ANALYTIC FUNCTIONS IN GENERAL ANALYSIS

Thesis by

Angus E. Taylor

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

1936

Introduction

The theory of functions of a complex variable is distinguished from the theory of functions of a real variable by its simplicity - a simplicity which is directly traceable to the complexity of the variable. Two of the remarkable simplicities of the theory are, first, that from the assumption that $f(z)$ is differentiable throughout the neighborhood of a point $z = z_0$ follows the existence of all higher derivatives and the convergence of the Taylor's series for $f(z)$; and secondly, that we are able to classify in simple terms the possible singularities of an analytic function.

It is the purpose of this work to generalize, insofar as is possible, the basic theorems of the classical theory, and to investigate in what measure the simplicities mentioned above are preserved when the arguments and function values lie in a Banach space. Of the three principally recognized points of view which are used in developing the theory of analytic functions we have used mainly the one due to Cauchy, which finds its natural extension in the ideas of Gateaux concerning differentials. Much of the work which we present was sketched in a memoir of Gateaux on functionals of continuous functions.* In addition we have developed the 'Weierstrassian' properties of analytic functions, using as a foundation the notion of polynomial as set forth by R.S. Martin.** Finally, a brief section is devoted to a generalization of the Cauchy-Riemann equations. Nothing has been done with the implicitly suggested theory of pairs of conjugate harmonic functions, however.

---

* R. Gateaux, (3) and (4). The numbers in parenthesis refer to the bibliography at the end of this thesis.
**R.S. Martin, (7) p. 18-53.

The study of differentials leads to an important result showing the relation of the Fréchet and Gateaux concepts of a differential.

The classification of singular points is a most difficult problem. We have dealt completely with removable singularities, and showed to some extent the departures from classical theory which are caused by the generalization here undertaken. A more detailed investigation should be carried out in special cases.

I freely express my admiration for the treatise of Professor W.F. Osgood, Lehrbuch der Funktionentheorie, to which I have had constant recourse in the writing of this thesis. Many of the proofs are directly carried over, with only the slight changes made necessary by the abstract nature of the quantities in hand.

To Professor A.D. Michal I am indebted for encouragement and advice at all times.

TABLE OF CONTENTS

CHAPTER I

Vector Spaces

CHAPTER II

Analytic Functions of a Complex Variable

CHAPTER III

Analytic Functions in General Analysis

Chapter I

Vector Spaces

1. Let E be a non-empty class of elements $x, y, \ldots$ of unspecified nature; and let A denote either the real or complex number system. Suppose that in E there are given:

(i) a binary relation called equality, and denoted by $=$, such that given an ordered pair of elements x, y from E, then either x bears the relation to y ($x = y$) or it does not ($x \neq y$).

(ii) a binary operation, or rule of combination, such that when x, y are in E there is a uniquely determined third element of E, called their sum, and denoted by $x + y$.

(iii) an operation which orders to an element a from A and an element x from E a uniquely determined element of E, denoted by $a \cdot x$ (the product of x by a).

(iv) an operation which orders to each element x of E a unique real number denoted by $\|x\|$ (the norm of x).

If then the system composed of E, A, and the operations and relation has the properties set forth in the postulates below, we shall call E an abstract vector space.* According as A is the real or complex number system we shall call E a real or a complex space.

(v) 1° The relation $=$ is an equivalence relation in E. That is, it is reflexive, symmetric, and transitive.

---

* This postulate system is not independent; since our main interest here is with the properties of the vector space, we have postulated these properties in the most convenient form for our use.

2° If $x_1 \equiv x_2$ and $y_1 \equiv y_2$ then $x_1 + y_1 \equiv x_2 + y_2$, $a \cdot x_1 \equiv a \cdot x_2$, $\|x_1\| = \|x_2\|$.

3° $x + y \equiv y + x$, $x + (y + z) \equiv (x + y) + z$

4° $1 \cdot x \equiv x$

$$a \cdot (x + y) \equiv a \cdot x + a \cdot y$$

$$(a + b) \cdot x \equiv a \cdot x + b \cdot x$$

$$(ab) \cdot x \equiv a \cdot (b \cdot x)$$

5° If $\|x\| = 0$ then $y + x \equiv y$ (all y)

$$\|x + y\| \leq \|x\| + \|y\|$$

$$\|a \cdot x\| = |a| \, \|x\|$$

With these postulates we can prove that $\|x\| \geqslant 0$, and that when E is divided into residue classes according to the equivalence relation $\equiv$ there is a unique class of equal elements the properties of whose representative z are

$$z + x \equiv x$$

$$z \equiv 0 \cdot x \qquad \text{(for all x)}$$

We denote for convenience all these representatives by 0 . The simple algebraic rules for manipulation of the elements of E follow readily, and will not be dwelt on in detail here. We remark that $\equiv$ does not at all necessarily indicate logical identity, and that, in fact, it usually denotes something other than identity in the specific instances which are of greatest interest.

We shall have to deal with spaces which satisfy a further postulate:

6° If $\{x_n\}$ is an infinite sequence of elements of E such that* $\lim_{n,m \to \infty} \|x_n - x_m\| = 0$, then there exists an element x in E such that

* We write x − y for x + −1·y . It is readily proved that if we let both x and y satisfy the conditions in 6°, then $x \equiv y$, so that the limit element is unique to within equal elements.

$\lim_{n\to\infty} \|x_n - x\| = 0$.

If E satisfies 6° it is said to be complete, and we call it a Banach space.

Let us suppose that E(R) is a real vector space. From it we construct a complex space E(C) in the following manner. E(C) shall be the class of element pairs $\{x,y\}$ where x, y are in E(R). We define

$$\{x_1, y_1\} = \{x_2, y_2\} \quad \text{if and only if} \quad x_1 = x_2 \ , \ y_1 = y_2 \ .$$

$$\{x_1, y_1\} + \{x_2, y_2\} = \{x_1 + x_2 , y_1 + y_2\}$$

$$(a + ib)\cdot\{x, y\} = \{a\cdot x - b\cdot y , b\cdot x + a\cdot y\}$$

$$\|\{x, y\}\| = \sqrt{\|x\|^2 + \|y\|^2}$$

It is easily seen that E(C), thus defined, is a complex vector space, with 'zero' element $\{0,0\}$. Since $\{x,y\} = \{x,0\} + i\cdot\{y,0\}$, and there is a one to one isomorphism between the space E(R) and the subclass of elements of E(C) of the form $\{x,0\}$, we may for convenience write, as in the case of ordinary complex numbers,

$$z = \{x,y\} = x + i\cdot y$$

Definition If E(R) is a real vector space, the complex vector space defined above will be called the couple-space associated with E(R).

It is almost at once obvious that E(C) will be complete if and only if E(R) is complete also, and that a variable quantity in E(C) will approach a limit if and only if its real and imaginary parts do likewise.

2. The norm $\|x\|$ furnishes us with a metric, the distance between two elements of E being defined as $\|x - y\|$. We can then introduce the basic concepts of topological and metric spaces. We shall not repeat the well known definitions of open and closed sets, limit points, continuity, and the like, but shall confine ourselves to the adoption of a standard terminology as regards point sets, especially curves and regions.

<u>Jordan Curve</u>: A set of points in E defined by a function of a real variable

$$x = \varphi(t) \qquad t_0 \leq t \leq t_1$$

where $\varphi(t)$ is continuous, and such that $\varphi(t) = \varphi(t')$ only if $t = t'$.

<u>Connected Point Set</u>: A set S of points of E such that if x_0, x_1 are any two points in S, there exists a Jordan curve, lying entirely in S, with end points x_0 and x_1.

<u>Domain</u>: An open, connected point set in E.

<u>Region</u>: A domain plus some, all, or none of its boundary points.

<u>Neighborhood of a Point</u>: A domain containing the point.

<u>Sphere</u> (open or closed): A set of points defined by $\|x - x_0\| < r$, or $\|x - x_0\| \leq r$; x_0 is called the center of the sphere, and r its radius. The set $\|x - x_0\| = r$ is called the surface of the sphere.

A concept of considerable importance for our purposes is that of compactness.

<u>Definition</u>: A set S in E is said to be compact if every infinite set in S gives rise to at least one limit point (not necessarily in S).

The axis of real numbers is a complete real vector space; its associated couple-space is the complex plane. In either of these spaces every bounded set is compact (the Bolzano-Weierstrass property). We shall need the following more specialized definitions concerning point sets in the plane.

<u>Normal Jordan Curve</u>: A Jordan curve in the plane is said to be normal if the interval $t_0 \leq t \leq t_1$ of the parameter can be broken up into a finite number of closed subintervals in each of which the curve has an equation of the form

$$y = f(x) \qquad \text{or} \qquad x = \varphi(y)$$

where f or φ denotes a continuous, one-valued function of its argument.

Regular Arc: A point set in the plane defined by

$$x = \varphi(t) \qquad y = \psi(t)$$

where φ and ψ are continuous, $t_0 \leq t \leq t_1$, with continuous first derivatives such that $\varphi'(t)^2 + \psi'(t)^2 > 0$, and where the two equations

$$\varphi(t) = \varphi(t') \quad , \quad \psi(t) = \psi(t')$$

admit but the one solution $t = t'$.

Regular Curve: A point set consisting of a finite number of regular arcs joined continuously end to end. (A regular curve may cut itself an infinite number of times; a simple regular curve, i.e. one which does not cut itself, is always a normal Jordan curve *).

Regular Region S: A closed region of the plane whose boundary consists of a finite number of regular curves, each of which cuts itself, or another curve of the boundary, in at most a finite number of points.

3. In this paragraph we shall develop briefly some of the fundamental notions and theorems on which our later work rests. These theorems are extensions of familiar results in the classical theory of functions; instead of numerically-valued functions of a numerical variable we have now to consider functions whose arguments range over a vector space E and whose values range over a second space E′. We describe this situation by saying that the function is on E to E′. By a function we always mean a one-valued function.

Theorem 1 Let $f(x)$ be defined and continuous in a closed set H of E, with values in E′. Then $f(x)$ is bounded and uniformly continuous in every compact set C extracted from H.**

---

* W.F. Osgood, (10) p.159.

** R. Gateaux, (4) p.2.

Proof: Suppose the theorem false. Then if G is a compact set contained in H, and $M > 0$ is arbitrary, there is an x in G such that $\|f(x)\| > M$. Taking $M = 1, 2, 3, \ldots$ successively we obtain an infinite sequence $\{x_n\}$ in G for which $\|f(x_n)\| \to \infty$. But since G is compact, and H is closed, $\{x_n\}$ must have at least one limit point x_0 in H, and we may assume that a subsequence $\{x_\nu\}$ has been chosen so that $x_\nu \to x_0$. Then

$$\|f(x_\nu)\| \leq \|f(x_0)\| + \|f(x_0) - f(x_\nu)\|$$

and as $\nu \to \infty$ the right hand member tends to $\|f(x_0)\|$, whereas the left member tends to infinity. This contradiction proves the assertion that f(x) remains bounded in G.

Similarly if we suppose that f(x) is not uniformly continuous in G then for some $\varepsilon > 0$, any $\delta > 0$, and some x, x' in G, we must have simultaneously the two inequalities

$$\|f(x) - f(x')\| > \varepsilon \qquad \|x - x'\| < \delta$$

Choose $\delta_n = \frac{1}{n}$, and denote the corresponding x, x' by x_n, x_n'. Then, by virtue of compactness we may assume the sequence $\{x_n\}$ so chosen that x_n converges to a point x_0 in H. Since

$$\|x_n' - x_0\| \leq \|x_n' - x_n\| + \|x_n - x_0\|$$

we conclude that $x_n' \to x_0$ also, and hence, in view of the inequality

$$\|f(x_n) - f(x_n')\| \leq \|f(x_n) - f(x_0)\| + \|f(x_0) - f(x_n')\|$$

and the continuity of f(x), that

$$\lim_{n \to \infty} \|f(x_n) - f(x_n')\| = 0$$

This involves a contradiction, and so proves the theorem.

<u>Theorem 2</u> Let $\{f_n(x)\}$ be a sequence of functions defined and continuous in a domain D of E, with values in E'. Let $f(x)$ on D to E be a function such that $\lim_{n \to \infty} f_n(x) = f(x)$, the convergence uniform in every compact subset of D. Then f(x) is continuous in D.

Proof: Suppose x_0 is a point of D at which $f(x)$ is discontinuous. Then for some $\varepsilon > 0$ we can find a sequence $\{x_\nu\}$ lying in a sphere (in D) about x_0, and a sequence of positive numbers $\{\delta_\nu\}$, tending to zero, such that for all ν

$$\| f(x_\nu) - f(x_0) \| > \varepsilon \qquad \text{and} \qquad \| x_\nu - x_0 \| < \delta_\nu .$$

The set composed of $\{x_n\}$ and x_0 is compact, and the sequence $\{f_n(x)\}$ therefore converges uniformly in this set. That is to say, we can determine N, independent of ν, so that

$$\| f_n(x_\nu) - f(x_\nu) \| < \frac{\varepsilon}{3} \qquad n \geq N, \quad \nu = 0,1,2,\ldots$$

But

$$\begin{aligned} \| f(x_\nu) - f(x_0) \| &\leq \| f(x_\nu) - f_n(x_\nu) \| + \| f_n(x_\nu) - f_n(x_0) \| + \| f_n(x_0) - f(x_0) \| \\ &\leq \tfrac{2}{3}\varepsilon + \| f_n(x_\nu) - f_n(x_0) \| \qquad n \geq N, \end{aligned}$$

and, since $f_n(x)$ is continuous, we can choose ν so large that the right hand member is less then ε, thus obtaining a contradiction.

<u>Theorem 3</u> Let $f(x,y)$ be a function with values in a space E_3, defined for x in a domain D of a space E_1, and y in a closed set F of a space E_2. Then if x_0 is in D, and G is a compact set in F, $f(x,y)$ is continuous at x_0, uniformly with respect to y in G.

Proof: Suppose the theorem false. Then there will exist a number $\varepsilon > 0$, elements $\{x_n\}$ in D, and elements $\{y_n\}$ in G such that the inequalities

$$\| x_n - x_0 \| < \frac{1}{n} \qquad \| f(x_n, y_n) - f(x_0, y_n) \| > \varepsilon$$

are valid for values of n = 1,2,,... Since G is compact, and contained in the closed set F, we may suppose that the points y_n converge to a point y_0 in F. Then the inequality

$$\begin{aligned} 0 < \varepsilon < \| f(x_n, y_n) - f(x_0, y_n) \| \leq \; & \| f(x_n, y_n) - f(x_0, y_0) \| \\ & + \| f(x_0, y_0) - f(x_0, y_n) \| \end{aligned}$$

is valid for n = 1,2,3,... But by the continuity of $f(x,y)$ the right member tends to zero with $\frac{1}{n}$, and we are led to a contradiction. Hence the theorem is true.

Definition: Let f(x) be a function on E to E$'$, defined in the neighborhood of a point x_0. If for each y in E the difference quotient

$$\frac{f(x_0+\tau y)-f(x_0)}{\tau}$$

regarded as a function of the number τ alone, approaches a limit as τ tends to zero, this limit, which is a function of y, is called the Gateaux differential of f(x) at x_0.* We use the notation

$$\delta f(x_0;y) = \lim_{\tau\to 0}\frac{f(x_0+\tau y)-f(x_0)}{\tau} = \left[\frac{d f(x_0+\tau y)}{d\tau}\right]_{\tau=0}$$

Evidently

$$\delta f(x_0+\tau y;y) = \frac{d}{d\tau} f(x_0+\tau y)$$

Definition: The function f(x) (defined as above) is said to have a Fréchet differential at x_0 if there exists a function $\psi(y)$ with the following properties:

1° $\psi(y)$ is linear - that is, defined and continuous throughout E, and

$$\psi(\alpha x+\beta y) = \alpha\psi(x)+\beta\psi(y) .$$

2° given $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $\|y\| < \delta$ implies

$$\|f(x_0+y)-f(x_0)-\psi(y)\| \leq \varepsilon\|y\|$$

For the Fréchet differential of f(x) at x_0 we use the notation $df(x_0;y)$; $d^2f(x_0;y_1,y_2)$ denotes the differential, with increment y_2, of $df(x;y_1)$ regarded as a function of x. It frequently occurs that in dealing with higher order differentials all the increments are set equal. When this is the case we abbreviate by writing $d^n f(x;y)$ instead of $d^n f(x;y,\ldots,y)$. Similar conventions of notation will be used for the Gateaux differentials.

---

* Of course the limit must be independent of the manner in which τ goes to zero; τ is restricted to real values if E is a real space, but may be complex if the space is complex.

Theorem 4 If $f(x)$, on E to E' is defined in the neighborhood of x_0, and has a Fréchet differential at x_0, it is continuous at x_0.

Proof: Let $\varepsilon > 0$ be given. Then when $\|y\|$ is sufficiently small, $f(x_0+y)$ is defined, and

$$\|f(x_0+y) - f(x_0)\| \leq \|f(x_0+y) - f(x_0) - df(x_0;y)\| + \|df(x_0;y)\|.$$

Since $df(x_0;y)$ is linear, it is continuous at $y=0$, and vanishes there. Hence we can choose δ, $0 < \delta < \frac{1}{2}$ so that $\|y\| < \delta$ implies $\|df(x_0;y)\| < \frac{\varepsilon}{2}$ and at the same time $\|f(x_0+y) - f(x_0) - df(x_0;y)\| \leq \varepsilon \|y\|$.

Then $\|f(x_0+y) - f(x_0)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ when $\|y\| < \delta$.

Theorem 5 If $\varphi(\alpha)$ is a function of a numerical variable, with values in E, such that $\varphi(\alpha)$ has a derivative at α_0, where $x_0 = \varphi(\alpha_0)$, and if $f(x)$ on E to E' admits a Fréchet differential at x_0, then $f(\varphi(\alpha))$ regarded as a function of α has a derivative at α_0, and

$$\left\{\frac{\partial f(\varphi(\alpha))}{\partial \alpha}\right\}_{\alpha=\alpha_0} = df(x_0;\varphi'(\alpha_0))$$

Proof: By hypothesis $\varphi(\alpha)$ is defined in a neighborhood of $\alpha = \alpha_0$, and, being differentiable, is continuous at α_0. Similarly $f(x)$ is continuous at x_0; we see that $f(\varphi(\alpha))$ is defined in the neighborhood of $\alpha = \alpha_0$. We must prove that the expression

$$\left\|\frac{f(\varphi(\alpha_0+\Delta\alpha)) - f(\varphi(\alpha_0))}{\Delta\alpha} - df(\varphi(\alpha_0); \varphi'(\alpha_0))\right\|$$

tends to zero with $\Delta\alpha$. Let

$$\Delta_\alpha f = f(\varphi(\alpha_0+\Delta\alpha)) - f(\varphi(\alpha_0))$$

$$\Delta_\alpha \varphi = \varphi(\alpha_0+\Delta\alpha) - \varphi(\alpha_0)$$

Then

$$\left\|\frac{\Delta_\alpha f}{\Delta\alpha} - df(x_0;\varphi'(\alpha_0))\right\| \leq \left\|\frac{\Delta_\alpha f}{\Delta\alpha} - df\left(x_0; \frac{\Delta_\alpha\varphi}{\Delta\alpha}\right)\right\| + \left\|df\left(x_0; \frac{\Delta_\alpha\varphi}{\Delta\alpha}\right) - df(x_0;\varphi'(\alpha_0))\right\|$$

But $\lim_{\Delta\alpha\to 0} \frac{\Delta_\alpha\varphi}{\Delta\alpha} = \varphi'(\alpha_0)$, and

$$\lim_{\Delta\alpha\to 0}\left\|\frac{\Delta_\alpha f}{\Delta\alpha} - df\left(x_0;\frac{\Delta_\alpha\varphi}{\Delta\alpha}\right)\right\| = \lim_{\Delta\alpha\to 0}\frac{\|\Delta_\alpha f - df(x_0;\Delta_\alpha\varphi)\|}{|\Delta\alpha|} = 0$$

by the property $2°$ of the Fréchet differential. The second expression on the right of the inequality also tends to zero with $\Delta\alpha$, since the Fréchet differential is continuous in the increment. This completes the proof.

Theorem 6 If $f(x)$ admits a Fréchet differential at x_0, it admits a Gateaux differential there, and the two are equal.

Proof: Define a function $g(y)$ on E to E' by the equation

$$df(x_0;y) = f(x_0+y) - f(x_0) + g(y)$$

Then

$$\lim_{y\to 0}\frac{\|g(y)\|}{\|y\|} = 0$$

Select now an arbitrary element $y \neq 0$ in E, and hold it fast. Since $df(x_0;y)$ is homogeneous of the first degree,

$$df(x_0;y) = \frac{f(x_0+\tau y)-f(x_0)}{\tau} + \frac{g(\tau y)}{\tau\|y\|}\cdot\|y\| .$$

From this we conclude that

$$\lim_{\tau\to 0}\frac{f(x_0+\tau y)-f(x_0)}{\tau} = df(x_0;y)$$

If $y = 0$ the last equation is obviously true. Hence we have proved the theorem.

Corollary If a function $f(x)$ admits a Fréchet differential at a point, this differential is unique.

If E is the space A of real or complex numbers, the Gateaux and Fréchet differentials reduce to the usual form, namely the product of the derivative by the increment, a function obviously linear in the increment.

Definition: By a polynomial $p(\alpha)$ on A to E we mean a function defined by an expression of the form

$$p(\alpha) \equiv a_0 + \alpha a_1 + \alpha^2 a_2 + \cdots + \alpha^n a_n$$

where $a_0, \ldots, a_n$ are constant elements of E. If $a_n \neq 0$ $p(\alpha)$ is said to be of degree n.

Definition: A function $f(x)$ on E to E$'$ is called a polynomial if

1° it is defined and continuous for each x in E,

2° there exists an integer n such that for every x,y in E $f(x + \alpha \cdot y)$ is a polynomial in α of degree $\leq n$.

The least integer n satisfying this condition is called the degree of $f(x)$.

The theory of polynomials springing from these two definitions is in itself of considerable interest and importance.* We shall be content to note the main results to which we shall have to refer later.

A homogeneous polynomial is defined as a polynomial $h(x)$ such that $h(\alpha x) = \alpha^n h(x)$ for all α and x; its degree is then precisely n.

Theorem 7 If $h(x)$ is a homogeneous polynomial of degree n on E to E$'$ there exists a unique 'multilinear' function $h'(x_1, \ldots, x_n)$ of n variables over E, to E$'$, with the properties

1° $h'(x_1, \ldots, x_n)$ is linear in each argument, and completely symmetric in the x's.

2° $h'(x, \ldots, x) \equiv h(x)$.

This function is called the polar of $h(x)$; it is expressible in terms of the n^{th} difference of $h(x)$:

$$h'(\Delta_1 x, \ldots \Delta_n x) \equiv \frac{1}{n!}\Delta^{(n)} h(0) \equiv \frac{1}{n!}\Delta^{(n)} h(x)$$

where

$$\Delta_1 h(x) = h(x + \Delta_1 x) - h(x)$$

$$\Delta^{(2)} h(x) = \Delta_2(\Delta_1 h(x)) = h(x + \Delta_1 x + \Delta_2 x) - h(x + \Delta_2 x) - h(x + \Delta_1 x) + h(x)$$

etc., the n^{th} difference being independent of x.

---

* See R.S. Martin, (7). For proofs of Theorems 7-10 see esp. pp.30-50.

<u>Theorem 8</u> A polynomial of degree n on E to E$'$ is uniquely representable as the sum of homogeneous polynomials of degree $\leq$ n.

<u>Theorem 9</u> If $h(x)$ is a homogeneous polynomial of degree n on E to E$'$, and if $h'(x_1,\ldots,x_n)$ is its polar, then there exist numbers M, M_p such that

$$\|h(x)\| \leq M\|x\|^n$$

$$\|h'(x_1,\ldots,x_n)\| \leq M_p\|x_1\|\cdots\|x_n\|$$

The least such numbers are called the moduli of $h(x)$ and its polar, respectively. Concerning these moduli we know:

$$M = \underset{\|x\|=1}{\text{l.u.b.}}\ \|h(x)\|$$

$$1 \leq M_p/M \leq \frac{n^n}{n!}$$

<u>Theorem 10</u> If $h_n(x)$ is a homogeneous polynomial of degree n it has a Fréchet differential $dh_n(x;y) = nh_n'(x,\ldots,x,y)$, where $h_n'(x_1,\ldots,x_n)$ is the polar of $h_n(x)$.

<u>Theorem 11</u> Let B be an aggregate of objects such that to each b in B is ordered a sequence of elements $\{x_n(b)\}$ of a vector space E. Let the sequence $\{x_n(b)\}$ converge to a limit $x(b)$, <u>uniformly</u> in b. If then $l(x)$ is a linear operation on E to E$'$, the sequence $\{l(x_n(b))\}$ converges uniformly to $l(x(b))$.

Proof: By the uniformity we can choose an integer N, independent of b, such that $n \geq N$ implies $\|x_n(b) - x(b)\| < \varepsilon/M$, where $\varepsilon > 0$ is given arbitrarily, in advance, and M is the modulus of $l(x)$. Then

$$\|l(x_n(b)) - l(x(b))\| \leq M\|x_n(b) - x(b)\| \leq \varepsilon$$

when $n \geq N$. This proves the theorem.

Chapter II

Analytic Functions of a Complex Variable

1. Our purpose in this chapter is to develop the theory of functions of a complex variable, when the function values lie in a complete complex vector space. This will furnish the groundwork for a more thoroughgoing generalization of the theory of analytic functions in the next chapter. Much of the work is a direct carry-over from classical analysis, and for that reason we have omitted proofs when there seemed to be no special warrant for the repetition of well known arguments.

Let $f(\alpha)$ be a function defined in a domain T of the complex plane, with values in a complex Banach space E. Then $f(\alpha)$ is said to be analytic in T if it has a derivative at each point of T; $f(\alpha)$ is said to be analytic at a point α_0 if it is analytic in some neighborhood of α_0.

For functions of several complex variables the definition is similar. Let $\alpha_1, \ldots \alpha_n$ $(\alpha_k = \alpha_k' + i\alpha_k'')$ be complex variables whose real and imaginary parts $(\alpha_1', \alpha_1'', \ldots \alpha_n', \alpha_n'')$ are coordinates of a point in a 2n-dimensional space; then a function $f(\alpha_1, \ldots, \alpha_n)$ with values in E is said to be analytic in a domain of this space if it admits first partial derivatives with respect to each α_k at each point of the domain.

2. The line integral along a curve in the plane of a function $f(\alpha)$ on the complex plane to the space E may be defined in a variety of essentially equivalent ways. We shall consider only continuous functions $f(\alpha)$, and shall base our definition on that of G.N. Watson.*

<u>Definition of the Integral:</u> Let C be a rectifiable Jordan curve in the α-plane, defined by $\alpha = \alpha(t)$ $t_0 \leq t \leq T$

* G.N. Watson, (12) pp.17-30.

and let $f(\alpha)$ be defined and continuous on C, with values in E. Let there be given an infinite sequence $\{\alpha_n\}$ of points on C, and denote by $\alpha_1^{(n)},\dots,\alpha_n^{(n)}$ the first n of these points when arranged in their proper order on C (according to increasing t). We impose the condition that all points of the sequence be distinct, and that given $\delta > 0$ it be possible to choose n_0 such that $n \geq n_0$ implies $0 < t_{r+1}^{(n)} - t_r^{(n)} \leq \delta$, $r = 0,1,\dots,n$, where $\alpha_r^{(n)} = \alpha(t_r^{(n)})$, $t_0^{(n)} = t_0$, $t_{n+1}^{(n)} = T$. By the integral of $f(\alpha)$ over C we mean the following limit:*

$$\int_C f(\alpha)\,d\alpha = \lim_{n\to\infty} \sum_{r=0}^{n} \left[\left(\alpha_{r+1}^{(n)} - \alpha_r^{(n)}\right) f\left(\alpha_r^{(n)}\right)\right]$$

In order to show that this definition is satisfactory we shall prove the following theorems.

Theorem A The integral of a continuous function $f(\alpha)$ exists.

Theorem B Corresponding to an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that when $t_1,\dots,t_m$ are any m values of the parameter t for which $0 \leq t_{r+1} - t_r \leq \delta$ ($r = 0,1,\dots,m$ and $t_{m+1} = T$), while τ_r is such that $t_r \leq \tau_r \leq t_{r+1}$, then

$$\left\| \int_C f(\alpha)\,d\alpha - \sum_{r=0}^{m} (\alpha_{r+1} - \alpha_r) f(\beta_r) \right\| < \varepsilon$$

where $\alpha_r = \alpha(t_r)$, $\beta_r = \alpha(\tau_r)$.

Theorem C The value of the integral $\int_C f(\alpha)\,d\alpha$ is independent of the choice of the sequence $\{\alpha_n\}$, provided that the conditions of the definition are fulfilled.

Proof of A: Since $f(\alpha)$ is continuous on C, and C is defined by the continuous function $\alpha(t)$, $f(\alpha(t))$ is a continuous, and therefore uniform-

---

* We are integrating over C in the positive sense, i.e. in the direction of increasing t. The parameter can always be chosen so as to make a given direction along C positive. Clearly a reversal of the direction of integration merely changes the sign of the integral.

ly continuous,* function of t in the interval $t_0 \leq t \leq T$. Accordingly, given $\varepsilon > 0$ we can choose $\delta_0 > 0$ independent of t so that $|t - t'| < \delta_0$ implies $\|f(\alpha(t)) - f(\alpha(t'))\| < \frac{\varepsilon}{L}$, where

$$L = \text{l.u.b.} \sum_{k=0}^{n} |\alpha(t_{k+1}) - \alpha(t_k)|$$

for all sets of points $t_0 < t_1 < \cdots\cdots < t_{n+1} = T$. (L is finite, since C is rectifiable). Next choose n so large that for our sequence $\{\alpha_n\}$, $0 < t_{k+1}^{(n)} - t_k^{(n)} \leq \delta_0$, $k = 0, 1, \ldots, n$.

Consider now the quantity

$$S_n = \sum_{k=0}^{n} \left[\left(\alpha_{k+1}^{(n)} - \alpha_k^{(n)}\right) f\left(\alpha_k^{(n)}\right) \right]$$

and let m be any integer greater than n. If we designate by $\alpha_k^{(n)} = \alpha_{1,k}, \alpha_{2,k}, \ldots \alpha_{m_k,k}, \alpha_{m_k+1,k} = \alpha_{k+1}^{(n)}$ those points (in order) of the set $\alpha_1^{(m)}, \ldots\ldots, \alpha_m^{(m)}$ which lie between $\alpha_k^{(n)}$ and $\alpha_{k+1}^{(n)}$, we have

$$\sum_{s=1}^{m_k} (\alpha_{s+1,k} - \alpha_{s,k}) = \alpha_{k+1}^{(n)} - \alpha_k^{(n)},$$

so that

$$S_n = \sum_{k=0}^{n} \left[\left\{ \sum_{s=1}^{m_k} (\alpha_{s+1,k} - \alpha_{s,k}) \right\} f\left(\alpha_k^{(n)}\right) \right]$$

Also

$$S_m = \sum_{k=0}^{n} \left[\sum_{s=1}^{m_k} \left\{ (\alpha_{s+1,k} - \alpha_{s,k}) f(\alpha_{s,k}) \right\} \right]$$

as follows at once from the definition of S_m and the points $\alpha_{s,k}$.

Therefore

$$\|S_n - S_m\| \leq \sum_{k=0}^{n} \sum_{s=1}^{m_k} |\alpha_{s+1,k} - \alpha_{s,k}| \, \|f(\alpha_k^{(n)}) - f(\alpha_{s,k})\|.$$

But the parameters $t_{s,k}$ of the points $\alpha_{s,k}$ satisfy the conditions $t_k^{(n)} \leq t_{s,k} \leq t_{k+1}^{(n)}$, whence $0 \leq t_{s,k} - t_k^{(n)} \leq \delta_0$, and $\|f(\alpha_k^{(n)}) - f(\alpha_{s,k})\| < \frac{\varepsilon}{L}$. Also

---

* cf. I, §3, Theorem 1.

$$\sum_{r=0}^{n}\sum_{s=1}^{m_r} |\alpha_{s+1,r}-\alpha_{s,r}| = \sum_{r=0}^{m} |\alpha_{r+1}^{(m)}-\alpha_r^{(m)}| \leq L$$

Thus finally $\|S_n - S_m\| < \varepsilon$ when m > n. It then follows by the completeness of the space E (I, §1, postulate 6) that there exists a unique limit S of the sequence $\{S_n\}$. This limit is, by definition, the integral.

Proof of B: Let $\varepsilon > 0$ be given, and choose δ so that $|t-t'| < \delta$ implies $\|f(\alpha(t')) - f(\alpha(t))\| < \frac{\varepsilon}{3L}$; then choose n so that $0 < t_{r+1}^{(n)} - t_r^{(n)} \leq \delta \quad r = 0, 1, 2, \dots n.$ We shall prove that this δ satisfies the demands of the theorem. To do this, assume numbers $t_1, \dots, t_m$ as specified, and denote by $t_{1,r}, \dots, t_{N_r,r}$ those among them (if such exist) which satisfy the inequality $t_r^{(n)} \leq t \leq t_{r+1}^{(n)}$. Denote also by $t_{0,r}$ that member of the set $t_0, \dots, t_{m+1}$ which immediately precedes $t_{1,r}$, and by $t_{N_r+1,r}$ that one which immediately follows $t_{N_r,r}$. If for some r no number t_p satisfies the inequality, define $t_{0,r}$ and $t_{1,r}$ as respectively the members of the set $t_0, \dots, t_{m+1}$ immediately preceding $t_r^{(n)}$, and equal to, or immediately following $t_{r+1}^{(n)}$. The numbers τ_r may now be written $\tau_{s,r}$, where $t_{s,r} \leq \tau_{s,r} \leq t_{s+1,r} \quad s = 0, \dots, N_r$, and correspondingly $\beta_{s,r} = \alpha(\tau_{s,r})$. Then

$$\sum_{p=0}^{m}\left[(\alpha_{p+1}-\alpha_p)f(\beta_p)\right] = \sum_{r=0}^{n}\Big[(\alpha_{1,r}-\alpha_r^{(n)})f(\beta_{0,r}) + (\alpha_{2,r}-\alpha_{1,r})f(\beta_{1,r}) + \cdots + (\alpha_{r+1}^{(n)}-\alpha_{N_r,r})f(\beta_{N_r,r})\Big]$$

and

$$\sum_{p=0}^{m}\left[(\alpha_{p+1}-\alpha_p)f(\beta_p)\right] - S_n = \sum_{r=0}^{n}\Big[(\alpha_{1,r}-\alpha_r^{(n)})\{f(\beta_{0,r})-f(\alpha_r^{(n)})\} + (\alpha_{2,r}-\alpha_{1,r})\{f(\beta_{1,r})-f(\alpha_r^{(n)})\} + \cdots + (\alpha_{r+1}^{(n)}-\alpha_{N_r,r})\{f(\beta_{N_r,r})-f(\alpha_r^{(n)})\}\Big]$$

Now
$$\tau_{s,n} \leq t_{s,n} + \delta < t_{n+1}^{(n)} + \delta < t_n^{(n)} + 2\delta$$
$$\tau_{s,n} \geq t_{s+1,n} - \delta \geq t_n^{(n)} - \delta$$
so that $|\tau_{s,n} - t_n^{(n)}| < 2\delta$, and hence, for $t' = \frac{1}{2}(\tau_{s,n} + t_n^{(n)})$, $|\tau_{s,n} - t'| < \delta$, $|t' - t_n^{(n)}| < \delta$. This leads to the inequality
$$\|f(\beta_{s,n}) - f(\alpha_n^{(n)})\| \leq \|f(\beta_{s,n}) - f(\alpha(t'))\| + \|f(\alpha(t')) - f(\alpha_n^{(n)})\|$$
$$< \frac{2\varepsilon}{3L}$$
in virtue of the uniform continuity of f. Thus
$$\Big\|\sum_{p=0}^{m} (\alpha_{p+1} - \alpha_p) f(\beta_p) - S_n\Big\| \leq \frac{2\varepsilon}{3L}\sum_{n=0}^{n}\Big[|\alpha_{1,n} - \alpha_n^{(n)}| + |\alpha_{2,n} - \alpha_{1,n}| +$$
$$+ \cdots\cdots + |\alpha_{n+1}^{(n)} - \alpha_{N_n,n}|\Big]$$
$$\leq \frac{2\varepsilon}{3}$$
by the rectifiability of C.

Now, as in the proof of A, $\|S_\nu - S_n\| < \frac{\varepsilon}{3}$ when $\nu > n$, so that also $\|\int_C f(\alpha)d\alpha - S_n\| < \frac{\varepsilon}{3}$. Thus finally, by the triangular inequality
$$\Big\|\int_C f(\alpha)d\alpha - \sum_{p=0}^{m}(\alpha_{p+1} - \alpha_p) f(\beta_p)\Big\| < \varepsilon ,$$
as was to be established.

Proof of C: This is an evident consequence of B, for if we are given a second sequence $\{\bar{\alpha}_n\}$ satisfying the conditions laid down in the definition of the integral, we may take $\bar{t}_0^{(m)}, \ldots\ldots, \bar{t}_{m+1}^{(m)}$ as the numbers $t_0, \ldots\ldots, t_{m+1}$ of B, and thus prove at once that
$$\lim_{m\to\infty} \sum_{n=0}^{m}\Big[(\bar{\alpha}_{n+1}^{(m)} - \bar{\alpha}_n^{(m)}) f(\bar{\alpha}_n^{(m)})\Big] = \int_C f(\alpha)d\alpha .$$

For dealing with inequalities we find it useful to define the expression

$$\int_C \|f(\alpha)\,d\alpha\| = \lim_{n\to\infty} \sum_{k=0}^{n} |\alpha_{k+1}^{(n)} - \alpha_k^{(n)}|\, \|f(\alpha_k^{(n)})\|$$

If we envisage C as mapped, by a distortion without stretching, on the real interval $0 \leq s \leq L$, where s is arc-length along C from α_0, and L is the length of C, then $\int_C \|f(\alpha)d\alpha\|$ may be regarded as the integral in the sense already defined, of the continuous function $\|f(\alpha(s))\|$ along this interval, and hence we know it will exist, and be independent of the particular sequence $\{\alpha_n\}$ which is used. By the triangular inequality we have

$$\left\|\int_C f(\alpha)\,d\alpha\right\| \leq \int_C \|f(\alpha)d\alpha\| = \int_C \|f(\alpha)\|\,ds$$

3. Fundamental for our theory is the extension of Cauchy's theorem:

Theorem 1 If $f(\alpha)$ is analytic in a domain T whose boundary consists of a closed, rectifiable normal Jordan curve C, and if $f(\alpha)$ is continuous inside and on the boundary of T, then $\int_C f(\alpha)\,d\alpha = 0$

A slightly different form of the theorem, convenient for our purposes, is the following.

Theorem 1a Let C denote the complete boundary of a regular region S, and let $f(\alpha)$ be continuous in the closed region S, and analytic inside S. Then $\int_C f(\alpha)\,d\alpha = 0$, the integral being extended over C in the positive sense.*

The second theorem is a consequence of the first, for a region S may be broken up into a finite number of domains of the kind specified

---

* The positive sense of C here means that sense for which the outer normal and the tangent at a point of C are oriented like the positive real and imaginary axes.

in Theorem 1. This is because a simple regular curve is a rectifiable, normal Jordan curve (cf. I, §2). When the integration is carried out for the boundaries of the component domains, those contributions which arise from curves not included in the boundary of S occur in pairs which cancel out.

For a proof of Theorem 1 we refer to the monograph by Watson (see footnote in II, §2), where it will be found that the treatment requires no essential modification, even though $f(\alpha)$ is not numerically-valued.

Other basic integral theorems are the following:

Theorem 2 If C is a rectifiable Jordan curve, and $\varphi(\tau)$ is a function with values in E, defined and continuous on C, then the integral

$$F(\alpha) = \int_C \frac{\varphi(\tau)}{\tau-\alpha}\, d\tau$$

defines a function analytic at every point α not on C. Its derivative is

$$F'(\alpha) = \int_C \frac{\varphi(\tau)}{(\tau-\alpha)^2}\, d\tau$$

Proof: Let α_0 be a point not on C. Then α_0 may be imbedded in a neighborhood T whose interior and boundary contain no point of C. If then $\alpha_0+\Delta\alpha$ is a point of T,

$$\frac{F(\alpha_0+\Delta\alpha)-F(\alpha_0)}{\Delta\alpha} = \int_C \frac{\varphi(\tau)}{(\tau-\alpha_0-\Delta\alpha)(\tau-\alpha_0)}\, d\tau$$

and

$$\left\| \frac{F(\alpha_0+\Delta\alpha)-F(\alpha_0)}{\Delta\alpha} - \int_C \frac{\varphi(\tau)\,d\tau}{(\tau-\alpha_0)^2} \right\| = \left\| \int_C \frac{\Delta\alpha\ \varphi(\tau)}{(\tau-\alpha_0-\Delta\alpha)(\tau-\alpha_0)^2}\, d\tau \right\|$$

$$\leq \frac{ML}{\rho^3}\, |\Delta\alpha|$$

where M is the maximum value of $\|\varphi(\tau)\|$ on C, L is the length of C, and ρ is the least distance from the boundary of T to the curve C. The conclusion follows.

Theorem 3 Let $f(\alpha)$ be analytic in a regular region S, and continuous in and on the boundary C of S. Then if α is an interior point of S,

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(\tau)}{\tau-\alpha}\,d\tau$$

the integration being in the positive sense, Further, $f(\alpha)$ pos~~s~~esses derivatives of all orders, given by

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_C \frac{f(\tau)}{(\tau-\alpha)^{n+1}}\,d\tau$$

The proof of this theorem follows well known lines, using Theorem 1a, and the methods of the preceding. We omit the details.

Theorem 4 Let $\varphi(\alpha,\tau)$ be a function of two complex variables, with values in E, such that

1° $\varphi(\alpha,\tau)$ is defined for α in a domain T, and τ on a rectifiable Jordan curve C,

2° $\varphi(\alpha,\tau)$ is analytic in T for each τ on C, and continuous in α and τ throughout the range under consideration.

Then the integral

$$f(\alpha) = \int_C \varphi(\alpha,\tau)\,d\tau$$

defines a function analytic in T, with derivative

$$f'(\alpha) = \int_C \varphi_\alpha(\alpha,\tau)\,d\tau$$

Proof: Let α_0 be an arbitrary point of T, and $\mathcal{L}$ a circle of radius r about α_0 as center, such that $\mathcal{L}$ and its interior lie in T. Then for $|\Delta\alpha|$ small enough, $\alpha_0+\Delta\alpha$ lies inside this circle, and

$$\varphi(\alpha,\tau) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\varphi(t,\tau)}{t-\alpha}\,dt \quad , \ \alpha = \alpha_0, \alpha_0+\Delta\alpha$$

$$\varphi_\alpha(\alpha_0,\tau) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\varphi(t,\tau)}{(t-\alpha_0)^2}\,dt$$

Accordingly,

$$\frac{f(\alpha_0+\Delta\alpha)-f(\alpha_0)}{\Delta\alpha} - \int_C \varphi_\alpha(\alpha_0,\tau)\,d\tau = \frac{1}{2\pi i}\int_C d\tau \int_{\mathcal{L}} \frac{\Delta\alpha\,\varphi(t,\tau)}{(t-\alpha_0)^2(t-\alpha_0-\Delta\alpha)}\,dt$$

But $\varphi(t,\tau)$ is continuous in t and τ on $\mathcal{L}$ and C respectively, which are closed point sets, and is therefore bounded, say $\|\varphi(t,\tau)\| \leq G$ on the curves in question. Also $|t-\alpha_0| = h$, and $|t-\alpha_0-\Delta\alpha| \geq h-|\Delta\alpha|$. Therefore, letting L be the length of C, we have

$$\left\| \frac{f(\alpha_0+\Delta\alpha)-f(\alpha_0)}{\Delta\alpha} - \int_C \varphi_\alpha(\alpha_0,\tau)\,d\tau \right\| \leq \frac{G\,|\Delta\alpha|}{h(h-|\Delta\alpha|)}\,L ,$$

from which we easily infer the truth of the theorem.

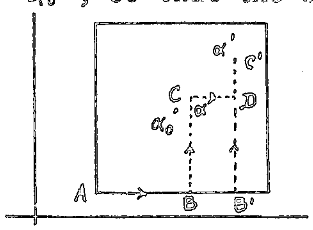
Liouville's theorem generalizes without difficulty:

<u>Theorem 5</u> If f(α) is analytic for all values of α, and if $\|f(\alpha)\| < M$, where M is a constant, then f(α) is a constant.*

Two further theorems of interest are the generalizations of the theorems due to Morera and Weierstrass:

<u>Theorem 6</u> Let T be a domain in which the function f(α), with values in E, is continuous. Suppose that the integral of f(α) vanishes when taken over an arbitrary closed path consisting of a rectangle with sides parallel to the axes, this rectangle lyong wholly within T. Then f(α) is analytic in T.

Proof: Let α_0 be any point in T, and imbed it in a small square, with center at α_0, so that the square lies wholly in T. Let α be any point



in the square, and define

$$F(\alpha) = \int_{\mathcal{L}} f(t)\,dt$$

---

* For proof see Osgood, (10) p.316.

where $\mathcal{L}$ is the path ABC. Then, because of the hypotheses of the theorem

$$F(\alpha') = F(\alpha) + \int_{(CDC')} f(t)\,dt$$

Thus

$$\frac{F(\alpha') - F(\alpha)}{\alpha' - \alpha} - f(\alpha) = \frac{1}{\alpha' - \alpha} \int_{(CDC')} \{ f(t) - f(\alpha) \}\,dt$$

$$\left\| \frac{F(\alpha') - F(\alpha)}{\alpha' - \alpha} - f(\alpha) \right\| \leq \frac{1}{|\alpha - \alpha'|} \int_{(CDC')} \| f(t) - f(\alpha) \| \, |dt|$$

But $f(\alpha)$ is continuous; $|t - \alpha| \leq |\alpha' - \alpha|$, and $\int_{CDC'} |dt| < 2|\alpha - \alpha'|$. Hence given $\varepsilon > 0$, we can choose δ so that $|\alpha' - \alpha| < \delta$ implies $\| f(t) - f(\alpha) \| < \frac{\varepsilon}{2}$. Then

$$\left\| \frac{F(\alpha') - F(\alpha)}{\alpha' - \alpha} - f(\alpha) \right\| < \varepsilon$$

Therefore $F(\alpha)$ is analytic in the square, with the derivative $f(\alpha)$ so that $f(\alpha)$ is also analytic in the square. This completes the proof.

<u>Theorem 7</u> Let $\{u_i(\alpha)\}$ be a sequence of functions analytic in a domain T, and such that the series

$$f(\alpha) = u_1(\alpha) + u_2(\alpha) + \cdots + u_n(\alpha) + \cdots.$$

converges uniformly in every regular region S lying in T. Then $f(\alpha)$ is analytic in T, and its derivative may be calculated by termwise differentiation of the series.

Proof: Let S be an arbitrary regular region in T. Then, since the $u_i(\alpha)$ are continuous in T, we conclude, by I, §3, Theorem 2, that $f(\alpha)$ is continuous in T; it is easy to prove that the series may be integrated term by term over the boundary C of S. Using Cauchy's theorem,

$$\int_C f(\alpha)\,d\alpha = \int_C u_1(\alpha)\,d\alpha + \int_C u_2(\alpha)\,d\alpha + \cdots \quad = 0$$

Therefore, by Theorem 6, $f(\alpha)$ is analytic in T.

Let S be an arbitrary regular region in T, and let α be any point in or on the boundary of S. A second regular region S' may be chosen, containing S entirely in its interior, so that, as t ranges over the boundary C' of S', and α over S, $|t-\alpha|$ has a positive lower bound. Thus the series

$$\frac{1}{2\pi i}\frac{f(t)}{(t-\alpha)^2} = \frac{1}{2\pi i}\frac{u_1(t)}{(t-\alpha)^2} + \frac{1}{2\pi i}\frac{u_2(t)}{(t-\alpha)^2} + \cdots\cdot$$

converges uniformly in α and t in their respective ranges. Integrating over C', we obtain, on referring to Theorem 3,

$$f'(\alpha) = u_1'(\alpha) + u_2'(\alpha) + \cdots$$

This series, moreover, converges uniformly in S,* so that the hypotheses of the theorem are fulfilled for this new series, and we may repeat the differentiation.

4. By a power series on A to E we mean an expression of the form

$$a_0 + \alpha a_1 + \alpha^2 a_2 + \cdots + \alpha^n a_n + \cdots$$

where α is a real or complex number, and the coefficients a_n are elements of E. We shall deal with complex variables, and shall always suppose that E is complete.

<u>Theorem 1</u>** If for a value $\alpha = \alpha_0$ the terms of the series are bounded:

$$\|\alpha_0^n a_n\| < G$$

---

* This statement is justified as follows: the class of functions $\varphi(t)$ defined and continuous on C', with values in E, forms a vector space $\bar{\bar{E}}$ with norm $\|\varphi\| = \max_{t \in C'} \|\varphi(t)\|$. By means of the integral there is defined a linear operation on $\bar{\bar{E}}$ to $\bar{E}$. We then use Theorem 11 of I, §3, where the class B is taken to be the set of points S.

** A series $\sum u_n$ of elements of E is said to converge absolutely if $\sum \|u_n\|$ converges. When E is complete, absolute convergence implies ordinary convergence, and also that the series converges to the same limit, no matter how the terms are rearranged.

then the series converges absolutely for all values of α such that $|\alpha|<|\alpha_0|$.

Proof: Take $\alpha_0 \neq 0$, and let α be a fixed number for which $|\alpha| < |\alpha_0|$.

Then

$$\|\alpha^n a_n\| = \left|\frac{\alpha}{\alpha_0}\right|^n \|\alpha_0^n a_n\| < G\left|\frac{\alpha}{\alpha_0}\right|^n$$

But

$$G\left(1 + \left|\frac{\alpha}{\alpha_0}\right| + \left|\frac{\alpha}{\alpha_0}\right|^2 + \cdots\right)$$

is obviously convergent, so the result follows.

In the usual manner we find that if the series does not converge for all α, there is a 'circle of convergence' such that inside this circle the series converges absolutely, while outside it diverges.

Theorem 2 Let T be a domain which together with its boundary lies wholly within the circle of convergence of the power series. Then the series converges uniformly in and on the boundary of T, and so defines an analytic function within the circle.

Proof: Let r be the radius of the circle of convergence. Then we may choose r_0, $0 < r_0 < r$, so that the concentric circle of radius r_0 also contains T and its boundary. Accordingly, if α is a point of T, $|\alpha| \leq r_0$, and

$$\|\alpha^n a_n\| \leq r_0^n \|a_n\|$$

But $\sum r_0^n \|a_n\|$ is a convergent series of constant terms, by Theorem 1, and it is not difficult to show that the power series converges uniformly in and on the boundary of T. The theorem is then a consequence of II, §3, Theorem 7.

Theorem 3 If the power series vanishes at the points of the infinite sequence of distinct points $\{\alpha_n\}$, where $\lim_{n\to\infty} \alpha_n = 0$, then all the coefficients are zero.

Proof: Since the power series defines an analytic function, it is continuous at the origin, and hence $f(0) = 0$. ($f(\alpha)$ being the function defined by the series). We have the equations

$$0 = a_0$$

$$0 = \alpha a_1 + \alpha^2 a_2 + \cdots \qquad \alpha = \alpha_1, \alpha_2, \ldots$$

But then

$$a_1 + \alpha a_2 + \alpha^2 a_3 + \cdots$$

is a convergent power series which defines a function continuous at 0; since it vanishes at $\alpha = \alpha_2, \alpha_3, \cdots$ it vanishes at 0. Hence $a_1 = 0$; Similarly we prove $a_2 = 0$, and so on.

The importance of power series is explained by the generalization of the classical Cauchy- Taylor expansion theorem:

<u>Theorem 4</u> Let $f(\alpha)$ be analytic in a domain T, and let α_0 be any point of T. Then we may write $f(\alpha)$ in a power series

$$f(\alpha) = f(\alpha_0) + (\alpha-\alpha_0)f'(\alpha_0) + \cdots + \frac{(\alpha-\alpha_0)^n}{n!}f^{(n)}(\alpha_0) + \cdots$$

This series converges and represents the function for all values of α inside the largest circle which can be drawn about α_0 such that it contains only points of T. This power series representation is unique.

Proof: We start from the algebraic identity

$$\frac{1}{t-\alpha} = \frac{1}{t-\alpha_0} + \frac{\alpha-\alpha_0}{(t-\alpha_0)^2} + \cdots + \frac{(\alpha-\alpha_0)^{n-1}}{(t-\alpha_0)^n} + \frac{(\alpha-\alpha_0)^n}{(t-\alpha_0)^n(t-\alpha)}$$

Let α be a point of T, and imbed both it and α_0 in a regular region S, with boundary C, contained in T. If then t is a point of C, integration of the preceding equation leads us to the result

$$f(\alpha) = f(\alpha_0) + (\alpha-\alpha_0)f'(\alpha_0) + \cdots + \frac{(\alpha-\alpha_0)^{n-1}}{(n-1)!}f^{(n-1)}(\alpha_0) + \frac{(\alpha-\alpha_0)^n}{2\pi i}\int_C \frac{f(t)}{(t-\alpha_0)^n(t-\alpha)}\,dt$$

We observe in passing that the integral defines a function $P_n(\alpha)$ analytic within S, by II, §3, Theorem 2. If now α is any point inside the circle described in the theorem, we can choose S to be a smaller concentric circle of radius R, which yet contains α in its interior. Then

$$\left\| \frac{(\alpha-\alpha_0)^n}{2\pi i} P_n(\alpha) \right\| \leq \left\{ \frac{|\alpha-\alpha_0|}{R} \right\}^n \frac{MR}{R-|\alpha|}$$

where $\|f(t)\| \leq M$ on C; but $R-|\alpha|>0$, $|\alpha-\alpha_0|<R$, so that this remainder term becomes vanishingly small with increasing n.

The uniqueness of the power series in $(\alpha-\alpha_0)$ is a consequence of Theorem 3.

5. The study of the behavior of analytic functions in the neighborhood of isolated singular points, or for very large values of the argument, presents more difficulty. It is here that we first begin to see the special character of the classical function theory, in the classification of singularities, and the characterization of functions in terms of them.

Definition: If $f(\alpha)$ is analytic in the neighborhood of a point α_0, that point alone excepted, α_0 is called an isolated singular point.

If $f(\alpha)$ has an isolated singularity at a point α_0, and if it is possible to make a new definition of $f(\alpha_0)$ in such a way that the function is analytic at α_0, then $f(\alpha)$ is said to have a removable singularity at α_0.

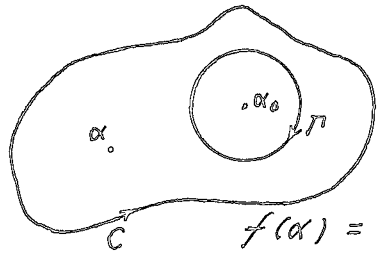
If α_0 is an isolated singular point of $f(\alpha)$, and if $\lim_{\alpha\to\alpha_0} \|f(\alpha)\| = \infty$ the singularity is called a pole.

Concerning removable singularities we have the result of Riemann:

Theorem 1 If α_0 is an isolated singular point of $f(\alpha)$, and if $\|f(\alpha)\|$ remains finite in the neighborhood of α_0, then the singularity is removable.

Proof: Let C be a simple, regular curve in the given neighborhood, with

α_0 in its interior. Select a point α inside C, and draw a small circle



Γ, of radius r, about α_0, excluding α. Then by II, §3, Theorem 3,

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-\alpha}\,dt + \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-\alpha}\,dt$$

the sense of integration being as indicated. Now $\|f(t)\| < G$ within C, and $|t-\alpha| \geq |\alpha-\alpha_0| - h > 0$. Hence

$$\left\| \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-\alpha}\,dt \right\| \leq \frac{1}{2\pi}\,\frac{G\,2\pi h}{|\alpha-\alpha_0|-h}$$

But the expression on the right tends to zero with r, while the left member of the inequality is independent of r; the second integral therefore vanishes, leaving the result

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-\alpha}\,dt$$

for every point $\alpha \neq \alpha_0$ inside C. But, by II, §3, Theorem 2, the integral on the right defines a function analytic at all points not on C, and hence in particular at $\alpha = \alpha_0$. It follows that $\lim\limits_{\alpha\to\alpha_0} f(\alpha)$ exists, and that the singularity is removable.

The 'point at infinity' of the complex plane may be introduced in the usual way. It is an ideal point regarding which we make the conventions:

1° The neighborhood of the point $\alpha = \infty$ is that portion of the plane lying outside an arbitrary closed Jordan curve. The point is in all these neighborhoods.

2° A domain T contains the point $\alpha = \infty$ if it contains some neighborhood of the point.

3° The point at infinity is the cluster point of a set if given a positive R, no matter how large, there are points of the set for which $|\alpha| > R$.

4° For the behavior of $f(\alpha)$ at $\alpha = \infty$ we agree to consult the behavior of $f(\frac{1}{\alpha})$ at $\alpha = 0$. We then define a function to be analytic at $\alpha = \infty$ if it is analytic for all sufficiently large values of α, and if $\lim_{\alpha \to \infty} f(\alpha)$ exists, whereupon we set $f(\infty)$ equal to this limit.

Obviously the 'extended' complex plane, with this ideal element adjoined, is a closed, compact set.

To illustrate some of the situations which present themselves in connection with singular points, we give some examples. Let E be the complex Banach space whose elements are complex-valued functions of a complex variable, defined and continuous on the unit circle. If $\varphi(z)$ is such a function we define its norm to be

$$\|\varphi\| = \max_{|z|=1} |\varphi(z)|$$

Suppose that $\psi(\alpha)$ is a numerically-valued function of α, analytic at a point α_0. Then the function on the α - plane to E:

$$f(\alpha) = e^{\psi(\alpha)z}$$

is analytic * at α_0, with derivative $f'(\alpha) = \psi'(\alpha) z\, e^{\psi(\alpha)z}$

The singularities of $f(\alpha)$ will be precisely the singular points of $\psi(\alpha)$. Now

$$\|f(\alpha)\| = \max_{|z|=1} |e^{\psi(\alpha)z}| = \max_{x^2+y^2=1} e^{xR(\psi)-yI(\psi)}$$

---

* To make sure of this we merely observe that $e^{\psi(\alpha)z}$, regarded as a function of two variables, is analytic in α for each z on the circle; that the derivative $\psi'(\alpha)z\,e^{\psi(\alpha)z}$ is continuous in z for each α; and finally that when α is restricted to lie in a closed region about α_0, $e^{\psi(\alpha)z}$ is uniformly continuous in α and z. This is sufficient to show that $f(\alpha)$ is differentiable according to the definition of II, §1.

where $\quad R(\psi) =$ real part of $\psi(\alpha)$

$\qquad\qquad I(\psi) =$ imaginary part of $\psi(\alpha)$.

If $\psi(\alpha) \neq 0$ we may choose $x = \frac{R(\psi)}{|\psi|}$, $y = -\frac{I(\psi)}{|\psi|}$, and thus see that

$$\|f(\alpha)\| \geq e^{|\psi(\alpha)|} \geq 1$$

an inequality which remains true when $\psi(\alpha) = 0$. From this we see that $\alpha = \alpha_0$ is a pole of $f(\alpha)$ if it is a pole of $\psi(\alpha)$. On the other hand if α_0 is an essential singularity of $\psi(\alpha)$, then by Weierstrass' theorem *, $\psi(\alpha)$ fluctuates and comes arbitrarily near all values as $\alpha \to \alpha_0$, so that $f(\alpha)$ will not have a pole at α_0. If we then agree to call a singularity which is neither removable, nor a pole, essential, we observe the following:

1° Weierstrass' theorem on essential singularities is not true in our theory. In particular the above function $f(\alpha)$ fails to assume values within the unit sphere about the origin in E.

2° Picard's theorem is likewise not true.**

Finally, by taking $\psi(\alpha) = \alpha$ we obtain the function $f(\alpha) = e^{\alpha z}$ such that $\|f(\alpha)\| \geq e^{|\alpha|}$. Thus $f(\alpha)$ is analytic for all finite values of α, and has a pole at $\alpha = \infty$. In classical theory this would lead us to infer that $f(\alpha)$ is a polynomial. Such is not the case in our present theory, however, for the Taylor's expansion of $f(\alpha)$ is

$$f(\alpha) = 1 + \alpha z + \frac{\alpha^2 z^2}{2!} + \cdots$$

which is not a polynomial in α.

To complete this paragraph we shall prove the following theorem

---

* cf. Osgood, (10) p.328.
** Osgood, (10) p.748.

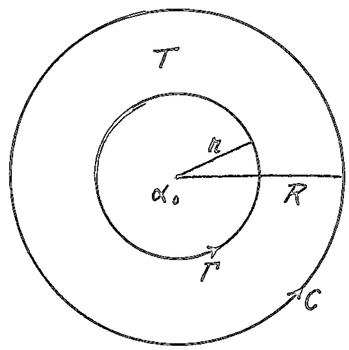
of Laurent.

<u>Theorem 2</u> Let $f(\alpha)$ be analytic in a ring-shaped domain T whose boundary consists of concentric circles Γ and C. Then $f(\alpha)$ is the sum of two functions

$$f(\alpha) = \varphi(\alpha) + \psi(\alpha)$$

where $\varphi(\alpha)$ is analytic inside C, and $\psi(\alpha)$ is analytic in the extended plane outside Γ, with the value 0 at ∞.

Proof: Let α_0 be the center of the two circles, whose radii we shall denote by r, R (see figure). Let S be a regular region lying inside T, bounded by circles $\bar{\Gamma}$ and $\bar{C}$ of radii $r+\delta_1$, $R-\delta_2$, respectively. Then by II, §3, Theorem 3, when α is in S,



$$f(\alpha) = \frac{1}{2\pi i}\int_{\bar{C}} \frac{f(t)}{t-\alpha}\,dt - \frac{1}{2\pi i}\int_{\bar{\Gamma}} \frac{f(t)}{t-\alpha}\,dt$$

both integrations being in the counterclockwise sense. But by II, §3, Theorem 2, these integrals define functions $\bar{\varphi}(\alpha), -\bar{\psi}(\alpha)$, analytic at all points not on $\bar{C}$ and $\bar{\Gamma}$ respectively. Now let $\delta_1 \to 0$. This does not affect the definition of $\bar{\psi}(\alpha)$ at points already attained, since $f(\alpha)$ and $\bar{\varphi}(\alpha)$ do not depend on $\bar{\Gamma}$, and $\bar{\psi}(\alpha) = f(\alpha) - \bar{\varphi}(\alpha)$. Hence we can extend $\bar{\psi}(\alpha)$ to a function $\psi(\alpha)$ defined and analytic outside Γ. Similarly $\bar{\varphi}(\alpha)$ is extended to a function $\varphi(\alpha)$ as required. For the analyticity of $\psi(\alpha)$ at ∞ we note that

$$\lim_{\alpha\to\infty} \int_{\bar{\Gamma}} \frac{f(t)}{t-\alpha}\,dt = 0$$

and refer to the convention laid down above.

<u>Corollary</u> Under the hypotheses of Theorem 2 $f(\alpha)$ may be represented

uniquely in the form

$$f(\alpha) = \sum_{-\infty}^{\infty}{}_n (\alpha-\alpha_0)^n C_n \quad , \quad r < |\alpha-\alpha_0| < R$$

$$C_n = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{f(t)}{(t-\alpha_0)^{n+1}}\,dt$$

where $\mathcal{L}$ is a regular closed curve in T, enclosing the point $\alpha = \alpha_0$.

Since $\varphi(\alpha)$ is analytic when $|\alpha-\alpha_0| < R$ it has an expansion

$$\varphi(\alpha) = \sum_{0}^{\infty}{}_n (\alpha-\alpha_0)^n C_n$$

Now if we set $\beta = \frac{1}{\alpha-\alpha_0}$, $\psi(\alpha) = \Psi(\beta)$, the latter function is analytic at $\beta = 0$, with the value 0 there. Hence

$$\psi(\alpha) = \sum_{1}^{\infty}{}_n \frac{1}{(\alpha-\alpha_0)^n} C_{-n} \qquad |\alpha-\alpha_0| > r$$

To evaluate the coefficients C_i we integrate the series expression for $\frac{f(t)}{(t-\alpha_0)^{n+1}}$ termwise over $\mathcal{L}$, as we may do, because of uniform convergence. The result is immediate.

Laurent's theorem affords us a useful method for studying the behavior of a function near an isolated singular point $\alpha = \alpha_0$. We may draw an arbitrarily small circle Γ about α_0, and a suitably small concentric circle C; in the region thus formed application of Laurent's theorem gives us

$$f(\alpha) = \sum_{0}^{\infty}{}_n (\alpha-\alpha_0)^n a_n + \sum_{1}^{\infty}{}_n \frac{1}{(\alpha-\alpha_0)^n} b_n \qquad 0 < |\alpha-\alpha_0| < R$$

It is readily seen that all the b's are zero if and only if the singularity is removable. In classical theory a pole is characterized by the fact that only a finite number of the b's are different from zero. Such is not the case in the present theory. For instance, in the example considered above, when $\psi(\alpha) \equiv \frac{1}{\alpha}$, $f(\alpha) = e^{\frac{1}{\alpha}z}$ has a pole at $\alpha = 0$, but its

Laurent expansion is

$$f(\alpha) = 1 + \frac{1}{\alpha} z + \frac{1}{\alpha^2}\frac{z^2}{2!} + \cdots\cdot$$

We shall say that α_0 is a pole of order m if in the Laurent expansion $b_n = 0$ when $n > m$. The expression

$$\sum_1^m {}_n \frac{1}{(\alpha-\alpha_0)^n} b_n$$

will in this case be called the <u>principal part</u> of the function $f(\alpha)$ at the pole.

6. The notion of a rational function, as the quotient of two polynomials with <u>abstract</u> values, is denied us, since we have not postulated division in the space E. Of course a function such as

$$f(\alpha) = \frac{P(\alpha)}{p(\alpha)} = \frac{\alpha^n a_0 + \alpha^{n-1} a_1 + \cdots + a_n}{\alpha^m \mu_0 + \cdots \quad \cdots + \mu_m}$$

where $P(\alpha)$ is an abstract polynomial, and $p(\alpha)$ a numerical polynomial, is a sort of rational function. Its only singularities are poles of finite order, occurring at the roots of $p(\alpha)$, and at $\alpha = \infty$ in case $n > m$. It admits a representation by rational fractions, as we may prove in the usual way, by subtracting the principal parts of the function at the poles and utilizing Liouville's theorem.

The problem of determining the nature of a function whose only singularities in the finite part of the plane are poles of finite order, is solved by application of Mittag-Leffler's theorem.

<u>Theorem 1</u> Let $\{\alpha_n\}$ be a sequence of points in the complex plane, such that $\alpha_n \to \infty$. Further let there be given a set of abstract polynomials, with coefficients in E:

$$g_n(\alpha) = \alpha A_1^{(n)} + \cdots + \alpha^{\nu_n} A_{\nu_n}^{(n)} \qquad \nu_n \geq 1$$

(we assume $A^{(n)}_{\nu_n} \neq 0$ unless from a certain value of n on, $g_n(\alpha) \equiv 0$.
Then there exists a single-valued function $f(\alpha)$ with pole of order ν_n at α_n and principal part $g_n\left(\frac{1}{\alpha - \alpha_n}\right)$ there, and otherwise analytic in the finite portion of the plane. The most general such function is then of the form $f(\alpha) + G(\alpha)$, where $G(\alpha)$ is an entire function, that is, a function analytic throughout the finite part of the plane.

For proof we refer to the treatment in the ordinary case.* As a corollary we have the following result:

<u>Theorem 2</u> If $f(\alpha)$ is an analytic function which has in the finite plane no other singularities than poles of finite order, then it has the form

$$f(\alpha) = \sum_{n=1}^{\infty} \left[g_n\left(\frac{1}{\alpha - \alpha_n}\right) - \gamma_n(\alpha) \right] + G(\alpha)$$

where $g_n\left(\frac{1}{\alpha - \alpha_n}\right)$ is the principal part of the function at the pole α_n, $\gamma_n(\alpha)$ is a suitable abstract polynomial, and $G(\alpha)$ is an entire function. In particular, if the number of poles is finite, then

$$f(\alpha) = \frac{P(\alpha)}{p(\alpha)} + G(\alpha)$$

where $\frac{P(\alpha)}{p(\alpha)}$ is a ' rational function' of the type discussed above.

There can not be more than a denumerable number of poles, for they must be isolated, clustering only (possibly) at infinity.

---

* cf. Osgood, (10) p.565-566.

Chapter III

Analytic Functions in General Analysis

1. In this chapter we shall deal with two complex vector spaces E, E', of which E' is complete, and with functions on E to E'. Greek letters will in general denote complex numbers, while letters $x, y, \ldots, z$, with or without subscripts, will denote quantities in E. Quantities in E' will enter only as function-values, and will not require special designation.

Definition If $f(x)$ is a function defined in a domain D of the space E, the values of $f(x)$ being in E', $f(x)$ is said to be analytic in D if it is continuous and has a Gateaux differential at each point of D. A function $f(x)$ is said to be analytic at a point x_0 if it is analytic in some neighborhood of x_0 .

The fundamental theorem, which enables us to utilize the results of the preceding chapter, may be stated as follows:

Theorem 1 Let $f(x)$ be defined and continuous in a domain D. Then a necessary and sufficient condition that $f(x)$ be analytic in D is that for each $n > 0$, $f(\alpha_1 x_1 + \cdots + \alpha_n x_n)$ be an analytic function of $\alpha_1, \ldots, \alpha_n$ (in the sense of chapter II) for all α's and x's such that $\alpha_1 x_1 + \cdots + \alpha_n x_n$ is in D.

Proof: The sufficiency of the condition is obvious. It is also necessary, for suppose that $[\alpha_1^0, \ldots, \alpha_n^0; x_1, \ldots, x_n]$ is a set such that $\alpha_1^0 x_1 + \cdots + \alpha_n^0 x_n$ is in D. Let i be an arbitrary integer, $1 \leq i \leq n$; then $\alpha_1^0 x_1 + \cdots + \alpha_{i-1}^0 x_{i-1} + \alpha_i x_i + \cdots + \alpha_n^0 x_n$ is in D when $|\alpha_i - \alpha_i^0| < r$, r being sufficiently small. Then $f(\alpha_1^0 x_1 + \cdots + \alpha_i x_i + \cdots + \alpha_n x_n)$ is analytic as a function of α_i in the circle $|\alpha_i - \alpha_i^0| < r$, since by hypothesis the Gateaux differential

$$\delta f(\alpha_1^0 x_1 + \cdots + \alpha_i x_i + \cdots + \alpha_n^0 x_n; x_i) = \left\{ \frac{\partial f(\alpha_1^0 x_1 + \cdots + (\alpha_i + h) x_i + \cdots + \alpha_n^0 x_n)}{\partial h} \right\}_{h=0}$$

exists under such conditions. This proves the theorem.

As a consequence of Theorem 1, and II, §3, Theorem 3 we conclude that an analytic function has Gateaux differentials of all orders.

<u>Theorem 2</u> If $f(x)$ is analytic in D it has Gateaux differentials of all orders there. The n^{th} differential is a completely symmetric function of the n increments.

We have to prove that for any n, any x in D, and any $y_1, \ldots, y_n$ in E, the differential $\delta^n f(x; y_1, \ldots, y_n)$ is defined, and is symmetric in the y's. Let $\alpha_1, \ldots, \alpha_n$ be complex variables, and consider $f(x + \alpha_1 y_1 + \cdots + \alpha_n y_n)$ which is an analytic function of $\alpha_1, \ldots, \alpha_n$ when all these variables are sufficiently small in absolute value. Consequently the partial derivatives of all orders exist and are continuous in the set $(\alpha_1, \ldots, \alpha_n)$ near $(0, \ldots, 0)$. The order of differentiation is then immaterial.* In particular

$$\delta^n f(x; y_1, \ldots, y_n) = \left\{ \frac{\partial^n f(x + \alpha_1 y_1 + \cdots + \alpha_n y_n)}{\partial \alpha_n \cdots \cdots \partial \alpha_1} \right\}_{(\alpha) = (0)}$$

exists, and has the same value for all arrangements of the y's.

Two important questions present themselves regarding the differential $\delta f(x; y)$. Is it an analytic function of x, for fixed y, and is it linear in y? The answer to both questions is affirmative, as we shall show.

Suppose, then, that x_0 is a point of D, and choose an arbitrary y in E, holding it fast. We may choose positive numbers r, r' such that $x + \tau y$ is in D when $\|x - x_0\| < r'$ and $|\tau| \leq r$. With these restrictions

---

* This theorem, standard in the differential calculus, remains true when when the function-values lie in a Banach space. The continuity in $(\alpha_1, \ldots \alpha_n)$ is a consequence of the continuity of $f(x)$ and the theory of analytic functions of several complex variables. cf. Osgood, (11) p.21.

$f(x + \tau y)$ is an analytic function of τ, and by II, §3, Theorem 3,

$$\delta f(x;y) = \frac{1}{2\pi i} \int_C \frac{f(x+\tau y)}{\tau^2}\, d\tau$$

C being a circle of radius r about the origin. Then

$$\|\delta f(x;y) - \delta f(x_0;y)\| \leq \frac{1}{2\pi} \int_C \left\| \frac{f(x+\tau y) - f(x_0+\tau y)}{\tau^2} d\tau \right\|$$

From this it follows that $\delta f(x;y)$ is continuous at x_0, provided that we can, for a given $\varepsilon > 0$, choose a δ such that $\|x - x_0\| < \delta$ implies the inequality

$$\|f(x+\tau y) - f(x_0+\tau y)\| < \varepsilon$$

for all τ on C. That we can actually do this is a direct consequence of I, §3, Theorem 3, since C is a compact, closed set.

The foregoing work proves that $\delta f(x;y)$ is analytic in D, for it is continuous and has a Gateaux differential at each point of D. In order to prove that the differential is a linear function of y, we first prove that it is additive and homogeneous of the first degree. We shall then prove that it is continuous at $y = 0$.

Let x_0 be a point of D and let α, β, y_1, y_2 be given arbitrarily. Then let $y = \alpha y_1 + \beta y_2$ and consider $f(x_0 + \tau y)$, which is an analytic function of τ at $\tau = 0$. Accordingly, by II, §4, Theorem 4,

$$f(x_0+\tau y) = f(x_0) + \tau \left[\frac{\partial f(x_0+\tau y)}{\partial \tau}\right]_{\tau=0} + \frac{\tau^2}{2!}\left[\frac{\partial^2 f(x_0+\tau y)}{\partial \tau^2}\right]_{\tau=0} + \cdots\cdots.$$

However, if we write $\xi = \tau\alpha$, $\eta = \tau\beta$, then $f(x_0 + \xi y_1 + \eta y_2)$ is an analytic function of ξ, η in the neighborhood of $\xi = \eta = 0$. We have then the expansion

$$f(x_0+\xi y_1+\eta y_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\xi^m \eta^n}{m!\,n!}\left[\frac{\partial^{m+n} f(x_0+\xi y_1+\eta y_2)}{\partial \xi^m\, \partial \eta^n}\right]_{\xi=\eta=0}$$

But the expansion of the left member as a power series in τ is unique,* and on equating coefficients of terms of the first power in τ we obtain

$$\left[\frac{\partial f(x_0+\tau y)}{\partial \tau}\right]_{\tau=0} = \alpha\left[\frac{\partial f(x_0+\xi y_1+\eta y_2)}{\partial \xi}\right]_{\xi=\eta=0} + \beta\left[\frac{\partial f(x_0+\xi y_1+\eta y_2)}{\partial \eta}\right]_{\xi=\eta=0}$$

This is precisely the relation

$$\delta f(x_0;\alpha y_1+\beta y_2) = \alpha\,\delta f(x_0;y_1) + \beta\,\delta f(x_0;y_2)$$

It remains to prove that $\delta f(x_0;y)$ is continuous. As in the proof that the differential is continuous as a function of x, choose $r > 0$ such that when C is a circle of radius r about $\tau = 0$,

$$\delta f(x_0;y) = \frac{1}{2\pi i}\int_C \frac{f(x_0+\tau y)}{\tau^2}\,d\tau$$

for all sufficiently small $\|y\|$. Recalling that $\int_C \frac{d\tau}{\tau^2} = 0$ we write

$$\delta f(x_0;y) = \frac{1}{2\pi i}\int_C \frac{f(x_0+\tau y) - f(x_0)}{\tau^2}\,d\tau$$

Let $\varepsilon > 0$ be given. Since f(x) is continuous at x_0 it is possible to choose $\delta > 0$ such that $\|y\| < \frac{\delta}{r}$ implies $\|f(x_0+\tau y)-f(x_0)\| < r\varepsilon$ when $|\tau| = r$. Then $\|\delta f(x_0;y)\| < \varepsilon$, and we are through.

We now have the theorem, summarizing the results so far obtained, and adding the new assertion that the n^{th} differential is a continuous function of its n + 1 arguments:

<u>Theorem 3</u> If f(x) is analytic in D, then for each n the differential

---

* The extension of the main results of II to functions of several complex variables presents no difficulties. In particular the Cauchy integral formula and the Cauchy-Taylor expansion theorem follow readily. The necessary definitions and theorems concerning multiple power series will be found in Osgood, (11), p.29-50. No essential modification is required.

$\delta^n f(x;y_1,\ldots,y_n)$ is an analytic function of x in D, when $y_1,\ldots,y_n$ are fixed. It is continuous in the set $(x,y_1,\ldots,y_n)$ at every point where it is defined. Therefore it is, for each x, a symmetric multilinear function of $y_1,\ldots,y_n$; in particular, $\delta^n f(x;y,\ldots,y)$ is a continuous function of x and y, homogeneous of degree n in y.

Proof: Since $\delta^n f(x;y_1,\ldots,y_n)$ is linear in each y_i, and continuous in x, it follows by a theorem of Kerner * that it is continuous in the set $(x, y_1,\ldots,y_n)$. A direct proof could be given, using a representation by Cauchy's integral formula for functions of several variables, in a manner similar to that of our proof that $\delta f(x;y)$ is continuous in x for fixed y. The other assertions of the theorem are evident in view of what has already been proved.

2. If f(x) is analytic at a point x_0 the radius of analyticity of f(x) at $x = x_0$ is defined to be the largest positive number ρ such that f(x) is analytic in the region defined by $\|x-x_0\|<\rho$.

<u>Theorem 1</u> If f(x) is analytic at x_0, with radius of analyticity ρ there, then f(x) may be expanded in the form

$$f(x) = f(x_0) + \delta f(x_0;x-x_0) + \cdots + \frac{1}{n!}\delta^n f(x_0;x-x_0) + \cdots$$

This series converges and defines the function for every x such that $\|x-x_0\|<\rho$.

Proof: Let such an x be chosen, and then pick a number $\rho_1 > 0$ so that $\|x-x_0\| \le \rho_1 < \rho$. We may then choose r so that $1 < r < \rho/\rho_1$. The function $\psi(\tau) = f(x_0+\tau y)$ $y = x - x_0$ is analytic inside a circle C of radius r with center at $\tau = 0$, and continuous inside and on the circle. Therefore, by II, §4, Theorem 4,

$$\psi(1) = \psi(0) + \psi'(0) + \frac{1}{2!}\psi''(0) + \cdots\cdots$$

or

$$f(x) = f(x_0) + \delta f(x_0;y) + \frac{1}{2!}\delta^2 f(x_0;y) + \cdots\cdots$$

---

* M. Kerner, (5) p.159 and (6) p.548.

as was to be proved.

We shall now study the nature of the convergence of this series somewhat more intensively.

Theorem 2 The series in Theorem 1 converges uniformly to f(x) in every compact set G extracted from the closed sphere $\|x - x_0\| \leq \theta\rho$, where θ, $0 < \theta < 1$, is arbitrary. Moreover, the series

$$\|f(x_0)\| + \|\delta f(x_0; x - x_0)\| + \frac{1}{2!}\|\delta^2 f(x_0; x - x_0)\| + \cdots\cdots$$

converges uniformly in G.

Proof: Without loss of generality we may assume that $x_0 = 0$. Then let θ $0 < \theta < 1$, be chosen arbitrarily, and held fast. Choosing r, $1 < r < \frac{1}{\theta}$, we have

$$\frac{1}{n!}\,\delta^n f(0; x) = \frac{1}{2\pi i}\int_C \frac{f(\tau x)}{\tau^{n+1}}\,d\tau$$

where C is a circle of radius r about $\tau = 0$, and $\|x\| \leq \theta\rho$. Let G be a compact set of points such that $\|x\| \leq \theta\rho$; then $\|f(\tau x)\|$ is bounded for x in G and τ on C. To prove this it suffices to show that the set of such points τx is compact (cf. I, §3, Theorem 1). Suppose, then, that $\{\tau x\}$ is an infinite aggregate, with x in G, and τ on C. The points τ have at least one limit point τ' on C, and we may select a sequence $\{\tau_n\}$ converging to τ'. The corresponding sequence $\{x_n\}$, lying in G, contains a subsequence converging to a point x', and $\|x'\| \leq \theta\rho$. This set, and the corresponding set of τ's yield a sequence, which for simplicity we may designate by $\{\tau_n x_n\}$, converging to $\tau' x'$. We have then $\|f(\tau x)\| < M$, and so

$$\left\|\frac{1}{n!}\,\delta^n f(0; x)\right\| \leq \frac{M}{r^n}$$

when x is in G. Since r > 1 the member on the right is the general term of a convergent series of constants. This proves the theorem.

So far we have not gained any information about the moduli of the homogeneous polynomials $\delta^n f(x\ ;y)$. Our first purpose in getting an appraisal of these moduli is to prove the important result that the Gateaux differentials with which we have been dealing are in fact Fréchet differentials. We have to prove that property 2° of the definition in I, §3 is fulfilled.

Let f(x) be analytic at x_0, with radius of analyticity ρ there. For definiteness choose a number $0 < \rho_1 < \rho$ such that $\|x - x_0\| \leq \rho_1$ implies

$$\|f(x) - f(x_0)\| < \frac{1}{2}$$

Then let us agree that for an arbitrary $y \neq 0$ we shall choose r so that $r\|y\| = \rho_1$. Then

$$\delta^n f(x_0 ; y) = \frac{n!}{2\pi i}\int_C \frac{f(x_0 + \tau y)}{\tau^{n+1}}\,d\tau$$

where C is a circle of radius r with center at $\tau = 0$. But, y being fixed, $\|f(x_0 + \tau y)\|$ is a continuous function of τ on C, and so has an attained maximum there:

$$\|f(x_0 + \tau y)\| \leq M(y, r)$$

This gives

$$\|\delta^n f(x_0 ; y)\| \leq \frac{n!\,M(y,r)}{r^n}$$

Now $\|f(x_0 + \tau y)\| \leq \|f(x_0)\| + \|f(x_0 + \tau y) - f(x_0)\|$

so that by the definition of M(y,r),

$$0 \leq M(y,r) \leq \|f(x_0)\| + \max_{|\tau| = r} \|f(x_0 + \tau y) - f(x_0)\|$$

$$\leq \|f(x_0)\| + \frac{1}{2} = G$$

where G is a constant. Thus we have

$$\|\delta^n f(x_0;y)\| \leq \frac{n!\,G}{\rho_1^n}\|y\|^n \quad ;$$

this inequality gives an upper bound for the modulus of $\delta^n f(x_0;y)$. Now when $\|y\| < \rho_1$ the series

$$\left(\frac{\|y\|}{\rho_1}\right)^2 + \left(\frac{\|y\|}{\rho_1}\right)^3 + \cdots\cdots$$

converges to the limit $\dfrac{1}{1-\frac{\|y\|}{\rho_1}}\,\dfrac{\|y\|^2}{\rho_1^2}$ .Hence, by Theorem 2

$$\|f(x_0+y) - f(x_0) - \delta f(x_0;y)\| \leq \frac{G}{1-\frac{\|y\|}{\rho_1}}\,\frac{\|y\|^2}{\rho_1^2}$$

and from this inequality it is easy to prove that $\delta f(x_0;y)$ is the Fréchet differential at x_0.

<u>Theorem 3</u> If $f(x)$ is analytic at a point x_0 it admits Fréchet differentials of all orders in the neighborhood of the point.

It should be noted that this theorem depends on the fact that both E and E' are <u>complex</u> vector spaces, E' being complete, so as to insure the existence of the integral of a continuous function.

3. Corresponding to II, §3, Theorem 7 we have the following generalization of Weierstrass' theorem:

<u>Theorem 1</u> Let $\{u_n(x)\}$ be a set of functions, each analytic in a domain D, and let the series

$$u_1(x) + u_2(x) + \cdots\cdots$$

converge uniformly in every compact set extracted from an arbitrary closed sphere lying in D. Then the series converges and defines a function $f(x)$ analytic in D. The differentials of $f(x)$ are obtained by termwise differentiation of this series.

Proof: The series obviously converges and defines a function $f(x)$ in D, which is continuous there, by I, §3, Theorem 2. We have to show that $f(x)$ has a differential at each point x_0 of D. Let y be an arbitrary fixed point of E, and consider the series

$$f(x_0+\tau y) = u_1(x_0+\tau y) + u_2(x_0+\tau y) + \cdots$$

where $|\tau| < h$ insures that $x_0 + \tau y$ is in D. Then the terms $u_n(x_0+\tau y)$ are analytic functions of τ when $|\tau| < h$, and therefore by II, §3, Theorem 7, $f(x_0 + \tau y)$ is also, if we can prove that the series converges uniformly in every closed subset of the above circle. Let S be such a set. Then the set of values assumed by $x_0 + \tau y$ as τ ranges over S is obviously compact, and lies in a closed sphere in D. The function $f(x_0 + \tau y)$ is therefore analytic at $\tau = 0$; that is, $\delta f(x_0;y)$ exists. The legitimacy of repeated termwise differentiation follows by application of II, §3, Theorem 7.

The theorem just proved is of importance in developing the properties of analytic functions from the Weierstrassian point of view. For this we shall define power series, utilizing the concept of polynomial set forth in I, §3.

By a power series is meant a formal expression

$$h_0(x) + h_1(x) + \cdots + h_n(x) + \cdots$$

where $h_n(x)$ is a homogeneous polynomial of degree n on E to E'. By the radius ρ of such a series we mean the largest positive number such that the series converges uniformly in every compact set extracted from the sphere $\|x\| \leq \theta\rho$ where $0 < \theta < 1$. The sphere $\|x\| < \rho$ is then called the sphere of convergence of the power series, and we say that the power series converges <u>regularly</u> in this sphere.*

<u>Theorem 2</u> A power series defines an analytic function within its sphere of convergence.

---

* See, however, the remarks in §7 on the shape of the region of convergence of a power series.

This theorem is a direct consequence of the one before, for a homogeneous polynomial $h_n(x)$ is analytic for all values of x, with the Fréchet differential $dh_n(x;y) = n h_n'(x,\dots,x,y)$. (See I, §3, Theorem 10)

Theorem 3 If a power series vanishes for all values of its argument in an arbitrarily small neighborhood of $x = \Theta$, then its individual terms vanish identically.

Proof: Suppose that

$$h_0(x) + h_1(x) + \cdots\cdots = 0$$

when $\|x\| < h$, and let x_0 be any point, other than $x = 0$, in E. Then $\|\lambda x_0\| < h$ if $|\lambda| < \frac{h}{\|x_0\|}$. For such values of λ,

$$h_0(x_0) + \lambda h_1(x_0) + \lambda^2 h_2(x_0) + \cdots = 0$$

Therefore, by II, §4, Theorem 3, the coefficients of this power series in λ must be zero. Since x_0 was arbitrary this completes the proof.

Theorem 4 The power series expansion of an analytic function is unique.

Proof: Let f(x) be an analytic function defined by the power series

$$f(x) = h_0(x) + h_1(x) + \cdots\cdots$$

for $\|x\| < \rho$. Then also, by theorems 1 and 2 of III, §2,

$$f(x) = f(0) + \delta f(0;x) + \frac{1}{2!}\delta^2 f(0;x) + \cdots\cdots$$

and this is a power series with radius at least as great as ρ. Subtracting, we have by Theorem 3,

$$h_n(x) = \frac{1}{n!}\delta^n f(0;x)$$

for all values of x.

Concerning the behavior of a power series on its sphere of convergence we have the following extension of the theorem of Abel:

Theorem 5 Let f(x) be the function defined by the power series $\sum_0^\infty h_n(x)$ inside its sphere of convergence, and let the series converge for the value $x = x_0$ on the sphere $\|x\| = \rho$. Then

$$\lim_{\lambda \to 1} f(\lambda x_0) = \sum_0^\infty h_n(x_0)$$

when the complex number λ approaches unity along a path included between two chords of the unit circle which pass through $\lambda = 1$.

Proof: It suffices to prove that the series $\sum_0^\infty h_n(\lambda x_0)$ converges uniformly in λ on any path of the specified kind terminating at $\lambda = 1$. To show this consider the partial remainders

$$S_{n,p} = h_n(x_0) + \cdots + h_p(x_0) \qquad p \geqq n$$

By hypothesis we can, if $\varepsilon > 0$ is given, choose n_0 so that $\|S_{n,p}\| < \varepsilon$ when $n_0 \leq n \leq p$. But

$$\sum_{k=n}^m h_k(\lambda x_0) = \sum_{k=n}^m \lambda^k h_k(x_0) = \lambda^n S_{n,n} + \lambda^{n+1}(S_{n,n+1} - S_{n,n}) + \cdots + \lambda^m(S_{n,m} - S_{n,m-1})$$

$$= (\lambda^n - \lambda^{n+1}) S_{n,n} + \cdots + (\lambda^{m-1} - \lambda^m) S_{n,m-1} + \lambda^m S_{n,m}$$

so that, if $n \geqq n_0$,

$$\left\| \sum_{k=n}^m h_k(\lambda x_0) \right\| \leq \varepsilon \left\{ |\lambda^n - \lambda^{n+1}| + \cdots + |\lambda^{m-1} - \lambda^m| + |\lambda|^m \right\}$$

$$\leq \varepsilon \left\{ |1-\lambda| \left[|\lambda|^n + \cdots + |\lambda|^{m-1} \right] + |\lambda|^m \right\}$$

$$\leq \varepsilon \left\{ \frac{|1-\lambda|}{1-|\lambda|} + 1 \right\}$$

since $|\lambda| < 1$. This inequality is enough to establish the uniform convergence in λ, if the path is such that $\frac{|1-\lambda|}{1-|\lambda|}$ is bounded as $\lambda \to 1$.

If we wish to impose the condition $|1-\lambda| \leq A(1-|\lambda|)$, where A is an arbitrary constant, $A > 1$, we find that it will be satisfied by restricting λ to lie inside that portion of the unit circle bounded by the curve

$$r = 2A \frac{1 + A\cos\theta}{1 - A^2} \qquad (\lambda - 1 = re^{i\theta})$$

in the complex plane. By choosing A sufficiently large we can bring the region of uniform convergence to include any path of the kind required by the theorem.

From §2, Theorem 2, and §3, Theorem 4 we see that a power series converges absolutely inside its sphere of convergence. This enables us to draw the following conclusion:

<u>Theorem 6</u> If the radius of convergence of the power series $\sum h_n(X)$ is ρ, then

$$\underline{\lim}_{n\to\infty} \|h_n(x)\|^{-\frac{1}{n}} \geq \frac{\rho}{\|X\|} \qquad \text{when} \quad \|X\| \neq 0$$

Proof: Let $\xi = \frac{\rho X}{\|X\|}$, so that $\|\xi\| = \rho$. Then the series

$$\sum_0^\infty \|h_n(\lambda\xi)\| = \sum_0^\infty |\lambda|^n \|h_n(\xi)\|$$

converges when $|\lambda| < 1$, so that

$$\underline{\lim}_{n\to\infty} \|h_n(\xi)\|^{-1/n} \geq 1$$

Therefore

$$\underline{\lim}_{n\to\infty} \|h_n(X)\|^{-1/n} \geq \frac{\rho}{\|X\|} .$$

4. The extension of Liouville's theorem is easily accomplished.

<u>Theorem 1</u> If f(x) is analytic at all points of E, and if $\|f(x)\|$ is bounded in E, then f(x) is a constant.

For let x_0 and x_1 be an arbitrary pair of points in E, and consider the function of the complex variable α, $f(x_0 + \alpha[x - x_0])$. By II, §3, Theorem 5 this function is a constant, and so has the same value at $\alpha = 0$ as at $\alpha = 1$. Thus $f(x_0) = f(x_1)$.

The following theorem regarding a function defined by means of integrals is of considerable importance.

<u>Theorem 2</u> Let C be a rectifiable Jordan curve in the complex plane. Let $y(\tau)$ be a function continuous on C, with values in E. Let f(x) be analytic in a domain D of E. Denote by G the set of points x in E such that $x + y(\tau)$ is in D for all τ on C. Then the integral

$$F(x,\alpha) = \int_C \frac{f(x+y(\tau))}{\tau-\alpha}\,d\tau$$

defines a function analytic in G for each α not on C, and analytic as a function of α at all points not on C, for each x in G. It is continuous in x and α together in these ranges, and

$$\frac{\partial F(x,\alpha)}{\partial\alpha} = \int_C \frac{f(x+y(\tau))}{(\tau-\alpha)^2}\,d\tau$$

$$\delta F(x,\alpha;z) = \int_C \frac{\delta f(x+y(\tau);z)}{\tau-\alpha}\,d\tau$$

Proof: We are assuming that G is non-empty. The assertions regarding $F(x,\alpha)$ as a function of α alone are consequences of II, §3, Theorem 2, since $f(x + y(\tau))$ is continuous on C.

The set G is open; for suppose it contains a point x_0. Then $x_0 + y(\tau)$ lies in D for all τ on C. Since D is open, with each τ there is associated a largest positive * δ_τ such that without exception $\|x - x_0 - y(\tau)\| < \delta_\tau$ (τ fixed) implies that x is in D. Let us suppose for a moment that the numbers δ_τ have a positive lower bound δ as τ ranges over C.

If then $\|x - x_0\| < \delta$, x lies in G, for when τ is on C

$$\|x + y(\tau) - x_0 - y(\tau)\| = \|x - x_0\| < \delta$$

so that $x + y(\tau)$ is always in D. We have yet to demonstrate the existence of δ. Suppose that l.u.b. $[\delta_\tau] = 0$. Then there exists a sequence $\{\tau_n\}$ on C such that $\delta_{\tau_n} \to 0$, and we may suppose that $\tau_n \to \tau'$ on C. Then $y(\tau_n) \to y(\tau')$. Now $\delta_{\tau'} \neq 0$, and we may choose n_0 so that for all larger values of n, $\|y(\tau_n) - y(\tau')\| < \frac{1}{3}\delta_{\tau'}$ and $\delta_{\tau_n} < \frac{1}{3}\delta_{\tau'}$.

---

* There is the exceptional possibility that $\delta_\tau = \infty$, i.e. D is identical with E. This causes no trouble, however.

Then

$$\|x - x_0 - y(\tau')\| \leq \|x - x_0 - y(\tau_n)\| + \tfrac{1}{3}\delta_{\tau'}$$

and to insure that x lie in D it suffices merely to require that $\|x - x_0 - y(\tau_n)\| < \tfrac{2}{3}\delta_{\tau'}$. Since $\delta_{\tau_n} < \tfrac{1}{3}\delta_{\tau'}$ this contradicts the definition of δ_{τ_n}, and hence proves that the numbers δ_τ have a positive lower bound.

We next prove that $F(x,\alpha)$ is continuous in the two variables. Let x_0 be in G, and α_0 a point not on C. We have

$$\|F(x,\alpha) - F(x_0,\alpha_0)\| = \left\| \int_C \left\{ \frac{f(x+y(\tau))}{\tau-\alpha} - \frac{f(x_0+y(\tau))}{\tau-\alpha_0} \right\} d\tau \right\|$$

$$\leq \int_C \frac{|\tau-\alpha_0|\,\|f(x+y(\tau)) - f(x_0+y(\tau))\| + |\alpha-\alpha_0|\,\|f(x_0+y(\tau))\|}{|\tau-\alpha|\,|\tau-\alpha_0|}\,|d\tau|$$

Now $f(x + y(\tau))$ is continuous in x at x_0, uniformly in τ for all τ on C; this follows by I, §3, Theorem 3. Also, $f(x_0 + y(\tau))$ is bounded on C. Since α_0 is not on C, there exists an $l > 0$ such that $|\tau - \alpha_0| \geq l$; and if we require $|\alpha - \alpha_0| < \tfrac{1}{2} l$, then $|\tau - \alpha| \geq \tfrac{1}{2} l$. Let $\varepsilon > 0$ be given, and choose $\delta < \tfrac{1}{2} l$ so that if $\|x - x_0\| < \delta$, the inequality

$$\|f(x+y(\tau)) - f(x_0+y(\tau))\| < \frac{\varepsilon l^2}{2(l+M)L}$$

holds for all τ on C, where L is the length of C, and $\|f(x_0+y(\tau))\| < M$ on C. With this we have

$$\|F(x,\alpha) - F(x_0,\alpha_0)\| \leq \frac{\varepsilon l}{l+M} + \frac{2|\alpha-\alpha_0|ML}{l^2} < \varepsilon$$

if $|\alpha - \alpha_0|$ is less than the smaller of the numbers δ, $\dfrac{\varepsilon l^2}{2(l+M)L}$.

It remains to prove that $F(x,\alpha)$ admits a differential at each point x_0 of G when α is not on C. Let z be any point of E. Then

$$\frac{f(x_0 + \lambda z + y(\tau))}{\tau - \alpha}$$

is an analytic function of λ in a certain neighborhood of $\lambda = 0$, for each τ on C, and it is continuous in λ, τ together. Therefore, by II, §3, Theorem 4 the function

$$F(x_0+\lambda z, \alpha) = \int_C \frac{f(x_0+\lambda z+y(\tau))}{\tau-\alpha}\, d\tau$$

is analytic at $\lambda = 0$ and its derivative may be obtained by differentiatior under the integral sign. Since $\delta F(x_0, \alpha; z) = \left[\frac{\partial F(x_0+\lambda z, \alpha)}{\partial \lambda}\right]_{\lambda=0}$, this completes the proof.

Theorem 3 Let $f(x,\tau)$ be defined for all values of x in a domain D of the space E, and τ on a rectifiable Jordan curve C in the complex plane. Let it be analytic in D for each τ on C, and continuous in both variables together. Then the integral

$$F(x) = \int_C f(x,\tau)\, d\tau$$

defines a function analytic in D, with the differential

$$\delta F(x;y) = \int_C \delta f(x,\tau; y)\, d\tau$$

The only point at which we need comment on the proof of this theorem is that concerning the continuity of F(x). This follows as soon as we have observed that $f(x,\tau)$ is continuous in x, uniformly in τ on the curve C, because of I, §3, Theorem 3. The theorem then is a direct consequence of II, §3, Theorem 4.

Theorem 2 finds an application in the following generalization of Riemann's theorem in II (§5, Theorem 1).

Theorem 4 If f(x) is analytic $0 < \|x-x_0\| < h$, and bounded in this range, then $\lim_{x \to x_0} f(x)$ exists, = A, and if we define $f(x_0) = A$ the function is then analytic at x_0 also.

Proof: For convenience denote by D the domain $0 < \|x-x_0\| < h$. Choose a fixed element y from E, such that $\|y\| = 1$, and consider the function $\psi(\tau) = f(x+\tau y)$, where x is a fixed element of D for which $0 < \|x - x_0\| < \frac{1}{2} h$. $\psi(\tau)$ is analytic for all values of τ such that $x + \tau y$ is in D, that is, for $0 \leq |\tau| < \|x - x_0\|$ and $\|x - x_0\| < |\tau| < h - \|x - x_0\|$. On the circle $\|x - x_0\| = |\tau|$ $\psi(\tau)$ can have at most one singularity, which may occur when $x + \tau y = x_0$. Under the hypotheses of the theorem $\psi(\tau)$ is bounded in the neighborhood of such a point, and so, by II, §5, Theorem 1, approaches a limit. Hence, allowing for a possible completion of the definition of $\psi(\tau)$ on the circle $|\tau| = \|x - x_0\|$,

$$\psi(0) = f(x) = \int_C \frac{f(x+\tau y)}{\tau} d\tau$$

where C is a circle about $\tau = 0$ of radius $\frac{1}{2} h$. This representation is valid $0 < \|x - x_0\| < \frac{1}{2} h$. But the integral

$$F(x) = \frac{1}{2\pi i} \int_C \frac{f(x+\tau y)}{\tau} d\tau$$

defines a function analytic without exception in the region $0 \leq \|x - x_0\| < \frac{1}{2} h$ as we see by reference to Theorem 2. Thus $f(x) = F(x)$ when $x \neq x_0$, and the theorem follows at once.

5. Isolated singular points of a function f(x) are defined precisely as in the case of a function of a complex variable (see II, §5). Likewise we define removable singularities and poles as before, and agree to call all other singularities essential. We have already proved the fundamental theorem regarding removable singularities (III, §4, Theorem 4). Next we prove a theorem analogous to that of Laurent.

Theorem 1 Let f(x) be analytic in the domain D: $0 \leq r < \|x\| < R$. Then f(x) may be expanded in the form

$$f(x) = \sum_{-\infty}^{\infty} p_n(x) \qquad r < \|x\| < R$$

where

$$p_n(x) = \frac{1}{2\pi i} \int_C \frac{f(\tau x)}{\tau^{n+1}} d\tau \qquad n = 0, \pm 1, \pm 2, \ldots$$

C being a circle of radius ρ, $\frac{r}{\|x\|} < \rho < \frac{R}{\|x\|}$ about the origin in the complex plane. The series converges uniformly in every compact set extracted from a closed subset of D.

Proof: For a fixed x in D, $f(\alpha x)$ is analytic in α when $\frac{r}{\|x\|} < |\alpha| < \frac{R}{\|x\|}$. Hence by II, §5, Theorem 2,

$$f(\alpha x) = \sum_{-\infty}^{\infty} \alpha^n p_n(x)$$

where $p_n(x)$ is given as above. Since the region of convergence includes the point $\alpha = 1$, we obtain the desired expansion. To complete the proof we observe that when a closed subset of D has been chosen, there exist numbers ρ_1 and ρ_2 such that $\frac{r}{\|x\|} < \rho_1 < 1 < \rho_2 < \frac{R}{\|x\|}$ for all the elements x of this set. Then if x is in a compact set we obtain from the integral representation the inequalities

$$\|p_n(x)\| \leq \frac{M_1}{\rho_1^{\,n}} \qquad n = -1, -2, \cdots$$

$$\|p_n(x)\| \leq \frac{M_2}{\rho_2^{\,n}} \qquad n = 0, 1, 2, \cdots\cdots$$

where M_1 and M_2 are upper bounds of $\|f(\sigma x)\|$ on circles of radii ρ_1 and ρ_2 respectively, as x ranges over the compact set. Since the series

$$M_1 \sum_{1}^{\infty} \rho_1^{\,n} + M_2 \sum_{0}^{\infty} \frac{1}{\rho_2^{\,n}}$$

converges, the proof is accomplished.

It is easily seen that the functions $p_n(x)$ are analytic when $r < \|x\| < R$, and that

$$p_n(\alpha x) = \alpha^n p_n(x)$$

when αx and x both satisfy this inequality.

It is to be expected that the most important singularities of analytic functions of an abstract variable will not be isolated. For functions of several complex variables, for instance, poles are never isolated, while essential singularities may or may not be. In general the singularities of such a function are not isolated, and their classification is a complicated problem. We shall accordingly limit our discussion to a few general considerations and the exhibition of examples.

When the space E' is merely the space of complex numbers, so that the values of f(x) are numerical, we call f(x) a functional. By III, §1, Theorem 1 and the classical theory of functions we conclude the following:

<u>Theorem 2</u> If f(x) is an analytic functional in a domain D, and does not vanish there, then $g(x) = \frac{1}{f(x)}$ is analytic in D, and

$$\delta g(x;y) = -\frac{\delta f(x;y)}{[f(x)]^2}$$

For functionals we have the theorem of Weierstrass pertaining to isolated essential singularities.

<u>Theorem 3</u> If f(x) is an analytic functional with an isolated essential singularity at $x = x_0$, it comes arbitrarily near any preassigned value in every neighborhood of that point.

Proof: Let A be an arbitrary complex number. Then, when ε and h are arbitrarily assigned positive numbers, there exists a point x such that $|f(x) - A| < \varepsilon$ and $0 < |x - x_0| < h$. For suppose this were not the case. Then

$$\left| \frac{1}{f(x) - A} \right| \leq \frac{1}{\varepsilon} \qquad \text{when} \qquad 0 < |x - x_0| < h$$

and so the analytic functional $g(x) = \frac{1}{f(x) - A}$ must have a removable singularity at $x = x_0$ (see III, §4, Theorem 4). Let $\lim_{x \to x_0} g(x) = B$. If $B = 0$, $\lim_{x \to x_0} f(x) = \infty$, whereas if $B \neq 0$, $\lim_{x \to x_0} f(x)$ exists, and both these eventualities contradict the hypotheses of the theorem.

Illuminating examples of functionals and the singularities which they may display are easily constructed by taking for E the space of complex-valued continuous functions $x(t)$ defined $0 \leq t \leq 1$, with the norm $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$.

If $g(x)$ is a functional analytic in a domain D of E, with values filling out a region S of the complex plane, and if $F(\alpha)$ is an ordinary function which is analytic in S, then $f(x) = F[g(x)]$ is an analytic functional of x in D. If $F(\alpha)$ has an isolated singularity at $\alpha = \alpha_0$, and G is the set of elements for which $g(x) = \alpha_0$, then the points of G are singular points of $f(x)$. For example, consider $F[g(x)]$, where

$$g(x) = \int_0^1 x(t)\,dt$$

The functional

$$F[g(x)] = \frac{1}{1-g(x)}$$

has a pole for all points $x(t)$ of E such that $\int_0^1 x(t)\,dt = 1$. These points clearly form a closed, non-isolated point set. Similarly the functional

$$f[g(x)] = e^{\frac{1}{1-g(x)}}$$

has an essential singularity at each point of the above set.

It is interesting to note that the above functional $g(x)$ is a homogeneous polynomial of degree one, which does not have a pole 'at infinity'; that is, $\|g(x)\|$ approaches no definite limit as $\|x\| \to \infty$. In fact, there exists a sequence $x_n(t)$ such that $\|x_n\| \to \infty$ and $g(x_n) = 0$. It is therefore impossible to characterize polynomials as entire functions with a pole 'at infinity'.

6. In I, §1 we defined the complex couple-space associated with a real vector space. If E(C) and E'(C) are two such couple-spaces, associated with the real vector spaces E(R) and E'(R), respectively, a function $f(z)$ on E(C) to E'(C) has the form

$$f(z) = f_1(x,y) + i f_2(x,y)$$

where $f_1(x,y)$ and $f_2(x,y)$ are functions of two variables over E(R), with values in E'(R).

Let us now suppose that E'(R) is complete, and that f(z) is defined in a domain D of E(C). Then we can discuss the analyticity of f(z) in terms of the properties of the functions f_1 and f_2. The fundamental proposition, a generalization of the classical theorem pertaining to the Cauchy-Riemann equations, is as follows:

Theorem 1 In order that f(z) be analytic in D it is necessary and sufficient that the functions $f_1(x,y)$, $f_2(x,y)$ be continuous and admit continuous first partial Gateaux differentials at all points of D, and that the equations

$$(1)\qquad \begin{aligned} \delta_x f_1(x,y;\xi) &= \delta_y f_2(x,y;\xi) \\ \delta_y f_1(x,y;\xi) &= -\delta_x f_2(x,y;\xi) \end{aligned}$$

be satisfied in D for an arbitrary ξ in E(R).

Proof: If f(z) is analytic in D it is continuous there, and the differential $\delta f(z;\Delta z)$ is linear in Δz, and continuous in the pair z, Δz. But

$$\delta f(z;\Delta z) = \lim_{\tau\to 0} \frac{f(z+\tau\Delta z)-f(z)}{\tau}$$

Hence in particular, taking $\Delta z = \Delta x + i\cdot 0$, $\tau = t$, where t is real,

$$\delta f(z;\Delta z) = \lim_{t\to 0}\left[\frac{f_1(x+t\Delta x,y)-f_1(x,y)+if_2(x+t\Delta x,y)-if_2(x,y)}{t}\right]$$

This limit will exist, however, only if the separate parts have limits. Therefore

$$\delta f(z;\Delta x) = \delta_x f_1(x,y;\Delta x) + i\,\delta_x f_2(x,y;\Delta x)$$

Similarly we obtain

$$\delta f(z;\Delta x) = \delta_y f_2(x,y;\Delta x) - i\,\delta_y f_1(x,y;\Delta x)$$

Since the left member of these equations is continuous, we see that the four terms on the right must be continuous in $(x, y, \Delta x)$ when $x + i \cdot y$ is in D and Δx is arbitrary in E(R). On equating corresponding parts we obtain equations (1). The continuity of f_1 and f_2 is a consequence of the continuity of $f(z)$.

To prove the sufficiency of the conditions suppose that $\Delta z = \Delta x + i\Delta y$ is an arbitrary element of E(C), and consider the expression

$$\frac{f(z+\tau\Delta z)-f(z)}{\tau} = \frac{f_1(x+s\Delta x-t\Delta y,\ y+t\Delta x+s\Delta y)-f_1(x,y)}{s+it}$$

$$+\,i\,\frac{f_2(x+s\Delta x-t\Delta y,\ y+t\Delta x+s\Delta y)-f_2(x,y)}{s+it}$$

where z is in D and $\tau = s + i \cdot t$ is a sufficiently small complex number. Next, consider the function

$$F_1(s,t,u,v) = f_1(x+s\Delta x-t\Delta y,\ y+u\Delta x+v\Delta y)$$

of four real variables, with values in E'(R). This function is continuous and admits continuous first partial derivatives near (0,0,0,0). It is then not difficult to show that it admits a total differential* at (0,0,0,0),

---

* We demonstrate the theorem, for simplicity, using only two variables; the general case is proved in a similar manner. Writing

$$f(s,t)-f(0,0)-sf_s(0,0)-tf_t(0,0) = \mathcal{E}(s,t)$$

we have

$$\frac{\|\mathcal{E}(s,t)\|}{|s|+|t|} \leq \frac{\|f(s,t)-f(s,0)-tf_t(s,0)\|}{|s|+|t|} + \frac{\|f(s,0)-f(0,0)-sf_s(0,0)\|}{|s|+|t|}$$

$$+ \frac{t\ \|f_t(s,0)-f_t(0,0)\|}{|s|+|t|}$$

$$\leq \frac{\left\|\int_0^t \{f_t(s,u)-f_t(s,0)\}\,du\right\|}{|s|+|t|} + \frac{\|f(s,0)-f(0,0)-sf_s(0,0)\|}{|s|} + \|f_t(s,0)-f_t(0,0)\|$$

and from this the result follows without difficulty as a result of the uniform continuity of $f_t(s,t)$.

that is

$$F_1(s,t,u,v) - F(0,0,0,0) = s F_{1s}(0,0,0,0) + t F_{1t}(0,0,0,0) + u F_{1u}(0,0,0,0)$$
$$+ v F_{1v}(0,0,0,0) + \mathcal{E}(s,t,u,v)$$

where $\displaystyle \lim_{(s,\cdots v)\to(0,\cdots 0)} \frac{\|\mathcal{E}(s,t,u,v)\|}{|s|+\cdots+|v|} = 0$

Therefore, when expressed in terms of Gateaux differentials, we have

$$f_1(x+s\Delta x-t\Delta y,\ y+t\Delta x+s\Delta y)-f_1(x,y) = s\,\delta_x f_1(x,y;\Delta x)-t\,\delta_x f_1(x,y;\Delta y)$$
$$+ t\,\delta_y f_1(x,y;\Delta x)+s\,\delta_y f_1(x,y;\Delta y)$$
$$+ \mathcal{E}(s,t,t,s)$$

There is a similar relation involving the function . $f_2(x,y)$ and an infinitesimal $\eta(s,t,t,s)$. On making use of the equations (1) we find that

$$\frac{f(z+\tau\Delta z)-f(z)}{\tau} = \Big[\{\delta_x f_1(x,y;\Delta x) - \delta_x f_2(x,y;\Delta y)\}$$
$$+i\{\delta_x f_1(x,y;\Delta y) + \delta_x f_2(x,y;\Delta x)\}\Big]\frac{s+it}{\tau}$$
$$+ \frac{\mathcal{E}+i\eta}{\tau}$$

But $\|\mathcal{E}+i\eta\| \leq \|\mathcal{E}\| + \|\eta\|$, and $|\tau| \geq \frac{1}{\sqrt{2}}(|s|+|t|)$. From this we conclude that

$$\lim_{\tau\to 0} \frac{\|\mathcal{E}+i\eta\|}{|\tau|} = 0 ,$$

and hence that f(z) has the differential

$$\delta f(z;\Delta z) = \delta_x f_1(x,y;\Delta x) - \delta_x f_2(x,y;\Delta y)$$
$$+ i\{\delta_x f_1(x,y;\Delta y) + \delta_x f_2(x,y;\Delta x)\}$$

Since f_1 and f_2 are continuous, so is f(z), and f(z) is analytic. This proves the theorem.

The known properties of $\delta f(z;\Delta z)$, as the differential of an analytic function, enable us to draw conclusions about the properties of the functions f_1, f_2, and their differentials. The various Gateaux differentials are in fact partial Fréchet differentials; their linearity is evident. We can, in fact, prove that $f_1(x,y)$ and $f_2(x,y)$ admit total Fréchet differentials. That is, we regard the element pair (x,y) as a single element in a composite space such that

$$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$$

$$a\cdot(x_1,y_1) = (a\cdot x_1, a\cdot y_1)$$

$$\|(x_1,y_1)\| = \|x_1\| + \|y_1\|$$

If we write

$$\delta f_1(x,y;\xi,\eta) = \delta_x f_1(x,y;\xi) + \delta_y f_1(x,y;\eta)$$

with a similar expression in $f_2(x,y)$, then

$$\delta f(z;\Delta z) = \delta f_1(x,y;\xi,\eta) + i\,\delta f_2(x,y;\xi,\eta)$$

where $\Delta z = \xi + i\eta$. From this, and the fact that $\|\Delta z\| \leq \|\xi\| + \|\eta\|$ it is easy to prove that $\delta f_1(x,y;\xi;\eta)$ is in fact the total Fréchet differential of $f_1(x,y)$.

Theorem 2 If $f_1(x,y)$ and $f_2(x,y)$ are two functions on $E(R)$ to $E'(R)$, where $E'(R)$ is complete, and

$$f(z) = f_1(x,y) + i\, f_2(x,y)$$

is analytic in a domain D of the couple-space $E(C)$, then f_1 and f_2 admit total Fréchet differentials in D, and these differentials, for fixed values of the increments, are such that

$$\delta f_1(x,y;\xi,\eta) + i\,\delta f_2(x,y;\xi,\eta)$$

is analytic in D. The partial Fréchet differentials of f_1 and f_2, of all orders, exist and are continuous in D. They have certain symmetry properties, such as

$$\delta^2_{xx} f_1(x,y;\xi_1,\xi_2) = \delta^2_{xx} f_1(x,y;\xi_2,\xi_1)$$

$$\delta^2_{xy} f_1(x,y;\xi,\eta) = \delta^2_{yx} f_1(x,y;\eta,\xi)$$

The proof requires no elaboration.

7. In this concluding paragraph we shall turn our attention to the relation of the foregoing work to that of other writers on the same subject. The memoirs of Gateaux, already mentioned, deal with special cases, and our work may be regarded as the logical abstraction and completion of these memoirs.

Professor Norbert Wiener* pointed out the validity of Cauchy's integral theorem for abstractly-valued functions of a complex variable, and so opened the way for the systematic developments which we have given in II.

The work of the Italian mathematician, Luigi Fantappié ** on 'analytic functionals' is not directly connected with our general theory. Fantappié considers operations $F[f(z)]$ such that to each function $f(z)$, analytic in the sense of Weierstrass (i.e. a complete monogenic function) there corresponds a certain complex number. Such a functional operation is said to be analytic if when $f(z,\alpha)$ depends analytically upon α in a certain domain, $F[f(z,\alpha)]$ is an ordinary analytic function of α. The difficulty which prevents such functionals from being included in our theory lies in the fact that the class (A) of complete monogenic functions is not a vector space. Various topological questions connected with this class have been investigated by Minetti ***, and it appears that certain of Fantappié's analytic functionals are not continuous according to the topology suggested in his work.

In spite of these matters the underlying notion of Fantappié's definition may serve as a definition of analyticity equivalent to the

---

* N. Wiener, (14) p.139.
** L. Fantappié, (1) and (2).
*** S. Minetti, (9) p. 118 ff.

one which we have used. Let us phrase it as follows:

Definition A function $f(x)$ on E to E' is said to be analytic in a domain D if

1° it is continuous in D

2° whenever $\varphi(\alpha)$ is an analytic function on C to E, and T is a domain of the plane such that $\varphi(\alpha)$ is analytic in T, and $\varphi(\alpha)$ lies in D when α is in T, then $f(\varphi(\alpha))$ is analytic in T.

This is clearly sufficient to insure the existence of the Gateaux differential. Condition 2° is also necessary, for if $f(x)$ is analytic, it admits a Fréchet differential in D. Therefore, by I, §3, Theorem 5, $f(\varphi(\alpha))$ admits a derivative

$$\frac{\partial f(\varphi(\alpha))}{\partial \alpha} = \delta f(\varphi(\alpha);\varphi'(\alpha))$$

whenever $\varphi(\alpha)$ is in D.

A definition of analyticity very similar to that which is expressed in our theorems on power series was used by R.S. Martin in his thesis, and subsequently by Professor A.D. Michal and others.* According to Martin a function on E to E' is analytic at a point x_0 if it admits an expansion in a series of homogeneous polynomials:

$$f(x) = \sum_{n=0}^{\infty} h_n(x-x_0)$$

If mh_n is the modulus of $h_n(x)$, and the series

$$\sum_{0}^{\infty} mh_n \lambda^n$$

is convergent when $|\lambda|<\rho$, ρ is called the radius of analyticity of the function at the point $x = x_0$. When the space E' is complete this implies

---

* R.S. Martin, (7) p. 58. A.D. Michal and A.H. Clifford, (8). A.D. Michal and R.S. Martin, (8a). The last two papers deal with various applications of the theory to problems in general analysis.

that the series defining f(x) is uniformly convergent when $\|x-x_0\| \leq \theta\rho$, $0 < \theta < 1$. It would therefore seem that this radius ρ is smaller than the radius of analyticity as we have defined it in III, §2, for a closed sphere is not in general compact, and uniform convergence in a compact set is all that is supposed in our work.

However, a function analytic in Martin's sense is analytic in the sense of this paper, and conversely; the only question is that of the radius of analyticity, for in the proof of III, §2, Theorem 2 we saw that the series of moduli

$$\sum_0^\infty mh_n \lambda^n$$

had a positive radius of convergence. A closer inspection reveals the likelihood that this radius is less than the radius of analyticity of f(x) at x_0. Let ρ be this latter radius. Then, letting $x - x_0 = \lambda\xi$ where $\|\xi\| = 1$, we know that the series

$$\sum_0^\infty |\lambda|^n \|h_n(\xi)\|$$

converges when $|\lambda| < \rho$, and is a continuous function of ξ. Now

$$\underset{\|\xi\|=1}{\text{l.u.b.}} \|h_n(\xi)\| = mh_n$$

and hence, if it should happen that a sequence $\{\xi_\nu\}$ exists such that

$$\lim_{\nu\to\infty} \|h_n(\xi_\nu)\| = mh_n$$

for all sufficiently large values of n, we could assert that the series

$$\sum_0^\infty |\lambda|^n mh_n$$

converges when $|\lambda| < \rho$. It is not at all obvious that such a sequence will exist.

It should be remarked that the notion of radius of analyticity is not as important in the general theory as it is in the classical case. This is because the region of convergence of a power series is not necessarily spherical, or circular, as is the case with numerical power series. For instance, in the example of III, §5, the functional

$$\frac{1}{1-g(x)} = 1 + g(x) + [g(x)]^2 + \ldots\ldots$$

where $g(x) = \int_0^1 x(t)\,dt$, has unit radius of analyticity at $x = 0$. Yet the power series defining the function converges and defines an analytic function when $|g(x)| < 1$. The open region defined by this inequality includes far more than the sphere $\|x\| < 1$; indeed one may construct elements of arbitrarily large norm lying in the region defined by $|g(x)| < 1$.

BIBLIOGRAPHY

(1) L. Fantappie, Memorie dei Lincei, vol.3, fasc.11 (1930)

(2) L. Fantappie, Jahresbericht der Deutschen Math. Verein. vol.43, (1933) p.1-24.

(3) R. Gateaux, Bulletin de la Societe Math. de France, vol.47, (1919) p.70-96.

(4) R. Gateaux, Bulletin de la Societe Math. de France, vol.50, (1922) p.1-21.

(5) M. Kerner, Studia Mathematica, vol.3, (1931) p.156-173.

(6) M. Kerner, Annals of Mathematics, vol.34, (1933) p.546-572.

(7) R.S. Martin, California Institute of Technology Thesis, (1932)

(8) A.D. Michal and A.H. Clifford, Comptes Rendus, vol.197, (1933) p.735-737.

(8a) A.D. Michal and R.S. Martin, Journal de Math. Pures et Appliquees, vol.13, (1934) p.69-91.

(9) S. Minetti, Rendiconti del Circolo Math. di Palermo, vol.59, (1935) p.97-136.

(10) W.F. Osgood, Lehrbuch der Funktionentheorie, vol.I 5 ed. (1928)

(11) W.F. Osgood, Lehrbuch der Funktionentheorie, vol.II 1, 2 ed. (1929)

(12) G.N. Watson, Complex Integration and Cauchy's Theorem, Cambridge Tract no. 15, (1914)

(13) N. Wiener, Fundamenta Mathematicae, vol.4, (1923) p.136-143.