DISCONTINUOUS DEFORMATION GRADIENTS IN PLANE FINITE ELASTOSTATICS OF INCOMPRESSIBLE MATERIALS

(I) GENERAL CONSIDERATIONS

(II) AN EXAMPLE

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Rohan Chandra Abeyaratne

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To My Mother and Father

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ABSTRACT

This investigation is concerned with the possibility of the change of type of the differential equations governing finite plane elastostatics for incompressible elastic materials, and the related issue of the existence of equilibrium fields with discontinuous deformation gradients. Explicit necessary and sufficient conditions on the deformation invariants and the material for the ellipticity of the plane displacement equations of equilibrium are established. The issue of the existence, locally, of "elastostatic shocks"— elastostatic fields with continuous displacements and discontinuous deformation gradients — is then investigated. It is shown that an elastostatic shock exists only if the governing field equations suffer a loss of ellipticity at some deformation. Conversely, if the governing field equations have lost ellipticity at a given deformation at some point, an elastostatic shock can exist, locally, at that point. The results obtained are valid for an arbitrary homogeneous, isotropic, incompressible, elastic material. In order to illustrate the occurrence of elastostatic shocks in a physical problem, a specific displacement boundary value problem is studied. Here, a particular class of isotropic, incompressible, elastic materials which allow for a loss of ellipticity is considered. It is shown that no solution which is smooth in the classical sense exists to this problem for certain ranges of the applied loading. Next, we admit solutions involving elastostatic shocks into the discussion and find that the problem may then be solved completely. When this is done, however, there results a lack of uniqueness of solutions to the boundary value problem. In order to resolve this non-uniqueness, dissipativity and stability are investigated.

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INTRODUCTION

It is known that the type of the system of partial differential equations governing finite elastostatics can change type from being elliptic to being non-elliptic at sufficiently large deformations for certain reasonable materials. This leads one to suspect that classically smooth solutions may cease to exist in certain situations for such materials, and raises the question of the possibility that the elastostatic field may exhibit certain discontinuities. The mathematical problem here is analogous to that describing the steady irrotational flow of an inviscid compressible fluid, where the governing equations are elliptic at a point where the flow is subsonic and hyperbolic where it is supersonic. It is well known that in such flows there may occur shocks — surfaces in the flow field across which certain physical quantities suffer jump discontinuities.

The present study is in two parts. In the first, we look at the general theory associated with the issues mentioned above in the context of plane deformations of incompressible elastic materials. In particular we determine the precise conditions under which ellipticity of the governing system of partial differential equations is lost and examine the conditions under which solutions exhibiting certain discontinuities, referred to as elastostatic shocks, can exist.

In the second part we look at a specific equilibrium boundary value problem in order to demonstrate how elastostatic shocks could actually occur in a physical situation. We show that this one-dimensional example has no classically smooth solution for certain ranges of the applied loading, but that solutions involving shocks can be constructed in such circumstances.

PART I

GENERAL CONSIDERATIONS

CHAPTER I

1.1 Introduction

In two recent papers [1], [2], Knowles and Sternberg looked into the question of the possible loss of ellipticity of the displacement equations of equilibrium of nonlinear elastostatics for compressible materials. In [1], three dimensional homogeneous deformations of a particular isotropic compressible elastic material were considered, and necessary and sufficient restrictions on the principal stretches for ellipticity to prevail were deduced. It was shown that for this material, a loss of ellipticity occurred at sufficiently severe local deformations. In [2] they established similar explicit necessary and sufficient conditions for an arbitrary homogeneous, isotropic, compressible elastic solid subjected to plane deformations.

These papers were motivated by some asymptotic studies of crack problems they had considered previously, in which certain difficulties encountered suggested that the problem may not admit a classically smooth solution.

In a subsequent paper [3] Knowles and Sternberg investigated the implications of a loss of ellipticity. The question of the existence of "elastostatic (or equilibrium) shocks" — solutions which possess finite jump discontinuities of the displacement gradient across certain surfaces while maintaining continuous displacements — was studied within the context of plane deformations of compressible elastic solids. It was established in [3] that a necessary condition for the existence of a piecewise homogeneous elastostatic shock was that the material lose strong ellipticity at some homogeneous deformation. The question of whether in fact a loss of ordinary ellipticity was necessary was left unanswered. In the particular case of weak elastostatic shocks it was shown that ordinary ellipticity must necessarily be lost at the preassigned deformation on one side of the shock.

Rice [5] had previously examined the phenomenon of "localization of deformation" for plastic materials. Localization is the bifurcation of an initially smooth state of deformation into one involving a zone of highly localized shearing. Localized deformations as described in [5] appear to have certain qualitative features in common with elastostatic shocks as described in [3]. In fact, within his setting, Rice shows that the onset of localization is first possible, in a program of deformation, when the displacement equations of equilibrium lose ellipticity.

In the present study we treat the corresponding issues for an arbitrary homogeneous incompressible elastic solid subjected to plane deformations. Some of the results established are appropriate only for isotropic materials. Explicit necessary and sufficient restrictions on the deformation invariants and the material are deduced which ensure ellipticity of the plane displacement equations of equilibrium. In the context of isotropic materials it is established that a loss of ordinary ellipticity at some homogeneous deformation is a necessary condition for the existence of a piecewise homogeneous elastostatic shock. It is further shown that a strict loss of ordinary ellipticity at a given homogeneous deformation is sufficient, but not necessary, to ensure the existence of a piecewise homogeneous elastostatic shock which bounds this preassigned deformation on one side.

In Section 2 we cite some relevant results from the theory of finite elastostatics for incompressible elastic solids which we then specialize to plane deformations. The notion of the "local amount of shear" associated with any plane volume preserving deformation is then described. In Section 3 the conventional notion of ellipticity is adapted to the displacement equations of equilibrium and necessary and sufficient conditions for ellipticity are then deduced. In the isotropic case these conditions are put into explicit form and a simple interpretation is given in terms of what we call the "local amount of shear". A loss of ellipticity is found to depend on a loss of invertibility of the shear stress-amount of shear relation in simple shear.

The notion of piecewise homogeneous elastostatic shocks developed in [3] for the compressible case is extended to the incompressible case in Section 4. In Section 5 we then consider <u>weak</u> elastostatic shocks in homogeneous, incompressible, <u>anisotropic</u> elastic solids and show that a loss of ellipticity at the pre-assigned deformation on one side of the shock is necessary for its existence. The jumps across the shock-line of various physically significant field quantities are also deduced.

In Section 6 we return to equilibrium shocks of <u>finite</u> strength in homogeneous, incompressible, <u>isotropic</u> elastic solids. We show that a strict failure of ordinary ellipticity at a given deformation is sufficient to ensure the existence of a piecewise homogeneous elastostatic shock which bounds this deformation on one side. Moreover we show that a failure of ordinary ellipticity at some homogeneous deformation is necessary for the existence of a shock of the type under consideration.

In Section 7 we discuss the dissipativity inequality first proposed by Knowles and Sternberg in [3] and later extended by Knowles [4] to three-dimensional deformations of both compressible and incompressible materials and explore some of its consequences. In particular its implications in the case of weak elastostatic shocks in anisotropic materials is examined.

Finally in Section 8 we illustrate some of the preceeding results by means of an example involving a particular hypothetical constitutive law.

CHAPTER 2

2.1 Preliminaries on Finite Plane Elastostatics

Let $\mathcal{R}$ be the three-dimensional open region occupied by the interior of a body in its undeformed configuration. A deformation of the body is described by a sufficiently smooth and invertible transformation

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \quad \text{on } \mathcal{R} \tag{2.1}$$

which maps $\mathcal{R}$ onto a domain $\mathcal{R}_*$. Here $\underset{\sim}{y}$ is the position vector after deformation of the particle which, in the undeformed configuration was located at $\underset{\sim}{x}$. We will assume for the moment that the displacement vector field $\underset{\sim}{u}(\underset{\sim}{x})$ is twice continuously differentiable on $\mathcal{R}$.

The deformation gradient tensor $\underset{\sim}{F}$ is defined by

$$\underset{\sim}{F} = \nabla \underset{\sim}{y} \quad \text{on } \mathcal{R} , \tag{2.2}$$

and since the material is presumed to be incompressible,

$$\det \underset{\sim}{F} = 1 \quad \text{on } \mathcal{R} , \tag{2.3}$$

where $\det \underset{\sim}{F}$ is the Jacobian of the mapping (2.1). The right and left Cauchy-Green tensors $\underset{\sim}{C}$ and $\underset{\sim}{G}$ are defined respectively by

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} , \quad \underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T . \tag{2.4}$$

Let $\underset{\sim}{\tau}$ be the Cauchy stress tensor field accompanying the deformation at hand. Assuming that $\underset{\sim}{\tau}$ is continuously differentiable on $\mathcal{R}_*$, the equilibrium equations are

$$\operatorname{div} \underset{\sim}{\tau} = \underset{\sim}{0} \ , \quad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T \quad \text{on } \mathcal{R}_* \ , \tag{2.5}$$

where body forces are presumed to be absent. The nominal (Piola) stress tensor corresponding to $\underset{\sim}{\tau}$ is given by

$$\underset{\sim}{\sigma} = \underset{\sim}{\tau}(\underset{\sim}{F}^T)^{-1} \ , \tag{2.6}$$

where use has been made of (2.3). Equations (2.2), (2.3), (2.5) and (2.6) lead to the equilibrium equations

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0} \ , \quad \underset{\sim}{\sigma}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{\sigma}^T \quad \text{on } \mathcal{R} \ . \tag{2.7}$$

We now turn to the constitutive relations and suppose that the body is homogeneous and elastic and possesses an elastic potential $W = \hat{W}(\underset{\sim}{F})$. W represents the strain energy density per unit undeformed volume. The nominal stresses are now given by

$$\underset{\sim}{\sigma} = \hat{W}_{\underset{\sim}{F}}(\underset{\sim}{F}) - p(\underset{\sim}{F}^T)^{-1} \ , \qquad (2.8)^1$$

where $p(\underset{\sim}{x})$ is a scalar field arising because of the incompressibility constraint. We assume for the moment that $p(\underset{\sim}{x})$ is continuously differentiable on $\mathcal{R}$. Alternatively, from (2.6), (2.8) we have

$$\underset{\sim}{\tau} = \hat{W}_{\underset{\sim}{F}}(\underset{\sim}{F})\underset{\sim}{F}^T - p\underset{\sim}{1} \ . \tag{2.9}$$

From (2.1)-(2.3), (2.7) and (2.8) we are led to

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[1] See Truesdell and Noll [6].

$$c_{ijk\ell}(\underset{\sim}{F})u_{k,\ell j} - p_{,j}F^{-1}_{ji} = 0 \quad \text{on } \Re \ , \tag{2.10}$$[1]

where

$$c_{ijk\ell}(\underset{\sim}{F}) = \frac{\partial^2 \hat{W}(\underset{\sim}{F})}{\partial F_{ij}\partial F_{k\ell}} \ . \tag{2.11}$$

Let $\lambda_1^2(\underset{\sim}{x})$, $\lambda_2^2(\underset{\sim}{x})$ and $\lambda_3^2(\underset{\sim}{x})$, where $\lambda_i > 0$, be the eigenvalues of the symmetric positive definite tensor field $\underset{\sim}{G}$ (or $\underset{\sim}{C}$). The principal scalar invariants of $\underset{\sim}{G}$ are

$$I_1 = \mathrm{tr}\,\underset{\sim}{G} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \ ,$$

$$I_2 = \frac{1}{2}[(\mathrm{tr}\,\underset{\sim}{G})^2 - (\mathrm{tr}\,\underset{\sim}{G}^2)] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \ , \tag{2.12}$$

$$I_3 = \det \underset{\sim}{G} = \lambda_1^2\lambda_2^2\lambda_3^2 \ ,$$

where tr denotes the trace. From (2.3), (2.4) and (2.12) it follows that

$$\lambda_1^2\lambda_2^2\lambda_3^2 = 1 \quad \text{on } \Re \ . \tag{2.13}$$

In the special case when the material is incompressible and isotropic, W depends on $\underset{\sim}{F}$ only through the invariants I_1 and I_2, whence we have

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[1] All tensor and vector components are taken with respect to a fixed rectangular cartesian frame. A comma followed by a subscript indicates differentiation with respect to the corresponding x-coordinate. Latin subscripts take the values 1, 2, 3 while greek subscripts take the values 1, 2. Repeated subscripts are summed over the appropriate range.

$$W = \overset{*}{W}(I_1, I_2) \quad . \tag{2.14}$$

Suppose now that the domain $\Re$ occupied by the undeformed body is a right cylinder with generators parallel to the x_3-axis. Let Π be the open region of the x_1-x_2 plane occupied by the interior of the middle cross-section of this cylinder. Suppose further that the deformation (2.1) is a plane deformation so that

$$\begin{aligned} y_\alpha &= x_\alpha + u_\alpha(x_1, x_2) \quad \text{on } \Pi \ , \\ y_3 &= x_3 \qquad\qquad \text{on } \Re \ . \end{aligned} \tag{2.15}$$

Π is then mapped onto a domain Π_* of the same plane, which would be the middle cross-section of the cylindrical region $\Re_*$. From here on we shall be exclusively concerned with plane deformations unless specifically stated otherwise. It follows from (2.2) and (2.15) that

$$\left.\begin{aligned} &F_{\alpha\beta} = y_{\alpha,\beta} \ , \\ &F_{\alpha 3} = F_{3\alpha} = 0 \ , \quad F_{33} = 1 \ . \end{aligned}\right\} \tag{2.16}$$

The nominal stresses are now given by

$$\begin{aligned} \sigma_{\alpha\beta} &= \frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}} - pF^{-1}_{\beta\alpha} \ , \\ \sigma_{33} &= \frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{33}} - p \ . \end{aligned} \tag{2.17}$$

If we assume that the elastic potential W is such that

$$\frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha 3}} = \frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{3\alpha}} = 0 \tag{2.18}$$

for every $\underset{\sim}{F}$ such that (2.16) holds, then we further have

$$\sigma_{3\alpha} = \sigma_{\alpha 3} = 0 \quad . \tag{2.19}$$

The assumption (2.18) holds true identically for isotropic materials in particular.

One sees readily from (2.7), (2.15)-(2.19) that for equilibrium in the x_3-direction it is necessary and sufficient that the scalar field $p(\underset{\sim}{x})$ be independent of x_3. Thus

$$p = p(x_1, x_2) \quad \text{on } \Pi \quad . \tag{2.20}$$

In the present circumstances (2.10) specializes to

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F})u_{\gamma,\beta\delta} - p_{,\beta}F^{-1}_{\beta\alpha} = 0 \quad \text{on } \Pi \quad . \tag{2.21}$$

Equation (2.21), together with the incompressibility condition (2.3), constitute the governing system of equations for the plane problem and we shall refer to them as the <u>displacement equations of equilibrium</u> in plane strain (despite the obvious inaccuracy of the title). They are three scalar equations involving the three functions $u_\alpha(x_1, x_2)$ and $p(x_1, x_2)$.

One sees readily from (2.4) and (2.15) that in any plane deformation, unity is an eigenvalue of the left and right Cauchy-Green tensors, whence we have

$$\lambda_3(\underset{\sim}{x}) = 1 \quad . \tag{2.22}$$

Equations (2.12) and (2.13) now specialize to

$$I_1 = \lambda_1^2 + \lambda_2^2 + 1 \ ,$$

$$I_2 = \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2 \ , \tag{2.23}$$

$$I_3 = \lambda_1^2\lambda_2^2 \ ,$$

and

$$\lambda_1^2\lambda_2^2 = 1 \ , \tag{2.24}$$

whence

$$I_1 = I_2 = \lambda_1^2 + \frac{1}{\lambda_1^2} + 1 \ . \tag{2.25}$$

If we now define I by

$$I = F_{\alpha\beta}F_{\alpha\beta} \ , \tag{2.26}$$

we have, because of (2.4), (2.12), (2.16) and (2.25) that

$$I = I_1 - 1 = I_2 - 1 = 2 + \left(\lambda_1 - \frac{1}{\lambda_1}\right)^2 \geq 2 \ . \tag{2.27}$$

In the special case when the material is isotropic, we have from (2.14) and (2.27) that, in plane deformations,

$$W = \overset{*}{W}(I+1\ ,\ I+1) \tag{2.29}$$

so that if we define the <u>Plane Strain Elastic Potential</u> $W(I)$ by

$$W(I) = \overset{*}{W}(I+1\ ,\ I+1)\ ,\qquad I \geq 2\ , \tag{2.30}$$

we have in the present context that $\hat{W}(\underset{\sim}{F}) = W(I)$ where $I = F_{\alpha\beta}F_{\alpha\beta}$. It follows from this that

$$\frac{\partial \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}} = 2F_{\alpha\beta}W'(I)\ , \tag{2.31}$$

and that

$$c_{\alpha\beta\gamma\delta}(\underset{\sim}{F}) = \frac{\partial^2 \hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}} = 2\delta_{\alpha\gamma}\delta_{\beta\delta}W'(I) + 4F_{\alpha\beta}F_{\gamma\delta}W''(I)\ . \tag{2.32}$$

From (2.4), (2.9) and (2.31) we conclude that

$$\tau_{\alpha\beta} = 2W'(I)G_{\alpha\beta} - p\delta_{\alpha\beta}\ . \tag{2.33}$$

It is apparent that the plane strain elastic potential $W(I)$ fully determines the in-plane stress components. This is not true, however, of the component τ_{33} .

Finally we recall that in this case the in-plane Baker-Ericksen inequality requires that

$$(\tau_1 - \tau_2)(\lambda_1 - \lambda_2) > 0 \quad \text{if } \lambda_1 \neq \lambda_2 \tag{2.34}$$[1]

for all pure homogeneous (plane) deformations of the form

$$\left.\begin{aligned} &y_\alpha = \lambda_\alpha x_\alpha \quad \text{(no sum)};\ \lambda_1\lambda_2 = 1\ ,\ \lambda_\alpha > 0\ ,\\ &y_3 = x_3\ , \end{aligned}\right. \tag{2.35}$$

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[1] See Truesdell and Noll [6].

where τ_α are the principal in-plane Cauchy stresses. From (2.4), (2.16), (2.26), (2.33) and (2.35) we have

$$\tau_\alpha = 2W'(I)\lambda_\alpha^2 - p \ , \quad I = \lambda_1^2 + \lambda_2^2 \ , \tag{2.36}$$

whence (2.34) may be equivalently written as

$$W'(\lambda_1^2+\lambda_2^2)(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2) > 0 \ , \quad \lambda_1, \lambda_2 > 0 \ , \ \lambda_1\lambda_2 = 1, \ \lambda_1 \neq \lambda_2 \tag{2.37}$$

or

$$W'(\lambda_1^2+\lambda_2^2) > 0 \ , \quad \lambda_1, \lambda_2 > 0 \ , \ \lambda_1\lambda_2 = 1 \ , \ \lambda_1 \neq \lambda_2 \ , \tag{2.38}$$

which in turn is equivalent to

$$W'(I) > 0 \quad \text{for} \quad I > 2 \ . \tag{2.39}$$

The infinitesimal shear modulus is easily shown to be $\overset{\circ}{\mu} = 2W'(2)$; if we assume that $\overset{\circ}{\mu} > 0$, we may replace (2.39) by

$$W'(I) > 0 \quad \text{for} \quad I \geq 2 \ . \tag{2.40}$$

Requiring that (2.40) hold for the material at hand is equivalent to requiring that the material have a positive (finite) shear modulus. Conversely, (2.40) implies (2.34), though it does not imply the full (three-dimensional) Baker-Ericksen inequalities.

2.2 Local Amount of Shear

We now establish that any plane volume preserving deformation can be decomposed locally into the product of a simple shear in a suitable direction followed or preceded by a suitable rotation.

To this end, let $\underset{\sim}{F}$ be a two-dimensional tensor such that $\det \underset{\sim}{F} = 1$. Define

$$k = \sqrt{I - 2} \quad , \quad I = F_{\alpha\beta} F_{\alpha\beta} \quad . \tag{2.41}$$[1]

Then we will show that there exist proper orthogonal tensors $\underset{\sim}{Q}_1$, $\underset{\sim}{Q}_2$, non-singular tensors $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$ with unit determinant (all two-dimensional) and rectangular cartesian frames X_1 , X_2 such that

$$\underset{\sim}{F} = \underset{\sim}{Q}_1 \underset{\sim}{K}_1 = \underset{\sim}{K}_2 \underset{\sim}{Q}_2 \tag{2.42}$$

and

$$\underset{\sim}{K}_1^{X_1} = \underset{\sim}{K}_2^{X_2} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad . \tag{2.43}$$[2]

Conversely, if (2.42) holds for some proper orthogonal tensors $\underset{\sim}{Q}_1$, $\underset{\sim}{Q}_2$ and tensors $\underset{\sim}{K}_1$, $\underset{\sim}{K}_2$ with unit determinant such that (2.43) is true in some rectangular cartesian frames X_1, X_2 , then we will show that k is necessarily given by $k = \pm\sqrt{I - 2}$.

In order to prove the first part of the result, let X be a principal frame for the symmetric positive definite tensor $\underset{\sim}{F}\underset{\sim}{F}^T$. Then

$$(\underset{\sim}{F}\underset{\sim}{F}^T)^X = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{bmatrix} \quad , \qquad \lambda > 0 \tag{2.44}$$

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[1]Since $\det \underset{\sim}{F} = 1$, we have that necessarily $I \geq 2$.

[2]$\underset{\sim}{K}_1^{X_1}$ is the matrix of components of the tensor $\underset{\sim}{K}_1$ in the frame $\underset{\sim}{X}_1$.

where we have made use of the fact that $\det \underset{\sim}{F} = 1$. Clearly we may assume $\lambda \geq 1$ with no loss of generality. Consider the rectangular cartesian coordinate frame X_2 obtained by rotating the frame X counterclockwise through an angle θ determined by

$$\sin\theta = -\frac{1}{\sqrt{1+\lambda^2}}, \quad \cos\theta = \frac{\lambda}{\sqrt{1+\lambda^2}}. \tag{2.45}$$

By the change of frame formula for tensors,

$$(\underset{\sim}{F}\underset{\sim}{F}^T)^{X_2} = \underset{\sim}{R}(\underset{\sim}{F}\underset{\sim}{F}^T)^{X}\underset{\sim}{R}^T,$$

where

$$\underset{\sim}{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

we compute $(\underset{\sim}{F}\underset{\sim}{F}^T)^{X_2}$ to find

$$(\underset{\sim}{F}\underset{\sim}{F}^T)^{X_2} = \begin{bmatrix} \lambda^2+\lambda^{-2}-1 & \lambda-\lambda^{-1} \\ \lambda-\lambda^{-1} & 1 \end{bmatrix}. \tag{2.46}$$

Let $\underset{\sim}{K}_2$ be the tensor with unit determinant defined by

$$\underset{\sim}{K}_2^{X_2} = \begin{bmatrix} 1 & \lambda-\lambda^{-1} \\ 0 & 1 \end{bmatrix}. \tag{2.47}$$

Then (2.46) and (2.47) imply that

$$\underset{\sim}{F}\underset{\sim}{F}^T = \underset{\sim}{K}_2\underset{\sim}{K}_2^T \quad . \tag{2.48}$$

Define the tensor $\underset{\sim}{Q}_2$ by

$$\underset{\sim}{Q}_2 = \underset{\sim}{K}_2^{-1}\underset{\sim}{F} \quad ; \tag{2.49}$$

(2.48) and (2.49) now lead to

$$\underset{\sim}{K}_2\underset{\sim}{Q}_2\underset{\sim}{Q}_2^T\underset{\sim}{K}_2^T = \underset{\sim}{K}_2\underset{\sim}{K}_2^T \quad .$$

Since $\underset{\sim}{K}_2$ is non-singular it thus follows that $\underset{\sim}{Q}_2\underset{\sim}{Q}_2^T = \underset{\sim}{1}$ whence $\underset{\sim}{Q}_2$ is orthogonal. But, from (2.49) it follows that $\det \underset{\sim}{Q}_2 = +1$ since $\det \underset{\sim}{K}_2 = \det \underset{\sim}{F} = 1$, so that in fact $\underset{\sim}{Q}_2$ is proper orthogonal.

Finally, since we are assuming $\lambda \geq 1$, it follows from (2.46) that $\lambda - \lambda^{-1} = \sqrt{F_{\alpha\beta}F_{\alpha\beta} - 2} = \sqrt{I - 2}$ whence from (2.41) $k = \lambda - \lambda^{-1}$. This establishes the left decomposition $\underset{\sim}{F} = \underset{\sim}{K}_2\underset{\sim}{Q}_2$. The right decomposition $\underset{\sim}{F} = \underset{\sim}{Q}_1\underset{\sim}{K}_1$ can be similarly established by considering $\underset{\sim}{F}^T\underset{\sim}{F}$ in place of $\underset{\sim}{F}\underset{\sim}{F}^T$.

The second part of the result is easily proved as follows. Suppose now that associated with the given tensor $\underset{\sim}{F}$ there exists some proper orthogonal tensor $\underset{\sim}{Q}_2$, some tensor $\underset{\sim}{K}_2$ with unit determinant and some rectangular cartesian frame X_2 such that

$$\underset{\sim}{F} = \underset{\sim}{K}_2\underset{\sim}{Q}_2 \quad , \tag{2.50}$$

$$\underset{\sim}{K}_2^{X_2} = \begin{bmatrix} 1 & k \\ \\ 0 & 1 \end{bmatrix} , \qquad (2.51)$$

for some real number k. Note that the tensors $\underset{\sim}{Q}_2$, $\underset{\sim}{K}_2$ and the frame X_2 are <u>not</u> required to be the particular ones used in the preceeding proof. Since $\underset{\sim}{Q}_2$ is orthogonal, it follows from (2.50) that

$$\underset{\sim}{F}\underset{\sim}{F}^T = \underset{\sim}{K}_2\underset{\sim}{K}_2^T , \qquad (2.52)$$

whence in particular, the traces of the two-dimensional tensors $\underset{\sim}{F}\underset{\sim}{F}^T$ and $\underset{\sim}{K}_2\underset{\sim}{K}_2^T$ are equal. By virtue of (2.51) we now have that necessarily $I = F_{\alpha\beta}F_{\alpha\beta} = 2 + k^2$, whence

$$k = \pm\sqrt{I - 2} .$$

The corresponding result for the decomposition $\underset{\sim}{F} = \underset{\sim}{Q}_1\underset{\sim}{K}_1$ may be similarly established.

Given any plane volume preserving deformation with deformation gradient $\underset{\sim}{F}(\underset{\sim}{x})$, we refer to $k(\underset{\sim}{x})$ defined by (2.41) as the associated <u>local amount of shear</u>. Therefore any <u>arbitrary</u> plane deformation of an incompressible material can be viewed <u>locally</u> as a simple shear in a suitable direction with local amount of shear $k(\underset{\sim}{x})$, followed or preceeded by a suitable rotation.

CHAPTER 3

3.1 Ellipticity of the Plane Displacement Equations of Equilibrium

We now introduce the relevant notion of ellipticity without restricting ourselves to isotropic materials.

Consider a cylindrical surface S with generators parallel to those of the undeformed body and lying wholly within $\mathcal{R}$. Let C be the curve along which S intersects Π. Assume that C has a continuous curvature, and let ξ be the arc length on C. Then C may be described by the non-singular parameterization

$$C: \; x_\alpha = \overset{\circ}{x}_\alpha(\xi) \; .$$

If ζ is a coordinate normal to C and $\underset{\sim}{N}(\xi)$ is a unit vector normal to C in the x_1-x_2 plane, then near a fixed point P on C we have the orthogonal curvilinear coordinate system (ξ, ζ), permitting us to write

$$x_\alpha = \overset{\circ}{x}_\alpha(\xi) + \zeta N_\alpha(\xi) \tag{3.1}$$

for any point (x_1, x_2) in a two-dimensional neighborhood of P. The mapping (3.1) is locally one to one, so that it has an inverse

$$\xi = f(x_1, x_2) \;\; , \;\; \zeta = g(x_1, x_2) \;\; , \tag{3.2}$$

and f and g are twice continuously differentiable in a neighborhood of P. Note that we may take

$$\underset{\sim}{N} = \frac{\nabla \underset{\sim}{g}}{|\nabla \underset{\sim}{g}|} \quad \text{on } C \ .$$

Now suppose that $(u_\alpha(x_1, x_2)\ ,\ p(x_1, x_2))$ is a solution of the plane displacement equations of equilibrium (2.3) and (2.21) such that u_α is once continuously differentiable and twice piecewise continuously differentiable on Π, while p is continuous and piecewise continuously differentiable on Π. We set

$$\hat{u}_\alpha(\xi, \zeta) = u_\alpha(\overset{\circ}{x}_1(\xi) + \zeta N_1(\xi)\ ,\ \overset{\circ}{x}_2(\xi) + \zeta N_2(\xi))\ ,$$

$$\hat{p}(\xi, \zeta) = p(\overset{\circ}{x}_1(\xi) + \zeta N_1(\xi)\ ,\ \overset{\circ}{x}_2(\xi) + \zeta N_2(\xi))\ ,$$

and further suppose that, in fact, the second order partial derivatives of $\hat{u}_\alpha$ are all continuous across C except possibly for the normal derivative $\frac{\partial^2 \hat{u}_\alpha}{\partial \zeta^2}$, and that the first order partial derivative $\frac{\partial \hat{p}}{\partial \xi}$ is continuous across C, while the normal derivative $\frac{\partial \hat{p}}{\partial \zeta}$ may suffer a jump discontinuity.

Let

$$U_\alpha = \left[\frac{\partial^2 \hat{u}_\alpha}{\partial \zeta^2}\right]\ , \quad q = \left[\frac{\partial \hat{p}}{\partial \zeta}\right] \tag{3.3}$$

where $[h]$ denotes the jump of a function h across C. Then one shows easily that

$$[u_{\alpha,\beta\gamma}] = U_\alpha \bar{N}_\beta \bar{N}_\gamma\ , \qquad (3.4)^1$$

where $\bar{\underset{\sim}{N}} = \nabla \underset{\sim}{g} = (\bar{\underset{\sim}{N}} \cdot \bar{\underset{\sim}{N}})^{1/2} \underset{\sim}{N}$. We have by the chain rule and (3.2) that

[1] See Section 1 of [1].

$$p_{,\alpha} = \frac{\partial \hat{p}}{\partial \xi}\frac{\partial f}{\partial x_\alpha} + \frac{\partial \hat{p}}{\partial \zeta}\frac{\partial g}{\partial x_\alpha} \ ,$$

which because of the presumed smoothness and (3.3) leads to

$$[p_{,\alpha}] = q\bar{N}_\alpha \ . \tag{3.5}$$

Taking jumps in the first two displacement equations of equilibrium (2.21), and making use of (3.4), (3.5) and the assumed smoothness we get

$$c_{\alpha\beta\gamma\delta}U_\gamma \bar{N}_\delta \bar{N}_\beta - F^{-1}_{\beta\alpha}\bar{N}_\beta q = 0 \quad \text{on } C \ . \tag{3.6}$$

If for all vectors $\underset{\sim}{\bar{N}}$ and nonsingular tensors $\underset{\sim}{F}$ with unit determinant, we define the matrix $Q_{\alpha\beta}(\underset{\sim}{\bar{N}}, \underset{\sim}{F})$ through

$$Q_{\alpha\gamma}(\underset{\sim}{\bar{N}}, \underset{\sim}{F}) = c_{\alpha\beta\gamma\delta}(\underset{\sim}{F})\bar{N}_\beta \bar{N}_\delta \ , \tag{3.7}$$

then $Q_{\alpha\beta}$ is symmetric by virtue of (2.11). Equation (3.6) can now be written in the form

$$Q_{\alpha\beta}U_\beta = qF^{-1}_{\beta\alpha}\bar{N}_\beta \quad \text{on } C \ . \tag{3.8}$$

We also need the "jump equation" associated with the remaining displacement equation of equilibrium (2.3). We compute $\frac{\partial}{\partial \zeta}(\det \underset{\sim}{F})$ to find

$$\frac{\partial}{\partial \zeta}(\det \underset{\sim}{F}) = (\det \underset{\sim}{F})F^{-1}_{\beta\alpha}\frac{\partial}{\partial \zeta}\left\{\frac{\partial \hat{u}_\alpha}{\partial \xi}\frac{\partial f}{\partial x_\beta} + \frac{\partial \hat{u}_\alpha}{\partial \zeta}\frac{\partial g}{\partial x_\beta}\right\} \ , \tag{3.9}$$

where use has been made of (3.2), the chain rule and a standard formula for the differentiation of a determinant. Taking jumps in (3.9) and making use of (2.3), (3.3) and the presumed smoothness leads to

$$\left[\frac{\partial}{\partial \zeta}(\det \underset{\sim}{F})\right] = F^{-1}_{\beta\alpha}\bar{N}_\beta U_\alpha \quad . \tag{3.10}$$

But by (2.3) the jump in $\det \underset{\sim}{F}$ must vanish, whence (3.10) simplifies to

$$F^{-1}_{\beta\alpha}\bar{N}_\beta U_\alpha = 0 \quad \text{on } C \ . \tag{3.11}$$

The system of jump equations associated with the displacement equations of equilibrium are (3.8) and (3.11), and may be regarded as three linear homogeneous algebraic equations for the jumps U_α and q.

We say that the system of plane displacement equations of equilibrium is <u>elliptic</u> at the solution (u_α, p) and at the point (x_1, x_2) if and only if, for all vectors $\underset{\sim}{\bar{N}} \neq \underset{\sim}{0}$, the system (3.8), (3.11) has only the trivial solution $U_\alpha = 0$, $q = 0$.

Consequently if the system is elliptic, the displacement field u_α will in fact be twice continuously differentiable at the point under consideration and the pressure p will be continuously differentiable there. If on the other hand there exists a non-trivial solution of (3.8), (3.11) for some vector $\underset{\sim}{\bar{N}}$, then $\underset{\sim}{\bar{N}}$ is normal to a characteristic curve in the undeformed configuration. These characteristic curves are the only possible carriers of discontinuities of the kind admitted here in u_α and p, and ellipticity precludes the existence of real characteristics.

If we set

$$m_\alpha = F^{-1}_{\beta\alpha}\bar{N}_\beta \ , \tag{3.12}$$

we can write the system of jump equations (3.8) and (3.11) as

$$\begin{bmatrix} Q_{11} & Q_{12} & -m_1 \\ Q_{21} & Q_{22} & -m_2 \\ m_1 & m_2 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ q \end{bmatrix} = \underset{\sim}{0} \ .$$

This system of linear homogeneous algebraic equations for U_α and q has only the trivial solution if and only if

$$\det \begin{bmatrix} Q_{11} & Q_{12} & -m_1 \\ Q_{21} & Q_{22} & -m_2 \\ m_1 & m_2 & 0 \end{bmatrix} \neq 0 \ , \tag{3.13}$$

or equivalently

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}Q_{\alpha\beta}m_\lambda m_\mu \neq 0 \ . \qquad (3.14)^1$$

Since $\underset{\sim}{F}$ has unit determinant, one shows easily that in plane strain

$$F^{-1}_{\beta\alpha} = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}F_{\gamma\delta} \ . \tag{3.15}$$

By virtue of (3.12) and (3.15) we may write (3.14) equivalently as

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}F_{\gamma\lambda}F_{\delta\mu}Q_{\gamma\delta}\bar{N}_\alpha\bar{N}_\beta \neq 0 \ . \tag{3.16}$$

Therefore, we have that <u>a necessary and sufficient condition for</u>

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[1] $\epsilon_{\alpha\beta}$ is the two-dimensional alternator. $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = 1$, $\epsilon_{21} = -1$.

the displacement equations of equilibrium to be elliptic at a solution (u_α, p) and at some point (x_1, x_2) is

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}F_{\gamma\lambda}F_{\delta\mu}Q_{\gamma\delta}(\bar{\underset{\sim}{N}}, \underset{\sim}{F})\bar{N}_\alpha\bar{N}_\beta \neq 0 \ , \tag{3.17}$$

for every vector $\bar{\underset{\sim}{N}} \neq \underset{\sim}{0}$. Finally, because of (3.7) it is clear that (3.17) is equivalent to

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}F_{\gamma\lambda}F_{\delta\mu}Q_{\gamma\delta}(\underset{\sim}{N}, \underset{\sim}{F})N_\alpha N_\beta \neq 0 \qquad \text{for all unit vectors } \underset{\sim}{N} \ . \tag{3.18}$$

3.2 Specialization to Isotropic Materials

When the material at hand is isotropic, we can use (2.4), (2.11), (2.32) and (3.7) to simplify the necessary and sufficient condition for ellipticity (3.18), which then gives

$$(\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}C_{\alpha\beta}N_\lambda N_\mu)W'(I) + 2(\epsilon_{\alpha\lambda}C_{\alpha\rho}N_\rho N_\lambda)^2 W''(I) \neq 0 \ , \tag{3.19}$$

for every unit vector $\underset{\sim}{N}$. Now let the frame be principal for $\underset{\sim}{C}$, so that

$$[C_{\alpha\beta}] = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} ,$$

and evaluate (3.19) in this frame. We then find

$$(\lambda_1^2 N_2^2 + \lambda_2^2 N_1^2)W'(I) + 2(\lambda_1^2 - \lambda_2^2)^2 N_1^2 N_2^2 W''(I) \neq 0 \tag{3.20}$$

for all unit vectors $\underset{\sim}{N}$, as being necessary and sufficient for ellipticity.

We will now show that the plane displacement equations of equilibrium are elliptic at a solution (u_α, p) and at a point (x_1, x_2) if and only if

$$\left.\begin{aligned} &W'(I) \neq 0 \\ \text{and}\qquad & \\ &\frac{2W''(I)}{W'(I)}(I-2)+1>0 \end{aligned}\right\} \tag{3.21}$$

at the point under consideration; i.e. that (3.21) is equivalent to (3.20).

To show this, we observe that since $\underset{\sim}{N}$ is a unit vector,

$$\lambda_1^2 N_2^2 + \lambda_2^2 N_1^2 = (\lambda_1^2 N_2^2 + \lambda_2^2 N_1^2)(N_1^2 + N_2^2) = \lambda_1^2 N_2^4 + \lambda_2^2 N_1^4 + (\lambda_1^2 + \lambda_2^2) N_1^2 N_2^2 \ , \tag{3.22}$$

so that (3.20) may be written as

$$\{\lambda_2^2 W'(I)\} N_1^4 + \{\lambda_1^2 W'(I)\} N_2^4 + \{(\lambda_1^2 + \lambda_2^2) W'(I) + 2(\lambda_1^2 - \lambda_2^2)^2 W''(I)\} N_1^2 N_2^2 \neq 0 \tag{3.23}$$

for all unit vectors $\underset{\sim}{N}$. If we set

$$\left.\begin{aligned} &E_{11} = \lambda_2^2 W'(I) \ , \quad E_{22} = \lambda_1^2 W'(I) \ , \\ &E_{21} = E_{12} = \frac{(\lambda_1^2 + \lambda_2^2)}{2} W'(I) + (\lambda_1^2 - \lambda_2^2)^2 W''(I) \ , \\ &z_\alpha = N_\alpha^2 \ , \end{aligned}\right\} \tag{3.24}$$

we can replace (3.23) by

$$E_{\alpha\beta} z_\alpha z_\beta \neq 0 \qquad \text{for all } \underset{\sim}{z} \neq \underset{\sim}{0} \ , \ z_\alpha \geq 0 \ . \tag{3.25}$$

It has been shown in Section 2 of reference [2] that (3.25) holds if and only if

$$E_{11}E_{22} > 0 \tag{3.26}$$

and

$$\frac{\epsilon\, E_{12}}{\sqrt{E_{11}E_{22}}} > -1 \quad , \tag{3.27}$$

where

$$\epsilon = \operatorname{sgn}\ E_{11} = \operatorname{sgn}\ E_{22} \quad . \tag{3.28}$$

Substituting from (3.24) into (3.26) we get $\lambda_1^2\lambda_2^2\{W'(I)\}^2 > 0$ which, because $\lambda_\alpha > 0$, is equivalent to

$$W'(I) \neq 0 \quad . \tag{3.29}$$

Using (3.24) and (3.28) in (3.27) leads to

$$2(\lambda_1 - \lambda_2)^2 \frac{W''(I)}{W'(I)} + 1 > 0 \quad ,$$

which because of (2.24) and (2.27) may in turn be written as

$$2(I - 2) \frac{W''(I)}{W'(I)} + 1 > 0 \quad . \tag{3.30}$$

Equations (3.29) and (3.30) are what we set out to establish.

A physical interpretation of the ellipticity condition (3.21) may be obtained in terms of the concept of the local amount of shear introduced in Section 2.2. Consider an isotropic, incompressible, homogeneous, elastic solid which has a positive shear modulus:[1]

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[1]See (2.40). A similar interpretation can clearly be given in terms of the local amount of shear even in the unrealistic case when (2.40) does not hold.

$$W'(I) > 0 \quad \text{for} \quad I \geq 2 \ . \tag{3.31}$$

The first of (3.21) is now trivially satisfied. If we define the function $\tau(\cdot)$ by

$$\tau(k) = 2kW'(2+k^2) \ , \quad |k| < \infty \ , \tag{3.32}$$

then $\tau(k)$ is easily shown to be the shear stress corresponding to an amount of shear k in a simple shear deformation. The graph of $\tau(k)$ vs. k described by (3.32) will be called the <u>response curve in simple shear</u>. Differentiating (3.32) with respect to k and observing that (3.31) holds leads to

$$\tau'(k) = 2W'(2+k^2) \left\{ 2k^2 \frac{W''(2+k^2)}{W'(2+k^2)} + 1 \right\} \ .$$

We therefore find that (3.21) is equivalent to

$$\tau'(k) > 0 \quad \text{for} \quad k = \sqrt{I-2} \ , \tag{3.33}$$

from which we conclude that <u>for an isotropic, incompressible elastic solid having a positive shear modulus, the plane displacement equations of equilibrium are elliptic at a solution (u_α, p) and a point (x_1, x_2) if and only if the slope of the response curve in simple shear at an amount of shear equal to the local amount of shear is positive</u>.

Suppose for example that the response of a particular homogeneous, isotropic, incompressible elastic solid in simple shear is as described by Fig. 2. Then in <u>any</u> plane deformation the displacement equations of equilibrium are elliptic at some point (x_1, x_2) and some solution if and only if the local amount of shear at that point $k(x_1, x_2)$, defined by (2.41), is such that $-k_0 < k(x_1, x_2) < k_0$.

It is apparent from the above discussion that a loss of ellipticity for materials of the type being considered is dependent upon a loss of invertibility of the shear stress — amount of shear relation in simple shear.

Finally, we note from (3.21) that the undeformed state is elliptic if and only if the infinitesimal shear modulus $\overset{\circ}{\mu} = 2W'(2) \neq 0$. This is precisely the condition for ellipticity of the linearized displacement equations of equilibrium for a homogeneous, isotropic, incompressible, elastic material.

3.3 Characteristic Curves

If the ellipticity condition (3.21) is violated, it follows that there exists a unit vector $\underset{\sim}{N}$ such that equality holds in (3.20). $\underset{\sim}{N}$ will then be normal to a (material) characteristic, and we now determine the number of possible characteristics and their inclinations. To this end, let

$$N_1 = -\sin\theta \quad , \quad N_2 = \cos\theta \quad , \tag{3.34}$$

so that θ is the local inclination of the material characteristic to the λ_1-principal axis of $\underset{\sim}{C}$. Substituting this in (3.20), with equality holding now, we find

$$(\lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta) W'(I) + 2(\lambda_1^2 - \lambda_2^2)^2 \sin^2\theta \cos^2\theta W''(I) = 0 \quad . \tag{3.35}$$

We seek solutions θ of this equation in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let us assume that the infinitesimal shear modulus of the material is positive:

$$\mathring{\mu} = 2W'(2) > 0 \quad . \tag{3.36}$$

We see immediately from (3.21) that, if the point under consideration is locally undeformed $(I = 2)$ in the given deformation, then the displacement equations of equilibrium are elliptic there. Consequently we need only consider $I > 2$ in our search for characteristics.

Suppose first that ellipticity is lost by virtue of the fact that the first of the ellipticity conditions (3.21) is violated.[1] Then

$$W'(I) = 0 \tag{3.37}$$

at the point (x_1, x_2) of interest at the given deformation. We then find from (3.35) that either $W''(I) = 0$ or $\theta = 0, \frac{\pi}{2}$. Using (2.41) and (3.32), we may state this result as follows. Let k be the local amount of shear. Then if $\tau(k) = 0$, the displacement equations of equilibrium are not elliptic for the given deformation at the point under consideration. Furthermore, we then have two (material) characteristics inclined at angles 0 and $\frac{\pi}{2}$ to a principal axis of $\underset{\sim}{C}$, except in the particular case when $\tau'(k) = 0$ as well, in which case any number of arbitrarily inclined characteristics may exist locally.

Now suppose that $W'(I) \neq 0$ at the point of interest and that ellipticity is lost by virtue of the fact that the second of (3.21) has been violated. Then

$$\frac{2W''(I)}{W'(I)}(I - 2) + 1 \leq 0 \quad . \tag{3.38}$$

Equation (3.35) can now be rearranged into the form of a quadratic

[1]Note from (2.40) that this possibility does not exist if the material has a positive shear modulus.

equation for $\cos 2\theta$.

$$\frac{(\lambda_1^2-\lambda_2^2)^2}{2}\frac{W''(I)}{W'(I)}\cos^2 2\theta - \frac{(\lambda_1^2-\lambda_2^2)}{2}\cos 2\theta$$

$$-\left\{\frac{\lambda_1^2+\lambda_2^2}{2}+\frac{(\lambda_1^2-\lambda_2^2)^2}{2}\frac{W''(I)}{W'(I)}\right\}=0 \quad . \qquad (3.39)$$

Formally we can write the solution of this after making use of (2.24) and (2.27) as

$$\cos 2\theta = \frac{1\pm(\{2(I-2)W''(I)/W'(I)+1\}\ \{2(I+2)W''(I)/W'(I)+1\})^{\frac{1}{2}}}{2(I^2-4)^{\frac{1}{2}}W''(I)/W'(I)} \qquad (3.40)$$

where with no loss of generality we have assumed that $\lambda_1 > \lambda_2$.

If (3.38) holds with equality (so that $\tau(k) \neq 0$, $\tau'(k) = 0$ at the local amount of shear k) we find two values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ from (3.40), whence two characteristics exist. Equation (3.40) now simplifies to

$$\cos 2\theta = -\sqrt{\frac{I-2}{I+2}} \ , \qquad (3.41)$$

which because of (2.24) and (2.27) (and since $\lambda_1 > \lambda_2$) leads to

$$\cos 2\theta = \frac{1-\lambda_1^2}{1+\lambda_1^2} \ , \qquad (3.42)$$

whence

$$\tan\theta = \pm\lambda_1 \quad . \qquad (3.43)$$

Suppose the spatial characteristic corresponding to this material characteristic is inclined at an angle α to the λ_1-principal axis of $\underset{\sim}{G}$. It can be shown that

$$\tan\alpha = \frac{\lambda_2}{\lambda_1}\tan\theta \quad , \tag{3.44}$$

so that (2.24), (3.43) and (3.44) give

$$\tan\alpha = \pm\frac{1}{\lambda_1} \quad . \tag{3.45}$$

Because of (2.33), α is also the inclination to the corresponding principal axis of the Cauchy stress tensor.

If however, strict inequality holds in (3.38) (so that $\tau(k) \neq 0$, $k\tau(k)\tau'(k) < 0$ at the local amount of shear k) (3.40) gives us four values of θ which in turn implies the existence of two pairs of characteristics. Clearly, each pair is positioned symmetrically with respect to the principal axes of $\underset{\sim}{C}$. In what follows we will have need for the inclinations α of the corresponding spatial characteristics to the λ_1-principal axis of $\underset{\sim}{G}$ $(\lambda_1 > \lambda_2)$. From (2.24), (2.27), (3.40) and (3.44) we have

$$\cos 2\alpha = \frac{-1 \pm (\{2(I-2)W''(I)/W'(I)+1\}\ \{2(I+2)W''(I)/W'(I)+1\})^{\frac{1}{2}}}{2(I^2-4)^{\frac{1}{2}}W''(I)/W'(I)} \quad . \tag{3.46}$$

CHAPTER 4

4.1 Weak Formulation of Problem

In the derivation of the classical field equations of elasticity the displacement field $\underset{\sim}{u}$ and stress field $\underset{\sim}{\sigma}$ are assumed to satisfy certain smoothness requirements. There are, however, some physical problems in which these conditions are not met, so that in order to study them one would be forced to relax the smoothness demanded of the field quantities. It may, for example, be necessary to require only that the displacement field $\underset{\sim}{u}(\underset{\sim}{x})$ be continuous and piecewise continuously differentiable on $\mathfrak{R}$, while the nominal stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ and the pressure field $p(\underset{\sim}{x})$ are to be piecewise continuous[1] on $\mathfrak{R}$. Clearly, the global balance laws continue to be meaningful even under these smoothness conditions, but one must re-examine the validity of the local field equations.

Of particular physical interest is the case wherein the field quantities possess the classical degree of smoothness[2] everywhere except on one or more regular surfaces within the body. This would, for example, describe an idealized model for shear bands. To formulate this problem, we suppose that there is a surface S in $\mathfrak{R}$ such that $\underset{\sim}{\sigma}$, $\underset{\sim}{F}$ and p are continuously differentiable everywhere in $\mathfrak{R}$ except on S, and such that $\underset{\sim}{\sigma}$, $\underset{\sim}{F}$ and p suffer finite jump discontinuities across it. The displacement $\underset{\sim}{u}(\underset{\sim}{x})$ is presumed to be continuous

[1] We return momentarily to the three-dimensional case in this section.

[2] See Section 2.1.

everywhere in $\mathcal{R}$. The possibility of the breakdown of ellipticity of the governing equations suggests that solutions of this type to the equations of finite elastostatics may emerge in some circumstances.

Going through the usual arguments,[1] one finds from the global equilibrium of forces that

$$\operatorname{div} \underset{\sim}{\sigma} = \underset{\sim}{0} \quad \text{on } \mathcal{R} - S \tag{4.1}$$

and

$$[\underset{\sim}{\sigma}]_{-}^{+} \underset{\sim}{N} = \underset{\sim}{0} \quad \text{on } S \ , \tag{4.2}$$

while from the global equilibrium of moments we have

$$\underset{\sim}{\sigma} \underset{\sim}{F}^{T} = \underset{\sim}{F} \underset{\sim}{\sigma}^{T} \quad \text{on } \mathcal{R} - S \tag{4.3}$$

and

$$\underset{\sim}{y}(\underset{\sim}{x}) \times [\underset{\sim}{\sigma}]_{-}^{+} \underset{\sim}{N} = \underset{\sim}{0} \quad \text{on } S \ . \tag{4.4}$$

Equation (4.2) says that the nominal tractions are continuous across S . Here $[\underset{\sim}{\sigma}]_{-}^{+} = \overset{+}{\underset{\sim}{\sigma}} - \overset{-}{\underset{\sim}{\sigma}}$ where $\overset{+}{\underset{\sim}{\sigma}}$ and $\overset{-}{\underset{\sim}{\sigma}}$ are the limiting values of $\underset{\sim}{\sigma}$ (presumed to exist) as a point on S is approached from each side, and $\underset{\sim}{N}$ is is a unit normal to S . Equations (4.2) and (4.4) are referred to as <u>jump conditions</u>. Note that (4.4) is trivially satisfied once (4.2) is.

The global version of mass balance likewise leads to

$$\det \underset{\sim}{F} = 1 \quad \text{on } \mathcal{R} - S \ , \tag{4.5}$$

while the associated jump condition is trivially satisfied in equilibrium

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[1] See Chadwick [7].

problems.

Such a surface S carrying jump discontinuities in $\underset{\sim}{F}$, $\underset{\sim}{\sigma}$ and p which conform with the jump condition (4.2), while maintaining continuous displacements across it is called an "equilibrium shock", or an "elastostatic shock" in the particular case when the body is composed of an elastic material.

4.2 Piecewise Homogeneous Elastostatic Shocks

To investigate many of the local issues related to elastostatic shocks, it is sufficient to consider the case in which S is a plane and the deformation gradient $\underset{\sim}{F}$ is constant on either side of S. From here on we shall be concerned with such a situation within the context of plane deformations[1] of an incompressible elastic solid, so that we may take S to be a plane parallel to the generators of the body.

The corresponding problem for a compressible elastic solid was investigated by Knowles and Sternberg [3]. In this section, we formulate the problem governing the existence of an elastostatic shock in the incompressible case in a manner entirely analogous to [3].

Suppose that the middle cross-section of the body we are dealing with occupies the entire x_1-x_2 plane Π in its undeformed configuration. Let X be a fixed rectangular cartesian coordinate frame and let $\mathscr{L}$ be the straight line through the origin of X with unit direction vector $\underset{\sim}{L}$. Thus

$$\mathscr{L}: \; x_\alpha = L_\alpha \xi \;, \quad -\infty < \xi < \infty \;. \tag{4.6}$$

[1] We leave the three-dimensional introduction to elastostatic shocks of the last section and return to plane deformations from here on.

Let $\underset{\sim}{N}$ be the unit vector normal to $\mathscr{L}$ obtained by a counterclockwise rotation of $\underset{\sim}{L}$. Let $\overset{+}{\Pi}$ and $\overset{-}{\Pi}$ be the two open half planes into which $\mathscr{L}$ divides Π, with $\overset{+}{\Pi}$ being the one into which $\underset{\sim}{N}$ points. (See Fig. 1.)

Now consider the piecewise homogeneous plane deformation

$$y_\alpha = \begin{cases} \overset{+}{F}_{\alpha\beta} x_\beta & \text{on } \overset{+}{\Pi}, \\ \overset{-}{F}_{\alpha\beta} x_\beta & \text{on } \overset{-}{\Pi}, \end{cases} \tag{4.7}$$

where $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$ are constant tensors such that

$$\det \overset{+}{\underset{\sim}{F}} = \det \overset{-}{\underset{\sim}{F}} = 1 . \tag{4.8}$$

The nominal stresses associated with the deformation (4.7) are

$$\left.\begin{aligned} \overset{+}{\sigma}_{\alpha\beta} &= \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \overset{+}{p}\overset{+}{F}{}^{-1}_{\beta\alpha} \quad \text{on } \overset{+}{\Pi}, \\ \overset{-}{\sigma}_{\alpha\beta} &= \frac{\partial \hat{W}(\overset{-}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \overset{-}{p}\overset{-}{F}{}^{-1}_{\beta\alpha} \quad \text{on } \overset{-}{\Pi}. \end{aligned}\right\} \tag{4.9}$$

Clearly, the equilibrium equations (4.1) are satisfied if and only if $\overset{+}{p}$ and $\overset{-}{p}$ are constants.

If we are to view the line $\mathscr{L}$ as the intersection of an equilibrium shock S with the cross-section Π, then according to Section 4.1 we need to impose displacement and traction continuity requirements across $\mathscr{L}$. Because of (4.7) the requirement of a continuous displacement field is equivalent to

$$\overset{+}{F}_{\alpha\beta} x_\beta = \overset{-}{F}_{\alpha\beta} x_\beta \quad \text{on } \mathscr{L}, \tag{4.10}$$

which in view of (4.6) reduces to

$$\overset{+}{F}_{\alpha\beta}L_\beta = \overset{-}{F}_{\alpha\beta}L_\beta \quad . \tag{4.11}$$

By (4.9), we have traction continuity (4.2) if and only if

$$\left\{\frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \overset{+}{p}\overset{+}{F}{}^{-1}_{\beta\alpha}\right\} N_\beta = \left\{\frac{\partial \hat{W}(\overset{-}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \overset{-}{p}\overset{-}{F}{}^{-1}_{\beta\alpha}\right\} N_\beta \quad . \tag{4.12}$$

If the deformation field (4.7), subject to (4.8), together with real constants $\overset{+}{p}$ and $\overset{-}{p}$ conform with (4.11) and (4.12), and if $\overset{+}{\underset{\sim}{F}} \neq \overset{-}{\underset{\sim}{F}}$,[1] then we refer to the corresponding elastostatic field as a piecewise homogeneous elastostatic shock. The line $\mathscr{L}$ will be referred to as the material shock-line. Figure 1(b) displays the images of the three rectangles shown in Fig. 1(a) under a typical mapping (4.7) in the presence of a piecewise homogeneous elastostatic shock.

In order to examine questions related to the existence of piecewise homogeneous elastostatic shocks we pose the following problem. Given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with $\det \overset{+}{\underset{\sim}{F}} = 1$ and a real constant $\overset{+}{p}$, determine a constant tensor $\overset{-}{\underset{\sim}{F}}$ with $\det \overset{-}{\underset{\sim}{F}} = 1$ $(\overset{-}{\underset{\sim}{F}} \neq \overset{+}{\underset{\sim}{F}})$ and a real constant $\overset{-}{p}$ such that (4.11) and (4.12) hold.

Equation (4.11) may be solved as follows. Let $\mathscr{L}_*$, which we shall refer to as the spatial shock-line, be the image of $\mathscr{L}$ under the mapping (4.7). Let $\overset{+}{\Pi}_*$ and $\overset{-}{\Pi}_*$ be the two half planes into which $\overset{+}{\Pi}$ and $\overset{-}{\Pi}$ map by virtue of (4.7). Suppose $\underset{\sim}{\ell}$ is the unit direction vector of $\mathscr{L}_*$ such that the unit normal $\underset{\sim}{n}$ to $\mathscr{L}_*$ obtained by rotating $\underset{\sim}{\ell}$ counterclockwise points into $\overset{+}{\Pi}_*$. (See Fig. 1.) Without any loss of

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[1] Note from (4.12) that if $\overset{+}{p} \neq \overset{-}{p}$ then necessarily $\overset{+}{\underset{\sim}{F}} \neq \overset{-}{\underset{\sim}{F}}$.

generality the inclinations Φ and ϕ of the shock-lines $\mathcal{L}$ and $\mathcal{L}_*$ relative to the x_1-axis may be confined to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

One can show readily that, given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with $\det \overset{+}{\underset{\sim}{F}} = 1$, (4.11) will hold for a tensor $\overset{-}{\underset{\sim}{F}}$ with unit determinant if and only if

$$\overset{-}{F}_{\alpha\beta} = (\delta_{\alpha\gamma} + \varkappa \ell_\alpha n_\gamma)\overset{+}{F}_{\gamma\beta} , \tag{4.13}$$

for some real number $\varkappa$. We omit the derivation of this result as it parallels exactly the corresponding derivation in the compressible case contained in [3]. Let X' be the rectangular cartesian frame obtained by rotating the frame X counterclockwise through an angle ϕ. The base vectors associated with X' are then $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$. Expressing (4.13) in the frame X' we have

$$\begin{bmatrix} \overset{-}{F}{}^{X'}_{11} & \overset{-}{F}{}^{X'}_{12} \\ \overset{-}{F}{}^{X'}_{21} & \overset{-}{F}{}^{X'}_{22} \end{bmatrix} = \begin{bmatrix} 1 & \varkappa \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overset{+}{F}{}^{X'}_{11} & \overset{+}{F}{}^{X'}_{12} \\ \overset{+}{F}{}^{X'}_{21} & \overset{+}{F}{}^{X'}_{22} \end{bmatrix} . \tag{4.14}$$

Accordingly, the deformation on $\overset{-}{\Pi}$ may be viewed as being equivalent to the deformation on $\overset{+}{\Pi}$ followed by a simple shear parallel to $\mathcal{L}_*$ with an amount of shear $\varkappa$.

We may now pose the following problem which is equivalent to the one posed earlier. Given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with unit determinant and a real constant $\overset{+}{p}$, determine real numbers $\overset{-}{p}$, $\varkappa (\neq 0)$ and $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that (4.12) holds with $\overset{-}{\underset{\sim}{F}}$ defined by (4.13). Here we have omitted $\varkappa = 0$ since, by (4.13) we see that this corresponds to the shockless state $\overset{-}{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}}$.

Finally, we note that since the traction continuity condition (4.12) imposes only two scalar restrictions on the three parameters ϕ , $\varkappa$ and $\bar{p}$, one would anticipate that if there exists an elastostatic shock corresponding to a given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$, then in fact there exists a one parameter family of shocks.

CHAPTER 5

5.1 Weak Piecewise Homogeneous Elastostatic Shocks

We now specialize the problem posed in the general setting of Section 4.2 to the first of two simpler cases. Here we confine attention to elastostatic shocks that are weak in the sense that the departure of $\bar{\underset{\sim}{F}}$ from $\overset{+}{\underset{\sim}{F}}$ is small. Motivated by the remarks at the end of the previous section, we assume here that there exists a one parameter family of shocks, corresponding to the given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$, depending on the parameter $\varkappa$ and sufficiently smooth near $\varkappa = 0$. Specifically, we suppose that there are functions $\phi(\varkappa)$, $\bar{p}(\varkappa)$ both twice continuously differentiable in a neighborhood of $\varkappa = 0$, such that $\bar{\underset{\sim}{F}}$ defined by (4.13) together with $\bar{p}(\varkappa)$ conforms with the traction continuity requirement (4.12). Since from (4.13) we have that $\bar{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}}$ when $\varkappa = 0$, we may use $\varkappa$ as a measure of the departure of $\bar{\underset{\sim}{F}}$ from $\overset{+}{\underset{\sim}{F}}$. Accordingly $\varkappa$ will be referred to as the <u>shock strength parameter</u>.

We first record the following kinematic results which are established in [3]. Let

$$c = |\overset{+}{\underset{\sim}{F}}\underset{\sim}{L}| = |\bar{\underset{\sim}{F}}\underset{\sim}{L}| \quad . \tag{5.1}$$

Then

$$\ell_\alpha = \frac{1}{c}\overset{+}{F}_{\alpha\beta}L_\beta = \frac{1}{c}\bar{F}_{\alpha\beta}L_\beta \quad , \tag{5.2}$$

$$n_\alpha = c\overset{+}{F}{}^{-1}_{\beta\alpha}N_\beta = c\bar{F}{}^{-1}_{\beta\alpha}N_\beta \quad , \tag{5.3}$$

$$\ell_\alpha \ell_\beta + n_\alpha n_\beta = \delta_{\alpha\beta} \quad , \quad L_\alpha = \epsilon_{\alpha\beta} N_\beta \quad . \tag{5.4}$$

If $\phi(\varkappa)$ and $\bar{p}(\varkappa)$ exist as described above, it follows that $\underset{\sim}{L}$, $\underset{\sim}{N}$, $\underset{\sim}{\ell}$, $\underset{\sim}{n}$, $\bar{\underset{\sim}{F}}$ and c are all $\varkappa$ dependent whence we write $\underset{\sim}{L}(\varkappa)$, $\underset{\sim}{N}(\varkappa)$, $\underset{\sim}{\ell}(\varkappa)$, $\underset{\sim}{n}(\varkappa)$, $\bar{\underset{\sim}{F}}(\varkappa)$ and $c(\varkappa)$.

Because of the presumed smoothness of $\phi(\varkappa)$ we have the following Taylor expansions, where a prime denotes differentiation with respect to $\varkappa$,

$$\left.\begin{aligned} \underset{\sim}{\ell}(\varkappa) &= \underset{\sim}{\ell}(0) + \underset{\sim}{\ell}'(0)\varkappa + o(\varkappa) \ , \\ \underset{\sim}{n}(\varkappa) &= \underset{\sim}{n}(0) + \underset{\sim}{n}'(0)\varkappa + o(\varkappa) \ , \\ \underset{\sim}{N}(\varkappa) &= \underset{\sim}{N}(0) + \underset{\sim}{N}'(0)\varkappa + o(\varkappa) \ . \end{aligned}\right\} \tag{5.5}$$ [1]

Equation (4.13) now gives

$$\bar{F}_{\alpha\beta}(\varkappa) = \overset{+}{F}_{\alpha\beta} + \varkappa \ell_\alpha(0) n_\gamma(0) \overset{+}{F}_{\gamma\beta} + o(\varkappa) \ , \tag{5.6}$$

$$\bar{F}^{-1}_{\alpha\beta}(\varkappa) = \overset{+}{F}{}^{-1}_{\alpha\beta} - \varkappa \ell_\gamma(0) n_\beta(0) \overset{+}{F}{}^{-1}_{\alpha\gamma} + o(\varkappa) \ , \tag{5.7}$$

where we have also used (5.4). This enables us to write the following Taylor expansion

$$\frac{\partial \hat{W}(\bar{\underset{\sim}{F}}(\varkappa))}{\partial F_{\alpha\beta}} = \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} + \varkappa \frac{\partial^2 \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta} \partial F_{\gamma\delta}} \ell_\alpha(0) n_\nu(0) \overset{+}{F}_{\nu\delta} + o(\varkappa) \ . \tag{5.8}$$

The Taylor expansion of $\bar{p}(\varkappa)$ leads to

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[1]Whenever we write $o(\varkappa)$, we mean $o(\varkappa)$ as $\varkappa \to 0$.

$$\bar{p}(\varkappa) = \bar{p}(0) + \varkappa \bar{p}'(0) + o(\varkappa) \quad . \tag{5.9}$$

Using (5.6), (5.7) and (5.9) and evaluating the traction continuity condition (4.12) to leading order, gives

$$\overset{+}{p} = \bar{p}(0) \quad . \tag{5.10}$$

Consequently we may write (5.9) as

$$\bar{p}(\varkappa) = \overset{+}{p} + \varkappa \bar{p}'(0) + o(\varkappa) \quad . \tag{5.11}$$

We now return to the traction continuity condition (4.12) and re-evaluate it to leading order using (2.11), (5.5), (5.7), (5.8) and (5.11). This leads to

$$\left\{ c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)n_\nu(0)\overset{+}{F}_{\nu\delta} - \bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\alpha} + \overset{+}{p}\ell_\gamma(0)n_\alpha(0)\overset{+}{F}{}^{-1}_{\beta\gamma}\right\} N_\beta(0) = 0 \quad , \tag{5.12}$$

which are two scalar equations for $\bar{p}'(0)$ and $\phi(0)$.

5.2 A Necessary Condition for the Existence of a Weak Shock

We now derive a necessary condition for equation (5.12) to have a solution $\bar{p}'(0)$, $\phi(0)$. We have from a Taylor expansion of (5.3) that

$$N_\beta(0) = \frac{1}{c(0)} \overset{+}{F}_{\nu\beta} n_\nu(0) \quad . \tag{5.13}$$

Using (5.13) in (5.12) leads to

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)n_\pi(0)n_\nu(0)\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\nu\beta} - \bar{p}'(0)n_\alpha(0) + \overset{+}{p}\ell_\gamma(0)n_\gamma(0)n_\alpha(0) = 0 \quad . \tag{5.14}$$

But since $\underset{\sim}{\ell}$ is perpendicular to $\underset{\sim}{n}$ we have $\ell_\alpha(0)n_\alpha(0) = 0$, whence (5.14) simplifies to

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)n_\pi(0)n_\nu(0)\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\nu\beta} - \bar{p}'(0)n_\alpha(0) = 0 \quad . \tag{5.15}$$

Multiplying (5.15) by $n_\alpha(0)$ and making use of the fact that $\underset{\sim}{n}$ is a unit vector leads to

$$\bar{p}'(0) = c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)n_\pi(0)n_\nu(0)n_\alpha(0)\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\nu\beta} \quad . \tag{5.16}$$

Alternatively, multiplying (5.15) by $\ell_\alpha(0)$ gives that

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)\ell_\alpha(0)n_\pi(0)n_\nu(0)\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\nu\beta} = 0 \quad , \tag{5.17}$$

by virtue of the fact that $\underset{\sim}{\ell}\cdot\underset{\sim}{n} = 0$. Using (5.3) and (5.4) in (5.17) leads to

$$\epsilon_{\pi\nu}\epsilon_{\lambda\mu}\overset{+}{F}_{\gamma\pi}\overset{+}{F}_{\alpha\lambda}\, c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})N_\beta(0)N_\delta(0)N_\nu(0)N_\mu(0) = 0 \quad , \tag{5.18}$$

which because of (3.7) can be equivalently written as

$$\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}\overset{+}{F}_{\gamma\lambda}\overset{+}{F}_{\delta\mu}Q_{\gamma\delta}(\underset{\sim}{N}(0), \overset{+}{\underset{\sim}{F}})N_\alpha(0)N_\beta(0) = 0 \quad . \tag{5.19}$$

Equation (5.19) must necessarily hold if a one parameter family of elastostatic shocks of the type being considered is to exist. On comparing with (3.18), we see that (5.19) implies a loss of ellipticity of the displacement equations of equilibrium on $\overset{+}{\Pi}$ at the given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$. We therefore have the following result:

> <u>Theorem 1.</u> A necessary condition for the existence of a one parameter family of elastostatic shocks, of the kind under consideration, is that the displacement equations of equilibrium suffer a loss of ellipticity at the given deformation and pressure on $\overset{+}{\Pi}$. Furthermore, in the weak shock

limit $(\varkappa \to 0)$ the material shock-line and the spatial shock-line tend respectively to a material and spatial characteristic associated with $\overset{+}{\Pi}$.

The corresponding result was obtained by Knowles and Sternberg [3] in the case of compressible elastic materials.

In the event that, corresponding to a given $\overset{+}{\underset{\sim}{F}}$ and $\overset{+}{p}$ a one parameter family of shocks of the type being considered exists, the jumps of various physical quantities across the shock can be easily determined to leading order in terms of the given $\overset{+}{\underset{\sim}{F}}$, $\overset{+}{p}$ and the presumably determinable (from (5.12)) $\phi(0)$, $\bar{p}'(0)$. We now determine some of these jumps.

(i) <u>The jump in energy density</u> $[W]_-^+$

The Taylor expansion of $\hat{W}(\bar{\underset{\sim}{F}}(\varkappa))$ about $\varkappa = 0$, together with (2.9) and (5.6) leads to

$$\hat{W}(\bar{\underset{\sim}{F}}(\varkappa)) = \hat{W}(\overset{+}{\underset{\sim}{F}}) + \varkappa\left\{\overset{+}{\tau}_{\alpha\beta} n_\beta(0)\ell_\alpha(0) + \overset{+}{p}\ell_\alpha(0)n_\alpha(0)\right\} + o(\varkappa) \quad . \tag{5.20}$$

Since $\underset{\sim}{\ell}$ is perpendicular to $\underset{\sim}{n}$ we can drop the last term in (5.20) to get

$$[W]_+^- = \varkappa\, t_\alpha(0)\ell_\alpha(0) + o(\varkappa) \quad , \tag{5.21}$$

where we have set

$$\underset{\sim}{t}(\varkappa) = \overset{+}{\underset{\sim}{\tau}}\underset{\sim}{n}(\varkappa) \quad . \tag{5.22}$$

As a consequence of (2.6), (4.2), (5.3) and displacement continuity, we see immediately that $\underset{\sim}{t}(\varkappa) = \overset{+}{\underset{\sim}{\tau}}\underset{\sim}{n}(\varkappa) = \bar{\underset{\sim}{\tau}}\underset{\sim}{n}(\varkappa)$ which implies the continuity of the Cauchy traction vector across $\mathcal{L}_*$.

(ii) The stress jumps $[\tau_{\alpha\beta}]^+_-$

From (2.9) we have that

$$\bar{\tau}_{\alpha\beta} = \frac{\partial \hat{W}(\bar{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} \bar{F}_{\beta\gamma} - \bar{p}\delta_{\alpha\beta} \quad , \tag{5.23}$$

$$\overset{+}{\tau}_{\alpha\beta} = \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} \overset{+}{F}_{\beta\gamma} - \bar{p}\delta_{\alpha\beta} \quad , \tag{5.24}$$

which together with (2.11), (5.6), (5.8), (5.11) and (5.22) leads to

$$[\tau_{\alpha\beta}]^-_+ = \varkappa\Big\{c_{\alpha\mu\gamma\delta}(\overset{+}{\underset{\sim}{F}})\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\beta\mu}\ell_\gamma(0)n_\pi(0) - \bar{p}'(0)\delta_{\alpha\beta} + \overset{+}{p}n_\alpha(0)\ell_\beta(0)$$

$$+ t_\alpha(0)\ell_\beta(0)\Big\} + o(\varkappa) \quad . \tag{5.25}$$

(iii) The jump in the normal stress acting on a plane perpendicular to $\mathscr{L}_*$ $[\tau^{X'}_{11}]^+_-$

Consider the plane perpendicular to the spatial shock so that the normal to this plane is $\underset{\sim}{\ell}$. The jump in the normal stress acting on this plane across the shock-line, $[\tau^{X'}_{11}]^+_-$, is

$$[\tau^{X'}_{11}]^+_- = \overset{+}{\tau}_{\alpha\beta}\ell_\alpha\ell_\beta - \bar{\tau}_{\alpha\beta}\ell_\alpha\ell_\beta \quad , \tag{5.26}$$

which because of (5.25) and the perpendicularity of the vectors $\underset{\sim}{\ell}$, and $\underset{\sim}{n}$ can be written to leading order as

$$[\tau^{X'}_{11}]^+_- = -\varkappa\Big\{ t_\alpha(0)\ell_\alpha(0) - \bar{p}'(0)$$

$$+ c_{\alpha\mu\gamma\delta}(\overset{+}{\underset{\sim}{F}})\overset{+}{F}_{\pi\delta}\overset{+}{F}_{\beta\mu}\ell_\gamma(0)\ell_\beta(0)\ell_\alpha(0)n_\pi(0)\Big\} + o(\varkappa) \quad . \tag{5.27}$$

In view of (5.4) and (5.16) this leads to

$$[\tau_{11}^{X'}]_{-}^{+} = -\varkappa\Big\{ t_{\alpha}(0)\ell_{\alpha}(0) + c_{\alpha\mu\gamma\delta}(\underset{\sim}{\overset{\pm}{F}})\overset{\pm}{F}_{\pi\delta}\overset{\pm}{F}_{\alpha\mu}\ell_{\gamma}(0)n_{\pi}(0)$$

$$- 2\bar{p}'(0)\Big\} + o(\varkappa) \quad ,$$

which together with (5.25) gives

$$[\tau_{11}^{X'}]_{-}^{+} = -\operatorname{tr}\underset{\sim}{\bar{\tau}}'(0) - \varkappa\bar{p}'(0) + o(\varkappa) \quad . \tag{5.28}$$

CHAPTER 6

6.1 Finite Elastostatic Shocks in Isotropic Incompressible Materials

We now return to shocks of finite strength, but assume the material at hand to be isotropic. Substituting (2.31) in (4.12) and making use of (2.4) and (5.3) leads to

$$\left\{2W'(\overset{+}{I})\overset{+}{G}_{\alpha\beta}-\overset{+}{p}\delta_{\alpha\beta}\right\}n_\beta=\left\{2W'(\overset{-}{I})\overset{-}{G}_{\alpha\beta}-\overset{-}{p}\delta_{\alpha\beta}\right\}n_\beta\ , \tag{6.1}$$

where

$$\overset{+}{I}=\overset{+}{F}_{\alpha\beta}\overset{+}{F}_{\alpha\beta}=\overset{+}{G}_{\alpha\alpha}\ ,\quad \overset{-}{I}=\overset{-}{F}_{\alpha\beta}\overset{-}{F}_{\alpha\beta}=\overset{-}{G}_{\alpha\alpha}\ . \tag{6.2}$$

Clearly, (6.1) is simply a statement of the fact that the Cauchy traction is continuous across the spatial shock. The original problem concerning the existence of elastostatic shocks can now be posed as follows: given a constant tensor $\overset{+}{\underset{\sim}{F}}$ with unit determinant and a real constant $\overset{+}{p}$, determine real numbers $\overset{-}{p}$, $\varkappa(\neq 0)$ and ϕ such that (6.1) holds with $\overset{-}{\underset{\sim}{G}}$ given by (4.13), (6.2).

If we express (6.1) in terms of its components in the frame X', we have

$$2\overset{+}{G}{}^{X'}_{12}W'(\overset{+}{I})=2\overset{-}{G}{}^{X'}_{12}W'(\overset{-}{I})\ , \tag{6.3}$$

$$2\overset{+}{G}{}^{X'}_{22}W'(\overset{+}{I})-\overset{+}{p}=2\overset{-}{G}{}^{X'}_{22}W'(\overset{-}{I})-\overset{-}{p}\ . \tag{6.4}$$

As observed earlier, (6.3) and (6.4) impose only two scalar restrictions on the three quantities ϕ, $\varkappa$ and $\overset{-}{p}$. Furthermore since $\overset{-}{p}$ enters only in (6.4), and there too only linearly, we may consider (6.3) and (6.4) separately i.e. if there are numbers $\varkappa$ and ϕ such that (6.3) holds, then there certainly is a third number $\bar{p}$ such that (6.4) holds as well. The existence of an elastostatic shock therefore depends on whether there are numbers $\varkappa$ and ϕ such that (6.3) holds.

To pursue this question further, we need the components of $\overset{+}{\underset{\sim}{G}}$ and $\overset{-}{\underset{\sim}{G}}$ in the frame X'. With no loss of generality let us take X to be a principal frame for $\overset{+}{\underset{\sim}{G}}$. Then

$$\overset{+}{\underset{\sim}{G}}{}^{X} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}, \quad \lambda_1\lambda_2 = 1 . \tag{6.5}$$

By the change of frame formula for tensors we have

$$\overset{+}{\underset{\sim}{G}}{}^{X'} = \underset{\sim}{R}\,\overset{+}{\underset{\sim}{G}}{}^{X}\underset{\sim}{R}^{T} , \tag{6.6}$$

where

$$\underset{\sim}{R} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} . \tag{6.7}$$

Using (6.5) and (6.7) in (6.6) gives

$$\overset{+}{\underset{\sim}{G}}{}^{X'} = \begin{bmatrix} \dfrac{\lambda_1^2+\lambda_2^2}{2} + \dfrac{\lambda_1^2-\lambda_2^2}{2}\cos 2\phi & -\dfrac{\lambda_1^2-\lambda_2^2}{2}\sin 2\phi \\ -\dfrac{\lambda_1^2-\lambda_2^2}{2}\sin 2\phi & \dfrac{\lambda_1^2+\lambda_2^2}{2} - \dfrac{\lambda_1^2-\lambda_2^2}{2}\cos 2\phi \end{bmatrix} . \tag{6.8}$$

If we now set

$$\beta = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2} , \tag{6.9}$$

we can write (6.8) because of the second of (6.5) as

$$\overset{+}{\underset{\sim}{G}}{}^{X'} = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1+\beta\cos 2\phi & -\beta\sin 2\phi \\ & \\ -\beta\sin 2\phi & 1-\beta\cos 2\phi \end{bmatrix} . \tag{6.10}$$

It is clear from (6.5) and (6.9), that the value of β alone suffices to determine $\overset{+}{\underset{\sim}{G}}{}^{X}$ completely, and in this sense <u>β is a measure of the deformation on $\overset{+}{\Pi}$.</u> Note that because $\lambda_\alpha > 0$, (6.9) implies that

$$1 > \beta > -1 . \tag{6.11}$$

Furthermore, we have $\beta = 0$ if and only if the part of the body occupying $\overset{+}{\Pi}$ in its undeformed configuration remains undeformed[1] under the mapping (4.7).

We now find from (4.14), (6.2) and (6.10) that

$$\overset{-}{\underset{\sim}{G}}{}^{X'} = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1+\beta\cos 2\phi - 2\varkappa\beta\sin 2\phi & -\beta\sin 2\phi \\ +\varkappa^2(1-\beta\cos 2\phi) & +\varkappa(1-\beta\cos 2\phi) \\ & \\ -\beta\sin 2\phi + \varkappa(1-\beta\cos 2\phi) & 1-\beta\cos 2\phi \end{bmatrix} , \tag{6.12}$$

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[1] i.e. $\overset{+}{\underset{\sim}{F}}$ is a proper orthogonal tensor.

and from (6.2), (6.10) and (6.12) that

$$\overset{+}{I} = \frac{2}{\sqrt{1-\beta^2}} \, , \tag{6.13}$$

$$\overset{-}{I} = \frac{2 - 2\varkappa\beta\sin 2\phi + \varkappa^2(1-\beta\cos 2\phi)}{\sqrt{1-\beta^2}} \, . \tag{6.14}$$

Returning to the traction continuity requirement (6.3) with (6.10), (6.12) - (6.14) we find

$$-\beta\sin 2\phi W'\left(\frac{2}{\sqrt{1-\beta^2}}\right) = \Big\{-\beta\sin 2\phi$$

$$+\varkappa(1-\beta\cos 2\phi)\Big\} W'\left(\frac{2 - 2\varkappa\beta\sin 2\phi + \varkappa^2(1-\beta\cos 2\phi)}{\sqrt{1-\beta^2}}\right) \, . \tag{6.15}$$

We may now pose the problem as follows: given a number β in $(-1,1)$, find numbers $\varkappa \neq 0$ and ϕ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that (6.15) holds.

If, for the given β in $(-1,1)$ and any ϕ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ there does not exist a root $\varkappa \neq 0$ to (6.15), the material is incapable of sustaining an elastostatic shock corresponding to the given deformation associated with β on $\overset{+}{\Pi}$. On the other hand if, for the given β in $(-1,1)$ and for some ϕ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ there is a root $\varkappa \neq 0$ to equation (6.15), then there exists a corresponding elastostatic shock. Therefore, we now investigate the possibility that (6.15) has a root $\varkappa \neq 0$ for all values of ϕ and β such that $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$, $|\beta| < 1$.

Finally we observe from (6.15) that if for some pair (ϕ, β),

$|\beta| < 1$ and $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$, there exists a root κ to equation (6.15), then

(i) $-\kappa$ is a root of (6.15) for the values $(-\phi, \beta)$, and

(ii) $-\kappa$ is a root of (6.15) for the values $(\frac{\pi}{2} - \phi, -\beta)$.

It therefore follows that, as far as the issue of existence is concerned, we may in fact restrict ϕ to $\left[0, \frac{\pi}{2}\right]$ and β to $[0, 1)$. If we define the set $\mathcal{G}$ by

$$\mathcal{G} = \{(\phi, \beta) \,|\, 0 \leq \phi \leq \frac{\pi}{2} \;,\; 0 \leq \beta < 1\} \;, \tag{6.16}$$

we need to look at the question of the existence of a root $\kappa \neq 0$ to equation (6.15) for every (ϕ, β) in $\mathcal{G}$.

6.2 Some General Results

We now establish some general results concerning the existence of elastostatic shocks, valid for an arbitrary homogeneous, isotropic, incompressible elastic solid which has a positive shear modulus.[1]

We first make the following preliminary observation. If $\beta = 0$ or $\phi = 0$ or $\phi = \frac{\pi}{2}$, the only root of (6.15) is $\kappa = 0$. This follows directly from (6.15) because of (2.40). This means that for a material of the type we are considering, no elastostatic shock is possible if the part of the body occupying $\overset{+}{\Pi}_*$ is undeformed, nor can any spatial shock-line be inclined at 0 or $\frac{\pi}{2}$ to the principal axes of $\overset{+}{\underset{\sim}{G}}$. We may now restrict attention to the interior $\overset{\circ}{\mathcal{G}}$ of the set $\mathcal{G}$:

$$\overset{\circ}{\mathcal{G}} = \{(\phi, \beta) \,|\, 0 < \phi < \frac{\pi}{2} \;,\; 0 < \beta < 1\} \;. \tag{6.17}$$

If we set

[1] We assume from here on that (2.40) holds.

$$b = b(\phi, \beta) = -\frac{\beta \sin 2\phi}{(1-\beta^2)^{\frac{1}{2}}} \quad \text{on } \overset{\circ}{\mathfrak{a}} , \tag{6.18}$$

$$c = c(\phi, \beta) = \frac{(1-\beta \cos 2\phi)}{(1-\beta^2)^{\frac{1}{2}}} \quad \text{on } \overset{\circ}{\mathfrak{a}} , \tag{6.19}$$

we can write (6.15) using (6.13), (6.18) and (6.19) as

$$bW'(\overset{+}{I}) = (b + c\varkappa)W'(\overset{+}{I} + 2b\varkappa + c\varkappa^2) \quad . \tag{6.20}$$

Clearly

$$b < 0 \quad \text{on } \overset{\circ}{\mathfrak{a}} , \tag{6.21}$$

$$c > 0 \quad \text{on } \overset{\circ}{\mathfrak{a}} . \tag{6.22}$$

Choose and <u>fix</u> a point $(\hat{\phi}, \hat{\beta})$ in $\overset{\circ}{\mathfrak{a}}$. At this fixed value of ϕ and β we define the function $h(\cdot)$ by

$$h(x) = (b + cx)W'(\overset{+}{I} + 2bx + cx^2) - bW'(\overset{+}{I}) \quad \text{for } |x| < \infty , \tag{6.23}$$[1]

where $\overset{+}{I}$, b and c are given by (6.13), (6.18) and (6.19) respectively evaluated at $(\hat{\phi}, \hat{\beta})$. If the plane strain elastic potential $W(I)$ is twice continuously differentiable on $I \geq 2$, as we have implicitly assumed, it follows that $h(x)$ is continuously differentiable on $(-\infty, \infty)$. If there exists an equilibrium shock corresponding to the homogeneous deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ and inclined at an angle $\hat{\phi}$ to the y_1-axis, it is necessary and sufficient that $h(x)$ have a zero at some

[1] From (6.13), (6.18), (6.19) and (6.22) we have that $\overset{+}{I} + 2bx + cx^2 = 2 + \frac{(b+cx)^2}{c} + \frac{(c-1)^2}{c} \geq 2$ for all $|x| < \infty$.

$x \neq 0$. The zero of $h(x)$ gives the shock strength $\varkappa$.

Because of (2.40), (6.21) and (6.22) we see from (6.23) that

$$\left.\begin{aligned} &h(0) = 0 \ , & (6.24)\\ &h(-\frac{b}{c}) = -bW'(\overset{+}{I}) > 0 \quad , \quad -\frac{b}{c} > 0 \ . & (6.25) \end{aligned}\right\}$$

It now follows from (6.24), (6.25) and the smoothness of $h(x)$ that

(i) if $h'(0) < 0$, then there exists a zero of $h(x)$ other than $x = 0$.

(ii) if there exists a zero other than $x = 0$ of $h(x)$, then there exists a number $\varkappa_* \neq 0$ such that $h'(\varkappa_*) = 0$. Furthermore, since $h'(-\frac{b}{c}) = cW'\left(\overset{+}{I} - \frac{b^2}{c}\right) > 0$ it follows that

$$\varkappa_* \neq -\frac{b}{c} \ . \tag{6.26}$$

Because of the remarks made before (6.24), we may interpret (i) and (ii) as follows:

(a) $h'(0) < 0$ is sufficient to ensure the existence of an elastostatic shock corresponding to the deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ with spatial shock inclination $\hat{\phi}$.

(b) If an elastostatic shock as just described is to exist, then it is necessary that $h'(\varkappa_*) = 0$ for some number $\varkappa_*$.

This leads to the main results of this section which we now establish. We first introduce the following terminology. Recall from (2.40) and (3.21), that a loss of ellipticity of the displacement equations of equilibrium can occur at some deformation and at some point if and only if

$$\frac{2W''(I)}{W'(I)}(I-2)+1 \leq 0 \quad . \tag{6.27}$$

If ellipticity is lost because in fact the strict inequality holds in (6.27), we say that a strict failure of ellipticity has occurred. Since four characteristic curves exist in this case (see Section 3.3) one may say that the displacement equations of equilibrium are hyperbolic at such a deformation on $\overset{+}{\Pi}$.

Theorem 2. A strict failure of ellipticity of the displacement equations of equilibrium, at the given homogeneous deformation and pressure on $\overset{+}{\Pi}$, is sufficient to ensure the existence of a corresponding elastostatic shock in a homogeneous, isotropic, incompressible, elastic solid with a positive shear modulus.

Proof:

By hypothesis, the given deformation gradient $\overset{+}{\underset{\sim}{F}}$ is such that the associated value of β , say $\hat{\beta}$, given by (2.4), (6.5) and (6.9) conforms with the inequality

$$\frac{2W''(\overset{+}{I})}{W'(\overset{+}{I})}(\overset{+}{I}-2)+1<0 \quad , \tag{6.28}$$

where by (6.13)

$$\overset{+}{I} = \frac{2}{\sqrt{1-\hat{\beta}^2}} \quad . \tag{6.29}$$

Note from (2.40) and (6.28) that necessarily $\overset{+}{I} \neq 2$, whence $\hat{\beta} \neq 0$. We now choose the value of $\hat{\phi}$ as

$$\hat{\phi} = \frac{1}{2}\cos^{-1}\left(\frac{1 - \sqrt{1 - \hat{\beta}^2}}{\hat{\beta}}\right) \quad . \tag{6.30}$$ [1]

We will show that corresponding to the given homogeneous deformation on $\overset{+}{\Pi}$, such that the associated value of β conforms with (6.28) and (6.29), there exists an elastostatic shock at the inclination $\hat{\phi}$ given by (6.30).

According to (6.23) and statement (a) following (6.26), we need only show that

$$2b^2 W''(\overset{+}{I}) + cW'(\overset{+}{I}) < 0 \quad \text{at} \quad (\hat{\phi}, \hat{\beta}) \ , \tag{6.31}$$

in order to establish this. Using (2.40) we may write (6.28) alternately as

$$2b^2 W''(\overset{+}{I}) + cW'(\overset{+}{I}) + \left(\frac{b^2}{\overset{+}{I} - 2} - c\right)W'(\overset{+}{I}) < 0 \ , \tag{6.32}$$

where b and c are defined by (6.18), (6.19) and evaluated at $(\hat{\phi}, \hat{\beta})$. Using (6.18), (6.19) and (6.29) in (6.32) we find

$$2b^2 W''(\overset{+}{I}) + cW'(\overset{+}{I}) - \frac{\hat{\beta}^2}{(1 - \hat{\beta}^2)(\overset{+}{I} - 2)}\left\{\cos 2\hat{\phi} - \frac{1 - \sqrt{1 - \hat{\beta}^2}}{\hat{\beta}}\right\}^2 W'(\overset{+}{I}) < 0 \ , \tag{6.33}$$

which because of (6.30) reduces to (6.31), which in turn establishes our result.

<u>Theorem 3</u>. A <u>necessary</u> condition for the existence of a piecewise homogeneous elastostatic shock in a homogeneous, isotropic, incompressible, elastic

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[1] Since $\hat{\beta} \neq 0$, $|\hat{\beta}| < 1$ this defines a real angle $\hat{\phi}$ in $(0, \frac{\pi}{2})$.

solid with positive shear modulus, is that the displacement equations of equilibrium suffer a loss of ellipticity at <u>some</u> homogeneous deformation.

Proof:

By hypothesis there is a point $(\hat{\phi}, \hat{\beta})$ in $\overset{\circ}{\mathcal{a}}$ such that there exists an associated elastostatic shock. By statement (b) following (6.26) then, there is a real number $\varkappa_*$ such that

$$h'(\varkappa_*) = 0 \tag{6.34}$$

where $h(x)$ is given by (6.23) with b, c and $\overset{+}{I}$ evaluated at $(\hat{\phi}, \hat{\beta})$. Equations (6.23) and (6.34) give that

$$cW'(\overset{+}{I} + 2b\varkappa_* + c\varkappa_*^2) + 2(b + c\varkappa_*)^2 W''(\overset{+}{I} + 2b\varkappa_* + c\varkappa_*^2) = 0 \quad . \tag{6.35}$$

Let

$$I_* = \overset{+}{I} + 2b\varkappa_* + c\varkappa_*^2 \quad , \tag{6.36}$$

so that we have from (6.35) that

$$(I_* - 2)\left\{cW'(I_*) + 2(b + c\varkappa_*)^2 W''(I_*)\right\} = 0 \quad .$$

It follows from this that

$$(I_* - 2)\left\{cW'(I_*) + 2(b + c\varkappa_*)^2 W''(I_*)\right\}$$

$$\leq \frac{\hat{\beta}^2}{(1 - \hat{\beta}^2)} \left\{\cos 2\hat{\phi} - \frac{1 - \sqrt{1 - \hat{\beta}^2}}{\hat{\beta}}\right\}^2 W'(I_*) \quad , \tag{6.37}$$

since by virtue of (2.40) and (6.11) the right hand side of (6.37) is non-negative. Using (6.13), (6.18), (6.19) and (6.36) in (6.37) leads to

$$(b+c\varkappa_*)^2\left\{2W''(I_*)(I_*-2)+W'(I_*)\right\} \le 0 \quad , \tag{6.38}$$

which because of (2.40) and (6.26) gives

$$\frac{2W''(I_*)}{W'(I_*)}(I_*-2)+1 \le 0 \quad . \tag{6.39}$$

This implies a loss of ellipticity of the displacement equations of equilibrium at a homogeneous deformation in which the deformation gradient $\underset{\sim}{F}$ is such that $F_{\alpha\beta}F_{\alpha\beta} = I_*$.

To summarize, we have shown that for the type of material at hand, a strict loss of ellipticity at the given deformation is sufficient to ensure the existence of a corresponding elastostatic shock. On the other hand, a loss of ellipticity at some homogeneous deformation is necessary, if an elastostatic shock is to exist.

We draw attention to the fact that Theorem 2 does <u>not</u> imply that if ellipticity is strictly lost at the given deformation then the corresponding configuration of the body must involve a shock. Rather, it claims that such a configuration is available. There is also a shock-less configuration available corresponding to the root $\varkappa = 0$ of (6.20). Likewise, a loss of ellipticity at the given deformation is <u>not necessary</u> for a corresponding elastostatic shock to exist. In the boundary value problem considered in the second part of this paper, we encounter configurations of the body involving elastostatic shocks such that the displacement equations of equilibrium are elliptic on <u>both</u> sides of the shock-line.

CHAPTER 7

7.1 Dissipativity Inequality

If we admit weak solutions into the discussion of a problem, (such as those of the type introduced in Sections 4.0 - 6.0), we would anticipate that since the admissible class of solutions has been greatly widened, there could possibly be many solutions to that problem. It is well known that this is indeed the case in the theory of quasi-linear hyperbolic partial differential equations. See for example Lax [8]. The specific boundary value problem considered in the next part of this paper confirms this to be the case in the present context as well.

In such circumstances, it is essential to introduce criteria which single out a physically admissible solution from among the many solutions admitted by the differential equations. The second law of thermodynamics appears to play such a role in gas dynamics. Lax [8] has examined "entropy conditions" which furnish such criteria in the context of hyperbolic systems of conservation laws.

An analogous "entropy condition" in the context of elastostatics was proposed by Knowles and Sternberg [3] and subsequently extended by Knowles [4]. A thermodynamic motivation for the proposed condition, in the case of compressible materials, was also given in [4]. In the three-dimensional case, a quasi-static time dependent family of equilibrium states was considered, the time merely playing the role of a history parameter, and it was then required that

$$\int_{\partial \mathcal{D}} \underset{\sim}{\sigma} \underset{\sim}{N} \cdot \underset{\sim}{v} \, dA - \frac{d}{dt} \int_{\mathcal{D}} \hat{W}(\underset{\sim}{F}) dv \geq 0 \qquad (7.1)^{1}$$

for every regular sub-domain $\mathcal{D}$ of $\mathcal{R}$, at each instant of the time interval considered. Here t is the time and $\underset{\sim}{v}$ the quasi-static particle velocity. Equation (7.1) gives expression to the idea that the rate at which elastic energy is being stored in $\mathcal{D}$ cannot exceed the rate at which work is being done on $\mathcal{D}$.

One shows easily that for a subdomain $\mathcal{D}$ of the body which is such that the field quantities have classical smoothness properties at each interior point, the global condition (7.1) holds with inequality replaced by equality by virtue of the field equations. This is indeed as one would expect, and accordingly (7.1) imposes no local restrictions at a point where the fields are smooth. If however an elastostatic shock is present in the domain $\mathcal{D}$, then (7.1) does <u>not</u> hold automatically and consequently, at each point on the shock it imposes a local restriction on the jumps of the field quantities.

Now consider a quasi-static family of plane strain piecewise homogeneous elastostatic shocks in a homogeneous, incompressible, elastic solid. It can be shown that, if at some instant t (7.1) holds with strict inequality for all sub-domains $\mathcal{D}$ which intersect $\mathcal{L}$, then

(i) the motion of the shock-line $\mathcal{L}$ at that instant is translatory in a direction not parallel to itself. Moreover, if we orient the shock-line $\mathcal{L}$ such that this translation is directed into $\overset{+}{\Pi}$, then at that instant

[1]Body forces were omitted from this discussion.

$$[\hat{W}(\underset{\sim}{F}) - \sigma_{\alpha\beta}N_\beta F_{\alpha\gamma}N_\gamma]^+_- > 0 \ . \tag{7.2}$$

Conversely, if at some time t the quasi-static family of solutions conforms with (i), then (7.1) holds with strict inequality at that instant.

On the other hand we can show that if at the instant t (7.1) holds with equality, then either

(ii) the shock-line $\mathscr{L}$ is instantaneously stationary at that moment, or

(iii) the shock-line $\mathscr{L}$ is instantaneously in a state of translation parallel to itself at that moment, or

(iv) the jump of $\hat{W}(\underset{\sim}{F}) - \sigma_{\alpha\beta}N_\beta F_{\alpha\gamma}N_\gamma$ across the shock is zero, (in which case the shock-line motion is not restricted to being translatory).

Conversely, if at some time t the quasi-static family of solutions conforms with one of (ii), (iii), and (iv), then (7.1) holds with equality at that instant.

Finally one can show that if at some time t (7.1) holds for all sub-domains $\mathscr{D}$, and if in addition it holds with equality for <u>some</u> sub-domain which intersects the shock, then in fact, at that instant (7.1) holds with equality for <u>all</u> sub-domains $\mathscr{D}$. We conclude from this that the preceding are the only possibilities. Therefore, if (7.1) holds it is <u>necessary</u> that one of (i) - (iv) hold. Conversely if one of (i) - (iv) holds this is <u>sufficient</u> to ensure that (7.1) hold.

One arrives at (i) - (iv) by applying to the incompressible case the parallel arguments used by Knowles and Sternberg in [3], or by specializing to this context the results of Knowles [4]. Since (7.1) implies that the presence of an elastostatic shock decreases, or at least does not increase, the stored energy in the body, we refer to (7.2) as the dissipativity inequality.

It is apparent from (i) - (iv) that the dissipativity requirement (7.1) may be viewed as restricting the admissible class of quasi-static motions. The only quasi-static motions admitted by it are those in which the value of $\{-\hat{W}(\underset{\sim}{F})+\underset{\sim}{F}\underset{\sim}{N}\cdot\underset{\sim}{\sigma}\underset{\sim}{N}\}$ at a particle does not decrease as the particle crosses the shock-line.

It may be remarked that the dissipativity inequality does not rule out any piecewise homogeneous elastostatic shock itself as being inadmissible, since any given piecewise homogeneous elastostatic shock can always be embedded in a suitable time dependent family of such shocks which conforms with the dissipativity inequality.

As one would expect, and as is verified by Knowles [4], these results remain true locally in the general case of a curved shock in a non-homogeneous elastic field, with the exception that the shock motion may no longer be restricted to translation. The latter property is clearly peculiar to piecewise homogeneous elastostatic shocks.

Using (2.4), (2.6), (4.13), (5.3), and evaluating the left hand side of (7.2) in the frame X' leads to

$$[\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}\sigma_{\alpha\gamma}N_{\beta}N_{\gamma}]_{-}^{+} = [\hat{W}(\underset{\sim}{F})]_{-}^{+} + \varkappa\tau_{12}^{X'} \ , \tag{7.3}$$

where we have also used the fact that $\tau_{12}^{X'}$ is continuous across $\mathcal{L}_{*}$. Therefore the inequality (7.2) may be written in the simpler form

$$[\hat{W}(\underset{\sim}{F})]_{-}^{+} + \varkappa\tau_{12}^{X'} > 0 \ , \tag{7.4}$$

where $\mathcal{L}$ is presumed to be oriented such that it moves into $\overset{+}{\Pi}$ as time t increases.

In the particular case when the material at hand is isotropic, we have from (2.33), (6.10) and (6.18) that

$$\tau_{12}^{X'} = 2bW'(\overset{+}{I}) \quad , \tag{7.5}$$

whence (7.4), by virtue of (6.14), (6.18) and (6.19), may be equivalently written as

$$W(\overset{+}{I}) - W(\overset{+}{I} + 2b\varkappa + c\varkappa^2) + 2b\varkappa W'(\overset{+}{I}) > 0 \quad . \tag{7.6}$$

Note, however, that (6.23) may alternatively be written as

$$h(x) = \frac{1}{2}\frac{\partial W}{\partial x}(\overset{+}{I} + 2bx + cx^2) - bW'(\overset{+}{I}) \quad , \tag{7.7}$$

whence (7.6) takes the simple form

$$\int_0^{\varkappa} h(x)dx < 0 \quad . \tag{7.8}$$

We will make use of this form of the dissipativity inequality in the example taken up in the next section.

Finally, we return to anisotropic, incompressible, elastic solids in order to determine the weak shock approximation to the value of the jump of $\{\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}\sigma_{\alpha\gamma}N_\beta N_\gamma\}$ across the shock-line. Recall from Section 5.1, where we first looked at weak elastostatic shocks, that we now assume that, given the deformation gradient $\overset{+}{\underset{\sim}{F}}$ with unit determinant and the pressure $\overset{+}{p}$, there exist functions $\phi(\varkappa)$ and $\bar{p}(\varkappa)$, both sufficiently smooth in a neighborhood of $\varkappa = 0$, such that $\bar{F}_{\alpha\beta}(\varkappa)$

defined by

$$\bar{F}_{\alpha\beta}(\varkappa) = \overset{+}{F}_{\alpha\beta} + \varkappa \ell_{\alpha}(\varkappa) n_{\gamma}(\varkappa) \overset{+}{F}_{\gamma\beta} \quad , \tag{7.9}$$

conforms with the traction continuity requirement (4.12). Observe from (7.9) that

$$\bar{F}^{-1}_{\alpha\beta}(\varkappa) = \overset{+}{F}{}^{-1}_{\alpha\beta} - \varkappa \ell_{\gamma}(\varkappa) n_{\beta}(\varkappa) \overset{+}{F}{}^{-1}_{\alpha\gamma} \quad . \tag{7.10}$$

It is first necessary to analyze the traction continuity condition (4.12). To this end set

$$\Delta_{\alpha}(\varkappa) = \overset{+}{\sigma}_{\alpha\beta} N_{\beta}(\varkappa) - \bar{\sigma}_{\alpha\beta}(\varkappa) N_{\beta}(\varkappa) \quad , \tag{7.11}$$

which because of (4.9), (5.3), (7.10) and the perpendicularity of $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$ leads to

$$\Delta_{\alpha}(\varkappa) = \left\{ \frac{\partial \hat{W}(\overset{+}{\underset{\sim}{F}})}{\partial F_{\alpha\beta}} - \frac{\partial \hat{W}(\bar{\underset{\sim}{F}}(\varkappa))}{\partial F_{\alpha\beta}} \right\} N_{\beta}(\varkappa) + (\bar{p}(\varkappa) - \overset{+}{p}) \overset{+}{F}{}^{-1}_{\beta\alpha} N_{\beta}(\varkappa) \quad . \tag{7.12}$$

Differentiating (7.12) with respect to $\varkappa$ and using (2.11), (5.10) and (7.9) gives

$$\Delta'_{\alpha}(0) = - c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}}) \overset{+}{F}_{\pi\delta} N_{\beta}(0) \ell_{\gamma}(0) n_{\pi}(0) + \bar{p}'(0) \overset{+}{F}{}^{-1}_{\beta\alpha} N_{\beta}(0) \quad . \tag{7.13}$$

The continuity of traction across the shock requires that

$$\Delta_{\alpha}(\varkappa) = 0 \tag{7.14}$$

for all sufficiently small $\varkappa$, from which it follows that in particular

$$\Delta'_{\alpha}(0) = 0 \quad . \tag{7.15}$$

From (7.13) and (7.15) we find that

$$c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\overset{+}{F}_{\pi\delta}\ell_{\gamma}(0)n_{\pi}(0)N_{\beta}(0) - \bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\alpha}N_{\beta}(0) = 0 \quad . \tag{7.16}$$

As one would expect, (7.16) is in fact the same as (5.12) because of (5.3). Differentiation of (7.12) twice with respect to $\varkappa$, together with the symmetry $c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta}$, the fact that $\underset{\sim}{\ell}\cdot\underset{\sim}{n} = 0$ and (2.11), (5.1) - (5.3), (5.10) and (7.9) leads to

$$\begin{aligned}\ell_{\alpha}(0)\Delta''_{\alpha}(0) = &- c^{2}(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_{\alpha}(0)\ell_{\gamma}(0)\ell_{\lambda}(0)N_{\beta}(0)N_{\delta}(0)N_{\mu}(0)\\ &- 4c(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_{\alpha}(0)\ell_{\gamma}(0)N_{\beta}(0)N'_{\delta}(0)\\ &- 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\gamma}N_{\beta}(0)\ell'_{\alpha}(0) + 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\alpha}N'_{\beta}(0)\ell_{\alpha}(0) \quad ,\end{aligned} \tag{7.17}$$

where we have set

$$d_{\alpha\beta\gamma\delta\lambda\mu}(\underset{\sim}{F}) = \frac{\partial^{3}\hat{W}(\underset{\sim}{F})}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}\partial F_{\lambda\mu}} \quad , \tag{7.18}$$

and $c(\varkappa)$ was defined in (5.1). Because of (7.14) we have that $\Delta''_{\alpha}(0) = 0$, whence we have from (7.17) that

$$\begin{aligned}&4c(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_{\alpha}(0)\ell_{\gamma}(0)N_{\beta}(0)N'_{\delta}(0)\\ &= - c^{2}(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_{\alpha}(0)\ell_{\gamma}(0)\ell_{\lambda}(0)N_{\beta}(0)N_{\delta}(0)N_{\mu}(0)\\ &- 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\gamma}N_{\beta}(0)\ell'_{\alpha}(0) + 2\bar{p}'(0)\overset{+}{F}{}^{-1}_{\beta\alpha}N'_{\beta}(0)\ell_{\alpha}(0) \quad .\end{aligned} \tag{7.19}$$

We now compute the jump in $\{\hat{W}(\underset{\sim}{F}) - \underset{\sim}{F}\underset{\sim}{N}\cdot\underset{\sim}{\sigma}\underset{\sim}{N}\}$ across the shock.

To this end, let

$$\eta(\varkappa) = \hat{W}(\bar{\underset{\sim}{F}}(\varkappa)) - \hat{W}(\overset{+}{\underset{\sim}{F}}) + \overset{+}{F}_{\alpha\beta} N_\beta(\varkappa) \overset{+}{\sigma}_{\alpha\gamma} N_\gamma(\varkappa)$$

$$- \bar{F}_{\alpha\beta}(\varkappa) N_\beta(\varkappa) \bar{\sigma}_{\alpha\gamma}(\varkappa) N_\gamma(\varkappa) \quad . \tag{7.20}$$

Because of traction continuity, the fact that $\underset{\sim}{\ell} \cdot \underset{\sim}{n} = 0$, (4.9), (5.3) and (7.9) we can write (7.20) as

$$\eta(\varkappa) = \hat{W}(\bar{\underset{\sim}{F}}(\varkappa)) - \hat{W}(\overset{+}{\underset{\sim}{F}}) - \varkappa c(\varkappa) \ell_\alpha(\varkappa) N_\gamma(\varkappa) \frac{\partial \hat{W}(\bar{\underset{\sim}{F}}(\varkappa))}{\partial F_{\alpha\gamma}} \quad . \tag{7.21}$$

Clearly,

$$\eta(0) = 0 \quad , \tag{7.22}$$

by virtue of (7.9). Differentiating (7.21) with respect to $\varkappa$ and using (2.11), (5.3) and (7.9) gives

$$\eta'(\varkappa) = - \varkappa c(\varkappa) c_{\alpha\beta\gamma\delta}(\bar{\underset{\sim}{F}}(\varkappa)) \ell_\gamma(\varkappa) N_\delta(\varkappa) \frac{d}{d\varkappa} \{ \varkappa c(\varkappa) \ell_\alpha(\varkappa) N_\beta(\varkappa) \} \quad , \tag{7.23}$$

from which we have that

$$\eta'(0) = 0 \quad . \tag{7.24}$$

Differentiating (7.23) with respect to $\varkappa$ and using (7.9) leads to

$$\eta''(0) = - c^2(0) c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}}) \ell_\alpha(0) \ell_\gamma(0) N_\beta(0) N_\delta(0) \quad , \tag{7.25}$$

which because of (5.3) and (7.16) gives

$$\eta''(0) = - \bar{p}'(0) n_\alpha(0) \ell_\alpha(0) \quad , \tag{7.26}$$

which in turn, because $\ell_\alpha(0)n_\alpha(0)=0$, implies that

$$\eta''(0)=0 \quad . \tag{7.27}$$

Finally, differentiating (7.23) twice with respect to $\varkappa$, using the symmetry $c_{\alpha\beta\gamma\delta}=c_{\gamma\delta\alpha\beta}$, (5.3), (7.9) and (7.18) leads to

$$\begin{aligned}\eta'''(0) = &- 2c^3(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0) \\ &- 6c^2(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)N'_\delta(0)N_\beta(0) \\ &- 6c(0)c_{\alpha\beta\gamma\delta}(\overset{+}{\underset{\sim}{F}})\ell_\gamma(0)N_\beta(0)N_\delta(0)\{c'(0)\ell_\alpha(0)+c(0)\ell'_\alpha(0)\}\end{aligned} \tag{7.28}$$

which on using (5.3), (7.19) and $\ell_\alpha(0)n_\alpha(0)=0$ implies that

$$\begin{aligned}\eta'''(0) = &-\tfrac{1}{2}c^3(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0) \\ &- 3\bar{p}'(0)\{n'_\alpha(0)\ell_\alpha(0)+n_\alpha(0)\ell'_\alpha(0)\}\end{aligned} \tag{7.29}$$

Since the vectors $\underset{\sim}{\ell}(\varkappa)$ and $\underset{\sim}{n}(\varkappa)$ are perpendicular to each other,

$$\ell_\alpha(\varkappa)n_\alpha(\varkappa)=0 \quad \text{for all sufficiently small } \varkappa \quad . \tag{7.30}$$

Differentiating (7.30) with respect to $\varkappa$ shows that

$$\ell'_\alpha(0)n_\alpha(0)+\ell_\alpha(0)n'_\alpha(0)=0 \quad , \tag{7.31}$$

so that finally we may write (7.29) as

$$\eta'''(0) = -\tfrac{1}{2}c^3(0)d_{\alpha\beta\gamma\delta\lambda\mu}(\overset{+}{\underset{\sim}{F}})\ell_\alpha(0)\ell_\gamma(0)\ell_\lambda(0)N_\beta(0)N_\delta(0)N_\mu(0) \quad . \tag{7.32}$$

Therefore (7.21), (7.22), (7.24) and (7.27) allow us to write

$$[\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}N_{\beta}\sigma_{\alpha\gamma}N_{\gamma}]_{+}^{-} = \frac{1}{6}\eta'''(0)\varkappa^{3} + o(\varkappa^{3}) \quad \text{as } \varkappa \to 0 \;, \tag{7.33}$$

where $\eta'''(0)$ is given by (7.32). We observe that the jump in $\{\hat{W}(\underset{\sim}{F}) - F_{\alpha\beta}N_{\beta}\sigma_{\alpha\gamma}N_{\gamma}\}$ across the shock is of the third order in the shock strength $\varkappa$, which is as in the case of compressible elastic solids. This is analogous to the situation in gas dynamics where the entropy jump is of the third order in the appropriate shock strength.[1]

[1]See references cited in [4].

CHAPTER 8

8.1 An Illustrative Example

For the purpose of illustrating the results of the previous sections and demonstrating how in a particular case one could in fact obtain even more information than has been indicated, we now specialize our constitutive law. Consider the hypothetical class of homogeneous, isotropic, incompressible, elastic solids for which the plane strain elastic potential is given by

$$W(I) = \mu k_0^2 \left\{ 1 - \exp\left(- \frac{(I-2)}{2k_0^2} \right) \right\} \quad , \ \mu > 0 \ , \qquad k_0 > 0 \ . \tag{8.1}$$

One sees immediately from (8.1) that (2.40) is satisfied whence this class of materials has a positive shear modulus.

According to (3.32), we have in simple shear, the shear stress-amount of shear relation

$$\tau(k) = \mu k \exp\left(- \frac{k^2}{2k_0^2} \right) \ . \tag{8.2}$$

A sketch of the response curve in shear defined by (8.2) is shown in Fig. 2. The significant feature of this for our purposes is that $\tau'(k)$ is positive for all k in the interval $(-k_0, k_0)$ and is non-positive otherwise. The implications of this as far as the issue of the ellipticity of the displacement equations of equilibrium are concerned were observed in Section 3.[1]

We now turn to the issue of piecewise homogeneous elastostatic shocks. Suppose we are given the deformation gradient $\overset{+}{\underset{\sim}{F}}$, and hence (through (2.4), (6.5) and (6.9)) the associated value of β (say $\hat{\beta}$), and the pressure $\overset{+}{p}$ on $\overset{+}{\Pi}$. We look at the question of the existence of a corresponding elastostatic shock with spatial shock-line inclination $\hat{\phi}$ to the y_1-axis. $\hat{\phi}$ and $\hat{\beta}$ are held fixed in this discussion, and as noted previously we may assume $(\hat{\phi}, \hat{\beta})$ to be in $\overset{\circ}{\mathcal{Q}}$, with no loss of generality. We recall that a corresponding piecewise homogeneous elastostatic shock exists if and only if the function

$$h(x) = (b + cx)W'(\overset{+}{I} + 2bx + cx^2) - bW'(\overset{+}{I}) \quad , \tag{8.3}$$

where

$$\left.\begin{aligned} b = -\frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2}}\sin 2\hat{\phi} \quad (<0)\ , \quad c = \frac{1-\hat{\beta}\cos 2\hat{\phi}}{\sqrt{1-\hat{\beta}^2}} \quad (>0)\ , \\ \overset{+}{I} = \frac{2}{\sqrt{1-\hat{\beta}^2}} \quad (>2)\ , \end{aligned}\right\} \tag{8.4}$$

has a zero at some $x \neq 0$. Using (8.1) in (8.3), we find, for the type of materials under consideration, that

$$h(x) = \frac{\mu}{2}\exp\left(-\frac{(\overset{+}{I}-2)}{2k_0^2}\right)\left\{(b+cx)\exp\left(-\frac{(2bx+cx^2)}{2k_0^2}\right) - b\right\} \quad . \tag{8.5}$$

Case (i) Suppose $\hat{\phi}$ and $\hat{\beta}$ are such that

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[1] Shortly after Equation (3.33).

$$b^2 > ck_0^2 \ . \tag{8.6}$$

Then $h'(0) < 0$. One shows easily from (8.4), (8.5) and (8.6) that in this case $h(x)$ has a unique zero (in addition to the one at the origin) at $x = \varkappa$, where $\varkappa$ is a positive number and is such that $\int_0^{\varkappa} h(x)dx < 0$.

It follows that, corresponding to the homogeneous deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ and to the inclination $\hat{\phi}$ compatible with (8.4) and (8.6), there exists a unique piecewise homogeneous elasto-static shock with positive shock strength $\varkappa$. Furthermore, suppose this piecewise homogeneous shock is embedded in a quasi-static family of shocks. Then if at the instant when the family of shocks coincides with this given shock the shock-line $\mathscr{L}$ is translating into $\overset{+}{\Pi}$, it conforms with the dissipativity inequality.

<u>Case (ii)</u> Suppose $\hat{\phi}$ and $\hat{\beta}$ are such that

$$b^2 < ck_0^2 \ . \tag{8.7}$$

Then $h'(0) > 0$. In this case, it is easily verified by virtue of (8.4), (8.5) and (8.7) that $h(x)$ has a unique zero (in addition to the one at the origin) at $x = \varkappa$, where $\varkappa$ is a negative number such that $\int_0^{\varkappa} h(x)dx > 0$.

It follows that corresponding to the homogeneous deformation associated with $\hat{\beta}$ on $\overset{+}{\Pi}$ and the inclination $\hat{\phi}$ compatible with (8.4) and (8.7), there exists a unique piecewise homogeneous elastostatic shock with negative shock strength $\varkappa$. Furthermore, suppose this piecewise homogeneous shock is embedded in a quasi-static family of shocks. Then, if at the instant when the family of shocks coincides with this given shock the shock-line $\mathscr{L}$ is translating into $\overset{-}{\Pi}$, it conforms with the dissipativity inequality.

Case (iii) Suppose $\hat{\phi}$ and $\hat{\beta}$ are such that

$$b^2 = ck_0^2 \quad . \tag{8.8}$$

Then $h'(0) = 0$. In this case the only zero of $h(x)$ is at the origin, from which we conclude that if the homogeneous deformation on $\overset{+}{\Pi}$ is such that the associated value of $\hat{\beta}$ and the (proposed) inclination $\hat{\phi}$ conform with (8.4) and (8.8), then there is no corresponding piecewise homogeneous elastostatic shock.

These results are best visualized on the ϕ - β plane. Using (8.4) we have that

$$b^2 - ck_0^2 = \frac{-\hat{\beta}^2\cos^2 2\hat{\phi} + k_0^2\hat{\beta}\sqrt{1-\hat{\beta}^2}\cos 2\hat{\phi} + \left(\hat{\beta}^2 - k_0^2\sqrt{1-\hat{\beta}^2}\right)}{(1-\hat{\beta}^2)} \quad . \tag{8.9}$$

Let Γ be the curve in the first quadrant of the ϕ - β plane whose equation is

$$\Gamma : \ \beta^2\cos^2 2\phi - k_0^2\beta\sqrt{1-\beta^2}\cos 2\phi - \left(\beta^2 - k_0^2\sqrt{1-\beta^2}\right) = 0 \quad . \tag{8.10}$$

Γ separates $\overset{\circ}{\mathfrak{a}}$ into two regions as shown in Fig.3. Case (i) refers to points in the hatched open region shown there, while Case (ii) refers to points in the unhatched open region. Points on Γ refer to Case (iii). One finds that Γ has a minimum point at (ϕ_e, β_e) where

$$\phi_e = \frac{1}{2}\cos^{-1}\left(\frac{k_0}{\sqrt{4+k_0^2}}\right) \quad , \quad \beta_e = \frac{k_0\sqrt{k_0^2+4}}{k_0^2+2} \quad . \tag{8.11}$$

From (2.41), (3.33), (8.2) and (8.4) we find that the displacement equations of equilibrium are elliptic on $\overset{+}{\Pi}$, if and only if the deformation there is such that the associated value of β is less than β_e. Suppose that the given deformation on $\overset{+}{\Pi}$ is such that the displacement equations of equilibrium are non-elliptic there. Then $\hat{\beta} \geq \beta_e$. The spatial characteristics associated with this deformation are inclined to the λ_1-principal axis of $\overset{+}{\underset{\sim}{G}}$ at angles α, which because of (3.46), (8.1) and (8.4) are given by

$$\cos 2\alpha = \frac{k_0^2\sqrt{1-\hat{\beta}^2} \pm \left(\left\{(k_0^2+2)\sqrt{1-\hat{\beta}^2}-2\right\}\left\{(k_0^2-2)\sqrt{1-\hat{\beta}^2}-2\right\}\right)^{\frac{1}{2}}}{2\hat{\beta}} \quad . \tag{8.12}$$

Note however, that the equation of the curve Γ, (8.10), can alternatively be written as

$$\Gamma: \quad \cos 2\phi = \frac{k_0^2\sqrt{1-\beta^2} \pm \left(\left\{(k_0^2+2)\sqrt{1-\beta^2}-2\right\}\left\{(k_0^2-2)\sqrt{1-\beta^2}-2\right\}\right)^{\frac{1}{2}}}{2\beta} \quad . \tag{8.13}$$

It is immediately evident from a comparison of (8.12) and (8.13) that, the abscissa of the points on Γ corresponding to $\hat{\beta} \geq \beta_e$ give the spatial characteristic inclinations corresponding to the deformation associated with $\hat{\beta}$.

We now summarize our findings for the particular class of materials at hand. Corresponding to any given homogeneous deformation on $\overset{+}{\Pi}$ we can have a piecewise homogeneous elastostatic shock (provided $\overset{+}{\underset{\sim}{F}}$ is not proper orthogonal, i.e. $\hat{\beta} \neq 0$).

If, at the given deformation, the displacement equations of equilibrium are elliptic on $\overset{+}{\Pi}$, so that $\hat{\beta} < \beta_e$, the spatial shock-line may be inclined at any angle ϕ provided it is not parallel to the principal axes of $\overset{+}{\underset{\sim}{G}}$ (i.e. $\phi \neq 0$, $\frac{\pi}{2}$) . One can show that for such an elastostatic shock, the displacement equations of equilibrium are non-elliptic on $\bar{\Pi}$. Furthermore the corresponding shock strength is negative and a quasi-static motion from such a configuration is compatible with the dissipativity inequality if the shock moves into $\bar{\Pi}$.

On the other hand if the displacement equations of equilibrium are non-elliptic at the given deformation on $\overset{+}{\Pi}$, so that $\hat{\beta} \geq \beta_e$, the spatial shock-line may be inclined at any angle ϕ provided it is not parallel to the principal axes of $\overset{+}{\underset{\sim}{G}}$ nor parallel to the spatial characteristic directions associated with the deformation on $\overset{+}{\Pi}$. In this case the sign of the shock strength and the admissible direction of quasi-static motion depends on the specific shock-line inclination (see Fig. 3). In particular note that the admissible direction of quasi-static shock motion, for dissipativity, is governed solely by <u>whether the spatial shock-line inclination is between or outside the inclinations of the 2 spatial characteristics</u> (in the relevant quadrant) associated with the deformation on $\overset{+}{\Pi}$. The ellipticity or non-ellipticity of the displacement equations of equilibrium on $\bar{\Pi}$ also turn out to depend on the specific shock-line inclination.

PART II

AN EXAMPLE

CHAPTER 9

9.1 Introduction

During the study of some crack problems [12], [13], [14], Knowles and Sternberg encountered certain difficulties which suggested that the problem may not admit a classically smooth solution. In order to clarify this situation, a series of preliminary studies were undertaken (references [1] - [4]) in which they looked at the question of the failure of ellipticity of the displacement equations of equilibrium and the related issue of the existence of weak solutions involving elastostatic shocks. They further proposed a dissipativity requirement in an attempt to single out a physically acceptable solution from among the many available weak solutions. Part I of the present study is also in this same spirit.

In order to illustrate the occurrence of elastostatic shocks in a boundary value problem, we consider a problem in finite plane strain for a hollow circular cylinder. We examine the case in which the outer surface is held fixed while the inner surface is twisted circumferentially. The cylinder is presumed to be composed of a homogeneous, isotropic, incompressible, elastic solid, subject to certain restrictions on its strain energy density. In particular, the strain energy density is chosen such that a failure of ellipticity of the displacement equations of equilibrium can occur at some deformations.

We demonstrate that, while for both sufficiently large and small values of the prescribed twist the problem admits a unique smooth solution, there are certain intermediate ranges of the prescribed twist at which <u>no</u> classically smooth solutions exist. We then show that there are however, an infinite number of <u>weak</u> solutions involving elastostatic shocks in this range of the applied twist.

We then consider the quasi-static problem in which the prescribed twist is gradually changed in time, and we explore the consequences of the dissipation inequality. It turns out that enforcing this inequality fails to single out a unique weak solution.

In an attempt to clarify this issue of non-uniqueness, we examine the stability of the various equilibrium solutions against circumferential perturbations of arbitrary magnitude. It turns out that the classical energy criterion for stability, without reference to the dissipation inequality, picks out a unique solution to the boundary value problem at every value of the prescribed twist. In the discussion of the various issues outlined above, we restrict attention to configurations involving not more than one elastostatic shock. As a consequence of the stability criterion, it turns out that an equilibrium solution involving more than one shock is, in fact, unstable.

Ericksen [9] had previously discussed the equilibrium of a bar composed of a material whose stress response in uniaxial tension is qualitatively similar to the shear stress response in simple shear of the class of materials considered here. There is a striking similarity between his results and ours; in fact, certain aspects of our study of the stability of weak solutions were suggested by the arguments in [9].

In Section 10 we set up the classical problem governing the twisting of a hollow cylinder composed of an arbitrary homogeneous, incompressible, isotropic, elastic solid. We then discuss the particular class of materials with which we will be concerned. In Section 11 we determine the solutions of this problem and construct the associated torque — twist curves. For sufficiently small and large values of the prescribed twist, we have a unique smooth solution at which the displacement equations of equilibrium are elliptic. Depending on the details of the geometry and constitutive law, it is also possible to have a unique, non-elliptic, smooth solution at certain — but not all — values of the twist in the intermediate range. In all cases there are ranges of values of the prescribed twist for which we find no solution. We then prove that, in fact, no smooth solutions exist in these ranges of the prescribed twist.

We next set up and solve, in Section 12, the problem in its weak formulation. We now find a solution corresponding to every value of the prescribed twist, but unfortunately, many solutions corresponding to certain values.

In Section 13 we make use of the dissipation inequality in an unsuccessful attempt to extract a unique solution from among these many solutions to the boundary value problem. Finally, in Section 14, we examine the stability of each of the available solutions against circumferential perturbations of arbitrary magnitude. We find that at every value of the prescribed twist there is precisely one stable solution to the boundary value problem in its weak formulation. For sufficiently small and large values of the applied twist, this unique stable solution is smooth and elliptic. For all intermediate values, the unique stable solution involves an elastostatic shock and is elliptic.

CHAPTER 10

10.1 Formulation of Problem

Suppose that the open region $\mathcal{R}$ occupied by the interior of a body in its undeformed configuration is a hollow right circular cylinder of internal and external radii a and b, respectively. Let Π be the open middle cross-section of the cylinder $\mathcal{R}$, and let O be the center of the annular region Π.

Suppose the inner surface of the cylinder is rotated circumferentially through an angle ϕ_0, while the outer lateral surface is held fixed. We assume that the resulting deformation maps the point with cylindrical coordinates (r, θ, z) in the undeformed configuration onto the point with cylindrical coordinates (ρ, ψ, ξ), where

$$\left.\begin{aligned} \rho &= \hat{\rho}(r,\theta,z) = r \ , \\ \psi &= \hat{\psi}(r,\theta,z) = \theta + \phi(r) \ , \\ \xi &= \hat{\xi}(r,\theta,z) = z \ . \end{aligned}\right\} \qquad (10.1)$$

This describes a plane deformation in which each particle moves circumferentially through an angle $\phi(r)$. Suitable tractions are presumed to be applied on the ends of the cylinder so as to maintain such a state of plane strain.

The deformation (10.1) may be equivalently expressed as follows. Let X be a fixed rectangular cartesian coordinate frame with its origin at O and formed by the orthonormal base vectors $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ such that $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ are in the plane of Π and $\underset{\sim}{e}_3$ is normal to Π.

If $\underset{\sim}{y}$ is the position vector after deformation of the particle which was located at $\underset{\sim}{x}$ in the undeformed configuration, we can write (10.1) equivalently as

$$\left.\begin{aligned} y_1 &= x_1 \cos\phi(r) - x_2 \sin\phi(r) \ , \\ y_2 &= x_2 \cos\phi(r) + x_1 \sin\phi(r) \ , \\ y_3 &= x_3 \ , \end{aligned}\right\} \tag{10.2}$$

where

$$r = \sqrt{x_1^2 + x_2^2} \ . \tag{10.3}$$

Here y_i and x_i are the components of the vectors $\underset{\sim}{y}$ and $\underset{\sim}{x}$ in the frame X. We will temporarily assume that the local angle of twist $\phi(r)$ is twice continuously differentiable on (a, b).

It is convenient to express the field quantities at any point (r, θ, z) in terms of components in the rectangular cartesian coordinate frame X' which is obtained by rotating the frame X through an angle $\hat{\psi}(r, \theta, z)$ about the $\underset{\sim}{e}_3$-axis. The matrix of components of the deformation gradient tensor $\underset{\sim}{F} = \nabla \underset{\sim}{y}$ in the frame X' is easily computed from (10.2), (10.3) and the change of frame formula for tensors to be

$$\underset{\sim}{F}^{X'} = \begin{bmatrix} \cos\phi(r) & -\sin\phi(r) & 0 \\ \sin\phi(r) + r\phi'(r)\cos\phi(r) & \cos\phi(r) - r\phi'(r)\sin\phi(r) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{10.4}$$

Note that the matrix $\underset{\sim}{F}^{X'}$ may be decomposed as follows;

$$\underset{\sim}{F}^{X'} = \begin{bmatrix} 1 & 0 & 0 \\ r\,\phi'(r) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi(r) & -\sin\phi(r) & 0 \\ \sin\phi(r) & \cos\phi(r) & 0 \\ 0 & 0 & 1 \end{bmatrix} , \qquad (10.5)$$

which implies that locally the deformation (10.1) is composed of a rigid rotation through an angle ϕ about the $\underset{\sim}{e}_3$-axis followed by a simple shear parallel to the circumferential direction with an amount of shear $r\phi'(r)$. Set

$$k(r) = r\phi'(r) , \qquad (10.6)$$

so that $k(r)$ is the local amount of shear.

Equation (10.4) indicates that $\det \underset{\sim}{F} = 1$, so that the deformation (10.1) is locally volume preserving. From (10.4) and (10.6) we have the components of the left Cauchy-Green tensor $\underset{\sim}{G} = \underset{\sim}{F}\,\underset{\sim}{F}^T$

$$\underset{\sim}{G}^{X'} = \begin{bmatrix} 1 & k(r) & 0 \\ k(r) & 1+k^2(r) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \qquad (10.7)$$

The principal invariants of $\underset{\sim}{G}$ are found from (10.7) to be

$$\left.\begin{aligned} I_1 &= \operatorname{tr} \underset{\sim}{G} = 3 + k^2(r) , \\ I_2 &= \frac{1}{2}\left\{(\operatorname{tr}\underset{\sim}{G})^2 - \operatorname{tr}\underset{\sim}{G}^2\right\} = 3 + k^2(r) , \\ I_3 &= \det \underset{\sim}{G} = 1 . \end{aligned}\right\} \qquad (10.8)$$

Suppose that the body is composed of a homogeneous, isotropic, incompressible, elastic solid which possesses an elastic potential $W = \overset{*}{W}(I_1, I_2)$. W represents the strain energy density per unit undeformed volume. The constitutive law for the Cauchy stress tensor $\underset{\sim}{\tau}$ is then

$$\underset{\sim}{\tau} = 2\left(\frac{\partial \overset{*}{W}}{\partial I_1} + I_1 \frac{\partial \overset{*}{W}}{\partial I_2}\right)\underset{\sim}{G} - 2\frac{\partial \overset{*}{W}}{\partial I_2}\underset{\sim}{G}^2 - p\underset{\sim}{1} \tag{10.9}$$[1]

where $p(\underset{\sim}{y})$ is a scalar field arising because of the constraint of incompressibility. We suppose for the moment that $p(\underset{\sim}{y})$ is continuously differentiable on $\mathcal{R}$. Using (10.7), (10.8) and (10.9) we find that the stresses induced by the deformation (10.1) are

$$\left.\begin{aligned}
&\tau_{11}^{X'} = 2W'(I) - q \;, \\
&\tau_{12}^{X'} = \tau_{21}^{X'} = 2kW'(I) \;, \\
&\tau_{22}^{X'} = 2(1+k^2)W'(I) - q \;, \\
&\tau_{13}^{X'} = \tau_{31}^{X'} = \tau_{23}^{X'} = \tau_{32}^{X'} = 0 \;, \\
&\tau_{33}^{X'} = 2W'(I) + 2k^2 \frac{\partial \overset{*}{W}}{\partial I_2}(I+1,\; I+1) - q \;,
\end{aligned}\right\} \tag{10.10}$$

where we have set

$$I = 2 + k^2(r) \;, \quad W(I) = \overset{*}{W}(I+1,\; I+1) \quad \text{for } I \geq 2 \tag{10.11}$$

and

[1]See Truesdell and Noll [6].

$$q = p - 2\,\frac{\partial \overset{*}{W}}{\partial I_2} \ . \tag{10.12}$$

Since the pressure p might depend on the coordinates ψ and ξ in addition to ρ, it follows that the Cauchy stress tensor $\underset{\sim}{\tau}$ might depend on all three of ρ, ψ, and ξ.

It is suggestive to introduce the notation

$$\left.\begin{aligned} \tau_{\rho\rho} &= \tau^{X'}_{11} \ , \\ \tau_{\psi\psi} &= \tau^{X'}_{22} \ , \\ \tau_{\rho\psi} &= \tau_{\psi\rho} = \tau^{X'}_{12} \ . \end{aligned}\right\} \tag{10.13}$$

The equilibrium equation[1] in the axial direction is easily shown to be satisfied if and only if q does not depend on y_3 (and hence ξ). It follows from (10.10), that the Cauchy stress tensor $\underset{\sim}{\tau}$ is also independent of the axial coordinate ξ. The remaining two equilibrium equations now take the form

$$\frac{\partial \tau_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \tau_{\rho\psi}}{\partial \psi} + \frac{1}{\rho}(\tau_{\rho\rho} - \tau_{\psi\psi}) = 0 \ , \tag{10.14}$$

$$\frac{\partial \tau_{\rho\psi}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \tau_{\psi\psi}}{\partial \psi} + \frac{2}{\rho}\tau_{\rho\psi} = 0 \ . \tag{10.15}$$

From (10.1), (10.6), (10.10), (10.11) and (10.13) we see that $\tau_{\rho\psi}$ is independent of the coordinate ψ, whence (10.14) and (10.15) specialize to

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[1] Body forces are presumed to be absent.

$$\frac{\partial\tau_{\rho\rho}}{\partial\rho} + \frac{\tau_{\rho\rho} - \tau_{\psi\psi}}{\rho} = 0 \ , \tag{10.16}$$

$$\frac{d}{d\rho}\left(\rho^2\tau_{\rho\psi}\right) + \rho\,\frac{\partial\tau_{\psi\psi}}{\partial\psi} = 0 \ . \tag{10.17}$$

Integrating (10.17) with respect to ψ leads to

$$\tau_{\psi\psi} = -\left\{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho^2\tau_{\rho\psi}\right)\right\}\psi + c(\rho) \ , \tag{10.18}$$

where $c(\rho)$ is a constant of integration depending on ρ alone. It is apparent from (10.18) that $\tau_{\psi\psi}$ is single valued only if

$$\frac{d}{d\rho}\left(\rho^2\tau_{\rho\psi}\right) = 0 \ . \tag{10.19}$$

It now follows from (10.18) and (10.19) that $\tau_{\psi\psi}$, and hence q, p and $\tau_{\rho\rho}$ as well, are independent of the angular coordinate ψ. Using (10.10) and (10.13) in (10.16) and (10.19), we obtain the governing system of ordinary differential equations

$$\frac{d}{d\rho}\left\{\rho^3\phi'(\rho)W'(2+\rho^2\phi'^2(\rho))\right\} = 0 \ , \tag{10.20}$$

$$2\frac{d}{d\rho}W'(2+\rho^2\phi'^2(\rho)) - 2\rho\phi'^2(\rho)W'(2+\rho^2\phi'^2(\rho)) = \frac{dq}{d\rho} \ , \tag{10.21}$$

for $\phi(\rho)$ and $q(\rho)$.

On integrating (10.20) with respect to ρ we find that

$$\rho^3\phi'(\rho)W'(2+\rho^2\phi'^2(\rho)) = -\frac{T}{4\pi} \ , \tag{10.22}$$

where T is a constant of integration. Likewise, integration of (10.21) with respect to ρ and making use of (10.22) gives

$$q(\rho) = 2W'(2+\rho^2\phi'^2(\rho)) + \frac{T}{2\pi}\int_a^\rho \frac{\phi'(\xi)}{\xi^2}\,d\xi + q_0 \quad , \tag{10.23}$$

where q_0 is a constant.

It is convenient to define the scalar valued function $f(\cdot)$ by

$$f(k) = 2kW'(2+k^2) \quad \text{for} \ -\infty<k<\infty \quad . \tag{10.24}$$ [1]

It is readily seen that, if an incompressible, isotropic, elastic solid is subjected to a simple shear deformation, the shear stress corresponding to an amount of shear k is $f(k)$. Accordingly the function $f(\cdot)$ may be interpreted as the <u>shear stress response function in simple shear</u>.

Equation (10.22) can now be written as

$$f(\rho\phi'(\rho)) = -\frac{T}{2\pi\rho^2} \quad \text{on} \ (a,b) \ , \tag{10.25}$$

which together with the boundary conditions

$$\phi(a) = \phi_0 \ , \tag{10.26}$$

$$\phi(b) = 0 \ , \tag{10.27}$$

constitutes the boundary value problem for $\phi(\rho)$. We wish to find a function $\phi(\rho)$, continuous on $[a,b]$ and twice continuously differentiable on (a,b), and a real number T such that (10.25) - (10.27) hold. We will refer to such a solution as a <u>smooth solution</u>. Note that once $\phi(\rho)$

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[1] In Part I of this paper we called this function $\tau(\cdot)$. It is convenient to re-label it here.

has been so determined, (10.23) gives $q(\rho)$ directly.

Finally, note from (10.6), (10.10), (10.13) and (10.22) that $\tau_{\rho\psi}(\rho) = -\dfrac{T}{2\pi\rho^2}$ so that T is the torque per unit axial length of the cylinder acting on the inner surface, measured positive in the counter-clockwise sense.

10.2 A Particular Class of Constitutive Laws

We now describe the particular class of homogeneous, isotropic, incompressible, elastic materials to which we will restrict attention in this study. It is adequate for our purposes to specify the response of the material in simple shear alone. Observe from (10.24) that the plane strain elastic potential $W(I)$ is completely determined by the function f, so that the response in simple shear, in fact, determines completely the in-plane response in all plane deformations.

Equation (10.24) implies that f is an odd function

$$f(k) = -f(-k) \quad \text{for } -\infty<k<\infty \ . \tag{10.28}$$

We presume that

(i) f is continuously differentiable on $(-\infty,\infty)$,

(ii) f is positive on $(0,\infty)$, whence it follows from (10.28) that

$$kf(k)>0 \quad \text{for } k\neq 0 \ , \tag{10.29}$$

(iii) there exist real numbers k_1 and k_2 $(0<k_1<k_2<\infty)$ such that

$$\left.\begin{aligned} &f'(k_1) = f'(k_2) = 0 \ , \\ &f'(k)>0 \quad \text{for } 0\leq k<k_1 \ , \ k_2<k<\infty \ , \\ &f'(k)<0 \quad \text{for } k_1<k<k_2 \ , \end{aligned}\right\} \tag{10.30}$$

(iv) $f(k) \to \infty$ as $k \to \infty$.

Since f is an odd function it now follows that $f'(-k_1) = f'(-k_2) = 0$, $f'(k) > 0$ for $-\infty < k < -k_2$, $-k_1 < k \leq 0$, $f'(k) < 0$ on $-k_2 < k < -k_1$ and $f(k) \to -\infty$ as $k \to -\infty$. Therefore, the function $f(k)$ has local maxima at $k = k_1$, $-k_2$ and local minima at $k = k_2$, $-k_1$ and is monotone in between. A sketch of such a function f is shown in Fig. 4, where we have set

$$f(k_1) = \tau_{max} , \quad f(k_2) = \tau_{min} . \tag{10.31}$$

Note that necessarily

$$\left.\begin{aligned} |f(k)| &\leq \tau_{max} \quad \text{for} \quad |k| \leq k_1 , \\ \tau_{min} \leq |f(k)| &\leq \tau_{max} \quad \text{for} \quad k_1 \leq |k| \leq k_2 , \\ |f(k)| &\geq \tau_{min} \quad \text{for} \quad |k| \geq k_2 . \end{aligned}\right\} \tag{10.32}$$

Recall from Section 2.1 of Part I that an isotropic, incompressible, elastic solid conforms with the in-plane Baker-Ericksen inequality if and only if $W'(I) > 0$ for $I > 2$. By virtue of (10.24) this is equivalent to $kf(k) > 0$ for $k \neq 0$. Because of (10.29) we see that the class of materials under consideration satisfies this condition.

Moreover, we have because of Section 3.2 of Part I that in any plane deformation, the plane strain displacement equations of equilibrium are elliptic at some point if and only if the associated local amount of shear[1] is less than k_1 or greater than k_2 . In the context of the problem considered here, we have by virtue of (10.6) and (10.11) that the

[1]See Section 2.2 of Part I.

displacement equations of equilibrium[1] are elliptic, at a solution corresponding to $\phi(r)$ and some point, if and only if $|r\phi'(r)|$ is less than k_1 or greater than k_2 at that point.

It is clear that for the particular class of materials just described, f has no single-valued inverse. The restrictions of f to certain subintervals of $(-\infty, \infty)$, on the other hand, do have unique inverses. Let F_1, F_2 and F_3 be the functions defined by

$$\left.\begin{aligned} F_1(f(k)) &= k \quad \text{for } |k| \le k_1 , \\ F_2(f(k)) &= k \quad \text{for } k_1 \le |k| \le k_2 , \\ F_3(f(k)) &= k \quad \text{for } k_2 \le |k| < \infty . \end{aligned}\right\} \tag{10.33}$$

By virtue of (10.32), it follows that F_1, F_2 and F_3 are defined on $[-\tau_{max}, \tau_{max}]$, $[-\tau_{max}, -\tau_{min}] \cup [\tau_{min}, \tau_{max}]$ and $(-\infty, -\tau_{min}] \cup [\tau_{min}, \infty)$ respectively, and that they are continuously differentiable on the corresponding open intervals.

The following properties of the inverses F_i $(i = 1, 2, 3)$ can be easily verified; we list them here for subsequent reference.

$$\left.\begin{aligned} f(F_1(\tau)) &= \tau \quad \text{for } |\tau| \le \tau_{max} , \\ f(F_2(\tau)) &= \tau \quad \text{for } \tau_{min} \le |\tau| \le \tau_{max} , \\ f(F_3(\tau)) &= \tau \quad \text{for } \tau_{min} \le |\tau| < \infty , \end{aligned}\right\} \tag{10.34}$$

[1] i.e. the system of partial differential equations (2.3), (2.21).

$$\left.\begin{aligned} &F_1'(\tau)>0 \quad \text{for} \quad |\tau|<\tau_{max} \ , \\ &F_2'(\tau)<0 \quad \text{for} \quad \tau_{min}<|\tau|<\tau_{max} \ , \\ &F_3'(\tau)>0 \quad \text{for} \quad \tau_{min}<|\tau|<\infty \ , \end{aligned}\right\} \qquad (10.35)$$

$$\left.\begin{aligned} &F_1(\tau_{max}) = F_2(\tau_{max}) = k_1 \ , \\ &F_2(\tau_{max}) = F_3(\tau_{max}) = k_2 \ , \end{aligned}\right\} \qquad (10.36)$$

$$\left.\begin{aligned} |F_1(\tau)| \le k_1 \quad &\text{for} \quad |\tau| \le \tau_{max} \ , \\ k_1 \le |F_2(\tau)| \le k_2 \quad &\text{for} \quad \tau_{min} \le |\tau| \le \tau_{max} \ , \\ |F_3(\tau)| \ge k_3 \quad &\text{for} \quad |\tau| \ge \tau_{min} \ , \end{aligned}\right\} \qquad (10.37)$$

$$F_3(|\tau|)>F_2(|\tau|)>F_1(|\tau|) \quad \text{for} \quad \tau_{min}<|\tau|<\tau_{max} \ , \qquad (10.38)$$

$$\left.\begin{aligned} &F_i(-\tau) = -F_i(\tau) \ , && (10.39) \\ &\tau F_i(\tau)>0 \, , (\tau \neq 0) \ , && (10.40) \\ &|F_i(\tau)| = F_i(|\tau|) \ , && (10.41) \end{aligned}\right\} \quad \begin{array}{l} i = 1, 2, 3 \text{ and} \\ \text{for } \tau \text{ in the} \\ \text{appropriate} \\ \text{interval} \end{array}$$

$$F_3(\tau) \to \pm\infty \quad \text{as} \quad \tau \to \pm\infty \ . \qquad (10.42)$$

CHAPTER 11

11.1 Smooth Solutions

We now return to the task of solving (10.25) - (10.27) for the special class of materials described in the previous section. To this end, we first establish the following preliminary result.

Lemma: There does not exist a solution $\phi(r)$ in the class $C^2(a,b)$ to the differential equation

$$f(r\phi'(r)) = -\frac{T}{2\pi r^2} \,, \tag{11.1}$$

where f is a continuously differentiable function conforming with (10.28) - (10.30) and T is a constant, such that at some radius s, $a < s < b$,

$$s\phi'(s) = \pm k_1 \ \text{or} \ \pm k_2 \ . \tag{11.2}$$

Proof: Suppose that there is such a solution $\phi(r)$. Differentiating (11.1) with respect to r and setting $r = s$ leads to

$$f'(s\phi'(s))\left\{\phi'(s) + s\phi''(s)\right\} = \frac{T}{\pi s^3} \,,$$

which because of (10.28), (10.30) and (11.2) yields

$$T = 0 \ . \tag{11.3}$$

The differential equation (11.1) now reads

$$f(r\phi'(r)) = 0 \,,$$

which because of (10.28) and (10.29) implies that

$$\phi'(r) = 0 \quad \text{on} \quad (a,b) \ . \tag{11.4}$$

Equation (11.4), however, contradicts the assumption (11.2). This establishes the lemma.

Now suppose that the prescribed twist ϕ_0 is a number in the interval

$$-\int_a^b \frac{1}{\xi} F_1\left(\frac{a^2 \tau_{max}}{\xi^2}\right) d\xi \leq \phi_0 \leq \int_a^b \frac{1}{\xi} F_1\left(\frac{a^2 \tau_{max}}{\xi^2}\right) d\xi \ . \tag{11.5}$$

On using (10.26), (10.27), (10.35) and (10.36) in (11.5) we have

$$-\int_a^b \frac{k_1}{\xi} d\xi < -\int_a^b \phi'(\xi) d\xi < \int_a^b \frac{k_1}{\xi} d\xi \ , \tag{11.6}$$

whence

$$\int_a^b \frac{\xi\phi'(\xi) + k_1}{\xi} d\xi > 0 \ , \ \int_a^b \frac{\xi\phi'(\xi) - k_1}{\xi} d\xi < 0 \ . \tag{11.7}$$

By the preceeding lemma we know that $r\phi'(r) \neq \pm k_1$ on (a,b), by virtue of which (11.7) implies that

$$k_1 > r\phi'(r) > -k_1 \quad \text{on} \quad (a,b) \ , \tag{11.8}$$

since the integrands in (11.7) are continuous on (a,b). Therefore, if ϕ_0 is a number such that (11.5) holds, then necessarily the solution to (10.25) - (10.27) must satisfy (11.8). But because of (10.33) and (10.34), we see that, (10.25) and (11.8) hold if and only if

$$r\phi'(r) = F_1\left(-\frac{T}{2\pi r^2}\right) . \tag{11.9}$$

Integrating this and using the boundary condition (10.27) together with (10.39) leads to

$$\phi(r) = \int_r^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi . \tag{11.10}$$

On enforcing the boundary condition (10.26), we have from (11.10) that

$$\phi_0 = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi . \tag{11.11}$$

Finally, we verify that (11.11) determines a unique number T for every given number ϕ_0 in the interval defined by (11.5). To this end, define the function Φ_1 by

$$\Phi_1(T) = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{for} \quad |T| \le 2\pi a^2 \tau_{max} . \tag{11.12}$$

On differentiating (11.12) with respect to T and making use of (10.35), we find that

$$\Phi_1'(T) > 0 \quad \text{for} \quad |T| \le 2\pi a^2 \tau_{max} , \tag{11.13}$$

whence Φ_1 is monotonically increasing on $[-2\pi a^2\tau_{max}, 2\pi a^2\tau_{max}]$. Thus, if ϕ_0 is a number such that

$$\Phi_1(-2\pi a^2\tau_{max}) \le \phi_0 \le \Phi_1(2\pi a^2\tau_{max}) , \tag{11.14}$$

then, $\phi_0 = \Phi_1(T)$ defines a unique number T. Note that (11.14), because of (11.12), is identical to (11.5).

Therefore, we conclude that, if the prescribed twist ϕ_0 is in the interval defined by (11.5), Equation (11.11) determines a unique real number T, which together with (11.10) gives the corresponding unique smooth solution to (10.25) - (10.27).

In an entirely analogous manner, we can show that, if the prescribed twist ϕ_0 is in the interval

$$\int_a^b \frac{1}{\xi} F_2\left(\frac{a^2 \tau_{max}}{\xi^2}\right) d\xi \leq |\phi_0| \leq \int_a^b \frac{1}{\xi} F_2\left(\frac{b^2 \tau_{min}}{\xi^2}\right) d\xi \ , \tag{11.15}$$

then, the relation

$$\phi_0 = \int_a^b \frac{1}{\xi} F_2\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{11.16}$$

determines a unique real number T, which together with

$$\phi(r) = \int_r^b \frac{1}{\xi} F_2\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{11.17}$$

is the corresponding unique smooth solution to (10.25) - (10.27).

Similarly, if the prescribed twist ϕ_0 is in the interval

$$|\phi_0| \geq \int_a^b \frac{1}{\xi} F_3\left(\frac{b^2 \tau_{min}}{\xi^2}\right) d\xi \ , \tag{11.18}$$

the relation

$$\phi_0 = \int_a^b \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{11.19}$$

determines a unique real number T, which together with

$$\phi(r) = \int_r^b \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi \tag{11.20}$$

is the corresponding unique smooth solution to (10.25) - (10.27).

We will refer to (11.10), (11.17) and (11.20) as (smooth) <u>solution 1</u>, <u>solution 2</u> and <u>solution 3</u> respectively. Equations (11.11), (11.16) and (11.19) are the corresponding <u>torque-twist relations</u>. One sees readily from (11.10), (11.17), (11.20) and the discussion on ellipticity in Section 10.2, that the displacement equations of equilibrium are elliptic everywhere in Π at solution 1 and solution 3, and that they are non-elliptic at solution 2.

Because of (10.35) we see that

$$\int_a^b \frac{1}{\xi} F_2\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi \leq \int_a^b \frac{1}{\xi} F_2\left(\frac{b^2\tau_{min}}{\xi^2}\right) d\xi \ , \tag{11.21}$$

if and only if

$$b^2\tau_{min} \leq a^2\tau_{max} \ . \tag{11.22}$$

Accordingly, it is only when (11.22) holds that there are values of ϕ_0 in the interval (11.15), and consequently that solution 2 exists. In this paper, we will consider in detail the case when the dimensions of the tube and the constitutive law of the material are such that

$$\frac{a^2}{b^2} > \frac{\tau_{min}}{\tau_{max}} \ , \tag{11.23}$$

and trace the other cases (which are in fact less complicated) through footnotes. The end result turns out to be the same in all cases. For a given material, one could view (11.23) as requiring the thickness of the tube to be sufficiently small. Since $\tau_{\rho\psi}(\rho) = -\dfrac{T}{2\pi\rho^2}$, we have $\dfrac{\tau_{\rho\psi}(a)}{\tau_{\rho\psi}(b)} = \dfrac{b^2}{a^2}$ in any equilibrium configuration of the body irrespective of the magnitude of the applied twist. Thus (11.23) can be equivalently written as

$$\frac{\tau_{\rho\psi}(a)}{\tau_{\rho\psi}(b)} < \frac{\tau_{max}}{\tau_{min}} \ .$$

The torque-twist relations (11.11), (11.16) and (11.19) are sketched in Fig. 5. Clearly, because of material isotropy, these curves are anti-symmetric.

We observe from the preceding calculations, and also from Fig. 5, that we have not as yet found any solutions to (10.25) - (10.27) if the prescribed twist lies in one of the intervals

$$\left.\begin{aligned} &\int_a^b \frac{1}{\xi} F_1\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi < |\phi_0| < \int_a^b \frac{1}{\xi} F_2\left(\frac{a^2\tau_{max}}{\xi^2}\right) d\xi \ , \\ &\int_a^b \frac{1}{\xi} F_2\left(\frac{b^2\tau_{min}}{\xi^2}\right) d\xi < |\phi_0| < \int_a^b \frac{1}{\xi} F_3\left(\frac{b^2\tau_{min}}{\xi^2}\right) d\xi \ . \end{aligned}\right\} \tag{11.24}$$

We show in the next section that there are, in fact, no smooth solutions when the prescribed twist lies in these ranges.

11.2 Non-existence of Smooth Solutions

We now establish a sequence of lemmas leading to a result which is in fact stronger than the one claimed at the end of the last section. We will show that there is no solution $\phi(r)$ to (10.25) - (10.27) which is continuously differentiable if the prescribed twist ϕ_0 is in one of the intervals defined by (11.24).

> Lemma 1: There is no continuously differentiable solution $\phi(r)$ to the differential equation (11.1), where T is a constant and f is a continuously differentiable function conforming with (10.28) - (10.30), for which (11.2) holds at some radius s, $a<s<b$.

Proof: Assume that there exists such a solution $\phi(r)$ and suppose that

$$k(s) = s\phi'(s) = +k_1 \quad . \tag{11.25}$$

By hypothesis $k(r) = r\phi'(r)$ is continuous on (a, b) so that, in particular, it is continuous at $r = s$. Therefore, given any number $\epsilon>0$, there exists a number $\delta = \delta(\epsilon)>0$ such that $|k(s) - k(r)|<\epsilon$ for all r such that $|r - s|<\delta(\epsilon)$. Using (11.25) we may write this as

$$|k_1 - k(r)|<\epsilon \quad \text{for all} \quad |r - s|<\delta(\epsilon) \quad . \tag{11.26}$$

Recall that $f(k)$ has a local maximum at $k = k_1$, so that there is a number $\eta>0$ such that

$$f(k_1) \geq f(k) \quad \text{for} \quad |k_1 - k|<\eta \quad . \tag{11.27}$$

Combining (11.26) with (11.27) gives

$$f(k_1) \geq f(k(r)) \quad \text{for} \quad |r - s|<\delta(\eta) \quad ,$$

which on using (10.6), (11.1) and (11.25) leads to

$$\frac{-T}{2\pi s^2} \geq -\frac{T}{2\pi r^2} \quad \text{for } |r-s| < \delta(\eta) \ . \tag{11.28}$$

Note from (11.1) and (11.25) that $T = -2\pi s^2 f(k_1)$, whence $T < 0$. Equation (11.28) now requires that

$$r^2 \geq s^2 \quad \text{for } s - \delta(\eta) < r < s + \delta(\eta) \ , \ \delta(\eta) > 0 \ , \tag{11.29}$$

which is impossible. Consequently there cannot exist a solution $\phi(r)$ with the properties we assumed.

The cases $s\phi'(s) = -k_1, \pm k_2$ can be dealt with similarly.

Lemma 2: Suppose that there exists a continuously differentiable solution $\phi(r)$ to (11.25) - (11.27), where T is a constant and f is as in Lemma 1. Then

(i) $|r\phi'(r)| < k_1$ on (a, b) if and only if ϕ_0 is in the interval (11.5).

(ii) $k_1 < |r\phi'(r)| < k_2$ on (a, b) if and only if ϕ_0 is in the interval (11.29).

(iii) $|r\phi'(r)| > k_2$ on (a, b) if and only if ϕ_0 is in the interval (11.18).

Proof: Considering part (i), suppose that ϕ_0 is in the interval (11.5). By virtue of lemma 1, the steps leading from (11.5) to (11.8), go through even when $\phi(\cdot)$ is merely continuously differentiable. Thus necessarily $|r\phi'(r)| < k_1$ on (a, b).

Conversely, suppose that $|r\phi'(r)| < k_1$ on (a, b). It follows from (10.25) and (11.25) that

$$\frac{|T|}{2\pi r^2} < \tau_{max} \quad \text{on } (a, b) \ , \tag{11.30}$$

whence

$$|T| \leq 2\pi a^2 \tau_{max} \quad . \tag{11.31}$$

Since $|r\phi'(r)| < k_1$ on (a, b), we have because of (10.33) that (10.25) holds only if

$$r\phi'(r) = F_1\left(- \frac{T}{2\pi r^2}\right) \quad . \tag{11.32}$$

Integrating (11.32) and using (10.26), (10.27) and (10.39) gives

$$\phi_0 = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad , \tag{11.33}$$

which by virtue of (10.40) and (10.41) leads to

$$|\phi_0| = \int_a^b \frac{1}{\xi} F_1\left(\frac{|T|}{2\pi\xi^2}\right) d\xi \quad . \tag{11.34}$$

Since by (10.35) $F_1(\cdot)$ is a monotone increasing function, it follows from (11.31) and (11.34) that

$$|\phi_0| \leq \int_a^b \frac{1}{\xi} F_1\left(\frac{a^2 \tau_{max}}{\xi^2}\right) d\xi \quad . \tag{11.35}$$

which completes the proof of part (i) of the lemma. Parts (ii) and (iii) can be similarly established.

<u>Lemma 3:</u> There does <u>not</u> exist a continuously differentiable solution $\phi(r)$ to (10.25) - (10.27), where T is a constant and f is as in Lemma 1, if the prescribed twist ϕ_0 is in one of the intervals (11.24).

Proof: This result follows immediately from Lemmas 1 and 2. For, suppose that there is such a solution $\phi(r)$. It follows from Lemma 2 that we must have

$$r\phi'(r) = \pm k_1 \text{ or } \pm k_2 \quad \text{at some } r, \ a < r < b \ . \tag{11.36}$$

But Lemma 1 says that this is impossible.

We have thus shown that for certain ranges of the prescribed twist, there is no solution in the classical sense to the problem under consideration.

CHAPTER 12

12.1 Weak Formulation of Problem

There are some problems of considerable physical interest in which the field quantities do not vary smoothly through the body. Rice gives some examples of such problems in [5]. We have observed that the problem under consideration here has no smooth solution for certain ranges of the applied loading. One possibility, which we shall not consider, is that the tube buckles, possibly into some unsymmetric state of plane strain at such a loading. An alternative possibility is that the tube remains in a configuration of axisymmetric plane strain, but that now the field quantities are no longer smooth and exhibit certain discontinuities. This latter possibility is suggested by the observation in Section 10.2 that the displacement equations of equilibrium may suffer a loss of ellipticity at certain deformations for the material at hand.

We now relax the smoothness demanded of the local twist $\phi(r)$ and the pressure field $q(r)$, in the hope that this will enable us to explain what happens when the prescribed twist ϕ_0 is in one of the intervals (11.24).

To this end, let $\bar{r}$ be a number in the interval $[a, b]$. If in fact $a<\bar{r}<b$, we will now require that $\phi(r)$ be merely twice continuously differentiable on the intervals $(a, \bar{r})$ and $(\bar{r}, b)$ and continuous on $[a, b]$. The stress field and pressure field induced by the deformation (10.1) are now required only to be continuously differentiable on $(a, \bar{r})$ and $(\bar{r}, b)$ while the traction is presumed to be continuous at $r=\bar{r}$. Accordingly, we have admitted the possibility of the existence of a

cylindrical <u>elastostatic shock</u>[1] of radius $\bar{r}$ co-axial with the cylindrical region $\mathcal{R}$.

The global balance laws, which continue to be meaningful, now reduce to the same differential equations obtained in Section 10.1 on $(a, \bar{r})$ and $(\bar{r}, b)$, together with jump conditions at $r = \bar{r}$. Accordingly we now have

$$\left.\begin{aligned} &\frac{d}{dr}\left\{r^2 f(r\phi'(r))\right\} = 0 \ , \qquad\qquad &(12.1)\\ &2\frac{d}{dr}W'(2 + r^2\phi'^2(r)) \\ &\qquad -2r\phi'^2(r)W'(2 + r^2\phi'^2(r)) = \frac{dq}{dr} \ , \qquad\qquad &(12.2)\end{aligned}\right\} \text{ on } \begin{aligned} &a < r < \bar{r} \ , \\ &\bar{r} < r < b \ , \end{aligned}$$

instead of (10.20) and (10.21). On integrating (12.1) we have

$$f(r\phi'(r)) = \begin{cases} -\dfrac{\overset{-}{T}}{2\pi r^2} & \text{on } (a, \bar{r}) \ , \\[2ex] -\dfrac{\overset{+}{T}}{2\pi r^2} & \text{on } (\bar{r}, b) \ , \end{cases} \qquad (12.3)$$

where $\overset{+}{T}$ and $\overset{-}{T}$ are (not necessarily equal) constants. Because of (10.10), (10.13), (10.24) and (12.3) we have that

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[1] We formulate the problem in the case when a single elastostatic shock exists. We will find that this suffices for our purposes, and more importantly, that a configuration involving more than one shock is necessarily unstable (in a sense to be made precise).

$$\tau_{\rho\psi}(r) = \begin{cases} -\dfrac{\bar{T}}{2\pi r^2} & \text{on } (a, \overline{r}) \ , \\[2ex] -\dfrac{\overset{+}{T}}{2\pi r^2} & \text{on } (\overline{r}, b) \ . \end{cases} \tag{12.4}$$

Clearly the tractions are continuous across $r = \overline{r}$ if and only if

$$\tau_{\rho\psi}(\overline{r}-) = \tau_{\rho\psi}(\overline{r}+) \tag{12.5}$$

and

$$\tau_{\rho\rho}(\overline{r}-) = \tau_{\rho\rho}(\overline{r}+) \ . \tag{12.6}$$

Equations (12.4) and (12.5) lead to

$$\bar{T} = \overset{+}{T} \equiv T \ . \tag{12.7}$$

We therefore have the following problem governing the local twist $\phi(r)$. Given a number ϕ_0, find a function $\phi(r)$ which is continuous on $[a, b]$ and twice continuously differentiable on $(a, \overline{r})$ and $(\overline{r}, b)$, and numbers T, $\overline{r}$ with $a \le \overline{r} \le b$, such that

$$f(r\phi'(r)) = -\frac{T}{2\pi r^2} \quad \text{on } a < r < \overline{r} \ , \ \overline{r} < r < b \ , \tag{12.8}$$

$$\phi(a) = \phi_0 \ , \tag{12.9}$$

$$\phi(b) = 0 \ . \tag{12.10}$$

Integrating (12.2) leads to

$$q(r) = \begin{cases} 2W'(2 + r^2\phi'^2(r)) + \int_r^{\bar{r}} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi + \bar{q}_0 & \text{on } (a, \bar{r}) \\ \\ 2W'(2 + r^2\phi'^2(r)) - \int_{\bar{r}}^{r} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi + \overset{+}{q}_0 & \text{on } (\bar{r}, b) \end{cases} \tag{12.11}$$

which, together with (10.10) and (10.13), gives

$$\tau_{\rho\rho}(r) = \begin{cases} -\int_r^{\bar{r}} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi - \bar{q}_0 & \text{on } (a, \bar{r}) \ , \quad (12.12) \\ \\ \int_{\bar{r}}^{r} 2\xi\phi'^2(\xi)W'(2 + \xi^2\phi'^2(\xi))d\xi - \overset{+}{q}_0 & \text{on } (\bar{r}, b) \ . \quad (12.13) \end{cases}$$

We see from (12.12) and (12.13) that the traction continuity condition (12.6) holds if and only if

$$\overset{+}{q}_0 = \bar{q}_0 \ . \tag{12.14}$$

Once (12.8) - (12.10) has been solved for the function $\phi(r)$, equation (12.11) together with (12.14) directly gives the pressure field $q(r)$.

12.2 Weak Solutions

We first observe that the Lemma at the beginning of Section 11.1 continues to hold if we replace (a, b) by (r_1, r_2) where r_1 and r_2 are any two numbers such that $a \leq r_1 < r_2 \leq b$. This result, with the particular choices $r_1 = a$, $r_2 = \bar{r}$ and $r_1 = \bar{r}$, $r_2 = b$, leads to the conclusion that all admissible solutions of (12.8) are necessarily such that

$$r\phi'(r) \neq \pm k_1 \, , \pm k_2 \quad \text{on} \quad a < r < \overline{r} \, , \; \overline{r} < r < b \, . \tag{12.15}$$

Since $\phi'(r)$ is continuous on $(a, \overline{r})$ it now follows that any admissible solution to (12.8) must be such that $r\phi'(r)$ takes on values exclusively in one of the intervals $(-\infty, -k_2)$, $(-k_2, -k_1)$, $(-k_1, k_1)$, (k_1, k_2) or (k_1, ∞), at all points in $(a, \overline{r})$. The same must be true on $(\overline{r}, b)$. Therefore we see, because of (10.33) and (10.39), that (12.8) holds if and only if

$$r\phi'(r) = \begin{cases} -F_i\left(\dfrac{T}{2\pi r^2}\right) & \text{on } (a, \overline{r}) \, , \\[2ex] -F_j\left(\dfrac{T}{2\pi r^2}\right) & \text{on } (\overline{r}, b) \, , \end{cases} \tag{12.16}$$

for some fixed $i, j = 1, 2, 3$.

Integrating (12.16) and using the boundary conditions (12.9) and (12.10) leads to

$$\phi(r) = \begin{cases} \phi_0 - \displaystyle\int_a^r \frac{1}{\xi} F_i\left(\frac{T}{2\pi\xi^2}\right) d\xi & \text{on } [a, \overline{r}) \, , \\[3ex] \displaystyle\int_r^b \frac{1}{\xi} F_j\left(\frac{T}{2\pi\xi^2}\right) d\xi & \text{on } (\overline{r}, b] \, . \end{cases} \tag{12.17}$$

Finally, we require that

$$\phi_0 = \int_a^{\overline{r}} \frac{1}{\xi} F_i\left(\frac{T}{2\pi\xi^2}\right) d\xi + \int_{\overline{r}}^b \frac{1}{\xi} F_j\left(\frac{T}{2\pi\xi^2}\right) d\xi \, , \tag{12.18}$$

since the local twist $\phi(r)$ given by (12.17) is supposed to be continuous at $r=\bar{r}$.

Collecting the preceeding results, we come to the following conclusion. Given a real number ϕ_0, if there exist real numbers T and $\bar{r}$, $a \leq \bar{r} \leq b$, such that (12.18) holds for some fixed choice of the subscripts $i,j=1,2,3$, then (12.17) with this choice of T, $\bar{r}$, i and j is a solution to (12.8) - (12.10) at the given ϕ_0.

Clearly in the case when $i=j=1,2,3$, (12.17) and (12.18) describe the smooth solutions we obtained in Section 11.1. This is not surprising, since any smooth solution of (10.25) - (10.27) is also a solution of the problem in its weak formulation. Likewise, in the particular cases when $\bar{r}=a$ and $\bar{r}=b$, (12.17) and (12.18) are readily seen to reduce again to these same smooth solutions. A solution defined by (12.17) and (12.18) is therefore <u>not</u> smooth only if $i \neq j$ and $a<\bar{r}<b$.

The existence of a solution (12.17) corresponding to the prescribed value of the twist ϕ_0 is contingent upon the existence of numbers T and $\bar{r}$, $a \leq \bar{r} \leq b$, such that (12.18) holds. We now examine this latter issue. First we note that, since (12.18) furnishes only one scalar restriction on the two numbers T and $\bar{r}$, we expect that if there are values of T and $\bar{r}$ conforming with (12.18), then there would in fact be many such values. If, therefore, we <u>momentarily</u> imagine specifying <u>both</u> ϕ_0 and T, we may pose the following question: at each fixed choice of the subscripts $i,j=1,2,3$, $i \neq j$, for what values of the pair (ϕ_0, T) will (12.18) determine a value for $\bar{r}$, $a \leq \bar{r} \leq b$? We will, with no loss of generality, restrict attention to the first quadrant of the ϕ_0- T plane. We will show that for each fixed choice of the subscripts $i,j=1,2,3$, $i \neq j$, there is a <u>simply connected closed region A_{ij} in the</u>

<u>first quadrant of the ϕ_0- T plane such that (12.18) determines a value for $\bar{r}$ if and only if (ϕ_0, T) is in A_{ij}. Furthermore, this value of $\bar{r}$ is unique.</u>

To this end, define the functions ϕ_{ij}, $i,j=1,2,3$, $i\neq j$, by

$$\phi_{ij}(\bar{r},T)=\int_a^{\bar{r}}\frac{1}{\xi}F_i\left(\frac{T}{2\pi\xi^2}\right)d\xi+\int_{\bar{r}}^{b}\frac{1}{\xi}F_j\left(\frac{T}{2\pi\xi^2}\right)d\xi \quad \text{on } B_{ij}\,, \tag{12.19}$$

where the domains of definition B_{ij} of the functions ϕ_{ij} on the $\bar{r}$ - T plane are given by

$$\begin{aligned}
B_{31}&=\left\{(\bar{r},T)\,|\,a\le\bar{r}\le b\,,\ 2\pi\bar{r}^2\tau_{min}\le T\le 2\pi\bar{r}^2\tau_{max}\right\}\,,\\
B_{13}&=\left\{(\bar{r},T)\,|\,a\le\bar{r}\le b\,,\ 2\pi b^2\tau_{min}\le T\le 2\pi a^2\tau_{max}\right\}\,,\\
B_{21}&=\left\{(\bar{r},T)\,|\,a\le\bar{r}\le b\,,\ 2\pi\bar{r}^2\tau_{min}\le T\le 2\pi a^2\tau_{max}\right\}\,,\\
B_{12}&=\left\{(\bar{r},T)\,|\,a\le\bar{r}\le b\,,\ 2\pi b^2\tau_{min}\le T\le 2\pi a^2\tau_{max}\right\}\,,\\
B_{32}&=\left\{(\bar{r},T)\,|\,a\le\bar{r}\le b\,,\ 2\pi b^2\tau_{min}\le T\le 2\pi\bar{r}^2\tau_{max}\right\}\,,\\
B_{23}&=\left\{(\bar{r},T)\,|\,a\le\bar{r}\le b\,,\ 2\pi b^2\tau_{min}\le T\le 2\pi a^2\tau_{max}\right\}\,.
\end{aligned} \tag{12.20}$$
[1]

We now consider the <u>case $i=3$, $j=1$</u> in detail. For each fixed value of $\bar{r}$ in $[a,b]$, it follows from (12.19), (12.20) that $\phi_0=\phi_{31}(\bar{r},T)$ defines a segment of a smooth curve on the ϕ_0- T plane for $2\pi\bar{r}^2\tau_{min}\le T\le 2\pi\bar{r}^2\tau_{max}$. Therefore, we have a family of such curves on the ϕ_0- T plane, each corresponding to a different value of $\bar{r}$ in

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[1] Because of (10.32), (10.33) one sees that these are the largest possible domains of definition of the functions ϕ_{ij}. In the case when $a^2\tau_{max}<b^2\tau_{min}$ – so that (11.23) does not hold – we see that B_{13}, B_{12} and B_{23} are empty. In this case, therefore, solutions (12.17) with $(i,j)=(1,3)$, $(1,2)$ and $(2,3)$ do not exist.

$[a, b]$, and all of them having their end points on the curves $\phi_0 = \phi_{31}\left(\sqrt{\frac{T}{2\pi\tau_{min}}}, T\right)$ and $\phi_0 = \phi_{31}\left(\sqrt{\frac{T}{2\pi\tau_{max}}}, T\right)$. Since by (10.38) and (12.19) we have

$$\frac{\partial \phi_{31}}{\partial \bar{r}}(\bar{r}, T) = \frac{1}{\bar{r}}\left\{F_3\left(\frac{T}{2\pi\bar{r}^2}\right) - F_1\left(\frac{T}{2\pi\bar{r}^2}\right)\right\} > 0 \quad \text{on } B_{31}, \tag{12.21}$$

it follows that the different members of this family of curves do not intersect each other. Furthermore, a curve corresponding to a larger value of $\bar{r}$ lies to the right of a curve corresponding to a smaller value of $\bar{r}$. And finally, since ϕ_{31} depends continuously on $\bar{r}$, these curves span a simply connected region, A_{31}, in the ϕ_0-T plane. From the above discussion it follows that A_{31} is the closed region bounded by the curves $\phi_0 = \phi_{31}(a, T)$, $\phi_0 = \phi_{31}(b, T)$, $\phi_0 = \phi_{31}\left(\sqrt{\frac{T}{2\pi\tau_{min}}}, T\right)$ and $\phi_0 = \phi_{31}\left(\sqrt{\frac{T}{2\pi\tau_{max}}}, T\right)$. A sketch of this region, together with the spanning family of curves, is shown in Fig. 6(i). The fact that a curve corresponding to a larger value of $\bar{r}$ is to the right of a curve associated with a smaller value of $\bar{r}$ is indicated in Fig. 6(i) by the arrow labelled "direction of increasing $\bar{r}$". Since there is exactly one of these curves passing through any point in A_{31}, it follows that there is a unique number $\bar{r}$ associated with every point (ϕ_0, T) in A_{31}, such that $\phi_0 = \phi_{31}(\bar{r}, T)$. This is what we set out to establish. We may express this analytically as follows: there exists a function $\hat{r}_{31}$, defined on A_{31}, such that $\bar{r}$ determined by

$$\bar{r} = \hat{r}_{31}(\phi_0, T) \tag{12.22}$$

conforms with $\phi_0 = \phi_{31}(\bar{r}, T)$, i.e. $\phi_0 = \phi_{31}\left(\hat{r}_{31}(\phi_0, T), T\right)$ on A_{31}.

Summarizing the results for this case, we have that, if ϕ_0 and T are numbers such that (ϕ_0, T) is in A_{31}, (the region PQRS in Fig. 6(i)), then there is a unique number $\overline{r}$, $a \le \overline{r} \le b$, such that (12.18) holds (with $i=3$, $j=1$). Equation (12.17), with these values of T, $\overline{r}$, i and j, is a solution to (12.8) - (12.10) at that value of ϕ_0.

The other cases — corresponding to the remaining choices of the subscripts i, j — may be likewise examined. In each case we find a simply connected closed region A_{ij}, shown in Figs. 6-8, such that, if ϕ_0 and T are numbers with (ϕ_0, T) in A_{ij}, then there is a unique number $\overline{r}$, $a \le \overline{r} \le b$, such that (12.18) holds for that choice of i, j. Equation (12.17) then provides the corresponding solution $\phi(r)$. Accordingly, in each case there exist functions $\hat{r}_{ij}$ defined on A_{ij}, such that

$$\overline{r} = \hat{r}_{ij}(\phi_0, T) \quad \text{on } A_{ij} \tag{12.23}$$

conforms with $\phi_0 = \phi_{ij}(\overline{r}, T)$.

The composite torque-twist diagram, wherein all of these admissible regions A_{ij} together with the torque-twist curves for the smooth solutions are sketched on one figure, is shown in Fig. 9. We observe that the admissible regions A_{ij} "fit" appropriately between the torque-twist curves associated with the smooth solutions (Fig. 5). Therefore corresponding to <u>any</u> given value of the twist ϕ_0 we now have a solution. However, we are now faced with the unsatisfactory situation in which there is an infinite number of admissible solutions at certain values of the prescribed twist ϕ_0.

We observe from Fig. 9 that at sufficiently small twists ϕ_0 $(\le \phi_S)$ and at sufficiently large twists ϕ_0 $(\ge \phi_Q)$ we have a unique solution, which is smooth. When the prescribed twist ϕ_0 is in one of the intervals $\phi_p < \phi_0 < \phi_M$, $\phi_N < \phi_0 < \phi_R$, we have an infinite number of solutions, all of which are weak solutions. In the remaining intervals, we have an infinite number of solutions, one of which is smooth, all the rest being weak solutions.

Observe from Figure 9 that even a knowledge of both ϕ_0 and T may be insufficient, in some cases, to determine a unique solution. For example, there are four solutions corresponding to any point in PMNK, one for each of the pairs, $(i, j) = (1, 2)$, $(2, 1)$, $(1, 3)$ and $(3, 1)$. We remark that at any point (ϕ_0, T) on PS or RQ there is in fact only one solution — the smooth one. One sees this from (12.17), (12.18), since all of the weak solutions at such a point have either $\bar{r} = a$ or $\bar{r} = b$ (see Figs. 6-8). Likewise, at any point on MN we only have smooth solution 2 or the weak solutions $(i, j) = (1, 3)$, $(3, 1)$.

Finally, we observe that it is convenient to visualize the various solutions as follows. Consider for example a weak solution with $i = 3$, $j = 1$. Let $\bar{r}$ denote the radius of the associated shock. Let A, B, C and D be points on a radial line in the cross-section Π of the tube in the undeformed configuration, see Fig. 10(a), such that A and D are at the inside and outside boundaries respectively, while B and C are points just inside and outside the shock-line. The solution at hand is given by (12.17) with $i = 3$, $j = 1$. If we use this to compute $r\phi'(r)$ and then plot the points with coordinates $\left(|r\phi'(r)|, \frac{T}{2\pi r^2}\right)$ (suppose $T > 0$) for each r in the intervals $a \leq r < \bar{r}$, $\bar{r} < r \leq b$, we obtain the curves A_1B_1 and C_1D_1 (typically) shown in Fig. 10(b). The graph of $f(k)$ has been superimposed on this diagram. The abscissa of any point on A_1B_1 or C_1D_1 gives the value of the local amount of shear $|r\phi'(r)|$ at the corresponding point in the tube, while the ordinate gives the corresponding shear stress $|\tau_{\rho\psi}|$. Observe from Fig. 10(b) how the local amount of shear varies continuously on either side of the shock but suffers a jump discontinuity across it. The shear stress $\tau_{\rho\psi}$, on the other hand, is seen to vary continuously throughout the tube. If we refer to the portions of the curve $f(k)$ vs. k between $0 \le k \le k_1$, $k_1 \le k \le k_2$ and $k_2 \le k < \infty$ as the first, second and third branches of $f(k)$ respectively, we see that this solution $(i = 3, j = 1)$ is associated with the third and first branches of $f(k)$, with the region inside the shock-line associated with the former branch. In general, <u>the weak solution (i, j) is associated with the i^{th} and j^{th} branches of $f(k)$, with the part of the tube inside the shock-line corresponding to the i^{th} branch.</u>

We see from this and Section 10.2 that the <u>type</u> of these weak solutions is mixed, in general. The displacement equations of equilibrium are elliptic on that part of Π for which $a < r < \bar{r}$ and non-elliptic where $\bar{r} < r < b$, at solutions with $(i, j) = (1, 2)$, $(3, 2)$, while they are elliptic where $\bar{r} < r < b$ and non-elliptic where $a < r < \bar{r}$, at the solutions $(i, j) = (2, 1)$, $(2, 3)$. In the case of the solutions corresponding to $(i, j) = (1, 3)$, $(3, 1)$, the displacement equations of equilibrium are elliptic everywhere in Π where $r \neq \bar{r}$.

CHAPTER 13

13.1 Dissipativity

The lack of uniqueness encountered in the preceeding section is not unexpected, since we had enlarged the admissible class of solutions there. In such circumstances, it is usually the case that not all of the solutions admitted by the differential equations are physically reasonable. In gas dynamics, for example, there are problems in which the differential equations admit solutions which are unacceptable since they are associated with a decrease of entropy. It is essential therefore to introduce additional criteria which will single out a physically admissible solution.

Knowles and Sternberg proposed such a criterion in [3], in the context of finite elastostatics, which they referred to as the dissipativity inequality. A thermodynamic motivation for this inequality, stemming from the Clausius-Duhem inequality, was given by Knowles in [4]. The dissipativity inequality is essentially an expression of the physically reasonable idea that the rate at which elastic energy is being stored in any part of the body, in some quasi-static process, cannot exceed the rate at which work is being done on that part.

We now examine the implications of the dissipativity inequality in the context of the present problem. While we could specialize the general dissipativity inequality given in [4] to our problem, it is illustrative (and equally easy) to derive it from first principles.

We now consider a quasi-static time dependent family of equilibrium solutions. The time t merely plays the role of a history parameter and no inertia effects are accounted for. Accordingly, we are concerned with a one parameter family of functions $\phi(r,t)$, depending on the parameter t in some time interval $\mathcal{T}$, such that at each t in $\mathcal{T}$, $\phi(r,t)$ is a solution to (12.8) - (12.10). The torque, twist and shock radius are all time dependent now, and we write $T(t)$, $\phi_0(t)$ and $\bar{r}(t)$. It is convenient to set

$$\left.\begin{aligned} \overset{+}{\Pi}(t) &= \{ r \,|\, \bar{r}(t) < r < b \} \;, \\ \overset{-}{\Pi}(t) &= \{ r \,|\, a < r < \bar{r}(t) \} \;, \end{aligned}\right\} \text{ for } t \text{ in } \mathcal{T} \;. \tag{13.1}$$

Then $\phi(\cdot,t)$ is continuous on $[a,b]$ and twice continuously differentiable on $\overset{+}{\Pi}$ and $\overset{-}{\Pi}$ at each t in $\mathcal{T}$. Furthermore

$$f(r\phi_r(r,t)) = -\frac{T(t)}{2\pi r^2} \quad \text{on } \overset{+}{\Pi}(t) \text{ and } \overset{-}{\Pi}(t) \;, \tag{13.2}$$ [1]

$$\phi(a,t) = \phi_0(t) \;, \tag{13.3}$$

$$\phi(b,t) = 0 \;, \tag{13.4}$$

at each t in $\mathcal{T}$. Here $\phi_0(t)$ is the prescribed twist, and we suppose it to be continuous and piecewise continuously differentiable on $\mathcal{T}$. In certain discussions, as we observed previously, it will be temporarily necessary to imagine that $T(t)$ is also specified. In such circumstances, we presume $T(t)$ to possess the same smoothness as $\phi_0(t)$ on $\mathcal{T}$.

It is convenient to set

$$k(r,t) = r\phi_r(r,t) \;, \tag{13.5}$$

---

[1] We use the notation $\phi_r = \dfrac{\partial\phi(r,t)}{\partial r}$ and $\phi_t = \dfrac{\partial\phi(r,t)}{\partial t}$.

$$\overset{+}{k}(t) = \bar{r}(t)\phi_r(\bar{r}(t)+, t) \quad , \tag{13.6}$$

$$\bar{k}(t) = \bar{r}(t)\phi_r(\bar{r}(t)-, t) \quad , \tag{13.7}$$

with the understanding that when $\bar{r}(t) = b$ we take[1] $\bar{r}(t)+ = b$ and when $\bar{r}(t) = a$ we take $\bar{r}(t)- = a$. $\overset{+}{k}$ and $\bar{k}$ represent the instantaneous local amounts of shear at points just outside and inside the shock, respectively.

We will now require that at each instant in $\mathcal{T}$, the rate at which the external forces on the tube are doing work should not be less than the rate of increase of the stored energy, i.e. we demand that

$$T(t)\frac{d}{dt}\phi_0(t) \geq \frac{d}{dt}\int_a^b W(2+k^2(r,t))2\pi r dr \qquad \text{for all } t \text{ in } \mathcal{T} . \tag{13.8}$$

We may evaluate the right hand side of (13.8), using (13.6) and (13.7), as follows.

$$\frac{d}{dt}\int_a^b W(2+k^2(r,t))2\pi r dr$$

$$= \frac{d}{dt}\int_a^{\bar{r}(t)} W(2+k^2(r,t))2\pi r dr + \frac{d}{dt}\int_{\bar{r}(t)}^b W(2+k^2(r,t))2\pi r dr \quad ,$$

$$= \{W(2+\bar{k}^2) - W(2+\overset{+}{k}{}^2)\}2\pi\bar{r}(t)\frac{d\bar{r}}{dt}(t) + \int_a^b 4\pi r k\frac{\partial k}{\partial t}W'(2+k^2)dr \quad . \tag{13.9}$$

Using (10.24), (13.2) and (13.5) in (13.9), gives

[1]This is admissible since we observed in Section 12.2 that when $\bar{r} = a$ or b, the solution is in fact smooth. Thus ϕ_r exists there.

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr$$

$$= \{ W(2+\bar{k}^2) - W(2+\overset{+}{k}{}^2)\} 2\pi\bar{r}(t)\frac{d\bar{r}}{dt} - T(t)\int_a^b \phi_{rt}(r,t)dr \quad . \tag{13.10}$$

However, because of (13.3) and (13.4), we have

$$\int_a^b \phi_{rt}(r,t)dr = \phi_t(b,t) - \phi_t(\bar{r}(t)+,t) + \phi_t(\bar{r}(t)-,t) - \phi_t(a,t) \quad ,$$

$$= -\frac{d}{dt}\phi_0(t) - \phi_t(\bar{r}(t)+,t) + \phi_t(\bar{r}(t)-,t) \quad , \tag{13.11}$$

so that we may write (13.10) as

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr = \{ W(2+\bar{k}^2) - W(2+\overset{+}{k}{}^2)\}\, 2\pi\bar{r}(t)\frac{d\bar{r}}{dt} + T(t)\frac{d}{dt}\phi_0(t)$$

$$+ T(t)\{\phi_t(\bar{r}(t)+,t) - \phi_t(\bar{r}(t)-,t)\} \quad . \tag{13.12}$$

Since the displacements are continuous across the shock, we have

$$\phi(\bar{r}(t)+,t) = \phi(\bar{r}(t)-,t) \qquad \text{for } t \text{ in } \mathcal{T} \quad , \tag{13.13}$$

which when differentiated with respect to t leads to

$$\phi_r(\bar{r}(t)+,t)\frac{d\bar{r}}{dt}(t) + \phi_t(\bar{r}(t)+,t) = \phi_r(\bar{r}(t)-,t)\frac{d\bar{r}}{dt}(t) + \phi_t(\bar{r}(t)-,t) \quad . \tag{13.14}$$

Using (13.6), (13.7) and (13.14) in (13.12) gives

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr = \{W(2+\bar{k}^2) - W(2+\overset{+}{k})^2\}2\pi\bar{r}\frac{d\bar{r}}{dt} + T(t)\frac{d}{dt}\phi_0(t)$$

$$+\frac{1}{\bar{r}(t)}T(t)\{\bar{k}(t) - \overset{+}{k}(t)\}\frac{d\bar{r}}{dt}(t) \quad , \tag{13.15}$$

which because of (13.2), (13.6) can be written as

$$\frac{d}{dt}\int_a^b W(2+k^2)2\pi r\,dr = T(t)\frac{d}{dt}\phi_0(t) + \{W(2+\bar{k}^2) - W(2+\overset{+}{k}{}^2)\}2\pi\bar{r}\frac{d\bar{r}}{dt}$$

$$+ 2\pi\bar{r}f(\overset{+}{k})(\overset{+}{k} - \bar{k})\frac{d\bar{r}}{dt}(t) \quad . \qquad (13.16)^{[1]}$$

The dissipativity requirement (13.8) can now be written as

$$\{W(2+\overset{+}{k}{}^2) - W(2+\bar{k}^2) - f(\overset{+}{k})(\overset{+}{k} - \bar{k})\}2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{on } \mathcal{T} \;,$$

or alternatively, because of (10.24), as

$$\left\{\int_{\bar{k}}^{\overset{+}{k}} f(\xi)d\xi - (\overset{+}{k} - \bar{k})f(\overset{+}{k})\right\}2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for all } t \text{ in } \mathcal{T} \;. \tag{13.17}$$

This is the form of the <u>dissipativity inequality</u> that we shall find useful.

It follows from the results of Section 12.2 that all admissible quasi-static families of equilibrium solutions are of the form

[1]Note from (13.2) that $f(\overset{+}{k}) = f(\bar{k})$.

$$\phi(r,t) = \begin{cases} \phi_0(t) - \int_a^r \frac{1}{\xi} F_i\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi & \text{on } \bar{\Pi}(t) , \\[2ex] \int_r^b \frac{1}{\xi} F_j\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi & \text{on } \overset{+}{\Pi}(t) , \end{cases} \tag{13.18}$$

subject to the restriction

$$\phi_0(t) = \int_a^{\bar{r}(t)} \frac{1}{\xi} F_i\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi + \int_{\bar{r}(t)}^b \frac{1}{\xi} F_j\left(\frac{T(t)}{2\pi\xi^2}\right) d\xi , \tag{13.19}$$

for some $i, j = 1, 2, 3$ and for t in $\mathcal{T}$. We now proceed to apply the dissipativity inequality (13.17) to the various families of solutions represented by (13.18), (13.19).

We first note that, if at some instant t we have a smooth solution, then (13.17) holds at that instant by virtue of the continuity of ϕ_r, i.e. since $\bar{k} = \overset{+}{k}$. Therefore, we may restrict attention to the cases for which $i \neq j$ in (13.18), (13.19), and to times in $\mathcal{T}$ for which

$$a < \bar{r}(t) < b . \tag{13.20}$$[1]

Equations (13.6), (13.7) and (13.18) now give

$$\left.\begin{aligned} \overset{+}{k}(t) &= -F_j\left(\frac{T(t)}{2\pi\bar{r}^2(t)}\right) , \\[2ex] \bar{k}(t) &= -F_i\left(\frac{T(t)}{2\pi\bar{r}^2(t)}\right) , \end{aligned}\right\} \tag{13.21}$$

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[1] See discussion following (12.18).

so that we may use (10.28), (10.39), (10.40), (10.41) (13.2), (13.6) and (13.21) to write (13.17) as

$$2\pi\bar{r}(t)\,\frac{d\bar{r}}{dt}(t)\int_{F_i\left(\frac{|T(t)|}{2\pi\bar{r}^2(t)}\right)}^{F_j\left(\frac{|T(t)|}{2\pi\bar{r}^2(t)}\right)} \left\{f(\xi) - \frac{|T(t)|}{2\pi\bar{r}^2(t)}\right\} d\xi \geq 0 \quad , \tag{13.22}$$

for all t in $\mathcal{T}$ for which (13.20) holds.

It is convenient to define the functions $A_1(\cdot)$ and $A_2(\cdot)$ by

$$A_1(\tau) = \int_{F_1(\tau)}^{F_2(\tau)} \left\{f(\xi) - \tau\right\} d\xi \quad \text{for } \tau_{min} \leq \tau \leq \tau_{max} \quad , \tag{13.23}$$

$$A_2(\tau) = \int_{F_2(\tau)}^{F_3(\tau)} \left\{\tau - f(\xi)\right\} d\xi \quad \text{for } \tau_{min} \leq \tau \leq \tau_{max} \quad . \tag{13.24}$$

These functions A_1 and A_2 have the following geometrical interpretation. If in Fig. 4 we draw a line parallel to the k-axis at a distance τ above it $(\tau_{min} \leq \tau \leq \tau_{max})$, then $A_1(\tau)$ and $A_2(\tau)$ are the areas of the two loops formed. It follows from this that

$$A_1(\tau_{max}) = A_2(\tau_{min}) = 0 \quad , \tag{13.25}$$

$$A_1(\tau) > 0 \quad \text{for } \tau_{min} \leq \tau < \tau_{max} \quad , \tag{13.26}$$

$$A_2(\tau) > 0 \quad \text{for } \tau_{min} < \tau \leq \tau_{max} \quad . \tag{13.27}$$

Using (13.23) and (13.24) in (13.22) leads to

$$\left\{A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) - A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right)\right\} 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for} \quad (i,j) = (3,1) \ , \tag{13.28}$$

$$-\left\{A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) - A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right)\right\} 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for} \quad (i,j) = (1,3) \ , \tag{13.29}$$

$$-A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for} \quad (i,j) = (2,1) \ , \tag{13.30}$$

$$A_1\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for} \quad (i,j) = (1,2) \ , \tag{13.31}$$

$$A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for} \quad (i,j) = (3,2) \ , \tag{13.32}$$

$$-A_2\left(\frac{|T|}{2\pi\bar{r}^2}\right) 2\pi\bar{r}\frac{d\bar{r}}{dt} \geq 0 \quad \text{for} \quad (i,j) = (2,3) \ , \tag{13.33}$$

in each of the different cases.

Now consider, for example, the <u>case $(i,j) = (2,1)$</u>, i.e. suppose that for all times sufficiently close to some t_1 in $\mathcal{T}$, the quasi-static family of solutions (13.18) has $i = 2$, $j = 1$. We then have from (10.32), (10.33), (13.1), (13.18), (13.26) and (13.30) that the dissipativity inequality is satisfied at a time t_1 for which (13.20) holds if and only if

$$\frac{d\bar{r}}{dt}(t_1) \leq 0 \ . \tag{13.34}$$

As previously observed, in the event that (13.20) does not hold, so that

$$\bar{r}(t_1) = a \ \text{or} \ b \ , \tag{13.35}$$

then the dissipativity inequality holds without need for any restrictions such as (13.34). The meaning of these restrictions is most transparent when viewed in the torque-twist diagram (see Fig. 11). With no loss of generality we suppose that $T(t)$ and $\phi_0(t)$ are non-negative for all times in $\mathcal{T}$. We shall refer to the piecewise smooth oriented[1] curve Γ in the torque-twist plane defined by $\phi_0 = \phi_0(t)$, $T = T(t)$ for t in $\mathcal{T}$ as the loading path. By hypothesis, for all values of t sufficiently close to t_1, the loading path Γ lies in A_{21}. Let $Z = (\phi_0(t_1), T(t_1))$ be the point on Γ corresponding to $t = t_1$.

Recall[2] that the region A_{21} is spanned by a one parameter family of curves $\phi_0 = \phi_{21}(\bar{r}, T)$, $a \leq \bar{r} \leq b$, and that a member of this family of curves corresponding to a larger value of the parameter $\bar{r}$ lies to the right of a curve corresponding to a smaller value. Let C be the particular member of this family with equation $\phi_0 = \phi_{21}(\bar{r}(t_1), T)$, so that C passes through Z., (see Fig. 11). It follows that the shock radius $\bar{r}$ corresponding to any point in A_{21} to the right of C is greater than $\bar{r}(t_1)$, while at a point to the left of C, it is less than $\bar{r}(t_1)$. Therefore, dissipativity — (13.34) — requires that the loading path Γ should be oriented at Z in such a way that it does not point to the right of C, provided Z is not a point on PS or MN. This is shown in Fig. 7(i) as well, wherein the concentrated source of arrows indicates the admissible orientations of a loading path through a typical point. This is true for all points in A_{21} except for those which lie on PS and MN. At a point on these curves the loading path may be arbitrarily

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[1] Γ is oriented in the direction of increasing time.

[2] See Fig. 7(i).

oriented, by virtue of (13.35).

Clearly we can analyze the other cases in an entirely analogous manner. We find that dissipativity is essentially equivalent to

$$\frac{d\bar{r}}{dt} \geq 0 \quad \text{if} \quad (i,j) = (1,2) \ , \tag{13.36}$$

$$\frac{d\bar{r}}{dt} \geq 0 \quad \text{if} \quad (i,j) = (3,2) \ , \tag{13.37}$$

$$\frac{d\bar{r}}{dt} \leq 0 \quad \text{if} \quad (i,j) = (2,3) \ , \tag{13.38}$$

and these are geometrically interpreted in Figs. 7 and 8 as before. The only exceptions to (13.36) - (13.38) are respectively at points on the curves PK , MN and MN, RQ and MN , LR , whereat the orientation is arbitrary.

Equations (13.28) and (13.29) — i.e. the cases $(i,j) = (3,1)$ and $(1,3)$ — can also be similarly examined, taking care now to note that $\{A_2(\tau) - A_1(\tau)\}$ is not always of the same sign. If we set

$$A(\tau) = A_2(\tau) - A_1(\tau) \quad \text{for} \quad \tau_{min} \leq \tau \leq \tau_{max} \ , \tag{13.39}$$

where A_1 and A_2 are as defined previously, we find because of (13.23)-(13.27) and (13.39) that

$$A(\tau_{min}) < 0 \ , \ A(\tau_{max}) > 0 \ , \tag{13.40}$$

$$\frac{dA}{d\tau}(\tau) > 0 \quad \text{for} \quad \tau_{min} \leq \tau \leq \tau_{max} \ . \tag{13.41}$$

Since $A(\tau)$ is continuous, it follows from (13.40) and (13.41) that there is a unique number τ_c in (τ_{min}, τ_{max}), such that

$$A(\tau_c) = 0 \quad , \tag{13.42}$$

$$A(\tau) > 0 \ \text{ on } \ (\tau_c, \tau_{max}] \ , \ A(\tau) < 0 \ \text{ on } \ [\tau_{min}, \tau_c) \ . \tag{13.43}$$

The number τ_c is shown in Fig. 4, where, since $A_1(\tau_c) = A_2(\tau_c)$, the two hatched regions are of equal area. The dissipativity conditions (13.28) and (13.29), because of (13.39), (13.42) and (13.43), are equivalent to

$$\left.\begin{array}{ll} \dfrac{d\bar{r}}{dt} \geq 0 & \text{if } \dfrac{|T|}{2\pi\bar{r}^2} > \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} \leq 0 & \text{if } \dfrac{|T|}{2\pi\bar{r}^2} < \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} & \text{is arbitrary if } \dfrac{|T|}{2\pi\bar{r}^2} = \tau_c \ , \end{array}\right\} (i,j) = (3,1) \ , \tag{13.44}$$

and

$$\left.\begin{array}{ll} \dfrac{d\bar{r}}{dt} \leq 0 & \text{if } \dfrac{|T|}{2\pi\bar{r}^2} > \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} \geq 0 & \text{if } \dfrac{|T|}{2\pi\bar{r}^2} < \tau_c \ , \\[2ex] \dfrac{d\bar{r}}{dt} & \text{is arbitrary if } \dfrac{|T|}{2\pi\bar{r}^2} = \tau_c \ , \end{array}\right\} (i,j) = (1,3) \ . \tag{13.45}$$

Consider the case $(i,j) = (3,1)$. One shows easily that $\phi_0 = \phi_{31}\left(\sqrt{\dfrac{T}{2\pi\tau_c}}, \ T\right)$ is a curve in A_{31} which qualitatively looks as shown in Fig. 6(i) — curve XY. Corresponding to any point on this curve, we have $\frac{T}{2\pi \bar{r}^2} = \tau_c$ while it is readily seen that at any point above or below XY $\frac{T}{2\pi \bar{r}^2}$ is, respectively, greater than or less than τ_c. Accordingly, the dissipativity inequality is equivalent to the first, second and third of (13.44) at points in A_{31}, respectively, above, below and on the curve XY. The arrows in Fig.6(i) indicate the admissible orientations of a loading path at some typical points in A_{31}. As before, the orientation at points on PS and QR is arbitrary. The solution $(i, j) = (1, 3)$ may be similarly interpreted, as shown[1] in Fig.6(ii).

13.2 Consequences of Dissipativity

The dissipativity inequality was introduced in the hope that it would single out a physically admissible solution from among the many available equilibrium solutions. We now demonstrate that, if we require the local twist $\phi(r, \cdot)$ to be continuous[2] on $\mathcal{T}$ at each r in $[a, b]$, and if we suppose that the body was in an undeformed configuration at some time, then a configuration corresponding to solutions $(i, j) = (1, 2)$, $(2, 1)$, $(2, 3)$ $(3, 2)$ or smooth solution 2 cannot be attained at any subsequent time.

First, omit the weak solutions $(i, j) = (1, 3)$ and $(3, 1)$ from discussion. We observe from Fig. 7 that any loading path in Fig. 9 conforming with the dissipativity inequality and starting from O is necessarily confined to the curve OP for all subsequent time. Note similarly,

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[1] An examination of the details of the curve $\phi_0 = \phi_{13}(\sqrt{T/2\pi\tau_c}, T)$ show that it is possible for this curve, depending on the specific geometry and constitutive law, to intersect a different pair of boundaries of A_{13} than shown in Fig. 6(ii). The figure is drawn for $b^2/a^2\tau_{min} < \tau_c < a^2/b^2\tau_{max}$.

[2] Note that despite the presumed continuity of $\phi_0(t)$ and $T(t)$, $\phi(r, \cdot)$ defined by (13.18) is not necessarily continuous on $\mathcal{T}$, since the subscripts i and j may change values at certain times.

from Figs. 8 and 9, that any admissible loading path starting from O' is likewise restricted to $O'R$ for all subsequent time. The only possible way of achieving a solution $(i,j) = (1,2)$, $(2,3)$, $(3,2)$, $(2,1)$ or solution 2 is then, by virtue of a loading path which is associated with one of the solutions $(i,j) = (3,1)$, $(1,3)$ for some time interval less than some time t_1, and with one of these solutions after time t_1. One sees readily from (13.18) however, that this involves a discontinuity in $\phi(r,\cdot)$ at the time t_1. Since we have disallowed this possibility, we now conclude that a configuration corresponding to <u>any solution associated with the second branch of the graph of $f(k)$ vs. k cannot be attained through a dissipative quasi-static deformation process</u>. These are, incidentally, the solutions at which the displacement equations of equilibrium are non-elliptic somewhere in Π.

However, even if we now discard the solutions associated with the second branch of f, we would not have overcome our troubles with non-uniqueness. For example, consider the solutions 1, 3 and $(i,j) = (3,1)$. The appropriate torque-twist diagram is shown in Fig. 12. If we imagine gradually <u>increasing</u> the applied twist ϕ_0 from zero, the only available loading path initially is OS. During the next stage, $\phi_S < \phi_0 < \phi_X$, dissipativity — see Fig. 6(i) — disallows all loading paths except SX. Once the applied twist ϕ_0 exceeds the value ϕ_X, however, the loading path could lie anywhere in PQYX, and we have no criterion for deciding which path to follow. Eventually, for $\phi_0 > \phi_Q$, we are restricted to the path QO'. Likewise, during a steady <u>decrease</u> of the applied twist the loading path would be restricted to $O'QY$, then allowed to follow an arbitrary path (consistent with dissipativity) in XYRS and finally restricted to SO. It is interesting to note that if in either case the loading path lies on the curve $OXYO'$, then the quasi-static process is dissipationless in the sense that (13.8) would hold with equality at every instant t.

It is therefore imperative that we seek an additional — or possibly an alternative — physical criterion, to the dissipativity inequality, that would sort out more completely the issue of non-uniqueness.

CHAPTER 14

14.1 Preliminaries on Stability

In the context of the initial value problem for nonlinear hyperbolic partial differential equations, Lax [10] says: "First of all, we exclude all solutions where entropy of a particle has been decreased. It is not clear, however, whether this insures the uniqueness of the solution of the initial value problem, especially if there are several space variables but even in the case of one space variable. Some additional principle is needed to pick out a unique solution, such as: (a) the weak solutions occuring in nature are limits of viscous flows (b) the weak solutions occuring in nature must be stable. It is commonly believed that (a) characterizes uniquely the solutions occuring in nature. But whether the same is true of postulate (b) is seriously doubted by some."

We now look into the possibility of using a stability criterion, instead of the dissipativity inequality, in order to single out a physically admissible solution to the boundary value problem under consideration. We draw attention to the fact that we will not make use of the partial success achieved through the dissipativity inequality, since we are at present examining the possibility of an alternative — rather than additional — criterion.

The notion of stability that we will use is a static one based on the energy criterion.[1] According to this, an equilibrium configuration of a body is stable if and only if the work done by the external loads in

[1] See page 195 of [11] for a discussion of this criterion.

every kinematically possible virtual displacement from this equilibrium configuration is less than the corresponding increase in the stored energy.

We first need to specify the manner in which the applied loading behaves during a virtual displacement. We consider two possibilities — stability under dead loading and stability with fixed boundaries. Suppose first, that we have dead loading on the inner surface of the tube while the outer surface is held fixed, so that the torque T remains constant during a virtual displacement. Let $\phi(r)$ be the equilibrium solution whose stability we wish to investigate. Define the potential energy functional $V\{\psi\}$ by

$$V\{\psi\} = \int_a^b \{W(2+r^2\psi'^2) - W(2+r^2\phi'^2)\} 2\pi r dr - T\{\psi(a) - \phi(a)\} \quad , \qquad (14.1)$$

for all functions $\psi(r)$ in some set χ. In order to interpret $\psi(r)$ as a virtual twist,[1] — measured from the undeformed configuration — we suppose that χ is the set of all functions which are defined, continuous and twice piecewise continuously differentiable on $[a, b]$ and are such that $\psi(b) = 0$. Since this limited degree of smoothness is all that is required of an equilibrium solution $\phi(r)$, it seems reasonable not to impose more severe smoothness requirements on the virtual displacement. The kinematical restriction of incompressibility is automatically satisfied by any purely circumferential virtual displacement. We now say that the equilibrium solution $\phi(r)$ is stable against arbitrarily large circumferential

[1] We restrict attention to purely circumferential virtual displacements.

perturbations[1] if and only if

$$V\{\psi\} > 0 \quad \text{for all } \psi \text{ in } \chi \text{ , } \psi \neq \phi \text{ .} \tag{14.2}$$

In order to establish the <u>instability</u> of some solution $\phi(r)$, it is clearly sufficient to show that $V\{\psi\} < 0$ for <u>some</u> ψ in χ . We now determine, from (14.2), a sufficient condition for stability, which will be useful for our purposes. We can rewrite (14.1), after making use of (10.25) and (10.27), as

$$V\{\psi\} = \int_a^b \{ W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi') \} 2\pi r dr \text{ ,} \tag{14.3}$$

for any ψ in χ . It follows that a sufficient condition for the stability of $\phi(r)$ is that for every ψ in χ , $\psi \neq \phi$,

$$W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi') \geq 0 \tag{14.4}$$

at each r in (a, b) where the left hand side exists, and

$$W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2) - f(r\phi')(r\psi' - r\phi') > 0 \tag{14.5}$$

at each r in some sub-interval of (a, b) where the left hand side exists. On the other hand, if at each r in (a, b) where $k(r) = r\phi'(r)$ exists, we have

$$W(2 + \varkappa^2) - W(2 + k^2(r)) - f(k(r))(\varkappa - k(r)) > 0 \tag{14.6}$$

for all <u>numbers</u> $\varkappa \neq k(r)$, it follows that (14.4), (14.5) holds. Equation

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[1] We shall merely say <u>stable</u> for brevity.

(14.6) is thus a sufficient condition for the stability of the solution $\phi(r)$.

Now consider the case in which the inner and outer surfaces of the tube are held fixed during a virtual displacement. The potential energy functional $V\{\psi\}$ is now defined by

$$V\{\psi\} = \int_a^b \{W(2 + r^2\psi'^2) - W(2 + r^2\phi'^2)\} 2\pi r dr \quad ,$$

for all functions ψ in some set χ. In this case we take χ to be the subset of the previous set of admissible virtual twists which conforms with $\psi(a) = \phi(a) = \phi_0$. By virtue of (10.25) and (10.27) we can again write $V\{\psi\}$ in the form given by (14.3), whence (14.6) continues to be a sufficient condition for stability. Instability of a solution may again be established by demonstrating that $V\{\psi\} < 0$ for some virtual twist ψ which is admissible.

14.2 Consequences of Stability

Following Ericksen [9], we first make note of a geometric property of the response curve in shear. Recall the functions $A_1(\tau)$ and $A_2(\tau)$ defined by (13.23) and (13.24), representing the areas of the loops formed by drawing a line, in Fig. 4, parallel to the k-axis at a distance τ, $\tau_{min} \leq \tau \leq \tau_{max}$, above it. Recall also that

$$\left.\begin{aligned} &A_1(\tau_c) = A_2(\tau_c) \;, \\ &A_1(\tau) > A_2(\tau) \quad \text{for } \tau_{min} \leq \tau < \tau_c \;, \\ &A_1(\tau) < A_2(\tau) \quad \text{for } \tau_c < \tau \leq \tau_{max} \;. \end{aligned}\right\} \qquad (14.7)$$

Keeping this in mind, one observes the following properties of $f(k)$ upon examining its graph (Fig. 4). If we set

$$k_3 = F_1(\tau_c) \ , \quad k_4 = F_3(\tau_c) \ , \tag{14.8}$$

then we may observe first that

(i) if k is any number such that either

$$|k| < k_3 \quad \text{or} \quad |k| > k_4 \ , \tag{14.9}$$

then

$$\int_k^{\varkappa} f(\xi)d\xi > f(k)(\varkappa - k) \qquad \text{for all } \varkappa \neq k \ . \tag{14.10}$$

Equation (14.10) is a statement of the geometric observation that, provided (14.9) holds, the area under the response curve from k to $\varkappa$, for any $\varkappa \neq k$ is greater than the area of the rectangle of the same width and of height $f(k)$. By virtue of (10.24), we can write (14.10) as

$$W(2+\varkappa^2) - W(2+k^2) - f(k)(\varkappa - k) > 0 \qquad \text{for all } \varkappa \neq k \ . \tag{14.11}$$

Next, it may be noted that

(ii) if k is any number such that

$$k_3 < |k| < k_4 \ , \tag{14.12}$$

then there exists some sub-interval $\mathcal{J}$ of $(-\infty, \infty)$ such that

$$\int_k^{\varkappa} f(\xi)d\xi < f(k)(\varkappa - k) \qquad \text{for } \varkappa \text{ in } \mathcal{J} \ , \tag{14.13}$$

whence by (10.24), we have

$$W(2+\varkappa^2) - W(2+k^2) - f(k)(\varkappa - k) < 0 \quad \text{for } \varkappa \text{ in } \mathcal{J} \; . \tag{14.14}$$

Alternatively, (i) and (ii) can be established analytically.

We now conclude, by virtue of (14.6), (14.9) and (14.11), that any equilibrium solution $\phi(r)$ for which

$$\left. \begin{aligned} &|k(r)| = |r\phi'(r)| < k_3 \; , \\ \text{or} \qquad & \\ &|k(r)| = |r\phi'(r)| > k_4 \; , \end{aligned} \right\} \tag{14.15}$$

everywhere on (a, b) where ϕ' exists, is stable. It is a trivial exercise to examine all the available equilibrium solutions $\phi(r)$ — given by (12.17) — and determine those that conform with (14.15). One finds that only the following do:

(i) Smooth Solution 1 with $|\phi_0| \le \int_a^b \frac{1}{\xi} F_1\left(\frac{a^2 \tau_c}{\xi^2}\right) d\xi$,

(ii) Smooth Solution 3 with $|\phi_0| \ge \int_a^b \frac{1}{\xi} F_3\left(\frac{b^2 \tau_c}{\xi^2}\right) d\xi$,

(iii) Weak Solution (3, 1) with $|T| = 2\pi\bar{r}^2\tau_c$, i.e. the solution (3, 1) with the torque given by $|\phi_0| = \phi_{31}(\sqrt{|T|/2\pi\tau_c}\, , |T|)$ and with shock radius $\bar{r} = \sqrt{|T|/2\pi\tau_c}$.

These solutions therefore are <u>stable</u>. On Fig. 12, these refer to the solutions associated with points on the curves OX, YO′ and XY respectively.

We will now show that all the other solutions are <u>unstable</u>. We do this by exhibiting particular admissible functions $\psi(r)$ which render $V\{\psi\}$ negative. It is readily shown that these remaining solutions— i.e. solutions (12.17) which do not conform with (14.15) — all have

$$k_3 < |r\phi'(r)| < k_4 \quad , \tag{14.16}$$

on some sub-interval of $[a, b]$. In each case $\psi(r)$ is chosen to take advantage of (14.12), (14.14) and (14.16). We first consider the case of dead loading.

Consider <u>Solution 1</u> with

$$\int_a^b \frac{1}{\xi} F_1\left(\frac{a^2 \tau_c}{\xi^2}\right) d\xi < |\phi_0| \le \int_a^b \frac{1}{\xi} F_1\left(\frac{a^2 \tau_{max}}{\xi^2}\right) d\xi \quad . \tag{14.17}$$

In this case recall that

$$r\phi'(r) = -F_1\left(\frac{T}{2\pi r^2}\right) \quad , \tag{14.18}$$

with T given by

$$\phi_0 = \int_a^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad . \tag{14.19}$$

By the monotonicity of the function F_1 it follows from (14.17) and (14.19) that

$$2\pi a^2 \tau_c < |T| \le 2\pi a^2 \tau_{max} \quad . \tag{14.20}$$

By virtue of (14.20) there is a number s, $a<s<b$, such that

$$|T| > 2\pi r^2 \tau_c \quad \text{for } a \leq r < s \ . \tag{14.21}$$

Note from (10.35), (10.41), (14.8), (14.18) and (14.21) that $|r\phi'(r)| > k_3$ on $[a, s)$. We now choose the function $\psi(r)$ in χ such that

$$r\psi'(r) = \begin{cases} -F_3\left(\dfrac{T}{2\pi r^2}\right) & \text{for } a<r<s \ , \\ \\ -F_1\left(\dfrac{T}{2\pi r^2}\right) & \text{for } s<r<b \ . \end{cases} \tag{14.22}$$

On using (14.18) and (14.22) in (14.3) we find by virtue of (10.24), (10.28), (10.34), (10.40) and (10.41) that

$$V\{\psi\} = \int_a^s 2\pi r \int_{F_1\left(\frac{|T|}{2\pi r^2}\right)}^{F_3\left(\frac{|T|}{2\pi r^2}\right)} \left\{ f(\xi) - \frac{|T|}{2\pi r^2} \right\} d\xi \, dr \tag{14.23}$$

This can be written as

$$V\{\psi\} = -\int_a^s A\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr \ , \tag{14.24}$$

because of (13.23), (13.24) and (13.39). Since (14.20) and (14.21) imply that

$$\tau_{max} \geq \frac{|T|}{2\pi r^2} > \tau_c \quad \text{for } a \leq r < s \ , \tag{14.25}$$

it now follows from (13.43), (14.24) and (14.25) that

$$V\{\psi\} < 0 \quad . \tag{14.26}$$

Therefore Solution 1 with (14.17) in effect is unstable. On Fig. 12, these solutions are associated with points on the curve XP (excluding X).

The instability of the other solutions may be established in an entirely analogous manner. We merely present the results.

Solution 3 with

$$\int_a^b \frac{1}{\xi} F_3\left(\frac{b^2 \tau_{min}}{\xi^2}\right) d\xi \leq |\phi_0| < \int_a^b \frac{1}{\xi} F_3\left(\frac{b^2 \tau_c}{\xi^2}\right) d\xi \quad . \tag{14.27}$$

Let s be a number, $a<s<b$, such that $|r\phi'(r)| < k_4$ on $(s, b]$. Choosing ψ in χ such that

$$r\psi'(r) = \begin{cases} -F_3\left(\frac{T}{2\pi r^2}\right) & \text{on } (a, s) \; , \\ \\ -F_1\left(\frac{T}{2\pi r^2}\right) & \text{on } (s, b) \; , \end{cases} \tag{14.28}$$

we find

$$V\{\psi\} = \int_s^b A\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr < 0 \quad . \tag{14.29}$$

This case is associated with points on RY — except R — in Fig. 12.

Solution 2

Choosing

$$\psi(r)=\int_r^b \frac{1}{\xi}F_3\left(\frac{T}{2\pi\xi^2}\right)d\xi \quad \text{on } [a,b] \ , \tag{14.30}$$

we find

$$V\{\psi\}=-\int_a^b A_2\left(\frac{|T|}{2\pi r^2}\right)2\pi r dr<0 \ . \tag{14.31}$$

This case is associated with points on MN in Fig. 5.

<u>Solution (2, 1)</u> with

$$\bar{r}>a \ . \tag{14.32}$$

Recall that Solution (2, 1) with $\bar{r}=a$ is identical with Solution 1, which we have already examined. An analogous comment is appropriate for the next four solutions as well. Choosing

$$\psi(r)=\int_r^b \frac{1}{\xi}F_1\left(\frac{T}{2\pi\xi^2}\right)d\xi \quad \text{on } [a,b] \ , \tag{14.33}$$

we find

$$V\{\psi\}=-\int_a^{\bar{r}} A_1\left(\frac{|T|}{2\pi r^2}\right)2\pi r dr<0 \ . \tag{14.34}$$

This case is associated with points in PMNS (excluding the curve PS) in Fig. 7(ii).

<u>Solution (1, 2)</u> with

$$\bar{r}<b \ . \tag{14.35}$$

Choosing

$$\psi(r) = \int_r^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{on } [a,b] \;, \tag{14.36}$$

we find

$$V\{\psi\} = -\int_{\bar{r}}^{b} A_1\left(\frac{|T|}{2\pi r^2}\right) 2\pi r\,dr < 0 \;. \tag{14.37}$$

This case is associated with points in PMNK (excluding the curve PK) in Fig. 7(ii).

<u>Solution (2, 3)</u> with

$$\bar{r} > a \;. \tag{14.38}$$

Choosing

$$\psi(r) = \int_r^b \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{on } [a,b] \;, \tag{14.39}$$

we find

$$V\{\psi\} = -\int_a^{\bar{r}} A_2\left(\frac{|T|}{2\pi r^2}\right) 2\pi r\,dr < 0 \;. \tag{14.40}$$

This case is associated with points in MLRN (excluding the curve LR) in Fig. 8(ii).

<u>Solution (3, 2)</u> with

$$\bar{r} < b \;. \tag{14.41}$$

Choosing

$$\psi(r) = \int_r^b \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{on } [a,b] \ , \tag{14.42}$$

we find

$$V\{\psi\} = -\int_{\bar{r}}^b A_2\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr < 0 \ . \tag{14.43}$$

This case is associated with points in MQRN (excluding the curve RQ) in Fig. 8(i).

Solution (1, 3) with

$$b > \bar{r} > a \ . \tag{14.44}$$

If $|T| \geq 2\pi\bar{r}^2\tau_c$ we choose

$$\psi(r) = \int_r^b \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{on } [a,b] \ , \tag{14.45}$$

and find

$$V\{\psi\} = -\int_a^{\bar{r}} A\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr < 0 \ . \tag{14.46}$$

If $|T| < 2\pi\bar{r}^2\tau_c$, we choose

$$\psi(r) = \int_r^b \frac{1}{\xi} F_1\left(\frac{T}{2\pi\xi^2}\right) d\xi \quad \text{on } [a,b] \ , \tag{14.47}$$

and find

$$V\{\psi\} = \int_{\bar{r}}^{b} A\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr < 0 \quad . \tag{14.48}$$

This case refers to points in PLRK (excluding the curves PK and RL) in Fig. 6(ii).

<u>Solution (3, 1)</u> with

$$|T| \neq 2\pi\bar{r}^2\tau_c \quad . \tag{14.49}$$

If $|T| > 2\pi\bar{r}^2\tau_c$, there is a number s, $\bar{r}<s<b$, such that $k_3 < |r\phi'(r)| < k_4$ on $(\bar{r}, s)$. Choosing ψ in χ such that

$$r\psi'(r) = \begin{cases} -F_3\left(\dfrac{T}{2\pi r^2}\right) & \text{on } (a, s) \;, \\ \\ -F_1\left(\dfrac{T}{2\pi r^2}\right) & \text{on } (s, b) \;, \end{cases} \tag{14.50}$$

we find

$$V\{\psi\} = -\int_{\bar{r}}^{s} A\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr < 0 \quad . \tag{14.51}$$

If $|T| < 2\pi\bar{r}^2\tau_c$ it follows that there is a number s, $a<s<\bar{r}$, such that $k_3 < |r\phi'(r)| < k_4$ on $(s, \bar{r})$. Choosing $\psi(r)$ as above (for this value of s) we find

$$V\{\psi\} = \int_{s}^{\bar{r}} A\left(\frac{|T|}{2\pi r^2}\right) 2\pi r dr < 0 \quad . \tag{14.52}$$

This case refers to points in PQRS (excluding the curves XS, XY and QY) in Fig. 6(i) and also in Fig. 12.

Instability in the case when the inner boundary is fixed may be established in a similar manner, taking care now to satisfy the boundary condition $\psi(a) = \phi(a) = \phi_0$. For example, consider Solution 1 with (14.17) in effect. Let $c(\epsilon)$ be the function defined implicitly by

$$\phi_0 = \int_a^{c(\epsilon)} \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi + \int_{c(\epsilon)}^{b} \frac{1}{\xi} F_1\left(\frac{T-\epsilon}{2\pi\xi^2}\right) d\xi \ . \tag{14.53}$$

By virtue of (10.38), (14.19) and the implicit function theorem, one can show that (14.53) does in fact define a function $c(\epsilon)$ which is twice[1] continuously differentiable in a neighborhood of $\epsilon = 0$, and that

$$c(0) = a \ , \quad c'(0) > 0 \ . \tag{14.54}$$

Thus $c(\epsilon) > a$ for sufficiently small positive ϵ.

Now consider the virtual twist $\psi(r)$, defined by

$$\psi(r) = \begin{cases} \phi_0 - \displaystyle\int_a^{r} \frac{1}{\xi} F_3\left(\frac{T}{2\pi\xi^2}\right) d\xi & \text{on } [a, c(\epsilon)] \ , \\ \\ \displaystyle\int_r^{b} \frac{1}{\xi} F_1\left(\frac{T-\epsilon}{2\pi\xi^2}\right) d\xi & \text{on } [c(\epsilon), b] \ , \end{cases} \tag{14.55}$$

for a sufficiently small $\epsilon > 0$. Note that ψ is in χ by virtue of (14.53). If we now set

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[1] Twice continuously differentiable when $T < 2\pi a^2 \tau_{max}$. The argument presented here can be readily modified in the case $T = 2\pi a^2 \tau_{max}$.

$$\left.\begin{aligned} V_1(\epsilon) &= \int_a^{c(\epsilon)} \{W(2+r^2\psi'^2) - W(2+r^2\phi'^2) - f(r\phi')(r\psi' - r\phi')\} 2\pi r dr \ , \\ V_2(\epsilon) &= \int_{c(\epsilon)}^{b} \{W(2+r^2\psi'^2) - W(2+r^2\phi'^2) - f(r\phi')(r\psi' - r\phi')\} 2\pi r dr \ , \end{aligned}\right\} \tag{14.56}$$

with ψ given by (14.55), we may write (14.3) as

$$V\{\psi\} = V_1(\epsilon) + V_2(\epsilon) \ . \tag{14.57}$$

We find from (14.18), (14.54), (14.55) and (14.56) that

$$V_2(0) = 0 \ , \quad V_2'(0) = 0 \ , \tag{14.58}$$

because of (10.24) and (10.28). Likewise we find

$$V_1(0) = 0 \ , \quad V_1'(0) = -A\left(\frac{|T|}{2\pi a^2}\right)c'(0) \ , \tag{14.59}$$

where we have also used (10.34), (10.40), (10.41), (13.23), (13.24), (13.39) and (14.54). Note because of (14.17), (14.19) and the monotonicity of F_1 that $\tau_c < \frac{|T|}{2\pi a^2} \le \tau_{max}$, whence by (13.43), (14.54) and (14.59) we have

$$V_1'(0) = -A\left(\frac{|T|}{2\pi a^2}\right)c'(0) < 0 \ . \tag{14.60}$$

On using (14.58), (14.59) in (14.57) we find

$$V\{\psi\} = -A\left(\frac{|T|}{2\pi a^2}\right)c'(0)\epsilon + o(\epsilon) \quad \text{as} \quad \epsilon \to 0 \ , \tag{14.61}$$

so that by (14.60)

$$V\{\psi\} < 0 \quad \text{for sufficiently small } \epsilon > 0 \ . \tag{14.62}$$

This establishes the instability of Solution 1 with (14.17) in effect in this case. The instability of the other solutions may be likewise established. This completes our instability analysis.

Collecting the preceeding results leads to the conclusion that the only stable solutions[1] are the ones given by (i), (ii), (iii) following equation (14.15). Recall that on the torque-twist diagram, Fig. 12, these are the solutions associated with the curve OXYO′. We therefore have that <u>there is a unique stable solution $\phi(r)$ to the boundary value problem in its weak formulation corresponding to every value of the applied twist ϕ_0</u>, i.e. there is a unique solution $\phi(r)$ to (12.8) - (12.10) which conforms with (14.2), (14.3). Note that at every value of ϕ_0, the displacement equations of equilibrium are elliptic on Π $(r \neq \bar{r})$ at this unique solution.

We now refer to a remark made in Section 12.1 that a configuration involving more than one elastostatic shock is unstable. In the case of a solution with a single shock we showed instability whenever (14.16) held. Clearly, it is (14.16) and not the number of shocks that is important in that argument.[2] It is readily established that (14.16) holds for every weak solution involving more than one elastostatic shock. This is most easily seen from a visualization of such a solution in the manner

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[1] These solutions exist irrespective of the geometric and constitutive details, i.e. even in the cases when (11.23) does not hold these are the only stable solutions.

[2] As remarked previously, the importance of (14.16) for instability is related to the property (14.12), (14.14).

explained in association with Fig. 10. This leads to the instability of such a solution.

Now consider a quasi-static loading of the body. If the loading is performed in a manner which involves large circumferential disturbances, one might expect that at each instant, the tube seeks out the configuration which is stable against such distrubances. On increasing the applied twist we would then expect the loading path to follow the curve OXYO′ in Fig. 12. We would first observe a smooth configuration of the tube. An elastostatic shock would then emerge at the inner boundary and gradually move outwards, disappearing upon reaching the outer boundary and giving way to a smooth configuration. On decreasing the applied twist, we would observe this process in reverse. Note from Section 13.2 that this loading path conforms with the dissipation inequality, even though we did not demand — here — that it should be so. In fact this is the dissipation-free path referred to previously.

The stability criterion which we have used is a very strong condition, requiring stability against arbitrarily large perturbations. Correspondingly the result achieved is also strong — picking out a single solution from among an infinite number of available solutions. One would anticipate that a weaker criterion, such as infinitesimal stability, would not suffice to determine a unique solution to the boundary value problem. However, it does not seem necessary to demand, as we have, a unique solution to the boundary value problem. It would be sufficient for physical reasonableness to demand a unique response corresponding to any prescribed loading. It is possible to have different paths on loading and unloading, resulting in such a unique quasi-static response without having a unique solution to the boundary value problem. Such a situation would be acceptable, especially because of the possibility of dissipation in the sense of Section 13. We have examined a criterion motivated by infinitesimal stability which in fact leads to such a situation. Of course solutions involving elastostatic shocks continue to play an essential part even then.

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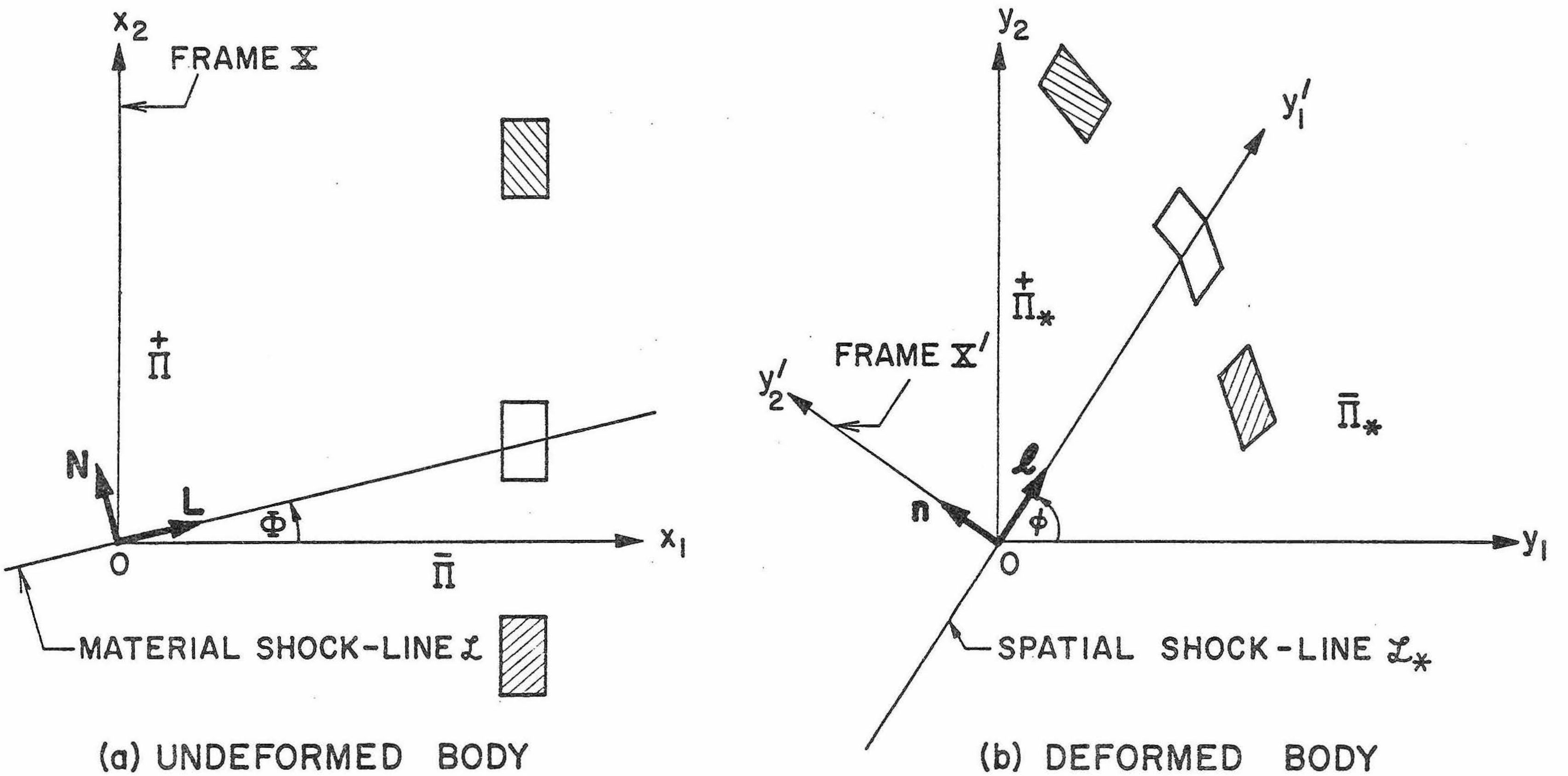


FIGURE 1. KINEMATICS OF PIECEWISE HOMOGENEOUS ELASTOSTATIC SHOCKS

$$\tau(k) = \mu k \exp(-k^2/2k_0^2)$$

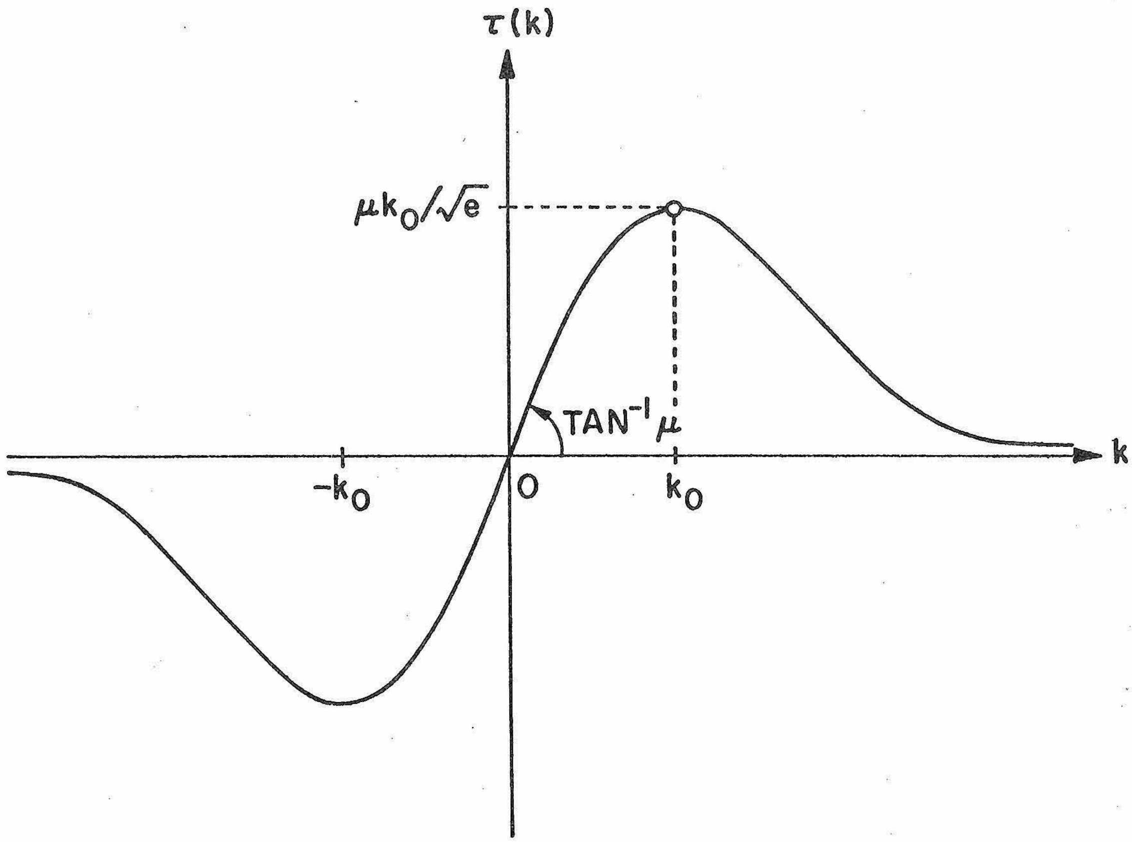
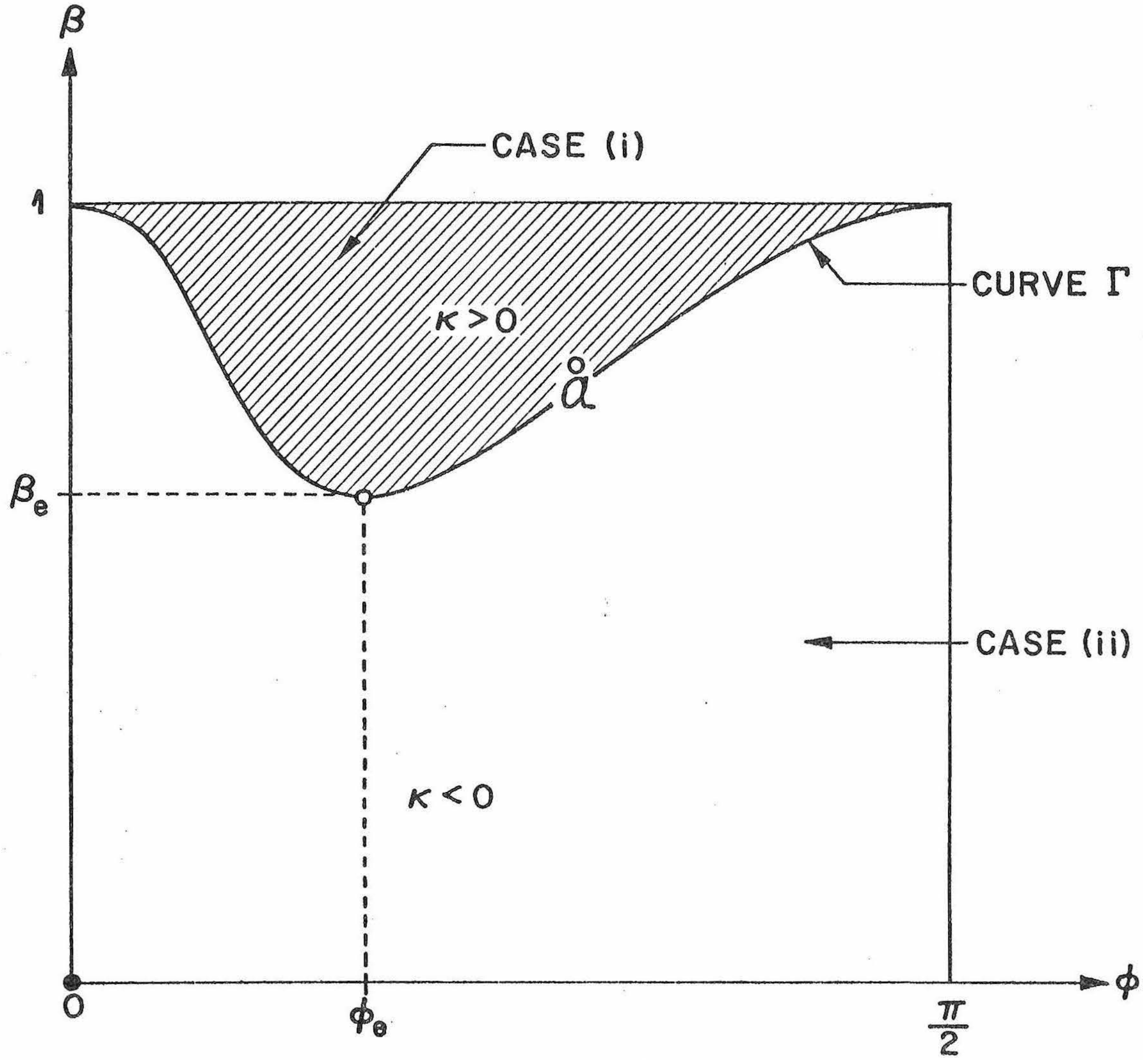


FIGURE 2. RESPONSE IN SIMPLE SHEAR.
SHEAR STRESS VS. AMOUNT OF SHEAR



$$\beta = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}$$

ADMISSIBLE SHOCK MOTION IS INTO $\overset{+}{\Pi}$

ADMISSIBLE SHOCK MOTION IS INTO $\overset{-}{\Pi}$

FIGURE 3. PLANE OF PARAMETERS ϕ AND β; ADMISSIBLE REGION $\mathring{a}$

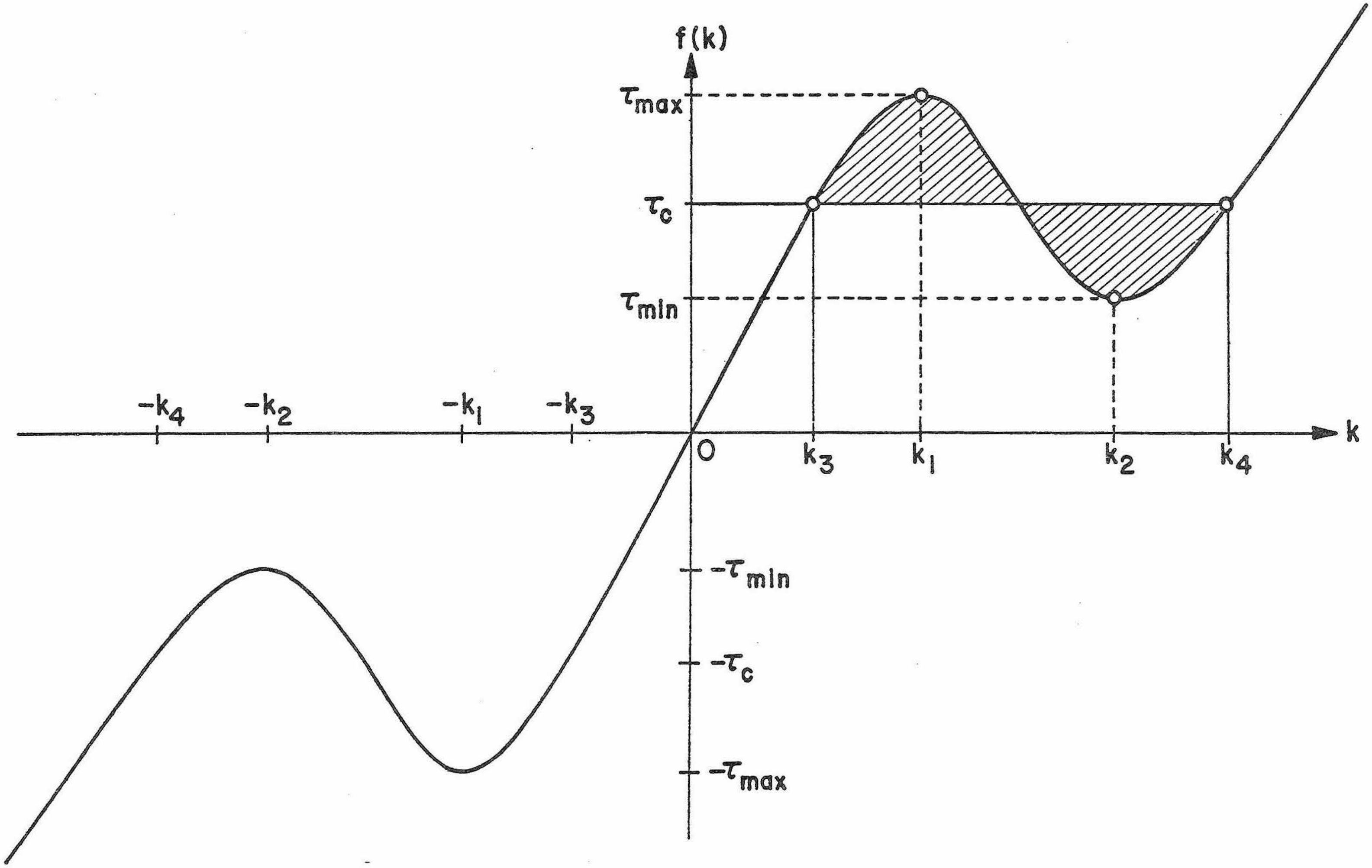
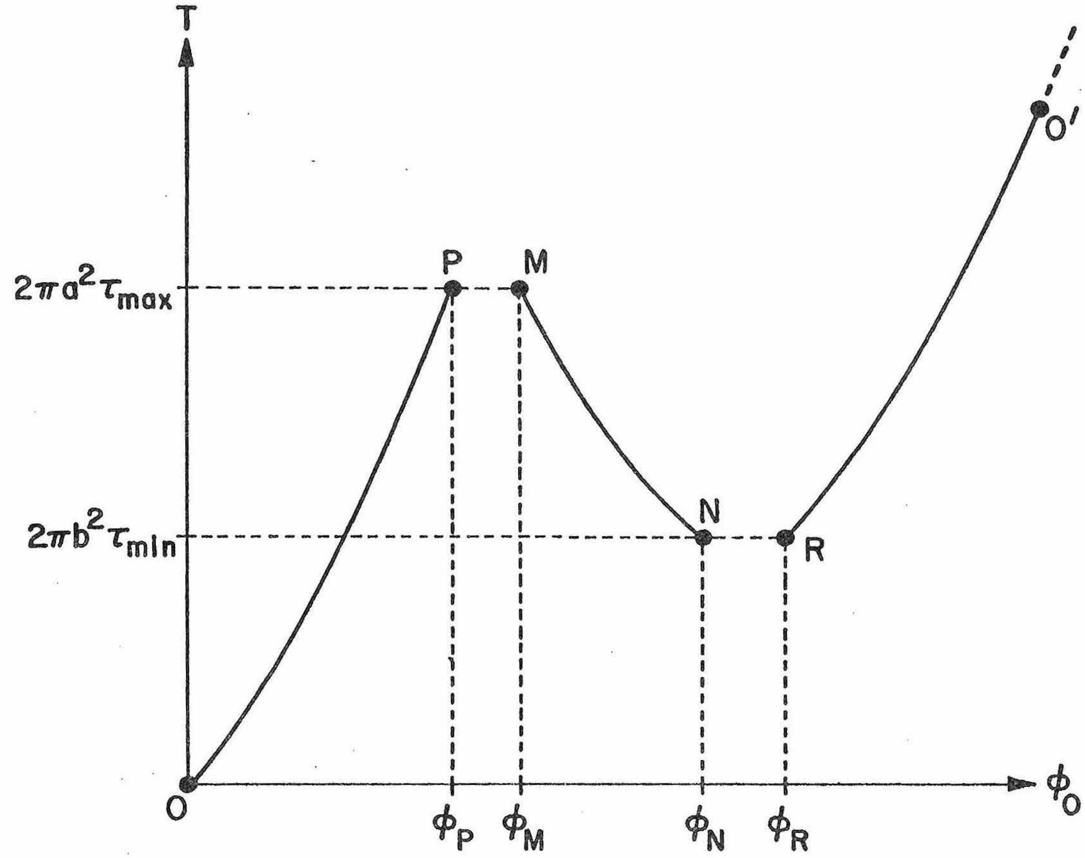


FIGURE 4. RESPONSE CURVE IN SIMPLE SHEAR FOR A HYPOTHETICAL MATERIAL. SHEAR STRESS VS. AMOUNT OF SHEAR



SOLUTION 1. OP : $\phi_0 = \int_a^b \frac{1}{\zeta} F_1 (T/2\pi\zeta^2)\, d\zeta$

SOLUTION 2. MN : $\phi_0 = \int_a^b \frac{1}{\zeta} F_2 (T/2\pi\zeta^2)\, d\zeta$

SOLUTION 3. RO′: $\phi_0 = \int_a^b \frac{1}{\zeta} F_3 (T/2\pi\zeta^2)\, d\zeta$

FIGURE 5. TORQUE VS. TWIST CURVES FOR SMOOTH SOLUTIONS

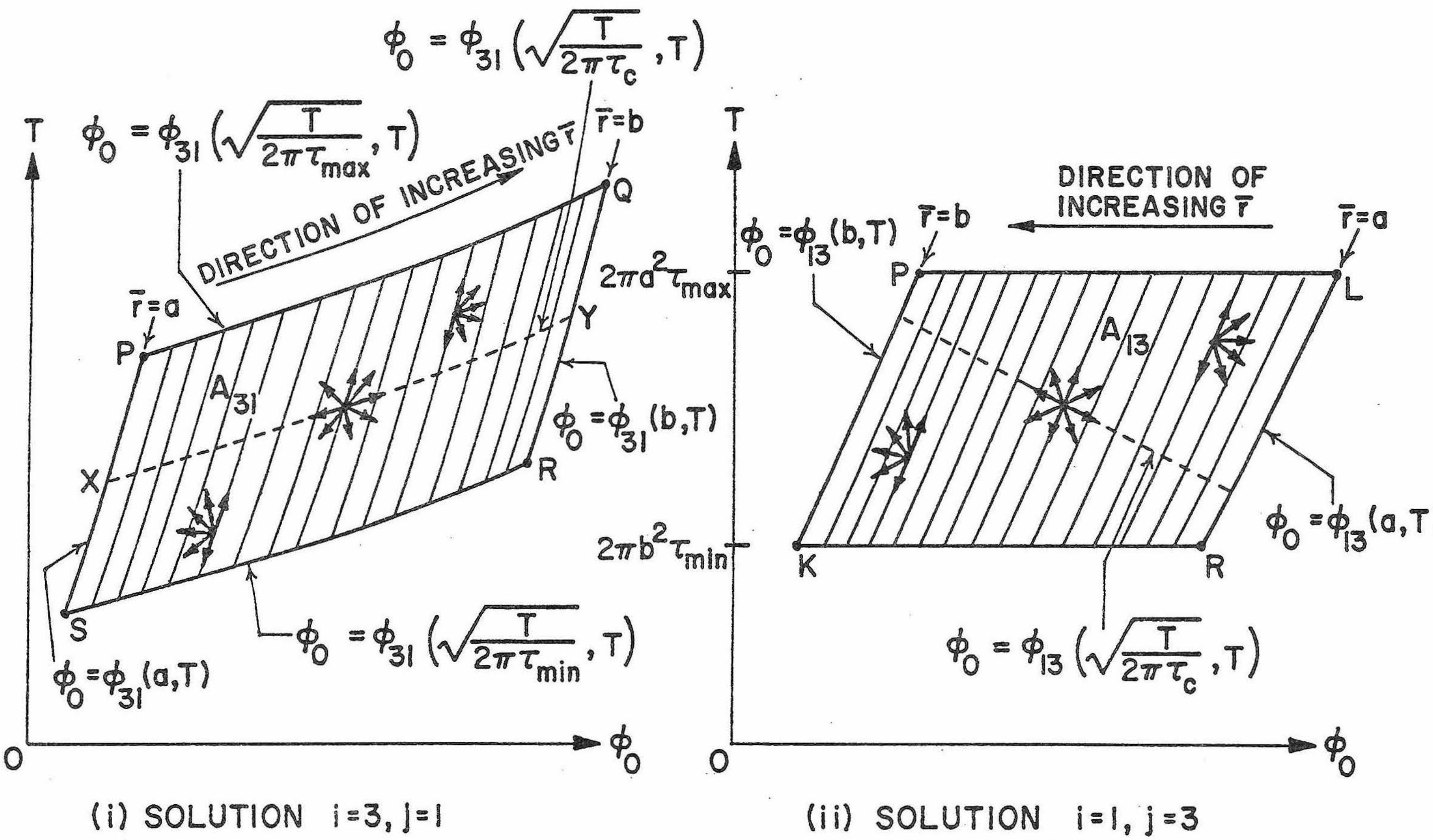
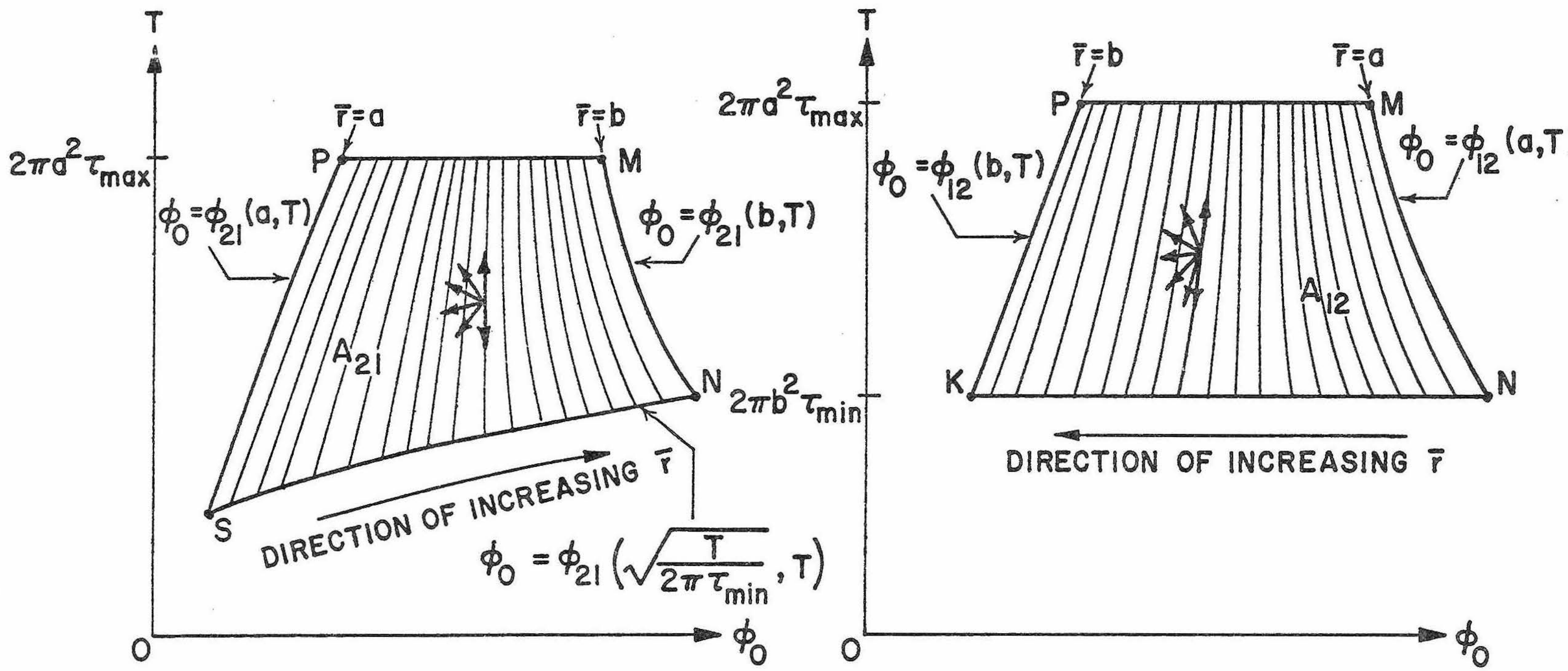
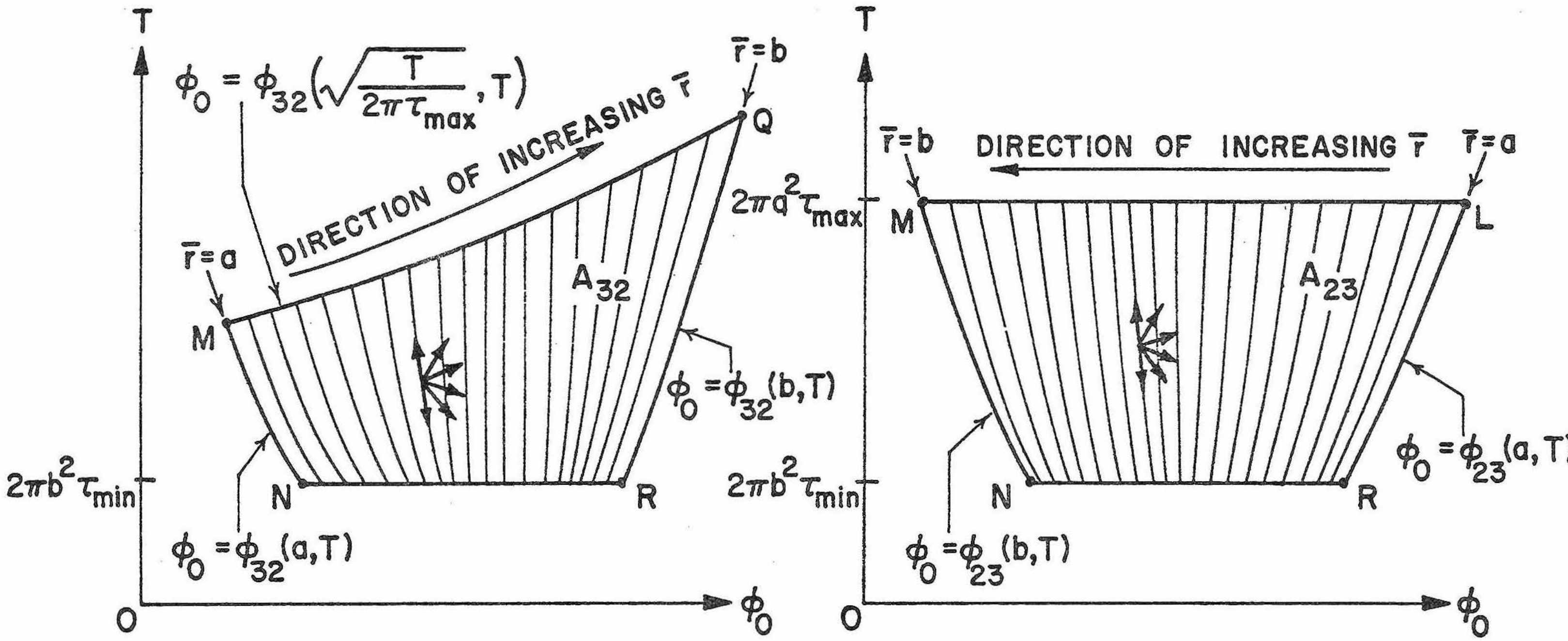


FIGURE 6. ADMISSIBLE REGIONS A_{ij} AND ORIENTATIONS OF LOADING PATHS



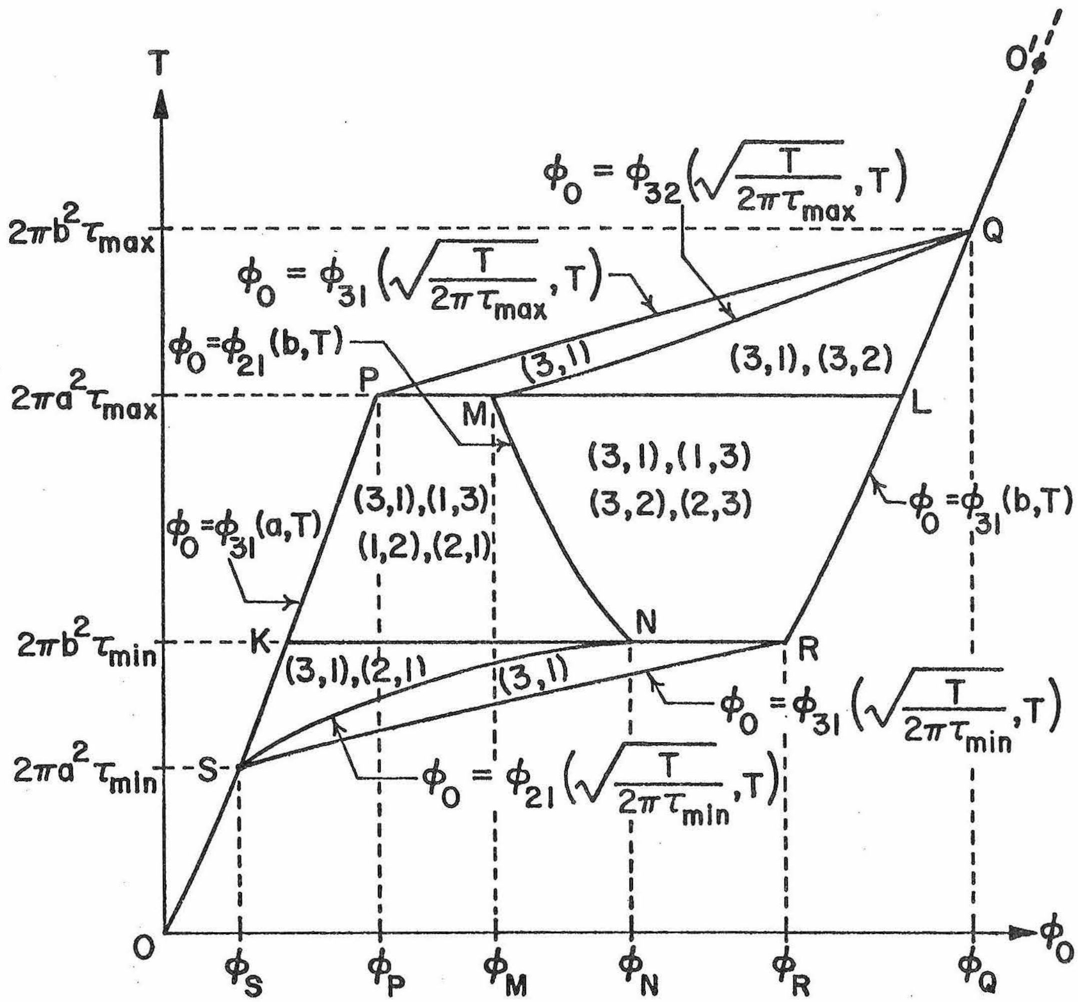
(i) SOLUTION $i=2,\ j=1$ (ii) SOLUTION $i=1,\ j=2$

FIGURE 7. ADMISSIBLE REGIONS A_{ij} AND ORIENTATIONS OF LOADING PATHS



(i) SOLUTION $i=3$, $j=2$ (ii) SOLUTION $i=2$, $j=3$

FIGURE 8. ADMISSIBLE REGIONS A_{ij} AND ORIENTATIONS OF LOADING PATHS



SOLUTION (i, j)	ADMISSIBLE REGION A_{ij}
(1,2)	PMNK
(2,1)	PMNS
(2,3)	MLRN
(3,2)	MQRN
(3,1)	PQRS
(1,3)	PLRK

FIGURE 9. ADMISSIBLE REGIONS OF TORQUE-TWIST PLANE

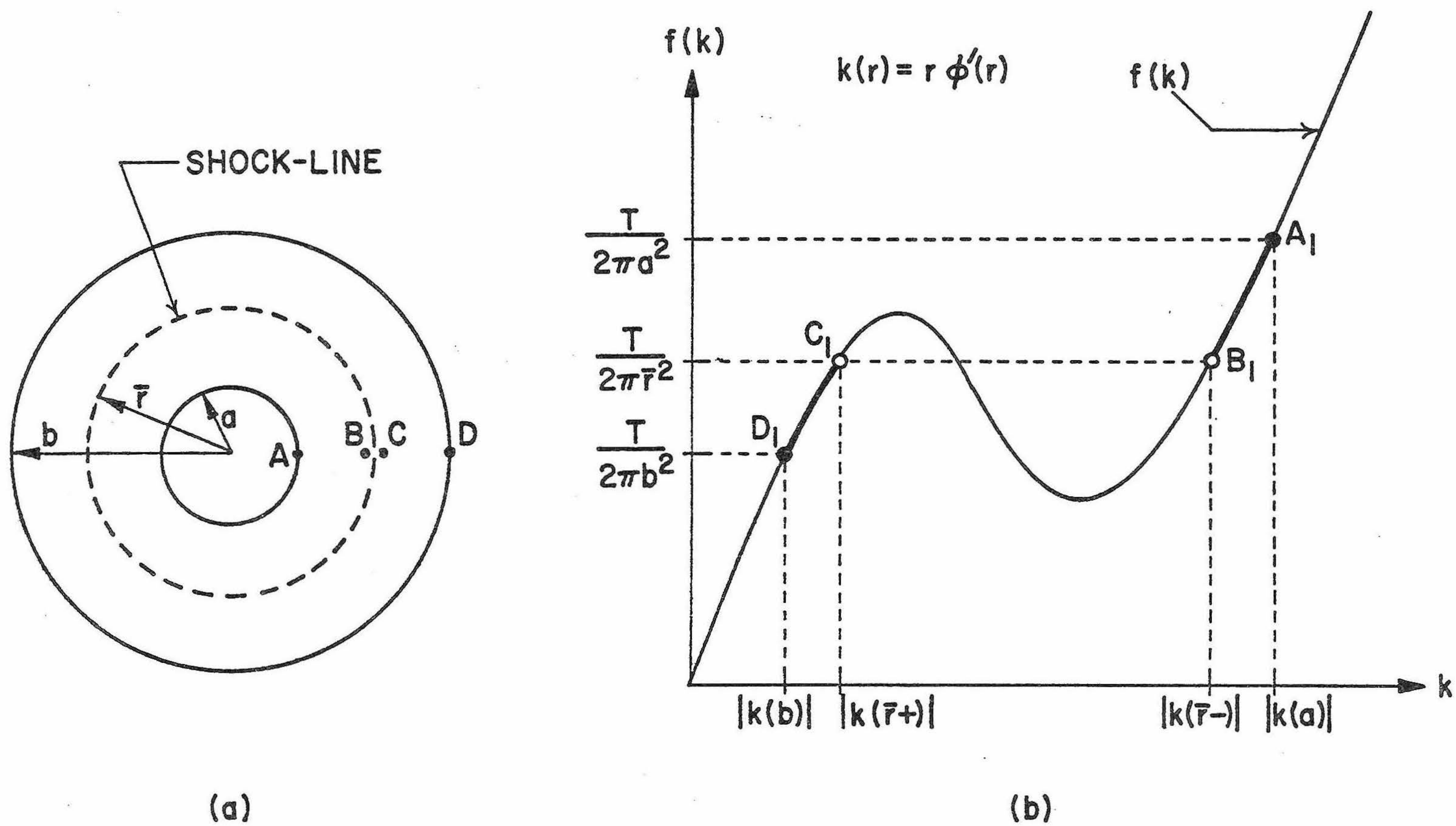


FIGURE 10. SHEAR STRESS VS. AMOUNT OF LOCAL SHEAR THROUGH THE TUBE.
SOLUTION $i=3$, $j=1$

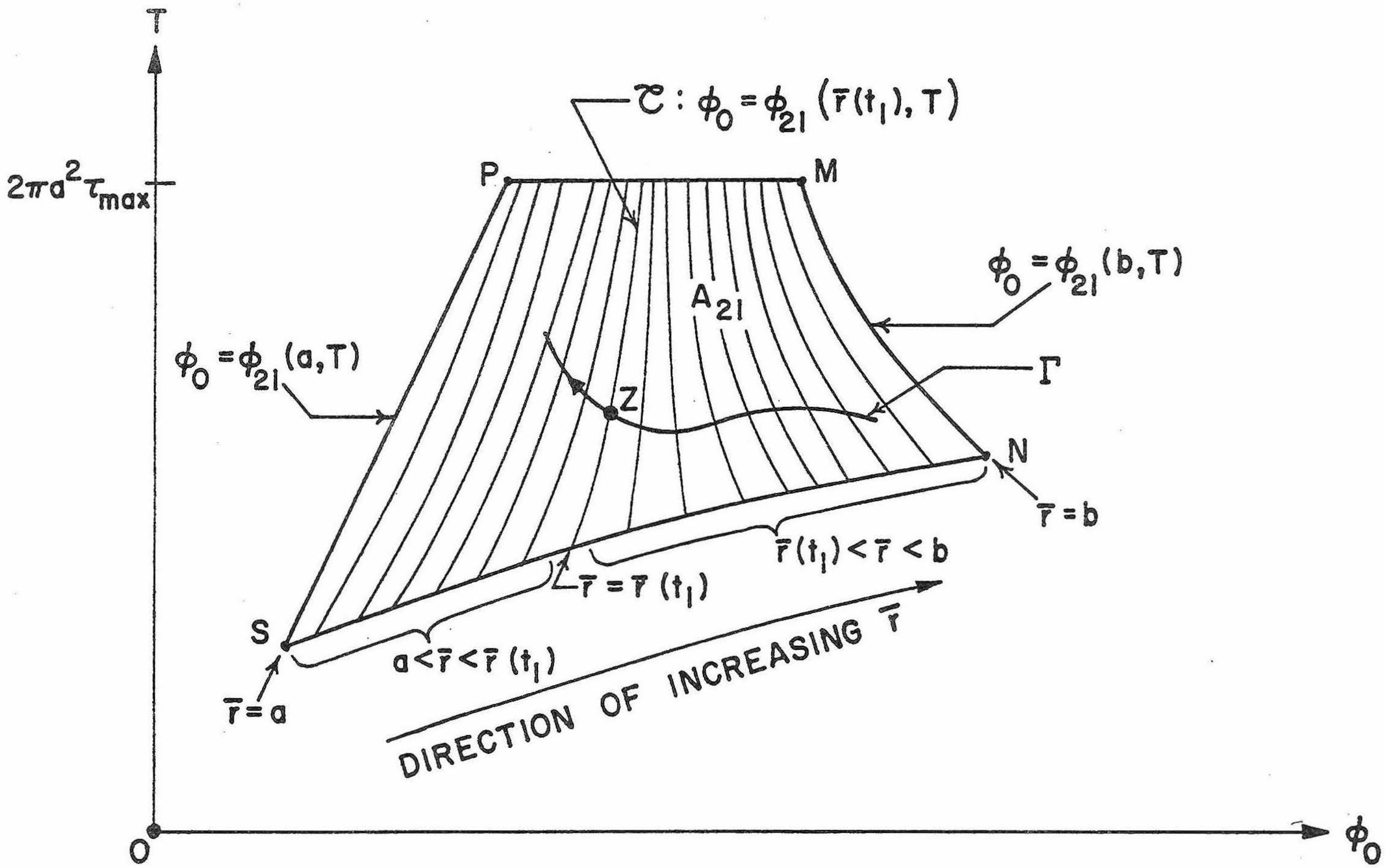
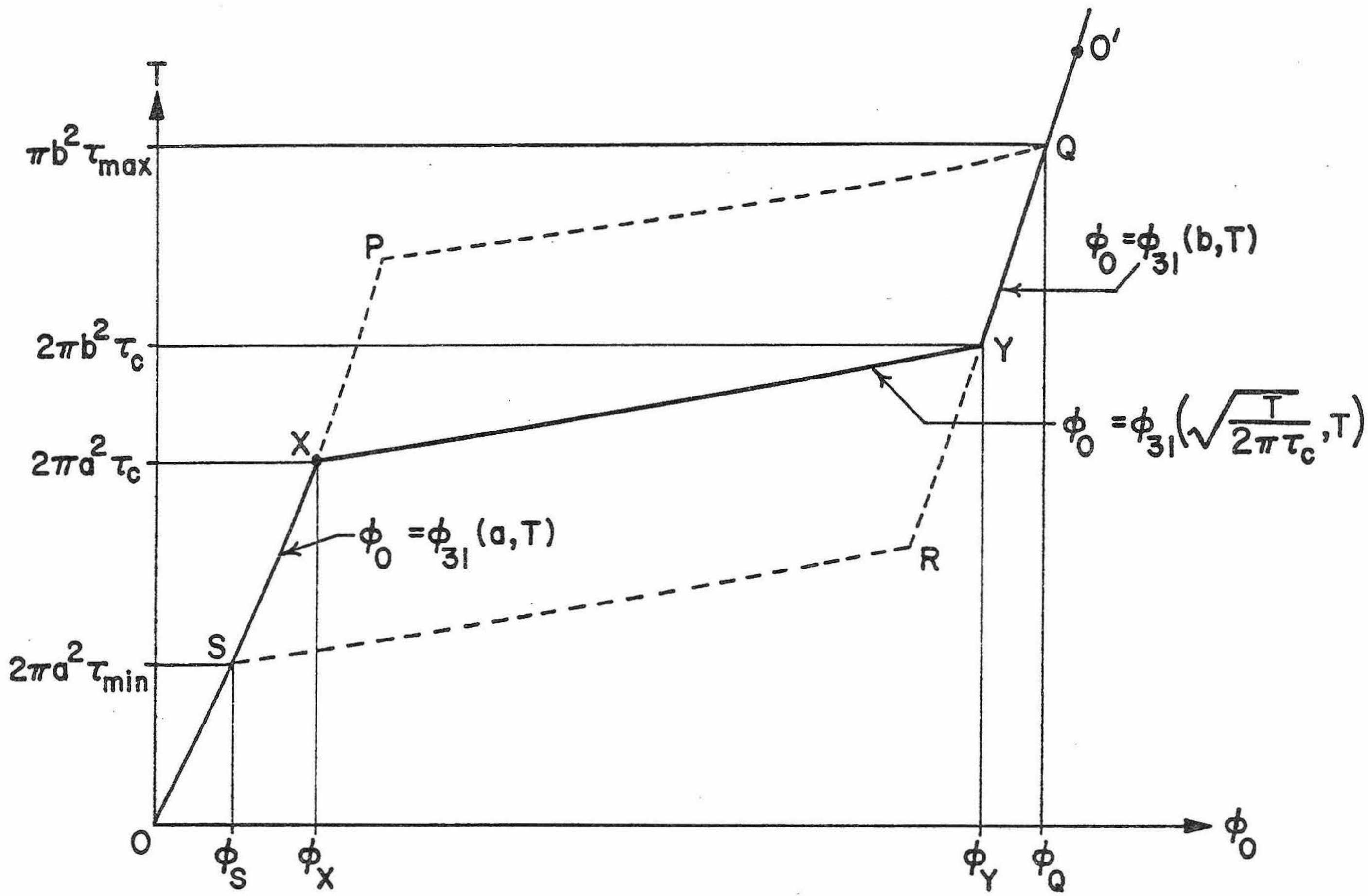


FIGURE II. ADMISSIBLE ORIENTATIONS OF A LOADING PATH FOR SOLUTION $i=2$, $j=1$



OXYO′-FINITELY STABLE TORQUE-TWIST CURVE

FIGURE 12. TORQUE-TWIST PLANE. SOLUTIONS 1, 3, (3, 1).