SCATTERING THEORY FOR THE LAPLACIAN IN PERTURBED CYLINDRICAL DOMAINS

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Abstract

In this work, the theory of scattering with two Hilbert spaces is applied to a certain selfadjoint elliptic operator acting in two different domains in Euclidean N-space, R^N. The wave operators and scattering operator are then constructed. The selfadjoint operator is the negative Laplacian acting on functions which satisfy a Dirichlet boundary condition.

The unperturbed operator, denoted by H_0, is defined in the Hilbert space $\mathcal{H}_0 = L_2(S)$, where S is a uniform cylindrical domain in R^N, $S = G \times R$, G a bounded domain in R^{N-1} with smooth boundary. For this operator, an eigenfunction expansion is derived which shows that H_0 has only absolutely continuous spectrum. The eigenfunction expansion is used to construct the resolvent operator, the spectral measure, and a spectral representation for H_0.

The perturbed operator, denoted by H, is defined in the Hilbert space $\mathcal{H} = L_2(\Omega)$, where Ω is a perturbed cylindrical domain in R^N with the property that there is a smooth diffeomorphism $\Phi: \overline{\Omega} \leftrightarrow \overline{S}$ which is the identity map outside a bounded region. The mapping Φ is used to construct a unitary operator J mapping $\mathcal{H}_0$ onto $\mathcal{H}$ which has the additional property that $JD(H_0) = D(H)$.

The following theorem is proved:

Theorem: Let π^{ac} be the orthogonal projection onto the subspace of absolute continuity of H. Then the wave operators

$$W_{\pm}(H, H_0; J) = \text{s-lim}_{t\to\pm\infty} e^{itH} J e^{-itH_0}$$

and

$$W_{\pm}(H_0, H; J^*) = \text{s-lim}_{t\to\pm\infty} e^{itH_0} J^* e^{-itH} \pi^{ac}$$

exist. The operators $W_{\pm}(H, H_0; J)$ map $\mathcal{H}_0$ isometrically onto $\mathcal{H}^{ac} = \pi^{ac}\mathcal{H}$ and provide a unitary equivalence between H_0 and H^{ac}, the part of H in $\mathcal{H}^{ac}$. Furthermore,

$$[W_{\pm}(H, H_0; J)]^* = W_{\pm}(H_0, H; J^*) \ . \quad \square$$

It is proved that the point spectrum of H is nowhere dense in R. A limiting absorption principle is proved for H which shows that H has no singular continuous spectrum. The limiting absorption principle is used to construct two sets of generalized eigenfunctions for H. The wave operators $W_{\pm}(H, H_0; J)$ are constructed in terms of these two sets of eigenfunctions. This construction and the above theorem yield the usual completeness and orthogonality results for the two sets of generalized eigenfunctions. It is noted that the construction of the resolvent operator, spectral measure, and a spectral representation for H_0 can be repeated for the operator H^{ac} and yields similar results. Finally, the channel structure of the problem is noted and the scattering operator

$$S(H, H_0; J) = W_+(H_0, H; J^*)W_-(H, H_0; J)$$

is constructed.

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Scattering Theory for the Laplacian in Perturbed Cylindrical Domains

It is only within the past two decades that scattering theory, long used by physicists in problems of quantum mechanics, has been put on a solid mathematical foundation. Mathematically, scattering theory is concerned with the unitary equivalence of two selfadjoint Hilbert space operators.

The following formulation of scattering, given by Kuroda in [18], is the single-channel one-space theory. There were earlier formulations by Cook [6] and Jauch [11], but they did not encompass as wide a class of problems as Kuroda's. Let H_0 and H be selfadjoint operators on the Hilbert space $\mathcal{H}$ with domains $D(H_0)$ and $D(H)$, respectively. Let π_0^{ac} and π^{ac} denote the orthogonal projections onto the subspaces of absolute continuity for H_0 and H, respectively, and let e^{-itH_0} and e^{-itH} denote the strongly continuous unitary groups generated by H_0 and H. Suppose the strong operator limits

$$W_{\pm} = s-\lim_{t\to\pm\infty} e^{itH} e^{-itH_0} \pi_0^{ac}$$

exist; these operators are called the (generalized) wave operators. Then $W_{\pm}$ are partial isometries with initial set $\pi^{ac}\mathcal{H}$. $W_{\pm}$ intertwine H_0 and H, i.e.

$$e^{itH}W_{\pm} = W_{\pm}e^{itH_0} ,$$

and (hence) the ranges $R(W_{\pm})$ of $W_{\pm}$ are reducing subspaces for H. The absolutely continuous part of H_0 is unitarily equivalent to the restriction of H to either of $R(W_{\pm})$. If $R(W_+) = R(W_-) = \pi^{ac}\mathcal{H}$, the wave operators are said to be complete. In this case, the absolutely continuous parts of H_0 and H are unitarily equivalent. The scattering operator S is defined by

$$S = W_+^* W_- ,$$

where W_+^* is the adjoint of W_+. S commutes with H_0, and, if the wave operators are complete, S is unitary on $\pi_0^{ac}\mathcal{H}$.

To see how this formulation came about, consider a typical scattering problem in physics. In quantum mechanics, the evolution of a system is determined by the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H\psi , \tag{0.1}$$

where $\psi = \psi(t)$, the state vector describing the physical properties of the system at time t, is a vector in the Hilbert space $\mathcal{H}$, and H, the Hamiltonian describing the total energy of the system, is a selfadjoint operator on $\mathcal{H}$ with domain $D(H)$. The solution of (0.1), with initial state $\psi(0)$ at time $t = 0$, is given by

$$\psi(t) = e^{-itH}\psi(0)$$

Suppose the Hamiltonian H corresponds to a particle travelling in a space in which a small obstacle is present. It seems reasonable to expect that if, at large positive and negative times, the particle is far from the obstacle, it will behave like a free particle. Take H_0 to be the Hamiltonian for a free particle. Suppose the H-system is in the state $e^{-itH}\psi(0)$ for all time t. Then, for large negative times, since the particle is almost free, it is reasonable to expect that there is a state $e^{-itH_0}\phi_-(0)$ in the H_0-system which approximates the H-state in the sense that

$$\| e^{-itH}\psi(0) - e^{-itH_0}\phi_-(0)\| \to 0 \quad \text{as} \quad t \to -\infty .$$

(In the case where the 'obstacle' is a Coulomb potential, this type of approximation is too much to expect. See Dollard [7].) Similarly, one expects that there is a state $e^{-itH_0}\phi_+(0)$ in the H_0-system which satisfies

$$\| e^{-itH}\psi(0) - e^{-itH_0}\phi_+(0)\| \to 0 \quad \text{as} \quad t \to \infty .$$

Note that, in Kuroda's terms, $\psi(0) = W_-\phi_-(0) = W_+\phi_+(0)$, and $\phi_+(0) = S\,\phi_-(0)$. In an experiment, where information about the H-system is needed, the experimenter sends a beam of particles, which are 'free' at the beginning of the experiment, toward the obstacle. He then measures the 'free' scattered particles; that is, he knows $\phi_-(0)$ and measures $\phi_+(0)$. The experimenter, thus, is extremely interested in constructing the scattering operator.

From a mathematical point of view, however, the wave operators, which provide the unitary equivalence of the two operators, are more interesting than the scattering operator. The obvious question to ask now is, what conditions are sufficient to guarantee the existence and completeness of the wave operators $W_{\pm}$? It must be noted that conditions guaranteeing the existence of the wave operators are much easier to come by than conditions guaranteeing the completeness. A result obtained by Rosenblum [21] and Kato [13, 14] is that if $H = H_0 + V$, where V is a trace class operator on $\mathcal{H}$, then $W_{\pm}$ exist and are complete. Kuroda [17] (see also Kato [15], p. 525) proved that the trace class is practically the only class for which this is true. Birman and Krein [4], and de Branges [5] proved that if $R_\zeta - R^\circ_\zeta$, where $R_\zeta = (H - \zeta)^{-1}$ and $R^\circ_\zeta = (H_0 - \zeta)^{-1}$, is of trace class for some ζ in the resolvent set of both H_0 and H , then $W_{\pm}$ exist and are complete.

In 1966, Wilcox [28] in his investigations into the application of scattering theory to problems in classical physics, found that the one-space theory of scattering was not the correct setting. In these problems, the operators H_0 and H are defined in two different Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively. There is, however, a bounded linear map J of $\mathcal{H}_0$ onto $\mathcal{H}$ which can be used to identify $\mathcal{H}_0$ with $\mathcal{H}$. Kato [16] introduced the wave operator.

$$(0.2) \qquad W_{\pm}(H, H_0; J) = \underset{t \to \pm\infty}{\text{s-lim}}\, e^{itH} J e^{-itH_0} \pi_0^{ac} .$$

which is a two-space operator. He showed that if $W_{\pm}(H, H_0; J)$ exist, then they intertwine H_0 and H , and are used to provide a unitary equivalence between a part of H_0 and a part of H. Note that the mapping J makes the two space theory much more complicated than the one-space theory. If the wave operators exist, the scattering operator S can be defined by

$$S(H, H_0; J) = W_+(H, H_0; J)^* W_-(H, H_0; J) \ . \tag{0.3}$$

Even if the wave operators exist and are complete, it is not necessarily true that the scattering operator is unitary on $\pi_0^{ac}\mathcal{H}_0$.

In 1968, Birman [3], and Birman and Belopolskii [2], gave conditions on the operators H_0, H, and J which are sufficient to guarantee the existence, completeness, and other desirable properties of the wave operators. Their results are stated in the following theorem:

Birman's Theorem: Let $\pi_0(\cdot)$ denote the spectral measure for H_0, i.e.

$$H_0 = \int_R \lambda \, d\pi_0(\lambda) \ ,$$

and $\pi(\cdot)$ the spectral measure for H. Suppose

1) J is a bounded invertible linear mapping of $\mathcal{H}_0$ onto $\mathcal{H}$,
2) $JD(H_0) = D(H)$, and for each bounded interval $\delta \subset R$,
3) $\pi(\delta)(HJ - JH_0)\pi_0(\delta)$ is a trace class operator, and
4) $(J^*J - I)\pi_0(\delta)$ is a compact operator.

Then the wave operators $W_\pm(H, H_0; J)$ and $W_\pm(H_0, H; J^*)$ exist and are complete. The wave operators $W_\pm(H, H_0; J)$ are partial isometries with initial set $\pi_0^{ac}\mathcal{H}_0$ and final set $\pi^{ac}\mathcal{H}$, and

$$[W_{\pm}(H, H_0; J)]^* = W_{\pm}(H_0, H; J^*) .$$

The absolutely continuous parts of H_0 and H are unitarily equivalent.

This powerful result of Birman was used by Wilcox and Schulenberger [22] to prove the completeness of the wave operators for the class of problems they were studying. In this thesis Birman's theorem is used to prove the existence and completeness of the wave operators for the two operators given by the negative Laplacian with a Dirichlet boundary condition in a uniform cylindrical domain and in a perturbed cylindrical domain in R^N .

Another approach to the problem of unitary equivalence and scattering which must be mentioned is the eigenfunction expansion method. In this case, the operators H_0 and H are (selfadjoint extensions of) differential operators in some domain(s) in R^N . Ikebe [10] used this approach in studying perturbations of the Laplacian in R^N by a potential. Povzner [20], Shenk [23], Thoe [26], Goldstein [9], and many others have also used this method.

A set of generalized eigenfunctions is constructed for H_0 . Associated with this set of eigenfunctions is a measurable space Ω and a positive measure μ on Ω . The generalized eigenfunctions are of the form $\phi(x, \xi)$, where x is in the domain in R^N in which H_0 is defined, and $\xi \in \Omega$. The set of eigenfunctions is complete in $\mathcal{H}_0$ in the sense that if $f \in \mathcal{H}_0$, then there is a $\hat{g} \in L_2(\Omega, \mu)$ such that

$$f(\cdot) = \int_{\Omega} \hat{g}(\xi)\phi(\cdot, \xi)d\mu(\xi) \tag{0.4}$$

(interpreted in a suitable sense). The eigenfunctions are 'orthogonal' in $\mathcal{H}_0$ in the sense that if $\hat{g} \in L_2(\Omega, \mu)$, then the function f given by the formula (0.4) is in $\mathcal{H}_0$, and the norm of $\hat{g}$ in $L_2(\Omega, \mu)$ is equal to the norm of f in $\mathcal{H}_0$. (This is not the usual Hilbert space concept of orthogonality.)

Two sets of generalized eigenfunctions $\phi_{\pm}(x, \xi)$ are constructed for H, usually by perturbing the eigenfunctions for H_0. The existence of the wave operator is proved by showing that the two sets of generalized eigenfunctions are orthogonal in $\pi^{ac}\mathcal{H}$ in the above sense. The completeness of the wave operators is proved by showing that any function $f \in \pi^{ac}\mathcal{H}$ can be represented as

$$f(\cdot) = \int_{\Omega} \hat{g}_{\pm}(\xi)\phi_{\pm}(\cdot, \xi)d\mu(\xi) \tag{0.5}$$

for some $\hat{g}_{\pm} \in L_2(\Omega, \mu)$ (i.e. by showing that the eigenfunctions $\phi_{\pm}(\cdot, \xi)$ are complete in $\pi^{ac}\mathcal{H}$). The wave operator is then constructed in terms of these expansions as follows:

$$W_{\pm} \int_{\Omega} \hat{g}(\xi)\phi(\cdot, \xi)d\xi = \int_{\Omega} \hat{g}(\xi)\phi_{\pm}(\cdot, \xi)d\xi$$

for $\hat{g} \in L_2(\Omega, \mu)$.

This approach to the problem of scattering yields more information about the operator H than do the abstract theories. In particular, it is usually the case that the point spectrum of H can be shown to be discrete (or even finite in some cases). Also, the singular continuous spectrum of H can usually show to be empty. Results of this sort are not derivable from the abstract theory.

This thesis is a combination of the abstract theory of Birman and the eigenfunction expansions of Ikebe. It is a generalization of a class of problems dealt with by Goldstein [9] utilizing a very different approach. In §1, the nonnegative selfadjoint extension of the Laplacian acting on functions which satisfy a Dirichlet boundary condition in a domain (open connected set) in R^N, for $N = 1, 2, \cdots$, is defined. Some regularity results are proved for this operator. In addition, the function spaces used in this paper are introduced along with some properties of these spaces.

In §2, the operator H_0 is defined to be the selfadjoint extension of the Laplacian as above, in the cylindrical domain $S = G \times R$ in R^N, where G is a bounded domain in R^{N-1} with smooth boundary. The coordinate system in R^N is chosen so that x_N is the longitudinal coordinate in S, i.e. $x = (x_1, \cdots, x_N) \in S \Leftrightarrow x_N \in R$ and $\tilde{x} \equiv (x_1, \cdots, x_{N-1}) \in G$. It is noted that the operator H_G, defined to be the selfadjoint extension of the Laplacian as in §1, in the $N-1$ dimensional domain $G \subset R^{N-1}$, has a complete set of eigenfunctions $\{\eta_n(\tilde{x})\}_{n=1}^{\infty}$ and corresponding eigenvalues $\{k_n^2\}_{n=1}^{\infty}$. Assume the eigenvalues are increasing. It is shown that the functions

$$\overset{\circ}{w}_n(x, \xi) = (2\pi)^{-1/2} e^{i\xi x_N} \eta_n(\tilde{x}), \; n = 1, 2, \cdots, \; \xi \in R, \; x \in S$$

form a complete set of generalized eigenfunctions for H_0. This set of eigenfunctions is used to construct the resolvent operator $R^{\circ}_{\zeta} = (H_0 - \zeta)^{-1}$ for H_0 . The resolvent, in turn is used to construct the spectral measure $\pi_0(\cdot)$ for H_0 , which shows that H_0 has only absolutely continuous spectrum. Finally, a spectral representation for H_0 is constructed from which it follows that the spectral multiplicity function for H_0 is piecewise constant and increases by two at each of the (transverse) eigenvalues k_n^2 .

In §3, the operator H is defined to be the selfadjoint extension of the Laplacian as in §1 in the perturbed cylindrical domain $\Omega \subset R^N$, where Ω has the property that there is a smooth diffeomorphism $\Phi:\overline{\Omega} \leftrightarrow \overline{S}$ which is the identity map outside some bounded set. A unitary mapping J from $\mathcal{H}_0 = L_2(S)$ onto $\mathcal{H} = L_2(\Omega)$ is defined by

$$Jf(x) = |D\Phi(x)|^{1/2} f(\Phi(x)), \ x \in \Omega, \ f \in \mathcal{H}_0 ,$$

where $|D\Phi(x)|$ is the Jacobian-determinant of Φ at x . It is shown that $JD(H_0) = D(H)$.

In §4, the abstract two-space theory of Birman is applied to this problem, yielding the existence and completeness of the wave operators (0.2).

In §5 and 6, the wave operators whose existence is proved in §4 are constructed, and the spectrum of H is investigated. In §5, it is shown that the point spectrum of H is nowhere dense in R . The limiting absorption principle (proved in the appendix) is stated and used to prove that the singular continuous spectrum of H is empty. Finally, two sets of generalized eigenfunctions, $\{w_n^{\pm}(x,\xi)\}_{n=1}^{\infty}$, $x \in \Omega$, $\xi \in R$, are constructed for H using the limiting absorption principle.

In §6 the wave operators $W_{\pm}(H, H_0; J)$ are constructed. The completeness and orthogonality of the two sets of generalized eigenfunctions, $\{w_n^{\pm}(x, \xi)\}$, for H follow from the existence and completeness of the wave operators. Finally, the scattering operator (0.2) is constructed and the multichannel aspects of the problem are mentioned.

§1 The Operator $-\Delta_D(\mathcal{D})$ in a General Domain $\mathcal{D}$ in R^N

Let $\mathcal{D}$ be a domain in R^N for some $N \geq 1$. In this section some function spaces and notations are introduced, and a precise definition is given of the selfadjoint extension of the negative Laplacian acting on functions defined in $\mathcal{D}$ which are zero on the boundary. Some regularity results for the operator are proved.

The following spaces will be used in the remainder of this paper. $L_2(\mathcal{D})$ is the Hilbert space of square-integrable complex valued functions defined in $\mathcal{D}$ with the L_2-inner product given by

$$\langle u, v \rangle_{L_2(\mathcal{D})} = \int_{\mathcal{D}} \overline{u(x)}\, v(x)dx, \quad u, v \in L_2(\mathcal{D}) .$$

If $u \in L_2(\mathcal{D})$, the support of u , supp u , is the complement of the largest open set $V \subset \mathcal{D}$ satisfying $\int_V |u(x)|^2 dx = 0$. $L_2^{loc}(\mathcal{D})$ is the space of all complex-valued functions u defined in $\mathcal{D}$ for which $u \in L_2(\mathcal{D} \cap M)$ for all bounded measurable subsets $M \subset \mathcal{D}$. (Note that functions in $L_2^{loc}(\mathcal{D})$ are in L_2 up to the boundary of $\mathcal{D}$.) The seminorms

$$\|u\|_{L_2(\mathcal{D}\cap B_r)} , \quad u \in L_2^{loc}(\mathcal{D}),\ r > 0 ,$$

where

$$B_r = \{x \in R^N : \ |x| < r\} ,$$

generate a topology in which $L_2^{loc}(\mathcal{D})$ is a Fréchet space.

For each integer $m \geq 0$, $H_m(\mathcal{D})$ $(H_m^{loc}(\mathcal{D}))$ is the space of all functions in $L_2(\mathcal{D})$ (in $L_2^{loc}(\mathcal{D})$) which have distribution derivatives in $L_2(\mathcal{D})$ (in $L_2^{loc}(\mathcal{D})$) of order less than or equal to m, i.e. $u \in H_m(\mathcal{D})$ ($u \in H_m^{loc}(\mathcal{D})$) iff $u \in L_2(\mathcal{D})$ ($u \in L_2^{loc}(\mathcal{D})$) and, for any multi-index $\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_N)$ of nonnegative integers with $|\alpha| = \sum_{j=1}^{N} \alpha_j \leq m$, there is a unique element in $L_2(\mathcal{D})$ (in $L_2^{loc}(\mathcal{D})$) denoted by $D^\alpha u$ such that

$$\int_{\mathcal{D}} \overline{u(x)} \frac{\partial^{|\alpha|} \phi(x)}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} dx = (-1)^{|\alpha|} \int_{\mathcal{D}} \overline{D^\alpha u(x)} \phi(x) dx$$

holds for all $\phi \in C_0^\infty(\mathcal{D})$. $H_m(\mathcal{D})$ is a Hilbert space with the m-norm defined by

$$||u||^2_{m,\mathcal{D}} = \sum_{|\alpha| \leq m} ||D^\alpha u||^2_{L_2(\mathcal{D})} , \quad u \in H_m(\mathcal{D}) .$$

$H_m^{loc}(\mathcal{D})$ is a Fréchet space in the topology generated by the semi-norms

$$||u||_{m, \mathcal{D} \cap B_r} , \quad u \in H_m^{loc}(\mathcal{D}), \; r > 0 .$$

(Note that functions in $H_m^{loc}(\mathcal{D})$ are in H_m up to the boundary of $\mathcal{D}$.) The space $\mathring{H}_m(\mathcal{D})$ is the closure in the m-norm of $C_0^\infty(\mathcal{D})$. $\mathring{H}_m^{loc}(\mathcal{D})$ is the closure in $H_m^{loc}(\mathcal{D})$ of $C_0^\infty(\mathcal{D})$. Note that

$$\mathring{H}_m^{loc}(\mathcal{D}) = H_m^{loc}(\mathcal{D}) \cap \{u: \phi u \in \mathring{H}_m(\mathcal{D}) \; \forall \phi \in C_0^\infty(R^N)\} .$$

Notational note: Many Hilbert spaces, norms, and inner products appear in the following pages. If $\mathcal{H}$ is a Hilbert space, the inner product and norm on $\mathcal{H}$ will often be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $||\cdot||_{\mathcal{H}}$, respectively. The inner product and norm in $H_m(\mathcal{D})$ will be denoted by $\langle\cdot,\cdot\rangle_{m,\mathcal{D}}$ and $||\cdot||_{m,\mathcal{D}}$, respectively, unless the domain $\mathcal{D}$ is obvious, in which case the subscript $\mathcal{D}$ will be omitted.

The negative Laplacian, $-\Delta = \frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2^2} - \cdots - \frac{\partial}{\partial x_N^2}$, can be applied pointwise to any function in $C_0^\infty(\mathcal{D})$, and yields a new function in $C_0^\infty(\mathcal{D})$. It can also be applied, in a distribution theoretic (or weak) sense, to larger classes of functions in $L_2(\mathcal{D})$ and in $L_2^{loc}(\mathcal{D})$.

<u>Definition 1.1</u>: Let $u \in L_2(\mathcal{D})$ $(u \in L_2^{loc}(\mathcal{D}))$, Then $-\Delta u$ is said to exist weakly and equal $v \in L_2(\mathcal{D})$ $(v \in L_2^{loc}(\mathcal{D}))$ iff

$$\int_{\mathcal{D}} \overline{u(x)}\ (-\Delta\phi)(x)\,dx = \int_{\mathcal{D}} \overline{v(x)}\ \phi(x)\,dx \qquad \forall\phi \in C_0^\infty(\mathcal{D}) .$$

Denote by $L_2(-\Delta;\ \mathcal{D})$ $(L_2^{loc}(-\Delta;\ \mathcal{D}))$ the set of all functions $u \in L_2(\mathcal{D})$ $(u \in L_2^{loc}(\mathcal{D}))$ for which $-\Delta u$ exists weakly in $L_2(\mathcal{D})$ $(L_2^{loc}(\mathcal{D}))$. (The operator $-\Delta$ with domain $L_2(-\Delta;\ \mathcal{D})$ is the adjoint of the operator given by $-\Delta$ with domain $C_0^\infty(\mathcal{D})$). Whenever $-\Delta$ is used from now on, it will denote the weak negative Laplacian applied to functions in $L_2(-\Delta;\ \mathcal{D})$ or $L_2^{loc}(-\Delta;\ \mathcal{D})$.

The operator $-\Delta$ with domain $L_2(-\Delta;\ \mathcal{D})$ is not selfadjoint. However, consider the following.

<u>Definition 1.2</u>: Let the operator $-\Delta(\mathcal{D})$ on the Hilbert space $L_2(\mathcal{D})$ be the restriction of $-\Delta$ to $\overset{\circ}{H}_1(\mathcal{D})$, i.e.

1) $D(-\Delta_D(\mathcal{D})) = L_2(-\Delta;\ \mathcal{D}) \cap \mathring{H}_1(\mathcal{D})$,

2) $-\Delta_D(\mathcal{D})u = -\Delta u$ for $u \in D(-\Delta_D(\mathcal{D}))$.

The following theorem was proved by Wilcox [27].

Theorem 1.3: The operator $-\Delta_D(\mathcal{D})$ is a nonnegative selfadjoint operator on $L_2(\mathcal{D})$. If $u \in \mathring{H}_1(\mathcal{D})$, then $u \in D(-\Delta_D(\mathcal{D}))$ and $-\Delta_D(\mathcal{D})u = v$ if and only if

$$\sum_{j=1}^{N} \int_{\mathcal{D}} \overline{\frac{\partial u}{\partial x_j}} \frac{\partial \phi}{\partial x_j}\, dx = \langle v, \phi \rangle_{L_2(\mathcal{D})} \qquad \forall\ \phi \in \mathring{H}_1(\mathcal{D}) .$$

This theorem has the obvious corollary:

Corollary 1.4: If $u \in D(-\Delta_D(\mathcal{D}))$, then

$$\| u\|^2_{1,\mathcal{D}} = \| u\|^2_{L_2(\mathcal{D})} + \langle -\Delta_D(\mathcal{D})u,\ u\rangle_{L_2(\mathcal{D})}$$

In the remainder of this section the operator $-\Delta_D(\mathcal{D})$ will be denoted simply by $-\Delta_D$. Some regularity results are proved in the remainder of this section under the added assumption that the boundary of $\mathcal{D}$ is smooth. Since the negative Laplacian is an elliptic operator, the following theorem applies (see Agmon, [1], p. 129).

Theorem 1.5: $D(-\Delta_D) \subset H_2^{loc}(\mathcal{D})$, and, given any $r > 0$, there is a constant K , depending only on r and $\mathcal{D}$, such that

$$\| u\|_{2,\mathcal{D}\cap B_r} \le K(\| u\|_{L_2(\mathcal{D})} + \| -\Delta_D u\|_{L_2(\mathcal{D})})$$

holds for all $u \in D(-\Delta_D)$.

This means that the 2-norm of any function u in $D(-\Delta_D)$ in a bounded subset of $\mathcal{D}$ is bounded by the graph norm of u. (The graph norm of a function $u \in D(-\Delta_D)$ is $\|u\|_{L_2(\mathcal{D})} + \|-\Delta_D u\|_{L_2(\mathcal{D})}$. $D(-\Delta_D)$ is a Banach space with this graph norm.) The following lemma now applies to functions in $D(-\Delta_D)$.

Lemma 1.6: If $u \in H_2^{loc}(\mathcal{D})$ and $\phi \in C_0^\infty(R^N)$, then $\phi u \in H_2(\mathcal{D})$, and

$$(1.1) \qquad -\Delta(\phi u) = \phi(-\Delta u) + u(-\Delta\phi) - 2\nabla\phi\cdot\nabla u ,$$

where ∇ is the (weak) gradient operator.

Proof: This is a simple case of Leibnitz rule (for a proof, see Agmon [1], p. 9).

Theorem 1.7: The set of functions in $D(-\Delta_D)$ with bounded support is dense, in the graph norm, in $D(-\Delta_D)$.

Proof: Let $u \in D(-\Delta_D)$. Let $\psi \in C_0^\infty(R^N)$ satisfy $0 \leq \psi \leq 1$, and $\psi(x) = 1$ if $|x| \leq 1$. Let $\psi_n(x) = \psi(x/n)$ for $n = 1, 2, \cdots$, $x \in R^N$. Then $\psi_n u \in \mathring{H}_1(\mathcal{D}) \cap H_2(\mathcal{D}) \subset D(-\Delta_D)$ for all n, and (1.1) implies

$$-\Delta_D(\psi_n u) = \psi_n(-\Delta_D u) + u(-\Delta\psi_n) - 2\nabla\psi_n \nabla u .$$

Hence,

$$\|u - \psi_n u\|^2_{L_2(\mathcal{D})} = \int_{\mathcal{D}} |1 - \psi_n(x)|^2 |u(x)|^2 dx = \int_{\mathcal{D}} |1 - \psi(x/n)|^2 |u(x)|^2 dx$$

$$\leq \int_{\mathcal{D}-B_n} |u(x)|^2 dx \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\|-\Delta_D u + \Delta_D(\psi_n u)\|_{L_2(\mathcal{D})} \leq \|-(1-\psi_n)\Delta_D u\|_{L_2(\mathcal{D})}$$

$$+ \|u(-\Delta\psi_n)\|_{L_2(\mathcal{D})} + 2\|\nabla\psi_n \cdot \nabla u\|_{L_2(\mathcal{D})} .$$

Since $\psi \in C_0^\infty(R^N)$, there is an $M > 0$ such that $|\Delta\psi_n(x)| \leq M$ and $|\nabla\psi_n(x)| \leq M$ for all $x \in R^N$. Also, $\Delta\psi_n(x) = 0$ and $\nabla\psi_n(x) = 0$ if $|x| < n$. Thus, $\|u(-\Delta\psi_n)\|_{L_2(\mathcal{D})} \leq M\|u\|_{L_2(\mathcal{D}-B_n)}$ and $\|\nabla\psi_n \cdot \nabla u\|_{L_2(\mathcal{D})} \leq M\|u\|_{1,\mathcal{D}-B_n}$. Since $u \in \mathring{H}_1(\mathcal{D})$, it follows that $\|-\Delta_D u + \Delta_D(\psi_n u)\|_{L_2(\mathcal{D})} \to 0$ as $n \to \infty$. Thus, $\psi_n u$ converges to u in the graph norm. □

The above theorem implies that the set of functions in $D(-\Delta_D)$ with bounded support is a core for $-\Delta_D$, i.e. $-\Delta_D$ restricted to such functions is a closable operator in $L_2(\mathcal{D})$ whose closure is $-\Delta_D$.

Finally, a local inequality is proved.

<u>Theorem 1.8</u>: If $r > 0$ and $\varepsilon > 0$, then there is a constant M, depending only on r, ε, and $\mathcal{D}$, such that if $u \in H_2^{loc}(\mathcal{D}) \cap \mathring{H}_1^{loc}(\mathcal{D})$, then

(1.2) $$\|u\|_{2,\mathcal{D}\cap B_r} \leq M(\|u\|_{L_2(\mathcal{D}\cap B_{r+\varepsilon})} + \|-\Delta u\|_{L_2(\mathcal{D}\cap B_{r+\varepsilon})}) .$$

Proof: Choose $\phi \in C_0^\infty(R^N)$ such that $\phi(x) = 1$ if $|x| \leq r$, and $\phi(x) = 0$ if $|x| > r + \varepsilon/2$. Then, since $\phi u \in H_2(\mathcal{D}) \cap \mathring{H}_1(\mathcal{D}) \subset D(-\Delta_D)$, theorem 1.5 implies that

$$(1.3) \qquad \|\phi u\|_{2,\mathcal{D}\cap B_r} \leq K(\|\phi u\|_{L_2(\mathcal{D})} + \|-\Delta_D(\phi u)\|_{L_2(\mathcal{D})})$$

for some constant K, depending on r and $\mathcal{D}$. Since $\phi(x) = 1$ if $|x| \leq r$, it follows that

$$(1.4) \qquad \|\phi u\|_{2,\mathcal{D}\cap B_r} = \|u\|_{2,\mathcal{D}\cap B_r} .$$

It follows from lemma 1.6 that $-\Delta_D(\phi u) = -\Delta(\phi u)$ is given by (1.1). Since $\phi \in C_0^\infty(R^N)$, there is a constant K_1, depending only on ϕ, such that

$$(1.5) \qquad \|\phi u\|_{L_2(\mathcal{D})} = \|\phi u\|_{L_2(\mathcal{D}\cap B_{r+\varepsilon/2})} \leq K_1\|u\|_{L_2(\mathcal{D}\cap B_{r+\varepsilon/2})} ,$$

and

$$(1.6) \qquad \|-\Delta_D(\phi u)\|_{L_2(\mathcal{D})} = \|-\Delta_D(\phi u)\|_{L_2(\mathcal{D}\cap B_{r+\varepsilon/2})}$$
$$\leq K_1\|-\Delta u\|_{L_2(\mathcal{D}\cap B_{r+\varepsilon/2})} + K_1\|u\|_{L_2(B_{r+\varepsilon/2})} + K_1\|u\|_{1,D\cap B_{r+\varepsilon/2}} .$$

Eidus [8] has shown that for such functions u, there is a constant K_2, depending only on r, ε, and $\mathcal{D}$, such that

$$(1.7) \qquad \|u\|_{1,\mathcal{D}\cap B_{r+\varepsilon/2}} \leq K_2(\|u\|_{L_2(D\cap B_{r+\varepsilon})} + \|-\Delta u\|_{L_2(\mathcal{D}\cap B_{r+\varepsilon})}) .$$

Combining (1.4), (1.5), (1.6), and (1.7) in (1.3) yields (1.2).

§2 The Operator H_0

Let $S = G \times R$ be the cylindrical domain in R^N defined in the introduction. The unperturbed operator H_0 is defined to be $-\Delta_D(S)$; it is a selfadjoint operator in the Hilbert space $\mathcal{H}_0 = L_2(S)$. The operator H_0 is investigated in this section.

Notational note: From now on, if $\mathcal{D}$ is any domain in R^N and $r > 0$,

$$\mathcal{D}_r = \{x \in \mathcal{D}: |x_N| < r\} .$$

It follows from the theory of elliptic operators in bounded domains that the selfadjoint operator H_G , defined to be $-\Delta_D(G)$ in the N-1 dimensional domain G , has a complete set of orthonormal eigenfunctions in $L_2(G)$. Denote the eigenvalues of H_G , ordered increasingly, by $\{k_n^2\}$, and the corresponding eigenfunctions by $\{\eta_n(\tilde{x})\}$. Note that for each n ,

$$\eta_n \in D(-\Delta_D(G)) = \mathring{H}_1(G) \cap H_2(G) \tag{2.1}$$

(that they are in $H_2(G)$ follows from theorem 1.5), and

$$-\Delta_D(G)\eta_n(\tilde{x}) = k_n^2\eta_n(\tilde{x}), \ x \in G \tag{2.2}$$

The elliptic theory also implies that the η_n are smooth functions.

Definition 2.1: For $n = 1, 2, \cdots$, let

$$w_n^\circ(x, \xi) = (2\pi)^{-1/2} e^{i\xi x_N}\eta_n(\tilde{x}), \ \xi \in R, \ x \in S . \tag{2.3}$$

These are generalized eigenfunctions for H_0 .

Theorem 2.2: For each n , the mappings

$$\xi \to w_n^\circ(\cdot,\ \xi),\ \xi \in R \tag{2.4}$$

and

$$\xi \to \frac{\partial}{\partial \xi} w_n^\circ(\cdot,\ \xi),\ \xi \in R \tag{2.5}$$

are continuous mappings of R into $H_2^{loc}(S) \cap \overset{\circ}{H}{}_1^{loc}(S)$. Furthermore,

$$-\Delta w_n^\circ(x,\ \xi) = (\xi^2 + k_n^2) w_n^\circ(x,\ \xi),\ \xi \in R,\ x \in S\ . \tag{2.6}$$

Proof: It follows from (2.1) and (2.3) that $w_n^\circ(\cdot,\ \xi) \in H_2^{loc}(S) \cap \overset{\circ}{H}{}_1^{loc}(S)$ for every $\xi \in R$. It is easy to verify, using (2.2) and (2.3), that (2.6) is satisfied.

Let $r > 0$, and ξ , $\nu \in R$. Then

$$\|w_n^\circ(\cdot,\xi) - w_n^\circ(\cdot,\ \nu)\|^2_{L_2(S_r)} = \int_{S_r} (2\pi)^{-1} |e^{i\xi x_N} - e^{i\nu x_N}|^2 |\eta_n(\tilde{x})|^2 dx$$

$$= (2\pi)^{-1} \int_{-r}^{r} |e^{i\xi x_N} - e^{i\nu x_N}|^2 dx_N \to 0 \quad \text{as} \quad \xi \to \nu\ .$$

Also,

$$\|-\Delta w_n^\circ(\cdot,\ \xi) + \Delta w_n^\circ(\cdot,\ \nu)\|_{L_2(S_r)} =$$

$$(2\pi)^{-1} \int_{-r}^{r} |(\xi^2 + k_n^2) e^{i\xi x_N} - (\nu^2 + k_n^2) e^{i\nu x_N}|^2 dx_N \to 0 \quad \text{as} \quad \xi \to \nu$$

It follows from the above and from theorem 1.8 that the mapping (2.4) is continuous from R into $H_2^{loc}(S) \cap \overset{\circ}{H}{}_1^{loc}(S)$.

Since $\dfrac{\partial w_n^\circ(x, \xi)}{\partial \xi} = i x_N w_n^\circ(x, \xi)$, an argument similar to the above proves that the mapping (2.5) is continuous from R into $H_2^{loc}(S) \cap \overset{\circ}{H}{}_1^{loc}(S)$. □

Definition 2.3: The Hilbert space $\mathcal{E}$ is defined by

$$\mathcal{E} = \sum_{n=1}^{\infty} \oplus \mathcal{E}_n, \quad \text{where} \quad \mathcal{E}_n = L_2(R) \quad \text{for} \quad n = 1, 2, \cdots,$$

and

$$f = \{f_n\} \in \mathcal{E} \Leftrightarrow f_n \in L_2(R) \quad \forall n \quad \text{and} \quad \|f\|_{\mathcal{E}}^2 = \sum_{n=1}^{\infty} \|f_n\|_{L_2(R)}^2 < \infty .$$

$\mathcal{E}$ is the direct sum of a countable number of copies of $L_2(R)$. The following theorem proves that the generalized eigenfunctions $w_n^\circ(x, \xi)$ are complete and orthogonal in H_0 .

Theorem 2.4: The operator T_0 from $\mathcal{H}_0$ to $\mathcal{E}$, defined by

$$(2.7) \qquad T_0 f(\xi) = \mathcal{E}\text{-}\lim_{r \to \infty} \{\int_{S_r} f(x)\overline{w_n^\circ(x, \xi)}dx\}, \ \xi \in R, \ f \in \mathcal{H}_0 ,$$

where $\mathcal{E}$-lim denotes the limit in the norm in $\mathcal{E}$, is unitary. The adjoint operator T_0^* from $\mathcal{E}$ onto $\mathcal{H}_0$ is given by

$$(2.8) \quad T_0^* h(x) = \mathcal{H}_0 - \lim_{\substack{M \to \infty \\ K \to \infty}} \sum_{n=1}^{M} \int_{-K}^{K} h_n(\xi) w_n^\circ(x, \xi) d\xi , \ x \in S,$$

$$h = \{h_n\} \in \mathcal{E} ,$$

where $\mathcal{H}_0$ - lim denotes the limit in the norm in $\mathcal{H}_0$.

Proof: Let $f \in \mathcal{H}_0$. Then for a.e. $x_N \in R$, the function $f(\tilde{x}, x_N) = f(x)$ is in $L_2(G)$, and, for each n , $|\int_G f(x)\overline{\eta_n(\tilde{x})}d\tilde{x}|^2 \leq \int_G |f(\tilde{x}, x_N)|^2 d\tilde{x}$. It follows that $\int_G f(\tilde{x}, x_N)\overline{\eta_n(\tilde{x})}\, d\tilde{x} \in L_2(R)$ for each n . Denote by F the operation of Fourier transforms on $L_2(R)$, i.e.

$$Fu(\xi) = L_2(R) - \lim_{M \to \infty} \int_{-M}^{M} (2\pi)^{-1/2} e^{-i\xi x_N} u(x)dx, \ \xi \in R, \ u \in L_2(R) .$$

Then $F \int_G f(\tilde{x}, x_N)\overline{\eta_n(\tilde{x})}d\tilde{x} \in L_2(R)$ for each n , and, using the completeness of the eigenfunctions $\eta_n(\tilde{x})$ in $L_2(G)$ and the unitarity of F ,

$$\sum_{n=1}^{\infty} || F\int_G f(\tilde{x}, x_N)\overline{\eta_n(\tilde{x})}d\tilde{x}||^2_{L_2(R)} = \sum_{n=1}^{\infty} || \int_G f(\tilde{x}, x_N)\overline{\eta_n(\tilde{x})}d\tilde{x}||^2_{L_2(R)}$$

$$= \sum_{n=1}^{\infty} \int_R | \int_G f(\tilde{x}, x_N)\overline{\eta_n(\tilde{x})}d\tilde{x}|^2 dx_N$$

$$= \int_R \sum_{n=1}^{\infty} |\int_G f(\tilde{x}, x_N)\overline{\eta_n(\tilde{x})}d\tilde{x}|^2 dx_N$$

$$= \int_R \int_G |f(\tilde{x}, x_N)|^2 d\tilde{x}dx_N = \int_S |f(x)|^2 dx = || f||^2_{\mathcal{H}_0}$$

Thus, the mapping

$$f \to \{F \int_G f(\tilde{x}, x_N)\overline{\eta_n(\tilde{x})}d\tilde{x}\} \tag{2.9}$$

is an isometric mapping of $\mathcal{H}_0$ into $\mathcal{E}$. But the map (2.9) and the map (2.7) are the same. Thus, T_0 is an isometry of $\mathcal{H}_0$ into $\mathcal{E}$.

Let $h \in \mathcal{E}$ have only a finite number of nonzero components, and suppose those nonzero components are in $C_0^\infty(R)$ (such functions h are dense in $\mathcal{E}$). Denote by Q, for the present, the operator defined by the right side of (2.8). Then

$$Qh(x) = \sum_{n=1}^{\infty} \int_R h_n(\xi) w_n^\circ(x, \xi)\, d\xi$$

$$= \sum_{n=1}^{\infty} (2\pi)^{-1/2} \int_R e^{i\xi x_N} h_n(\xi)\, d\xi\, \eta_n(\tilde{x})$$

$$= \sum_{n=1}^{\infty} F^* h_n(x_N)\eta_n(\tilde{x}), \quad x \in S,$$

where F^* is the inverse Fourier transform on $L_2(R)$. Using the orthogonality of the η_n's in $L_2(G)$ (note that the sum is finite),

$$\|Qh\|^2_{\mathcal{H}_0} = \int_S |Qh(x)|^2 dx = \int_R \int_G \left| \sum_{n=1}^{\infty} (F^* h_n)(x_N)\eta_n(\tilde{x}) \right|^2 d\tilde{x}\, dx_N$$

$$= \int_R \sum_{n=1}^{\infty} |F^* h_n(x_N)|^2 dx_N = \sum_{n=1}^{\infty} \int_R |F^* h_n(x_N)|^2 dx_N$$

$$= \sum_{n=1}^{\infty} \|F^* h_n\|^2_{L_2(R)} = \sum_{n=1}^{\infty} \|h_n\|^2_{L_2(R)} = \|h\|_{\mathcal{E}}.$$

Thus, Q is an isometry on $\mathcal{E}$ into $\mathcal{H}_0$. Finally, it is easy to verify for the class of $h \in \mathcal{E}$ above, that $T_0 Q h = h$. Since this class of functions is dense in $\mathcal{E}$, $T_0 Q = I$. Thus, T_0 and Q are unitary and $Q = T_0^* = T_0^{-1}$. □

From now on, $T_0 f$ will be denoted by $\{\hat{f}_n(\xi)\}$ when $f \in \mathcal{H}_0$. Note that

(2.10) $$\hat{f}_n(\xi) = F \int_G f(x)\overline{\eta_n(\tilde{x})}\, d\tilde{x} = F \langle \eta_n, f(\cdot, x_N) \rangle_{L_2(G)}, f \in \mathcal{H}_0 .$$

The following theorem gives a representation for H_0 in $\mathcal{E}$.

<u>Theorem</u> 2.5: If $u \in D(H_0)$, then

$$T_0 H_0 u = \{(\xi^2 + k_n^2)\hat{u}_n(\xi)\}, \ \xi \in R ,$$

and

$$(2.11) \quad T_0 D(H_0) = \{h = \{h_n\} \in \mathcal{E}: \sum_{n=1}^{\infty} \int_R (\xi^2 + k_n^2)^2 |h_n(\xi)|^2 d\xi < \infty \} .$$

<u>Proof</u>: Denote by V the mapping in $\mathcal{E}$ whose domain $D(V)$ is given by the right side of (2.11) and which is defined by

$$Vh(\xi) = \{(\xi^2 + k_n^2)h_n(\xi)\}, \ \xi \in R, \ h \in D(V) .$$

Then V is the direct sum of a countable number of selfadjoint operators (multiplication of $h_n(\xi) \in \mathcal{E}_n = L_2(R)$ by $\xi^2 + k_n^2$) and therefore is selfadjoint on $D(V)$. Thus, $T_0^* V T_0$ is a selfadjoint operator on $\mathcal{H}_0$.

Let $f \in D(H_0)$ have bounded support. Then, for sufficiently large r , supp $f \subset S_{r-1}$. Let $\phi \in C_0^\infty(R^N)$ satisfy $\phi(x) = 1$ if $x \in S_r$. Then, since $\overset{\circ}{w}_n(\cdot, \xi) \in H_2^{loc}(S) \cap \overset{\circ}{H}{}_1^{loc}(S)$, $\phi(\cdot)\overset{\circ}{w}_n(\cdot, \xi) \in D(H_0)$, and $H_0\phi(\cdot)\overset{\circ}{w}_n(\cdot, \xi) = -\Delta\phi(\cdot)\overset{\circ}{w}_n(\cdot, \xi)$ is given by a formula similar to (1.1). Hence, in (2.7),

$$T_0 H_0 f(\xi) = \{\int_{S_r} H_0 f(x) \overline{\overset{\circ}{w}_n(x, \xi)} dx\} = \{\int_{S_r} H_0 f(x) \overline{\phi(x)\overset{\circ}{w}_n(x, \xi)} dx\}$$

$$= \{\langle H_0 f, \phi(\cdot)\overset{\circ}{w}_n(\cdot, \xi)\rangle_{\mathcal{H}_0}\} = \{\langle f, H_0\phi(\cdot)\overset{\circ}{w}_n(\cdot, \xi)\rangle_{\mathcal{H}_0}\}$$

$$= \{ \int_{S_r} f(x) \overline{-\Delta w_n^\circ(x, \xi) d\xi} \} = \{(\xi^2 + k_n^2)\hat{f}_n(\xi)\} \in \mathcal{E} .$$

Thus, $T_0 f \in D(V)$ and $T_0 H_0 f = VT_0 f$, or $H_0 f = T_0^* VT_0 f$. Since such functions form a core for H_0 (theorem 1.7), and since H_0 and $T_0^* VT_0$ agree on this set of functions, it follows that $T_0^* VT_0$ is a selfadjoint extension of H_0. But H_0 is selfadjoint, so $H_0 = T_0^* VT_0$, and $T_0 H_0 = VT_0$. □

If $f \in \mathcal{H}_0$, then theorem 2.4 implies that

$$\text{(2.12)} \qquad f(x) = \mathcal{H}_0 - \lim_{\substack{M \to \infty \\ K \to \infty}} \sum_{n=1}^{M} \int_{-K}^{K} \hat{f}_n(\xi)\, w_n^\circ(x, \xi) d\xi, \qquad x \in S .$$

This notation will be shortened to

$$\text{(2.13)} \qquad f(x) = \sum_{n=1}^{\infty} \int_R \hat{f}_n(\xi)\, w_n^\circ(x, \xi) d\xi, \qquad x \in S, \quad f \in \mathcal{H}_0$$

where this is understood to imply the limits as in (2.12). By theorem 2.5, if $f \in D(H_0)$, then

$$\text{(2.14)} \quad H_0 f(x) = \sum_{n=1}^{\infty} \int_R (\xi^2 + k_n^2)\hat{f}_n(\xi)\, w_n^\circ(x, \xi) d\xi, \quad x \in S .$$

Theorem 2.5 has the following corollary:

<u>Corollary</u> 2.6: If $f \in D(H_0)$, then the representation (2.14) for f is also valid in $H_2^{loc}(S)$, i.e.

$$f(x) = H_2^{loc}(S) - \lim_{M \to \infty,\ K \to \infty} \sum_{n=1}^{M} \int_{-K}^{K} \hat{f}_n(\xi) w_n^\circ(x, \xi) d\xi, \quad x \in S .$$

Proof: If $f \in D(H_0)$, then f has the representation (2.13) and $H_0 f$ has the representation (2.14) in $\mathcal{H}_0$. Using theorems 1.4, 2.4, and 2.5, given any $r > 0$, there is a constant K such that

$$\|f - \sum_{n=1}^{M} \int_{-T}^{T} \hat{f}_n(\xi) w_n^{\circ}(\cdot,\ \xi) d\xi\|_{2,S_r} \leq K(\|f - \sum_{n=1}^{M} \int_{-T}^{T} \hat{f}_n(\xi) w_n^{\circ}(\cdot,\ \xi) d\xi\|_{\mathcal{H}_0}$$

(2.15)

$$+ \|H_0 f - \sum_{n=1}^{M} \int_{-T}^{T} (\xi^2 + k_n^2)\hat{f}_n(\xi) w_n^{\circ}(\cdot,\ \xi) d\xi\|_{\mathcal{H}_0}) .$$

Since both terms on the right of (2.15) go to zero as M and T go to ∞, the corollary is proved. □

With the above representation of H_0 in $\mathcal{E}$, the resolvent operator $R_\zeta^{\circ} = (H_0 - \zeta)^{-1}$ can be easily constructed.

Theorem 2.7: Let $\rho(H_0)$ denote the resolvent set of H_0. Then if $u \in \mathcal{H}_0$, and $\zeta \in \rho(H_0)$,

$$R_\zeta^{\circ} u = T_0^{*}\{ \frac{\hat{u}_n(\xi)}{\xi^2 + k_n^2 - \zeta} \}$$

Proof: Let $v = R_\zeta^{\circ} u \in D(H_0)$. Then $(H_0 - \zeta)v = u$, and $T_0(H_0 - \zeta)v = T_0 u$. Using theorem 2.5, it follows that for each n,

$$(\xi^2 + k_n^2 - \zeta)\hat{v}_n(\xi) = \hat{u}_n(\xi) \quad \text{a.e. } \xi \in R ,$$

or

$$\hat{v}_n(\xi) = \frac{\hat{u}_n(\xi)}{\xi^2 + k_n^2 - \zeta} \quad \text{a.e. } \xi \in R .$$

But $R^\circ_\zeta u = v = T_0^* T_0 v = T_0^*\{\hat{v}_n(\xi)\}$. □

The following theorem of Stone ([25], p. 183) is used to construct the spectral measure $\pi_0(\cdot)$ for H_0 from the resolvent of H_0 .

Stone's Theorem: Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$, and let $\pi(\cdot)$ denote its spectral measure. If $f, g \in \mathcal{H}$ and $-\infty < a < b < \infty$, then

$$1/2 \langle g,[\pi(b) + \pi(b-)]f \rangle_{\mathcal{H}} - 1/2 \langle g,[\pi(a) + \pi(a-)]f \rangle_{\mathcal{H}}$$

$$= \lim_{\sigma \to 0+} \frac{1}{2\pi i} \int_a^b \langle g,[R_{\lambda+i\sigma} - R_{\lambda-i\sigma}]f \rangle_{\mathcal{H}}\, d\lambda ,$$

where $R_{\lambda \pm i\sigma} = (H - \lambda \mp i\sigma)^{-1}$.

Theorem 2.8: If $f \in \mathcal{H}_0$, and $-\infty < a < b < \infty$, then

$$1/2 \langle f,\{\pi_0(b) + \pi_0(b-) - \pi_0(a) - \pi_0(a-)\}f \rangle_{\mathcal{H}_0} \tag{2.16}$$

$$= \sum_{n=1}^{\infty} \int_{a \le \xi^2 + k_n^2 \le b} |\hat{f}_n(\xi)|^2 d\xi .$$

Proof: Theorem 2.7 implies that if $f \in \mathcal{H}_0$ and $\zeta \in \rho(H_0)$, then

$$\langle f, R^\circ_\zeta f \rangle_{\mathcal{H}_0} = \langle T_0 f, T_0 R^\circ_\zeta f \rangle_{\mathcal{E}} = \sum_{n=1}^{\infty} \int_R \frac{|\hat{f}_n(\xi)|^2}{\xi^2 + k_n^2 - \zeta}\, d\xi .$$

If $\lambda \in R$ and $\sigma > 0$, then

$$\langle f, [R_{\lambda+i\sigma} - R_{\lambda-i\sigma}]f \rangle_{\mathcal{H}_0} = \sum_{n=1}^{\infty} \int_R \frac{2i\sigma}{(\xi^2 + k_n^2 - \lambda)^2 + \sigma^2} |\hat{f}_n(\xi)|^2 d\xi \; .$$

By Stone's theorem, the left side of (2.16) equals

$$\lim_{\sigma \to 0+} \frac{1}{\pi} \int_a^b \sum_{n=1}^{\infty} \int_R \frac{\sigma}{(\xi^2 + k_n^2 - \lambda)^2 + \sigma^2} |\hat{f}_n(\xi)|^2 d\xi \, d\lambda$$

$$= \lim_{\sigma \to 0+} \sum_{n=1}^{\infty} \int_R |\hat{f}_n(\xi)|^2 \{\frac{1}{\pi} \int_a^b \frac{\sigma}{(\xi^2 + k_n^2 - \lambda)^2 + \sigma^2} d\lambda\} d\xi$$

$$= \sum_{n=1}^{\infty} \int_R |\hat{f}_n(\xi)|^2 \, \chi_{[a,b]}(\xi^2 + k_n^2) d\xi$$

$$= \sum_{n=1}^{\infty} \int_{a \le \xi^2 + k_n^2 \le b} |\hat{f}_n(\xi)|^2 d\xi \; ,$$

where $\chi_{[a,b]}(\cdot)$ denotes the characteristic function of the interval $[a,b]$. □

Corollary 2.9: If $f, g \in \mathcal{H}_0$ and $-\infty \le a < b \le \infty$, then

$$1/2 \langle g, \{\pi_0(b) + \pi_0(b-) - \pi_0(a) - \pi_0(a-)\} f \rangle_{\mathcal{H}_0}$$

(2.17) $$= \sum_{n=1}^{\infty} \int_{a \le \xi^2 + k_n^2 \le b} \overline{\hat{g}_n(\xi)} \, \hat{f}_n(\xi) d\xi \; ,$$

where $\pi_0(\infty) = \pi_0(\infty-) = I$ and $\pi_0(-\infty) = \pi_0(-\infty-) = 0$.

Proof: The result (2.16) extends to $-\infty \le a < b \le \infty$ by taking limits on both sides as $a \to -\infty$ and $b \to \infty$. The corollary follows by polarization. □

Corollary 2.10: 1) If $f \in \mathcal{H}_0$ and $\lambda \in R$, then

(2.18) $$\pi_0(\lambda)f(x) = \sum_{n=1}^{\infty} \int_{\xi^2+k_n^2<\lambda} \hat{f}_n(\xi)\, w_n^\circ(x, \xi)\,d\xi, \quad x \in S .$$

2) $\pi_0(\lambda)$ is absolutely continuous

Proof: Letting $a \nearrow b$ in (2.17) proves that $\pi_0(b) = \pi_0(b-)$, i.e. $\pi_0(\lambda)$ is continuous. Letting $a \to -\infty$ in (2.17) and setting $b = \lambda$ yields

$$\langle g, \pi_0(\lambda)f\rangle_{\mathcal{H}_0} = \sum_{n=1}^{\infty} \int_{\xi^2+k_n^2<\lambda} \overline{\hat{g}_n(\xi)\hat{f}_n}(\xi)\,d\xi$$

$$= \sum_{n=1}^{\infty} \int_R \overline{\hat{g}_n(\xi)}\, X_{(-\infty,\lambda)}(\xi^2 + k_n^2)\hat{f}_n(\xi)\,d\xi = \langle T_0 g, T_0\pi_0(\lambda)f\rangle_{\mathcal{E}} .$$

It follows that $T_0\pi_0(\lambda)f(\xi) = \{X_{(-\infty,\lambda)}(\xi^2 + k_n^2)\hat{f}_n(\xi)\}$, and

$$\pi_0(\lambda)f(x) = \sum_{n=1}^{\infty} \int_R (\pi_0(\lambda)f)^\wedge_n(\xi)w_n^\circ(x, \xi)\,d\xi = \sum_{n=1}^{\infty} \int_{\xi^2+k_n^2<\lambda} \hat{f}_n(\xi)w_n^\circ(x, \xi)\,d\xi.$$

This is (2.18). It follows from this representation that $\pi_0(\lambda)$ is absolutely continuous. □

Note that the sum on the right sides of (2.16) and (2.18) are finite, and the integrations are over bounded subsets of R.

With the spectral measure for H_0 constructed, it is now possible to construct a spectral representation for H_0. Since this will not be used in the remainder of the paper, it will only be sketched here.

For each $n = 1, 2, \cdots$, define the mapping U_n of $\mathcal{E}_n = L_2(R)$

onto the direct sum $L_2((k_n^2, \infty)) \oplus L_2((k_n^2, \infty))$ by

$$U_n h_n(\lambda) = 2^{-1/2}(\lambda - k_n^2)^{-1/4} \begin{bmatrix} h_n(\sqrt{\lambda - k_n^2}) \\ h_n(-\sqrt{\lambda - k_n^2}) \end{bmatrix} , \quad \lambda > k_n^2 .$$

Then U_n is unitary and $U_n(\xi^2 + k_n^2)h_n(\xi) = \lambda\, U_n h_n(\lambda)$. Let $U = \sum_{n=1}^{\infty} \oplus U_n$ be the direct sum of the U_n's . Then U is a unitary map of $\mathcal{E}$ onto the direct sum $\sum_{n=1}^{\infty} \oplus L_2((k_n^2, \infty)) \oplus L_2((k_n^2, \infty))$, and $UT_0H_0f\,(\lambda) = \lambda\, U_0T_0f\,(\lambda)$. Thus, UT_0 is a spectral representation for H_0 . The spectral multiplicity function $m_0(\lambda)$ is piecewise constant; it has the value 0 for $\lambda < k_1^2$, and the value 2n for $k_n^2 \leq \lambda < k_{n+1}^2$ for $n = 1, 2, \cdots$.

§3 The Operator H

Let Ω be the perturbed cylindrical domain defined in the introduction. The perturbed operator H is defined to be $-\Delta_D(\Omega)$; it is a selfadjoint operator on the Hilbert space $\mathcal{H} = L_2(\Omega)$. Then for the (C^∞) diffeomorphism $\Phi: \overline{\Omega} \leftrightarrow \overline{S}$ there is an $h > 0$ such that

$$x \in \overline{\Omega},\ |x_N| > h \Rightarrow \Phi(x) = x .$$

In this section, the mapping Φ is used to construct a unitary map J of $\mathcal{H}_0$ onto $\mathcal{H}$. Some further properties of J are also noted.

Definition 3.1: If $f \in L_2^{loc}(S)$, define

$$Jf(x) = |D\Phi(x)|^{1/2} f(\Phi(x)),\quad x \in \Omega ,$$

where $|D\Phi(x)|$ is the Jacobian-determinant of Φ at x .

Note that if $f \in L_2^{loc}(S)$ and if $\Omega' \subset \Omega$ is a measurable subset of Ω , then

$$(3.1)\quad \int_{\Omega'} |Jf(x)|^2 dx = \int_{\Omega'} |f(\Phi(x))|^2\, |D\Phi(x)|\, dx = \int_{\Phi(\Omega')} |f(x)|^2 dx$$

Lemma 3.2: The mapping $J: f \to Jf$ is a continuous linear mapping of $L_2^{loc}(S)$ onto $L_2^{loc}(\Omega)$ and

$$J^{-1}g(x) = |D\Phi^{-1}(x)|^{1/2}\, g(\Phi^{-1}(x)),\quad x \in S,\ g \in L_2^{loc}(\Omega) .$$

Proof: Equation (3.1) shows that J is a continuous mapping of $L_2^{loc}(S)$ into $L_2^{loc}(\Omega)$. Similarly, the mapping (3.2) is a continuous mapping of $L_2^{loc}(\Omega)$ into $L_2^{loc}(S)$. Since

(3.3) $D\Phi(\Phi^{-1}(x)) \cdot D\Phi^{-1}(x) = I$(the $N \times N$ identity matrix), $x \in S$,

and

$$D\Phi^{-1}(\Phi(x)) \cdot D\Phi(x) = I , \quad x \in \Omega ,$$

it follows that J and the mapping (3.2) are inverse to each other. Thus, J maps $L_2^{loc}(S)$ onto $L_2^{loc}(\Omega)$, and (3.2) is its inverse. □

Corollary 3.3: The restriction of J to $\mathcal{H}_0$ is a unitary map of $\mathcal{H}_0$ onto $\mathcal{H}$ with inverse map given by (3.2).

Proof: From (3.1), it follows (by setting $\Omega' = \Omega$) that J is an isometry of $\mathcal{H}_0$ into $\mathcal{H}$. Similarly, J^{-1} is an isometry of $\mathcal{H}$ into $\mathcal{H}_0$. Finally $J^{-1}J = I$ on $\mathcal{H}_0$ and $JJ^{-1} = I$ on $\mathcal{H}$. □

Lemma 3.3: (a) $J: C_0^\infty(S) \leftrightarrow C_0^\infty(\Omega)$.

(b) $J: H_m(S) \leftrightarrow H_m(\Omega)$ is continuous for $m = 1, 2$.

(c) $J: H_m^{loc}(S) \leftrightarrow H_m^{loc}(\Omega)$ is continuous for $m = 1, 2$.

(d) $J: \mathring{H}_1(S) \leftrightarrow \mathring{H}_1(\Omega)$.

Proof: Since Φ is C^∞, it follows that $J: C_0^\infty(S) \to C_0^\infty(\Omega)$, $J: H_m(S) \to H_m(\Omega)$, and $J: H_m^{loc}(S) \to H_m^{loc}(\Omega)$. But Φ^{-1} is also C^∞, so it follows that the ranges of the mappings above are those indicated. Since $Jf(x) = f(x)$ if $|x_N| > h$, the continuity properties in (b) and (c) need only be proved for functions with bounded support. Hence, (b) $\Rightarrow$ (c) .

Let $f \in H_1(S)$. Then for $i = 1, 2, \cdots, N$,

$$D_i\, Jf\,(x) = D_i(|D\Phi(x)|^{1/2} f(\Phi(x)))$$

$$= (D_i|D\Phi(x)|^{1/2})\; f(\Phi(x)) + |D\Phi(x)|^{1/2}\; D_i(f(\Phi(x)))$$

$$= (D_i|D\Phi(x)|^{1/2}) f(\Phi(x)) + \sum_{k=1}^{N} \frac{\partial f}{\partial x_K}(\Phi(x))\; D_i\Phi_k(x), \quad x \in \Omega\ .$$

Thus there are constants K, K_1, K_2 such that

$$\|D_i Jf\|_{\mathcal{H}} \leq K\, \|f \circ \Phi\|_{\mathcal{H}} + K \sum_{k=1}^{N} \left\|\frac{\partial f}{\partial x_k} \circ \Phi\right\|_{\mathcal{H}}$$

$$\leq K_1\, \|f\|_{\mathcal{H}_0} + K_1 \sum_{k=1}^{N} \left\|\frac{\partial f}{\partial x_k}\right\|_{\mathcal{H}_0}$$

$$\leq K_2\, \|f\|_{1,S}$$

Hence, $\|Jf\|_{1,\Omega} \leq K_2 \|f\|_{1,S}$.

A similar, but more involved computation, shows that there is a constant K_3 such that

$$\|Jf\|_{2,\Omega} \leq K_3\, \|f\|_{2,\,S}\ .$$

Note that (d) follows from (a) and (b). □

Lemma 3.4: If $f \in L_2^{loc}(S)$, $g \in L_2^{loc}(\Omega)$, and $\Omega' \subset \Omega$ is a bounded measurable subset of Ω , then

$$\int_{\Omega'} g(x)\ Jf(x)\ dx = \int_{\Phi(\Omega')} (J^{-1}g)(x)\ f(x)dx$$

Proof: Using the definitions of J and J^{-1}, and (3.3),

$$\int_{\Omega'} g(x)\ Jf(x)dx = \int_{\Omega'} g(x)\ f(\Phi(x))\ |D\Phi(x)|^{1/2}dx$$

$$= \int_{\Omega'} \frac{g(x)}{|D\Phi(x)|^{1/2}} f(\Phi(x))\ |D\Phi(x)|\ dx$$

$$= \int_{\Phi(\Omega')} \frac{g(\Phi^{-1}(x))}{|D\Phi(\Phi^{-1}(x))|^{1/2}} f(x)dx$$

$$= \int_{\Phi(\Omega')} |D\Phi^{-1}(x)|^{1/2}\ g(\Phi^{-1}(x))\ f(x)dx$$

$$= \int_{\Phi(\Omega')} (J^{-1}g)(x)\ f(x)dx\ . \quad \square$$

The final theorem of this section is important in applying Birman's theorem.

Theorem 3.5: $JD(H_0) = D(H)$.

Proof: Let $u \in D(H_0)$, and let $\phi \in C_0^\infty(R^N)$ satisfy $\phi(x) = 1$ if $x \in \Omega$ and $|x_N| \leq h + 1$. Then it follows from the results of §1 that $\phi u \in D(H_0)$, and hence $(1 - \phi)u \in D(H_0)$. Since $(1 - \phi)u$ is supported outside of S_h , it follows that $J(1 - \phi)u = (1 - \phi)u$ outside of Ω_h , and $J(1 - \phi)u = 0$ inside Ω_h . Then, since H and H_0 agree on functions supported outside Ω_h , it follows that $J(1 - \phi)u \in D(H)$.

It follows from §1 that $\phi u \in H_2(S) \cap \overset{\circ}{H}_1(S)$. By lemma 3.3, $J\phi u \in H_2(\Omega) \cap \overset{\circ}{H}_1(\Omega) \subset D(H)$. Thus, $Ju = J(1 - \phi)u + J\phi u \in D(H)$.

Hence, $JD(H_0) \subset D(H)$.

The same argument can be made to show that $J^{-1}D(H) \subset D(H_0)$. Thus, $JD(H_0) = D(H)$. □

§4 The Existence and Completeness of the Wave Operators

The main result of this section is the following theorem.

Theorem 4.1: The wave operators $W_{\pm}(H, H_0; J)$ and $W_{\pm}(H_0, H; J^*)$ exist. $W_{\pm}(H, H_0; J)$ are isometries of $\mathcal{H}_0$ onto $\pi^{ac}\mathcal{H}$ and provide a unitary equivalence between H_0 and the absolutely continuous part of H . Moreover,

$$[W_{\pm}(H, H_0; J)]^* = W_{\pm}(H_0, H; J^*) .$$

To prove this theorem, all that is needed is to verify conditions 1-4 of Birman's theorem (see the introduction). Since J is unitary, conditions 1 and 4 are satisfied. Theorem 3.5 shows that condition 2 is satisfied. Only condition 3 remains to be verified. In fact, a stronger result will be proved, namely that $(HJ - JH_0)\,\pi_0(\delta)$ is trace class for any bounded interval $\delta \subset R$. In order to prove this, the following version of a theorem of Stinespring [24] is used.

Stinespring's theorem: Let μ be a regular measure on R of the form $d\mu(\xi) = \rho(\xi)d\xi$, where ρ is a bounded measurable function with compact support. Let $\mathcal{H}$ be a Hilbert space, and let $M(\xi)$ be a continuous function from R into $\mathcal{H}$. Let T be the transformation from $L_2(R)$ into $\mathcal{H}$ given by

$$Tf = \int_R f(\xi)M(\xi)d\mu(\xi),\quad f \in L_2(R) . \tag{4.1}$$

If $\frac{\partial M(\xi)}{\partial \xi}$ is a continuous function from R into $\mathcal{H}$ then T is trace class.

Proof of theorem 4.1: Let $\delta \subset R$ be a bounded interval, and let $f \in \mathcal{H}_0$. Then corollary 2.10 implies that

$$(4.2)\qquad \pi_0(\delta)f(x) = \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)\, w_n^\circ(x, \xi)d\xi, \quad x \in S,$$

where

$$\delta_n = \{\xi \in R\colon \xi^2 + k_n^2 \in \delta\} \quad \text{for} \quad n = 1, 2, \cdots .$$

Since δ is bounded, the sum on the right of (4.2) is finite, and the integrations are over bounded sets. Since $\pi_0(\delta)f \in D(H_0)$, it follows that $J\pi_0(\delta)f \in D(H)$. Let $\phi \in C_0^\infty(\Omega)$. Then using the unitarity of J on $\mathcal{H}_0$ and lemma 3.4,

$$\langle \phi, J\pi_0(\delta)f\rangle_{\mathcal{H}} = \langle J^*\phi, \pi_0(\delta)f\rangle_{\mathcal{H}_0} = \langle J^{-1}\phi, \pi_0(\delta)f\rangle_{\mathcal{H}_0}$$

$$= \int_S \overline{(J^{-1}\phi)}(x)\left[\sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)\, w_n^\circ(x, \xi)d\xi\right]dx$$

$$= \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)\left[\int_S \overline{(J^{-1}\phi)}(x)\, w_n^\circ(x, \xi)dx\right]d\xi$$

$$= \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)\left[\int_\Omega \overline{\phi(x)}\, Jw_n^\circ(x, \xi)dx\right]d\xi$$

$$= \int_\Omega \overline{\phi(x)}\left[\sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)Jw_n^\circ(x, \xi)d\xi\right]dx .$$

$$= \langle \phi, \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)\, Jw_n^\circ(\cdot, \xi)d\xi\rangle_{\mathcal{H}} .$$

Since $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$, it follows that

$$J\pi_0(\delta)f(x) = \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi) Jw_n^\circ(x, \xi)d\xi, \quad x \in \Omega .$$

Hence,

$$JH_0\pi_0(\delta)f(x) = \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi) \ (\xi^2 + k_n^2) Jw_n^\circ(x, \xi)d\xi, \quad x \in \Omega .$$

Then, since $Jw_n^\circ(\cdot,\xi) \in H_2^{loc}(\Omega) \cap \overset{\circ}{H}{}_1^{loc}(\Omega) \subseteq L_2^{loc}(-\Delta;\Omega)$, definition 1.1 yields,

$$\langle \phi, HJ\pi_0(\delta)f \rangle_{\mathcal{H}} = \langle H\phi, J\pi_0(\delta)f \rangle_{\mathcal{H}}$$

$$= \int_\Omega \overline{(H\phi)(x)} \left[\sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi) \ Jw_n^\circ(x, \xi)d\xi \right] dx$$

$$= \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi) \left[\int_\Omega \overline{H\phi(x)} \ Jw_n^\circ(x, \xi)dx \right] d\xi$$

$$= \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi) \left[\int_\Omega \overline{\phi(x)} \ (-\Delta Jw_n^\circ)(x, \xi)dx \right] d\xi$$

$$= \int_\Omega \overline{\phi(x)} \left[\sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)(-\Delta Jw_n^\circ)(x, \xi)d\xi \right] dx .$$

Thus,

$$H\pi_0(\delta)f(x) = \sum_{n=1}^{\infty} \int_{\delta_n} \hat{f}_n(\xi)(-\Delta Jw_n^\circ)(x,\xi)d\xi, \quad \underline{x} \in \Omega .$$

Combining these results,

$$(4.3) \quad (HJ-JH_0)\pi_0(\delta)f(x) = \sum_{n=1}^{\infty} \int_{\xi^2+k_n^2\in\delta} \hat{f}_n(\xi)(-\Delta-\xi^2 - k_n^2)Jw_n^\circ(x, \xi)d\xi$$

$$= \sum_{n=1}^{\infty} \int_R \hat{f}_n(\xi)[(-\Delta - \xi^2 - k_n^2)Jw_n^\circ(x, \xi)]X_\delta(\xi^2 + k_n^2)d\xi, \quad x \in \Omega$$

For $n = 1, 2, \ldots,$ define

$$(4.4) \qquad M_n(x, \xi) = (-\Delta - \xi^2 - k_n^2)Jw_n^\circ(x, \xi), \quad x \in \Omega, \ \xi \in R .$$

Consider the mapping from $\mathcal{E}$ into $\mathcal{H}$ given by

$$(4.5) \qquad (HJ - JH_0)\pi_0(\delta)T_0^* h(x) =$$

$$\sum_{n=1}^{\infty} \int_R h_n(\xi)M_n(x, \xi)X_\delta(\xi^2 + k_n^2)d\xi, \quad x \in \Omega, \ h \in \mathcal{E} .$$

If it can be shown that this mapping is trace class, then, since T_0 is unitary, it will follow that the mapping (4.3) is trace class.

Note that there are only a finite number of terms on the right side of (4.5). In order to prove that the mapping (4.5) is trace class, it is sufficient to prove that the mapping from $\mathcal{E}$ into $\mathcal{H}$ given by

$$(4.6) \qquad h \to \int_R h_n(\xi)\, M_n(x, \xi)X_\delta(\xi^2 + k_n^2)d\xi, \quad x \in \Omega, \quad h \in \mathcal{E} ,$$

is trace class for any n .

The mapping (4.6) is of the form (4.1), where

$$\rho(\xi) = X_\delta(\xi^2 + k_n^2), \quad \text{and} \quad M(\xi) = M_n(\cdot, \xi), \text{ for } \xi \in R$$

To apply Stinespring's theorem, it must be shown that $M_n(\cdot, \xi)$ and $\frac{\partial M_n}{\partial \xi}(\cdot, \xi)$ are continuous functions from R into $\mathcal{H}$.

By theorem 2.2, the mappings $\xi \to w_n^\circ(\cdot, \xi)$ and $\xi \to \frac{\partial w_n^\circ(\cdot, \xi)}{\partial \xi}$ are continuous from R into $H_2^{loc}(S) \cap \mathring{H}_1^{loc}(S)$. It follows from lemma 3.3 that the mappings $\xi \to Jw_n^\circ(\cdot, \xi)$ and $\xi \to J\dfrac{\partial w_n^\circ(\cdot, \xi)}{\partial \xi}$ are continuous from R into $H_2^{loc}(\Omega) \cap \overset{\circ}{H}{}_1^{loc}(\Omega)$. Hence from the definition (4.4) of $M_n(x, \xi)$ the mapping $\xi \to M_n(\cdot, \xi)$ is continuous from R into $L_2^{loc}(\Omega)$. Since $Jw_n^\circ(x, \xi) = w_n^\circ(x, \xi)$ if $|x_N| > h$, it follows that $M_n(x, \xi) = 0$ if $|x_N| > h$, so that $M_n(\cdot, \xi) \in L_2(\Omega) = \mathcal{H}$ and $M_n(\cdot, \xi)$ is a continuous function from R into $\mathcal{H}$.

It follows also from (4.4) that

$$\frac{\partial M_n(x,\xi)}{\partial \xi} = -2\xi Jw_n^\circ(x, \xi) + (-\Delta - \xi^2 - k_n^2)\frac{\partial Jw_n^\circ(x,\xi))}{\partial \xi}$$

$$= -2\xi\, Jw_n^\circ(x, \xi) + (-\Delta - \xi^2 - k_n^2)J\,\frac{\partial w_n^\circ(x, \xi)}{\partial \xi},, \quad x \in \Omega$$

(here the fact that $\dfrac{\partial Jw_n^\circ(x, \xi)}{\partial \xi} = i\Phi_N(x)Jw_n^\circ(x, \xi) = J\dfrac{\partial w_n^\circ(x, \xi)}{\partial \xi}$ is used). Consequently the mapping $\xi \to \dfrac{\partial M_n(x, \xi)}{\partial \xi}$ is a continuous mapping of R into $L_2^{loc}(\Omega)$. Again, since $\dfrac{\partial M_n(x, \xi)}{\partial \xi} = 0$ if $|x_N|>h$, $\dfrac{\partial M_n(\cdot, \xi)}{\partial \xi}$ is a continuous function from R into $\mathcal{H}$. Thus, Stinespring's theorem applies, and the mapping (4.6) is trace class. This completes the proof. □

§5 The Spectral Properties of H

In this section the spectrum of H is examined. It is shown that the eigenvalues of H have finite multiplicty, and the point spectrum of H is nowhere dense in R . A limiting absorption principle is stated for H (it is proved in the appendix). It is used to show that H has no singular continuous spectrum and to construct two sets of generalized eigenfunctions for H .

Let $K = H_2^{loc}(\Omega) \cap \mathring{H}_1^{loc}(\Omega)$, and equip K with the topology of $H_2^{loc}(\Omega)$. Then K is a Fréchet space consisting of functions which are 'locally' in D(H) , i.e. $u \in K \Leftrightarrow \phi u \in D(H)$ for every $\phi \in C_0^\infty(R^N)$.

<u>Definition 5.1</u>: If $u \in K$, then for $n = 1, 2, \cdots$, define

$$u_n^{\pm}(x_N) = \int_G u(x)\overline{\eta_n(\tilde{x})}\,d\tilde{x}, \quad \pm x_N > h .$$

<u>Theorem 5.2</u>: If $u \in K$, then

$$u(x) = \sum_{n=1}^{\infty} u_n^{\pm}(x_N)\eta_n(\tilde{x}), \quad x \in \Omega , \pm x_N > h \tag{5.1}$$

in K .

<u>Proof</u>: Let $\Omega' = \{x \in \Omega: a < x_N < b\}$ for some $h < a < b$. Let $\phi \in C_0^\infty(R^N)$ satisfy $\phi(x) = 0$ if $x \in \Omega_h$, $\phi(x) = 1$ if $x \in \Omega'$. Then $\phi u \in D(H)$ and $J^*\phi u = \phi u$ outside S_h . It follows from cor-

ollary 2.6 and (2.10) that

$$u(x) = \phi(x)u(x) = (J^*\phi u)(x) = \sum_{n=1}^{\infty} \int_R (J^*\phi u)_n\hat{}(\xi) w_n^{\circ}(x, \xi)d\xi$$

$$= \sum_{n=1}^{\infty} [\int_G (J^*\phi u)(\tilde{y}, x_N)\eta_n(\tilde{y})dy]\eta_n(\tilde{x}) = \sum_{n=1}^{\infty} u_n^+(x_N)\eta_n(\tilde{x})$$

in $H_2(\Omega')$.

If $\Omega' = \{x \in \Omega: -b < x_N < -a\}$ for some $h < a < b$, a similar argument shows that

$$u(x) = \sum_{n=1}^{\infty} u_n^-(x_N)\eta_n(\tilde{x})$$

in $H_2(\Omega')$.

Since, in both cases, a and b are arbitrary, the theorem is proved. □

The above theorem says that functions in K can be expanded, in the uniform part of Ω , in terms of the transverse eigenfunctions $\eta_n(\tilde{x})$.

<u>Lemma 5.3</u>: If $u \in \mathcal{H}$ is an eigenfunction for H with the corresponding eigenvalue λ , then there are constants $C_n^{\pm}$ for $n = 1, 2, \cdots$, such that

$$u(x) = \sum_{\substack{n=1 \\ k_n^2>\lambda}}^{\infty} C_n^{\pm} e^{-\sqrt{k_n^2 - \lambda}\,|x_N|} \eta_n(\tilde{x}), \quad x \in \Omega,\ \pm x_N > h$$

in K .

<u>Proof</u>: Since u is an eigenfunction for H ,

$$Hu = -\Delta u = \lambda u .$$

Since $u \in D(H)$, it follows from theorems 1.5 that $u \in \mathcal{K}$. Thus, u has an expansion (5.1) in $\mathcal{K}$. The functions $u_n^{\pm}(x_N)$ satisfy

$$\frac{d^2 u_n^{\pm}(x_N)}{dx_N^2} = \int_G \frac{\partial^2 u(x)}{\partial x_N^2} \overline{\eta_n(\tilde{x})} d\tilde{x}$$

$$= \int_G \left(-\sum_{j=1}^{N-1} \frac{\partial^2 u(x)}{\partial x_j^2} - \lambda u(x)\right)\overline{\eta_n(\tilde{x})} d\tilde{x} - \int_G \left(-\sum_{j=1}^{N} \frac{\partial^2 u(x)}{\partial x_j^2} - \lambda u(x)\right)\overline{\eta_n(\tilde{x})} dx$$

$$= \int_G \left(-\sum_{j=1}^{N-1} \frac{\partial^2 u(x)}{\partial x_j^2}\right)\overline{\eta_n(\tilde{x})} d\tilde{x} - \lambda u_n^{\pm}(x_N) - \int_G [(H-\lambda)u(x)]\eta_n(\tilde{x}) d\tilde{x}$$

$$= \int_G u(x)\, \overline{H_G \eta_n(\tilde{x})}\, d\tilde{x} - \lambda\, u_n^{\pm}(x_N)$$

$$= \int_G u(x) k_n^2\, \overline{\eta_n(\tilde{x})} d\tilde{x} - \lambda\, u_n^{\pm}(x_N) = (k_n^2 - \lambda) u_n^{\pm}(x_N)$$

for $\pm x_N > h$.

Thus, there are constants $C_n^{\pm}$ and $D_n^{\pm}$ such that

$$u_n^{\pm}(x_N) = C_n^{\pm} e^{-\sqrt{k_n^2-\lambda}\,|x_N|} + D_n^{\pm} e^{\sqrt{k_n^2-\lambda}\,|x_N|} .$$

Since $u \in \mathcal{H}$, $u_n^{\pm}(x_N)$ must die out as $x_N \to \pm\infty$. Thus, $D_n^{\pm} = 0$ for all n and $C_n^{\pm} = 0$ if $k_n^2 - \lambda < 0$. □

<u>Theorem 5.4</u>: If the bounded interval $[a,b] \subset [k_n^2, k_{n+1}^2)$ for some n , or if $[a,b] \subset (-\infty, k_1^2)$, then there are at most a finite

number of eigenvalues for H in $[a,b]$, and each eigenvalue in $[a,b]$ has finite multiplicity.

Proof: Since H is nonnegative, assume $b \geq 0$. Let $\mathcal{S}_{[a,b]}$ denote the closed span of the set $\{f \in \mathcal{H} : Hf = \lambda f$ for some $\lambda \in [a,b]\}$, and let $S_{[a,b]}$ denote the unit ball in $\mathcal{S}_{[a,b]}$. The theorem is proved by proving that $S_{[a,b]}$ is precompact and hence, $\mathcal{S}_{[a,b]}$ is finite dimensional.

Let $\{u_n\}$ be a sequence in $S_{[a,b]}$. Then, by corollary 1.4,

$$||u_n||^2_{1,\Omega} = ||u_n||^2_{\mathcal{H}} + \langle Hu_n, u_n\rangle \leq (1 + b)||u_n||^2_{\mathcal{H}} \leq 1 + b .$$

By Rellich's compactness theorem, for each $m = 1, 2, \cdots,$ there is a subsequence $\{u^m_n\}$ of $\{u_n\}$ such that $\{u^m_n\}$ is Cauchy in $L_2(\Omega_{h+m})$. Choose the sequences $\{u^m_n\}$ such that $\{u^{m+1}_n\} \subset \{u^m_n\}$ for $m = 1, 2, \cdots$. Let $\{u_{n_j}\}$ be the diagonal sequence, $u_{n_j} = u^j_j$ for $j = 1, 2, \cdots$. The sequence $\{u^j_j\}$ is Cauchy in any bounded portion of Ω.

Let $\eta = \min\{\sqrt{k_n^2 - b} : k_n^2 > b,\ n = 1, \cdots\} > 0$. Let $t > 0$. Then, using (5.2), if $u \in S_{[a,b]}$,

$$(5.3)\quad ||u||^2_{L_2(\Omega - \Omega_{h+t})} = \sum_{\substack{n=1 \\ k_n^2>\lambda}}^{\infty} (|c^+_n|^2 + |c^-_n|^2) \int_{h+t}^{\infty} e^{-2\sqrt{k_n^2-\lambda}\,|x_N|}\, d|x_N|$$

$$= \sum_{\substack{n=1 \\ k_n^2>\lambda}}^{\infty} (|c^+_n|^2 + |c^-_n|^2)\, \frac{e^{-2\sqrt{k_n^2-\lambda}\,(h+t)}}{2\sqrt{k_n^2-\lambda}}$$

$$= \sum_{\substack{n=1 \\ k_n^2>\lambda}}^{\infty} e^{-2\sqrt{k_n^2-\lambda}\, t} (|c_n^+|^2 + |c_n^-|^2) \frac{e^{-2\sqrt{k_n^2-\lambda}\, h}}{2\sqrt{k_n^2-\lambda}}$$

$$\leq e^{-2\eta t} \sum_{\substack{n=1 \\ k_n^2>\lambda}}^{\infty} (|c_n^+|^2 + |c_n^-|^2 \frac{e^{-2\sqrt{k_n^2-\lambda}\, h}}{2\sqrt{k_n^2-\lambda}} = e^{-2\eta t} ||u||^2_{L_2(\Omega-\Omega_h)}$$

$$\leq e^{-2\eta t} ||u||^2_{\mathcal{H}} \leq e^{-2\eta t} .$$

Thus, functions in $S_{[a,b]}$ die out exponentially as $x_N \to \pm\infty$.

Let $\varepsilon > 0$. Choose m_0 so that $e^{-2\eta m_0} < \varepsilon/3$. Choose M such that $n, k > M$ imply that

$$|| u_n^{m_0} - u_k^{m_0} ||_{L_2(\Omega_{h+m_0})} < \varepsilon/3 .$$

Then if $k, j > M$,

$$|| u_{n_k} - u_{n_j} ||_{\mathcal{H}} \leq || u_{n_K} - u_{n_j} ||_{L_2(\Omega_{h+m_0})} + || u_{n_k} ||_{L_2(\Omega-\Omega_{h+m_0})} + || u_{n_j} ||_{L_2(\Omega-\Omega_{h+m_0})}$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Hence, the sequence $\{u_{n_j}\}$ is Cauchy in $\mathcal{H}$ so that $S_{[a,b]}$ is precompact. □

Theorem 5.4 says the point spectrum of H is nowhere dense in R. The transverse eigenvalues k_n^2 may be accumulation points (from the left only) of the point spectrum.

<u>Definition 5.5</u>: Let $\Lambda = \{\lambda \in R: \lambda \neq k_n^2$ for any n and λ is not an eigenvalue for $H\}$, $C^{\pm} = \{\zeta \in C: \pm\mathrm{Im}\,\zeta > 0\}$, $\Lambda^{\pm} = C^{\pm} \cup \Lambda$.

Theorem 5.6 (limiting Absorption Principle): Let $r > h$. The mapping

$$(5.4) \qquad (\zeta, u) \to R_\zeta u$$

is a continuous mapping from $C^{\pm} \times L_2(\Omega_r)$ into K possessing a unique continuous extension to $\Lambda^{\pm} \times L_2(\Omega_r)$ (into K) .

This theorem is proved in the appendix. It is used in the following theorem to rule out singular continuous spectrum for H . Eidus [8] proved that the mapping (5.4) is continuous in $\zeta \in C^{\pm}$ for fixed $f \in L_2(\Omega_r)$ and has a unique continuous extension to $\Lambda^{\pm}$.

Theorem 5.7. H has no singular continuous spectrum.

Proof: Let $\mathcal{H}^c$ denote the subspace of continuity of H , i.e. $f \in \mathcal{H}^c$ if and only if $f \in \mathcal{H}$ and the function $\langle f, \pi(\lambda)f \rangle$, defined for $\lambda \in R$, is continuous. Then $\mathcal{H}^{ac} \subseteq \mathcal{H}^c$, and H has no singular continuous spectrum iff $\mathcal{H}^c = \mathcal{H}^{ac}$, (see Kato [15] p. 516). The theorem is proved by showing that $\mathcal{H}^c \subset \mathcal{H}^{ac}$.

Let $[a,b] \subset \Lambda$ be a bounded interval, let $f \in C_0^\infty(\Omega)$, and let $r > 0$ satisfy $\operatorname{supp} f \subseteq \Omega_r$. Then, using Stone's theorem, the continuity of $\pi(\cdot)$ on Λ , and theorem 5.6,

$$\langle f, \pi([a,b])f \rangle_{\mathcal{H}} = \lim_{\sigma \to 0+} \frac{1}{2\pi i} \int_a^b \langle f, R_{\lambda+i\sigma} f - R_{\lambda-i\sigma} f \rangle_{\mathcal{H}} d\lambda$$

$$= \lim_{\sigma \to 0+} \frac{1}{2\pi i} \int_a^b \langle f, R_{\lambda+i\sigma} f - R_{\lambda-i\sigma} f \rangle_{L_2(\Omega_r)} d\lambda$$

$$= \frac{1}{2\pi i} \int_a^b \langle f, R_{\lambda+i0} f - R_{\lambda-i0} f \rangle_{L_2(\Omega_r)} d\lambda ,$$

where $R_{\lambda\pm i0}f$ are the limits in K of $R_{\lambda\pm i\sigma}f$ as $\sigma \to 0+$. The left and right side of the above equation extend uniquely to the ring of all Borel subsets of [a,b], i.e. if $M \subseteq [a,b]$ is a Borel set then,

$$\langle f, \pi(M)f \rangle_{\mathcal{H}} = \frac{1}{2\pi i} \int_M \langle f, R_{\lambda+i0}f - R_{\lambda-i0}f \rangle_{\mathcal{H}} d\lambda .$$

If $M \subset R$ is a Borel set of Lebesgue measure zero, then $\langle \pi([a,b])f, \pi(M)\pi([a,b])f \rangle_{\mathcal{H}} = \langle f, \pi(M \cap [a,b])f \rangle = 0$. Thus, $\pi([a,b])f \in \mathcal{H}^{ac}$. Since $C_0^\infty(\Omega)$ is dense in $\mathcal{H}$ and $\mathcal{H}^{ac}$ is a closed subspace of $\mathcal{H}$, it follows that $\pi([a,b])f \in \mathcal{H}^{ac}$ for every $f \in \mathcal{H}$.

Let $f \in \mathcal{H}^c$. Then for any bounded interval $(a,b) \subset R$, $\pi([a, b])f = \pi((a,b))f = \lim_{\delta \to 0} \pi([a+\delta, b-\delta])f$. If $(a,b) \subset \Lambda$ is a bounded interval, then $\pi((a,b))f = \pi([a,b])f = \lim_{\delta \to 0} \pi([a+\delta, b-\delta])f \in \mathcal{H}^{ac}$. From theorem 5.4 it is seen that there exist a countable number of disjoint intervals δ_n whose interiors lie in Λ such that $R = \bigcup_{n=1}^{\infty} \delta_n$. Since the measure $\pi(\cdot)f$ is additive, $f = \pi(R)f$ $= \pi(\bigcup_{n=1}^{\infty} \delta_n)f = \sum_{n=1}^{\infty} \pi(\delta_n)f \in \mathcal{H}^{ac}$, since $\pi(\delta_n)f \in \mathcal{H}^{ac}$ for each n . Thus, $\mathcal{H}^c \subseteq \mathcal{H}^{ac}$, and H has no singular continuous spectrum. □

In the remainder of this section, two sets of generalized eigenfunctions for H are constructed. Recall from §4 that the mapping $\xi \to M_n(\cdot, \xi)$, where $M_n(\cdot, \xi)$ is given by (4.4), is continuous from R into $\mathcal{H}$, and $M_n(\cdot, \xi)$ is supported in Ω_h for every $\xi \in R$. This mapping, thus, can be considered as a continuous mapping of R into $L_2(\Omega_h)$.

Definition 5.8: For $n = 1, 2, \cdots$, define

$$(5.5) \qquad V_n(\cdot, \xi, \zeta) = -R_\zeta M_n(\cdot, \xi), \; \xi \in R, \; \zeta \in C - R .$$

Theorem 5.9: The limits

$$(5.6) \qquad V_n^{\pm}(\cdot, \xi, \lambda) = \mathcal{K}\text{-}\lim_{\sigma \to 0+} V_n(\cdot, \xi, \lambda \mp i\sigma)$$

exist in $\mathcal{K}$ for every $\xi \in R$ and $\lambda \in \Lambda$. Furthermore,

$$(5.7) \qquad (-\Delta - \lambda)\, V_n^{\pm}(\cdot, \xi, \lambda) = -M_n(\cdot, \xi)$$

for all $\xi \in R$. The mapping

$$(5.8) \qquad (\xi, \lambda, \sigma) \to V_n(\cdot, \xi, \lambda \mp i\sigma)$$

with boundary values $V_n^{\pm}(\cdot, \xi, \lambda)$, is continuous from $R \times \Lambda \times [0, \infty)$ into $\mathcal{K}$.

Proof: Theorem 5.6 implies the existence of the limit (5.6) and the continuity of the mapping (5.8). It follows from (5.5) that

$$(H - \lambda) V_n(\cdot, \xi, \lambda \mp i\sigma) = -M_n(\cdot, \xi) \mp i\sigma V_n(\cdot, \xi, \lambda \mp i\sigma)$$

in $\mathcal{H}$. Taking limits of both sides in $L_2^{loc}(\Omega)$ as $\sigma \to 0+$ yields (5.5) . □

Definition 5.10: For $n = 1, 2, \cdots$, define

$$w_n^{\pm}(x, \xi) = J\hat{w}_n(x, \xi) + V_n^{\pm}(x, \xi, \xi^2 + k_n^2), \; x \in \Omega \text{ , and }$$
$$\xi \in R \ni \xi^2 + k_n^2 \in \Lambda .$$

Note that $w_n^{\pm}(x, \xi)$ is defined only for a.e. $\xi \in R$.

Lemma 5.11: For $n = 1, 2, \cdots$, $w_n^{\pm}(\cdot, \xi) \in K$ for a.e. $\xi \in R$, and

$$(-\Delta - \xi^2 - k_n^2)w_n^{\pm}(\cdot, \xi) = 0 \quad \text{a.e.} \quad \xi \in R .$$

Proof: This follows from theorem 5.9 and the definition (4.4) of $M_n(\cdot, \xi)$. □

These two sets of functions $\{w_n^{\pm}(x, \xi)\}$ in $H_2^{loc}(\mathcal{H}) \cap \mathring{H}_1^{loc}(\mathcal{H})$ are generalized eigenfunctions for H . In the next section, after the wave operators have been constructed, the completeness and orthogonality of these two sets of eigenfunctions in $\mathcal{H}^{ac} = \pi^{ac}\mathcal{H}$ are proved.

§6 Construction of the Wave Operators and Scattering Operator

In this section the wave operators $W_{\pm} = W_{\pm}(H, H_0; J)$, whose existence and completeness were proved in §4, are constructed in terms of the two sets of generalized eigenfunctions $\{w_n^{\pm}(x, \xi)\}$. The representation of the wave operators implies the completeness and orthogonality of these two sets of eigenfunctions. Finally the multichannel character of the problem is mentioned, and the scattering operator $S(H, H_0, J)$, denoted by $\mathcal{S}$, is constructed.

Lemma 6.1: Let $f \in \mathcal{H}_0$ satisfy

$$f(x) = \int_{\delta} \hat{f}_n(\xi)\, w_n^{\circ}(x, \xi) d\xi, \quad x \in S$$

for some bounded interval $\delta \subset R$ and some n . Then

$$W_{\pm} f(x) = \int_{\delta} \hat{f}_n(\xi)\, w_n^{\pm}(x, \xi) d\xi, \quad x \in \Omega$$

Proof: The proof will be given for W_+ . The proof for W_- is the same. Let $W(t) = e^{itH} J e^{-itH_0}$ for $t \in R$. Then $W_+ = \operatorname*{s-lim}_{t\to\infty} W(t)$. Let $g \in C_0^{\infty}(\Omega)$. Then, using the Abelian limit,

$$(6.1)\quad \langle g, W_+ f \rangle_{\mathcal{H}} = \lim_{t\to\infty} \langle g, W(t) f \rangle_{\mathcal{H}} = \lim_{\sigma\to 0+} \sigma \int_0^{\infty} e^{-\sigma t} \langle g, W(t) f \rangle_{\mathcal{H}} dt .$$

Since $f \in D(H_0)$, and $J: D(H_0) \to D(H)$, it follows that $\frac{d}{dt} W(t) f$ exists and $\frac{d}{dt} W(t) f = e^{itH}(HJ - JH_0) e^{-itH_0} f$. Integrating by parts in (6.1) yields

$$(6.2)\quad \langle g, W_+ f\rangle_{\mathcal{H}} = \langle g, W(0)f\rangle_{\mathcal{H}} + \lim_{\sigma\to 0+} i\int_0^\infty e^{-\sigma t}\langle g, e^{itH}(HJ - JH_0)e^{-itH_0}f\rangle_{\mathcal{H}} dt$$

It follows from theorems 2.4 and 2.5 that

$$e^{-itH_0}f(x) = \int_\delta \hat{f}_n(\xi)\; e^{-i(\xi^2+k_n^2)t}\; w_n^\circ(x, \xi)d\xi, \quad x \in \Omega .$$

Using this and the representation (4.3),

$$(HJ - JH_0)e^{-itH}f(x) = \int_\delta \hat{f}_n(\xi)\; e^{-i(\xi^2+k_n^2)t}\; M_n(x, \xi)d\xi .$$

Since the mapping $\xi \to M_n(\cdot, \xi)$ is continuous from R into $\mathcal{H}$, it follows that

$$(6.3)\quad e^{itH}(HJ-JH_0)e^{-itH_0}f(x) = \int_\delta \hat{f}_n(\xi)e^{-i(\xi^2+k_n^2)t}e^{itH}M_n(x, \xi)d\xi .$$

Using (6.3) in (6.2), and definitions 5.8 and 5.10,

$$\langle g, W_+ f\rangle_{\mathcal{H}} = \langle g, Jf\rangle_{\mathcal{H}} + \lim_{\sigma\to 0+} i\int_0^\infty e^{-\sigma t}\int_\Omega \overline{g(x)}\int_\delta \hat{f}_n(\xi)e^{-i(\xi^2+k_n^2)t}e^{itH}M_n(x,\xi)d\xi dx\; dt$$

$$= \langle g, Jf\rangle_{\mathcal{H}} + \lim_{\sigma\to 0+} i\int_\delta \hat{f}_n(\xi)\int_\Omega \overline{g(x)}\;[\int_0^\infty e^{-i(\xi^2+k_n^2-i\sigma)t}\; e^{itH}M_n(x, \xi)dt]dx\; d\xi$$

$$= \langle g, Jf\rangle_{\mathcal{H}} + \lim_{\sigma\to 0+} i\int_\delta \hat{f}_n(\xi)\int_\Omega \overline{g(x)}\; i\, R_{\xi^2+k_n^2-i\sigma}\, M_n(x,\xi)dx\; d\xi$$

$$= \langle g, Jf\rangle_{\mathcal{H}} + \lim_{\sigma\to 0+} \int_\delta \hat{F}_n(\xi)\int_\Omega \overline{g(x)}\; V_n(x,\xi,\xi^2+k_n^2-i\sigma)dx\; d\xi$$

$$= \langle g, Jf\rangle_{\mathcal{H}} + \int_\delta \hat{f}_n(\xi)\int_\Omega \overline{g(x)}\; V_n^+(x,\xi)dx\; d\xi$$

$$= \langle g, Jf\rangle_{\mathcal{H}} + \int_\Omega \overline{g(x)}\int_\delta \hat{f}_n(\xi)V_n^+(x,\xi)d\xi\; dx$$

$$= \int_\Omega \overline{g(x)}\int_\delta \hat{f}_n(\xi)[Jw_n^\circ(x,\xi) + V_n^+(x,\xi)]d\xi\; dx$$

$$= \langle g, \int_\delta \hat{f}_n(\xi) w_n^{+}(\cdot,\xi)\, d\xi \rangle_{\mathcal{H}}$$

Here the fact that $\int_0^\infty e^{-i(\xi^2+k_n^2-i\sigma)t}\, e^{itH} M_n(\cdot,\xi)\, dt = iR_{\xi^2+k_n^2-i\sigma} M_n(\cdot,\xi)$

is used (see [19], p. 247). Since $C_0^\infty(\Omega)$ is dense in $\mathcal{H}$, the lemma is proved. □

Theorem 6.2: Let $f \in \mathcal{H}_0$. Then

(6.4) $$W_{\pm} f(x) = \sum_{n=1}^{\infty} \int_R \hat{f}_n(\xi)\, w_n^{\pm}(x, \xi)\, d\xi, \quad x \in \Omega$$

in $\mathcal{H}$.

Proof: Let $\delta \subset R$ be an interval. Then it follows from Corollary 2.10 and lemma 6.1 that

(6.5) $$W_{+} \pi_0(\delta) f(x) = \sum_{n=1}^{\infty} \int_{\xi^2+k_n^2 \in \delta} \hat{f}_n(\xi)\, w_n^{\pm}(x, \xi)\, d\xi .$$

Since $W_{\pm}$ is an isometry and since $\|\pi(\delta)f - f\|_{\mathcal{H}} \to 0$ as the interval δ increases to R, (6.4) follows from (6.5) by letting δ increase to R. □

Corollary 6.3: The two sets of generalized eigenfunctions $\{w_n^{\pm}(x, \xi)\}$ are complete and orthogonal in $\mathcal{H}^{ac}$.

Proof: Let $u \in \mathcal{H}^{ac}$. Then, since $W_{\pm}$ is complete, there is an $f_{\pm} \in \mathcal{H}_0$ such that $W_{\pm} f_{\pm} = u$. Since $f_{\pm} \in \mathcal{H}_0$, it has the expansion

$$f_{\pm}(x) = \sum_{n=1}^{\infty} \int_R (f_{\pm})_n^{\wedge}(\xi)\, w_n^{\circ}(x, \xi)\, d\xi, \quad x \in S .$$

Then, from theorem 6.2,

$$u(x) = W_{\pm} f_{\pm}(x) = \sum_{n=1}^{\infty} (f_{\pm})_n^{\wedge}(\xi)\, w_n^{\pm}(x, \xi)\, d\xi, \quad x \in \Omega .$$

Thus, any function in $\mathcal{H}^{ac}$ can be expanded in terms of $w_n^{\pm}(x, \xi)$. This means the two sets of generalized eigenfunctions are complete in $\mathcal{H}^{ac}$.

Let $h = \{h_n\} \in \mathcal{E}$. Then $T_0^* h \in \mathcal{H}_0$ has the expansion (2.8). By theorem 6.2,

$$(6.6) \qquad W_{\pm} T_0^* h(x) = \sum_{n=1}^{\infty} \int_R h_n(\xi) w_n^{\pm}(x, \xi)\, d\xi, \quad x \in \Omega$$

is in $\mathcal{H}^{ac}$. Thus, given any $h \in \mathcal{E}$, the expansion on the right of (6.6) yields a function in $\mathcal{H}^{ac}$. This means the two sets of generalized eigenfunctions are orthogonal. □

It follows from the above theorems that if $f \in \mathcal{H}^{ac}$, then

$$f(x) = \sum_{n=1}^{\infty} \int_R (W_{\pm}^* f)_n^{\wedge}(\xi) w_n^{\pm}(x, \xi)\, d\xi, \quad x \in \Omega .$$

The lemma below indicates that $(W_{\pm}^* f)_n^{\wedge}(\xi)$, denoted from now on by $\hat{f}_n^{\pm}(\xi)$, is given by the expected inversion formula.

<u>Lemma 6.4</u>: Let $f \in \mathcal{H}^{ac}$. Then

$$\hat{f}_n^{\pm}(\xi) = \int_\Omega f(x)\, \overline{w_n^{\pm}(x, \xi)}\, dx \quad \text{a.e. } \xi \in R$$

<u>Proof</u>: Since $W_{\pm}^*$ is defined on all of $\mathcal{H}$, and since

$W_\pm^* W_\pm = I$ on $\mathcal{H}_0$, theorem 2.4 implies that

$$W_\pm^* f(x) = \sum_{n=1}^{\infty} \int_R (W_\pm^*)_n^\wedge(\xi) w_n^\circ(x, \xi) d\xi, \quad x \in S ,$$

holds for every $f \in \mathcal{H}$. Thus, the statement of this lemma is equivalent to

$$(W_\pm^* f)_n^\wedge(\xi) = \int_\Omega f(x) \overline{w_n^\pm(x, \xi)} dx \quad \text{a.e. } \xi \in R$$

for all $f \in \mathcal{H}$.

Let $f \in C_0^\infty(\Omega)$. Let $h = \{h_n\} \in \mathcal{E}$ such that all but one component of h are zero, and for that component, $h_m \in C_0^\infty(R)$. Then

$$\langle W_\pm T_0^* h, f \rangle_{\mathcal{H}} = \int_\Omega \int_R \overline{h_m(\xi)} \, \overline{w_m^\pm(x, \xi)} d\xi \, f(x) dx$$

$$= \int_R \overline{h_m(\xi)} \int_\Omega f(x) \overline{w_m^\pm(x, \xi)} dx \, d\xi$$

$$= \langle T_0^* h, W_\pm^* f \rangle_{\mathcal{H}_0} = \langle h, T_0 W_\pm^* f \rangle_{\mathcal{E}} = \int_R \overline{h_m(\xi)} \, (W_\pm^* f)_m^\wedge(\xi) d\xi$$

Since this holds for all $h_m \in C_0^\infty(R)$, it follows that $(W_\pm^* f)_n^\wedge(\xi) = \int_\Omega f(x) \overline{w_m^\pm(x, \xi)} dx$ a.e. $\xi \in R$. □

Since two eigenfunction expansions exist for each $f \in \mathcal{H}^{ac}$, the theorems in §2 can be repeated now for functions in $\mathcal{H}^{ac}$. Thus, the resolvent of H in $\mathcal{H}^{ac}$ is given by

$$R_\zeta u(x) = \sum_{n=1}^{\infty} \int_R \frac{\hat{u}_n^\pm(\xi)}{\xi^2 + k_n^2 - \zeta} w_n^\pm(x, \xi) d\xi, \quad x \in \Omega$$

for $u \in \mathcal{H}^{ac}$, $\zeta \in \rho(H)$. If $\lambda \in R$, then

$$\pi(\lambda)f(x) = \sum_{n=1}^{\infty} \int_{\xi^2+k_n^2<\lambda} \hat{f}_n^{\pm}(\xi)\, w_n^{\pm}(x, \xi)d\xi, \quad x \in \Omega$$

for any $f \in \mathcal{H}^{ac}$. The spectral multiplicity function $m_{ac}(\lambda)$ for H^{ac} is piecewise constant, has the value 0 for $\lambda < k_1^2$ and the value $2n$ for $k_n^2 \leq \lambda < k_{n+1}^2$. (This is already known from §4).

Now that the wave operators have been constructed, consider the scattering operator S defined by (0.3).

Lemma 6.5: Let $f \in \mathcal{H}_0$ satisfy

$$f(x) = \int_R \hat{f}_n(\xi)\, w_n^{\circ}(x, \xi)d\xi, \quad x \in S$$

for some n . Then

(6.7) $$Sf(x) = \sum_{k=1}^{\infty} \int_R \{\int_\Omega [\int_R \hat{f}_n(\sigma)\, w_n^-(y, \sigma)d\sigma]\overline{w_k^+(y, \xi)}dy\; w_k^{\circ}(x, \xi)d\xi$$

$x \in S$.

Proof: It follows from theorem 6.2 that

(6.8) $$W_-f(x) = \int_R \hat{f}_n(\sigma)w_n^-(x, \sigma)d\sigma, \quad x \in \Omega ,$$

and

(6.9) $$W_+^*W_-f(x) = \sum_{k=1}^{\infty} \int_R \widehat{(W_-f)}_k^+(\xi)\, w_k^{\circ}(x, \xi)d\xi, \quad x \in S .$$

Lemma 6.2 and (6.8) imply

(6.10) $$\widehat{(W_-f)}_k^+(\xi) = \int_\Omega (W_-f)(y)\, \overline{w_k^+(y, \xi)}\, dy .$$

$$= \int_{\Omega} [\int_{R} \hat{f}_n(\sigma) w_n^-(y, \sigma) d\sigma] \overline{w_k^+(y, \xi)} dy .$$

Substituting (6.10) into (6.9) yields (6.7). □

Let $\mathcal{H}_0^{(n)} = \{f \in \mathcal{H}_0 : f(x) = \int_R \hat{f}_n(\xi) w_n(x, \xi) d\xi, \quad x \in S\}$ and $\mathcal{H}_{\pm}^{(n)} = \{f \in \mathcal{H} : f(x) = \int_R \hat{f}_n^{\pm}(\xi) w_n^{\pm}(x, \xi) d\xi, \ x \in \Omega\}$ for $n = 1, 2, \cdots$. It follows from theorems 2.3 and 2.4 that $\mathcal{H}_0^{(n)}$ is a reducing subspace for H_0 . Theorem 6.2 implies that $W_{\pm} \mathcal{H}_0^{(n)} = \mathcal{H}_{\pm}^{(n)}$. Since $W_{\pm}$ is an intertwining operator for H_0 and H , it follows that $\mathcal{H}_{\pm}^{(n)}$ are reducing subspaces for H .

Theorem 6.2 implies that if $f \in \mathcal{H}_-^{(n)}$, then, for large negative time, $e^{-itH} f$ behaves like $e^{-itH_0} W_-^* f$, where $W_-^* f \in \mathcal{H}_0^{(n)}$. However, for large positive times, $e^{-itH} f$ behaves like $e^{-itH_0} S W_-^* f$, and $S W_-^* f$ has components in each of the subspaces $\mathcal{H}_0^{(m)}$, $m = 1, 2, \cdots$. Thus, a function in $\mathcal{H}$ which is in the 'channel' associated with the reducing subspace $\mathcal{H}_0^{(n)}$ at large negative times is scattered at large positive times into all the channels associated with the subspaces $\mathcal{H}_0^{(m)}$, $m = 1, 2, \cdots$. Using lemma 6.5, a channel scattering operator S_{nm} can be defined which maps functions in the n-th channel $\mathcal{H}_0^{(n)}$ in $\mathcal{H}_0$ into the component of the scattered functions in the m-th channel $\mathcal{H}_0^{(m)}$.

<u>Corollary 6.4</u>: If $f \in \mathcal{H}_0$, then

$$S_{nm} f(x) = \int_R \{\int_{\Omega} [\int_R \hat{f}_n(\sigma) w_n^-(y, \sigma) d\sigma] \overline{w_m^+(y, \xi)} dy\} w_m^{\circ}(x, \xi) d\xi$$

$x \in \Omega$.

This problem, then, can be thought of as a multichannel scattering problem. Associated with the operator H_0 are a set of channels, or reducing subspaces, $\mathcal{H}_0^{(n)}$, which are complete in the sense that the direct sum of the $\mathcal{H}_0^{(n)}$'s is $\mathcal{H}_0 = \pi_0^{ac}\mathcal{H}$, and are orthogonal. Two sets of channels are associated with the operator H. One set $\{\mathcal{H}_-^{(n)} : n = 1, 2, \cdots\}$ corresponds to the channels in $\mathcal{H}_0$ at large negative times, and the other set $\{\mathcal{H}_+^{(n)} : n = 1, 2, \cdots\}$ corresponds to the channels in $\mathcal{H}_0$ at large positive times. The scattering problem is as follows: suppose at large negative time the H-system is in the channel $\mathcal{H}_-^{(n)}$, which corresponds to the channel $\mathcal{H}_0^{(n)}$ in the H_0-system. What will be the components of the state vector of the system at large positive times in the channel $\mathcal{H}_+^{(m)}$?

The multichannel theory of scattering has not been developed mathematically as completely as has the single-channel theory. For a more detailed formulation of a multichannel scattering theory, see Jauch [12].

Appendix: Proof of the Limiting Abosrption Principle

In this section, the limiting absorption principle (theorem 5.6) is proved. Refer to §5 for the definitions of K and $u_n^{\pm}(\cdot)$.

Lemma A.1 Let $u \in K$ satisfy

$$-\Delta u(x) = \zeta u(x), \quad x \in \Omega - \Omega_c$$

for some $\zeta \in C^+ \cup R$ $(\zeta \in C^- \cup R)$ and some $c > h$. Then there are constants $C_n^{\pm}$ and $D_n^{\pm}$ for $n = 1, 2, \cdots$, such that

$$\text{(A.1)} \qquad u_n^{\pm}(x_N) = C_n^{\pm} e^{-\sqrt{k_n^2-\zeta}\,|x_N|} + D_n^{\pm} e^{\sqrt{k_n^2-\zeta}\,|x_N|}, \quad \pm x_N > c ,$$

where $\mathrm{Re}\sqrt{k_n^2-\zeta} \geq 0$ and $\mathrm{Im}\sqrt{k_n^2-\zeta} \leq 0$ $(\mathrm{Im}\sqrt{k_n^2-\zeta} \geq 0)$.

Proof: It follows as in the proof of lemma 5.3 that $u_n^{\pm}(x_N)$ satisfy

$$\frac{d^2u_n^{\pm}}{dx_N^2}(x_N) = (k_n^2 - \zeta)u_n^{\pm}(x_N), \quad \pm x_N > c .$$

Thus, there are constants $C_n^{\pm}$ and $D_n^{\pm}$ such that (A.1) holds. □

Lemma A.2 Let $f \in \mathcal{H}$ be supported in Ω_r for some $r > h$. Then, if $\zeta \in \rho(H)$, there are constants $C_{n,\zeta}^{\pm}$ for $n = 1, 2, \cdots$, such that

$$\text{(A.2)} \qquad (R_\zeta f)_n^{\pm}(x_N) = C_{n,\zeta}^{\pm} e^{-\sqrt{k_n^2-\zeta}\,|x_N|}, \quad \pm x_N > r ,$$

where $\mathrm{Re}\sqrt{k_n^2-\zeta} > 0$.

Proof: Note that $R_\zeta f \in D(H) \subset K$, so that $(R_\zeta f)_n^{\pm}(x_N)$ is defined. It follows from lemma A.1 that $(R_\zeta f)_n^{\pm}(x_N)$ has the form (A.1) for $\pm x_N > r$. Since $R_\zeta f \in \mathcal{H}$, it follows that $(R_\zeta f)_n^{\pm}(x_N)$ must die out as $x_N \to \pm\infty$. Thus, $D_n^{\pm} = 0$ for all n . □

Definition A.3: The function $u \in K$ satisfies the incoming (outgoing) radiation condition in Ω iff there are constants $C_n^{\pm}$ for $n = 1, 2, \cdots$, an $r > h$ and a $\lambda \in R$ such that

$$(A.3) \qquad u_n^{\pm}(x_N) = C_n^{\pm} e^{-\sqrt{k_n^2-\lambda}\,|x_N|}, \quad \pm x_N > r ,$$

where $\mathrm{Re}\sqrt{k_n^2-\lambda} \geq 0$ and $\mathrm{Im}\sqrt{k_n^2-\lambda} \leq 0$ $(\mathrm{Im}\sqrt{k_n^2-\lambda} \geq 0)$. □

(The terms 'incoming' and 'outgoing' are arbitrary in this problem. They have no corresponding physical interpretation.)

Note that in lemma A.2, if $\zeta \in C^+$, then $\mathrm{Im}\sqrt{k_n^2-\zeta} < 0$, and if $\zeta \in C^-$, then $\mathrm{Im}\sqrt{k_n^2-\zeta} > 0$.

Lemma A.4: If $u \in K$ satisfies

$$(A.4) \qquad -\Delta u = \lambda u \quad \text{in } \Omega$$

for some $\lambda \in \Lambda$, and if u also satisfies the incoming (outgoing) radiation condition in Ω , then $u = 0$.

Proof: The theorem is proved by showing that the assumptions on u imply that $u \in \mathcal{H}$. Since $\lambda \in \Lambda$, and u satisfies $Hu = \lambda u$, u

must be zero. Assume u satisfies the incoming radiation condition (the proof for the outgoing radiation condition is the same). Then, for some $r > h$, u satisfies (A.3), where λ is given by (A.4). Since $u \in K$, it follows that u has an expansion (5.2) with $u_n^{\pm}(x_N)$ given by (A.3). Then

(A.5) $$\frac{\partial u}{\partial x_N}(x) = \mp \sum_{n=1}^{\infty} C_n^{\pm} \sqrt{k_n^2-\lambda}\, e^{-\sqrt{k_n^2-\lambda}\,|x_N|}\, \eta_n(\tilde{x}), \quad x \in \Omega,\ \pm x_N > r .$$

Since u satisfies (A.4),

(A.6)
$$\begin{aligned} 0 &= \int_{\Omega_{r+1}} (\lambda u(x)\overline{u(x)} - u(x)\overline{\lambda u(x)})\,dx \\ &= \int_{\Omega_{r+1}} (-\Delta u(x))\overline{u(x)} - u(x)(\overline{-\Delta u(x)})\}dx \\ &= \int_{\partial\Omega_{r+1}} \sum_{j=1}^{N} \left(-\frac{\partial u}{\partial x_j}\bar{u} + u\frac{\partial \bar{u}}{\partial x_j}\right) n_j\,dx \end{aligned}$$

$$= \int_G \left\{-\frac{\partial u(\tilde{x},r+1)}{\partial x_N}\,\bar{u}(\tilde{x},r+1) + u(\tilde{x},r+1)\,\frac{\partial \bar{u}(\tilde{x},r+1)}{\partial x_N}\right\} d\tilde{x}$$

$$- \int_G \left\{-\frac{\partial u(\tilde{x},-r-1)}{\partial x_N}\,\bar{u}(\tilde{x},-r-1) + u(\tilde{x},-r-1)\,\frac{\partial \bar{u}(\tilde{x},-r-1)}{\partial x_N}\right\} d\tilde{x}$$

where n_j is the j-th component of the normal to $\partial\Omega_{r+1}$. Substituting (5.2) and (A.5) in (A.6) yields

$$0 = \sum_{n=1}^{\infty} (|C_n^+|^2 + |C_n^-|^2)(\sqrt{k_n^2-\lambda} - \overline{\sqrt{k_n^2-\lambda}})\, e^{-2\mathrm{Re}\sqrt{k_n^2-\lambda}\,(r+1)}$$

$$= 2 \sum_{\substack{n=1 \\ k_n^2<\lambda}}^{\infty} (|C_n^+|^2 + |C_n^-|^2)\sqrt{k_n^2-\lambda} ,$$

since $\sqrt{k_n^2-\lambda} = \overline{\sqrt{k_n^2-\lambda}}$ when $\lambda \le k_n^2$, and $\sqrt{k_n^2-\lambda} = -\overline{\sqrt{k_n^2-\lambda}}$ when $k_n^2 < \lambda$. Since $\sqrt{k_n^2-\lambda} = i\ \mathrm{Im}\sqrt{k_n^2-\lambda}$ and $\mathrm{Im}\sqrt{k_n^2-\lambda} < 0$ for all n such that $k_n^2 < \lambda$, it follows that $C_n^+ = C_n^- = 0$ if $k_n^2 < \lambda$. Thus,

$$u(x) = \sum_{\substack{n=1 \\ k_n^2>\lambda}}^{\infty} C_n^{\pm} e^{-\sqrt{k_n^2-\lambda}\,|x_N|} \eta_n(\tilde{x}),\ x \in \ ,\ \pm x_N > r\ .$$

But then $u(x)$ dies out exponentially as $x_N \to \pm\infty$. Hence, $u \in L_2(\Omega) = \mathcal{H}$. □

It is interesting to note that lemma A.4 plays an essential role in the proof of the limiting absorption principle. This is another of the many instances where uniqueness of a limit implies existence of the limit.

Lemma A.5: Let $[a,b] \subset \Lambda$ be a bounded interval, $\sigma_0 > 0$, and $r > h$. Then, for each $r' > 0$, there is a constant $M_{r'}$ such that

$$\| R_{\lambda+i\sigma} f \|_{2,\Omega_{r'}} \le M_{r'} \| f \|_{\mathcal{H}}$$

for all $\lambda \in [a,b]$, $0 < \sigma < \sigma_0$, and $f \in \mathcal{H}$ with support in Ω_r .

Proof: It is clearly sufficient to consider only $r' > r$ and $f \in \mathcal{H}$ with $\| f \|_{\mathcal{H}} = 1$. Suppose the lemma is false. Then there are sequences $\{\lambda_n\}$ in $[a,b]$, $\{\sigma_n\}$ in $(0, \sigma_0)$, and $\{f_n\}$ in $\mathcal{H}$ with $\| f_n \| = 1$, such that, denoting $\lambda_n + i\sigma_n$ by ζ_n ,

$$\| R_{\zeta_n} f_n \|_{2,\Omega_{r'}} \ge n\ .$$

Assume that ζ_n converges (this may be done by taking a subsequence). Since R_ζ is analytic in C^+, ζ_n must converge to some $\lambda \in [a,b]$. Note that theorem 1.8 implies that there is a constant K such that

$$\|R_{\zeta_n} f_n\|_{2,\Omega r'} \leq K\{\|R_{\zeta_n} f_n\|_{L_2(\Omega_{r'+1})} + \|-\Delta R_{\zeta_n} f_n\|_{L_2(\Omega_{r'+1})}\}$$

$$\leq (1 + |\zeta_n|) K\|R_{\zeta_n} f_n\|_{L_2(\Omega_{r'+1})} + K\|f_n\|_{L_2(\Omega_{r'+1})} \quad \text{for all } n .$$

Let

$$u_n = \frac{1}{\|R_{\zeta_n} f_n\|_{2,\Omega_{r'}}} R_{\zeta_n} f_n, \text{ and } F_n = \frac{1}{\|R_{\zeta_n} f_n\|_{2,\Omega_{r'}}} f_n .$$

Then $\|u_n\|_{2,\Omega_{r'}} = 1$ and $\|F_n\|_{\mathcal{H}} \to 0$ as $h \to \infty$. It follows by Rellich's compactness theorem that there is a subsequence (again denoted by $\{u_n\}$) which is Cauchy in $L_2(\Omega_{h'})$, where $r < h' < r'$. From lemma A.2,

$$u_n(x) = \sum_{m=1}^{\infty} C^{\pm}_{m,\zeta_n} e^{-\sqrt{k_m^2-\zeta_n}|x_N|} \eta_m(\tilde{x}), \quad x \in \Omega, \pm x_N > r ,$$

where $\mathrm{Re}\sqrt{k_m^2-\zeta_n} > 0$ and $\mathrm{Im}\sqrt{k_m^2-\zeta_n} \leq 0$. Using a calculation similar to (5.3), it is seen that

$$\text{(A.7)} \qquad \|u_n\|_{L_2(\Omega_{h'+t}-\Omega_{r+t})} \leq \|u_n\|_{L_2(\Omega_{h'}-\Omega_{r'})}, \quad \text{for } t > 0 .$$

Since $\{u_n\}$ converges in $L_2(\Omega_{h'})$, it follows from (A.7) that $\{u_n\}$ converges in $L_2(\Omega_c)$ for all $c > h$. Since $-\Delta u_n = \zeta_n u_n + F_n$, it follows that $\{-\Delta u_n\}$ is also Cauchy in $L_2(\Omega_c)$ for all $c > h$.

Thus, from theorem 1.8, it follows that $\{u_n\}$ is Cauchy in $H_2^{loc}(\Omega)$; hence, $u_n \to u \in K$ as $n \to \infty$. Since $-\Delta u_n = \zeta_n u_n + F_n$, and $||F_n||_{\mathcal{H}} \to 0$ as $n \to \infty$, it follows that $-\Delta u = \lambda u$. Lemma A.1 implies that $(u)_m^{\pm}(x_N)$ has the form (A.1). Since $u_n \to u$ in $H_2^{loc}(\Omega)$, it follows that $(u_n)_m^{\pm}(x_N) \to (u)_m^{\pm}(x_N)$ in $L_2^{loc}(R)$. Thus, $D_m^{\pm} = 0\ \forall m$ and $(u)_m^{\pm}(x_N) = C_m^{\pm} e^{-\sqrt{k_n^2-\lambda}|x_N|}$ for $\pm x_N > r$ where $\mathrm{Re}\sqrt{k_n^2-\lambda} \geq 0$ and $\mathrm{Im}\sqrt{k_n^2-\lambda} \leq 0$. This means that u satisfies the incoming radiation condition , and $-\Delta u = \lambda u$ in Ω . By lemma A.4, $u = 0$. But $u_n \to u$ in $H_2^{loc}(\Omega)$ and $||u_n||_{2,\Omega_r} = 1$ for all n . This is an obvious contradiction. □

Lemma A.6: Let $[a,b] \subset \Lambda$ be a bounded interval, $\sigma_0 > 0$, and $r > h$. Then the mapping

$$(\lambda,\ \sigma,\ f) \to R_{\lambda+i\sigma} f$$

is a uniformly continuous mapping from $[a,b] \times (0,\ \sigma_0) \times$ unit ball in $L_2(\Omega_r)$ into K .

Proof: Suppose the lemma is false. Then for some $r' > r$ and some $\varepsilon > 0$, there exist sequences $\{\lambda_n\}$ and $\{\nu_n\}$ in $[a,b]$, $\{\sigma_n\}$ and $\{\tau_n\}$ in $(0,\ \sigma_0)$, and $\{f_n\}$ and $\{g_n\}$ in the unit ball of $L_2(\Omega_r)$ such that

(A.8) $\quad |\lambda_n - \nu_n| < 1/n,\ |\sigma_n - \tau_n| < 1/n,$ and $||f_n - g_n||_{L_2(\Omega_r)} < 1/n$,

(A.9) $$||R_{\lambda_n + i\sigma_n} f_n - R_{\mu_n + i\tau_n} g_n||_{2,\Omega_{r'}} \geq \varepsilon .$$

Assume that the sequences $\{\lambda_n\}$, $\{\nu_n\}$, $\{\sigma_n\}$, and $\{\tau_n\}$ converge (this may be done by taking subsequences). Denote $\lambda_n + i\sigma_n$ by ζ_n, $\nu_n + i\tau_n$ by μ_n. It follows from (A.8) that $\lambda_n \to \lambda$, $\nu_n \to \lambda$, $\sigma_n \to \sigma$ and $\tau_n \to \sigma$ as $n \to \infty$, where $\lambda \in [a,b]$, $\sigma \in [0,\sigma_0]$. As in lemma A.5, the analyticity of R_ζ in C^+ implies that $\sigma = 0$. Thus, $\zeta_n \to \lambda$ and $\mu_n \to \lambda$ as $n \to \infty$.

By lemma A.5, there is an M such that

$$||R_{\zeta_n} f_n||_{L_2(\Omega_{r'+1})} \leq M \quad \text{and} \quad ||R_{\mu_n} g_n||_{L_2(\Omega_{r'+1})} \leq M$$

for $n = 1, 2, \cdots$. As in the proof of lemma A.5, it can be shown that there are subsequences, denoted again by $\{\zeta_n\}$, $\{\mu_n\}$, $\{f_n\}$, and $\{g_n\}$, such that $\{R_{\zeta_n} f_n\}$ and $\{R_{\mu_n} g_n\}$ are Cauchy in $L_2(\Omega_r)$, and $R_{\zeta_n} f_n \to u_\lambda$ and $R_{\zeta_n} g_n \to v_\lambda$ in K, where u_λ and v_λ satisfy the incoming radiation condition. Since $-\Delta R_{\zeta_n} f_n = \zeta_n R_{\zeta_n} f_n + f_n$ and $-\Delta R_{\mu_n} g_n = \mu_n R_{\mu_n} g_n + g_n$, it follows that

$$-\Delta(R_{\zeta_n} f_n - R_{\mu_n} g_n) = \zeta_n R_{\zeta_n} f_n - \mu_n R_{\mu_n} g_n + f_n - g_n .$$

Since $||f_n - g_n||_{L_2(\Omega_r)} < 1/n$, it follows that $-\Delta(u_\lambda - v_\lambda) = \lambda(u_\lambda - v_\lambda)$. Thus, $u_\lambda - v_\lambda \in K$ satisfies the incoming radiation condition and $-\Delta(u_\lambda - v_\lambda) = \lambda(u_\lambda - v_\lambda)$. By lemma A.4, $u_\lambda - v_\lambda = 0$, or $u_\lambda = v_\lambda$. Thus, $R_{\zeta_n} f_n \to u_\lambda$ and $R_{\mu_n} g_n \to u_\lambda$ in K. But this contradicts (A.9). □

Lemma A.5 and A.6 can also be proved for $0 > \sigma > -\sigma_0$ with the outgoing radiation condition replacing the incoming radiation condition.

Completion of proof of theorem 5.6: Lemma A.6 implies that the mapping (5.4) is continuous from $C^+ \times L_2(\Omega_r)$ into K, and is uniformly continuous on $\{\lambda + i\sigma\colon a < \lambda < b,\ 0 < \sigma < \sigma_0\} \times$ unit ball of $L_2(\Omega_r)$, where $[a,b] \subset \Lambda$ and $\sigma_0 > 0$. Since K is complete (Frechet), it follows that this mapping can be extended continuously to $\{\lambda + i\sigma\colon a < \lambda < b,\ 0 \leq \sigma \leq \sigma_0\} \times$ unit ball in $L_2(\Omega_r)$ into K, for $[a,b] \subset \Lambda$ and $\sigma_0 > 0$. From the proofs of lemma A.5 and A.6, the boundary values of this mapping satisfy the incoming radiation condition and hence, by lemma A.4, are unique.

A similar argument proves that the mapping (5.5) on $C^- \times L_2(\Omega_r)$ can be extended to $\Lambda^- \times L_2(\Omega_r)$.

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