EARTHQUAKE RESPONSE OF BUILDING-FOUNDATION SYSTEMS

Thesis by

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To My Parents

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ABSTRACT

The influence of a deformable foundation on the response of buildings to earthquake motion is examined. The study is divided into two parts; the vibration of the base of the building on the foundation medium, and the response of the whole building-foundation system.

Studied first are the forced horizontal, rocking and vertical harmonic oscillations of a rigid disc bonded to an elastic half-space, which is considered as a mathematical model for the soil. The problem, formulated in terms of dual integral equations, is reduced to a system of Fredholm integral equations of the second kind. For the limiting static case these equations yield a closed form solution in agreement with that obtained by others.

Using the force-deflection relations for the base, the equations of motion of linear building-foundation systems are solved by both direct and transform methods. It is shown that, under assumptions which appear to be physically reasonable, the earthquake response of the interaction system reduces to the linear superposition of the responses of damped, linear one-degree-of-freedom oscillators subjected to modified excitations. This result is valid even for systems that do not possess classical normal modes. Explicit approximations in terms of the parameters of the system are obtained for the dynamic properties of the one-degree-of-freedom oscillator which is equivalent to a single-story building-foundation system. For multi-story buildings it is shown that the effect of an elastic foundation, as measured by the change in the natural frequencies of the building, is negligible for modes higher than the first for many types of building structures.

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INTRODUCTION

There are two aspects of the problem of building-foundation interaction during earthquakes which are of major significance to earthquake engineering. First, the response to earthquake motion of a structure founded on a deformable soil will not be the same as if the structure were supported on a rigid foundation. Second, the ground motion recorded on the base of the structure will be different from that which would have been recorded had there been no building. The practical importance of these effects depends on the properties of the soil-structure system. In terms of the dynamic properties of the system, this dynamic coupling, or interaction between a building and the surrounding soil, will generally have the effect of (1) reducing the fundamental frequency of the system from that of the structure on a rigid base, and (2) dissipating part of the vibrational energy of the building by wave radiation into the foundation medium. There will also be energy losses due to internal friction of the soil. Because of these effects, the response of a structure on a soft foundation to a given earthquake excitation will, in general, be different from that of the same structure supported on a rigid ground. It is the influence of a flexible foundation on the response of structures to earthquake motion that is the general subject of this thesis.

It is convenient to divide the studies of building-foundation interaction into two distinct parts; the first concerned exclusively with the vibration of the base of the building on the foundation medium, and the second dealing with the response of the entire system. The base of the building is idealized as a rigid circular plate and the soil is modeled by a homogeneous, isotropic, elastic half-space. Under these conditions, the essential features of the problem reduce to the forced vibrations of a massless disc bonded to an elastic half-space, which are studied in Chapter I. Of special interest is the relation between the forces applied to the rigid plate and the resulting displacements.

Once the force-deflection relation for the base is determined, the response of a linear building foundation system to a prescribed earthquake excitation can be evaluated. This is the subject of Chapter II. Using both direct and transform methods, it is shown that, under assumptions which appear to be physically reasonable, the response of the interaction system can be expressed as a linear combination of the responses of one-degree-of-freedom oscillators subjected to modified excitations. This result is shown to be valid even for systems that do not possess classical normal modes. The advantages of this representation include the physical insight it gives into the dynamics of the building-foundation system and the ease of calculations, which are equivalent to those for simple structures.

A summary of the main results obtained in this investigation is contained in Chapter III.

I. FORCED VIBRATIONS OF A RIGID DISC PERFECTLY BONDED TO AN ELASTIC HALF-SPACE

A. Introduction

The problem of forced oscillations of a rigid footing on an elastic half-space is a mixed boundary-value problem in which either the displacements, or certain displacements and tractions are prescribed under the footing, and the tractions are specified to be zero over the remainder of the surface of the elastic half-space. Two distinct classes of mixed boundary-value problems may be considered depending on the type of contact between the footing and the half-space. A complete mixed boundary value problem, for which all the components of the displacement under the footing are specified, occurs if the rigid footing is perfectly bonded to the free surface of the elastic half-space. Perfect bond, or adhesive contact, is defined as the type of attachment in which there is complete continuity between the displacements and stresses of the footing and the underlying half-space in the zone of contact. A relaxed mixed boundary-value problem results if it is assumed that at least one of the components of the surface traction under the footing is zero. To have a well-posed problem, a corresponding number of components of the displacement under the footing is left unconstrained. Thus, for vertical and rocking oscillations the contact may be assumed to be frictionless whereas for horizontal vibrations the normal component of the surface traction under the footing is taken to be zero. Correspondingly, for the first two problems the horizontal displacements under the disc are not prescribed, while for the horizontal oscillations the vertical displacements under the disc are left unconstrained.

Considerable attention has been given to the solution of the problem of forced oscillations of a rigid footing on an elastic half-space. Reissner [1], Quinlan[2], Sung[3], Arnold et al[4], Bycroft[5] and Thomson and Kobori[6] have approached this problem by assuming the dynamic stress distribution at the contact region to be either constant, linear, parabolic or proportional to the static stress distribution. Under these assumptions only stresses are specified and a mixed boundary-value problem does not arise. The relaxed mixed boundary value problem has been considered by a number of investigators. Robertson[7], Awojobi and Grootenhuis[8], Lysmer[9], Shah[10] and Luco and Westmann[11] solved the problem of a smooth rigid disc undergoing vertical oscillations. Zakorko and Rostovtsev[12] considered the cases of vertical and rocking oscillations while Gladwell[13], Luco and Westmann[11] and Veletsos and Wei[14] solved the cases of horizontal and rocking vibrations of the disc. The latter problem has also been considered by Awojobi[15]. Karasudhi, Keer and Lee[16] have treated the vertical, rocking and horizontal oscillations of a smooth rigid strip footing while Elorduy, Szekely and Nieto[17] considered the vertical and rocking vibrations of a smooth, rigid rectangular footing. Torsional oscillations of a rigid disc on an elastic half-space have been studied by Reissner and Sagoci[18], Ufliand[19], Collins[20], Robertson[21], Thomas[22] and Stallybrass[23]. This may be interpreted either as a relaxed or a complete mixed boundary-value problem because there is only one non-vanishing component of stress and displacement in cylindrical coordinates throughout the half-space. The complete mixed boundary-value problem for a strip footing has been examined by Luco[24], who studied the vertical, rocking and horizontal oscillation of a rigid strip perfectly bonded to the free surface of an elastic half-space. To date, only the static solution has been obtained for the complete boundary value problem for a rigid disc[25-29].

Both the relaxed and the complete mixed boundary-value problems can be formulated in terms of a system of dual integral equations. For circular and strip footings, a standard technique[30-33] has been used by which these equations are transformed with the aid of auxiliary functions into a system of Abel type integral equations whose solution in turn leads to a system of Fredholm integral equations of the second kind in the auxiliary functions. Quantities of interest may be calculated directly from the auxiliary functions.

In this investigation an analysis is made of the complete dynamic mixed boundary-value problem for a rigid disc on an elastic half-space. An extension of the method used in the solution of the relaxed problem will be employed to transform the corresponding system of coupled, dual integral equations into a system of Cauchy type singular integral equations in auxiliary functions, the solution of the dominant part of which results in a system of Fredholm integral equations of the second kind. Simplified forms of these equations are obtained for an incompressible material and for the particular case of the relaxed mixed boundary-value problem. The stresses and displacements on the surface of the half-space can be determined directly from the auxiliary functions.

The static solution of the complete mixed boundary value problem obtained by Mossakovski[25] and Ufliand[26] and later by Keer[27], Spence[28] and Gladwell[29] includes a factor of the form $\exp\,[ik\ln(\frac{1-r'}{1+r})]$ where r' is the radius and k is a constant. This frequency independent factor also occurs in the solution to the dynamic problem. Consequently, it is possible to obtain a new system of Fredholm integral equations of the second kind in terms of auxiliary functions, not involving singularities, which can be solved numerically for arbitrary values of the frequency of oscillation. For the limiting static case the terms containing integrals disappear, and therefore, an explicit solution may be obtained. This solution is in agreement with the solutions found by Mossakovski, Ufliand, Keer, Spence and Gladwell[25-29].

B. Formulation of the Problem

In the following analysis, a cylindrical polar coordinate system r, θ, z will be employed; the r-θ plane coincides with the half-space surface and the z - axis is directed into the half-space. The elastic, homogeneous, isotropic half-space is characterized by the density ρ, the shear modulus μ and Poisson's ratio σ (or equivalently by ρ and the Lame' constants μ and λ). No body forces are present in the system. The massless rigid disc of radius a is placed on the plane $z = 0$ with its center coinciding with the origin of the coordinate system. The motion of the disc is produced by the actions of a vertical force, a horizontal force and a moment, all with harmonic time dependence. The complete system and the applied forces are shown in Fig. 1. The problem is formulated within the scope of classical elastodynamics.

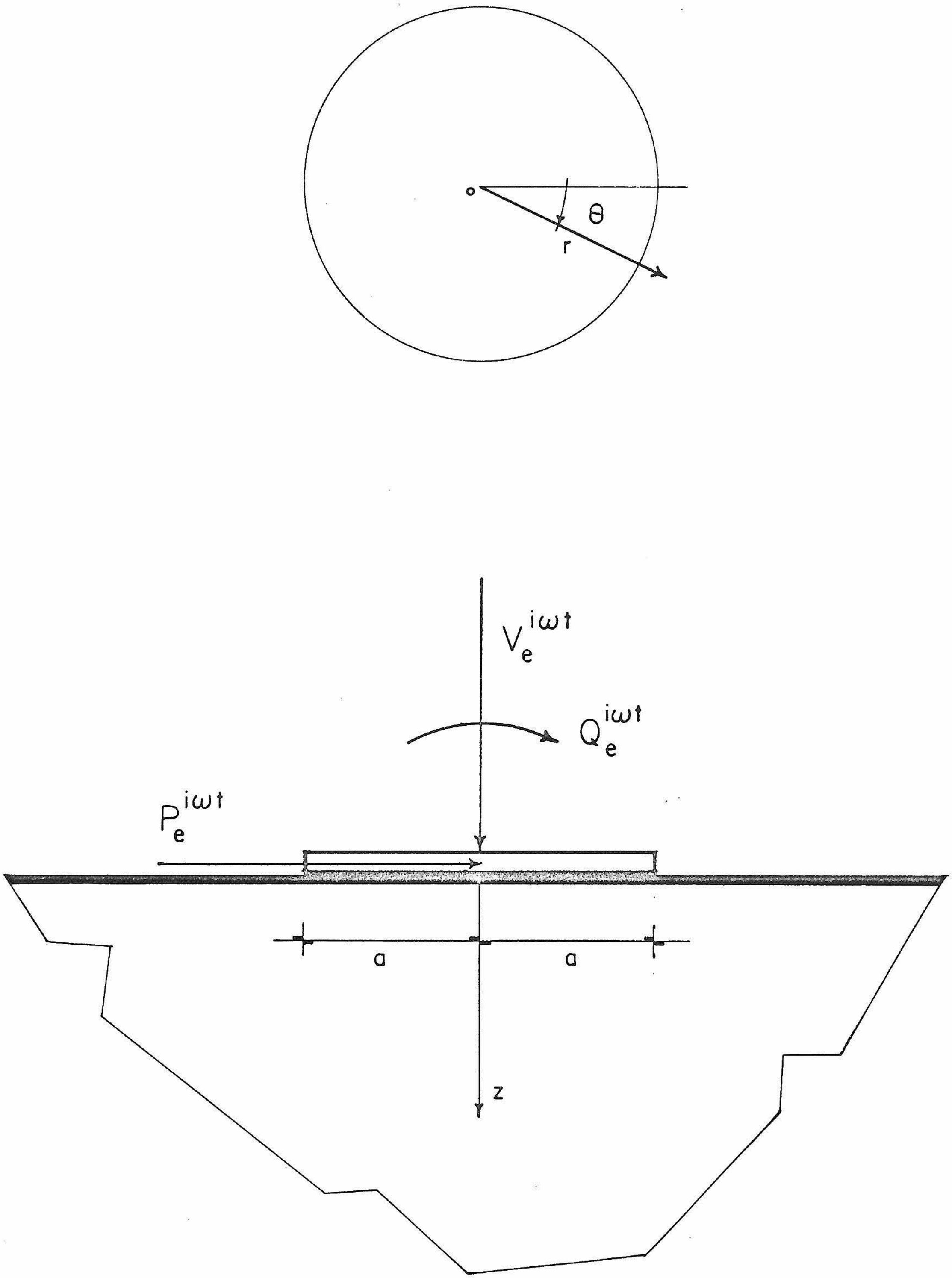


Fig. 1. Diagram of footing and coordinate system

For steady-state vibrations with circular frequency ω, the equations of motion that must be satisfied by the displacement vector $\underline{u}\exp(i\omega t) = (u_r, u_\theta, u_z)\exp(i\omega t)$ are given by[35]

$$(\lambda + 2\mu)\,\text{grad div}\,\underline{u} - \mu\,\text{curl curl}\,\underline{u} + \rho\omega^2\underline{u} = 0 \tag{2.1}$$

in their cylindrical polar coordinate formulation.

Suppose that the disc experiences vertical displacement $\Delta_v \exp(i\omega t)$, rotation $\Gamma_0 \exp(i\omega t)$ about the axis $\theta = \pi/2$ and horizontal motion $\Delta_h \exp(i\omega t)$ in the direction $\theta = 0$. Δ_v and Δ_h are the constant amplitudes of the vertical and horizontal displacements of the disc, respectively, and Γ_0 is the amplitude of the angle of rocking. The displacement boundary conditions are then

$$\left.\begin{aligned} u_r(ar', \theta, 0) &= \Delta_h \cos\theta && (2.2a)\\ u_\theta(ar', \theta, 0) &= -\Delta_h \sin\theta && (2.2b)\\ u_z(ar', \theta, 0) &= \Delta_v + a\Gamma_0 r' \cos\theta && (2.2c) \end{aligned}\right\}\quad 0 \le r' \le 1$$

where $r' = r/a$. The corresponding stresses σ_{zr}, $\sigma_{z\theta}$, σ_{zz} must satisfy

$$\left.\begin{aligned} \sigma_{zz}(ar', \theta, 0)\ e^{i\omega t} &= 0 \qquad (2.2d)\\ \sigma_{z\theta}(ar', \theta, 0)\ e^{i\omega t} &= 0 \qquad (2.2e)\\ \sigma_{zz}(ar', \theta, 0)\ e^{i\omega t} &= 0 \qquad (2.2f) \end{aligned}\right\} \quad 1 < r' < \infty$$

In addition, equilibrium of the massless disc requires that the external forces balance the forces resulting from the surface tractions acting in the zone of contact.

Finally, since the vibrations are generated in a limited zone of the boundary, it is required that only outgoing waves be present at infinity.

The vertical harmonic force $V \exp(i\omega t)$ applied to the disc shown in Fig. 1 results only in vertical harmonic vibration of the disc, whereas the horizontal force $P \exp(i\omega t)$ produces a harmonic rocking motion of the disc in addition to the horizontal displacement, and conversely, the rocking moment $Q \exp(i\omega t)$ produces a horizontal oscillation of the disc in addition to the rotational vibration. For this reason, it is convenient to study the vertical vibration of the disc separately from the coupled horizontal and rocking oscillations.

The coupled horizontal and rocking oscillations of the disc are considered in Section C. The boundary conditions for this problem are given by Eqs. (2.2) with $\Delta_v = 0$. The vertical vibration of the disc is studied in Section D and the corresponding boundary conditions are obtained by setting Δ_h and Γ_0 equal to zero in Eqs. (2.2).

C. Forced Horizontal and Rocking Oscillations of the Disc

1. Derivation of the System of Coupled Dual Integral Equations

Bycroft[5], following Sezawa[36], has shown that the equations of motion (2.1) have a particular solution of the form

$$u_r(ar', \theta, az') = au_r^*(r', z') \cos \theta \tag{2.3a}$$

$$u_\theta(ar', \theta, az') = au_\theta^*(r', z') \sin \theta \tag{2.3b}$$

$$u_z(ar', \theta, az') = au_z^*(r', z') \cos \theta \tag{2.3c}$$

where $z' = z/a$. In this solution

$$u_r^*(r', z') + u_\theta^*(r', z') = -2\int_0^\infty [k\, F_1(k, z') - \frac{1}{v_2} C(k)e^{-v_2 z'}] \cdot J_2(kr')\,dk \tag{2.3d}$$

$$u_r^*(r', z') - u_\theta^*(r'z') = 2\int_0^\infty [kF_1(k, z') + \frac{1}{v_2} C(k)e^{-v_2 z'}] \cdot J_0(kr')\,dk \tag{2.3e}$$

$$u_z^*(r', z') = 2\int_0^\infty F_2(k, z') J_1(kr')\,dk \tag{2.3f}$$

where $F_1(k, z')$ and $F_2(k, z')$ are given by

$$F_1(k, z') = -A(k)e^{-v_1 z'} + v_2 B(k)e^{-v_2 z'} \tag{2.3g}$$

$$F_2(k, z') = v_1 A(k)e^{-v_1 z'} - k^2 B(k)e^{-v_2 z'} \tag{2.3h}$$

and

$$v_1 = (k^2 - \gamma^2 a_0^2)^{\frac{1}{2}} \tag{2.3i}$$

$$v_2 = (k^2 - a_0^2)^{\frac{1}{2}} \quad . \tag{2.3j}$$

In these equations $a_0 = \omega a (\rho/\mu)^{\frac{1}{2}}$ is a dimensionless frequency, and $\gamma = [(1-2\sigma)/2(1-\sigma)]^{\frac{1}{2}}$ is the ratio of the equivoluminal (or shear) wave velocity to the dilatational wave velocity in the half-space material. It is required that Re $v_1, v_2 \geqq 0$ for the displacements at infinity to remain bounded. The unknown functions A(k), B(k) and C(k) are determined by the boundary conditions.

The corresponding stresses $\sigma_{zr}, \sigma_{z\theta}, \sigma_{zz}$ are

$$\sigma_{zr}(ar', \theta, az')e^{i\omega t} = \mu \sigma_{zr}^*(r', z') \cos\theta \, e^{i\omega t} \tag{2.4a}$$

$$\sigma_{z\theta}(ar', \theta, az')e^{i\omega t} = \mu \sigma_{z\theta}^*(r', z') \sin\theta \, e^{i\omega t} \tag{2.4b}$$

$$\sigma_{zz}(ar', \theta, az')e^{i\omega t} = \mu \sigma_{zz}^*(r', z') \cos\theta \, e^{i\omega t} \tag{2.4c}$$

in which σ_{zr}^*, $\sigma_{z\theta}^*$, σ_{zz}^* are given by

$$\sigma_{zr}^*(r', z') + \sigma_{z\theta}^*(r', z') = 2 \int_0^\infty [k F_3(k, z') - C(k) e^{-v_2 z'}] J_2(kr') dk \tag{2.4d}$$

$$\sigma_{zr}^*(r', z') - \sigma_{z\theta}^*(r', z') = -2 \int_0^\infty [k F_3(k, z') + C(k) e^{-v_2 z'}] J_0(kr') dk \tag{2.4e}$$

$$\sigma_{zz}^{*}(r', z') = 2\int_0^\infty F_4(k, z')\, J_1\,(kr')\,dk \tag{2.4f}$$

and the functions F_3 and F_4 are

$$F_3(k, z') = -2v_1 A(k)e^{-v_1 z'} + (2k^2 - a_0^2)\, B(k)e^{-v_2 z'} \tag{2.4g}$$

$$F_4(k, z') = -(2k^2 - a_0^2)\, A(k)e^{-v_1 z'} + 2v_2 B(k)e^{-v_2 z'}\ . \tag{2.4h}$$

Applying the equations for the displacements and stresses given above it is found that the boundary conditions (2.2) will be satisfied provided that

$$\left.\begin{aligned}
&\lim_{z'\to 0} \int_0^\infty [kF_1(k, z') + \frac{1}{v_2} C(k)e^{-v_2 z'}]\, J_0(kr')dk = \frac{\Delta h}{a} && (2.5a)\\
&\lim_{z'\to 0} \int_0^\infty [kF_1(k, z') - \frac{1}{v_2} C(k)e^{-v_2 z'}]\, J_2\,(kr')dk = 0 && (2.5b)\\
&\lim_{z'\to 0} \int_0^\infty F_2\,(k, z')\, J_1(kr')\,dk = \tfrac{1}{2}\Gamma_0\, r' && (2.5c)
\end{aligned}\right\}\ 0 \le r \le 1$$

$$\left.\begin{aligned}
&\lim_{z'\to 0} \int_0^\infty [kF_3(k, z') + C(k)e^{-v_2 z'}]\, J_0(kr')\,dk = 0 && (2.5d)\\
&\lim_{z'\to 0} \int_0^\infty [kF_3(k, z') - C(k)e^{-v_2 z'}]\, J_2(kr')\, dk = 0 && (2.5e)\\
&\lim_{z'\to 0} \int_0^\infty F_4(k, z')\, J_1(kr')\,dk = 0 && (2.5f)
\end{aligned}\right\}\ 1 < r' < \infty\ .$$

The boundary conditions have been stated as limits so that certain integrals that will appear later in the analysis remain bounded.

Equations (2.5) may be transformed into a system of dual integral equations suitable for subsequent analysis by replacing the functions $A(k)$, $B(k)$ in Eqs. (2.4g) and (2.4h) by the new functions $D(k)$, $E(k)$ defined by

$$-(2k^2 - a_0^2)\,A(k) + 2v_2k^2\,B(k) = D(k) \qquad (2.6a)$$

$$-2v_1\,A(k) + (2k^2 - a_0^2)\,B(k) = k^{-1}E(k) \qquad (2.6b)$$

and by the introduction of the functions $H_1(k)$, $H_2(k)$, $H_3(k)$, $H_4(k)$ which are defined, following Robertson [7], by

$$H_1(k) = \frac{-a_0^2\,v_2\,k}{(1-\sigma)[(2k^2-a_0^2)^2 - 4v_1v_2k^2]} - 1 \qquad (2.7a)$$

$$H_2(k) = \frac{2v_1v_2k^2 - (2k^2-a_0)k^2}{(1-\sigma)[(2k^2-a_0^2)^2 - 4v_1v_2\,k^2]} - \gamma \qquad (2.7b)$$

$$H_3(k) = \frac{1}{1-\sigma}\left(\frac{k}{v_2} - 1\right) \qquad (2.7c)$$

$$H_4(k) = \frac{-\,a_0^2\,v_1k}{(1-\sigma)[(2k^2-a_0^2)^2 - 4v_1v_2k^2]} - 1\ . \qquad (2.7d)$$

These functions have the property that they all tend to zero as a_0 goes to zero. Then, in the static case $H_i(k) = 0$, $i = 1, 2, 3, 4$.

After substituting, reordering and taking limits inside the integrals, with the exception of those indicated below, Eqs. (2.5) become

$$\left.\begin{aligned}
&\int_0^\infty k^{-1}E(k)\,J_0(kr')\,dk = f_0(r') &\quad (2.8a)\\
&\int_0^\infty k^{-1}\,C(k)\,J_2(kr')dk = f_2(r') &\quad (2.8b)\\
&\int_0^\infty k^{-1}\,D(k)\,J_1\,(kr')\,dk = f_1\,(r') &\quad (2.8c)
\end{aligned}\right\}\quad 0 \le r' \le 1$$

$$\left.\begin{aligned}
&\int_0^\infty [E(k) + C(k)]\,J_0\,(kr')\,dk = 0 &\quad (2.8d)\\
&\int_0^\infty [E(k) - C(k)]\,J_2\,(kr')\,dk = 0 &\quad (2.8e)\\
&\int_0^\infty D(k)\,J_1\,(kr')\,dk = 0 &\quad (2.8f)
\end{aligned}\right\}\quad 1 < r' < \infty$$

where $f_0(r')$, $f_1(r')$, $f_2(r')$ are defined by

$$f_0(r') = \frac{\Delta_h}{a(1-\sigma)} + \gamma \lim_{z'\to 0} \int_0^\infty e^{-kz'} k^{-1}\,D(k)J_0(kr')\,dk - \int_0^\infty k^{-1}\left\{H_1(k)\,E(k) - H_2(k)\,D(k) + [1+H_3(k)]\,C(k)\right\} J_0(kr')dk \qquad (2.8g)$$

$$f_1(r') = -\frac{\Gamma_0 r'}{2(1-\sigma)} + \gamma \lim_{z'\to 0} \int_0^\infty e^{-kz'} k^{-1}\,E(k)\,J_1\,(kr')\,dk + \int_0^\infty k^{-1}\,[-H_4(k)\,D(k) + H_2(k)\,E(k)]\,J_1(kr')\,dk \qquad (2.8h)$$

$$f_2(r') = (1-\sigma)\Big\{ -\gamma \lim_{z'\to 0} \int_0^\infty e^{-kz'} D(k) J_2(kr')\,dk \; +$$

$$\int_0^\infty k^{-1} \Big\{[1+H_1(k)]\,E(k) - H_2(k)D(k) - H_3(k)\,C(k)\Big\} J_2(kr')dk\Big\}. \quad (2.8i)$$

Equations (2.8a-f) are a system of three coupled, dual integral equations in the unknowns $C(k)$, $D(k)$ and $E(k)$ whose solution is the subject of the following section.

2. Reduction of the Three Simultaneous Pairs of Integral Equations to a System of Fredholm Integral Equations of the Second Kind

Dual integral equations such as those appearing in (2.8a-f) have been treated extensively in the literature. Those involving only one pair were first discussed systematically by Titchmarsh[37] and also by Busbridge[38], Copson[31], Sneddon[32] and Noble[39] among others. The problem of solving a system with an arbitrary (but finite) number of simultaneous dual integral equations of the same type has been considered by Erdogan and Bahar[40]. They reduced the problem to the solution of an infinite set of linear algebraic equations. The special case of two pairs of dual integral equations for which the order of the Bessel functions between one pair and the other differs by two was discussed by Westmann[34]. He was able to find a closed form solution to the problem by using a linear combination of the auxiliary functions introduced by Copson[31]. The problem of two pairs of integral equations with Hankel kernels of order 0 and 1 and prescribed right hand sides has been considered by Erdogan[41] and for a special case by Spence[28]. Erdogan transformed the problem into the solution of two simultaneous Cauchy type singular integral equations, the dominant part of which could be solved exactly. Spence reduced the problem to the solution of a singular Fredholm integral equation of the second kind which could be solved exactly by the Wiener-Hopf technique if the right hand sides of the dual integral equations were polynomials. Gladwell[29], extending the work of Spence, solved a system of three dual integral equations, equivalent to Eqs. (2.8) with $H_i(k) = 0$, $i = 1, 2, 3, 4$. However, because of its restriction to polynomials, Spence's method or the extended version derived by Gladwell cannot be used to solve Eqs. (2.8) when $H_i(k) \neq 0$. Instead, an extension of the method used by Copson[31] and Westmann[34] will be employed. Assuming a special form of the solution in terms of auxiliary functions, the system of equations (2.8) will be transformed formally into a system of Cauchy type singular integral equations in the auxiliary functions. A system of Fredholm integral equations of the second kind will then be obtained by solving the dominant part of these equations. For the static problem, the resulting Fredholm equations reduce to simple expressions from which a complete closed form solution may be obtained.

The solution of Eqs. (2.8) is assumed to be of the form

$$E(k) = \sqrt{\frac{2}{\pi}}\, k^{3/2} \int_0^1 x^{\frac{1}{2}}\, \varphi_0(x)\, J_{-\frac{1}{2}}(kx)dx \tag{2.9a}$$

$$C(k) = C_0\Big[\sqrt{\frac{2}{\pi}}\, k^{3/2} \int_0^1 x^{\frac{1}{2}}\, J_{-\frac{1}{2}}(kx)\, dx \; + \\ \sqrt{\frac{2}{\pi}}\, k^{3/2} \int_0^1 x^{3/2} \varphi_2(x)\, J_{3/2}(kx)\, dx\Big] \tag{2.9b}$$

$$D(k) = \sqrt{\frac{2}{\pi}}\, k^{3/2} \int_0^1 x^{\frac{1}{2}} \varphi_1(x)\, J_{\frac{1}{2}}(kx)\, dx \tag{2.9c}$$

where $\varphi_0(x)$, $\varphi_1(x)$, $\varphi_2(x)$ are the unknown auxiliary functions and C_0 is an unknown constant. All these quantities may depend on Poisson's ratio and the frequency of oscillation. With this representation of the solution it may be shown that Eqs. (2.8d) and (2.8f) are automatically satisfied and Eq. (2.8e) will be satisfied if C_0 is given by

$$C_0 = \int_0^1 \varphi_0(x)\, dx \, . \tag{2.10}$$

Substitution of (2.9a) - (2.9c) into Eqs. (2.8a) - (2.8c) leads to three Abel type integral equations[32] whose solutions are given by

$$\varphi_0(r') = \frac{d}{dr'} \int_0^{r'} \frac{x f_0(x)}{(r'^2 - x^2)^{\frac{1}{2}}}\, dx \; ; \quad 0 \le r' \le 1 \tag{2.11a}$$

$$C_0\varphi_2(r') = \frac{1}{r'^3}\frac{d}{dr'}\int_0^{r'}\frac{x^3 f_2(x)}{(r'^2-x^2)^{\frac{1}{2}}}dx \; ; \; 0 \le r' \le 1 \qquad (2.11b)$$

$$\varphi_1(r') = \frac{1}{r'}\frac{d}{dr'}\int_0^{r'}\frac{x^2 f_1(x)}{(r'^2-x^2)^{\frac{1}{2}}}dx \qquad ; \; 0 \le r' \le 1 \; . \qquad (2.11c)$$

After replacing $f_0(r')$, $f_1(r')$, $f_2(r')$ by Eqs. (2.8g) - (2.8i), substituting again Eqs. (2.9a) - (2.9c) and reordering, Eqs. (2.11a) - (2.11c) become

$$\varphi_0(r') = \frac{\Delta_h - aC_0}{a(1-\sigma)} + \frac{2}{\pi}\gamma\int_0^1 \varphi_1(x)\,dx \lim_{z'\to 0}\int_0^\infty e^{-kz'}\sin(kx)\cos(kr')dk -$$

$$\frac{2}{\pi}\int_0^1 \varphi_0(x)dx\int_0^\infty H_1(k)\cos(kx)\cos(kr')dk \; +$$

$$\frac{2}{\pi}\int_0^1 \varphi_1(x)dx\int_0^\infty H_2(k)\sin(kx)\cos(kr')dk$$

$$-\frac{2}{\pi}C_0\int_0^\infty k^{-1}H_3(k)\sin k\cos(kr')dk \; -$$

$$\frac{C_0}{1-\sigma}\sqrt{\frac{2}{\pi}}\int_0^1 x^{3/2}\varphi_2(x)dx\int_0^\infty k^{\frac{1}{2}}J_{3/2}(kx)\cos(kr')dk$$

$$-\sqrt{\frac{2}{\pi}}C_0\int_0^1 x^{3/2}\varphi_2(x)dx\int_0^\infty k^{\frac{1}{2}}H_3(k)J_{3/2}(kx)\cos(kr')dk; \; 0 \le r' \le 1 \qquad (2.12a)$$

$$r'\varphi_2(r')C_0 = (1-\sigma)\Big\{\sqrt{\frac{2}{\pi}}\int_0^1 \varphi_0(x)dx\int_0^\infty (kr')^{\frac{1}{2}}\cos(kx)J_{3/2}(kr')dk$$

$$+\sqrt{\frac{2}{\pi}}\int_0^1 \varphi_0(x)dx\int_0^\infty H_1(k)(kr')^{\frac{1}{2}}\cos(kx)J_{3/2}(kr')dk$$

$$-\sqrt{\frac{2}{\pi}}\gamma\int_0^1 \varphi_1(x)dx\lim_{z'\to 0}\int_0^\infty e^{-kz'}(kr')^{\frac{1}{2}}\sin(kx)J_{3/2}(kr')dk$$

$$-\sqrt{\frac{2}{\pi}}\int_0^1 \varphi_1(x)dx\int_0^\infty H_2(k)(kr')^{\frac{1}{2}}\sin(kx)J_{3/2}(kr')dk \; -$$

$$\sqrt{\frac{2}{\pi}}\; C_0 \int_0^\infty k^{-1}\; H_3(k)\; \sin(k)\,(kr')^{\frac{1}{2}}\; J_{3/2}(kr')dk$$

$$- C_0 \int_0^1 x\,\varphi_2(x)dx \int_0^\infty k(xr')^{\frac{1}{2}}\, H_3(k)\, J_{3/2}(kx)\, J_{3/2}(kr')dk\Big\} \quad 0 \le r' \le 1$$

$$(2.12b)$$

$$\varphi_1(r') = - \frac{1}{1-\sigma}\,\Gamma_0\, r' + \frac{2}{\pi}\,\gamma \int_0^1 \varphi_0(x)dx \lim_{z'\to 0} \int_0^\infty e^{-kz'} \cos(kx)\sin(kr')dk$$

$$- \frac{2}{\pi}\int_0^1 \varphi_1(x)dx \int_0^\infty H_4(k)\sin(kx)\,\sin(kr')dk \;+$$

$$\frac{2}{\pi} \int_0^1 \varphi_0(x)\,dx \int_0^\infty H_2(k)\cos(kx)\sin(kr')dk; \quad 0 \le r' \le 1 \qquad (2.12c)$$

A more useful form of Eqs. (2.12) may be found by extending the functions $\varphi_0(r')$, $\varphi_1(r')$ and $\varphi_2(r')$ into the interval $[-1, 0)$. Thus, by defining

$$\varphi_0(r') = \varphi_0(-r'), \quad \varphi_1(r') = -\varphi_1(-r'), \quad \varphi_2(r') = -\varphi_2(-r'); \; -1 \le r' < 0 \qquad (2.13)$$

it may be shown that[42]

$$\int_0^1 \varphi_1(x)\,dx \lim_{z'\to 0} \int_0^\infty e^{-kz'} \sin(kx)\cos(kr')dk = \frac{1}{2} \int_{-1}^1 \frac{\varphi_1(x)}{x-r'}\, dx \qquad (2.14a)$$

$$\sqrt{\frac{2}{\pi}} \int_0^1 \varphi_1(x)dx \lim_{z'\to 0} \int_0^\infty e^{-kz'} (kr')^{\frac{1}{2}} \sin(kx)\, J_{3/2}(kr')dk$$

$$= \frac{1}{\pi}\,\frac{1}{r'} \int_0^1 \log\frac{x+r'}{|x-r'|}\,\varphi_1(x)dx - \frac{1}{\pi}\int_{-1}^1 \frac{\varphi_1(x)}{x-r'}\, dx \qquad (2.14b)$$

$$\int_0^1 \varphi_0(x)dx \lim_{z'\to 0} \int_0^\infty e^{-kz'} \cos(kx)\sin(kr')dk = -\tfrac{1}{2}\int_{-1}^{1} \frac{\varphi_0(x)}{x-r'} dx \,. \quad (2.14c)$$

When these forms are inserted into Eqs. (2.12) and two improper integrals appearing in these equations are evaluated[42], they become

$$\varphi_0(r') - \frac{\gamma}{\pi}\int_{-1}^{1} \frac{\varphi_1(x)}{x-r'}dx = \frac{\Delta_h - aC_0}{a(1-\sigma)} - \frac{C_0}{1-\sigma}\int_{r'}^{1} \varphi_2(x)dx -$$

$$\frac{C_0}{1-\sigma} r'\varphi_2(r') - a_0\left\{\int_0^1 K_{00}(x,r')\varphi_0(x)dx - \int_0^1 K_{01}(x,r')\varphi_1(x)dx + \right.$$

$$\left. C_0\int_0^1 K_{02}(x,r')\varphi_2(x)dx + C_0K_0(r')\right\} \;;\quad 0 \le r' \le 1 \qquad (2.15a)$$

$$r'^2\varphi_2(r')C_0 = (1-\sigma)\left\{\left[\int_0^{r'} \varphi_0(x)dx - \frac{\gamma}{\pi}\int_0^1 \varphi_1(x)\log\frac{x+r}{x-r'}dx\right] - \right.$$

$$r'\left[\varphi_0(r') - \frac{\gamma}{\pi}\int_{-1}^{1}\frac{\varphi_1(x)}{x-r'}\right] - a_0\left[C_0\int_0^1 K_{22}^{o}(x,r')\varphi_2(x)dx - \right.$$

$$\int_0^1 K_{20}^{o}(x,r')\varphi_0(x)dx + \int_0^1 K_{21}^{o}(x,r')\varphi_1(x)dx +$$

$$\left.\left. C_0K_2^{o}(r')\right]\right\} \;;\quad 0 \le r' \le 1 \qquad (2.15b)$$

$$\varphi_1(r') + \frac{\gamma}{\pi}\int_{-1}^{1} \frac{\varphi_0(x)}{x-r'}\, dx = -\frac{\Gamma_0 r'}{1-\sigma} - a_0\Big[\int_0^1 K_{11}(x,r')\varphi_1(x)dx$$

$$-\int_0^1 K_{10}(x,r')\varphi_0(x)dx\Big] \; ; \quad 0 \le r' \le 1 \qquad (2.15c)$$

The kernels appearing in these equations are defined by

$$K_{00}(x,r') = \frac{2}{\pi}\int_0^\infty L_1(t)\cos(txa_0)\cos(tr'a_0)\,dt \qquad (2.15d)$$

$$K_{01}(x,r') = \frac{2}{\pi}\int_0^\infty L_2(t)\sin(txa_0)\cos(tr'a_0)dt \qquad (2.15e)$$

$$K_{02}(x,r') = \sqrt{\frac{2}{\pi}}\, x\int_0^\infty (txa_0)^{\frac{1}{2}} L_3(t) J_{3/2}(txa_0)\cos(tr'a_0)dt \qquad (2.15f)$$

$$K_0(r') = \frac{2}{\pi}\int_0^\infty (ta_0)^{-1} L_3(t)\sin(ta_0)\cos(tr'a_0)dt \qquad (2.15g)$$

$$K_{22}^{o}(x,r') = xr'\int_0^\infty a_0 t\,(xr')^{\frac{1}{2}} L_3(t) J_{3/2}(txa_0) J_{3/2}(tr'a_0)dt \qquad (2.15h)$$

$$K_{20}^{o}(x,r') = \sqrt{\frac{2}{\pi}}\, r'\int_0^\infty (tr'a_0)^{\frac{1}{2}} L_1(t)\cos(txa_0) J_{3/2}(tr'a_0)dt \qquad (2.15i)$$

$$K_{21}^{o}(x,r') = \sqrt{\frac{2}{\pi}}\, r'\int_0^\infty (tr'a_0)^{\frac{1}{2}} L_2(t)\sin(txa_0) J_{3/2}(tr'a_0)dt \qquad (2.15j)$$

$$K_2^{o}(r) = \sqrt{\frac{2}{\pi}}\, r'\int_0^\infty (ta_0)^{-1} L_3(t)\sin(ta_0)(tr'a_0) J_{3/2}(tr'a_0)dt \qquad (2.15k)$$

$$K_{11}(x, r') = \frac{2}{\pi} \int_0^\infty L_4(t) \sin(txa_0) \sin(tr'a_0) dt \tag{2.15l}$$

$$K_{10}(x, r') = \frac{2}{\pi} \int_0^\infty L_2(t) \cos(txa_0) \sin(tr'a_0) dt \tag{2.15m}$$

in which

$$L_1(t) = \frac{-v'_2 t}{(1-\sigma)[(2t^2-1)^2 - 4v'_1 v'_2 t^2]} - 1 \tag{2.15n}$$

$$L_2(t) = \frac{2v'_1 v'_2 t^2 - (2t^2-1)t^2}{(1-\sigma)[(2t^2-1)^2 - 4v'_1 v'_2 t^2]} - \gamma \tag{2.15o}$$

$$L_3(t) = \frac{1}{1-\sigma} \left(\frac{t}{v_2'} - 1\right) \tag{2.15p}$$

$$L_4(t) = \frac{-v'_1 t}{(1-\sigma)[(2t^2-1)^2 - 4v'_1 v'_2 t^2]} - 1 . \tag{2.15q}$$

The branches of

$$v'_1 = (t^2 - \gamma^2)^{\frac{1}{2}} \tag{2.15r}$$

$$v'_2 = (t^2 - 1)^{\frac{1}{2}} \tag{2.15s}$$

must be chosen so that Re $v'_1, v'_2 \geqq 0$

It may be shown that the functions $L_i(t)$ have the property

$$L_i(t) = 0(t^{-2}) \text{ as } t \to \infty , \ i = 1, 2, 3, 4 .$$

Three of these functions also have a simple pole on the positive real axis since the Rayleigh function $F(t) = (2t^2 - 1)^2 - 4v'_1 v'_2 t^2$ has a simple zero on that axis. Therefore, the integrals appearing in Eqs. (2.15d) - (2.15m) are all convergent, either in the regular sense or in the sense of a Cauchy Principal Integral.

Equation (2.15b) may be simplified considerably if use is made of Eq. (2.15a). Integrating (2.15a) with respect to r' between the limits 0 and r' and substituting the result, together with Eq. (2.15a) itself, into Eq. (2.15b) gives

$$2\int_0^{r'} x\,\varphi_2(x)dx + a_0\int_0^1 \tilde{K}_{22}(x, r')\varphi_2(x)dx = -a_0\,\tilde{K}_2(r');\ 0 \le r' \le 1 \qquad (2.16a)$$

where

$$\tilde{K}_{22}(x, r') = \int_0^\infty a_0 t(xr')^{\frac{1}{2}} L_3(t)\, J_{3/2}(txa_0)\, J_{3/2}(tr'a_0)dt \qquad (2.16b)$$

$$\tilde{K}_2(r') = \sqrt{\frac{\pi}{2}} \int_0^\infty (ta_0)^{-1} L_3(t)\sin(ta_0)\, J_{3/2}(tr'a_0)dt\ . \qquad (2.16c)$$

The following Fredholm integral equation of the second kind is obtained by differentiating Eq. (2.16a) with respect to r' and dividing the result by r':

$$\varphi_2(r') + a_0\int_0^1 K_{22}(x, r')\varphi_2(x)dx = -a_0\, K_2(r');\quad 0 \le r' \le 1 \qquad (2.17a)$$

in which

$$K_{22}(x, r') = \frac{2}{\pi} \int_0^{\infty} (txa_0)\, L_3(t) \left[\frac{\sin(txa_0)}{(txa_0)} - \cos(txa_0) \right] \sin(tr'a_0) dt \quad (2.17b)$$

$$K_2(r') = \frac{2}{\pi} \int_0^{\infty} L_3(t) \sin(ta_0) \sin(tr'a_0)\, dt\,. \quad (2.17c)$$

In this way the problem has been reduced to the study of the system of coupled integral equations (2.15a), (2.15c) and (2.17a) in the unknowns $\varphi_0(r')$, $\varphi_1(r')$ and $\varphi_2(r')$. Equation (2.17a), however, can be solved independently of the remaining two equations as it involves only the unknown function $\varphi_2(r')$. Equations (2.15a) and (2.15c) may then be interpreted as a system of Cauchy type integral equations in the two unknowns $\varphi_0(r')$, $\varphi_1(r')$, and be rewritten as

$$\varphi_0(r') - \frac{\gamma}{\pi} \int_{-1}^{1} \frac{\varphi_1(x)}{x-r'} dx = \frac{\Delta_h - aC_0}{a(1-\sigma)} - \frac{C_0}{1-\sigma} g(r')$$

$$- a_0 \sum_{m=0}^{1} \int_0^1 K_{0m}(x, r')\varphi_m(x)dx; \quad 0 \le r' \le 1 \quad (2.18a)$$

$$\varphi_1(r') + \frac{\gamma}{\pi} \int_{-1}^{1} \frac{\varphi_0(x)}{x-r'} dx = -\frac{\Gamma_0 r'}{1-\sigma} - a_0 \sum_{m=0}^{1} \int_0^1 K_{1m}(x, r')\varphi_m(x)dx;$$

$$0 \le r' \le 1 \quad (2.18b)$$

where

$$g(r') = \int_{r'}^{1} \varphi_2(x)dx + r'\varphi_2(r') + (1-\sigma)a_0 \int_0^1 K_{02}(x, r')\varphi_2(x)dx \quad . \qquad (2.18c)$$

For the particular case of an incompressible material ($\sigma = \frac{1}{2}$ or equivalently $\gamma = 0$), Eqs. (2.18a) and (2.18b) reduce to a system of Fredholm integral equations of the second kind, as the terms containing a Cauchy type singularity are then eliminated from the equations.

<u>Reduction to the relaxed problem</u>

The solution to the relaxed mixed boundary-value problems corresponding to the complete mixed boundary-value problems which are the subject of this analysis, may be derived from Eqs. (2.17a), (2, 18a) and (2.18b). For horizontal vibrations of the disc produced under relaxed conditions, it is assumed that the normal traction is zero everywhere on the surface of the elastic half-space and that the vertical displacements under the disc are unconstrained. These conditions will be satisfied if Eq. (2.18b) is disregarded and $\varphi_1(r')$ is set equal to zero. There results

$$\left.\begin{aligned} &\varphi_2(r') + a_0 \int_0^1 K_{22}(x, r')\, \varphi_2(x)dx = - a_0 K_2(r') && (2.19a) \\ &\varphi_0(r') + a_0 \int_0^1 K_{00}(x, r')\varphi_0(x)dx = \frac{\Delta_h - aC_0}{a(1-\sigma)} - \frac{C_0}{1-\sigma}\, g(r') && (2.19b) \end{aligned}\right\} 0 \le r' \le 1 \, .$$

This is a system of two Fredholm integral equations of the second kind, coupled only through the term $g(r')$. This formulation permits the complete determination of $\varphi_2(r')$ and $\varphi_0(r')$ in terms of the unknown constant C_0, obtained in turn from Eq. (2.10). The integral equations (2.19) involve only finite integrations on the unknown functions $\varphi_0(r')$, $\varphi_2(r')$, whereas the corresponding equations presented by Gladwell[13] for the same problem include improper integrals on the unknown functions. Luco and Westmann[11] have obtained recently a pair of equations similar to Eqs. (2.19). The only essential difference being that in their analysis, the two functions corresponding to φ_0 and φ_2 appear in both integral equations.

For rocking vibrations produced under relaxed conditions, it is assumed that the contact is frictionless and consequently the horizontal displacements under the disc are unconstrained. These conditions will be satisfied if Eqs. (2.17a) and (2.18a) are disregarded and the functions φ_0 and φ_2 are set equal to zero. This gives

$$\varphi_1(r') + a_0 \int_0^1 K_{11}(x, r')\, \varphi_1(x)dx = -\frac{\Gamma_0 r'}{1-\sigma} \quad ; \quad 0 \le r' \le 1 \tag{2.20}$$

that is, a Fredholm integral equation of the second kind which permits the complete determination of $\varphi_1(r')$. Equation (2.20) is equivalent to the corresponding equation obtained by Gladwell[13].

Simplification of the general equations

The general case has been reduced to the solution of the uncoupled Fredholm integral equation (2.17a) and to the pair of simultaneous Cauchy type singular equations (2.18a) and (2.18b). The left hand sides of these latter two equations may be uncoupled by introducing the two new functions $\Psi_1(r')$, $\Psi_2(r')$ defined by

$$\Psi_m(r') = \varphi_1(r') + (-1)^{m+1}\varphi_0(r') \quad ; \quad m = 1,2 . \tag{2.21}$$

With this substitution, Eqs. (2.18a) and (2.18b) become

$$\Psi_m(r') + (-1)^m i\frac{\gamma}{\pi}\int_{-1}^{1} \frac{\Psi_m(x)}{x-r'}\, dx = -\frac{\Gamma_0 r'}{1-\sigma} + (-1)^{m+1}\, i\,\frac{(\Delta_h - a\,C_0)}{a(1-\sigma)} +$$

$$a_0 \sum_{s=1}^{2} \int_0^1 M_{ms}(x, r')\, \Psi_s(x)dx + (-1)^m \frac{i}{1-\sigma}\, C_0\, g(r'); \quad m=1,2, \quad 0\le r'\le 1 \tag{2.22a}$$

where

$$M_{ms}(x, r') = \tfrac{1}{2}[K_{11}(x, r') + (-1)^{s+1} K_{10}(x, r') + (-1)^{m+1} K_{01}(x, r') +$$

$$(-1)^{m+s+1} K_{00}(x, r')\,] \; ; \; m, s = 1, 2 \, . \qquad (2.22b)$$

Equations of this type, sometimes referred to as singular equations of the Carleman type, have been studied extensively[43,44,45]. Treating the right hand sides of Eqs. (2.22a) as known functions, each of these equations can be solved independently. Their solution leads to the following system of Fredholm integral equations:

$$\Psi_m(r') = \frac{X_m(r')}{(1-\sigma)\sqrt{1-\gamma^2}} \left[(-1)^{m+1}\, i\,(\frac{\Delta h}{a} - C_0) - \Gamma_0 r' + (-1)^{m+1} 2i\, k\Gamma_0\right] +$$

$$\frac{1}{1-\gamma^2}\left[a_0 \sum_{s=1}^{2} \int_0^1 N_{ms}(x, r')\,\Psi_s(x)dx + (-1)^m 2i\, C_0\, G_m(r')\right]; \; m=1, 2,$$

$$0 \le r' \le 1 \qquad (2.23a)$$

where the functions $X_m(r')$ are defined by

$$X_m(r') = \exp\left[(-1)^m\, i\, k \ln(\frac{1-r'}{1+r'})\right]; \quad m = 1, 2 \, . \qquad (2.23b)$$

The constant $k = \frac{1}{2\pi} \ln(\frac{1-\gamma}{1+\gamma})$, the kernels $N_{ms}(x, r')$ and the functions $G_m(r')$ are given by

$$N_{ms}(x, r') = M_{ms}(x, r') + (-1)^{m+1}\, i\, \frac{\gamma}{\pi}\, X_m(r') \int_{-1}^{1} \frac{X_{3-m}(t)}{t-r'}\, M_{ms}(x, t)dt;$$

$$m, s = 1, 2 \qquad (2.23c)$$

$$G_m(r') = g(r') + (-1)^{m+1}\, i\frac{\gamma}{\pi}\, X_m(r') \int_{-1}^{1} \frac{X_{3-m}(t)}{t-r'}\, g(t)dt, \quad m=1,2. \quad (2.23d)$$

It is possible to factor the functions $X_m(r')$ from Eqs. (2.23a). Defining the functions $\theta_1(r)$ and $\theta_2(r')$ by

$$\theta_m(r') = \Psi_m(r')\, X_{3-m}(r') \;;\; m = 1,2 \quad (2.24)$$

and making use of the integral

$$\int_{-1}^{1} \frac{X_m(t)}{t - r'}\, dt = (-1)^m \frac{\pi i}{\gamma} \left[X_m(r') - \sqrt{1-\gamma^2} \right]; \; m = 1,2 \quad (2.25)$$

equations (2.23a) become

$$\theta_m(r') = \frac{1}{(1-\sigma)\sqrt{1-\gamma^2}} \left[(-1)^{m+1}\, i\, (\frac{\Delta h}{a} - C_0) - \Gamma_0 r' + (-1)^{m+1} 2ik\Gamma_0 \right] \quad (2.26a)$$

$$+ \frac{1}{1-\gamma^2} \left[a_0 \sum_{s=1}^{2} \int_0^1 T_{ms}(x,r')\,\theta_s(x)dx + (-1)^m 2iC_0\, T_m(r') \right]; m=1,2\,,$$

$$0 \le r' \le 1$$

in which

$$T_{ms}(x,r') = X_s(x) \left\{ \sqrt{1-\gamma^2}\; M_{ms}(x,r') + (-1)^{m+1}\, i\, \frac{\gamma}{\pi} \int_{-1}^{1} X_{3-m}(t) \cdot \frac{[M_{ms}(x,t) - M_{ms}(x,r')]}{t - r'}\, dt \right\}; \; m,s = 1,2 \quad (2.26b)$$

$$T_m(r') = \sqrt{1-\gamma^2}\;\; g(r') + (-1)^{m+1}\, i\frac{\gamma}{\pi} \int_{-1}^{1} X_{3-m}(t) \frac{[g(t) - g(r')]}{t - r'}\, dt, \quad m=1,2\,. \quad (2.26c)$$

It may be shown that the integrands appearing in Eqs. (2.26b) and (2.26c) have removable singularities at $t = r'$, and therefore, the integrals themselves are amenable to numerical evaluation. Before calculating these integrals it is convenient however, to use contour integration to obtain alternative simpler expressions for the improper integrals which define the kernels $M_{ms}(x, r')$.

The system of Fredholm integral equations of the second kind (2.26a) may be solved by standard numerical procedures[46, 47], thus yielding a solution for the functions $\theta_1(r')$ and $\theta_2(r')$ in terms of the unknown constant C_0. The functions $\varphi_0(r')$ and $\varphi_1(r')$ may then be found from

$$\varphi_0(r') = \frac{i}{2} [-\theta_1(r') X_1(r') + \theta_2(r') X_2(r')] \tag{2.27a}$$

$$\varphi_1(r') = \frac{1}{2} [\theta_1(r') X_1(r') + \theta_2(r') X_2(r')] \tag{2.27b}$$

C_0 is determined from Eqs. (2.27a) and (2.10).

Equations (2.27), (2.26) and (2.23b) show that the frequency independent functions $\begin{matrix}\sin \\ \cos\end{matrix}\left[k \ln \frac{1-r'}{1+r'}\right]$, characteristic of the static, complete mixed boundary value problem occur also in the solution to the corresponding dynamic problem.

Physical quantities of interest, such as stresses, impedances and displacements may be expressed in terms of the functions

$\varphi_0(r')$, $\varphi_1(r')$, $\varphi_2(r')$. Such expressions are given in the following section.

3. Stresses under the Disc

The stresses under the disc are obtained by taking the limit as $z' \to 0$ of Eqs. (2.4d) - (2.4f) and by substituting in the resulting equations the values of $A(k)$ and $B(k)$ given by Eqs. (2.6), following which the functions $E(k)$, $C(k)$, $D(k)$ are expressed in terms of $\varphi_0(r')$, $\varphi_1(r')$, $\varphi_2(r')$ and C_0 from Eqs. (2.9). Thus,

$$\sigma^*_{zr}(ar', 0) + \sigma^*_{z\theta}(ar', 0) = \frac{4}{\pi} r' \frac{d}{dr'} \frac{1}{r'^2} \int_{r'}^{1} \frac{x[\varphi_0(x) - C_0]}{(x^2 - r'^2)^{\frac{1}{2}}} dx +$$

$$\frac{4}{\pi} r' \frac{d}{dr'} \int_{r'}^{1} \frac{\varphi_2(x)}{(x^2 - r'^2)^{\frac{1}{2}}} dx; \quad 0 \le r' < 1 \tag{2.28a}$$

$$\sigma^*_{zr}(ar', 0) - \sigma^*_{z\theta}(ar', 0) = \frac{4}{\pi} \frac{1}{r'} \frac{d}{dr'} \int_{r'}^{1} \frac{x\varphi_0(x)}{(x^2 - r'^2)^{\frac{1}{2}}} dx - \frac{4}{\pi} C_0 (1 - r'^2)^{-\frac{1}{2}}$$

$$- \frac{4}{\pi} \frac{1}{r'} \frac{d}{dr'} r'^2 \int_{r'}^{1} \frac{\varphi_2(x)}{(x^2 - r'^2)^{\frac{1}{2}}} dx; \quad 0 \le r' < 1 \tag{2.28b}$$

$$\sigma^*_{zz}(ar', 0) = \frac{4}{\pi} \frac{d}{dr'} \int_{r'}^{1} \frac{\varphi_1(x)}{(x^2 - r'^2)^{\frac{1}{2}}} dx; \quad 0 \le r' < 1 \,. \tag{2.28c}$$

The total horizontal force $P \exp(i\omega t)$ applied to the disc in the direction $\theta = 0$ is given by

$$P = -\int_0^{2\pi} \int_0^a (\sigma_{zr} \cos\theta - \sigma_{z\theta} \sin\theta)\, r\, dr\, d\theta \tag{2.29a}$$

where σ_{zr} and $\sigma_{z\theta}$ are evaluated at $z' = 0$. After substitutions from Eqs. (2.4a), (2.4b), (2.28a), (2.28b) it is found that

$$P = 8\mu a^2 \int_0^1 \varphi_0(x)\, dx\,. \tag{2.29b}$$

The moment $Q \exp(i\omega t)$ applied about the axis $\theta = \pi/2$ is given by

$$Q = -\int_0^{2\pi} \int_0^a \sigma_{zz} \cos\theta\, r^2\, dr\, d\theta \tag{2.30a}$$

where σ_{zz} is evaluated at $z' = 0$. After substitution from Eqs. (2.4c) and (2.28c), Eq. (2.30a) leads to

$$Q = -8\mu a^3 \int_0^1 x\, \varphi_1(x)\, dx\,. \tag{2.30b}$$

Equations (2.29b) and (2.30b) may be written as

$$\left\{ \begin{matrix} \dfrac{P}{\mu a^2} \\ \\ \dfrac{Q}{\mu a^3} \end{matrix} \right\} = \begin{bmatrix} K_{hh}(ia_0, \sigma) & K_{hm}(ia_0, \sigma) \\ \\ K_{mh}(ia_0, \sigma) & K_{mm}(ia_0, \sigma) \end{bmatrix} \left\{ \begin{matrix} \dfrac{\Delta_h}{a} \\ \\ \Gamma_0 \end{matrix} \right\} \tag{2.31}$$

where K_{hh}, K_{hm}, K_{mh}, K_{mm} represent the dimensionless impedances of the problem. The two functions K_{hm} and K_{mh}

must be equal as can be shown by the use of reciprocity theorems[35]. The impedance functions appearing in Eq. (2.31) are complex, thus indicating that the applied forces and the corresponding displacements are not in phase.

4. Limiting Static Problem

In the static case $a_0 = 0$. Under this condition, Eqs. (2.17a), (2.26a), (2.27) lead to

$$\varphi_0(r') = \frac{1}{(1-\sigma)\sqrt{1-\gamma^2}}\left\{\left(\frac{\Delta h}{a} - C_0 + 2k\Gamma_0\right)\cos\left[k \ln \frac{1-r'}{1+r'}\right]\right.$$

$$\left. + \Gamma_0 r' \sin\left[k \ln \frac{1-r'}{1+r'}\right]\right\}; \quad 0 \le r' \le 1 \tag{2.32a}$$

$$\varphi_1(r') = \frac{1}{(1-\sigma)\sqrt{1-\gamma^2}}\left\{-\Gamma_0 r' \cos\left[k \ln \frac{1-r'}{1+r'}\right]\right.$$

$$\left. + \left(\frac{\Delta h}{a} - C_0 + 2k\Gamma_0\right)\sin\left[k \ln \frac{1-r'}{1+r'}\right]\right\}; \quad 0 \le r' \le 1 \tag{2.32b}$$

$$\varphi_2(r') = 0 \qquad 0 \le r' \le 1. \tag{2.32c}$$

Substitutions of these expressions for the functions $\varphi_0(r')$, $\varphi_1(r')$, $\varphi_2(r')$ into Eqs. (2.28) gives the stress distribution underneath the disc in terms of the as yet undetermined constant C_0. Substitution of Eq. (2.32a) into Eq. (2.10) yields

$$(1-\sigma)\sqrt{1-\gamma^2}\ C_0 = (\frac{\Delta h}{a} - C_0 + 2k\Gamma_0)\, I_c^{\,o} + \Gamma_0 I_s^{\,1} \tag{2.33a}$$

where $I_c^{\,o}$ and $I_s^{\,1}$ are given by[42]

$$I_c^{\,o} = \int_0^1 \cos\left[k \ln \frac{1-r'}{1+r'}\right] dr' = \int_0^\infty \operatorname{sech}^2\theta \cos(\delta\theta)\, d\theta = \frac{\pi\delta\sqrt{1-\gamma^2}}{2\gamma} \tag{2.33b}$$

$$I_s^{\,1} = \int_0^1 r' \sin\left[k \ln \frac{1-r'}{1+r'}\right] dr' = \int_0^\infty \tanh\theta \operatorname{sech}^2\theta \sin(\delta\theta) = \frac{\pi\delta^2\sqrt{1-\gamma^2}}{4\gamma} \tag{2.33c}$$

in which

$$\delta = \frac{1}{\pi} \ln(3-4\sigma) . \tag{2.33d}$$

Using the expressions for $I_c^{\,o}$ and $I_s^{\,o}$ in Eq. (2.33a),

$$C_0 = \frac{\frac{\Delta h}{a} - \frac{1}{2\pi}\Gamma_0 \ln(3-4\sigma)}{1 + \frac{1-2\sigma}{\ln(3-r\sigma)}} . \tag{2.33e}$$

With this, the horizontal force P is obtained immediately from Eqs. (2.29b) and (2.10);

$$P = \frac{4\mu a^2 \left[\frac{2\Delta h}{a} - \frac{1}{\pi}\Gamma_0 \ln(3-4\sigma)\right]}{1 + \frac{1-2\sigma}{\ln(3-4\sigma)}} . \tag{2.34}$$

The moment Q is obtained by replacing $\varphi_1(r')$, as given by Eq. (2.32b), into Eq. (2.30b). This results in

$$Q = -\frac{8\mu a^3}{(1-\sigma)\sqrt{1-\gamma^2}}\left[-\Gamma_0 I_c{}^2 + \left(\frac{\Delta_h}{a} - C_0 - \frac{1}{\pi}\Gamma_0 \ln(3-4\sigma) I_s{}^1\right)\right] \tag{2.35a}$$

where $I_c{}^2$ is given by[42]

$$I_c{}^2 = \int_0^1 r'^2 \cos\left[k \ln \frac{1-r'}{1+r'}\right] dr' = \frac{1}{2}\int_0^\infty \operatorname{sech}^4\theta \cos(\delta\theta) d\theta - \frac{\delta}{2} I_s{}^2$$

$$= \frac{\pi\delta\sqrt{1-\gamma^2}}{\gamma}\left(\frac{1}{3} - \frac{\delta^2}{6}\right) . \tag{2.35b}$$

Substitution of Eqs. (2.35b), (2.33c) - (2.33e) into Eq. (2.35a) leads to

$$Q = \frac{4\mu a^3}{1+\dfrac{1-2\sigma}{\ln(3-4\sigma)}}\left\{-\frac{1}{\pi}\ln(3-4\sigma)\frac{\Delta_h}{a} + \left[\frac{\left[4+\frac{1}{\pi^2}(\ln(3-4\sigma))^2\right]\ln(3-4\sigma)}{6(1-2\sigma)}\right.\right.$$

$$\left.\left. + \frac{2}{3}\left[1+\frac{1}{\pi^2}(\ln(3-4\sigma))^2\right]\right]\Gamma_0\right\} . \tag{2.36}$$

Equations (2.34) and (2.36) may be written in the form of Eq. (2.31),

$$\begin{Bmatrix} \dfrac{P}{\mu a^2} \\ \\ \dfrac{Q}{\mu a^3} \end{Bmatrix} = \frac{4}{1+\dfrac{1-2\sigma}{\ln(3-4\sigma)}} \begin{bmatrix} 2 & -\frac{1}{\pi}\ln(3-4\sigma) \\ & \left\{\left[4+\frac{1}{\pi^2}(\ln(3-4\sigma))^2\right]\dfrac{\ln(3-4\sigma)}{6(1-2\sigma)}\right. \\ -\frac{1}{\pi}\ln(3-4\sigma) & \left. +\frac{2}{3}\left[1+\frac{1}{\pi^2}(\ln(3-4\sigma))^2\right]\right\} \end{bmatrix} \begin{Bmatrix} \dfrac{\Delta_h}{a} \\ \\ \Gamma_0 \end{Bmatrix} . \tag{2.37}$$

Equation (2.37) is equivalent to the corresponding equations obtained by Gladwell[29], except for a misprint in his expression for K_{mm}.

The flexibility matrix presented by Gladwell is the inversion of the stiffness matrix of Eq. (2.37).

D. Vertical Vibration of the Disc

1. Derivation of the Integral Equations

The vertical vibration of a disc perfectly bonded to an elastic half-space is an axially symmetric, mixed boundary value problem in elastodynamics. To formulate the problem in terms of dual integral equations, use will be made in this analysis of the solution of the equations of motion (2.1) given by Bycroft[5]. This solution, which is independent of the angular coordinate θ, may be expressed as

$$u_r(ar', \theta, az') = a\int_0^\infty k^2[-A_1(k)e^{-v_1 z'} + v_2 B_1(k)e^{-v_2 z'}]J_1(kr')dk \quad (2.38a)$$

$$u_\theta(ar', \theta, az') = 0 \quad (2.38b)$$

$$u_z(ar', \theta, az') = a\int_0^\infty k[-v_1 A_1(k)e^{-v_1 z'} + k^2 B_1(k)e^{-v_2 z'}]J_0(kr')dk \ . \quad (2.38c)$$

The corresponding stresses σ_{zr}, $\sigma_{z\theta}$, σ_{zz} are

$$\sigma_{zr}(ar', \theta, az')e^{i\omega t} = \mu\int_0^\infty k^2[2v_1 A_1(k)e^{-v_1 z'} - (2k^2 - a_0^2)B_1(k)e^{-v_2 z'}] \cdot J_1(kr')dk\, e^{i\omega t} \quad (2.39a)$$

$$\sigma_{z\theta}(ar', \theta, az')e^{i\omega t} = 0 \tag{2.39b}$$

$$\sigma_{zz}(ar', \theta, az')e^{i\omega t} = \mu \int_0^\infty k[(2k^2 - a_0^2)A_1(k)e^{-v_1 z'} - 2k^2 v_2 B_1(k)e^{-v_2 z'}] J_0(kr')dke^{i\omega t} . \tag{2.39c}$$

This solution will satisfy the boundary conditions given by Eqs. (2.2), with $\Delta_h = 0$, $\Gamma_0 = 0$, provided the following equations are satisfied

$$\left.\begin{aligned} &\lim_{z'\to 0} \int_0^\infty k[-v_1 A_1(k)e^{-v_1 z'} + k^2 B_1(k)e^{-v_2 z'}] J_0(kr')dk = \frac{\Delta_v}{a} && (2.40a) \\ &\lim_{z'\to 0} \int_0^\infty k^2 [-A_1(k)e^{-v_1 z'} + v_2 B_1(k)e^{-v_2 z'}] J_1(kr')dk = 0 && (2.40b) \end{aligned}\right\} 0 \le r' \le 1$$

$$\left.\begin{aligned} &\lim_{z'\to 0} \int_0^\infty k[(2k^2-a_0^2)A_1(k)e^{-v_1 z'} - 2k^2 v_2 B_1(k)e^{-v_2 z'}] J_0(kr')dk = 0 && (2.40c) \\ &\lim_{z'\to 0} \int_0^\infty k^2 [2v_1 A_1(k)e^{-v_1 z'} - (2k^2-a_0^2)B_1(k)e^{-v_2 z'}] J_1(kr')dk = 0 && (2.40d) \end{aligned}\right\} 1 < r' < \infty .$$

These equations may be cast in a form more suitable for further analysis by replacing the undetermined functions $A_1(k)$, $B_1(k)$ in Eqs. (2.40) by the functions $D_1(k)$, $E_1(k)$ defined by

$$2v_1 A_1(k) - (2k^2-a_0^2) B_1(k) = k^{-2} D_1(k) \tag{2.41a}$$

$$(2k^2-a_0^2) A_1(k) - 2k^2 v_2 B_1(k) = k^{-1} E_1(k) \tag{2.41b}$$

and by introducing the functions $H_1(k)$, $H_2(k)$, $H_4(k)$ defined in Eqs. (2.7). After substituting, reordering and taking limits inside the integrals, except in those indicated below, Eqs. (2.40) become

$$\left.\begin{aligned} &\int_0^\infty k^{-1} E_1(k)\, J_0(kr')\, dk = h_0(r') \qquad (2.42a) \\ &\int_0^\infty k^{-1} D_1(k)\, J_1(kr')\, dk = h_1(r') \qquad (2.42b) \end{aligned}\right\} \quad 0 \le r' \le 1$$

$$\left.\begin{aligned} &\int_0^\infty E_1(k)\, J_0(kr')\, dk = 0 \qquad (2.42c) \\ &\int_0^\infty D_1(k)\, J_1(kr')\, dk = 0 \qquad (2.42d) \end{aligned}\right\} \quad 1 < r' < \infty$$

where the functions $h_0(r')$ and $h_1(r')$ are defined by

$$h_0(r') = -\frac{\Delta_V}{a(1-\sigma)} + \gamma \lim_{z'\to 0} \int_0^\infty e^{-kz'} k^{-1} D_1(k)\, J_0(kr')dk$$

$$+ \int_0^\infty k^{-1} [-H_4(k) E_1(k) + H_2(k) D_1(k)]\, J_0(kr')dk \qquad (2.42e)$$

$$h_1(r') = \gamma \lim_{z'\to 0} \int_0^\infty e^{-kz'} k^{-1} E_1(k)\, J_1(kr)dk + \int_0^\infty k^{-1} [-H_1(k) D_1(k) +$$

$$H_2(k) E_1(k)]\, J_0(kr')dk\,. \qquad (2.42f)$$

These are two pairs of simultaneous dual integral equations, similar to those obtained for the problem of horizontal and rocking oscillations of the disc.

2. <u>Reduction of the Two Simultaneous Pairs of Integral Equations to a System of Fredholm Integral Equations of the Second Kind</u>

Following the approach developed in section C, the solution of Eqs. (2.42) is assumed to be of the form

$$E_1'(k) = \sqrt{\frac{2}{\pi}}\, k^{3/2} \int_0^1 x^{\frac{1}{2}}\, \varphi_0(x)\, J_{-\frac{1}{2}}(kx)dx \tag{2.43a}$$

$$D_1(k) = \sqrt{\frac{2}{\pi}}\, k^{3/2} \int_0^1 x^{\frac{1}{2}}\, \varphi_1(x)\, J_{\frac{1}{2}}(kx)dx\ . \tag{2.43b}$$

With this representation of the solution, Eqs. (2.42c) and (2.42d) are satisfied identically whereas Eqs. (2.42a) and (2.42b), in a manner similar to that followed with Eqs. (2.8a) and (2.8c), lead to the system of Cauchy type singular integral equations

$$\varphi_0(r') - \frac{\gamma}{\pi}\int_{-1}^{1} \frac{\varphi_1(x)}{x-r'}\,dx = -\frac{\Delta_v}{a(1-\sigma)} - a_0\int_0^1 [K_{00}(x,r')\varphi_0(x) -$$

$$K_{01}(x,r')\varphi_1(x)]\,dx; \quad 0 \le r' \le 1 \tag{2.44a}$$

$$\varphi_1(r') + \frac{\gamma}{\pi}\int_{-1}^{1} \frac{\varphi_0(x)}{x-r'}\,dx = -a_0\int_0^1 [K_{11}(x,r')\varphi_1(x) - K_{10}(x,r')\varphi_0(x)]dx;$$

$$0 \le r' \le 1\ . \tag{2.44b}$$

The kernels K_{00}, K_{10}, K_{11} and K_{10} are defined by Eqs. (2.15d), (2.15e), (2.15l) and (2.15m), respectively.

The problem is thus reduced to the study of the system of Cauchy type singular integral equations (2.44) in the unknown functions $\varphi_0(r')$ and $\varphi_1(r')$. For the limiting static case ($a_0=0$), these equations are identical to the corresponding equations obtained by Keer[27] for the same problem. Keer used the method developed by Green and Zerna[48] and Collins[49], as applied to axially symmetric problems in potential theory.

For an incompressible material ($\gamma=0$), Eqs. (2.44) reduce to a system of two coupled Fredholm integral equations of the second kind, thus permitting the complete determination of $\varphi_0(r')$ and $\varphi_1(r')$ by means of standard numerical techniques.

Reduction to the relaxed problem

The solution to the related relaxed mixed boundary value problem may be derived from Eqs. (2.44). For the relaxed problem, it is assumed that the tangential traction is zero everywhere on the surface of the elastic half-space and that the horizontal displacements under the disc are unconstrained. These conditions are satisfied by setting $\varphi_1(r')$ equal to zero and disregarding Eq. (2.44b). This gives

$$\varphi_0(r') + a_0 \int_0^1 K_{00}(x, r')\varphi_0(x)dx = \frac{\Delta_v}{a(1-\sigma)} . \tag{2.45}$$

This Fredholm integral equation of the second kind is equivalent to the corresponding equation obtained by Robertson[7] for the same problem.

Simplification of the general equations

By solving the dominant part of a system of Cauchy type singular integral equations, Eqs. (2.44) may be transformed into two Fredholm integral equations of the second kind. However, because of the analogy between Eqs. (2.44) and (2.18), the solution to (2.44) may be derived from the solution of Eqs. (2.18). Thus, the function $\varphi_0(r')$, $\varphi_1(r')$ may be expressed as in Eqs. (2.27):

$$\varphi_0(r') = \frac{i}{2}[- \theta_1(r')\, X_1(r') + \theta_2(r')\, X_2(r')] \tag{2.46a}$$

$$\varphi_1(r') = \tfrac{1}{2}[\theta_1(r')\, X_1(r') + \theta_2(r')\, X_2(r')] \tag{2.46b}$$

where, as before

$$X_m(r') = \exp[(-1)^m i k \ln(\frac{1-r'}{1+r'})]; \quad m = 1, 2 \tag{2.46c}$$

and the functions $\theta_1(r')$ and $\theta_2(r')$ satisfy a system of two simultaneous Fredholm integral equations of the second kind obtained by setting $g(r')$ and Γ_0 equal to zero and by replacing

$\Delta_h/a - C_0$ by $-\Delta_v/a$ in Eqs. (2.26a), that is

$$\theta_m(r') = \frac{(-1)^m\, i}{(1-\sigma)\sqrt{1-\gamma^2}}\ \frac{\Delta_v}{a} + \frac{a_o}{1-\gamma^2} \sum_{s=1}^{2} \int_0^1 T_{ms}(x, r')\, \theta_s(x)dx,$$

$$m = 1,2\ ;\quad 0 \le r' \le 1 \qquad (2.47)$$

where the kernels $T_{ms}(x, r')$ are defined by Eqs. (2.26b).

From Eqs. (2.46) and (2.47) it may be observed that the solution of the present problem also contains the factor

$$\begin{matrix}\cos\\ \sin\end{matrix} \left[k \ln \frac{1-r'}{1+r'} \right] .$$

Equations (2.47) may be solved by standard numerical procedures, thus yielding a complete solution for the functions $\theta_1(r')$ and $\theta_2(r')$ in terms of the dimensionless amplitude Δ_v/a. The functions $\varphi_0(r')$ and $\varphi_1(r')$ may then be obtained from Eqs. (2.46). Stresses, displacements impedances and other physical quantities of interest can be expressed in terms of $\varphi_0(r')$ and $\varphi_1(r')$. Such expressions are given in the following section.

3. Stresses under the Disc

The nonvanishing stresses under the disc are obtained by taking the limit as $z' \to 0$ of Eqs. (2.39), (2.39c) and by replacing $A_1(k)$ and $B_1(k)$ in the resulting equations by the functions $D_1(k)$, $E_1(k)$

defined by Eqs. (2.41). These functions are in turn expressed in terms of $\varphi_0(r')$ and $\varphi_1(r')$ from Eqs. (2.43). Performing these operations

$$\sigma_{zr}(ar', \theta, 0) = -\frac{2}{\pi}\mu \frac{d}{dr'}\int_{r'}^{1} \frac{\varphi_1(x)}{(x^2 - r'^2)^{\frac{1}{2}}}\,dx; \quad 0 \leq r' < 1 \qquad (2.48a)$$

$$\sigma_{zz}(ar', \theta, 0) = -\frac{2}{\pi}\mu \frac{1}{r'} \frac{d}{dr'}\int_{r'}^{1} \frac{x\varphi_0(x)}{(x^2 - r'^2)^{\frac{1}{2}}}\,dx; \quad 0 \leq r' < 1\,. \qquad (2.48b)$$

The total vertical force on the contact region, $V \exp(i\omega t)$, is

$$V = -\int_0^{2\pi}\int_0^a \sigma_{zz}(r, \theta, 0)\, r\, dr\, d\theta \qquad (2.49a)$$

which, upon substitution of Eq. (2.48b), becomes

$$V = -4\mu a^2 \int_0^1 \varphi_0(x)\,dx\,. \qquad (2.49b)$$

4. Limiting Static Problem

For the static case $(a_0 = 0)$, Eqs. (2.46) and (2.47) give

$$\varphi_0(r') = -\frac{\Delta_V}{a(1-\sigma)\sqrt{1-\gamma^2}} \cos\left[k \ln\left(\frac{1-r'}{1+r'}\right)\right]; \quad 0 \leq r' \leq 1 \qquad (2.50a)$$

$$\varphi_1(r') = -\frac{\Delta_V}{a(1-\sigma)\sqrt{1-\gamma^2}} \sin\left[k \ln\left(\frac{1-r'}{1+r'}\right)\right]; \quad 0 \leq r' \leq 1\,. \qquad (2.50b)$$

Substitution of these equations into Eqs. (2.48) gives for the stresses under the disc

$$\sigma_{zr}(ar', \theta, 0) = \frac{2}{\pi} \frac{\mu\Delta_v}{a(1-\sigma)\sqrt{1-\gamma^2}} \frac{d}{dr'} \int_{r'}^{1} \frac{\sin[k \ln(\frac{1-x}{1+x})]}{(x^2-r'^2)^{\frac{1}{2}}} dx \; ;$$

$$0 \leq r' < 1 \qquad (2.51a)$$

$$\sigma_{zz}(ar', \theta, 0) = \frac{2}{\pi} \frac{\mu\Delta_v}{a(1-\sigma)\sqrt{1-\gamma^2}} \frac{1}{r'} \frac{d}{dr'} \int_{r'}^{1} \frac{x \cos[k \ln(\frac{1-x}{1+x})]}{(x^2-r'^2)^{\frac{1}{2}}} dx \; ;$$

$$0 \leq r' \leq 1 \, . \qquad (2.51b)$$

These equations are identical to the corresponding ones obtained by Spence[28] who has shown that both stresses are singular as $r' \to 1$ behaving like $\left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} [k \ln(\frac{1-r'}{1+r'})] \right\} (1-r'^2)^{\frac{1}{2}}$, respectively.

The vertical force V is obtained by substituting Eq. (2.50a) into (2.49b) and making use of Eq. (2.33b).

$$V = \frac{4\mu\Delta_v a}{1-2\sigma} \ln(3-4\sigma) \, . \qquad (2.52)$$

This result is in agreement with the result obtained by Spence[28].

II. DYNAMICS OF BUILDING-SOIL INTERACTION

A. Introduction

It is well known that the behavior of a structure during an earthquake can be affected by the properties of the ground upon which it is founded. Conversely, the ground motion recorded in the vicinity of a building can differ from that which would be recorded at the site in the absence of the building. This dynamic coupling, or interaction between a building and the surrounding soil will result, generally, in (1) a reduction of the fundamental frequency of the system from that of the building founded on a rigid base, and (2) an energy loss, or damping, due to wave radiation into the soil medium. There will also be energy losses due to internal friction of the soil.

The response of a system to dynamic loading depends fundamentally on the natural frequencies and the amount of damping in the system. Therefore, the effects of dynamic coupling between a building and the ground will be determined by the extent to which these parameters are modified when soil-structure interaction is taken into consideration.

Observations of buildings during earthquakes have shown that the responses are influenced by their supporting media, especially when the soils are soft[50-53,74]. For special structures, interaction effects can be important even for relatively hard soils since the relevant parameter is not the stiffness of the soil, per se, but a dimensionless ratio of the stiffness of the building to the soil stiffness.

Thus, dynamic coupling may be appreciable for a very rigid structure, such as a nuclear reactor container, even when it is founded on relatively firm soil.

The influence of a flexible foundation medium upon the response of a building subjected to dynamic loading has been analyzed in several recent studies[54-69]. The foundation medium has been represented by constant, discrete parameter models[54,57,66], and by a homogeneous, isotropic, elastic half-space[56,61,64,67]. Some authors, for instance Sato and Yamaguchi[58], Thomson[57], Parmelee[61] and Hradilek[67] have studied the steady-state response to sinusoidal excitation of single and multi-story structures, while others, e.g., Housner and Merritt[54], Parmelee et al.[66] and Castellani[65], have used actual earthquakes or earthquake-type ground accelerations to obtain the response. Linearity of the buildings and their foundations has been assumed in all of these investigations. Kobori and Minai[70] and more recently Isenberg[71] have studied the effects of interaction for elastic buildings founded on elastic/perfectly-plastic soils.

Standard numerical methods for the step-by-step solution of the differential equations of motion can be applied when the foundation medium is represented by discrete models consisting, for example, of simple springs and viscous dampers. This discrete representation is no longer valid, in general, when an elastic half-space is used as a mathematical model for the soil, although it has been shown by Hsieh[72] that, under certain assumptions, the elastic half-space may be replaced by linear springs and dashpots. These elements are frequency dependent, however, and therefore can not be used for analyzing the transient response without resorting to operational methods.

In one approach the transient response of interacting systems has been calculated using the principle of superposition. For instance, Parmelee _et al._[64] expanded the input function into a Fourier series and obtained the steady-state response of the system for each harmonic. The transient response of the system was then approximated (initial transients in the harmonic solutions were neglected) by a linear combination of these responses.

An alternative method has also been presented by Parmelee _et al._[66]. Most of the dynamic properties of the springs and dashpots representing the elastic half-space model of the foundation were observed to remain almost constant within the frequency range of interest. Thus, the authors approximated these properties with average, constant values and obtained a system of ordinary differential equations with constant coefficients which was then solved numerically.

Another approach, the finite element method, has also been used to obtain the transient response of interacting systems[69,71]. The principal advantage of this approach is its ability to represent embedded structures of complex geometry and non-homogeneous soils.

Sandi[58], Rosenberg[60] and Castellani[65] have presented transform methods for analysis of the soil-structure interaction problem for linear systems. Sandi and Castellani, using Laplace transforms, and Rosenberg, using the Fourier integral, obtained relations between the base displacements and the incident earthquake motion.

B. Object and Scope

The objectives of the present study are: (1) to develop alternative methods of analysis for the problem of dynamic coupling of linear, building-foundation systems, and (2) to apply these methods to the investigation of the response of a few sample structures subjected to both harmonic and transient excitation.

Both direct and transform methods will be used to show that, under assumptions which appear to be physically reasonable, the response of the interacting system can be expressed as a linear combination of the response of simple linear oscillators subjected to modified excitations. This result will be shown to be valid even for systems that do not have classical normal modes. The major advantages of this representation are that it gives physical insight into the dynamics of the building-foundation system and makes the calculations involved equivalent to those for simple structures.

In the analysis, the foundation medium will be modeled by a linear, homogeneous, isotropic, elastic half-space. It will also be assumed that the base of the building is a rigid circular plate resting on the surface of the ground. Thus, any effects resulting from more complex geometry or from more complex material behavior will not be included.

The formulation of the problem and the solution of the corresponding equations of motion are given in Section C. Section D is devoted to the study of the transient and harmonic response of several single and multi-story, idealized building-foundation systems.

C. Method of Analysis

1. Formulation of the Problem

The idealized building-foundation system under investigation is shown in Fig. 2. It consists of a linear, viscously damped n-story structure with one degree of freedom per floor, resting on the surface of the half-space. For fixed-base response, the superstructure has a stiffness matrix K, mass matrix M, and damping matrix C, satisfying the condition $M^{-1}KM^{-1}C = M^{-1}CM^{-1}K$. O'Kelly[73] has shown this to be a necessary and sufficient condition for the superstructure to admit decomposition into classical normal modes. The base is assumed to be a single rigid plate of negligible thickness and no slippage is allowed between the base and the soil. Formulated this way, the building-foundation system has n + 2 significant degrees of freedom, namely, horizontal translation of each floor mass, horizontal translation of the base mass, and rotation of the system in the plane of motion.

The system, initially at rest, will be subjected to seismic motion represented by plane, horizontal shear waves traveling vertically upward. No scattering will result as the waves are normally incident on the flat foundation. In addition to this type of transient excitation, harmonic motion also will be considered.

The model for the building-foundation system shown in Fig. 2 has also been used by Tajimi[62], Parmelee et al.[66] and others.

It is noteworthy that since the superstructure admits decomposition into classical normal modes, there is a simpler mathematical model which is dynamically equivalent to the building foundation

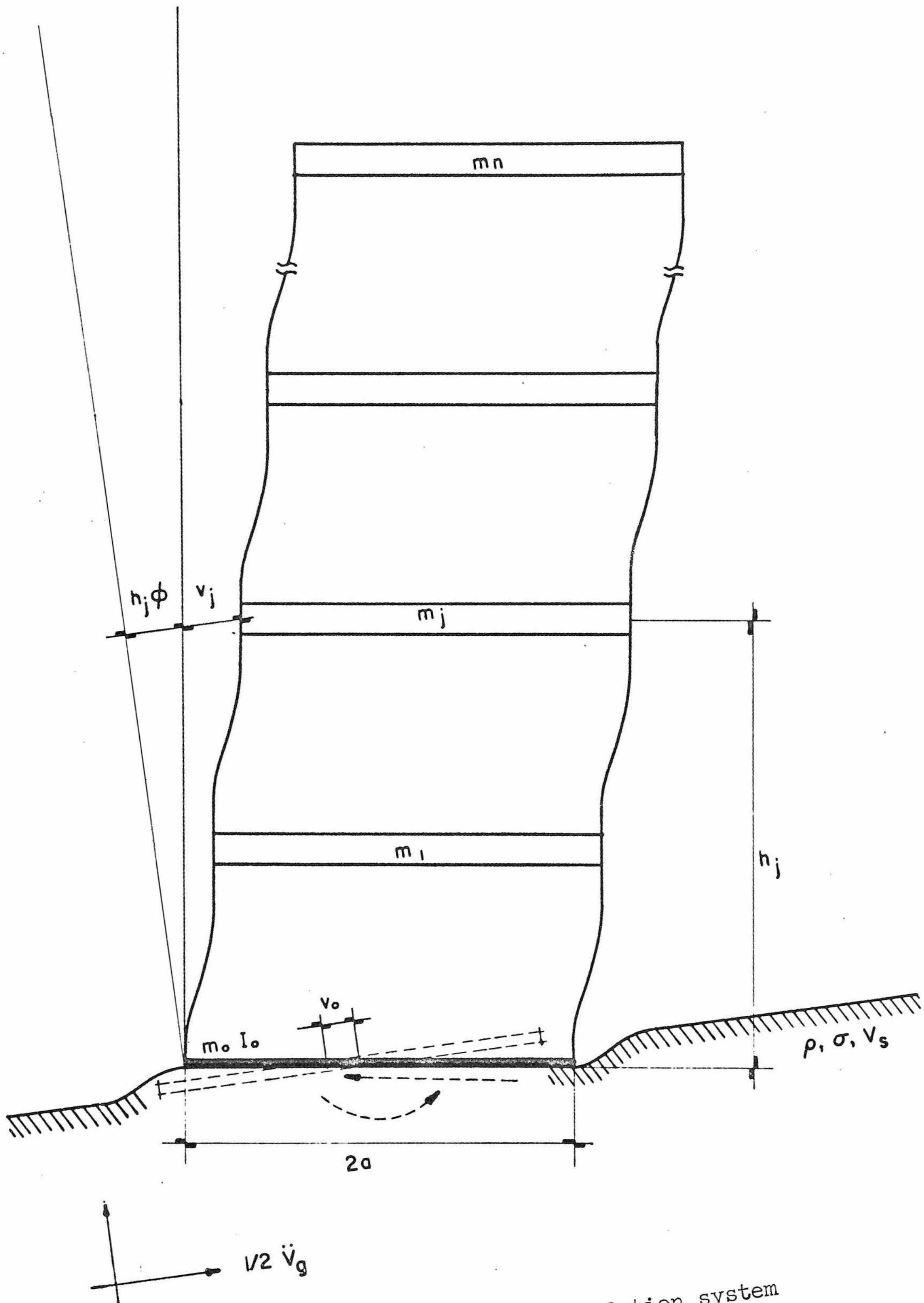


Fig. 2. Model of building-foundation system

system under study. This model, which is shown in Fig. 3, consists of n simple, damped oscillators attached to a base, identical to that of the system shown in Fig. 2. Each oscillator is described by its natural frequency ω_j, critical damping ratio η_j, mass M_j and height H_j defined by the corresponding modal quantities. In addition, the sum of the centroidal moments of inertia of the n masses is the same for both systems.

Equations of motion

Assuming small displacements, the equations of motion of the building-foundation model shown in Fig. 2 are

$$M\,\underline{\ddot{v}}^t + C\,\underline{\dot{v}} + K\,\underline{v} = 0 \tag{3.1a}$$

$$\sum_{j=1}^{n} m_j \ddot{v}_j^{\,t} + m_o(\ddot{v}_o + \ddot{v}_g) + P(t) = 0 \tag{3.1b}$$

$$\sum_{j=1}^{n} m_j h_j \ddot{v}_j^{\,t} + I_t \ddot{\varphi} + Q(t) = 0 \; . \tag{3.1c}$$

In these equations, $\underline{v} = \{v_j\}$, a column vector, v_j = horizontal displacement of the superstructure at the j^{th} floor relative to the base mass, excluding rotations; v_g = free-field, surface displacement due to the incident earthquake wave and its total reflection; v_o = translation of the base mass relative to the free-field motion; φ = rotation of the base mass; h_j = height of the j^{th} story above the base mass; $v_j^{\,t}$ = total horizontal displacement of the j^{th} mass with respect to a fixed vertical axis, i.e., $v_j^{\,t} = v_g + v_o + h_j\varphi + v_j$; m_j = mass of

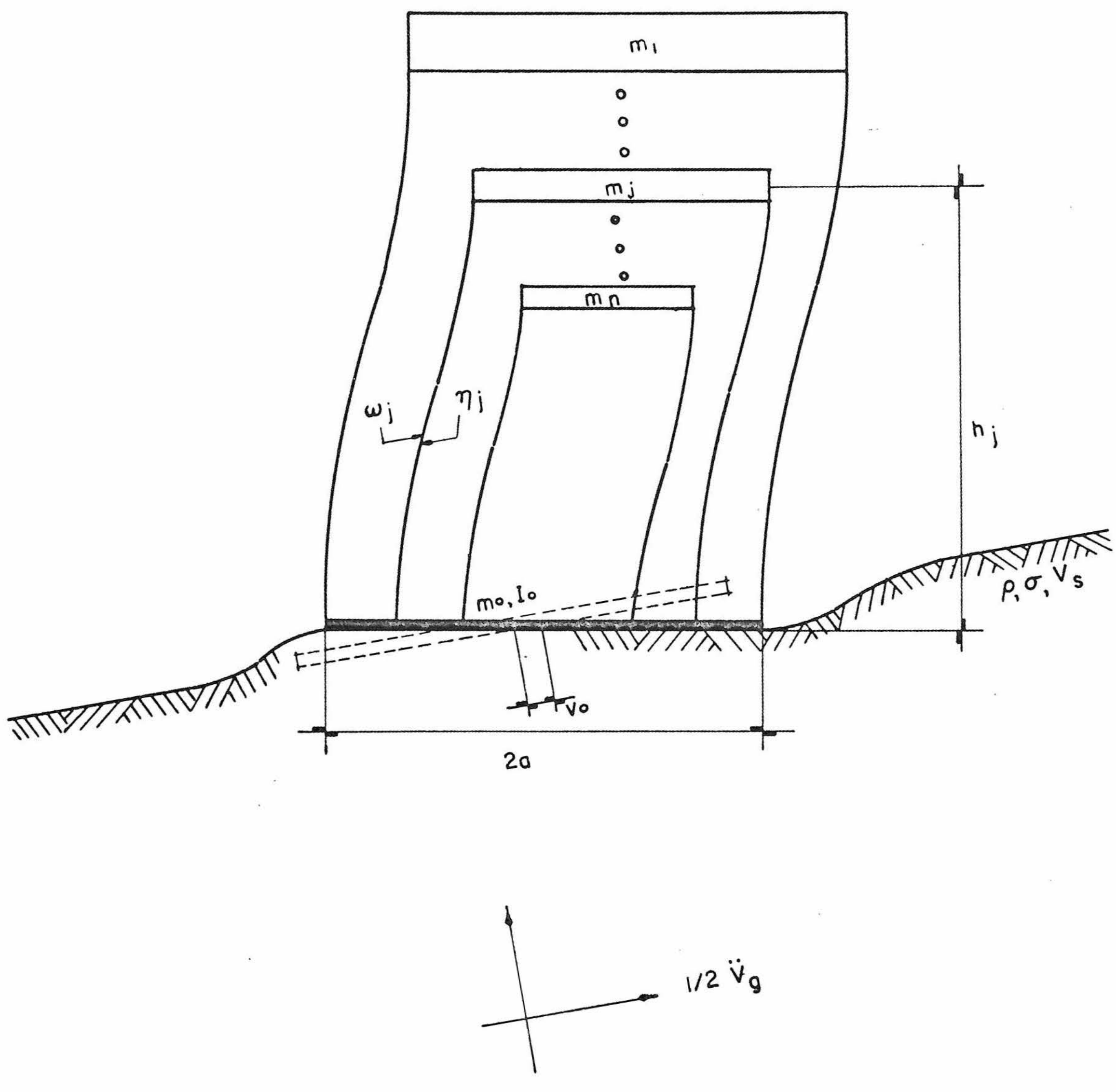


Fig. 3. Mathematical model of building-foundation system

the j^{th} floor; m_o = base mass, I_t = sum of the centroidal moments of inertia of the m + 1 masses; and P(t) and Q(t) are the interaction force and moment, respectively, between the base mass and the soil.

In this idealization of the earthquake motion, the free-field acceleration at the surface, $\ddot{v}_g$, is twice the amplitude of the incoming wave, and the motion at depth is the sum of the incident and reflected waves.

2. Relation Between the Interaction Forces and the Base Displacements

Laplace transforms will be used to solve Eqs. (3.1) for the unknown displacements v_o and v_j and the rotation φ. Before proceeding with the solution, it is necessary to express the interaction forces P(t) and Q(t) in terms of the displacements $v_o(t)$ and $\varphi(t)$. The functions P(t) and Q(t) are related to $v_o(t)$ and $\varphi(t)$ through convolution integrals due to the frequency dependence of the resistance of the half-space. Upon substitution into Eqs. (3.1), this leads to a system of linear integro-differential equations. The relation in the transformed space, however, is given by an impedance matrix similar to that of Eq. (2.31).

For the case of a rigid circular base, the necessary relations may be derived from the results obtained in Chapter I, which deals with the steady-state harmonic oscillations of a rigid disc perfectly bonded to an elastic half-space. In the present case, a solution is sought to the problem of transient horizontal and rocking vibrations of the disc. For this problem, the equations of motion that must be satisfied by the displacement vector $\underline{u} = (u_r, u_\theta, u_z)$ are[35]

$$(\lambda + 2\mu) \text{ grad div } \underline{u} - \mu \text{ curl curl } \underline{u} = \rho \frac{\partial^2 \underline{u}}{\partial t^2} \ . \tag{3.2}$$

The corresponding displacement boundary conditions under the base are

$$\left.\begin{aligned} u_r(ar',\theta,o,t) &= v_o(t) \cos\theta & (3.3a)\\ u_\theta(ar',\theta,0,t) &= - v_o(t) \sin\theta & (3.3b)\\ u_z(ar',\theta,0,t) &= ar'\varphi(t) \cos\theta & (3.3c)\end{aligned}\right\} \quad \begin{aligned} 0 \le r' \le 1 \\ t > 0 \end{aligned} \ .$$

As for the problem of harmonic oscillations of the disc, the tractions are specified to be zero over the remainder of the surface. In addition, it is assumed that the system is initially at rest, i.e.,

$$\underline{u}(ar',\theta,z,t) = 0 \qquad \text{at } t = 0 \ .$$

Taking Laplace transforms of Eqs. (3.2) and (3.3) results, respectively, in

$$(\lambda + 2\mu) \text{ grad div } \overline{\underline{u}} - \mu \text{ curl curl } \overline{\underline{v}} - \rho s^2 \overline{\underline{u}} = 0 \tag{3.4}$$

and

$$\left.\begin{aligned} \overline{u}_r(ar',\theta,0,s) &= \overline{v}_o(s) \cos\theta & (3.5a)\\ \overline{u}_\theta(ar',\theta,0,s) &= - \overline{v}_o(s) \sin\theta & (3.5b)\\ \overline{u}_z(ar',\theta,0,s) &= ar'\overline{\varphi}(s) \cos\theta & (3.5c)\end{aligned}\right\} \quad 0 \le r' \le 1$$

in which a bar over a function denotes the Laplace transform of that function and s is the parameter of the transform.

Comparison of Eqs. (3.4) and (3.5) with Eqs. (2.1) and (2.2a)-(2.2c), respectively suggests that the solution to the transient problem may be found directly from the solution to the harmonic problem considered in Chapter I. After detailed examination of the mathematical analysis developed in Chapter I. section C it is possible to obtain an expression analogous to Eq. (2.31) relating, in transformed space, the interaction forces $P(t)$ and $Q(t)$ to the displacements $v_o(t)$ and $\varphi(t)$:

$$\left\{ \begin{array}{c} \dfrac{\bar{P}(s)}{\mu a^2} \\ \\ \dfrac{\bar{Q}(s)}{\mu a^3} \end{array} \right\} = \begin{bmatrix} K_{hh}(s_o,\sigma) & K_{hm}(s_o,\sigma) \\ \\ K_{mh}(s_o,\sigma) & K_{mm}(s_o,\sigma) \end{bmatrix} \left\{ \begin{array}{c} \dfrac{\bar{v}_O(s)}{a} \\ \\ \bar{\varphi}(s) \end{array} \right\} \qquad (3.6)$$

In Eq. (3.6), $\bar{P}(s)$ and $\bar{Q}(s)$ are the Laplace transforms of the force $P(t)$ and moment $Q(t)$, respectively, and K_{hh}, K_{hm}, K_{mh}, K_{mm} represent, as before the dimensionless impedances of the problem. The functions K_{hh}, K_{hm}, K_{mh} and K_{mm} are given by the same equations that determine the corresponding functions for the steady-state harmonic vibrations of the disc. In this case, however, the frequency parameter ia_o is replaced by the complex dimensionless variable $s_o = sa/V_s$ where V_s is the shear wave velocity of the foundation medium.

3. Solution of the Equations of Motion

Since the superstructure admits decomposition into classical normal modes, the transformed version of Eq. (3.1a) may be solved for v_j in terms of the interaction displacements v_o and φ and the free-field earthquake displacement v_g:

$$\bar{v}_j = -\sum_{k=1}^{n} \frac{\beta_k(\bar{\ddot{v}}_g + \bar{\ddot{v}}_o) + \gamma_k\bar{\ddot{\varphi}}}{s^2 + 2\eta_j\omega_j s + \omega_j^2} X_{jk} \qquad (3.7a)$$

where

X_{jk} = j^{th} component of $\underline{X}_k$ = modal displacement of the j^{th} mass in the k^{th} mode of the superstructure, if it were supported on a rigid foundation

$$\beta_k = \frac{\underline{X}_k^T M \underline{1}}{\underline{X}_k^T M \underline{X}_k} \; ; \; \underline{1} = \{1\} \tag{3.7b}$$

$$\gamma_k = \frac{\underline{X}_k^T M \underline{h}}{\underline{X}_k^T M \underline{X}_k} \; ; \; \underline{h} = \{h_j\} \tag{3.7c}$$

ω_k = undamped natural frequency of the k^{th} mode of the superstructure, given by

$$\omega_k^2 = \frac{\underline{X}_k^T K \underline{X}_k}{\underline{X}_k^T M \underline{X}_k} \tag{3.7d}$$

η_k = critical damping ratio of the k^{th} mode of the superstructure, defined by

$$2\eta_k \omega_k = \frac{\underline{X}_k^T C \underline{X}_k}{\underline{X}_k^T M \underline{X}_k} \; . \tag{3.7e}$$

Substituting Eq. (3.7a) into the transformed versions of Eqs. (3.1b) and (3.1c) leads to

$$s^2[(\sum_j \overline{F}_j M_j) + m_o]\, \overline{v}_o + s^2(\sum_j \overline{F}_j Z_j)\, \overline{\varphi} + \overline{P}(s) = -(\sum_j \overline{F}_j M_j + m_o)\, \overline{\ddot{v}}_g \tag{3.8a}$$

$$s^2(\sum_j \overline{F}_j Z_j)\, \overline{v}_o + s^2[(\sum_j \overline{F}_j I_j) + I_t]\, \overline{\varphi} + \overline{Q}(s) = -(\sum_j \overline{F}_j Z_j)\, \overline{\ddot{v}}_g \tag{3.8b}$$

in which

$$\bar{F}_j(s) = \frac{\omega_j^2 + 2\eta_j\omega_j s}{s^2 + 2\eta_j\omega_j s + \omega_j^2} \tag{3.9a}$$

$$M_j = \beta_j \sum_i m_i X_{ij} = \frac{(\sum_i m_i X_{ij})^2}{\sum_i m_i X_{ij}^2} \tag{3.9b}$$

$$Z_j = \gamma_j \sum_i m_i X_{ij} = \frac{(\sum_i m_i X_{ij})(\sum_i m_i h_i X_{ij})}{\sum_i m_i X_{ij}^2} \tag{3.9c}$$

$$I_j = \gamma_j \sum_i m_i h_i X_{ij} = \frac{(\sum_i m_i h_i X_{ij})^2}{\sum_i m_i X_{ij}^2} \; . \tag{3.9d}$$

Summation throughout will be from 1 to n unless otherwise indicated.

Substitution of Eq. (3.6) into Eqs. (3.8) leads to a system of linear algebraic equations in the unknowns $\bar{v}_o$ and $\bar{\varphi}$ whose solution is

$$\bar{v}_o = -\frac{\bar{\ddot{v}}_g}{\Delta}\Big[s^2 \sum_{\substack{j,k \\ j\neq k}} (M_j I_k - Z_j Z_k)\hat{F}_{jk} + (s^2 I_t + \mu a^3 K_{mm})(\sum_j \hat{F}_j M_j)$$

$$+ s^2 m_o(\sum_j \hat{F}_j I_j) - \mu a^2 K_{hm}(\sum_j \hat{F}_j Z_j) \tag{3.10a}$$

$$+ m_o(s^2 I_t + \mu a^3 K_{mm}) \prod_{k=1}^{n} (s^2 + 2\eta_k\omega_k s + \omega_k^2)\Big]$$

$$\bar{\varphi} = -\frac{\bar{\ddot{v}}_g}{\Delta}\Big\{ \mu a K_{hh}(\sum_j \hat{F}_j Z_j) - \mu a^2 K_{hm}[(\sum_j \hat{F}_j M_j)$$

$$+ m_o \prod_{k=1}^{n} (s^2 + 2\eta_k\omega_k s + \omega_k^2)]\Big\} \tag{3.10b}$$

in which

$$\hat{F}_j(s) = (\omega_j^2 + 2\eta_j\omega_j s) \prod_{\substack{k=1 \\ k\neq j}}^{n} (s^2 + 2\eta_k \omega_k s + \omega_k^2) \qquad (3.10c)$$

$$\hat{F}_{jk}(s) = (\omega_j^2 + 2\eta_j \omega_j s)(\omega_k^2 + 2\eta_k\omega_k s) \prod_{\substack{\ell=1 \\ \ell\neq j, \ell\neq k}}^{n} (s^2 + 2\eta_\ell\omega_\ell s + \omega_\ell^2) \qquad (3.10d)$$

and

$$\Delta = s^4 \sum_{\substack{j,k \\ j\neq k}} (M_j I_k - Z_j Z_k)\hat{F}_{jk} + s^2(s^2 I_t + \mu a^3\ Kmm)(\sum_j \hat{F}_j M_j)$$

$$+ s^2(s^2 m_o + \mu a\ K_{hh})(\sum_j \hat{F}_j I_j) - 2s^2\ \mu a^2\ K_{hm}(\sum_j \hat{F}_j Z_j)$$

$$+ \prod_{k=1}^{n} (s^2 + 2\eta_k\omega_k s + \omega_k^2)[s^4 m_o I_t + \mu a^3\ s^2 m_o\ K_{mm}$$

$$+ \mu a^2\ s^2\ K_{hh}\ I_t + \mu^2 a^4(K_{hh}K_{mm} + K_{hm}^2)] \ . \qquad (3.10e)$$

The displacements $\bar{v}_j$ may then be found by substituting Eqs. (3.10) into Eq. (3.7a):

$$\bar{v}_j = - \frac{\bar{\ddot{v}}_g}{\Delta} \sum_\ell \bar{\xi}_\ell(s)\ X_{j\ell}\ ;\ j = 1,\cdots,n \qquad (3.11a)$$

in which

$$\bar{\xi}_\ell(s) = - \mu a^2\ s^2\ K_{hm} \sum_{\substack{k \\ k\neq\ell}} (\beta_\ell Z_k - \gamma_\ell M_k)\ \hat{\Pi}_{k\ell}(s)$$

$$+ \mu a\ s^2\ K_{hh} \sum_{\substack{k \\ k\neq\ell}} (\beta_\ell I_k - \gamma_\ell Z_k)\ \hat{\Pi}_{k\ell}(s) + (\mu a\ s^2\ K_{hh}\ I_t$$

$$+ \mu^2\ a^4\ K_{hh}\ K_{mm})\ \hat{\Pi}_\ell(s)\beta_\ell + \mu a^2\ s^2\ m_o\ K_{hm}\ \hat{\Pi}_\ell(s)\gamma_\ell \qquad (3.11b)$$

and $$\hat{\Pi}_{k\ell}(s) = (\omega_k^2 + 2\eta_k\,\omega_k s) \prod_{\substack{m=1\\ m\neq\ell,\, m\neq k}}^{n} (s^2 + 2\eta_m\,\omega_m s + \omega_m^2) \tag{3.11c}$$

$$\hat{\Pi}_\ell(s) = \prod_{\substack{m=1\\ m\neq\ell}}^{n} (s^2 + 2\eta_m\,\omega_m s + \omega_m^2)\ . \tag{3.11d}$$

Equations (3.10a), (3.10b) and (3.11a) may be rewritten for simplicity:

$$\bar{v}_o = \bar{\ddot{v}}_g \frac{\Delta_o}{\Delta} \tag{3.12a}$$

$$\bar{\varphi} = \bar{\ddot{v}}_g \frac{\Delta_\varphi}{\Delta} \tag{3.12b}$$

$$\bar{v}_j = \bar{\ddot{v}}_g \frac{\Delta_j}{\Delta}\ ;\ j = 1,2,\cdots,n^*\ . \tag{3.12c}$$

These equations provide an explicit solution in transformed space to the equations of motion (3.1) in terms of the transform of the incident earthquake, the physical quantities defining the building and its foundation, and the impedance functions $K_{hh}(s_o,\sigma)$, $K_{hm}(s_o,\sigma)$, $K_{mh}(s_o\sigma)$ and $K_{mm}(s_o,\sigma)$.

<u>Inversion of Laplace transforms</u>

The unknown displacements $v_o(t)$, $\varphi(t)$ and $v_j(t)$ may now be found by inverting their corresponding Laplace transforms. Thus,

$$\begin{Bmatrix} v_o(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \frac{1}{2\pi i} \int_C \frac{\bar{\ddot{v}}_g(s)}{\Delta(s)} \begin{Bmatrix} \Delta_o(s) \\ \Delta_\varphi(s) \\ \Delta_j(s) \end{Bmatrix} e^{st}\, ds \tag{3.13}$$

*The range of the variable subscripts will be given only when they first appear.

where the integrals are evaluated over C, the Bromwich contour.

By a direct application of the convolution theorem of Laplace transformations[75], Eq. (3.13) becomes

$$\begin{Bmatrix} v_o(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \int_0^t \begin{Bmatrix} h_o(t-\tau) \\ h_\varphi(t-\tau) \\ h_j(t-\tau) \end{Bmatrix} \ddot{v}_g(\tau)\, d\tau \qquad (3.14a)$$

in which

$$h_k(t) = \frac{1}{2\pi i} \int_C \frac{\Delta_k(s)}{\Delta(s)}\, e^{st}\, ds; \; k=0,\varphi,j \qquad (3.14b)$$

are the impulse response functions of the system.

Contour integration is next used to solve, in closed form, the integrals appearing in Eq. (3.14b). For purposes of clarity, these integrals will be evaluated first for the case of linear impedances (equivalent to an idealized discrete foundation, represented by linear springs and dashpots). Secondly, the actual foundation of the system under investigation, i.e., the elastic half-space, will be considered.

(i) <u>Analysis for a discrete foundation</u>

Hsieh[72] showed that the steady-state, forced harmonic motion of a rigid plate on the elastic half-space can be modeled using a simpler system. For each of the four degrees of freedom of the plate the elastic medium is replaced by a simple oscillator consisting of a linear spring whose stiffness depends upon the frequency of oscillation, and a linear viscous dashpot which is also frequency dependent.

Available numerical results(5,11,14) indicate that most of the dynamic properties of the springs and dashpots representing the elastic half-space model of the foundation remain nearly constant within the frequency range of interest. It is then reasonable to assume as a first approximation that the linear springs and viscous dampers have constant properties. This is the approach used by Parmelee et al.(66) to study, by numerical integration of the equations of motion, the earthquake response of selected multi-story buildings resting on an elastic half-space.

With the assumption of constant properties, the functions K_{hh}, K_{hm}, K_{mh} and K_{mm} become linear in s_o (or s). The functions $\Delta_o(s)$, $\Delta_\varphi(s)$ and $\Delta_j(s)$ in the numerators of Eqs. (3.12) then become polynomials of degree 2n, while the function $\Delta(s)$ in the denominator gives a polynomial of degree 2n + 4. Hence, the only singularities of the integrand in Eq. (3.14b) correspond to the n + 2 pairs of complex conjugate roots of the polynomial $\Delta(s)$. Each of these pairs is associated with a resonant frequency of the system.

With this information the integral in Eq. (3.14b) may now be evaluated by contour integration. This integral vanishes for $t < 0$ because there are no singularities on the right hand plane and in addition

$$\frac{\Delta_k(s)}{\Delta(s)} = O(s^{-4}) \quad \text{as} \quad s \to \infty . \tag{3.15}$$

For $t \geq 0$ the integration is performed around the contour Γ shown in Fig. 4. By Cauchy's theorem of the residues(75):

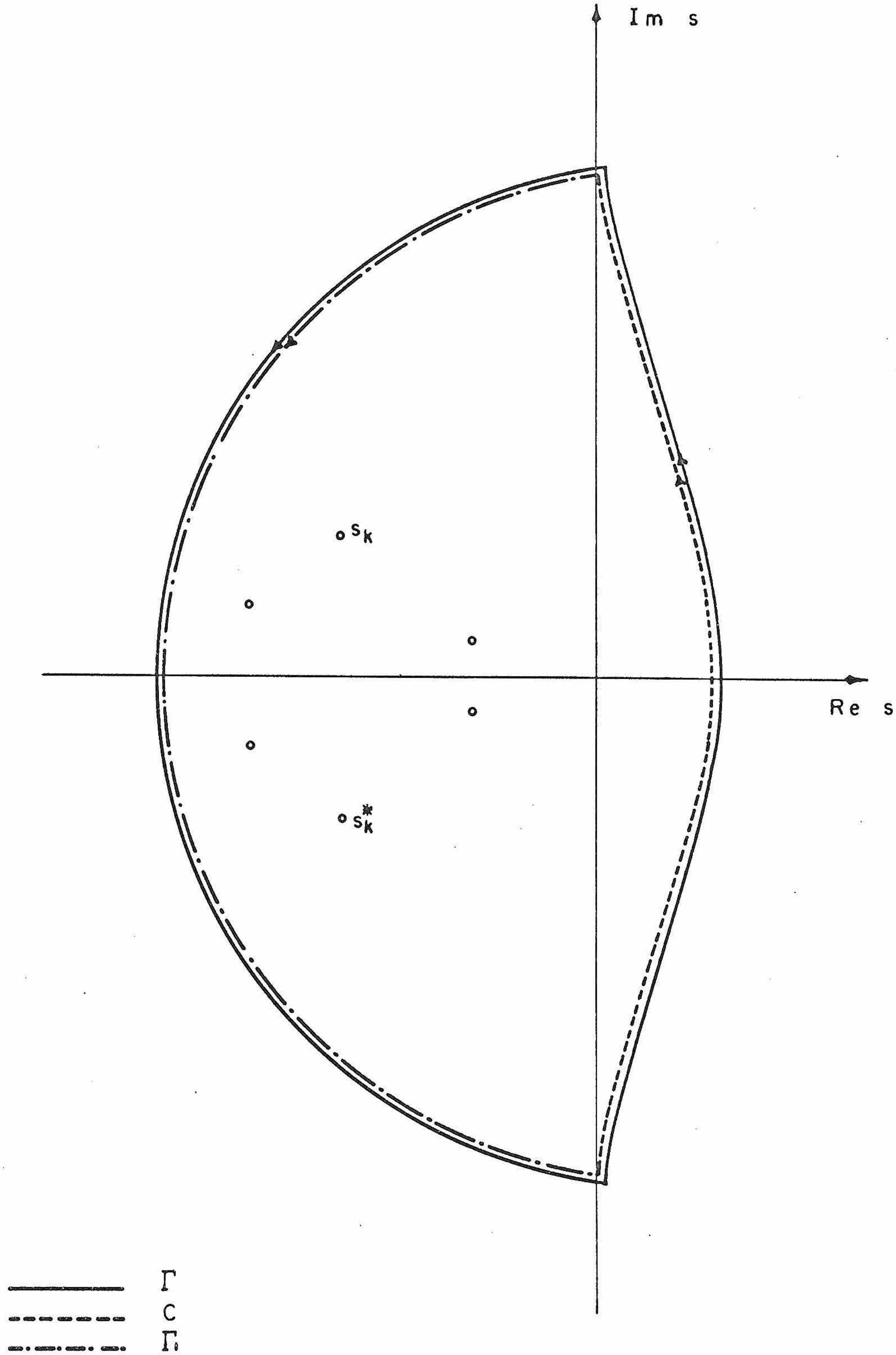


Fig. 4. Contour for integral of Eq. (3.14b)

$$h_k(t) = \Sigma \text{ Residues } - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\Delta_k(s)}{\Delta(s)} e^{st} \, ds . \tag{3.16}$$

The integral on the right hand side of Eq. (3.16) vanishes in view of Eq. (3.15). Under the assumption that $\Delta(s)$ does not have repeated roots (whose presence would merely modify the expressions for the residues) Eq. (3.16) gives

$$h_k(t) = \sum_{\ell=1}^{n+2} \left[\frac{\Delta_k(s_\ell)}{\Delta'(s_\ell)} e^{s_\ell t} + \frac{\Delta_k(s_\ell^*)}{\Delta'(s_\ell^*)} e^{s_\ell^* t} \right] \tag{3.17a}$$

$$= 2 \sum_{\ell=1}^{n+2} \left[\operatorname{Re} \frac{\Delta_k(s_\ell)}{\Delta'(s_\ell)} e^{s_\ell t} \right] \tag{3.17b}$$

where s_ℓ is a root of $\Delta(s)$ and s_ℓ^* is its conjugate; and $\Delta'(s_\ell)$ is the first derivative of $\Delta(s)$ with respect to s evaluated at s_ℓ. $\Delta'(s_\ell)$ can be evaluated explicitly from the preceding analysis.

Introducing the real constants $a_{\ell k}$, $b_{\ell k}$, σ_ℓ and β_ℓ defined by

$$a_{\ell k} + i b_{\ell k} = 2 \frac{\Delta_k(s_\ell)}{\Delta'(s_\ell)} \tag{3.18a}$$

$$-\sigma_\ell + i\beta_\ell = s_\ell \tag{3.18b}$$

Eq. (3.17b) becomes

$$h_k(t) = \sum_{\ell=1}^{n+2} e^{-\sigma_\ell t} (a_{\ell k} \cos \beta_\ell t - b_{\ell k} \sin \beta_\ell t) \tag{3.19}$$

that is, an equation for the impulse response functions of the system in terms of elementary functions and real constants that can be evaluated explicitly from Eqs. (3.18).

(ii) Analysis for a continuous foundation

Equation (3.19) shows that the impulse response functions of a building-foundation system supported on a discrete foundation are given by a linear combination of n + 2 pairs of terms, corresponding to the n + 2 pairs of complex conjugate roots of $\Delta(s)$ in Eq. (3.14b). There are as many pairs of roots of $\Delta(s)$ as there are degrees of freedom in the system and associated with each pair of roots there is a resonant frequency.

When the discrete foundation is replaced by the elastic half-space the impedance functions K_{hh}, K_{hm}, K_{mh} and K_{mm} no longer have the precise functional form k + cs (k and c are constants). On physical grounds, however, it is expected that the building-foundation system should still exhibit resonant frequencies and that the number of these frequencies should not be affected by the substitution of the half-space.

To preserve these features of the problem, it is assumed that $\Delta(s)$ will again have n + 2 pairs of non-repeated complex conjugate roots; one pair to be associated with each resonant frequency. It will also be assumed that the impedance functions K_{hh}, K_{hm}, K_{mh} and K_{mm} are analytic away from infinity and are such that $\Delta_k(s)/\Delta(s)$ in Eq. (3.14b) goes to zero as s approaches infinity.

With these assumptions it may be seen that the contour integration performed in the preceding section remains valid when the discrete foundation is replaced by an elastic half-space. Hence, the impulse response functions of the corresponding building-foundation system are again given by Eq. (3.19). In this case, however, the quantities $a_{\ell k}$, $b_{\ell k}$, σ_ℓ, β_ℓ, s_ℓ, $\Delta_k(s_\ell)$ and $\Delta'(s_\ell)$ must be obtained by using the impedance functions in the half-space and they will be different, in general, from those found for the case of the discrete foundation .

Expressions for the displacements

Expressions for the displacements $v_o(t)$, $\varphi(t)$ and $v_j(t)$ may be obtained by substituting Eq. (3.19) into Eq. (3.14a),

$$\begin{Bmatrix} v_o(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \sum_{\ell=1}^{n+2} \begin{Bmatrix} a_{\ell o} \\ a_{\ell \varphi} \\ a_{\ell j} \end{Bmatrix} \int_o^t e^{-\sigma_\ell (t-\tau)} \cos \beta_\ell (t-\tau)\, \ddot{v}_g(\tau)\, d\tau$$

$$- \sum_{\ell=1}^{n+2} \begin{Bmatrix} b_{\ell o} \\ b_{\ell \varphi} \\ b_{\ell j} \end{Bmatrix} \int_o^t e^{-\sigma_\ell (t-\tau)} \sin \beta_\ell (t-\tau)\, \ddot{v}_g(\tau)\, d\tau \,. \qquad (3.20)$$

An alternative form of these expressions, convenient because it lends itself to a simple physical interpretation, is found by integrating by parts the terms in Eqs. (3.20) involving cosines:

$$\begin{Bmatrix} v_o(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = -\sum_{\ell=1}^{n+2} \begin{Bmatrix} a_{\ell o} \\ a_{\ell \varphi} \\ a_{\ell j} \end{Bmatrix} \int_o^t e^{-\sigma_\ell (t-\tau)} \sin \beta_\ell (t-\tau) \left[\frac{\sigma_\ell^{\,2}+\beta_\ell^{\,2}}{\beta_\ell} \, \dot{v}_g(\tau) + \frac{\sigma_\ell}{\beta_\ell} \, \ddot{v}_g(\tau) \right] d\tau$$

$$-\sum_{\ell=1}^{n+2} \begin{Bmatrix} b_{\ell o} \\ b_{\ell \varphi} \\ b_{\ell j} \end{Bmatrix} \int_o^t e^{-\sigma_\ell (t-\tau)} \sin \beta_\ell (t-\tau) \; \ddot{v}_g(\tau) \; d\tau \qquad (3.21)$$

or

$$\begin{Bmatrix} v_o(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = -\sum_{\ell=1}^{n+2} \frac{1}{\tilde{\omega}_\ell \sqrt{1 - \tilde{\eta}_\ell^{\,2}}} \int_o^t e^{-\tilde{\eta}_\ell \tilde{\omega}_\ell (t-\tau)} \sin \tilde{\omega}_\ell \sqrt{1 - \tilde{\eta}_\ell^{\,2}} \; (t-\tau) \begin{Bmatrix} \ddot{v}_{\ell o}^{\,e}(\tau) \\ \ddot{v}_{\ell \varphi}^{\,e}(\tau) \\ \ddot{v}_{\ell j}^{\,e}(\tau) \end{Bmatrix} d\tau$$

(3.22a)

in which $\tilde{\omega}_\ell$ and $\tilde{\eta}_\ell$ are defined by

$$\tilde{\omega}_\ell = (\sigma_\ell^{\,2} + \beta_\ell^{\,2})^{\frac{1}{2}} \qquad (3.22b)$$

$$\tilde{\eta}_\ell = \sigma_\ell / (\sigma_\ell^{\,2} + \beta_\ell^{\,2})^{\frac{1}{2}} \qquad (3.22c)$$

and

$$\begin{Bmatrix} \ddot{v}_{\ell o}^{\,e}(t) \\ \ddot{v}_{\ell \varphi}^{\,e}(t) \\ \ddot{v}_{\ell j}^{\,e}(t) \end{Bmatrix} = \begin{Bmatrix} a_{\ell o} \\ a_{\ell \varphi} \\ a_{\ell j} \end{Bmatrix} [\tilde{\omega}_\ell^{\,2} \dot{v}_g(t) + \tilde{\eta}_\ell \tilde{\omega}_\ell \ddot{v}_g(t)] + \begin{Bmatrix} b_{\ell o} \\ b_{\ell \varphi} \\ b_{\ell j} \end{Bmatrix} \tilde{\omega}_\ell \sqrt{1 - \tilde{\eta}_\ell^{\,2}} \; \ddot{v}_g(t). \qquad (3.22d)$$

In the above derivation use has been made of the equation

$$\sum_{\ell=1}^{n+2} \begin{Bmatrix} a_{\ell o} \\ a_{\ell\varphi} \\ a_{\ell j} \end{Bmatrix} = 0 \qquad (3.23)$$

which is obtained from the requirement that the velocities $\dot{v}_o(t)$, $\dot{\varphi}(t)$ and $\dot{v}_j(t)$ vanish at $t = 0$.

Interpretation of the solution

The expressions for the displacements $v_o(t)$, $\varphi(t)$ and $v_j(t)$ in Eq. (3.22a) show that the transient response of the building-foundation system may be obtained as a linear combination of the response of $n + 2$ simple, viscously damped, linear oscillators resting on a rigid ground (provided the roots of $\Delta(s_\ell)$ are distinct). Each oscillator, described by its undamped natural frequency $\tilde{\omega}_\ell$ and fraction of critical damping $\tilde{\eta}_\ell$, experiences an acceleration at its base given by $\ddot{v}^e_{\ell k}$. The subscript k takes the values 0, φ and j corresponding to the displacements $v_o(t)$, $\varphi(t)$ and $v_j(t)$, respectively. The main advantages of this representation are the physical insight it gives into the dynamics of the building-foundation system, and the simplicity of the computations which are reduced to those of a simple oscillator.

The result described above is valid even for building-foundation systems that do not have classical normal modes as no assumption about the existence of such modes was made in the derivation of Eq. (3.22a).

An alternative form of the solution

An alternative form of the solution of the equations of motion (3.1) may be obtained from Eq. (3.13) by selecting the imaginary axis as the Bromwich contour. This is possible because the building-foundation system under investigation is stable, and therefore, any singularities occurring in the transformed space must be either to the left of the imaginary axis, or if on that axis, they can be at most simple poles. Thus, after introducing the variable transformation $s = i\omega$, Eq. (3.13) becomes

$$\begin{Bmatrix} v_o(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\ddot{\bar{v}}_g(i\omega)}{\Delta(i\omega)} \begin{Bmatrix} \Delta_o(i\omega) \\ \Delta_\varphi(i\omega) \\ \Delta_j(i\omega) \end{Bmatrix} e^{i\omega t}\, d\omega \qquad (3.24)$$

The integral operator appearing in Eq. (3.24) represents a Fourier integral that may be evaluated by the Fast Fourier Transform (FFT) technique. The formulation in Eq. (3.24) coupled with the FFT should prove to be very useful because of the high computational efficiency of the FFT algorithm.[76]

Equation (3.24) could have been obtained directly by applying the Fourier operator to the equations of motion (3.1) and using the inverse Fourier theorem. This, together with the FFT was used recently by Liu and Fagel[83] to obtain the response of a single-story building-foundation system.

It should be noted that in using Eq. (3.24) only values of the impedance functions K_{hh}, K_{hm}, K_{mh} and K_{mm} corresponding to the real frequency a_o are required.

Steady-state response

Although the methods developed above apply to the transient response of building-foundation systems, expressions for the steady-state response may be obtained readily. If the free-field surface motion $v_g(t)$ equals $\bar{v}_g \exp(i\omega t)$, where $\bar{v}_g$ is the amplitude of the motion and ω the frequency of oscillation, then the corresponding displacements $v_o(t)$, $\varphi(t)$ and $v_j(t)$ are

$$\begin{Bmatrix} v_o(t) \\ \varphi(t) \\ v_j(t) \end{Bmatrix} = \begin{Bmatrix} \bar{v}_o \\ \bar{\varphi} \\ \bar{v}_j \end{Bmatrix} e^{i\omega t} \tag{3.25}$$

where the complex quantities $\bar{v}_o$, $\bar{\varphi}$ and $\bar{v}_j$ are given by Eqs. (3.12), with the provision that s be replaced by $i\omega$ and $\ddot{v}_g$ by $-\omega^2 \bar{v}_g$.

Random response

The preceding analyses have implicitly assumed that the free-field earthquake acceleration $\ddot{v}_g$ is a deterministic function of time. However, since the problem is linear, a complete analysis of the response of the building-foundation system to a random free-field acceleration may be obtained from Eqs. (3.14a), (3.22a) or (3.25). For instance, Eq. (3.14a) may be used to show that if $\ddot{v}_g(t)$ is Gaussian, then so are $v_o(t)$, $\varphi(t)$ and $v_j(t)$[78]. The corresponding mean and covariance functions, which completely specify a Gaussian response process, may also be obtained from Eq. (3.14a). Thus, for example

$$E[v_o(t)] = \int_0^t E[\ddot{v}_g(\tau)]\, h_o(t - \tau)\, d\tau \tag{3.26a}$$

and

$$\kappa_{v_o v_o}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} \kappa_{\ddot{v}_g \ddot{v}_g}(\tau_1,\tau_2)\, h_o(t_1 - \tau_1)\, h_o(t_2 - \tau_2)\, d\tau_1\, d\tau_2 \tag{3.26b}$$

give, respectively, the mean and the covariance function of the relative base displacement $v_o(t)$ in terms of the corresponding properties of the free-field acceleration $\ddot{v}_g(t)$. Similar expressions can be obtained for the displacements $\varphi(t)$ and $v_j(t)$.

4. An Application of Foss's Method to Systems with Discrete Foundations

The foregoing sections of this chapter have been devoted to the solution of the equations of motion of a building supported on an elastic half-space. Equations (3.1) were solved by the Laplace operational method as it was found that the relation between the interaction forces and the displacements could be most conveniently expressed in the transformed space (see Eq. (3.6)). When the soil is represented by linear, discrete elements, Eqs. (3.1) reduce to a system of second order ordinary differential equations with constant coefficients which may be solved by several methods. The Laplace operational approach used earlier provides but one example. Another method for the solution of these equations will next be considered. This method also gives the response of a building supported on a discrete foundation as a linear combination of the responses of simple oscillators resting on a rigid ground.

Equations of motion

The equations of motion of buildings resting upon foundations which can be represented by lumped parameter, time-invariant, linear models may be written as

$$M_o \ddot{\underline{X}} + C_o \dot{\underline{X}} + K_o \underline{X} = - \underline{f}\, \ddot{v}_g(t) \tag{3.27}$$

where M_o, C_o and K_o are $N \times N$ $(N = n + 2)$ symmetric matrices with K_o non-singular, $\underline{X}$ is the displacement vector; $\underline{f}$ is a known vector, and $\ddot{v}_g(t)$ is the free-field earthquake acceleration.

The classical normal mode method of analysis cannot be used to solve Eq. (3.27) as the system does not, in general, possess normal modes in N-space. Foss[77] however, has shown that systems which cannot be uncoupled in N-space may still be solvable by modal methods on transforming them to 2N-space. To include systems which cannot be solved by modal techniques in either N or 2N-space, O'Kelly[73] developed a general theory of vibration for lumped parameter, time-invariant, damped linear systems. Although this method is always applicable, it is more convenient to use Foss's formulation in 2N-space when possible.

Equation (3.27) will be solved by the modal method proposed by Foss, as the corresponding building-foundation systems generally satisfy the condition necessary for its applicability.

Method of solution

Equation (3.27) may be combined with the identity equation

$$M_o\dot{\underline{X}} - M_o\dot{\underline{X}} = 0 \tag{3.28}$$

to obtain a system of first order differential equations in 2N unknowns:

$$R\dot{\underline{Z}} + S\underline{Z} = -\underline{F}\,\ddot{v}_g(t) \tag{3.29a}$$

in which

$$R = \begin{bmatrix} 0 & M_o \\ M_o & C_o \end{bmatrix} \; ; \quad S = \begin{bmatrix} -M_o & 0 \\ 0 & K_o \end{bmatrix} \tag{3.29b}$$

$$\underline{F} = \begin{Bmatrix} \underline{0} \\ \underline{f} \end{Bmatrix} \; ; \quad \underline{Z} = \begin{Bmatrix} \dot{\underline{X}} \\ \underline{X} \end{Bmatrix} .$$

Equation (3.29a) may be uncoupled and solved by superposition provided the eigenvalues of $S^{-1}R$ are distinct. Proceeding on this assumption, the matrix

$$U = -S^{-1}R = \begin{bmatrix} 0 & I \\ -C_oM_o & -C_oK_o^{-1} \end{bmatrix} ; \text{ a } 2N \times 2N \text{ matrix} \tag{3.30}$$

may be diagonalized by a similarity transformation Φ, the columns of which are the eigenvectors of R and S. From the fact that R and S are symmetric,

$$\Phi^T R\, \Phi = \tilde{R}, \text{ a diagonal } 2N \times 2N \text{ matrix} \tag{3.31a}$$

$$\Phi^T S\, \Phi = \tilde{S}, \text{ a diagonal } 2N \times 2N \text{ matrix} \tag{3.31b}$$

Equations (3.31) are the orthogonality conditions in 2N-space and may be expanded in terms of N-space quantities. From the form of Eqs. (3.29) it is easy to see that the i^{th} column, $\underline{\Phi}_i$, of Φ may be partitioned

$$\underline{\Phi}_i = \begin{Bmatrix} \alpha_i \ \underline{\varphi}_i \\ \\ \underline{\varphi}_i \end{Bmatrix} ; \ i=1,2,\cdots,2N \,. \tag{3.32}$$

$\underline{\varphi}_i$ is an $N \times 1$ column vector and α_i is an eigenvalue of U.

Equations (3.29) may be uncoupled by making use of the orthogonality conditions (3.31). After solving each uncoupled equation, the following solution was obtained by Foss[77] for a system which is initially at rest

$$\underline{Z} = \begin{Bmatrix} \dot{\underline{X}} \\ \\ \underline{X} \end{Bmatrix} = -\sum_{k=1}^{2N} \left(\frac{G_k}{\tilde{R}_{kk}} \int_0^t e^{\alpha_k(t-\tau)} \ \ddot{v}_g(\tau) \ d\tau \right) \begin{Bmatrix} \alpha_k \ \underline{\varphi}_k \\ \\ \underline{\varphi}_k \end{Bmatrix} . \tag{3.33}$$

In this equation, G_k is an element of the $2N \times 1$ column vector $\underline{G} = \Phi^T \underline{F}$ and $\tilde{R}_{kk}$ is the k^{th} diagonal element of the matrix R.

A more convenient form of the solution may be obtained by noting that the eigenvalues α_k occur in complex conjugate pairs as do the corresponding eigenvectors. Thus, the equation for the displacements $\underline{X}$ in Eq. (3.33) becomes

$$\underline{X} = -2 \sum_{k=1}^{2N} \left(\mathrm{Re} \ \frac{G_k}{\tilde{R}_{kk}} \ \underline{\varphi}_k \int_0^t e^{\alpha_k(t-\tau)} \ \ddot{v}_g(\tau) \ d\tau \right) . \tag{3.34}$$

Proceeding as in Section 3, it is easy to show that Eq. (3.34) may be written in the form

$$\underline{X} = -\sum_{k=1}^{N} \frac{1}{\tilde{\theta}_k \sqrt{1 - \tilde{\lambda}_k}} \int_0^t e^{-\tilde{\lambda}_k \tilde{\theta}_k (t-\tau)} \sin[\tilde{\theta}_k \sqrt{1 - \tilde{\lambda}_k^2}\,(t-\tau)]\ddot{\underline{v}}^e(\tau)d\tau \quad (3.35a)$$

where the constants $\tilde{\theta}_k$ and $\tilde{\lambda}_k$, and the equivalent input acceleration vector $\ddot{\underline{v}}_k^{\,e}(t)$ are defined by

$$\tilde{\theta}_k = [(\mathrm{Im}\ \alpha_k)^2 + (\mathrm{Re}\ \alpha_k)^2]^{\frac{1}{2}} \quad (3.35b)$$

$$\tilde{\lambda}_k = \frac{-\ \mathrm{Re}\ \alpha_k}{[(\mathrm{Im}\ \alpha_k)^2 + (\mathrm{Re}\ \alpha_k)^2]^{\frac{1}{2}}} \quad (3.35c)$$

and

$$\ddot{\underline{v}}_k^{\,e}(t) = [\tilde{\theta}_k^{\,2}\ \dot{v}_g(t) + \tilde{\lambda}_k\tilde{\theta}_k\ddot{v}_g(t)]\ \mathrm{Re}\left(-\ 2\ \frac{G_k}{\tilde{R}_{kk}}\ \underline{\varphi}_k\right)$$
$$+\ \tilde{\theta}_k \sqrt{1 - \tilde{\lambda}_k^2}\ \ddot{v}_g(t)\ \mathrm{Im}\left(-\ \frac{2G_k}{\tilde{R}_{kk}}\ \underline{\varphi}_k\right)\ . \quad (3.35d)$$

These are formulas that can be evaluated explicitly after finding the eigenvalues and eigenvectors of the matrix U.

The result obtained above for a lumped parameter, time-invariant damped linear system may be stated as follows. The response of the system may be obtained as a linear combination of the responses of N simple oscillators subjected to modified excitations, provided the eigenvalues of the matrix U are distinct. This result is valid even for systems that do not have classical normal modes.

For the particular case of a building supported on a discrete foundation Eq. (3.35) may be shown to be equivalent to the corresponding equations obtained by the Laplace operational method (Eqs. (3.22)). The essential difference between the two formulations lies in the method of determining the natural frequencies, critical damping ratios and base accelerations of the equivalent linear oscillators. Given a choice, it is more convenient to use Eqs. (3.35) for numerical calculations since, in general, it is easier to solve an eigenvalue problem than to obtain the roots of the corresponding frequency equation. It will be recalled, however, that Eqs. (3.35) can only be used for discrete foundations. Eqs. (3.22) and (3.24), on the other hand, are applicable for buildings supported on either a discrete foundation or on the elastic half-space.

5. A Note on the Assumption of Classical Normal Modes of the Superstructure

In the description of the system shown in Fig. 2, the damping matrix C was specified such that the superstructure would admit decomposition into classical normal modes. This restriction was imposed so that Eq. (3.1a) could be uncoupled and solved explicitly for the displacements v_j in terms of the free-field acceleration $\ddot{v}_g$ and the unknown accelerations $\ddot{v}_o$ and $\ddot{\varphi}$. The transformed expression for the displacements v_j was then substituted into the transformed equations corresponding to Eqs. (3.1b) and (3.1c). This gave a system of two algebraic equations in the unknown functions $\overline{v}_o$ and $\overline{\varphi}$ which were then solved explicitly. It may now be seen that essentially the same method can be used to analyze building-foundation systems having arbitrary damping matrices C. It is sufficient to use Foss's formulation to uncouple Eq. (3.1a). Explicit solutions for the displacements v_j can then be found in terms of $\ddot{v}_g$, $\ddot{v}_o$ and $\ddot{\varphi}$.

D. Applications

1. Introduction

The steady-state harmonic and earthquake response of several idealized single and multi-story building-foundation systems will be obtained herein by the methods developed in the foregoing sections. This will help to illustrate the use of such methods and to determine the conditions under which the interaction effects become important. In addition, explicit formulas will be presented that permit treating the problem of the earthquake response of a single-story building resting on an elastic foundation as one of a single mass system on a rigid soil.

2. Dynamic Soil Coefficients

In order to apply the methods developed in Section C, it is first necessary to evaluate the impedance functions K_{hh}, K_{hm}, K_{mh} and K_{mm}. As may be observed from Eq. (3.6), these functions relate the stress resultants of the contact area to the displacements experienced by a rigid disc undergoing oscillations on the surface of a semi-infinite elastic medium. The functions K_{hh} and K_{mm} have been evaluated numerically[5,11,14] for the case of steady-state harmonic oscillations of a rigid disc with relaxed boundary conditions, for values of the frequency parameter a_o up to 10. The function K_{hm} vanishes identically for this case as there is no coupling between the interaction forces produced by the rocking and the horizontal translational motions. The corresponding dynamic values of K_{hh}, K_{hm}, K_{mh} and K_{mm} for the perfectly bonded disc are not available. On the other hand, dynamic values for the infinite rigid strip have been obtained by Luco[24], who showed that the difference between the welded and the frictionless case is significant only for large values of the frequency parameter a_o and small values of Poisson's ratio σ.

It will be assumed in this study that the qualitative result obtained for the infinite strip also holds for the disc, and therefore, that the dynamic force-displacement relations for the frictionless disc can be used as approximations for the corresponding complete mixed boundary value problem. This assumption may be partially verified by comparing the natural frequencies of single-story building-foundation systems corresponding to the two types of bond assumed between the base of the building and the underlying half-space.

The single-story building-foundation system used for this calculation is shown in Fig. 5. It consists of a linear, viscously damped single-story structure with one degree of freedom, resting on the surface of the half-space. For fixed base response, the structure has a stiffness k_1, mass m_1, natural frequency $\omega_1 = (k_1/m_1)^{\frac{1}{2}}$ and damping coefficient c_1. The building has a height h_1 above the base mass.

For the purposes of the present analysis it is convenient to consider only undamped building-foundation systems with massless bases.

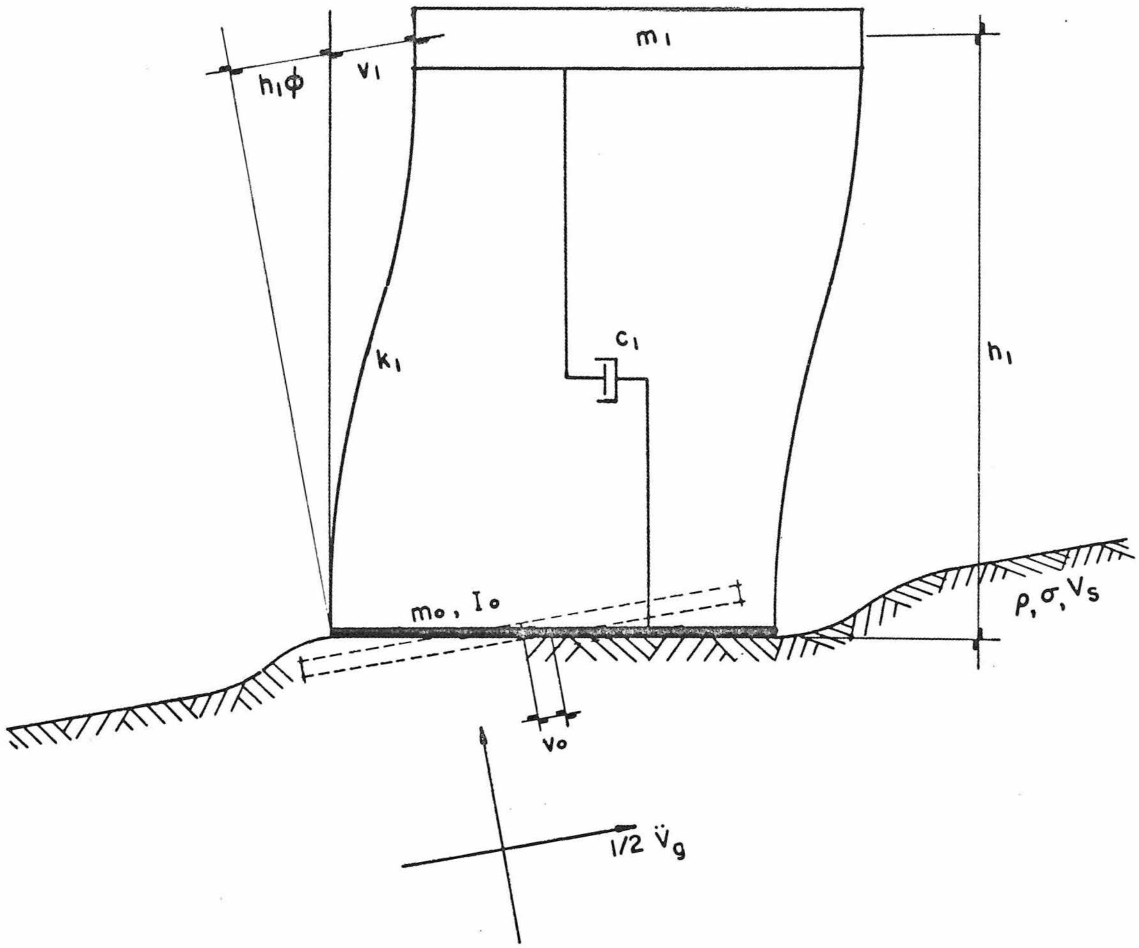


Fig. 5. Single-story building-foundation system

With these restrictions the corresponding frequency equation can then be solved in closed form:

$$\frac{\tilde{\omega}_1^2}{\omega_1^2} = \frac{1 - \dfrac{K_{hm}^2}{K_{hh} K_{mm}}}{1 - \dfrac{K_{hm}^2}{K_{hh} K_{mm}} + \dfrac{k_1}{\mu a}\left[\dfrac{1}{K_{hh}} + 2\dfrac{K_{hm}}{K_{hh} K_{mm}}\left(\dfrac{h_1}{a}\right) + \dfrac{1}{K_{mm}}\left(\dfrac{h_1}{a}\right)^2\right]} \quad (3.36)$$

For the numerical evaluation of Eq. (3.36), the force-displacement relations K_{hh}, K_{hm}, K_{mh} and K_{mm} will be approximated by the corresponding static values. Thus, for a perfectly bonded disc these functions are given by Eq. (2.36) whereas for a frictionless disc, $K_{hh} = 8/(2-\sigma)$, $K_{hm} = K_{mh} = 0$ and $K_{mm} = 8/3(1-\sigma)$[13].

Equation (3.36) has been calculated for values of the stiffness ratio, $k_1/\mu a$, between 0.01 and 2, of the slenderness ratio, h_1/a, between 0.5 and 4 and of Poisson's ratio, σ, between 0 and 0.5, for both a bonded and a frictionless base. The difference in the values of the corresponding natural frequencies is in no case greater than 5%. For instance, for a system defined by $k_1/\mu a = 0.5$, $h_1/a = 2$ and $\sigma = 0$, the frequency ratio $\tilde{\omega}_1/\omega_1$ is equal to 0.725 if the base is perfectly bonded to the soil and 0.730 for a frictionless base. In general, the discrepancy is greatest for $\sigma = 0$, whereas no difference occurs for an incompressible material ($\sigma = \frac{1}{2}$).

<u>Interpretation of the dynamic force-displacement relations</u>

Returning to the dynamic problem, Bycroft[5] and Gladwell[13] have shown that for steady-state oscillations of the disc, the functions K_{hh} and K_{mm} can be expressed formally as

$$K_{hh}(ia_o) = k_{hh}(a_o) + ia_o\ c_{hh}(a_o) \tag{3.37a}$$

$$K_{mm}(ia_o) = k_{mm}(a_o) + ia_o\ c_{mm}(a_o) \tag{3.37b}$$

in which the functions k_{hh}, c_{hh}, k_{mm} and c_{mm} are real. These functions can be given a simple physical interpretation. That is, it may be shown[72] that k_{hh} and k_{mm} are related to the stiffnesses of frequency dependent linear springs whereas c_{hh} and c_{mm} are associated with viscous dampers which are also frequency dependent. As an illustration, the exact relationship is given here for one of these pairs.

After setting K_{hm} equal to zero, Eq. (3.37a) and the first of Eqs. (2.31) lead to

$$P_1(t) = \mu a\ \ k_{hh}(a_o)\ v_o(t) + \frac{\mu a^2}{V_s}\, c_{hh}(a_o)\ \dot{v}_o(t) \tag{3.38a}$$

where $P_1(t) = P\ \exp(i\omega t)$ and $v_o(t) = \Delta_h \exp(i\omega t)$.

Equation (3.38a) shows explicitly that the force $P_1(t)$ depends linearly on the displacement $v_o(t)$ and the velocity $\dot{v}_o(t)$. This is equivalent to having a system consisting of a linear spring and a viscous damper whose properties are frequency dependent. The corresponding stiffness k and the damping coefficient c are

$$k = \mu a\ \ k_{hh}(a_o) \tag{3.38b}$$

$$c = \frac{\mu a^2}{V_s}\, c_{hh}(a_o)\ . \tag{3.38c}$$

Impedance functions for transient vibrations

In studying the earthquake response of building-foundation systems, it is necessary to evaluate the functions K_{hh} and K_{mm} in terms of the parameter s_o, a complex number, rather than a_o. No such numerical solutions have been found to date. It is possible however, to obtain $K_{hh}(s_o)$ and $K_{mm}(s_o)$ by analytic continuation from the known solutions for $K_{hh}(ia_o)$ and $K_{mm}(ia_o)$.

To explain the concept of analytic continuation, let D_1 and D_2 be two domains which have in common a set of points forming a domain D, and let $f_1(s_o)$ be an analytic function defined in D_1. If there exists a function $f_2(s_o)$ analytic in D_2 which is equal to $f_1(s_o)$ at each point of D, then $f_2(s_o)$ is unique. $f_2(s_o)$ is called the analytic continuation of $f_1(s_o)$ into the domain D_2[75].

Gladwell[13] has shown that the functions $K_{hh}(ia_o)$ and $K_{mm}(ia_o)$ can be expanded formally in terms of power series. Assuming that these series are convergent, their sums will be analytic functions at every point interior to their corresponding circles of convergence, R_1 and R_2[75]. Since the function $K_{hh}(s_o)$ defined by

$$K_{hh}(s_o) = K_{hh}(ia_o)\Big|_{ia_o=s_o} \tag{3.39a}$$

coincides with the function $K_{hh}(ia_o)$ on that part of the imaginary axis which is within its circle of convergence, it then follows that $K_{hh}(s_o)$ is the analytic continuation of $K_{hh}(ia_o)$ into the region interior to R_1. Thus, provided s_o is interior to R_1, it is possible to find $K_{hh}(s_o)$ by merely replacing ia_o by s_o in the corresponding expression for $K_{hh}(ia_o)$. Similarly, $K_{mm}(s_o)$ may be found from

$$K_{mm}(s_o) = K_{mm}(ia_o)\Big|_{ia_o=s_o} \quad . \tag{3.39b}$$

Alternative expressions for $K_{hh}(s_o)$ and $K_{mm}(s_o)$ may be obtained by combining Eqs. (3.37) and (3.39)

$$K_{hh}(s_o) = k_{hh}(a_o)\Big|_{ia_o=s_o} + s_o\, c_{hh}(a_o)\Big|_{ia_o=s_o} \tag{3.40a}$$

$$K_{mm}(s_o) = k_{mm}(a_o)\Big|_{ia_o=s_o} + s_o\, c_{mm}(a_o)\Big|_{ia_o=s_o} \quad . \tag{3.40b}$$

<u>Approximate formulas for K_{hh} and K_{mm}</u>

Equations (3.22) and (3.18) show that the functions $K_{hh}(s_o)$ and $K_{mm}(s_o)$ have to be evaluated only at discrete values of s_o, corresponding to the various resonant frequencies of the building-foundation system. Equations (3.22) and (3.18) also show that the imaginary part, a_o, of s_o is related to the resonant frequencies of the system, whereas the real part, $-\sigma_o$, of s_o is a measure of the amount of damping associated with a given resonant frequency. Considering that: (a) σ_o is in general small compared to a_o, (b) a_o is for most practical applications less than about 2 and (c) the functions k_{hh}, c_{hh}, k_{mm} and c_{mm} involve only even powers of s_o, it is reasonable to use an approximate form of Eqs. (3.40) for the functions $K_{hh}(s_o)$ and $K_{mm}(s_o)$ which neglects the higher order terms in σ_o, namely

$$K_{hh}(s_o) = k_{hh}(a_o) + s_o\, c_{hh}(a_o) \tag{3.41a}$$

$$K_{mm}(s_o) = k_{mm}(a_o) + s_o\, c_{mm}(a_o)\,. \tag{3.41b}$$

These equations have the advantage of giving a representation for the functions $K_{hh}(s_o)$ and $K_{mm}(s_o)$ which involve only the real functions $k_{hh}(a_o)$, $c_{hh}(a_o)$, $k_{mm}(a_o)$ and $c_{mm}(a_o)$ arising in the problem of steady-state harmonic oscillations of a rigid disc on the elastic half-space. Numerical values for these functions will be adapted from the numerical results obtained by Luco and Westmann(11).

3. Dynamic Response of Single-Story Building-Foundation Systems

The dynamic response of the idealized single-story building-foundation system shown in Fig. 5 will be studied here. Dimensionless expressions for the transformed displacements $\bar{v}_o$, $\bar{\varphi}$ and $\bar{v}_1$ can be obtained from Eqs. (3.10), (3.11) and (3.41). After setting n = 1 and $K_{hm} = 0$, these equations lead to

$$\frac{\bar{v}_o}{\bar{v}_g} = -\frac{s_o^2}{\Delta_d}\left\{\left(1 + \frac{s_o^2\, b_m}{\beta_m\, \sigma_m} + \zeta_m\, s_o\right)\left[\left(1 + 2\eta_1 \frac{s_o}{a_1}\right)(b_1 + b_h)\,\frac{1}{\beta_h \sigma_h} + \frac{s_o^2\, b_h}{a_1^2 \beta_h \sigma_h}\right]\right.$$
$$\left. + \frac{b_1\, b_h\, \alpha_1^2\, s_o^2}{\beta_h\, \beta_m\, \sigma_h\, \sigma_m}\left(1 + 2\eta_1 \frac{s_o}{a_1}\right)\right\} \tag{3.42a}$$

$$\frac{\bar{\varphi} h_1}{\bar{v}_g} = -\frac{s_o^2}{\Delta_d}\,(1 + s_o\, \zeta_h)\left(1 + 2\eta_1 \frac{s_o}{a_1}\right)\frac{b_1\, \alpha_1^2}{\beta_m\, \sigma_m} \tag{3.42b}$$

$$\frac{\bar{v}_1}{\bar{v}_g} = -\frac{s_o^{\,2}}{a_1^2\,\Delta_d}\,(1 + s_o\,\zeta_h)\left(1 + \frac{s_o^{\,2}\,b_m}{\beta_m\,\sigma_m} + s_o\,\zeta_m\right) \qquad (3.42c)$$

in which

$$\begin{aligned}\Delta_d = {} & s_o^{\,2}\,b_1\left(1 + 2\eta_1\,\frac{s_o}{a_1}\right)\left[\frac{1}{\beta_h\,\sigma_h}\left(1 + \frac{s_o^{\,2}\,b_m}{\beta_m\,\sigma_m} + s_o\,\zeta_m\right)\right. \\ & \left. + \frac{\alpha_1^{\,2}}{\beta_m\,\sigma_m}\left(1 + \frac{s_o^{\,2}\,b_h}{\beta_h\,\sigma_h} + s_o\,\zeta_h\right)\right] \\ & + \left(1 + 2\eta_1\,\frac{s_o}{a_1} + \frac{s_o^{\,2}}{a_1^{\,2}}\right)\left[\frac{s_o^{\,4}\,b_h\,b_m}{\beta_h\,\beta_m\,\sigma_h\,\sigma_m} + \frac{s_o^{\,2}\,b_h}{\beta_h\,\sigma_h}\,(1 + s_o\,\zeta_m)\right. \\ & \left. + \frac{s_o^{\,2}\,b_m}{\beta_m\,\sigma_m}\,(1 + s_o\,\zeta_h) + 1 + s_o(\zeta_m + \zeta_h) + s_o^{\,2}\,\zeta_h\,\zeta_m\right] .\end{aligned} \qquad (3.42d)$$

The functions β_h, β_m, σ_h, σ_m, ζ_h and ζ_m are defined by

$$k_{hh}(a_o,\sigma) = \frac{8}{2 - \sigma}\,\beta_h(a_o,\sigma) = \sigma_h\,\beta_h(a_o,\sigma) \qquad (3.43a)$$

$$c_{hh}(a_o,\sigma) = \zeta_h(a_o,\sigma)\,k_{hh}(a_o,\sigma) \qquad (3.43b)$$

$$k_{mm}(a_o,\sigma) = \frac{8}{3(1 - \sigma)}\,\beta_m(a_o,\sigma) = \sigma_m\beta_m(a_o,\sigma) \qquad (3.43c)$$

$$c_{mm}(a_o,\sigma) = \zeta_m(a_o,\sigma)\,k_{mm}(a_o,\sigma)\ . \qquad (3.43d)$$

The constants σ_h and σ_m are the static values of the stiffness coefficients k_{hh} and k_{mm}, respectively, whereas the functions β_h and β_m measure the deviation of k_{hh} and k_{mm} from their static values. Thus, for $a_o = 0$, β_h and β_m are equal to unity. ζ_h and ζ_m are related to the amount of energy lost by radiation into the elastic half-space due to horizontal translation and rotation of the base, respectively. These parameters, however, do not represent ratios of critical damping.

The functions β_h, β_m, ζ_h and ζ_m are shown in Fig. 6 for three values of Poisson's ratio as functions of the parameter a_o, for values of a_o from 0 to 2. Although this range of a_o is sufficient for most practical applications, these functions can be calculated for values of a_o up to 10 from the results presented by Luco and Westmann[11].

The dimensionless parameters a_1, α_1, η_1, b_1, b_h and b_m appearing in Eqs. (3.42) are defined by

$$a_1 = \frac{\omega_1 a}{V_s} \tag{3.44a}$$

$$\alpha_1 = \frac{h_1}{a} \tag{3.44b}$$

$$\eta_1 = \frac{c_1}{2m_1 \omega_1} \tag{3.44c}$$

$$b_1 = \frac{m_1}{\rho a^3} \tag{3.44d}$$

$$b_h = \frac{m_o}{\rho a^3} \tag{3.44e}$$

and

$$b_m = \frac{I_t}{\rho a^5} . \tag{3.44f}$$

Equations (3.42) show that the dynamic behavior of the building-foundation system under study is governed by a set of seven dimensionless variables, i.e., the response quantities $\overline{v}_o/\overline{v}_g$, $\overline{\varphi}h_1/\overline{v}_g$ and

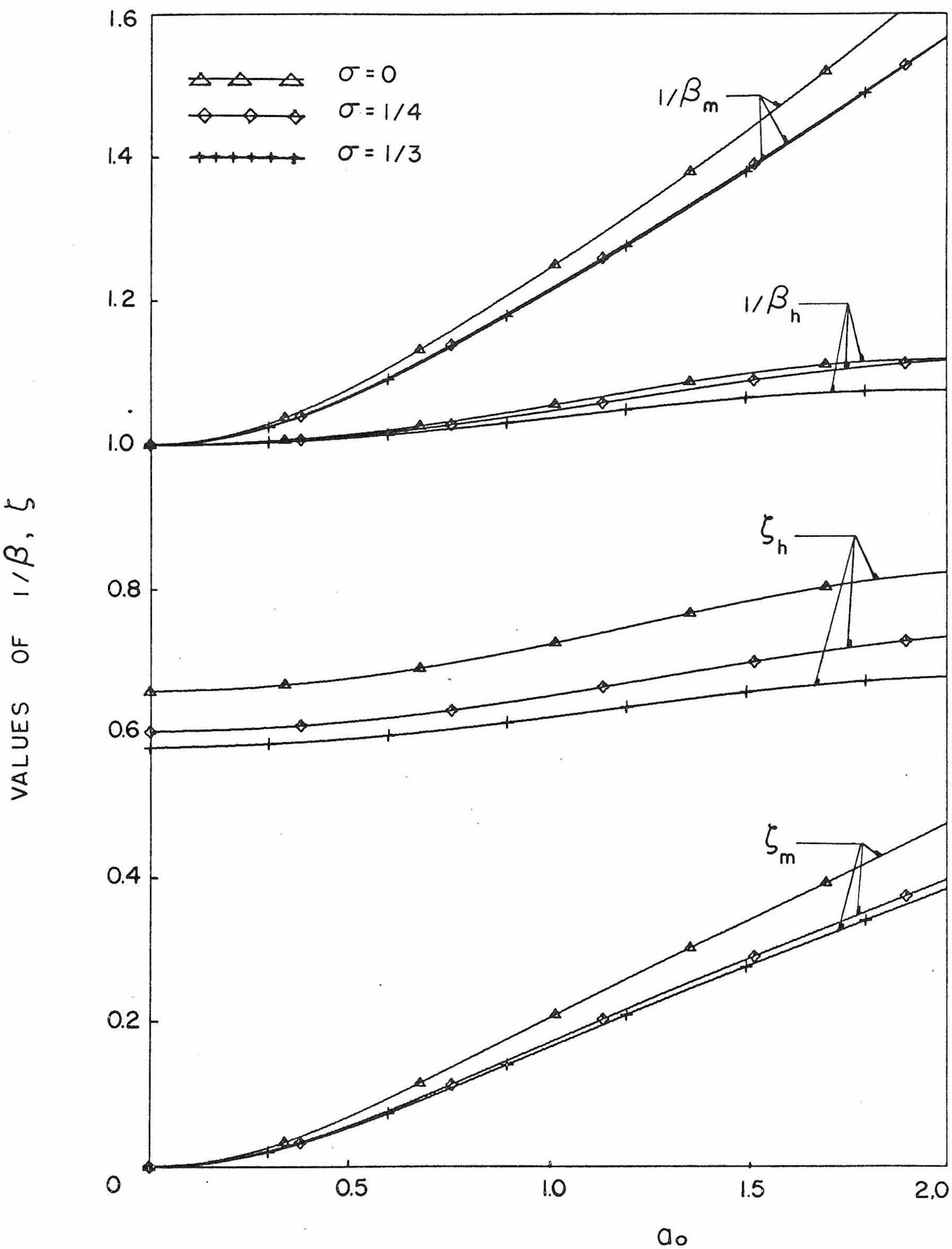


Fig. 6. Dynamic soil coefficients

$\bar{v}_1/\bar{v}_g$, as functions of s_o, are defined completely by the parameters a_1, α_1, η_1, b_1, b_h, b_m and σ.

Relations between the mass ratios and the frequencies of rigid motion of the base mass

A simple relation can be found between the mass ratios b_h and b_m and the dimensionless frequencies a_h and a_m, defined respectively by

$$a_h = \frac{\omega_h a}{V_s} \tag{3.45a}$$

$$a_m = \frac{\omega_m a}{V_s} \tag{3.45b}$$

in which

$$\omega_h^2 = \frac{\mu a \sigma_h}{m_o} \tag{3.45c}$$

$$\omega_m^2 = \frac{\mu a^3 \sigma_m}{I_t} \tag{3.45d}$$

that is, ω_h is the natural frequency of horizontal oscillations of the base mass under the assumption that $K_{hh} = \sigma_h$ and ω_m is the natural frequency of rotational vibration of a rigid disc of radius a and centroidal moment of inertia I_t if it is assumed that $K_{mm} = \sigma_m$.

The required relations are obtained by combining Eqs. (3.44e), (3.44f) and (3.45),

$$a_h^2 = \frac{\sigma_h}{b_h} \tag{3.46a}$$

$$a_m^2 = \frac{\sigma_m}{b_m} . \tag{3.46b}$$

These equations show that the frequencies a_h and a_m are independent of the stiffness of the foundation.

Equations (3.46a) and (3.46b) give approximate formulas for the loci of the resonant frequencies corresponding to the problems of horizontal oscillations and rocking, respectively, of a rigid circular footing on an elastic half-space.

Total horizontal displacement of the base mass

Equations (3.42) give expressions for the Laplace transforms of the functions v_o, φ and v_1, which represent respectively, the horizontal translation of the base mass relative to the free-field motion, the rotation of the base mass, and the relative displacement of the top mass relative to the base mass, excluding rotation. It is also useful to have an explicit representation for the total horizontal displacement of the base mass y_o,

$$y_o = v_o + v_g \tag{3.47a}$$

which upon transformation and substitution of Eq. (3.42a) yields

$$\frac{\bar{y}_o}{\bar{v}_g} = \frac{1 + s_o \zeta_h}{\Delta_d} \left[\left(1 + 2\eta_1 \frac{s_o}{a_1}\right) \frac{s_o^2 b_1 \alpha_1^2}{\beta_m \sigma_m} \right.$$
$$\left. + \left(1 + 2\eta_1 \frac{s_o}{a_1} + \frac{s_o^2}{a_1^2}\right)\left(1 + \frac{s_o^2 b_m}{\beta_m \sigma_m} + s_o \zeta_m\right)\right] . \tag{3.47b}$$

Systems subjected to base constraints

Equations (3.42) and (3.47b) have been derived for the building-foundation system shown in Fig. 5. As shown in the figure, the base mass will both rotate and move horizontally with respect to the free-field displacement. Physical constraints, however, may be imposed on the base mass which prevent one of these motions, for example, a system founded on piles might not allow rocking of the base. The dynamic behavior of the resulting system can still be described by Eqs. (3.42) and (3.47b); it is only necessary to set $1/\beta_m$ and ζ_m equal to zero on these equations if the base is not allowed to rotate. Similarly, terms in Eqs. (3.42) and (3.47b) containing $1/\beta_h$ or ζ_h must be eliminated if the base can not move horizontally with respect to the free-field displacement.

Steady-state response

The response of the single-story interaction system shown in Fig. 5 to the free-field harmonic motion

$$v_g(t) = \bar{v}_g\, e^{i\omega t} \tag{3.48a}$$

will be studied in this section. $\bar{v}_g$ is the amplitude of the motion and ω is the frequency of oscillation.

The corresponding displacements $v_o(t)$, $y_o(t)$, $\varphi(t)$ and $v_1(t)$ are

$$\begin{Bmatrix} v_o(t) \\ y_o(t) \\ \varphi(t) \\ v_1(t) \end{Bmatrix} = \begin{Bmatrix} \bar{v}_o \\ \bar{y}_o \\ \bar{\varphi} \\ \bar{v}_1 \end{Bmatrix} e^{i\omega t} \tag{3.48b}$$

where the complex quantities $\bar{v}_o$, $\bar{y}_o$, $\bar{\varphi}$, $\bar{v}_1$ are given by Eqs. (3.42) and (3.47b) in which s_o is replaced by ia_o ($a_o = \omega a/V_s$).

(i) Systems with negligible base masses

Numerical evaluation of the steady-state response of the interaction system will be obtained first for the limiting case in which b_h and b_m vanish. This corresponds to having a system with a negligible base mass and with a negligible centroidal moment of inertia of the top mass. Systems with b_h, b_m different from zero will be examined subsequently.

The critical damping ratio η_1 of the structure will be taken to be zero for all numerical calculations. In this manner, all the energy dissipated by the system will be due to wave radiation into the elastic half-space. Also, calculations will be presented only for one value of Poisson's ratio ($\sigma = 1/4$) since similar results are expected for other values.

Having fixed the values of b_h, b_m, η_1 and σ, the frequency response of the system will depend solely on the parameters a_1, α_1 and b_1, defined by Eqs. (3.44). Of these, only a_1 is a function of the soil stiffness. In fact, the rigidity of the soil, as measured by its shear wave velocity, V_s, only enters the problem in conjunction with ω_1. Therefore, the dynamic coupling between a building of this type and the surrounding ground will depend on the relative stiffness between the superstructure and its foundation, and not on the rigidity of the soil per se.

Numerical results

Calculations have been carried out for several combinations of the parameters a_1, α_1 and b_1 to assess their influence on the frequency response of the system. The values of the parameters used in the calculations are: (1) $a_1 = 0.4, 0.5, 0.7, 0.9$; (2) $\alpha_1 = 1.0, 1.5, 2.0$; and (3) $b_1 = 0.5, 1.0, 1.5$. These values of a_1, α_1 and b_1 are intended to approximate those of real structures. For instance, $a_1 = 0.7$ might correspond to a concrete nuclear reactor containment vessel of radius a = 60 ft and natural frequency $f_1 = 4$ cps founded on a soil whose shear wave velocity, V_s, is equal to 2150 ft/sec.

The frequency response of the building-foundation systems characterized by $\alpha_1 = 1.5$, $b_1 = 1$ and $a_1 = 0.4, 0.5, 0.7, 0.9$ is depicted in Fig. 7. Shown in the figure are three sets of curves, each illustrating the variation of a specific translational magnification factor, $|\overline{y}_o|/\overline{v}_g$, $|\overline{\varphi}h_1|/\overline{v}_g$ or $|\overline{v}_1|/\overline{v}_g$, obtained from Eqs. (3.42) and (3.47b) as a function of the exciting frequency ratio ω/ω_1. The symbol $|\overline{y}_o|$ refers to the amplitude of the total horizontal displacement of the base, whereas $|\overline{\varphi}h_1|$ and $|\overline{v}_1|$ denote respectively the amplitude of the horizontal translation produced by rocking and the flexural displacement of the top mass.

Whereas a single-story undamped system founded on a rigid base ($a_1 = 0$) exhibits an infinite relative story displacement at $\omega/\omega_1 = 1$, Fig. 7(a) shows that for non-zero values of a_1, $|\overline{v}_1|/\overline{v}_g$ reaches a finite maximum at a value of $\omega/\omega_1 = \tilde{\omega}_1/\omega_1$ which decreases monotonically from unity for increasing values of a_1. The reduction in the peak values of the response indicates the presence of damping in the system, which is due to wave radiation into the half-space. This damping, or energy loss becomes larger for increasing values of a_1 and is sensitive

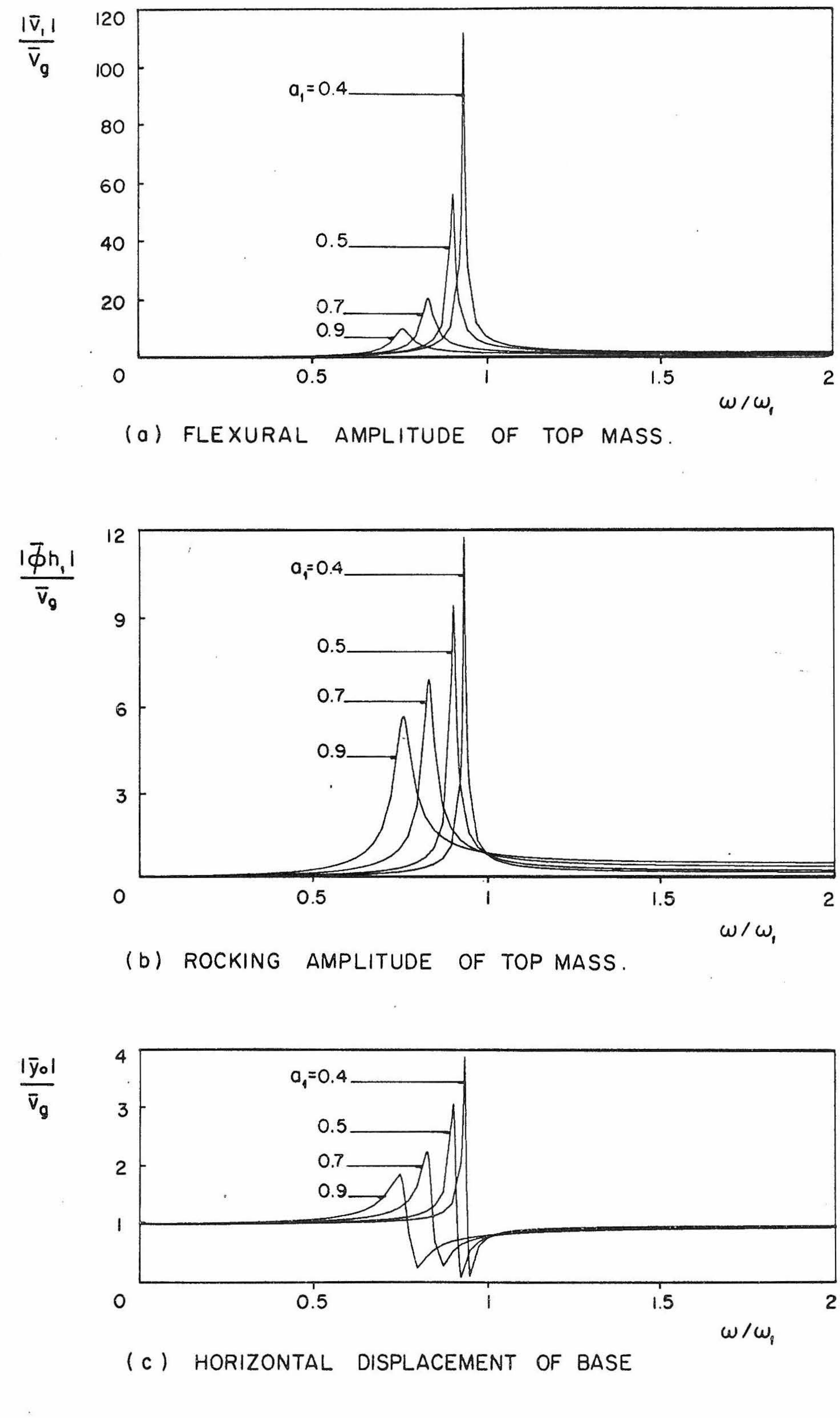


Fig. 7. Frequency response of single-story building-foundation system ($\alpha_1 = 1.5$, $b_1 = 1$, $\sigma = 1/4$; $b_h = b_m = \eta_1 = 0$)

to small variations of this parameter.

The response curves for the horizontal translation of the top mass as produced by rocking are plotted in Fig. 7(b). Contrary to the expectation that the peak values of $|\bar{\varphi}h_1|/\bar{v}_g$ would increase monotonically with a_1, Fig. 7(b) shows that the largest values of $|\bar{\varphi}h_1|/\bar{v}_g$ occur when a_1 is small and decrease as a_1 increases. For the limiting rigid foundation ($a_1 = 0$), the amplification factor $|\bar{\varphi}h_1|/\bar{v}_g$ vanishes identically except at ω/ω_1 where it has an infinite singularity. The ratio $|\bar{\varphi}h_1|/|\bar{v}_1|$, however, does go to zero.

The response curves for the amplification factor of the horizontal translation of the base are shown in Fig. 7(c). These curves are to be compared with the line $|\bar{y}_o|/\bar{v}_g = 1$ (except at ω/ω_1 where there is an infinite singularity) corresponding to a building founded on a rigid soil. The crinkle exhibited by the response curves near the resonant frequency $\tilde{\omega}_1$ can be explained by rewriting Eq. (3.47b) in the form

$$\frac{\bar{y}_o(\frac{\omega}{\omega_1})}{\bar{v}_g} = (1 + i\zeta_h\ a_1\ \frac{\omega}{\omega_1})\ \frac{\Delta_e(\frac{\omega}{\omega_1})}{\Delta_d(\frac{\omega}{\omega_1})} \tag{3.49a}$$

in which Δ_e is defined by

$$\Delta_e\ (\frac{\omega}{\omega_1}) = \left[\Delta_d\ \ (\frac{\omega}{\omega_1})\right]_{1/\beta_h=\zeta_h=0}\ . \tag{3.49b}$$

For the purpose of this discussion, it is assumed that the amount of damping in the system is small enough so that its effect on the value of the resonant frequency can be neglected. Under this assumption, and recalling that $b_h = b_m = 0$, Eqs. (3.42d) and (3.49b) give

$$\Delta_d\left(\frac{\omega}{\omega_1}\right) = 1 - \left(\frac{\omega}{\omega_1}\right)^2 \left[1 + a_1^2\, b_1 \left(\frac{1}{\beta_h\, \sigma_h} + \frac{\alpha_1^2}{\beta_m\, \sigma_m}\right)\right] \tag{3.50a}$$

$$\Delta_e\left(\frac{\omega}{\omega_1}\right) = 1 - \left(\frac{\omega}{\omega_1}\right)^2 \left[1 + \frac{a_1^2\, b_1 \alpha_1^2}{\beta_m\, \sigma_m}\right]. \tag{3.50b}$$

After substituting Eqs. (3.50) into (3.49) it becomes apparent that the response curve for $|\overline{y}_o|/\overline{v}_g$ for the undamped system will exhibit both an infinite peak and a minimum value (zero), the maximum occurring at a frequency ω_1^* given by $\Delta_d\left(\frac{\omega_1^*}{\omega_1}\right) = 0$ and the minimum at a frequency ω_1^{**} defined by $\Delta_e\left(\frac{\omega_1^{**}}{\omega_1}\right) = 0$. ω_1^* is the natural frequency of the undamped system whereas ω_1^{**} is the natural frequency of a system like the one above, except that the base mass can only rotate.

Based on this analysis of the undamped system, it is reasonable to expect that the peak in the response curve for $|\overline{y}_o|/\overline{v}_g$ will be attained near the resonant frequency $\tilde{\omega}_1$ of the original system whereas the minimum value of $|\overline{y}_o|/\overline{v}_g$ will occur near the resonant frequency that the system would have if its base were allowed to rotate, but were restricted to have the same displacement in the horizontal direction as the free-field surface.

As Fig. 7(c) indicates, the minimum value of $|\overline{y}_o|/\overline{v}_g$ will generally be different from zero because of the presence of damping in the system. There is, however, one particular case for which this minimum does assume a zero value. Namely, Eq. (3.47b) shows that if the system is not allowed to rotate, then $|\bar{y}_o|/\bar{v}_g$ vanishes at a point corresponding to a frequency of excitation $\omega = \omega_1$, that is, the single-story building-foundation system shown in Fig. 5 exhibits characteristics of a vibration absorber provided the system is prevented from rocking. This property remains valid when the mass ratios b_h and b_m are not equal to zero as only the condition $\eta_1 = 0$ need be maintained.

Approximate formulas for the resonant frequency and peak values of the response

The numerical calculations so far presented indicate that single-story systems founded on elastic soils have resonant frequencies $\tilde{\omega}_1$ smaller than the corresponding frequencies ω_1 of the buildings on rigid foundations. To determine $\tilde{\omega}_1$ it is necessary to obtain the frequency response of the interaction system defined by the parameters a_1, b_1 and α_1. Hence, a number of these responses must be obtained in order to study the effect of the individual parameters a_1, b_1 and α_1 on the resonant frequency $\tilde{\omega}_1$. It is therefore of considerable practical interest that $\tilde{\omega}_1$ can be approximated by the natural frequency $\omega_1{}^*$ of the corresponding undamped system, as a closed form solution in terms of a_1, b_1 and α_1 is available for $\omega_1{}^*$.

Figure 7(a) shows that the peak values of the amplification factor $|\bar{v}_1|/\bar{v}_g$ are greater than about 10 for all the cases considered in the figure. Interpreting Fig. 7(a) as the response of a single mass system on a rigid foundation, these values of the amplification factor would correspond to critical damping ratios of about 5% or less.

This small amount of damping allows for the approximation $\tilde{\omega}_1 = \omega_1{}^*$. An explicit solution is obtained for $\tilde{\omega}_1$ by recalling that $\omega_1{}^*$ is defined by $\Delta_d\left(\frac{\omega_1{}^*}{\omega_1}\right) = 0$, where $\Delta_d\left(\frac{\omega}{\omega_1}\right)$ is given by Eq. (3.50a). Thus,

$$\frac{\tilde{\omega}_1}{\omega_1} = \frac{1}{\left[1 + a_1{}^2\, b_1\left(\frac{1}{\beta_h\, \sigma_h} + \frac{\alpha_1{}^2}{\beta_m\, \sigma_m}\right)\right]^{\frac{1}{2}}} \cdot \tag{3.51a}$$

Successive approximations are required to obtain $\tilde{\omega}_1/\omega_1$ from this equation as the functions β_h and β_m have to be evaluated at $a_o = \tilde{a}_1$ ($\tilde{a}_1 = a_1\tilde{\omega}_1/\omega_1$).

Equation (3.51a) shows that the resonant frequency ratio $\tilde{\omega}_1/\omega_1$ is smaller than unity and is essentially a monotonically decreasing function of the two parameters $a_1{}^2\, b_1$ and $\alpha_1{}^2$.

With the resonant frequency ratio established, approximate expressions for the peak values of the amplication factors $|\bar{v}_1|/\bar{v}_g$, $|\bar{\varphi}h_1|/\bar{v}_g$ and $|\bar{v}_o|/\bar{v}_g$ can be obtained by evaluating Eqs. (3.42) at $s_o = i\tilde{a}_1$. Retaining only first-order terms in η_1, ζ_h and ζ_m, one gets

$$\max\frac{|\bar{v}_1|}{\bar{v}_g} = \frac{\left[1 + a_1{}^2\, b_1\left(\frac{1}{\beta_h\, \sigma_h} + \frac{\alpha_1{}^2}{\beta_m\, \sigma_m}\right)\right]^{\frac{1}{2}}}{2\eta_1 + a_1{}^3\, b_1\left(\frac{\zeta_h}{\beta_h\, \sigma_h} + \frac{\zeta_m\alpha_1{}^2}{\beta_m\, \sigma_m}\right)} \tag{3.51b}$$

$$\max\frac{|\bar{\varphi}h_1|}{\bar{v}_g} = \frac{a_1{}^2\, b_1\, \alpha_1{}^2}{\beta_m\, \sigma_m}\max\frac{|\bar{v}_1|}{\bar{v}_g} \tag{3.51c}$$

$$\max \frac{|\bar{v}_o|}{\bar{v}_g} = \frac{a_1{}^2\, b_1}{\beta_h\, \sigma_h} \max \frac{|\bar{v}_1|}{\bar{v}_g} \;. \tag{3.51d}$$

Equation (3.51b) shows explicitly that the maximum relative story displacement is finite even when there is no damping in the superstructure. In fact, when η_1 is equal to zero, the peak value of $|\bar{v}_1|/\bar{v}_g$ is inversely proportional to $a_1{}^3\, b_1$ for small values of a_1 whereas it approaches zero as $1/a_1{}^2\, b^{1/2}$ as a_1 becomes large. Equation (3.51b) shows in addition that when η_1 does not vanish, the peak value of the relative story displacement may be smaller or larger than the corresponding value for the system on a rigid soil ($1/2\eta_1$ to first order in η_1), depending on the parameters a_1, b_1 and $\alpha_1{}^2$.

That the amplification ratios $|\bar{\varphi}h_1|/\bar{v}_g$ and $|\bar{v}_o|/\bar{v}_g$ vanish for rigid soils can be observed from Eqs. (3.51c) and (3.51d), respectively. An unexpected result, however, is obtained when η_1 vanishes, for then the peak values of $|\bar{\varphi}h_1|/\bar{v}_g$ and $|\bar{v}_o|/\bar{v}_g$ become inversely proportional to a_1 for small values of a_1.

Verification of the approximate formulas

A few numerical calculations have been performed in order to study the approximate formulas (3.51) as well as the influence of the individual parameters a_1, b_1 and α_1 on the response of the single-story interaction system under investigation (Figure 5). The peak values of the amplification factors $|\bar{v}_1|/\bar{v}_g$, $|\bar{\varphi}h_1|/\bar{v}_g$ and $|\bar{v}_o|/\bar{v}_g$ and the corresponding resonant frequencies have been evaluated using both Eqs. (3.42) and (3.51) and are presented in Table 1 for several values of a_1, b_1 and α_1.

TABLE 1

Resonant Frequencies and Amplification Factors of Single-Story Interaction System

$(b_h = b_m = \eta_1 = 0, \; \sigma = 1/4)$

a_1 (1)	$\tilde{\omega}_1/\omega_1$ Exact (2)	$\tilde{\omega}_1/\omega_1$ Approx. (3)	$\lvert\bar{v}_1\rvert/\bar{v}_g$ Exact (4)	$\lvert\bar{v}_1\rvert/\bar{v}_g$ Approx. (5)	$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$ Exact (6)	$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$ Approx. (7)	$\lvert\bar{v}_o\rvert/\bar{v}_g$ Exact (8)	$\lvert\bar{v}_o\rvert/\bar{v}_g$ Approx. (9)
.4	.937	.936	111.7	107.0	11.75	11.26	3.84	3.77
.5	.906	.905	56.24	53.08	9.40	8.87	2.99	2.93
.7	.835	.832	20.25	18.64	6.83	6.29	2.08	2.03
.9	.761	.755	9.70	8.79	5.54	5.02	1.62	1.59

(a) $b_1 = 1.0, \; \alpha_1 = 1.5$

b_1 (1)	$\tilde{\omega}_1/\omega_1$ Exact (2)	$\tilde{\omega}_1/\omega_1$ Approx. (3)	$\lvert\bar{v}_1\rvert/\bar{v}_g$ Exact (4)	$\lvert\bar{v}_1\rvert/\bar{v}_g$ Approx. (5)	$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$ Exact (6)	$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$ Approx. (7)	$\lvert\bar{v}_o\rvert/\bar{v}_g$ Exact (8)	$\lvert\bar{v}_o\rvert/\bar{v}_g$ Approx. (9)
.5	.949	.949	105.9	99.25	8.83	8.33	2.79	2.75
1.0	.906	.905	56.24	53.08	9.40	8.87	2.99	2.93
1.5	.868	.866	39.64	37.59	9.90	9.38	3.19	3.11

(b) $a_1 = 0.5, \; \alpha_1 = 1.5$

α_1 (1)	$\tilde{\omega}_1/\omega_1$ Exact (2)	$\tilde{\omega}_1/\omega_1$ Approx. (3)	$\lvert\bar{v}_1\rvert/\bar{v}_g$ Exact (4)	$\lvert\bar{v}_1\rvert/\bar{v}_g$ Approx. (5)	$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$ Exact (6)	$\lvert\bar{\varphi}h_1\rvert/\bar{v}_g$ Approx. (7)	$\lvert\bar{v}_o\rvert/\bar{v}_g$ Exact (8)	$\lvert\bar{v}_o\rvert/\bar{v}_g$ Approx. (9)
1.0	.943	.941	60.35	56.41	4.50	4.21	3.20	3.12
1.5	.906	.905	56.24	53.08	9.40	8.87	2.99	2.93
2.0	.861	.860	52.40	50.17	15.49	14.83	2.80	2.77

(c) $a_1 = 0.5, \; b_1 = 1.0$

Estimates for the resonant frequency calculated from Eq. (3.51a) fall within one percent of the exact values computed from Eqs. (3.42), whereas an error of ten percent or smaller is obtained for the peak values of the response. This difference, although small and satisfactory for practical applications, is to be expected as the peak values are more sensitive than the resonant frequency to the damping coefficients η_1, ζ_h and ζ_m whose second powers have been neglected in the derivation of Eqs. (3.51). These equations have the advantage of providing relatively simple expressions that show explicitly the effect of the individual parameters a_1, b_1, α_1 and η_1 on the maximum values of the response quantities of interest and the corresponding resonant frequencies.

The following trends are worthy of note in Table 1.

1. The resonant frequency ratio $\tilde{\omega}_1/\omega_1$ is smaller than unity and decreases for increasing values of the parameters a_1, b_1 and α_1.
2. The peak values of the flexural amplification factor $|\bar{v}_1|/\bar{v}_g$ become increasingly small for growing values of a_1 and b_1 but are most sensitive to variations in a_1. No strong dependence of $|\bar{v}_1|/\bar{v}_g$ on α_1 is observed.
3. The maximum value of $|\bar{\varphi}h_1|/\bar{v}_g$ is approximately proportional to α_1^2, decreases for increasing values of a_1 and is almost insensitive to the mass ratio b_1.
4. Whereas the peak value of the base amplification factor $|\bar{v}_0|/\bar{v}_g$ does not show a strong dependence on b_1 and α_1,

it is quite sensitive to variations in a_1, becoming smaller as a_1 increases.

The trends described above can be observed directly from Eqs. (3.51). These equations also show that the ratio of the maximum displacement of the top mass caused by rocking to the corresponding flexural displacement is approximately proportional to $a_1^2 b_1 \alpha_1^2$, and similarly, that the maximum value of $|\overline{v}_o|/|\overline{v}_1|$ is almost a linear function of the parameter $a_1^2 b_1$.

Limiting rigid structure

The frequency ratio $\tilde{\omega}_1/\omega_1$ goes to zero as the frequency parameter a_1 approaches infinity. The resulting rigid structure, however, is still capable of undergoing rigid body oscillations on the surface of the elastic half-space; the amplitude of these oscillations being a function of the frequency of excitation. The peak values of the amplitudes $|\overline{\varphi h_1}|/\overline{v}_g$ and $|\overline{v}_o|/\overline{v}_g$ and the corresponding resonant frequency, $\tilde{a}_1$, can be obtained as the limits of Eqs. (3.51) as a_1 tends to infinity,

$$\tilde{a}_1^2 = \frac{1}{b_1\left(\dfrac{1}{\beta_h \sigma_h} + \dfrac{\alpha_1^2}{\beta_m \sigma_m}\right)} \tag{3.52a}$$

$$\max \frac{|\overline{\varphi h_1}|}{\overline{v}_g} = \frac{\alpha_1^2 \tilde{a}_1}{\beta_m \sigma_m\left(\dfrac{\zeta_h}{\beta_h \sigma_h} + \dfrac{\zeta_m \alpha_1^2}{\beta_m \sigma_m}\right)} \tag{3.52b}$$

$$\max \frac{|\bar{v}_o|}{\bar{v}_g} = \frac{\tilde{a}_1}{\beta_h\ \sigma_h \left(\dfrac{\zeta_h}{\beta_h\ \sigma_h} + \dfrac{\zeta_m\ \alpha_1^{\,2}}{\beta_m\ \sigma_m} \right)} \ . \tag{3.52c}$$

Clearly, $|\bar{v}_1|/\bar{v}_g$ vanishes identically for this case.

(ii) Influence of the base mass on the response of the system

It is the purpose of the following analysis to show that the peak values of the response of the single-story interaction system depicted in Fig. 5 vary slowly with respect to the parameters b_h and b_m. This in turn implies that the results obtained in the previous section for systems with negligible base masses and negligible centroidal moment of inertia of the top mass, give satisfactory approximations for systems for which b_h and b_m depart appreciably from zero.

In computing the mass ratios b_h and b_m, it should be noted that whereas the base of a real building is partially embedded into the ground, the idealized model used in this investigation is resting on the surface. Hence, only the buoyant mass of the embedded portion of the base should be included in b_h and b_m.

The change in the resonant frequencies of a building-foundation system for increasing values of b_h and b_m provides an estimate of the effect of these parameters on the dynamic coupling between the building and its foundation. In general, the system shown in Fig. 5 will exhibit three distinct resonant frequencies corresponding to the three degrees of freedom of the system. Only one resonant frequency, i.e., $\tilde{\omega}_1$, is observed, however, when the parameters b_h and b_m vanish; the other two being located at infinity. It is of interest to investigate to what extent $\tilde{\omega}_1$ is modified and how the additional frequencies are brought in from infinity as b_h and b_m become greater than zero.

Effect of b_h and b_m on the resonant frequencies of the system

The resonant frequencies of the system under study correspond approximately to the values of ω/ω_1 which minimize the function $|\Delta_d|$, where Δ_d is defined by Eq. (3.42d) and $s_o = ia_1\omega/\omega_1$. Approximate values for the resonant frequencies can be obtained from the equation

$$\Delta_d = 0 \tag{3.53}$$

if it is assumed, as before, that the coefficients of damping η_1, ζ_h and ζ_m may be neglected without affecting significantly the values of these frequencies.

An explicit formula for the modified frequency $\tilde{\omega}_1$ may be derived from Eq. (3.53) by retaining only first-order terms in b_h and b_m,

$$\frac{\tilde{\omega}_1}{\omega_1} = \frac{1}{\left[1 + a_1^2 b_1\left(\frac{1}{\beta_h \sigma_h} + \frac{\alpha_1^2}{\beta_m \sigma_m}\right)\right]^{\frac{1}{2}}} \left\{ 1 - \frac{1}{2} \frac{a_1^4 b_1 \left(\frac{b_h}{\beta_h^2 \sigma_h^2} + \frac{b_m \alpha_1^2}{\beta_m^2 \sigma_m^2} \right)}{\left[1 + a_1^2 b_1\left(\frac{1}{\beta_h \sigma_h} + \frac{\alpha_1^2}{\beta_m \sigma_m}\right)\right]^2} \right\}. \tag{3.54}$$

Equation (3.54) shows that the fundamental resonant frequency of the system decreases as b_h and b_m assume non-zero values. In fact, the equation shows that this reduction may be small even for relatively large values of b_h and b_m. For instance, corresponding to $a_1 = 0.5$, $b_1 = 1.5$, $\alpha_1 = 1.5$ and $\sigma = 0.25$, the frequency ratio $\tilde{\omega}_1/\omega_1$ is equal to 0.866 for vanishing values of b_h and b_m whereas an additional reduction of only about 2.5% is obtained when $b_h = 3$ and $b_m = 4$.

The behavior of the remaining resonant frequencies of the system corresponding to non-vanishing values of b_h and b_m can be studied most conveniently by considering b_h and b_m separately. Suppose, for instance that b_h vanishes but b_m does not. Then, neglecting η_1, ζ_h and ζ_m, Eqs. (3.53) and (3.42d) lead to the frequency equation

$$- a_1^2\, b_1 \left(\frac{\omega}{\omega_1}\right)^2 \left\{ \frac{1}{\beta_h\, \sigma_h} \left[1 - \frac{a_1^2\, b_m}{\beta_m\, \sigma_m} \left(\frac{\omega}{\omega_1}\right)^2 \right] + \frac{\alpha_1^2}{\beta_m\, \sigma_m} \right\}$$

$$+ \left[1 - \left(\frac{\omega}{\omega_1}\right)^2 \right] \left[1 - \frac{a_1^2\, b_m}{\beta_m\, \sigma_m} \left(\frac{\omega}{\omega_1}\right)^2 \right] = 0 \tag{3.55}$$

whose solution gives approximate values for the resonant frequencies of the interaction system under examination. Two resonant frequencies can in general be determined from Eq. (3.55), the third one being at infinity.

Rather than solving Eq. (3.55) for ω/ω_1 in terms of b_m, which would involve finding the solution of a cubic equation, it is found more suitable to invert the problem by solving Eq. (3.55) for b_m in terms of the resonant frequency ratio $\tilde{\omega}/\omega_1$. This gives

$$b_m = \frac{\beta_m \sigma_m}{a_1^2 (\frac{\tilde{\omega}}{\omega_1})^2} \quad \frac{1 - (\frac{\tilde{\omega}}{\omega_1})^2 \left[1 + a_1^2 \, b_1 \left(\frac{1}{\beta_h \, \sigma_h} + \frac{\alpha_1^2}{\beta_m \, \sigma_m}\right)\right]}{1 - (\frac{\tilde{\omega}}{\omega_1})^2 \left[1 + \frac{a_1^2 \, b_1}{\beta_h \, \sigma_h}\right]} . \qquad (3.56)$$

A schematic representation of Eq. (3.56) is given in Fig. 8, with b_m and $\tilde{\omega}/\omega_1$ plotted along the horizontal and vertical axes, respectively. As shown in the figure, Eq. (3.56) defines two distinct branches, each corresponding to one resonant frequency. The lower branch, representing the fundamental resonant frequency of the system, starts at a value $(\tilde{\omega}_1)_o/\omega_1$ given by Eq. (3.51a) and decreases asymptotically to zero for increasing values of b_m. The upper branch, which corresponds to the second resonant frequency, $\tilde{\omega}_2$, starts at infinity and becomes asymptotic to the horizontal line

$$\frac{\tilde{\omega}}{\omega_1} = \frac{1}{\left[1 + \frac{a_1^2 \, b_1}{\beta_h \, \sigma_h}\right]} \qquad (3.57)$$

as b_m gets large.

The asymptotic behavior of b_m for large values of $\tilde{\omega}_2/\omega_1$, or equivalently, the asymptotic behavior of $\tilde{\omega}_2/\omega_1$ for small values of b_m, can be found from Eq. (3.56),

$$\tilde{a}_2^2 = \frac{\beta_m \, \sigma_m}{b_m} \left[1 + \frac{\frac{a_1^2 \, b_1 \, \alpha_1^2}{\beta_m \, \sigma_m}}{1 + \frac{a_1^2 \, b_1}{\beta_h \, \sigma_h}}\right] \qquad (3.58)$$

where $\tilde{a}_2 = a_1 \, \tilde{\omega}_2/\omega_1$.

$$\frac{(\tilde{\omega}_1)_0}{\omega_1} = \frac{1}{\left[1 + a_1^2 b_1 \left(\frac{1}{\beta_h \sigma_h} + \frac{\alpha_1^2}{\beta_m \sigma_m}\right)\right]^{1/2}}$$

$$\frac{(\tilde{\omega}_2)_\infty}{\omega_1} = \frac{1}{\left[1 + \frac{a_1^2 b_1}{\beta_h \sigma_h}\right]^{1/2}}$$

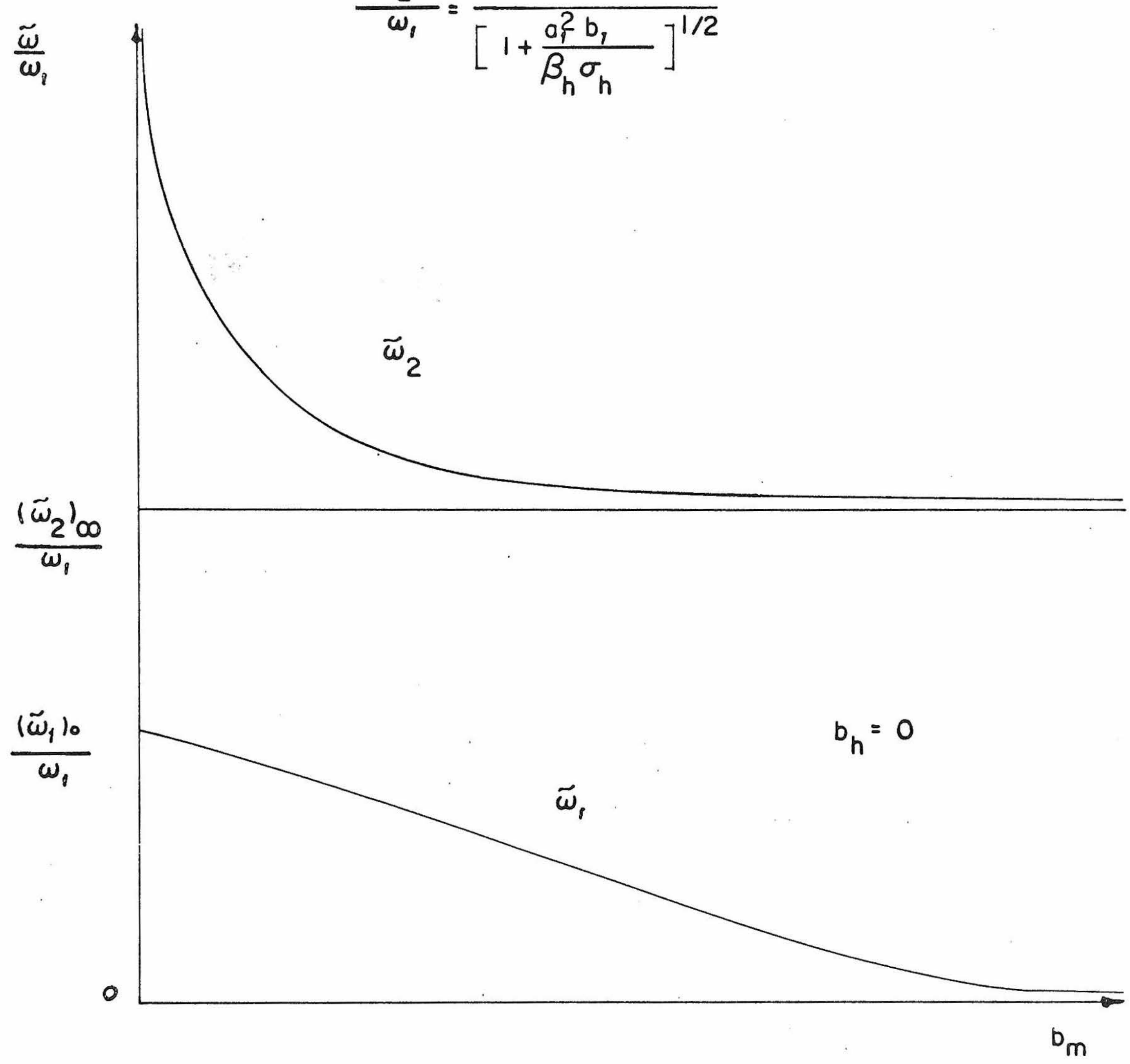


Fig. 8. Resonant frequencies of single-story building-foundation system (schematic diagram)

Equations (3.58) and (3.46b) show that $\tilde{a}_2$ may be interpreted as the dimensionless resonant frequency of a rigid disc undergoing rocking oscillations on an elastic half-space, provided the dimensionless centroidal moment of inertia of the disc is defined by

$$b_m = \frac{I_t}{\rho a^5} \left[1 + \frac{\dfrac{a_1^2 \, b_1 \alpha_1^2}{\sigma_m}}{1 + \dfrac{a_1^2 \, b_1}{\sigma_h}} \right]^{-1} . \tag{3.59}$$

The functions β_h and β_m have been set equal to unity in Eq. (3.59).

Verification of approximate formulas

As a partial check upon the results of this section the resonant frequencies of a typical set of single-story interaction systems, defined by $a_1 = 0.5$, $b_1 = 1$, $\alpha_1 = 2$ and $b_h = 0$ have been calculated from Eqs. (3.54), (3.56) and (3.58) for several values of the parameter b_m and compared with the corresponding exact values obtained from Eq. (3.42c). The results, which are presented in Fig. 9, show a very close agreement between the values of $\tilde{\omega}_1$ obtained from the exact and the approximate formulas. Slightly less accurate values are obtained for $\tilde{\omega}_2$. From the agreement between the exact and the approximate results it is concluded that Eqs. (3.54), (3.56) and (3.58) give a satisfactory description of the behavior of the resonant frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ as functions of the parameter b_m. The degree of accuracy of these equations depends on the particular values of the parameters a_1, b_1 and α_1.

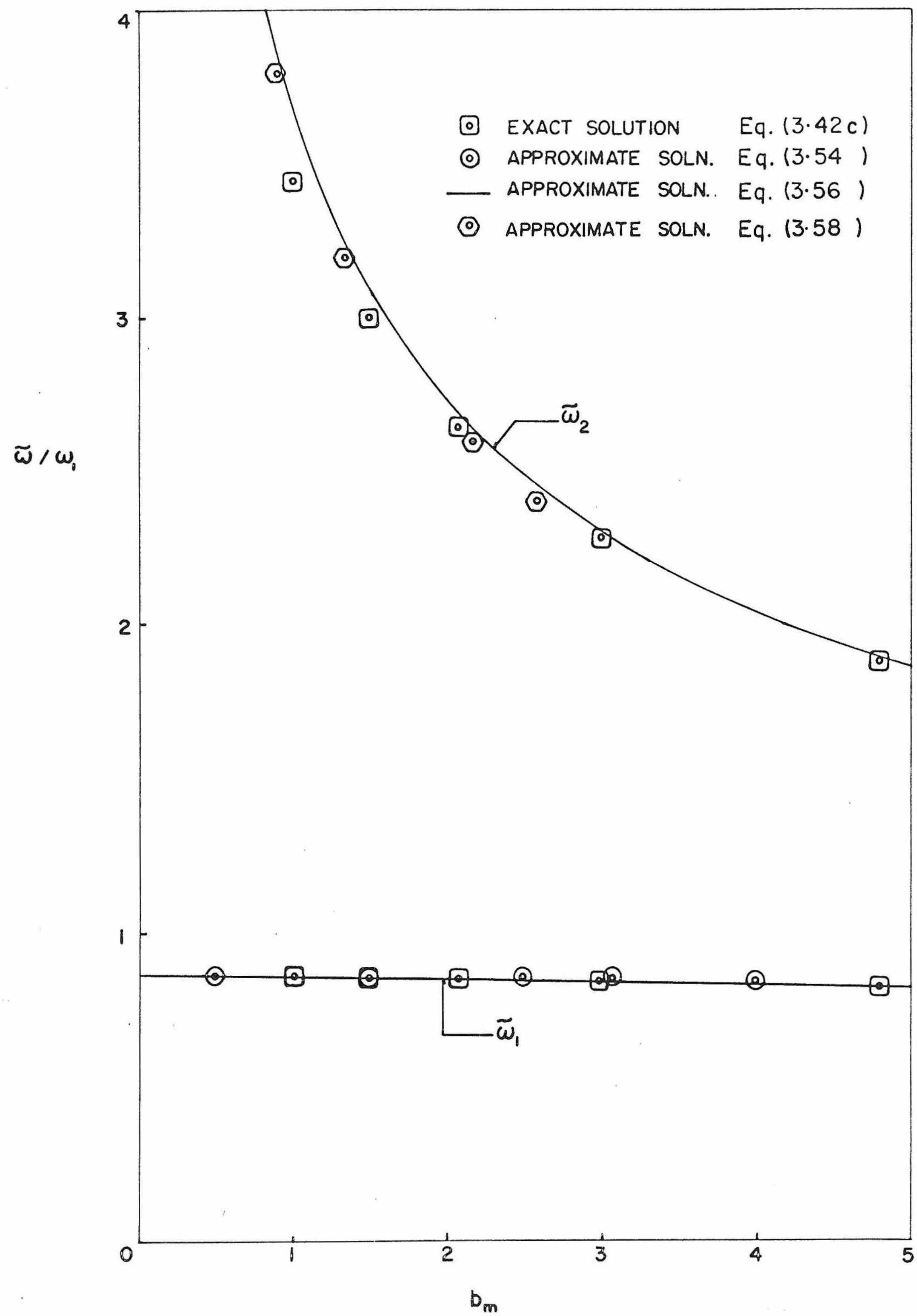


Fig. 9. Resonant frequencies of single-story building-foundation system ($a_1 = 0.5$, $b_1 = 1$, $\alpha_1 = 2$, $\sigma = 1/4$, $\eta_1 = b_h = 0$)

Behavior of resonant frequencies as functions of b_h and b_m

To study the behavior of the three resonant frequencies of the system shown in Fig. 5 for non-vanishing values of the parameters b_h and b_m, it is found convenient to write the frequency equation (3.53) in a form similar to that of Eq. (3.56):

$$b_h = \frac{\beta_h \ \sigma_h}{a_1^2 (\frac{\tilde{\omega}}{\omega_1})^2} \ \frac{[\Delta_d]_{b_h=0}}{[\Delta_d]_{b_h=0, 1/\beta_h=0}} \tag{3.60}$$

where Δ_d is defined by Eq. (3.42d) with $\eta_1 = \zeta_h = \zeta_m = 0$ and $s_o = ia_1 \tilde{\omega}/\omega_1$.

The meaning of Eq. (3.60) can be explained most conveniently by referring to its schematical representation depicted in Fig. 10. As shown in the figure, Eq. (3.60) describes the behavior of the three resonant frequencies of the system with respect to b_h for a fixed value of b_m. In fact, Fig. 10 shows that all the resonant frequencies decrease monotonically for increasing values of b_h, each becoming asymptotic to a horizontal line. The initial values and the corresponding asymptotic values of $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\omega}_3$ can be found from Eq. (3.60). The two finite initial values of $\tilde{\omega}/\omega_1$ are the roots of the frequency equation

$$[\Delta_d]_{b_h=0} \tag{3.61a}$$

whereas the non-vanishing asymptotic values of $\tilde{\omega}$ correspond to the roots of

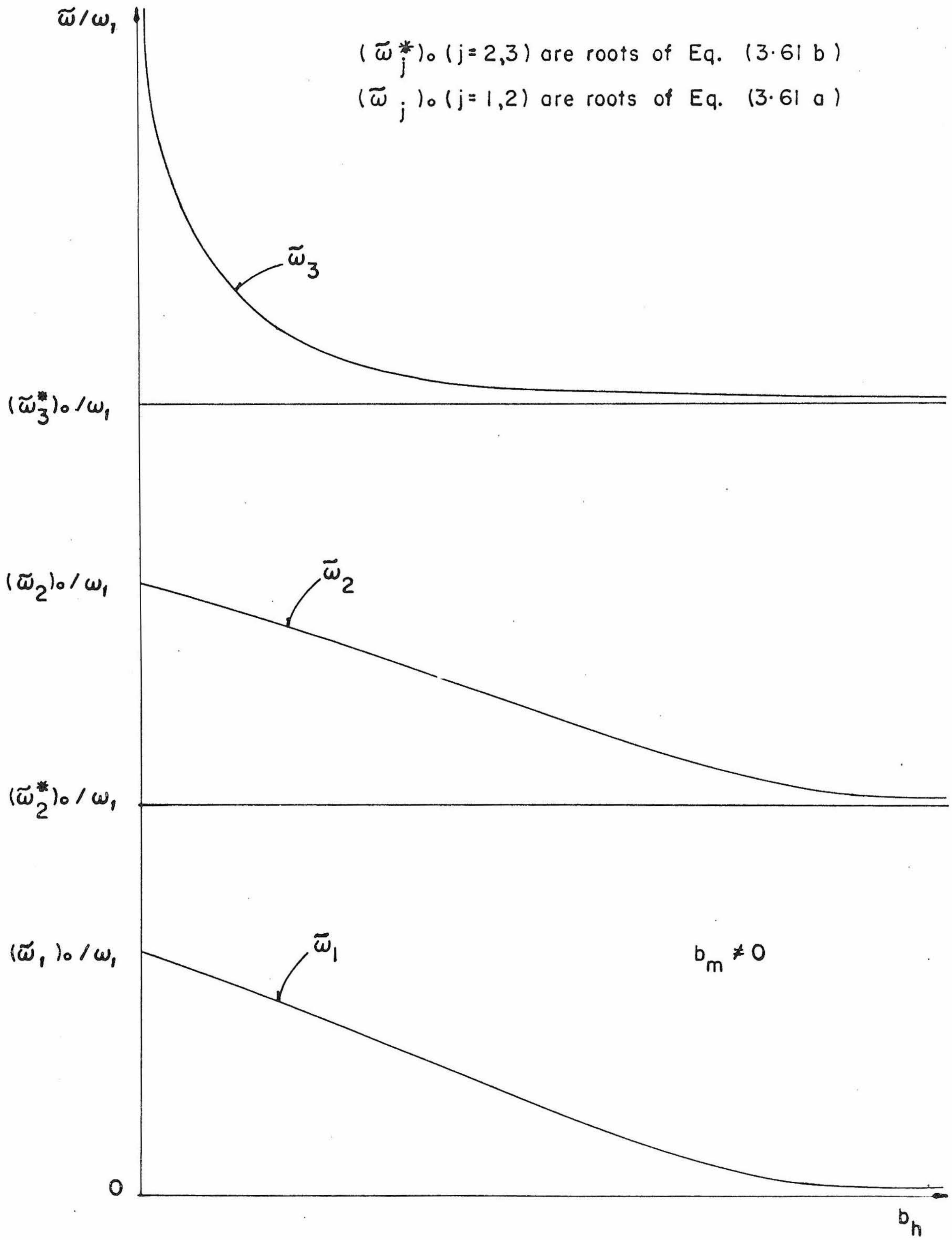


Fig. 10. Resonant frequencies of single-story building-foundation system (schematic diagram)

$$[\Delta_d]_{b_h=0,1/\beta_h=0} = 0 \,. \tag{3.61b}$$

Since Eqs. (3.61a) and (3.55) are identical it is verified that the roots of Eq. (3.61a) correspond to the resonant frequencies of a single-story interaction system for which b_h vanishes. The roots of Eq. (3.61b) are the resonant frequencies of a system like the one above, except that the base mass can only rotate.

It can be shown that Eq. (3.60) remains valid for multi-story building-foundation systems provided Δ_d is interpreted as the frequency equation of the corresponding system. This result can be used to show that all the resonant frequencies of a multi-story building-foundation system decrease for increasing values of b_m and b_h.

Frequency response

In addition to modifying the resonant frequencies of a single-story interaction system, the parameter b_m (and b_h) will also affect the peak values of the amplification factors of the response. This is illustrated in Fig. 11, which gives the frequency response of the building foundation system defined by $a_1 = 0.5$, $b_1 = 1$, $\alpha_1 = 2$ and $b_h = 0$ for several values of b_m. Plotted in the figure are the amplification factors $|\bar{y}_o|/\bar{v}_g$, $|\bar{\varphi}h_1|/\bar{v}_g$ and $|\bar{v}_1|/\bar{v}_g$, obtained from Eqs. (3.42) in terms of the frequency ratio ω/ω_1.

Two distinct resonant frequencies of the response can be recognized in Fig. 11 for each non-zero value of b_m. For the values of b_m considered in the figure, however, the peak values of the response corresponding to the second resonant frequencies are small, compared to

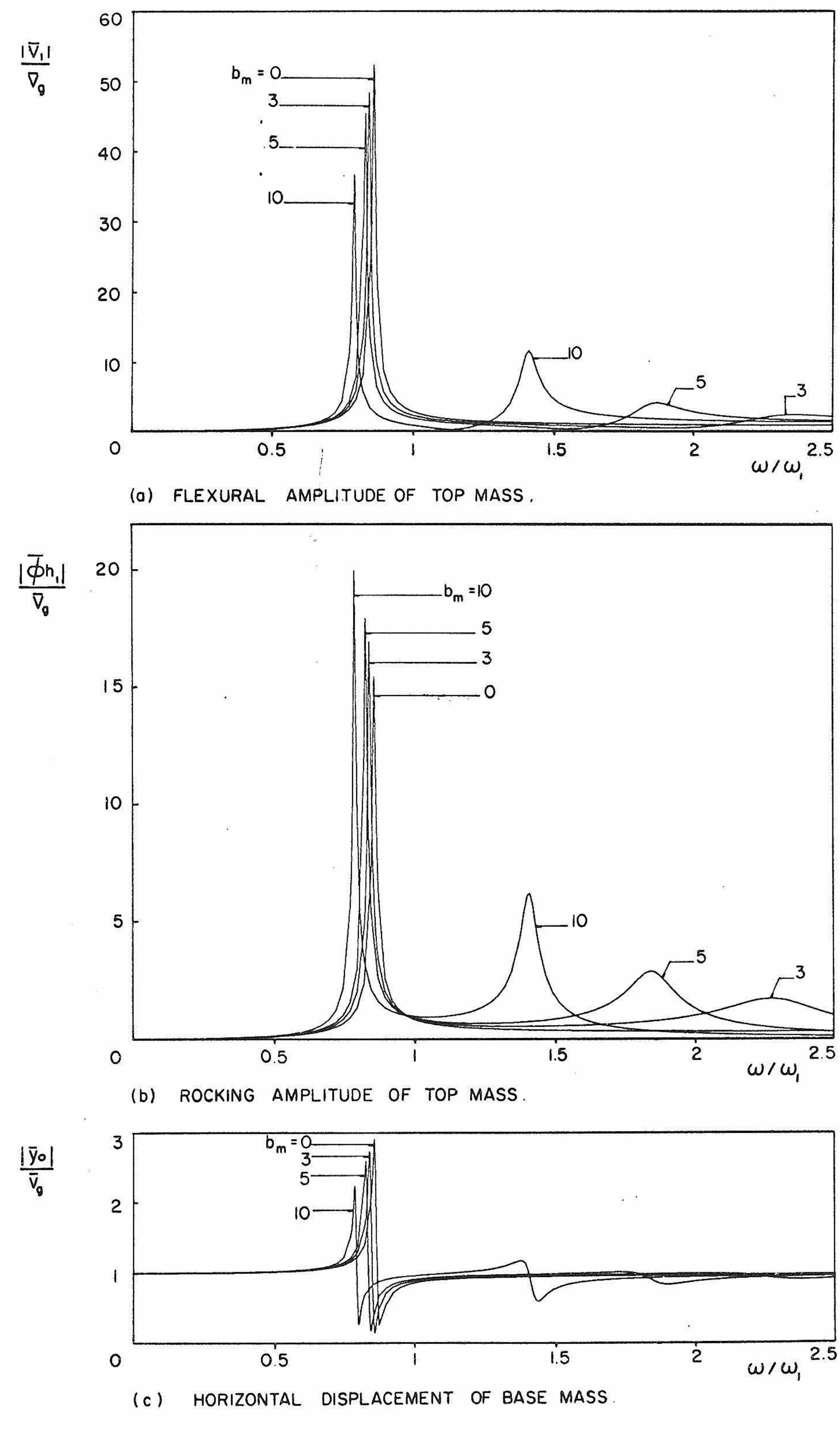


Fig. 11. Frequency response of single-story building-foundation system
($a_1 = 0.5$, $b_1 = 1$, $\alpha_1 = 2$, $\sigma = 1/4$, $b_h = \eta_1 = 0$)

those obtained for the fundamental mode of vibration, thus indicating that a larger amount of effective damping is associated with the second mode of vibration than with the first. It may be noted also that values of b_m up to about 3 do not affect significantly the maximum values of the amplification factors corresponding to the fundamental resonant frequency.

The foregoing observations suggest that, depending on the values of a_1, b_1 and α_1, the response of a building-foundation system having values of b_h and b_m which depart appreciably from zero, may not be significantly different from the response of the system with vanishing values of b_h and b_m.

Earthquake response

This section is devoted to studying the response of an idealized single-story building-foundation system to the free-field earthquake motion $\ddot{v}_g(t)$. The system, shown in Fig. 5, is taken to be initially at rest.

The equations for the Laplace transforms of the response quantities of interest, given by Eqs. (3.42), can be expressed for simplicity as

$$\begin{Bmatrix} \bar{v}_o \\ \bar{y}_o \\ \bar{\varphi} h_1 \\ \bar{v}_1 \end{Bmatrix} = \frac{\bar{\ddot{v}}_g}{\Delta_d} \begin{Bmatrix} \Delta_o \\ \Delta_y \\ \Delta_\varphi \\ \Delta_1 \end{Bmatrix} \quad . \tag{3.62}$$

The corresponding displacements are obtained by inverting

Eq. (3.62). After making use of the convolution theorem of Laplace transformations, Eq. (3.62) gives

$$\begin{Bmatrix} v_o(t) \\ y_o(t) \\ h_1\varphi(t) \\ v_1(t) \end{Bmatrix} = \int_0^t \begin{Bmatrix} h_o(t-\tau) \\ h_y(t-\tau) \\ h_\varphi(t-\tau) \\ v_1(t-\tau) \end{Bmatrix} \ddot{v}_g(\tau)\, d\tau \qquad (3.63a)$$

in which the impulse response functions $h_k(t)$ are defined by

$$h_k(t) = \frac{1}{2\pi i}\int_C \frac{\Delta_k(s)}{\Delta_d(s)}\, e^{st}\, ds \ ; \ k=0,y,\varphi,1 \qquad (3.63b)$$

where C is the Bromwich contour.

Contour integration can be used to solve, in closed form, the integrals appearing in Eq. (3.63b) for given values of the parameters a_1, b_1, α_1, η_1, b_h, b_m and σ. The resulting equations can in turn be substituted back into Eq. (3.63a) to obtain expressions for the displacements v_o, y_o, φh_1 and v_1, similar to those given by Eqs. (3.22).

It is not generally practicable to obtain explicit formulas for the impulse response functions $h_k(t)$ in terms of the system parameters themselves, because of the difficulty in finding explicit solutions for the zeros of the function Δ_d. It is, however, of considerable practical interest that approximate, closed form solutions can be found for the zeros of Δ_d when b_h and b_m vanish. This permits obtaining explicit formulas for the response quantities in terms of the parameters a_1, b_1, α_1, η_1 and σ. Furthermore, as was shown in the preceding section, the results obtained for vanishing values of b_h and b_m can be used to estimate the response of systems for which b_h and b_m are non-zero. The range of values of b_h and b_m for which this approximation holds depends fundamentally on the values of the parameters a_1, b_1 and α_1.

Approximate solution

It seems reasonable to assume that only one of the pairs of complex conjugate roots of the equation

$$\Delta_d = 0 \tag{3.64}$$

need be taken into consideration when b_h and b_m are small, i.e., the system has only one significant natural frequency. The other two pairs are associated with high frequencies and large amounts of damping, and therefore do not affect materially the response of the system.

Approximate values for the significant roots $(s_o)_{1,2}$ of Eq. (3.64) can be obtained by retaining only first-order terms in η_1, ζ_h and ζ_m. For the case when b_h and b_m vanish, one gets

$$(s_o)_{1,2} = -\frac{a_1\eta_1 + \dfrac{a_1^4 b_1}{2}\left(\dfrac{\zeta_h}{\beta_h\sigma_h} + \dfrac{\zeta_m\alpha_1^2}{\beta_m\sigma_m}\right)}{\left[1 + a_1^2 b_1\left(\dfrac{1}{\beta_h\sigma_h} + \dfrac{\alpha_1^2}{\beta_m\sigma_m}\right)\right]^2} \pm i\,\frac{a_1}{\left[1 + a_1^2 b_1\left(\dfrac{1}{\beta_h\sigma_h} + \dfrac{\alpha_1^2}{\beta_m\sigma_m}\right)\right]^{\frac{1}{2}}}. \tag{3.65}$$

The functions β_h, β_m, ζ_h and ζ_m in Eq. (3.65) are to be evaluated at a frequency $a_o = \mathrm{Im}(s_o)_{1,2}$. Thus, several iterations may be necessary to evaluate $(s_o)_{1,2}$.

After determining the roots $(s_o)_{1,2}$, the impulse response functions $h_k(t)$ defined by Eq. (3.63b) can then be obtained using contour integration and the residue theorem. Substituting the resulting equations into Eq. (3.63a) leads to

$$\begin{Bmatrix} v_o(t) \\ h_1\varphi(t) \\ v_1(t) \end{Bmatrix} = -\frac{1}{\tilde{\omega}_1\left[1+a_1^2 b_1\left(\frac{1}{\beta_h\sigma_h}+\frac{\alpha_1^2}{\beta_m\sigma_m}\right)\right]}\int_0^t e^{-\tilde{\eta}_1\tilde{\omega}_1(t-\tau)}\sin\tilde{\omega}_1(t-\tau)\begin{Bmatrix} \ddot{v}_o^{\,e}(\tau) \\ \ddot{v}_\varphi^{\,e}(\tau) \\ \ddot{v}_1^{\,e}(\tau) \end{Bmatrix} d\tau. \tag{3.66a}$$

The equivalent undamped natural frequency $\tilde{\omega}_1$ and equivalent critical damping ratio $\tilde{\eta}_1$ are given by

$$\tilde{\omega}_1 = \frac{\omega_1}{\left[1 + a_1^2\, b_1\left(\frac{1}{\beta_h\,\sigma_h} + \frac{\alpha_1^2}{\beta_m\,\sigma_m}\right)\right]^{\frac{1}{2}}} \tag{3.66b}$$

$$\tilde{\eta}_1 = \frac{\eta_1 + \frac{a_1^3\, b_1}{2}\left(\frac{\zeta_h}{\beta_h\,\sigma_h} + \frac{\zeta_m\,\alpha_1^2}{\beta_m\,\sigma_m}\right)}{\left[1 + a_1^2\, b_1\left(\frac{1}{\beta_h\,\sigma_h} + \frac{\alpha_1^2}{\beta_m\,\sigma_m}\right)\right]^{\frac{3}{2}}} \tag{3.66c}$$

and $\ddot{v}_o^{\,e}$, $\ddot{v}_\varphi^{\,e}$, $\ddot{v}_1^{\,e}$ are defined by

$$\begin{Bmatrix} \ddot{v}_o^{\,e}(t) \\ \ddot{v}_\varphi^{\,e}(t) \\ \ddot{v}_1^{\,e}(t) \end{Bmatrix} = \begin{Bmatrix} \frac{a_1^2 b_1}{\beta_h\,\sigma_h}\left[\ddot{v}_g(t) - \tilde{\omega}_1\dot{v}_g(t)\left(2\eta_1 + \zeta_m\,\frac{a_1^3 b_1 \alpha_1^2}{\beta_m\sigma_m} - \left(1 + \frac{a_1^2 b_1 \alpha_1^2}{\beta_m\sigma_m}\right)\zeta_h a_1\right)\right] \\ \frac{a_1^2 b_1 \alpha_1^2}{\beta_m\,\sigma_m}\left[\ddot{v}_g(t) - \tilde{\omega}_1\dot{v}_g(t)\left(2\eta_1 + \zeta_h\,\frac{a_1^3 b_1}{\beta_h\,\sigma_h} - \left(1 + \frac{a_1^2 b_1}{\beta_h\,\sigma_h}\right)\zeta_m a_1\right)\right] \\ \ddot{v}_g(t) + \tilde{\omega}_1\dot{v}_g(t)\left[2\eta_1 a_1^2 b_1\left(\frac{1}{\beta_n\sigma_n} + \frac{\alpha_1^2}{\beta_m\sigma_m}\right) + a_1^3 b_1\left(\frac{\zeta_m\alpha_1^2}{\beta_m\,\sigma_m} + \frac{\zeta_h}{\beta_h\sigma_h}\right)\right] \end{Bmatrix} \tag{3.66d}$$

An approximate form of Eq. (3.66d) can be obtained by neglecting the damping coefficients η_1, ζ_h and ζ_m. With this, Eq. (3.66a) becomes

$$\begin{Bmatrix} v_o(t) \\ h_1\varphi(t) \\ v_1(t) \end{Bmatrix} = - \frac{1}{\tilde{\omega}_1 \left[1 + a_1^2 \, b_1 \left(\frac{1}{\beta_h \, \sigma_h} + \frac{\alpha_1^2}{\beta_m \, \sigma_m} \right) \right]} \begin{Bmatrix} \dfrac{a_1^2 \, b_1}{\beta_h \, \sigma_h} \\ \dfrac{a_1^2 \, b_1 \, \alpha_1^2}{\beta_m \, \sigma_m} \\ 1 \end{Bmatrix}$$

$$\int_0^t e^{-\tilde{\eta}_1 \tilde{\omega}_1 (t-\tau)} \sin \tilde{\omega}_1 (t-\tau) \, \ddot{v}_g(\tau) \, d\tau \, . \qquad (3.67)$$

Interpretation of the solution

The individual equations in Eq. (3.67) are almost identical to those obtained for the response of linear oscillators on rigid foundations. Thus, to a first-order approximation in η_1 the following result is established by Eq. (3.67): the earthquake response of the single-story building-foundation system shown in Fig. 5 is equivalent to the response of a one degree-of-freedom damped oscillator resting upon a rigid ground. This equivalent oscillator, defined by its undamped natural frequency $\tilde{\omega}_1$ (3.66b) and critical damping ratio $\tilde{\eta}_1$ (3.66c), is subjected to the equivalent acceleration

$$\ddot{v}_g^{\,e}(t) = \left(\frac{\tilde{\omega}_1}{\omega_1} \right)^2 \ddot{v}_g(t) \, . \qquad (3.68)$$

The deformation of the equivalent oscillator is identical to the relative story displacement v_1 of the original system. Also, the displacements $v_o(t)$ and $h_1\varphi(t)$ are

$$v_o(t) = \frac{a_1^2 b_1}{\beta_h \sigma_h} v_1(t) \tag{3.69a}$$

$$h_1\varphi(t) = \frac{a_1^2 b_1 \alpha_1^2}{\beta_m \sigma_m} v_1(t) . \tag{3.69b}$$

The practical implication of the foregoing is that the earthquake response spectrum of a single-story building-foundation system may be obtained from the standard spectra available for fixed-base, one degree-of-freedom oscillators. It is only necessary to evaluate the natural frequency $\tilde{\omega}_1$ and critical damping ratio $\tilde{\eta}_1$ of the equivalent oscillator from Eqs. (3.66b) and (3.66c), respectively, and to multiply the free-field earthquake acceleration $\ddot{v}_g(t)$ by $(\tilde{\omega}_1/\omega_1)^2$ to obtain the equivalent input acceleration $\ddot{v}_g^{\,e}(t)$.

From Eqs. (3.66b) and (3.68) it is seen that the effective natural frequency of the single-story building-foundation system, as well as the amplitude of the equivalent input acceleration, always decrease as a result of the dynamic coupling between the building and the soil. In contrast, it is expected that the effective damping in the system will, in general, be increased by soil-structure interaction. Equation (3.66c) shows, however, that the opposite effect also is possible. Whether $\tilde{\eta}_1$ is less than or greater than η_1 is determined by the values of the system parameters a_1, b_1, α_1, σ and η_1.

It is worth noting that there is an exact correspondence to first order in $\tilde{\eta}_1$ between the equivalent linear oscillator described above and the approximate results obtained before for the steady-state harmonic response of the original single-story building-foundation system (Eqs. (3.51)).

Response of the equivalent oscillator to white noise excitation

The earthquake response of a single-story building is modified by the dynamic coupling between the building and its foundation. Whether there will be an increase or decrease of the response will depend upon the values of $\tilde{\omega}_1$ and $\tilde{\eta}_1$ and upon the detailed time history of the particular earthquake under consideration.

It is of interest to obtain an estimate of the effect of soil-structure interaction on the earthquake response of buildings without referring to a particular earthquake. This can be accomplished most simply by considering idealized earthquakes represented by a white noise excitation. In this case, the weakly-stationary mean-square response of the one-degree-of-freedom oscillator representing the single-story building-foundation system, is[78]

$$E[v_1^2(t)] = \frac{\pi}{2} \frac{\tilde{S}_f}{\tilde{\eta}_1 \tilde{\omega}_1^3} \tag{3.70a}$$

where $\tilde{S}_f$ is the constant spectral density of the equivalent excitation. The relation between $\tilde{S}_f$ and S_f, the constant spectral density of the free-field surface excitation, is obtained from Eq. (3.68),

$$\tilde{S}_f = \left(\frac{\tilde{\omega}_1}{\omega_1}\right)^2 S_f \ . \tag{3.70b}$$

To examine the effect of the flexibility of the foundation on the earthquake response of the system it is convenient to define

$$R = \frac{\left\{E[v_1^2(t)]\right\}_{\text{flexible foundation}}}{\left\{E[v_1^2(t)]\right\}_{\text{rigid foundation}}} . \qquad (3.71a)$$

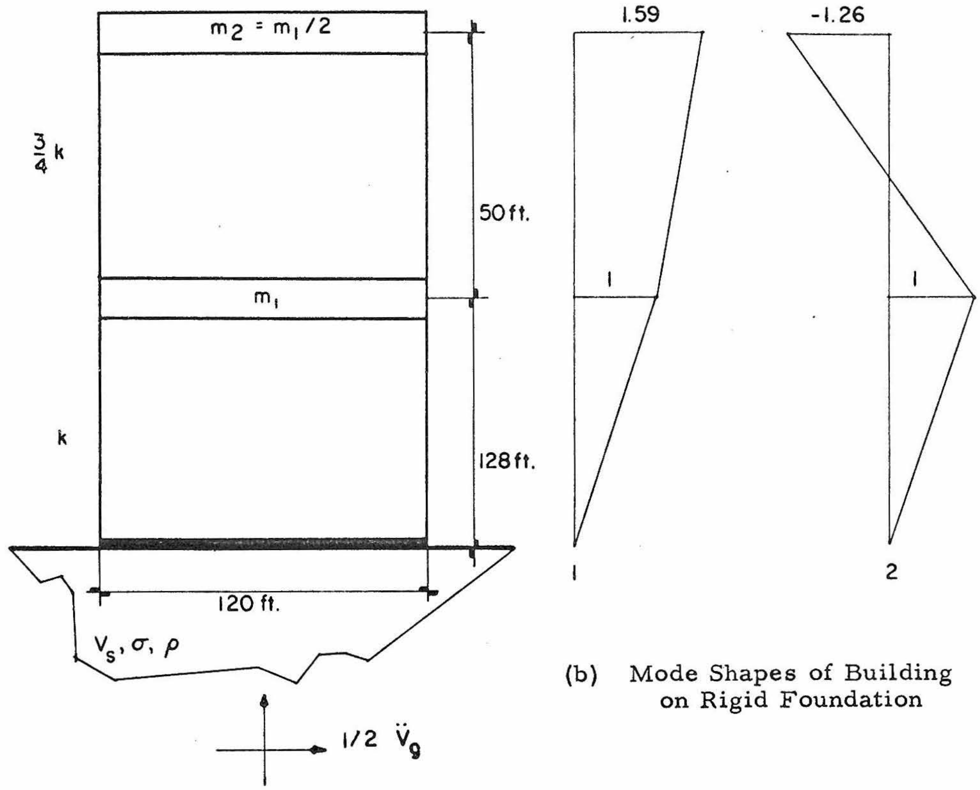
The ratio R can be evaluated explicitly by using Eqs. (3.66b), (3.66c) and (3.70),

$$R = \frac{2\eta_1 \left[1 + a_1^2 \, b_1 \left(\frac{1}{\beta_h \, \sigma_h} + \frac{\alpha_1^2}{\beta_m \, \sigma_m}\right)\right]^2}{2\eta_1 + a_1^3 \, b_1 \left(\frac{\zeta_h}{\beta_h \, \sigma_h} + \frac{\zeta_m \, \alpha_1^2}{\beta_m \, \sigma_m}\right)} . \qquad (3.71b)$$

This equation gives an approximate formula in terms of the system parameters a_1, b_1, α_1, η_1, and σ that permits analyzing the effect of a deformable soil on the earthquake response of the single-story building-foundation system. From Eq. (3.71a) it is seen that a reduction in the response of the system compared to that of the building on a rigid soil is indicated if $R < 1$. Conversely, an increase in the response will be obtained if $R > 1$. Without knowledge of the parameters of a particular system, the value of R cannot be established.

4. Earthquake Response of Two-Story Building-Foundation Systems

The earthquake response of the idealized two-story building-foundation system shown in Fig. 12 will be studied in this section to illustrate the use of the methods of analysis developed in section C for the case of multi-story buildings. The values of the parameters



(a) Model

(b) Mode Shapes of Building on Rigid Foundation

$\omega_1 = 25.15$ rad/sec ($T_1 = 0.25$ sec)	$I_t = 16.48 \times 10^8$ lb-sec^2/ft
$\omega_2 = 55.30$ rad/sec	$r = 60$ ft.
$m_1 = .475 \times 10^6$ lb-sec^2/ft	$\rho = 3.73$ lb-sec^2/ft^4 ($\gamma_s = 120$ lb/ft^3)
$m_2 = .2375 \times 10^6$ lb-sec^2/ft	$\sigma = 1/4$
$m_0 = 1.07 \times 10^6$ lb-sec^2/ft	Undamped Building

(c) Values of the Parameters

Fig. 12. Idealized model of two-story building-foundation system

used for the model have been selected to represent, approximately, a concrete nuclear power plant[68]. Figure 12 also shows the mode shapes of the two-story building if it were supported on a rigid foundation.

The response of the idealized system will be obtained from Eqs. (3.22) as a linear combination of the individual response of four simple linear oscillators resting on a rigid ground. To study the influence of the flexibility of the soil on the response of the system, several values of shear wave velocity of the foundation medium, V_s, will be considered.

Presentation and discussion of results

The natural frequencies $\tilde{\omega}_\ell$ and critical damping ratios $\tilde{\eta}_\ell$ of the four equivalent oscillators defined by Eq. (3.22a) have been obtained from Eqs. (3.10e), (3.22b) and (3.22c) for several values of the shear wave velocity of the elastic medium. The results of the calculations, presented in Table 2, show that whereas the fundamental frequency of the system, $\tilde{\omega}_1$, is reduced considerably as the soil becomes soft, $\tilde{\omega}_2$ remains almost constant for all values of V_s. The frequencies $\tilde{\omega}_3$ and $\tilde{\omega}_4$, which arise with the introduction of rocking and relative lateral motion of the base, decrease monotonically from infinity for decreasing values of V_s. For softer soils, $\tilde{\omega}_3$ becomes less than $\tilde{\omega}_2$. Table 2 also shows that the amount of damping associated with the frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ is negligible for hard soils but increases as the soil gets softer; $\tilde{\eta}_1$ increasing monotonically to about 4 percent and $\tilde{\eta}_2$ reaching a maximum value of about one percent corresponding to V_s = 1500 ft/sec. For this value of V_s, $\tilde{\eta}_1$ is equal to 2.6 percent. In contrast with

TABLE 2

Resonant Frequencies and Critical Damping Ratios of Two-Story System

V_s (ft/sec)	$\tilde{\omega}_\ell$ (rad/sec)				$\tilde{\eta}_\ell$ (%)			
ℓ	1	2	3	4	1	2	3	4
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
800	8.87	54.84	23.86	87.73	4.26	.88	58.69	16.14
1000	10.76	54.85	29.85	89.25	3.77	.99	58.70	19.86
1200	12.47	54.86	35.86	90.79	3.27	1.05	58.74	23.55
1500	14.70	54.86	44.93	93.66	2.57	1.06	58.86	27.61
2000	17.54	54.82	60.14	101.6	1.65	.90	59.27	31.53
2700	20.12	54.86	81.29	117.3	.87	.53	59.93	34.47
3500	21.83	54.97	105.2	138.4	.44	.27	60.43	36.03
5000	23.39	55.12	149.8	182.8	.16	.09	60.92	37.22
8000	24.43	55.22	238.8	279.5	.04	.02	61.25	37.24
10000	24.68	55.25	298.2	345.4	.02	.01	61.33	37.37
15000	24.94	55.28	446.7	512.0	.006	.003	61.41	37.53
∞	25.15	55.30	∞	∞	0	0	—	—

$\tilde{\eta}_1$ and $\tilde{\eta}_2$, the critical damping ratios $\tilde{\eta}_3$ and $\tilde{\eta}_4$ are large even for hard soils.

The participation factors $a_{\ell k}$ and $b_{\ell k}$ appearing in Eqs. (3.22) have been computed from Eq. (3.18a) and are given in Table 3 for several values of V_s. The corresponding values for a rigid soil are also included in the table. The coefficients $a_{\ell k}$ serve to measure the extent to which a system fails to be classical, as these coefficients vanish identically for systems with classical normal modes.

With $\tilde{\omega}_\ell$, $\tilde{\eta}_\ell$, $a_{\ell k}$ and $b_{\ell k}$ established, the earthquake response of the two-story building-foundation system may then be obtained by means of standard numerical techniques for evaluating the response of single-degree-of-freedom linear oscillators subject to base motion.

Suppose, for instance, that the system is subjected to the free-field acceleration $\ddot{v}_g(t)$ depicted in Fig. 13a, which represents the corrected version of the N33E component of the earthquake motion (first event) recorded at the SCE Power Plant, San Onofre, California on April 8, 1968[79]. In this example the shear wave velocity of the soil is taken to be equal to 1500 ft/sec, but for purposes of comparison, the response of the building also will be obtained for a rigid soil.

The velocity trace $\dot{v}_g(t)$ of the input motion and the equivalent input acceleration $\ddot{v}_{11}^e(t)$ calculated from Eq. (3.22d) for the flexible soil are shown in Figs. 13b and 13c, respectively. It may be seen that the accelerations $\ddot{v}_{11}^e(t)$ and $\ddot{v}_g(t)$ are nearly proportional, thus indicating that the contribution of $\dot{v}_g(t)$ to $\ddot{v}_{11}^e(t)$ is small and therefore, could be neglected for a first approximation. The remaining equivalent

TABLE 3

Participation Factors $b_{\ell k}$, $a_{\ell k}$ for Two-Story System

V_s (ft/sec) (1)	ℓ / k (2)	$b_{\ell k}$ (sec)				$a_{\ell k}$ (sec)			
		$0(10^{-1})$ (3)	$\varphi(10^{-4})$ (4)	$1(10^{-1})$ (5)	$2(10^{-1})$ (6)	$0(10^{-1})$ (7)	$\varphi(10^{-4})$ (8)	$1(10^{-1})$ (9)	$2(10^{-1})$ (10)
1000	1	.0773	4.922	.1474	.2277	.0326	.0962	-.0050	-.0083
	2	.0068	.0248	.0073	-.0149	.0178	.0448	.0066	-.0275
	3	.3924	-2.409	-.0316	-.0479	-.0277	.0343	-.0297	.0066
	4	.0036	.0984	-.0175	-.0202	-.0101	-.2790	.0421	.0549
1200	1	.0619	3.916	.1707	.2641	.0256	.0836	-.0047	-.0079
	2	.0149	.0350	.0104	-.0219	.0243	.0482	.0085	-.0329
	3	.3205	-1.946	-.0391	-.0510	-.0255	-.0554	-.0346	.0096
	4	.0063	.1390	-.0240	-.0285	-.0144	-.3078	.0455	.0603
1500	1	.0464	2.894	.2005	.3111	.0186	.0665	-.0039	-.0066
	2	.0311	.0481	.0164	-.0341	.0270	.0432	.0108	-.0373
	3	.2408	-1.474	-.0514	-.0505	-.0206	.0867	-.0393	.0114
	4	.0121	.2065	-.0343	-.0420	-.0247	-.3424	.0496	.0666
2000	1	.0307	1.875	.2371	.3698	.0116	.0423	-.0021	-.0039
	2	.0497	.0509	.0268	-.0527	.0090	.0218	.0107	-.0328
	3	.1542	-.974	-.0687	-.0477	.0016	.1250	-.0373	.0032
	4	.0173	.2536	-.0399	-.0505	-.0418	-.3604	.0512	.0693
5000	1	.0063	.352	.3012	.4764	.0015	.0022	.0003	.0003
	2	.0066	.0119	.0367	-.0534	-.0107	.0007	.0010	-.0027
	3	.0620	-.133	-.0673	-.0602	.0474	.0530	-.0090	-.0075
	4	-.0040	.0683	-.0092	-.0125	-.0576	-.1230	.0164	.0225
∞	1	0	0	.3152	.5013	0	0	0	0
	2	0	0	.0375	-.0472	0	0	0	0
	3	0	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0	0

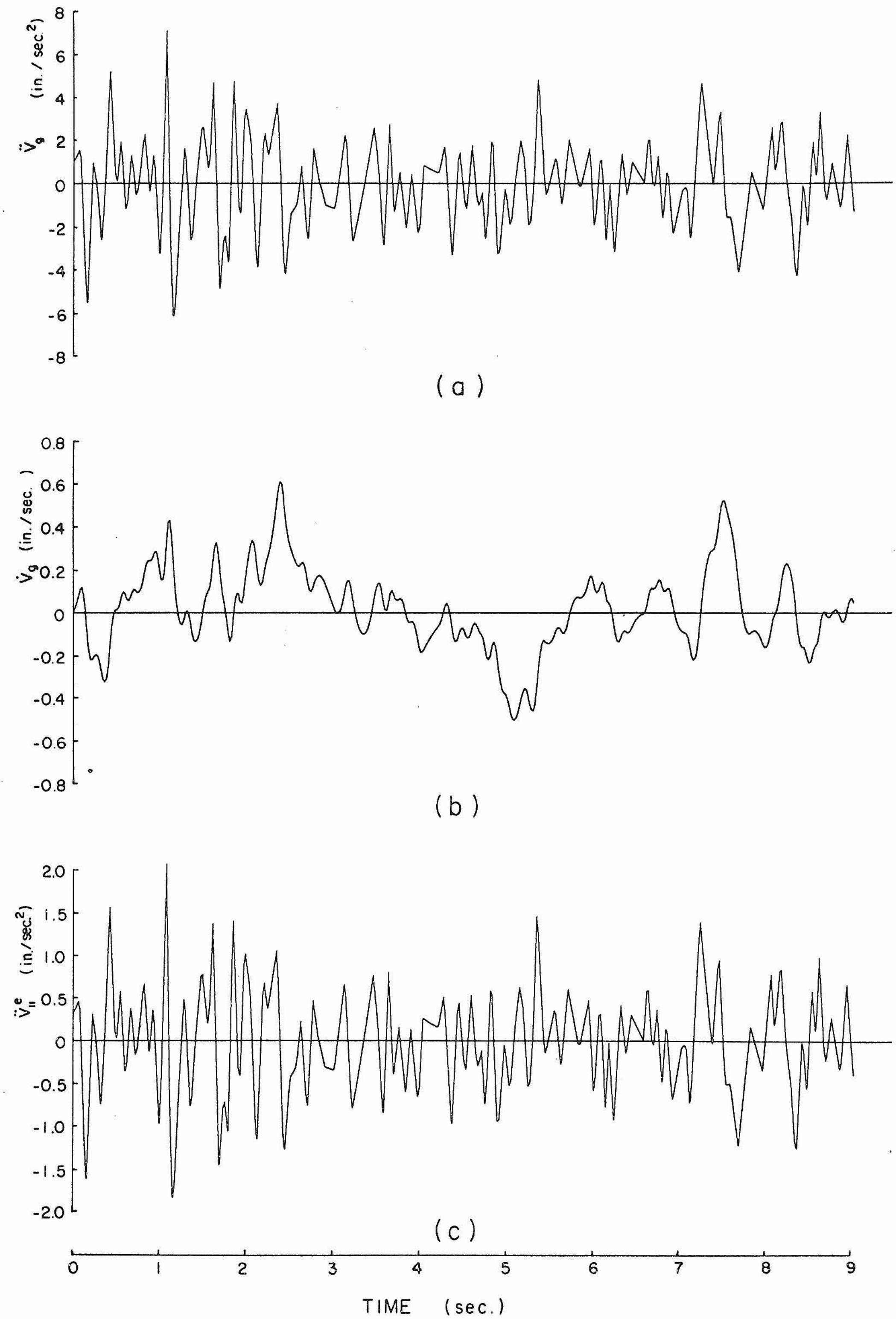


Fig. 13. Earthquake excitation for two-story building-foundation system

accelerations can also be obtained from Eq. (3.22d). The accelerations $\ddot{v}_{\ell k}^{e}(t)$, however, need not be evaluated explicitly as it is more advantageous to calculate first the response of the equivalent simple oscillators to the input motions $\ddot{v}_g(t)$ and $\tilde{\omega}_1\dot{v}_g(t)$ and then combine the individual responses linearly to obtain the response quantities $v_o(t)$, $\varphi(t)$, $v_1(t)$ and $v_2(t)$, as prescribed by Eqs. (3.22).

Figure 14 gives the time history for the displacements v_1, v_2, $h_1\varphi$ and v_o. To study the effect of the terms associated with the frequencies $\tilde{\omega}_3$ and $\tilde{\omega}_4$ on the response of the system, two families of curves have been included in Fig. 14, one obtained by omitting the terms on the right hand side of Eq. (3.22a) which contain the frequencies $\tilde{\omega}_3$ and $\tilde{\omega}_4$, and the other which includes all four terms. As the figure shows, this effect is only significant for the horizontal displacement of the base. Figure 14 also shows that: (1) the building vibrates primarily with a frequency $\tilde{\omega}_1$, the fundamental resonant frequency of the building-foundation system, and (2) the displacement of the first story due to rocking is about twice as large as the corresponding flexural displacement.

On comparing the relative displacements $v_1(t)$ and $v_2(t)$ of the building on a flexible foundation with the corresponding displacements obtained for a rigid soil (Fig. 15), it may be seen that the deformable foundation has the main effect of reducing both the dominant frequency of vibration and the maximum amplitude of the flexural displacements. Also it is obvious that the displacements $v_o(t)$ and $h_1\varphi(t)$ depart appreciably from zero for the flexible foundation, but they vanish identically for a rigid soil.

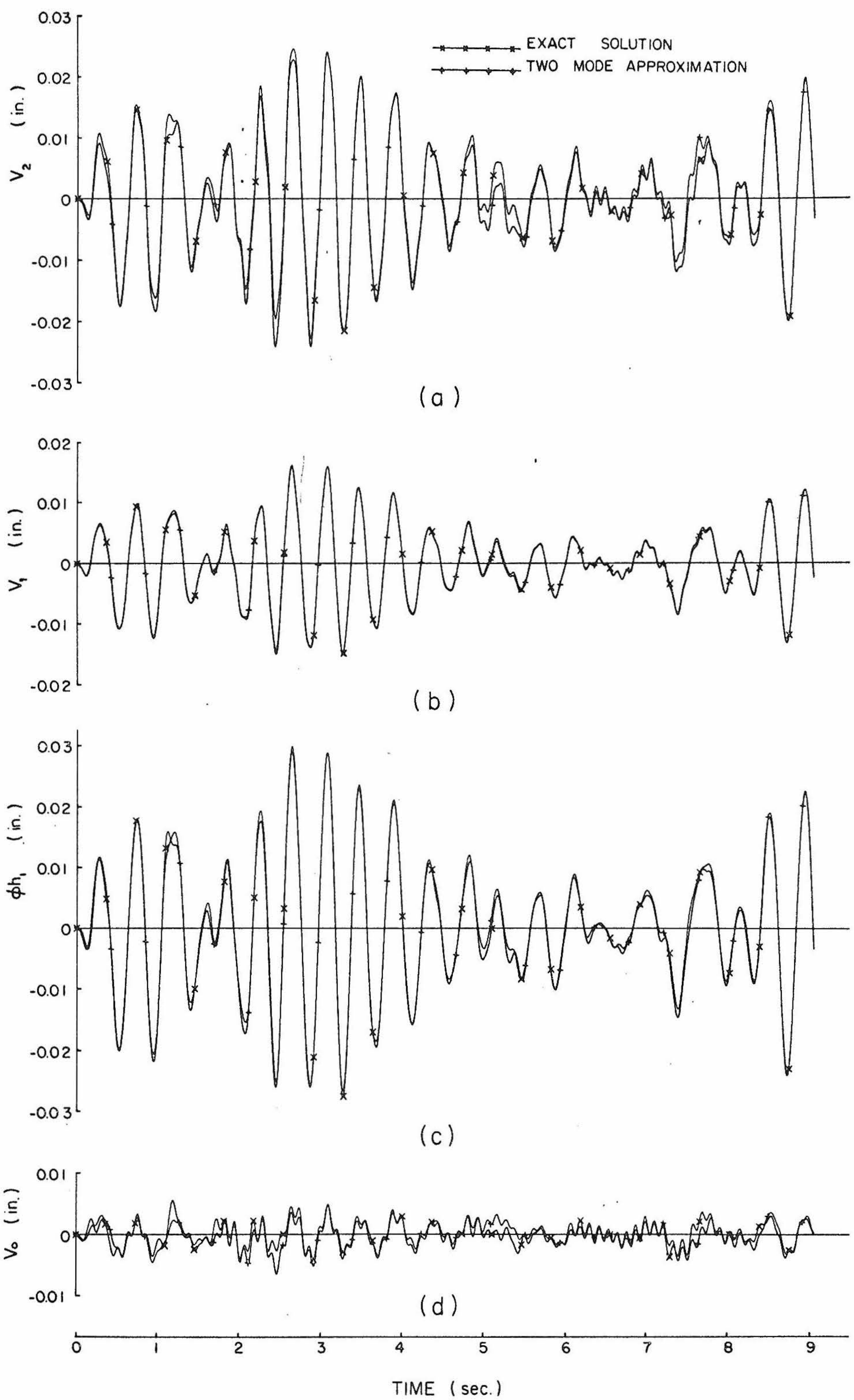


Fig. 14. Earthquake response of two-story building on elastic half-space

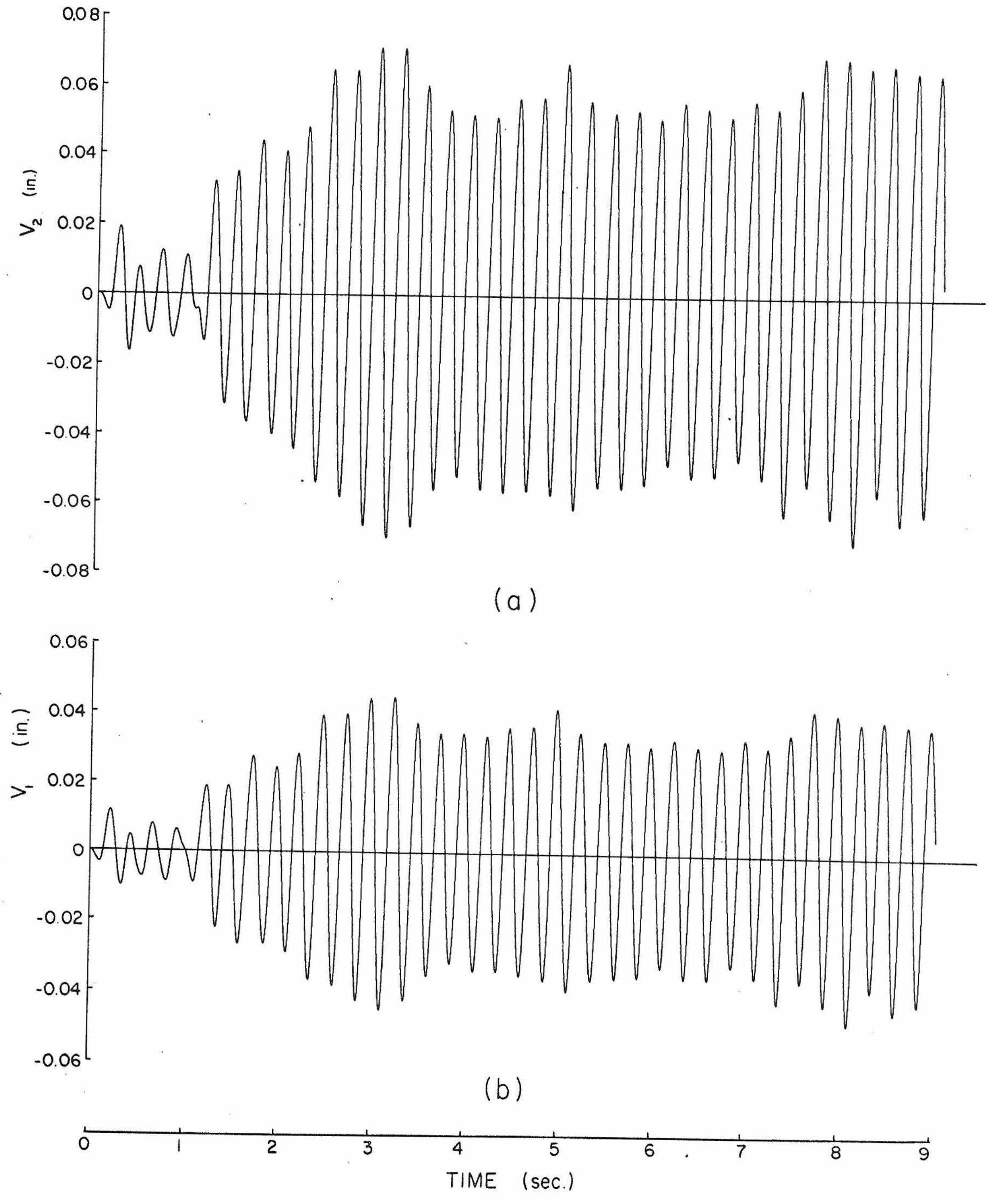


Fig. 15. Earthquake response of two-story building on rigid ground

Approximate formulas for first modal response

A comparison of Figs. 14 and 15, together with the approximate formulas (3.67) and (3.68) for the transient response of a single-story building-foundation system suggests that the mere knowledge of the fundamental resonant frequency $\tilde{\omega}_1$ and the critical damping ratio $\tilde{\eta}_1$ of a building-foundation system may be of considerable use in estimating the effect of a flexible foundation on the earthquake response of the fundamental mode of the building. Approximate values for $\tilde{\omega}_1$ and $\tilde{\eta}_1$ may be obtained from Eqs. (3.66b) and (3.66c) provided the parameters b_1 and α_1 appearing in these equations are defined by

$$b_1 = \frac{M_1}{\rho a^3} \tag{3.72a}$$

$$\alpha_1 = \frac{H_1}{a} \tag{3.72b}$$

where M_1, the first modal mass, is given by Eq. (3.9b) and H_1 is specified by

$$H_1 = \frac{Z_1}{M_1} \tag{3.72c}$$

in which Z_1 is given by Eq. (3.9c). The frequency parameter a_1 is defined as in Eq. (3.44a).

The frequency $\tilde{\omega}_1$ and damping ratio $\tilde{\eta}_1$ are given in Table 4 as discrete functions of the shear wave velocity of the foundation medium. Shown in the table are the exact values of $\tilde{\omega}_1$ and $\tilde{\eta}_1$ reproduced from Table 2 and the corresponding approximate values calculated from Eqs. (3.66b) and (3.66c). A remarkable agreement is obtained between the

TABLE 4

Fundamental Resonant Frequency and Critical Damping Ratio of Two-Story System

V_s (ft/sec) (1)	$\tilde{\omega}_1$ (rad/sec)		$\tilde{\eta}_1$ (%)	
	Exact (2)	Approximate (3)	Exact (4)	Approximate (5)
800	8.87	9.16	4.26	4.47
1000	10.76	11.07	3.77	3.50
1200	12.47	12.77	3.27	3.33
1500	14.70	14.97	2.57	2.57
2000	17.54	17.74	1.65	1.61
2700	20.12	20.23	.87	.84
3500	21.83	21.89	.44	.43
5000	23.39	23.41	.16	.16
8000	24.43	24.43	.04	.04
10000	24.68	24.68	.02	.02
15000	24.94	24.94	.006	.006

exact and the approximate values of $\tilde{\omega}_1$ and $\tilde{\eta}_1$ for the sample structure considered herein. Satisfactory results also would be expected for other systems.

5. Natural Frequencies of Multi-Story Building-Foundation Systems

In this section the effect of foundation compliance on the resonant frequencies of multi-story buildings is investigated. The resonant frequencies of an idealized ten-story undamped building resting upon an elastic half-space are calculated first and more general systems are examined subsequently.

The natural frequencies and mode shapes of the ten-story building are taken as equal to the values calculated by Housner and Brady[80] for case 10 c, a two-bay steel frame attached to a rigid foundation, with infinitely rigid floor girders, a story height of 12 ft, a bay width of 20 ft, tributary floor area of 40 ft by 40 ft, and a lumped weight per floor of 160,000 lb. The fundamental mode of the building is illustrated in Fig. 16.

Because the analytical methods presented in Section C were developed for circular bases, it is assumed that the ten-story building has a circular massless base with the same area as that of the actual building. The centroidal moments of inertia of the floors are neglected compared to those about the base of the building. Because of this assumption, together with that of a massless base, the building-foundation system has only ten significant resonant frequencies.

The foundation medium is taken to have a unit weight of 120 lb/

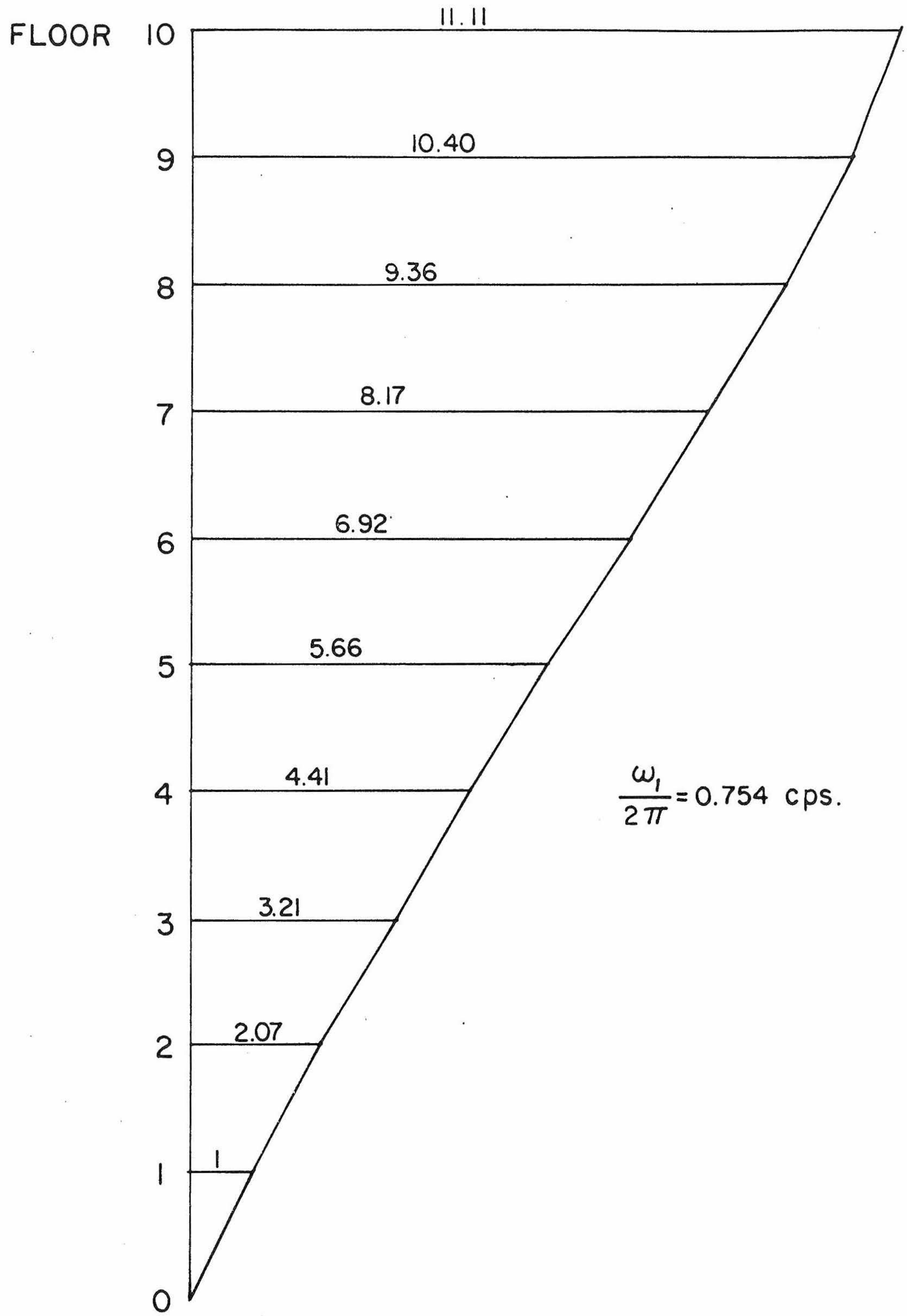


Fig. 16. Fundamental mode of ten-story building

ft^3 and a Poisson's ratio of 0.25. Several values of the shear wave velocity are considered in the numerical calculations, ranging from 300 ft/sec to the limiting rigid condition.

With the building and its foundation specified completely, the resonant frequencies and critical damping ratios of the interaction system can be obtained from the frequency equation

$$\Delta = 0 \tag{3.73}$$

where Δ is defined in Eq. (3.10e).

Presentation and discussion of results

The resonant frequencies of the system have been calculated from Eq. (3.73) for two different types of base motion: (1) rocking and horizontal translation, and (2) rocking only. The corresponding fundamental frequencies are presented in Table 5 for several values of the shear wave velocity of the elastic foundation. Also shown in Table 5 are approximate values of the fundamental frequency of the system calculated from Eq. (3.66b). To use this equation for multi-story buildings the parameters a_1, b_1 and α_1 have been defined by Eqs. (3.44a), (3.72a) and (3.72b), respectively. Excellent agreement is obtained between the exact and the approximate values of the fundamental resonant frequency of the ten-story building-foundation system.

A comparison of columns (2) and (3) of Table 5 shows that for soft soil an appreciable reduction in the fundamental frequency of the system takes place when rocking of the base is permitted. However, the additional change resulting from the horizontal translation of the base

TABLE 5

Fundamental Resonant Frequency (cps) of Ten-Story Building on the Elastic Half-Space

V_s (ft/sec) (1)	Rocking and Hor. Transl. (Exact) (2)	Rocking only (Exact) (3)	Rocking and Hor. Transl. (Approximate) (4)
∞	.754	.754	.754
5000	.753	.753	.753
2000	.750	.750	.750
1500	.747	.747	.747
1000	.738	.739	.738
700	.722	.724	.723
500	.695	.698	.696
400	.667	.671	.668
300	.615	.620	.618

is small compared to the initial reduction. This behavior may be explained by referring to Eq. (3.66b). Used as an approximation for the fundamental frequency of multi-story building-foundation systems, Eq. (3.66b) shows that $\tilde{\omega}_1/\omega_1$ depends fundamentally upon the compliance factor $a_1{}^2b_1$, Poisson's ratio σ, and the square of the slenderness ratio H_1/a. It may be seen, however, that H_1/a appears only in the term which represents rocking of the system. This implies that the effect of the horizontal translation of the base on the fundamental frequency will be significant only for systems for which $(H_1/a)^2$ is small. Thus, for sufficiently tall buildings this effect could be neglected, compared to that of rocking.

Table 5 shows that the fundamental resonant frequency of the ten-story building can be reduced significantly as the soil becomes soft. This trend, however, was not observed for the higher modes of the building. The second and higher resonant frequencies calculated from Eq. (3.73) for all values of the shear wave velocity of the soil, did not differ by more than one percent from the corresponding frequencies of the building on a rigid foundation. It will be recalled that the second resonant frequency of the two-story building studied in the preceding section (Table 2) also remained essentially unchanged, independently of the stiffness of the soil. Thus, it becomes of interest to determine whether more general systems show the same type of behavior.

Vibration tests have shown that many tall buildings have fundamental modes resembling straight lines[81]. It thus seems reasonable to use classical linear models whose fundamental mode shapes are given by straight lines in studying the dynamic coupling between tall buildings and their foundations. Only the rocking of the base will be considered since it has already been shown that the effect of the horizontal translation of the base on the fundamental frequency of the system is negligible for sufficiently tall buildings. Under these conditions it can be shown that the contributions of the second and higher modes to the overturning moment at the base of the building vanish identically[82]. Because only the fundamental mode has a non-vanishing base moment, and therefore a tendency to rotate, it is concluded that the second and higher natural frequencies will remain unchanged regardless of the stiffness of the soil. That the rotation of the base does not occur in the higher modes was previously found by Tajimi[62] from the corresponding frequency equation.

As noted above, the second and higher natural frequencies of a building whose fundamental mode is given by a straight line are not influenced by the properties of the ground upon which it is founded, provided its base is only allowed to rotate. It seems reasonable to expect that the higher natural frequencies should remain nearly constant even when the first mode of the building is not given by a straight line and both rocking and horizontal displacement of the base are allowed to take place. This is in accordance with the corresponding result obtained for the shear beams and bending beams often used to model tall buildings. It is well known that the effect of the type of constraint at the supports of single-spanned beams becomes less important for the higher modes of vibration, e.g., whereas the ratio of the fundamental frequency of a free-free bending beam to that of a cantilever beam is equal to 6.35, the corresponding ratio reduces to 1.50 for the fifth mode of vibration.

From the above discussion it is concluded that the effect of an elastic foundation, as measured by the change in the natural frequencies of a building as the underlying soil becomes softer, is negligible for modes higher than the first for many types of building structures. It is observed that only the fundamental frequency of a building decreases significantly as the soil becomes softer and that, except for short buildings, the reduction in the fundamental frequency is primarily due to rocking and to a lesser extent to the horizontal translation of the base of the structure.

Evaluation of the properties of a discrete foundation

It will be recalled that the elastic half-space which is used to represent the soil is sometimes approximated by a simpler discrete foundation consisting of linear constant springs and viscous dampers. The preceding paragraph suggests that the constant properties of these elements should be evaluated at $a_o = \tilde{\omega}_1 a/V_s$, where $\tilde{\omega}_1$ is the fundamental frequency of the interaction system. The earthquake response of the simplified system could then be obtained by using Foss's formulation, as described in Section C, or by other methods for solving systems of ordinary differential equations with constant coefficients.

III. SUMMARY AND CONCLUSIONS

The thesis investigation on the dynamics of soil-structure interaction was divided into two parts for convenience of analysis and presentation. In Chapter I, the forced horizontal, rocking and vertical harmonic oscillations of a rigid disc perfectly bonded to an elastic half-space were studied. The problem was formulated in terms of a system of dual integral equations which was transformed, with the aid of auxiliary functions, into a system of Cauchy type singular integral equations; the solution of the dominant part of which led to a system of Fredholm integral equations of the second kind in the auxiliary functions. Simplified forms of these equations were obtained for an incompressible material and for the relaxed mixed boundary-value problems corresponding to the complete mixed boundary-value problems examined. The stresses under the area of contact and the corresponding resultant forces were determined directly from the auxiliary functions. For the limiting static, complete mixed boundary-value problems, the Fredholm integral equations of the second kind reduced to simple expressions, thus yielding a solution in closed form which is in agreement with that obtained by other investigators. It was found, in addition, that the factor of the form $\begin{matrix}\cos\\ \sin\end{matrix}\left[k \ln \frac{1 - r'}{1 + r'}\right]$ which occurs in the static problem also enters into the solution of the corresponding dynamic problem.

The effect of a deformable foundation on the response of a building to earthquake excitation was studied in Chapter II. The base of the building was idealized as a rigid circular plate attached to the surface of the ground, and the soil was modeled by a homogeneous, isotropic, elastic half-space. Using the force-deflection relations for the base derived in Chapter I, the equations of motion of an n-story building-foundation system were solved by both direct and transform methods. It was shown that, under certain reasonable assumptions based on physical grounds, the earthquake response of the building-foundation system can be obtained as a linear combination of the responses to modified excitations of n + 2 one-degree-of-freedom, viscously damped, linear oscillators resting on a rigid ground. The undamped natural frequencies and fractions of critical damping of the equivalent rigid based oscillators are determined from the roots of the frequency equation of the building-foundation system. The modified excitations for the individual oscillators are obtained by linear superposition of the acceleration and the velocity traces of the original earthquake excitation. This result was shown to be valid even for systems that do not possess classical normal modes. The main advantages of this representation are the physical insight it gives into the dynamics of the building-foundation system and the simplicity of the calculations, which are reduced to those of a simple oscillator.

For the special case of a single-story building on a flexible foundation, approximate explicit formulas were obtained for the effective natural frequency, $\tilde{\omega}_1$, critical damping ratio, $\tilde{\eta}_1$, and the amplitude of the modified excitation in terms of the dimensionless parameters which govern the behavior of the system. It was found that whereas the effective natural frequency of the single-story building, as well as the amplitude of the equivalent input acceleration, always decrease as a result of the dynamic coupling between the building and the soil, the effective damping in the system can be increased or decreased by soil-structure interaction, depending on the parameters of the system. Whether the earthquake response of the building on a deformable soil will increase or decrease with respect to that of the same building supported on a rigid ground will depend upon the values of $\tilde{\omega}_1$ and $\tilde{\eta}_1$, and upon the detailed time history of the earthquake under consideration.

For multi-story buildings it was shown that the effect of an elastic foundation, as measured by the change in the natural frequencies of the building, is negligible for modes higher than the first for many types of building structures. It was noted that only the fundamental frequency of a building decreases significantly as the soil becomes softer and that, except for short buildings, the reduction in the fundamental frequency is primarily due to rocking and to a lesser extent to the horizontal translation of the base of the structure.

To confirm the applicability of the results obtained in this study, it is recommended as the next step to test the analytical solutions against the measured response of actual buildings to strong earthquakes and the forces of dynamic tests.

REFERENCES

1. Reissner, E., "Stationäre, axialsymmetrische durch eine schüttelnde Masse erregte Schwingungen eines homogenen elastischen Halbraumes," Ingenieur-Archiv, Vol. 7, Dec. 1936, pp. 381-396.

2. Quinlan, P. M., "The Elastic Theory of Soil Dynamics," Symposium on Dynamic Testing of Soils, ASTM, special tech.-publ. No. 156, 1953, pp. 3-34.

3. Sung, T. Y., "Vibrations on Semi-infinite Solids due to Periodic Surface Loading," Symposium on Dynamic Testing of Soils, ASTM, special tech.-publ. No. 156, 1953, pp. 35-68.

4. Arnold, R. N., Bycroft, G. N., and Warburton, G. B., "Forced Vibration of a Body on an Infinite Elastic Solid," Journal of Applied Mechanics, Vol. 22, No. 3, 1955, pp. 391-400.

5. Bycroft, G. N., "Forced Vibrations of a Rigid Circular Plate on a Semi-infinite Elastic Half-space and on Elastic Stratum," Philos. Trans. Roy. Soc. London, Ser. A, Vol. 248, 1956, pp. 327-368.

6. Thomson, W. T. and Kobori, T., "Dynamical Compliance of Rectangular Foundations on an Elastic Half-space," Journal of Applied Mechanics, Vol. 30, Dec. 1963, pp. 579-584.

7. Robertson, I. A., "Forced Vertical Vibration of a Rigid Circular Disc on a Semi-infinite Elastic Solid," Proc. Camb. Phil. Soc., Vol. 62, 1966, pp. 547-553.

8. Awojobi, A. O. and Grootenhuis, P., "Vibration of Rigid Bodies on Semi-infinite Elastic Media," Proc. Roy. Soc. London, Ser. A, Vol. 287, 1965, pp. 27-63.

9. Lysmer, J., "Vertical Motion of Rigid Footings," Ph.D. Thesis, University of Michigan, 1965.

10. Shah, P. M., "On the Dynamic Response of Foundation Systems," Ph.D. Thesis, Rice University, 1968.

11. Luco, J. E., and Westmann, R. A., "Dynamic Response of Circular Footings," to be published in the Journ. of the Eng. Mech. Div., ASCE.

12. Zakorko, V. N. and Rostovtsev, N. A., "Dynamic Contact Problem of Steady Vibrations of an Elastic Half-space," Journal of Applied Mathematics and Mechanics, Vol. 29, 1965, pp. 644-653.

13. Gladwell, G. M. L., "Forced Tangential and Rotatory Vibration of a Rigid Circular Disc on a Semi-infinite Solid," Int. J. Engng. Sci., Vol. 6, 1968, pp. 591-607.

14. Veletsos, A. S. and Wei, Y. T., "Lateral and Rocking Vibration of Footings," Structural Research at Rice, Report No. 8, Rice University, Jan. 1971.

15. Awojobi, A. O., "Harmonic Rocking of a Rigid Rectangular Body on a Semi-infinite Elastic Medium," Journal of Applied Mechanics, Vol. 33, 1966, pp. 547-552.

16. Karasudhi, P., Keer, L. M., and Keer, S. L., "Vibratory Motion of a Body on an Elastic Half Plane," Journal of Applied Mechanics, Vol. 35, E, No. 4, 1968, pp. 697-705.

17. Elorduy, J., Nieto, J. A., and Szekely, E. M., "Dynamic Response of Bases of Arbitrary Shape Subjected to Periodic Vertical Loading," Proc. Intl. Symp. on Wave Propagation and Dyn. Properties of Earth Materials, 1967, pp. 105-121.

18. Reissner, E. and Sagoci, H. F., "Forced Torsional Oscillations of an Elastic Half-space," Journal of Applied Physics, Vol. 15, No. 9, 1944, pp. 652-654.

19. Ufliand, Ia. S., "On Torsional Vibrations of Half-space," Journal of Applied Mathematics and Mechanics, Vol. 25, No. 1, 1961, pp. 228-233.

20. Collins, W. D., "The Forced Torsional Oscillations of an Elastic Half-space and an Elastic Stratum," Proc. London Math. Soc., Ser. 3, Vol. 12, No. 46, 1962, pp. 226-244.

21. Robertson, I. A., "On a Proposed Determination of the Shear Modulus of an Isotropic Elastic Half-space by the Forced Torsional Oscillations of a Circular Disc," Appl. Sci. Res., Vol. 17, 1967, pp. 305-312.

22. Thomas, D. P., "Torsional Oscillations of an Elastic Half-space," J. Mech. Appl. Math., Vol. 21, 1968, pp. 51-65.

23. Stallybrass, M. P., "A Variational Approach to a Class of Mixed Boundary-value Problems in the Forced Oscillations of an Elastic Medium," Proc. 4th U.S. Natl. Cong. Appl. Mech., 1962, pp. 391-400.

24. Luco, J. E., "Applications of Singular Integral Equations to the Problem of Forced Vibrations of a Rigid Footing," Ph.D. Thesis, University of California, Los Angeles, 1969.

25. Mossakovski, V. I., "The Fundamental General Problem of the Theory of Elasticity for a Half-space with a Circular Curve Determining Boundary Conditions," Prikl. Mat. Mekh., Vol. 18, 1954, pp. 187-196 (in Russian).

26. Ufliand, Ia. S., "Contact Problem in the Theory of Elasticity for a Circular Cylindrical Punch in the Presence of Adhesion," Prikl. Mat. Mekh., Vol. 20, 1956, pp. 578-587.

27. Keer, L. M., "Mixed Boundary-value Problems for an Elastic Half-space," Proc. Camb. Phil. Soc., Vol. 63, 1967, pp. 1379-1386.

28. Spence, D. A., "Self Similar Solutions to Adhesive Contact Problems with Incremental Loading," Proc. Roy. Soc. London, Ser. A., Vol. 305, 1968, pp. 55-80.

29. Gladwell, G. M. L., "A Contact Problem for a Circular Cylindrical Punch in Adhesive Contact with an Elastic Half-space: The Case of Rocking, and Translation Parallel to the Plane," Int. J. Engng. Sci., Vol. 7, 1969, pp. 295-307.

30. Noble, B., "Certain Dual Integral Equations," Journ. of Math and Physics, Vol. 37, 1958, pp. 128-136.

31. Copson, E. T., "On Certain Dual Integral Equations," Proc. Glasgow Math. Assoc., Vol. 5, 1961, pp. 21-24.

32. Sneddon, I. N., Mixed Boundary Value Problems in Potential Theory, North-Holland Publishing Co., Amsterdam, 1966.

33. Lowengrub, N. and Sneddon, I. N., "The Solution of a Pair of Dual Integral Equations," Proc. Edinb. Math. Soc., 1962, pp. 33-46.

34. Westmann, R. A., "Simultaneous Pairs of Dual Integral Equations," SIAM Review, Vol. 7, No. 3, 1965, pp. 341-348.

35. Fung, Y. C., Foundations of Solid Mechanics, Prentice-Hall, N.J., 1965.

36. Sezawa, K., "Further Studies on Rayleigh Waves Having Some Azimuthal Distribution," Bull. Earth. Res. Inst. Tokyo, Vol. 6, No. 2, 1929.

37. Titschmarsh, E. C., Introduction to the Theory of Fourier Integrals, 2nd. ed., Clarendon Press, Oxford, 1948.

38. Busbridge, I. W., "Dual Integral Equations," Proc. London Math. Soc., Vol. 44, Series 2, 1938, pp. 115-129.

39. Noble, B., "The Solution of Bessel Function Dual Integral Equations by a Multiplying-factor Method," Proc. Camb. Phil. Soc., Vol. 59, 1963, pp. 351-362.

40. Erdogan, F. and Bahar, L. Y., "On the Solution of Simultaneous Dual Integral Equations," J. Soc. Indust. Appl. Math., Vol. 12, 1964, pp. 666-675.

41. Erdogan, F., "Simultaneous Dual Integral Equations with Trigonometric and Bessel Kernds," ZAMM, Vol. 48, 1968, pp. 217-225.

42. Gradshteyn, I. S. and Ryzhik, I. M., Table of Integrals, Series, and Products, Academic Press, New York and London, 1965.

43. Muskhelishvili, N. I., Singular Integral Equations, P. Noordhoff, Groningen, The Netherlands, 1963.

44. Mikhlin, S. G., Integral Equations, Pergamon Press, New York, 1957.

45. Tricomi, F. G., Integral Equations, Interscience Publishers, Inc., New York, 1957.

46. Fox, L., and Goodwin, E. T., "The Numerical Solution of Non-singular Linear Integral Equations," Phil. Trans. Roy. Soc. London, Series A, Vol. 245, 1953, pp. 501-534.

47. Baker, C. T. H., Fox, L., Mayers, D. F., and Wright, K., "Numerical Solution of Fredholm Integral Equations of First Kind," Computer Journal, Vol. 7, 1964, pp. 141-148.

48. Green, A. E. and Zerna, W., Theoretical Elasticity, Oxford Univ. Press, London, 1954.

49. Collins, W. D., "Some Axially Symmetric Stress Distributions in Elastic Solids Containing Penny-shaped Cracks. I. Cracks in an Infinite Solid and a Thick Plate," Proc. Roy. Soc. London, Series A, Vol. 266, 1962, pp. 359-386.

50. Kano, K., "Relation Between the Earthquake Damage of Non-wooden Buildings and the Nature of the Ground," Bull. Earth. Res. Inst. Tokyo, Vol. 27, 1949, pp. 97-100.

51. Suyehiro, K., "Engineering Seismology, Notes on American Lectures," Proceedings ASCE, Vol. 58, No. 4, Part 2, May 1932, 110 pp.

52. Tanabashi, R. and Ishizaki, H., "Earthquake Damages and Elastic Properties of the Ground," Bulletin No. 4, Disaster Prevention Research Institute, Kyoto University, Japan, 1953.

53. Ishizaki, H. and Hatakeyama, N., "Experimental and Numerical Studies on Vibrations of Buildings," Proc. 2nd World Conf. on Earthquake Engng., Japan, 1960, Vol. 2, pp. 1263-1284.

54. Merritt, R. G. and Housner, G. W., "Effect of Foundation Compliance on Earthquake Stresses in Multi-story Buildings," Bull. Seism. Soc. Amer., Vol. 44, No. 4, 1954, pp. 551-569.

55. Housner, G. W., "Interaction of Building and Ground During an Earthquake," Bull. Seism. Soc. Amer., Vol. 47, No. 3, 1957, pp. 179-186.

56. Sato, Y., and Yamaguchi, R., "Vibration of a Building Upon an Elastic Foundation," Bull. Earth. Res. Inst. Tokyo, V. 38, 1960, pp. 369-383.

57. Thomson, W., "A Survey of Ground-building Vibrations," Proc. 2nd. World Conf. on Earthquake Engng., Japan, 1960, Vol. 2, pp. 833-846.

58. Sandi, H., "A Theoretical Investigation of the Interaction Between Ground and Structure During Earthquakes," Proc. 2nd World Conf. on Earthquake Engng., Japan, 1960, Vol. 2, pp. 1327-1343.

59. Fleming, J. F., Screwvala, F. N., and Kondner, R. L., "Foundation Superstructure Interaction under Earthquake Motion," Proc. 3rd. World Conf. on Earthquake Engng., New Zealand, Vol. 1, 1965, pp. 122-130.

60. Rosenberg, L. A., "On the Interaction Between Ground and Structure During Earthquakes," Bull. of the International Institute of Seismology and Earthquake Engng., Vol. 2, 1965, pp. 29-36.

61. Parmelee, R. A., "Building-Foundation Interaction Effects," Jour. of the Eng. Mech. Div., ASCE, Vol. 93, No. EM2, 1967, pp. 131-152.

62. Tajimi, H., "Discussion: Building-Foundation Interaction Effects, Proc. Paper 5200," Jour. of the Eng. Mech. Div., ASCE, V. 93, No. EM6, 1967, pp. 294-298.

63. Scavuzzo, R. J., "Foundation-Structure Interaction in the Analysis of Wave Motions," Bull. Seism. Soc. of Amer., Vol. 57, No. 4, 1967, pp. 735-746.

64. Parmelee, R. A., Perelman, D. S., Lee, S. L., and Keer, L. M., "Seismic Response of Structure-Foundation Systems," Jour. of the Eng. Mech. Div., ASCE, V. 94, No. EM6, 1968, pp. 1295-1315.

65. Castellani, A., "Foundation Compliance Effects on Earthquake Response Spectra," Jour. of the Soil Mech. and Found. Div., ASCE, V. 96, No. SM4, 1970, pp. 1335-1355.

66. Parmelee, R. A., Perelman, D. S., and Lee, S. L., "Seismic Response of Multiple-story Structure on Flexible Foundations," Bull. Seism. Soc. of Amer., Vol. 59, No. 3, 1969, pp. 1061-1070.

67. Hradilek, P. J., "Dynamic Soil-Structure Interaction," M.S. Thesis, University of California, Los Angeles, 1970.

68. Sravuzzo, R. J., Raftopoulos, D. D., and Bailey, J. L., "Lateral Structure-Foundation Interaction of Nuclear Power Plants with Large Base Masses," Technical Report No. 3, The Research Foundation, The University of Toledo, Sept. 1969.

69. Chiapetta, R., "Effect of Soil-Structure Interaction on the Response of Reactor Structures to Seismic Ground Motion," IIT Research Institute Report for The Research Foundation, The University of Toledo, April 1970.

70. Kobori, T. and Minai, R., "Study on Unstationary Vibration of Building Structure with Plastic Deformation of Substructure," Proc. 2nd. World Conf. on Earthquake Engng., Japan, 1960, Vol. 2, pp. 1085-1104.

71. Isenberg, J., "Interaction Between Soil and Nuclear Reactor Foundations During Earthquakes," Report prepared for The Research Foundation of The University of Toledo, June 1970.

72. Hsieh, T. K., "Foundation Vibrations," Proceedings, Institution of Civil Engineers, Vol. 22, 1962, pp. 211-225.

73. O'Kelly, M. E. J., "Vibration of Viscously Damped Linear Dynamic Systems," Ph.D. Thesis, California Institute of Technology, 1964.

74. Duke, C. M., Luco, J. E., Carriveau, A. R., Hradilek, P. J., Lastrico, R., and Ostrom, D., "Strong Earthquake Motions and Site Conditions: Hollywood," Bull. Seism. Soc. Amer., Vol. 60, No. 4, 1970, pp. 1271-1289.

75. Carrier, G. F., Krook, M., and Pearson, C. E., Functions of a Complex Variable: Theory and Technique, McGraw-Hill, New York, 1966.

76. Bergland, G. D., "A Guided Tour of the Fast Fourier Transform," IEEE Spectrum, Vol. 6, July 1969, pp. 41-52.

77. Foss, K. A., "Co-Ordinates which Uncouple the Equations of Motion of Damped Linear Dynamic Systems," Journal of Applied Mechanics, Vol. 25, 1958, pp. 361-364.

78. Lin, Y. K., Probabilistic Theory of Structural Dynamics, McGraw-Hill, New York, 1967.

79. Strong-Motion Instrumental Data on the Borrego Mountain Earthquake of 9 April 1968, Report prepared by the Seismological Field Survey, Coast and Geodetic Survey and by the Earthquake Engineering Research Laboratory, California Institute of Technology, August 1968.

80. Housner, G. W. and Brady, A. G., "Natural Periods of Vibration of Buildings," Journ. of the Eng. Mech. Div., ASCE, Vol. 89, No. EM4, 1963, pp. 31-65.

81. Jennings, P. C., "Spectrum Techniques for Tall Buildings," Proc. 4th World Conf. on Earthquake Engng., Santiago, Chile, 1969, Vol. 2, pp. A3.61-A3.74.

82. Bielak, J., "Base Moment for a Class of Linear Systems," Journ. of the Eng. Mech. Div., ASCE, Vol. 95, No. EM5, 1969, pp. 1053-1062.

83. Liu, S. C. and Fagel, L. W., "Earthquake Interaction by the Fast Fourier Transform," to be published in the Journ. of the Eng. Mech. Div., ASCE.