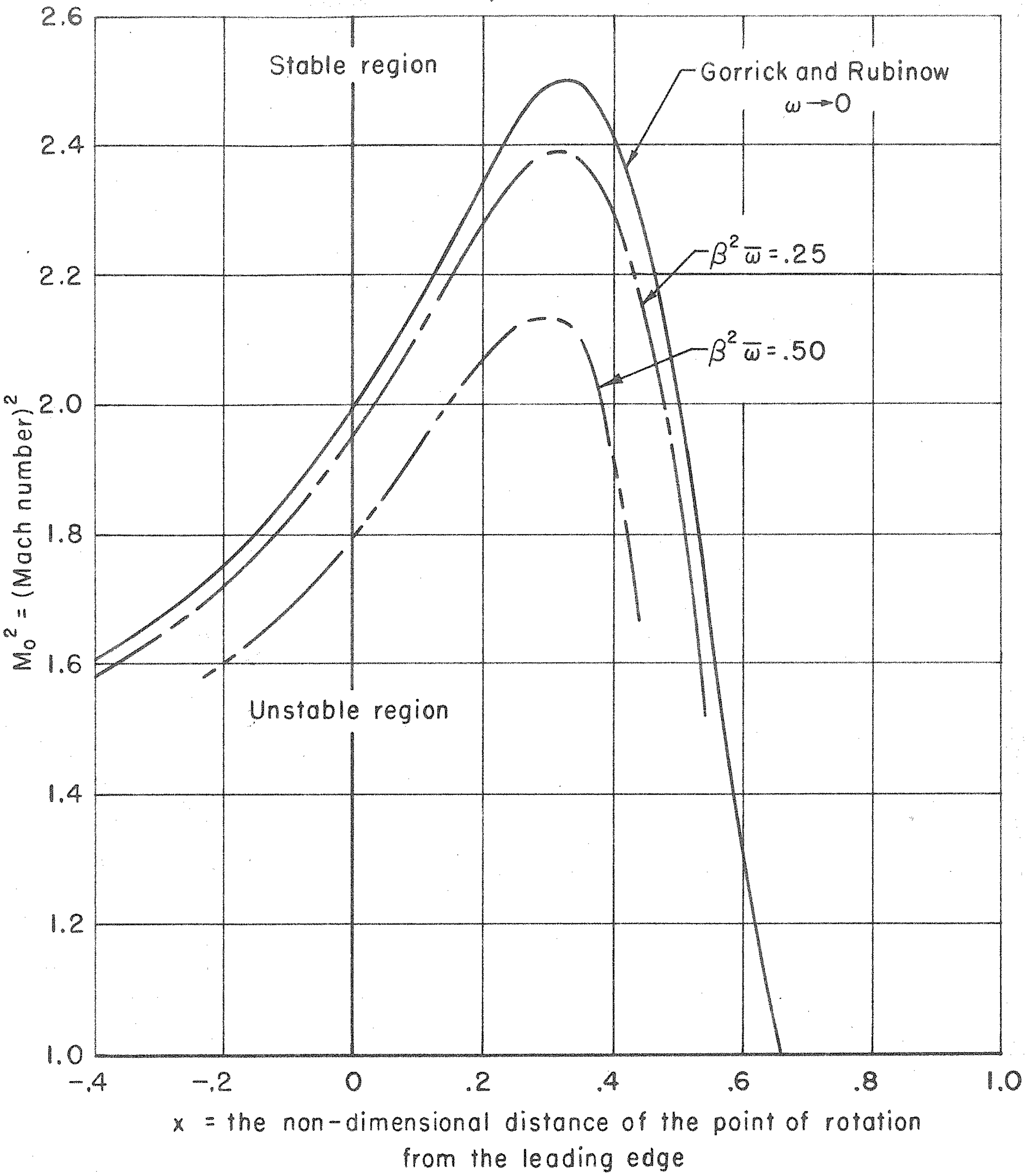

Fig. 4 — Region of instability for an oscillating airfoil as given by linearized solution valid to fourth order in frequency

SECTION IV. THE SECOND-ORDER SOLUTION IN ANGLE OF ATTACK FOR AN AIRFOIL OF ZERO THICKNESS

In this Section the second-order solution in angle of attack, for an oscillating flat airfoil, will be considered. The second-order solution in frequency, as determined in Section III, will be used in the iteration process thereby limiting the final result to second-order in angle of attack and frequency.

The General Solution

Another difficulty arises as soon as the first-order potential [Eq. (21) or the first two terms of Eq. (22)] is substituted into the right hand side [H (ϕ^1)] of the non-homogeneous wave equation which determines ϕ^2. Consider the second derivative of the first-order term (in frequency) of the linearized potential ϕ^1 (i.e., ϕ^1_1) as given by Eq. (21).

$$\phi^1_1 = V_0 \alpha \left(y - \frac{x}{\beta} \right) .$$

Thus,

$$(\phi^1_1)_y = V_0 \alpha \ ,$$

and hence,

$$(\phi^1_1)_{yy} = 0 \ .$$

The second derivative is then zero ahead of the line $x = y\,\beta$ since ϕ^1_1 is defined as zero ahead of the Mach line from the leading edge, and is also seen to be zero downstream from this Mach line. However, since $(\phi^1_1)_y$ is a discontinuous function [$\phi^1_{1y} = 0$ for $x < y\,\beta$ and $\phi^1_{1y} = V_o\alpha$ for $x > y\,\beta$] $(\phi^1_1)_{yy}$ is infinite along the Mach line from the leading edge. The second derivative is then given as zero everywhere except at the Mach line from the leading edge at which point it is represented by a delta function. Although it is possible to work with this velocity potential another method of attack which eliminates the discontinuous derivatives was chosen. The method was that employed by Van Dyke(14), and is also mentioned on page 365 of Courant and Fredericks(26), for the stationary flow problem.

In attacking the steady flow problem Van Dyke generalized the airfoil shape so that the ordinate (See Fig. 5) of the airfoil is given as $y = \varepsilon f(x)$. The function $f(x)$ is undefined, except that the surface and the slope of the surface are required to be continuous and to go to zero at the leading edge (i.e., $f(x) = f'(x) = 0$ at $x = 0$). Now there is no discontinuity in velocity occurring at the leading edge, or

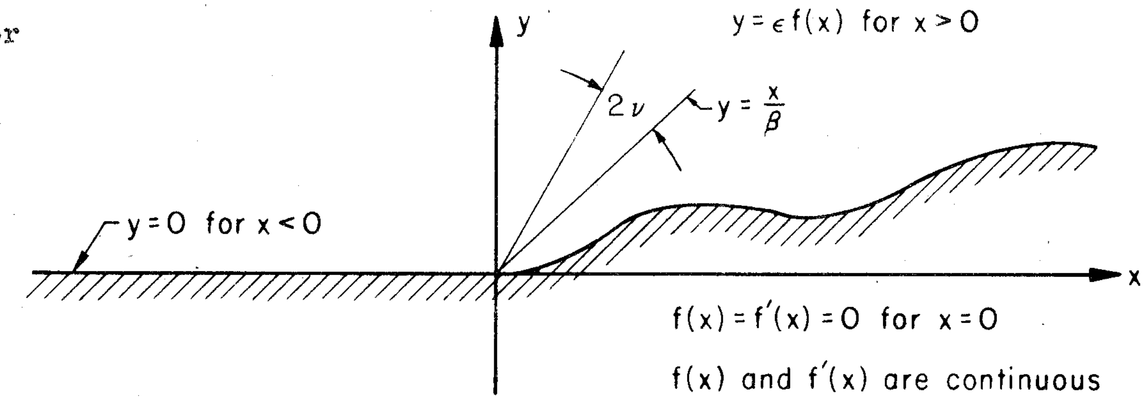

Fig. 5

for that matter anywhere in the flow field. The iteration process was then utilized and an expression for the pressure was determined. This solution showed that the second-order pressure at any point on the airfoil depended only on the slope $\varepsilon f'(x)$ of the airfoil at that point. Therefore, suppose that $\varepsilon f(x)$ was required to represent a flat airfoil everywhere except in the region $0 \leq x \leq \delta$ where the original restrictions on $f(x)$ are still enforced. The general solution is still valid for this problem. Now the solution remains valid no matter how small

δ is allowed to become, and as $\delta \rightarrow 0$ the pressure is known everywhere on the airfoil except for an infinitesimal distance aft of the leading edge. In the limit where $\delta = 0$ the pressure is known everywhere except at the point $x = 0$. This is the flat plate solution. Van Dyke has further shown that, in the limit, the velocity components and the pressure are known everywhere in the flow field except in the fan shaped region $y = (\frac{x}{\beta} + \nu)^{+}_{-} \nu$ extending from the leading edge of the airfoil. The expression for ν was shown to be $\nu = \frac{(\gamma + 1) M^2}{2 \beta^2} \varepsilon$. The angle ν is equal to the angle that the shock (as given by second-order theory) lies ahead of the Mach wave, or for an expansion 2ν represents the width of the Prandtl-Meyer expansion fan. The iteration process, as applied to the case of an airfoil of zero thickness by means of the ''general solution'' attack, was thus shown to be successful except within the previously mentioned fan like region of order ε lying near the Mach line from the leading edge.

A similar procedure will now be applied to the problem of an oscillating flat plate. A first-order [5] velocity potential ϕ_2^1 (f, f'), such that ϕ_2^1 (f, f') becomes the previously determined first-order solution ϕ_2^1 (x, y) when $f = f(x - y\beta) = x - y\beta$, must be determined. A general potential which satisfies these requirements is found to be

$$\phi_2' = -\alpha V_0 \left\{\frac{f}{\beta}\right\} + \dot{\alpha}\left\{\frac{M_0^2 x f}{\beta^3} + \frac{x_0 \bar{c} f}{\beta} - \left(\frac{M_0^2}{\beta^3} + \frac{1}{\beta}\right)\int_0^{x-y\beta} f(\xi)\, d\xi\right\} . \qquad (28)$$

When $f(x - y\beta) = x - y\beta$ the general expression for the velocity potential ϕ_2^1 (f) becomes,

$$\phi_2' = \alpha V_0\left(y - \frac{x}{\beta}\right) + \dot{\alpha}\left\{\left(\frac{1-2M_0^2}{2\beta}\right)y^2 + xy + \frac{x^2}{2\beta^3} - x_0\left(y - \frac{x}{\beta}\right)\right\} ,$$

[5] For the remainder of this report order will refer to angle of attack unless otherwise stated.

the particular velocity potential ϕ_2^1 (x, y, t) as given by Eq. (21). That Eq. (28) also satisfies the boundary conditions and the wave equation is shown below. Since f (x - y β) = f' (x - y β) = 0 at x = 0 it follows that ϕ_2^1 and $(\phi_2^1)_x$ are zero at x = 0. This is the boundary condition on the plane x = 0 as required by Eq. (14). Then since $\frac{\partial f}{\partial x} = f'$ and $\frac{\partial f}{\partial y} = -\beta f'$ the derivative of Eq. (28) with respect to y is,

$$\left(\phi'_{2_y}\right)_{y=0} = \alpha V_0 f' + \dot{\alpha}\left\{-\frac{M_0^2 X f'}{\beta^2} + \left(\frac{M_0^2}{\beta^2}+1\right) f - f' X_0 \bar{c}\right\}. \qquad (29)$$

Hence, when f (ξ) = ξ , (ξ = x - y β),

$$\left(\phi'_{2_y}\right)_{y=0} = V_0 \alpha + \dot{\alpha}(X - X_0 \bar{c}) ,$$

which is the tangency condition as required by Eq. (17d). Finally, $\Box \phi_2^1$ is seen to be zero.

$$\Box \phi'_2 = \beta^2 \phi'_{2_{xx}} - \phi'_{2_{yy}} + \frac{2M_0^2}{V_0} \phi'_{2_{xt}} + \frac{M_0^2}{V_0^2} \phi'_{2_{tt}} ,$$

$$= \beta^2\left[-\alpha V_0 \frac{f''}{\beta} + \dot{\alpha}\left\{\frac{M_0^2 X f''}{\beta^3} + \frac{f'}{\beta^3} + \frac{X_0 \bar{c} f''}{\beta}\right\}\right] - \left[-V_0 \alpha \beta f''\right.$$

$$\left. + \dot{\alpha}\left\{\frac{M_0^2 X f''}{\beta} - \left(\frac{M_0^2}{\beta} + \beta\right) f' + \beta X_0 \bar{c} f''\right\}\right] + \frac{2M_0^2}{V_0}\left[-\alpha V_0 \frac{f'}{\beta}\right] + O(\bar{\omega}^2),$$

$$= 0 + O(\bar{\omega}^2) .$$

Thus, the general potential Eq. (28) satisfies the conditions,

$$\Box \phi'_2 = 0 , \qquad (30A)$$

$$\phi'_2 = (\phi'_2)_X = 0 \quad at \quad X = 0 , \qquad (30B)$$

$$\left.(\phi'_2)_y\right]_{y=0} = \alpha V_0 f' + \dot{\alpha}\left\{-\frac{M_0^2 X f'}{\beta^2} + \left(\frac{M_0^2}{\beta^2}+1\right) f - X_0 \bar{c} f'\right\}. \qquad (30C)$$

Also this general potential reduces to the desired specific potential Eq. (21), and the requirements on the general potential Eq. (30) reduce to the requirements on Eqs.(17) and (21) when f (ξ) = ξ . It is of interest to note that in order to solve the problem of an oscillating flat plate the general potential selected must satisfy a boundary condition Eq. (30c) which has no physical meaning until the problem is specialized by letting $f(\xi) = \xi$.

Now having determined a satisfactory first-order general potential ϕ_2^1 a general second-order potential ϕ_2^2 must be found which will satisfy Eqs. (9), (13), and (14).

Eq. (9):

$$\Box \phi_2^2 = -\frac{M_0^2}{V_0^2}\Big\{(\gamma-1)(\phi'_{2xx}+\phi'_{2yy})(\phi'_{2x}+\frac{1}{V_0}\phi'_{2t})$$

$$+2\phi'_{2x}(\phi'_{2xx}+\frac{1}{V_0}\phi'_{2xt})+2\phi'_{2y}(\phi'_{2xy}+\frac{1}{V_0}\phi'_{2yt})\Big\}.$$

Eq. (13):

$$(\phi^2_{2y})_{y=0} = (\alpha V_0 f'\phi'_{2x})_{y=0} + \left[(\phi'_{2y})_{y=0} - (\phi'_{2y})_{surface}\right].$$

and Eq. (14):

$$\phi_2^2(0,y,t) = 0 \ ,$$

$$\phi^2_{2x}(0,y,t) = 0 \ .$$

Substituting Eq. (28) into Eq. (9) it follows that

$$\Box\phi_2^2 = -\frac{\alpha^2 V_0(\gamma+1)M_0^4}{\beta^2}f'f'' + \alpha\dot{\alpha}\Big\{\frac{2M_0^6(\gamma+1)}{\beta^4}xf'f'' + \frac{2M_0^4(\gamma+1)X_0\bar{C}}{\beta^2}f'f''$$

$$-\frac{2M_0^4(\gamma+1)}{\beta^2}ff'' + \frac{M_0^4(3-2M_0^2)(\gamma+1)}{\beta^4}(f')^2\Big\}. \qquad (31)$$

Now let $\phi_2^2 = \phi+\sigma$ where ϕ is a particular solution which satisfies the non-homogeneous wave equation Eq. (31) and σ is the correction potential which makes ϕ_2^2 satisfy the boundary conditions [Eqs. (13) and (14)]. A function ϕ which satisfies Eq. (31) is,

$$\phi = \alpha^2 V_0\Big\{-\frac{M_0^4(\gamma+1)}{4\beta^3}y(f')^2 + \frac{M_0^2(\gamma+1)}{4\beta^4}\int_0^{x-y\beta}(f')^2 d\xi + \frac{M_0^2(\gamma+1)X_0\bar{C}}{4\beta^2}(f')^2\Big\}$$

$$+\alpha\dot{\alpha}\Big\{\frac{M_0^6(\gamma+1)}{2\beta^5}xy(f')^2 - \frac{M_0^4(\gamma+1)}{\beta^4}xff'\Big\} \qquad (32)$$

That ϕ does satisfy Eq. (31) may be verified by direct substitution, it is also apparent that $\phi = \phi_x = 0$ at $x = 0$.

Now σ must satisfy the equations shown below.

$$\Box\,\sigma = 0 \tag{33A}$$

$$\sigma = \sigma_x = 0 \qquad at \quad x = 0 \tag{33B}$$

$$(\sigma_y)_{y=0} = -(\phi_y)_{y=0} + (\alpha f' \phi'_{2x})_{y=0} + \left[(\phi'_{2y})_{y=0} - (\phi'_{2y})_{SUR.} \right] \tag{33C}$$

For the oscillating airfoil the ordinate of the surface is

$$y = \alpha\,(x - X_0 \bar{c})\ ,$$

hence, for the general solution the ordinate will be defined as,

$$y = \alpha f'(x - X_0 \bar{c})\ .$$

From Eq. (32) ϕ_y is given by the expression,

$$\phi_y = \alpha^2 V_0 \left\{ -\frac{M_0^4(\gamma+1)}{4\beta^3}(f')^2 - \frac{M_0^2(\gamma+1)(f')^2}{4\beta^3} + \frac{M_0^4(\gamma+1)\,y f' f''}{2\beta^2} - \frac{M_0^2(\gamma+1)X_0\bar{c} f' f''}{2\beta} \right\}$$
$$+ \alpha\dot{\alpha}\left\{ \frac{M_0^6(\gamma+1)}{2\beta^5}\left[x(f')^2 - 2xy\beta f' f'' \right] + \frac{M_0^4(\gamma+1)}{\beta^3} x \left[f f'' + (f')^2 \right] \right\} ,$$

and hence, the function - $(\phi_y)_{y=0}$ as required by Eq. (33C) is seen to be,

$$-(\phi_y)_{y=0} = \alpha^2 \frac{V_0(M_0^4 + M_0^2)(\gamma+1)}{4\beta^3}(f')^2 + \underline{\alpha^2 V_0 \frac{M_0^2(\gamma+1)}{2\beta} X_0 \bar{c} f' f''}$$
$$- \alpha\dot{\alpha}\frac{M_0^4(\gamma+1)(3M_0^2 - 2)}{2\beta^5} x (f')^2 - \underline{\alpha\dot{\alpha}\frac{M_0^4(\gamma+1)}{\beta^3} x f f''}\ . \tag{34}$$

The two underlined terms in Eq. (34) are actually unnecessary in the solution since, when f (ξ) is set equal to ξ, f'' is zero. However, these terms will be retained in this paper so that the general solution will be complete. The term α f' $(\phi_2^1)_x$ of Eq. (33C) is determined by differentiating Eq. (28) with respect to x and is found to be,

$$\alpha f'(\phi'_{2x})_{y=0} = -\alpha^2 \frac{V_0}{\beta}(f')^2 + \alpha\dot{\alpha}\left\{ \frac{M_0^2 x}{\beta^3}(f')^2 - \frac{f f'}{\beta} + \frac{X_0 \bar{c}}{\beta}(f')^2 \right\} . \tag{35}$$

The final term of Eq. (33C) may be determined by expanding f and f' in a Maclaurin series - i.e.,

$$\alpha f(x-y\beta)\Big]_{y=\alpha f'(x-X_o\bar{c})} = \alpha f(x) - \alpha^2\beta(x-X_o\bar{c})[f'(x)]^2 + O(\alpha^3) \tag{36}$$

$$\alpha f'(x-y\beta)\Big]_{y=\alpha f'(x-X_o\bar{c})} = \alpha f'(x) - \alpha^2\beta(x-X_o\bar{c})f'f''(x) + O(\alpha^3)$$

Thus,

$$\left[(\phi'_{2y})_{y=0} - (\phi'_{2y})_{y=\alpha f'(x+X_o\bar{c})}\right] = V_o\alpha^2\beta f f'' - \alpha^2 V_o\beta X_o\bar{c} f''$$
$$+ \alpha\dot{\alpha}\Big\{-\frac{M_o^2}{\beta}x f f'' + \Big(\frac{M_o^2}{\beta}+\beta\Big)f f' - X_o\bar{c}\beta f f'' + \frac{M_o^2}{\beta}x X_o\bar{c} f''$$
$$- \Big(\frac{M_o^2}{\beta}+\beta\Big)X_o\bar{c} f' + \beta X_o^2\bar{c}^2 f''\Big\}. \tag{37}$$

Substituting Eqs. (34), (35) and (37) into Eq. (33C) the boundary condition on $(\sigma_y)_{y=0}$ is complete and it only remains to determine σ.

The development of σ is reasonably simple once the behaviour of the general function f under the operations indicated by Eq. (33) is understood. Some relations which are useful in the derivation are listed below, where $f = f(x - y\beta)$, $f' = \frac{\partial f}{\partial(x - y\beta)}$ and g is any function of f or its derivatives (i.e., $g = ff'$, $(f')^2$ etc.),

$$\beta^2 g_{xx} - g_{yy} = 0,$$

$$\beta^2[(x-y\beta)g]_{xx} - [(x-y\beta)g]_{yy} = 0 .$$

If $\quad \psi = \frac{A}{\beta}\int_0^{x-y\beta} g(\xi)\,d\xi,$

then $\quad (\psi_y)_{y=0} = Ag.$

If $\quad \psi = -A/\beta\,(x-y\beta)\int_0^{x-y\beta} g(\xi)\,d\xi + A/\beta\int_0^{x-y\beta}\int_0^{\xi} g(\eta)\,d\eta\,d\xi,$

then $\quad (\psi_y)_{y=0} = Axg.$

With the use of such relations the undetermined function σ is found to be,

$$\sigma = \alpha^2 V_o\Big\{ff' - \int_0^{x-y\beta} f(\xi)\,d\xi\Big[\frac{(\gamma+1)(M_o^4+M_o^2)-4M_o^2\beta^2}{4\beta^4}\Big] + X_o\bar{c}f'$$
$$- X_o\bar{c}\int_0^{x-y\beta} f'f''\,d\xi\Big[\frac{M_o^2(\gamma+1)}{2\beta^2}\Big]\Big\} + \alpha\dot{\alpha}\Big\{y f f'\Big(\frac{2M_o^2}{\beta}\Big)$$

$$
\begin{aligned}
& x\int_0^{x-y\beta} f\,d\xi \left[\frac{(\gamma+1)(M_0^6+M_0^4)-4\beta^2 M_0^2}{2\beta^6}\right] - x_0\bar{c}\,x\,f'\left(\frac{2M_0^2}{\beta^2}\right) \\
& + x x_0 \bar{c}\int_0^{x-y\beta} f' f''\,d\xi\left[\frac{M_0^4(\gamma+1)}{\beta^4}\right] + (x-y\beta)\int_0^{x-y\beta} f''\,d\xi\left[\frac{2M_0^2}{\beta^2}\right] \\
& - \int_0^{x-y\beta}\int_0^{\xi} f''(n)\,dn\,d\xi\left[\frac{2M_0^2}{\beta^2}\right] + \int_0^{x-y\beta} f f'\,d\xi\left[\frac{2M_0^2}{\beta^2}\right] \\
& + \left[\frac{2M_0^4}{\beta^4} - (\gamma+1)\frac{M_0^6+M_0^4}{2\beta^6}\right]\left[(x-y\beta)\int_0^{x-y\beta}(f')^2\,d\xi - \int_0^{x-y\beta}\int_0^{\xi}[f'(n)]^2\,dn\,d\xi\right] \\
& + \left[\frac{M_0^4(\gamma+1)}{\beta^4}\right]\left[\int_0^{x-y\beta}\int_0^{\xi} f'(n)f''(n)\,dn\,d\xi - (x-y\beta)\int_0^{x-y\beta} f' f''\,d\xi\right]\Bigg\}.
\end{aligned}
\tag{38}
$$

The complete solution Φ_2^2 valid to second-order in frequency and angle of attack is

$$\Phi_2^2 = \Phi_2^1 + \phi + \sigma \tag{39}$$

where ϕ_2^1, ϕ and σ are given by equations (28), (32), and (38).

The Case of an Airfoil of Zero Thickness

Now consider the particular solution for the problem of an oscillating airfoil of zero thickness, i.e., for an airfoil whose ordinate is defined as,

$$y = \alpha(x - x_0\bar{c}).$$

A second-order potential is defined such that the potential satisfies the following conditions,

$$\Box\Phi_2^2 = H(\phi_2^1) \qquad \text{(See Eq. 9)},$$

$$\left[\frac{\Phi_{2y}^2 - \dot{\alpha}(x - x_0\bar{c})}{1 + \frac{1}{V_0}\Phi_{2x}^2}\right]_{y=\alpha(x+x_0\bar{c})} = \alpha \qquad \text{(See Eq. 11)},$$

$$\Phi_2^2 = \Phi_{2x}^2 = 0 \quad \text{at} \quad x = 0 \qquad \text{(See Eq. 14)},$$

The solution Φ_2^2 (x, y, t) is obtained by setting f (x - y β) = x - y β in the general solution Φ_2^2[Eq. (39)]. When this is done the second-order potential is found to be,

$$
\begin{aligned}
\Phi_2^2 = {} & V_0\alpha(y - x/\beta) + \dot{\alpha}\left\{y^2\left(\frac{1-2M_0^2}{2\beta}\right) + xy + \frac{x^2}{2\beta^3} - x_0\bar{c}(y - x/\beta)\right\} \\
& + V_0\alpha^2\left\{x\left[\frac{4\beta^2-(\gamma+1)M_0^4}{4\beta^4}\right] - \frac{y}{\beta}\right\} + \alpha\dot{\alpha}\left\{\frac{(x-y\beta)^2}{2}\left[\frac{2M_0^4+M_0^2-2}{\beta^4}\right.\right. \\
& \left.+ (\gamma+1)\left(\frac{2M_0^6-3M_0^4}{2\beta^6}\right)\right] + 2x_0\bar{c}(x-y\beta) - x_0\bar{c}x^2\left(\frac{M_0^2}{\beta^2}\right) \\
& \left. + (x^2 - xy\beta)\left[(\gamma+1)\left(\frac{M_0^6+M_0^4}{2\beta^6}\right) - \frac{2M_0^4}{\beta^4}\right] + (yx - y^2\beta)\left(\frac{2M_0^2}{\beta}\right)\right\} .
\end{aligned} \tag{40}
$$

Hence, the derivative evaluated at the surface, i.e., at $y = \alpha\,(x - x_0\,\bar{C})$ are found to be,

$$
\begin{aligned}
\frac{1}{V_0}\Phi_x = {} & -\frac{\alpha}{\beta} + \frac{\dot{\alpha}}{V_0}\left(\frac{x}{\beta^3} + \frac{x_0\bar{c}}{\beta}\right) + \alpha^2\left[\frac{4\beta^2 - M_0^4(\gamma+1)}{4\beta^4}\right] + \frac{\dot{\alpha}}{V_0}\alpha\left\{-x_0\bar{c}\left(\frac{M_0^2+1}{\beta^2}\right)\right. \\
& \left. + x\left[-\left(\frac{M_0^4+M_0^2+1}{\beta^4}\right) + \frac{(\gamma+1)(3M_0^2)}{2\beta^6}\right]\right\} , \\
\frac{1}{V_0}\Phi_y = {} & \alpha + \frac{\dot{\alpha}}{V_0}(x - x_0\bar{c}) - \frac{1}{\beta^2}\alpha^2 + \frac{\dot{\alpha}}{V_0}\alpha\left\{\frac{x}{\beta^3} + \frac{x_0\bar{c}}{\beta}\right\} , \\
\frac{1}{V_0^2}\Phi_t = {} & -\frac{\dot{\alpha}}{V_0}\frac{x}{\beta} + \frac{\alpha}{V_0}\dot{\alpha}\left\{x\left[\frac{4M_0^2\beta^2 - M_0^4(\gamma+1)}{2\beta^4}\right] + x_0\bar{c}\left[\frac{(\gamma+1)M_0^2 + 2\beta^2}{2\beta^2}\right]\right\} .
\end{aligned} \tag{41}
$$

Now the pressure coefficient is given by Eq. (16),

$$
C_p = -2\left\{\frac{1}{V_0}\Phi_x + \frac{1}{V_0^2}\Phi_t - \frac{1}{2V_0^2}\left[\beta^2(\Phi_x)^2 - (\Phi_y)^2 + \frac{2M_0^2}{V_0^2}\Phi_x\Phi_t + \frac{M_0^2}{V_0^2}(\Phi_t)^2\right]\right\} .
$$

Substituting the results of Eq. (41) into the expression for the pressure coefficient, C_p is given in the form,

$$
\begin{aligned}
C_p = \frac{2}{\beta}\Bigg\{ & \alpha - \frac{\dot{\alpha}}{V_0}\left[x_0\bar{c} + \frac{x}{\beta^2}(2 - M_0^2)\right] + \alpha^2\left[\frac{(\gamma+1)M_0^4 - 4\beta^2}{4\beta^3}\right] \\
& + \frac{\dot{\alpha}}{V_0}\alpha\left[\frac{x}{2\beta^5}\left(\gamma(M_0^6 - 4M_0^4) - 3M_0^6 + 10M_0^4 - 10M_0^2\right]\Bigg\} .
\end{aligned} \tag{42}
$$

This pressure coefficient is the pressure coefficient for the upper surface of the flat plate shown in Fig. 1. The pressure on the lower surface is equal and opposite to that on the upper surface and is 180° out of phase, i.e.,

$$C_{p_u} = C_p(\sin \omega t, \cos \omega t),$$

$$C_{p_l} = -C_p(\sin[\omega t+\pi], \cos[\omega t+\pi]) = -C_p(-\sin \omega t, -\cos \omega t).$$

The pressure difference between the upper and lower surfaces is,

$$C_{p_T} = C_{p_l} - C_{p_u}.$$

The pressure difference is given by the equation,

$$C_{p_T} = \frac{4\alpha_m}{\beta}\left\{\cos \omega t + \frac{\bar{\omega}}{2}\sin \omega t\left[2X_0\bar{c}\beta^2 + 2X(2-M_0^2)\right]\right\}. \quad (43)$$

It may be seen from Eq. (43) that the pressure difference to second-order is exactly the same as the previously determined result [See Eq. (25)] valid to first-order in angle of attack. Thus, the second-order terms in angle of attack cancel out for an airfoil of zero thickness. This is similar to Busemann's result for the flat plate in a steady flow. There is no need to indicate the second-order expressions for the lift and pitching moment coefficients since they are identical to Eqs. (26) and (27) the equations for the linearized lift and pitching moment.

SECTION V. THE SECOND-ORDER SOLUTION FOR AN AIRFOIL OF GENERAL SHAPE

It was seen in Section IV that the second-order effects do not enter into the equations for the lift and pitching moment of an airfoil of zero thickness. In this Section the effects of thickness will be considered. In part one an oscillating wedge will be considered. In part two a generalized double wedge is investigated. Part three extends this solution for a double wedge to include the problem of an arbitrary airfoil section. In part four a specific airfoil (modified double wedge) is considered, and the pressure lift and pitching moment are determined. The second-order lift and pitching moment are then compared with the exact linearized solution of von Borbély.

The Oscillating Wedge

The oscillating wedge which is to be considered has a thickness ratio 2ε ($\frac{\text{thickness}}{\text{chord}} = 2\varepsilon$), See Fig. 6.

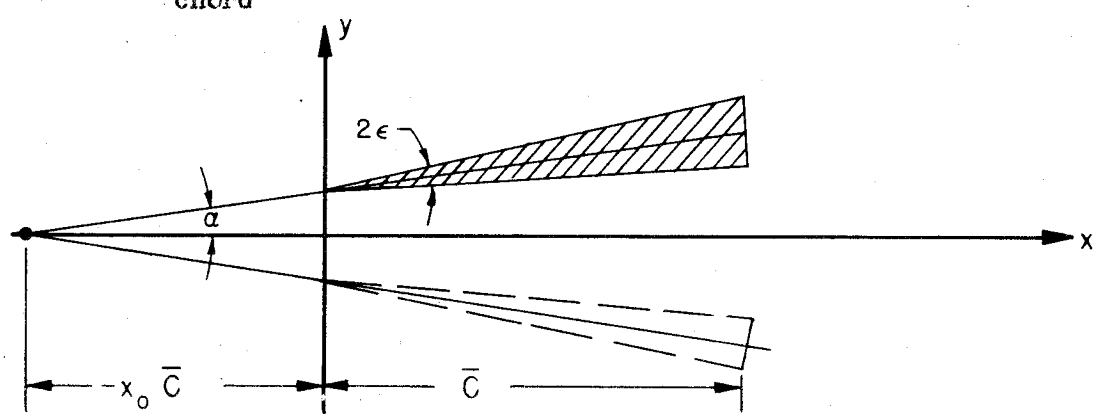

Fig. 6

The ordinate for the upper surface is given as,

$$y = \alpha(x - x_o\bar{c}) + \varepsilon x$$

$$\frac{dy}{dx} = (\alpha + \varepsilon) \quad ; \quad \frac{dy}{dt} = \dot{\alpha}(x - x_o\bar{c}) \tag{44}$$

The general problem is quite similar to the zero thickness problem considered in Section IV. In fact the problem and the method of solution are so similar that only the major steps and the final results will be given for the case of the wedge.

The first-order potential ϕ_2^1 for the wedge is,

$$\phi_2' = -V_0(\alpha+\varepsilon)\frac{f}{\beta} + \dot{\alpha}\left\{\frac{M_0^2 X f}{\beta^3} + \frac{X_0 \bar{C} f}{\beta} - \left(\frac{M_0^2}{\beta^3} + \frac{1}{\beta}\right)\int_0^{X-y\beta} f(\xi)\,d\xi\right\} \cdot \tag{45}$$

Hence,

$$\left(\phi'_{2y}\right)_{y=0} = V_0(\alpha+\varepsilon)f' + \dot{\alpha}\left\{-\frac{M_0^2 X f'}{\beta^2} + \left(\frac{M_0^2}{\beta^2}+1\right)f - X_0\bar{C}f'\right\},$$

and

$$\left[\phi'_{2y}(X,y,t)\right]_{y=0} = V_0(\alpha+\varepsilon) + \dot{\alpha}(X - X_0\bar{C}) = V_0\frac{dY}{dX} + \frac{dY}{dt},$$

where $f(\xi) = \xi$.

Thus, the general first-order potential satisfies the conditions,

$$\Box\phi_2' = 0 + O(\bar{\omega}^2),$$

$$\phi_2' = (\phi'_{2X}) = 0 \quad at \quad X = 0,$$

$$\left(\phi'_{2y}\right)_{y=0} = V_0\frac{dY}{dX} + \frac{dY}{dt},$$

and is the correct first-order potential for the oscillating wedge.

Substituting ϕ_2^1 (Eq. (45)) into Eq. (9) the non-homogeneous wave equation is developed.

$$\phi_2^2 = H(\phi_2') = -\frac{V_0(\gamma+1)M_0^4 f'f''}{\beta^2}(\alpha+\varepsilon)^2 \tag{46}$$
$$+ \dot{\alpha}(\alpha+\varepsilon)\left\{-\frac{2M_0^6(\gamma+1)}{\beta^4}X f'f'' + \frac{2M_0^4(\gamma+1)}{\beta^2}X_0\bar{C}f'f''\right.$$
$$\left. - \frac{2M_0^4(\gamma+1)}{\beta^2}ff'' + \frac{M_0^4(3-2M_0^2)(\gamma+1)}{\beta^4}(f')^2\right\}.$$

Now let the second-order solution ϕ_2^2 be composed of a particular solution ψ and a correction potential θ. The particular solution which satisfies the non-homogeneous wave equation [Eq. (46)] is then found to be,

$$\psi = -(\alpha+\varepsilon)^2 \frac{V_0 M_0^4(\gamma+1)}{4\beta^3} y(f')^2 \tag{47}$$

$$+ \alpha(\alpha+2\varepsilon)V_0 \left\{ \frac{M_0^2(\gamma+1)}{4\beta^4} \int_0^{x-y\beta} (f')^2 d\xi + \frac{M_0^2(\gamma+1)}{4\beta^2} X_0 \bar{c} (f')^2 \right\}$$

$$+ \dot{\alpha}(\alpha+\varepsilon) \left\{ \frac{M_0^6(\gamma+1)}{2\beta^5} xy(f')^2 - \frac{M_0^4(\gamma+1)}{\beta^4} x f f' \right\} .$$

As before $\psi = \psi_x = 0$ at $x = 0$ hence, the correction potential θ is determined by the following conditions,

$$\square\, \theta = 0 \;, \tag{48A}$$

$$\theta = \theta_x = 0 \quad at \quad X = 0 \;, \tag{48B}$$

$$(\theta_y)_{y=0} = -(\psi_y)_{y=0} + (\alpha+\varepsilon) f' \left[(\phi_2')_x \right]_{y=0}$$

$$+ \left[(\phi_{2y}')_{y=0} - (\phi_{2y}')_{y=-(\alpha+\varepsilon)f - X_0 \bar{c} f'} \right] . \tag{48C}$$

The particular functions which make up the right side of Eq. (48C) may now be determined from Eqs. (45) and (47).

$$f'(\alpha+\varepsilon)\left(\phi_{2x}'\right)_{y=0} = -(\alpha+\varepsilon)^2 \frac{V_0}{\beta}(f')^2 + \dot{\alpha}(\alpha+\varepsilon)\left\{ \frac{M_0^2 x (f')^2}{\beta^3} - \frac{ff'}{\beta} + \frac{X_0 \bar{c} f' f'}{\beta} \right\}, \tag{49}$$

$$-(\psi_y)_{y=0} = +(\alpha+\varepsilon)^2 \frac{V_0 M_0^4(\gamma+1)(f')^2}{4\beta^3} + \alpha V_0(\alpha+2\varepsilon)\left\{ \frac{M_0^2(\gamma+1)(f')^2}{4\beta^3} \right.$$

$$\left. + \frac{M_0^2(\gamma+1)X_0\bar{c}}{2\beta} f' f'' \right\} - \dot{\alpha}(\alpha+\varepsilon)\left\{ \frac{M_0^4(\gamma+1)}{\beta^3} x f f'' \right.$$

$$\left. + \frac{M_0^4(3M_0^2-2)(\gamma+1)}{2\beta^5} x (f')^2 \right\} . \tag{50}$$

After expanding the general functions f and f' in a Maclaurin Series, i.e., $(\alpha f)_{y=(\alpha+\varepsilon)f - \alpha X_0 \bar{c}} = \alpha f - \alpha(\alpha+\varepsilon)\beta f(x) f'(x) + \alpha^2 X_0 \bar{c} \beta f(x) f'(x)$

$$(\alpha f')_{y=(\alpha+\varepsilon)f - \alpha X_0 \bar{c}} = \alpha f(x) - \alpha(\alpha+\varepsilon)\beta f f'' + \alpha^2 X_0 \bar{c} \beta f f'' \tag{51}$$

the final term in Eq. (48C) is seen to be,

$$\left[(\phi_{2y}')_{y=0} - (\phi_{2y}')_{y=sur.} \right] = V_0(\alpha+\varepsilon)^2 f f'' \beta - \alpha(\alpha+\varepsilon) V_0 \beta X_0 \bar{c} f''$$

$$+ \dot{\alpha}(\alpha+\varepsilon)\left\{ -\frac{M_0^2 x}{\beta} f f'' + \left(\frac{M_0^2}{\beta} + \beta \right) f f' - X_0 \beta \bar{c} f f'' \right\} \tag{52}$$

continued.

$$\alpha\dot{\alpha}\left\{+\frac{M_0^2}{\beta}X_0X\bar{c}f'' - \left(\frac{M_0^2}{\beta}+\beta\right)X_0\bar{c}f' + \beta X_0^2\bar{c}^2 f''\right\} .$$

The correction potential θ that fulfills the conditions of Eq. (48) is,

$$\begin{aligned}
\theta = {} & V_0(\alpha+\epsilon)^2\left[\int_0^{x-y\beta}(f')^2 d\xi\left(1-\frac{4\beta^2-(\gamma+1)(M_0^4+M_0^2)}{4\beta^4}\right)-ff'\right]+V_0\alpha(\alpha+\epsilon)X_0\bar{c}f' \\
& -V_0\alpha(\alpha+2\epsilon)X_0\bar{c}\int_0^{x-y\beta}f'f''d\xi\left(\frac{M_0^2(\gamma+1)}{2\beta^2}\right)+V_0\epsilon^2\int_0^{x-y\beta}(f')^2\left(\frac{M_0^2(\gamma+1)}{4\beta^4}\right) \\
& +\dot{\alpha}(\alpha+\epsilon)\Bigg\{-2\int_0^{x-y\beta}ff'd\xi-\frac{X_0\bar{c}}{\beta^2}\int_0^{x-y\beta}(f')^2d\xi+xX_0\bar{c}\int_0^{x-y\beta}f'f''d\xi\left(\frac{M_0^4(\gamma+1)}{\beta^4}\right) \\
& +\left[(x-y\beta)\int_0^{x-y\beta}(f')^2d\xi-\int_0^{x-y\beta}\int_0^{\xi}[f'(\eta)]^2d\eta\, d\xi\right]\left[(\gamma+1)\left(\frac{2M_0^6-3M_0^4}{2\beta^6}\right)+\frac{M_0^2-2M_0^4}{\beta^4}\right] \\
& +\left[(x-y\beta)\int_0^{x-y\beta}ff''d\xi-\int_0^{x-y\beta}\int_0^{\xi}ff''d\eta\, d\xi\right]\left[\frac{M_0^2}{\beta^2}+(\gamma+1)\frac{M_0^4}{\beta^3}\right]+X_0\bar{c}\int_0^{x-y\beta}f'd\xi\left(\frac{M_0^2}{\beta^2}\right) \qquad (53) \\
& +x\int_0^{x-y\beta}(f')^2d\xi\left[(\gamma+1)\left(\frac{M_0^6+M_0^4}{2\beta^6}\right)-\frac{2M_0^4}{\beta^4}\right]+X_0\bar{c}\int_0^{x-y\beta}ff''d\xi+yff'\left(\frac{2M_0^2}{\beta}\right) \\
& -X_0\bar{c}(x-y\beta)f'\frac{M_0^2}{\beta^2}\Bigg\}+\alpha\dot{\alpha}\Bigg\{-X_0\bar{c}Xf'\frac{2M_0^2}{\beta^2}+\left(\frac{M_0^2}{\beta^2}+1\right)X_0\bar{c}f \\
& +\left[\int_0^{x-y\beta}\int_0^{\xi}f'f''d\eta\, d\xi-(x-y\beta)\int_0^{x-y\beta}f'f''d\xi\right]\left[\frac{M_0^4(\gamma+1)}{\beta^4}\right]+\frac{2M_0^2}{\beta^2}\int_0^{x-y\beta}ff'd\xi \\
& -f\left(\frac{2M_0^2}{\beta^2}\right)-X_0^2\bar{c}^2f'+(x-y\beta)f'\frac{2M_0^2}{\beta^2}\Bigg\}+\dot{\alpha}\epsilon\Bigg\{-\int_0^{x-y\beta}\int_0^{\xi}ff''d\eta\, d\xi\left(\frac{M_0^2}{\beta^2}\right) \\
& +\left[\int_0^{x-y\beta}\int_0^{\xi}(f')^2d\eta\, d\xi-(x-y\beta)\int_0^{x-y\beta}(f')^2d\xi\right]\frac{M_0^2}{\beta^2}+(x-y\beta)\frac{M_0^2}{\beta^2}\int_0^{x-y\beta}ff''d\xi \\
& +\frac{M_0^2}{\beta^2}\int_0^{x-y\beta}ff'd\xi-X_0\bar{c}X\frac{M_0^2}{\beta^2}f'+x\frac{M_0^2}{\beta^2}\int_0^{x-y\beta}ff''d\xi+x\frac{M_0^2}{\beta^2}\int_0^{x-y\beta}(f')^2d\xi\Bigg\}
\end{aligned}$$

Thus, the complete second-order velocity potential is obtained.

$$\Phi_2^2 = \Phi_2^1 + \psi + \theta$$

where ϕ_2^1, ψ and θ are given by Eqs. (45), (47), and (53). This completes the general solution. Now consider the solution (in x, y, t) to the oscillating wedge problem.

Setting f $(x - y\,\beta) = x - y\,\beta$, the potential Φ_2^2 may be determined. Then the derivatives evaluated on the surface of the wedge $\left[y = (\alpha + \mathcal{E})\,x - x_o\,\alpha\,\bar{c}\right]$ are found to be,

$$\frac{1}{V_o}\Phi^2_{2x} = -\frac{1}{\beta}(\alpha+\varepsilon) + \frac{\dot{\alpha}}{V_o}\left(\frac{X}{\beta^3} + \frac{X_o\bar{c}}{\beta}\right) + (\alpha+\varepsilon)^2\left[\frac{4\beta^2 - M_o^4(\gamma+1)}{4\beta^4}\right]$$

$$+\frac{\dot{\alpha}}{V_o}(\alpha+\varepsilon)\left\{-X\left[\frac{M_o^4+M_o^2+1}{\beta^4} - (\gamma+1)\frac{3M_o^4}{2\beta^6}\right] - X_o\bar{c}\left(\frac{M_o^2+1}{\beta^2}\right)\right\}, \qquad (54)$$

$$\frac{1}{V_o}\Phi^2_{2y} = (\alpha+\varepsilon) + \frac{\dot{\alpha}}{V_o}(X - X_o\bar{c}) - \frac{1}{\beta^2}(\alpha+\varepsilon)^2 + \frac{\dot{\alpha}}{V_o}(\alpha+\varepsilon)\left(\frac{X}{\beta^3} + \frac{X_o\bar{c}}{\beta}\right), \qquad (55)$$

$$\frac{1}{V_o^2}\Phi^2_{2t} = -\frac{\dot{\alpha}X}{V_o\beta} + \frac{\dot{\alpha}}{V_o}(\alpha+\varepsilon)\left\{X\left[\frac{4M_o^2\beta^2 - M_o^4(\gamma+1)}{2\beta^4}\right] + X_o\bar{c}\left[\frac{(\gamma+1)M_o^2 + 2\beta^2}{2\beta^2}\right]\right\}. \qquad (56)$$

The pressure coefficient as given by Eq. (16) is,

$$C_p = -2\left[\frac{1}{V_o}\Phi^2_{2x} + \frac{1}{V_o^2}\Phi^2_{2t} + \frac{1}{2}\left[-\frac{\beta^2}{V_o^2}\left(\Phi'_{2x}\right)^2 + \frac{1}{V_o^2}\left(\Phi'_{2y}\right)^2 - \frac{2M_o}{V_o^3}\Phi'_{2x}\Phi'_{2t} - \frac{M_o^2}{V_o^4}\left(\Phi'_{2t}\right)^2\right]\right].$$

Which after substituting the values of ϕ^2_{2x}, ϕ^2_{2y} and ϕ^2_{2t}, from Eqs. (54), (55), and (56), becomes,

$$C_p = \frac{2}{\beta}\left\{(\alpha+\varepsilon) - \frac{\dot{\alpha}}{V_o}\left[X_o\bar{c} + \frac{X}{\beta^2}(2 - M_o^2)\right] + (\alpha+\varepsilon)^2\left[\frac{(\gamma+1)M_o^4 - 4\beta^2}{4\beta^3}\right]\right.$$

$$\left. + \frac{\dot{\alpha}}{V_o}(\alpha+\varepsilon)\left[\frac{X}{2\beta^5}\left(\gamma(M_o^6 - 4M_o^4) - 3M_o^6 + 10M_o^4 - 10M_o^2\right) - \frac{X_o\bar{c}}{2\beta}\left(M_o^2(\gamma+1) - 4\right)\right]\right\}. \qquad (57)$$

This is the pressure coefficient for the upper surface of the wedge. As before the pressure coefficient on the lower surface is equal to and 180° out of phase with, that of the upper surface. The total pressure difference C_{p_T} is then given as,

$$C_{p_T} = C_{p_\ell} - C_{p_u},$$

$$= C_p\left(\cos[\omega t + \pi],\ \sin[\omega t + \pi]\right) - C_p.$$

Or redefining α to be the actual angle of attack of the plate,

$$C_{p_T} = C_p - C_p(-\cos\omega t,\ -\sin\omega t).$$

This, after the substitution of C_p from Eq. (57), becomes;

$$C_{p_T}=\frac{4\alpha_M}{\beta}\left\{\cos\omega t\left[1+\frac{\varepsilon}{\beta}\left(\frac{(\gamma+1)M_0^4-4\beta^2}{2\beta^2}\right)\right]+\frac{\bar{\omega}}{2}\sin\omega t\left[X_0\beta^2\left(2+\frac{\varepsilon}{\beta}\left[M_0^2(\gamma+1)-4\right]\right)\right.\right.$$

$$\left.\left.+\frac{X}{\bar{C}}\left(4-2M_0^2+\frac{\varepsilon}{\beta}\left[\frac{\gamma(4M_0^4-M_0^6)+3M_0^6-10M_0^4+10M_0^2}{\beta^2}\right]\right)\right]\right\}. \qquad (58)$$

Where the angle of attack now represents the true angle of attack of the airfoil.

The lift coefficient

$$C_L=\frac{1}{\bar{C}}\int_0^{\bar{C}} C_{p_T}\,dX$$

may now be determined from Eq. (58) and is found to be,

$$C_L=\frac{4\alpha_M}{\beta}\left\{\cos\omega t\left[1+\frac{\varepsilon}{\beta}\left(\frac{(\gamma+1)M_0^4-4\beta^2}{2\beta^2}\right)\right]+\frac{\bar{\omega}}{2}\sin\omega t\left[X_0\beta^2\left(2+\frac{\varepsilon}{\beta}\left[M_0^2(\gamma+1)-4\right]\right)\right.\right.$$

$$\left.\left.\left(+2-M_0^2+\frac{\varepsilon}{\beta}\left[\frac{\gamma(4M_0^4-M_0^6)+3M_0^6-10M_0^4+10M_0^2}{2\beta^2}\right]\right)\right]\right\}. \qquad (59)$$

The pitching moment coefficient

$$C_M=\frac{1}{\bar{C}^2}\int_0^{\bar{C}}(X-X_0\bar{C})C_{p_T}\,dX$$

is found to be

$$C_M=-\frac{2\alpha_M}{\beta}\left\{\cos\omega t(1-2X_0)\left(1+\frac{\varepsilon}{\beta}\left[\frac{(\gamma+1)M_0^4-4\beta^2}{2\beta^2}\right]\right)+\frac{\bar{\omega}}{3}\sin\omega t\left[4-2M_0^2\right.\right.$$

$$+\frac{\varepsilon}{\beta}\left(\frac{\gamma(4M_0^4-M_0^6)+3M_0^6-10M_0^4+10M_0^2}{\beta^2}\right)+3X_0\left[2M_0^2-3\right. \qquad (60)$$

$$\left.\left.\left.+\frac{\varepsilon}{\beta}\left(\frac{\gamma(2M_0^6-6M_0^4+M_0^2)-2M_0^6+4M_0^4-M_0^2-4}{2\beta^2}\right)\right]-3\beta^2\left(2+\frac{\varepsilon}{\beta}\left[M_0^2(\gamma+1)-4\right]\right)X_0^2\right]\right\}.$$

This completes the analysis of the oscillating wedge. The discussion of the second-order effect on the lift and pitching moment for the wedge will be taken up in part four of this Section.

The Double Wedge Airfoil

The problem under investigation in this part is the effect of a sharp bend or break in the upper surface of the wedge. As a convenient reference term this problem has been entitled the double wedge problem since the solution will certainly be applicable to such an airfoil shape. From the results determined in Sections III and IV, it is apparent that the pressure at any point on this airfoil will consist of four terms, i.e.,

$$C_p = \frac{4}{\beta}\left[A\alpha + B\dot{\alpha} + C\alpha^2 + D\alpha\dot{\alpha}\right].$$

Of these four terms three are known apriori: The first term A is the term depending on the first-order solution in angle of attack and on the first-order in frequency ($\alpha^1\ \omega^1$). It must be the linearized solution previously determined and hence, A = instantaneous slope at that point. The second term B ($\alpha^1\ \omega^2$) is also a linearized term but is second-order in frequency. It is given in Eq. (57) as a function of x_o, $\bar{C}$, V_o and M_o. The third term C ($\alpha^2\ \omega^1$) is a term which corresponds to Busemann's second-order solution multiplied by the square of the instantaneous slope. Therefore, it appears that only the term D ($\alpha^2\ \omega^2$) actually remains to be determined.

Consider the oscillating wedge as shown in Fig. 7. This problem must be solved in

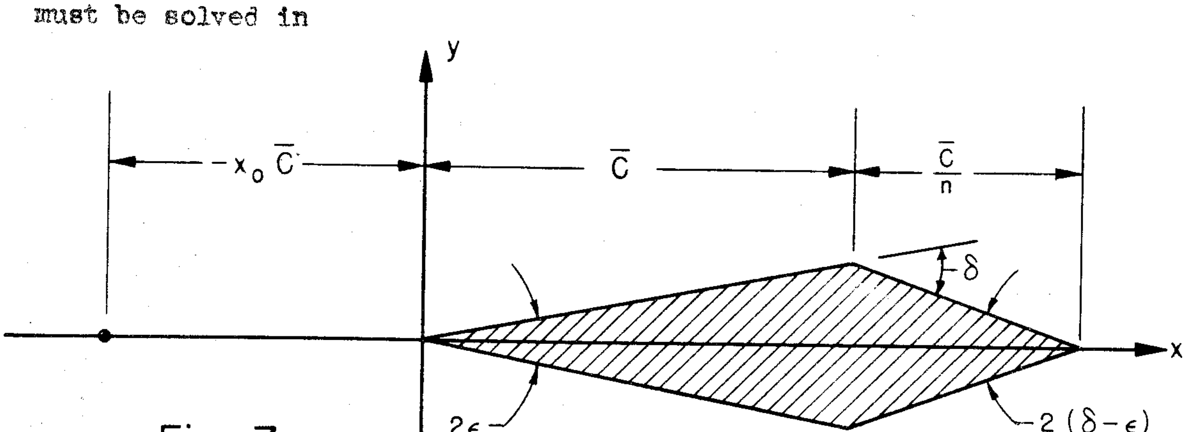

Fig. 7

two parts, one part representing the solution for $\frac{x}{\beta} \leq y \leq \frac{x}{\beta} + \frac{\bar{C}}{\beta}$, and one part representing the solution for $\frac{x}{\beta} + \frac{\bar{C}}{\beta} \leq y \leq \frac{x}{\beta} + \frac{\bar{C}}{\beta}(1 + \frac{1}{n})$. Of course, the previous wedge solution will take care of the solution for the diverging section of the wedge (i.e., $x \leq \bar{C}$). In addition, it will be necessary only to consider the effect, on the pressure, of a discontinuity in the slope of an oscillating flat plate therefore ε will be set equal to zero.

The Problem

For $x < \bar{C}$,

$$\left.\begin{aligned} &\Box \phi_2^1 = 0 , \\ &\left(\phi_{2y}^1\right)_{y=0} = V_0 \alpha f' + \dot{\alpha}\left\{-x_0 \bar{C} f' + \left(\frac{M_0^2}{\beta^2}+1\right) f - \frac{M_0^2 x f'}{\beta^2}\right\}, \\ &\phi_2^1 = \phi_{2x}^1 = 0 \quad at \quad x = 0 . \end{aligned}\right\} \quad 1^{st} \text{ Order} \qquad (61)$$

$$\left.\begin{aligned} &\Box \phi_2^2 = H(\phi_2^1) , \\ &\left[\frac{\Phi_{2y}^2 - \frac{dy}{dt}}{V_0 + \Phi_{2x}^2}\right]_{surface} = \frac{dy}{dx} , \\ &\phi_2^2 = \phi_{2x}^2 = 0 \quad at \quad x = 0 . \end{aligned}\right\} \quad 2^{nd} \text{ Order} \qquad (62)$$

Where the complete second-order potential is $\Phi_2^2 \left[\Phi_2^2 = \phi_2^1 + \phi_2^2\right]$. The problem posed by Eqs. (61) and (62) is identical to that solved by the potential for the flow past the oscillating airfoil of zero thickness Eq. (39).

<u>For $x > \bar{C}$,</u> the complete second-order potential is Ψ_2^2 ($\Psi_2^2 = \psi_2^1 + \psi_2^2$). A new function g will now be defined such that $g = g_x = 0$ at $x = \bar{C}$. Later for the specific problem of the broken flat plate $g(\xi)$ will specifically be defined as $g(\xi) = g(x - y\beta - \bar{C})$;

thus $\frac{\partial g}{\partial x} = g'$ and $\frac{\partial g}{\partial y} = -\beta\, g'$. The first-order potential must satisfy the conditions,

$$\left.\begin{aligned} &\Box \psi_2^1 = 0 \ , \\ &\psi^1_{2y}\Big)_{y=0} = V_0(\alpha f' + \delta g') + \dot{\alpha}\left\{ -x_0 \bar{c} f' + \left(\frac{M_0^2}{\beta^2} + 1\right) f - \frac{M_0^2 x}{\beta^2} f' \right\} \\ &\psi_2^1 = \phi_2^1 \ \& \ \psi^1_{2x} = \phi^1_{2x} \quad \text{at } x = \bar{c} \ . \end{aligned}\right\} \ 1^{st} \text{ Order} \qquad (63)$$

The last condition requires that there be no discontinuity in the potential or its x derivative anywhere in the flow. In particular there can be no discontinuities near the Mach line from the break in the plate. The second-order potential ψ_2^2 must satisfy the conditions,

$$\left.\begin{aligned} &\Box \psi_2^2 = H(\psi_2^1) \ , \\ &\left[\frac{\Psi^2_{2y} - \frac{dy}{dt}}{V_0 + \Psi^2_{2x}}\right]_{surface} = \frac{dy}{dx} \ , \\ &\psi_2^2 = \phi_2^2 \ \& \ \psi^2_{2x} = \phi^2_{2x} \quad \text{at } x = \bar{c} \end{aligned}\right\} \ 2^{nd} \text{ Order} \qquad (64)$$

Again the boundary conditions allow no discontinuities in the flow. Now, note that this problem is very similar to the double wedge problem in α and ε. In fact if $\delta\, g$ were replaced by $\varepsilon\, f$ the problems would be identical except for the requirements on ϕ and ϕ_x. However, it is apparent that this boundary condition would be satisfied in both cases. The determination of the solution for $x > \bar{c}$ is much simplified by this fact. A first-order potential is immediately seen to be,

$$\psi_2^1 = -\frac{1}{\beta}(\alpha f + \delta g) + \dot{\alpha}\left\{ + \frac{M_0^2 x f}{\beta^3} + \frac{x_0 \bar{c} f}{\beta} - \left(\frac{M_0^2}{\beta^3} + \frac{1}{\beta}\right) \int_0^{x - y\beta} f(\xi)\, d\xi \right\} \ . \qquad (65)$$

The second-order solution which will now be determined requires some changes but is handled in essentially the same manner. From Eq. (65) the function H (ψ_2^1) as given in Eq. (9) is,

$$H(\psi_2^1) = -V_0\frac{(\gamma+1)M_0^4}{\beta^2}\left\{\alpha^2 ff'' + \alpha\delta(f'g''+g'f'') + \delta^2 g'g''\right\}$$
$$+\dot{\alpha}(\alpha f'+\delta g')\left\{\frac{2M_0^6(\gamma+1)}{\beta^4}xf'' + \frac{(3M_0^4-2M_0^6)(\gamma+1)}{\beta^4}f'\right\}$$
$$-\dot{\alpha}(\alpha f''+\delta g'')\left\{\frac{2M_0^4(\gamma+1)}{\beta^2}\right\}f + \dot{\alpha}\delta x(f'g''-g'f'')\left\{\frac{M_0^6(\gamma+1)}{\beta^4}\right\} \tag{66}$$
$$-\dot{\alpha}\left[(\alpha f'+\delta g')f''+(\alpha f''+\delta g'')f'\right]\left\{x_0\bar{c}\,\frac{M_0^4(\gamma+1)}{2\beta^2}\right\}.$$

Thus σ the particular solution is found to be,

$$\sigma = -(\alpha f'+\delta g')\frac{V_0M_0^4(\gamma+1)y}{4\beta^3} + V_0\alpha f'(\alpha f'+2\delta g')\left\{\frac{M_0^2(\gamma+1)x_0\bar{c}}{4\beta^2}\right\}$$
$$+\alpha V_0\left\{\alpha\int_0^{x-y\beta}(f')^2 d\xi + 2\delta\int_0^{x-y\beta}f'g'd\xi\right\}\left\{\frac{M_0^2(\gamma+1)}{4\beta^4}\right\} \tag{67}$$
$$+\dot{\alpha}(\alpha f'+\delta g')\left\{\frac{M_0^6(\gamma+1)}{2\beta^5}xyf' - \frac{M_0^4(\gamma+1)}{\beta^4}xf\right\}.$$

It will be noted that Eq. (67) is very nearly the same as Eq. (50), the particular solution for the wedge; in fact for f $(\xi) = \xi$ and g $(\xi) = \xi - \bar{C}$ the two expressions are identical. The second-order boundary condition to be satisfied by the correction potential θ $(\psi_2^2 = \sigma + \theta)$ is,

$$(\theta_y)_{y=0} = -(\sigma_y)_{y=0} + \left[(\psi_{2y}^1)_{y=0} - (\psi_{2y}^1)_{sur.}\right] + (\alpha f'+\delta g')(\psi_{2x}^1). \tag{68}$$

The terms on the right side of Eq. (68) may be evaluated from Eqs. (66) and (67).

$$\psi_{2x}^1(\alpha f'+\delta g') = -\frac{1}{\beta}(\alpha f'+\delta g') + \dot{\alpha}(\alpha f'+\delta g')\left\{\frac{M_0^2}{\beta^3}xf' - \frac{f}{\beta} + \frac{x_0\bar{c}f'}{\beta}\right\}, \tag{69}$$

$$-(\sigma_y)_{y=0} = (\alpha f' + \delta g')^2 \frac{V_0 M_0^4 (\gamma+1)}{4\beta^3} + \alpha V_0 (\alpha f' + 2\delta g')(f') \frac{M_0^2(\gamma+1)}{4\beta^3}$$

$$+ V_0\left[\alpha^2 f' f'' + \alpha\delta(g' f'' + f' g'')\right]\frac{M_0^2(\gamma+1)X_0\bar{C}}{2\beta} \tag{70}$$

$$-\dot{\alpha}(\alpha f' + \delta g')\left[x f' \frac{M_0^4(\gamma+1)(3M_0^2-2)}{2\beta^5}\right] - \dot{\alpha}(\alpha f'' + \delta g'')\left[x f \frac{M_0^4(\gamma+1)}{\beta^4}\right].$$

and

$$\left[(\psi'_{2y})_{y=0} - (\psi'_{2y})_{SUR}\right] = V_0\beta\left[\alpha^2 f f'' + 2\delta\alpha(g f'' + f g'') + \delta^2 g g''\right]$$

$$+ \dot{\alpha}(\alpha f + \delta g)\left[\left(\frac{M_0^2}{\beta} + \beta\right) f' - \frac{M_0^2}{\beta} x f''\right] - \alpha(\alpha f'' + \delta g'')\beta X_0 \bar{C} f' \tag{71}$$

$$-\dot{\alpha}(\alpha f + \delta g)\beta X_0 \bar{C} f'' + \alpha\dot{\alpha}\left[\beta X_0^2 \bar{C}^2 f' f'' + \frac{M_0^2}{\beta} x X_0 \bar{C} f' f'' - \left(\frac{M_0^2}{\beta^2} + 1\right)\beta X_0 \bar{C}(f')^2\right].$$

The correction potential θ which must satisfy the homogeneous wave equation and Eq. (68) can now be determined. However, since the solution is quite lengthy let it suffice to say that the correction potential θ is essentially the same as that for the correction potential (Eq. (53)) for the oscillating wedge when ε f is replaced by δ g. In fact when f and g are replaced by $x - y\beta$ and $x - y\beta - \bar{C}$ the two functions become identical.

The pressure on the oscillating double wedge may now be determined from Eq. (57).

<u>For $x < \bar{C}$</u>, the pressure coefficient C_p is given by Eq. (57) when ε is set equal to zero.

$$C_{p_u} = \frac{2}{\beta}\left\{(\alpha) - \left(\frac{\dot{\alpha}}{V_0}\right)\left[X_0\bar{C} + \frac{X}{\beta^2}(2 - M_0^2)\right] + (\alpha^2)\left[\frac{A}{2\beta}\right]\right.$$

$$\left. + \left(\frac{\dot{\alpha}}{V_0}\alpha\right)\left[\frac{X}{2\beta}D - \frac{X_0\bar{C}}{2\beta}B\right]\right\}. \tag{72A}$$

Where

$$A = \frac{(\gamma+1)M_0^4 - 4\beta^2}{2\beta^2} ,$$

$$B = M_0^2(\gamma+1) - 4 ,$$

and

$$D = \frac{\gamma(4M_0^4 - M_0^6) + 3M_0^6 - 10M_0^4 + 10M_0^2}{2\beta^2} .$$

For $\gamma = 1.40$ A, B and D are plotted as a function of Mach number on Fig. 8.

For $x > \bar{C}$

The pressure coefficient C_p is given by Eq. (57) when ε is set equal to $-\delta$ ($-\delta$ since the bend, see Fig. 7), causes a negative change of slope).

$$C_{p_u} = \frac{2}{\beta}\left\{ (\alpha-\delta) - \left(\frac{\dot{\alpha}}{V_0}\right)\left[X_0\bar{C} + \frac{X}{\beta^2}(2-M_0^2)\right] + (\alpha-\delta)^2\left[\frac{1}{2\beta}A\right] + \left[\frac{\dot{\alpha}}{V_0}(\alpha-\delta)\right]\left[\frac{X}{2\beta}D - \frac{X_0\bar{C}}{2\beta}B\right]\right\} . \tag{72B}$$

Thus the effect of a corner on the pressure has been determined. Consider the four terms in the expression for the pressure.

A: $\alpha^1 \omega^1$ (first-order in α and ω)

The pressure depends only on the instantaneous slope of the airfoil at the point in question. For a symmetrical airfoil the effects of thickness will cancel. This would be the well-known linearized solution for an oscillating plate as ω approached zero.

B: $\alpha^1 \omega^2$ (first-order in α second-order in ω)

The pressure depends on the rate of change of angle of attack $\dot{\alpha}$ multiplied by a function of x, x_0, M_0^2 and γ. This is also a part of the linearized solution and thickness has no effect.

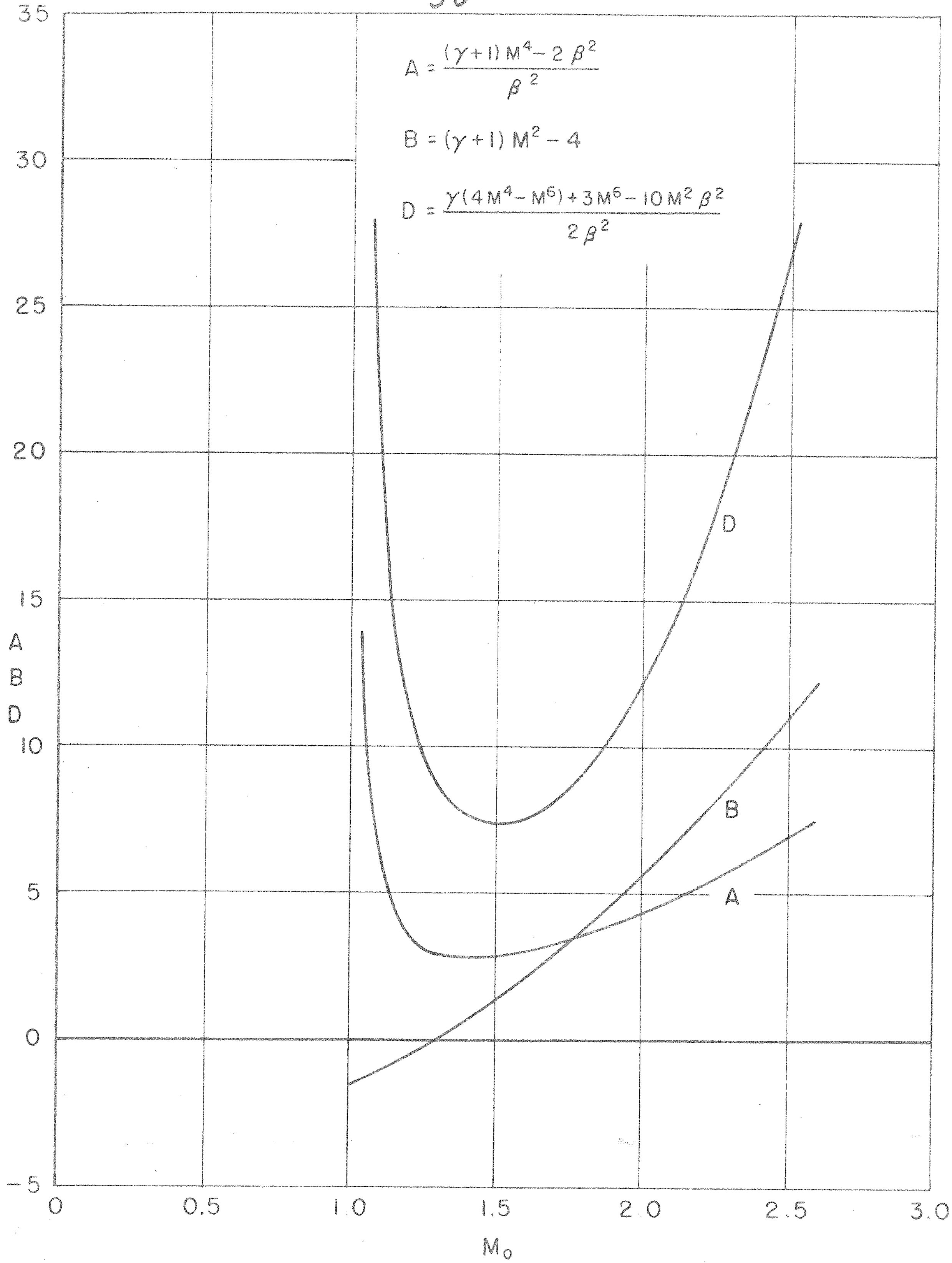

Fig. 8 — Graphical representation of the functions A, B and D versus Mach number for $\gamma = 1.40$

C: $\alpha^2 \omega^1$ (second-order in α, first-order in ω)

The pressure depends on a function of Mach number multiplied by the square of the instantaneous slope. This is Busemann's second-order solution and is the result that would be expected as ω approaches zero.

D: $\alpha^2 \omega^2$ (second-order in α and ω)

The pressure depends on the instantaneous slope, the rate of change of angle of attack α, and a function of x_o, x, M_o and γ.

This completes the special study of the double wedge airfoil and as will be seen in part three it has to all practical purposes completed the study of an oscillating airfoil of general shape.

The General Airfoil

Consider the airfoil as shown in Fig. 9.

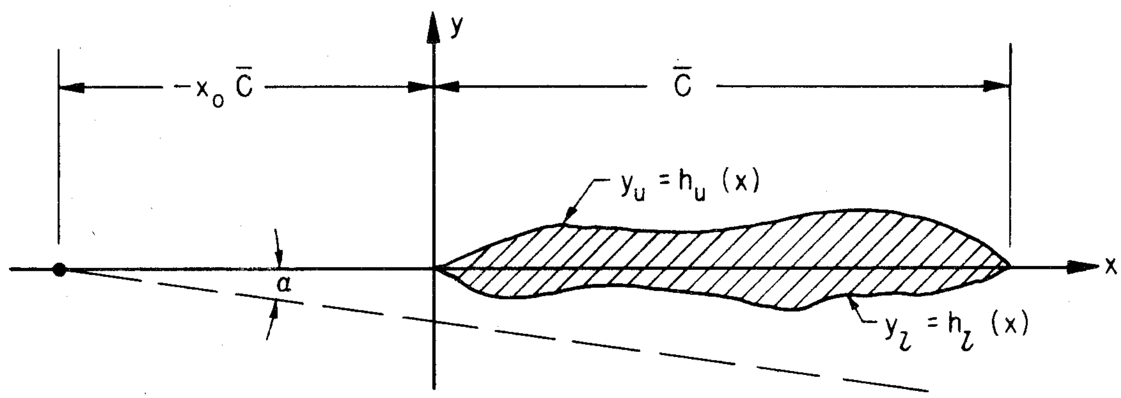

Fig. 9

Consider the pressure at any point x_1 on the upper surface. From the results determined in part two for the oscillating double wedge it is apparent that by a series of n steps the pressure on the surface of an airfoil with n breaks could also be calculated.

Then, independent of the shape of the airfoil ahead of the surface in question the pressure at that point would be the same as that given in the discussion of the pressure on the double wedge airfoil. Thus, it is apparent that the pressure at any point is independent of the shape of the remainder of the airfoil and depends only on x_o, M_o and γ, and on the slope and the rate of change of slope at that point. Hence, the pressure on the general airfoil of Fig. 9 may be expressed in the form, (on the upper surface)

$$C_{p_u} = \frac{2}{\beta}\left\{\left(\alpha - \frac{dy_u}{dx}\right) - \left(\frac{\dot{\alpha}}{V_o}\right)\left[x_o\bar{c} + \frac{x}{\beta^2}(2-M_o^2)\right] + \left(\alpha - \frac{dy_u}{dx}\right)^2\left(\frac{A}{2\beta}\right)\right.$$
$$\left. + \left(\frac{\dot{\alpha}}{V_o}\right)\left(\alpha - \frac{dy_u}{dx}\right)\left[\frac{x}{2\beta}D - \frac{x_o\bar{c}}{2\beta}B\right]\right\}, \qquad (73A)$$

(on the lower surface)

$$C_{p_l} = \frac{2}{\beta}\left\{\left(-\alpha + \frac{dy_l}{dx}\right) + \left(\frac{\dot{\alpha}}{V_o}\right)\left[x_o\bar{c} + \frac{x}{\beta^2}(2-M_o^2)\right] + \left(\alpha + \frac{dy_l}{dx}\right)\left(\frac{A}{2\beta}\right)\right.$$
$$\left. - \left(\frac{\dot{\alpha}}{V_o}\right)\left(\alpha + \frac{dy_l}{dx}\right)\left[\frac{x}{2\beta}D - \frac{x_o\bar{c}}{2\beta}B\right]\right\}. \qquad (73B)$$

Eqs. (73) give the pressure on the upper and lower surfaces of an oscillating airfoil of general shape. The lift, drag and pitching moment coefficients may be calculated from these equations in the usual manner. This result completes the general second-order investigation of the oscillating airfoil problem.

Lift and Pitching Moment Coefficients For a Modified Double Wedge Airfoil

Consider the specific problem of the modified double wedge airfoil as shown in Fig. 10.

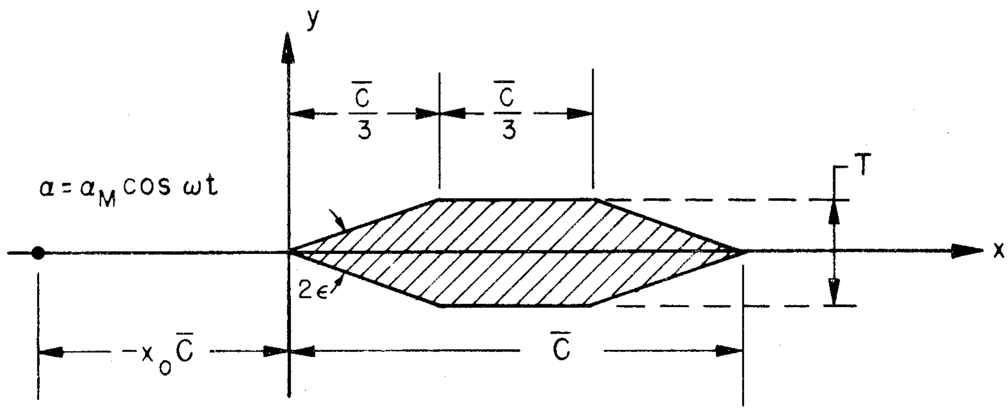

Fig. 10

From Eq. (73) the pressure difference ($C_{p_T} = C_{p_u} - C_{p_L}$) between the the upper and lower surfaces of the modified double wedge is found to be

$(0 < x < \frac{\bar{c}}{3})$

$$C_{p_T} = \frac{4\alpha_M}{\beta}\left\{\cos \omega t\left[1 + \frac{A\varepsilon}{\beta}\right] + \frac{\bar{\omega}}{2}\sin \omega t\left[2X_0\beta^2 + 2\frac{x}{\bar{c}}(2-M_0^2) + \frac{\varepsilon}{\beta}\left(\frac{x}{\bar{c}}D + X_0\beta^2 B\right)\right]\right\}, \tag{74A}$$

$(\frac{\bar{c}}{3} < x < \frac{2\bar{c}}{3})$

$$C_{p_T} = \frac{4\alpha_M}{\beta}\left\{\cos \omega t + \frac{\bar{\omega}}{2}\sin \omega t\left[2X_0\beta^2 + 2\frac{x}{\bar{c}}(2-M_0^2)\right]\right\}, \tag{74B}$$

$(\frac{2\bar{c}}{3} < x < \bar{c})$

$$C_{p_T} = \frac{4\alpha_M}{\beta}\left\{\cos \omega t\left[1 - \frac{A\varepsilon}{\beta}\right] + \frac{\bar{\omega}}{2}\sin \omega t\left[2X_0\beta^2 + 2\frac{x}{\bar{c}}(2-M_0^2) - \frac{\varepsilon}{\beta}\left(\frac{x}{\bar{c}}D + X_0\beta^2 B\right)\right]\right\}. \tag{74C}$$

Hence, the lift coefficient $C_L = \frac{1}{\bar{c}}\int_0^{\bar{c}} C_{p_T}\, dx$ is,

$$C_L = \frac{4\alpha_M}{\beta}\left\{\cos\omega t + \frac{\bar{\omega}}{2}\sin\omega t\left[2X_0\beta^2 + (2-M_0^2) - \frac{2\varepsilon}{9\beta}D\right]\right\}. \tag{75}$$

While the pitching moment coefficient $C_m = \frac{1}{\bar{C}^2}\int_0^{\bar{C}} (x - x_o\,\bar{C})\,C_{p_T}\,dx$ may be expressed in the form,

$$C_M = -\frac{2\alpha_M}{\beta}\left\{\cos\omega t\left[1-2X_0-\frac{4\varepsilon}{9\beta}A\right] + \frac{\bar{\omega}}{3}\sin\omega t\left[2\left(2-M_0^2-\frac{\varepsilon}{\beta}D\right)\right.\right.$$

$$\left.\left. + X_0\left(6\beta^2-3+2\frac{\varepsilon}{\beta}\left[D-B\beta^2\right]-6X_0^2\beta^2\right]\right\}. \tag{76}$$

Now consider the importance of the second-order terms in angle of attack. For the lift coefficient (where $C_L = \frac{4\,\alpha_M}{\beta}\left[P\cos\omega t + \frac{\bar{\omega}}{2}Q\sin\omega t\right]$) a plot of the function Q versus Mach number is presented on Fig. 11 for three thickness ratios ($\frac{T}{\bar{C}} = 0, .05, .10$). It appears that the second-order effects in angle of attack (See Fig. 11) are most predominant at the low Mach numbers. However since the higher order terms in frequency are small for $M_o < 1.4$ the relative importance of the second-order terms in angle of attack is approximately constant throughout the Mach number range $1.1 < M_o < 2.0$.

For the moment coefficient where $C_m = -\frac{2\,\alpha_M}{\beta}(R\cos\omega t + \frac{\bar{\omega}}{3}S\sin\omega t)$ the functions R and S are plotted versus Mach number for three thickness ratios ($\frac{T}{\bar{C}} = 0, .05, .10$) on Figs. 12 and 13. Again in the case of the moment coefficient the second-order terms have about the same relative importance throughout the Mach number range $1.1 < M_o < 2.0$ since the higher order terms in frequency are of smaller magnitude at the higher Mach numbers. The second-order terms are especially important for the

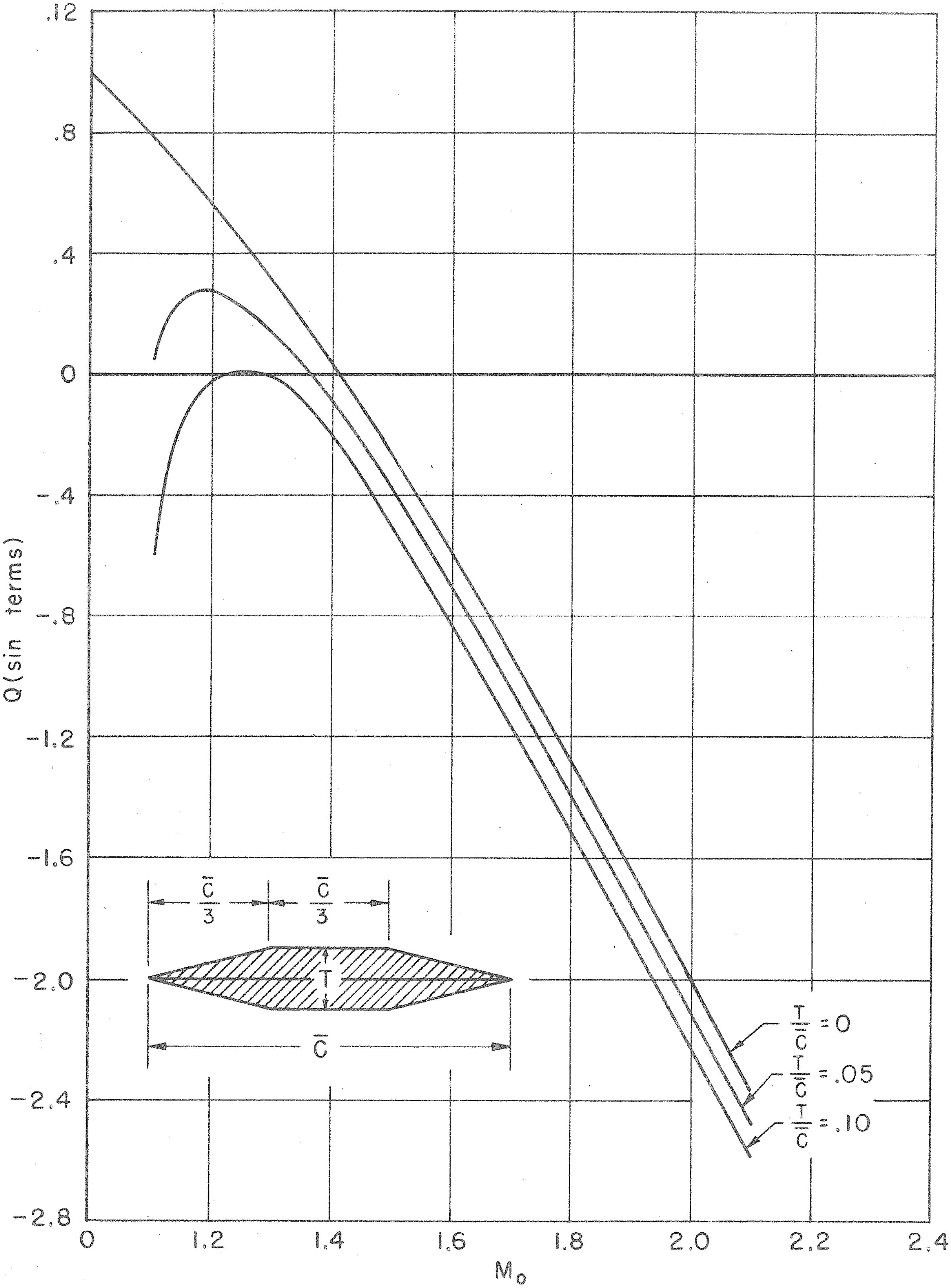

Fig. 11 — The effect of thickness on the second order terms in frequency for a modified double wedge airfoil rotating about its leading edge

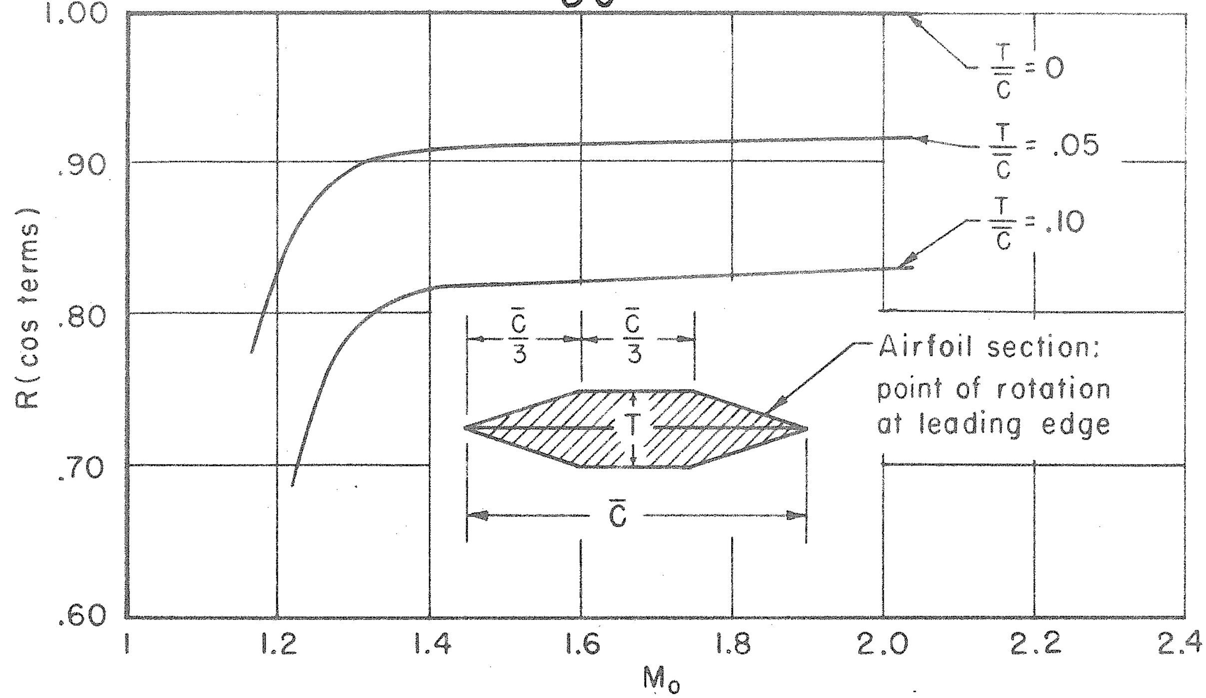

Fig. 12 — Comparison of the even terms in the second order solution for the pitching moment of a modified double wedge airfoil at three thickness ratios

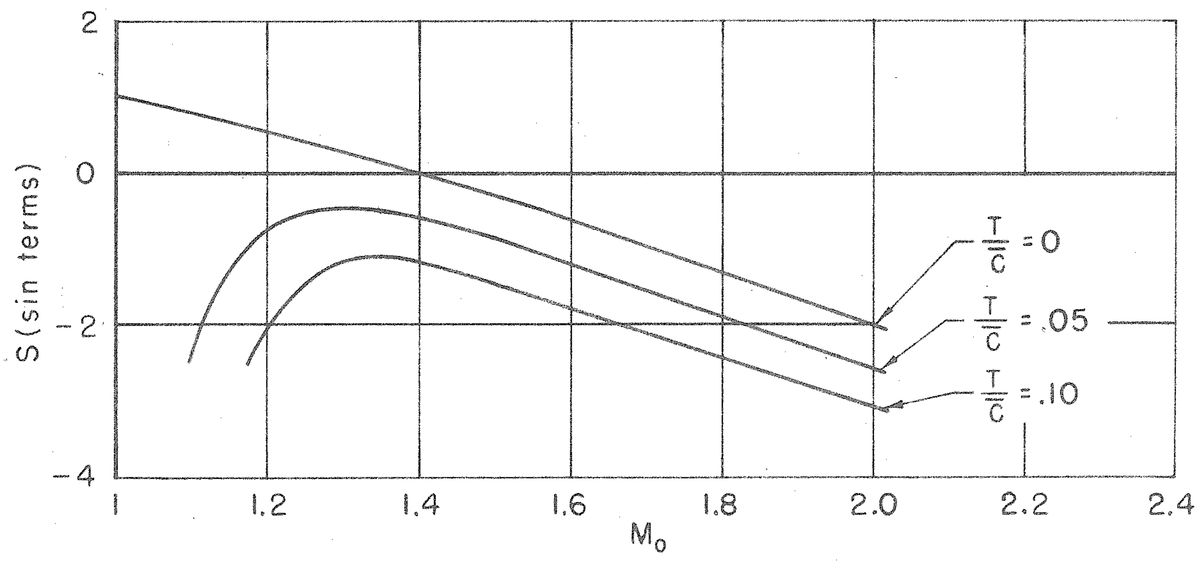

Fig. 13 — Comparison of the odd terms in the second order solution for the pitching moment of a modified double wedge airfoil at three thickness ratios

odd (sin) terms in frequency. In Fig. 13 it is seen that the second-order terms in angle of attack actually are of greater magnitude than the first-order terms for most of the Mach number range. It is this fact which causes the unstable tendencies, noted by Garrick and Rubinow, to be eliminated and the motion to be stable when the effects of thickness are considered. When the Mach number is less than $\sqrt{2}$ (for this case of $x_o = 0$) the function S is seen to be positive for the case of a flat plate ($\frac{\overline{T}}{\overline{C}} = 0$). This term being positive corresponds to negative damping - i.e., an unstable motion. However, when the effect of thickness is considered (See Fig. 13), the function S is seen to be negative throughout the whole Mach number range. This is further indicated in Fig. 14 where the region of instability shrinks rapidly to zero with increasing $\frac{\overline{T}}{\overline{C}}$ and disappears entirely for $\frac{\overline{T}}{\overline{C}} \geq .03$.

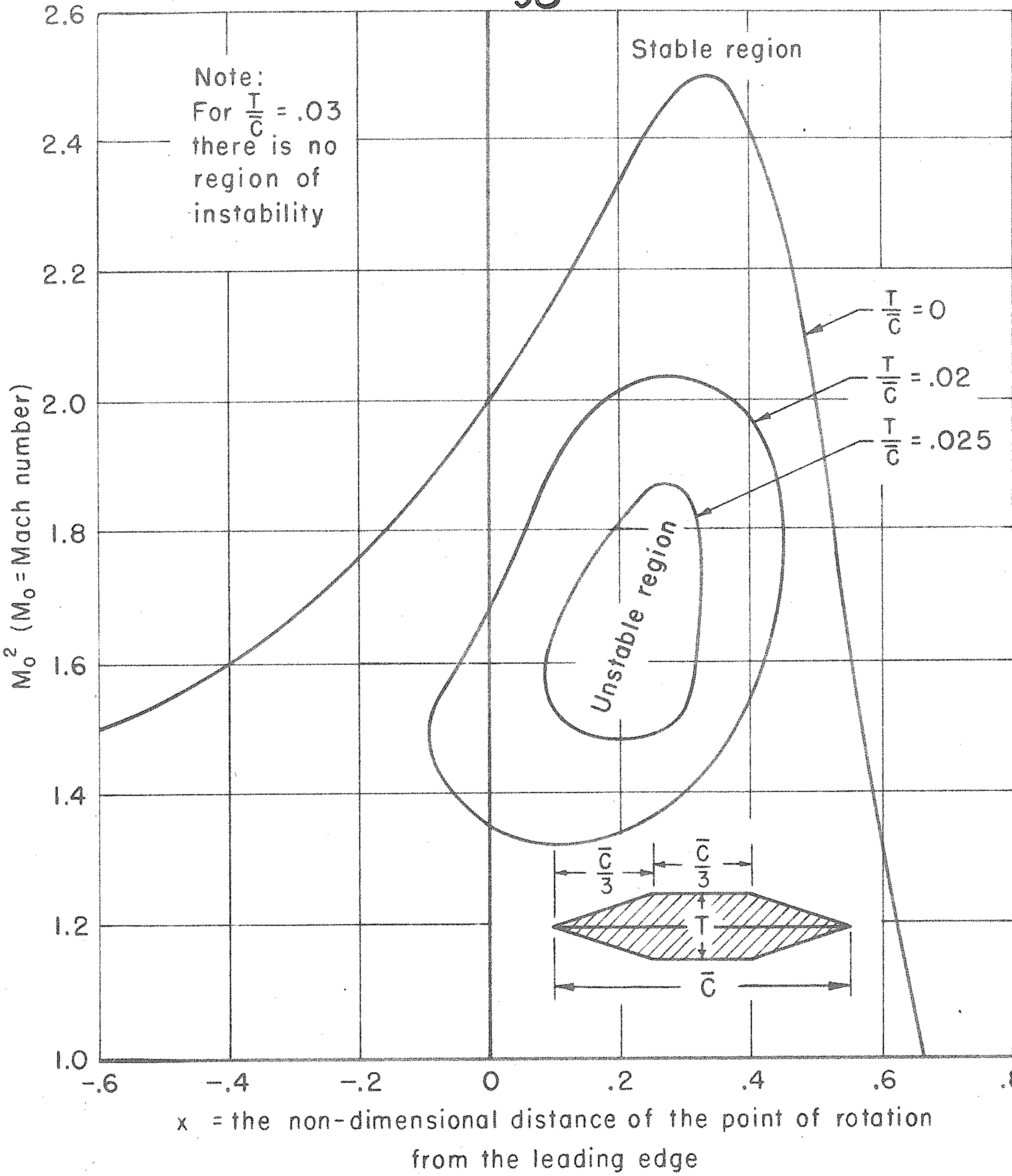

Fig. 14 — Region of instability for an oscillating double wedge airfoil as a function of Mach number (M_0) and thickness ratio ($T/\bar{C}$)

SECTION VI. CONCLUDING REMARKS

Two extensions of the study presented in this paper are apparent. Either a solution valid to third-order in angle of attack and second-order in frequency or a second-order solution in angle of attack valid to fourth- or sixth-order in frequency could be attempted.

A third-order solution in angle of attack would require that the effects of entropy be considered, since the effects of entropy are proportional to the cube of the deflection angle for small deflections. To include the effects of changes in entropy it would be necessary to consider the exact position and slope of the shock wave. This flow region is however, exactly the region in which the iteration procedure fails to give a solution. Hence, it does not seem possible to include the entropy changes in the iteration procedure. It has been pointed out by Laitone[27] that for reasonably small deflections (say less than ten degrees) the entropy effects constitute only a small portion of the total third-order effects and thus, may be neglected. With this assumption, the third-order solution could be determined from the third step of the iteration procedure.

The extension of the second-order solution to higher order in frequency could probably be justified if more exact pressure distributions are needed in the investigation of flutter. The extension to sixth-order in frequency, while quite tedious, should give a solution which would be within a few percent of the exact second-order solution in angle of attack whenever the reduced frequency $(\bar{\omega})$ is less than 1.30.

Second-order solutions to other non-stationary problems (such as, an airfoil experiencing a constant rate of change of angle of attack or an airfoil experiencing a constant acceleration in forward velocity, etc.) can be attempted by the same method utilized in this report. That is by the calculation of a general first-order velocity potential and then the utilization of the iteration procedure.

REFERENCES

1. Possio, Camillo; L' Azione Aerodinamica sul Profilo Oscillante alle Velocita Ultrasonore, Acta Pontif. Acad. Sci., Vol. I, No. II (1937). Available as Cornell Aeronautical Laboratory Translation: The Aerodynamic Action on an Oscillating Profile at Supersonic Velocities.

2. von Borbély, S.; Über die Luftkräfte die auf einen harmonisch schwingenden zweidimensionalen Flügel bei Überschallgeschwindigkeit wirken, ZWB FB 1701, (1942). Also available as British RTP-2019.

3. Garrick, I. E. and Rubinow, S. I.; Flutter and Oscillating Air Force Calculations for an Airfoil in a Two-Dimensional Supersonic Flow, NACA TN 1158 (1946).

4. Garrick, I. E. and Rubinow, S. I.; Theoretical Study of Air Forces on an Oscillating or Steady Thin Wing in a Supersonic Main Stream, NACA TN 1383 (1947).

5. Temple, G. and Jahn, H. A.; Flutter at Supersonic Speeds, R.A.E. Report SME 3314 (1945).

6. Miles, J. W.; The Aerodynamic Forces on an Oscillating Airfoil at Supersonic Speeds, Jour. of Aero. Sci., Vol. 14, pp. 351-358 (1947).

7. Stewart, H. J. and Li, Ting-Yi; Periodic Motions of a Rectangular Wing Moving at Supersonic Speed, Jour. of Aero. Sci., Vol. 17, No. 9, (1950).

8. Miles, J. W.; Transient Loading of Wide Delta Airfoils at Supersonic Speeds, U.S. Naval Ordnance Test Station Tech. Memo. RRB-37 (1950).

9. Miles, J. W.; The Oscillating Rectangular Airfoil at Supersonic Speeds, U.S. Naval Ordnance Test Station Report 1170 (1949).

10. Hipsch, H. M.; Harmonic Oscillations of a Narrow Delta Wing in Supersonic Flow. Thesis, California Institute of Technology, Pasadena (1951).

11. Froelich, J. E.; Non-Stationary Motion of Purely Supersonic Wings. Thesis, California Institute of Technology, Pasadena (1950).

12. Parkinson, G. V.; Non-Uniform Motion of a Wide Delta Wing. Thesis, California Institute of Technology, Pasadena (1951).

13. Busemann, A.; Infinitesimale Kegelige Überschallströmung. D. A. L., pp.445-570 (December 4, 1942). Translated as NACA TM 1100.

14. Van Dyke, Milton D.; A Study of Second-Order Supersonic Flow. Thesis, California Institute of Technology, Pasadena (1948). Also available as NACA TN 2200 (1951).

15. Guderley, G.; The Lift of a Two-Dimensional Flat Plate Oscillating with a Small Frequency, unpublished note, (December, 1947).

16. Prandtl, L.; Allgemeine Überlegungen über die Strömung zusammendruckbarer Flüssigkeiten. Atti dei Convegni 5, Reale Accademia D' Italia (Volta Congress), pp. 169-197, Rome (1936). Also translated as NACA TM 805 and as British RTP 1872.

17. Görtler, H.; Gasströmungen mit Übergang von Unterschall-zu Überschallgeschwindigkeit. Z.A.M.M., Vol. 20, pp. 254-262 (1940).

18. Hantzsche, W. and Wendt, H.; Der Kompressibilitätseinfluss für dünne wenig gekrümmte Profile bei Unterschallgeschwindigkeit. Z.A.M.M., Vol. 22, pp. 72-86 (1942).

19. Hantzsche, W.; Die Prandtl - Glauertsche Näherung als Grundlage für ein Iterationsverfahren zur Berechnung kompressibler Unterschallströmungen. Z.A.M.M., Vol. 23, pp. 185-199, (1943).

20. Imai, Isao; Two-Dimensional Aerofoil Theory for Compressible Fluids. (Japanese, English abstract), Aero. Res. Inst. Tokyo Imp. Univ., Report Number 294 (1944).

21. Imai, Isao and Oyama, Seiichi; The Third-Approximation of the Thin-Wing-Expansion for Compressible Fluids. (Japanese) Rep. Inst. Sci. Technol., Tokyo, Vol. 2, pp. 37-39 (1948).

22. Kaplan, Carl; The Flow of a Compressible Fluid Past a Curved Surface. NACA TR 768 (1943).

23. Kaplan, Carl; The Flow of a Compressible Fluid Past a Circular Arc Profile. NACA TR 794 (1944).

24. Schmieden, C. and Kawalki, K. H.; Beitrage zum Unströmungsproblem bei höhen Geschwindigkeiten. Bericht S 13/1, Teil der Lilienthal-Gesellschaft für Luftfahrtforschung, pp. 140-152 (1942). Translated as NACA TM 1233.

25. Watkins, C. E.; Effect of Aspect Ratio on the Air Forces and Moments of Harmonically Oscillating Thin Rectangular Wings in Supersonic Potential Flow, NACA TN 2064, (1950).

26. Courant, R. and Friedrichs, K. O.; Supersonic Flow and Shock Waves. Interscience, New York (1948).

27. Laitone, E. V.; Exact and Approximate Solutions of Two-Dimensional Oblique Shock Flow. Journal of Aero. Sci., Vol. 14, Number 1, (1947).