AN ASYMPTOTIC STUDY OF A CLASS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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ERRATA

Page 8 Eq. (2.8), $\frac{d^2v}{dx^2}$ replaces $\frac{d^2y}{dx^2}$

Page 16 Eq. (2.37), A_1 replaces A

Page 23 Eq. (3.26), insert "=" after $\phi_{ko}(x)$

Page 43 Eq. (6.15), $e^{\eta-x}$ replaces $e^{\eta-x}$

Page 66 2nd line from top, $\nu^{+\frac{1}{6}}$ replaces $\nu^{-\frac{1}{6}}$

Page 68 2nd line from bottom, (7.18) replaces (7.17)

Page 69 5th line from top, $e^{-\eta-x}$ replaces $e^{\eta-x}$

ABSTRACT

The asymptotic behavior, with respect to the large parameter μ, of two solutions of $\frac{d^2y}{dx^2} + \mu^2\left[c^2 - x^2 + f(x,\mu^{-1})\right]y = 0$ is given where $f(x,\mu^{-1}) = O(\mu^{-1})$ is subjected to suitable hypotheses and all variables are real. These solutions are approximated to by the parabolic cylinder functions $D_\nu(\pm\sqrt{2\mu}\, t)$ with $\nu = \mu\frac{a^2}{2} - \frac{1}{2}$ and $t = \phi(x,\mu^{-1})$. The function $\phi(x,\mu^{-1})$ and the quantity a^2 are constructed as a part of the analysis in which the parameter μ is restricted in such a way that ν is bounded away from the positive integers. The relative error of the approximating functions is uniform for all x.

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SECTION 1

INTRODUCTION

Investigations of the asymptotic solutions of ordinary differential equations containing a large parameter are not new. A very good account of these investigations is contained in R. E. Langer's Symposium Lecture of April 6, 1934 [1]* in which he discusses the work that had been done up to that time. Moreover, he includes a very complete set of references. Of particular interest is the work that had been done on functions with transition points.

A region in which the solutions of a differential equation change from monotonic to oscillating is called a transition region. Although the actual transition does not take place at a precisely defined point, it is often convenient to determine one point of this region and call it the transition point. Different asymptotic formulas for the same function are obtained in intervals separated by the transition points, and it is important to know how these asymptotic formulas in the separate intervals are related. This knowledge is usually supplied by "connecting formulas" which are valid in the transition region. Where the intervals of validity overlap, the "connecting formulas" are asymptotically equivalent to the formulas previously obtained. In this regard the work of Langer [2] should be mentioned in which he treated

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* Numbers in brackets apply to references listed in the bibliography.

a class of differential equations in the complex plane and containing a complex parameter under rather general hypotheses. His results include the case of one simple transition point. Also, Langer [3] considered certain differential equations in which the dominating term in the coefficient of the dependent variable has a double zero, and [4], he considered the case of a simple transition point directly. In this latter work the variable is real and the parameter complex. Finally, in 1950, functions with one simple transition point, in which all variables are complex, were investigated by T. M. Cherry [5] who treated the problem by a new method.

Generally speaking, all the methods used, including Cherry's, are based upon the observation that "similar" differential equations are satisfied by "similar" functions. The use of the word "similar" will become clear in the context. Consequently, the differential equation being investigated is compared with a similar differential equation, the solutions of which, along with their properties, are usually well known. Then, the solutions of the two equations are compared in such a manner that the values of the known functions may be used as approximations to the values of the unknown functions.

The novelty of Cherry's method is the way he uses a certain transformation by means of which the differential equation being studied may be compared with the "most simple" similar equation. The construction of the transformation is, naturally, an essential part of the problem. Although uniform approximation to the unknown functions by means of the known functions (with respect to the independent variable as the parameter tends to infinity) has been obtained by other mathematicians, the outstanding feature of Cherry's approach is the ease with which he obtains this desirable property which is enjoyed by his approximations.

Various possibilities of Cherry's method were recognized by Professor Arthur Erdélyi who outlined how it could be applied to functions with two transition points in his seminar in Asymptotic Theory at the California Institute of Technology in the spring of 1951. The development of this idea will be carried out in this dissertation. Naturally, all results pertaining to a single transition point may be applied twice in the case of two transition points provided the coefficients of the differential equation meet the required hypotheses. However, the present investigation copes with the case of two transition points directly and gives very simple formulas for the asymptotic solutions of the differential equation. It is felt that Professor Cherry's paper is one of the more significant contributions to this type of research.

These investigations are not to be confused with the asymptotic development of one special function, say the Bessel function, which is treated by methods especially suitable to the function. With respect to such developments it should be noted that N. Schwid [6] applied the results of Langer [2] to Weber's equation (Section 2.3) and obtained asymptotic formulas for the parabolic cylinder functions. Schwid's results are subsequently used in the present investigation.

Real functions of a real variable with two transition points, which satisfy a certain class of ordinary linear differential equations of the second order containing a real, positive, large parameter, will be studied. The asymptotic behavior of these functions with respect to the large parameter is desired. The class of differential equations has the form

$$\frac{d^2y}{dx^2} + \mu^2 \left[c^2 - x^2 + f(x,\mu^{-1})\right] y = 0 \tag{1.1}$$

where μ is the large parameter, c is a positive constant, and f is subjected to suitable hypotheses among which is the requirement that $f = O(\mu^{-1})$. However, the investigation will include the equation

$$\frac{d^2Y}{dx^2} + P(x,\mu^{-1}) \frac{dY}{dx} + \left[\mu^2 h_o(x) + \mu Q(x,\mu^{-1})\right] Y = 0 \tag{1.2}$$

where $h_o(x)$, like $c^2 - x^2$, has, on the real axis, only two simple zeros and is positive between the zeros. P,Q are also subjected to suitable hypotheses; in particular P,Q are $O(1)$ for large μ. Appropriate transformations will reduce (1.2) to (1.1).

In equation (1.1), the outstanding characteristic, when μ is large, is the presence of the function $c^2 - x^2$. The natural comparison equation is

$$\frac{d^2u}{dt^2} + \mu^2(a^2 - t^2)u = 0 \tag{1.3}$$

The solutions of (1.1) are approximated to by the real valued parabolic cylinder functions $D_\nu(\sqrt{2\mu}\, t)$ and $D_\nu(-\sqrt{2\mu}\, t)$ which are solutions of (1.3), with $t = \phi(x,\mu^{-1})$, the previously mentioned transformation, and $\nu = \frac{\mu a^2}{2} - \frac{1}{2}$.

When ν is not a positive integer or zero, $D_\nu(\sqrt{2\mu}\, t)$ and $D_\nu(-\sqrt{2\mu}\, t)$ form a fundamental set of solutions of the comparison equation (1.3) since their Wronskian does not vanish, but when ν is a positive integer or zero, $D_\nu(\sqrt{2\mu}\ t)$ is a constant multiple of $D_\nu(-\sqrt{2\mu}\ t)$, (Section 2.3). Comparison of the solutions of (1.1) and (1.3) requires asymptotic formulas for $D_\nu(\sqrt{2\mu}\ t)$ and $D_\nu(-\sqrt{2\mu}\ t)$ when ν is large. From these formulas (Section 6) it is seen that when ν is not a positive integer or zero, $D_\nu(\pm\sqrt{2\mu}\ t)$ is unbounded as t tends to $\mp\infty$. This behavior changes radically when ν is a positive integer or zero for then $D_\nu(\pm\sqrt{2\mu}\ t)$ tends to zero as $|t|$ tends to ∞.

The comparison of the solutions of (1.1) and (1.3) will be carried out with ν bounded away from the positive integers so that this exceptional behavior of the parabolic cylinder functions will not occur. Actually in connection with the differential equation (1.1) there are two distinct problems: (i) the general case, which is investigated in this thesis, and (ii) the special case, that is to say, the characteristic value problem where the differential equation has a solution which tends to zero as $|x|$ tends to ∞. This occurs for certain values of μ such that $\nu = \mu \frac{a^2}{2} - \frac{1}{2}$ becomes arbitrarily close to, or equals a positive integer.

Work on problem (ii) will be continued, and it is hoped that satisfactory results can be given in the near future. Also, applications of the results, particularly to suitable forms of the one-dimensional Schroedinger equation, will be made in the future. It is believed that solutions of the Schroedinger equation may be obtained in more useful forms than those obtained by the presently used WKB method (the historical development of this is also described in Langer [1]) and possibly more quickly.

A summary of the results of this paper is given in Section 8. The asymptotic formulas and conditions under which they are valid are precisely stated there.

<u>Notation</u>. Capital letters A, A_j, K, K_j refer to constants which are independent of μ and x or t, and the same letter appearing twice may represent a different constant each time. Arguments of the functional symbols are usually suppressed, or only one of the two variables may be shown when it is of primary interest. It is usually supposed that μ is "sufficiently large." Attention is called to the "+,o,-" subscript notation which is explained in Section 2.2.

SECTION 2

DESCRIPTION OF THE PROBLEM AND PRELIMINARY TRANSFORMATIONS

2.1 Description of the Problem.

An investigation will be made of the asymptotic formulas for the solutions of the class of differential equations described by

$$\frac{d^2Y}{dx^2} + P(x,\mu^{-1})\,\frac{dY}{dx} + \left[\mu^2 h_o(x) + \mu Q(x,\mu^{-1})\right] Y = 0 \tag{2.1}$$

where μ is a sufficiently large, positive parameter, and P,Q are analytic functions of x and μ^{-1}, real and regular for all real x and $o < \mu_o \leq \mu$. The <u>essential</u> <u>feature</u> is that the analytic function $h_o(x)$, which is real and regular for all real x, has, on the real axis, only two simple zeros $b_1 < b_2$ and is positive in the interval between the zeros.

Equation (2.1) may be converted by the transformation

$$Y = y\,\exp\left(-\frac{1}{2}\int P dx\right) \tag{2.2}$$

into

$$\frac{d^2y}{dx^2} + \mu^2\left[h_o(x) + Q\mu^{-1} - \left(\frac{P^2}{4} + \frac{1}{2}\frac{dP}{dx}\right)\mu^{-2}\right]y = 0 \tag{2.3}$$

and, since P,Q are regular functions of μ^{-1} for $\mu_o \leq \mu$, this last equation may be put into the form

$$\frac{d^2y}{dx^2} + \mu^2 h(x,\mu^{-1})y = 0 \tag{2.4}$$

where $h(x,\mu^{-1})$ is an analytic function of x and μ^{-1}, real and regular for all real x and $\mu_o \leq \mu$. It possesses the expansion

$$h(x,\mu^{-1}) = \sum_{n=o}^{\infty} h_n(x)\mu^{-n} \tag{2.5}$$

in which $h_o(x)$ is the function described above, and the $h_n(x)$, $o < n$, are analytic functions of x, real and regular for all real x.

When the transformation of the independent <u>and</u> dependent variables

$$t = \phi(x), \quad u(t) = v(x)\, \phi'(x)^{\frac{1}{2}}, \tag{2.6}$$

in which $0 < \phi'(x)$ for all real x, is applied to

$$\frac{d^2u}{dt^2} + H(t)u = 0 \tag{2.7}$$

the form of this equation is preserved, the result being

$$\frac{d^2v}{dx^2} + \left\{H\left[\phi(x)\right]\phi'(x)^2 + \frac{1}{2}\,\frac{\phi'''(x)}{\phi'(x)} - \frac{3}{4}\,\frac{\phi''(x)^2}{\phi'(x)^2}\right\} v = 0 \tag{2.8}$$

Since (2.4) is of the same form as (2.7), a transformation can be constructed, independent of μ, by means of which the coefficient of the new dependent variable has the following property: in its expansion in powers of μ^{-1}, the coefficient of μ^2 may be any desired function which, like $h_o(x)$, has two real, simple zeros and is positive between the zeros. The simplest function of this type is $c^2 - z^2$ where c is a positive constant. Consequently, the differential equation (2.4) may be converted, by a transformation which will be determined in Section 2.2, into the equation

$$\frac{d^2w}{dz^2} + \mu^2\left[c^2 - z^2 + f(z,\mu^{-1})\right]w = 0 \tag{2.9}$$

where $f(z,\mu^{-1})$ is an analytic function of z and μ^{-1}, real and regular for all real z and $\mu_o \leq \mu$. It possesses the expansion

$$f(z,\mu^{-1}) = \sum_{n=1}^{\infty} f_n(z)\mu^{-n} \tag{2.10}$$

The present investigation will therefore apply to the three equivalent differential equations (2.1), (2.4) and (2.9). However, only the simplest of these three equations, equation (2.9), will be treated in detail. The desired results are approximations for the solutions of (2.9) whose relative error is $O(\mu^{-m})$, m being an arbitrary positive integer. This error will be uniform for all real z with the exception of two arbitrarily small intervals including the points $z = \pm c$, respectively. The lengths of these intervals depend on μ.

In order to apply the results to equations (2.1) and (2.4), the preliminary transformations previously discussed must first be used in order to convert these equations into (2.9). It is necessary to develop such a transformation for equation (2.4).

2.2 The Preliminary Transformation for Equation (2.4).

Introduce

$$x = \phi(z), \quad y(x) = w(z)\, \phi'(z)^{1/2} \tag{2.11}$$

into equation (2.4) which becomes, in view of (2.6,7,8),

$$\frac{d^2w}{dz^2} + \mu^2 \left\{ h\left[\phi(z)\right]\phi'(z)^2 + \left(\frac{1}{2}\frac{\phi'''(z)}{\phi'(z)} - \frac{3}{4}\frac{\phi''(z)^2}{\phi'(z)^2}\right)\mu^{-2} \right\} w = 0 \tag{2.12}$$

Now by (2.5) $h\left[\phi(z)\right] = h_o\left[\phi(z)\right] + \sum_{n=1}^{\infty} h_n\left[\phi(z)\right]\mu^{-n}$. It will be shown that a positive constant c can be determined so that

$$h_o\left[\phi(z)\right]\phi'(z)^2 = c^2 - z^2 \tag{2.13}$$

has an analytic solution ϕ which is real and regular for all real z. Then the real zeros of the two functions $c^2 - z^2$ and $h_o\left[\phi(z)\right]$ will coincide so that $\phi'(z)$ will vanish for no real z. Consequently, it is assumed that positive roots are taken, implying $0 < \phi'(z)$ for all real

z. Then (2.12) may be put into the form (2.9) as desired. It is evident that $\phi(z)$ will be independent of μ. Equation (2.13) is equivalent to

$$h_o(x)\left(\frac{dx}{dz}\right)^2 = c^2 - z^2 \tag{2.14}$$

and, since $h_o(x)$ is positive for $b_1 < x < b_2$, equation (2.14) may be written in the two forms

$$\sqrt{c^2 - z^2}\,dz = \sqrt{h_o(x)}\,dx, \quad b_1 < x < b_2$$

$$\sqrt{z^2 - c^2}\,dz = \sqrt{-h_o(x)}\,dx, \quad x < b_1 \text{ or } b_2 < x.$$

Since the zeros of $h_o(x)$ correspond to those of $c^2 - z^2$, integrals of these equations are

$$\int_z^{-c} \sqrt{\zeta^2 - c^2}\,d\zeta = \int_x^{b_1} \sqrt{-h_o(\xi)}\,d\xi, \quad x \le b_1, \quad z \le -c$$

$$\int_{-c}^{z} \sqrt{c^2 - \zeta^2}\,d\zeta = \int_{b_1}^{x} \sqrt{h_o(\xi)}\,d\xi, \quad b_1 \le x \le b_2, \quad -c \le z \le c$$

$$\int_{c}^{z} \sqrt{\zeta^2 - c^2}\,d\zeta = \int_{b_2}^{x} \sqrt{-h_o(\xi)}\,d\xi, \quad b_2 \le x, \quad c \le z$$

which give the results

$$\frac{z}{c^2}\sqrt{z^2 - c^2} + \operatorname{Cosh}^{-1} - \frac{z}{c} = -\frac{2}{c^2}\int_x^{b_1} \sqrt{-h_o(\xi)}\,d\xi \tag{2.15}$$

where $x \le b_1$, $z \le -c$, $0 \le \operatorname{Cosh}^{-1} - \frac{z}{c}$,

$$\frac{z}{c^2}\sqrt{c^2 - z^2} + \cos^{-1} - \frac{z}{c} = \frac{2}{c^2}\int_{b_1}^{x} \sqrt{h_o(\xi)}\,d\xi \tag{2.16}$$

where $b_1 \le x \le b_2$, $-c \le z \le c$, $0 \le \cos^{-1} - \frac{z}{c} \le \pi$, and

$$\frac{z}{c^2}\sqrt{z^2 - c^2} - \mathrm{Cosh}^{-1}\frac{z}{c} = \frac{2}{c^2}\int_{b_2}^{x}\sqrt{-h_o(\xi)}\,d\xi \qquad (2.17)$$

where $b_2 \leq x$, $c \leq z$, $0 \leq \mathrm{Cosh}^{-1}\frac{z}{c}$.

Since $x = b_2$ and $z = c$ must correspond, these values may be substituted into (2.16), and the positive constant c is determined by

$$c^2 = \frac{2}{\pi}\int_{b_1}^{b_2}\sqrt{h_o(\xi)}\,d\xi \qquad (2.18)$$

With this value for c, it will be shown that equations (2.15,16,17) determine an analytic function $z = \Psi(x)$ which is real and regular for all real x with the property that $0 < \Psi'(x)$ for all real x.

It follows from (2.14) that the quotient $\frac{c^2 - z^2}{h_o(x)}$ is positive for all real x and z since the zeros of numerator and denominator have been made to correspond in such a way that the numerator and denominator always have the same sign. Therefore, when x and z are real $\phi' = \frac{dx}{dz}$ and $\Psi' = \frac{dz}{dx}$ do not vanish and are always positive. That is, the functions ϕ and Ψ will be monotonic increasing and inverse to one another. Moreover, $z = \mp c$ only when real $x = b_1$, b_2.

From (2.16) it follows that

$$\frac{z}{c^2}\sqrt{c^2 - z^2} + \cos^{-1}-\frac{z}{c} = \pi + \frac{2}{c^2}\int_{b_2}^{x}\sqrt{h_o(\xi)}\,d\xi$$

where π is introduced by using (2.18), and since $\cos^{-1}-\frac{z}{c} = \pi - \cos^{-1}\frac{z}{c}$ with $0 \leq \cos^{-1}\frac{z}{c} \leq \pi$, this last equation becomes

$$\frac{z}{c^2}\sqrt{c^2 - z^2} - \cos^{-1}\frac{z}{c} = \frac{2}{c^2}\int_{b_2}^{x}\sqrt{h_o(\xi)}\,d\xi \qquad (2.19)$$

where $b_1 \leq x \leq b_2$, $-c \leq z \leq c$. This is equivalent to

$$\int_c^z \sqrt{c^2 - \zeta^2}\, d\zeta = \int_{b_2}^x \sqrt{h_o(\xi)}\, d\xi, \quad b_1 \le x \le b_2 \tag{2.20}$$

Since the analytic function $h_o(\xi)$ is regular for all real ξ and has simple zeros $b_1 < b_2$, the following expansion is valid about b_2:

$\sqrt{h_o(\xi)} = \sqrt{-h_o'(b_2)}\,(b_2 - \xi)^{1/2} + c_1(b_2 - \xi)^{3/2} + \ldots$, with $h_o'(b_2) < 0$.

Also $\sqrt{c^2 - \zeta^2}$ may be expanded about $\zeta = c$ giving

$$\sqrt{c^2 - \zeta^2} = \sqrt{2c}\,(c - \zeta)^{1/2} + d_1(c - \zeta)^{3/2} + \ldots.$$

so that equation (2.20) gives

$$\frac{2}{3}\sqrt{2c}\,(c-z)^{3/2} + \frac{2}{5}\,d_1(c-z)^{5/2} + \ldots = \frac{2}{3}\sqrt{-h_o'(b_2)}\,(b_2-x)^{3/2} + \frac{2}{5}\,c_1(b_2-x)^{5/2} + \ldots \tag{2.21}$$

It follows from (2.21) that one may take

$$\arg \frac{c - z}{b_2 - x} \approx 0 \text{ when } x \text{ is sufficiently near } b_2 \tag{2.22}$$

When $x < b_2$ take $\arg(b_2 - x) = 0$. Then, from (2.22), $\arg(c - z) = 0$ and $z < c$. Let x be continued into the complex x-plane from the left of b_2 around a semi-circular arc to the right of b_2, either above or below b_2. Then $b_2 < x$ and $\arg(b_2 - x) = \pm\pi$ so that, from (2.22), $\arg(c - z) = \pm\pi$. It follows that $c < z$, $1 < \frac{z}{c}$, and $\cos^{-1}\frac{z}{c}$ is pure imaginary. The following relations are then valid:

$\sqrt{c^2 - z^2} = \sqrt{z^2 - c^2}\, e^{\pm i\frac{\pi}{2}}$, $c < z$, and $h_o(\xi) = -h_o(\xi)\, e^{\pm i\pi}$, $b_2 < \xi$.

These are introduced into (2.19) giving

$$\frac{z}{c^2}\sqrt{z^2 - c^2} - e^{\mp i\frac{\pi}{2}}\cos^{-1}\frac{z}{c} = \frac{2}{c^2}\int_{b_2}^x \sqrt{-h_o(\xi)}\, d\xi,$$

and since $\cos^{-1}\frac{z}{c}$ is pure imaginary, $e^{\mp i\frac{\pi}{2}}\cos^{-1}\frac{z}{c} = \operatorname{Cosh}^{-1}\frac{z}{c}$, so that the last equation becomes

$$\frac{z}{c^2}\sqrt{z^2 - c^2} - \mathrm{Cosh}^{-1}\frac{z}{c} = \frac{2}{c^2}\int_{b_2}^{x}\sqrt{-h_o(\xi)}\,d\xi, \quad b_2 < x, \quad c < z$$

which is identical with (2.17). Thus, by continuing x around b_2, either above or below b_2, equation (2.16) goes over into (2.17).

When equation (2.16) is investigated at the points $x = b_1$ and $z = -c$, a similar discussion shows that, by continuing x into the complex x-plane from the right of b_1 around a semi-circular path to the left of b_1, either above or below b_1, equation (2.16) goes over into (2.15).

In equations (2.15,16,17) let the left members be denoted by $F_-(z)$, $F_o(z)$ and $F_+(z)$ and the right members by W_-, W_o and W_+, respectively, so that these equations become

$$W_- = F_-(z), \quad W_o = F_o(z), \quad W_+ = F_+(z) \tag{2.23}$$

where the subscripts +, o or - are used in the following manner:

Subscript notation. The subscripts +, o or - shall mean that the quantity bearing these subscripts is associated with the interval of the real axis to the right, between, or to the left, respectively, of the zeros of the function of interest.

Sometimes the subscript o is not intended to convey the meaning of this subscript notation. This is evident in $h_o(x)$. From (2.15,16,17) it follows that

$$F_-'(z) = \frac{2}{c^2}\sqrt{z^2 - c^2}, \quad F_o'(z) = \frac{2}{c^2}\sqrt{c^2 - z^2}, \quad F_+'(z) = \frac{2}{c^2}\sqrt{z^2 - c^2} \tag{2.24}$$

and that F_-, F_o and F_+ are analytic functions of z, real and regular for all real z except $z = \pm c$. It is seen from (2.24) that F'_-, F'_o, and F'_+ vanish only for $z = \pm c$, i.e. only when $x = b_1, b_2$ by a previous remark. This means that equations (2.23) may be inverted in the neighborhood of any $z \neq \pm c$ giving

$$z = H_-(W_-), \quad z = H_o(W_o), \quad z = H_+(W_+) \tag{2.25}$$

where the H's are analytic functions of the W's, real and regular when real $z \neq \mp c$, or when $x \neq b_1, b_2$. But since the W's are given by the right members of (2.15,16,17), they have expansions similar to the right member of (2.21) and are therefore analytic functions of x, real and regular for all real x except $x = b_1, b_2$. Consequently, the H's are analytic functions of x, real and regular for all real x except $x = b_1, b_2$, and they will be denoted by $\psi(x)$. Equations (2,25) then become

$$z = \psi_-(x), \quad z = \psi_o(x), \quad z = \psi_+(x) \tag{2.26}$$

where ψ_- is defined by (2.15), ψ_o by (2.16), and ψ_+ by (2.17). However, it has previously been shown that (2.16) may be continued into (2.15) and (2.17). This means that ψ_o may be continued into ψ_- and ψ_+. Therefore, these three functions are analytic continuations of each other and represent a single function $\psi(x)$. Moreover, the functions are defined and have first derivatives at the points $x = b_1, b_2$ so that these points must belong to the region of regularity of $\psi(x)$.

Hence $z = \psi(x)$ is an analytic function of x, real and regular for all real x, with the property that $0 < \psi'(x)$ for all real x as was previously pointed out. This function is the solution of equation

(2.13). Its inverse, $x = \phi(z)$, is the desired transformation (2.11), and is an analytic function, real and regular for all real z, with the property that $0 < \phi'(z)$ for all real z. Referring to the statement following equation (2.13), it is easily seen that equation (2.12) may be reduced to the form

$$\frac{d^2w}{dz^2} + \mu^2\left[c^2 - z^2 + f(z,\mu^{-1})\right]w = 0 \tag{2.27}$$

where $f(z,\mu^{-1})$ is an analytic function of z and μ^{-1}, real and regular for all real z and $\mu_o \leq \mu$. This function possesses the expansion

$$f(z,\mu^{-1}) = \sum_{n=1}^{\infty} f_n(z)\mu^{-n} \tag{2.28}$$

and will be subjected to suitable hypotheses. The investigation will henceforth be confined to an equation of this form.

2.3 The Comparison Equation and Its Solutions.

Equation (2.27) will be compared with the equation

$$\frac{d^2u}{dt^2} + \mu^2(a^2 - t^2)u = 0 \tag{2.29}$$

which may be transformed into Weber's equation

$$\frac{d^2w}{dz^2} + (\nu + \frac{1}{2} - \frac{z^2}{4})w = 0 \tag{2.30}$$

where $\nu = \mu\,\frac{a^2}{2} - \frac{1}{2}$, by the transformation

$$t = \frac{z}{\sqrt{2\mu}}, \quad u(t) = w(z) \tag{2.31}$$

It should be noticed that the coefficients of the dependent variables in (2.27) and (2.29) have a common property. In their expansion in powers of μ^{-1}, the coefficients of μ^2 have only two simple zeros on the real axis and are positive between the zeros.

The parabolic cylinder functions $D_\nu(\pm z)$, $\underline{D}_{-\nu-1}(\pm iz)$ are solutions of Weber's equation so that solutions of (2.29), using (2.31), are $D_\nu(\pm\sqrt{2\mu}\, t)$, $\underline{D}_{-\nu-1}(\pm i\sqrt{2\mu}\, t)$.

Some properties of the parabolic cylinder functions $D_\nu(\pm z)$ and $\underline{D}_{-\nu-1}(\pm iz)$ are [7,8]:

(i) They are integral functions of z

(ii) $$D_\nu(z) = \frac{\Gamma(\nu+1)}{\sqrt{2\pi}}\left\{e^{\frac{\nu}{2}\pi i}\,\underline{D}_{-\nu-1}(iz) + e^{-\frac{\nu}{2}\pi i}\,\underline{D}_{-\nu-1}(-iz)\right\} \tag{2.32}$$

$$= e^{\nu\pi i}\, D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\, e^{\frac{\nu+1}{2}\pi i}\,\underline{D}_{-\nu-1}(-iz) \tag{2.33}$$

(iii) $$\underline{D}_{-\nu-1}(iz) = \frac{\sqrt{2\pi}}{\Gamma(\nu+1)}\, e^{\frac{\nu}{2}\pi i}\, D_\nu(-z) - e^{\nu\pi i}\,\underline{D}_{-\nu-1}(-iz) \tag{2.34}$$

(iv) They have the Wronskians

$$D_\nu(z)\frac{d}{dz}D_\nu(-z) - D_\nu(-z)\frac{d}{dz}D_\nu(z) = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} \tag{2.35}$$

$$D_\nu(z)\frac{d}{dz}\underline{D}_{-\nu-1}(iz) - \underline{D}_{-\nu-1}(iz)\frac{d}{dz}D_\nu(z) = -ie^{-\frac{\nu}{2}\pi i} \tag{2.36}$$

(v) If ν is real, $D_\nu(z)$ has $[\nu+1]$ real zeros where $[\nu+1]$ denotes the greatest positive integer less than $\nu+1$ or zero.

From (2.35) it follows that $D_\nu(z)$, $D_\nu(-z)$ are linearly independent if ν is not a positive integer or zero. It follows from (2.36) that $D_\nu(z)$, $\underline{D}_{-\nu-1}(iz)$ and $D_\nu(-z)$, $\underline{D}_{-\nu-1}(-iz)$ are linearly independent for all values of ν.

When ν is not a positive integer or zero, the general solution of the comparison equation is

$$u(t) = A_1 D_\nu(\sqrt{2\mu}\, t) + A_2 D_\nu(-\sqrt{2\mu}\, t) \tag{2.37}$$

where

$$v = \mu \frac{a^2}{2} - \frac{1}{2} \tag{2.38}$$

In addition to the foregoing properties of the parabolic cylinder functions, various asymptotic formulas and bounds for these functions are needed. These will be discussed in Section 6.

SECTION 3

HYPOTHESES FOR $f(x,\mu^{-1})$ AND CONSTRUCTION OF THE TRANSFORMATION

The differential equation to be investigated, (2.9) and (2.27), in which the variables are renamed, becomes

$$\frac{d^2y}{dx^2} + \mu^2\left[c^2 - x^2 + f(x,\mu^{-1})\right]y = 0 \tag{3.1}$$

where the function $f(x,\mu^{-1})$ is subjected to the following

Hypotheses:

(i) $f(x,\mu^{-1})$ is an analytic function of x and μ^{-1}, real and regular for all real x and $\mu_0 \leq \mu$, and possesses an expansion of the form

$$f(x,\mu^{-1}) = \sum_{n=1}^{\infty} f_n(x)\,\mu^{-n} \tag{3.2}$$

where the $f_n(x)$ are analytic functions of x, real and regular for all real x.

(ii) When x is real and $x \to \pm\infty$, $f(x,\mu^{-1})$ may be put into the form

$$f(x,\mu^{-1}) = \alpha_\mu x^2 + \beta_\mu x + \gamma_\mu + O(x^{-\eta}), \quad 0 < \eta < 1 \tag{3.3}$$

where $$\alpha_\mu = \sum_{n=1}^{\infty} \alpha_n \mu^{-n}, \quad \beta_\mu = \sum_{n=1}^{\infty} \beta_n \mu^{-n}, \quad \gamma_\mu = \sum_{n=1}^{\infty} \gamma_n \mu^{-n} \tag{3.4}$$

the series for α_μ, β_μ and γ_μ being convergent.

From hypothesis (ii) it follows immediately that in (3.2)

$$f_n(x) = \alpha_n x^2 + \beta_n x + \gamma_n + O(x^{-\eta}) \text{ as } x \to \pm\infty \tag{3.5}$$

In order to compare (3.1) with the comparison equation

$$\frac{d^2u}{dt^2} + \mu^2(a^2 - t^2)u = 0 \tag{3.6}$$

this comparison equation will be expressed in terms of the variables x and v by putting

$$t = \phi(x), \quad u(t) = v(x)\, \phi'(x)^{\frac{1}{2}} \tag{3.7}$$

and using equations (2.6,7,8). The result is

$$\frac{d^2v}{dx^2} + \left\{\mu^2(a^2 - \phi^2)\, \phi'^2 + \frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2}\right\} v = 0 \tag{3.8}$$

It is now required to construct the transformation $t = \phi(x)$ and to determine the quantity a^2 so that equations (3.1) and (3.8) will be approximately the same. The required transformation may be taken in the form

$$t = \phi(x) \equiv \phi(x,\mu^{-1}) = x + \sum_{n=1}^{\infty} \phi_n(x)\, \mu^{-n} \tag{3.9}$$

The choice $a^2 = c^2$ would seem to be the natural one at first glance. However, it is necessary to choose a^2 so that $\phi(x,\mu^{-1})$ has no real singularities. This requirement is met by taking

$$a^2 = c^2 + \sum_{n=1}^{\infty} a_n \mu^{-n} \tag{3.10}$$

in which the a_n will be determined subsequently. Let the function $g(x,\mu^{-1})$ be defined by

$$g(x,\mu^{-1}) = \mu^2(a^2-\phi^2)\, \phi'^2 + \frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2} - \mu^2(c^2-x^2) - \mu^2 f(x,\mu^{-1}) \tag{3.11}$$

and note that the differential equations (3.1) and (3.8) would be identical if $g(x,\mu^{-1})$ were required to be zero, and that $g(x,\mu^{-1})$ measures the discrepancy between these two equations. The equation $g(x,\mu^{-1}) = 0$ is a non-linear differential equation for $\phi(x,\mu^{-1})$. This equation determines a system of equations for the ϕ_n of (3.9) and the a_n of (3.10). Such formal series do not necessarily converge, but their first m + 1 terms, $x + \sum_{n=1}^{m} \phi_n(x)\,\mu^{-n}$ and $c^2 + \sum_{n=1}^{m} a_n\mu^{-n}$, where m is arbitrary, will be computed and retained since they make the expansion of $g(x,\mu^{-1})$ in powers of μ^{-1} begin with the term μ^{-m+1}. In order to determine ϕ_n, a_n, $m + 1 \leq n$, the remaining system of equations for ϕ_n and a_n will be modified so that (i) the series (3.9) and (3.10) converge, (ii) $g(x,\mu^{-1}) = O(\mu^{-m+1})$, and (iii) g tends to zero as x tends to $\pm\infty$. Consequently, the differential equations (3.1) and (3.8) will approximate each other very closely for large μ and <u>all</u> real x.

Introducing (3.2,9,10) into (3.11) and using the Cauchy product for power series gives the result

$$
\begin{aligned}
g(x,\mu^{-1}) &= \mu^2\left[\sum_{n=1}^{\infty} a_n\mu^{-n} - 2x\sum_{n=1}^{\infty}\phi_n\mu^{-n} - \sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}\phi_k\phi_{n-k}\Big)\mu^{-n}\right] + \\
&\quad + \mu^2\left[c^2 - x^2 + \sum_{n=1}^{\infty} a_n\mu^{-n} - 2x\sum_{n=1}^{\infty}\phi_n\mu^{-n} - \sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}\phi_k\phi_{n-k}\Big)\mu^{-n}\right]\cdot \\
&\quad \cdot\left[2\sum_{n=1}^{\infty}\phi_n'\,\mu^{-n} + \sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}\phi_k'\,\phi_{n-k}'\Big)\mu^{-n}\right] + \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty}\phi_n'''\mu^{-n}\Big(1 + \sum_{n=1}^{\infty}\phi_n'\,\mu^{-n}\Big)^{-1} - \frac{3}{4}\left[\sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}\phi_k''\,\phi_{n-k}''\Big)\mu^{-n}\right]\cdot \\
&\quad \cdot\Big(1 + \sum_{n=1}^{\infty}\phi_n'\,\mu^{-n}\Big)^{-2} - \mu^2\sum_{n=1}^{\infty} f_n\,\mu^{-n} \qquad (3.12)
\end{aligned}
$$

in which the coefficients of μ^{-n}, $n = -1,0,1,\ldots, m - 2$ are set equal to zero, giving a system of differential equations which determine the

ϕ_n and the a_n, $n = 1, \ldots, m$.

$$\mu: \qquad 2(c^2 - x^2)\phi_1' - 2x\phi_1 = f_1 - a_1 \tag{3.13}$$

$$\mu^0: \qquad 2(c^2 - x^2)\phi_2' - 2x\phi_2 = f_2 - a_2 - 2a_1\phi_1' - $$

$$-(c^2 - x^2)\phi_1'^2 + \phi_1^2 + 4x\phi_1\phi_1' \tag{3.14}$$

and in general, for ϕ_k

$$\mu^{-k+2}: \; 2(c^2 - x^2)\phi_k' - 2x\phi_k = f_k - a_k + P_k - p_k \tag{3.15}$$

where

$$P_k(x) = \sum_{j=1}^{k-1}\left[\phi_j\phi_{k-j} - (c^2 - x^2)\,\phi_j'\phi_{k-j}' - 2a_j\phi_{k-j}' + 4x\phi_j\phi_{k-j}'\right] +$$

$$+ \sum_{j=1}^{k-2}\;\sum_{r=1}^{k-j-1}\left[2\phi_j'\phi_r\phi_{k-j-r} - a_j\phi_r'\phi_{k-j-r}' + 2x\phi_j\phi_r'\phi_{k-j-r}'\right] +$$

$$+ \sum_{j=2}^{k-2}\left(\sum_{r=1}^{j-1}\phi_r\phi_{j-r}\right)\left(\sum_{h=1}^{k-j-1}\phi_h'\phi_{k-j-h}'\right) \tag{3.16}$$

$$p_k(x) = \begin{Bmatrix}\text{Coefficient of } \mu^{-k+2} \text{ in the expansion} \\ \text{in powers of } \mu^{-1} \text{ of } \dfrac{1}{2}\dfrac{\phi'''}{\phi'} - \dfrac{3}{4}\dfrac{\phi''^2}{\phi'^2}\end{Bmatrix} \tag{3.17}$$

It is most easily seen from (3.13) that

$$P_1(x) \equiv 0 \tag{3.18}$$

and from (3.16) that $P_k(x)$ is a polynomial with constant coefficients in ϕ_n, ϕ_n', a_n and x, $n = 1, \ldots, k-1$. From the terms in (3.12) containing second and third derivatives it follows that

$$p_1(x) \equiv p_2(x) \equiv 0 \tag{3.19}$$

and also from these same terms in (3.12) that $p_k(x)$ has the form

$$p_k(x) = \left\{\begin{array}{l}\text{Polynomial with constant coefficients in} \\ \phi_n', \phi_n'', \phi_n''', \; n = 1, \ldots\ldots, k-2. \\ \text{It is quadratic in } \phi_n'' \text{ and linear in } \phi_n'''. \end{array}\right\} \tag{3.20}$$

Equations (3.13,14,15) may be used to successively determine $\phi_1, \ldots\ldots, \phi_m$ and $a_1, \ldots\ldots, a_m$ when appropriate boundary conditions are imposed. These conditions will be chosen so that the ϕ_k will have no real singularities. It is sufficient to discuss the general equation (3.15) which may be put in the two forms

$$\sqrt{c^2-x^2}\;\phi_k' - \frac{x\phi_k}{\sqrt{c^2-x^2}} = \frac{f_k(x)+P_k(x)-p_k(x)}{2\sqrt{c^2-x^2}} - \frac{a_k}{2\sqrt{c^2-x^2}}, \quad |x| < c \tag{3.21}$$

$$\sqrt{x^2-c^2}\;\phi'_k + \frac{x\phi_k}{\sqrt{x^2-c^2}} = -\frac{f_k(x)+P_k(x)-p_k(x)}{2\sqrt{x^2-c^2}} + \frac{a_k}{2\sqrt{x^2-c^2}}, \quad c < |x| \tag{3.22}$$

Integrating (3.21) gives the result

$$\sqrt{c^2-x^2}\;\phi_k = \int_{-c}^{x} \frac{f_k(\xi)+P_k(\xi)-p_k(\xi)}{2\sqrt{c^2-\xi^2}}\,d\xi - \frac{a_k}{2}\cos^{-1}\!-\frac{x}{c} \tag{3.23}$$

$$\text{where } -c \le x \le c, \quad 0 \le \cos^{-1}\!-\frac{x}{c} \le \pi$$

from which a_k can be determined by putting $x = c$ giving

$$a_k = \frac{1}{\pi}\int_{-c}^{c} \frac{f_k(\xi) + P_k(\xi) - p_k(\xi)}{\sqrt{c^2 - \xi^2}}\,d\xi \tag{3.24}$$

With this value for a_k and the subscript notation of Section 2.2, where the function of interest is $c^2 - x^2$, it follows from (3.22) that

$$\phi_{k-}(x) = \frac{1}{\sqrt{x^2-c^2}}\int_{x}^{-c} \frac{f_k(\xi)+P_k(\xi)-p_k(\xi)}{2\sqrt{\xi^2-c^2}}\,d\xi - \frac{a_k}{2\sqrt{x^2-c^2}}\,\text{Cosh}^{-1}\!-\frac{x}{c} \tag{3.25}$$

$$\text{where } x < -c, \quad 0 < \text{Cosh}^{-1}\!-\frac{x}{c}$$

from (3.23) that

$$\phi_{ko}(x) \frac{1}{\sqrt{c^2-x^2}} \int_{-c}^{x} \frac{f_k(\xi)+P_k(\xi)-p_k(\xi)}{2\sqrt{c^2-\xi^2}} d\xi - \frac{a_k}{2\sqrt{c^2-x^2}} \cos^{-1}-\frac{x}{c} \qquad (3.26)$$

$$\text{where } -c < x < c, \quad 0 < \cos^{-1}-\frac{x}{c} < \pi$$

and again from (3.22) that

$$\phi_{k+}(x) = \frac{-1}{\sqrt{x^2-c^2}} \int_{c}^{x} \frac{f_k(\xi)+P_k(\xi)-p_k(\xi)}{2\sqrt{\xi^2-c^2}} d\xi + \frac{a_k}{2\sqrt{x^2-c^2}} \operatorname{Cosh}^{-1}\frac{x}{c} \qquad (3.27)$$

$$\text{where } c < x, \quad 0 < \operatorname{Cosh}^{-1}\frac{x}{c}$$

The functions ϕ_{k-}, ϕ_{ko}, ϕ_{k+} assume indeterminate forms at $x = \pm c$, but they may be defined there by their limiting values, obtained after applying L'Hospital's rule. These definitions are

$$\phi_{k-}(-c) = \phi_{ko}(-c) = \frac{f_k(-c) + P_k(-c) - p_k(-c) - a_k}{2c} \qquad (3.28)$$

$$\phi_{ko}(c) = \phi_{k+}(c) = -\frac{f_k(c) + P_k(c) - p_k(c) - a_k}{2c} \qquad (3.29)$$

It is easily shown that ϕ_{k-}, ϕ_{ko}, and ϕ_{k+} are analytic continuations of each other and are therefore elements of an analytic function $\phi_k(x)$, real and regular for all real x. This assertion is first established for $\phi_1(x)$. All the functions involved in the definitions of ϕ_{1-}, ϕ_{1o}, and ϕ_{1+} (3.25,26,27) are analytic functions of x, real and regular for all real x, except $x = \pm c$, and if x is continued into the complex x-plane along a semi-circular path around $\pm c$, whether above or below $\pm c$, ϕ_{1o} goes over into $\phi_{1\pm}$, respectively. These functions are defined at $x = \pm c$ by (3.29) and (3.28) so that $\pm c$ must belong to their region of regularity. This implies that ϕ_{1-}, ϕ_{10}, and ϕ_{1+} are elements of an analytic function $\phi_1(x)$, real and regular for all real x. The assertion follows for $\phi_k(x)$ from (3.25,26,27) by induction.

Having determined the ϕ_k, a_k, $k = 1, \ldots, m$, the system of differential equations (3.15) is modified, as mentioned on page 20, in order to determine the remaining ϕ_k, a_k. Thus, when $m + 1 \leq k$, the quantity $p_k(x)$ in (3.15) is deleted so that ϕ_k, a_k are given by (3.24,25,26,27) after crossing out the $p_k(\xi)$ in these equations. This modification is equivalent to deleting all terms in (3.12) which contain second and third derivatives, when the differential equations for the ϕ_k, $m + 1 \leq k$, are formed. As can be seen from (3.17) this is also equivalent to using only the first m - 1 terms in the expansion of $\frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2}$ in powers of μ^{-1} for the determination of <u>all</u> the ϕ_k, a_k. Since a convergence discussion of the series for ϕ and a^2 is difficult, this required result will be obtained from properties of the functions themselves.

A differential equation which determines ϕ and a^2, and hence <u>all</u> the ϕ_k, a_k, is easily constructed. For this purpose, define

$$F_k(x) = f_k(x) - p_k(x), \quad k = 1, \ldots, m \tag{3.30}$$

$$F_k(x) = f_k(x), \qquad m + 1 \leq k \tag{3.31}$$

$$F(x,\mu^{-1}) = \sum_{n=1}^{\infty} F_n(x)\mu^{-n} \equiv f(x,\mu^{-1}) - \left\{\begin{array}{l}\text{first m-1 terms in} \\ \text{the expansion of} \\ \frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2} \\ \text{in powers of } \mu^{-1}\end{array}\right\} \tag{3.32}$$

and introduce (3.32) into (3.11) which gives, because of the above modification, the following differential equation for ϕ:

$$\mu^2(a^2 - \phi^2)\,\phi'^2 - \mu^2(c^2 - x^2) - \mu^2 F(x,\mu^{-1}) = 0 \tag{3.33}$$

Moreover, once the $\phi_1,\dots,\phi_m$ and $a_1,\dots,a_m$ have been determined, $F(x,\mu^{-1})$ is a <u>known</u> analytic function of x and μ^{-1}, real and regular for all real x and $\mu_0 \le \mu$.

It will be shown that (i) the solution $\phi(x,\mu^{-1})$ of (3.33) exists and is an analytic function of x and μ^{-1}, real and regular for all real x and $\mu_0 \le \mu$, and its expansion in powers of μ^{-1} has as coefficients the ϕ_k previously determined for $k = 1,\dots, m$, (ii) a^2 can be determined from (3.33) and is an analytic function of μ^{-1}, real and regular for $\mu_0 \le \mu$, having as coefficients in its expansion in powers of μ^{-1} the a_k already computed, $k = 1,\dots, m$, and (iii) $0 < \phi'$ for all real x and $\mu_0 \le \mu$ so that $t = \phi(x,\mu^{-1})$ is a reversible transformation with respect to x.

From (3.33) and (3.11) it follows that

$$\begin{aligned} g(x,\mu^{-1}) &= \frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2} + \mu^2(F - f) \\ &= \frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2} + \mu^2 \sum_{n=1}^{\infty} (F_n - f_n)\,\mu^{-n} \end{aligned}$$

and using (3.30,31) this last equation becomes

$$g(x,\mu^{-1}) = \frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2} - \mu^2 \sum_{n=1}^{m} p_n \mu^{-n} \tag{3.34}$$

SECTION 4

PROPERTIES OF THE TRANSFORMATION $t = \phi(x, \bar{\mu}')$

4.1 Behavior of the ϕ_k, $k = 1, \ldots, m$ as $x \to \pm\infty$.

For this purpose Hypothesis (ii) will be used. A detailed discussion will be given for $\phi_1(x)$. From equations (3.27,18,19)

$$\sqrt{x^2 - c^2}\, \phi_{1+}(x) = -\frac{1}{2}\int_c^x \frac{f_1(\xi)d\xi}{\sqrt{\xi^2 - c^2}} + \frac{a_1}{2}\,\mathrm{Cosh}^{-1}\frac{x}{c}, \qquad c < x \tag{4.1}$$

and, making use of (3.5), this equation may be put into the form

$$\sqrt{x^2 - c^2}\, \phi_{1+}(x) = -\frac{1}{2}\int_c^x \frac{\alpha_1\xi^2 + \beta_1\xi + \gamma_1}{\sqrt{\xi^2 - c^2}}\,d\xi + \frac{a_1}{2}\,\mathrm{Cosh}^{-1}\frac{x}{c} -$$

$$-\frac{1}{2}\int_c^\infty \frac{f_1(\xi) - \alpha_1\xi^2 - \beta_1\xi - \gamma_1}{\sqrt{\xi^2 - c^2}}\,d\xi + \frac{1}{2}\int_x^\infty \frac{O(\xi^{-\eta})}{\sqrt{\xi^2 - c^2}}\,d\xi \tag{4.2}$$

giving, after integration, the result

$$\phi_{1+}(x) = -\frac{\alpha_1}{4}x - \frac{\beta_1}{2} + \omega_1\frac{\mathrm{Cosh}^{-1}\frac{x}{c}}{\sqrt{x^2 - c^2}} + \frac{\delta_{1+}}{\sqrt{x^2 - c^2}} + O(x^{-1-\eta}), \qquad \text{as } x \to \infty \tag{4.3}$$

where $$\delta_{1+} = -\frac{1}{2}\int_c^\infty \frac{f_1(\xi) - \alpha_1\xi^2 - \beta_1\xi - \gamma_1}{\sqrt{\xi^2 - c^2}}\,d\xi \tag{4.4}$$

and $$\omega_1 = \frac{a_1}{2} - \frac{\alpha_1 c^2}{2} - \frac{\gamma_1}{2} \tag{4.5}$$

Differentiating (4.3) gives

$$\phi'_{1+}(x) = -\frac{\alpha_1}{4} - \omega_1 x\frac{\mathrm{Cosh}^{-1}\frac{x}{c}}{(x^2 - c^2)^{3/2}} + \frac{\omega_1}{x^2 - c^2} - \frac{\delta_{1+}x}{(x^2 - c^2)^{3/2}} + O(x^{-2-\eta}) \tag{4.6}$$

It follows from (3.16), but is most easily seen in the right member of (3.14), that $P_2(x)$ is given by

$$P_2(x) = -2a_1\phi_1' + (x^2 - c^2)\phi_1'^2 + \phi_1^2 + 4x\phi_1\phi_1' \tag{4.7}$$

In order to compute P_2 up to terms of order $O(x^{-\eta})$, $\phi_1'^2$ must be computed up to $O(x^{-2-\eta})$, ϕ_1^2 up to $O(x^{-\eta})$, and $\phi_1\phi_1'$ up to $O(x^{-1-\eta})$. Making these calculations with (4.3) and (4.6), and introducing the results into (4.7) gives

$$P_{2+}(x) = \frac{3}{8}\alpha_1^2x^2 + \frac{3}{4}\alpha_1\beta_1 x - \frac{a_1}{4}\alpha_1 + \frac{5}{16}\alpha_1^2c^2 + \frac{3}{4}\alpha_1\gamma_1 + \frac{\beta_1^2}{4} + O(x^{-\eta}) \tag{4.8}$$

in which ω_1 has been eliminated by using (4.5).

From equations (3.25), (3.18) and (3.19)

$$\sqrt{x^2-c^2}\,\phi_{1-}(x) = \frac{1}{2}\int_x^{-c}\frac{f_1(\xi)d\xi}{\sqrt{\xi^2-c^2}} - \frac{a_1}{2}\operatorname{Cosh}^{-1}-\frac{x}{c}, \quad x < -c \tag{4.9}$$

into which (3.5) is introduced, the result, after integration, being

$$\phi_{1-}(x) = -\frac{\alpha_1}{4}x - \frac{\beta_1}{2} - \omega_1\frac{\operatorname{Cosh}^{-1}-\frac{x}{c}}{\sqrt{x^2-c^2}} + \frac{\delta_{1-}}{\sqrt{x^2-c^2}} + O(x^{-1-\eta}) \text{ as } x \to -\infty \tag{4.10}$$

where ω_1 is given by (4.5) and

$$\delta_{1-} = \frac{1}{2}\int_{-\infty}^{-c}\frac{f_1(\xi) - \alpha_1\xi^2 - \beta_1\xi - \gamma_1}{\sqrt{\xi^2 - c^2}}\,d\xi \tag{4.11}$$

Substituting ϕ_{1-}, ϕ_{1-}' into (4.7), and following the procedure stated below (4.7), gives the result

$$P_{2-}(x) = \frac{3}{8}\alpha_1^2x^2 + \frac{3}{4}\alpha_1\beta_1 x - \frac{a_1}{4}\alpha_1 + \frac{5}{16}\alpha_1^2c^2 + \frac{3}{4}\alpha_1\gamma_1 + \frac{\beta_1^2}{4} + O(x^{-\eta}) \tag{4.12}$$

where, again, ω_1 has been eliminated by (4.5). Equations (4.8) and

(4.12) show that P_{2+} and P_{2-} are identical, that is, $P_2(x)$ has the form

$$P_2(x) = A_2x^2 + B_2x + C_2 + O(x^{-\eta}), \quad \text{as } x \to \pm\infty \tag{4.13}$$

which is of the same form as the $f_2(x)$ given by (3.5). Therefore, the ϕ_2 given by (3.25) and (3.27), using (3.19), will have the same form as ϕ_1. Because of the form of ϕ_{1+} in (4.3), the assumption is made that

$$\phi_{j+}(x) = b_jx + c_j + \omega_j \frac{\operatorname{Cosh}^{-1}\frac{x}{c}}{\sqrt{x^2 - c^2}} + \frac{\delta_{j+}}{\sqrt{x^2 - c^2}} + O(x^{-1-\eta}) \tag{4.14}$$

$$\text{as } x \to \infty, \quad j = 1, \ldots, k - 1$$

and the form of $\phi_{k+}(x)$ will be established for all k by induction. The functions ϕ_{j+} and their first derivatives are substituted into (3.16), the computations being similar to the procedure outlined below (4.7). After a straightforward, but long calculation the result has the form

$$P_{k+}(x) = A_kx^2 + B_kx + C_k + O(x^{-\eta}) \tag{4.15}$$

Differentiating (4.14) gives the results

$$\phi'_{j+}(x) = b_j + O(x^{-1-\eta}), \quad \phi''_{j+}(x) = O(x^{-2-\eta}), \quad \phi'''_{j+}(x) = O(x^{-3-\eta}) \tag{4.16}$$

which are introduced into (3.20) giving

$$p_{k+}(x) = O(x^{-3-\eta}) \tag{4.17}$$

If (4.17), (4.15) and (3.5) are substituted into (3.27), the result has the same form as the right member of (4.2) so that

$$\phi_{k+}(x) = b_kx + c_k + \omega_k \frac{\operatorname{Cosh}^{-1}\frac{x}{c}}{\sqrt{x^2-c^2}} + \frac{\delta_{k+}}{\sqrt{x^2-c^2}} + O(x^{-1-\eta}), \quad \text{as } x \to \infty \tag{4.18}$$

Equation (4.18) establishes by induction the form of $\phi_{k+}(x)$, when $x \to \infty$, for all k in view of (4.3), (4.14), and (4.18). By noting the form of ϕ_{1-} in (4.10) it can be established in a similar fashion that the form of $\phi_{k-}(x)$, when $x \to -\infty$, for all k is

$$\phi_{k-}(x) = b_k x + c_k - \omega_k \frac{\text{Cosh}^{-1} - \frac{x}{c}}{\sqrt{x^2 - c^2}} + \frac{\delta_{k-}}{\sqrt{x^2 - c^2}} + O(x^{-1-\eta}), \text{ as } x \to -\infty \quad (4.19)$$

The derivatives of ϕ_k from (4.18) and (4.19) have the forms given by (4.16). When these derivatives are introduced into (3.20) the result is

$$p_k(x) = O(x^{-3-\eta}), \quad \text{as } x \to \pm\infty \quad (4.20)$$

4.2 The Solution $\phi(x)$ of Equation (3.33).

Equation (3.33) may be written

$$(a^2 - \phi^2)\phi'^2 - (c^2 - x^2) - F(x, \bar{\mu}') = 0 \quad (4.21)$$

It is necessary to discuss the behavior of the function

$$c^2 - x^2 + F(x, \bar{\mu}') \quad (4.22)$$

which occurs in (4.21). It follows from (3.2,32) and (3.30,31) that (4.22) is equivalent to

$$c^2 - x^2 + F \equiv c^2 - x^2 + f + (F - f) = c^2 - x^2 + f - \sum_{n=1}^{m} p_n \mu^{-n} \quad (4.23)$$

and from (4.20) that

$$\sum_{n=1}^{m} p_n \mu^{-n} = O(x^{-3-\eta}) \text{ as } x \to \pm\infty, \quad \mu_o \text{ sufficiently large} \quad (4.24)$$

Equations (3.3) and (4.24) are substituted into (4.23) giving the result

$$c^2 - x^2 + F = c^2 - x^2 + \alpha_\mu x^2 + \beta_\mu x + \gamma_\mu + O(x^{-\eta}) \text{ as } x \to \pm\infty \qquad (4.25)$$

For μ_0 sufficiently large it follows from (3.4) that $|\alpha_\mu| < 1$, $|\beta_\mu| < 1$ which, with (4.25), give the inequality

$$c^2 - x^2 + F < 0, \quad c << |x|, \quad \mu_0 \leq \mu \qquad (4.26)$$

Since the analytic function $F(x,\mu^{-1})$ is real and regular for all real x and $\mu_0 \leq \mu$, and possesses the expansion $F(x,\mu^{-1}) = \sum_{n=1}^{\infty} F_n(x)\mu^{-n}$ by (3.32), it follows that μF is bounded for $\mu_0 \leq \mu$, $X_1 \leq x \leq X_2$ where the numbers X_1, X_2 are arbitrary, but in the present discussion are chosen such that $X_1 \leq -2c$, $2c \leq X_2$. Consequently, for μ_0 large enough

$$|F| < c^2, \; X_1 \leq x \leq X_2, \quad \mu_0 \leq \mu \qquad (4.27)$$

Then, it follows from (4.26) and (4.27) that the function $c^2 - x^2 + F$ is negative at least when $2c^2 < x^2$, and from (4.27) that this function is positive for $|x|$ small enough. Moreover, $0 < c^2 + F < 2c^2$ by (4.27) so that the function $c^2 - x^2 + F$ of (4.22) has precisely two real zeros which evidently depend on μ and tend to $\pm$ c as μ tends to ∞. The equation $c^2 - x^2 + F(x,\mu^{-1}) = 0$ implies $x = x(\mu^{-1})$ is an analytic function of μ^{-1}, real and regular for $\mu_0 \leq \mu$, and consequently, the zeros may be represented by the expressions

$$x_1 = -c + \sum_{n=1}^{\infty} x_{1n}\mu^{-n}, \quad x_2 = c + \sum_{n=1}^{\infty} x_{2n}\mu^{-n} \qquad (4.28)$$

These results may be summed up in the statement that when μ_0 is sufficiently large and $\mu_0 \leq \mu$, the function $c^2 - x^2 + F(x,\mu^{-1})$ of (4.22), which occurs in the differential equation (4.21), has two real zeros given by (4.28) and has the property that $c^2 - x^2 + F < 0$, $x < x_1$ or $x_2 < x$ and $0 < c^2 - x^2 + F$, $x_1 < x < x_2$.

For a <u>fixed</u> value of μ, a discussion of the solution $\phi(x,\mu^{-1})$ of the differential equation (4.21) is identical with the discussion of the preliminary transformation in Section 2.2. For, $c^2 - x^2 + F(x,\mu^{-1})$ is then a function of x alone with two real zeros and is positive between them so that (4.21) is of the same form as the differential equation (2.13). Therefore, the analysis of Section 2.2 applies to the solution $\phi(x,\mu^{-1})$ of (4.21), and it follows that for fixed μ the transformation $t = \phi(x,\mu^{-1})$ is an analytic function of x, real and regular for all real x. However, all quantities in (4.21) and the limits of integration (4.28) depend on μ and are analytic functions of μ^{-1}, real and regular for $\mu_0 \leq \mu$. Consequently, $t = \phi(x,\mu^{-1})$ is an analytic function of x <u>and</u> μ^{-1}, real and regular for all real x and $\mu_0 \leq \mu$, with the property that $0 < \phi'(x,\mu^{-1})$ for all real x and $\mu_0 \leq \mu$. From equations (2.15,16,17,18), in which the necessary changes have been made, it follows that the elements ϕ_-, ϕ_0, ϕ_+ of the function $\phi(x,\mu^{-1})$ and the quantity a^2 are defined, respectively, by the equations

$$\frac{\phi}{a^2}\sqrt{\phi^2 - a^2} + \operatorname{Cosh}^{-1}\frac{-\phi}{a} = -\frac{2}{a^2}\int_x^{x_1}\left[\xi^2 - c^2 - F(\xi,\mu^{-1})\right]^{\frac{1}{2}}d\xi \qquad (4.29)$$

where $x \leq x_1$, $\phi \leq -a$, $0 \leq \operatorname{Cosh}^{-1}\frac{-\phi}{a}$

$$\frac{\phi}{a^2}\sqrt{a^2 - \phi^2} + \cos^{-1}\frac{-\phi}{a} = \frac{2}{a^2}\int_{x_1}^{x}\left[c^2 - \xi^2 + F(\xi,\mu^{-1})\right]^{\frac{1}{2}}d\xi \qquad (4.30)$$

where $x_1 \leq x \leq x_2$, $-a \leq \phi \leq a$, $0 \leq \cos^{-1}\frac{-\phi}{a} \leq \pi$

$$\frac{\phi}{a^2}\sqrt{\phi^2 - a^2} - \operatorname{Cosh}^{-1}\frac{\phi}{a} = \frac{2}{a^2}\int_{x_2}^{x}\left[\xi^2 - c^2 - F(\xi,\bar{\mu}^{1})\right]^{\frac{1}{2}}d\xi \qquad (4.31)$$

where $x_2 \leq x$, $a \leq \phi$, $0 \leq \operatorname{Cosh}^{-1}\frac{\phi}{a}$

$$a^2 = \frac{2}{\pi}\int_{x_1}^{x_2}\left[c^2 - \xi^2 + F(\xi,\bar{\mu}^{1})\right]^{\frac{1}{2}}d\xi \qquad (4.32)$$

where x_1, x_2 in these equations are given by (4.28).

Finally, if ϕ, ϕ', and a^2 are expanded, uniquely, in powers of $\bar{\mu}^{1}$ and substituted into the differential equation (4.21) or (3.33), the coefficients of μ^{-k} must vanish so that the quantities ϕ_k, a_k, $k = 1,\ldots, m$, originally determined, are recovered. Moreover, the series $\phi = x + \sum_{n=1}^{\infty} \phi_n\mu^{-n}$, $a^2 = c^2 + \sum_{n=1}^{\infty} a_n\mu^{-n}$ converge. This completely establishes the three assertions on page 25.

4.3 Behavior of ϕ as $x \to \pm\infty$.

Introduce (4.25) into the integrands of (4.29,31) which become

$$\left[\xi^2 - c^2 - F(\xi,\bar{\mu}^{1})\right]^{\frac{1}{2}} = \sqrt{1-\alpha_\mu}\ \xi\left[1 - \frac{\beta_\mu}{(1-\alpha_\mu)\xi} - \frac{\gamma_\mu + c^2}{(1-\alpha_\mu)\xi^2} + O(\xi^{-2-\eta})\right]^{\frac{1}{2}}$$

since $|\alpha_\mu| < 1$ for μ_o sufficiently large. Because $|\xi|$ is large, this equation gives

$$\left[\xi^2-c^2-F(\xi,\bar{\mu}^{1})\right]^{\frac{1}{2}} = \sqrt{1-\alpha_\mu}\ \xi - \frac{\beta_\mu}{2\sqrt{1-\alpha_\mu}} - \frac{\gamma_\mu + c^2}{2\sqrt{1-\alpha_\mu}\ \xi} -$$

$$- \frac{1}{8}\frac{\beta_\mu^{\,2}}{(1-\alpha_\mu)^{3/2}\xi} + O(\xi^{-1-\eta}) \qquad (4.33)$$

so that (4.29,31), after integration, become, respectively,

$$\emptyset\sqrt{\emptyset^2-a^2} + a^2\text{Cosh}^{-1}-\frac{\emptyset}{a} = \sqrt{1-\alpha_\mu}\, x^2 - \frac{\beta_\mu x}{\sqrt{1-\alpha_\mu}} - \frac{4(1-\alpha_\mu)(\gamma_\mu+c^2)+\beta_\mu^2}{4(1-\alpha_\mu)^{3/2}}\log x +$$

$$+ e_{\mu-} + O(x^{-\eta}) \qquad \text{as } x \to -\infty \qquad (4.34)$$

where
$$e_{\mu-} = \lim_{x\to-\infty}\Big[\emptyset\sqrt{\emptyset^2-a^2} + a^2\,\text{Cosh}^{-1}-\frac{\emptyset}{a} - \sqrt{1-\alpha_\mu}\, x^2 +$$

$$+ \frac{\beta_\mu x}{\sqrt{1-\alpha_\mu}} + \frac{4(1-\alpha_\mu)(\gamma_\mu+c^2)+\beta_\mu^2}{4(1-\alpha_\mu)^{3/2}}\log x\Big] \qquad (4.35)$$

$$\emptyset\sqrt{\emptyset^2-a^2} - a^2\text{Cosh}^{-1}\frac{\emptyset}{a} = \sqrt{1-\alpha_\mu}\, x^2 - \frac{\beta_\mu x}{\sqrt{1-\alpha_\mu}} - \frac{4(1-\alpha_\mu)(\gamma_\mu+c^2)+\beta_\mu^2}{4(1-\alpha_\mu)^{3/2}}\log x +$$

$$+ e_{\mu+} + O(x^{-\eta}) \qquad \text{as } x \to \infty \qquad (4.36)$$

where
$$e_{\mu+} = \lim_{x\to\infty}\Big[\emptyset\sqrt{\emptyset^2-a^2} - a^2\text{Cosh}^{-1}\frac{\emptyset}{a} - \sqrt{1-\alpha_\mu}\, x^2 +$$

$$+ \frac{\beta_\mu x}{\sqrt{1-\alpha_\mu}} + \frac{4(1-\alpha_\mu)(\gamma_\mu+c^2)+\beta_\mu^2}{4(1-\alpha_\mu)^{3/2}}\log x\Big] \qquad (4.37)$$

The left members in equations (4.34,36) are large if and only if $|\emptyset|$ is large. But this takes place only when $|x|$ is large, and then these left members may be written $\emptyset^2\left[1 + O(\emptyset^{-1-\eta})\right]$. Using this result and taking the square root of (4.34,36) gives

$$\emptyset\left[1+O(\emptyset^{-1-\eta})\right] = (1-\alpha_\mu)^{\frac{1}{4}} x\left[1 - \frac{\beta_\mu}{2(1-\alpha_\mu)x} + O(x^{-1-\eta})\right] \qquad \text{as } x \to \pm\infty \; (4.38)$$

It follows from this equation that $\emptyset$ and x are of the same order of magnitude, that is

$$\frac{\phi}{x} = O(1) \text{ as } x \to \pm\infty \tag{4.39}$$

Consequently, (4.38) gives the result

$$\phi = (1 - \alpha_\mu)^{\frac{1}{4}} x - \frac{\beta_\mu}{2(1-\alpha_\mu)^{3/4}} + O(x^{-\eta}) \quad \text{as } x \to \pm\infty \tag{4.40}$$

The derivatives of ϕ, from (4.40), are

$$\phi' = (1 - \alpha_\mu)^{\frac{1}{4}} + O(x^{-1-\eta}), \quad \phi'' = O(x^{-2-\eta}),$$

$$\phi''' = O(x^{-3-\eta}) \qquad \text{as } x \to \pm\infty \tag{4.41}$$

Returning now to the expression for $g(x,\mu^{-1})$ given by (3.34), it is evident that g is an analytic function of x and μ^{-1}, real and regular for all real x and $\mu_o \leq \mu$. This follows from the fact that ϕ and its derivatives are analytic functions of x and μ^{-1}, real and regular for all real x and $\mu_o \leq \mu$, and that the derivatives of the ϕ_n, which occur in the p_n, are analytic functions of x, real and regular for all real x, with $p_1 \equiv p_2 \equiv 0$ by (3.19). Moreover, the first m terms in the expansion of g in powers of μ^{-1} cancel since ϕ was constructed to insure that this would occur. That is, $\mu^{m-1} g$ is an analytic function of x and μ^{-1}, real and regular for all real x and $\mu_o \leq \mu$. When real x is bounded, all the functions involved in g of (3.34) are bounded, since $0 < \phi'$ for all real x. Consequently,

$$|g(x,\mu^{-1})| \leq A\,\mu^{-m+1}, \quad X_1 \leq x \leq X_2, \quad \mu_o \leq \mu \tag{4.42}$$

where the X_1, X_2 of (4.27) are used. When $|x|$ is large, it follows from (3.34) and (3.19), (4.41,24) that

$$|g(x,\mu^{-1})| \le A_\mu |x|^{-3-\eta}$$

where A_μ is a function of μ, and since $\mu^{m-1}g$ is regular for $\mu_o \le \mu$

$$|\mu^{m-1}g(x,\mu^{-1})| \le \mu^{m-1}A_\mu |x|^{-3-\eta} \le A_m |x|^{-3-\eta} \text{ as } x \to \pm\infty \qquad (4.43)$$

A suitable adjustment of the constants in (4.42,43) gives the results

$$|g(x,\mu^{-1})| \le A_m \mu^{-m+1}, \quad -c-\delta \le x \le c+\delta, \quad \mu_o \le \mu \qquad (4.44)$$

$$|g(x,\mu^{-1})| \le A_m \mu^{-m+1}|x|^{-3-\eta}, \quad x \le -c-\delta \text{ or } c+\delta \le x, \quad \mu_o \le \mu \qquad (4.45)$$

where the A_m may depend on m, and δ is an arbitrary positive number.

SECTION 5

DEVELOPMENT OF THE INTEGRAL EQUATION

Equation (3.1) will now be compared with (3.6), that is, with the transformed form of (3.6) given by (3.8). Introducing (3.11) into (3.1), the result is

$$\frac{d^2y}{dx^2} + \left\{\mu^2(a^2 - \phi^2)\phi'^2 + \frac{1}{2}\frac{\phi'''}{\phi'} - \frac{3}{4}\frac{\phi''^2}{\phi'^2}\right\} y = g(x, \mu^{-1})y \tag{5.1}$$

in which the left member is identical with (3.8). Equation (3.8) has fundamental sets of solutions given by

$$v_1(x) = D_\nu\left[\sqrt{2\mu}\,\phi(x)\right]\,\phi'(x)^{-\frac{1}{2}}, \quad v_2(x) = D_{-\nu-1}\left[i\sqrt{2\mu}\,\phi(x)\right]\,\phi'(x)^{-\frac{1}{2}} \tag{5.2}$$

and

$$\bar{v}_1(x) = D_\nu\left[-\sqrt{2\mu}\,\phi(x)\right]\,\phi'(x)^{-\frac{1}{2}}, \quad \bar{v}_2(x) = D_{-\nu-1}\left[-i\sqrt{2\mu}\,\phi(x)\right]\,\phi'(x)^{-\frac{1}{2}} \tag{5.3}$$

which follow from (3.7) and Section 2.3. These pairs of solutions are fundamental systems for the homogeneous equation belonging to (5.1).

Now let $v(x)$ be any solution of (3.8). Then, formally, the general solution of (5.1), and hence of (3.1), is given by

$$y(x) = v(x) + \frac{1}{\Delta}\int_{x_0}^{x}\left[v_2(x)v_1(\xi) - v_1(x)v_2(\xi)\right]g(\xi)y(\xi)d\xi \tag{5.4}$$

where x_0 is an arbitrary constant and

$$\Delta = v_1(x)v_2'(x) - v_2(x)v_1'(x) \tag{5.5}$$

A similar equation is obtained by barring all the v's and the Δ in conformity with (5.3). From (5.5) and (5.2)

$$\Delta = D_\nu\left[\sqrt{2\mu}\,\phi(x)\right]\ \phi'(x)^{-\frac{1}{2}} \frac{d}{dx} \underline{D}_{\nu-1}\left[i\sqrt{2\mu}\,\phi(x)\right] \phi'(x)^{-\frac{1}{2}} -$$

$$- \underline{D}_{\nu-1}\left[i\sqrt{2\mu}\,\phi(x)\right]\ \phi'(x)^{-\frac{1}{2}} \frac{d}{dx} D_\nu\left[\sqrt{2\mu}\,\phi(x)\right]\ \phi'(x)^{-\frac{1}{2}} \qquad (5.6)$$

so that

$$\Delta = \frac{D_\nu}{\phi'^{3/2}}\left[\phi'^{\frac{1}{2}}\frac{d}{dx}\underline{D}_{\nu-1} - \underline{D}_{\nu-1}\frac{d}{dx}\phi'^{\frac{1}{2}}\right] - \frac{\underline{D}_{\nu-1}}{\phi'^{3/2}}\left[\phi'^{\frac{1}{2}}\frac{d}{dx}D_\nu - D_\nu\frac{d}{dx}\phi'^{\frac{1}{2}}\right] =$$

$$= \frac{D_\nu}{\phi'}\frac{d}{dx}\underline{D}_{\nu-1} - \frac{\underline{D}_{\nu-1}}{\phi'}\frac{d}{dx}D_\nu = \sqrt{2\mu}\left\{D_\nu\frac{d\,\underline{D}_{\nu-1}}{d(\sqrt{2\mu}\,\phi)} - \underline{D}_{\nu-1}\frac{d\,D_\nu}{d(\sqrt{2\mu}\,\phi)}\right\}$$

The last expression in brackets is the Wronskian of the parabolic cylinder functions of argument $\sqrt{2\mu}\,\phi$. Therefore, by (2.36),

$$\Delta = \sqrt{2\mu}\left\{-i\ e^{-i\pi\frac{\nu}{2}}\right\} \qquad (5.7)$$

In exactly the same way, from (5.5) and (5.3)

$$\overline{\Delta} = -\sqrt{2\mu}\left\{-i\ e^{-i\pi\frac{\nu}{2}}\right\} \qquad (5.8)$$

Introducing (5.2) and (5.7) into (5.4) gives the result

$$y(x) = v(x) + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}}\int_{x_0}^{x}\left\{\underline{D}_{\nu-1}\left[i\sqrt{2\mu}\,\phi(x)\right]D_\nu\left[\sqrt{2\mu}\,\phi(\xi)\right] -\right.$$

$$\left. - D_\nu\left[\sqrt{2\mu}\,\phi(x)\right]\underline{D}_{\nu-1}\left[i\sqrt{2\mu}\,\phi(\xi)\right]\right\}\ \frac{g(\xi)y(\xi)}{\phi'(x)^{\frac{1}{2}}\phi'(\xi)^{\frac{1}{2}}}\,d\xi \qquad (5.9)$$

with a similar equation resulting from the use of the barred quantities of (5.3) and (5.8).

After the existence of solutions of these integral equations has been established, estimates of the integrals will be required. However, the analysis will be facilitated if the solutions of (5.9) and $\overline{(5.9)}$, hence of (3.1), can be expressed in terms of variables which make the functions involved as simple as possible. This can be accomplished by reversing the transformation $t = \phi(x)$ of (3.9) which is allowable since $0 < \phi'(x)$ for all real x. For this purpose the following definitions are made:

$$\begin{aligned} x &= \Psi(t) \equiv \phi^{-1}(t), \quad \Psi'(t) = \frac{1}{\phi'(x)} \\ \xi &= \Psi(s) \equiv \phi^{-1}(s), \quad \Psi'(s) = \frac{1}{\phi'(\xi)} \end{aligned} \tag{5.10}$$

where $x = \Psi(t)$ is the inverse of $t = \phi(x)$. Equation (5.9) then becomes

$$\frac{y[\Psi(t)]}{\Psi'(t)^{\frac{1}{2}}} = \frac{v[\Psi(t)]}{\Psi'(t)^{\frac{1}{2}}} + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_{t_0}^{t} \Big[\underline{D}_{-\nu-1}(i\sqrt{2\mu}\, t) D_{\nu}(\sqrt{2\mu}\, s) -$$

$$- D_{\nu}(\sqrt{2\mu}\, t) \underline{D}_{-\nu-1}(i\sqrt{2\mu}\, s)\Big] g[\Psi(s)]\, \Psi'(s)^2\, \frac{y[\Psi(s)]}{\Psi'(s)^{\frac{1}{2}}}\, ds$$

where, for convenience, further definitions are made:

$$G(s) = g[\Psi(s)]\, \Psi'(s)^2 \equiv \frac{g(\xi)}{\phi'(\xi)^2} \tag{5.11}$$

$$w(t) = \frac{y[\Psi(t)]}{\Psi'(t)^{\frac{1}{2}}} \equiv y[\Psi(t)]\, \phi'[\Psi(t)]^{\frac{1}{2}} \tag{5.12}$$

$$u(t) = \frac{v[\Psi(t)]}{\Psi'(t)^{\frac{1}{2}}} \equiv v[\Psi(t)]\, \phi'[\Psi(t)]^{\frac{1}{2}} \tag{5.13}$$

and, with these definitions, the integral equation becomes

$$w(t) = u(t) + \frac{i\, e^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_{t_o}^{t} \Big[D_{-\nu-1}(i\sqrt{2\mu}\, t) D_\nu(\sqrt{2\mu}\, s) -$$

$$- D_\nu(\sqrt{2\mu}\, t) \underline{D}_{-\nu-1}(i\sqrt{2\mu}\, s) \Big] G(s) w(s) ds \qquad (5.14)$$

Equations (5.10), (5.13) together are simply the inverse of the transformation (3.7), by means of which (3.6) was converted into (3.8). Consequently, $w(t)$ satisfies the equation

$$\frac{d^2 w}{dt^2} + \Big[\mu^2 (a^2 - t^2) - G(t) \Big] w = 0 \qquad (5.15)$$

and $u(t)$ is a solution of (3.6), which was discussed in Section 2.3.

Since (5.14) and (5.9) are equivalent equations, the existence of a solution of the simpler equation (5.14) will be established, and then an estimate of the integral will be made.

Remark: Because the analysis will be carried out in terms of the variable t, it might seem more reasonable to construct originally the transformation $x = \Psi(t)$ of (5.10) and hence, convert the differential equation (3.1) into (5.15) at the very beginning. However, the function $\Psi(t)$ is not nearly as easily determined as $\phi(x)$, the required manipulations being much more complicated.

The behavior of $G(s)$ in (5.14) is required, and for this, the results given by (4.44,45) will be used. It follows from equations (4.29,30,31) that the points $x = x_1, x_2$ correspond to the points $t \equiv \phi = -a, a$. Then from (5.11) and (4.44) it follows that

$$|G(s)| \le A_m \mu^{-m+1}, \quad -a - \frac{1}{\sqrt{\mu}} \le s \le a + \frac{1}{\sqrt{\mu}} \qquad (5.16)$$

since $\phi'[\Psi(s)]$ is bounded and positive for s in the above interval, and from (5.11) and (4.45) that

$$\left|G(s)\right| \le \bar{A}_m \mu^{-m+1} \frac{|\xi|^{-3-\eta}}{\phi'(\xi)^2} \le A_m \mu^{-m+1} |s|^{-3-\eta},$$

$$s \le -a - \frac{1}{\sqrt{\mu}}, \quad a + \frac{1}{\sqrt{\mu}} \le s \qquad (5.17)$$

since, by (4.39), x and $\phi \equiv t$, or ξ and s, are of the same order and $\phi'(\xi) = (1 - \alpha_\mu)^{\frac{1}{4}} + O(\xi^{-1-\eta}) \equiv (1 - \alpha_\mu)^{\frac{1}{4}} + O(s^{-1-\eta})$.

It will be seen in Section 6 that $D_\nu(\sqrt{2\mu}\ t)$ tends exponentially to zero as t tends to infinity. Consequently, t_o will be taken as $+\infty$ in equation (5.14) and $u(t) = D_\nu(\sqrt{2\mu}\ t)$. In the corresponding "barred" equation, t_o will be taken as $-\infty$ and $\bar{u}(t) = D_\nu(-\sqrt{2\mu}\ t)$. Making these changes, (5.14) becomes

$$w_1(t) = D_\nu(\sqrt{2\mu}\ t) + \frac{i\ e^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^{\infty} \Big[D_\nu(\sqrt{2\mu}\ t)\underline{D}_{\nu-1}(i\sqrt{2\mu}\ s) - \underline{D}_{\nu-1}(i\sqrt{2\mu}\ t)D_\nu(\sqrt{2\mu}\ s)\Big]\, G(s)w_1(s)ds \qquad (5.18)$$

It is easily shown that the corresponding "barred" equation is

$$w_2(t) = D_\nu(-\sqrt{2\mu}\ t) + \frac{i\ e^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_{-\infty}^{t} \Big[D_\nu(-\sqrt{2\mu}\ t)\underline{D}_{\nu-1}(-i\sqrt{2\mu}\ s) - \underline{D}_{\nu-1}(-i\sqrt{2\mu}\ t)D_\nu(-\sqrt{2\mu}\ s)\Big]\, G(s)w_2(s)ds \qquad (5.19)$$

If in (5.19) the transformation $s = -\sigma$, $t = -\tau$ is introduced, the result is

$$w_2(-\tau) = D_\nu(\sqrt{2\mu}\,\tau) + \frac{i\,e^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_\tau^\infty \Big[D_\nu(\sqrt{2\mu}\,\tau)D_{-\nu-1}(i\sqrt{2\mu}\,\sigma) - D_{-\nu-1}(i\sqrt{2\mu}\,\tau)D_\nu(\sqrt{2\mu}\,\sigma)\Big]\,G(-\sigma)w_2(-\sigma)d\sigma \quad (5.20)$$

Therefore, by associating $w_2(-\tau)$ with $w_1(t)$, it is sufficient to treat equation (5.18) only. This will be done subsequently, but it is first necessary to discuss the parabolic cylinder functions that occur in the above integral equations.

SECTION 6

ASYMPTOTIC FORMULAS AND BOUNDS FOR THE PARABOLIC CYLINDER FUNCTIONS

6.1 A Summary of the Results of this Section.

The formulas which will be derived in this section are expressed in terms of the quantity

$$\eta_x = \frac{2\nu+1}{2}\left\{\frac{x}{\sqrt{4\nu+2}}\sqrt{\frac{x^2}{4\nu+2}-1} - \log\left(\frac{x}{\sqrt{4\nu+2}}+\sqrt{\frac{x^2}{4\nu+2}-1}\right)\right\}, \sqrt{4\nu+2} \le x \qquad (6.1)$$

Notation. The symbol $[Q]$ shall be used with the meaning

$$[Q] \equiv Q + \sum_{n=1}^{\infty} E_n \nu^{-n}$$

in which the E_n are functions of ν and x. They are bounded for all real x and $o < \nu_o \le \nu$. The expression $\alpha \sim \beta$ is used in the sense that $\frac{\alpha}{\beta} \to 1$.

$$D_\nu(x) \sim \frac{[1]\,\nu^{\frac{\nu}{2}} e^{-\frac{\nu}{2}} e^{-\eta_x}}{\sqrt{2}\left(\frac{x^2}{4\nu+2}-1\right)^{1/4}} \qquad \sqrt{4\nu+2}+\sqrt{2} \le x \qquad (6.2)$$

$$\underline{D}_{\nu-1}(ix) \sim \frac{\left[e^{-\frac{\nu+1}{2}\pi i}\right] e^{\frac{\nu}{2}} e^{\eta_x}}{\sqrt{2}\,\nu^{\frac{\nu}{2}+\frac{1}{2}}\left(\frac{x^2}{4\nu+2}-1\right)^{1/4}} \qquad \sqrt{4\nu+2}+\sqrt{2} \le x \qquad (6.3)$$

$$|D_\nu(x)| \le A\,\nu^{\frac{\nu+1}{2}} e^{-\frac{\nu}{2}} e^{-\eta_x} \qquad \sqrt{4\nu+2} \le x \le \sqrt{4\nu+2}+\sqrt{2} \qquad (6.4)$$

$$|\underline{D}_{\nu-1}(ix)| \le A\,\nu^{-\frac{\nu}{2}-\frac{1}{3}} e^{\frac{\nu}{2}} e^{\eta_x} \qquad \sqrt{4\nu+2} \le x \le \sqrt{4\nu+2}+\sqrt{2} \qquad (6.5)$$

$$\left|\underline{D}_{\nu-1}(ix)\right| \le A\, e^{\frac{\nu}{2}}\, \nu^{-\frac{\nu}{2}-\frac{1}{3}} \qquad \sqrt{4\nu+2} - \sqrt{2} \le x \le \sqrt{4\nu+2} \tag{6.6}$$

$$\left|D_{\nu}(x)\right| \le A\, e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}+\frac{1}{6}} \qquad \sqrt{4\nu+2} - \sqrt{2} \le x \le \sqrt{4\nu+2} \tag{6.7}$$

$$D_{\nu}(x) = \left(\frac{2}{\nu}\right)^{\frac{1}{8}} \frac{M_{\nu}}{\left(1 - \frac{x^2}{4\nu}\right)^{\frac{1}{4}}} \left\{\cos \phi_o' + O(\nu^{-\frac{1}{4}})\right\} \quad -\sqrt{4\nu+2} + \sqrt{2} \le x \le \sqrt{4\nu+2} - \sqrt{2} \tag{6.8}$$

$$\text{where } M_{\nu} = 2^{\frac{3}{8}}\, e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}+\frac{1}{8}} \tag{6.9}$$

$$\phi_o' = \frac{x}{4}\sqrt{4\nu - x^2} + \left(\nu - \frac{1}{2}\right)\sin^{-1}\frac{x}{2\sqrt{\nu}} - \frac{\nu\pi}{2} \tag{6.10}$$

$$\left|D_{\nu}(x)\right| \le A\, e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}+\frac{1}{8}} \qquad -\sqrt{4\nu+2} + \sqrt{2} \le x \le \sqrt{4\nu+2} - \sqrt{2} \tag{6.11}$$

$$\left|\underline{D}_{\nu-1}(ix)\right| \le A\, e^{\frac{\nu}{2}}\, \nu^{-\frac{\nu}{2}-\frac{3}{8}} \qquad -\sqrt{4\nu+2} + \sqrt{2} \le x \le \sqrt{4\nu+2} - \sqrt{2} \tag{6.12}$$

$$\left|D_{\nu}(x)\right| \le A\, e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}+\frac{1}{6}} \qquad -\sqrt{4\nu+2} \le x \le -\sqrt{4\nu+2} + \sqrt{2} \tag{6.13}$$

$$\left|D_{\nu}(x)\right| \le e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}+\frac{1}{2}} \left\{A_1 e^{-\eta_- x} + A_2\, \nu^{-\frac{1}{3}} |\sin \nu\pi|\, e^{\eta_- x}\right\}$$
$$-\sqrt{4\nu+2} - \sqrt{2} \le x \le -\sqrt{4\nu+2} \tag{6.14}$$

$$D_{\nu}(x) \sim \frac{e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}}}{\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}}} \left\{\frac{e^{\nu\pi i}}{\sqrt{2}}\, [1]\, e^{-\eta_- x} - \sqrt{2}\sin \nu\pi\, [1]\, e^{\eta_- x}\right\}$$
$$x \le -\sqrt{4\nu+2} - \sqrt{2} \tag{6.15}$$

6.2 Bounds for $D_{\nu}(x)$, $\underline{D}_{\nu-1}(ix)$ in the Interval $\sqrt{4\nu+2} - \sqrt{2} \le x \le \sqrt{4\nu+2} + \sqrt{2}$.

The results of Schwid [6] will be used. These results are stated for functions $w_1(z)$ and $w_2(z)$ which satisfy the equation

$$\frac{d^2w}{dz^2} + (2\nu + 1 - z^2)w = 0 \tag{6.16}$$

The parabolic cylinder functions satisfy Weber's equation (2.30), that is,

$$\frac{d^2D}{dx^2} + (\nu + \frac{1}{2} - \frac{x^2}{4})D = 0 \tag{6.17}$$

which is transformed into (6.16) by the relation

$$z = \frac{x}{\sqrt{2}} \tag{6.18}$$

Consequently, the parabolic cylinder functions can be expressed as a linear combination of w_1 and w_2. The relations (39) in [6] give the results

$$D_{-\nu-1}(ix) = \frac{2^{\frac{\nu-1}{2}}}{\Gamma(\nu+1)} \left\{ \Gamma(\frac{\nu+1}{2})\, w_1(\frac{x}{\sqrt{2}}) - 2i\Gamma(\frac{\nu}{2}+1)\, w_2(\frac{x}{\sqrt{2}}) \right\} \tag{6.19}$$

$$D_{-\nu-1}(-ix) = \frac{2^{\frac{\nu-1}{2}}}{\Gamma(\nu+1)} \left\{ \Gamma(\frac{\nu+1}{2})\, w_1(\frac{x}{\sqrt{2}}) + 2i\Gamma(\frac{\nu}{2}+1)\, w_2(\frac{x}{\sqrt{2}}) \right\} \tag{6.20}$$

Introducing these expressions into (2.32) gives

$$D_\nu(x) = \frac{2^{\frac{\nu}{2}}}{\sqrt{\pi}} \left\{ \Gamma(\frac{\nu+1}{2}) \cos\frac{\nu\pi}{2}\, w_1(\frac{x}{\sqrt{2}}) + 2\Gamma(\frac{\nu}{2}+1) \sin\frac{\nu\pi}{2}\, w_2(\frac{x}{\sqrt{2}}) \right\} \tag{6.21}$$

Before Schwid's formulas for w_1 and w_2 are substituted into (6.19,21), the gamma functions which are present will be replaced by their asymptotic representations. The following formula can be deduced from [7] Chapter XII, page 263, example 44:

$$\Gamma(z + a) = \sqrt{2\pi}\, e^{-z} z^{z+a-\frac{1}{2}} \left[1 + O(z^{-1})\right], \quad |\arg z| < \pi \tag{6.22}$$

Making appropriate choices of z and a and introducing the resulting expressions into (6.19,21) gives

$$D_{-v-1}(ix) = 2^{-\frac{1}{2}} e^{\frac{v}{2}} v^{-\frac{v}{2}-\frac{1}{2}} \left\{ [1] \; w_1\left(\frac{x}{\sqrt{2}}\right) - i \, [1] \sqrt{2v} \; w_2\left(\frac{x}{\sqrt{2}}\right) \right\} \tag{6.23}$$

$$D_v(x) = 2^{\frac{1}{2}} e^{-\frac{v}{2}} v^{\frac{v}{2}} \left\{ [1] \cos\frac{v\pi}{2} \; w_1\left(\frac{x}{\sqrt{2}}\right) + [1] \sqrt{2v} \sin\frac{v\pi}{2} \; w_2\left(\frac{x}{\sqrt{2}}\right) \right\} \tag{6.24}$$

The interval under consideration is

$$\sqrt{4v+2} - \sqrt{2} \leq x \leq \sqrt{4v+2} + \sqrt{2} \tag{6.25}$$

and because of (6.18), formulas for $w_1(z)$, $w_2(z)$ are required for

$$\sqrt{2v+1} - 1 \leq z \leq \sqrt{2v+1} + 1 \tag{6.26}$$

Schwid's results in Theorem V are stated for bounded ξ, but Theorem 6 in Langer [2] contains these results as a special case. This latter theorem states that they are valid for general values of ξ. However, when the error term is large in comparison with the other terms, these formulas are of little use here. Schwid states that

$$w_1(z) = \sqrt{\frac{2\pi}{3}} \, e^{\frac{i\pi}{6}} \left(\frac{\Phi}{\phi}\right)^{\frac{1}{2}} \sqrt{2v+1} \left\{ C_{11}^2 \, J_{-\frac{1}{3}}(\xi) + C_{12}^2 \, J_{\frac{1}{3}}(\xi) + O(v^{-1}) \right\} \tag{6.27}$$

$$w_2(z) = \sqrt{\frac{2\pi}{3}} \, e^{\frac{i\pi}{6}} \left(\frac{\Phi}{\phi}\right)^{\frac{1}{2}} \sqrt{2v+1} \left\{ C_{21}^2 \, J_{-\frac{1}{3}}(\xi) + C_{22}^2 \, J_{\frac{1}{3}}(\xi) + O(v^{-1}) \right\} \tag{6.28}$$

where

$$\xi = \frac{2v+1}{4} (\sin 2\theta - 2\theta), \qquad \frac{\Phi}{\phi} = \frac{\sin 2\theta - 2\theta}{4 \sin\theta} \tag{6.29}$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{2v+1}} \tag{6.30}$$

and the C^2_{jk} are bounded functions of v alone which are listed in [6] Table 4.

The numbers z occurring in (6.26) may be obtained by putting

$$z = \sqrt{2v + 1} + \beta, \quad -1 \leq \beta \leq 1 \tag{6.31}$$

From (6.31) and (6.30), it follows that

$$\theta = \cos^{-1}\left(1 + \frac{\beta}{\sqrt{2v+1}}\right) = \left(\frac{-2\beta}{\sqrt{2v+1}}\right)^{\frac{1}{2}}\left[1 + O(v^{-\frac{1}{2}})\right] \tag{6.32}$$

and from (6.32) and (6.29) that

$$\xi = \frac{i}{3}(2v + 1)^{\frac{1}{4}}(2\beta)^{\frac{3}{2}}\left[1 + O(v^{-\frac{1}{2}})\right] \tag{6.33}$$

$$\left(\frac{\Phi}{\phi}\right)^{\frac{1}{2}} = \sqrt{\frac{2\beta}{3}}\,(2v + 1)^{-\frac{1}{4}}\left[1 + O(v^{-\frac{1}{2}})\right] \tag{6.34}$$

Asymptotic formulas for the Bessel functions [7] are

$$J_{\pm\frac{1}{3}}(\xi) = \frac{e^{i(\xi \mp \frac{\pi}{6} - \frac{\pi}{4})}}{\sqrt{2\pi\xi}}\left[1+O(\xi^{-1})\right] + \frac{e^{-i(\xi \mp \frac{\pi}{6} - \frac{\pi}{4})}}{\sqrt{2\pi\xi}}\left[1+O(\xi^{-1})\right] \text{ as } \xi \to \infty \tag{6.35}$$

and

$$J_{\pm\frac{1}{3}}(\xi) = O(\xi^{\pm\frac{1}{3}}) \text{ as } \xi \to 0 \tag{6.36}$$

<u>Remarks</u>. It follows easily from (6.33) that ξ is a bounded function of v in part of the interval $-1 \leq \beta \leq 1$ and unbounded in the remainder. When β is negative, ξ is real by (6.33). Then, for unbounded ξ, $J_{\pm\frac{1}{3}}(\xi) = O(\xi^{-\frac{1}{2}})$ by (6.35). At worst, when $\beta = -1$, $J_{\pm\frac{1}{3}}(\xi) = O(v^{-\frac{1}{8}})$ and the error term in (6.27,28) remains small in comparison with $J_{\pm\frac{1}{3}}(\xi)$. When β is positive, ξ is pure imaginary by (6.33). Then, when ξ is unbounded, the function $e^{-i\xi}$ which occurs in (6.35) is large. That is, the error in (6.27,28) is again small in comparison

with $J_{\pm\frac{1}{3}}(\xi)$. Consequently, equations (6.27,28) give useful results in the interval of (6.26,31).

Equations (6.27,28) may be put into the form

$$w_1(z) = \sqrt{\frac{2\pi}{3}}\, e^{i\frac{\pi}{6}}\, \nu^{\frac{1}{6}}\, e^{-i\xi}\, S_1(z,\nu)\, T_1(z,\nu) \tag{6.37}$$

$$w_2(z) = \sqrt{\frac{2\pi}{3}}\, e^{i\frac{\pi}{6}}\, \nu^{-\frac{1}{3}}\, e^{-i\xi}\, S_2(z,\nu)\, T_2(z,\nu) \tag{6.38}$$

where
$$S_1(z,\nu) = \left(\frac{\Phi}{\phi}\right)^{\frac{1}{2}} \frac{\sqrt{2\nu + 1}}{\nu^{\frac{1}{6}}\xi^{\frac{1}{3}}}, \quad S_2(z,\nu) = \left(\frac{\Phi}{\phi}\right)^{\frac{1}{2}} \frac{\nu^{\frac{1}{3}}}{\xi^{\frac{1}{3}}} \tag{6.39}$$

$$T_1(z,\nu) = \xi^{\frac{1}{3}}\, e^{i\xi} \left\{ C_{11}^2\, J_{-\frac{1}{3}}(\xi) + C_{12}^2\, J_{\frac{1}{3}}(\xi) + O(\nu^{-1}) \right\} \tag{6.40}$$

$$T_2(z,\nu) = \xi^{\frac{1}{3}}\, e^{i\xi} \left\{ \overline{C}_{21}^2\, J_{-\frac{1}{3}}(\xi) + \overline{C}_{22}^2\, J_{\frac{1}{3}}(\xi) + O(\nu^{-\frac{1}{2}}) \right\} \tag{6.41}$$

in which $\overline{C}_{2k}^2 \equiv \sqrt{2\nu + 1}\, C_{2k}^2$, [6] Table 4, is a bounded function of ν alone.

From (6.33,34) it follows that S_1 and S_2 are bounded and from (6.35,36) that T_1 and T_2 are bounded. Consequently, bounds for $w_1(z)$ and $w_2(z)$ are

$$|w_1(z)| \le A\nu^{\frac{1}{6}}\, |e^{-i\xi}|, \quad |w_2(z)| \le A\nu^{-\frac{1}{3}}\, |e^{-i\xi}|,$$

$$\sqrt{2\nu + 1} - 1 \le z \le \sqrt{2\nu + 1} + 1 \tag{6.42}$$

From (6.33), ξ is real when $-1 \le \beta \le 0$ so that $|e^{-i\xi}| = 1$. Then, from (6.42) and (6.31)

$$|w_1(z)| \le A\nu^{\frac{1}{6}}, \quad |w_2(z)| \le A\nu^{-\frac{1}{3}}, \quad \sqrt{2\nu + 1} - 1 \le z \le \sqrt{2\nu + 1} \tag{6.43}$$

Introducing these results into (6.23,24) and using (6.18) gives (6.6,7).

When $0 < \beta \leq 1$, it follows from (6.32,33) that θ and ξ are pure imaginary. The transformation (6.18) is inserted into (6.30) in which the relation $\theta = i\alpha_x$, where α_x is real and positive, is used. The result is

$$\alpha_x = \text{Cosh}^{-1} \frac{x}{\sqrt{4\nu + 2}} \tag{6.44}$$

Then $(6.29)_1$ becomes

$$\xi = \frac{i}{4}(2\nu + 1)\ (\text{Sinh } 2\alpha_x - 2\alpha_x) \equiv i\eta_x \tag{6.45}$$

where
$$\eta_x = \frac{2\nu + 1}{4}\ (\text{Sinh } 2\alpha_x - 2\alpha_x) \tag{6.46}$$

and (6.1) follows from (6.44,46). Equations (6.42) become, in terms of η_x ,

$$\left|w_1\left(\frac{x}{\sqrt{2}}\right)\right| \leq A\nu^{\frac{1}{6}}\, e^{\eta_x}, \quad \left|w_2\left(\frac{x}{\sqrt{2}}\right)\right| \leq A\nu^{-\frac{1}{3}}\, e^{\eta_x}, \quad \sqrt{4\nu+2} \leq x \leq \sqrt{4\nu+2} + \sqrt{2} \tag{6.47}$$

which, when introduced into (6.23), give (6.5).

A satisfactory bound for $D_\nu(x)$ cannot be obtained from (6.27,28) in the interval of (6.47) so that the required bound will now be derived from Whittaker's integral [7] which is

$$D_\nu(x) = -\frac{\Gamma(\nu+1)}{2\pi i}\, e^{-\frac{x^2}{4}} \int_{\infty}^{(0^+)} \exp\left(-xt - \frac{t^2}{2}\right)(-t)^{-\nu-1}\, dt,$$

$$|\arg(-t)| \leq \pi \tag{6.48}$$

The function $f(t) = -xt - \frac{t^2}{2} - (\nu + \frac{1}{2}) \log(-t)$ has saddlepoints given by $f'(t) = -\frac{1}{t}(t^2 + xt + \nu + \frac{1}{2}) = 0$ which are $t = -\frac{x}{2} \pm \frac{1}{2}\sqrt{x^2-(4\nu+2)}$. The saddlepoint of interest is $t = -\frac{x}{2} + \frac{1}{2}\sqrt{x^2-(4\nu+2)}$. Put

$$t_o = \frac{x}{2} - \frac{1}{2}\sqrt{x^2 - (4\nu + 2)}, \quad \sqrt{4\nu + 2} \leq x \tag{6.49}$$

so that t_o is non-negative. The integration in (6.48) is carried out as follows:

$$\int_{\infty}^{(o^+)} = \int_{\infty}^{t_o} + \int_{t_o}^{t_o e^{2i\pi}} + \int_{t_o e^{2i\pi}}^{\infty} \tag{6.50}$$

In these integrals substitute $-t = se^{-i\pi}$, $-t = t_o e^{i\theta}(-\pi \leq \theta \leq \pi)$, and $-t = se^{i\pi}$, respectively. Then

$$\int_{\infty}^{t_o} = -e^{(\nu+1)i\pi} \int_{t_o}^{\infty} \exp(-xs - \frac{s^2}{2})\, s^{-\nu-1} ds$$

and

$$\int_{t_o e^{2i\pi}}^{\infty} = e^{-(\nu+1)i\pi} \int_{t_o}^{\infty} \exp(-xs - \frac{s^2}{2})\, s^{-\nu-1} ds$$

so that

$$\int_{\infty}^{t_o} + \int_{t_o e^{2i\pi}}^{\infty} = -2i \sin(\nu + 1)\pi \int_{t_o}^{\infty} \exp(-xs - \frac{s^2}{2}) s^{-\nu-1} ds$$

which gives the result

$$\left| \int_{\infty}^{t_o} + \int_{t_o e^{2i\pi}}^{\infty} \right| \leq \frac{2}{\nu t_o^{\nu}} \exp(-xt_o - \frac{t_o^2}{2}) \tag{6.51}$$

The second integral on the right of (6.50) is

$$\int_{t_o}^{t_o e^{2i\pi}} = -\int_{-\pi}^{\pi} \exp\left[f(-t_o e^{i\theta})\right] \sqrt{t_o}\, e^{i\frac{\theta}{2}}\, i d\theta$$

The real part of the function $f(-t_o e^{i\theta})$ has a maximum at the saddlepoint, that is, when $\theta = 0$. Then this integral gives the result

$$\left| \int_{t_o}^{t_o e^{2i\pi}} \right| \leq 2\pi\, t_o^{-\nu} \exp(xt_o - \frac{t_o^2}{2}) \tag{6.52}$$

Since (6.52) is larger than (6.51), the bound for $D_\nu(x)$ of (6.48) is

$$|D_\nu(x)| \leq A\Gamma(\nu + 1)t_o^{-\nu} \exp(-\frac{x^2}{4} + xt_o - \frac{t_o^2}{2}), \quad \sqrt{4\nu + 2} \leq x \tag{6.53}$$

where the constant A is less than 3π for large ν. From (6.49) $t_o^{-1} = \frac{x + \sqrt{x^2-(4\nu+2)}}{2\nu + 1}$ which with t_o is introduced into (6.53). The result may be put in the form

$$|D_\nu(x)| \leq A \frac{\Gamma(\nu+1)}{(\nu+\frac{1}{2})^{\frac{\nu}{2}}} e^{\frac{\nu}{2}} \left[\frac{x + \sqrt{x^2-(4\nu+2)}}{\sqrt{4\nu+2}}\right]^\nu \exp\left[- \frac{2\nu+1}{2} \frac{x}{\sqrt{4\nu+2}}\sqrt{\frac{x^2}{4\nu+2} - 1}\right]$$

and finally

$$|D_\nu(x)| \leq A\Gamma(\nu+1) \frac{e^{\frac{\nu}{2}}}{\nu^{\frac{\nu}{2}}} \exp\left\{-\eta_x - \frac{1}{2} \log \left[\frac{x + \sqrt{x^2 -(4\nu+2)}}{\sqrt{4\nu+2}}\right]\right\}, \sqrt{4\nu+2} \leq x \tag{6.54}$$

where η_x is given by (6.1). Replacing the gamma function by its asymptotic form (6.22) and omitting the second term in the exponent gives (6.4).

6.3 Asymptotic Formulas for $D_\nu(x)$, $D_{-\nu-1}(ix)$ in the Interval $\sqrt{4\nu+2} + \sqrt{2} \leq x$.

The results of Schwid [6] Theorem I may be used, since asymptotic forms for $w_1(z)$, $w_2(z)$ are required in the interval

$$\sqrt{2\nu + 1} + 1 \leq z \tag{6.55}$$

because of (6.18). In this interval ξ is unbounded as can be seen from (6.33). These results are

$$w_1(z) \sim \frac{1}{2\left(\frac{z^2}{2v+1} - 1\right)^{1/4}} \left\{ e^{\frac{v}{2} i\pi} [1] \; e^{i\xi} - 2\left[\sin \frac{v\pi}{2}\right] e^{-i\xi} \right\} \tag{6.56}$$

$$w_2(z) \sim \frac{1}{2\left(\frac{z^2}{2v+1} - 1\right)^{1/4}} \left\{ \frac{e^{\frac{v-1}{2} i\pi}}{\sqrt{2v+1}} [1] \; e^{i\xi} + \frac{2}{\sqrt{2v+1}} \left[\cos v \frac{\pi}{2}\right] e^{-i\xi} \right\} \tag{6.57}$$

where the coefficients B^{20}_{jk} from [6] Table 2 have been used, since by (6.30,29) arg $\xi = \frac{\pi}{2}$ so that h = o from Schwid's relation (14). However, the coefficients of $e^{i\xi}$ in (6.56,57), which are listed in this Table 2, contain the ambiguous symbols $(-1)^{\frac{v}{2}}$ and $(-1)^{\frac{v-1}{2}}$. They must mean $e^{\frac{v}{2} i\pi}$, $e^{\frac{v-1}{2} i\pi}$, respectively, as in (6.56,57). This interpretation is deduced from (6.27,28) since these equations are valid for unbounded ξ, as previously mentioned, and are therefore valid in the interval (6.55). When the asymptotic formulas for the Bessel functions (6.35) are inserted in (6.27,28) these equations are asymptotically equivalent to (6.56,57) only if the above meaning is attached to the ambiguous symbols.

Using the relation $z = \frac{x}{\sqrt{2}}$ from (6.18) and the definition of η_x from equations (6.46) and (6.1), equations (6.56,57) become

$$w_1\left(\frac{x}{\sqrt{2}}\right) \sim \frac{1}{2\left(\frac{x^2}{4v+2} - 1\right)^{1/4}} \left\{ e^{\frac{v}{2} i\pi} [1] \; e^{-\eta_x} - 2\left[\sin v \frac{\pi}{2}\right] e^{\eta_x} \right\} \tag{6.58}$$

$$w_2\left(\frac{x}{\sqrt{2}}\right) \sim \frac{1}{2\left(\frac{x^2}{4v+2} - 1\right)^{1/4}} \left\{ \frac{e^{\frac{v-1}{2} i\pi}}{\sqrt{2v+1}} [1] \; e^{-\eta_x} + \frac{2}{\sqrt{2v+1}} \left[\cos v \frac{\pi}{2}\right] e^{\eta_x} \right\} \tag{6.59}$$

These functions are substituted into (6.23) giving the equation

$$\underline{D}_{\nu-1}(ix) \sim \frac{e^{\frac{\nu}{2}}}{2\sqrt{2}\ \nu^{\frac{\nu}{2}+\frac{1}{2}}\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}}} \left\{ e^{\frac{\nu}{2}i\pi}\left([1] - [1]\right) e^{-\eta_x} - \right.$$

$$\left. - 2i\left(\left[\cos\frac{\nu\pi}{2}\right] - i\left[\sin\frac{\nu\pi}{2}\right]\right) e^{\eta_x} \right.$$

and since the first term in the braces is asymptotically negligible equation (6.3) is obtained. Now put (6.58,59) into equation (6.24) from which it follows that

$$D_\nu(x) \sim \frac{\nu^{\frac{\nu}{2}}\ e^{-\frac{\nu}{2}}}{\sqrt{2}\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}}} \left\{ e^{\frac{\nu}{2}i\pi}\left([1]\ \cos\frac{\nu\pi}{2} - i\ [1]\ \sin\frac{\nu\pi}{2}\right) e^{-\eta_x} + \right.$$

$$\left. + 2\left(\sin\frac{\nu\pi}{2}\left[\cos\frac{\nu\pi}{2}\right] - \left[\sin\frac{\nu\pi}{2}\right]\cos\frac{\nu\pi}{2}\right) e^{\eta_x} \right\}$$

and since $D_\nu(x) \to 0$ as $x \to \infty$, the coefficient of e^{η_x} must vanish, not only asymptotically but exactly. This leaves the result (6.2).

6.4 Asymptotic Formulas and Bounds for $D_\nu(x)$, $\underline{D}_{\nu-1}(ix)$ in the Interval $-\sqrt{4\nu+2} + \sqrt{2} \leq x \leq \sqrt{4\nu+2} - \sqrt{2}$.

The results of Schwid's Theorem 3 hold in the right half plane and within the circle $|x| = \sqrt{4\nu + 2}$. Watson [9] derived corresponding results which hold within the circle $|x| = 2\sqrt{\nu}$, and consequently yield a single pair of formulas which cover the interval under consideration. Watson's formula for $D_\nu(x)$ is

$$D_\nu(x) = \frac{\Gamma(\nu+1)\ e^{\frac{\nu}{2}}}{\sqrt{\pi}\ \nu^{\frac{\nu+1}{2}}\left(1 - \frac{x^2}{4\nu}\right)^{\frac{1}{4}}} \left\{ \cos \phi_0' + O\left(\frac{2x^2}{\sqrt{\nu}\ (4\nu-x^2)^{\frac{3}{2}}}\right) \right\} \qquad (6.60)$$

where ϕ_0' is given by (6.10). As x approaches $\pm 2\sqrt{v}$ the error becomes large, and it is necessary to check the order term in (6.60) at the end points of the interval being considered. At these points the order term is $O\left(\frac{v+1-\sqrt{2v+1}}{\sqrt{v}(\sqrt{2v+1}-1)^{3/2}}\right) = O(v^{-\frac{1}{4}})$, and therefore (6.60) is valid throughout the interval.

From Watson's result ([9] p. 140) a bound for $D_{-v-1}(ix)$ is easily obtained. It should be noted in this reference that when $-2\sqrt{v} < x < 2\sqrt{v}$, $\gamma = i\beta$, so that Watson's asymptotic formula for $D_{-v-1}(ix)$ yields

$$\left|D_{-v-1}(ix)\right| \leq A \frac{e^{\frac{v}{2}} v^{-\frac{v+1}{2}}}{(1-\frac{x^2}{4v})^{\frac{1}{4}}}, \quad -\sqrt{4v+2}+\sqrt{2} \leq x \leq \sqrt{4v+2}-\sqrt{2} \qquad (6.61)$$

The error term in Watson's formula for $D_{-v-1}(ix)$ is identical with that in (6.60), and consequently, the bound (6.61) is also valid throughout the interval under consideration. Also, only a bound for $D_{-v-1}(ix)$ is needed in this investigation so that the asymptotic formula is not stated.

The asymptotic formula (6.60) shows that $D_v(x)$ oscillates in the interval under consideration. The coefficient of the braces of (6.60) takes its greatest value at the endpoints of the interval. This value is

$$\frac{\Gamma(v+1)\, e^{\frac{v}{2}}\, v^{\frac{1}{4}}}{\sqrt{\pi}\, v^{\frac{v+1}{2}}(\sqrt{2v+1}-1)^{\frac{1}{4}}} = 2^{\frac{3}{8}} e^{-\frac{v}{2}} v^{\frac{v}{2}+\frac{1}{8}}\left[1 + O(v^{-\frac{1}{2}})\right]$$

which, essentially, is used in the following

Definition:

$$M_\nu = 2^{\frac{3}{8}} e^{-\frac{\nu}{2}} \nu^{\frac{\nu}{2}+\frac{1}{8}} \qquad (6.62)$$

M_ν will be called the modulus of $D_\nu(x)$ when $-\sqrt{4\nu+2}+\sqrt{2} \le x \le \sqrt{4\nu+2}-\sqrt{2}$.

Using M_ν, equation (6.60) gives (6.8). Finally, $(1-\frac{x^2}{4\nu})^{\frac{1}{4}}$ is smallest when x is an endpoint of the interval in (6.62), taking the value $(\frac{\sqrt{2\nu+1}-1}{\nu})^{\frac{1}{4}}$. Introducing this value into (6.8) gives (6.11) and into (6.61) gives (6.12).

6.5 Asymptotic Formulas and Bounds for $D_\nu(x)$ in the Interval $x \le -\sqrt{4\nu+2}+\sqrt{2}$.

These relations may be obtained by expressing $D_\nu(x)$ in terms of $D_\nu(-x)$ and $\underline{D}_{-\nu-1}(-ix)$, in which $\sqrt{4\nu+2}-\sqrt{2} \le -x$, and using the formulas already developed in the interval $\sqrt{4\nu+2}-\sqrt{2} \le -x$. For this purpose, equation (2.33) is put into the form

$$D_\nu(x) = e^{\nu\pi i} D_\nu(-x) - \sqrt{\frac{2}{\pi}} \sin \nu\pi \, \Gamma(\nu+1) \, e^{\frac{\nu+1}{2}\pi i} \underline{D}_{-\nu-1}(-ix) \qquad (6.63)$$

Then, equation (6.13) follows from (6.63) into which (6.6,7) are inserted. Similarly, (6.14) follows from (6.63) and (6.4,5). Finally, equation (6.15) follows from (6.63) and (6.2,3).

SECTION 7

SOLUTION OF THE INTEGRAL EQUATION

7.1 Notation and Remarks.

Equation (5.18) may be put into the form

$$w(t) = D_\nu(\sqrt{2\mu}\, t) + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^\infty K(\mu,t,s)G(s)w(s)ds \tag{7.1}$$

where

(i) $$K(\mu,t,s) = D_\nu(\sqrt{2\mu}\, t)D_{-\nu-1}(i\sqrt{2\mu}\, s) - D_\nu(\sqrt{2\mu}\, s)D_{-\nu-1}(i\sqrt{2\mu}\, t) \tag{7.2}$$

(ii) $\nu = \mu \frac{a^2}{2} - \frac{1}{2}$ from (2.38) is bounded away from the positive integers

(iii) from (5.16,17) it follows that

$$\int_{a+\frac{1}{\sqrt{\mu}}}^\infty \left|\frac{G(s)}{s}\right|ds \le K_\mu \equiv K\,\mu^{-m+1} \tag{7.3}$$

$$\int_{-\infty}^{-a-\frac{1}{\sqrt{\mu}}} \left|\frac{G(s)}{s}\right|ds \le K_\mu \equiv K\,\mu^{-m+1} \tag{7.4}$$

$$|G(s)| \le K_\mu \equiv K\,\mu^{-m+1}, \quad |s| \le a + \frac{1}{\sqrt{\mu}} \tag{7.5}$$

Remarks:

(i) The arbitrary integer m designates the number of approximating functions computed for equation (3.9) which may be written $\phi(x,\mu^{-1}) = x + \phi_1\mu^{-1} + \dots + \phi_m\mu^{-m} + O(\mu^{-m-1})$. However, when the number of functions used is 0 or 1, it follows from (3.34) and

(5.16,17) that in (7.3,4,5) the right members may be replaced by $K\,\mu^{-1}$.

(ii) For brevity, the asymptotic formulas and bounds for the parabolic cylinder functions will be expressed in terms of x and y. Therefore, the following notation will be freely used:

$$x = \sqrt{2\mu}\, t, \quad y = \sqrt{2\mu}\, s \tag{7.6}$$

in which x is not the x occurring in the transformation $\phi(x,\mu^{-1})$.

(iii) With this notation, the transition points $t = \pm a$ of equation (5.15) correspond to the transition points $x = \pm\sqrt{4\nu + 2}$ of Weber's equation, (2.30).

(iv) The asymptotic behavior of w(t) will be described in the three intervals $t \leq -a - \frac{1}{\sqrt{\mu}}$, $-a + \frac{1}{\sqrt{\mu}} \leq t \leq a - \frac{1}{\sqrt{\mu}}$, and $a + \frac{1}{\sqrt{\mu}} \leq t$, but the asymptotic behavior in the transition region will not be obtained.

(v) In the notation of (7.6), substitute (2.33,34) into (7.2). The result is

$$K(\mu,\frac{x}{\sqrt{2\mu}},\frac{y}{\sqrt{2\mu}}) = \left[D_\nu(-x)\underline{D}_{\nu-1}(-iy) - D_\nu(-y)\underline{D}_{\nu-1}(-ix)\right](2e^{(\nu+\frac{1}{2})\pi i}\sin \nu\pi - 1) \tag{7.7}$$

7.2 The Solution in the Interval

$$a + \frac{1}{\sqrt{\mu}} \leq t \quad \text{or} \quad \sqrt{4\nu + 2} + \sqrt{2} \leq x \tag{7.8}$$

Since (6.2) indicates $D_\nu(x)$ never vanishes in the interval (7.8), the integral equation (7.1) may be treated in the form

$$\frac{w(t)}{D_\nu(\sqrt{2\mu}\ t)} = 1 + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^\infty \frac{D_\nu(\sqrt{2\mu}\ s)}{D_\nu(\sqrt{2\mu}\ t)} K(\mu,t,s)G(s) \frac{w(s)}{D_\nu(\sqrt{2\mu}\ s)}\, ds \tag{7.9}$$

For brevity, the following definitions are made:

$$W(t) = \frac{w(t)}{D_\nu(\sqrt{2\mu}\ t)} \tag{7.10}$$

$$\overline{K}(\mu,t,s) = \frac{D_\nu(\sqrt{2\mu}\ s)}{D_\nu(\sqrt{2\mu}\ t)} K(\mu,t,s) \tag{7.11}$$

Using these definitions, equation (7.9) becomes

$$W(t) = 1 + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^\infty \overline{K}(\mu,t,s)G(s)W(s)ds \tag{7.12}$$

In order to carry out the usual iteration process, the functions $W_k(t)$ are defined by the relations

$$W_k(t) = 1 + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^\infty \overline{K}(\mu,t,s)\ G(s)W_{k-1}(s)ds,$$

$$W_o(t) \equiv 0, \quad k = 1,2,\ldots \tag{7.13}$$

From these relations it follows that

$$W_1(t) \equiv 1 \tag{7.14}$$

and that

$$W_{k+1}(t) - W_k(t) = \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^\infty \overline{K}(\mu,t,s)G(s)\left[W_k(s) - W_{k-1}(s)\right]\, ds \tag{7.15}$$

From (7.11,2,6) it follows that

$$\overline{K}(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}}) = D_\nu(y)\ \underline{D}_{-\nu-1}(iy) - D_\nu(y)^2\ \frac{\underline{D}_{-\nu-1}(ix)}{D_\nu(x)} \tag{7.16}$$

and from (6.2,3) that

$$\left|\bar{K}\left(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}}\right)\right| \leq \frac{\nu^{-\frac{1}{2}}}{\left(\frac{y^2}{4\nu+2} - 1\right)^{\frac{1}{2}}} \left\{A_1 + A_2\, e^{-2\eta_y + 2\eta_x}\right\}$$

From (6.1) it is easily seen that η_x is an increasing function of x so that the above bound is simply increased by replacing η_y by η_x. For $t \leq s$ implies $x \leq y$, and therefore $\eta_x \leq \eta_y$. The result is

$$\left|\bar{K}\left(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}}\right)\right| \leq \frac{A\,\nu^{-\frac{1}{2}}}{y\left(\frac{1}{4\nu+2} - \frac{1}{y^2}\right)^{\frac{1}{2}}} \tag{7.17}$$

The quantity in parentheses in the denominator of (7.17) is smallest when $y = \sqrt{4\nu+2} + \sqrt{2}$. Introducing this value gives the result

$$\left|\bar{K}\left(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}}\right)\right| \leq \frac{A\nu^{\frac{1}{4}}}{y} \tag{7.18}$$

which becomes, using (7.6)

$$\left|\bar{K}(\mu,t,s)\right| \leq \frac{A\nu^{\frac{1}{4}}}{\sqrt{2\mu}\; s} \leq \frac{A}{\mu^{\frac{1}{4}} s} \tag{7.19}$$

since μ, ν are of the same order by (2.38).

The iteration in (7.15) may now be performed. The steps are

$$\left|W_2(t) - W_1(t)\right| \leq AK_\mu \mu^{-\frac{3}{4}} \tag{7.20}$$

which follows from (7.14,19,3). Now assume

$$\left|W_k(t) - W_{k-1}(t)\right| \leq \left(AK_\mu \mu^{-\frac{3}{4}}\right)^{k-1} \tag{7.21}$$

and then (7.15) gives the result

$$\left|W_{k+1}(t) - W_k(t)\right| \leq \left(AK_\mu \mu^{-\frac{3}{4}}\right)^{k} \tag{7.22}$$

In virtue of (7.20,21,22), equation (7.22) is established by induction for all k. Hence, the sequence $W_k(t)$ converges uniformly in the interval of (7.8) provided $AK_\mu \mu^{-\frac{3}{4}} < 1$ which is certainly true when $\mu_o \leq \mu$ is sufficiently large. Then

$$W(t) = \lim_{k\to\infty} W_k(t) \tag{7.23}$$

is the solution of (7.12), since the uniform convergence permits the appropriate limits to be taken on both sides of (7.13). It also follows from (7.22,23) that

$$W(t) = \lim_{n\to\infty} W_n(t) = W_o(t) + \sum_{k=o}^{\infty} \left[W_{k+1}(t) - W_k(t)\right],$$

but $W_o(t) \equiv 0$ by (7.13) so that

$$\left|W(t)\right| \leq \sum_{k=o}^{\infty} \left|W_{k+1}(t) - W_k(t)\right| \leq \frac{1}{1 - AK_\mu \mu^{-\frac{3}{4}}} \leq A_1 \tag{7.24}$$

Introducing this bound for $W(t)$ into (7.12) gives the desired result

$$\left|W(t) - 1\right| \leq AA_1 \mu^{-\frac{3}{4}} \int_t^\infty \left|\frac{G(s)}{s}\right| ds \leq A\mu^{-m+\frac{1}{4}} \tag{7.25}$$

From this equation and (7.10) it follows that

$$\frac{w(t)}{D_\nu(\sqrt{2\mu}\, t)} = 1 + O(\mu^{-m+\frac{1}{4}}), \quad a + \frac{1}{\sqrt{\mu}} \leq t \tag{7.26}$$

7.3 The Solution in the First Transition Interval.

This interval will be treated in the two parts

$$\text{(a)} \quad a \leq t \leq a + \frac{1}{\sqrt{\mu}}, \qquad \text{(b)} \quad a - \frac{1}{\sqrt{\mu}} \leq t \leq a$$

Part (a): $a \le t \le a + \frac{1}{\sqrt{\mu}}$ or $\sqrt{4\nu + 2} \le x \le \sqrt{4\nu + 2} + \sqrt{2}$ (7.27)

Let the bound for $D_\nu(x)$, given by (6.4), be denoted by $B_\nu(x)$. Then equation (7.1) may be put into the form

$$\frac{w(t)}{B_\nu(\sqrt{2\mu}\, t)} = \frac{D_\nu(\sqrt{2\mu}\, t)}{B_\nu(\sqrt{2\mu}\, t)} + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^{a+\frac{1}{\sqrt{\mu}}} \frac{B_\nu(\sqrt{2\mu}\, s)}{B_\nu(\sqrt{2\mu}\, t)} K(\mu,t,s)G(s)\frac{w(s)}{B_\nu(\sqrt{2\mu}\, s)}ds +$$

$$+ \frac{D_\nu(\sqrt{2\mu a^2} + \sqrt{2})}{B_\nu(\sqrt{2\mu}\, t)} \left[\frac{w(a + \frac{1}{\sqrt{\mu}})}{D_\nu(\sqrt{2\mu a^2} + \sqrt{2})} - 1 \right] \quad (7.28)$$

The existence of a continuous solution $w(t)$ is assured by the classical theory of the Volterra integral equation, since all the functions involved are continuous, and the interval is finite. From (7.2) and (6.4,5)

$$\left| \frac{B_\nu(y)}{B_\nu(x)} K(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}}) \right| \le B_\nu(y) \left| D_{\nu-1}(iy) \right| + B_\nu(y)^2 \frac{|D_{\nu-1}(ix)|}{B_\nu(x)}$$

$$\le \nu^{\frac{1}{6}}(A_1 + A_2 e^{-2\eta_y + 2\eta_x}) \le A\nu^{\frac{1}{6}} \quad (7.29)$$

where the last inequality is obtained from the same argument as the one leading to equation (7.17). It follows from (6.4) that

$$\left| \frac{D_\nu(\sqrt{2\mu a^2} + \sqrt{2})}{B_\nu(\sqrt{2\mu}\, t)} \right| \le \frac{B_\nu(\sqrt{4\nu + 2} + \sqrt{2})}{B_\nu(x)} = \exp(\eta_x - \eta_{\sqrt{4\nu+2}+\sqrt{2}}) \le 1 \quad (7.30)$$

so that with (6.4) and (7.29,5,30,26), equation (7.28) gives the result

$$\left| \frac{w(t)}{B_\nu(\sqrt{2\mu}\, t)} \right| \le 1 + A_1 \nu^{\frac{1}{6}} \mu^{-\frac{1}{2}} K_\mu \sup \left| \frac{w(t)}{B_\nu(\sqrt{2\mu}\, t)} \right| (a + \frac{1}{\sqrt{\mu}} - a) + A_2 \mu^{-m + \frac{1}{4}}$$

Since μ, ν are of the same order, $A_1 \nu^{\frac{1}{6}} \mu^{-1} K_\mu = A_1 \mu^{-\frac{5}{6}} K_\mu$, and this quantity is less than one for μ_o sufficiently large. The last equation then gives the bound

$$\sup \left| \frac{w(t)}{B_\nu(\sqrt{2\mu}\ t)} \right| \leq \frac{1 + A_2 \mu^{-m+\frac{1}{4}}}{1 - A_1 K_\mu \mu^{-\frac{5}{6}}} \leq A \tag{7.31}$$

Introducing (7.31) and the foregoing bounds into (7.28) gives the result

$$\left| \frac{w(t)}{B_\nu(\sqrt{2\mu}\ t)} - \frac{D_\nu(\sqrt{2\mu}\ t)}{B_\nu(\sqrt{2\mu}\ t)} \right| \leq A_1 \mu^{-\frac{5}{6}} K_\mu + A_2 \mu^{-m+\frac{1}{4}} \leq A \mu^{-m+\frac{1}{4}},$$

$$a \leq t \leq a + \frac{1}{\sqrt{\mu}} \tag{7.32}$$

<u>Part (b)</u>: $a - \frac{1}{\sqrt{\mu}} \leq t \leq a$ or $\sqrt{4\nu + 2} - \sqrt{2} \leq x \leq \sqrt{4\nu + 2}$ (7.33)

Using the M_ν of (6.9), equation (7.1) may be put into the form

$$\frac{w(t)}{M_\nu} = \frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu} + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^a K(\mu,t,s)G(s)\frac{w(s)}{M_\nu}\,ds \ +$$

$$+ \frac{B_\nu(\sqrt{2\mu a^2})}{M_\nu} \left[\frac{w(a)}{B_\nu(\sqrt{2\mu a^2})} - \frac{D_\nu(\sqrt{2\mu a^2})}{B_\nu(\sqrt{2\mu a^2})} \right] \tag{7.34}$$

From (6.7,9)
$$\left| \frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu} \right| \leq A\nu^{\frac{1}{24}} \tag{7.35}$$

From (7.2) and (6.6,7)
$$|K(\mu,t,s)| \leq A\nu^{-\frac{1}{6}} \tag{7.36}$$

From (6.4,1)
$$B_\nu(\sqrt{4\nu + 2}) = A\,\nu^{\frac{\nu}{2}+\frac{1}{2}} e^{-\frac{\nu}{2}}$$

so that
$$\frac{B_\nu(\sqrt{2\mu a^2})}{M_\nu} = \frac{B_\nu(\sqrt{4\nu + 2})}{M_\nu} \leq A\,\nu^{\frac{3}{8}} \tag{7.37}$$

Introducing these bounds, together with (7.5,32), into (7.34) gives

$$\left|\frac{w(t)}{M_\nu}\right| \leq A_1 \nu^{\frac{1}{24}} + A_2\, \nu^{-\frac{1}{6}} \mu^{-\frac{1}{2}} K_\mu \sup \left|\frac{w(t)}{M_\nu}\right| \left[a - (a - \frac{1}{\sqrt{\mu}})\right] + A_3\, \nu^{\frac{3}{8}} \mu^{-m+\frac{1}{4}}$$

Since μ, ν are of the same order, $A_2\, \nu^{-\frac{1}{6}} \mu^{-1} K_\mu = A_2\, \mu^{-\frac{7}{6}} K_\mu$ which is less than one for μ_o large enough, so that

$$\sup \left|\frac{w(t)}{M_\nu}\right| \leq \frac{A_1 \mu^{\frac{1}{24}} + A_3 \mu^{-m+\frac{5}{8}}}{1 - A_2 K_\mu \mu^{-\frac{7}{6}}} \leq A\, \mu^{\frac{1}{24}} \tag{7.38}$$

Using this inequality, (7.34) gives the result

$$\left|\frac{w(t)}{M_\nu} - \frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu}\right| \leq A_1 \nu^{-\frac{1}{6}} K_\mu\, \mu^{-\frac{23}{24}} + A_2\, \nu^{\frac{3}{8}}\, \mu^{-m+\frac{1}{4}} \leq A\, \mu^{-m+\frac{5}{8}}, \qquad a - \frac{1}{\sqrt{\mu}} \leq t \leq a \tag{7.39}$$

This completes the treatment of the first transition interval.

7.4 The Solution in the Interval

$$-a + \frac{1}{\sqrt{\mu}} \leq t \leq a - \frac{1}{\sqrt{\mu}} \quad \text{or} \quad -\sqrt{4\nu + 2} + \sqrt{2} \leq x \leq \sqrt{4\nu + 2} - \sqrt{2} \tag{7.40}$$

It follows from the differential equation (5.15) that $w(t)$ is an oscillating function in this interval. The zeros of $w(t)$ cannot be expected to coincide with those of $D_\nu(\sqrt{2\mu}\ t)$, (6.8), and consequently the quotient, $\dfrac{w(t)}{D_\nu(\sqrt{2\mu}\ t)}$ is unbounded in this interval. For this reason, $w(t)$ and the error will be compared with the "modulus" M_ν. Equation (7.1) becomes

$$\frac{w(t)}{M_\nu} = \frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu} + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^{a-\frac{1}{\sqrt{\mu}}} K(\mu,t,s)G(s)\,\frac{w(s)}{M_\nu}\,ds + $$

$$+ \frac{w(a - \frac{1}{\sqrt{\mu}})}{M_\nu} - \frac{D_\nu(\sqrt{2\mu a^2} - \sqrt{2})}{M_\nu} \qquad (7.41)$$

From (6.11), $\left|\frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu}\right| \leq A_1$ and from (6.11,12), $|K(\mu,t,s)| \leq A_2 \nu^{-\frac{1}{4}}$

Inserting these bounds together with (7.5,39) into (7.41) gives the relation

$$\left|\frac{w(t)}{M_\nu}\right| \leq A_1 + A_2\ \nu^{-\frac{1}{4}}\ \mu^{-\frac{1}{2}}\ K_\mu\ \sup\left|\frac{w(t)}{M_\nu}\right|\ (2a) + A_3\ \mu^{-m+\frac{5}{8}}$$

and because μ,ν are of the same order, $A_2\ \nu^{-\frac{1}{4}}\ \mu^{-\frac{1}{2}} K_\mu = A_2\ \mu^{-\frac{3}{4}} K_\mu$ which is less than one for μ_o large enough. This gives the result

$$\sup\left|\frac{w(t)}{M_\nu}\right| \leq \frac{A_1 + A_3\ \mu^{-m+\frac{5}{8}}}{1 - A_2\ K_\mu\ \mu^{-\frac{3}{4}}} \leq A \qquad (7.42)$$

and again using (7.41) gives

$$\left|\frac{w(t)}{M_\nu} - \frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu}\right| \leq A_1 K\ \mu^{-m+\frac{1}{4}} + A_2\ \mu^{-m+\frac{5}{8}} \qquad (7.43)$$

which may be written

$$\frac{w(t)}{M_\nu} = \frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu} + O(\mu^{-m+\frac{5}{8}}), \quad -a + \frac{1}{\sqrt{\mu}} \leq t \leq a - \frac{1}{\sqrt{\mu}} \qquad (7.44)$$

7.5 The Solution in the Second Transition Interval.

This interval will also be treated in the two parts

(a) $-a \le t \le -a + \frac{1}{\sqrt{\mu}}$ (b) $-a - \frac{1}{\sqrt{\mu}} \le t \le -a$

Part (a): $-a \le t \le -a + \frac{1}{\sqrt{\mu}}$ or $-\sqrt{4\nu + 2} \le x \le -\sqrt{4\nu + 2} + \sqrt{2}$ (7.45)

Equation (7.1) is put into the form

$$\frac{w(t)}{M_\nu} = \frac{D_\nu(\sqrt{2\mu}\, t)}{M_\nu} + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^{-a+\frac{1}{\sqrt{\mu}}} K(\mu,t,s)G(s)\, \frac{w(s)}{M_\nu}\, ds \; +$$

$$+ \; \frac{w(-a + \frac{1}{\sqrt{\mu}})}{M_\nu} - \frac{D_\nu(-\sqrt{2\mu a^2} + \sqrt{2})}{M_\nu} \qquad (7.46)$$

From (6.9,13)

$$\left| \frac{D_\nu(\sqrt{2\mu}\, t)}{M_\nu} \right| \le A\, \nu^{\frac{1}{24}} \qquad (7.47)$$

and from (7.7) and (6.6,7)

$$\left| K(\mu, \frac{x}{\sqrt{2\mu}}\ \frac{y}{\sqrt{2\mu}}) \right| \le A\, \nu^{-\frac{1}{6}} \qquad (7.48)$$

These bounds, and (7.5,43), with (7.46) give the relation

$$\left|\frac{w(t)}{M_\nu}\right| \le A_1\, \nu^{\frac{1}{24}} + A_2\, \nu^{-\frac{1}{6}}\mu^{-\frac{1}{2}} K_\mu \sup \left|\frac{w(t)}{M_\nu}\right| \left[-a + \frac{1}{\sqrt{\mu}} - (-a)\right] + A_3 \mu^{-m+\frac{5}{8}}$$

Since μ, ν are of the same order, $A_2\, \nu^{-\frac{1}{6}}\mu^{-1} K_\mu = A_2\, \mu^{-\frac{7}{6}} K_\mu$ which is less than one for μ_0 sufficiently large. Therefore, this last equation gives the result

$$\sup \left|\frac{w(t)}{M_\nu}\right| \le \frac{A_1 \nu^{\frac{1}{24}}}{1 - A_2\, K_\mu\, \mu^{-\frac{7}{6}}} \le A\, \mu^{\frac{1}{24}} \qquad (7.49)$$

Inserting this last result into (7.46) gives

$$\left| \frac{w(t)}{M_\nu} - \frac{D_\nu(\sqrt{2\mu}\ t)}{M_\nu} \right| \leq A_1 K \nu^{-\frac{1}{6}} \mu^{-m+\frac{1}{24}} + A_2 \mu^{-m+\frac{5}{8}} \leq A \mu^{-m+\frac{5}{8}}$$

$$-a \leq t \leq -a + \frac{1}{\sqrt{\mu}} \qquad (7.50)$$

It is necessary to distinguish between integral and non-integral ν in the remaining discussion of the integral equation. The behavior of $D_\nu(x)$ changes radically when $x < -\sqrt{4\nu + 2}$ according as one or the other subsists, as is easily seen from equation (6.63). Consequently, the restriction that ν be bounded away from the positive integers will be used in the subsequent analysis.

Part (b): $-a - \frac{1}{\sqrt{\mu}} \leq t \leq -a$ or $-\sqrt{4\nu + 2} - \sqrt{2} \leq x \leq -\sqrt{4\nu + 2}$ (7.51)

Let the bound for $D_\nu(x)$, (6.14), be denoted by $C_\nu(x)$. Since $\sin \nu\pi$ is bounded away from zero, it then follows from (6.14) that

$$|D_\nu(x)| \leq C_\nu(x) \equiv A\, e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}+\frac{1}{2}}\, e^{\eta - x} \qquad (7.52)$$

Making use of the quantity $C_\nu(x)$, equation (7.1) may be put into the form

$$\frac{w(t)}{C_\nu(\sqrt{2\mu}\ t)} = \frac{D_\nu(\sqrt{2\mu}\ t)}{C_\nu(\sqrt{2\mu}\ t)} + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}} \int_t^{-a} \frac{C_\nu(\sqrt{2\mu}\ s)}{C_\nu(\sqrt{2\mu}\ t)} K(\mu,t,s)G(s) \frac{w(s)}{C_\nu(\sqrt{2\mu}\ s)}\, ds +$$

$$+ \frac{M_\nu}{C_\nu(\sqrt{2\mu}\ t)} \left[\frac{w(-a)}{M_\nu} - \frac{D_\nu(-\sqrt{2\mu a^2})}{M_\nu} \right] \qquad (7.53)$$

From (7.7), (6.4,5) and (7.52)

$$\left| \frac{C_\nu(y)}{C_\nu(x)} K(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}}) \right| \le \nu^{\frac{1}{6}} \left[A_1 \exp(-2\eta_{-x} + 2\eta_{-y}) + A_2\right]$$

It follows from $t \le s$ that $x \le y$, $-y \le -x$ so that $\eta_{-y} \le \eta_{-x}$. Therefore, replacing η_{-y} by η_{-x} merely increases the bound. Consequently,

$$\left| \frac{C_\nu(y)}{C_\nu(x)} K(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}}) \right| \le A\, \nu^{\frac{1}{6}} \qquad (7.54)$$

It follows from (6.9) and (7.52) that

$$\frac{M_\nu}{C_\nu(\sqrt{2\mu}\, t)} \le \frac{M_\nu}{C_\nu(-\sqrt{2\mu a^2})} \le A\, \nu^{-\frac{3}{8}} \qquad (7.55)$$

since $-\sqrt{2\mu a^2}$ corresponds to $-\sqrt{4\nu+2}$ and $\eta_{-x} = 0$ at this point by (6.1). Substituting these bounds and (7.5,50) into (7.53) gives the result

$$\left| \frac{w(t)}{C_\nu(\sqrt{2\mu}\, t)} \right| \le 1 + A_1 \nu^{\frac{1}{6}} \mu^{-\frac{1}{2}} K_\mu \sup \left| \frac{w(t)}{C_\nu(\sqrt{2\mu}\, t)} \right| \left[-a - (-a - \frac{1}{\sqrt{\mu}})\right] + A_2 \nu^{-\frac{3}{8}} \mu^{-m+\frac{5}{8}}$$

and since $A_1 \nu^{\frac{1}{6}} \mu^{-1} K_\mu = A_1 \mu^{-\frac{5}{6}} K_\mu < 1$ for μ_o sufficiently large

$$\sup \left| \frac{w(t)}{C_\nu(\sqrt{2\mu}\, t)} \right| \le \frac{1 + A_2 \mu^{-m+\frac{1}{4}}}{1 - A_1 \mu^{-\frac{5}{6}} K_\mu} \le A \qquad (7.56)$$

Putting (7.56) into (7.53) gives the desired inequality that

$$\left| \frac{w(t)}{C_\nu(\sqrt{2\mu}\, t)} - \frac{D_\nu(\sqrt{2\mu}\, t)}{C_\nu(\sqrt{2\mu}\, t)} \right| \le A_1 \nu^{\frac{1}{6}} K \mu^{-m} + A_2 \nu^{-\frac{3}{8}} \mu^{-m+\frac{5}{8}} \le A\, \mu^{-m+\frac{1}{4}},$$

$$-a - \frac{1}{\sqrt{\mu}} \le t \le -a \qquad (7.57)$$

This completes the treatment of the second transition interval.

7.6 The Solution in the Interval

$$t \leq -a - \frac{1}{\sqrt{\mu}} \quad \text{or} \quad x \leq -\sqrt{4\nu + 2} - \sqrt{2} \tag{7.58}$$

From (6.15), bounds for $D_\nu(x)$ and $\frac{1}{D_\nu(x)}$ may be obtained. Since $\sin \nu\pi$ is bounded from zero

$$\left|D_\nu(x)\right| \leq \frac{A\, e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}}\, e^{\eta_{-x}}}{\left(\frac{x^2}{4\nu + 2} - 1\right)^{\frac{1}{4}}} \tag{7.59}$$

and

$$\frac{1}{|D_\nu(x)|} \leq \frac{A\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}}}{e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}}} \; \frac{1}{\left|\sqrt{2}\sin \nu\pi\,[1]\, e^{\eta_{-x}}\right| - \left|\frac{e^{\nu\pi i}}{\sqrt{2}}\,[1]\, e^{-\eta_{-x}}\right|}$$

$$\leq \frac{A_1\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}}}{e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}}}\; e^{-\eta_{-x}}\; \frac{1}{1 - A_2\, e^{-2\eta_{-x}}}$$

so that the bound for $\frac{1}{D_\nu(x)}$ is

$$\frac{1}{|D_\nu(x)|} \leq A\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}} \frac{e^{-\eta_{-x}}}{e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}}} \tag{7.60}$$

since $A_2\, e^{-2\eta_{-x}} < 1$ for μ_o sufficiently large. This assertion is valid because $e^{-\eta_{-x}}$ is greatest when $x = -\sqrt{4\nu + 2} - \sqrt{2}$, and for this point η_{-x} is an unbounded function of ν in virtue of (6.45) and the remark following equation (6.36). Since $D_\nu(x)$ never vanishes in the interval (7.58), as can be seen from its asymptotic formula (6.15), equation (7.1) is put into the form

$$\frac{w(t)}{D_\nu(\sqrt{2\mu}\ t)} = 1 + \frac{ie^{i\pi\frac{\nu}{2}}}{\sqrt{2\mu}}\int_t^{-a-\frac{1}{\sqrt{\mu}}} \frac{D_\nu(\sqrt{2\mu}\ s)}{D_\nu(\sqrt{2\mu}\ t)} K(\mu,t,s)G(s) \frac{w(s)}{D_\nu(\sqrt{2\mu}\ s)} ds +$$

$$+ \frac{C_\nu(-\sqrt{2\mu a^2} - \sqrt{2})}{D_\nu(\sqrt{2\mu}\ t)}\left[\frac{w(-a - \frac{1}{\sqrt{\mu}})}{C_\nu(-\sqrt{2\mu a^2} - \sqrt{2})} - \frac{D_\nu(-\sqrt{2\mu a^2} - \sqrt{2})}{C_\nu(-\sqrt{2\mu a^2} - \sqrt{2})}\right] \quad (7.61)$$

From (7.7) and (6.2,3)

$$\left|K(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}})\right| \le \frac{\nu^{-\frac{1}{2}}}{2(\frac{x^2}{4\nu+2} - 1)^{\frac{1}{4}}(\frac{y^2}{4\nu+2} - 1)^{\frac{1}{4}}} \cdot$$

$$\cdot\left[A_1 \exp(-\eta_{-x} + \eta_{-y}) + A_2 \exp(-\eta_{-y} + \eta_{-x})\right] \quad (7.62)$$

and this equation combines with (7.59,60) to give

$$\left|\frac{D_\nu(y)}{D_\nu(x)} K(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}})\right| \le \frac{\nu^{-\frac{1}{2}}}{(\frac{y^2}{4\nu+2} - 1)^{\frac{1}{2}}}\left[A_1 \exp(-2\eta_{-x} + 2\eta_{-y}) + A_2\right]$$

From $t \le s$, it follows that $x \le y$, $-y \le -x$ so that $\eta_{-y} \le \eta_{-x}$. Hence, the right member of the above relation is simply increased if η_{-y} is replaced by η_{-x}. Consequently,

$$\left|\frac{D_\nu(y)}{D_\nu(x)} K(\mu, \frac{x}{\sqrt{2\mu}}, \frac{y}{\sqrt{2\mu}})\right| \le \frac{A\ \nu^{-\frac{1}{2}}}{(\frac{y^2}{4\nu+2} - 1)^{\frac{1}{2}}} \le \frac{A\ \nu^{\frac{1}{4}}}{(-y)} \quad (7.63)$$

in which the last quantity on the right is obtained from the same argument as the one leading to equation (7.17). Therefore,

$$\left|\frac{D_\nu(\sqrt{2\mu}\ s)}{D_\nu(\sqrt{2\mu}\ t)} K(\mu,t,s)\right| \le \frac{A}{\mu^{\frac{1}{4}}(-s)} \quad (7.64)$$

A bound for $\frac{C_\nu(-\sqrt{2\mu a^2} - \sqrt{2})}{D_\nu(\sqrt{2\mu}\ t)}$ is needed which, in terms of x, is

$$\left| \frac{C_\nu(-\sqrt{4\nu+2} - \sqrt{2})}{D_\nu(x)} \right| \le \frac{C_\nu(-\sqrt{4\nu+2} - \sqrt{2})}{\left|D_\nu(-\sqrt{4\nu+2} - \sqrt{2})\right|} \le A\ \nu^{\frac{3}{8}} \qquad (7.65)$$

This follows from (7.52,60) provided (7.60) increases as (-x) decreases. But this is easily seen when the notation of (6.46) is used, since

$$\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}} e^{\eta_{-x}} \equiv \exp\left[- \frac{2\nu+1}{4} (\text{Sinh } 2\alpha_{-x} - 2\alpha_{-x}) + \right.$$
$$\left. + \frac{1}{4} \log \text{Sinh } \alpha_{-x}\right]$$

and

$$\frac{d}{d(-x)} \left[- \frac{2\nu+1}{4} (\text{Sinh } 2\alpha_{-x} - 2\alpha_{-x}) + \frac{1}{4} \log \text{Sinh } \alpha_{-x}\right] =$$

$$= \left[-(2\nu+1) \text{Sinh}^2 \alpha_{-x} + \frac{1}{4} \frac{\text{Cosh } \alpha_{-x}}{\text{Sinh } \alpha_{-x}}\right] \frac{d\alpha_{-x}}{d(-x)}$$

$$= \left\{- \frac{1}{2} \left[x^2 - (4\nu+ 2)\right] + \frac{1}{4} \frac{-x}{\sqrt{x^2 - (4\nu+2)}}\right\} \frac{d\alpha_{-x}}{d(-x)}$$

The braces have the value

$$-(1 + 2\sqrt{2\nu+1}) + \frac{1}{4} \frac{\sqrt{2\nu+1} + 1}{(1 + 2\sqrt{2\nu+1})^{\frac{1}{2}}} < 0$$

when $-x = \sqrt{4\nu+2} + \sqrt{2}$, and $0 < \frac{d\alpha_{-x}}{d(-x)}$ always, so that the above derivative is negative at $-x = \sqrt{4\nu+2} + \sqrt{2}$ and decreases as x decreases, remaining negative. Hence, $\left(\frac{x^2}{4\nu+2} - 1\right)^{\frac{1}{4}} e^{-\eta_{-x}}$ increases when -x decreases as required.

Introducing (7.64,4,65,57) into (7.61) gives the relation

$$\left|\frac{w(t)}{D_\nu(\sqrt{2\mu}\; t)}\right| \le 1 + A_1\mu^{-\frac{3}{4}} \sup\left|\frac{w(t)}{D_\nu(\sqrt{2\mu}\; t)}\right| K_\mu + A_2\, \nu^{\frac{3}{8}}\, \mu^{-m+\frac{1}{4}}$$

and since $A_1\mu^{-\frac{3}{4}} K_\mu < 1$ for μ_o sufficiently large

$$\sup\left|\frac{w(t)}{D_\nu(\sqrt{2\mu}\; t)}\right| \le \frac{1 + A_2\nu^{\frac{3}{8}}\mu^{-m+\frac{1}{4}}}{1 - A_1\, K_\mu\, \mu^{-\frac{3}{4}}} \le A \tag{7.66}$$

Inserting this bound into (7.61) gives

$$\left|\frac{w(t)}{D_\nu(\sqrt{2\mu}\; t)} - 1\right| \le A_1\mu^{-\frac{3}{4}}\int_t^{-a-\frac{1}{\sqrt{\mu}}} \left|\frac{G(s)}{s}\right| ds + A_2\, \nu^{\frac{3}{8}}\, \mu^{-m+\frac{1}{4}} \le A\mu^{-m+\frac{5}{8}} \tag{7.67}$$

and finally

$$\frac{w(t)}{D_\nu(\sqrt{2\mu}\; t)} = 1 + O(\mu^{-m+\frac{5}{8}}), \quad t \le -a - \frac{1}{\sqrt{\mu}} \tag{7.68}$$

SECTION 8

SUMMARY

All the results and the conditions under which they are valid will be summarized here.

In this investigation uniform asymptotic formulas have been obtained for two solutions of the differential equation

$$\frac{d^2y}{dx^2} + \mu^2 \left[c^2 - x^2 + f(x,\mu^{-1}) \right] y = 0 \qquad (3.1)$$

in which μ, a large positive parameter, is restricted as in (2.38) below, c is an arbitrary positive constant, and $f(x,\mu^{-1})$ is subjected to the following

Hypotheses:

(i) $f(x,\mu^{-1})$ is an analytic function of x and μ^{-1}, real and regular for all real x and $o < \mu_o \leq \mu$, where μ_o is sufficiently large. This function possesses the expansion

$$f(x,\mu^{-1}) = \sum_{n=1}^{\infty} f_n(x)\, \mu^{-n} \qquad (3.2)$$

where the f_n are analytic functions of x, real and regular for all real x.

(ii) When x is real and $x \to \pm\infty$, $f(x,\mu^{-1})$ may be put into the form

$$f(x,\mu^{-1}) = \alpha_\mu x^2 + \beta_\mu x + \gamma_\mu + O(x^{-\eta}), \quad 0 < \eta < 1 \qquad (3.3)$$

where $$\alpha_\mu = \sum_{n=1}^{\infty} \alpha_n \mu^{-n}, \quad \beta_\mu = \sum_{n=1}^{\infty} \beta_n \mu^{-n}, \quad \gamma_\mu = \sum_{n=1}^{\infty} \gamma_n \mu^{-n} \tag{3.4}$$

the series for α_μ, β_μ and γ_μ being convergent.

Asymptotic formulas for two solutions of (3.1) which may be deduced from the analysis of Sections 5 and 7 are

$$y_1(x)\phi'(x)^{\frac{1}{2}} = D_\nu\left[\sqrt{2\mu}\,\phi(x)\right]\left[1 + O(\mu^{-m+\frac{1}{4}})\right], \quad c + \delta \leq x \tag{8.1}$$

$$y_1(x)\phi'(x)^{\frac{1}{2}} = D_\nu\left[\sqrt{2\mu}\,\phi(x)\right] + M_\nu O(\mu^{-m+\frac{5}{8}}), \quad -c + \delta \leq x \leq c - \delta \tag{8.2}$$

in which $$M_\nu = 2^{\frac{3}{8}}\, e^{-\frac{\nu}{2}}\, \nu^{\frac{\nu}{2}+\frac{1}{8}} \tag{8.3}$$

$$y_1(x)\phi'(x)^{\frac{1}{2}} = D_\nu\left[\sqrt{2\mu}\,\phi(x)\right]\left[1 + O(\mu^{-m+\frac{5}{8}})\right], \quad x \leq -c - \delta \tag{8.4}$$

and

$$y_2(x)\phi'(x)^{\frac{1}{2}} = D_\nu\left[-\sqrt{2\mu}\,\phi(x)\right]\left[1+O(\mu^{-m+\frac{5}{8}})\right], \quad c + \delta \leq x \tag{8.5}$$

$$y_2(x)\phi'(x)^{\frac{1}{2}} = D_\nu\left[-\sqrt{2\mu}\,\phi(x)\right] + M_\nu O(\mu^{-m+\frac{5}{8}}), \quad -c + \delta \leq x \leq c - \delta \tag{8.6}$$

in which M_ν is given by (8.3)

$$y_2(x)\phi'(x)^{\frac{1}{2}} = D_\nu\left[-\sqrt{2\mu}\,\phi(x)\right]\left[1+O(\mu^{-m+\frac{1}{4}})\right], \quad x \leq -c - \delta \tag{8.7}$$

where

$$\delta = \frac{1}{\sqrt{\mu}} + O(\mu^{-1}) \tag{8.8}$$

$$\nu = \mu\frac{a^2}{2} - \frac{1}{2} \tag{2.38}$$

in which the values of μ must be such that ν is bounded away from the positive integers.

$$a^2 = c^2 + \sum_{n=1}^{\infty} a_n \mu^{-n} = c^2 + a_1 \mu^{-1} + \dots + a_m \mu^{-m} + O(\mu^{-m-1}) \qquad (3.10)$$

in which the a_n, n=1,..., m are given by (3.24).

$$\phi(x) \equiv \phi(x, \mu^{-1}) = x + \sum_{n=1}^{\infty} \phi_n(x) \mu^{-n} = x + \phi_1 \mu^{-1} + \dots + \phi_m \mu^{-m} + O(\mu^{-m-1}) \qquad (3.9)$$

in which the ϕ_n, n=1,..., m are given by (3.25,26,27) and termwise differentiation is valid.

Remark: The arbitrary integer m designates the degree of the approximation made to the comparison equation (3.6) by the functions ϕ_n and the numbers a_n. In the case of the 0th and 1st approximations, equations (8.1,2,4,5,6,7) may be used with $\underline{m = 2}$ in the order symbol.

As $x \to \pm\infty$, $\phi(x)$ and $\phi'(x)$ have the forms

$$\phi(x) = (1 - \alpha_\mu)^{\frac{1}{4}} x - \frac{\beta_\mu}{2(1 - \alpha_\mu)^{3/4}} + O(x^{-\eta}) \qquad (4.40)$$

$$\phi'(x) = (1 - \alpha_\mu)^{\frac{1}{4}} + O(x^{-1-\eta}) \qquad (4.41)$$

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