THE CANCELLATION OF RANDOM DISTURBANCES IN AUTOMATIC CONTROL SYSTEMS

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ABSTRACT

A disturbance-cancelling feedback transfer function is proposed in a preliminary study for linear systems with constant coefficients. This idea is then experimentally demonstrated.

In the main theory for more general systems, expressions are obtained for a computer which generates a cancelling input from measurements of the disturbances. The cases where there are fewer measurements than disturbances and also noisy measurements are treated. Three schemes and a basis for comparison are given. An example is calculated.

TABLE OF CONTENTS

I. INTRODUCTION

Much of the present automatic control design for engineering systems is based on the assumption that the pertinent system properties are either known or may be specified precisely when the system is designed. The accuracy of the system performance under this assumption, i.e., the smallness of the dynamic error, then depends solely on the design of the system and is the central design objective. Until recently the system accuracy requirement may not have been so strict as to invalidate the basic assumption. However, when the required accuracy is extreme as in some of the modern guided engineering systems, the basic assumption is not valid; it is necessary to recognize that the properties of the individual manufactured systems differ from each other as well as the design specification, and furthermore that any property in any one system will in general vary in time. These deviations in the system properties limit the ultimate performance accuracy unless they are recognized.

At every time instant the properties of an individual engineering system have definite values, of course; there are no intrinsic uncertainties. The difficulty is to know these definite values at the time of the control design. This is an impossibility if there are scatter and random drifts. Actually this is just the situation encountered; manufacturing tolerances introduce scatter and aging of the device introduces drift. Furthermore the system properties may be affected by an environment that changes in some random manner. *Therefore in any real situation the only* means to have sufficiently accurate information about the properties of a system for very accurate control purposes is to continuously measure the deviations in the pertinent properties of each individual engineering system during its operation. For very accurate controlled engineering systems these continuous measurements are a necessity. It is the purpose of this thesis to explore and formulate a theory for such systems.

II. PRELIMINARY STUDY; LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

Since a control system which continuously senses and measures its properties† during its operation is quite novel, it seems desirable to first describe a rather naive treatment devoid of any mathematical or conceptual complication. This preliminary study will then suggest possible improvements and lead to more advanced ideas. In Part II then, a study, both theoretical and experimental, of linear systems with nominally constant coefficients incorporating continuous measurement of pertinent system properties will be described.

2.1 System Description and Objective

The starting point of the discussion assumes that a control system governed by linear equations with constant coefficients has been designed, which satisfies the usual requirements of stability, speed of response, and small dynamic error for certain types of inputs.[1,3,4] Undoubtedly one or more feedback loops have been used to meet the performance specifications. Designate the transfer function of this system design as $G^*(p)$. The function $G^*(p)$ is the ratio of the Laplace transform of the output to the transform of the input with all initial conditions set to zero; p is the transform parameter. A less elegant but equally adequate definition is that $G^*(p)$ is the shorthand expression for the input-output relation in which p is the operator $\frac{d}{dt}$ and t is time.

† For treatment of a related type of system see Ref. 1 Chap. 15, and Ref. 2.

However any actual system built to have the desired transfer function $G^*(p)$ will necessarily have somewhat different properties because of tolerances in manufacture and aging of the device. Furthermore because of environmental changes and possible abuse these properties will change with time. Designate the transfer function of this nonstandard actual system $G(p)$. $G(p)$ then has the same form as $G^*(p)$ but somewhat different coefficients which also change with time.

For some control applications fluctuation of certain system components is not objectionable. An actuator which alone is incapable of precise control because of drifting properties may render precise control when used with a feedback loop made up of precision elements. Just the use of feedback then represents a significant advance toward precise control. However the more ambitious objective sought here is beyond the capabilities of fixed transfer functions in feedback loops as ordinarily used. It is proposed that a disturbance-cancelling feedback loop incorporating a variable transfer function be added to the nonstandard system characterized by $G(p)$ such that the system so modified always has the standard transfer function $G^*(p)$. Such a disturbance-cancelling system always responds, statically or dynamically, in the standard predictable manner.

In Fig. 1 the nonstandard system with fluctuating properties is indicated in a block diagram by its transfer function $G(p)$. Suppose that a feedback loop having the transfer function $H(p)$ is added, as indicated in Fig. 2. The resulting system transfer function then is

$$\frac{G(p)}{1 - G(p)H(p)}$$

By setting this equal to the standard transfer function $G^*(p)$, the condition on H(p) is obtained below.

$$\frac{G(p)}{1 - G(p)H(p)} = G^*(p)$$

$$H(p) = \frac{1}{G(p)} - \frac{1}{G^*(p)} \qquad (2.1)$$

Obviously the instantaneous properties of the nonstandard system must be continuously measured and compared with the standard properties. From this information the disturbance-cancelling transfer function is generated by an appropriate computer. Fig. 3 shows the arrangement. The properties of the nonstandard system are determined from its response to test signals. The regular system input must not have components of the nature of the test signals; this restricts the choice of the latter. Filter No. 1 passes the regular system input unaltered but infinitely attenuates the test signals; filter No. 2 does the opposite.

2.2 Experimental Demonstration

In order to demonstrate the ideas discussed in the previous section an electrical disturbance-cancelling system was devised. A simple resistance-capacitance network in series with a buffer amplifier (see Fig. 4) was chosen as the standard control system. The standard transfer function then is

$$G^*(p) = \frac{-K_1}{\tau^* p + 1}$$

where the standard time-constant τ^* is the resistance-capacitance product. A disturbance was intentionally introduced in the time-constant by varying the capacitance. Therefore the disturbed control system transfer function is

$$G(p) = \frac{-K_1}{\tau p + 1}$$

Only a positive disturbance in τ was used to simplify the instrumentation.

From Eq. (2.1) the feedback transfer function which must be generated to cancel the disturbance is

$$H(p) = \frac{1}{G(p)} - \frac{1}{G^*(p)} = -\frac{1}{K_1}(\tau - \tau^*)\, p \qquad (2.2)$$

This was done with a network of the type shown in Fig. 5, which has the transfer function,

$$\frac{\tau' p}{\tau' p + 1}$$

where τ' is the resistance-capacitance product. Cancellation of the unwanted denominator will be discussed later.

The diagram for the example system is shown in Fig. 6. The transfer function designated $H'(p)$ is

$$H'(p) = -K_4 \frac{R - R^*}{R_1} \frac{R_1 C_1 p}{R_1 C_1 p + 1} = -K_4 \frac{(R - R^*)\, C_1 p}{R_1 C_1 p + 1} \qquad (2.3)$$

Comparison of the numerator of Eq. (2.3) and Eq. (2.2) shows that the resistance R must be made proportional to τ. The servo shown schematically in Fig. 6 generates this resistance by positioning the arm of potentiometer R_1 which is ganged to the servo position-feedback potentiometer R_2 as indicated by the dashed line.

For any transfer function F(p) the ratio of the output amplitude to the input amplitude for a sinusoidal input of angular frequency ω is $|F(i\omega)|$. Therefore the output amplitude of G(p) for a sinusoidal test signal of amplitude E_t and angular frequency ω_t is

$$E_t \left| G(i\omega_t) \right| = \frac{E_t K_1}{\left| i\omega_t \tau + 1 \right|} \qquad (2.4)$$

The test signal angular frequency ω_t is chosen high enough to be beyond the frequency range of the regular system input and such that $\omega_t \tau \gg 1$. Then

$$E_t \left| G(i\omega_t) \right| \doteq \frac{E_t K_1}{\omega_t \tau}$$

The high-pass filter in Fig. 6 passes the test signal unaltered in amplitude into the rectifier which amplifies it by a factor K_2 and converts it to a d.c. voltage E_1. Thus

$$E_1 = \frac{E_t K_1 K_2}{\omega_t \tau} \qquad (2.5)$$

For the servo in equilibrium (no signal at its mixer),

$$E_o = \frac{\theta}{2\pi} E_1 \tag{2.6}$$

But because the potentiometers are ganged,

$$\frac{\theta}{2\pi} = \frac{R}{R_1} \tag{2.7}$$

From Eqs. (2.5), (2.6), and (2.7),

$$R = \frac{E_o \omega_t R_1}{E_t K_1 K_2} \tau$$

Then define (2.8)

$$R^* = \frac{E_o \omega_t R_1}{E_t K_1 K_2} \tau^*$$

The servo thus generates a shaft position θ and a corresponding resistance R proportional to the disturbed time-constant τ. The standard time-constant τ^* is set into the system by rotating the case of potentiometer R_1 as indicated in Fig. 6, Fig. 9, and Eqs. (2.8). Putting Eqs. (2.8) into (2.3) gives

$$H'(p) = -\frac{E_o K_4 \omega_t \tau_1}{E_t K_1 K_2} \frac{(\tau - \tau^*) p}{\tau_1 p + 1} \tag{2.9}$$

where $\tau_1 = R_1 C_1$.

The regular system input was assumed to extend over a frequency range up to an arbitrary upper limit of 500 cycles-per-second. The low-pass filter should ideally pass the regular system input unaltered. But over the system input frequency range the filter has the approximate transfer function

$$F_f(p) = \frac{1}{\tau_2 p + 1} \tag{2.10}$$

where $\tau_2 = \dfrac{L_1 + L_3 + L_5}{R_5}$

In order to cancel the undesired time lags τ_1 in $H'(p)$ and τ_2 in $F_f(p)$ the phase compensator was introduced. Its transfer function is

$$F_c(p) = K_3 \alpha \frac{\tau_3 p + 1}{\alpha \tau_3 p + 1}$$

where $\alpha = \dfrac{R_6}{R_6 + R_7}$ and $\tau_3 = R_7 C_{10}$. For low frequencies,

$$F_c(p) = K_3 \alpha \left[(1 - \alpha)\, \tau_3 p + 1 \right] \tag{2.11}$$

Thus the transfer function for the nonideal low-pass filter, $H'(p)$, and phase compensator for low frequencies is

$$H'(p)\, F_f(p)\, F_c(p) = - \frac{E_o K_3 K_4 \omega_t \tau_1 \alpha}{E_t K_1 K_2} \; \frac{(1 - \alpha)\, \tau_3 p + 1}{(\tau_1 + \tau_2)\, p + 1} \, (\tau - \tau^*)\, p \tag{2.12}$$

By adjusting the quantities in Eq. (2.12) so that

$$\frac{(1 - \alpha)\,\tau_3}{\tau_1 + \tau_2} = 1 \qquad (2.13)$$

and

$$\frac{E_o K_3 K_4 \omega_t \tau_1 \alpha}{E_t K_2} = 1 \qquad (2.14)$$

then

$$H'(p)\, F_f(p)\, F_c(p) = -\frac{1}{K_1}\,(\tau - \tau^*)\, p = H(p) \qquad (2.15)$$

The required feedback transfer function, H(p), as indicated in Eq. (2.2) is thus realized.

Notice, however, that the unwanted time lag τ_1 in the feedback loop and τ_2 in the forward loop are cancelled by a single phase compensator for reasons of instrumentation economy. Thus although the example system has the correct transfer function from the output of G(p) around the loop to the mixer, the unwanted time lag τ_2 from the output of G(p) to the system output is over-compensated by the amount τ_1. A simple phase correction of the experimental data was required to account for this forward loop over-compensation.

2.3 Experimental Results

The system parameter settings are given in Fig. 6. In the interest of obtaining the best system performance the parameters were not initially set to satisfy the phase compensation condition given by Eq. (2.13) and the loop gain condition, Eqs. (2.14). Instead the system parameters were set to give minimum variation in the input-output relation while varying τ for a particular setting of $\tau^* = .18$ milliseconds. The parameters then were not changed during the remainder of the experiment. Using the values thus determined, Eqs. (2.13) and (2.14) are checked below. An additional time lag $\tau_4 = 6.4$ microseconds was discovered in the amplifiers indicated in Fig. 6 as having gains $-K_1$, K_3, and $-K_4$. Over the system input frequency range their actual transfer functions are approximated by

$$-\frac{K_1}{\tau_4 p + 1}, \quad \frac{K_3}{\tau_4 p + 1}, \quad \text{and} \quad -\frac{K_4}{\tau_4 p + 1}$$

The phase compensator must cancel these three additional time lags also, so Eq. (2.13) is modified to include them as follows.

$$\frac{(1 - \alpha)\, \tau_3}{\tau_1 + \tau_2 + 3\tau_4} = 1 \qquad (2.13')$$

Substituting the experimental settings from Fig. 6 into the phase compensation condition, Eq. (2.13') gives

$$\frac{(1 - \alpha)\,\tau_3}{\tau_1 + \tau_2 + 3\tau_4} = \frac{(1 - .241) \times 133 \times 10^{-6}}{(20 + 47.7 + 19.2) \times 10^{-6}} = 1.16$$

The actual phase compensation required is thus 16 percent greater than predicted. This excess is attributed primarily to additional small time lags due to stray capacity.

A similar check of the loop gain condition, Eq. (2.14), gives

$$\frac{E_o K_3 K_4 \omega_t \tau_1 \alpha}{E_t K_2} = \frac{20 \times 3.40 \times 5.08 \times 2\pi \times 10^4 \times 20 \times 10^{-6} \times .241}{4\sqrt{2} \times 20} = .925$$

Instrument error is probably responsible for the 7.5 percent discrepancy.

To evaluate the example system a 500 cycle-per-second sinusoidal system input was used in each of ten runs. During any one run the input remained constant while τ was varied from .2 to .4 milliseconds in .02 millisecond steps. The input was applied to the horizontal deflection plates of an oscilloscope and the output to the vertical plates as well as to an a.c. voltmeter. During each run the a.c. output voltage and the absolute value of the sine of the phase angle φ between the input and output were recorded for each value of τ. The quantity $|\sin\varphi|$ is equal to $\frac{a}{b}$ where a and b are the dimensions of the ellipse on the oscilloscope shown in Fig. 10.

During Run 1 the feedback loop was opened to record the effect of variation of τ on the measured quantities. The data appear with calculated quantities on p. 92. The normalized output is obtained by dividing the output voltage by the first reading of the run. A calculated phase shift, φ_{calc}, is obtained as

$$\varphi_{calc} = \arg. G(i\omega)$$

$$= \arg. \left(\frac{1}{i\omega\tau + 1} \right)$$ where ω is the angular frequency of the system input. However because of the forward loop over-compensation described at the end of Sec. 2.2 the measured phase shift, φ, differs from φ_{calc} by an average of 5.1 degrees. This 5.1 degree correction was applied to the succeeding data of Runs 2 through 10 to obtain a corrected phase angle, φ_{cor}.

In Runs 2 through 10 the feedback loop was closed, and the capability of the system to cancel disturbances in τ was checked for nine values of the standard time constant τ^* from .12 to .28 milliseconds. After introduction of a disturbance in τ, readings of output voltage and $|\sin \varphi|$ were taken after the transient in the disturbance-cancelling action subsided. (This transient, extending over a fraction of a second, is discussed briefly later.) The data appear on pp. 93 to 101. The measured phase angle φ is corrected for the forward loop over-compensation to give φ_{cor}. The phase angle appropriate to the particular setting of τ^* appears in the last column on the data sheets for comparison with φ_{cor}, and is given by

$$\varphi_{calc} = \arg. \quad G^*(i\omega)$$

$$= \arg. \left(\frac{1}{i\omega\tau^* + 1} \right)$$

The significant experimental result is the excellent degree to which the output was held constant, in both magnitude and phase angle, while τ was varied. For no value of τ^* was there a perceptible variation of φ with τ. For two values of τ^*, .24 milliseconds in Run 8 and .28 milliseconds in Run 10, there was a variation in output voltage of .4 percent. The experimental results are shown graphically in Fig. 11.

The measured phase angle, after correction for forward loop overcompensation, should be equal to the phase angle appropriate to the τ^* setting; that is, φ_{cor} and φ_{calc} should be equal. The phase angle error, $\varphi_{cor} - \varphi_{calc}$ is tabulated below.

Run	τ^* Milliseconds	$\varphi_{cor} - \varphi_{calc}$ Degrees
2	.12	-1.3
3	.14	-1.0
4	.16	- .7
5	.18	- .8
6	.20	- .7
7	.22	- .4
8	.24	- .8
9	.26	.0
10	.28	-1.8

This tabulation serves as a calibration of the τ^* scale shown in Fig. 9.

In any practical disturbance-cancelling system the transient in the cancellation must be short compared to a characteristic time of the disturbance. The cancellation transient for three step disturbances in τ was recorded for the example system. The output voltage variation with time appears on the oscillograms in Fig. 12. The upper envelope of the 500 cycle-per-second output signal is shown. Since the cancellation transient is independent of τ^*, an arbitrary setting was used. The transient for this example reflects primarily the lag in the action of the servo which positions the feedback potentiometer R_1.

2.4 Necessity for Modification of Concept for Complicated Systems

It is evident from the preceding sections that the generation of a disturbance-cancelling feedback transfer function from the system response to test signals leads to a rather complicated system. The concept, although shown to give good results for the very simple example, would be difficult if not impossible to implement for complicated engineering systems. This naive and simple idea must be modified to widen its scope.

It is desirable that the more general theory for the cancellation of disturbances in control systems should not be limited to systems for which the governing equations are linear. However we observe that a well-designed engineering system will not have its properties altered to a large degree by aging, drift, and environmental fluctuations. Furthermore a good manufacturing process ensures close adherence to the standard so that the resulting scatter of system properties will be small. Hence well-engineered and well-made systems will not deviate far from the standard. Consequently for systems governed by nonlinear equations we may linearize with respect to deviations from the standard, neglecting second and higher order terms in the deviations. These linearized equations will not have constant coefficients, however; the more general theory which follows must therefore treat linear systems with time-varying coefficients.

In the previous sections the nature of the disturbance was presumed to be a deviation of some parameter of the control system proper. In practice these disturbances will be encountered along with disturbances in the environment of the system which act through the latter to produce an error. In the equations expressing the system error the disturbances which cause it will be indistinguishable as to their origin. In fact the boundary between the control system and its environment may be difficult to define in some cases. Therefore in the theory which follows, more general disturbances will be treated - namely, any disturbances which degrade the system performance by causing an error in the system output, regardless whether the origin of these disturbances is considered to be in the control system proper, its environment, or both.

The information basic to the cancellation of the disturbances in an automatic control system is the magnitude of these disturbances. The means of obtaining these magnitudes - whether by the use of test signals, direct measurement, or some other - is separate from a theory for the use of this information. Only the latter is our concern here; attention in the following sections will therefore be directed toward the use of this information. The instrumentation problem involved in obtaining it is dismissed as something separate.

The theory must however take into account the following two real situations:

A complicated engineering system has a great number of elements, each of which may fluctuate or differ from the standard and cause an error in the system output. Measurement of each of a great number of disturbances may not be feasible because of the bulk and complexity of the required sensing and measuring equipment. However certain overall system properties may be measured instead, where a deviation in such an overall property reflects the deviations of a number (in general, all) of the basic system disturbances. The theory will therefore include the possibility of having fewer measurements than disturbances.

Each disturbance, a physical quantity, must be sensed and measured by some imperfect device and transmitted to that part of the control system which digests the disturbance measurements. The theory which follows must allow for interference or the introduction of noise into these information channels.

The preceding paragraphs are a rough indication of the content of the theory to be gradually built up in Part III, the main part of the present investigation.

III. MAIN THEORY; LINEAR SYSTEMS WITH TIME-VARYING COEFFICIENTS

3.1 System Description and Objective

The engineering system to be treated here is one governed by a set of linear ordinary differential equations with time-varying coefficients. The longitudinal guidance of a long-range rocket described by H. S. Tsien[1] is an example of the automatic control of such an engineering system. The problem undertaken here will be more vivid and the motivation for many of the assumptions will be clearer if the reader is familiar with this example.

According to the above, the equations expressing the intrinsic physical laws governing the system may be written in the form

$$\frac{d\xi_j}{dt} - \sum_{k=1}^{\mu} \beta_{jk}\xi_k = \sum_{i=1}^{n+1} \gamma_{ji} y_i \qquad j=1, \ldots, \mu \tag{3.1}$$

where the $\xi_j(t)$ are the μ dependent variables describing the state or configuration of the engineering system, the first n $y_i(t)$ are the spurious unwanted inputs, and the coefficients $\beta_{jk}(t)$ and $\gamma_{ji}(t)$ are known functions of time. The input $y_{n+1}(t)$ is of a different nature than the other y's; it is an extra input, created by and under the control of the cyberneticist. The object is to cancel the effect on the system output of these spurious inputs, $y_i(t)$, $i = 1, \ldots, n$, by appropriate generation of the cancelling input, $y_{n+1}(t)$. In the case of the guided rocket the spurious inputs are atmospheric disturbances (in part), their effect on the system output is a range error, and the cancelling input is a correction of the programmed elevator setting.

Notice that the $\xi_j(t)$ as defined by Eq. (3.1) describe the effect on the system of the unwanted disturbances only. By virtue of the linearity of the governing equations this effect of the disturbances is separated from the effect of the useful inputs which are introduced to accomplish the purpose of the engineering system. In the event that the governing equations are nonlinear but the effect of the disturbances is small, a set of equations like Eq. (3.1) approximately describing this effect may be obtained by linearization.

Eq. (3.1) is first order, but higher order equations may be reduced to this form. For instance $\frac{d^2\xi_j}{dt^2}$ may be replaced by $\frac{d\xi_\ell}{dt}$ by adding the equation

$$\frac{d\xi_j}{dt} - \xi_\ell = 0$$

to the set.

Suppose that the system is in operation for the time interval t_1 to t_2, but that the state of the system at the end of this interval only is of interest. The state of the system is specified by the ξ_j's evaluated at $t = t_2$; define the system output (more specifically, the error) due to the disturbances as

$$\varepsilon \equiv \sum_{j=1}^{\rho} \Omega_j \, \xi_j(t_2) \tag{3.2}$$

The range error of a guided rocket may be written in this manner. The nature of the output specifies the Ω_j's.

In order to find the expression for the error ε in terms of the disturbances and cancelling input, we introduce the adjoint functions[5] $\lambda_j(t)$, which satisfy the set of homogeneous equations,

$$\frac{d\lambda_j}{dt} + \sum_{k=1}^{\rho} \beta_{kj} \lambda_k = 0 \qquad j=1, \ldots, \rho \tag{3.3}$$

Multiplying Eq. (3.1) by λ_j and Eq. (3.3) by ξ_j, then adding and summing over j, we obtain

$$\frac{d}{dt}\sum_{j=1}^{\rho} \lambda_j\xi_j + \sum_{j=1}^{\rho}\sum_{k=1}^{\rho} (\beta_{kj}\lambda_k\xi_j - \beta_{jk}\lambda_j\xi_k) = \sum_{i=1}^{n+1}\sum_{j=1}^{\rho} \gamma_{ji}\lambda_j y_i$$

The two parts of the double sum on the left cancel each other so that

$$\frac{d}{dt}\sum_{j=1}^{\rho} \lambda_j\xi_j = \sum_{i=1}^{n+1} a_i y_i \quad \text{where} \quad a_i(t) = \sum_{j=1}^{\rho} \gamma_{ji}\lambda_j$$

Integrating the above expression over the time interval t_1 to t_2 we obtain

$$\sum_{j=1}^{\rho} \lambda_j(t_2)\, \xi_j(t_2) = \sum_{j=1}^{\rho} \lambda_j(t_1)\, \xi_j(t_1) + \int_{t_1}^{t_2} \sum_{i=1}^{n+1} a_i y_i \, dt \tag{3.4}$$

Notice that Eq. (3.3) does not specify the λ's completely. From the infinity of sets of solutions which satisfy Eq. (3.3) choose the set such that $\lambda_j(t_2) = \Omega_j$ for $j=1, \ldots, \rho$. The left side of Eq. (3.4) then is the error ε.

If the system is put into operation in an undisturbed condition $\xi_j(t_1) = 0$ for all j, and Eq. (3.4) becomes the desired expression relating the error to the disturbances and the cancelling input:

$$\varepsilon = \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} a_i(t)\, y_i(t) + b(t)\, x(t) \right) dt \qquad (3.5)$$

In Eq. (3.5) the cancelling input $y_{n+1}(t)$ has been given the new symbol $x(t)$, and $a_{n+1}(t)$ has been replaced by $b(t)$.

Eq. (3.5) might have been the starting point of our discussion by simply writing the error for a linear system as a superposition of responses to impulsive disturbances. That is,

$$\varepsilon(t_2) = \varepsilon(t_1) + \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} a_i(t,t_2)\, y_i(t) + b(t,t_2)\, x(t) \right) dt$$

where the $a_i(t,t_2)$ and $b(t,t_2)$ are respectively the responses to unit impulses for the corresponding disturbances and cancelling input. For systems with time-varying coefficients these responses depend on two parameters: t, the time when the impulse is applied, and t_2-t, the time elapsed since application of the impulse. (For systems with constant coefficients these responses depend on only one parameter, t_2-t.)

If the system is put into operation in an undisturbed condition, then $\varepsilon(t_1) = 0$, and if t_2 is a fixed parameter so that dependence on t_2 may be suppressed, then this error expression is the same as Eq. (3.5).

The disturbances $y_i(t)$, $i = 1, \ldots, n$, are unknown functions of time. As information available for use in the generation of a cancelling input suppose m measurements, $\mu_i(t)$, are continuously made, where

$$\mu_i(t) = c_{io}(t)\, x(t) + \sum_{j=1}^{n} c_{ij}(t)\, y_j(t) \qquad i=1, \ldots, m \qquad (3.6)$$

in which the c's are known and may be functions of time. The linear combination of the disturbances and also the cancelling input is suggested by the guided rocket problem and is more general than separate measurements of some of the disturbances (i.e., the special case $c_{ij} = 1$ for $i = j$, $= 0$ for $i \neq j$). For instance μ_1 might be the difference between an acceleration of an actual rocket and the standard rocket at some time instant in the flight. The dependence of this acceleration deviation on the atmospheric and other deviations (the y_i) and the elevator correction (x) is of the form of Eq. (3.6) - after linearization, if necessary.

3.2 Determinate Case; Equal Number of Measurements and Disturbances (m=n)

In this case Eq. (3.6) may be solved for the y_i in terms of the μ_i with x as a parameter and substituted into Eq. (3.5). The integrand after this substitution is a function only of the measurements $\mu_i(t)$ and the cancelling input x(t). In order to obtain a vanishing error the integrand must be made to vanish at every time instant, since the measurements are not known in advance. This specifies the cancelling input x(t) as a function of the measurements $\mu_i(t)$ to ensure a vanishing system error. It is presumed of course that Eq. (3.6) and the integrand of Eq. (3.5) are linearly independent.

3.3 Indeterminate Case; Fewer Measurements Than Disturbances $(m < n)$

If there are fewer measurements than disturbances the latter are undetermined by Eq. (3.6). The problem now is to generate a purposeful input x(t) from the insufficient number of measurements. It is this problem which concerns the remainder of Part III; its consideration is demanded by that practical situation where there are many disturbances in the system, more than may be feasibly measured.

It is possible to generate a purposeful input x(t) from the indeterminate measurement information provided something about the statistics of the disturbances is known or may be estimated. Three criteria for the generation of the cancelling input from the disturbance measurements will be presented; the first criterion is described in this section, and the second and third in Secs. 3.6 and 3.8.

Suppose the joint probability density of the disturbances, $P_o(y_1, \ldots, y_n; t)$, is known for every time instant in the interval t_1 to t_2. Then $P_o(y_1, \ldots, y_n; t)dy_1 \ldots dy_n$ is the fraction of an ensemble of systems for which the disturbances simultaneously lie in the intervals y_1 to $y_1 + dy_1, \ldots, y_n$ to $y_n + dy_n$ at time t. The fact that P_o depends upon time indicates the varying nature of the statistics of the disturbances; we are not restricting the development to time-stationary random variables. Consider the y_i to be the axes of an n-dimensional space in which we know P_o at every time instant from t_1 to t_2. At some time instant during the operation of a single control system the m measurements μ_i at that time instant together with x as a parameter define a hypersurface in the y_i space through Eq. (3.6). We thus know that this single engineering system is a member of a limited ensenble whose members lie on this hypersurface. The first criterion for the generation of the cancelling input x is that the average error expression integrand for this limited ensemble must vanish. It should be noted that this limited ensemble at some time instant is defined on the basis of the measurements only at the same time instant; past measurement information is discarded.

We now define some notation and symbols to be used presently.

Let f designate the joint probability density[6] of the random variables appearing in parentheses behind it. Thus $f(x_1, \ldots, x_n)$ is the joint probability density of the random variables $x_1, \ldots, x_n$. It is an abbreviation and an exception to the usual functional notation.

Let $f(x_1, \ldots, x_m | x_{m+1}, \ldots, x_n)$ designate the conditional joint probability density[6] of the m random variables $x_1, \ldots, x_m$, given parametric values of the remaining n-m random variables $x_{m+1}, \ldots, x_n$.

- a single bar over a function of random variables indicates the a priori ensemble average of this function of random variables. It is obtained by multiplying the function of random variables by the a priori joint probability density of these random variables and integrating over all possible values of each of the latter.

$\overline{\overline{\qquad\qquad}}$ - a double bar over a function of random variables at some time instant indicates the a posteriori ensemble average of this function of random variables, given a set of measurements (Eq. (3.6)) at that time instant. This average is obtained by multiplying the function of the random variables by the conditional joint probability density of the random variables, given the measurements, and integrating over all possible values of the random variables but not the measurements. Such an average at some time instant then is a function of the measurements at the same time instant. This average is defined and used only when the random variables over which the average is carried are all evaluated at the same time instant.

Using the error expression (Eq. (3.5)) and the notation just defined our criterion for the generation of the cancelling input becomes,

$$\overline{\overline{\sum_{i=1}^{n} a_i y_i + bx}} = 0$$

Since the cancelling input is not a random variable but a parameter which is generated from measurements which are the same by definition for all members of the ensemble included in the double bar average, the double bar need not be extended over x. Thus the criterion for the generation of the cancelling input is

$$\overline{\overline{\sum_{i=1}^{n} a_i y_i + bx}} = 0 \qquad (3.7)$$

Even if the a priori averages, $\overline{y_i}$, of each of the disturbances is zero the a posteriori averages, $\overline{\overline{y_i}}$, - the averages, given the measurements - will not in general be zero. The cancelling input thus cancels this a posteriori bias. Such a cancelling input is justified at the moment only on the basis of its intuitive appeal. Ultimate acceptance or rejection of a cancelling input generated according to this criterion for a particular problem must be based on a comparison of the cancellation effectiveness for each of the three criteria to be presented.

It will be convenient to express the average in Eq. (3.7) in a new set of random variables, η_i, related to the original random variables, y_i, according to

$$\eta_i = \sum_{j=1}^{n} c_{ij} \, y_j \qquad i=1, \ldots, n \qquad (3.8)$$

where c_{ij} is the same as in Eq. (3.6) for $i = 1, \ldots, m$; for $i = m + 1, \ldots, n$ the c's may be chosen arbitrarily beyond the requirement that the set given by Eq. (3.8) must be linearly independent. Let the inverse relations be

$$y_i = \sum_{j=1}^{n} d_{ij}\,\eta_j \qquad i=1, \ldots, n \tag{3.9}$$

The joint probability density P of the new random variables is[6]

$$P(\eta_1, \ldots, \eta_n; t) \equiv P_o(y_1, \ldots, y_n; t)\,\frac{\partial(y_1, \ldots, y_n)}{\partial(\eta_1, \ldots, \eta_n)} \tag{3.10}$$

after Eq. (3.9) has been substituted in the right side.

Changing variables in Eq. (3.7) gives

$$\overline{\overline{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\, d_{ij}\,\eta_j}} + bx = 0$$

and if $\sum_{i=1}^{n} a_i\, d_{ij} \equiv \alpha_j$ this becomes

$$\overline{\overline{\sum_{j=1}^{n} \alpha_j\,\eta_j}} + bx = 0 \tag{3.11}$$

Comparing Eqs. (3.6) and (3.8) we see that the m measurements together with the cancelling input x regarded as a parameter fix $\eta_1, \ldots, \eta_m$. The remaining random variables over which the average in Eq. (3.11) must be carried are $\eta_{m+1}, \ldots, \eta_n$. Therefore Eq. (3.11) may be written

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{n} \alpha_i \eta_i \, f(\eta_{m+1}, \ldots, \eta_n \mid \eta_1, \ldots, \eta_m) \, d\eta_{m+1} \ldots d\eta_n + bx = 0$$

where $\eta_i = \mu_i - c_{io} x$ for $i = 1, \ldots, m$. But we have the identity[6],

$$f(\eta_{m+1}, \ldots, \eta_n \mid \eta_1, \ldots, \eta_m) = \frac{f(\eta_1, \ldots, \eta_n)}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\eta_1, \ldots, \eta_n) d\eta_{m+1} \ldots d\eta_n}$$

where in this case,

$$f(\eta_1, \ldots, \eta_n) = P(\eta_1, \ldots, \eta_n; t)$$

Combining the last three equations gives the expression for the generation of the cancelling input x(t) from the m measurements $\mu_i(t)$ at the same time instant:

$$\frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{j=1}^{n}\alpha_j\eta_j\,P(\eta_1,\ \ldots,\ \eta_n;\ t)\ d\eta_{m+1}\cdots d\eta_n}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}P(\eta_1,\ \ldots,\ \eta_n;\ t)\ d\eta_{m+1}\cdots d\eta_n} + bx = 0$$

where $\eta_i = \mu_i - c_{io}x$ for $i = 1, \ldots, m$ (3.12)

Eq. (3.12) is mechanized by a computer, and therefore is called the computer equation. When the function P is specified for a particular problem, Eq. (3.12) is solved by any suitable means for x as a function of the m measurements, μ_i, and time t as a parameter. The computer then is constructed so that at any time instant it carries out the logical processes - the manipulation of the μ_i - dictated by this equation; i.e., it continuously solves for x(t), using the m measurements $\mu_i(t)$ continuously supplied to it. The computer ideally generates the cancelling input instantaneously from the measurements. Practically this means that the computing time or computer lag must be short compared to a time characteristic of the random disturbances.

The flow of information for the entire system is shown in the block diagram in Fig. 13. The mixing of the n random disturbances and cancelling input according to Eq. (3.6) is inherent. (See the explanation following Eq. (3.6).) The i th transducer senses μ_i, the i th quantity to be measured, and converts it to a form - an electrical voltage, perhaps - acceptable to the computer. The transducers and the information channels between the transducers and the computer are assumed perfect and undisturbed in this section; the computer receives the measurements without time lag or error of any kind.

3.4 Noisy Measurements

In Sec. 3.3 it was assumed that the computer receives perfect measurement information. Let μ_i, the i th combination of the disturbances and cancelling input, be designated the i th signal where the term is used generally here to include disturbances and a cancelling input of any physical form, electrical and otherwise. In Sec. 3.3 then, the computer receives the m signals without error of any kind.

Suppose now that before the i th signal reaches the computer some noise, n_i, is added to it. The computer now receives the signal-plus-noise, μ_i', where

$$\mu_i' \equiv \mu_i + n_i \qquad (3.13)$$

The m $\mu_i'(t)$ are now the measurements. The noise may have its origin in the transducer which senses the signal as a physical variable and converts it to a form acceptable by the computer, or it may be introduced in the transmission of the signal from the transducer to the computer, or it may be the combination of both. Whatever its origin, let n_i be the difference between the i th input to the computer, μ_i', and the i th signal, μ_i. See Fig. 14. Let

$$N(\mu_1', \ldots, \mu_m'; \mu_1, \ldots, \mu_m, t)$$

be the known conditional joint probability density of the signals-plus-noise, given the signals. The noises considered here are random variables of a nature similar to the y's or η's; they are not time stationary random variables; hence the dependence of N on t. Notice that the noises are not necessarily independent of each other or the signals; N depends on the 2m parameters, μ_1', ..., μ_m', μ_1, ..., μ_m, and also time. If the noises are independent of the signals then N will depend on the m parameters, $(\mu_1'-\mu_1)$, ..., $(\mu_m'-\mu_m)$, and time. If in addition they are independent of each other then N may be written as a product of m functions, the i th of which depends on $(\mu_i'-\mu_i)$ and time.

The criterion for generating the cancelling input x(t) from the measurements received by the computer is

$$\overline{\overline{\sum_{j=1}^{n} \alpha_j \eta_j}} + bx = 0 \tag{3.11}$$

Because of the unknown noise components the m measurements, μ_i', do not fix η_1, ..., η_m as in the last section. Recalling the definition of the double bar average we write Eq. (3.11) as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{n} \alpha_j \eta_j \, f(\eta_1, \ldots, \eta_n \mid \mu_1', \ldots, \mu_m') \, d\eta_1 \ldots d\eta_n + bx = 0 \tag{3.14}$$

To obtain f, the conditional joint probability density of $\eta_1, \ldots, \eta_n$, given $\mu_1', \ldots, \mu_m'$, in Eq. (3.14) write the identities[6],

$$f(\eta_1, \ldots, \eta_n \mid \mu_1', \ldots, \mu_m') = \frac{f(\eta_1, \ldots, \eta_n, \mu_1', \ldots, \mu_m')}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\eta_1, \ldots, \eta_n, \mu_1', \ldots, \mu_m')\, d\eta_1 \ldots d\eta_n} \tag{3.15}$$

and

$$f(\eta_1, \ldots, \eta_n, \mu_1', \ldots, \mu_m') = f(\eta_1, \ldots, \eta_n) f(\mu_1', \ldots, \mu_m' \mid \eta_1, \ldots, \eta_n) \tag{3.16}$$

By definition,

$$P(\eta_1, \ldots, \eta_n; t) = f(\eta_1, \ldots, \eta_n) \tag{3.17}$$

$$N(\mu_1', \ldots, \mu_m'; \mu_1, \ldots, \mu_m, t) = f(\mu_1', \ldots, \mu_m' \mid \mu_1, \ldots, \mu_m)$$

But from Eqs. (3.6) and (3.8),

$$\mu_i = \eta_i + c_{io}x$$

Then

$$N(\mu_1', \ldots, \mu_m'; \eta_1+c_{1o}x, \ldots, \eta_m+c_{mo}x,t) = f(\mu_1', \ldots, \mu_m' | \eta_1, \ldots, \eta_m)$$

Specification of parametric values for $\eta_{m+1}, \ldots, \eta_n$ does not affect the conditional joint probability density of $\mu_1', \ldots, \mu_m'$ so that

$$N(\mu_1', \ldots, \mu_m'; \eta_1+c_{1o}x, \ldots, \eta_m+c_{mo}x,t) = f(\mu_1', \ldots, \mu_m' | \eta_1, \ldots, \eta_n) \tag{3.18}$$

Combining Eqs. (3.15) through (3.18) gives

$$f(\eta_1, \ldots, \eta_n | \mu_1', \ldots, \mu_m') = \frac{P(\eta_1, \ldots,\eta_n;t)\ N(\mu_1', \ldots, \mu_m';\eta_1+c_{1o}x, \ldots, \eta_m+c_{mo}x,t)}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\eta_1, \ldots, \eta_n;t)\ N(\mu_1', \ldots,\mu_m';\eta_1+c_{1o}x, \ldots,\eta_m+c_{mo}x,t)d\eta_1 \ldots d\eta_n} \tag{3.19}$$

Eq. (3.14) into which Eq. (3.19) has been substituted is the expression for the generation of the cancelling input x(t) at every time instant from the m signals-plus-noise, $\mu_i'(t)$ at the same time instant; this is the computer equation.

Fig. 14 shows the flow of information in a block diagram of the system. The mixing of the n random disturbances, $y_i(t)$, and the cancelling input, x(t), according to Eq. (3.6) to yield the m signals, $\mu_i(t)$, is inherent in the system. The transducers (not shown in Fig. 14) convert the signals to a form acceptable to the computer to which the transducer outputs are then transmitted. In this operation random errors or noises are added, the sum of these in any one channel being the noise, $n_i(t)$. Eqs. (3.14) and (3.19) together with P and N for the problem at hand, when solved for x as a function of the μ_i' and t as a parameter, specify the construction of the computer so that it continuously generates the cancelling input x(t) from the m noisy measurements, $\mu_i'(t)$, continuously supplied to it.

If there is no noise then N is a product of Dirac delta-functions,

$$N(\mu_1', \ldots, \mu_m'; \mu_1, \ldots, \mu_m, t) = \delta(\mu_1' - \mu_1)\ldots\delta(\mu_m' - \mu_m)$$

Substitution of this into Eqs. (3.14) and (3.19) yields Eq. (3.12), the result of the last section.

3.5 Example; Calculation of the Cancelling Input

In order to show the nature of the computer equation (Eq. (3.14) into which Eq. (3.19) has been substituted) for one common case, this equation will be obtained for an example in which the disturbances and noise have the often-encountered Gaussian distributions. The result of this calculation will then suggest the next criterion for the generation of a cancelling input from the measurement information.

Let $z(t)$ be the output of a control system governed by the equation,

$$\left(\tau_1\tau_2 + y_1(t)\right)\frac{d^2z}{dt^2} + \left((\tau_1 + \tau_2) + y_2(t)\right)\frac{dz}{dt} + \left(1 + y_3(t)\right) z = x(t)$$

subject to the initial conditions,

$$z(t)\Big|_{t=0} = Z\,, \qquad \frac{dz}{dt}\Big|_{t=0} = \dot{Z}$$

The τ's are constants; $y_1(t)$, $y_2(t)$, and $y_3(t)$ are small random disturbances such that $y_1(t) \ll \tau_1\tau_2$, $y_2(t) \ll (\tau_1 + \tau_2)$, and $y_3(t) \ll 1$; $x(t)$ is the cancelling input introduced to cancel the effect of the disturbances on the output and is of the same small order of magnitude.

Let $z(t) = z_o(t) + \varepsilon(t)$ where $\varepsilon(t) \ll z_o(t)$ and is the effect of the disturbances. Then $z_o(t)$ satisfies the equation,

$$\tau_1\tau_2\frac{d^2z_o}{dt^2} + (\tau_1 + \tau_2)\frac{dz_o}{dt} + z_o = 0$$

with the initial conditions,

$$z_o(t)\Big|_{t=0} = Z , \quad \frac{dz_o}{dt}\Big|_{t=0} = \dot{Z}$$

and $\varepsilon(t)$ satisfies the linearized perturbation equation,

$$\tau_1\tau_2 \frac{d^2\varepsilon}{dt^2} + (\tau_1 + \tau_2) \frac{d\varepsilon}{dt} + \varepsilon = - \frac{d^2 z_o}{dt^2} y_1(t) - \frac{dz_o}{dt} y_2(t) - z_o y_3(t) + x(t)$$

with the initial conditions,

$$\varepsilon(t)\Big|_{t=0} = 0 , \quad \frac{d\varepsilon}{dt}\Big|_{t=0} = 0$$

If $h(t)$ is the response to a unit impulse for the above equation then

$$\varepsilon(t_2) = \int_0^{t_2} h(t_2 - t)\left(- \frac{d^2 z_o}{dt^2} y_1 - \frac{dz_o}{dt} y_2 - z_o y_3 + x\right) dt$$

or

$$\varepsilon = \int_0^{t_2} (a_1 y_1 + a_2 y_2 + a_3 y_3 + bx)\, dt \qquad (3.20)$$

Thus the expression for the error at some time instant t_2 is expressed in the general form treated in the previous theory. For this example,

$$h(t) = -\frac{e^{-\frac{t}{\tau_1}}}{\tau_2 - \tau_1} + \frac{e^{-\frac{t}{\tau_2}}}{\tau_2 - \tau_1}$$

and

$$z_o(t) = -\frac{\tau_1 Z + \tau_1\tau_2\dot{Z}}{\tau_2 - \tau_1} e^{-\frac{t}{\tau_1}} + \frac{\tau_2 Z + \tau_1\tau_2\dot{Z}}{\tau_2 - \tau_1} e^{-\frac{t}{\tau_2}}$$

Then in Eq. (3.20),

$$a_i(t) = -(A_{i1} + A_{i2}\, e^{\gamma t} + A_{i3}\, e^{-\gamma t}) \qquad i = 1,2,3$$

where $\gamma = \frac{1}{\tau_1} - \frac{1}{\tau_2}$ and

$$A_{11} = \frac{\tau_1 Z + \tau_1\tau_2\dot{Z}}{\tau_1^2(\tau_2-\tau_1)^2} e^{-\frac{t_2}{\tau_1}} + \frac{\tau_2 Z + \tau_1\tau_2\dot{Z}}{\tau_2^2(\tau_2-\tau_1)^2} e^{-\frac{t_2}{\tau_2}}$$

$$A_{12} = -\frac{\tau_2 Z + \tau_1\tau_2\dot{Z}}{\tau_2^2(\tau_2-\tau_1)^2} e^{-\frac{t_2}{\tau_1}}$$

$$A_{13} = -\frac{\tau_1 Z + \tau_1\tau_2\dot{Z}}{\tau_1^2(\tau_2-\tau_1)^2} e^{-\frac{t_2}{\tau_2}} \qquad (3.21)$$

$$A_{21} = -\frac{\tau_1 Z + \tau_1\tau_2\dot{Z}}{\tau_1\ (\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_1}} - \frac{\tau_2 Z + \tau_1\tau_2\dot{Z}}{\tau_2\ (\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_2}}$$

$$A_{22} = \frac{\tau_2 Z + \tau_1\tau_2\dot{Z}}{\tau_2\ (\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_1}}$$

$$A_{23} = \frac{\tau_1 Z + \tau_1\tau_2\dot{Z}}{\tau_1\ (\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_2}}$$

$$A_{31} = \frac{\tau_1 Z + \tau_1\tau_2\dot{Z}}{(\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_1}} + \frac{\tau_2 Z + \tau_1\tau_2\dot{Z}}{(\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_2}}$$

$$A_{32} = -\frac{\tau_2 Z + \tau_1\tau_2\dot{Z}}{(\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_1}}$$

$$A_{33} = -\frac{\tau_1 Z + \tau_1\tau_2\dot{Z}}{(\tau_2-\tau_1)^2}\, e^{-\frac{t_2}{\tau_2}} \qquad \text{(3.21 Cont'd)}$$

and

$$b(t) = B_1\, e^{\frac{t}{\tau_1}} + B_2\, e^{\frac{t}{\tau_2}}$$

where

$$B_1 = -\frac{e^{-\frac{t_2}{\tau_1}}}{\tau_2-\tau_1} \quad \text{and} \quad B_2 = \frac{e^{-\frac{t_2}{\tau_2}}}{\tau_2-\tau_1} \qquad \text{(3.21 Cont'd)}$$

The quantities tabulated below appear in later expressions.

$$\int_0^{t_2} a_i^2(t)\, dt = (A_{i2}^2 + 2A_{i2}A_{i3})\, t_2 + \frac{A_{i2}^2}{2\gamma}\left(e^{2\gamma t_2} - 1\right)$$

$$- \frac{A_{i3}^2}{2\gamma}\left(e^{-2\gamma t_2} - 1\right) + \frac{2A_{i1}A_{i2}}{\gamma}\left(e^{\gamma t_2} - 1\right)$$

$$+ \frac{2A_{i1}A_{i3}}{\gamma}\left(e^{-\gamma t_2} - 1\right)$$

$$\int_0^{t_2} a_i(t)\, b(t)\, dt = -\tau_1(A_{i1}B_1 + A_{i2}B_2)\left(e^{\frac{t_2}{\tau_1}} - 1\right)$$

$$+ \frac{A_{i2}B_1}{\gamma+\frac{1}{\tau_1}}\left(e^{(\gamma+\frac{1}{\tau_1})t_2} - 1\right) + \tau_2(A_{i3}B_1 + A_{i1}B_2)\left(e^{\frac{t_2}{\tau_2}} - 1\right)$$

$$+ \frac{A_{i3}B_2}{-\gamma+\frac{1}{\tau_2}}\left(e^{(-\gamma+\frac{1}{\tau_2})t_2} - 1\right) \qquad (3.22)$$

$$\int_0^{t_2} b^2(t)\,dt = \frac{\tau_1 B_1^{\,2}}{2}\left(e^{2\frac{t_2}{\tau_1}} - 1\right) + \frac{\tau_2 B_2^{\,2}}{2}\left(e^{2\frac{t_2}{\tau_2}} - 1\right)$$

$$+ \frac{2B_1B_2}{\frac{1}{\tau_1}+\frac{1}{\tau_2}}\left(e^{\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)t_2} - 1\right)$$

(3.22 Cont'd)

Suppose that each of the three y_i's are measured separately; then $m = n = 3$, and $c_{ij} = 1$ for $i = j$, $= 0$ for $i \neq j$ in Eq. (3.6). No change of variable is necessary; $\eta_i = y_i = \mu_i$ and $\alpha_i = a_i$ in Eqs. (3.8) and (3.11).

Let the disturbances be independent, time-stationary, and Gaussian[1,6] with zero means so that

$$P(\eta_1, \eta_2, \eta_3; t) = P_o(y_1, y_2, y_3; t) = \prod_{i=1}^{3} \frac{1}{\sigma_i\sqrt{2\pi}}\, e^{-\frac{y_i^{\,2}}{2\sigma_i^{\,2}}}$$

(3.23)

and suppose that the measurements each contain noise (Sec. 3.4), these noises also being independent, time-stationary, and Gaussian with zero means. Then

$$N(\mu_1',\mu_2',\mu_3';\mu_1,\mu_2,\mu_3,t) = \prod_{i=1}^{3} \frac{1}{\sigma_i'\sqrt{2\pi}}\, e^{-\frac{(\mu_i'-\mu_i)^2}{2\sigma_i'^2}} \qquad (3.24)$$

In Eqs. (3.23) and (3.24) σ_i is the mean deviation of the i th disturbance, y_i, and σ_i' is the mean deviation of the i th noise, $n_i = \mu_i' - \mu_i$; $\prod$ designates the continued product.

Upon substituting Eqs. (3.23) and (3.24) into Eqs. (3.14) and (3.19) the cancelling input is obtained as

$$x(t) = -\sum_{i=1}^{3} \frac{\sigma_i^2}{\sigma_i^2+\sigma_i'^2}\, \frac{a_i(t)}{b(t)}\, \mu_i'(t) \qquad (3.25)$$

where $a_i(t)$ and $b(t)$ are as given in Eq. (3.21). If there is no noise then each of the σ_i' is zero, and Eq. (3.25) causes the integrand to vanish in Eq. (3.20) to give no system error. If for some measurement there is extreme noise so that $\frac{\sigma_i^2}{\sigma_i'^2} = 0$ then that measurement is unreliable and is discarded. The factor $\frac{\sigma_i^2}{\sigma_i^2+\sigma_i'^2}$ thus appears to assess the reliability of the corresponding measurement.

Eq. (3.25) is the computer equation for this example, and from it the system block diagram, Fig. 15, is constructed. The computer multiplies the i th measurement, $\mu_i'(t)$, by the time-varying factor,

$$\frac{\sigma_i^2}{\sigma_i^2+\sigma_i'^2} \; \frac{a_i(t)}{b(t)}$$

then adds the results for each of the three channels to generate the cancelling input, x(t). These time-varying factors are information built into the computer or stored in the computer memory before operation of the system.

The nature of the functional dependence of the cancelling input x on the measurements μ_i' depends on the probability densities P and N. These latter functions represent fairly detailed information about the statistics of the disturbances and noises, and in any practical situation they must be found by statistical experiment which may be difficult and tedious. In the example just described the cancelling input is a linear combination of the measurements, a result of the Gaussian probability densities. This result suggests an alternative. Let the cancelling input be a linear combination of the measurements for every problem, without regard for the probability densities of the disturbances and noises. The weighting of each measurement may then be adjusted to give the least mean square system error. This idea is exploited in the next few sections; in Sec. 3.6 the weighting of each measurement is constant during the time interval of system operation, and in Sec. 3.8 it varies.

3.6 Cancelling Input, Second Criterion

Reviewing briefly, we have the expression for the control system error as

$$\varepsilon = \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} a_i y_i + bx \right) dt \qquad (3.5)$$

where the $y_i(t)$ are the n random disturbances which degrade the system performance, x(t) is the cancelling input which is to be generated in some manner to eliminate or reduce the effect of the y_i, and $a_i(t)$ and b(t) are known functions of time specified by the nature of the controlled engineering system. As information available for the generation of x we have m quantities, $\mu_k(t)$,

$$\mu_k = \sum_{i=1}^{n} c_{ki}\, y_i \qquad k = 1, \ldots, m \qquad (3.26)$$

which are continuously measured and transmitted to the computer. (We suppose now that the quantities to be measured, the μ_i, are linear combinations of the disturbances but not the cancelling input, x; i.e., we have put $c_{io} = 0$ in Eq. (3.6), to simplify the development which follows.) But in the process of measuring each μ_k and transmitting it to the computer some noise, n_k, is added to it. The computer thus receives the m quantities, μ_k', where

$$\mu_k' = \mu_k + n_k = \sum_{i=1}^{n} c_{ki}\, y_i + n_k \qquad k = 1, \ldots, m \tag{3.27}$$

As suggested by the example in the last section let the computer generate the cancelling input as a linear combination of the measurements, the μ_k'; i.e., let

$$x(t) = \sum_{k=1}^{m} g_k\, \mu_k'(t) \tag{3.28}$$

where the g's are those which result in the least mean square system error. In this section the g's are constant throughout the time interval of system operation; the purpose of this section then is to find those constant g's which result in the least mean square system error. With the result of this calculation Eq. (3.28) is the computer equation for generating the cancelling input from the measurement information according to the second criterion.

For convenience in the calculation to follow we rewrite Eq. (3.27) as

$$\mu_k' = \sum_{i=1}^{n+m} c_{ki}\, y_i \qquad k = 1, \ldots, m \tag{3.29}$$

where $n_i \equiv y_{n+i}$ and $c_{ki} = 0$ for all $i > n$ except $i = n + k$. For $i = n + k$, $c_{k(n+k)} = 1$.

We also rewrite Eq. (3.5) as

$$\varepsilon = \int_{t_1}^{t_2} \left(\sum_{i=1}^{n+m} a_i y_i + bx \right) dt \tag{3.30}$$

where $a_i = 0$ for $i > n$. Combining Eqs. (3.28), (3.29), and (3.30) gives

$$\varepsilon = \int_{t_1}^{t_2} \sum_{i=1}^{n+m} \left(a_i + b \sum_{k=1}^{m} g_k\, c_{ki} \right) y_i \, dt$$

The system error squared is

$$\varepsilon^2 = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \left(a_i + b \sum_{k=1}^{m} g_k c_{ki} \right) \left(a_j + b \sum_{\ell=1}^{m} g_\ell c_{\ell j} \right)^* y_i y_j^* \, dt\, dt'$$

where the asterisk to the upper right of a quantity means that it is evaluated at time t', and its absence means that the quantity is evaluated at time t. The mean square error for the ensemble then is

$$\overline{\varepsilon^2} = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \left(a_i + b \sum_{k=1}^{m} g_k c_{ki} \right) \left(a_j + b \sum_{\ell=1}^{m} g_\ell c_{\ell j} \right)^* R_{ij}(t,t') \, dt\, dt' \tag{3.31}$$

where $\overline{y_i\, y_j^*} = \overline{y_i(t)\, y_j(t')} \equiv R_{ij}(t,t')$

The R's are correlation functions, and are the statistical information necessary for generating the cancelling input according to the second criterion. The only unknowns in Eq. (3.31) are the g's which are constants; the double integration over t and t' may be regarded as having been carried out, so that Eq. (3.31) is an algebraic function of the g's. Those g's which minimize $\overline{\varepsilon^2}$ are specified simply by putting

$$\frac{\partial \overline{\varepsilon^2}}{\partial g_p} = 0 \quad \text{for each } p = 1, \ldots, m$$

Thus

$$0 = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \Bigg(\Big(a_i + b\sum_{k=1}^{m} g_k\, c_{ki}\Big)(bc_{pj})^* R_{ij}\,(t,t')$$

$$+ (bc_{pi})\Big(a_j + b\sum_{\ell=1}^{m} g_\ell c_{\ell j}\Big)^* R_{ij}(t,t')\Bigg)\ dt\ dt'$$

$$p = 1, \ldots, m$$

If in the second term of the double summation we interchange the dummy variables of integration, t and t', and also interchange the summation indices, i and j, we obtain

$$0 = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} (a_i + b \sum_{k=1}^{m} g_k c_{ki}) b^* c_{pj} * \left(R_{ij}(t,t') + R_{ji}(t',t) \right) dt\, dt'$$

$$p = 1, \ldots, m$$

But

$$R_{ij}(t,t') = \overline{y_i(t)\, y_j(t')} = \overline{y_j(t')\, y_i(t)} = R_{ji}(t',t)$$

So

$$\sum_{k=1}^{m} g_k \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \int_{t_1}^{t_2} \int_{t_1}^{t_2} b(t) b(t') c_{ki}(t) c_{pj}(t') R_{ij}(t,t')\, dt\, dt'$$

$$= - \sum_{i=1}^{n} \sum_{j=1}^{n+m} \int_{t_1}^{t_2} \int_{t_1}^{t_2} a_i(t) b(t') c_{pj}(t') R_{ij}(t,t')\, dt\, dt'$$

$$p = 1, \ldots, m \qquad (3.32)$$

The summation over i on the right is extended only to n since $a_i = 0$ for $i > n$. Eq. (3.32) is a set of m linear equations for the m g's which minimize $\overline{\varepsilon^2}$.

Those constant g's resulting from the solution of Eq. (3.32), when used with Eq. (3.28), specify the cancelling input as a function of the measurements at every time instant in the interval t_1 to t_2 according to the second criterion.

Fig. 16 shows the block diagram for the system. Since the g's are constants the computer performs the very simple operation of weighting and adding the m measurements, the weighting of each remaining the same throughout the interval of system operation. The statistical information necessary for the calculation of this weighting - the g's - is the correlation functions. Since these functions are averages they appear to be less specific (and somewhat different) information than the probability densities P and N, and may possibly be more readily obtained.

3.7 Example; Calculation of the Cancelling Input Using the Second Criterion

We wish to particularize Eq. (3.32) for the g's for the example given in Sec. 3.5. For this example, m = 3, n = 3, and the a's and b are given in Eq. (3.21). In Eq. (3.29) $y_4 = n_1$, $y_5 = n_2$, and $y_6 = n_3$ so that

$$c_{11} = c_{14} = c_{22} = c_{25} = c_{33} = c_{36} = 1$$

and all other c's are zero. Because each of the disturbances and noises is independent only the autocorrelation functions of the disturbances, R_{11}, R_{22}, and R_{33}, and of the noises, R_{44}, R_{55}, and R_{66}, are different from zero. Suppose for this example these autocorrelation functions are

$$R_{11}(t,t') = \overline{y_1(t)\, y_1(t')} = \sigma_1^{\,2}\, e^{-\frac{2|t-t'|}{T_1}}$$

$$R_{22}(t,t') = \overline{y_2(t)\, y_2(t')} = \sigma_2^{\,2}\, e^{-\frac{2|t-t'|}{T_2}}$$

$$R_{33}(t,t') = \overline{y_3(t)\, y_3(t')} = \sigma_3^{\,2}\, e^{-\frac{2|t-t'|}{T_3}}$$

$$R_{44}(t,t') = \overline{y_4(t)\, y_4(t')} = \overline{n_1(t)\, n_1(t')} = \sigma_1'^{\,2}\, e^{-\frac{2|t-t'|}{T_1'}}$$

$$R_{55}(t,t') = \overline{y_5(t)\, y_5(t')} = \overline{n_2(t)\, n_2(t')} = \sigma_2'^{\,2}\, e^{-\frac{2|t-t'|}{T_2'}}$$

$$R_{66}(t,t') = \overline{y_6(t)\, y_6(t')} = \overline{n_3(t)\, n_3(t')} = \sigma_3'^{\,2}\, e^{-\frac{2|t-t'|}{T_3'}} \tag{3.33}$$

where the σ's are defined by Eqs. (3.23) and (3.24), and the T's are indicative of the memories of the disturbances and the T''s of the noises.

For the example Eq. (3.32) gives

$$g_p = - \frac{\int_0^{t_2}\int_0^{t_2} a_p(t)\ b(t')\ R_{pp}(t,t')\ dt\ dt'}{\int_0^{t_2}\int_0^{t_2} b(t)\ b(t') \left(R_{pp}(t,t') + R_{(p+3)(p+3)}(t,t') \right) dt\ dt'}$$

$$p = 1,2,3 \qquad (3.34)$$

Using the a's, b, and R's given by Eqs. (3.21) and (3.33) in the above expression gives the g's. But suppose that the memory times of the disturbances and noises are short compared to the time required for appreciable variation of the a's, b, and the time interval of system operation; i.e., that the T's and T''s in Eq. (3.33) are small compared to τ_1, τ_2, and t_2 in Eq. (3.21). If only first order terms are retained the resulting g's are those obtained by replacing the autocorrelation functions in Eqs. (3.33) and (3.34) by Dirac δ-functions. We now make this approximation; let

$$R_{11}(t,t') = \sigma_1^{\ 2}T_1\ \delta(t-t')$$
$$R_{22}(t,t') = \sigma_2^{\ 2}T_2\ \delta(t-t')$$
$$R_{33}(t,t') = \sigma_3^{\ 2}T_3\ \delta(t-t')$$
$$R_{44}(t,t') = \sigma_1'^{\,2}T_1'\ \delta(t-t')$$
$$R_{55}(t,t') = \sigma_2'^{\,2}T_2'\ \delta(t-t')$$
$$R_{66}(t,t') = \sigma_3'^{\,2}T_3'\ \delta(t-t') \qquad (3.35)$$

(The coefficients of the δ-functions are adjusted so that the area under the curve of an actual autocorrelation function in Eq. (3.33), plotted versus t for parametric t', say, is the same as the area under the corresponding approximation to this autocorrelation function in Eq. (3.35).)

Substituting Eq. (3.35) into (3.34) gives

$$g_p = - \frac{\sigma_p^{\,2} T_p}{\sigma_p^{\,2} T_p + \sigma_p^{\prime\,2} T_p^{\prime}} \, \frac{\int_0^{t_2} a_p(t) b(t) dt}{\int_0^{t_2} b^2(t) \, dt} \qquad p=1,2,3 \quad (3.36)$$

and putting this result into Eq. (3.28) yields the cancelling input as

$$x(t) = - \sum_{i=1}^{3} \frac{\sigma_i^{\,2} T_i}{\sigma_i^{\,2} T_i + \sigma_i^{\prime\,2} T_i^{\prime}} \, \frac{\int_0^{t_2} a_i(t) b(t) dt}{\int_0^{t_2} b^2(t) \, dt} \, \mu_i^{\prime}(t) \qquad (3.37)$$

where the integrals are as tabulated in Eq. (3.22). Fig. 17 shows the block diagram for the system.

3.8 Cancelling Input, Third Criterion

In this section we again propose that the computer generate the cancelling input as a linear combination of the measurements, where this combination now varies with time. That is, let

$$x(t) = \sum_{k=1}^{m} h_k(t)\, \mu_k'(t) \tag{3.38}$$

so that

$$\overline{\varepsilon^2} = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} (a_i + b\sum_{k=1}^{m} h_k c_{ki})(a_j + b\sum_{\ell=1}^{m} h_\ell c_{\ell j})^{*} R_{ij}(t,t')\; dt\, dt' \tag{3.39}$$

as in Sec. 3.6 except that the h's vary during the time interval t_1 to t_2. The purpose of this section is to find those variable h's which minimize $\overline{\varepsilon^2}$. With these h's available, Eq. (3.38) is the computer equation for continuously generating the cancelling input from the measurement information according to the third criterion.

Suppose that the h's which appear in Eq. (3.39) are those which minimize $\overline{\varepsilon^2}$. If to each $h_k(t)$ we add an arbitrary function $\varepsilon_k h_k'(t)$ where ε_k is a parameter, then $\overline{\varepsilon^2}$ must be minimum when $\varepsilon_1 = \ldots = \varepsilon_m = 0$.

Adding these arbitrary functions we obtain

$$\overline{\varepsilon^2} = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \left(a_i + b \sum_{k=1}^{m} (h_k + \varepsilon_k h_k') c_{ki} \right) X$$

$$\left(a_j + b \sum_{\ell=1}^{m} (h_\ell + \varepsilon_\ell h_\ell') c_{\ell j} \right)^* R_{ij}(t,t') \, dt \, dt'$$

The condition for minimum $\overline{\varepsilon^2}$ is expressed by putting

$$\left. \frac{\partial \overline{\varepsilon^2}}{\partial \varepsilon_p} \right|_{\varepsilon_1 = \ldots = \varepsilon_m = 0} = 0 \qquad p = 1, \ldots, m$$

Thus

$$0 = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \Big((a_i + b \sum_{k=1}^{m} h_k c_{ki})(bh_p' c_{pj})^* R_{ij}(t,t')$$

$$+ (bh_p' c_{pi})(a_j + b \sum_{\ell=1}^{m} h_\ell c_{\ell j})^* R_{ij}(t,t') \Big) \, dt \, dt'$$

for each p = 1, ..., m and arbitrary h''s.

Interchanging dummy variables of integration t and t' and also the summation indices i and j in the first term of the double summation and using the fact that $R_{ji}(t',t) = R_{ij}(t,t')$ (see Sec. 3.6) gives

$$0 = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} bh_p' \, c_{pi}\left(a_j + b\sum_{\ell=1}^{m} h_\ell \, c_{\ell j}\right)^* R_{ij}(t,t') \, dt \, dt'$$

$$p = 1, \ldots, m$$

or

$$0 = \int_{t_1}^{t_2} bh_p' \sum_{i=1}^{n+m} c_{pi} \left(\int_{t_1}^{t_2} \sum_{j=1}^{n+m} \left(a_j + b\sum_{\ell=1}^{m} h_\ell \, c_{\ell j}\right)^* R_{ij}(t,t') \, dt' \right) dt$$

$$p = 1, \ldots, m \qquad (3.40)$$

But $h_p'(t)$ is an arbitrary function of t so that in order to satisfy Eq. (3.40) the rest of the integrand must be identically zero for any t. Thus

$$\sum_{i=1}^{n+m} c_{pi} \int_{t_1}^{t_2} \sum_{j=1}^{n+m} \left(a_j + b\sum_{\ell=1}^{m} h_\ell \, c_{\ell j}\right)^* R_{ij}(t,t') \, dt' = 0$$

$$p = 1, \ldots, m$$

or

$$\sum_{i=1}^{n+m} c_{pi}(t) \sum_{j=1}^{n+m} \sum_{\ell=1}^{m} \int_{t_1}^{t_2} h_\ell(t') \; b(t') \; c_{\ell j}(t') \; R_{ij}(t,t') \; dt'$$

$$= - \sum_{i=1}^{n+m} c_{pi}(t) \sum_{j=1}^{n} \int_{t_1}^{t_2} a_j(t') \; R_{ij}(t,t') \; dt' \qquad (3.41)$$

for $p = 1, \ldots, m$ and any t. The summation over j on the right is extended only to n since $a_j = 0$ for $j > n$.

Eq. (3.41) is a set of m integral equations of the first kind for the m h's. Little may be said about a general inversion of this set; the obtaining of solutions in most cases must be by series expansions or numerical computation. However, a case of practical importance may be treated very simply. Suppose that the memory times of the disturbances are short compared to the times required for appreciable variation of the a's, b, c's, presumably the unknown h's, and also the time interval $t_2 - t_1$. Then $R_{ij}(t,t')$ is non-zero only when t and t' are nearly equal, and Eq. (3.41) may be written

$$\sum_{i=1}^{n+m} c_{pi}(t) \sum_{j=1}^{n+m} \sum_{\ell=1}^{m} h_{\ell}(t)\, b(t)\, c_{\ell j}(t) \int_{t_1}^{t_2} R_{ij}(t,t')\, dt'$$

$$= - \sum_{i=1}^{n+m} c_{pi}(t) \sum_{j=1}^{n} a_j(t) \int_{t_1}^{t_2} R_{ij}(t,t')\, dt'$$

$$p = 1, \ldots, m \tag{3.42}$$

Note that for any t not too close to the ends of the time interval t_1 to t_2,

$$\int_{t_1}^{t_2} R_{ij}(t,t')\, dt' = \int_{-\infty}^{\infty} R_{ij}(t,t')\, dt' \tag{3.43}$$

This is true under the assumption of short memory times for the disturbances and noises. Then R_{ij} is non-zero only for t' near t, hence non-zero only inside the interval t_1 to t_2 as long as t is not too close to t_1 or t_2 - or more precisely as long as $t - t_1$ or $t_2 - t$ is not as small as the memory times of the disturbances and noises. Since these memory times have been assumed to be small compared to the interval t_1 to t_2, Eq. (3.41) holds for most of the interval. Modification of Eq. (3.42) using (3.43) yields

$$b(t) \sum_{\ell=1}^{m} h_{\ell}(t) \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} c_{pi}(t)\, c_{\ell j}(t) \int_{-\infty}^{\infty} R_{ij}(t,t')\, dt'$$

$$= - \sum_{i=1}^{n+m} \sum_{j=1}^{n} c_{pi}(t)\, a_j(t) \int_{-\infty}^{\infty} R_{ij}(t,t')\, dt'$$

$$p = 1, \ldots, m \qquad (3.44)$$

for short memory times. The variation of

$$\int_{-\infty}^{\infty} R_{ij}(t,t')\, dt'$$

with time t reflects the variation of the statistics of the disturbances and noises. This variation is presumed to be no more rapid than the variation of the a's, b, and c's so that the h's resulting from the solution of Eq. (3.44) vary slowly compared to the variation of $R_{ij}(t,t')$, as assumed earlier. It should be noted that

$$\int_{t_1}^{t_2} R_{ij}(t,t')\, dt'$$

varies rapidly for t near t_1 or t_2 so that the h's calculated from Eq. (3.43) also vary rapidly for t near t_1 or t_2. This rapid variation of the h's near the ends of the time interval t_1 to t_2 is in contradiction to our original assumption about the h's; hence the modification of Eq. (3.42). The effect of this modification on the system mean square error is small since Eq. (3.44) modifies the h's only over a small fraction of the time interval of system operation t_1 to t_2 (of the order of the ratio of the disturbance and noise memory times to this $t_2 - t_1$ interval).

Eq. (3.44) constitutes a set of m linear algebraic equations for the m h's which may be simply solved.

Those variable h's resulting from the solution of Eq. (3.41) or (3.44), when used with Eq. (3.38), specify the cancelling input as a function of the measurements at every time instant in the interval t_1 to t_2 according to the third criterion. Fig. 18 shows the system block diagram.

3.9 Example; Calculation of the Cancelling Input Using the Third Criterion

In this section we wish to apply the result of Sec. 3.8 to the example given in Sec. 3.5. That is, we wish to calculate those variable h's which when used in the computer equation, Eq. (3.38), give the least mean square error for this example. In Sec. 3.7 we suppose that the memory times of the disturbances and noises for this example were short compared to the times required for appreciable variation of the a's and b, and the time interval $t_2 - t_1$; Eq. (3.44) is therefore applicable. Using Eq. (3.44) for this example (see parameter listing in Sec. 3.7) gives

$$h_p(t) = - \frac{a_p(t) \int_{-\infty}^{\infty} R_{pp}(t,t')\, dt'}{\int_{-\infty}^{\infty} \left(R_{pp}(t,t') + R_{(p+3)(p+3)}(t,t') \right) dt'}$$

$$p = 1,2,3$$

Substituting for the autocorrelation functions (Eq. (3.33) or (3.35)) in the above gives

$$h_p(t) = \frac{- \sigma_p{}^2 T_p}{\sigma_p{}^2 T_p + \sigma_p{}'^2 T_p{}'} \quad \frac{a_p(t)}{b(t)} \qquad p = 1,2,3$$

and putting this result into Eq. (3.38) yields the computer equation,

$$x(t) = -\sum_{i=1}^{3} \frac{\sigma_i^{2}T_i}{\sigma_i^{2}T_i + \sigma_i'^{2}T_i'} \frac{a_i(t)}{b(t)} \mu_i'(t) \qquad (3.45)$$

The system block diagram is shown in Fig. 19.

3.10 Comparison of Criteria; Mean Square Error

In the previous sections of Part III three criteria have been presented for the use of the disturbance measurements to generate a cancelling input. The cyberneticist designing a disturbance-cancelling system must choose one of these (or find a criterion better than each of the three presented here). On what basis should he make his choice? It is proposed that that criterion which gives the least mean square system error be used to generate the cancelling input. While this is not an unusual basis for such a choice it should be recognized that it may not always be appropriate. Frequent large errors yield a large mean square error, and frequent small errors, a small mean square error. Smallness of the mean square error therefore is a good basis for judging the performance of an engineering system when frequent large errors are objectionable while frequent small errors are not. This is certainly the situation in many cases. However when frequent small errors are not relatively unobjectionable, when errors of one sign are much more objectionable than the other, when an error of a magnitude less than some limit is completely acceptable and greater than this limit completely unacceptable, then smallness of the mean square error may not be an appropriate basis for judgment of system performance. In this section, then, the mean square system error is compared for each of the three criteria (and for no cancelling input at all) under the assumption that this is an appropriate basis for comparison.

The mean square error for no cancelling input is the same as that for the second or third criterion in which the g's or h's in the computer equation are put identically to zero. Since the g's or h's calculated according to the second or third criterion are those which give the least mean square error, any set of g's or h's not all zero resulting from this calculation must give a mean square system error less than that for no cancelling input at all. However any attempt to show that the first criterion always yields a mean square error no greater than that for no cancelling input is in vain; this will be shown presently.

From the formulation of the second and third criteria we know immediately that the third always leads to a smaller mean square error than the second (except in that improbable case where the h's calculated according to the third criterion are constants which must then be the same as the g's calculated according to the second criterion). But because the first criterion is based on the instantaneous joint probability densities of the disturbances and noises (P and N in Eq. (3.19)) and is quite different from the second and third which depend on correlation functions, any general comparison of the mean square error using the first criterion with that resulting from the second or third appears rather difficult. However in any given problem in which the necessary statistical information is provided, the mean square error may be calculated and compared for each of the three criteria and the one yielding the least chosen. This comparison will now be made for the example system treated in Secs. 3.5, 3.7, and 3.9. The mean square error for no cancelling input at all is included in this comparison to show the relative reduction of the mean square error resulting from the use of each criterion and to verify for one example some of the statements in the preceding paragraphs.

For no cancelling input at all x = 0 in Eq. (3.20) so that the system error is

$$\varepsilon = \int_{o}^{t_2} \sum_{i=1}^{3} a_i y_i \, dt$$

Squaring,

$$\varepsilon^2 = \int_{o}^{t_2} \int_{o}^{t_2} \sum_{i=1}^{3} \sum_{j=1}^{3} a_i a_j^* y_i y_j^* \, dt \, dt'$$

and averaging,

$$\overline{\varepsilon^2} = \int_0^{t_2}\int_0^{t_2} \sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j^* R_{ij}(t,t') \, dt \, dt'$$

and using the fact that the y_i are independent random variables so that only the autocorrelation functions are different from zero gives

$$\overline{\varepsilon^2} = \int_0^{t_2}\int_0^{t_2} \sum_{i=1}^{3} a_i a_i^* R_{ii}(t,t') \, dt \, dt'$$

Substituting for the R's from Eq. (3.35) yields the mean square error,

$$\overline{\varepsilon^2} = \sum_{i=1}^{3} \sigma_i^2 T_i \int_0^{t_2} a_i^2(t) \, dt \qquad (3.46)$$

for no cancelling input.

In order to calculate the mean square error for the first criterion Eq. (3.14) and (3.19) are solved for x in terms of the μ_i', using P and N for the problem at hand; this is the computer equation. For our example,

$$x = -\sum_{i=1}^{3} \frac{\sigma_i^2}{\sigma_i^2 + \sigma_i'^2} \frac{a_i}{b} \mu_i' \tag{3.25}$$

Then use of Eqs. (3.6) and (3.13) with the computer equation yields x as a function of the y's and n's, the disturbances and noises. Thus

$$x = -\sum_{i=1}^{3} \frac{\sigma_i^2}{\sigma_i^2+\sigma_i'^2} \frac{a_i}{b} (y_i + n_i)$$

Substitution of this in the error expression, Eq. (3.5) or Eq. (3.20) for this example, gives

$$\varepsilon = \int_0^{t_2} \sum_{i=1}^{3}\left(a_i \frac{\sigma_i'^2}{\sigma_i^2 + \sigma_i'^2} y_i - a_i \frac{\sigma_i^2}{\sigma_i^2 + \sigma_i'^2} n_i \right) dt$$

Squaring and averaging yields

$$\overline{\varepsilon^2} = \sum_{i=1}^{3} \left[\left(\frac{\sigma_i'^2}{\sigma_i^2 + \sigma_i'^2} \right)^2 \int_0^{t_2}\int_0^{t_2} a_i a_i^* R_{ii}(t,t')\, dt\, dt' \right.$$

$$\left. + \left(\frac{\sigma_i^2}{\sigma_i^2 + \sigma_i'^2} \right)^2 \int_0^{t_2}\int_0^{t_2} a_i a_i^* R_{(i+3)(i+3)}(t,t')\, dt\, dt' \right]$$

where the notation of Eq. (3.33) has been used. Substituting for the autocorrelation functions from Eq. (3.35) gives the mean square error,

$$\overline{\varepsilon^2} = \sum_{i=1}^{3} \frac{1 + \dfrac{\sigma_i^2 T_i'}{\sigma_i'^2 T_i}}{\left(1 + \dfrac{\sigma_i^2}{\sigma_i'^2}\right)^2} \, \sigma_i^2 T_i \int_0^{t_2} a_i^2(t)\, dt \qquad (3.47)$$

for the cancelling input generated according to the first criterion. We note that as long as

$$\frac{T_i'}{T_i} < 2 + \frac{\sigma_i^2}{\sigma_i'^2}$$

the i th term in Eq. (3.47) is less than the corresponding term in Eq. (3.46). If

$$\frac{T_i'}{T_i} > 2 + \frac{\sigma_i^2}{\sigma_i'^2}$$

the opposite is true. Although this latter condition would not be anticipated often in practice it demonstrates the possibility that a cancelling input generated according to the first criterion can give a greater mean square error than no cancelling input at all. This defect of the first criterion seems attributable to its formulation on the basis of the instantaneous relationship of the disturbances and noises (i.e., P and N) without regard for the relationship of these random variables from one instant to the next (i.e., the correlation functions for this example).

To compute the mean square error for this example according to the second criterion, Eqs. (3.35), (3.36), and the parameters given for this example at the beginning of Sec. 3.7 are substituted into Eq. (3.31). The result is

$$\overline{\varepsilon^2} = \sum_{i=1}^{3} \sigma_i^2 T_i \left[\int_0^{t_2} a_i^2(t)dt - \frac{\sigma_i^2 T_i}{\sigma_i^2 T_i + \sigma_i'^2 T_i'} \frac{\left(\int_0^{t_2} a_i(t)b(t)dt \right)^2}{\int_0^{t_2} b^2(t)dt} \right] \tag{3.48}$$

for the cancelling input generated according to the second criterion. Since the second term in the square bracket represents the subtraction of a non-negative quantity from the first, this mean square error is never more than that given by Eq. (3.46) for no cancelling input at all.

For the third criterion Eqs. (3.35), (3.45), and the parameters given at the beginning of Sec. 3.7 are substituted into Eq. (3.39). The result is

$$\overline{\varepsilon^2} = \sum_{i=1}^{3} \frac{1}{1 + \dfrac{\sigma_i^{\,2} T_i}{\sigma_i{}'^{2} T_i{}'}} \sigma_i^{\,2} T_i \int_0^{t_2} a_i^{\,2}(t)\, dt \tag{3.49}$$

for the cancelling input generated according to the third criterion. Because the factor

$$\frac{1}{1 + \dfrac{\sigma_i^{\,2} T_i}{\sigma_i{}'^{2} T_i{}'}} \leqq 1$$

the mean square error above is never more than that given by Eq. (3.46) for no cancelling input at all.

To compare the mean square error for the third criterion (Eq. (3.49)) with the first (Eq. (3.47)) we write

$$2 \leqq \frac{T}{T'} + \frac{T'}{T}$$

so that

$$1 + 2\,\frac{\sigma_i^{\,2}}{\sigma_i'^{\,2}} + \left(\frac{\sigma_i^{\,2}}{\sigma_i'^{\,2}}\right)^2 \leqq 1 + \left(\frac{T}{T'} + \frac{T'}{T}\right)\frac{\sigma_i^{\,2}}{\sigma_i'^{\,2}} + \left(\frac{\sigma_i^{\,2}}{\sigma_i'^{\,2}}\right)^2$$

or

$$\frac{1}{1 + \dfrac{\sigma_i^{\,2} T_i}{\sigma_i'^{\,2} T_i'}} \leqq \frac{1 + \dfrac{\sigma_i^{\,2} T_i'}{\sigma_i'^{\,2} T_i}}{\left(1 + \dfrac{\sigma_i^{\,2}}{\sigma_i'^{\,2}}\right)^2}$$

Thus for this example the mean square error for the third criterion is less than or equal to that for the first. This result is expected since the computer equations for each are of the same form but with parameters (i.e., the h's) for the third adjusted to give the least mean square error.

In comparing the mean square error for the third criterion to that for the second (Eq. (3.48)) we use Schwarz's inequality for continuous $a_i(t)$ and $b(t)$:

$$\left(\int_0^{t_2} a_i(t)\, b(t)\, dt \right)^2 \leqq \int_0^{t_2} a_i^{\,2}(t)\, dt \int_0^{t_2} b^2(t)\, dt$$

This inequality and Eqs. (3.48) and (3.49) show that for this example the mean square error for the third criterion is less than or equal to that for the second, as anticipated.

For this example the cancelling input generated according to the third criterion thus yields the least mean square error and is the system that should be adopted. This system is shown in Fig. 19.

3.11 Remarks on Obtaining the Required Statistical Information

The design of the computer which generates the cancelling input from the measurements is specified by the computer equation. This equation depends on statistical data of the disturbances - joint probability densities for the first criterion and correlation functions for the second and third - which must be obtained by appropriate statistical experiment before the computer may be designed. It is not intended to discuss here the design of experiments to yield these probability densities and correlation functions; this section will serve only to point out certain aspects of the problem and to refer to pertinent literature.

Much of the literature treats time-stationary random processes where observation of an ensemble of systems may be replaced by observation of a single system for a long period of time. Here we have not restricted the development to time-stationary processes. For instance the joint probability density of the atmospheric disturbances in the flight of a long-range rocket certainly varies if the vehicle leaves the atmosphere during part of its flight. But it still is not necessary to fire a suitably large number of test vehicles for ensemble measurements. At any fixed point on the known standard trajectory the atmospheric disturbances are time-stationary; but they vary from point to point, so that as the vehicle moves along its trajectory it encounters non-stationary disturbances. If there is no relation between the time of initiation of the flights and the disturbances then it is only necessary to obtain the joint probability density of the atmospheric disturbances by appropriately long observations at fixed points along the standard trajectory. By changing the parametric dependence of this joint probability density on the location along the trajectory to the corresponding time that a standard vehicle is at this location, the desired time-varying joint probability density is obtained. For this example then, a stationary aspect of the disturbances allows an ensemble observation to be replaced by a time observation, even though the desired statistical information is apparently non-stationary. The same technique of substituting a time observation for an ensemble observation may be employed to obtain the correlation functions for the atmospheric disturbances in this example.

In many cases the noises in the information channels to the computer will be independent of the disturbances being measured. Then the statistical data for the noises may be obtained from observations of the outputs of these information channels (i.e., the computer inputs, $\mu_i'(t)$) in the laboratory with no test input disturbances being applied. In cases where the noises depend on the disturbance inputs but the memory times of the noises are short compared to those of the disturbances, then the noises really depend only on the instantaneous levels of the disturbances rather than the statistical nature of their variation. The statistical data for the noises may then be obtained approximately from observations of the information channel outputs while applying steady disturbance test inputs. The most difficult (and probably infrequently encountered) case where the noise and disturbance memory times are of the same order of magnitude, the statistical data for the noises must be obtained while disturbance test inputs of the statistical nature to be encountered by the system are applied.

The matter of arguing from the experimental data to the desired statistical quantity is the problem of statistical inference; see Ref. 6 which lists further references extensively. For random processes associated with electrical equipment see Ref. 7, and with aeronautics, Ref. 8. Unfortunately most of the literature treats time-stationary random processes; for non-stationary processes the cyberneticist is left with the definition of the desired statistical quantity and his ingenuity.

IV. CONCLUDING REMARKS

4.1 Summary; Further Research

The preceding development is motivated by the conviction that the design of automatic control systems under the assumption that the system properties are known or may be specified precisely is inadequate when performance accuracy requirements are strict. It is pointed out that these properties have random deviations that degrade the system performance to an extent that becomes more serious as the required performance accuracy becomes extreme. For these extremely accurate control systems the continuous measurement of the deviations of the properties of each individual system is a necessity; the theory presented treats the use of these measurements to cancel the effect of the deviations on the system performance. But this theory is not yet complete; in the following paragraphs some areas for further research are designated.

When the third criterion is used to generate a cancelling input from a number of noisy measurements equal to the number of disturbances, the computer then serves as a filter to reduce the effect of these noise components added to the disturbance measurements. The theory of Sec. 3.8 restricts the computer equation (the input-output relation of the filter) to a variable-coefficient linear relation of the inputs and output but not time-derivatives of these. A more elegant computer equation would include these time-derivatives. The result would be a multiple-input filter for linear systems with time-varying coefficients, related to

N. Wiener's optimum linear filter[1,9] with constant coefficients for time-stationary inputs and noise, and to the extension[10] of Wiener's work to time-varying filters for non-stationary inputs and noise. This effort would be part of a very important search for more sophisticated use of the measurement information to generate a cancelling input.

Previous to Sec. 3.4 the computer was assumed to receive perfect measurement information. Then in Sec. 3.4 the measurement information was corrupted by the addition of noise by the transducers or in the information channels between the transducers and the computer. The investigation might be extended to include the effects of time-lags in the transducers; i.e., the transducer may have some transfer function other than a constant. In order for the transducer outputs to be useful, such time-lags should, of course, be small compared to the characteristic memory times of the disturbances being measured; it is suggested that the effect of these time-lags be quantitatively investigated.

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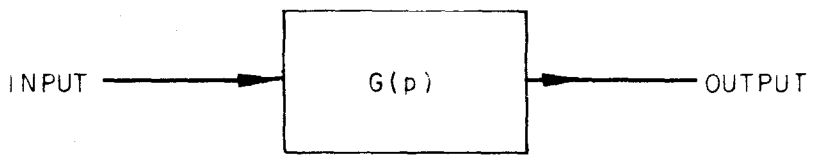


FIG. 1

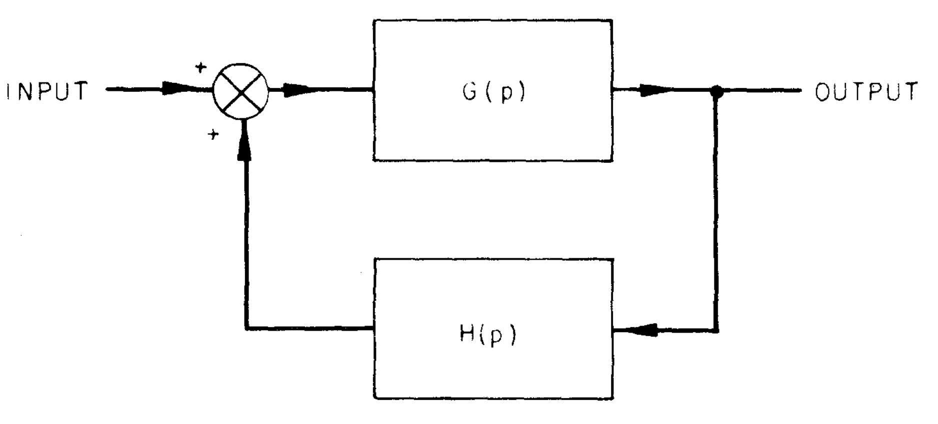


FIG. 2

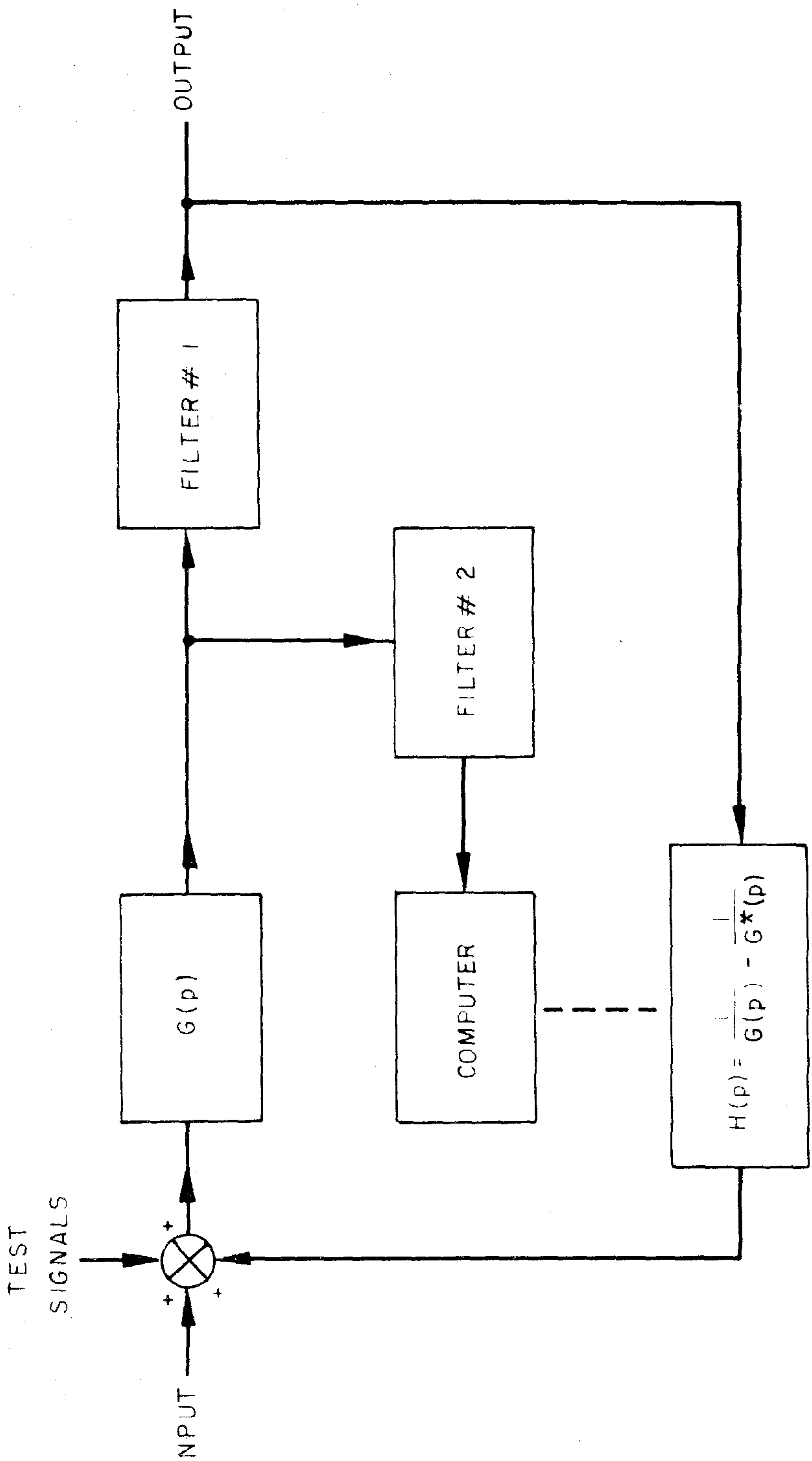

OUTPUT
FILTER # 1
FILTER # 2
G(p)
COMPUTER
H(p) = 1/G(p) - 1/G*(p)
TEST SIGNALS
INPUT

FIG. 3

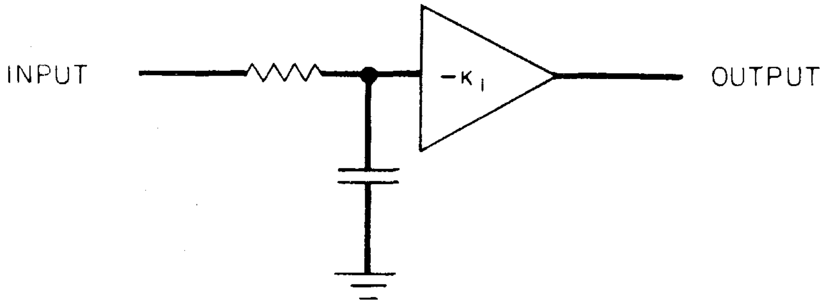


FIG. 4

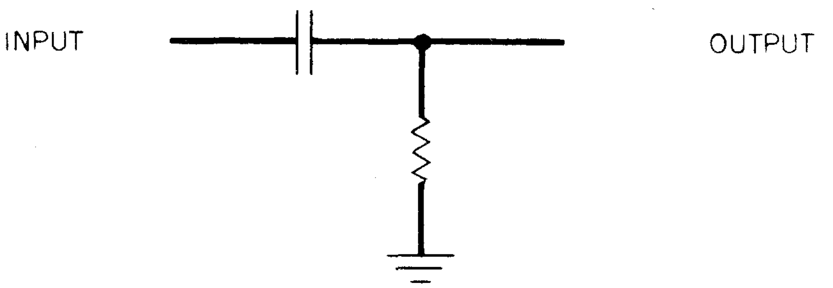


FIG. 5

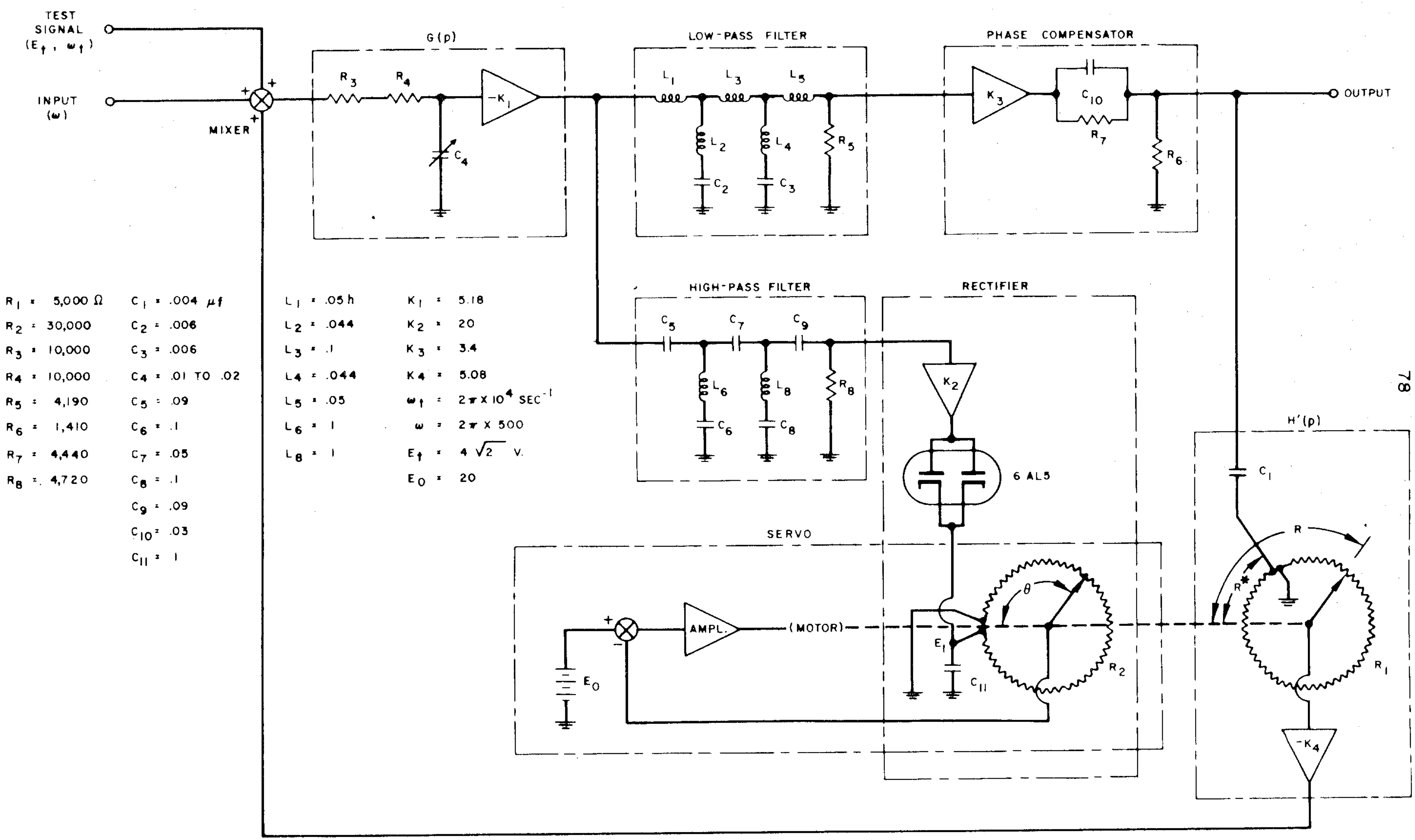
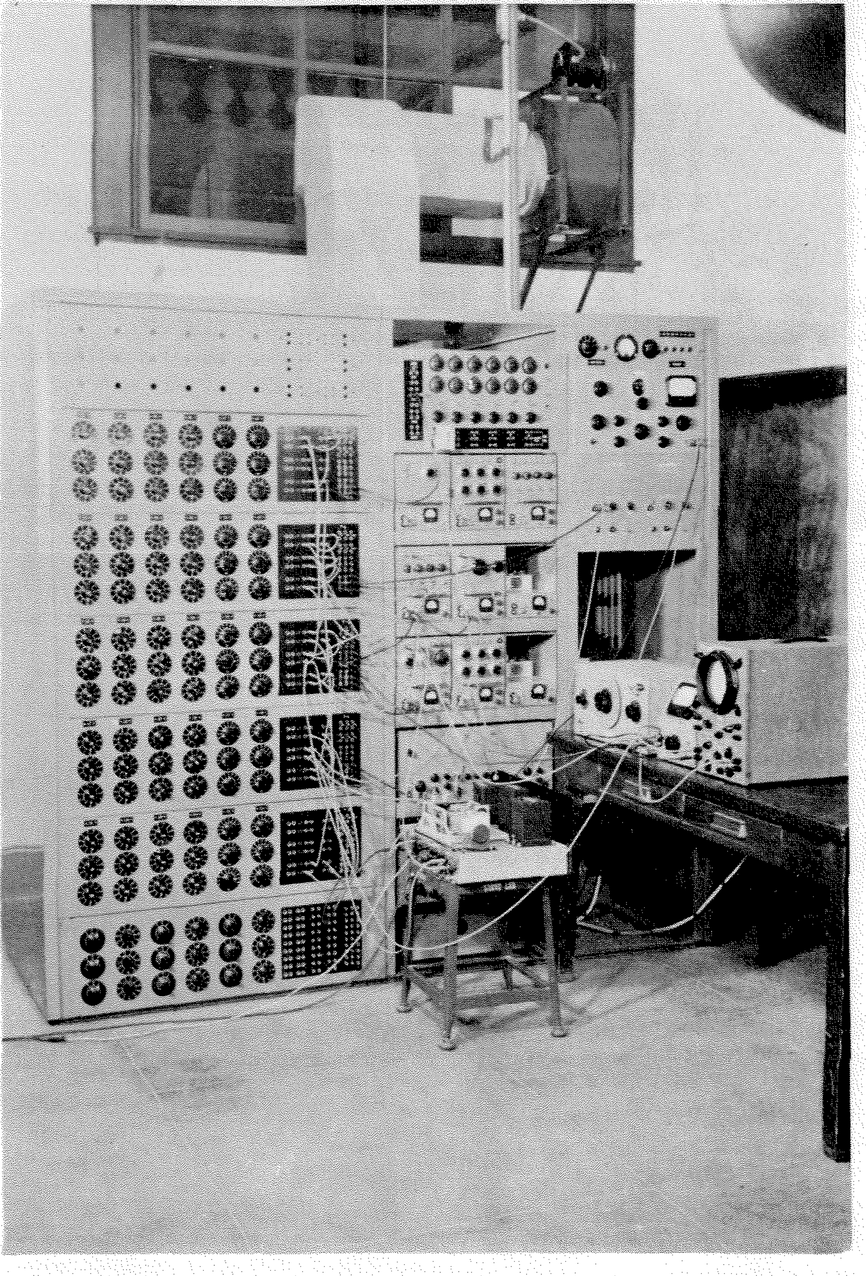
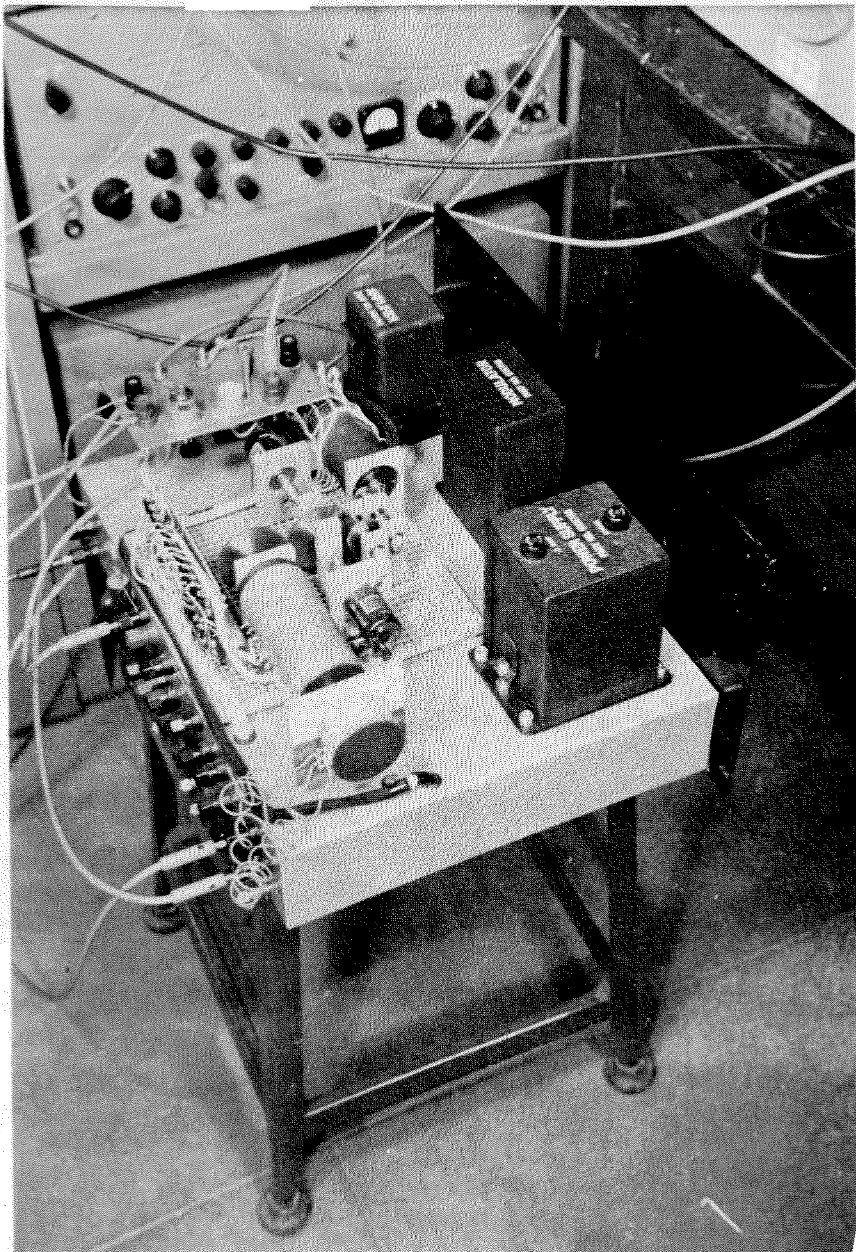

TEST SIGNAL (E_t, ω_t)
INPUT (ω)
MIXER
G(p)
R_3
R_4
C_4
$-K_1$
LOW-PASS FILTER
L_1
L_3
L_5
L_2
L_4
R_5
C_2
C_3
PHASE COMPENSATOR
K_3
C_{10}
R_7
R_6
OUTPUT
HIGH-PASS FILTER
C_5
C_7
C_9
L_6
L_8
R_8
C_6
C_8
RECTIFIER
K_2
6 AL5
SERVO
AMPL.
(MOTOR)
E_0
E_1
C_{11}
θ
R_2
H′(p)
C_1
R
R^*
R_1
$-K_4$
R_1 = 5,000 Ω
R_2 = 30,000
R_3 = 10,000
R_4 = 10,000
R_5 = 4,190
R_6 = 1,410
R_7 = 4,440
R_8 = 4,720
C_1 = .004 μf
C_2 = .006
C_3 = .006
C_4 = .01 TO .02
C_5 = .09
C_6 = .1
C_7 = .05
C_8 = .1
C_9 = .09
C_{10} = .03
C_{11} = 1
L_1 = .05 h
L_2 = .044
L_3 = .1
L_4 = .044
L_5 = .05
L_6 = 1
L_8 = 1
K_1 = 5.18
K_2 = 20
K_3 = 3.4
K_4 = 5.08
ω_t = 2π X 10^4 SEC^{-1}
ω = 2π X 500
E_t = 4 $\sqrt{2}$ V.
E_0 = 20

FIGURE 6



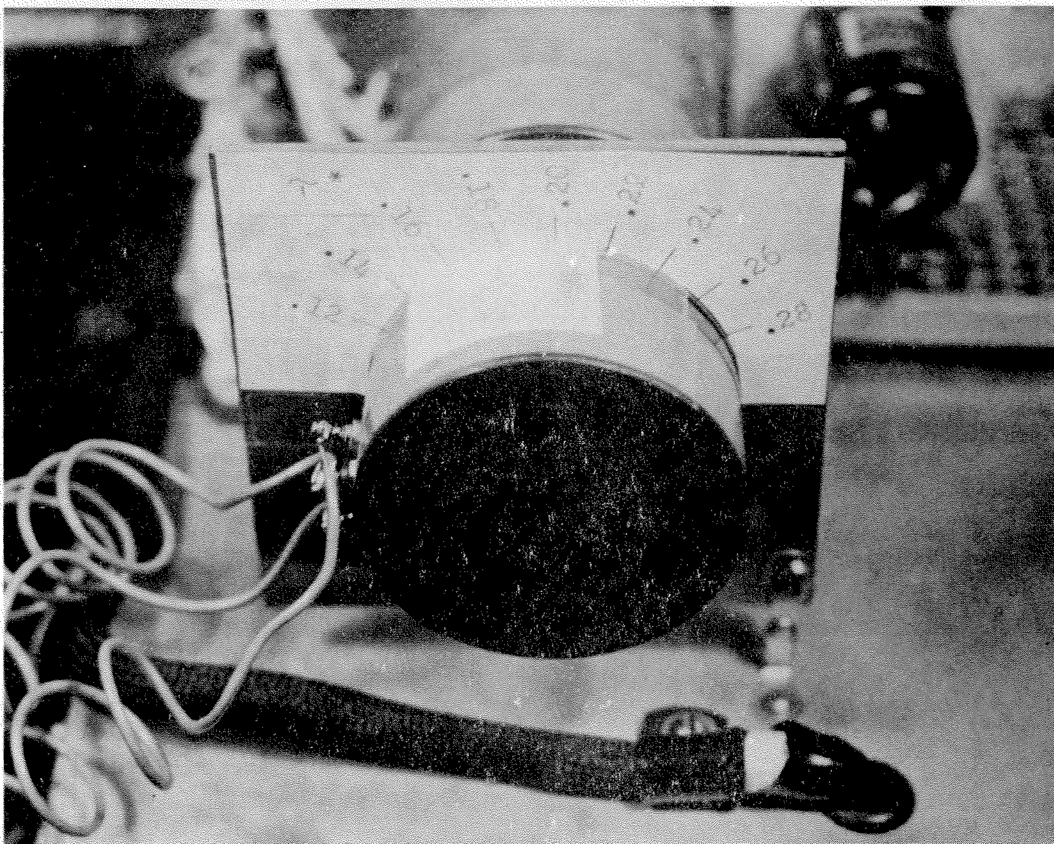
EXPERIMENTAL SETUP ON CALTECH ELECTRIC ANALOG COMPUTER

FIG. 7



RECTIFIER AND SERVO

FIG. 8



FEEDBACK POTENTIOMETER R_1 WITH τ^* SCALE IN MILLISECONDS

FIG. 9

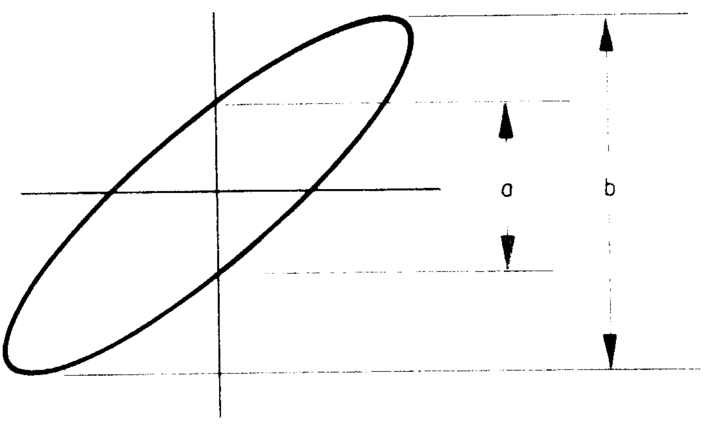
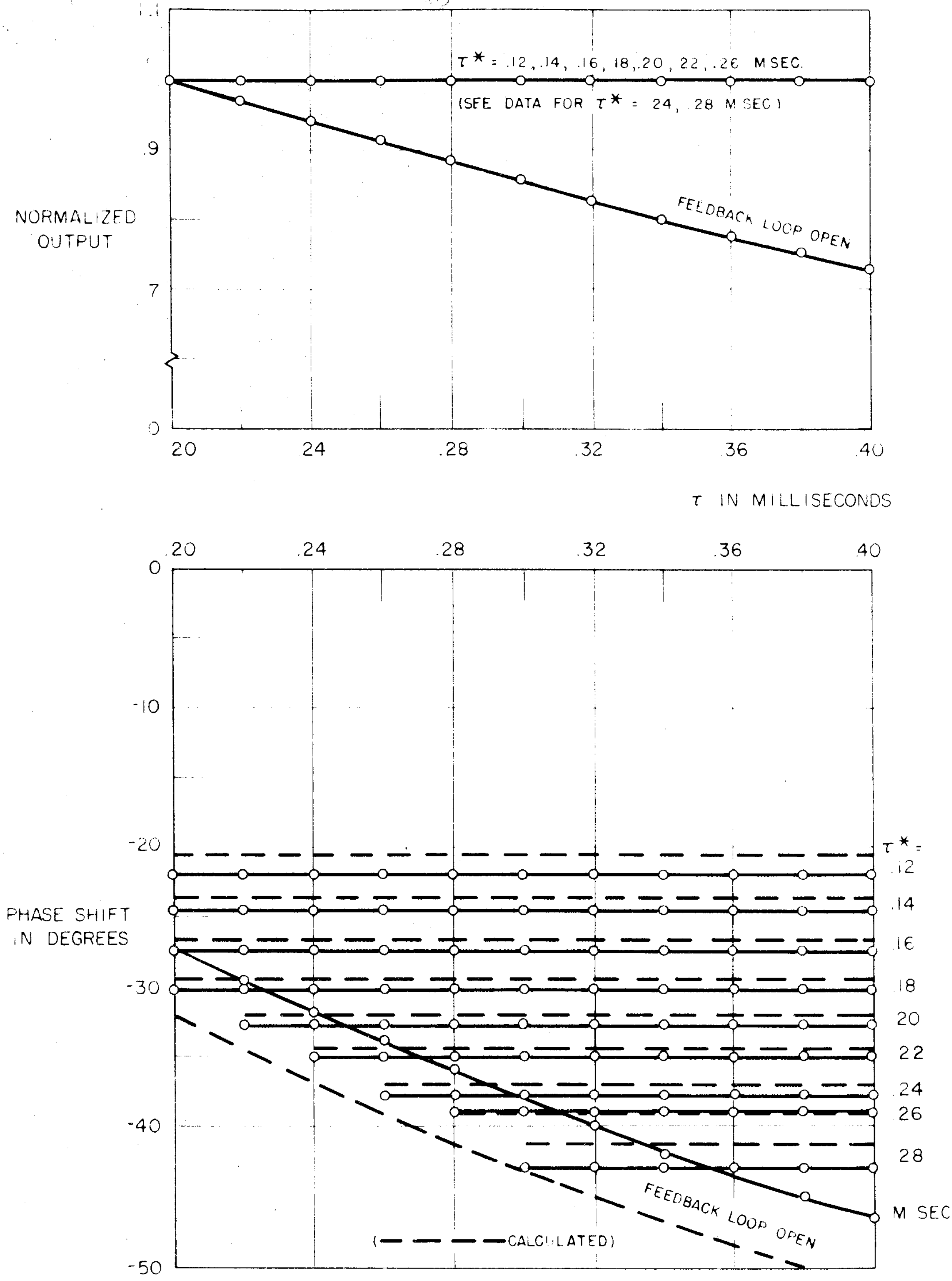
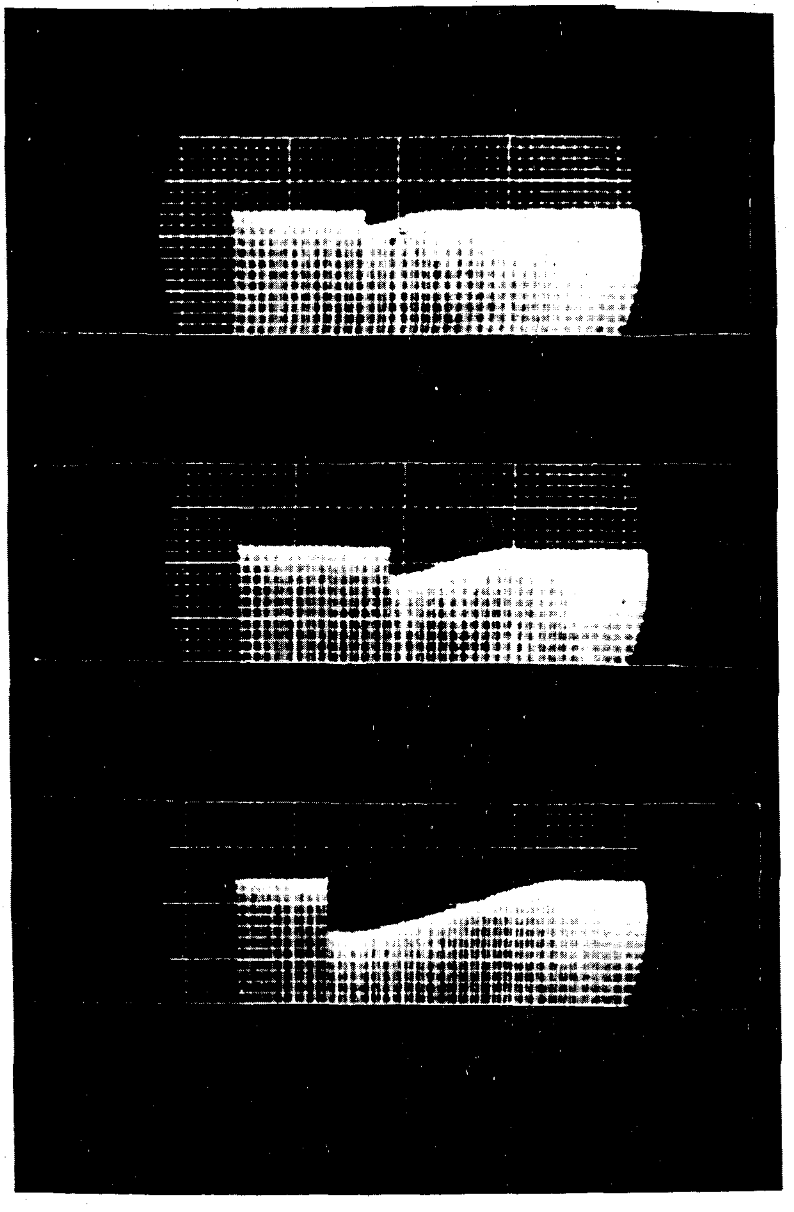


FIG. 10



τ* = .12, .14, .16, .18, .20, .22, .26 M SEC.
(SEE DATA FOR τ* = .24, .28 M SEC.)
FEEDBACK LOOP OPEN
NORMALIZED OUTPUT
1.1
.9
.7
0
.20
.24
.28
.32
.36
.40
τ IN MILLISECONDS
τ* =
.12
.14
.16
.18
.20
.22
.24
.26
.28
M SEC
PHASE SHIFT IN DEGREES
0
-10
-20
-30
-40
-50
FEEDBACK LOOP OPEN
(CALCULATED)

FIG. 11



CANCELLATION TRANSIENT OSCILLOGRAMS

HORIZONTAL SCALE: 30 DIVISIONS = 1 SECOND

STEP IN τ: UPPER, .20 TO .25 MILLISECONDS

CENTER, .20 TO .30

LOWER, .20 TO .40

τ^* = .18 MILLISECONDS

FIG. 12

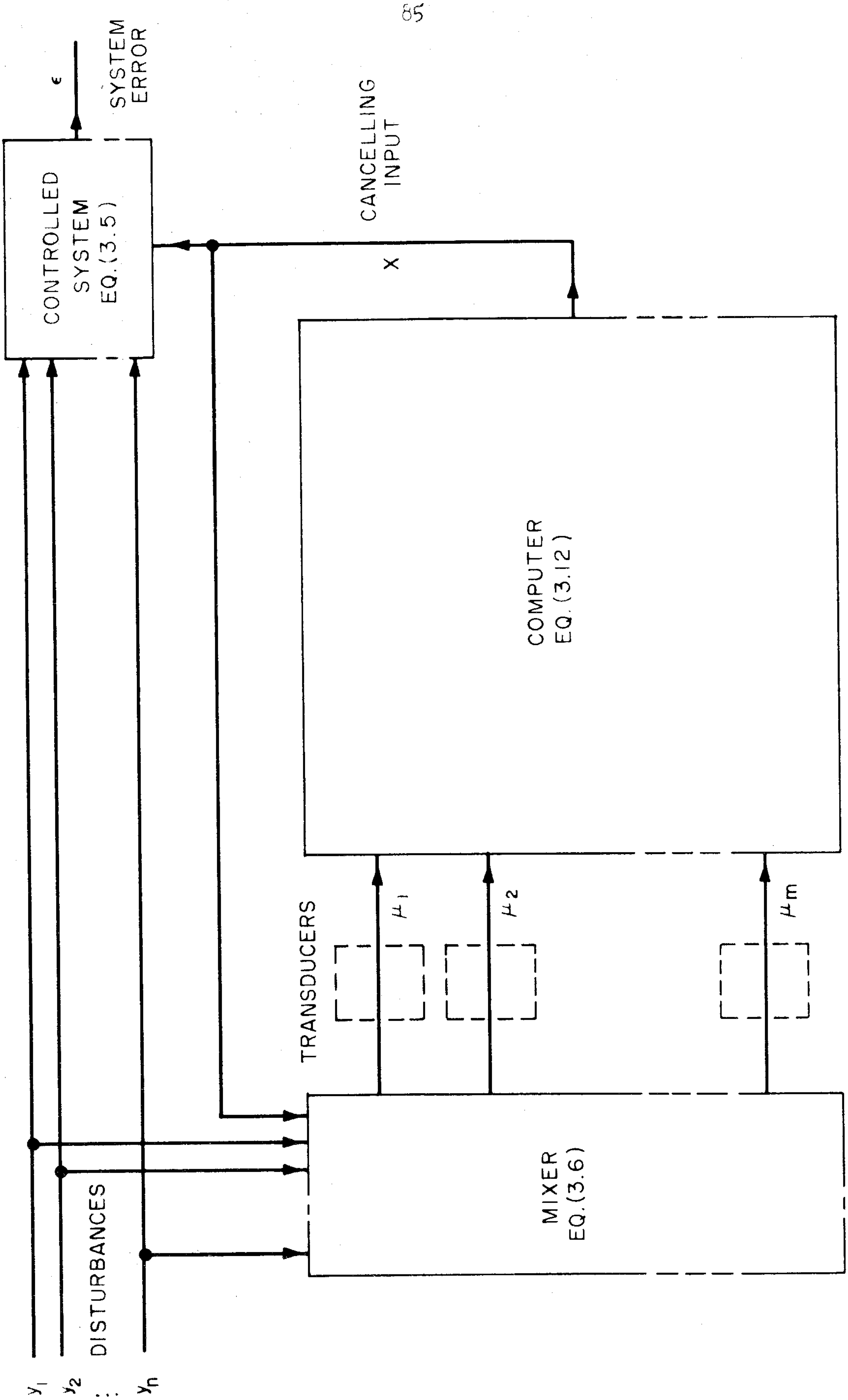
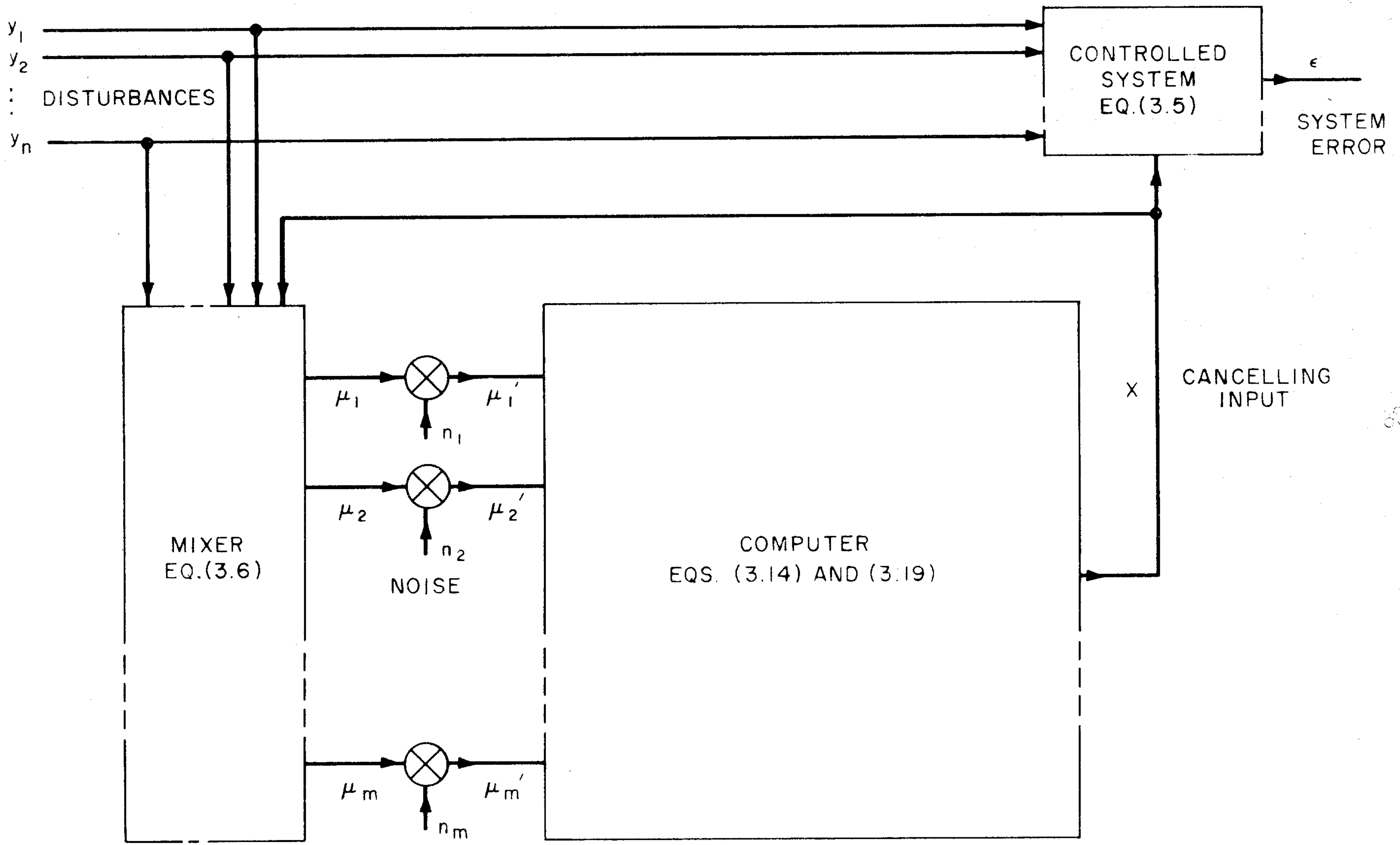
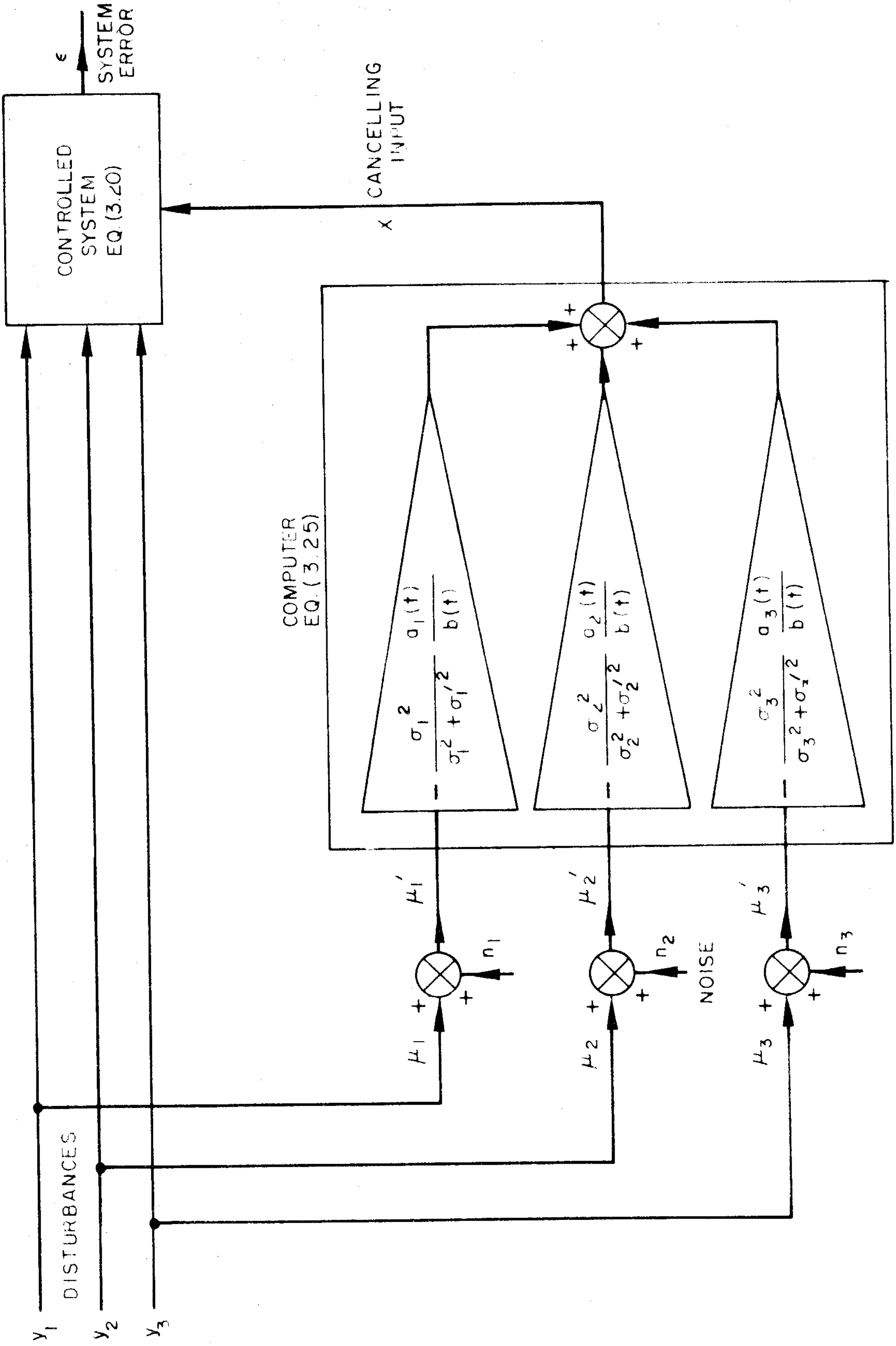

SYSTEM ERROR
ε
CONTROLLED SYSTEM EQ.(3.5)
CANCELLING INPUT
X
COMPUTER EQ.(3.12)
TRANSDUCERS
μ1
μ2
μm
MIXER EQ.(3.6)
DISTURBANCES
y1
y2
yn

FIG. 13



y1
y2
DISTURBANCES
yn
CONTROLLED SYSTEM EQ.(3.5)
ε
SYSTEM ERROR
MIXER EQ.(3.6)
μ1
μ2
μm
n1
n2
nm
NOISE
μ1'
μ2'
μm'
COMPUTER EQS. (3.14) AND (3.19)
X
CANCELLING INPUT

FIG. 14



CONTROLLED SYSTEM EQ.(3.20)
SYSTEM ERROR
CANCELLING INPUT
COMPUTER EQ.(3.25)
NOISE
DISTURBANCES

FIG. 15

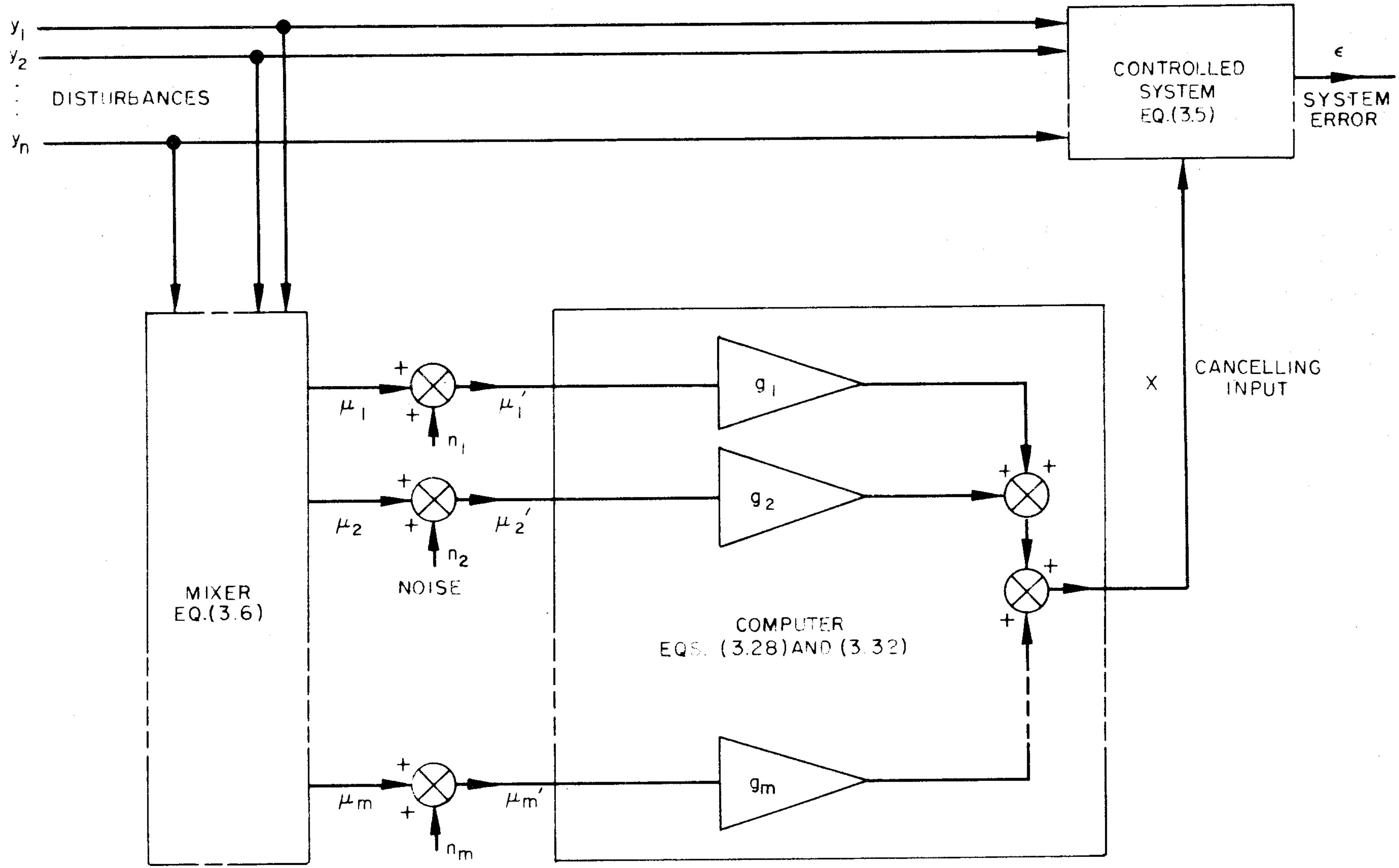
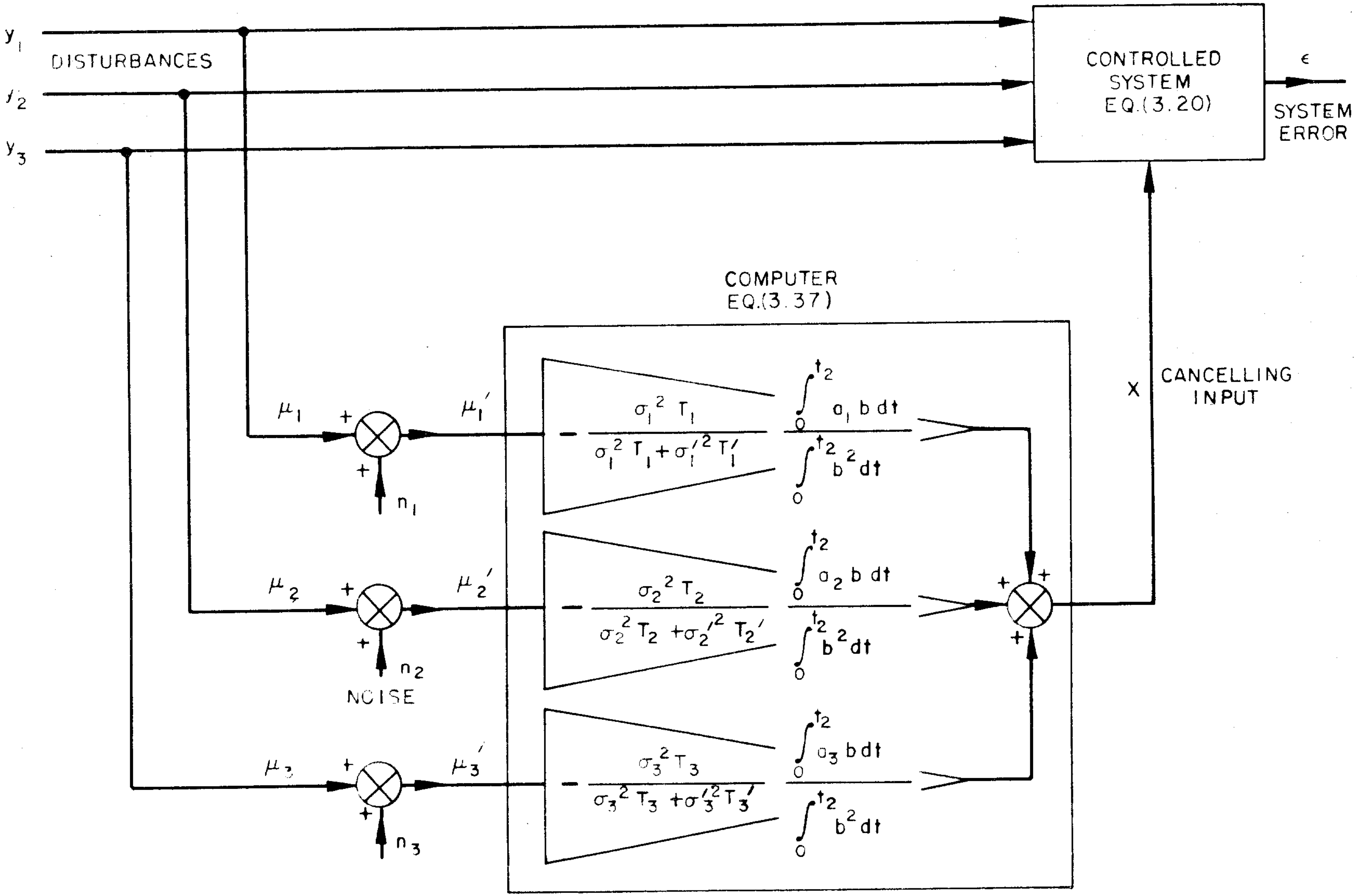

y1
y2
yn
DISTURBANCES
CONTROLLED SYSTEM EQ.(3.5)
ϵ
SYSTEM ERROR
MIXER EQ.(3.6)
μ1
μ2
μm
n1
n2
nm
NOISE
μ1′
μ2′
μm′
g1
g2
gm
COMPUTER EQS. (3.28) AND (3.32)
X
CANCELLING INPUT

FIG. 16



y1
DISTURBANCES
y2
y3
CONTROLLED
SYSTEM
EQ.(3.20)
ε
SYSTEM
ERROR
COMPUTER
EQ.(3.37)
X
CANCELLING
INPUT
μ1
μ1′
n1
μ2
μ2′
n2
NOISE
μ3
μ3′
n3

FIG 17

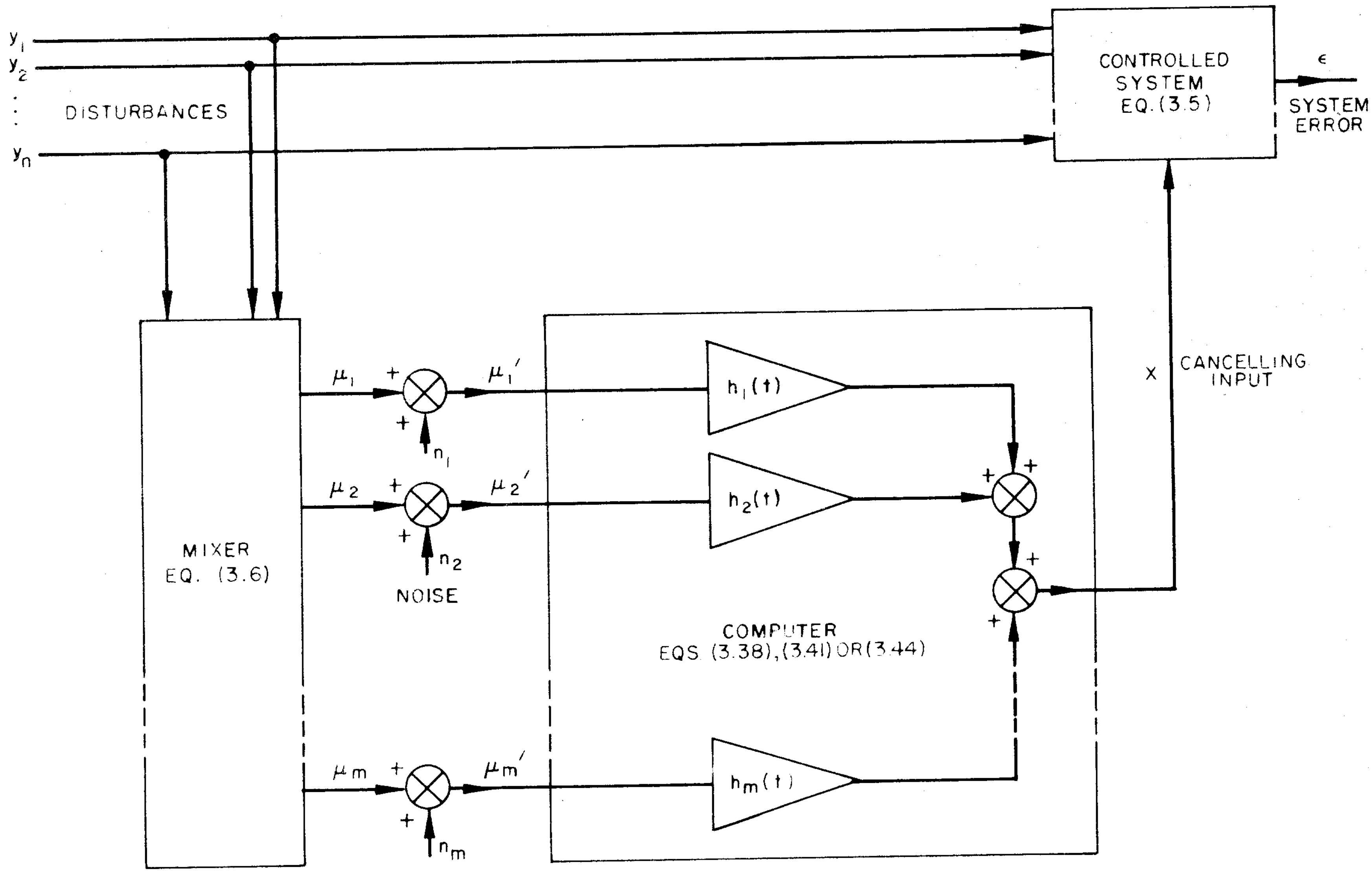
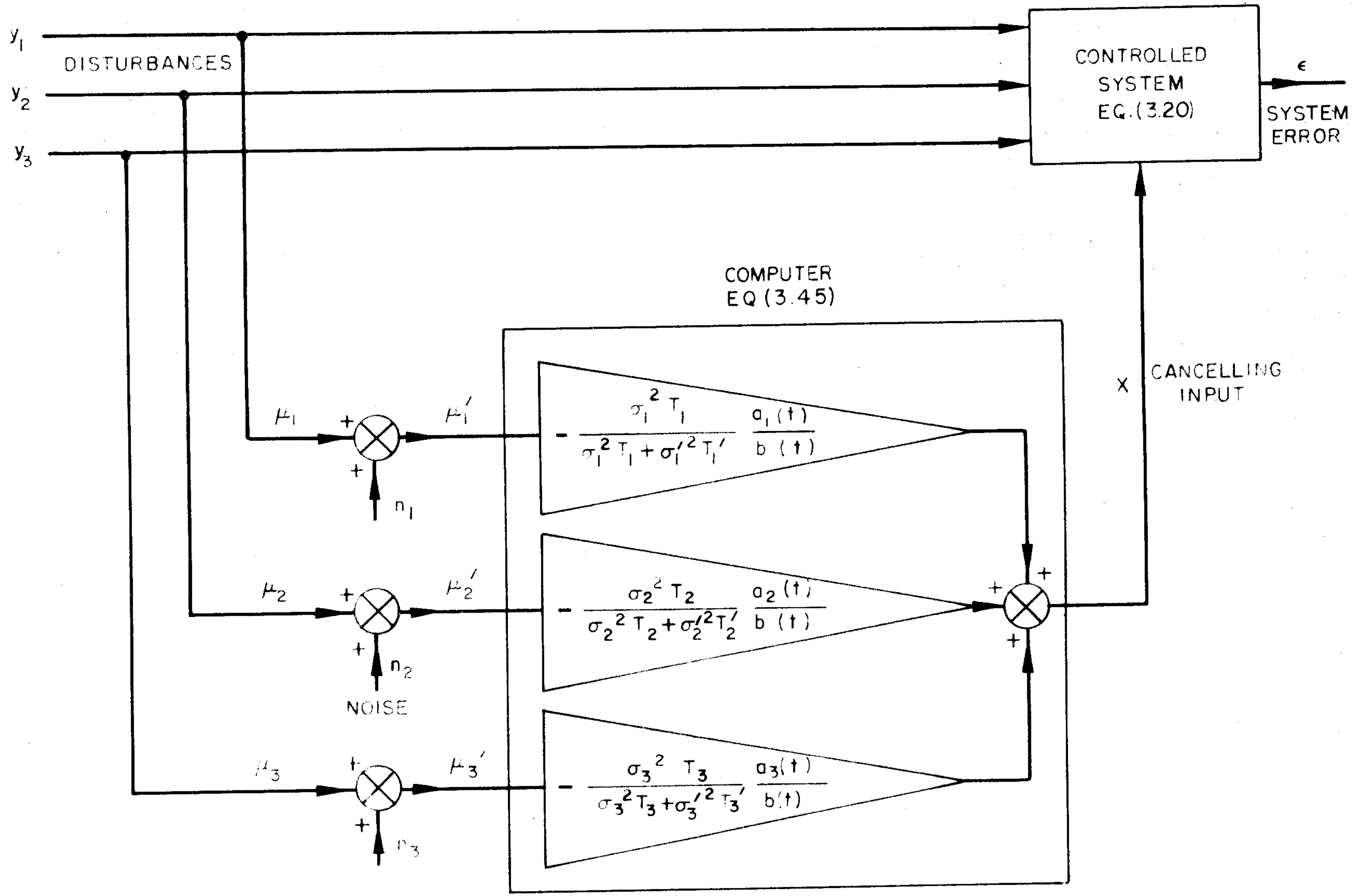

y1
y2
yn
DISTURBANCES
CONTROLLED SYSTEM EQ.(3.5)
ϵ
SYSTEM ERROR
X
CANCELLING INPUT
MIXER EQ. (3.6)
μ1
μ2
μm
μ1'
μ2'
μm'
n1
n2
nm
NOISE
h1(t)
h2(t)
hm(t)
COMPUTER EQS (3.38),(3.41) OR (3.44)

FIG 18



y_1
y_2
y_3
DISTURBANCES
CONTROLLED SYSTEM EQ.(3.20)
ϵ
SYSTEM ERROR
X
CANCELLING INPUT
COMPUTER EQ (3.45)
μ_1
μ_1'
n_1
μ_2
μ_2'
n_2
NOISE
μ_3
μ_3'
n_3
$-\frac{\sigma_1^2 T_1}{\sigma_1^2 T_1 + \sigma_1'^2 T_1'} \frac{a_1(t)}{b(t)}$
$-\frac{\sigma_2^2 T_2}{\sigma_2^2 T_2 + \sigma_2'^2 T_2'} \frac{a_2(t)}{b(t)}$
$-\frac{\sigma_3^2 T_3}{\sigma_3^2 T_3 + \sigma_3'^2 T_3'} \frac{a_3(t)}{b(t)}$

FIG. 19

RUN 1

Feedback Loop Open

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \sin \varphi \rvert$	φ degrees	φ_{calc} degrees	$\varphi-\varphi_{calc}$ degrees
.20	1.40	1.00	9.1/20 = .455	-27.1	-32.1	5.0
.22	1.36	.971	9.6/19.4 = .495	-29.7	-34.7	5.0
.24	1.32	.943	10.0/18.9 = .529	-31.9	-37.0	5.1
.26	1.28	.914	10.2/18.3 = .557	-33.8	-39.2	5.4
.28	1.24	.886	10.4/17.7 = .588	-36.0	-41.3	5.3
.30	1.20	.857	10.6/17.2 = .616	-38.0	-43.3	5.3
.32	1.16	.829	10.7/16.6 = .645	-40.2	-45.2	5.0
.34	1.12	.800	10.7/16.0 = .669	-42.0	-46.9	4.9
.36	1.09	.779	10.7/15.6 = .686	-43.3	-48.5	5.2
.38	1.06	.757	10.7/15.1 = .709	-45.2	-50.0	4.8
.40	1.02	.729	10.6/14.6 = .726	-46.6	-51.5	4.9

Average $\varphi-\varphi_{calc} = 5.1$

RUN 2

$\tau^* = .12$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \sin\varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.20	1.40	1.00	5.8/20 = .290	-16.9	-22.0	-20.7
.22						
.24						
.26						
.28						
.30						
.32						
.34						
.36						
.38						
.40						

$\varphi_{cor} - \varphi_{calc} = -1.3$

RUN 3

$\tau^* = .14$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \text{SIN } \varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.20	1.40	1.00	6.7/20 = .335	-19.6	-24.7	-23.7
.22						
.24						
.26						
.28						
.30						
.32						
.34						
.36						
.38						
.40						

$\varphi_{cor} - \varphi_{calc} = -1.0$

RUN 4

$\tau^* = .16$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \sin\varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.20	1.40	1.00	7.6/20 = .380	-22.3	-27.4	-26.7
.22						
.24						
.26						
.28						
.30						
.32						
.34						
.36						
.38						
.40						

$\varphi_{cor} - \varphi_{calc} = -.7$

RUN 5

$\tau^* = .18$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \text{SIN } \varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.20	1.40	1.00	8.5/20 = .425	-25.2	-30.3	-29.5
.22						
.24						
.26						
.28						
.30						
.32						
.34						
.36						
.38						
.40						

$\varphi_{cor} - \varphi_{calc} = -.8$

RUN 6

$\tau^* = .20$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \sin \varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.22	1.40	1.00	9.3/20 = .456	-27.7	-32.8	-32.1
.24						
.26						
.28						
.30						
.32						
.34						
.36						
.38						
.40						

$\varphi_{cor} - \varphi_{calc} = -.7$

RUN 7

τ^* = .22 Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \sin \varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.24	1.40	1.00	10.0/20 = .500	-30	-35.1	-34.7
.26						
.28						
.30						
.32						
.34						
.36						
.38						
.40						

$\varphi_{cor} - \varphi_{calc} = -.4$

RUN 8

$\tau^* = .24$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \text{SIN } \varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.26	1.40 - †	1.00	10.8/20 = .540	-32.7	-37.8	-37.0
.28						
.30						
.32	1.39	.996				
.34						
.36						
.38						
.40						

† 1.395 was used for calculation purposes

$\varphi_{cor} - \varphi_{calc} = -.8$

RUN 9

$\tau^* = .26$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \text{SIN } \varphi \rvert$	φ degrees	φ degrees	φ degrees
.28	1.40	1.00	11.2/20 = .560	-34.1	-39.2	-39.2
.30						
.32						
.34						
.36						
.38						
.40						

$\varphi_{cor} - \varphi_{calc} = .0$

RUN 10

$\tau^* = .28$ Milliseconds

τ milliseconds	OUTPUT volts a.c.	NORMALIZED OUTPUT	$\lvert \sin \varphi \rvert$	φ degrees	φ_{cor} degrees	φ_{calc} degrees
.30	1.39	1.00	12.3/20 = .615	-38.0	-43.1	-41.3
.32	|	|	|	|	|	|
.34	|	|	|	|	|	|
.36	1.39 - †	.996	|	|	|	|
.38	|	|	|	|	|	|
.40	|	|	|	|	|	|

† 1.385 was used for calculation purposes

$\varphi_{cor} - \varphi_{calc} = -1.8$