VARIETIES OF ALGEBRAS WHOSE CONGRUENCE LATTICES SATISFY LATTICE IDENTITIES

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ABSTRACT

Given a variety $\mathcal{K}$ of algebras, among the interesting questions we can ask about the members of $\mathcal{K}$ is the following: does there exist a lattice identity $\mathcal{S}$ such that for each algebra $A \in \mathcal{K}$, the congruence lattice $\Theta(A)$ satisfies $\mathcal{S}$? This thesis deals with questions of this type.

First, the thesis shows that the congruence lattices of relatively free unary algebras satisfy no nontrivial lattice identities.

It is also shown that the class of congruence lattices of semilattices satisfies no nontrivial lattice identities. As a consequence it is shown that if $\mathcal{K}$ is a semigroup variety all of whose congruence lattices satisfy some fixed nontrivial lattice identity, then all the members of $\mathcal{K}$ are groups with exponent dividing a fixed finite number. In particular, the congruence lattices of members of $\mathcal{K}$ are modular.

Finally, it is shown that the varieties whose congruence lattices satisfy one of a class of lattice identities of a fairly general form are in fact congruence modular.

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INTRODUCTION

A variety of algebras is a class of algebras closed under the formation of homomorphic images, subalgebras, and direct products. Equivalently, a class $\mathcal{K}$ is a variety if and only if $\mathcal{K}$ is the class of all algebras of a given similarity type satisfying some set of identities. In particular, given a class $\mathcal{K}_0$ of similar algebras, we can form the variety $\mathcal{K}_0^e$ consisting of those algebras which satisfy each identity which holds in every algebra of $\mathcal{K}_0$. Clearly $\mathcal{K}_0 \subseteq \mathcal{K}_0^e$, and $\mathcal{K}_0$ is a variety if and only if $\mathcal{K}_0 = \mathcal{K}_0^e$.

If $\mathcal{K}$ is a variety and A an algebra in $\mathcal{K}$, it may in general occur that A will satisfy some identities not satisfied by all the algebras in $\mathcal{K}$. This leads us to the notion of a $\mathcal{K}$-free algebra. An algebra $A \in \mathcal{K}$ is said to be $\mathcal{K}$-free if there exists a subset $X \subseteq A$ such that (i) X generates A, and (ii) every map of X into an algebra in $\mathcal{K}$ can be extended to a homomorphism. A fundamental theorem of Birkhoff states that given a variety $\mathcal{K}$ and an arbitrary set X, there exists a $\mathcal{K}$-free algebra generated by X, which we will denote $F_{\mathcal{K}}(X)$.

For example, the class of all lattices forms a variety, and given a set X, we can form the free lattice FL(X). In addition to lattices, the following varieties will be of particular interest in this thesis: semilattices, semigroups, distributive lattices, modular lattices.

Given a variety $\mathcal{K}$ of algebras, among the interesting questions we can ask about the members of $\mathcal{K}$ is the following: does there exist a lattice identity δ such that for each algebra $A \in \mathcal{K}$, the congruence lattice $\Theta(A)$ satisfies δ? The classical theorems of this type state that the congruence lattice (normal subgroup lattice) of a group is modular, that the congruence lattice (lattice of two-sided ideals) of a ring is modular, and that the congruence lattice of a lattice is distributive. This thesis will deal with some questions regarding when the congruence lattices of algebras in a variety are all modular. (In this case, the variety is called <u>congruence modular</u>.) The results will be described more fully below.

In Chapter I the basic theorems on varieties of algebras and free algebras are introduced. It is then shown that the congruence lattices of relatively free unary algebras satisfy no nontrivial lattice identities.

Chapter II presents joint work of the author with Ralph Freese. The main result of this chapter is that the class of congruence lattices of semilattices satisfies no nontrivial lattice identities. It is also shown that the class of subalgebra lattices of semilattices satisfies no nontrivial lattice identities. As a consequence it is shown that if $\mathcal{V}$ is a semigroup variety all of whose congruence lattices satisfy some fixed nontrivial lattice identity, then all the members of $\mathcal{V}$ are groups with exponent dividing a fixed finite number. In particular, the congruence lattices of members of $\mathcal{V}$ are modular.

The final result of Chapter II suggests a closer study of conditions which imply congruence modularity. Necessary and sufficient conditions for congruence modularity were given by Day [4], and it was shown by Wille that the condition that the congruence lattices of the algebras in a variety satisfy a given lattice identity is determined by a weak Mal'cev condition [28, see also 22, 25]. In Chapter III it is shown that varieties whose congruence lattices satisfy one of a class of lattice identities of a fairly general form (see Theorem 3.1) are in fact congruence modular. A similar theorem is proved for congruence distributivity. It should be noted that there are no known examples of varieties whose congruences satisfy some nontrivial lattice identity which are not congruence modular.

CHAPTER I

CONGRUENCE LATTICES OF RELATIVELY FREE UNARY ALGEBRAS

An algebra A is an ordered pair $\langle A_0, F\rangle$ where A_0 is some set and F is an indexed collection of operations on A_0, i.e., $F = \{f_i : i \in I\}$ for some set I, and each $f \in F$ is a mapping of $A_0^{n(f)}$ into A_0 for some finite integer n(f). As is customary, we will often write $x \in A$ to mean $x \in A_0$. The algebras $A = \langle A_0, \{f_i : i \in I\}\rangle$ and $B = \langle B_0, \{g_j : j \in J\}\rangle$ will be called similar (or of the same similarity type) if $I = J$ and $n(f_i) = n(g_i)$ for all $i \in I$. General information about abstract algebras may be found in [9] or [11].

Let X be an arbitrary set and $\mathcal{C}$ a class of similar algebras. If A is any algebra in $\mathcal{C}$, then the operations of A are indexed by the set I, and we may denote the rank function as n(i) $(i \in I)$ since it is independent of which algebra in $\mathcal{C}$ we select. We define polynomials of the type of $\mathcal{C}$ over X to be the smallest set of formal expressions such that:

(1.1) for each $x \in X$, x is a polynomial;

(1.2) if $p_1, \ldots, p_{n(i)}$ are polynomials, then $f_i(p_1, \ldots, p_{n(i)})$ is a polynomial.

It follows from the definition that the polynomials over X form an algebra of the similarity type of $\mathcal{C}$, called the word algebra over X, and denoted W(X).

In view of the inductive definition of a polynomial, each polynomial involves only finitely many members of X. These "variables" may be arranged into a sequence, say $x = \langle x_1, \ldots, x_k \rangle \in X^k$. Then we may regard polynomials as functions p(x) on any fixed algebra A of $\mathcal{C}$ according to the rule: if $a = \langle a_1, \ldots, a_k \rangle \in A_0^k$, then

(1.3) $x_i(a) = a_i$,

(1.4) $f_i(p_1, \ldots, p_{n(i)})(a) = f_i(p_1(a), \ldots, p_{n(i)}(a))$.

If a polynomial p involves k or fewer members of X, then p is called a k-variable polynomial.

If p(x) and q(x) are polynomials over X, an algebra A of $\mathcal{C}$ is said to satisfy the identity $p(x) = q(x)$ if for every $a \in A_0^k$ we have $p(a) = q(a)$. The class $\mathcal{C}$ is said to satisfy an identity if every algebra in $\mathcal{C}$ satisfies it.

A class $\mathcal{K}$ of algebras is a variety (or equational class) if there exists a set Δ of identities such that $\mathcal{K}$ is the class of all algebras of a given similarity type which satisfy all the identities of Δ.

We review some basic notions for the study of algebras. Let A^γ $(\gamma \in \Gamma)$ be an indexed set of similar algebras. The direct product $\bigtimes_{\gamma \in \Gamma} A^\gamma$ is the algebra $B = \langle B_0, F \rangle$ where B_0 is the full cartesian product $\bigtimes_{\gamma \in \Gamma} A_0^\gamma$, and the operations are defined componentwise, i.e., if $b_1, \ldots, b_{n(i)} \in B_0$, then $(f_i(b_1, \ldots, b_{n(i)}))^\gamma = f_i^\gamma(b_1^\gamma, \ldots, b_{n(i)}^\gamma)$.

Given an algebra $A = \langle A_0, F \rangle$, a subset B_0 of A_0 is called a subalgebra of A if B_0 is closed under the operations of F, i.e., if $b \in B_0^{n(i)}$ implies $f_i(b) \in B_0$ for all $i \in I$.

Given two similar algebras $A = \langle A_0, \{f_i : i \in I\}\rangle$ and $B = \langle B_0, \{g_i : i \in I\}\rangle$, we say that a mapping ξ of A_0 into B_0 is a homomorphism if for all $i \in I$ and for all $a_1, \ldots, a_{n(i)} \in A_0$,

$$\xi(f_i(a_1, \ldots, a_{n(i)})) = g_i(\xi(a_1), \ldots, \xi(a_{n(i)})) .$$

One easily verifies that varieties are closed under the formation of direct products, subalgebras, and homomorphic images. A fundamental theorem on varieties states the converse: if $\mathcal{K}$ is a class of algebras closed under the formation of direct products, subalgebras, and homomorphic images, then $\mathcal{K}$ is a variety [2].

Varieties of algebras were first studied as such by Birkhoff [2]. Central to the study of varieties is the study of relatively free algebras, i.e., algebras which are $\mathcal{K}$-free for some variety $\mathcal{K}$. It was soon realized that the existence of $\mathcal{H}$-free algebras held for slightly more general classes $\mathcal{H}$, today called universal Horn classes. We shall sketch the proof of this theorem below.

Let $\mathcal{C}$ be a class of similar algebras. An algebra A is said to be $\mathcal{C}$-free if $A \in \mathcal{C}$ and there exists a subset $X \subseteq A$ such that (i) X generates A, and (ii) every map of X into an algebra in $\mathcal{C}$ can be extended to a homomorphism.

A universal Horn sentence is a sentence of the form

$$(1.5) \quad \forall\ x \in A_0^n : p_1(x) = q_1(x) \text{ and } \ldots \text{ and } p_k(x) = q_k(x) \Rightarrow p_{k+1}(x) = q_{k+1}(x)$$

where each p_i, q_i is an n-variable polynomial. An identity $p(x) = q(x)$

will be considered as a special case of a universal Horn sentence, since it may be written as

$$(1.6) \quad \forall\, x \in A_0^n : x_0 = x_0 \Rightarrow p(x) = q(x) .$$

A class $\mathcal{H}$ of algebras is a universal Horn class (or quasi-variety) if $\mathcal{H}$ is the class of all algebras of a given similarity type satisfying some collection of universal Horn sentences. Thus, in particular, every variety is a universal Horn class. Universal Horn classes have been characterized in terms of intrinsic properties by Mal'cev [19].

A class of algebras is said to be nontrivial if it contains an algebra with more than one element.

Given a set S, let $\Pi(S)$ denote the set of partitions (equivalence relations) on S. There is a natural partial ordering on equivalence relations: if $\pi, \rho \in \Pi(S)$, let $\pi \leq \rho$ if $x\,\pi\,y$ implies $x\,\rho\,y$. Under this ordering $\Pi(S)$ forms a complete lattice with unit element $1 = S^2$ and null element $0 = \{\langle x, y\rangle \in S^2 : x = y\}$. Furthermore, if we consider partitions as subsets of S^2, the meet of a set of partitions is precisely their set intersection, i.e., $x \bigwedge_{j \in J} \rho_j\, y$ if and only if $x\,\rho_j\,y$ for all $j \in J$.

If $\rho, \pi \in \Pi(S)$ we define the relational product $\rho;\pi$ by $r\,\rho;\pi\,s$ if there exists $t \in S$ such that $r\,\rho\,t\,\pi\,s$. The formation of relational products is easily seen to be an associative operation. The relational product $p;\pi$ is not in general a transitive relation, however. We define $(\rho;\pi)^n$ inductively: $(\rho;\pi)^0$ is the identity relation, and

$(\rho;\pi)^{k+1} = (\rho;\pi)^k;\rho;\pi$. In $\Pi(S)$ the join operation is given by

(1.7) $r \bigvee_{j\in J} \rho_j\ s$ if there exist $t_1, \ldots, t_{n-1} \in S$ and $j(1), \ldots, j(n) \in J$ such that $r = t_0\ \rho_{j(1)}\ t_1\ \rho_{j(2)}\ t_2 \cdots t_{n-1}\ \rho_{j(n)}\ t_n = s$.

In particular,

$$\rho \vee \pi = \bigcup_{n\geq 0} (\rho;\pi)^n .$$

If $\rho \in \Pi(S)$ and $s \in S$, then

$$s/\rho = \{x \in S : x\,\rho\, s\} .$$

Throughout we shall use round symbols $\cap$, $\cup$ for the set operations intersection and union, respectively, and sharp symbols $\wedge$, $\vee$ for the lattice operations meet and join. General information about lattices may be found in [1] or [3].

A congruence relation on an algebra A is an equivalence relation on A_0 which preserves the operations of F. More formally, an equivalence relation θ is a congruence relation if for all $i \in I$ and for all $a, b \in A_0^{n(i)}$, if $a_1\ \theta\ b_1, \ldots, a_{n(i)}\ \theta\ b_{n(i)}$, then $f_i(a_1, \ldots, a_{n(i)})\ \theta\ f_i(b_1, \ldots, b_{n(i)})$. Thus for any congruence relation θ we can form the quotient algebra $A/\theta = \langle A_0/\theta, F\rangle$ where $A_0/\theta = \{x/\theta : x \in A_0\}$, and the mapping $x \to x/\theta$ of A onto A/θ is a homomorphism. Conversely, if $\xi : A \to B$ is a homomorphism, then

(1.8) $\ker \xi = \{\langle x, y\rangle \in A_0^2 : \xi(x) = \xi(y)\}$

is a congruence relation on A, called the kernel of ξ, and $\xi(A)$ is isomorphic to $A/\ker \xi$.

The congruence relations on A form a complete sublattice $\Theta(A)$ of the partition lattice $\Pi(A_0)$. In particular, the meet and join of a set of congruence relations are just their meet and join as equivalence relations.

If $\xi : A \to B$ is a homomorphism, then $\Theta(\xi(A))$ is isomorphic to the quotient sublattice $1/\ker \xi$ of $\Theta(A)$.

If Γ is a collection of congruence relations on A, let $\bigwedge \Gamma$ denote $\bigwedge_{\varphi \in \Gamma} \varphi$. There is a natural imbedding of $A/\bigwedge \Gamma$ into $\times_{\varphi \in \Gamma} A/\varphi$, namely we map $x \in A_0$ into the vector $\langle x/\varphi \rangle_{\varphi \in \Gamma}$. Then $A/\bigwedge\Gamma$ is called a subdirect product of $\{A/\varphi\}_{\varphi \in \Gamma}$.

<u>Theorem 1.1</u>: Let $\mathcal{C}$ be a nontrivial class of algebras closed under the formation of subdirect products. Then for any set X there exists a $\mathcal{C}$-free algebra generated by X.

<u>Proof</u>: Let W(X) be the word algebra over X of the similarity type of the algebras of $\mathcal{C}$. Let

$$\Gamma = \{\varphi \in \Theta(W(X)) : W(X)/\varphi \in \mathcal{C}\},$$

and let $\theta = \bigwedge \Gamma$. We will show that $W(X)/\theta$ is $\mathcal{C}$-free and generated by the classes x/θ $(x \in X)$. Since $\mathcal{C}$ is nontrivial $x/\theta \neq y/\theta$ for $x \neq y$, and since X generates W(X), (i) of the definition of a $\mathcal{C}$-free algebra is satisfied. Any map of X into an algebra in $\mathcal{C}$ can be extended to a homomorphism on W(X). The kernel of this homomorphism will be in Γ and hence will contain θ, so that $W(X)/\theta$ satisfies (ii). Finally, we can represent $W(X)/\theta$ as a subdirect product of the $W(X)/\varphi$ $(\varphi \in \Gamma)$. Since $\mathcal{C}$ is closed under the formation of subdirect products, $W(X)/\theta \in \mathcal{C}$.

It is clear from the construction that $f(x) = g(x)$ holds in a $\mathcal{C}$-free algebra if and only if $f(x) = g(x)$ is an identity in $\mathcal{C}$, i.e., $f(x) = g(x)$ holds in every algebra of $\mathcal{C}$.

Corollary [7, 18]: Let $\mathcal{H}$ be a nontrivial universal Horn class. Then for any set X there exists an $\mathcal{H}$-free algebra generated by X.

Proof: Let A be a subdirect product of algebras in $\mathcal{H}$. The left-hand side of an implication in a universal Horn sentence holds in A if and only if it holds in all the subdirect factors, in which case the right-hand side will hold in all the factors, and hence in A. Thus $A \in \mathcal{H}$.

(R. Lyndon has characterized the logical sentences which are preserved under the formation of subdirect products [15].)

If $\mathcal{C}$ is a class of algebras closed under the formation of subdirect products, then the $\mathcal{C}$-free algebra generated by X will be denoted $F_{\mathcal{C}}(X)$, and where there is no possible ambiguity we will write $F(X)$ for $F_{\mathcal{C}}(X)$.

A lattice identity is an inclusion $u \leq v$, where u and v are lattice polynomials. Note that $u \leq v$ is equivalent to the equation $u \wedge v = u$, and that an equation $u' = v'$ is equivalent to the conjunction of the two inclusions $u' \leq v'$ and $u' \leq v'$. A lattice identity $u \leq v$ is said to be nontrivial if $u \leq v$ fails in some lattice, or equivalently, if $u \leq v$ fails on the generators of a free lattice. The procedure for deciding whether or not a given lattice identity is nontrivial is due to

Whitman [26].

Theorem 1.2: Let $s_i(i = 1, \ldots, m)$, $t_j(j = 1, \ldots, n) \in FL(X)$, and let $x, y \in X$. Then

(1.9) $x \leq y$ iff $x = y$;

(1.10) $\bigwedge_{i=1}^{m} s_i \leq y$ iff there exists $k \in \{1, \ldots, m\}$ such that $s_k \leq y$;

(1.11) $x \leq \bigvee_{j=1}^{n} t_j$ iff there exists $p \in \{1, \ldots, n\}$ such that $x \leq t_p$;

(1.12) $s_1 \leq \bigwedge_{j=1}^{n} t_j$ iff $s_1 \leq t_j$ for all $j \in \{1, \ldots, n\}$;

(1.13) $\bigvee_{i=1}^{m} s_i \leq t_1$ iff $s_i \leq t_1$ for all $i \in \{1, \ldots, m\}$;

(1.14) $\bigwedge_{i=1}^{m} s_i \leq \bigvee_{j=1}^{n} t_j$ iff there exists $k \in \{1, \ldots, m\}$ such that $s_k \leq \bigvee_{j=1}^{n} t_j$, or there exists $p \in \{1, \ldots, n\}$ such that $\bigwedge_{i=1}^{m} s_i \leq t_p$.

Conditions (1.12) and (1.13) hold in any lattice, and conditions (1.9), (1.10) and (1.11) hold in any relatively free lattice [14]. The significance of (1.14) is discussed in [5].

If $\mathcal{C}$ is a class of algebras, let $\Theta(\mathcal{C})$ denote the class of all congruence lattices $\Theta(A)$ for $A \in \mathcal{C}$. We say that $\Theta(\mathcal{C})$ satisfies a nontrivial lattice identity δ if every member of $\Theta(\mathcal{C})$ satisfies δ.

Let $\mathcal{C}^e$ denote the variety of algebras generated by a class $\mathcal{C}$, i.e., $\mathcal{C}^e$ consists of those algebras which satisfy each identity which holds in every algebra of $\mathcal{C}$.

Theorem 1.3: Let $\mathcal{C}$ be a class of algebras closed under the formation of subdirect products, and let δ be a nontrivial lattice identity. Then the following are equivalent:

(1.15) $\Theta(\mathcal{C})$ satisfies δ ;

(1.16) $\Theta(\mathcal{C}^e)$ satisfies δ ;

(1.17) $\Theta(F_{\mathcal{C}}(X))$ satisfies δ for every set X ;

(1.18) $\Theta(F_{\mathcal{C}}(Y))$ satisfies δ where Y is countably infinite ;

(1.19) $\Theta(F_{\mathcal{C}}(E))$ satisfies δ for every finite set E.

Proof: Since $\mathcal{C} \subseteq \mathcal{C}^e$, (1.16) implies (1.15). Clearly (1.15) implies (1.17) and (1.17) implies (1.18). If E is finite and Y is infinite, then $F_{\mathcal{C}}(E)$ is a homomorphic image of $F_{\mathcal{C}}(Y)$, so (1.18) implies (1.19). It remains to show that (1.19) implies (1.16).

First we show that for any set X, $F_{\mathcal{C}}(X)$ is isomorphic to $F_{\mathcal{C}^e}(X)$. For suppose $W(X)/\psi \in \mathcal{C}^e$. Let θ be as in the proof of theorem 1.1. If $\psi \not\geq \theta$, then there exist p(x), q(x) $\in W(X)$ such that $p(x)/\theta = q(x)/\theta$ while $p(x)/\psi \neq q(x)/\psi$. But $p(x)/\theta = q(x)/\theta$ means that $p(x) = q(x)$ is an identity in every algebra of $\mathcal{C}$, and hence in every algebra of $\mathcal{C}^e$. Since $W(X)/\psi \in \mathcal{C}^e$, we must have

$$p(x)/\psi = p(x/\psi) = q(x/\psi) = q(x)/\psi$$

where $x/\psi = (x_1/\psi, \ldots, x_n/\psi)$, a contradiction. Hence $\psi \geq \theta$. The proof of theorem 1.1 now shows that $F_{\mathcal{C}}(X) \cong W(X)/\theta \cong F_{\mathcal{C}^e}(X)$.

Assume $\Theta(F(E))$ satisfies the nontrivial lattice identity $u \leq v$ for every finite set E. Thus u and v are lattice polynomials in variables $x_1, \ldots, x_n$ say, and $u \not\leq v$ in FL(X), where $X = \{x_1, \ldots, x_n\}$. Let $A \in \mathcal{C}^e$ and let $\theta = (\theta_1, \ldots, \theta_n) \in (\Theta(A))^n$. Let $a, b \in A$ and suppose $a\, u(\theta) b$. We can decompose this relation into a finite number of simpler relations by successively applying the following processes:

(1.10) Replace $c\; w_1 \wedge w_2(\theta)\, d$ by the relations $c\; w_1(\theta)\, d$ and $c\; w_2(\theta)\, d$.

(1.21) If $c\; w_1 \vee w_2(\theta)\, d$, then there exist elements $c_1, \ldots, c_{k-1} \in A$ such that

$$c = c_0\; w_1(\theta)\; c_1\; w_2(\theta)\; c_2\; w_1(\theta)\; c_3 \ldots c_k = d .$$

Replace $c\; w_1 \vee w_2(\theta)\, d$ by the relations $c\; w_1(\theta)\; c_1$ and

$c_1\; w_2(\theta)\; c_2$ and ... and $c_{k-1}\; w_p(\theta)\, d$, where $p = k \pmod 2$.

We may continue until we are left with a finite list of relations of the form $c\; \theta_i\; d$ where $c, d \in A$ and $i \in \{1, \ldots, n\}$. Let Y be the set of members of A involved in the list Λ of relations obtained from our original relation $a\, u(\theta)\, b$. Then Y is finite. Let Y* be a new set of symbols

$$Y^* = \{y^* : y \in Y\} ,$$

and form the $\mathcal{C}$-free algebra F(Y*). Let τ be the homomorphism of F(Y*) into A generated by extending the mapping $\tau_0(y^*) = y$ for $y^* \in Y^*$. Let $\varphi_i (1 \leq i \leq n)$ be the congruence relation on F(Y*) generated by identifying all the members of Y* whose images under τ are identified by θ_i in the list Λ, i.e.,

$$\varphi_i = \bigvee \{ \theta(c^*, d^*) \ : \ c \ \theta_i \ d \text{ belongs to the list } \Lambda \} \ .$$

Let $\varphi = (\varphi_1, \ldots, \varphi_n) \in (\Theta(F(Y^*)))^n$. Then clearly $a^* \ u(\varphi) \ b^*$. Since $\Theta(F(Y^*))$ satisfies $u \leqslant v$, we have $a^* \ v(\varphi) \ b^*$. We now show that for any lattice word w and any $s, t \in F(Y^*)$ we have $s \ w(\varphi) \ t$ implies $\tau(s) \ w(\theta) \ \tau(t)$. From this we will be able to conclude that $a \ v(\theta) \ b$, as desired.

Let $\kappa_i = \{ (s, t) \in (F(Y^*))^2 \ : \ \tau(s) \ \theta_i \ \tau(t) \}$. Then κ_i is a congruence relation on $F(Y^*)$; in fact, if π_i is the canonical homomorphism of A onto A/θ_i, then κ_i is the kernel of the homomorphism $\pi_i \tau$ of $F(Y^*)$ into A/θ_i. Furthermore, if $c^*, d^* \in Y^*$ and $c \ \theta_i \ d$ belongs to the list Λ, then $c^* \ \kappa_i \ d^*$. Hence $\varphi_i \leqslant \kappa_i$, i.e., $s \ \varphi_i \ t$ implies $\tau(s) \ \theta_i \ \tau(t)$.

We now use induction on the complexity of the lattice word w to show that $s \ w(\varphi) \ t$ implies $\tau(s) \ w(\theta) \ \tau(t)$. If $w = w_1 \wedge w_2$, then $s \ w(\varphi) \ t$ implies $s \ w_1(\varphi) \ t$ and $s \ w_2(\varphi) \ t$. By the inductive hypothesis, $\tau(s) \ w_1(\theta) \ \tau(t)$ and $\tau(s) \ w_2(\theta) \ \tau(t)$, and thus $\tau(s) \ w(\theta) \ \tau(t)$. If $w = w_1 \vee w_2$ and $s \ w(\varphi) \ t$, then there exist $s_1, \ldots, s_{k-1} \in F(Y^*)$ such that

$$s = s_0 \ w_1(\varphi) \ s_1 \ w_2(\varphi) \ s_2 \ w_1(\varphi) \ s_3 \ \ldots \ s_k = t \ .$$

By induction we obtain

$$\tau(s) = \tau(s_0) \ w_1(\theta) \ \tau(s_1) \ w_2(\theta) \ t(s_2) \ w_1(\theta) \ \tau(s_3) \ \ldots \ \tau(s_k) = \tau(t) \ ,$$

and thus $\tau(s) \ w(\theta) \ \tau(t)$. This completes the proof of theorem 1.3.

The argument of theorem 1.3 is derived from an argument of Mal'cev [17]. Related results appear in [25].

The remaining theorems in this thesis will be stated in terms of varieties of algebras. In view of theorem 1.3, corresponding theorems hold for arbitrary classes of algebras closed under the formation of subdirect products.

A fundamental theorem of Whitman states that every lattice can be imbedded in a partition lattice [27]. It follows that the class of all partition lattices satisfies no nontrivial lattice identities. In fact, every nontrivial lattice identity fails in $\Pi(S)$ for some finite set S [23].

By a unary algebra we mean an algebra $A = \langle A_0, F\rangle$ such that each $f \in F$ is unary, i.e., $n(i) = 1$ for all $i \in I$.

<u>Theorem 1.4</u>: Let $\mathcal{K}$ be a non-trivial variety of unary algebras. Then for any set X, the partition lattice $\Pi(X)$ is isomorphic to a sublattice of the congruence lattice $\Theta(F(X))$, where $F(X)$ is the $\mathcal{K}$-free algebra generated by X.

<u>Proof</u>: We may assume $|X| > 1$. For $Y \subseteq X$ let $S(Y)$ denote the subalgebra of $F(X)$ generated by Y. Suppose $z \in S(x) \cap S(y)$ for some $x, y \in X$ $(x \neq y)$. Then there exist unary polynomials (compositions) f and g such that $z = f(x) = g(y)$. It follows that $f(a) = g(b)$ is an identity in $\mathcal{K}$. Hence $z = f(x) = g(x)$ for all $x \in X$, and $z \in \bigcap_{x \in X} S(x)$. Let $Z = \bigcap_{x \in X} S(x)$, and let $T(x) = S(x) \setminus Z$. Then $T(x) \neq \emptyset$ since $\mathcal{K}$ is non-trivial. Thus $F(X) = Z \cup \bigcup_{x \in X} T(x)$, and the union is disjoint. We now exhibit a monomorphism of $\Pi(X)$ into $\Theta(F(X))$. Suppose $\pi \in \Pi(X)$, and let E_i $(i \in I)$ denote the equivalence

classes of π. The natural map of X onto I given by

$$\varphi_\pi(x) = i \qquad \text{if } x \in E_i$$

can be extended to an epimorphism of F(X) onto F(I). Note that if $\varphi_\pi(u) = \varphi_\pi(v)$ where $u = f(x)$ and $v = g(y)$, then $f(i) = g(j)$ where $x \in E_i$, $y \in E_j$. If $i = j$ then $f(a) = g(a)$ is an identity in $\mathcal{K}$, and thus $v = f(y)$. If $i \neq j$ then $f(a) = g(b)$ is an identity in $\mathcal{K}$, so that $u = f(x) = g(y) = v$. Thus the kernel of φ_π is

$$\kappa_\pi = \{(f(x), f(y)) : x \,\pi\, y \text{ and } f \in \mathcal{F}\}$$

where $\mathcal{F}$ denotes all unary polynomials of $\mathcal{K}$. In particular, μ_π restricted to Z is the identity relation on Z. It is easy to see that the map taking π to μ_π preserves order and the join operation. To conclude that the map is a lattice monomorphism of $\Pi(X)$ into $\Theta(F(X))$ it remains to show that

$$\mu_\pi \wedge \mu_\rho \subseteq \mu_{\pi \wedge \rho} .$$

Suppose $(u, v) \in \mu_\pi \wedge \mu_\rho$. Then $u = f(x)$ and $v = f(y)$ where $x \,\pi\, y$, and $u = g(s)$ and $v = g(t)$ where $s \,\rho\, t$. If $x \neq s$ or $y \neq t$, then $u \in Z$ or $v \in Z$, and hence $u = v$. If $x = s$ and $y = t$, then $x \,\rho\, y$ and $(u, v) \in \mu_{\pi \wedge \rho}$.

<u>Corollary:</u> Let $\mathcal{K}$ be a nontrivial variety of unary algebras. Then the congruence lattices of algebras in $\mathcal{K}$ generate the variety of all lattices.

CHAPTER II

CONGRUENCE LATTICES OF SEMILATTICES

A standard method of proving that a class of lattices satisfies no nontrivial lattice identities is to show that all partition lattices are contained as sublattices of lattices in the class. The congruence lattices of semilattices, however, are known to satisfy the universal Horn sentence:

(2.1) $\forall x, y, z: \quad x \wedge y = x \wedge z \Rightarrow x \wedge (y \vee z) = x \wedge y,$

as we will show in theorem 2.2. It follows that the partition lattice on a three-element set (the five-element two dimensional lattice) is not isomorphic to a sublattice on the congruence lattice of a semilattice, and in fact is not a homomorphic image of a sublattice of the congruence lattice of a finite semilattice. Nonetheless we shall show in this chapter that the congruence lattices of semilattices satisfy no nontrivial lattice identities. This solves problem 6 of [24]. Using a theorem of T. Evans [8], we also show that if $\mathcal{V}$ is a variety of semigroups all of whose congruence lattices satisfy some fixed non-trivial lattice identity, then all the members of $\mathcal{V}$ are groups with exponent dividing a fixed finite number.

A semilattice is a commutative idempotent semigroup. We may impose a partial ordering on a semilattice S by defining

$$x \leq y \text{ if } xy = x.$$

Under this ordering, any two elements $x, y \in S$ have a greatest lower bound, namely their product xy. S is called a meet semilattice. It may be that x and y have a least upper bound $w \in S$; if so, we define

$$x + y = w.$$

Thus + is a partial operation on S, and $x + y$ is called the join of x and y. If S is finite, and if x and y have a common upper bound, then $x + y$ exists and

$$x + y = \Pi\{z \in S: z \geq x \text{ and } z \geq y\}.$$

The least element of a semilattice, if it exists, is denoted by 0; the greatest element, if it exists, by 1.

A dual ideal of a semilattice S is a set $D \subseteq S$ satisfying

(2.2) $\qquad d_1, d_2 \in D$ implies $d_1 d_2 \in D$;

(2.3) $\qquad x \geq d \in D$ implies $x \in D$.

We will denote the principal dual ideal above x by $1/x$, i.e.,

$$1/x = \{z \in S: z \geq x\}.$$

For reference we note that if $x + y$ is defined, then

$$1/x \cap 1/y = 1/x + y.$$

If S and T are semilattices, then $S \times T$ will denote the (external) direct product of S and T.

The following theorem is basic to the study of semilattice congruences.

Theorem 2.1 [21]: Let $\underset{\frown}{2}$ denote the two-element semilattice. If S is any semilattice and D is a dual ideal of X, then the mapping μ: $S \rightarrow \underset{\frown}{2}$ defined by

$$\mu(x) = \begin{cases} 1 \text{ if } x \in D \\ 0 \text{ if } x \notin D \end{cases}$$

is a homomorphisn. Thus $\underset{\frown}{2}$ is the only subdirectly irreducible semilattice, and the dual of $\Theta(S)$ is a point lattice ($\Theta(S)$ is a copoint lattice).

Proof: For $\mu(x)\ \mu(y) = 1$ iff $\mu(x) = 1$ and $\mu(y) = 1$ iff $x \in D$ and $y \in D$ iff $xy \in D$ iff $\mu(xy) = 1$.

Theorem 2.2 [21]: If S is a semilattice, then $\Theta(S)$ satisfies

(2.1)′ $\qquad \varphi \wedge \theta = \varphi \wedge \psi$ implies $\varphi \wedge (\theta \vee \psi) = \varphi \wedge \theta.$

Proof: Let $\varphi \wedge \theta = \varphi \wedge \psi = \kappa$. Suppose $x, y \in S$ and $x\ \varphi \wedge (\theta \vee \psi) y$. Then there exist elements $x_1, \cdots, x_{k-1} \in S$ such that

(2.4) $\qquad x\ \varphi\ y;$

(2.5) $\qquad x = x_0 \theta\ x_1 \psi\ x_2 \theta\ x_3 \cdots x_k = y.$

Note that for each i ($0 \leq i \leq k$) we have $yx_i\ \varphi\ xyx_i$. From (2.5) we have $yx_1 \theta\ xyx_1$. Thus $yx_1\ \varphi \wedge \theta\ xyx_1$, so $yx_1\ \kappa\ xyx_1$. Similarly, $yx_2\ \psi\ yx_1\ \kappa\ xyx_1\ \psi\ xyx_2$, so $yx_2\ \psi\ xyx_2$, hence $yx_2\ \kappa\ xyx_2$. We continue, until for x_k ($= y$) we obtain $y\ \kappa\ xy$. Symmetrically we obtain $x\ \kappa\ xy$, so that $x\ \kappa\ y$. Thus $\varphi \wedge (\theta \vee \psi) = \kappa$.

We now state the main result of this chapter.

Theorem 2.3: Let δ be a nontrivial lattice identity. Then there exists a finite semilattice $S(\delta)$ such that δ fails in the congruence lattice $\Theta(S(\delta))$.

The theorem is an immediate consequence of lemmas 2.1 and 2.4 to be proven below.

Lemma 2.1: Let S be a finite meet semilattice, and let $\mathcal{S}(S)$ be the lattice of (partial) join-subalgebras of S, with $0 \in S$ considered as a distinguished element. Then the congruence lattice $\Theta(S)$ is dually isomorphic to $\mathcal{S}(S)$.

A partial join subalgebra of S is a subset containing 0 and closed under joins, whenever they exist.

Proof: The dual atoms of $\Theta(S)$ are the partitions $\psi_d = (1/d)(S - 1/d)$ for $0 \neq d \in S$. On the other hand the atoms of $\mathcal{S}(S)$ are the subalgebras $\xi_d = \{0, d\}$ for $0 \neq d \in S$. We want to show that the mapping $\psi_d \to \xi_d$ induces a dual ismorphism of $\Theta(S)$ onto $\mathcal{S}(S)$. Since $\Theta(S)$ is a copoint lattice and $\mathcal{S}(S)$ is a point lattice, it is sufficient to show that their closure operations are duals under the mapping, i.e., that

$$\psi_c \geq \psi_{d_1} \wedge \cdots \wedge \psi_{d_k}$$

if and only if

$$\xi_c \leq \xi_{d_1} \vee \cdots \vee \xi_{d_k}.$$

This is equivalent to

$$\psi_c \geq \psi_{d_1} \wedge \cdots \wedge \psi_{d_k}$$

if and only if

$$c \in \langle d_1, \cdots, d_k \rangle$$

where $\langle d_1, \cdots, d_k \rangle$ denotes the join subalgebra generated by $\{d_1, \cdots, d_k\}$. Notice that the equivalence classes of $\psi_{d_1} \wedge \cdots \wedge \psi_{d_k}$ are

$$\left(\bigcap_{j \in I} 1/d_i - \bigcup_{j \in I^c} 1/d_j\right)$$

for $I \subseteq \{1, \cdots, k\}$. If $\psi_c \geq \psi_{d_1} \wedge \cdots \wedge \psi_{d_k}$ then each of these classes is contained in either $1/c$ or $S - 1/c$. Considered the $\psi_{d_1} \wedge \cdots \wedge \psi_{d_k}$ - class which contains c. Then c is the least element of that class, and thus

$$c = \sum_{i \in I} d_i \text{ for some } I \subseteq \{1, \cdots, k\}.$$

Hence $c \in \langle d_1, \cdots, d_k \rangle$. Conversely, if

$$c \in \langle d_1, \cdots d_k \rangle,$$

then

$$c = \sum_{i \in I} d_i$$

$I \subseteq \{1, \cdots, k\}$. Thus the congruence

$$\bigwedge_{i \in I} \psi_{d_i}$$

has one class equal to $1/c$ and the rest contained in $S - 1/c$. Hence

$$\psi_c \geq \bigwedge_{i \in I} \psi_{d_i} \geq \psi_{d_1} \wedge \cdots \wedge \psi_{d_k}.$$

This completes the proof of lemma 2.1.

Suppose $u \leq v$ is a nontrivial lattice identity, i.e., $u \leq v$ does not hold in a free lattice. Then we construct a finite semilattice $S(u)$ (depending only on u) such that $u \leq v$ fails in $\mathcal{S}(S(u))$. Combined with lemma 2.1, this will prove theorem 2.3.

Let $X = \{x, y, z \cdots\}$ be a countable set, and let $FL(X)$ denote the free lattice on X. For each element $u \in FL(X)$ we will define a finite semilattice $S(u)$. First of all we write each $u \in FL(X)$ in canonical form. Then we define

(2.6) $$S(x) = \underset{\frown}{2} \text{ for } x \in X$$

(2.7) $$S(u_1 \vee u_2) = S(u_1) \times S(u_2)$$

(2.8) $$S(u_1 \wedge u_2) = S(u_1) \times S(u_2) - \Gamma$$

where

$$\Gamma = 1/(1,0) \cup 1/(0,1) - \{(1,1)\}.$$

Let us look more carefully at the construction. If $S(u_1)$ and $S(u_2)$ are lattices, then $S(u_1) \times S(u_2) - \Gamma$ is meet-closed and has a unit element; hence it is a lattice. It follows by induction that $S(u)$ is a lattice for each $u \in FL(X)$. We need to know how to compute joins in $S(u)$. In $S(u_1 \vee u_2)$ joins are of course taken componentwise. In $S(u_1 \wedge u_2)$ we have

$$(2.9)\quad (r_1, r_2) + (s_1, s_2) = \begin{cases} (r_1+s_1, r_2+s_2) & \text{if } r_1+s_1 \neq 1 \text{ and } r_2+s_2 \neq 1 \\ (1, 1) & \text{if } r_1+s_1 = 1 \text{ or } r_2+s_2 = 1. \end{cases}$$

In any $S(u)$ let us denote $(1, 1)$ by 1.

For each $u \in FL(X)$ we now define a homomorphism φ_u of $FL(X)$ into $\mathcal{S}(S(u))$. We do this by associating with each $y \in X$ a join-subalgebra $\varphi_u(y)$ of $S(u)$, and extending this map to a homomorphism in the (unique) natural way. Once again we proceed inductively, with $u \in FL(X)$ written in canonical form. For $y \in X$ we set

$$(2.10)\qquad \varphi_x(y) = \begin{cases} S(x) & \text{if } y = x \\ \{0\} & \text{if } y \neq x \end{cases}$$

$$(2.11)\qquad \varphi_{u_1 \vee u_2}(y) = \{(r_1, r_2): r_1 \in \varphi_{u_1}(y),\ r_2 \in \varphi_{u_2}(y)\}$$

$$(2.12)\quad \varphi_{u_1 \wedge u_2}(y) = \{(r_1, r_2): r_1 \in \varphi_{u_1}(y) - \{1\},\ r_2 \in \varphi_{u_2}(y) - \{1\}\}$$

$$\cup\ \Lambda\ (\varphi_{u_1}(y),\ \varphi_{u_2}(y))$$

where
$$\Lambda(A,B) = \begin{cases} \emptyset & \text{if } 1 \notin A \text{ and } 1 \notin B \\ \{1\} & \text{if } 1 \in A \text{ or } 1 \in B. \end{cases}$$

Our computations will be based upon the following lemma.

Lemma 2.2: If $w \in FL(X)$, then

(2.13) $$\varphi_{u_1 \vee u_2}(w) = \varphi_{u_1}(w) \times \varphi_{u_2}(w)$$

(2.14) $$\varphi_{u_1 \wedge u_2}(w) - \{1\} = \{(r,s) \in \varphi_{u_1}(w) \times \varphi_{u_2}(w): r \neq 1 \text{ and } s \neq 1\}.$$

Proof: We prove (2.14); the proof of (2.13) is similar but easier. We proceed by induction on the length of w. For $w = y \in X$ the lemma is immediate from the definitions. Now note that since $0 \in T$ for every $T \in \mathcal{S}(S(u)$, we have

$$T_1 \vee T_2 = \{t_1 + t_2: t_1 \in T_1,\ t_2 \in T_2\}.$$

Hence if $w = w_1 \vee w_2$, then by (2.9) we have

$$\varphi_{u_1 \wedge u_2}(w) - \{1\} = \varphi_{u_1 \wedge u_2}(w_1) \vee \varphi_{u_1 \wedge u_2}(w_2) - \{1\}.$$
$$= \{(r_1, s_1) + (r_2, s_2): (r_1, s_1) \in \varphi_{u_1 \wedge u_2}(w_1),$$
$$(r_2, s_2) \in \varphi_{u_1 \wedge u_2}(w_2),\ r_1 + r_2 \neq 1,\ s_1 + s_2 \neq 1\}.$$

By the inductive hypothesis we have

$$(r_i, s_i) \in \varphi_{u_1 \wedge u_2}(w_i) - \{1\} = \varphi_{u_1}(w_i) - \{1\} \times \varphi_{u_2}(w_i) - \{1\}$$

for i = 1, 2 and hence

$$\varphi_{u_1 \wedge u_2}(w) - \{1\} = \{(r, s) \in \varphi_{u_1}(w) \times \varphi_{u_2}(w) : r \neq 1 \text{ and } s \neq 1\}.$$

On the other hand, if $w = w_1 \wedge w_2$, then

$$\varphi_{u_1 \wedge u_2}(w) - \{1\} = \varphi_{u_1 \wedge u_2}(w_1) - \{1\} \cap \varphi_{u_1 \wedge u_2}(w_2) - \{1\}$$

and the conclusion of the lemma follows.

Lemma 2.3. If $w \in FL(X)$, then $1 \in \varphi_w(w)$.

Proof: As usual we proceed by induction on the length of w. If $w = y \in X$ the lemma follows from the definitions. If $w = w_1 \vee w_2$, then $1 \in \varphi_{w_i}(w_i)$ $(i = 1, 2)$, and thus by (2.13) we have

$$(1, 0) \in \varphi_w(w_1) \text{ and } (0, 1) \in \varphi_w(w_2)$$

from which it follows that

$$1 \in \varphi_w(w_1) \vee \varphi_w(w_2) = \varphi_w(w).$$

If $w = w_1 \wedge w_2$, we can again assume $1 \in \varphi_{w_i}(w_i)$ for $i = 1, 2$. We need to show that

$$1 \in \varphi_{w_1 \wedge w_2}(w_i).$$

We prove a slightly stronger statement: if

$$1 \in \varphi_{w_1}(u),$$

then

$$1 \in \varphi_{w_1 \wedge w_2}(u).$$

If $u = y \in X$ this is immediate. Suppose $u = u_1 \vee u_2$. Then

$$1 \in \varphi_{w_1}(u_1) \vee \varphi_{w_1}(u_2)$$

and hence

$$1 = t_1 + t_2, \text{ where } t_i \in \varphi_w(u_i).$$

If $t_1 \neq 1$, $t_2 \neq 1$, then by (2.14) we have

$$1 = (t_1, 0) + (t_2, 0) \in \varphi_{w_1 \wedge w_2}(u_1) \vee \varphi_{w_1 \wedge w_2}(u_2)$$

$$= \varphi_{w_1 \wedge w_2}(u).$$

If $t_i = 1$ for some i then by induction $1 \in \varphi_{w_1}(u_i)$ implies

$$1 \in \varphi_{w_1 \wedge w_2}(u_i) \subseteq \varphi_{w_1 \wedge w_2}(u).$$

Suppose $u = u_1 \wedge u_2$. Then

$$1 \in \varphi_{w_1}(u_1) \cap \varphi_{w_1}(u_2).$$

By induction

$$1 \in \varphi_{w_1 \wedge w_2}(u_i)$$

for $i = 1, 2$ and we are done.

<u>Lemma 2.4.</u> If $u \not\leq v$ in FL(X), then $1 \in \varphi_u(v)$.

Assume we have proven lemma 2.4. Then lemmas 2.3 and 2.4 combine to yield: $1 \in \varphi_u(v)$ if and only if $u \leq v$ in FL(X). Hence $\varphi_u(u) \subseteq \varphi_u(v)$ if and only if $u \leq v$ in FL(X), and theorem 2.3 follows.

Proof of lemma 2.4. Suppose the lemma is false. Let u be a word of minimum length such that $1 \in \varphi_u(v')$ for some v' such that $u \not\leq v'$ in FL(X). Let v be of minimal length such that $u \not\leq v$ and $1 \in \varphi_u(v)$. We will show that these conditions lead to a contradiction. The cases

$$u \in X, \text{ or } u = u_1 \vee u_2,$$

and

$$u = u_1 \wedge u_2, \ v \in X \text{ or } v = v_1 \wedge v_2$$

are easy to handle. Let us assume, then that $u = u_1 \wedge u_2$ and $v = v_1 \vee v_2$. Then since $u \not\leq v$ we have

$$u \not\leq v_1 \text{ and } u \not\leq v_2 \text{ and } u_1 \not\leq v \text{ and } u_2 \not\leq v.$$

Since

$$1 \in \varphi_u(v) = \varphi_u(r_1) \vee \varphi_u(r_2),$$

there exist $t_i \in \varphi_u(v_i)$ such that $t_1 + t_2 = 1$. If $t_i = 1$ for some i then by the minimal length of v we have $u \leq v_i$, a contradiction. Thus $t_i \neq 1$ and by (2.14) we can write $t_i = (r_i, s_i)$ where $r_i \in \varphi_{u_1}(v_i)$ and $s_i \in \varphi_{v_2}(v_i)$. Now either $r_1 + r_2 = 1$ in $S(u_1)$, which means

$$1 \in \varphi_{u_1}(v_1) \vee \varphi_{u_1}(v_2) = \varphi_{u_1}(v)$$

and hence $u \leq v$, or $s_1 + s_2 = 1$ and $u_2 \leq v$. Both these statements are contradictions.

Since the semilattices S(u) constructed above are in fact lattices, they are join semilattices. Thus, the above proof shows that any nontrivial lattice identity fails in the subalgebra lattice of some finite semilattice.

Now the congruence lattices of lattices satisfy every nontrivial lattice identity, while those of semilattices satisfy no identity. It is reasonable then to ask if there is some "natural" restricted class $\mathbb{C}$ of semilattices such that the congruence lattices of semilattices in $\mathbb{C}$ satisfy some lattice identity.

One such class is known [6]. A simple argument based on lemma 2.1 shows that $\Theta(S)$ is nonmodular if and only if S contains a pair of noncomparable elements with a common upper bound. Hence $\Theta(S)$ is either nonmodular, or else it is isomorphic to the Boolean algebra of subsets of some set.

On the other hand, the semilattices S(u) constructed in the proof of theorem 2.3 are in fact lattices; in particular, the join of every pair of elements is defined. It follows from theorem 2.1 that S(u) can be imbedded as a join semilattice into a Boolean algebra B(u). Considering B(u) as a meet semilattice, we see that every nontrivial lattice identity fails in the (semilattice) congruence lattice of some finite Boolean algebra.

We can now prove an interesting corollary about varieties of semigroups. Let R denote the two-element semigroup with multiplication law $xy = y$; L the two-element semigroup with multiplication law $xy = x$; and C the two element semigroup with constant multiplication. The following theorem is due to T. Evans [8].

Theorem 2.4: If a nontrivial variety of semigroups does not contain R, L, C, or $\underset{\frown}{2}$, then it is a subvariety of $\mathcal{B}_n$, the variety of groups of exponent dividing n, for some finite n.

Proof: Let $\mathcal{V}$ be a nontrivial variety of semigroups not containing R, L, C, or $\underset{\frown}{2}$. Consider $F_{\mathcal{V}}(1)$. Either

(2.15) $\qquad F(1)$ is infinite cyclic, or

(2.16) $\qquad x^n = x^k$ for some $n > k > 1$, or

(2.17) $\qquad x^n = x$ for some $n > 1$.

If (2.15) holds, let θ be the congruence which collapses $\{x^k : k > 1\}$. Then $F(1)/\theta \cong C$. If (2.16) holds, let S be the subalgebra of $F(1)$ generated by x^{k-1}, and let φ be the congruence on S which collapses $\{x^{j(k-1)} : j > 1\}$. Then $S/\varphi \cong C$. Hence we can assume (2.17).

Consider $F(2)$. If $n = 2$ ($x^2 = x$), then the elements x, xyx generate $\underset{\frown}{2}$ if they are distinct. Hence we must have $xyx = x$. Then the elements x, xy generate R if they are distinct. If they are not distinct, then $\mathcal{V}$ satisfies $xy = x$, and $F(2) \cong L$. Hence $n > 2$.

Now if $x^{n-1} = y^{n-1}$, then x^{n-1} is a unit element ($yx^{n-1} = y^n = y$) and every element is invertible ($yy^{n-2} = y^{n-1} = x^{n-1}$), hence $\mathcal{V} \subseteq B_{n-1}$.

Suppose $x^{n-1} \neq y^{n-1}$. Then x^{n-1} and y^{n-1} are distinct idempotents in F(2) $((x^{n-1})^2 = x^n x^{n-2} = xx^{n-2} = x^{n-1})$ which we can denote e, f. Now the elements e, $(efe)^{n-1}$ generate $\underset{\wedge}{2}$ if they are distinct, hence we can assume $(efe)^{n-1} = e$. Then the elements $(ef)^{n-1}$, e generate R unless $(ef)^{n-1} = e$. Symmetrically $(fef)^{n-1} = f$ and the elements $(ef)^{n-1}$, f generate L unless $(ef)^{n-1} = f$. But then $e = (ef)^{n-1} = f$, contrary to hypothesis.

Now if T is a semigroup in the variety generated by R, L, or C, then $\Theta(T)$ is just the partition lattice on T. Hence theorems 2.3 and 2.4 combine to give the following corollary.

Corollary. If $\mathcal{V}$ is a semigroup variety of all whose congruence lattices satisfy some fixed nontrivial lattice identity, then $\mathcal{V}$ is a subvariety of $\mathcal{B}_n$ for some finite n.

CHAPTER III

VARIETIES WHOSE CONGRUENCES SATISFY CERTAIN LATTICE IDENTITIES

In this chapter we show that varieties whose congruence lattices satisfy one of a class of lattice identities of a fairly general form (see theorem 3.1) are in fact congruence modular. A similar theorem is proved for congruence distributivity.

Throughout this chapter $\mathcal{K}$ will represent an arbitrary variety of algebras.

If $w \in FL(X)$, then $var(w)$ is the set of members of X which appear in the canonical expression of w.

If $\rho \in \Pi(S)$ and $S \subseteq U$, then $\hat{\rho}$ is the member of $\Pi(U)$ given by

$$\hat{\rho} = \{(x, y) : x\rho y \text{ in } S \text{ or } x = y\}.$$

If $\varphi : X \rightarrow \Pi(S)$ and $S \subseteq U$, then we define $\hat{\varphi} : X \rightarrow \Pi(U)$ by

$$\hat{\varphi}(x) = \hat{\rho} \quad \text{where} \quad \rho = \varphi(x).$$

If $s, t \in S$, then we let $\xi(s, t)$ denote the principal equivalence relation

$$\xi(s, t) = \{(x, y) : \{x, y\} = \{s, t\} \text{ or } x = y\}.$$

The main purpose of this chapter is to prove the following theorem.

<u>Theorem 3.1:</u> In FL(X) suppose $\sigma_0, \cdots, \sigma_k (k \geq 1)$ are joins of members of X. Let $w \in FL(X)$ be such that $w \not\leq \sigma_0$ and $\sigma_0 \wedge w \not\leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i)$. Then if $\Theta(\mathcal{K})$ satisfies the lattice equation

$$\sigma_0 \wedge w \leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i) \tag{3.1}$$

$\Theta(\mathcal{K})$ is modular.

Before proving the theorem we make a couple of comments.

(1) If we write the modular law as

$$x \wedge [(x \wedge y) \vee z] \leq (x \wedge y) \vee (x \wedge z)$$

then it is in the form of (3.1).

(2) If k = 1 the relation (3.1) becomes $\sigma_0 \wedge w \leq \sigma_0 \wedge \sigma_1$, which is equivalent to $\sigma_0 \wedge w \leq \sigma_1$. Now in FL(X) we have $\sigma_0 \wedge w \not\leq \sigma_1$; applying Whitman's conditions this is equivalent to $w \not\leq \sigma_1$ and $\mathrm{var}(\sigma_0) \setminus \mathrm{var}(\sigma_1) \neq \emptyset$. We show that as a lattice identity $\sigma_0 \wedge w \leq \sigma_1$ implies $x = y$, or equivalently, that $\sigma_0 \wedge w \leq \sigma_1$ fails in the two-element lattice $\underset{\frown}{2} = \{0, 1\}$. Consider the homomorphism μ: $FL(X) \to \underset{\frown}{2}$ defined by

$$\mu(x) = 0 \quad \text{if} \quad x \in \mathrm{var}(\sigma_1)$$

$$\mu(x) = 1 \quad \text{if} \quad x \in X \setminus \mathrm{var}(\sigma_1).$$

Clearly $\mu(\sigma_0) = 1$ and $\mu(\sigma_1) = 0$. We want to show that $\mu(w) = 1$. In fact, we have $\mu(v) = 1$ if and only if $v \not\leq \sigma_1$. Recall that if $y \in X$ then $y \not\leq \sigma_1$ iff $y \notin \mathrm{var}(\sigma_1)$; $v_1 \vee v_2 \not\leq \sigma_1$ iff $v_1 \not\leq \sigma_1$ or

$v_2 \not\leq \sigma_1$; $v_1 \wedge v_2 \not\leq \sigma_1$ iff $v_1 \not\leq \sigma_1$ and $v_2 \not\leq \sigma_1$ (because σ_1 is a join of members of X). The claim now follows by induction on the complexity of v.

The proof of theorem 3.1 requires several lemmas. We give a different proof of lemma 3.1 from that of a stronger theorem in [10], one which we will be able to modify for use in lemma 3.3.

Lemma 3.1: Let $\sigma_1, \cdots, \sigma_m$ be joins of members of X and let $w \in FL(X)$ be such that $w \not\leq \sigma_i$ $(i = 1, \cdots, m)$. Then there exist a finite set $S(w)$, elements $s_1, s_2 \in S(w)$, and a homomorphism φ_w: $FL(X) \rightarrow \Pi(S(w))$ such that

$$(3.2) \qquad (s_1, s_2) \in \varphi_w(w)$$

$$(3.3) \qquad (s_1, s_2) \notin \varphi_w(\sigma_i) \quad (i = 1, \cdots, m).$$

Proof: We induct on the complexity of w. For $w = x \in X$ set $S(x) = \{s_1, s_2\}$ and let

$$\varphi_x(x) = (s_1, s_2)$$

$$\varphi_x(y) = (s_1)(s_2) \text{ for } y \in X \setminus \{x\}$$

and extend this map to a homomorphism. Clearly (3.2) and (3.3) are satisfied.

Suppose $w = w_1 \vee w_2$. Then the condition $w \not\leq \sigma_i$ $(i = 1, \cdots, m)$ means that for each i there is $k(i) \in \{1, 2\}$ such that $w_{k(i)} \not\leq \sigma_i$. Then by induction there exist (disjoint) finite sets $S(w_1)$ and $T(w_2)$ and homomorphisms $\varphi_{w_1}: FL(X) \rightarrow \Pi(S(w_1))$ and $\varphi_{w_2}: FL(X) \rightarrow \Pi(T(w_2))$ satisfying:

$$(s_1, s_2) \in \varphi_{w_1}(w_1) \text{ and } (s_1, s_2) \notin \varphi_{w_1}(\sigma_i) \text{ if } k(i) = 1$$

$$(t_1, t_2) \in \varphi_{w_2}(w_2) \text{ and } (t_1, t_2) \notin \sigma_{w_2}(\sigma_j) \text{ if } k(j) = 2.$$

Now we set $s_2 = t_1$ and let $U(w) = S(w_1) \bigcup T(w_2)$. For $x \in X$ we set

$$\varphi_w(x) = \hat{\varphi}_{w_1}(x) \vee \hat{\varphi}_{w_2}(x),$$

and extend this mapping to a homorphism. Clearly

$$\hat{\varphi}_{w_i}(v) \leq \varphi_w(v)$$

for all $v \in FL(X)$ $(i = 1, 2)$. Thus we have

$$s_1 \, \varphi_w(w_1) \, s_2 = t_1 \, \varphi_w(w_2) \, t_2$$

and $(s_1, t_2) \in \varphi_w(w)$. Observe that if

$$\rho = \rho_1 \vee \cdots \vee \rho_n$$

then

$$\hat{\rho} = \hat{\rho}_1 \vee \cdots \vee \hat{\rho}_n.$$

Thus

$$\varphi_w(\sigma_i) = \bigvee_{x \in \mathrm{var}(\sigma_i)} \varphi_w(x)$$

$$= \bigvee_{x \in \mathrm{var}(\sigma_i)} \hat{\varphi}_{w_1}(x) \vee \bigvee_{x \in \mathrm{var}(\sigma_i)} \hat{\varphi}_{w_2}(x)$$

$$= \hat{\varphi}_{w_1}(\sigma_i) \vee \hat{\varphi}_{w_2}(\sigma_i).$$

Suppose k(i) = 1. Then

$$t_2 \nleq s_1/\varphi_{w_1}(\sigma_i) = s_1/\varphi_w(\sigma_i).$$

Similarly, if k(j) = 2, then $s_1 \nleq t_2/\varphi_w(\sigma_j)$.

Suppose $w = w_1 \wedge w_2$. Then $w \nleq \sigma_i$ $(i = 1, \cdots, m)$ implies $w_1 \nleq \sigma_i$ and $w_2 \nleq \sigma_i$ (these are in fact equivalent because of the form of the σ_i). By induction there exist (disjoint) finite sets $S(w_1)$ and $T(w_2)$ and homomorphisms $\varphi_{w_1}: FL(X) \to \Pi(S(w_1))$ and $\varphi_{w_2}: FL(X) \to \Pi(T(w_2))$ satisfying:

$$(s_1, s_2) \in \varphi_{w_1}(w_1) \text{ and } (s_1, s_2) \notin \varphi_{w_1}(\sigma_i) \; (i = 1, \cdots, m)$$

$$(t_1, t_2) \in \varphi_{w_2}(w_2) \text{ and } (t_1, t_2) \notin \varphi_{w_2}(\sigma_i) \; (i = 1, \cdots, m)$$

Now we set $s_1 = t_1$ and $s_2 = t_2$, and let $U(w) = S(w_1) \cup T(w_2)$. For $x \in X$ we set

$$\varphi_w(x) = \hat{\varphi}_{w_1}(x) \vee \hat{\varphi}_{w_2}(x)$$

and extend this mapping to a homomorphism. As before we get

$$\hat{\varphi}_{w_i}(v) \leq \varphi_w(v)$$

for all $v \in FL(X)$ $(i = 1, 2)$. Thus $(s_1, s_2) \in \varphi_w(w)$. Also arguing as before we get

$$\varphi_w(\sigma_i) = \hat{\varphi}_{w_1}(\sigma_i) \vee \hat{\varphi}_{w_2}(\sigma_i)$$

so that, in particular,

$$s_1/\varphi_w(\sigma_i) = s_1/\varphi_{w_1}(\sigma_i) \cup t_1/\varphi_{w_2}(\sigma_i)$$

which does not contain s_2 or t_2. Thus $(s_1, s_2) \notin \varphi_w(\sigma_i)$.

This completes the proof of lemma 3.1.

Lemma 3.2: Let $\sigma_0, \cdots, \sigma_k$ $(k \geq 1)$ be joins of members of X. Then the following are equivalent in FL(X):

(3.4) $$w \not\leq \sigma_0 \text{ and } \sigma_0 \wedge w \not\leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i)$$

(3.5) $$w \not\leq \sigma_i \ (i = 0, \cdots, k) \text{ and } \operatorname{var}(\sigma_0) \setminus \bigcup_{i=1}^{k} \operatorname{var}(\sigma_i) \neq \emptyset.$$

Proof: Applying Whitman's conditions repeatedly, and dropping superfluous relations, we see that

$$w \not\leq \sigma_0 \text{ and } \sigma_0 \wedge w \not\leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i)$$

iff

$$w \not\leq \sigma_0 \text{ and } \sigma_0 \not\leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i) \text{ and } \sigma_0 \wedge w \not\leq \sigma_0 \wedge \sigma_i \ (i = 1, \cdots, k)$$

iff

$$\begin{cases} w \not\leq \sigma_0 \text{ and there exists } y_0 \in \operatorname{var}(\sigma_0) \text{ such that } y_0 \not\leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i) \\ \qquad \text{and } \sigma_0 \wedge w \not\leq \sigma_i \ (i = 1, \cdots, k) \end{cases}$$

iff

$$\begin{cases} w \not\leq \sigma_i \ (i = 0, \cdots, k) \text{ and } \sigma_0 \not\leq \sigma_i \ (i = 1, \cdots, k) \\ \text{and there exists } y_0 \in var(\sigma_0) \text{ such that } y_0 \not\leq (\sigma_0 \wedge \sigma_i) \ (i = 1, \cdots, k) \end{cases}$$

iff

$$w \not\leq \sigma_i \ (i = 0, \cdots, k) \text{ and there exists } y_0 \in var(\sigma_0) \setminus \bigcup_{i=1}^{k} var(\sigma_i).$$

Lemma 3.3 : Let $\sigma_0, \cdots, \sigma_k (k \not\geq 1)$ be joins of members of X and let $w \in FL(X)$. Suppose $w \not\leq \sigma_i$ $(i = 0, \cdots, k)$ and $var(\sigma_0) \setminus \bigcup_{i=1}^{k} var(\sigma_i) \neq \emptyset$. Then there exists a finite set $S(w)$, elements $s_1, s_2 \in S(w)$, and a homomorphism φ_w: $FL(X) \to \Pi(S(w))$ such that

(3.6) $$(s_1, s_2) \in \varphi_w(w),$$

(3.7) $$s_1/\varphi_w(\sigma_0) = s_2/\varphi_w(\sigma_0) = \{s_1, s_2\}$$

(3.8) $$(s_1, s_2) \notin \varphi_w(\sigma_i) \text{ for } i = 1, \cdots, k.$$

Proof: We will first prove lemma 3.3 with (3.7) replaced by

(3.7)′ $$s_1/\varphi_w(\sigma_0) = \{s_1\} \text{ and } s_2/\varphi_w(\sigma_0) = \{s_2\}.$$

Then the original statement can be derived as follows. Let φ_w be a homomorphism satisfying (3.6), (3.7)′, and (3.8). Let

$$y_0 \in var(\sigma_0) \setminus \bigcup_{i=1}^{k} var(\sigma_i).$$

Set

$$\psi_w(y_0) = \varphi_w(y_0) \vee \xi(s_1, s_2)$$

$$\psi_w(x) = \varphi_w(x) \text{ if } x \in X \setminus \{y_0\}.$$

and extend to a homomorphism. Then clearly ψ_w satisfies (3.6), (3.7), (3.8).

Again we induct on the complexity of w. For $w = x \in X$ the homomorphism φ_x defined in lemma 3.1 satisfies (3.6), (3.7)′, (3.8). Suppose $w = w_1 \wedge w_2$. Then w_1 and w_2 satisfy the hypotheses of lemma 3.2, and by induction there exist homomorphisms φ_{w_1}: $FL(X) \to \Pi(S(w_1))$ and $\varphi_{w_2}: FL(X) \to \Pi(T(w_2))$ satisfying (3.6), (3.7)′, (3.8) for w_1 and w_2, respectively. The construction used in this case in lemma 3.1 preserves the properties (3.6), (3.7)′, (3.8) using just the arguments given there.

Suppose $w = w_1 \vee w_2$. Then we must alter the construction as follows. Since $w_2 \vee w_2 \not\leq \sigma_i$ $(i = 0, \cdots, k)$, for each i there is $\eta(i) \in \{1, 2\}$ such that $w_{\eta(i)} \not\leq \sigma_i$. For convenience we can assume $\eta(0) = 1$. Now by induction (for w_1) and lemma 3.1 (for w_2) there exist (disjoint) finite sets $S(w_1)$ and $T(w_2)$ and homomorphisms $\varphi_{w_1}: FL(X) \to \Pi(S(w_1))$ and $\varphi_{w_2}: FL(X) \to \Pi(T(w_2))$ satisfying:

$$(s_1, s_2) \in \varphi_{w_1}(w_1) \text{ and } s_1/\varphi_{w_1}(\sigma_0) = \{s_1\}$$

$$\text{and } s_2/\varphi_{w_1}(\sigma_0) = \{s_2\}$$

$$\text{and } (s_1, s_2) \notin \varphi_{w_1}(\sigma_i) \text{ if } \eta(i) = 1.$$

$$(t_1, t_2) \in \varphi_{w_2}(w_2) \quad \text{and } (t_1, t_2) \notin \varphi_{w_2}(\sigma_j) \text{ if } \eta(j) = 2.$$

Now let $S'(w_1)$ be another copy of $S(w_1)$, disjoint from $S(w_1)$ and $T(w_2)$. Set $s_2 = t_1$ and $t_2 = s_1'$, and let $U(w) = S(w_1) \cup T(w_2) \cup S'(w_1)$. For $x \in X$ we set

$$\varphi_w(x) = \hat{\varphi}_{w_1}(x) \vee \hat{\varphi}_{w_2}(x) \vee \hat{\varphi}'_{w_1}(x)$$

and extend to a homomorphism. Arguing as in lemma 3.1,

$$\hat{\varphi}_{w_i}(v) \leq \varphi_w(v)$$

$$\varphi_w(\sigma_i) = \hat{\varphi}_{w_1}(\sigma_i) \vee \hat{\varphi}_{w_2}(\sigma_i) \vee \hat{\varphi}'_{w_1}(\sigma_i)$$

from which we readily obtain (3.6), (3.7)′, (3.8).

Now with a partition $\rho \in \Pi(S)$ we may associate a congruence relation $\theta(\rho) \in \Theta(F_{\mathcal{K}}(S))$ given by

$$\theta(\rho) = \vee\{\theta(s,t)\colon s, t \in S \text{ and } s\rho t\}.$$

Then given a homomorphism $\varphi\colon FL(X) \to \Pi(S)$ we may construct a homomorphism $\psi\colon FL(X) \to \Theta(F_{\mathcal{K}}(S))$ by defining

$$\psi(x) = \theta(\varphi(x)) \text{ for } x \in X$$

and extending naturally.

<u>Lemma 3.4:</u> (1) If $w \in FL(X)$, then $\psi(w) \geq \theta(\varphi(w))$.

(2) If σ is a join of members of X, then $\psi(\sigma) = \theta(\varphi(\sigma))$.

Proof: Observe that for $\rho, \pi \in \Pi(S)$ we have

$$\theta(\rho \vee \pi) = \theta(\rho) \vee \theta(\pi)$$
$$\theta(\rho \wedge \pi) \leq \theta(\rho) \wedge \theta(\pi).$$

Now we prove (1) by induction on the complexity of w. If $w = w_1 \vee w_2$, then

$$\begin{aligned}\psi(w) &= \psi(w_1) \vee \psi(w_2) \\ &\geq \theta(\varphi(w_1)) \vee \theta(\varphi(w_2)) \\ &= \theta(\varphi(w_1) \vee \varphi(w_2)) \\ &= \theta(\varphi(w))\end{aligned}$$

with equality holding in case w is a σ, thus proving (2). If $w = w_1 \wedge w_2$, a similar calculation yields (1).

The next lemma provides a more useful description of $\theta(\rho)$ for $\rho \in \Pi(S)$. We employ the following canonical homomorphisms:

η: $W(S) \to F(S)$ and $\eta(s) = s$ for $s \in S$,

η': $W(S/\rho) \to F(S/\rho)$ and $\eta'(s/\rho) = s/\rho$ for $s \in S$,

ρ^*: $W(S) \to W(S/\rho)$ and $\rho^*(s) = s/\rho$ for $s \in S$,

θ^*: $F(S) \to F(S/\rho)$ and $\theta^*(s) = s/\rho$ for $s \in S$,

$$\begin{array}{ccc} W(S) & \xrightarrow{\eta} & F(S) \\ \rho^* \downarrow & & \downarrow \theta^* \\ W(S/\rho) & \xrightarrow{\eta'} & F(S/\rho) \end{array}$$

Note that the kernel of θ^* is $\theta(\rho)$.

Lemma 3.5: Let $\rho \in \Pi(S)$ and $t_1, t_2 \in F_{\mathcal{K}}(S)$. Then $t_1\ \theta(\rho)\ t_2$ if and only if there exist $f_1 \in \eta^{-1}(t_1)$, $f_2 \in \eta^{-1}(t_2)$ such that $\eta'\ \rho^*(f_1) = \eta'\ \rho^*(f_2)$, i.e., such that $f_1(s/\rho) = f_2(s/\rho)$ is an identity in $\mathcal{K}$.

Proof: We must show that $\eta'\ \rho^* = \theta^*\eta$. Let

$$T = \{t \in W(S): \eta'\rho^*(t) = \theta^*\eta\ (t)\}.$$

Clearly $S \subseteq T$, and since by definition $\eta, \eta', \rho^*, \theta^*$ are homomorphisms, T is a subalgebra of W(S). Hence T = W(S).

Thus, if $f, g \in \eta^{-1}(t)$, then $\eta'\rho^*(f) = \theta^*(t) = \eta'\rho^*(g)$, so that we may in fact choose any $f_1 \in \eta^{-1}(t_1)$, $f_2 \in \eta^{-1}(t_2)$. The lemma now follows since $\ker(\theta^*) = \theta(\rho)$.

Lemma 3.6[4]: In $F_{\mathcal{K}}(4)$ set

$$\alpha = \theta(x_1, x_2) \vee \theta(x_3, x_4)$$

$$\beta = \theta(x_3, x_4)$$

$$\gamma = \theta(x_1, x_3) \vee \theta(x_2, x_4).$$

Then $\Theta(\mathcal{K})$ is modular if and only if $(x_1, x_2) \in \beta \vee (\alpha \wedge \gamma)$.

Proof: If $\Theta(\mathcal{K})$ is modular, then since $\alpha \geq \beta$ we have

$$(x_1, x_2) \in \alpha \wedge (\beta \vee \gamma) = \beta \vee (\alpha \wedge \gamma).$$

Conversely, suppose $(x_1, x_2) \in \beta \vee (\alpha \wedge \gamma)$. Then there exist elements $t_1, \cdots, t_{n-1} \in F(4)$ such that

$$x_1 = t_0\ \beta\ t_1\ \alpha\wedge\gamma\ \ t_2\ \beta\ t_3 \cdots t_n = x_2.$$

Applying lemma 3.5, there exist four-variable polynomials $f_0, \cdots, f_n$ such that the following identities hold in $\mathcal{K}$:

(3.9) $$f_0 = x_1, \; f_n = x_2;$$

(3.10) $$f_{i-1}(x_1, x_2, x_3, x_3) = f_1(x_1, x_2, x_3, x_3) \quad \text{if } i \text{ is odd};$$

(3.11) $$f_{i-1}(x_1, x_1, x_3, x_3) = f_i(x_1, x_1, x_3, x_3)$$

$$\text{and } f_{i-1}(x_1, x_2, x_1, x_2) = f_i(x_1, x_2, x_1, x_2) \quad \text{if } i \text{ is even.}$$

These conditions are equivalent to

(3.12) $$f_0 = x_1, \; f_n = x_2;$$

(3.13) $$\text{for all } i \; (0 \leq i \leq n)\text{: } f_i(x_1, x_1, x_3, x_3) = x_1;$$

(3.14) $$f_{i-1}(x_1, x_2, x_3, x_3) = f_i(x_1, x_2, x_3, x_3) \quad \text{if } i \text{ is odd};$$

(3.15) $$f_{i-1}(x_1, x_2, x_1, x_2) = f_i(x_1, x_2, x_1, x_2) \quad \text{if } i \text{ is even};$$

Now let $A \in \mathcal{K}$ and let $\varphi, \theta, \psi \in \Theta(A)$ with $\varphi \geq \theta$. We want to show that $\varphi \wedge (\theta \vee \psi) \leq \theta \vee (\varphi \wedge \psi)$. Let $\Delta_k = (\psi;\theta)^k; \psi$ so that

$$\theta \vee \psi = \bigcup_{k \geq 0} \Delta_k$$

and

$$\varphi \wedge (\theta \vee \psi) = \bigcup_{k \geq 0} (\varphi \cap \Delta_k).$$

Now

$$\varphi \cap \Delta_0 = \varphi \cap \psi \subseteq \theta \vee (\varphi \wedge \psi).$$

Assume

$$\varphi \cap \Delta_k \subseteq \theta \vee (\varphi \wedge \psi),$$

and suppose

$$a \ \ \varphi \cap \Delta_{k+1} \ d,$$

say

$$a \ \ \Delta_k \ b \ \theta \ c \ \psi \ d.$$

Note that

$$c \ \Delta_k \ d \ \text{ and } \ b \ \varphi \ c.$$

Let $u_i = f_i(a, d, b, c)$ for $0 \leq i \leq n$. Then $u_0 = a$, $u_n = d$. If i is odd, then by (3.14) we have

$$u_{i-1} = f_{i-1}(a, d, b, c) \ \theta \ f_{i-1}(a, d, b, b) = f_i(a, d, b, b) \ \theta \ f_i(a, d, b, c) = u_i.$$

Similarly, by (3.13) for all i $(0 \leq i \leq n)$ we have

$$u_i \ \varphi \ f_i(a, a, b, b) = a = f_i(a, a, a, a) \ \varphi \ f_i(a, d, a, d).$$

Also, for all i $(0 \leq i \leq n)$ since $a \ \Delta_k \ b$ and $c \ \Delta_k \ d$ we have

$$u_i \ \Delta_k \ f_i(a, d, a, d).$$

Thus, if i is even,

$$u_{i-1} \ \varphi \cap \Delta_k \ f_{i-1}(a, d, a, d) = f_i(a, d, a, d) \ \varphi \cap \Delta_k \ u_i.$$

By the inductive hypothesis,

$$u_{i-1}\ \theta \vee (\varphi \wedge \psi)\ u_i.$$

Combining these relations, we see that

$$(a, d) = (u_0, u_n) \in \theta \vee (\varphi \wedge \psi).$$

Thus

$$\varphi \cap \Delta_{k+1} \subseteq \theta \vee (\varphi \wedge \psi)$$

and by induction

$$\varphi \cap \Delta_m \subseteq \theta \vee (\varphi \wedge \psi)$$

holds for all m. Hence

$$\varphi \wedge (\theta \vee \psi) = \bigcup_{k \geq 0} (\varphi \wedge \Delta_k) \leq \theta \vee (\varphi \wedge \psi),$$

and $\Theta(A)$ is modular. Since A was arbitrary in $\mathcal{K}$, $\Theta(\mathcal{K})$ is modular.

<u>Proof of theorem 1:</u> Let $\Theta(\mathcal{K})$ satisfy the equation (3.1). Define $S(w)$ and φ_w: $FL(X) \to \Pi(S, w))$ as in lemma 3.3. Then $\Theta\ (F_{\mathcal{K}}(S(w)))$ satisfies (3.1) so that

$$\begin{aligned}(s_1, s_2) &\in \varphi_w(\sigma_0) \wedge \varphi_w(w)\\ &\subseteq \theta(\varphi_w(\sigma_0 \wedge w))\\ &\leq \psi(\sigma_0 \wedge w)\end{aligned}$$

$$\leq \psi\left(\bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i)\right)$$

$$= \bigvee_{i=1}^{k} (\psi(\sigma_0) \wedge \psi(\sigma_i))$$

$$= \bigvee_{i=1}^{k} [\theta(\varphi_w(\sigma_0)) \wedge \theta(\varphi_w(\sigma_i))].$$

Thus there exist elements $t_i \in F(S(w))$ $(i = 0, \cdots, n)$ satisfying

(3.16) $$t_0 = s_1, \; t_n = s_2$$

(3.17) $$t_{i-1} \; \theta(\varphi_w(\sigma_0)) \; t_i$$

(3.18) For each i $(1 \leq i \leq n)$ there exists $j(i) \in \{1, \cdots, k\}$ such that

$$t_{i-1} \; \theta(\varphi_w(\sigma_{j(i)}))t_i.$$

Now let ξ_j be the homomorphism of $F(S(w))$ onto $F(4)$ determined by

$$\xi_j(s_1) = x_1$$

$$\xi_j(s_2) = x_2$$

$$\xi_j(s) = x_3 \text{ if } s \in (s_1/\varphi_w(\sigma_j) \setminus \{s_1\})$$

$$\xi_j(s) = x_4 \text{ otherwise.}$$

Note that, in view of (3.8) of lemma 3.3

$$\xi_j(s) = x_4 \text{ if } s \in (s_2/\varphi_w(\sigma_j) \setminus \{s_2\}).$$

We will now show that $(x_1, x_2) \in \beta \vee (\alpha \wedge \gamma)$ where α, β, γ are as in lemma 3.6. Clearly,

$$\xi_j(t_0) = x_1 \text{ and } \xi_j(t_n) = x_2$$

by (3.16) above. Since for all j $(1 \leq j \leq k)$

$$\xi_j(s_i) = x_i \ (i = 1, 2)$$

$$\xi_j(s) \in \{x_3, x_4\} \text{ for } s \in S(w) \setminus \{s_1, s_2\},$$

we have, for any pair $j, j' \in \{1, \cdots, k\}$,

$$\xi_j(s) \ \beta \ \xi_{j'}(s) \text{ for all } s \in S(w).$$

Fix j and j', and let

$$T = \{t \in F(S(w)) : \xi_j(t) \ \beta \ \xi_{j'}(t)\}.$$

Since the ξ_j's are homomorphisms, T is a subalgebra of F(S(w)). Also $S(w) \subseteq T$; hence $T = F(S(w))$. In particular,

$$\xi_{j(i)}(t_i) \ \beta \ \xi_{j(i+1)}(t_i) \ (0 \leq i \leq n-1).$$

For $x, y \in F(S(w))$ define

$$x \ \eta_j \ y \text{ if } \xi_j(x) \ \alpha \ \xi_j(y).$$

Clearly η_j is a congruence relation on F(S(w)). From (3.7) of lemma 3.3 and the definition of ξ_j we verify that for $s, s' \in S(w)$,

$$(s, s') \in \varphi_w(\sigma_0) \text{ implies } \xi_j(s) \ \alpha \ \xi_j(s').$$

Thus

$$\varphi_w(\sigma_0) \subseteq \eta_j$$

and

$$\theta(\varphi_w(\sigma_0)) \leq \eta_j.$$

Then (3.17) yields

$$t_{i-1}\ \eta_j\ t_i,\ \text{i.e.,}\ \xi_j(t_{i-1})\ \alpha\ \xi_j(t_i).$$

Applying the same argument to γ and (3.18), we obtain

$$\xi_{j(i)}(t_{i-1})\ \gamma\ \xi_{j(i)}(t_i).$$

Thus we have

$$x_1 = \xi_{j(1)}(t_0)\ \alpha\wedge\gamma\ \xi_{j(1)}(t_1)\ \beta\ \xi_{j(2)}(t_1)$$

$$\alpha\wedge\gamma\ \xi_{j(2)}(t_2)\cdots\xi_{j(n)}(t_n) = x_2.$$

By lemma 3.6, $\Theta(\mathcal{K})$ is modular.

The following application of the arguments of theorem 3.1 yields an interesting corollary.

<u>Theorem 3.2:</u> The following conditions are equivalent:

(3.19) $\Theta(\mathcal{K})$ is modular.

(3.20) $\Theta(\mathcal{K})$ satisfies the equation.

(*) $x\wedge[(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)] \leq (x\wedge y)\vee(x\wedge z).$

(3.21) There exists a positive integer n and five-variable polynomials $f_0, \cdots, f_n$ in the word algebra of $\mathcal{K}$ such that

(i) $$f_0 = p, \ f_n = q$$

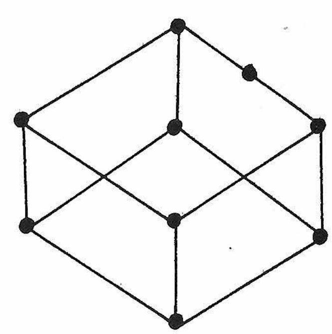
(ii) $$f_i(p,p,r,r,r) = p \ (0 \leq i \leq n)$$

(iii) $$f_{i-1}(p,q,p,p,q) = f_i(p,q,p,p,q) \quad \text{if } i \text{ is odd;}$$

$$f_{i-1}(p,q,p,q,q) = f_i(p,q,p,q,q) \quad \text{if } i \text{ is even.}$$

Corollary: Congruence modularity is determined by two-variable identities.

The conditions for congruence modularity given by Day [4] involve three-variable equations. McKenzie has shown that a lattice L satisfies the equation (*) if and only if the lattice (3.22) is not a sublattice of L[20].



(3.22)

Proof of theorem 3.2: Modularity implies (*), so (3. 19) implies (3.20). To see that (3.20) implies (3.21), we apply the Mal'cev argument with

$$\varphi(x) = (p, q)\,(r, s, t)$$

$$\varphi(y) = (p, r, s)\,(q, t)$$

$$\varphi(z) = (p, r)(q, s, t)$$

To see that (3.21) implies (3.19), we set

$$\xi_i(p) = x_1 \qquad \xi_i(q) = x_2$$

$$\xi_i(r) = x_3 \qquad \xi_i(t) = x_4 \quad (i = 1, 2)$$

$$\xi_1(s) = x_3 \qquad \xi_2(s) = x_4$$

and apply the argument in the proof of theorem 3.1 with p, q playing the roles of s_1, s_2 respectively.

We can prove a similar theorem for congruence distributivity.

<u>Theorem 3.3:</u> In FL(X) suppose $\sigma_0, \cdots, \sigma_k$ $(k \geq 1)$ are joins of members of X. Suppose $w \in FL(X)$ is such that $var(w) \cap var(\sigma_0) = \emptyset$ and

$$\sigma_0 \wedge w \not\leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i).$$

Then if $\Theta(\mathcal{K})$ satisfies the lattice equation

$$\sigma_0 \wedge w \leq \bigvee_{i=1}^{k} (\sigma_0 \wedge \sigma_i)$$

$\Theta(\mathcal{K})$ is distributive.

The proof is a modification of theorem 3.1. The construction used in the proof if lemma 3.3 can be applied to give lemma 3.7.

Lemma 3.7: Let w, σ_i (i = 0, ⋯, k) be as in theorem 3.3. Then there exist a finite set S(w), elements s_1, $s_2 \in S(w)$, and a homomorphism φ_w: FL(X) → Π(S(w)) such that

(3.23) $$(s_1, s_2) \in \varphi_w(w)$$

(3.24) $$\varphi_w(\sigma_0) = \not{\xi}(s_1, s_2)$$

(3.25) $$(s_1, s_2) \notin \varphi_w(\sigma_i) \text{ for } i = 1, \cdots, k.$$

Lemma 3.8 [12]. In $F_{\mathcal{K}}(3)$ set

$$\alpha = \theta(x_1, x_2)$$

$$\beta = \theta(x_1, x_3)$$

$$\gamma = \theta(x_2, x_3)$$

Then $\Theta(\mathcal{K})$ is distributive if and only if $(x_1, x_2) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$.

Proof: If $\Theta(\mathcal{K})$ is distributive, then

$$(x_1, x_2) \in \alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

Conversely, suppose

$$(x_1, x_2) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

Then, applying lemma 3.5, there exists a positive integer n and three-variable polynomials $f_0, \cdots, f_n$ in the word algebra of $\mathcal{K}$ such that

(3.26) $$f_0 = x_1, \ f_n = x_2;$$

(3.27) $$f_i(x_1, x_1, x_3) = x_1 \text{ for } 0 \leq i \leq n;$$

(3.28) $$f_{i-1}(x_1, x_2, x_1) = f_i(x_1, x_2, x_1) \quad \text{if } i \text{ is odd};$$

$$f_{i-1}(x_1, x_2, x_2) = f_i(x_1, x_2, x_2) \quad \text{if } i \text{ is even}.$$

Now let A be any algebra in $\mathcal{K}$, let $s, t \in A$ and let $\varphi, \theta, \psi \in \Theta(A)$. Suppose $(s,t) \in \varphi \wedge (\theta \vee \psi)$. Then there exist elements $s_1, \cdots, s_{2m-1} \in A$ such that

(3.29) $$s \ \varphi \ t$$

(3.30) $$s = s_0 \ \theta \ s_1 \ \psi \ s_2 \ \theta \ s_3 \cdots s_{2m-1} \ \psi \ s_{2m} = t.$$

We now consider the elements $f_i(s, t, s_j)$ $(0 \leq i \leq n, \ 0 \leq j \leq 2m)$. For j odd (3.30) yields $f_i(s, t, s_{j-1}) \ \theta \ f_i(s, t, s_j)$; similarly, for j even $f_i(s, t, s_{j-1}) \ \psi \ f_i(s, t, s_j)$. Also, in view of (3.27) and (3.29) we have $f_i(s, t, s_j) \ \varphi \ f_i(s, s, s_j) = s$ for all i, j. Combining these relations and using (3.28) we form the sequence

$$s = f_1 \ (s, t, s) \ \varphi \wedge \theta \ f_1 \ (s, t, s_1) \ \varphi \wedge \psi \ f_1(s, t, s_2) \cdots$$

$$\cdots f_1(s, t, s_{2m-1}) \ \varphi \wedge \psi \ f_1(s, t, t) = f_2(s, t, t)$$

$$\varphi \wedge \psi \ f_2(s, t, s_{2m-1}) \cdots f_2 \ (s, t, s_1) \ \varphi \wedge \theta \ \ f_2 \ (s, t, s)$$

$$= f_3 \ (s, t, s) \ \varphi \wedge \theta \ f_3 \ (s, t, s_1) \ \cdots \ f_n \ (s, t, z) = t$$

where z is either s or t depending upon whether n is odd or even. Thus Θ (A) is distributive. Since A was arbitrary in $\mathcal{K}$, $\Theta(\mathcal{K})$ is distributive.

<u>Proof of theorem 3.3</u>: Proceeding as before, we obtain the conditions

(3.31) $$t_0 = s_1,\ t_n = s_2$$

(3.32) $$t_i\theta(\varphi_w(\sigma_0))s_1 \quad (0 \leq i \leq n)$$

(3.33) for each i $(1 \leq i \leq n)$ there exists $j(i) \in \{1, \cdots, k\}$ such that

$$t_{i-1}\ \theta(\varphi_w(\sigma_{j(i)}))t_i,$$

which are easily seen to be equivalent to (3.16), (3.17), (3.18). We now define homomorphisms ξ_i, η_i, τ_i, and μ_i $(1 \leq i \leq n)$ of F(S(w)) onto F(3) by extending the mappings

$$\xi_i(s) = \begin{cases} x_1 \text{ if } s \in s_1/\varphi_w(\sigma_{j(i)}) \\ x_2 \text{ if } s \in s_2/\varphi_w(\sigma_{j(i)}) \\ x_3 \text{ otherwise} \end{cases}$$

$$\eta_i(s) = \begin{cases} x_1 \text{ if } s \in s_1/\varphi_w(\sigma_{j(i)}) \cap s_1/\varphi_w(\sigma_{j(i+1)}) \\ x_2 \text{ if } s \in s_2/\varphi_w(\sigma_{j(i)}) \\ x_3 \text{ otherwise} \end{cases}$$

$$\tau_i(s) = \begin{cases} x_1 \text{ if } s \in s_1/\varphi_w(\sigma_{j(i)}) \cap s_1/\varphi_w(\sigma_{j(i+1)}) \\ x_2 \text{ if } s \in s_2/\varphi_w(\sigma_{j(i)}) \cap s_2/\varphi_w(\sigma_{j(i+1)}) \\ x_3 \text{ otherwise} \end{cases}$$

$$\mu_i(s) = \begin{cases} x_1 \text{ if } s \in s_1/\varphi_w(\sigma_{j(i+1)}) \\ x_2 \text{ if } s \in s_2/\varphi_w(\sigma_{j(i)}) \cap s_2/\varphi_w(\sigma_{j(i+1)}) \\ x_3 \text{ otherwise.} \end{cases}$$

Note

$$\xi_i(s_p) = \eta_i(s_p) = \tau_i(s_p) = \mu_i(s_p) = x_p \quad (p = 1, 2).$$

In view of (3.24) we have for $s, s' \in S(w)$,

$$(s, s') \in \varphi_w(\sigma_0) \quad \text{implies} \quad \zeta(s) \; \alpha \; \zeta(s')$$

for $\zeta \in \{\xi_i, \eta_i, \tau_i, \mu_i\}$. Arguing as with α in the proof of theorem 3.1, (3.32) yields

$$\zeta(t_m) \; \alpha \; \zeta(s_1) = x_1.$$

Hence

$$\zeta(t_m) \; \alpha \; \zeta'(t_m)$$

for

$$0 \leq m \leq n \quad \text{and} \quad \zeta, \zeta' \in \{\xi_i, \eta_i, \tau_i, \mu_i, \xi_{i'}, \eta_{i'}, \tau_{i'}, \mu_{i'}\}.$$

From the definitions of ξ_i, η_i, τ_i, μ_i we see that

$$\xi_i(t) \; \beta \; \eta_i(t) \; \gamma \; \tau_i(t) \; \beta \; \mu_i(t) \; \gamma \; \xi_{i+1}(t)$$

for all $t \in F(S(w))$. Let λ_i be the kernel of ξ_i, i.e., for x, y, $\in$ F(S(w)),

$$x \, \lambda_i \, y \text{ if } \xi_i(x) = \xi_i(y).$$

From the definition of ξ_i we see that

$$\varphi_w(\sigma_{j(i)}) \leq \lambda_i$$

whence

$$\theta(\varphi_w(\sigma_{j(i)})) \leq \lambda_i.$$

In particular,

$$\xi_i(t_{i-1}) = \xi_i(t_i).$$

Combining these relations yields

$$\xi_i(t_{i-1}) = \xi_i(t_i) \; \alpha\wedge\beta \quad \eta_i(t_i)$$

$$\alpha\wedge\gamma \quad \tau_i(t_i) \quad \alpha\wedge\beta \quad \mu_i(t_i)$$

$$\alpha\wedge\gamma \quad \xi_{i+1}(t_i).$$

Thus $(x_1, x_2) \in (\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$, and $\Theta(\mathcal{K})$ is distributive.

<u>Corollary:</u> Let $\mathcal{M}_n$ denote the n+2-element two-dimensional lattice, and $\mathcal{N}_5$ the five-element nonmodular lattice. If $\Theta(\mathcal{K})$ is contained in the lattice variety generated by $\mathcal{M}_n$ for some finite n, or in the lattice variety generated by $\mathcal{N}_5$, then $\Theta(\mathcal{K})$ is distributive.

<u>Proof:</u> By [13], $\mathcal{M}_n$ satisfies the identity

$$(3.34) \qquad x \wedge \bigvee_{1 \leq i \leq j \leq n} (y_i \vee y_j) \leq \bigvee_{i \leq i \leq n} (x \wedge y_i).$$

Also, $\mathcal{N}_5$ satisfies each of these identities for $n > 2$.

REFERENCES

[1] G. Birkhoff, Lattice Theory, 3rd edition, Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc., Providence, R. I., 1967.

[2] G. Birkhoff, On the structure of abstract algebras, Proc. Cambridge Philos. Soc. 31 (1935), 433-454.

[3] P. Crawley and R. P. Dilworth, Algebraic Theory of Lattices, Prentice-Hall, to appear.

[4] A. Day, A characterization of modularity for congruence lattices of algebras, Canad. Math. Bull. 12 (1969), 167-173.

[5] A. Day, A simple solution to the word problem for lattices, Bull. Amer. Math. Soc. 13 (1970), 253-254.

[6] R. A. Dean and R. H. Oehmke, Idempotent semigroups with distributive right congruence lattices, Pacific J. Math. 14 (1964), 1187-1209.

[7] R. P. Dilworth, Lattices with unique complements, Trans. Amer. Math. Soc. 57 (1945), 123-154.

[8] T. Evans, The lattice of semigroup varieties, Semigroup Forum 2 (1971), 1-43.

[9] G. Grätzer, Universal Algebra, Van Nostrand, Princeton, N. J., 1967.

[10] A. Hales, Partition representations of free lattices, Proc. Amer. Math. Soc. 24 (1970), 517-520.

[11] B. Jónsson, Topics in Universal Algebra, Lecture Notes in Mathematics, Vol. 250, Springer-Verlag, New York, 1972.

[12] B. Jónsson, Algebras whose congruence lattices are distributive, Math. Scand. 21 (1967), 110-121.

[13] B. Jónsson, Equational classes of lattices, Math. Scand. 22 (1968), 187-196.

[14] B. Jónsson, Relatively free lattices, Coll. Math. 21 (1970), 191-196.

[15] R. Lyndon, Properties preserved in subdirect products, Pacific J. Math. 9 (1959), 155-164.

[16] A. Mal'cev, The Metamathematics of Algebraic Systems, North-Holland, Amsterdam, 1971. Translated by B. F. Wells, III. (Hereafter referred to as MAS.)

[17] A. Mal'cev, On the general theory of algebraic systems, Mat. Sb. (N.S.) 35 (1954), 3-20. (Amer. Math. Soc. Translations, Ser. 2, Vol. 27, 3-20.)

[18] A. Mal'cev, Quasiprimitive classes of abstract algebras, Dokl. Akad. Nauk. SSSR 108 (1956), 187-189. (MAS, Chap. IV, 27-31.)

[19] A. Mal'cev, A few remarks on quasivarieties of algebraic systems, Algebra i Logika Sem. 5 (1966), 3-9. (MAS, Chap. XXXI, 416-421.)

[20] R. McKenzie, Equational bases and non-modular varieties, to appear.

[21] D. Papert, Congruence relations in semilattices, Jour. London Math. Soc. 39 (1964), 723-729.

[22] A. Pixley, Local Mal'cev conditions, Canad. Math. Bull. 15 (1972).

[23] D. Sachs, Identities in finite partition lattices, Proc. Amer. Math. Soc. 12 (1961), 944-945.

[24] E. T. Schmidt, Kongruenzrelationen Algebraischer Strukturen, Veb Deutscher Verlag, Berlin, 1969.

[25] W. Taylor, Characterizing Mal'cev conditions, to appear.

[26] P. Whitman, Free lattices, Ann. Math. 42 (1941), 325-330.

[27] P. Whitman, Lattices, equivalence relations and subgroups, Bull. Amer. Math. Soc. 52 (1946), 507-522.

[28] R. Wille, Kongruenzklassengeometrien, Lecture Notes in Mathematics, Vol. 113, Springer-Verlag, New York, 1972.