INVERSION AND REPRESENTATION THEOREMS FOR THE LAPLACE TRANSFORMATION

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ABSTRACT

A study is made of the Laplace transformation on Banach-valued functions of a real variable, with particular reference to inversion and representation theories. First a new type of integral for Banach-valued functions of a real variable, the "Improper Bochner" integral is defined. The relations between the Bochner, Improper Bochner, Riemann-Graves, and Riemann-Stieltjes integrals are studied. Next, inversion theorems are proved for a new "real" inversion operator when the integral in the Laplace transformation is each of the above-mentioned types. Lastly, representation of Banach-valued functions by Laplace integrals of functions in $B_p([0,\infty); \mathfrak{X})$, $1 < p \leq \infty$, is studied, and theorems are proved giving necessary and sufficient conditions. The theorems are very like those proved, for numerically-valued functions, by D. V. Widder in his book "The Laplace Transform" (Princeton, 1941) page 312. The classes $H_p(\alpha; \mathfrak{X})$, $1 \leq p < \infty$, are also studied in this section as is the representation of numerically-valued functions by Laplace-Stieltjes integrals.

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INTRODUCTION

The theory of the Laplace transformation on real or complex functions of a real variable is one which has been, for a number of years, of considerable interest to both pure and applied mathematicians. The interest of a number of these mathematicians has centered around inversion and representation theories, the former of these theories being of interest from both the utilitarian and function-theoretic points of view, and the latter from a primarily function-theoretic viewpoint. Historically, the Laplace transformation arrived on the scene considerably before large developments in abstract spaces, and consequently was defined only on numerically-valued functions. However, the extension of the definition of the transformation to the domain of Banach-valued functions of a real variable, that is functions on $[0,\infty)$ to a Banach space, is one of the several extensions that were made, and it is with this extension that we shall be dealing here. In many respects the theory resulting from this extension is now approaching the degree of completeness enjoyed by the theory for numerically-valued functions. However, in the respects of inversion and especially representation theories, the approach is not so close. In particular there are, as yet, no representation theorems for Banach-valued functions of what might be called the "Widder" type. This terminology requires some explanation.

Let $f(\lambda)$ be a numerically-valued function of the real or complex variable λ, and let $L_{k,\tau}[f(\lambda)]$ denote any fixed but arbitrary inversion operator for the Laplace transformation. Then it has been shown for many such inversion operators that if $L_{\kappa,\tau}[f(\lambda)]$ exists, and either $\int_0^\infty |L_{\kappa,\tau}[f(\lambda)]|^p \, d\tau \leq M$, p fixed, $1 < p < \infty$, $\kappa > \kappa_0$, or $\operatorname{ess.\,sup}_{0 \leq \tau < \infty} |L_{\kappa,\tau}[f(\lambda)]| \leq M$, $\kappa > \kappa_0$, where M is independent of κ, $f(\lambda)$ is equal for $\lambda > 0$ to the Laplace transform of a function in $L_p(0,\infty)$ or $L_\infty(0,\infty)$ respectively. A theory of this type for a particular inversion operator is what we call a "Widder" type theory. For examples see Widder * [12, ch. 7, § 15].

The task of developing "Widder" type representation theorems for Banach-valued functions is the one we have set ourselves.

Since a "Widder" type representation theory is stated in terms of a specific inversion operator, the opportunity was also presented both to enlarge the inversion theory for Banach-valued functions, and to study a new inversion operator for the Laplace transformation. We have grasped this opportunity and have developed the theory in terms of a new "real" inversion operator. A real inversion operator is one that utilizes the values of the generating function arising only from real values of the independent variable. Several of these are known; for examples see Widder [12, ch. 7, § 6; ch. 8 § 25] , or Hirschman [7]. The new operator in question is defined by the formula

$$\text{I} \quad L_{\kappa,\tau}[f(\lambda)] = \frac{\kappa e^{2\kappa}}{\pi\tau} \int_0^\infty \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta+1)/\tau) \, d\eta \, .$$

If $f(\lambda) = \int_0^\infty e^{-\lambda\tau} \varphi(\tau) \, d\tau$, then under certain conditions,

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* Numbers in square brackets refer to the bibliography at the end of the dissertation.

$$\lim_{\kappa \to \infty} L_{\kappa,\tau}[f(\lambda)] = \varphi(\tau).$$

The fact that the representation theorems will be stated relative to this particular operator is no real restriction, for the method is quite general, and will work equally well with any inversion operator for which the theorems are true in the numerically-valued case.

This operator was originally given by A. Erdélyi [3]. However the resulting inversion and representation theories were not developed there. These theories were developed, by the author, originally for the numerically-valued case and have been accepted for publication; see Rooney [11].

There is another operator related to I, which is given by the formula

$$\text{II} \quad L_{\kappa,\tau}[f(\lambda)] = \frac{\kappa e^{2\kappa}}{\pi\tau} \int_0^\infty \sin(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta+1)/\tau)\, d\eta .$$

Both I and II are special cases of a third operator

$$\text{III} \quad L_{\kappa,\tau}[f(\lambda)] = (2\tau K_\nu(2\kappa))^{-1}\kappa \int_0^\infty \eta^{\nu/2} J_\nu(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta+1)/\tau)\, d\eta,$$

which can also be found in Prof. Erdélyi's paper. The inversion and representation theories for these last operators have also been investigated, and were found to be similar in every respect to those for operator I. To avoid inessential difficulties we shall restrict our attention to operator I.

Operator I has some points of resemblance to Phragmén's operator [12; ch. 7, § 2] in that both are "real" inversion operators, involve only the values of f(λ) for large real values of λ, and involve only elementary functions. However, Phragmén's operator is not an integral operator.

We have, perhaps inevitably, been drawn into certain subjects which, while of great interest and importance, are subsidiary to the main theme of this dissertation. Chief among these are certain difficulties concerning the relations between various integrals of Banach-valued functions. These difficulties are resolved in Chapter I, wherein are also the main theorems we shall need concerning these various integrals.

Chapter II is devoted to inversion theory. This theory is developed for several different kinds of integrals. Chapter III is given over to the representation theory. This chapter may be considered to contain the principal results of the dissertation.

Chapter I

1. Introduction.

In this chapter we collect and elaborate certain theorems concerning integrals of Banach-valued functions of real variables. The main tool in this regard is the Bochner integral, which is an analogue for this type of function, of the Lebesgue integral. Its theory is developed in sections 2 and 3. Nearly all of the material of these two sections is abstracted, verbatim or paraphrased, from Hille [6]. Consequently, for such theorems no proof is offered. The reader who wishes to see proofs of these theorems should look in Hille [6]. In other places where known results are used, we shall give references to these results.

Section 4 introduces a slight generalization of the Bochner integral which we call the "Improper Bochner Integral". It is the analogue of the improper Lebesgue integral.

Less powerful, but nevertheless important, tools are the Riemann-Graves and Riemann-Stieltjes integrals, which correspond, in the numerically-valued case, to the Riemann and Riemann-Stieltjes integrals respectively. Their theory is outlined in sections 5 and 6.

In section 7 we develop theorems giving sufficient conditions for various of these integrals to be equivalent. Finally in section 8, we develop a weak sequential form of the Banach-Steinhaus theorem.

The notation to be employed in this and subsequent chapters is, in the main, that of Hille [6]. That is, Banach spaces will be denoted by German capital letters, and their elements by English lower case letters. The space of bounded linear functionals over a Banach space will be denoted by "starring" the symbol for that space, and its elements will be denoted by the "starred" elements of the Banach space. Real or complex numbers will be denoted by Greek letters. We shall denote the zero vector of a Banach space by θ, and the void set of a collection of sets by $\emptyset$. Several exceptions to this rule will be made, mostly in cases where long usage has prescribed symbols, which clash with the above notation, for certain quantities, e.g. L_p, e, etc. One notable exception is that we shall often use English letters for subscripts. Other exceptions will be seen to occur.

One other point is worthy of notice. Whenever we use the word "limit" we mean the limit in the strong sense. Other types of limits will be prefaced by explaining words.

2. Functions and Measure: Let E_k be the k-dimensional Euclidean space, Σ a measurable set in E_k, and $x(\alpha)$ a function on Σ to the Banach space $\mathfrak{X}$.

Definition 1.2.1: Let $x(\alpha)$ and $x_n(\alpha)$ be functions on Σ to $\mathfrak{X}$. The sequence $\{x_n(\alpha)\}$ converges to $x(\alpha)$

(i) almost uniformly if to every $\varepsilon > 0$, there is a set Σ_ε with $m(\Sigma_\varepsilon) < \varepsilon$ such that $\{x_n(\alpha)\}$ converges uniformly to $x(\alpha)$ on $\Sigma - \Sigma_\varepsilon$.

(ii) almost everywhere if there exists a null set $\Sigma_0 \subset \Sigma$ such that $\|x(\alpha) - x_n(\alpha)\| \to 0$ for α in $\Sigma - \Sigma_0$.

Theorem 1.2.1: The two types of convergence are related as follows: (i) implies (ii), and if $m(\Sigma) < \infty$, (ii) implies (i).

Definition 1.2.2:

(i) $x(\alpha)$ is said to be finitely-valued in Σ if it is constant on each of a finite number of disjoint measurable sets Σ_j with $\bigcup_j \Sigma_j = \Sigma$.

(ii) $x(\alpha)$ is said to be countably valued if it assumes at most a countable set of values in $\mathfrak{X}$ each on a separate measurable set Σ_j.

Definition 1.2.3.:

(i) $x(\alpha)$ is said to be weakly measurable in Σ if $x^*[x(\alpha)]$ is measurable (Lebesgue) in Σ for every $x^* \varepsilon \mathfrak{X}^*$.

(ii) $x(\alpha)$ is strongly measurable in Σ if there exists a sequence of countably-valued functions converging almost uniformly in Σ to $x(\alpha)$.

Theorem 1.2.2:

(i) If $x(\alpha)$ and $y(\alpha)$ are strongly measurable in Σ and γ_1 and γ_2 are constants, then $\gamma_1 x(\alpha) + \gamma_2 y(\alpha)$ is strongly measurable.

(ii) If $\varphi(\alpha)$ is a finite numerically-valued function which is measurable (Lebesgue), then $\varphi(\alpha)\cdot x(\alpha)$ is strongly measurable if $x(\alpha)$ has this property.

(iii) If $x(\alpha)$ is the limit almost everywhere of a sequence of strongly measurable functions, then $x(\alpha)$ is strongly measurable.

Definition 1.2.4: A function $x(\alpha)$ on the closed interval $[\xi_1, \xi_2]$ to the space $\mathfrak{X}$ is of

(i) bounded variation if $\sup \| \sum_i [x(\beta_i) - x(\alpha_i)] \| < \infty$ for every choice of a finite number of non-overlapping intervals (α_i, β_i) in $[\xi_1, \xi_2]$;

(ii) strongly bounded variation if $\sup \sum_i \| x(\alpha_i) - x(\alpha_{i-1}) \| < \infty$ where all possible partitions of $[\xi_1, \xi_2]$ are allowed. The two suprema are known as the total and strong total variations respectively.

Definition 1.2.5: A set Ω of complex numbers will be called a domain if Ω is an open connected set. The closure of a domain will be called a closed domain.

Definition 1.2.6: If ζ is a complex variable, and $x(\zeta)$ is a function on the open domain Ω of the complex plane to $\mathfrak{X}$, then $x(\zeta)$ will be called holomorphic in Ω if $x^*(x(\zeta))$ is holomorphic in Cauchy's sense for every x^* in $\mathfrak{X}^*$.

3. Integration.

Definition 1.3.1: A countably-valued function $x(\alpha)$ on Σ to $\mathfrak{X}$ is integrable (Bochner) if and only if $\|x(\alpha)\|$ is integrable (Lebesgue). By definition

$$(B)\int_{\Sigma} x(\alpha)d\alpha = \sum_{j=1}^{\infty} x_j\, m(\Sigma_j).$$

The series converges since

$$\left\|\sum_{j=1}^{\infty} x_j\, m(\Sigma_j)\right\| \leq \sum_{j=1}^{\infty} \|x_j\|\, m(\Sigma_j) = \int_{\Sigma} \|x(\alpha)\|\, d\alpha .$$

Consequently

$$\left\|(B)\int_{\Sigma} x(\alpha)d\alpha\right\| \leq \int_{\Sigma}\|x(\alpha)\|\, d\alpha .$$

Definition 1.3.2: A function $x(\alpha)$ on Σ to $\mathfrak{X}$ is integrable (Bochner) if and only if there exists a sequence of countably-valued functions converging almost uniformly to $x(\alpha)$, and such that

$$\lim_{m,n\to\infty}\int_{\Sigma} \|x_m(\alpha) - x_n(\alpha)\|\, d\alpha = 0.$$

By definition

$$(B)\int_{\Sigma} x(\alpha)d\alpha = \lim_{n\to\infty}(B)\int_{\Sigma} x_n(\alpha)d\alpha .$$

We shall drop the "(B)" from the integral when there is no danger of confusion.

Under the postulated conditions, the integral exists uniquely. That is, for every sequence $\{x_n\}$ of countably-valued functions with the above postulated properties, $\lim_{n \to \infty}$ (B) $\int_{\Sigma} x_n(\alpha)d\alpha$ exists, and has the same value.

Theorem 1.3.1: A necessary and sufficient condition that $x(\alpha)$ on Σ to $\mathfrak{X}$ be integrable (Bochner) is that $x(\alpha)$ be strongly measurable, and that

$$\int_{\Sigma} \|x(\alpha)\| \, d\alpha < \infty.$$

Definition 1.3.3: A function $x(\alpha)$ on Σ to $\mathfrak{X}$ belongs to $B_p(\Sigma ; \mathfrak{X})$, $1 \leq p < \infty$, if $x(\alpha)$ is strongly measurable on Σ, and $\int_{\Sigma} \|x(\alpha)\|^p \, d\alpha < \infty$. $x(\alpha)$ belongs to $B_{\infty}(\Sigma , \mathfrak{X})$ if $x(\alpha)$ is strongly measurable and is bounded except in a null set.*

Theorem 1.3.2: If $\mathfrak{X}$ is a real or complex Banach space, then $B_p(\Sigma ; \mathfrak{X})$ is a real or complex Banach space under the norm

$$\| x(\cdot)\|_p = \left\{ \int_{\Sigma} \| x(\alpha)\|^p \, d\alpha \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

$$\| x(\cdot)\|_{\infty} = \operatorname{ess.sup}_{\Sigma} \| x(\alpha)\| .$$

Theorem 1.3.3: If $x(\alpha)$ is in $B(\Sigma ; \mathfrak{X})$, then

$$\left\| \int_{\Sigma} x(\alpha)d\alpha \right\| \leq \int_{\Sigma} \|x(\alpha)\| \, d\alpha .$$

Theorem 1.3.4: If $x(\alpha)$ is in $B(\Sigma ; \mathfrak{X})$ and x^* is in $\mathfrak{X}^*$, then $x^*(x(\alpha))$ is in $L(\Sigma)$, and

$$x^*\left(\int_{\Sigma} x(\alpha)d\alpha\right) = \int_{\Sigma} x^*(x(\alpha))d\alpha .$$

Theorem 1.3.5: If $x(\alpha)$ is in $B(E_1 ; \mathfrak{X})$, then for almost all $\mathfrak{z}$ in E_1,

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* We shall often use $B(\Sigma ; \mathfrak{X})$ for $B_1(\Sigma ; \mathfrak{X})$.

(i) $$\lim_{\gamma \to 0} \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} \| x(\alpha) - x(\xi) \| \, d\alpha = 0,$$

and in particular

(ii) $$\lim_{\gamma \to 0} \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} x(\alpha) d\alpha = x(\xi).$$

Definition 1.3.4: We shall call the set of ξ where formula (i) of the preceding theorem holds true the Lebesgue set of $x(\alpha)$.

Theorem 1.3.6: If $x(\alpha)$ is in $B(E_1; \mathfrak{X})$, and if ξ is in the Lebesgue set of $x(\alpha)$, then the Lebesgue set of $x(\alpha)$ is equal to the Lebesgue set of $\| x(\alpha) - x(\xi) \|$.

Theorem 1.3.7: If $x(\alpha, \beta)$ is a strongly measurable function of $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$, then $x(\alpha, \beta)$ is in $B(E_{m+n}; \mathfrak{X})$ if there is a function $y(\alpha, \beta)$ such that $y(\alpha, \beta) = x(\alpha, \beta)$ for almost all (α, β) and $\int_{E_n} \left\{ \int_{E_m} \| y(\alpha, \beta) \| d\alpha \right\} d\beta$ exists. In this case

$$\int_{E_{m+n}} x(\alpha, \beta) d\alpha \, d\beta = \int_{E_{m+n}} y(\alpha, \beta) d\alpha \, d\beta = \int_{E_n} \left\{ \int_{E_m} y(\alpha, \beta) d\alpha \right\} d\beta$$

$$= \int_{E_m} \left\{ \int_{E_n} y(\alpha, \beta) d\beta \right\} d\alpha .$$

Theorem 1.3.8: If $x_n(\alpha)$ are in $B(\Sigma; \mathfrak{X})$ for all n, and the sequence converges almost uniformly to a limit function $x(\alpha)$, and if there exists a numerically-valued function $\varphi(\alpha)$ in $L(\Sigma)$ such that $\| x_n(\alpha) \| \leq \varphi(\alpha)$ for all α in Σ, then $x(\alpha)$ is in $B(\Sigma; \mathfrak{X})$, and

$$\lim_{n \to \infty} \int_{\Sigma} x_n(\alpha) d\alpha = \int_{\Sigma} x(\alpha) d\alpha .$$

4. Improper Bochner Integral:

Definition 1.4.1: Let $x(\alpha)$ be in $B([\lambda, \omega]; \mathfrak{X})$ for a fixed λ and all $\omega > \lambda$. If $\int_{\lambda}^{\omega} x(\alpha)d\alpha$ converges (in the strong sense) to a limit y, as $\omega \to \infty$, that is to say if for any $\varepsilon > 0$ there is an $\omega_0(\varepsilon)$ such that

$\left\| y - \int_{\lambda}^{\omega} x(\alpha)d\alpha \right\| < \varepsilon$ for every $\omega > \omega_0$, then we say that the improper Bochner integral of $x(\alpha)$ over $[\lambda, \infty)$ exists and we put $\int_{\lambda}^{\to\infty} x(\alpha)d\alpha = y = \lim_{\omega \to \infty} \int_{\lambda}^{\omega} x(\alpha)d\alpha$.

We shall prove two theorems concerning interchange of integrations when one of the integrals involved is an improper integral. For this we need the following two lemmas.

Lemma 1.4.1: Let $x(\alpha)$ be a strongly measurable function on the finite closed interval $[\xi, \eta]$ to the Banach space $\mathfrak{X}$. Then $x(\alpha)$ is the almost uniform limit of finitely-valued functions on this interval.

Proof: By assumption there exists a sequence of countably-valued functions $x_n(\alpha)$, $(n = 1,2,\ldots)$ such that for every $\varepsilon, \varepsilon' > 0$ there is a set $\Sigma_1 \subseteq [\xi, \eta]$ and an integer $N(\varepsilon)$ such that $m(\Sigma_1) < \frac{\varepsilon'}{2}$ and $\| x_n(\alpha) - x(\alpha) \| < \varepsilon$ for $n > N(\varepsilon)$ and all α in $[\xi, \eta] - \Sigma_1$. Since $x_n(\alpha)$ is countably-valued, there exist sets $\Sigma_{n,i}$ such that $x_n(\Sigma_{n,i}) = x_{n,i}$, $\Sigma_{n,i} \cap \Sigma_{n,j} = \emptyset$, $i \neq j$, and $\sum_{i=1}^{\infty} m(\Sigma_{n,i}) = \eta - \xi$.

Thus for every $\varepsilon' > 0$, $M(\varepsilon', n)$ exists such that

$$\sum_{i=M}^{\infty} m(\Sigma_{n,i}) < \frac{\varepsilon'}{2^{n+1}} .$$

Let $\Sigma_2 = \Sigma_1 \cup (\bigcup_{n=1}^{\infty} \bigcup_{i \geqslant M} \Sigma_{n,i})$. Then

$$m(\Sigma_2) \leq m(\Sigma_1) + \sum_{n=1}^{\infty} (\sum_{i \geqslant M} m(\Sigma_{n,i})) < \frac{\varepsilon'}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon'}{2^{n+1}} = \varepsilon'.$$

Further, let $x_n'(\alpha) = x_n(\alpha)$ in $[\xi, \eta] - \Sigma_2$

$= \theta$ in Σ_2.

Then the x_n' are finitely-valued and on

$[\xi, \eta] - \Sigma_2$, $\|x_n'(\alpha) - x(\alpha)\| < \varepsilon$, $n > N(\varepsilon)$, since

$x_n' = x_n$ on $[\xi, \eta] - \Sigma_2$.

Lemma 1.4.2: If

1. $x_\lambda(\alpha)$ are in $B([0,\infty); \mathfrak{X})$ for each $\lambda > 0$,

2. $\lim_{\lambda \to \infty} x_\lambda(\alpha) = x(\alpha)$ uniformly for α in $[0, \omega]$, each $\omega > 0$,

3. $\varphi(\alpha)$ in $L(0,\infty)$ exists such that $\|x_\lambda(\alpha)\| \leq \varphi(\alpha)$ for all $\lambda > 0$ and all α, $0 \leq \alpha < \infty$;

then

(i) $x(\alpha)$ is in $B([0,\infty); \mathfrak{X})$,

(ii) $\lim_{\lambda \to \infty} \int_0^{\infty} x_\lambda(\alpha) d\alpha = \int_0^{\infty} x(\alpha) d\alpha$.

Proof:

(i) Since $x(\alpha)$ is the strong limit almost everywhere of strongly measurable functions, $x(\alpha)$ is strongly measurable by theorem 1.2.2 (iii). Further, since $\|x_\lambda(\alpha)\| \leq \varphi(\alpha)$, $\|x(\alpha)\| \leq \varphi(\alpha)$, and thus $\|x(\alpha)\|$ is in $L(0,\infty)$. Thus by theorem 1.3.1, $x(\alpha)$ is in $B([0,\infty);\mathfrak{X})$.

(ii) Since $\|x_\lambda(\alpha)\| \leq \varphi(\alpha)$, and $\|x(\alpha)\| \leq \varphi(\alpha)$,

$$\int_{\omega_0}^{\infty} \|x_\lambda(\alpha)\| \, d\alpha \leq \int_{\omega_0}^{\infty} \varphi(\alpha)d\alpha \text{ , and}$$

$$\int_{\omega_0}^{\infty} \|x(\alpha)\| \quad d\alpha \leq \int_{\omega_0}^{\infty} \varphi(\alpha)d\alpha \qquad \text{for every} \quad \omega_0 > 0.$$

Also, since $\varphi(\alpha)$ is in $L(0,\infty)$, for every $\varepsilon > 0$, $\omega(\varepsilon)$ exists such that $\int_{\omega_1}^{\infty} \varphi(\alpha)d\alpha < \frac{\varepsilon}{3}$ for $\omega_1 > \omega(\varepsilon)$, and thus

$$\int_{\omega_1}^{\infty} \|x_\lambda(\alpha)\| \quad d\alpha < \frac{\varepsilon}{3}, \text{ and } \int_{\omega_1}^{\infty} \|x(\alpha)\| \; d\alpha < \frac{\varepsilon}{3} \text{ for}$$

$\omega_1 > \omega(\varepsilon)$.

Choose an $\omega_1 > \omega(\varepsilon)$.

By 3., $\lambda_0(\varepsilon)$ exists such that for $\lambda > \lambda_0$ and α in $[0, \omega_1]$,

$\|x_\lambda(\alpha) - x(\alpha)\| < \dfrac{\varepsilon}{3\,\omega_1}$.

Thus, for $\lambda > \lambda_0$

$$\left\| \int_0^{\infty} x_\lambda(\alpha)d\alpha - \int_0^{\infty} x(\alpha)d\alpha \right\| =$$

$$\left\| \int_0^{\omega_1} (x_\lambda(\alpha) - x(\alpha))d\alpha + \int_{\omega_1}^{\infty} x_\lambda(\alpha)d\alpha - \int_{\omega_1}^{\infty} x(\alpha)d\alpha \right\|$$

$$\leq \int_0^{\omega_1} \|x_\lambda(\alpha) - x(\alpha)\| \, d\alpha + \int_{\omega_1}^{\infty} \|x_\lambda(\alpha)\| \, d\alpha + \int_{\omega_1}^{\infty} \|x(\alpha)\| \, d\alpha$$

$$< \varepsilon .$$

The differences between this last theorem and theorem 1.3.8 should be noted.

Let $E_{\omega,\zeta} = \left\{ (\alpha, \beta) \mid 0 \leq \alpha \leq \omega \;; 0 \leq \beta \leq \zeta \right\}$.

Theorem 1.4.1: If

1. $x(\alpha, \beta)$ is in $B(E_{\omega,\zeta}; \mathfrak{X})$ for a fixed ζ and all $\omega > 0$,
2. $\int_0^{\to\infty} x(\alpha, \beta) d\alpha$ converges uniformly with respect to β, $0 \leq \beta \leq \zeta$,

then

(i) $\int_0^{\to\infty} \int_0^{\zeta} x(\alpha, \beta) d\beta \; d\alpha$ exists,

(ii) $\int_0^{\to\infty} x(\alpha, \beta) d\alpha$ is in $B([0, \zeta]; \mathfrak{X})$,

(iii) $\int_0^{\to\infty} \int_0^{\zeta} x(\alpha, \beta) d\beta \; d\alpha = \int_0^{\zeta} \int_0^{\to\infty} x(\alpha, \beta) d\alpha \; d\beta$.

Proof:

(i) It is sufficient to show that for every $\varepsilon > 0$, $\omega(\varepsilon)$ exists such that

$$\left\| \int_{\omega_1}^{\omega_2} \int_0^{\zeta} x(\alpha, \beta) d\beta \; d\alpha \right\| < \varepsilon \quad \text{for} \quad \omega_2 > \omega_1 > \omega(\varepsilon).$$

By 2., for every $\varepsilon > 0$, $\omega_3(\varepsilon)$ exists such that

$$\left\| \int_{\omega_1}^{\omega_2} x(\alpha, \beta) d\alpha \right\| < \varepsilon \quad \text{if} \quad \omega_2 > \omega_1 > \omega_3(\varepsilon).$$

Let $\omega(\varepsilon) = \omega_3(\frac{\varepsilon}{\zeta})$. Then if $\omega_2 > \omega_1 > \omega(\varepsilon)$,

$$\left\| \int_{\omega_1}^{\omega_2} \int_0^{\zeta} x(\alpha , \beta) d\beta \, d\alpha \right\| = \left\| \int_0^{\zeta} \int_{\omega_1}^{\omega_2} x(\alpha , \beta) d\alpha \, d\beta \right\|$$

$$\int_0^{\zeta} \left\| \int_{\omega_1}^{\omega_2} x(\alpha , \beta) d\alpha \right\| d\beta < \frac{\varepsilon}{\zeta} \cdot \zeta = \varepsilon .$$ The interchange of integrations is justified by theorem 1.3.7.

(ii) By theorem 1.2.2 (iii)

$\int_0^{\to\infty} x(\alpha , \beta) d\alpha$ is a strongly measurable function of β , $0 \leq \beta \leq \zeta$. Thus by theorem 1.3.1 it is sufficient to show that $\left\| \int_0^{\to\infty} x(\alpha , \beta) d\alpha \right\|$ is in $L(0, \zeta)$. By lemma 1.4.1,

$\int_0^{\to\infty} x(\alpha , \beta) d\alpha$ is the almost uniform limit of finitely-valued functions, and consequently so is

$\int_{\omega}^{\to\infty} x(\alpha , \beta) d\alpha$. Further, by 1. and theorem 1.3.1,

$\left\| \int_0^{\omega} x(\alpha , \beta) d\alpha \right\| \in L(0, \zeta)$ for each $\omega > 0$. By 2., for each $\omega > 0$, $\omega(\varepsilon)$ exists such that for $\omega_1 > \omega(\varepsilon)$

$\left\| \int_{\omega_1}^{\to\infty} x(\alpha , \beta) d\alpha \right\| < \varepsilon$. Since $\int_{\omega_1}^{\to\infty} x(\alpha , \beta) d\alpha$ is the almost uniform limit of finitely-valued functions,

$\left\| \int_{\omega_1}^{\to\infty} x(\alpha , \beta) d\alpha \right\|$ is the almost uniform limit of step functions and, is thus measurable.

Thus $\left\| \int_{\omega_1}^{\to\infty} x(\alpha , \beta) d\alpha \right\|$, being a bounded measurable function, is in $L(0,\zeta)$ and thus so is $\left\| \int_0^{\to\infty} x(\alpha,\beta)\,d\alpha \right\|$.

(iii) By (i), for each $\varepsilon > 0$, $\omega(\varepsilon)$ exists such that for $\omega_1 > \omega(\varepsilon)$
$\left\| \int_{\omega_1}^{\to\infty} \int_0^{\zeta} x(\alpha,\beta)\,d\beta\,d\alpha \right\| < \frac{\varepsilon}{2}$. Thus

$$\Gamma = \left\| \int_0^{\zeta} \int_0^{\to\infty} x(\alpha,\beta)\,d\alpha\,d\beta - \int_0^{\to\infty} \int_0^{\zeta} x(\alpha,\beta)\,d\beta\,d\alpha \right\|$$

$$< \left\| \int_0^{\zeta} \int_0^{\to\infty} x(\alpha,\beta)\,d\alpha\,d\beta - \int_0^{\omega_1} \int_0^{\zeta} x(\alpha,\beta)\,d\beta\,d\alpha \right\| + \frac{\varepsilon}{2}$$

$$= \left\| \int_0^{\zeta} \int_{\omega_1}^{\to\infty} x(\alpha,\beta)\,d\alpha\,d\beta \right\| + \varepsilon/2,$$ by 1. and theorem 1.3.7.

But by 2., for each $\varepsilon > 0$, $\omega_2(\varepsilon)$ exists such that for $\omega_3 > \omega_2(\varepsilon)$
$\left\| \int_{\omega_3}^{\to\infty} x(\alpha,\beta)\,d\alpha \right\| < \varepsilon$.

Choose $\omega_1 > \max\left[\omega(\varepsilon), \omega_2\left(\frac{\varepsilon}{2\zeta}\right)\right]$. Then

$$\Gamma < \left\| \int_0^{\zeta} \int_{\omega_1}^{\to\infty} x(\alpha,\beta)\,d\alpha\,d\beta \right\| + \frac{\varepsilon}{2}$$

$$\leq \int_0^{\zeta} \left\| \int_{\omega_1}^{\to\infty} x(\alpha,\beta)\,d\alpha \right\| d\beta < \frac{\varepsilon}{2\zeta}\cdot\zeta + \frac{\varepsilon}{2} = \varepsilon .$$

Letting $\varepsilon \to 0$, the conclusion is reached.

Let $E_\omega = \left\{ (\alpha,\beta) \;\middle|\; 0 \leq \alpha \leq \omega;\ 0 \leq \beta < \infty \right\}$.

Theorem 1.4.2: If

1. $x(\alpha,\beta)$ is in $B(E_\omega; \mathfrak{X})$ all $\omega > 0$,

2. $\int_0^{\to\infty} x(\alpha,\beta)\,d\alpha$ converges uniformly in β, $0 \leq \beta \leq \eta$, for each $\eta > 0$,

3. $\varphi(\beta)$ exists, in $L(0,\infty)$ such that $\left\| \int_0^{\omega} x(\alpha,\beta)\,d\alpha \right\| < \varphi(\beta)$ all $\omega > 0$ and all β, $0 \leq \beta < \infty$,

Then

(i) $\int_0^{\to\infty} x(\alpha,\beta)\,d\alpha$ is in $B([0,\infty);\mathfrak{X})$,

(ii) $\int_0^{\to\infty} \int_0^{\infty} x(\alpha,\beta)\,d\alpha\,d\beta$ converges,

(iii) $\int_0^{\infty} \int_0^{\to\infty} x(\alpha,\beta)\,d\alpha\,d\beta = \int_0^{\to\infty} \int_0^{\infty} x(\alpha,\beta)\,d\beta\,d\alpha$.

Proof: Let $x_\omega(\beta) = \int_0^{\omega} x(\alpha,\beta)\,d\alpha$. Then, by 2. and 3., $x_\omega(\beta)$ satisfies all the hypotheses of lemma 1.4.2.

Thus $\lim_{\omega\to\infty} \int_0^{\infty} \int_0^{\omega} x(\alpha,\beta)\,d\alpha\,d\beta = \lim_{\omega\to\infty} \int_0^{\infty} x_\omega(\beta)\,d\beta$, and

$\int_0^{\infty} \lim_{\omega\to\infty} x_\omega(\beta)\,d\beta = \int_0^{\infty} \int_0^{\to\infty} x(\alpha,\beta)\,d\alpha\,d\beta$ exist and are equal. But

$$\lim_{\omega\to\infty} \int_0^{\infty} \int_0^{\omega} x(\alpha,\beta)\,d\alpha\,d\beta = \lim_{\omega\to\infty} \int_0^{\omega} \int_0^{\infty} x(\alpha,\beta)\,d\beta\,d\alpha$$

$= \int_0^{\to\infty} \int_0^{\infty} x(\alpha,\beta)\,d\beta\,d\alpha$, the interchange of integrations being permitted by (1) and theorem 1.3.7.

5. Riemann-Graves Integral.

Let $x(\alpha)$ be a bounded function on the finite closed interval $[\xi,\eta]$ to the Banach space $\mathfrak{X}$.

Subdivide $[\xi,\eta]$ into n subintervals Δ_i by points ξ_i, $\xi = \xi_0 < \xi_1 < \dots < \xi_n = \eta$. Let $\delta_i = \xi_i - \xi_{i-1}$, and let α_i be an arbitrary interior point of Δ_i. We shall denote the subdivision of $[\xi,\eta]$ together with the points α_i by Π, which we shall call a partition of $[\xi,\eta]$. Let $N(\Pi) = \max_i \delta_i$.

Let $G(\Pi) = \sum_{i=1}^{n} x(\alpha_i)\,\delta_i$. By $\lim_{N(\Pi)\to 0} G(\Pi)$ we mean the limit, if it exists, of $\sum_{i=1}^{n} x(\alpha_i)\,\delta_i$ as $n \to \infty$ in such a way that $\max_i \delta_i \to 0$.

Definition 1.5.1: If for every sequence of partitions with $N(\Pi) \to 0$, $\lim_{N(\Pi)\to 0} G(\Pi)$ exists and equals the same vector y, then we say that $x(\alpha)$ is Riemann-Graves integrable over $[\xi,\eta]$, and we denote this common limit y by

$$(R)\int_{\xi}^{\eta} x(\alpha)\,d\alpha .$$

We shall drop the "(R)" where there is no danger of confusion.

It should be noted that the integral is defined for bounded functions only.

If $x(\alpha)$ is Riemann-Graves integrable over $[\xi,\eta]$ for every $\eta > \xi$, and

$\lim_{\eta \to \infty} (R)\int_{\xi}^{\eta} x(\alpha)d\alpha$ exists in the strong sense, we shall denote this limit by

$$(R) \int_{\xi}^{\infty} x(\alpha)d\alpha \quad ,$$

and say that $x(\alpha)$ is Riemann-Graves integrable over the range $[\xi, \infty)$.

Definition 1.5.2: We shall call $O_x(\xi, \eta) \equiv \sup_{\xi \le \alpha_1, \alpha_2 \le \eta} \| x(\alpha_1) - x(\alpha_2) \|$ the oscillation of $x(\alpha)$ on the closed interval $[\xi, \eta]$. If α is an interior point of $[\xi, \eta]$, we call $O_x(\alpha) \equiv \lim_{\delta \to 0} O_x(\alpha - \delta, \alpha + \delta)$ the oscillation of $x(\alpha)$ at the point α. If α is an end point of the interval we use $O_x(\alpha, \alpha + \delta)$, or $O_x(\alpha - \delta, \alpha)$.

Lemma 1.5.1: If $O_x(\alpha) = 0$, then $x(\alpha)$ is continuous at α.

Proof: Consider first that α is an interior point of the interval. By hypothesis, for each $\varepsilon > 0$, $\delta(\varepsilon)$ exists such that $O_x(\alpha - \delta, \alpha + \delta) < \varepsilon$. That is $\sup_{\alpha - \delta \le \alpha_1, \alpha_2 \le \alpha + \delta} \| x(\alpha_1) - x(\alpha_2) \|$ so that $\| x(\alpha_1) - x(\alpha_2) \| < \varepsilon$ for $\alpha - \delta \le \alpha_1, \alpha_2 \le \alpha + \delta$. Set $\alpha_2 = \alpha$, and $\alpha_1 = \beta$; then we have $\| x(\alpha) - x(\beta) \|$ for $|\alpha - \beta| < \delta(\varepsilon)$. The extension to the case of α an end point is obvious, as is the converse of the lemma.

Let Γ_x be the set of discontinuities of $x(\alpha)$ in $[\xi, \eta]$.

Theorem 1.5.1: If $x(\alpha)$ is a bounded function on $[\xi, \eta]$ to $\mathfrak{X}$, and if $m(\Gamma_x) = 0$, then x is Riemann-Graves integrable on $[\xi, \eta]$.

Proof: See Graves [5].

It should be noted that the condition $m(\Gamma_x) = 0$ is a sufficient, but by no means necessary, condition that a bounded function be Riemann-

Graves integrable. There are functions which are everywhere discontinuous and are yet Riemann-Graves integrable. For example, see Graves [5].

6. Riemann-Stieltjes Integral.

Definition 1.6.1: Let $\varphi(\alpha)$ be a bounded numerically-valued function on the finite closed interval $[\xi, \eta]$, and let $a(\alpha)$ be a function on $[\xi, \eta]$ to the Banach space $\mathfrak{X}$.

Let Π be a partition of $[\xi, \eta]$ as in § 5. Let $H(\Pi) = \sum_{i=1}^{n} \varphi(\alpha_i)\left[a(\alpha_i) - a(\alpha_{i-1})\right]$. Then if for every sequence of partitions with $N(\Pi) \to 0$, $\lim_{N(\Pi) \to 0} H(\Pi)$ exists and equals the same vector y, we say that $\varphi(\alpha)$ is Riemann-Stieltjes integrable over $[\xi, \eta]$ with respect to $a(\alpha)$, and we denote this common limit y by

$$\int_{\xi}^{\eta} \varphi(\alpha) da(\alpha).$$

If $\int_{\xi}^{\eta} \varphi(\alpha) da(\alpha)$ exists for every $\eta > \xi$, and $\lim_{\eta \to \infty} \int_{\xi}^{\eta} \varphi(\alpha) da(\alpha)$ exists (in the strong sense), then we shall denote this limit by

$$\int_{\xi}^{\infty} \varphi(\alpha) da(\alpha).$$

Theorem 1.6.1: If $\varphi(\alpha)$ is a continuous numerically-valued function on the finite interval $[\xi, \eta]$, and if $a(\alpha)$ is a function of bounded variation on $[\xi, \eta]$ to the Banach space $\mathfrak{X}$, then $\int_{\xi}^{\eta} \varphi(\alpha) da(\alpha)$ exists.

Proof: See Hille [6]; page 52.

7. Relations Between Integrals

(A) Bochner and Improper Bochner Integrals.

The relation between these two types of integrals is provided by the following theorem.

Theorem 1.7.1: If

1. $x(\alpha)$ is in $B([0,\omega];\mathfrak{X})$ all $\omega > 0$,
2. $\|x(\alpha)\|$ is in $L(0,\infty)$,

then

(i) $x(\alpha)$ is in $B([0,\infty);\mathfrak{X})$,

(ii) $\int_0^{\to\infty} x(\alpha)d\alpha$ converges,

(iii) $\int_0^{\infty} x(\alpha)d\alpha = \int_0^{\to\infty} x(\alpha)d\alpha$.

Proof:

(i) By theorem 1.3.1, it is sufficient to show that $x(\alpha)$ is strongly measurable over $[0,\infty)$.

By 1., $x(\alpha)$ is strongly measurable over $[0,\omega]$ for each $\omega>0$.

Let
$$1_\omega(\alpha) = 1 \qquad 0 \leq \alpha \leq \omega$$
$$= 0 \qquad \alpha > \omega$$

Obviously 1_ω is Lebesgue measurable for each value of ω. Then $x_\omega(\alpha) = 1_\omega(\alpha)x(\alpha)$ is strongly measurable over $[0,\omega]$ by theorem 1.2.3 (ii), and since $x_\omega(\alpha) = \theta$ for $\alpha>\omega$, $x_\omega(\alpha)$ is strongly measurable over $[0,\infty)$.

Obviously $x(\alpha) = \lim\limits_{\omega\to\infty} x_\omega(\alpha)$, where the limit is in the strong sense, and thus, by theorem 1.2.3 (iii) $x(\alpha)$ is strongly measurable over $[0,\infty)$.

(ii) Since $\| x(\alpha)\|$ is in $L(0,\infty)$ we have

$$\left\| \int_{\omega_1}^{\omega_2} x(\alpha)d\alpha \right\| \leq \int_{\omega_1}^{\omega_2} \| x(\alpha)\| \, d\alpha \to 0 \text{ as } \omega_1, \omega_2 \to \infty.$$

$$\text{(iii)} \left\| \int_0^{\to\infty} x(\alpha)d\alpha - \int_0^{\infty} x(\alpha)d\alpha \right\| = \left\| \int_{\omega_1}^{\to\infty} x(\alpha)d\alpha - \int_{\omega_1}^{\infty} x(\alpha)d\alpha \right\|$$

$$\left\| \int_{\omega_1}^{\to\infty} x(\alpha)d\alpha \right\| + \int_{\omega_1}^{\infty} \| x(\alpha)\| \, d\alpha \to 0 \text{ as } \omega_1 \to \infty.$$

(B) Bochner, Riemann-Graves, and Improper Bochner Integrals.

Sufficient conditions for a function to be integrable in both the Bochner and Riemann-Graves sense, and for these integrals to be equal, are given by the following theorem.

Theorem 1.7.2: If $x(\alpha)$ is a bounded function on the finite interval $[\xi,\eta]$ to $\mathfrak{X}$, and the set Γ of discontinuities of $x(\alpha)$ has Lebesgue measure zero, then $x(\alpha)$ is both Bochner and Riemann-Graves integrable over $[\xi,\eta]$, and further,

$$(B) \int_{\xi}^{\eta} x(\alpha)d\alpha = (R) \int_{\xi}^{\eta} x(\alpha)d\alpha \quad .$$

Proof: By theorem 1.5.1, the Riemann-Graves integral of $x(\alpha)$ exists.

Let $\{\Pi_n\}$ be a sequence of partitions of $[\xi,\eta]$ with $N(\Pi_n)$ tending to zero. Let $G(\Pi_n) = \sum_i x(\alpha_{i,n}) \delta_{i,n}$ be the associated Riemann sums. In these sums we may assume that the $\alpha_{i,n}$

are chosen so that they do not fall in the set Γ. Let $x_n(\alpha) = \sum_i x(\alpha_{i,n}) \varphi_{i,n}$ where $\varphi_{i,n}$ is the characteristic function of the interval $\Delta_{i,n}$. Then if β is a point of $[\xi, \eta]$ not in Γ, $\lim_{n\to\infty} \| x(\beta) - x_n(\beta)\| = 0$. For, if $\Delta_{i,n}$ be the interval in which β lies, $\| x(\beta) - x_n(\beta)\| = \|x(\beta) - x(\alpha_{i,n})\|$, and since $x(\alpha)$ is continuous at $\alpha = \beta$ and $|\alpha_{i,n} - \beta| \to 0$ as $n \to \infty$, $\| x(\beta) - x_n(\beta)\| \to 0$.

Then, by theorem 1.2.1, $\lim x_n(\alpha) = x(\alpha)$ almost uniformly, so that, since $x_n(\alpha)$ are finitely-valued functions, $x(\alpha)$ is strongly measurable. But, by hypothesis, $x(\alpha)$ is bounded, so that, by theorem 1.3.1, $x(\alpha)$ is in $B([\xi, \eta]; \mathfrak{X})$.

We have then,

$$(B)\int_\xi^\eta x(\alpha)d\alpha = \lim_{n\to\infty} \int x_n(\alpha)d\alpha = \lim_{n\to\infty} \sum_i x(\alpha_{i,n})\, \delta_{i,n}$$

$$= \lim_{n\to\infty} G(\Pi_n) = (R)\int_\xi^\eta x(\alpha)d\alpha \, .$$

Corollary: If $x(\alpha)$ is a function on $[\xi, \infty)$ to $\mathfrak{X}$, if $x(\alpha)$ is bounded on the intervals $[\xi, \eta]$ for every $\eta > \xi$, and if the set Γ of discontinuities of $x(\alpha)$ in $[\xi, \infty)$ has Lebesgue measure zero, then

$$(R)\int_\xi^\infty x(\alpha)d\alpha = \int_\xi^{\to\infty} x(\alpha)d\alpha$$

if either of the integrals exist.

Proof: By the preceding theorem,

$$(R)\int_\xi^\eta x(\alpha)d\alpha = (B)\int_\xi^\eta x(\alpha)d\alpha \, , \text{ for every } \eta > \xi \, , \text{ and}$$

taking limits on both sides we have

$$(R) \int_{\xi}^{\infty} x(\alpha)d\alpha = \int_{\xi}^{\to\infty} x(\alpha)d\alpha \qquad \text{if either integral exists.}$$

(C) Stieltjes Integral.

The relation of the Stieltjes integral to other integrals depends on the following lemma.

Lemma 1.7.1: If $a(\alpha)$ is a function of strongly bounded variation on $[\xi, \eta]$ to $\mathfrak{X}$, then the set Γ of discontinuities of $a(\alpha)$ is at most countable.

Proof: Let M be the strong total variation of $a(\alpha)$ over $[\xi, \eta]$, and $\Gamma_n = \left\{ \alpha \mid \xi \leq \alpha \leq \eta \quad ; O_a(\alpha) > \frac{M}{2^n} \right\}$. Then $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, and Γ_n contains fewer than 2^n elements. Thus Γ is at most countable.

We can now prove a theorem relating Stieltjes, Riemann-Graves, and Bochner integrals.

Theorem 1.7.3: If $a(\alpha)$ is a function of strongly bounded variation on $[\xi, \eta]$ to $\mathfrak{X}$, and $\varphi(\alpha)$ is a numerically-valued function with continuous first derivative on $[\xi, \eta]$, then

$$\int_{\xi}^{\eta} \varphi(\alpha)da(\alpha) = \varphi(\eta)a(\eta) - \varphi(\xi)a(\xi) - \int_{\xi}^{\eta} \varphi'(\alpha)a(\alpha)d\alpha .$$

The integral on the right hand side may be taken in either the Riemann-Graves, or Bochner sense.

Proof: The proof is the same as that for numerically-valued functions once it is noted that since $a(\alpha)$ is of strongly bounded variation it is bounded, and by the preceding lemma its set of discontinuities has measure zero, so that, by theorem 1.5.1, the integral exists in the Riemann-Graves sense, and by theorem 1.7.1, it exists in the Bochner sense.

Corollary: If $a(\alpha)$ is a function on $[\xi,\infty)$ to $\mathfrak{X}$ which is of strongly bounded variation on $[\xi,\eta]$ for each $\eta>\xi$, if $\varphi(\alpha)$ is a numerically-valued function on $[\xi,\infty)$ which has a continuous first derivative on $[\xi,\infty)$, and if $\lim\limits_{\eta\to\infty}\varphi(\eta)a(\eta)$ exists and equals y, then

$$\int_\xi^\infty \varphi(\alpha)da(\alpha) = y - \varphi(\xi)a(\xi) - (R)\int_\xi^\infty \varphi'(\alpha)a(\alpha)d\alpha$$
$$= y - \varphi(\xi)a(\xi) - \int_\xi^{\to\infty} \varphi'(\alpha)a(\alpha)d\alpha ,$$

if any one of the three integrals exist.

Proof: The proof is obvious.

8. Weak Convergence of Operators.

The following theorem, which is, in a sense, a weak sequential analogue of the Banach-Steinhaus theorem, will be used in Chapter III in connection with certain representation theorems.

Theorem 1.8.1: If $\{T_\sigma\}$, $0 < \sigma < \infty$, is a set of linear transformations on a separable Banach space $\mathfrak{X}$ to a reflexive Banach space $\mathfrak{Y}$, and if $\|T_\sigma\| \leq M$ independent of σ, for all $\sigma > 0$, then there exists an increasing unbounded sequence $\{\sigma_i\}$, and a linear transformation T on $\mathfrak{X}$ to $\mathfrak{Y}$ with $\|T\| \leq M$, such that

$$\lim_{i \to \infty} y^*(T_{\sigma_i}(x)) = y^*(T(x)),$$

for every x in $\mathfrak{X}$ and every y^* in $\mathfrak{Y}^*$.

Proof: Let $D = \{x_n\}$ be a countable set dense in $\mathfrak{X}$. Since $\mathfrak{Y}$ is reflexive, it has, by Gantmakher and Šmulian [4], a weakly compact unit sphere, so that there exists an increasing unbounded sequence $\{\sigma_{i,1}\}$ and an element y_1 of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$,

$$\lim_{i \to \infty} y^*(T_{\sigma_{i,1}}(x_1)) = y^*(y_1).$$

Further, there exists an increasing unbounded sequence $\{\sigma_{i,2}\} \subseteq \{\sigma_{i,1}\}$ and an element y_2 in $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$,

$$\lim_{i \to \infty} y^*(T_{\sigma_{i,2}}(x_2)) = y^*(y_2).$$

Inductively, there exists an increasing unbounded sequence $\{\sigma_{i,n}\} \subseteq \{\sigma_{i,n-1}\}$ and an element y_n of $\mathfrak{Y}$ such that for every y^* in $\mathfrak{Y}^*$

$$\lim_{i \to \infty} y^*(T_{\sigma_{i,n}}(x_n)) = y^*(y_n).$$

Thus, using the diagonal sequence, we have, for every y^* in $\mathfrak{Y}^*$,

$$\lim_{i \to \infty} y^*(T_{\sigma_{i,i}}(x_j)) = y^*(y_j).$$

Further, $y^*(y_j) = \lim_{i \to \infty} y^*(T_{\sigma_{i,i}}(x_j)) \leq \| y^* \| \; M \; \| x_j \|$,

so that, by Hille [6; thm. 2.12.3], $\| y_j \| \leq M \| x_j \|$.

We define $\sigma_i = \sigma_{i,i}$, and $T(x_n) = y_n$. Obviously we have for T on D, $\| T \| \leq M$.

Let x be an arbitrary element of $\mathfrak{X}$. Then there is a sequence $\{ x_{n_j} \} \subseteq D$ such that $\lim_{j \to \infty} x_{n_j} = x$. Further, if $y_{n_j} = T(x_{n_j})$, $\lim_{j \to \infty} y_{n_j}$ exists; for,

$$\| y_{n_j} - y_{n_k} \| = \| T(x_{n_j} - x_{n_k}) \| \leq M \| x_{n_j} - x_{n_k} \| \to 0 \text{ as } j, k \to \infty.$$

Further if $\{ \bar{x}_{n_\ell} \} \subseteq D$ is any other sequence whose limit is x, then, if $\bar{y}_{n_\ell} = T(\bar{x}_{n_\ell})$,

$$\| y_{n_j} - \bar{y}_{n_\ell} \| = \| T(x_{n_j} - \bar{x}_{n_\ell}) \| \leq M \| x_{n_j} - \bar{x}_{n_\ell} \|$$

$M(\| x - x_{n_j} \| + \| x - \bar{x}_{n_\ell} \|) \to 0$ as $j, \ell \to \infty$, so that y_{n_j} and $\bar{y}_{n_\ell}$ have the same limit. We define $T(x) = \lim_{j \to \infty} y_{n_j}$.

It is evident that T is bounded and linear on $\mathfrak{X}$, and, in fact, $\| T \| \leq M$.

Also we have $\lim\limits_{i \to \infty} y^*(T_{\sigma_i}(x)) = y^*(T(x))$ for every y^* in $\mathcal{Y}^*$. For,

$$|y^*(T_{\sigma_i}(x)) - y^*(T(x))|$$

$$= |y^*(T_{\sigma_i}(x - x_{n_j})) + (y^*(T_{\sigma_i}(x_{n_j})) - y^*(T(x_{n_j}))) + y^*(T(x-x_{n_j}))|$$

$$\leq 2\|y^*\| M \|x - x_{n_j}\| + |y^*(T_{\sigma_{i,i}}(x_{n_j})) - y^*(T(x_{n_j}))| \to 0$$

as $i,j \to \infty$. Thus the theorem is proved.

Chapter II

1. Introduction

In this chapter we shall consider the inversion of the Laplace transformation of Banach-valued functions of a real variable. In particular we shall consider four kinds of Laplace integrals, namely

$$\text{I} \quad f(\lambda) = (B) \int_0^{\infty} e^{-\lambda\tau} x(\tau)d\tau$$

$$\text{II} \quad f(\lambda) = \int_0^{\to\infty} e^{-\lambda\tau} x(\tau)d\tau$$

$$\text{III} \quad f(\lambda) = (R) \int_0^{\infty} e^{-\lambda\tau} x(\tau)d\tau$$

$$\text{IV} \quad f(\lambda) = \int_0^{\infty} e^{-\lambda\tau} da(\tau).$$

We shall consider the inversion theory with respect to a certain "real" inversion operator

$$L_{K,\tau}\left[f(\lambda)\right] = \frac{K e^{2K}}{\pi\tau} (B) \int_0^{\infty} \eta^{-\frac{1}{2}} \cos(2\eta^{\frac{1}{2}}) f(K(\eta+1)/\tau)d\eta .$$

We shall also use the "Improper Bochner" generalization of this operator, namely

$$\vec{L}_{K,\tau}\left[f(\lambda)\right] = \frac{K e^{2K}}{\pi\tau} \int_0^{\to\infty} \eta^{-\frac{1}{2}} \cos(2\eta^{\frac{1}{2}}) f(K(\eta+1)/\tau)d\eta .$$

Whether we use $L_{K,\tau}\left[f(\lambda)\right]$, or $\vec{L}_{K,\tau}\left[f(\lambda)\right]$ does not depend, as one might suppose, on whether we are inverting transformation I or II. On the contrary, the use of $L_{K,\tau}\left[f(\lambda)\right]$ or $\vec{L}_{K,\tau}\left[f(\lambda)\right]$

depends upon the behaviour of $f(\lambda)$ in the neighbourhood of $\lambda = \infty$, that is on the behaviour of $x(\tau)$ in the neighbourhood of $\tau = 0$, while whether $x(\tau)$ has a transformation of type I or II depends on the behaviour of $x(\tau)$ in the neighbourhood of $\tau = \infty$.

In section 2 we prove several preliminary lemmas concerned with the evaluation of certain singular integrals.

Section 3 contains the inversion theory for transformations of type I, and section 4 for transformations of types II and III.

Finally, in section 5, we prove an inversion theorem for transformations of type IV.

2. Some Preliminary Lemmas:

For the various inversion and representation theories we shall need the following lemmas.

Lemma 2.2.1: If

1. $x(\alpha)$ is in $B([0,\omega];\mathfrak{X})$ for each $\omega > 0$,

2. $\int_0^{\to\infty} e^{-\lambda\alpha} x(\alpha)d\alpha$ converges for $\lambda = \lambda_0$.

Then

(i) for each $\tau > 0$ and for all $\kappa > \lambda_0\tau$

$$I_\kappa = e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_0^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} x(\alpha)d\alpha \quad \text{converges,}$$

(ii) for each $\tau > 0$ in the Lebesgue set of $x(\alpha)$,

$\lim_{\kappa\to\infty} I_\kappa = x(\tau)$.

Proof:

(i) Let $\kappa > \lambda_0\tau$, let $\omega_2 > \omega_1$, and let

$M = \sup_{0<\omega<\infty} \left\| \int_0^{\omega} e^{-\lambda_0\alpha} x(\alpha)d\alpha \right\|$. By 2., $M < \infty$.

Thus, $\left\| \int_{\omega_1}^{\omega_2} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} x(\alpha)\, d\alpha \right\| =$

$$\Big\| e^{-\kappa(\frac{\omega_2}{\tau}+\frac{\tau}{\omega_2})+\lambda_0\omega_2}\,\omega_2^{-\frac{1}{2}}\int_0^{\omega_2} e^{-\lambda_0\beta}\,x(\beta)\,d\beta$$

$$- e^{-\kappa(\frac{\omega_1}{\tau}+\frac{\tau}{\omega_1})+\lambda_0\omega_1}\,\omega_1^{-\frac{1}{2}}\int_0^{\omega_1} e^{-\lambda_0\beta}\,x(\beta)\,d\beta$$

$$- \int_{\omega_1}^{\omega_2}\Big\{\frac{d}{d\alpha}\big(e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})+\lambda_0\alpha}\,\alpha^{-\frac{1}{2}}\Big\}\int_0^{\alpha} e^{-\lambda_0\beta}\,x(\beta)\,d\beta\,d\alpha\Big\|$$

$$\leq M\big(e^{-\kappa(\frac{\omega_2}{\tau}+\frac{\tau}{\omega_2})+\lambda_0\omega_2}\,\omega_2^{-\frac{1}{2}} + e^{-\kappa(\frac{\omega_1}{\tau}+\frac{\tau}{\omega_1})+\lambda_0\omega_1}\,\omega_1^{-\frac{1}{2}}\big)$$

$$+ \int_{\omega_2}^{\omega_1}\Big\{\frac{d}{d\alpha}\big(e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})+\lambda_0\alpha}\,\alpha^{-\frac{1}{2}}\big)\Big\}\Big\|\int_0^{\alpha} e^{-\lambda_0\beta}\,x(\beta)\,d\beta\Big\|\,d\alpha$$

$$\leq M\big(e^{-\kappa(\frac{\omega_2}{\tau}+\frac{\tau}{\omega_2})+\lambda_0\omega_2}\,\omega_2^{-\frac{1}{2}} + e^{-\kappa(\frac{\omega_1}{\tau}+\frac{\tau}{\omega_1})+\lambda_0\omega_1}\,\omega_1^{-\frac{1}{2}}\big)$$

$$+ M\int_{\omega_2}^{\omega_1}\Big\{\frac{d}{d\alpha}\big(e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})+\lambda_0\alpha}\,\alpha^{-\frac{1}{2}}\Big\}\,d\alpha$$

$$= 2M\big(e^{-\kappa(\frac{\omega_1}{\tau}+\frac{\tau}{\omega_1})+\lambda_0\omega_1}\,\omega_1^{-\frac{1}{2}}\big)$$

$$\longrightarrow 0$$

as $\omega_1, \omega_2 \to \infty$.

(ii) It should be noted that

$$e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}}\int_0^{\infty} e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\,\alpha^{-\frac{1}{2}}\,d\alpha =$$

$$= 2\, e^{2\kappa}\pi^{-\frac{1}{2}} \int_0^{\infty} e^{-(\beta^2+\frac{\kappa^2}{\beta^2})}\, d\beta \qquad \text{(where } \beta^2 = \frac{\kappa\alpha}{\tau}\text{)}$$

$= 1$ By Peirce [8; page 63, formula 495].

Let τ be in the Lebesque set of x(α), and let $\omega > \tau$.

Let $\kappa > \lambda_0\tau$.

Then,

$$\left\| I_\kappa - x(\tau) \right\|$$

$$= \left\| e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_0^{\to\infty} e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\, x(\alpha)\, d\alpha - x(\tau) \right\|$$

$$= \left\| e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_0^{\to\infty} e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}(x(\alpha) - x(\tau))\, d\alpha \right\|$$

$$\leq e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_0^{\omega} e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}} \left\| x(\alpha) - x(\tau) \right\| d\alpha$$

$$+ \left\| e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_\omega^{\to\infty} e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\, x(\alpha)\, d\alpha \right\|$$

$$+ \left\| x(\tau) \right\| e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_\omega^{\infty} e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\, d\alpha$$

$$= J_1 + J_2 + J_3 .$$

Let κ_0 be a positive number, and $\kappa_0 < \kappa$. Then

$$J_3 = \|x(\tau)\|\, e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_\omega^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}}\, d\alpha$$

$$= \|x(\tau)\|\, e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_\omega^\infty e^{-\kappa_0\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} e^{-(\kappa-\kappa_0)\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\, d\alpha$$

$$\leq \|x(\tau)\|\, e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{-(\kappa-\kappa_0)\left(\frac{\omega}{\tau}+\frac{\tau}{\omega}\right)} \int_\omega^\infty e^{-\kappa_0\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}}\, d\alpha$$

$$= \|x(\tau)\| \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{\kappa_0\left(\frac{\omega}{\tau}+\frac{\tau}{\omega}\right)} e^{-\kappa\left(\frac{\omega}{\tau}+\frac{\tau}{\omega}-2\right)} \int_\omega^\infty e^{-\kappa_0\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}}\, d\alpha$$

$\rightarrow 0$ as $\kappa \rightarrow \infty$, since $\frac{\alpha}{\tau} + \frac{\tau}{\alpha}$ attains its minimum value of 2 at $\alpha = \tau$, and $\omega > \tau$.

$$J_2 = \left\| e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_\omega^{\rightarrow\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}}\, x(\alpha)\, d\alpha \right\|$$

$$= \left\| e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_\omega^\infty \left\{ \frac{d}{d\alpha}\left(e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)+\lambda_0\alpha} \alpha^{-\frac{1}{2}}\right)\right\} \int_\omega^\alpha e^{-\lambda_0\beta} x(\beta)\, d\beta \right\|$$

$$\leq 2M\, e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_\omega^\infty \left\{ -\frac{d}{d\alpha}\left(e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)+\lambda_0\alpha} \alpha^{-\frac{1}{2}}\right\} d\alpha$$

$$= 2M\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{-\kappa\left(\frac{\omega}{\tau}+\frac{\tau}{\omega}-2\right)+\lambda_0\omega}\, \omega^{-\frac{1}{2}}$$

$\rightarrow 0$ as $\kappa \rightarrow \infty$.

By Widder [12; theorem 2b, corollary 2b.1, page 278] $J_1 \rightarrow 0$ as $\kappa \rightarrow \infty$ if τ is in the Lebesgue set of $\|x(\alpha) - x(\tau)\|$. However, by theorem 1.3.6, this is exactly the Lebesgue set of $x(\alpha)$. Thus the lemma is proved.

Corollary: If $e^{-\lambda_0\alpha}$ $x(\alpha)$ is in $B([0,\infty);\mathfrak{X})$, then

(i) for each $\tau > 0$ and all $\kappa > \lambda_0\tau$, $e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}x(\alpha)$ is in $B([0,\infty);\mathfrak{X})$,

(ii) $\lim_{\kappa\to\infty} I_\kappa = x(\tau)$ for each τ in the Lebesgue set of $x(\alpha)$, where

$$I_\kappa = e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}}\int_0^\infty e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha .$$

Proof:

(i) This follows from the fact that for $\kappa > \lambda_0\tau$

$$e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\|x(\alpha)\| \leq K e^{-\lambda_0\alpha}\|x(\alpha)\| .$$

(ii) (ii) now follows from theorem 1.7.1 and the preceding lemma.

Lemma 2.2.2: If

1. $x(\alpha)$ is in $B([0,\omega];\mathfrak{X})$ for each $\omega > 0$,

2. $\int_0^{\to\infty} e^{-\lambda\alpha}x(\alpha)\,d\alpha$ converges for $\lambda = \lambda_0$,

then

(i) $I_\kappa = e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}}\int_0^{\to\infty} e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{3}{2}}x(\alpha)\,d\alpha$ converges for each $\tau > 0$ and all $\kappa > \lambda_0\tau$,

(ii) for each $\tau > 0$ in the Lebesgue set of $x(\alpha)$,

$\lim_{\kappa\to\infty} I_\kappa = x(\tau)$.

Proof: The proof is almost identical with that of the preceding lemma.

3. Inversion of Transformations of Type I.

Conditions for the inversion of $f(\lambda) = (B)\int_0^\infty e^{-\lambda\tau}\, x(\tau)d\tau$,

are provided by the following theorem and its corollary. The theorem gives conditions for the inversion by $\overrightarrow{L}_{\kappa,\tau}[f(\lambda)]$, and the corollary for $L_{\kappa,\tau}[f(\lambda)]$.

Theorem 2.3.1: If $e^{-\lambda\alpha}x(\alpha)$ is in $\overrightarrow{B}([0,\infty);\mathfrak{X})$ for all $\lambda > \gamma > 0$, then for each $\tau > 0$ and all $\kappa > \gamma\tau$, $\overrightarrow{L}_{\kappa,\tau}[f(\lambda)]$ exists, and

$$\lim_{\kappa\to\infty} L_{\kappa,\tau}[f(\lambda)] = x(\tau)$$

at every point, $\tau > 0$ of the Lebesgue set of $x(\tau)$.

Proof: We shall show that $\overrightarrow{L}_{\kappa,\tau}[f(\lambda)]$

$= \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}x(\alpha)d\alpha$, and the conclusion will then follow from the corollary to lemma 2.2.1.

Let κ, τ be fixed, positive, and $\frac{\kappa}{\tau} > \gamma$. Then,

$$I_\omega = \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\omega \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\, f(\kappa(\eta+1)/\tau)\,d\eta$$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\omega \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\,d\eta\int_0^\infty e^{-\kappa(\eta+1)\alpha/\tau}x(\alpha)\,d\alpha$$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\infty e^{-\kappa\alpha/\tau}x(\alpha)\,d\alpha\int_{-\sqrt{\omega}}^{\sqrt{\omega}} e^{-\frac{\kappa\alpha}{\tau}\zeta^2+2i\kappa\zeta}\,d\zeta$$

$$= \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha\left\{\frac{1}{\sqrt{\pi}}\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2}\,d\theta\right\}$$

(where $\theta = \sqrt{\frac{\kappa\alpha}{\tau}}\,\eta - i\sqrt{\frac{\kappa\tau}{\alpha}}$).

Thus,

$$J_\omega = \left\| \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} \chi(\alpha)\, d\alpha - I_\omega \right\|$$

$$\leq \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa} \int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} \|\chi(\alpha)\| \left| 1 - \frac{1}{\sqrt{\pi}} \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2} d\theta \right| d\alpha$$

$$= \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa} \left\{ \int_0^\varepsilon + \int_\varepsilon^\infty \right\} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} \|\chi(\alpha)\| \left| 1 - \frac{1}{\sqrt{\pi}} \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2} d\theta \right| d\alpha$$

$$= J_\omega^{(1)} + J_\omega^{(2)}.$$

Now if $\alpha \geq \varepsilon > 0$,

$$e^{-\kappa\tau/\alpha} \alpha^{-\frac{1}{2}} \left| 1 - \frac{1}{\sqrt{\pi}} \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\kappa\alpha\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\kappa\alpha\omega}{\tau}}} e^{-\theta^2} d\theta \right|$$

$$= e^{-\kappa\tau/\alpha} \alpha^{-\frac{1}{2}} \left| \left\{ \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^\infty + \int_{i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^\infty \right\} e^{-\theta^2} d\theta \right|$$

$$= e^{-\kappa\tau/\alpha} \alpha^{-\frac{1}{2}} \left| \left\{ \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{\sqrt{\frac{\alpha\kappa\omega}{\tau}}} + \int_{i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{\sqrt{\frac{\alpha\kappa\omega}{\tau}}} + 2\int_{\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^\infty \right\} e^{-\theta^2} d\theta \right|$$

$$\leq 2 e^{-\kappa\tau/\alpha} \alpha^{-\frac{1}{2}} \left\{ e^{-\frac{\alpha\kappa\omega}{\tau}} \int_0^{\sqrt{\frac{\kappa\tau}{\alpha}}} e^{\varphi^2} d\varphi + \int_{\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^\infty e^{-\theta^2} d\theta \right\}$$

$$\leq 2 e^{-\frac{\alpha\kappa\omega}{\tau}} \alpha^{-1} \sqrt{\kappa\tau} + 2 e^{-\kappa\tau/\alpha} \alpha^{-\frac{1}{2}} \int_{\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^\infty e^{-\theta^2} d\theta$$

$$\leq e^{-\frac{\varepsilon\kappa\omega}{\tau}} \varepsilon^{-1} \sqrt{\kappa\tau} + 2 e^{-\kappa\tau/\varepsilon} \varepsilon^{-\frac{1}{2}} \int_{\sqrt{\frac{\varepsilon\kappa\omega}{\tau}}}^\infty e^{-\theta^2} d\theta .$$

Thus $0 \leq J_\omega^{(2)} \leq$

$$\left(e^{-\frac{\varepsilon\kappa\omega}{\tau}}\varepsilon^{-1}\sqrt{\kappa\tau}+2e^{-\frac{\kappa\tau}{\varepsilon}}\varepsilon^{-\frac{1}{2}}\int_{\sqrt{\frac{\varepsilon\kappa\omega}{\tau}}}^{\infty}e^{-\theta^2}d\theta\right)\sqrt{\frac{\kappa}{\pi\tau}}\,e^{2\kappa}\int_0^\infty e^{-\kappa\alpha/\tau}\|x(\alpha)\|\,d\alpha$$

$\to 0$ as $\omega \to \infty$.

Further, since

$$\lim_{\omega\to\infty}\left[1-\frac{1}{\sqrt{\pi}}\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}e^{-\theta^2}d\theta\right]=0 \qquad \text{almost everywhere}$$

for α in $[0,\varepsilon]$, we have, by theorem 1.2.1, that the limit equals zero almost uniformly in α for α in $[0,\varepsilon]$. Also, since

$$\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}e^{-\theta^2}d\theta=\frac{e^{\frac{\kappa\tau}{\alpha}-\frac{\alpha\kappa\omega}{\tau}}}{\frac{\kappa\tau}{\alpha}+\frac{\alpha\kappa\omega}{\tau}}\left\{-\sin(2\kappa\sqrt{\omega})\sqrt{\frac{\kappa\tau}{\alpha}}-\cos(2\kappa\sqrt{\omega})\sqrt{\frac{\alpha\kappa\omega}{\tau}}\right\}$$

$$-\frac{1}{2}\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}e^{-\theta^2}\theta^{-2}d\theta \qquad \text{and}$$

$$\left|\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}e^{-\theta^2}\theta^{-2}d\theta\right|\leq M\,e^{\kappa\tau/\alpha}\alpha^{\frac{1}{2}}$$

where M is independent of α and ω, so that

$$e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}\|x(\alpha)\|\left|1-\frac{1}{\sqrt{\pi}}\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}e^{-\theta^2}d\theta\right|\leq N\,e^{-\kappa\alpha/\tau}\|x(\alpha)\|,$$

where N is independent of α and ω, and this bound is an integrable function of α. Thus, by theorem 1.3.8, $\lim_{\omega\to\infty} J_\omega^{(2)} = 0$, and the theorem is proved.

Corollary: If $e^{-\lambda\alpha}\alpha^{-\frac{1}{2}}x(\alpha)$ is in $B([0,\infty);\mathfrak{X})$ for all $\lambda > \gamma > 0$, then

(i) $f(\lambda)$ exists for $\lambda > \gamma$,

(ii) $L_{\kappa,\tau}[f(\lambda)]$ exists for each $\tau > 0$ and all $\kappa > \gamma\tau$,

(iii) for each $\tau > 0$ in the Lebesgue set of $x(\alpha)$

$$\lim_{\kappa\to\infty} L_{\kappa,\tau}[f(\lambda)] = x(\tau).$$

Proof: Since $e^{-\lambda\alpha} = e^{-(\frac{\lambda+\gamma}{2})\alpha}e^{-(\frac{\lambda-\gamma}{2})}$, and for and sufficiently large α, $e^{-(\frac{\lambda-\gamma}{2})\alpha} < \alpha^{-\frac{1}{2}}$, $f(\lambda)$ exists for $\lambda > \gamma$.

Further,

$$\frac{2\kappa^{\frac{1}{2}}e^{2\kappa}}{\pi\tau^{1/2}}\int_0^\infty e^{-\kappa\alpha/\tau}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha\int_0^\infty e^{-\xi^2}\cos(2(\kappa\tau/\alpha)^{\frac{1}{2}}\xi)\,d\xi$$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\infty e^{-\kappa\alpha/\tau}x(\alpha)\,d\alpha\int_0^\infty e^{-\kappa\eta\alpha/\tau}\eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\,d\eta$$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\infty \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\,d\eta\int_0^\infty e^{-\kappa(\eta+1)\alpha/\tau}x(\alpha)\,d\alpha$$

$= L_{\kappa,\tau}[f(\lambda)]$ exists for each $\tau > 0$ and all $\kappa > \gamma\tau$, since $\left|\int_0^\infty e^{-\xi^2}\cos(2(\kappa\tau/\alpha)^{\frac{1}{2}}\xi)\,d\xi\right| \le \frac{1}{2}\sqrt{\pi}$.

Finally (iii) follows from theorems 1.7.1 and 2.3.1, since theorem 1.7.1 tells us that when both $L_{\kappa,\tau}\,[f(\lambda)]$ and $\vec{L}_{\kappa,\tau}\,[f(\lambda)]$ exist, they are equal.

4. Inversion of Transformations of Types II and III.

Theorem 2.4.1: If $x(\alpha)$ is in $B([0,\omega];\mathfrak{X})$ for each $\omega > 0$, and if

$$f(\lambda) = \int_0^{\to\infty} e^{-\lambda\alpha}\, x(\alpha)\,d\alpha$$

converges uniformly in λ for $\lambda > \gamma > 0$, then for each $\tau > 0$ and all $\kappa > \gamma\tau$, $\vec{L}_{\kappa,\tau}[f(\lambda)]$ exists, and

$$\lim_{\kappa\to\infty} \vec{L}_{\kappa,\tau}[f(\lambda)] = x(\tau)$$

at every point $\tau > 0$ of the Lebesgue set of $x(\alpha)$.

Proof: We shall show that $\vec{L}_{\kappa,\tau}[f(\lambda)] =$

$$e^{2\kappa}\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}}\int_0^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha$$

, and the conclusion will follow from lemma 2.2.1.

Let κ, τ be fixed, positive and $\kappa > \gamma\tau$. Choose λ_0, $\gamma < \lambda_0 < \kappa/\tau$. Since

$\int_0^{\to\infty} e^{-\lambda_0\alpha}\, x(\alpha)\,d\alpha$ converges, $\int_0^{\omega} e^{-\lambda_0\alpha}\, x(\alpha)\,d\alpha$ is bounded.

Let $M = \sup_{0\le\omega<\infty} \left\|\int_0^{\omega} e^{-\lambda_0\alpha}\, x(\alpha)\,d\alpha\right\|$. Then $M < \infty$. Consider

$$I_\omega = \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^{\omega} \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\, f(\kappa(\eta+1)/\tau)\,d\eta$$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^{\omega} \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\,d\eta\int_0^{\to\infty} e^{-\kappa(\eta+1)\alpha/\tau}\, x(\alpha)\,d\alpha$$

Since $\int_0^{\to\infty} e^{-\lambda\alpha} x(\alpha)\,d\alpha$ converges uniformly for $\lambda > \gamma$, and $\kappa/\tau > \gamma$, we may, by theorem 1.4.1, interchange the order of the integrations. Thus

$$I_\omega = \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^{\to\infty} e^{-\kappa\alpha/\tau}x(\alpha)\,d\alpha\int_0^{\omega} e^{-\kappa\eta\alpha/\tau}\eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\,d\eta$$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^{\to\infty} e^{-\kappa\alpha/\tau}x(\alpha)\,d\alpha\int_{-\sqrt{\omega}}^{\sqrt{\omega}} e^{-\frac{\kappa\alpha}{\tau}\zeta^2+2i\kappa\zeta}\,d\zeta \quad \text{(where } \zeta^2 = \eta \text{)}$$

$$= \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha\left\{\frac{1}{\sqrt{\pi}}\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2}\,d\theta\right\}.$$

Then

$$J_\omega = \left\| \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha - I_\omega\right\|$$

$$\leq \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^{\varepsilon} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}\|x(\alpha)\|\left\{1-\frac{1}{\sqrt{\pi}}\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2}\,d\theta\right\}d\alpha$$

$$+\left\| \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_{\varepsilon}^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}x(\alpha)\left\{1-\frac{1}{\sqrt{\pi}}\int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2}\,d\theta\right\}\right\|$$

$$= J_\omega^{(1)} + J_\omega^{(2)}.$$

Now $J_\omega^{(1)}$ is the same as the $J_\omega^{(1)}$ of theorem 2.3.1, so that for every $\varepsilon > 0$, $J_\omega^{(1)} \to 0$ as $\omega \to \infty$. Also

$$J_\omega^{(3)} = \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa} \int_\varepsilon^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} \chi(\alpha) \left\{ 1 - \frac{1}{\sqrt{\pi}} \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2} d\theta \right\} d\alpha$$

converges uniformly in ω for $\omega > 0$. For, for every $\omega_2 > \omega_1 > \varepsilon$,

$$\left\| \int_{\omega_1}^{\omega_2} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} \chi(\alpha) \left\{ 1 - \frac{1}{\sqrt{\pi}} \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}-\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}} e^{-\theta^2} d\theta \right\} d\alpha \right\|$$

$$= \left\| \int_{\omega_1}^{\omega_2} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)} \alpha^{-\frac{1}{2}} \chi(\alpha) \left[\left\{ \int_{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{-i\sqrt{\frac{\kappa\tau}{\alpha}}+\infty} + \int_{i\sqrt{\frac{\kappa\tau}{\alpha}}+\sqrt{\frac{\alpha\kappa\omega}{\tau}}}^{i\sqrt{\frac{\kappa\tau}{\alpha}}+\infty} \right\} e^{-\theta^2} d\theta \right] d\alpha \right\|$$

$$= \left\| 2\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \int_{\omega_1}^{\omega_2} e^{-\kappa\alpha/\tau} \chi(\alpha)\, d\alpha \int_{\sqrt{\omega}}^{\infty} e^{-\frac{\kappa\alpha}{\tau}\varphi^2} \cos(2\kappa\varphi)\, d\varphi \right\|$$

$$= 2\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} \Big\| e^{-\left(\frac{\kappa}{\tau}-\lambda_0\right)\omega_2} \left(\int_{\sqrt{\omega}}^{\infty} e^{-\frac{\kappa\omega_2}{\tau}\varphi^2} \cos(2\kappa\varphi)\, d\varphi \right) \int_{\omega_1}^{\omega_2} e^{-\lambda_0\alpha} \chi(\alpha)\, d\alpha$$

$$+ \int_{\omega_1}^{\omega_2} e^{-\left(\frac{\kappa}{\tau}-\lambda_0\right)\alpha} \Big[\left(\frac{\kappa}{\tau}-\lambda_0\right) \int_{\sqrt{\omega}}^{\infty} e^{-\frac{\alpha\kappa}{\tau}\varphi^2} \cos(2\kappa\varphi)\, d\varphi \; +$$

$$\frac{\kappa}{\tau} \int_{\sqrt{\omega}}^{\infty} e^{-\frac{\alpha\kappa}{\tau}\varphi^2} \varphi^2 \cos(2\kappa\varphi)\, d\varphi \Big] \int_{\omega_1}^{\alpha} e^{-\lambda_0\beta} \chi(\beta)\, d\beta\, d\alpha \Big\|$$

$$\leq 4\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} M e^{-\left(\frac{\kappa}{\tau}-\lambda_0\right)\omega_2} \int_0^{\infty} e^{-\kappa\omega_2\varphi^2/\tau}\, d\varphi$$

$$+ 4\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} M \int_{\omega_1}^{\omega_2} e^{-\left(\frac{\kappa}{\tau}-\lambda_0\right)\alpha} \left[\left(\frac{\kappa}{\tau}-\lambda_0\right) \int_0^{\infty} e^{-\frac{\alpha\kappa}{\tau}\varphi^2} d\varphi + \int_0^{\infty} e^{-\frac{\alpha\kappa}{\tau}\varphi^2} \varphi^2\, d\varphi \cdot \frac{\kappa}{\tau} \right] d\alpha$$

$$= 4\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} M \left\{ e^{-\left(\frac{\kappa}{\tau}-\lambda_0\right)\omega_2} \int_0^{\infty} e^{-\frac{\kappa\omega_2}{\tau}\varphi^2} d\varphi + \int_{\omega_2}^{\omega_1} \left[\frac{d}{d\alpha} \left\{ e^{-\left(\frac{\kappa}{\tau}-\lambda_0\right)\alpha} \int_0^{\infty} e^{-\frac{\alpha\kappa}{\tau}\varphi^2} d\varphi \right\} \right] d\alpha \right\}$$

$$= 4\left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} M e^{-\left(\frac{\kappa}{\tau}-\lambda_0\right)\omega_1} \int_0^{\infty} e^{-\frac{\omega_1\kappa}{\tau}\varphi^2} d\varphi \to 0 \quad \text{as } \omega_1 \text{ and } \omega_2 \to \infty.$$

Thus we may choose ε large enough that $J_\omega^{(2)} = \| J_\omega^{(3)} \| < \varepsilon'$ for any $\varepsilon' > 0$, and thus $\lim_{\omega \to \infty} J_\omega = 0$, and the theorem is proved.

Theorem 2.4.2: If

1. $x(\alpha)$ is Riemann-Graves integrable over $[0, \omega]$ for each $\omega > 0$,

2. The set Γ of discontinuities of $x(\alpha)$, $0 \leq \alpha < \infty$, has Lebesgue measure zero,

3. $f(\lambda) = (R) \int_0^\infty e^{-\lambda\alpha} x(\alpha) d\alpha$ converges uniformly for $\lambda > \gamma > 0$, then

$\vec{L}_{\kappa,\tau}[f(\lambda)]$ exists, and for $\tau > 0$

$\lim_{\kappa \to \infty} \vec{L}_{\kappa,\tau}[f(\lambda)] = x(\tau)$ almost everywhere.

Proof: By 1., 2., and the corollary to theorem 1.7.2, $x(\alpha)$ is in $B([0, \omega]; \mathfrak{X})$ for each $\omega > 0$, and

$$(R) \int_0^\infty e^{-\lambda\alpha} x(\alpha) d\alpha = \int_0^{\to\infty} e^{-\lambda\alpha} x(\alpha) d\alpha .$$

Consequently we can make use of the previous theorem, which yields the above conclusions.

5. Inversion of Transformations of Type IV.

Let $f(\lambda) = \int_0^\infty e^{-\lambda\alpha}\, da(\alpha)$. Conditions for the inversion of this type of transformation are given by the following theorem. To avoid complications, we restrict ourselves to the operator $L_{\kappa,\tau}[f(\lambda)]$.

Theorem 2.5.1: If

1. $a(\alpha)$ is of strongly bounded variation on $[0,\omega]$ for each $\omega > 0$, and $a(0) = \theta$

2. $\int_0^\infty e^{-\lambda\alpha}\, da(\alpha)$ converges for $\lambda > \lambda_0$, and uniformly for $\lambda > \lambda_1$,

3. $\varphi(\lambda)$ exists such that $\varphi(\lambda^2)$ is in $L(0,\infty)$, and $\int_0^\omega e^{-\lambda\alpha}\, da(\alpha) \leq \varphi(\lambda)$, for all $\omega > 0$, and all $\lambda > \lambda_1$,

then $\lim\limits_{\kappa\to\infty} \int_0^\tau L_{\kappa,\tau}[f(\lambda)]\, d\tau = a(\tau)$ almost everywhere.

Proof: By theorem 1.7.3, corollary, and 2.,

$$f(\lambda) = \int_0^\infty e^{-\lambda\alpha}\, da(\alpha) = \lambda \int_0^{\to\infty} e^{-\lambda\alpha}\, a(\alpha)\, d\alpha .$$

Thus, $L_{\kappa,\tau}[f(\lambda)]$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau} \int_0^\infty \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}})(\kappa(\eta+1)/\tau)\, d\eta \int_0^{\to\infty} e^{-\kappa(\eta+1)\alpha/\tau} a(\alpha)\, d\alpha.$$

Choose $\kappa > \lambda_1 \tau$. Then by 2. and 3., $L_{\kappa,\tau}[f(\lambda)]$ exists and satisfies the hypotheses of theorem 1.4.2. We have then

$$L_{\kappa,\tau}[f(\lambda)] = \frac{\kappa^2 e^{2\kappa}}{\pi \tau^2}\int_0^{\to\infty} e^{-\kappa\alpha/\tau} a(\alpha)\,d\alpha \int_0^\infty e^{-\kappa\eta\alpha/\tau}\eta^{-\frac{1}{2}}(\eta+1)\cos(2\kappa\eta^{\frac{1}{2}})\,d\eta$$

$$= \frac{2\kappa^{\frac{3}{2}} e^{2\kappa}}{\pi \tau^{3/2}}\int_0^{\to\infty} e^{-\kappa\alpha/\tau}\alpha^{-\frac{1}{2}} a(\alpha)\,d\alpha \int_0^\infty e^{-\zeta^2}\left(\frac{\tau\zeta^2}{\kappa\alpha}+1\right)\cos\left(2\left(\frac{\kappa\tau}{\alpha}\right)^{\frac{1}{2}}\zeta\right)d\zeta$$

$$= \frac{2\kappa^{\frac{3}{2}} e^{2\kappa}}{\pi \tau^{3/2}}\int_0^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}\left(1+\frac{\tau}{2\kappa\alpha}-\frac{\tau^2}{\alpha^2}\right)d\alpha$$

$$= e^{2\kappa}\left(\frac{\kappa}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty}\left\{\frac{d}{d\tau}\left(e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\tau^{\frac{1}{2}}\alpha^{-\frac{3}{2}}\right\}a(\alpha)\,d\alpha.$$

Thus by 2. and theorem 1.4.1,

$$\int_0^\tau L_{\kappa,\tau}[f(\lambda)]\,d\tau = e^{2\kappa}\left(\frac{\kappa\tau}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty} e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{3}{2}}a(\alpha)\,d\alpha \to a(\alpha)$$

almost everywhere, as $\kappa \to \infty$, by lemma 2.2.2.

Chapter III

1. Introduction.

We consider here the representation of Banach-valued functions $f(\lambda)$ of a real or complex variable λ by Laplace integrals of Banach-valued functions of a real variable. We shall consider representations in the forms

$$\text{I.}\quad f(\lambda) = (B)\int_0^\infty e^{-\lambda\tau}\, x(\tau)\,d\tau\ ,$$

$$\text{II.}\quad f(\lambda) = \int_0^\infty e^{-\lambda\tau}\, da(\tau).$$

In the latter case however, we shall restrict ourselves to numerically-valued functions. The reason for this is the lack, at the present time, of theorems for Banach-valued functions corresponding to those theorems for real functions which derive from Helly's selection principle.

Our tool in this task will be the "real" inversion operator which we used in the last chapter.

Section 2 of this chapter contains certain preliminary lemmas which yield conditions ensuring the existence of $L_{\kappa,\tau}\left[f(\lambda)\right]$.

In section 3 we derive the "Fundamental Theorem". This Theorem shows that under certain conditions the Laplace transform (i.e. the function found by an integral of type I above) of $L_{\kappa,\tau}\left[f(\lambda)\right]$ has $f(\lambda)$ for its limit as κ tends to infinity.

Section 4 contains the conditions that $f(\lambda)$ be represented as the Laplace integral of a function of $B_p([0,\infty);\mathfrak{X})$, p fixed, $1 < p \leq \infty$. The cases $1 < p < \infty$ and $p = \infty$ are treated separately there, as they are of a very different nature.

Section 5 contains a representation theory for the case that $f(\lambda)$ belongs to the class $H_p(\alpha;\mathfrak{X})$, p fixed $1 \leq p < \infty$, this class being defined there. The cases $p = 1$, and $1 < p < \infty$ are treated separately, again because of their very different nature.

Lastly, section 6 contains the conditions that $f(\lambda)$ be represented as a Laplace-Stieltjes transform, (an integral of type II above), but as mentioned before, we restrict ourselves to numerically-valued functions.

2. Preliminary Lemmas.

We first prove the two following lemmas which are preliminary to the "Fundamental Theorem" which is proved in the next section.

Lemma 3.2.1: If

1. $\lambda^{-1}\varphi(\lambda)$ is in $L(\delta,\infty)$ for all $\delta > 0$,

2. $\psi(\xi) = \int_\xi^\infty \eta^{-1}|\varphi(\eta)|\,d\eta = O(\xi^{-m})$ with $m > 0$, as $\xi \to \infty$, and $\psi(\xi) = O(e^{\gamma/\xi})$ with $\gamma \geqslant 0$, as $\xi \to 0+$,

3. $m + n > 0$,

then

(i) $\lambda^{n-1}\varphi(\lambda^{-1})$ is in $L(0,\omega)$ for all $\omega > 0$,

(ii) $\theta(\tau) = \int_0^\tau \eta^{n-1}|\varphi(\eta^{-1})|\,d\eta = O(\tau^{m+n})$ as $\tau \to 0+$,

(iii) $\theta(\tau) = O(\tau^n e^{\gamma\tau})$ as $\tau \to \infty$, if either $\gamma > 0$ or $\gamma = 0$ and $n \geqslant 0$,

$= O(1)$ as $\tau \to \infty$ if $\gamma = 0$ and $n < 0$,

(iv) $\int_0^\infty e^{-\lambda\alpha}\alpha^{n-1}\varphi(\alpha^{-1})\,d\alpha$ exists for $\lambda > \gamma$, and is $O(\lambda^{-m-n})$ as $\lambda \to \infty$.

Proof:

(i) Clearly $\xi^{n-1}|\varphi(\xi^{-1})|$ is in $L(\delta,\omega)$ for all $\omega > \delta > 0$.

Thus, $\int_\varepsilon^\tau \xi^{n-1}|\varphi(\xi^{-1})|\,d\xi = \int_\varepsilon^\tau \xi^n\,d\psi(\xi^{-1})$

$= \tau^n\psi(\tau^{-1}) - \varepsilon^n\psi(\varepsilon^{-1}) - n\int_\varepsilon^\tau \xi^{n-1}\psi(\xi^{-1})\,d\xi$, and by 2., the right hand side tends to a finite limit as $\varepsilon \to 0$, since, by 3., $m + n > 0$. Thus

$$\theta(\tau) = \tau^{n}\psi(\tau^{-1}) - n\int_0^{\tau} \xi^{n-1}\psi(\xi^{-1})d\xi \quad .$$

(ii) $$\theta(\tau) = \tau^{n}\psi(\tau^{-1}) - n\int_0^{\tau} \xi^{n-1}\psi(\xi^{-1})d\xi = O(\tau^{m+n})$$

by the last equation and 2.

(iii) Let either $\gamma > 0$, or $n \geq 0$. Since $m + n > 0$, as $\tau \to \infty$,

$$\theta(\tau) = \tau^{n}O(e^{\gamma\tau}) - n\int_0^{\tau} \xi^{n-1}O(e^{\gamma\xi})d\xi = O(\tau^{n}e^{\gamma\tau}).$$

If $\gamma = 0$, $n < 0$, $\theta(\tau)$ is clearly bounded.

(iv) Clearly $e^{-\lambda\alpha}\alpha^{n-1}|\varphi(\alpha^{-1})|$ is in $L(\delta, \omega)$ for all $\omega > \delta > 0$.

$$\int_{\delta}^{\omega} e^{-\lambda\alpha}\alpha^{n-1}|\varphi(\alpha^{-1})|\,d\alpha = e^{-\lambda\omega}\theta(\omega) - e^{-\lambda\delta}\theta(\delta)$$
$$+ \lambda\int_{\delta}^{\omega} e^{-\lambda\alpha}\theta(\alpha)d\alpha \quad .$$

Convergence as $\delta \to 0$ follows from (ii) since $m + n > 0$. Convergence as $\omega \to \infty$ follows from (iii) if $\lambda > \gamma$. Moreover, from Widder [12, page 181, theorem 1], the integral is $O(\lambda^{-m-n})$ as $\lambda \to \infty$, (and is $O((\lambda - \gamma)^{-n})$ as $\lambda \to \gamma+$ if $n \geq 0$).

Lemma 3.2.2: If $f(\lambda)$ is in $B([\delta, \omega]; \mathfrak{X})$ for all $\omega > \delta > 0$, and if $\|f(\lambda)\|$ satisfies all the requirements of $\varphi(\lambda)$ of lemma 3.1.1 with $m > \frac{1}{2}$, then for each $\kappa > 0$, and almost all $\tau > 0$,

$$L_{\kappa,\tau}[f(\lambda)] = \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^{\infty} \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})f(\kappa(\eta+1)/\tau)d\eta$$

exists.

In particular, $L_{\kappa,\tau}[f(\lambda)]$ exists when $\kappa, \tau > 0$ and κ/τ is in the Lebesgue set of $f(\lambda)$.

Proof: It is sufficient to show that the integral

$$\int_0^\infty \eta^{-\frac{1}{2}} \cos(2\kappa\eta^{\frac{1}{2}})\, f(\kappa(\eta+1)/\tau)\, d\eta$$

converges at the origin and at infinity.

If κ/τ is in the Lebesgue set of $f(\lambda)$, we have

$$\omega(\gamma) = \int_0^\gamma \| f(\kappa(\eta+1)/\tau) - f(\kappa/\tau)\| \, d\eta = o(\gamma).$$

Thus $\int_\varepsilon^\delta \eta^{-\frac{1}{2}} \| \cos(2\kappa\eta^{\frac{1}{2}}) f(\kappa(\eta+1)/\tau)\| \, d\eta$

$$\leq \int_\varepsilon^\delta \eta^{-\frac{1}{2}} d\eta \; f(\tfrac{\kappa}{\tau}) + \int_\varepsilon^\delta \eta^{-\frac{1}{2}} \| f(\kappa(\eta+1)/\tau) - f(\kappa/\tau)\| \, d\eta$$

$$= o(1) + \int_\varepsilon^\delta \eta^{-\frac{1}{2}} d\omega(\eta) = o(1) + \int_\varepsilon^\delta \tfrac{1}{2}\eta^{-\frac{3}{2}} \omega(\eta)\, d\eta$$

$$= o(1) + \frac{1}{2}\int_\varepsilon^\delta \eta^{-\frac{3}{2}} o(\eta)\, d\eta = o(1)$$ as $\varepsilon, \delta \to 0$, and the integral converges at the origin.

From lemma 3.2.1 we have

$$\int_0^\varepsilon \zeta^{-\frac{3}{2}} \| f(\zeta^{-1})\| \, d\zeta < \infty.$$

Here we put $\zeta^{-1} = \kappa(\eta+1)/\tau$, and choose $\varepsilon < \tau/\kappa$. We then have,

$$\left(\frac{\kappa}{\tau}\right)^{\frac{1}{2}} \int_{\frac{\tau}{\kappa\varepsilon}-1}^\infty (1+\eta)^{-\frac{1}{2}} \| f(\kappa(\eta+1)/\tau)\| \, d\eta < \infty,$$

and the integral converges at infinity.

Lemma 3.2.3: If

1. $\lambda^{-1}\varphi(\lambda)$ is in $L(\delta, \infty)$ for all $\delta > 0$,

2. $\psi(\xi) = \int_{\xi}^{\infty} \lambda^{-1}|\varphi(\lambda)|\, d\lambda = O(\xi^{-m})$ with $m > 0$, as $\xi \to \infty$ and $\psi(\xi) = O(e^{\gamma/\xi})$ with $\gamma \geq 0$ as $\xi \to 0+$,

then, for each $\varepsilon > 0$,

$$\int_{\xi}^{\infty} \lambda^{-1} e^{-\varepsilon\lambda} |\varphi(\lambda)|\, d\lambda = o(\xi^{-n}) \text{ for every } n > 0, \text{ as } \xi \to \infty$$

$$= O(e^{\gamma/\xi}), \text{ as } \xi \to 0+.$$

Proof:

$$\int_{\xi}^{\infty} \lambda^{-1} e^{-\varepsilon\lambda} |\varphi(\lambda)|\, d\lambda \leq e^{-\varepsilon\xi} \int_{\xi}^{\infty} \lambda^{-1}|\varphi(\lambda)|\, d\lambda$$

$$= e^{-\varepsilon\xi}\psi(\xi) = o(\xi^{-n}) \text{ as } \xi \to \infty$$

$$= O(e^{\gamma/\xi}) \text{ as } \xi \to 0+.$$

3. Fundamental Theorem.

The following theorem is fundamental in the representation theory.

Theorem 3.3.1: If

1. $\lambda^{-1} f(\lambda)$ is in $B([\delta, \infty); \mathfrak{X})$ for all $\delta > 0$,

2. $\psi(\xi) = \int_{\xi}^{\infty} \eta^{-1} \| f(\eta) \| \, d\eta = O(\xi^{-m})$, with $m > \frac{1}{2}$,

as $\xi \to \infty$, and $\psi(\xi) = O(e^{\gamma/\xi})$, with $\gamma \geq 0$, as $\xi \to 0+$,

3. $e^{-\zeta\tau} L_{\kappa,\tau}[f(\lambda)]$ is in $B([0, \infty); \mathfrak{X})$ for $\zeta > \gamma_1$, and all $\kappa > \kappa_0$,

then

$$\lim_{\kappa \to \infty} \int_0^{\infty} e^{-\zeta\tau} L_{\kappa,\tau}[f(\lambda)] \, d\tau = f(\zeta) \text{ at every point } \zeta > \gamma_1$$

of the Lebesgue set of $f(\lambda)$.

Proof: $L_{\kappa,\tau}[f(\lambda)]$ exists by lemma 3.1.2, and has a Laplace transform when $\zeta > \gamma_1$ by 3. To prove the assertion we shall use theorem 1.3.6 and lemma 2.2.1, corollary.

Operating formally we have

$$\int_0^{\infty} e^{-\zeta\tau} L_{\kappa,\tau}[f(\lambda)] \, d\tau =$$

$$\frac{\kappa e^{2\kappa}}{\pi}\int_0^\infty e^{-\zeta\tau}\tau^{-1}d\tau\int_0^\infty \eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})\, f(\kappa(\eta+1)/\tau)d\eta$$

$$= \frac{2\kappa e^{2\kappa}}{\pi}\int_0^\infty e^{-\zeta\tau}\tau^{-1}d\tau\int_0^\infty \cos(2\kappa\xi)\, f(\kappa(\xi^2+1)/\tau)d\xi$$

$$= \frac{2\kappa e^{2\kappa}}{\pi}\int_0^\infty \cos(2\kappa\xi)d\xi\int_0^\infty e^{-\zeta\tau}\tau^{-1}\, f(\kappa(\xi^2+1)/\tau)d\tau$$

$$= \frac{2\kappa e^{2\kappa}}{\pi}\int_0^\infty \cos(2\kappa\xi)d\xi\int_0^\infty e^{-\kappa\zeta\beta(\xi^2+1)}\beta^{-1} f(\beta^{-1})d\beta$$

$$= \frac{2\kappa e^{2\kappa}}{\pi}\int_0^\infty e^{-\kappa\zeta\beta}\beta^{-1} f(\beta^{-1})d\beta\int_0^\infty e^{-\kappa\zeta\beta\xi^2}\cos(2\kappa\xi)d\xi$$

$$= \frac{2\kappa e^{2\kappa}}{\pi}\int_0^\infty e^{-\kappa\zeta\beta}\beta^{-\frac{3}{2}} f(\beta^{-1})d\beta\int_0^\infty e^{-\alpha^2}\cos(2(\kappa/\zeta\beta)^{\frac{1}{2}}\alpha)d\alpha$$

$$= e^{2\kappa}\left(\frac{\kappa}{\pi\zeta}\right)^{\frac{1}{2}}\int_0^\infty e^{-\kappa(\zeta\beta+(\zeta\beta)^{-1})}\beta^{-\frac{3}{2}} f(\beta^{-1})d\beta$$

$\to f(\zeta)$ as $\kappa \to \infty$.

These formal calculations will be justified if the two interchanges of integrations are justified and the conditions of lemma 2.1.1 are met.

For the first interchange of integrations it is sufficient to show that

$$\int_0^\infty |\cos(2\kappa\xi)|\, d\xi\int_0^\infty e^{-\kappa\zeta\beta(\xi^2+1)}\beta^{-1}\,\|f(\beta^{-1})\|\, d\beta < \infty.$$

But by 2. and lemma 3.2.2, if $\kappa\zeta > \gamma$, the inner integral is $O(\xi^{-2m})$ as $\xi \to \infty$, and $m > \frac{1}{2}$. Thus by theorem 1.3.9 the interchange is justified.

For the second interchange it is sufficient to show that

$$\int_0^\infty e^{-\kappa\zeta\beta} \beta^{-\frac{3}{2}} \| f(\beta^{-1}) \| \, d\beta \int_0^\infty e^{-\alpha^2} |\cos(2(\kappa/\zeta\beta)^{\frac{1}{2}}\alpha)| \, d\alpha < \infty.$$

But this is true since the inner integral is less than $\frac{1}{2}\sqrt{\pi}$, and since $\int_0^\infty e^{-\kappa\zeta\beta} \beta^{-\frac{3}{2}} \| f(\beta^{-1}) \| \, d\beta$ converges by 1., 2., and lemma 3.2.1(iv).

4. Representation Theorems for $L_{\kappa,\tau}[f(\lambda)]$ in $B_p([0,\infty);\mathfrak{X})$.

In this section we find conditions that a function $f(\lambda)$ on $[\alpha,\infty)$ to a Banach space $\mathfrak{X}$ be represented as the Laplace integral of a function in $B_p([0,\infty);\mathfrak{X})$, where $1 < p \leq \infty$.

In order to obtain such conditions for these general classes of functions, we find it necessary, in the cases $1 < p < \infty$, to postulate some sort of compactness condition on $B_p([0,\infty);\mathfrak{X})$. We have chosen the weakest condition at present known, namely weak compactness of the unit sphere in $B_p([0,\infty);\mathfrak{X})$. By Bochner and Taylor [1], and Pettis [9] a necessary and sufficient condition for this compactness is that $\mathfrak{X}$ be reflexive, and this is the manner in which we have set the condition. It is well known that $B_1([0,\infty);\mathfrak{X})$ has never a weakly compact unit sphere. Thus, to obtain a representation theorem for B_1 it would be necessary to postulate some convergence condition on $L_{\kappa,\tau}[f(\lambda)]$, and we would obtain a theorem very like that of Widder [12; page 318]. We have not chosen to do this, since the results are quite obvious.

The first theorem of this section gives sufficient, and in the cases $p > 2$ necessary, conditions that $f(\lambda)$ be represented as a Laplace integral of a function in $B_p([0,\infty);\mathfrak{X})$, $1 < p < \infty$. The fact that these conditions are not necessary for $p \leq 2$ is a consequence of the fact that, as the following example shows, $f(\lambda)$ may be the Laplace transform of a function in $B_p([0,\infty);\mathfrak{X})$, $p \leq 2$, and yet

$L_{\kappa,\tau}\ [f(\lambda)]$ may not exist. For example, let $x(\alpha) = (\Gamma(\frac{1}{3}))^{-1}\alpha^{-2/3}e^{-\alpha}$. Then

$$f(\lambda) = \int_0^\infty e^{-\lambda\alpha}x(\alpha)d\alpha = (\lambda+1)^{-1/3}.$$

But $L_{\kappa,\tau}\ [f(\lambda)]$ does not exist since

$$\frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\infty \left|\eta^{-\frac{1}{2}}\cos(2\kappa\eta^{\frac{1}{2}})f(\kappa(\eta+1)/\tau)\right| d\eta$$

$$= \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\infty \left|\cos(2\kappa\eta^{\frac{1}{2}})\eta^{-\frac{1}{2}}(\tfrac{\kappa}{\tau}(\eta+1)+1)^{-\frac{1}{3}}\right| d\eta = \infty.$$

In order to cope with this phenomenon, we resort to what is essentially Cauchy's method of summation. This yields, in theorem 3.4.2, necessary and sufficient conditions that $f(\lambda)$ be represented as the Laplace integral of a function in $B_p([0,\infty);\mathfrak{X})$, $1 < p < \infty$.

The case of $B_\infty([0,\infty);\mathfrak{X})$ is treated in the final theorem of this section. It will be noted that the methods used are very different from those of the two previous theorems.

Theorem 3.4.1: If $\mathfrak{X}$ is a reflexive Banach space, then the following conditions are sufficient for $f(\lambda)$ to be equal almost everywhere for $\lambda > 0$ to the Laplace integral of a function in $B_p([0,\infty);\mathfrak{X})$, $1<p<\infty$.

1. $\lambda^{-1}f(\lambda)$ is in $B([\delta,\infty);\mathfrak{X})$ for all $\delta > 0$,
2. $\int_\xi^\infty \eta^{-1}\|f(\eta)\|\,d\eta = O(\xi^{-m})$ with $m > \frac{1}{2}$, as $\xi \to \infty$
   $= O(e^{\gamma/\xi})$ with $\gamma > 0$, as $\xi \to 0+$,
3. $\|L_{\kappa,\cdot}\ [f(\lambda)]\|_p \leq M_p$, p fixed, $1 < p < \infty$, $\kappa > \kappa_0$.

Conditions 1. and 3. are necessary for every p, $1 < p < \infty$, and 2. is necessary if $p > 2$.

Proof:

Necessity: Suppose $f(\lambda) = \int_0^\infty e^{-\lambda\tau} x(\tau)\,d\tau$ a.e., and $x(\tau)$ is in $B_p([0,\infty);\mathfrak{X})$. Then using Hölders inequality we have, almost everywhere

$$\lambda^{-1}\|f(\lambda)\| \leq \lambda^{-1}\int_0^\infty e^{-\lambda\tau}\|x(\tau)\|\,d\tau \leq \lambda^{-1}\left\{\int_0^\infty e^{-q\lambda\tau}d\tau\right\}^{\frac{1}{q}}\left\{\int_0^\infty \|x(\tau)\|^p d\tau\right\}^{\frac{1}{p}}$$

$$= A\lambda^{-1-\frac{1}{q}},$$ so that 1. is necessary.

Thus $\int_3^\infty \lambda^{-1}\|f(\lambda)\|\,d\lambda \leq A\int_3^\infty \lambda^{-(1+\frac{1}{q})}\,d\lambda = B\lambda^{-\frac{1}{q}}$,

so that 2. is necessary if $\frac{1}{q} > \frac{1}{2}$, i.e. is if $p > 2$.

From theorem 2.3.1, we have

$$L_{\kappa,\tau}[f(\lambda)] = \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha,$$

so that,

$$\|L_{\kappa,\tau}[f(\lambda)]\| \leq \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}}e^{2\kappa}\int_0^\infty e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\|x(\alpha)\|\,d\alpha$$

$$= \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}}e^{2\kappa}\int_0^\infty e^{-\frac{\kappa}{p}(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2p}}\|x(\alpha)\|\,e^{-\frac{\kappa}{q}(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2q}}\,d\alpha$$

$$\leq \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}}e^{2\kappa}\left\{\int_0^\infty e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\|x(\alpha)\|^p d\alpha\right\}^{\frac{1}{p}}\left\{\int_0^\infty e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}d\alpha\right\}^{\frac{1}{q}}$$

$$= \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}(1-\frac{1}{q})}e^{2(1-1/q)\kappa}\left\{\int_0^\infty e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\|x(\alpha)\|^p d\alpha\right\}^{\frac{1}{p}}$$

$$= \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2p}}e^{2\kappa/p}\left\{\int_0^\infty e^{-\kappa(\frac{\alpha}{\tau}+\frac{\tau}{\alpha})}\alpha^{-\frac{1}{2}}\|x(\alpha)\|^p d\alpha\right\}^{\frac{1}{p}}.$$

Thus, $\int_0^\infty \| L_{\kappa,\tau}[f(\lambda)]\|^p d\tau$

$$\leqslant \left(\frac{\kappa}{\pi}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty d\tau \int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}(\tau\alpha)^{-\frac{1}{2}}\|x(\alpha)\|^p d\alpha$$

$$= \left(\frac{\kappa}{\pi}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty \|x(\alpha)\|^p d\alpha \int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}(\tau\alpha)^{-\frac{1}{2}} d\alpha$$

$$= \int_0^\infty \|x(\alpha)\|^p d\alpha$$

Hence $\| L_{\kappa,\cdot}[f(\lambda)]\|_p \leq \|x(\cdot)\|_p$, so that 3. is necessary.

Sufficiency: From 1., 2., 3., and theorem 3.3.1, we have for almost all $\zeta > 0$,

$$f(\zeta) = \lim_{\kappa\to\infty}\int_0^\infty e^{-\zeta\tau} L_{\kappa,\tau}[f(\lambda)]\, d\tau \; .$$

By Bochner and Taylor [1], and Pettis [9], if $\mathfrak{X}$ is a reflexive Banach space, $B_p([0,\infty);\mathfrak{X})$ is a reflexive space for $1 < p < \infty$, and by Gantmakher and Šmulian [4], a reflexive Banach space has a weakly compact unit sphere. Thus $B_p([0,\infty);\mathfrak{X})$ has a weakly compact unit sphere, so that there exists an element $x(\cdot)$ of $B_p([0,\infty);\mathfrak{X})$ and an increasing unbounded sequence $\{\kappa_i\}$ such that for every functional y^* on $B_p([0,\infty);\mathfrak{X})$, (i.e. for every y^* in $B_p^*([0,\infty);\mathfrak{X})$), $\lim_{i\to\infty} y^*(L_{\kappa_i,\cdot}[f(\lambda)]) = y^*(x(\cdot))$.

Let x^* be an arbitrary element of $\mathfrak{X}^*$. Then if $g(\cdot)$ is an element of $B_p([0,\infty);\mathfrak{X})$,

$$x^*\left(\int_0^\infty e^{-\zeta\alpha} g(\alpha)\, d\alpha\right) = \int_0^\infty e^{-\zeta\alpha} x^*(g(\alpha))\, d\alpha = y_\zeta^*(g(\cdot))$$

defines an element y_ζ^* of $B_p([0,\infty);\mathfrak{X})$ for each $\zeta > 0$. For, y^* is obviously linear, and using Hölder's inequality we have

$$|y_\zeta^*(g(\cdot))| = \left|\int_0^\infty e^{-\zeta\alpha} x^*(g(\alpha))d\alpha\right| \leq \left(\int_0^\infty e^{-q\zeta\alpha} d\alpha\right)^{\frac{1}{q}}\left\{\int_0^\infty \|x^*(g(\alpha))\|^p\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{(q\zeta)^{\frac{1}{q}}}\|x^*\|\left\{\int_0^\infty \|g(\alpha)\|^p d\alpha\right\}^{\frac{1}{p}} = \frac{\|x^*\|}{(q\zeta)^{\frac{1}{q}}}\|g(\cdot)\|_p,$$ so that y^*

is bounded for each $\zeta > 0$.

Thus we have, for each x^* in $\mathfrak{X}^*$ and for almost all $\zeta > 0$,

$$x^*(f(\zeta)) = x^*\left(\lim_{i\to\infty}\int_0^\infty e^{-\zeta\tau} L_{\kappa_i,\tau}[f(\lambda)]\,d\tau\right)$$

$$= \lim_{i\to\infty}\int_0^\infty e^{-\zeta\tau} x^*(L_{\kappa_i,\tau}[f(\lambda)])\,d\tau$$

$$= \lim_{i\to\infty} y_\zeta^*(L_{\kappa_i,\cdot}[f(\lambda)]) = y_\zeta^*(x(\cdot))$$

$$= \int_0^\infty e^{-\zeta\tau} x^*(x(\tau))\,d\tau = x^*\left(\int_0^\infty e^{-\zeta\tau} x(\tau)\,d\tau\right),$$

and thus, for almost all $\zeta > 0$,

$$f(\zeta) = \int_0^\infty e^{-\zeta\tau} x(\tau)\,d\tau .$$

To obtain necessary and sufficient conditions we define

$$L^\varepsilon_{\kappa,\tau}[f(\lambda)] = L_{\kappa,\tau}[e^{-\varepsilon\lambda} f(\lambda)] .$$

The following theorem yields the mentioned conditions.

Theorem 3.4.2: If $\mathfrak{X}$ is a reflexive Banach space, then the following conditions are necessary and sufficient for $f(\lambda)$ to be equal almost everywhere for $\lambda > 0$ to the Laplace integral of a function in $B_p([0,\infty);\mathfrak{X})$.

1. $\lambda^{-1}f(\lambda)$ is in $B([\delta,\infty);\mathfrak{X})$, $\delta>0$,

2. $$\int_{\xi}^{\infty} \lambda^{-1}\|f(\lambda)\|\,d\lambda \quad \begin{aligned} &= O(\xi^{-m}) \text{ with } m>0, \text{ as } \xi\to\infty \\ &= O(e^{\gamma/\xi}) \text{ with } \gamma>0, \text{ as } \xi\to 0+, \end{aligned}$$

3. $\| L^{\varepsilon}_{\kappa,\cdot}[f(\lambda)]\|_p \leq M_p$, where M_p is independent of κ and ε, p fixed, $1<p<\infty$, $\kappa>\kappa_0$.

Proof:

Necessity: 1. was proved necessary in the previous theorem as was 2. The proof of the necessity of 3. is almost exactly the same as in the previous theorem.

Sufficiency: By 1., 2., 3., lemma 3.2.3, and theorem 3.3.1,

$$e^{-\varepsilon\zeta} f(\zeta) = \lim_{\kappa\to\infty}\int_0^{\infty} e^{-\zeta\tau} L^{\varepsilon}_{\kappa,\tau}[f(\lambda)]\,d\tau .$$

As in the previous theorem, $B_p([0,\infty);\mathfrak{X})$ has a weakly compact unit sphere, so that for each $\varepsilon>0$, there exists an element $x_{\varepsilon}(\cdot)$ of $B_p([0,\infty);\mathfrak{X})$ and an increasing unbounded sequence $\{{}_{\varepsilon}\kappa_i\}$ such that for every y^* in $B_p^*([0,\infty);\mathfrak{X})$

$$\lim_{i\to\infty} y^*(L^{\varepsilon}_{{}_{\varepsilon}\kappa_i,\cdot}[f(\lambda)]) = y^*(x_{\varepsilon}(\cdot)).$$

Further, since

$y^*(L^{\varepsilon}_{\varepsilon K_i}, \cdot\,[f(\lambda)]) \leq \|y^*\| \cdot \| L^{\varepsilon}_{\varepsilon K_i}, \cdot\,[f(\lambda)]\|_p \leq \|y^*\| M_p$, we have $\|x_\varepsilon(\cdot)\|_p \leq M_p$.

Let x^* be an arbitrary element of $\mathfrak{X}^*$, and define y^*_ζ of $B^*_p([0,\infty);\mathfrak{X})$ as in the previous theorem. Then for each $\varepsilon > 0$, and almost all $\zeta > 0$

$$x^*(e^{-\varepsilon\zeta} f(\zeta)) = x^*(\lim_{i\to\infty} \int_0^\infty e^{-\zeta\tau} L^{\varepsilon}_{\varepsilon K_i,\tau}[f(\lambda)]\, d\tau)$$

$$= \lim_{i\to\infty} \int_0^\infty e^{-\zeta\tau} x^*(L^{\varepsilon}_{\varepsilon K_i,\tau}[f(\lambda)])\, d\tau$$

$$= \lim_{i\to\infty} y^*_\zeta(L^{\varepsilon}_{\varepsilon K_i,\tau}[f(\lambda)]) = y^*_\zeta(x_\varepsilon(\cdot))$$

$$= \int_0^\infty e^{-\zeta\tau} x^*(x_\varepsilon(\tau))\, d\tau = x^*(\int_0^\infty e^{-\zeta\tau} x_\varepsilon(\tau)\, d\tau).$$

Thus for almost all $\zeta > 0$,

$$e^{-\varepsilon\zeta} f(\zeta) = \int_0^\infty e^{-\zeta\tau} x_\varepsilon(\tau)\, d\tau .$$

Now since $\|x_\varepsilon(\cdot)\|_p \leq M_p$ for all $\varepsilon > 0$, and since $B_p([0,\infty);\mathfrak{X})$ has a weakly compact unit sphere, there exists an element $x(\cdot)$ of $B_p([0,\infty);\mathfrak{X})$ and a sequence $\{\varepsilon_i\}$ with $\lim_{i\to\infty} \varepsilon_i = 0$, such that for every y^* in $B^*_p([0,\infty);\mathfrak{X})$

$$\lim_{i\to\infty} y^*(x_{\varepsilon_i}(\cdot)) = y^*(x(\cdot)).$$

For each i there is a set $\Sigma_i \subseteq (0,\infty)$, whose measure is zero, such that for ζ in Σ_i,

$$e^{-\varepsilon_i\zeta} f(\zeta) \neq \int_0^\infty e^{-\varepsilon_i\tau} x_{\varepsilon_i}(\tau)\, d\tau .$$

Let $\Sigma = \bigcup_i \Sigma_i$. Then Σ has measure zero.

Let x^* be an arbitrary element of $\mathfrak{X}^*$, and define y^*_ζ of $B^*_p([0,\infty);\mathfrak{X})$ as previously. Then for every ζ not in Σ,

$$x^*(f(\zeta)) = \lim_{i\to\infty} x^*(e^{-\varepsilon_i\zeta} f(\zeta)) = \lim_{i\to\infty} x^*\left(\int_0^\infty e^{-\zeta\tau} x_{\varepsilon_i}(\tau)d\tau\right)$$

$$= \lim_{i\to\infty} \int_0^\infty e^{-\zeta\tau} x^*(x_{\varepsilon_i}(\tau))d\tau = \lim_{i\to\infty} y^*_\zeta(x_{\varepsilon_i}(\cdot))$$

$= y^*(x(\cdot)) = x^*\left(\int_0^\infty e^{-\zeta\tau} x(\tau)d\tau\right)$, so that for almost all $\zeta > 0$,

$$f(\zeta) = \int_0^\infty e^{-\zeta\tau} x(\tau)d\tau .$$

The following theorem deals with $B_\infty([0,\infty);\mathfrak{X})$.

Theorem 3.4.3: If $\mathfrak{X}$ is a uniformly convex Banach space, then the following conditions are necessary and sufficient that $f(\lambda)$ be equal almost everywhere for $\lambda > 0$ to a Laplace integral of a function in $B_\infty([0,\infty);\mathfrak{X})$.

1. $\lambda^{-1}f(\lambda)$ is in $B([\delta,\infty);\mathfrak{X})$ for all $\delta > 0$,
2. $\int_\xi^\infty \eta^{-1}\|f(\eta)\|\,d\eta = O(\xi^{-m})$ with $m > \frac{1}{2}$, as $\xi \to \infty$
   $= O(e^{\gamma/\xi})$ with $\gamma > 0$, as $\xi \to 0+$,
3. $\|L_{\kappa,\cdot}[f(\lambda)]\|_\infty \leq M_\infty$, $\kappa > \kappa_0$.

Proof:

Necessity: Suppose $f(\lambda) = \int_0^\infty e^{-\lambda\tau} x(\tau)d\tau$, where $x(\tau)$ is in $B_\infty([0,\infty);\mathfrak{X})$. Then

$$\lambda^{-1}\|f(\lambda)\| \leqslant \lambda^{-1}\int_0^\infty e^{-\lambda\tau}\|x(\tau)\|\,d\tau \leqslant \lambda^{-1}\|x(\cdot)\|_\infty \cdot$$

$\cdot \int_0^\infty e^{-\zeta\tau}\,d\tau \leqslant \lambda^{-2}\|x(\cdot)\|_\infty$, so that 1. is necessary. Further, then

$\int_\xi^\infty \lambda^{-1}\|f(\lambda)\|\,d\lambda \leqslant \xi^{-1}\|x(\cdot)\|_\infty$, so that 2. is necessary.

Finally, from theorem 2.3.1, we have

$$L_{\kappa,\tau}\left[f(\lambda)\right] = \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}x(\alpha)\,d\alpha \quad ,$$

so that

$$\|L_{\kappa,\tau}[f(\lambda)]\| \leqslant \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}\|x(\alpha)\|\,d\alpha$$

$$\leqslant \|x(\cdot)\|_\infty \left(\frac{\kappa}{\pi\tau}\right)^{\frac{1}{2}} e^{2\kappa}\int_0^\infty e^{-\kappa\left(\frac{\alpha}{\tau}+\frac{\tau}{\alpha}\right)}\alpha^{-\frac{1}{2}}\,d\alpha = \|x(\cdot)\|_\infty \,.$$

Thus $\|L_{\kappa,\cdot}\,[f(\lambda)]\|_\infty \leqslant \|x(\cdot)\|_\infty$.

Sufficiency: By 1., 2., 3., and theorem 3.3.1, we have, for almost all $\zeta > 0$,

$$f(\zeta) = \lim_{\kappa\to\infty}\int_0^\infty e^{-\zeta\tau}\,L_{\kappa,\tau}[f(\lambda)]\,d\tau \,.$$

By Pettis [10], a uniformly convex Banach space is reflexive, so that $\mathfrak{X}$ is reflexive.

Let φ be in $L_1(0,\infty)$. Define

$$T_\kappa(\varphi) = \int_0^\infty \varphi(\tau)L_{\kappa,\tau}[f(\lambda)]\,d\tau \,.$$

Then, $\|T_\kappa(\varphi)\| \leq \int_0^\infty |\varphi(\tau)| \cdot \|L_{\kappa,\tau}[f(\lambda)]\| \, d\tau$

$$\leq \|L_{\kappa,\cdot}[f(\lambda)]\|_\infty \int_0^\infty |\varphi(\tau)| \, d\tau \leq M_\infty \|\varphi(\cdot)\|_1$$

Thus $\{T_\kappa\}$ is a set of linear transformations on a separable space, $L_1(0,\infty)$, to a reflexive Banach space $\mathfrak{X}$, and $\|T_\kappa\| \leq M_\infty$. Thus, by theorem 1.8.1, there is an increasing unbounded sequence $\{\kappa_i\}$, and a linear transformation T on L_1 to $\mathfrak{X}$, such that for every functional x^* in $\mathfrak{X}^*$, and every φ in L_1,

$$\lim_{i\to\infty} x^*(T_{\kappa_i}(\varphi)) = x^*(T(\varphi)).$$

But by Dunford [2], every bounded linear transformation on $L_1(0,\infty)$ to a uniformly convex Banach space $\mathfrak{X}$ is of the form

$$T(\varphi) = \int_0^\infty \varphi(\tau)x(\tau)d\tau \quad \text{, where } x(\tau) \text{ is in } B_\infty([0,\infty);\mathfrak{X}).$$

Thus $x(\tau)$ in $B_\infty([0,\infty);\mathfrak{X})$ exists so that $T(\varphi)$ has the above form, and then we must have, for every x^* in $\mathfrak{X}^*$,

$$\lim_{i\to\infty} x^*\left(\int_0^\infty \varphi(\tau)L_{\kappa_i,\tau}[f(\lambda)]\,d\tau\right) = \lim_{i\to\infty} x^*(T_{\kappa_i}(\varphi))$$

$$= x^*(T(\varphi)) = x^*\left(\int_0^\infty \varphi(\tau)x(\tau)d\tau\right).$$

Let $\varphi(\tau) = e^{-\zeta\tau}$, $\zeta > 0$. Then, for almost all $\zeta > 0$,

$$x^*(f(\zeta)) = x^*\left(\lim_{i\to\infty} \int_0^\infty e^{-\zeta\tau} L_{\kappa_i,\tau}[f(\lambda)]\, d\tau\right)$$

$$= \lim_{i\to\infty} x^*\left(\int_0^\infty e^{-\zeta\tau} L_{\kappa_i,\tau}[f(\lambda)]\, d\tau\right)$$

$$= \lim_{i\to\infty} x^*(T_{\kappa_i}(e^{-\zeta\tau})) = x^*(T(e^{-\zeta\tau}))$$

$$= x^*\left(\int_0^\infty e^{-\zeta\tau} x(\tau) d\tau\right),$$

so that for almost all $\zeta > 0$,

$$f(\zeta) = \int_0^\infty e^{-\zeta\tau} x(\tau) d\tau .$$

5. Representation Theorems for $f(\lambda)$ in $H_p(\alpha\,;\mathfrak{X})$.

The class $H_p(\alpha\,;\mathfrak{X})$ is defined as follows.

Definition 3.5.1: $f(\lambda)$ will be said to belong to the class $H_p(\alpha\,;\mathfrak{X})$, p fixed, $1 \leq p < \infty$, if

(i) $f(\lambda)$ is a function on the complex numbers to the Banach space $\mathfrak{X}$ which is holomorphic* for $\operatorname{Re}\lambda > \alpha$.

(ii) $$\sup_{\rho>\alpha}\left\{\int_{-\infty}^{\infty}\|f(\rho+i\eta)\|^p\,d\eta\right\}^{\frac{1}{p}} = \|f\|_p < \infty.$$

(iii) $\lim\limits_{\rho\to\alpha} f(\rho+i\eta) \equiv f(\alpha+i\eta)$ exists for almost all values of η, and $f(\alpha+i\eta)$ is in $B_p((-\infty,\infty);\mathfrak{X})$.

For a discussion of the dependence of (iii) on (i) and (ii) see Hille [6].

The following two theorems give the conditions under which a function in $H_p(\alpha\,;\mathfrak{X})$ can be represented as a Laplace integral.

Theorem 3.5.1: If $f(\lambda)$ is in $H_1(\alpha\,;\mathfrak{X})$ where $\alpha > 0$, then $\lim\limits_{\kappa\to\infty} L_{\kappa,\tau}\left[f(\lambda)\right]$ exists and equals

$$g(\tau) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} f(\mu)\,d\mu \quad, \text{ and}$$

$$f(\lambda) = \int_0^{\infty} e^{-\lambda\tau} g(\tau)\,d\tau \;.$$

Proof: By Hille [6], page 213, theorem 10.4.1, we have for $\operatorname{Re}\lambda > \alpha$

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* See definition 1.2.5.

$$f(\lambda) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{f(\mu)}{\lambda - \mu}\, d\mu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(\alpha + i\eta)}{\lambda - (\alpha + i\eta)}\, d\eta \;.$$

Thus $L_{\kappa,\tau}\left[f(\lambda)\right] = \frac{\kappa e^{2\kappa}}{\pi\tau}\int_0^\infty \zeta^{-\frac{1}{2}} \cos(2\kappa\zeta^{\frac{1}{2}}) f(\kappa(\zeta + 1)/\tau)\, d\zeta$

$$= \frac{2\kappa e^{2\kappa}}{\pi\tau}\int_0^\infty \cos(2\kappa\xi)\, f(\kappa(\xi^2+1)/\tau)\, d\xi$$

$$= \frac{\kappa e^{2\kappa}}{\pi^2\tau}\int_0^\infty \cos(2\kappa\xi)\, d\xi \int_{-\infty}^\infty \frac{f(\alpha+i\eta)\, d\eta}{\frac{\kappa}{\tau}(\xi^2+1) - (\alpha+i\eta)}$$

$$= \frac{e^{2\kappa}}{\pi^2}\int_{-\infty}^\infty f(\alpha+i\eta)\, d\eta \int_0^\infty \frac{\cos(2\kappa\xi)\, d\xi}{\xi^2 + (1 - \frac{\tau}{\kappa}(\alpha+i\eta))}$$

The interchange of integrations is valid for $\kappa > \tau\alpha$ by Theorem 1.3.6 since

$$\int_{-\infty}^\infty \| f(\alpha + i\eta)\|\, d\eta \int_0^\infty \left| \frac{\cos(2\kappa\xi)}{\xi^2 + (1 - \frac{\tau}{\kappa}(\alpha+i\eta))} \right| d\xi$$

$$\leq \int_{-\infty}^\infty \| f(\alpha+i\eta)\|\, d\eta \int_0^\infty \frac{d\xi}{\xi^2 + (1 - \frac{\tau\alpha}{\kappa})} \leq \frac{\pi}{2(1-\frac{\tau\alpha}{\kappa})}\, \| f\|_1$$

Thus $L_{\kappa,\tau}\left[f(\lambda)\right] = \frac{e^{2\kappa}}{\pi}\int_{-\infty}^\infty e^{-2\kappa(1 - \frac{\tau}{\kappa}(\alpha + i\eta))^{\frac{1}{2}}} (1 - \frac{\tau}{\kappa}(\alpha + i\eta))^{-\frac{1}{2}} \cdot$

$f(\alpha + i\eta)\, d\eta$.

Obviously, for each τ and η,

$$\lim_{\kappa\to\infty} e^{2\kappa} \cdot e^{-2\kappa(1 - \frac{\tau}{\kappa}(\alpha + i\eta))^{\frac{1}{2}}} \cdot (1 - \frac{\tau}{\kappa}(\alpha + i\eta))^{-\frac{1}{2}} = e^{\tau(\alpha + i\eta)}.$$

Further, for each τ this limit exists uniformly in η for $-\infty < \eta < \infty$. For, a lengthy, but straightforward, calculation shows that the maximum value of

$$\left| e^{2\kappa(1-(1-\frac{\tau}{\kappa}(\alpha+i\eta))^{\frac{1}{2}})} \cdot (1-\frac{\tau}{\kappa}(\alpha+i\eta))^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)} \right|$$

occurs at $\eta = 0$. Thus since $\lim_{\kappa\to\infty} e^{2\kappa(1-(1-\frac{\tau\alpha}{\kappa})^{\frac{1}{2}})} \cdot (1-\frac{\tau\alpha}{\kappa})^{-\frac{1}{2}} = e^{\alpha\tau}$, for each $\varepsilon > 0$, $\kappa_0(\varepsilon)$ exists such that for $\kappa > \kappa_0(\varepsilon)$

$$\left| e^{2\kappa(1-(1-\frac{\tau\alpha}{\kappa})^{\frac{1}{2}})}(1-\frac{\tau\alpha}{\kappa})^{-\frac{1}{2}} - e^{\alpha\tau} \right| < \varepsilon.$$

Then for $\kappa > \kappa_0(\varepsilon)$,

$$\left| e^{2\kappa(1-(1-\frac{\tau}{\kappa}(\alpha+i\eta))^{\frac{1}{2}})}(1-\frac{\tau}{\kappa}(\alpha+i\eta))^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)} \right|$$

$$\leq \left| e^{2\kappa(1-(1-\frac{\tau\alpha}{\kappa})^{\frac{1}{2}})}(1-\frac{\tau\alpha}{\kappa})^{-\frac{1}{2}} - e^{\alpha\tau} \right| < \varepsilon$$

Now choose $\kappa > \kappa_0(\frac{2\pi\varepsilon}{\|f\|_1})$. Then

$$\left\| L_{\kappa,\tau}[f(\lambda)] - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\tau(\alpha+i\eta)} f(\alpha+i\eta)\,d\eta \right\|$$

$$= \left\| \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{2\kappa(1-(1-\frac{\tau}{\kappa}(\alpha+i\eta))^{\frac{1}{2}})}(1-\frac{\tau}{\kappa}(\alpha+i\eta))^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)} \right) f(\alpha+i\eta)\,d\eta \right\|$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| e^{2\kappa(1-(1-\frac{\tau}{\kappa}(\alpha+i\eta))^{\frac{1}{2}})}(1-\frac{\tau}{\kappa}(\alpha+i\eta))^{-\frac{1}{2}} - e^{\tau(\alpha+i\eta)} \right| \| f(\alpha+i\eta)\|\,d\eta$$

$$\leq \frac{1}{2\pi}\left| e^{2\kappa(1-(1-\frac{\tau\alpha}{\kappa})^{\frac{1}{2}})}(1-\frac{\tau\alpha}{\kappa})^{-\frac{1}{2}} - e^{\tau\alpha} \right| \int_{-\infty}^{\infty} \| f(\alpha+i\eta)\|\,d\eta$$

$$< \frac{1}{2\pi} \cdot \frac{2\pi\varepsilon}{\|f\|_1} \cdot \|f\|_1 = \varepsilon.$$

Thus $\lim_{\kappa\to\infty} L_{\kappa,\tau}[f(\lambda)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} f(\mu)\,d\mu = g(\mu)$.

Hence we have

$$\int_0^\infty e^{-\lambda\tau} g(\tau)d\tau = \frac{1}{2\pi i}\int_0^\infty e^{-\lambda\tau}d\tau \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\tau\mu} f(\mu)d\mu$$

$$= \frac{1}{2\pi}\int_0^\infty e^{-\lambda\tau}d\tau \int_{-\infty}^{\infty} e^{\tau(\alpha+i\eta)} f(\alpha+i\eta)d\eta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha+i\eta)d\eta \int_0^\infty e^{-(\lambda-(\alpha+i\eta))\tau}d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(\alpha+i\eta)}{\lambda-(\alpha+i\eta)}d\eta = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{f(\mu)}{\lambda-\mu}d\mu$$

$$= f(\lambda).$$

The interchange of integrations is valid for $\mathrm{Re}\,\lambda > \alpha$ since

$$\int_{-\infty}^{\infty} \| e^{\tau(\alpha+i\eta)} f(\alpha+i\eta)\| d\eta = e^{\alpha\tau}\int_{-\infty}^{\infty} \| f(\alpha+i\eta)\| d\eta .$$

To deal with the cases $p > 1$, we must take cognizance of the fact that

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\mu\tau} f(\mu)d\mu$$

may not exist. But, if $\frac{1}{p} + \frac{1}{p'} = 1$, and $\beta p' > 1$, $\lambda^{-\beta} f(\lambda)$ is in $H_1(\alpha ; \mathfrak{X})$ so that we may apply the previous theorem to it, with the following results.

Theorem 3.4.2: If

1. $f(\lambda)$ is in $H_p(\alpha ; \mathfrak{X})$ $p > 1$, $\alpha > 0$,
2. $\frac{1}{p} + \frac{1}{p'} = 1$,
3. $\beta p' > 1$,

then $f(\lambda) = \lambda^\beta \int_0^\infty e^{-\lambda\tau} g_\beta(\tau)d\tau$ where

$$g_{\beta}(\tau) = \lim_{\kappa \to \infty} L_{\kappa,\tau} \left[\lambda^{-\beta} f(\lambda)\right] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\tau\lambda} \lambda^{-\beta} f(\lambda) d\lambda \; .$$

Proof: $\lambda^{-\beta} f(\lambda)$ is in $H_1(\alpha\, ; \mathfrak{X})$, for by applying Hölder's inequality we have

$$\int_{-\infty}^{\infty} \| (\alpha + i\eta)^{-\beta} \; f(\alpha + i\eta) \| \; d\eta < \left\{ \int_{-\infty}^{\infty} \| f(\alpha + i\eta) \|^{p} d\eta \right\}^{\frac{1}{p}} .$$

$$\left\{ \int_{-\infty}^{\infty} |\alpha + i\eta|^{-\beta p'} \; d\eta \right\}^{\frac{1}{p'}} < \infty .$$

Thus applying the previous theorem, we have

$$\lim_{\kappa \to \infty} L_{\kappa,\tau} \left[\lambda^{-\beta} f(\lambda)\right] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\tau\mu} \mu^{-\beta} f(\mu) d\mu = g_{\beta}(\tau),$$

and $f(\lambda) = \lambda^{\beta} \int_{0}^{\infty} e^{-\lambda\tau} \; g_{\beta}(\tau) d\tau$.

6. A Special Representation Theorem for a Class of Numerically-valued Functions.

The following theorem gives sufficient conditions for a numerically-valued function to be represented as a Laplace-Stieltjes integral.

Theorem 3.6.2: If $f(\lambda)$ is a numerically-valued function which satisfies conditions 1. and 2. of theorem 3.3.1, and

3. $\int_0^{\tau} \left| L_{\kappa,\tau}\left[f(\lambda)\right]\right| d\tau \leq M, \quad \kappa > \kappa_0, \; 0 < \tau < \infty,$

then there exists an $\alpha(\tau)$, of bounded variation in $[0, \omega]$ all $\omega > 0$, such that

$$f(\lambda) = \int_0^{\infty} e^{-\lambda\tau}\, d\alpha(\tau), \text{ almost everywhere.}$$

Proof: By Widder [12], page 31, theorem 16.4, there exists an increasing and unbounded sequence of numbers $\{\kappa_i\}$, and a function $\alpha(\tau)$, of bounded variation in $[0, \omega]$ all $\omega > 0$, such that

$$\lim_{i \to \infty} \int_0^{\infty} e^{-\zeta\tau} L_{\kappa_i,\tau}\left[f(\lambda)\right] d\tau = \int_0^{\infty} e^{-\zeta\tau}\, d\alpha(\tau).$$

But because of 1., 2., and 3., $f(\lambda)$ satisfies all the postulates of theorem 3.3.1, so that for almost all

$$\lim_{i \to \infty} \int_0^{\infty} e^{-\zeta\tau} L_{\kappa_i,\tau}\left[f(\lambda)\right] d\tau = f(\zeta).$$ Thus we have

$$f(\zeta) = \int_0^{\infty} e^{-\zeta\tau}\, d\alpha(\tau) \text{ almost everywhere.}$$

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