A STUDY OF
SECOND-ORDER SUPERSONIC FLOW

Thesis by
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Summary

An attempt is made to develop a second approximation to the solution of problems of supersonic flow which can be solved by existing first-order theory. The method of attack adopted is an iteration procedure using the linearized solution as the first step.

Several simple problems are studied first in order to understand the limitations of the method. These suggest certain conjectures regarding convergence. A second-order solution is found for the cone which represents a considerable improvement over the linearized result.

For plane and axially-symmetric flows it is discovered that a particular integral of the iteration equation can be written down at once in terms of the first-order solution. This reduces the second-order problem to the form of the first-order problem, so that it is effectively solved. Comparison with solutions by the method of characteristics indicates that the method is useful for bodies of revolution which have continuous slope.

For full three-dimensional flow, only a partial particular integral has been found. As an example of a more general problem, the solution is derived for a cone at an angle. The possibility of treating other bodies of revolution at angle of attack and three-dimensional wings is discussed briefly.

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Errata

Page	Line	Correction
18	--	Add numbers to legend: 1 .... 2 - - - - - 3 — — — 4 ———
29	Eq. (2.17)	Replace $\frac{M^2(N+1)-3}{6\beta^2}\epsilon^2(x-\beta y)^2$ by $\ldots(x-\beta y)^3$
35	Eq. (2.22)	Replace $(N-1)$ by $(N+1)$
55	8	Replace $\frac{1}{4}\phi_r^2$ by $\frac{1}{4}\frac{1}{r}\phi_r^2$
55	12, 13	Replace $+\frac{3}{2}r\phi_{xr}^2$ by $-\frac{3}{2}r\phi_{xr}^2$
71	10-16	Replace this paragraph by: "Behind a discontinuity in curvature or any higher derivative, however, it can be shown that the solution gives the correct two-dimensional behavior."
71	21	Replace "a derivative of any order" by "slope."
72	8*	Delete sentence beginning "Evidently."
76	5*-4*	Replace "for the last .. involve N," by "for the terms in the first line, and with N = 0,"
80	Both tables	Add above each table: $z = \zeta = Z$
83	Eq. (4.17)	Replace $(5N+3)\frac{1}{T}$ by $(5N+3)\frac{1-T^2}{T}$
89	--	Replace "Eq. 4.25a" by "Eq. 4.25b"

* Count from bottom

Addendum

Treatment of Bodies of Revolution with Corners

As discussed in section 23, the method as it stands fails behind a discontinuity in slope. However, this defect can be readily corrected. The proper procedure would be to determine the solution for the case when the corner has been slightly rounded, and then pass to the limit of a sharp corner. It can be shown that this is completely equivalent to the following simpler procedure.

The particular solution given by Eq. (3.15) is discontinuous across the Mach wave from the corner, since ϕ_x and ϕ_r are discontinuous. The complete potential must be continuous, so that the correction potential χ must include an additional term which cancels the jump in ψ. Such a term is given by the solution discussed in section 23, with $k = \frac{1}{2}$:

$$\chi(x,r) = \begin{cases} 0 & x-\beta r < 0 \\ C\,F\left(\frac{1}{2},\frac{1}{2};1;-\frac{x-\beta r}{2\beta r}\right) & x-\beta r > 0 \end{cases}$$

The constant C is to be chosen so as to cancel the discontinuity. Using the analytic continuation of the hypergeometric function, this potential and its derivatives can be expressed in terms of complete elliptic integrals:

$$\chi = \frac{2C}{\pi}\sqrt{\frac{a}{r}}\,\sqrt{\frac{2t}{1+t}}\,K\left(\frac{1-t}{1+t}\right)$$

$$\chi_x = -\frac{C}{\pi}\,\frac{1}{x}\sqrt{\frac{a}{r}}\,\frac{1}{1-t}\sqrt{\frac{2t}{1-t}}\left[K\left(\frac{1-t}{1+t}\right) - E\left(\frac{1-t}{1+t}\right)\right]$$

$$\chi_r = -\frac{\beta C}{\pi}\,\frac{1}{x}\sqrt{\frac{a}{r}}\,\frac{1}{1-t}\sqrt{\frac{2t}{1-t}}\left[\frac{1}{t}E\left(\frac{1-t}{1+t}\right) - K\left(\frac{1-t}{1+t}\right)\right]$$

Here t is the conical variable (Eq. 1.14), and $K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of the first and second kinds. The origin of coordinates is located with respect to the corner as shown in Fig. 3.7.

With this modification, the procedure described in section 22 yields a solution valid behind the corner. For example, when revised in this way, the second-order solution shown in Fig. 3.9 coincides with the solution obtained by the method of characteristics. Consequently, the last paragraph on page 74 can now be ignored.

Introduction

As the linearized theory of supersonic flow approaches full development, the question arises as to whether more exact approximations are practical. If viscous effects are large, refinement of the perfect fluid solution is useless. Otherwise, however, higher approximations are known to yield a closer approach to reality. In intermediate cases, an improved solution is desirable in order to assess the relative effects of viscosity and non-linearity.

The prototype of a higher-order solution for supersonic flow is Busemann's series for the surface pressure in plane flow. This simple result is of considerable value in analyzing supersonic airfoil sections. Two terms of the series prove sufficient for almost all requirements; the extension to third and fourth order is chiefly of academic interest.

The aim of the present study is, therefore, to find a second approximation, analogous to Busemann's result, for supersonic flow past bodies which can be treated by existing first-order theory. The natural method of attack, and apparently the only practical one, is by means of an iteration process, taking the usual linearized result as the first step. Several writers have applied this procedure to plane subsonic flow. In supersonic flow, as usual, the solution is simpler, so that other problems can be solved.

I. The Iteration Procedure

1. Basic Assumptions

The problem to be considered is that of steady three-dimensional supersonic flow past one or more slender bodies. As indicated in Fig. 1.1, the bodies are assumed either to be pointed or to extend upstream indefinitely as cylinders parallel to the free-stream direction. Wind axes are intro-

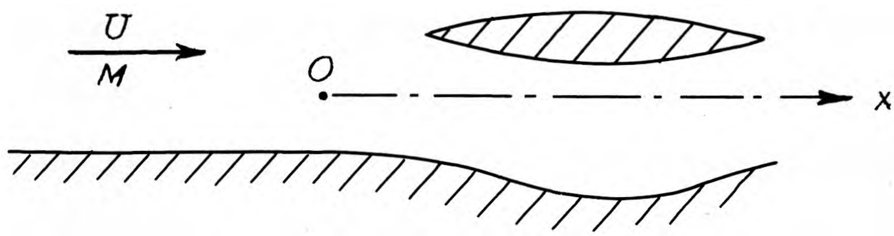

Fig. 1.1. The Problem

duced, so that far upstream the flow is uniform and parallel to the x-axis, with velocity U and Mach number M. For convenience, the origin is chosen so that variations in body shape occur only in the right half plane.

The bodies are slender, which means that at any point the component of U normal to the surface is small compared with U itself. The symbol ϵ will be used throughout as a measure of this smallness. Thus the ordinates of a body will be written as ϵ times a function of order unity. Used in this way, ϵ serves to distinguish terms of various orders of magnitude.

It will be assumed that the full linearized solution to the problem is available. Then the aim of this investigation is to provide a second approximation to the exact non-linear solution. The linearized solution is defined as the result obtained by keeping only linear perturbation quantities in the equation of motion. Similarly, the second-order solution is the result of retaining squares and products of perturbation quantities. In addition, however, certain of the triple products are in some cases found to be as important as one or more double products, and are therefore also retained. It may be emphasized that the second-order solution will not generally consist simply of terms of order ϵ and ϵ^2, although this is the case for plane flow. For example, the second-order solution for flow past a body of revolution contains terms as high as ϵ^4.

A velocity potential will be assumed to exist. This assumption is always valid for the first- and second-order solutions, since the rotation is found to be at most of the order of terms neglected. In some cases, such as the plane corner and the cone, a velocity potential exists to any degree of approximation.

2. The Exact Perturbation Equation

If a velocity potential Ω exists, the equation of motion in cartesian coordinates is (Ref. 1, eq. 39)

$$\left(1-\frac{\Omega_x^2}{c^2}\right)\Omega_{xx} + \left(1-\frac{\Omega_y^2}{c^2}\right)\Omega_{yy} + \left(1-\frac{\Omega_z^2}{c^2}\right)\Omega_{zz}$$
$$-2\Omega_{yz}\frac{\Omega_y\Omega_z}{c^2} - 2\Omega_{zx}\frac{\Omega_z\Omega_x}{c^2} - 2\Omega_{xy}\frac{\Omega_x\Omega_y}{c^2} = 0 \qquad (1.1)$$

The local speed of sound c is related to c_o, its value in the uniform stream, by

$$c^2 = c_o^2 - \frac{\gamma-1}{2}\left(\Omega_x^2+\Omega_y^2+\Omega_z^2-U^2\right) \qquad (1.2)$$

A perturbation potential Φ is now introduced in the usual way. For convenience, however, Φ is normalized through division by the free-stream velocity U. Hence the perturbation velocity is the gradient of Φ multiplied by U. Then

$$\Omega = U(x+\Phi) \qquad (1.3)$$

Multiplying the equation of motion by $\frac{c^2}{c_o^2}$ and introducing the perturbation potential gives, after some manipulation

$$\Phi_{yy} + \Phi_{zz} - (M^2-1)\Phi_{xx} = M^2\left[\begin{array}{l} \frac{\gamma-1}{2}(2\Phi_x+\Phi_x^2+\Phi_y^2+\Phi_z^2)(\Phi_{xx}+\Phi_{yy}+\Phi_{zz}) \\ +2\Phi_{xx}\Phi_x + \Phi_{xx}\Phi_x^2 + \Phi_{yy}\Phi_y^2 + \Phi_{zz}\Phi_z^2 \\ +2\Phi_{xy}\Phi_y(1+\Phi_x) + 2\Phi_{yz}\Phi_y\Phi_z + 2\Phi_{xz}\Phi_z(1+\Phi_x) \end{array}\right. \qquad (1.4)$$

where $M = \frac{U}{c_o}$. This is the three-dimensional equivalent of equation (168) of Ref. 1. Henceforth the notation

$$\beta = \sqrt{M^2-1}$$

will be used.

3. Solution by Iteration

The perturbation equation (1.4) is completely equivalent to the original potential equation (1.1). Simplifying assumptions must therefore be introduced in order to solve it. If it is assumed that squares and products of the derivatives of Φ can be neglected, the right-hand side of (1.4) disappears, leaving the wave equation

$$\square\Phi \equiv \Phi_{yy} + \Phi_{zz} - \beta^2\Phi_{xx} = 0 \qquad (1.5)$$

which is the basis of the linearized theory. The linearized solution will henceforth be designated by $\Phi^{(1)}$ (and later, for convenience, by ϕ).

More exact solution of (1.4) by means of iteration was first suggested by Prandtl (Ref. 2). The method has been applied to plane subsonic flow by Hantzsche and Wendt (Ref. 3), Imai and Oyama (Ref. 4), and Kaplan (Ref. 5). The procedure for this case is described by Sauer (Ref. 1, p. 140), and for three-dimensional supersonic flow remains essentially the same.

The linearized solution $\Phi^{(1)}$, subject to proper boundary conditions, is taken as the first approximation. This solution is then substituted into the right-hand side of (1.4), which becomes

$$\Box\Phi \equiv \Phi_{yy} + \Phi_{zz} - \beta^2\Phi_{xx} = F(x, y, z) \tag{1.6}$$

where F is a known function of the independent variables. This is again a linear equation, the non-homogeneous wave equation. A solution $\Phi^{(2)}$, subject to proper boundary conditions, can be sought by standard methods. The procedure can be repeated by substituting $\Phi^{(2)}$ into the right-hand side and solving again. Continuing this process yields a sequence of solutions $\Phi^{(n)}$ which under proper conditions presumably converges to the exact solution.

This procedure bears a superficial resemblance to the Picard process for hyperbolic equations in two independent variables (Ref. 6, vol. II, p. 317) with, however, an essential difference. In the Picard process, the characteristic lines of the differential equation are known at the outset, since F does not depend upon the highest-order derivatives. Here, on the other hand, the characteristic surfaces (the Mach cones in physical terms) are initially unknown. Because of the fundamental role played by the characteristics in the theory of hyperbolic equations (see, for example, Ref. 7, chap. II) it might be anticipated that the characteristics should be revised at each step of the iteration. Each step but the first would then involve equations with non-constant coefficients. The subsonic counterpart of such a procedure is known to converge under reasonable conditions (Ref. 6, vol. II, pp. 288-89).

However, the procedure outlined previously makes no provision for such revision. At each stage of the iteration, the equation (1.6) has the original characteristics of the undisturbed flow. As a result, the equation has constant coefficients, which greatly facilitates solution. Fortunately, it will be found that the procedure nevertheless converges under conditions of continuity which are satisfied in most cases of practical importance.

4. The Second-Order Iteration Equation

Henceforth, only the first two steps of the iteration process will be considered in detail. It is then convenient to regard the second approximation as consisting of the first approximation plus a smaller additional term. Hence we write

$$\Phi^{(2)} = \phi + \varphi \qquad \text{where} \qquad \begin{aligned} \phi &\equiv \Phi^{(1)} \\ \varphi &\equiv \Phi^{(2)} - \Phi^{(1)} \end{aligned} \tag{1.7}$$

Now $\phi \equiv \Phi^{(1)}$ is a solution of the homogeneous wave equation (1.5), so that substituting into the perturbation equation (1.4) shows that φ , as well as $\Phi^{(2)}$, is a solution of the iteration equation

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2 \left[\begin{array}{l} \frac{\gamma-1}{2}(2\phi_x+\phi_x^2+\phi_y^2+\phi_z^2)(\phi_{xx}+\phi_{yy}+\phi_{zz}) \\ +2\phi_{xx}\phi_x + \phi_{xx}\phi_x^2 + \phi_{yy}\phi_y^2 + \phi_{zz}\phi_z^2 \\ +2\phi_{xy}\phi_y(1+\phi_x) + 2\phi_{yz}\phi_y\phi_z + 2\phi_{xz}\phi_z(1+\phi_x) \end{array} \right] \quad (1.8)$$

Since ϕ satisfies equation (1.5), the term $(\phi_{xx}+\phi_{yy}+\phi_{zz})$ in the right-hand side of (1.8) can be replaced by $M^2\phi_{xx}$, and the equation for φ becomes

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2 \left[\begin{array}{l} \frac{\gamma-1}{2}M^2\phi_{xx}(2\phi_x+\phi_x^2+\phi_y^2+\phi_z^2) + 2\phi_{xx}\phi_x \\ + \phi_{xx}\phi_x^2 + \phi_{yy}\phi_y^2 + \phi_{zz}\phi_z^2 \\ +2\phi_{xy}\phi_y(1+\phi_x) + 2\phi_{yz}\phi_y\phi_z + 2\phi_{xz}\phi_z(1+\phi_x) \end{array} \right] \quad (1.9)$$

Here the right-hand side contains not only squares and products of perturbation quantities, but also cubes and triple products. The latter can be omitted for plane flow or flow past planar systems, since they contribute terms of smaller order (equal to those found in the next iteration). Otherwise, certain of the triple products should be retained, since their contribution is as great as that of one or more of the double products, and greater than any found from a third approximation. It will be seen later that those triple products should be retained which involve only derivatives normal to the free stream. Those which involve x-derivatives can always be neglected, so that the equation becomes

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2 \left[\begin{aligned} &\{2+(\gamma-1)M^2\}\phi_{xx}\phi_x + 2\phi_{xy}\phi_y + 2\phi_{xz}\phi_z \\ &+\phi_{yy}\phi_y^2 + 2\phi_{yz}\phi_y\phi_z + \phi_{zz}\phi_z^2 \end{aligned} \right. \tag{1.10}$$

Here the triple products which may be important are grouped in the second line.

The adiabatic exponent γ will be found to appear always in the form $(\gamma+1)$ and, in fact, in the form $\frac{(\gamma+1)M^2}{2\beta^2}$. It is therefore convenient to introduce a single symbol for this combination:

$$N = \frac{(\gamma+1)\ M^2}{2\ \beta^2} \tag{1.11}$$

which will be used henceforth in place of γ. Making this substitution, the iteration equation becomes finally

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2 \left[\begin{aligned} &2(N-1)\beta^2\phi_{xx}\phi_x + 2\phi_{xy}\phi_y + 2\phi_{xz}\phi_z \\ &+\phi_{yy}\phi_y^2 + 2\phi_{yz}\phi_y\phi_z + \phi_{zz}\phi_z^2 \end{aligned} \right. \tag{1.12}$$

5. Iteration Equation in Other Coordinates

In cylindrical coordinates equation (1.12) becomes

$$\varphi_{rr} + \frac{\varphi_r}{r} + \frac{\varphi_{\theta\theta}}{r^2} - \beta^2\varphi_{xx} = M^2 \left[\begin{aligned} &2(N-1)\beta^2\phi_{xx}\phi_x + 2\phi_{xr}\phi_r + 2\phi_{x\theta}\frac{\phi_\theta}{r^2} \\ &+\phi_{rr}\phi_r^2 + 2\phi_{r\theta}\frac{\phi_r\phi_\theta}{r^2} - \frac{\phi_r\phi_\theta^2}{r^3} \\ &\qquad\qquad + \phi_{\theta\theta}\frac{\phi_\theta^2}{r^4} \\ &+O\left(\phi_{xx}\phi_x^2, \phi_{xx}\phi_r^2, \phi_{xx}\frac{\phi_\theta^2}{r^2}, \phi_{x\theta}\frac{\phi_x\phi_\theta}{r^2}, \phi_{xr}\phi_x\phi_r\right) \end{aligned} \right. \tag{1.13}$$

The terms whose form is indicated in the last line are those triple products which will be found to be negligible.

For conical flows it is convenient to introduce non-orthogonal coordinates (x, t, θ) where

$$t = \frac{\beta r}{x} \tag{1.14}$$

If the body itself is conical, the perturbation potential is reduced to a function of two variables (Ref. 8) by introducing the conical perturbation potential $\bar{\Phi}$

$$\Phi(x, t, \theta) = x\,\bar{\Phi}(t, \theta) \tag{1.15}$$

with corresponding definitions for $\bar{\phi}$ and $\bar{\varphi}$. The derivatives are given by

$$\begin{array}{lll} \Phi_x = \bar{\Phi} - t\bar{\Phi}_t & \Phi_{xx} = \frac{t^2}{x}\bar{\Phi}_{tt} & \Phi_{xr} = -\frac{\beta t}{x}\bar{\Phi}_{tt} \\ \Phi_r = \beta\bar{\Phi}_t & \Phi_{rr} = \frac{\beta^2}{x}\bar{\Phi}_{tt} & \Phi_{x\theta} = \bar{\Phi}_\theta - t\bar{\Phi}_{t\theta} \\ \Phi_\theta = x\bar{\Phi}_\theta & \Phi_{\theta\theta} = x\bar{\Phi}_{\theta\theta} & \Phi_{r\theta} = \beta\bar{\Phi}_{t\theta} \end{array} \tag{1.16}$$

with the same relations connecting ϕ and $\bar{\phi}$, φ and $\bar{\varphi}$. The iteration equation (1.12) becomes

$$(1-t^2)\bar{\varphi}_{tt} + \frac{\bar{\varphi}_t}{t} + \frac{\bar{\varphi}_{\theta\theta}}{t^2} = M^2 \left[\begin{aligned} & 2(N-1)t^2\bar{\Phi}_{tt}(\bar{\Phi}-t\bar{\Phi}_t) - 2t\bar{\Phi}_{tt}\bar{\Phi}_t + \frac{2}{t^2}\bar{\Phi}_\theta(\bar{\Phi}_\theta - t\bar{\Phi}_{t\theta}) \\ & +\beta^2\bar{\Phi}_{tt}\bar{\Phi}_t^2 + 2\frac{\beta^2}{t^2}\bar{\Phi}_{t\theta}\bar{\Phi}_t\bar{\Phi}_\theta - \frac{\beta^2}{t^3}\bar{\Phi}_t\bar{\Phi}_\theta^2 \\ & \qquad + \frac{\beta^4}{t^4}\bar{\Phi}_{\theta\theta}\bar{\Phi}_\theta^2 \\ & + O\Big\{ t^2\bar{\Phi}_{tt}(\bar{\Phi}-t\bar{\Phi}_t)^2,\ t^2\bar{\Phi}_{tt}\bar{\Phi}_t^2,\ \bar{\Phi}_{tt}\bar{\Phi}_\theta^2 \\ & \qquad \frac{1}{t^2}(\bar{\Phi}_\theta - t\bar{\Phi}_{t\theta})(\bar{\Phi}-t\bar{\Phi}_t),\ t\bar{\Phi}_{tt}\bar{\Phi}_t(\bar{\Phi}-t\bar{\Phi}_t) \Big\} \end{aligned} \right. \tag{1.17}$$

Here the grouping of terms corresponds to that in (1.13).

6. Alternate Solution by Power Expansion

Another method of solving equation (1.4) by successive approximations is to assume that the exact solution can be expanded in powers of some small parameter λ. Here λ is related to the slenderness parameter ϵ, but may be taken equal to it only for plane flow. (This case is discussed in Ref. 9, p. 158, and is the procedure actually followed in Refs. 3, 4, 5a, and 5b.) Thus the perturbation potential is written as

$$\Phi = \lambda \overset{(1)}{\Phi} + \lambda^2 \overset{(2)}{\Phi} + \lambda^3 \overset{(3)}{\Phi} + \cdots$$

Substituting into equation (1.4) and equating like powers of λ yields a sequence of equations

$$\overset{(1)}{\Phi}_{yy} + \overset{(1)}{\Phi}_{zz} - \beta^2\overset{(1)}{\Phi}_{xx} = 0$$

$$\overset{(2)}{\Phi}_{yy} + \overset{(2)}{\Phi}_{zz} - \beta^2\overset{(2)}{\Phi}_{xx} = 2M^2\left[(N-1)\beta^2\overset{(1)}{\Phi}_{xx}\overset{(1)}{\Phi}_x + \overset{(1)}{\Phi}_{xy}\overset{(1)}{\Phi}_y + \overset{(1)}{\Phi}_{xz}\overset{(1)}{\Phi}_z\right]$$

. . . .

which can be solved successively. The first equation is identical with (1.5), so that the two methods are equivalent in the first approximation. However, the second equation is not identical with (1.12), since no triple products appear in the right-hand side. As mentioned before, the triple products will in some cases be found to yield terms of the same magnitude as those due to the double products, and therefore increase the accuracy of the second approximation. Consequently, although both methods may converge under proper conditions, the iteration procedure gives as good, and in some cases a better, second approximation.

7. Boundary Conditions

Physical considerations indicate that the flow should satisfy the following conditions:

1. The resultant velocity is everywhere tangent to the surface of the body.
2. All perturbations vanish identically everywhere upstream of the plane $x = 0$.

The theory of hyperbolic differential equations shows that these conditions are just sufficient to determine the solution. For supersonic flow, the exact equation (1.4) and the various wave equations by which is approximated (1.5 and 1.6) are of hyperbolic type, with the streamwise coordinate x assuming the role of a time-like variable (Ref. 7, p. 84). The two physical conditions listed above correspond mathematically to the case of mixed boundary conditions (Ref. 6, vol. II, p. 172). The requirement of tangent flow imposes one condition along time-like surfaces (the surfaces of the bodies). The vanishing of perturbations upstream imposes two more conditions along a space-like surface -- that Φ and Φ_x vanish on the plane $x = 0$. These three conditions -- two prescribed on a space-like surface and one on a time-like surface -- lead to a determinate solution for a second-order hyperbolic equation (see Ref. 7, p. 85).

The tangency condition may be written

$$\frac{\Phi_c}{1+\Phi_x} = \begin{bmatrix}\text{Slope with} \\ \text{respect to} \\ \text{free stream}\end{bmatrix} \qquad \text{On the surface of the body}$$

where Φ_c is the cross-wind component of the normal velocity at the surface of the body, given in vector notation by $\overline{U}\cdot\Phi_n$ if Φ_n is the normal derivative of Φ. In plane flow $\Phi_c = \Phi_y$ and in axially-symmetric flow $\Phi_c = \Phi_r$.

At the nth step of the iteration, this condition becomes

$$\frac{\Phi_c^{(n)}}{1+\Phi_x^{(n)}} = [\text{Slope}]$$

To the accuracy of the nth approximation, this can be replaced by

$$\frac{\Phi_c}{1+\Phi_x^{(n-1)}} = [\text{Slope}]$$

Hence for the first and second approximations, the tangency conditions are

$$\left.\begin{aligned} \phi_c &= [\text{Slope}] \\ \varphi_c &= (1+\phi_x)[\text{Slope}] - \phi_c \end{aligned}\right\} \text{On the Surface} \qquad \begin{aligned} &(1.18a) \\ &(1.18b) \end{aligned}$$

Note that the first of these cannot be used to eliminate ϕ_c from the second, since it may hold only to first order, not to second order.

A planar system is defined to be a system for which the first-order tangency condition can be applied at a plane parallel to the free stream, rather than on the surface of the body (Ref. 10, p. 52). Thin flat wings are planar systems, while slender pointed bodies of revolution are not. For planar systems, the second-order tangency condition can also be satisfied at the plane, provided that the value of ϕ_c is calculated at the surface of the body (ϕ_x may be calculated at either place). That is, for planar systems the tangency conditions are

$$[\phi_c]_{plane} = [\text{Slope}]_{surface} \qquad (1.19a)$$

$$[\varphi_c]_{plane} = [1+\phi_x]_{\substack{\text{plane or}\\ \text{surface}}} [\text{Slope}]_{surface} - [\phi_c]_{surface} \qquad (1.19b)$$

Corresponding results hold for quasi-cylindrical bodies, which are bodies of revolution whose radius varies so slightly that the tangency conditions can be satisfied on a circular cylinder parallel to the free stream.

The remaining boundary conditions are

$$\Phi(0,y,z) = \Phi_x(0,y,z) = 0$$

These conditions are satisfied by the first-order solution alone, and must therefore be satisfied also by the additional second-order potential alone. Thus

$$\phi(0,y,z) = \phi_x(0,y,z) = 0 \tag{1.20a}$$

$$\varphi(0,y,z) = \varphi_x(0,y,z) = 0 \tag{1.20b}$$

It should be understood that the boundary conditions (1.18) or (1.19) and (1.20) need be satisfied only to the order of terms which are being retained in any given approximation. In practice, however, equation (1.20) will usually be satisfied exactly in each step of the iteration.

8. Determination of Pressure

When the potential field is known, and hence the velocity q at every point, the pressure coefficient can be calculated from the Bernoulli equation

$$C_p \equiv \frac{p-p_0}{\frac{1}{2}\rho_0 U^2} = \frac{2}{\gamma M^2}\left\{\left[1+\frac{\gamma-1}{2}M^2\left(1-\frac{q^2}{U^2}\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\} \tag{1.21}$$

This assumes isentropic flow, which is valid at least for the second approximation, since changes in entropy will be found to introduce only terms of orders neglected therein.

It is the practice in linearized theory to linearize also the pressure equation for the sake of consistency. If (u, v, w) are the perturbation velocity components in any orthogonal coordinate system, u being directed along the streamwise x-axis, then

$$q^2 = (U+u)^2 + v^2 + w^2$$

Substituting into (1.21) and expanding in ascending powers of the perturbation velocities gives

$$\begin{aligned} C_p = & -2\frac{u}{U} - \frac{v^2+w^2}{U^2} + \beta^2\frac{u^2}{U^2} + M^2\left[\left(1-\frac{2-\gamma}{3}M^2\right)\frac{u^3}{U^3} + \frac{u}{U}\frac{v^2+w^2}{U^2}\right] \\ & + \frac{M^2}{4}\left[\left\{1+2(2-\gamma)M^2+\frac{(2-\gamma)(3-2\gamma)}{3}M^4\right\}\frac{u^4}{U^4} + 2\frac{u^2}{U^2}\frac{v^2+w^2}{U^2} + \left(\frac{v^2+w^2}{U^2}\right)^2\right] \\ & + \cdots \end{aligned} \tag{1.22}$$

In linearized theory only the first term is ordinarily retained. This is satisfactory for plane flow or flow past planar systems, since the contribution of the remaining terms is truly of higher order. In fact, for plane flow past a single body it happens that the next two terms cancel. However, for slender bodies such as a cone, orders of magnitude are not so clearly distinguished. Busemann suggests (Ref. 8) that the second term is then sufficiently large compared with the first that it should be used also, and this view is supported by Lighthill (Ref. 11). But for the sake of consistency it might seem logical to retain the third term, which also involves squares of perturbation quantities. Having gone this far, it may be simpler to use the exact relation (1.21).

Each of these four possibilities is shown in Fig. 1.2 in comparison with the exact solution (Ref. 12) for flow past a cone of five degree semi-vertex angle. The series (1.22) is seen to alternate in this case. It converges so slowly, however, that linearizing the pressure relation (curve 1) introduces much greater errors than linearizing only the equation of motion (curve 4). Even if one or both of the quadratic terms are retained (curve 2 or 3) the series contributes discrepancies nearly as great as those due directly to non-linearity.

The point of view to be adopted here is that calculating the velocities and calculating the pressure are two essentially distinct operations. Each should be done as accurately as practicable. Linearization may be necessary in order to solve for the velocities, but the pressure relation need not then be linearized simply for the sake of consistency. For it may happen that the errors thus introduced are greater than those which result from the original linearization. Indeed, this is the case for the cone at moderate Mach numbers, and will be found to be true also of the second-order solution. Moreover, so many terms must be retained for the

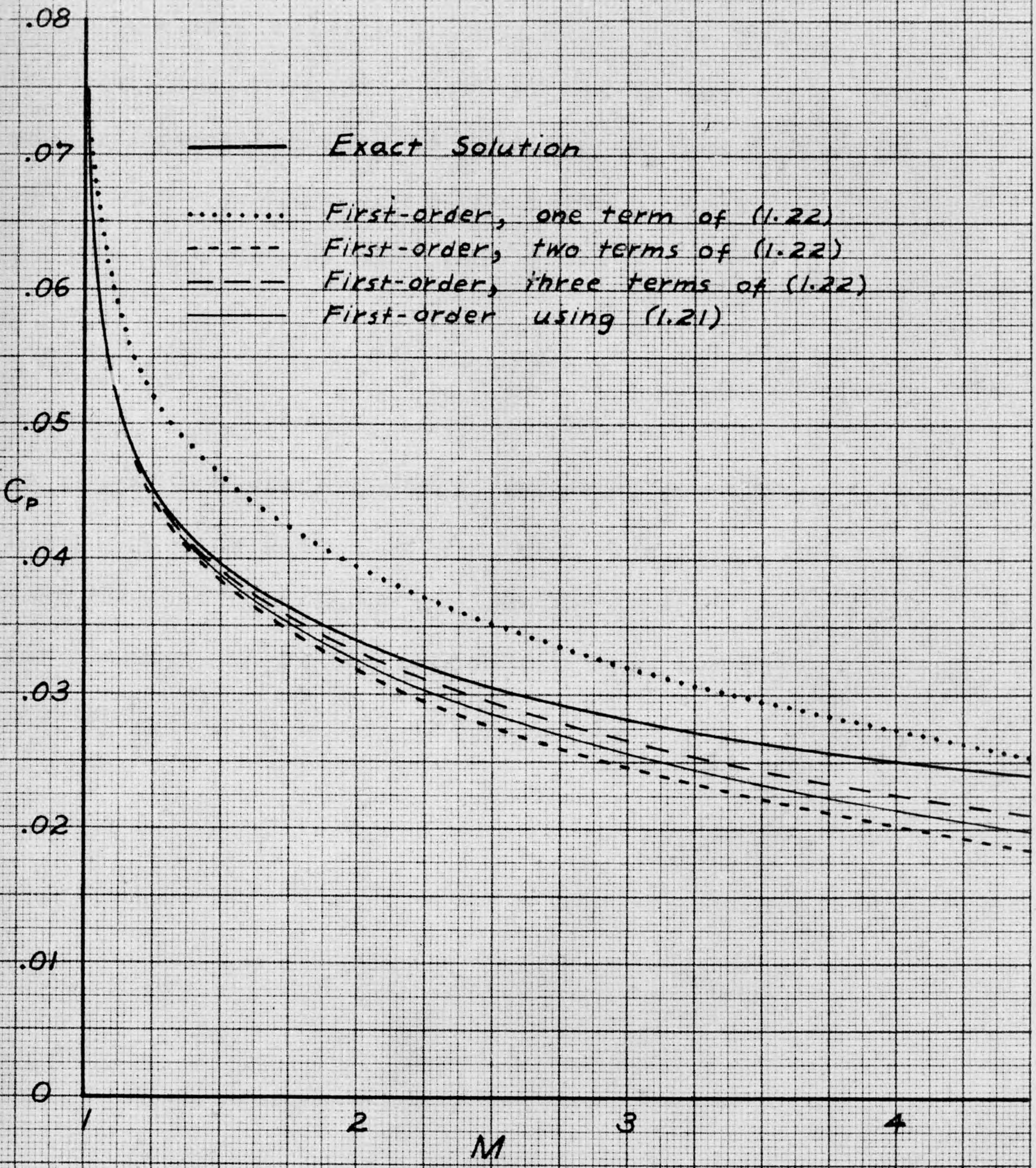

Fig. 1.2. Comparison of First-order Solutions for 5° Cone Using Various Pressure Relations.

second approximation that it is usually simpler to use the exact pressure relation.

For some purposes, however, it is desirable to have a simple expression for the pressure which does not involve $\frac{\gamma}{\gamma-1}$ -powers. Arranging the terms of equation (1.22) in descending order of magnitude gives

$$C_p = -2\frac{u}{U} - \frac{v^2+w^2}{U^2} + \beta^2\frac{u^2}{U^2} + M^2\frac{u}{U}\frac{v^2+w^2}{U^2} + \frac{M^2}{4}\left(\frac{v^2+w^2}{U^2}\right)^2 + \dots \qquad (1.23)$$

In the second approximation, only the first term is required for plane flow past a single body, the first three terms must be used for planar systems or general plane flow, while for slender non-planar bodies the last two terms are also required.

II. Some Simple Solutions

In this chapter several simple solutions of the second-order iteration equation will be investigated in detail. This will permit the nature of the iteration process to be analyzed, particularly with regard to its convergence.

9. Flow Past a Slightly Curved Wall

Consider flow past a plane wall which at some point begins to deviate slightly from a plane (Fig. 2.1). The wall can be represented by

$$y = \epsilon g(x) \tag{2.1}$$

where ϵ is a parameter small compared with unity, and $g(x)$ is a function of order unity which vanishes for

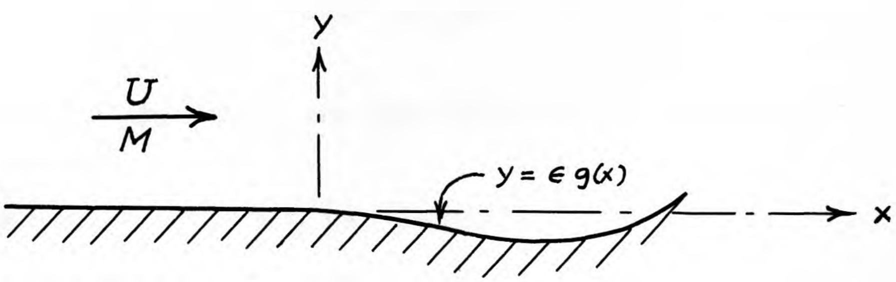

Fig. 2.1. Flow Past a Curved Wall

$x \leq 0$, and possesses whatever degree of continuity may be found necessary.

This is a planar system, so that according to Section 7 both the first- and second-order tangency conditions may be satisfied at the plane $y = 0$. The first-order problem is then given by (1.5), (1.19a), and (1.20a):

$$\begin{aligned} \phi_{yy} - \beta^2\phi_{xx} &= 0 \\ \phi_y(x,0) &= -\epsilon g'(x) \\ \phi(0,y) = \phi_x(0,y) &= 0 \end{aligned} \tag{2.2}$$

The solution is

$$\phi = -\frac{\epsilon}{\beta} g(x-\beta y) \tag{2.3}$$

The iteration equation (1.12) for the additional second-order potential reduces to

$$\begin{aligned} \varphi_{yy} - \beta^2\varphi_{xx} &= 2M^2\left[(N-1)\beta^2\phi_{xx}\phi_x + \phi_{xy}\phi_y\right] \\ &= M^2 N\epsilon^2\left[\{g'(x-\beta y)\}^2\right]' \end{aligned} \tag{2.4a}$$

The value of $\phi_c = \phi_y$ on the surface of the wall is

$$\begin{aligned} [\phi_y]_{surface} &= \epsilon g'\{x - \beta\epsilon g(x)\} \\ &= \epsilon g'(x) - \beta\epsilon^2 g(x) g''(x) + \ldots \end{aligned}$$

so that the boundary conditions (1.19b) and (1.20b) become

$$\varphi_y(x,0) = \epsilon^2\left[\beta g(x) g''(x) - \frac{1}{\beta}\{g'(x)\}^2\right]$$

$$\varphi(0,y) = \varphi_x(0,y) = 0 \qquad (2.4b)$$

By means of the impulse method (Ref. 6, vol. II, p. 164) it can be shown that the solution of the problem

$$u_{yy} - u_{tt} = f(y,t) \qquad \begin{matrix} y \geq 0 \\ t \geq 0 \end{matrix}$$

$$u_y(0,t) = h(t) \qquad (2.5)$$

$$u(y,0) = u_t(y,0) = 0$$

is given by

$$u(y,t) = -\int_0^{t-y} h(\tau)d\tau - \frac{1}{2}\left[\int_0^{t-y} d\tau \int_0^{y+(t-\tau)} f(\tau,\eta)d\eta + \int_0^{t-y} d\tau \int_0^{(t-\tau)-y} f(\tau,\eta)d\eta + \int_{(t-y)}^{t} d\tau \int_{y-(t-\tau)}^{y+(t-\tau)} f(\tau,\eta)d\eta\right] \qquad (2.6)$$

Using this result, and integrating by parts, using the fact that $g(x)$ vanishes identically for $x \leq 0$, the solution of equations (2.4) is found to be

$$\varphi = -\epsilon^2\left[g(x-\beta y)g'(x-\beta y) + \frac{M^2N}{2\beta}y\{g'(x-\beta y)\}^2 + \frac{M^2(N-2)}{2\beta^2}\int_0^{x-\beta y}\{g'(\xi)\}^2 d\xi\right] \qquad (2.7)$$

Adding ϕ from (2.3) gives the complete second-order perturbation potential

$$\Phi^{(2)} = -\frac{\epsilon}{\beta}g(x-\beta y) - \epsilon^2\left[g(x-\beta y)g'(x-\beta y) + \frac{M^2N}{2\beta}y\{g'(x-\beta y)\}^2 + \frac{M^2(N-2)}{2\beta^2}\int_0^{x-\beta y}\{g'(\xi)\}^2 d\xi\right] \qquad (2.8)$$

On the surface of the wall the streamwise velocity perturbation is

$$\left[\frac{u}{U}\right]_{wall} = \left[\Phi_x^{(2)}\right]_{y=\epsilon g(x)} = -\frac{\epsilon}{\beta} g'(x) - \frac{(M^2N-2)}{2\beta^2}\epsilon^2\{g'(x)\}^2$$

The pressure coefficient at the wall can now be calculated from (1.23) which, upon replacing N by its value from (1.11), gives

$$\left[C_P\right]_{wall} = \frac{2}{\beta}\epsilon g'(x) + \frac{(\gamma+1)M^4-4\beta^2}{2\beta^4}\{\epsilon g'(x)\}^2 + \ldots \qquad (2.9)$$

This is the well-known result of Busemann (Ref. 13). To second order, the surface pressure coefficient depends only upon the local slope.

10. The Role of the Characteristics

In Section 3 it was pointed out that because of the underlying significance of the characteristic surfaces for solutions of hyperbolic equations, it might be expected that the characteristics would have to be revised successively at each stage of the iteration. However, an iteration process was chosen which permits no such revision. It is therefore pertinent to inquire in this simple solution what role has been played by the original and the revised characteristics.

Only one of the two families of characteristics will be considered. The original characteristics of this family are the lines of slope

$$\frac{dy}{dx} = \frac{1}{\beta} \tag{2.10}$$

These are the downstream Mach lines of the undisturbed flow, and are also characteristics of equation (1.5) in the mathematical sense (Ref. 6, vol. II, chap. 5; Ref. 7, chap. II).

It can readily be shown that if the first-order streamwise perturbation velocity at any point in a flow is $u^{(1)}$, then the revised local values of Mach number and β are given by

$$\begin{aligned} M^{(1)} &= M\left[1 + \beta^2 (N-1) \frac{u^{(1)}}{U}\right] \\ \beta^{(1)} \equiv \sqrt{M^{(1)2} - 1} &= \beta\left[1 + M^2 (N-1) \frac{u^{(1)}}{U}\right] \end{aligned} \tag{2.11}$$

Using this result together with the first-order solution (2.3), the revised downstream Mach lines are found to have the slope

$$\frac{dy}{dx} = \frac{1}{\beta}\left[1 + \frac{M^2 N}{\beta} \epsilon g'(x - \beta y)\right] \tag{2.12}$$

These are not the mathematical characteristics of the iteration equation (2.4a) for the reason that fractions of the highest-order derivatives have there been transferred to the right-hand side and regarded as known. Mathematically, the characteristics continue to be given by (2.10).

Physically, the characteristics are lines along which discontinuities in velocity derivatives are propagated, and this definition is completely equivalent to the mathematical one (Ref. 6, vol. II, p. 297). Therefore in the second-order solution derived above, discontinuities in acceleration must occur along the original characteristics.

Suppose, however, that no such discontinuities occur. For flow past a single body the downstream characteristics are also lines along which the velocity is constant, provided that shock waves do not appear. Setting

$$d\Phi_x = \Phi_{xx}\, dx + \Phi_{xy}\, dy = 0$$

$$d\Phi_y = \Phi_{xy}\, dx + \Phi_{yy}\, dy = 0$$

it is seen that the velocity is constant if

$$\frac{dy}{dx} = -\frac{\Phi_{xx}}{\Phi_{xy}} = -\frac{\Phi_{xy}}{\Phi_{yy}}$$

For the second approximation (2.8) the velocity is constant along lines of slope

$$\frac{dy}{dx} = -\frac{\Phi^{(2)}_{xx}}{\Phi^{(2)}_{xy}} = -\frac{\Phi^{(2)}_{xy}}{\Phi^{(2)}_{yy}} = \frac{1}{\beta}\left[1 + \frac{M^2N}{\beta}\,\epsilon\, g'(x-\beta y)\right]$$

which according to (2.12) are the revised characteristics. Consequently, although the characteristics have not been revised in the mathematical sense, the solution behaves physically as if they had, so long as discontinuities do not occur. The question of discontinuities will be considered in the next section.

The connection between the original and revised characteristics can be interpreted physically. The right-hand side of the iteration equation (2.4a) may be regarded as due to supersonic sources distributed throughout the flow field. The influence of this source distribution spreads downstream along both families of original characteristics. The resulting velocity changes are just such that the second-order velocities become constant along the revised rather than the original characteristics.

Finally, it is interesting to note that the second-order potential is constant on lines which bisect the original and revised characteristics. For setting

$$d\Phi^{(2)} = \Phi_x^{(2)}\,dx + \Phi_y^{(2)}\,dy = 0$$

$\Phi^{(2)}$ is found to be constant along lines of slope

$$\frac{dy}{dx} = \frac{1}{\beta}\left[1 + \frac{M^2 N}{2\beta}\,\epsilon\, g'(x-\beta y)\right] \qquad (2.13)$$

11. Flow Past a Corner and a Parabolic Bend

A simple case in which discontinuities may occur is that of flow past a sharp corner. The exact solution is known to involve an oblique shock wave with attendant velocity discontinuities for compression, and a continuous Prandtl-Meyer fan for expansion.

Denoting the tangent of the deflection angle by ϵ, positive for compression (Fig. 2.2), the function $g(x)$

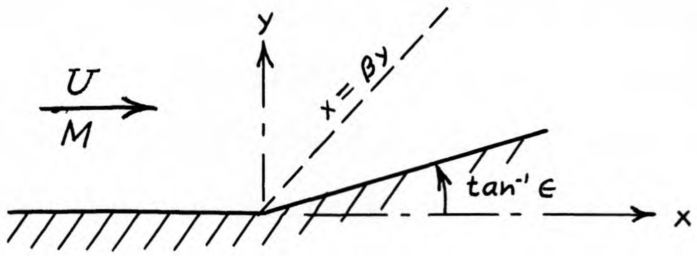

Fig. 2.2. Flow Past a Corner

appearing in (2.1) is

$$g(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \end{cases} \tag{2.14}$$

From (2.8) the second-order perturbation potential is found to be

$$\Phi^{(2)}(x, y) = -\frac{\epsilon}{\beta}(x-\beta y) + \frac{\epsilon^2}{\beta^2}(x-\beta y) - \frac{M^2 N}{2\beta^2}\epsilon^2 x \tag{2.15}$$

to the right of the line $x = \quad y$, and zero to the left. Consequently in either compression ($\epsilon > 0$) or expansion ($\epsilon < 0$) the second-order potential suffers a discontinuous drop along the Mach line from the corner, of strength proportional to x. Such a discontinuity cannot be admitted, which indicates that the iteration process fails in this region.

In the case of compression, the solution can be corrected by analytically continuing the perturbation potential upstream until it can be joined continuously to the free-stream potential. (This is permissible since the line of discontinuity is not actually a characteristic.) From the result of equation (2.13) the juncture is seen to occur along the line from the corner which bisects the upstream

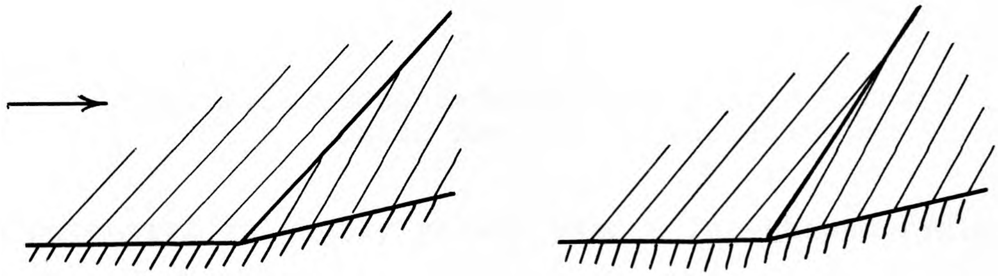

Fig. 2.3. Mach Lines Before and After Adjustment of Potential Discontinuity

and downstream Mach directions, as indicated in Fig. 2.3. The adjusted discontinuity corresponds to a shock wave, for it is known that an oblique shock bisects the Mach directions to first order (Ref. 7, p. 354). In the case of expansion, this type of correction cannot be justified, since it would involve continuation of the free-stream potential across a true characteristic. Instead, a Prandtl-Meyer fan must be inserted.

Evidently the iteration process converges except within an angular region of order ϵ lying near the Mach line from the corner. In particular, the pressure is given correctly everywhere on the surface of the wall.

It is enlightening to observe that the alternative method of iteration, in which the characteristics are successively revised, fails to converge in the same region.

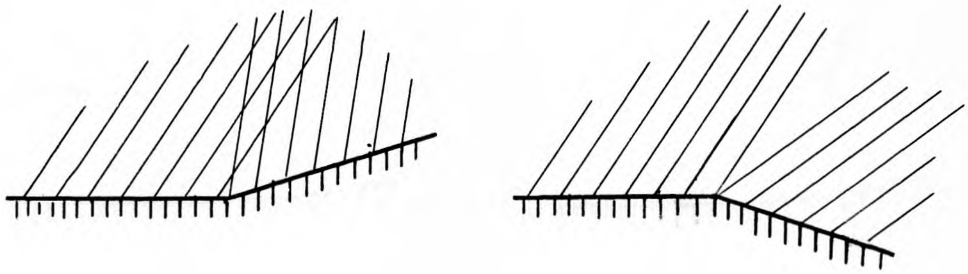

Fig. 2.4. Second-Order Flow Past a Corner Using Revised Characteristics

The potential is doubly valued over a fan-shaped region in the case of compression, and is left undefined over a similar region in the case of expansion (Fig. 2.4). The same artificial corrections are necessary to complete the solution.

Consider next flow past a parabolic bend which is represented by

$$y = \frac{1}{2} \epsilon x^2 \tag{2.16}$$

From (2.8) the second-order perturbation potential is found to be

$$\Phi^{(2)} = -\frac{\epsilon}{2\beta}(x-\beta y)^2 - \frac{M^2(N+1)-3}{6\beta^2}\epsilon^2(x-\beta y)^2 - \frac{M^2 N}{2\beta}\epsilon^2 y(x-\beta y)^2 \tag{2.17}$$

The potential and also the velocities are continuous, so that the previous difficulties do not occur. The acceleration is discontinuous across the original characteristic $x = y$, which in this case happens to be also a revised characteristic. However, a new complication arises. It is

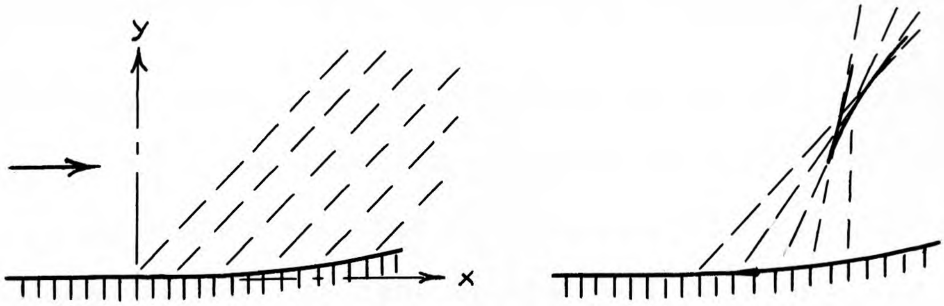

Fig. 2.5. First- and Second-Order Flow Past a Parabolic Bend

well known that in the exact solution for the compressive case, the characteristics form an envelope, as shown in Fig. 2.5. Inside the cusp the potential is triple-valued (Ref. 7, p. 111), so that a shock wave must be inserted. This envelope must also arise in the second approximation, since the characteristics are no longer parallel. However, the second-order potential given by (2.17) is single-valued, so that it cannot predict the formation of an envelope. Again the iteration process fails in a part of the flow field.

It can be seen that the alternative iteration process, using revised characteristics, will produce an envelope.

12. Convergence for Plane Flow

The examples just considered demonstrate that the question of convergence must be carefully investigated. Unfortunately, rigorous proofs of sufficient conditions for convergence have not been obtained, even in the case of plane flow. However, the above examples suggest certain conjectures regarding convergence. These will be stated, and some arguments for their plausibility advanced.

For flow past a slightly curved plane wall represented by $y = \epsilon g(x)$ the solution obtained by iteration using the <u>revised</u> characteristics is conjectured to converge in any bounded region adjacent to the wall provided that

(a) ϵ is sufficiently small

(b) $g(x)$ is continuously differentiable.

If $g(x)$ has only a piecewise continuous derivative, the convergence holds except possibly in fan-shaped regions springing from each corner, which lie near the original Mach line and subtend an angle of order ϵ.

For the iteration process actually adopted, in which the characteristics are not revised, the first n steps are conjectured to form part of a convergent process provided that

(a) ϵ is sufficiently small

(b') $g(x)$ has continuous derivatives up to (n-1)st order if the potential is required; nth order if the velocities are required.

If (b') is satisfied only piecewise, the result holds except possibly in fan-shaped regions springing from each corner.

In the first case, condition (a) is necessary in order to insure that the solution be unique, as is clear from the example of the parabolic wall. The above examples also show that condition (b) is necessary.

If the sufficiency of these two conditions is assumed, their connection with condition (b') in the second case can be illustrated by analogy with a mathematical model* which retains the essential difference between the two iteration processes -- namely, that the correct characteristics are not used in the method actually adopted. Consider the first-order problem given by (2.2):

$$\phi_{yy} - \phi_{xx} = 0$$

$$\phi_y(x,0) = -\epsilon g'(x)$$
$$\phi(0,y) = \phi_x(0,y) = 0$$

where we have taken $\beta = 1$ for convenience. The solution (2.3) was

$$\phi = -\epsilon g(x-y)$$

Now suppose we attempt to solve this problem using characteristics which differ from the true characteristics by $O(\epsilon)$. Thus we consider the equivalent problem

$$\phi_{yy} - (1-\epsilon)\phi_{xx} = \epsilon\phi_{xx}$$

$$\phi_y(x,0) = -\epsilon g'(x) \qquad (2.18)$$
$$\phi(0,y) = \phi_x(0,y) = 0$$

---

*Suggested by Dr. C. R. De Prima

and solve by iteration. In the first approximation the right-hand side can be neglected, so that

$$\phi_{yy} - (1-\epsilon)\,\phi_{xx} = 0$$

which has the solution, subject to the boundary conditions

$$\phi^{(I)} = -\epsilon\, g(x-\sqrt{1-\epsilon}\,y)$$

Substituting this into the right-hand side of (2.18) gives the iteration equation for the second approximation:

$$\phi_{yy} - (1-\epsilon)\,\phi_{xx} = -\epsilon^2 g''(x-\sqrt{1-\epsilon}\,y)$$

Using the result of equations (2.5) and (2.6), the solution subject to the boundary conditions is found to be

$$\phi^{(II)} = -\epsilon\, g(x-\sqrt{1-\epsilon}\,y) + \tfrac{1}{2}\epsilon^2 y\, g'(x-\sqrt{1-\epsilon}\,y)$$

But this is just the Taylor series expansion, correct to $O(\epsilon^2)$, of the true solution (2.3). Subsequent iterations add additional terms to the expansion. Hence despite the use of slightly incorrect characteristics, the iteration process converges to the correct solution. The connection between conditions (b) and (b') is thus seen to be that the existence of sufficiently many continuous derivatives compensates for the fact that the wrong characteristics are used.

13. Flow Past a Cone

Consider flow past a slender cone of semi-vertex angle $\tan^{-1}\epsilon$ (Fig. 2.6). The flow is conical and axially-symmet-

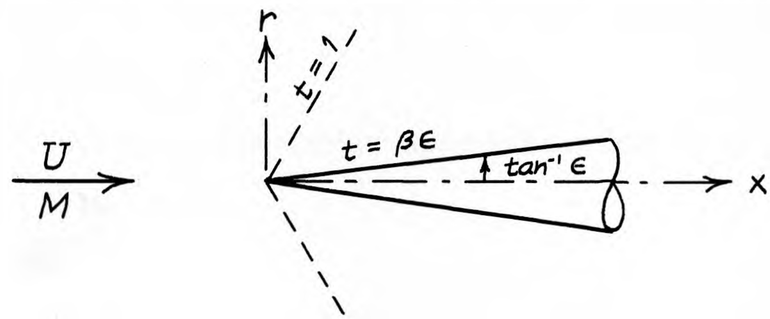

Fig. 2.6. Flow Past a Cone

ric, so that the iteration equation is given by (1.17) with θ-derivatives omitted. Conical potentials will be used exclusively, so that for ease of notation the bars can be omitted, with the understanding that velocity components must be calculated from (1.16). Including the boundary conditions (1.18a) and (1.19a), the first-order problem is

$$\begin{aligned} (1-t^2)\,\phi_{tt} + \frac{\phi_t}{t} &= 0 \\ \phi_t(\beta\epsilon) &= \frac{\epsilon}{\beta} \\ \phi(\infty) = \phi_t(\infty) &= 0 \end{aligned} \tag{2.19}$$

The equation can be immediately integrated to give the well-known result

$$\phi = -\epsilon^2\left(\operatorname{sech}^{-1} t - \sqrt{1-t^2}\right) \tag{2.20}$$

which is understood to vanish except within the downstream

Mach cone ($0 \le t \le 1$). The tangency condition has been satisfied only to the degree of approximation indicated in (2.20). At the Mach cone ($t = 1$) the velocity perturbations vanish, so that no shock wave deflection is predicted (Ref. 7, p. 403).

Substituting this first approximation into the iteration equation (1.17) gives

$$(1-t^2)\varphi_{tt} + \frac{\varphi_t}{t} = \epsilon^4 M^2 \left[2\frac{1}{t^2} + 2(N-1)\frac{\operatorname{sech}^{-1}t}{\sqrt{1-t^2}} - \beta^2\epsilon^2\frac{\sqrt{1-t^2}}{t^4} \right] \qquad (2.21a)$$

and from (1.18b) and (1.19b) the corresponding boundary conditions are

$$\varphi_t(\beta\epsilon) = \frac{\epsilon}{\beta}(1-\epsilon^2 \operatorname{sech}^{-1}\beta\epsilon) - \frac{\epsilon}{\beta}\sqrt{1-\beta^2\epsilon^2}$$
$$\varphi(\infty) = \varphi_t(\infty) = 0 \qquad (2.21b)$$

Now (2.21a) is a linear first-order differential equation for φ_t, and can be solved using the integrating factor $\frac{t}{\sqrt{1-t^2}}$. The various integrals encountered can invariably be treated by integrating by parts one or more times. Integrating again gives

$$\varphi = \epsilon^4 M^2 \Big[B + C(\operatorname{sech}^{-1}t - \sqrt{1-t^2}) + (\operatorname{sech}^{-1}t)^2 - (N-1)\sqrt{1-t^2}\,\operatorname{sech}^{-1}t - \frac{1}{4}\beta^2\epsilon^2\frac{\sqrt{1-t^2}}{t^2} \Big] + \ldots O\{\epsilon^6(\operatorname{sech}^{-1}t)^3\} \qquad (2.22)$$

where B and C are constants of integration. Setting

$B = 0$ satisfies the boundary condition of unperturbed upstream flow. Then the complete conical second-order perturbation potential is

$$\bar{\Phi}^{(2)} = -\epsilon^2(\operatorname{sech}^{-1}t - \sqrt{1-t^2}) + \epsilon^4 M^2\Big[C\,(\operatorname{sech}^{-1}t - \sqrt{1-t^2}) + (\operatorname{sech}^{-1}t)^2 - (N-1)\sqrt{1-t^2}\,\operatorname{sech}^{-1}t - \tfrac{1}{4}\beta^2\epsilon^2\frac{\sqrt{1-t^2}}{t^2}\Big] \qquad (2.23)$$

and from (1.16) the streamwise and radial velocity perturbations are

$$\frac{u}{U} = -\epsilon^2\operatorname{sech}^{-1}t + \epsilon^4 M^2\Big[C\operatorname{sech}^{-1}t + (\operatorname{sech}^{-1}t)^2 - (N-1)\frac{\operatorname{sech}^{-1}t}{\sqrt{1-t^2}} - (N+1) - \tfrac{3}{4}\beta^2\epsilon^2\frac{\sqrt{1-t^2}}{t^2}\Big]$$

$$\frac{1}{\beta}\frac{v}{U} = \epsilon^2\frac{\sqrt{1-t^2}}{t} + \epsilon^4 M^2\Big[-C\frac{\sqrt{1-t^2}}{t} - 2\frac{\sqrt{1-t^2}\operatorname{sech}^{-1}t}{t} + (N+1)\frac{1}{t} + (N-1)\frac{t\operatorname{sech}^{-1}t}{\sqrt{1-t^2}} + \tfrac{1}{2}\beta^2\epsilon^2\frac{\sqrt{1-t^2}}{t^3}\Big] \qquad (2.24)$$

The constant C must be adjusted so as to satisfy the tangency condition. In actual computation it is easier to adjust C numerically in exactly this fashion, rather than to calculate it from the cumbersome expression which could be written down. The pressure coefficient at any point can then be calculated from (1.21).

The last term in the bracket in (2.21a) is the triple product $\beta^2\phi_{tt}\phi_t^2$ which is retained in the second-order iteration equation (1.17). Its retention is now justified by noting that its contribution -- the last term in (2.22) -- is of the same order as the other terms near the surface of the cone $(t=\beta\epsilon)$. Actually it also contributes a second term, which has been neglected since it is at most of order $\epsilon^6 \operatorname{sech}^{-1}\beta\epsilon$. It can also be verified that the other triple products, whose form is indicated in the last two lines of (1.17), are in fact negligible, since they contribute at most terms of order $\epsilon^6(\operatorname{sech}^{-1}\beta\epsilon)^2$. Consideration of a further iteration indicates that a third approximation would add terms no greater than $\epsilon^6(\operatorname{sech}^{-1}\beta\epsilon)^3$, which is greater than the terms just neglected.

The second-order result for surface pressure coefficient is compared in Fig. 2.7 with the exact solution (Ref. 12) for cones of five, ten, and fifteen degree semi-vertex angles. The usual first-order results based upon one and two terms of the series for the pressure coefficient (1.23) are also shown for comparison. The second-order solution is seen to provide a much better approximation over a useful range of Mach number. It is clearly not suitable for very high Mach numbers, for the terms in (2.22) become meaningless when the Mach angle is smaller than the cone angle.

For low supersonic Mach numbers, the second-order result coincides with the exact solution to within the accuracy of plotting down to the point at which the shock wave detaches. It is surprising that the agreement continues to improve below the Mach number at which the flow near the surface becomes subsonic. Thus, for the fifteen degree cone, conical flow exists only above $M = 1.1193$, is com-

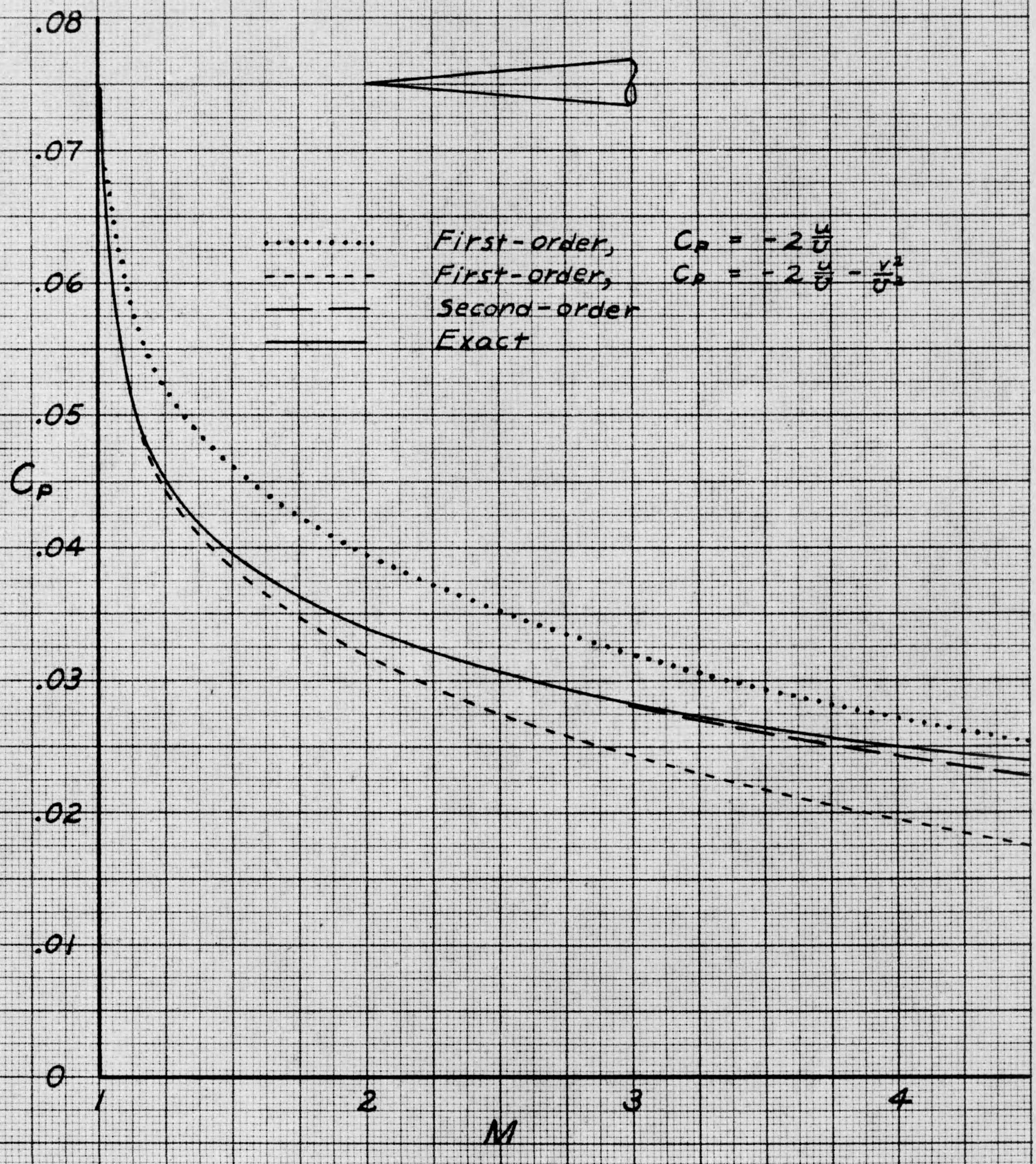

Fig. 2.7. Comparison of Various Approximations for Pressure on a Cone

(a) 5° Semi-vertex Angle

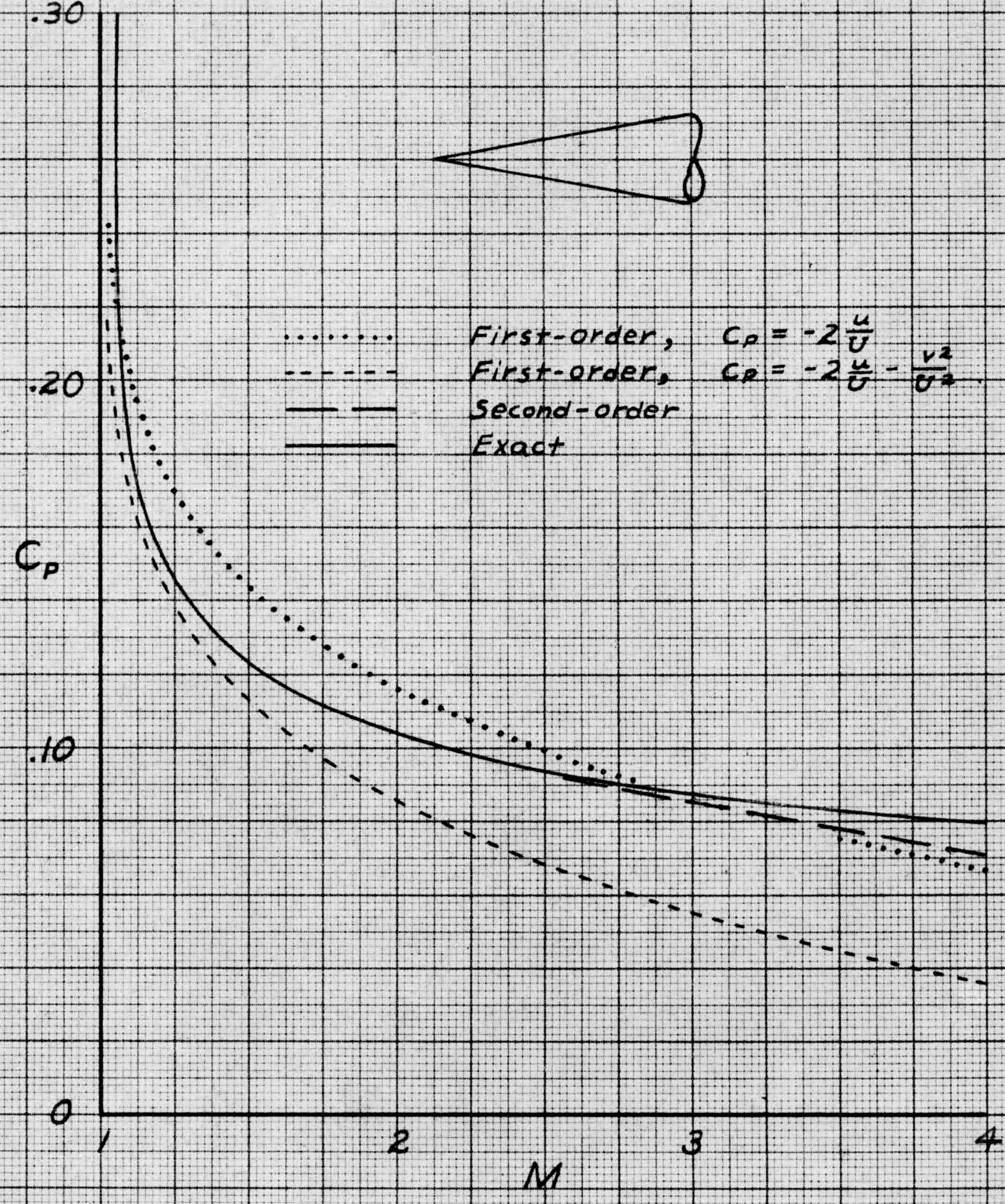

Fig. 2.7. Continued.

(b) 10° Semi-vertex Angle

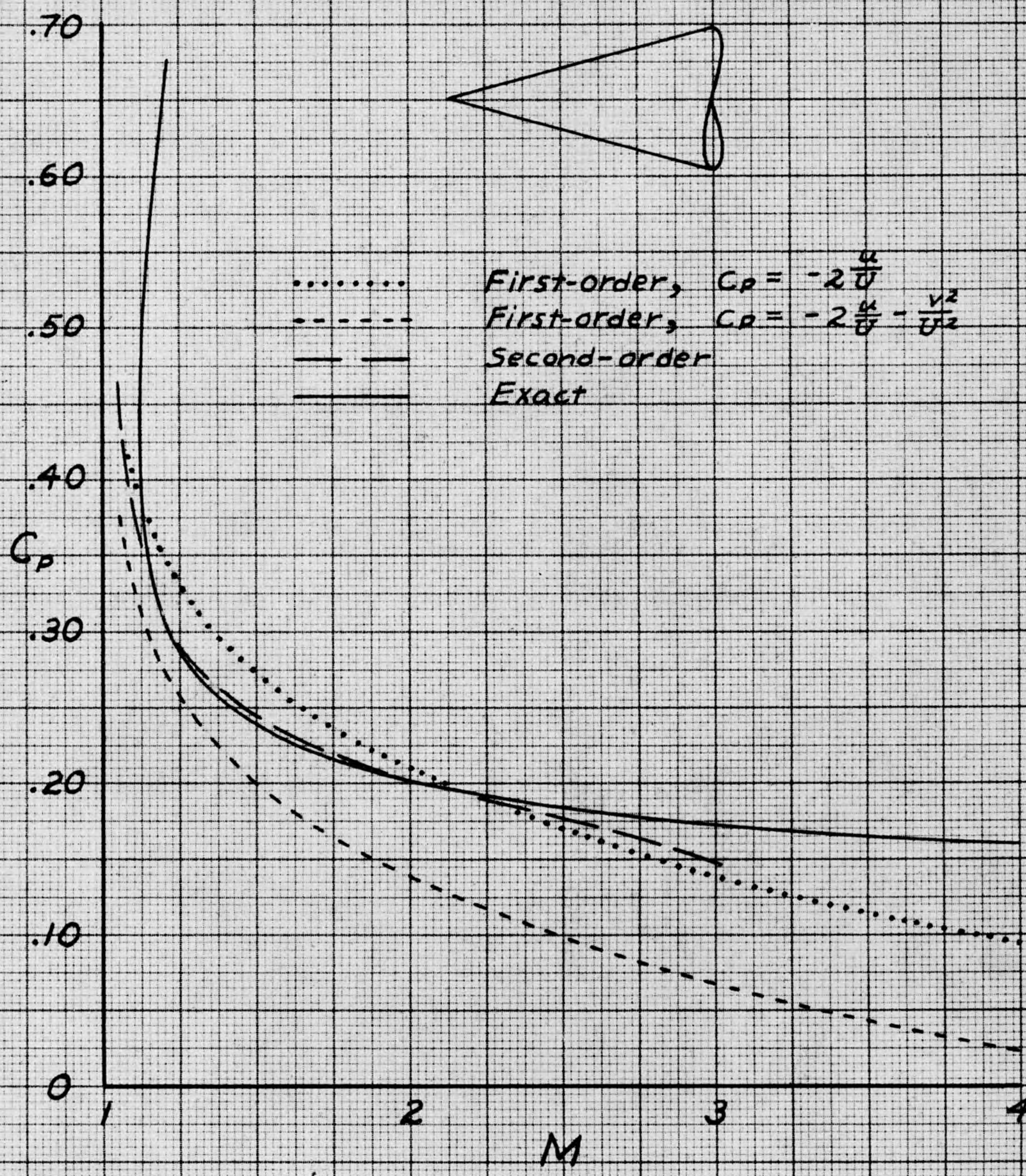

Fig. 2.7. Concluded.

(c) 15° Semi-vertex Angle

pletely subsonic behind the shock wave until $M = 1.1230$, and becomes completely supersonic at $M = 1.2187$. Within this range, the various determinations of surface pressure coefficient are as follows:

Mach Number	1.1382	1.1916	1.2186
Exact Value	.35834	.31302	.29996
Second-Order Result	.34216	.31350	.30289
1st-Order, 1 Term of (1.23)	.37615	.35093	.34058
1st-Order, 2 Terms of (1.23)	.30435	.27914	.26879

14. Series for Surface Pressure Coefficient

For some purposes it may be desirable to develop a series expansion for the pressure coefficient at the surface of the cone. This can be achieved by expanding $\bar{\Phi}^{(2)}$ in powers of t and $\log\frac{2}{t}$ for small t, using the expansion

$$\operatorname{sech}^{-1} t = \log\frac{2}{t} - \frac{1}{4}t^2 - \frac{3}{32}t^4 - \dots \quad (0<t<1) \tag{2.25}$$

From the tangency condition (2.21b) the constant C can be shown to be

$$C = \frac{2M^2-1}{M^2}\log\frac{2}{\beta\epsilon} - \left(N+1+\frac{1}{2M^2}\right) + \dots \tag{2.26}$$

and the velocity perturbations on the surface are

$$\frac{u}{U} = -\epsilon^2 \log \frac{2}{\beta\epsilon} - \epsilon^4\left[\beta^2\left(\log\frac{2}{\beta\epsilon}\right)^2 - \frac{4M^2+1}{2}\log\frac{2}{\beta\epsilon} + \left(M^2N + \frac{6M^2+1}{4}\right)\right] + \cdots$$

$$\frac{v}{U} = \epsilon - \epsilon^3 \log\frac{2}{\beta\epsilon} + \cdots \qquad (2.27)$$

Then from (1.23) the surface pressure coefficient is, if N is replaced by its value from (1.11)

$$c_p = \epsilon^2\left[2\log\frac{2}{\beta\epsilon} - 1\right] + \epsilon^4\left[3\beta^2\left(\log\frac{2}{\beta\epsilon}\right)^2 - (5M^2-1)\log\frac{2}{\beta\epsilon} + \left\{(N+1)\frac{M^4}{\beta^2} + \frac{13}{4}M^2 + \frac{1}{2}\right\}\right] \quad (2.28)$$
$$+ \cdots \quad O\left(\epsilon^6 \log^3\frac{2}{\beta\epsilon}\right)$$

This result was obtained by Broderick (Ref. 14) using a method which will be discussed in Section 21.

This series is compared with the previous form of the second-order solution in Fig. 2.8. For the five degree cone, the series agrees with the exact solution over a considerably wider range of Mach number than does the original form, but this must be considered accidental. For the larger cones, the expansion in series is seen to have reduced the accuracy, so that for the fifteen degree cone it represents no improvement over the first-order solution. The reason must be that the iteration process itself converges more rapidly than do the subsequent expansions, particularly (1.23), which are required to reduce it to series form. Hence terminating all expansions at terms of the order of those retained in the iteration process results in an

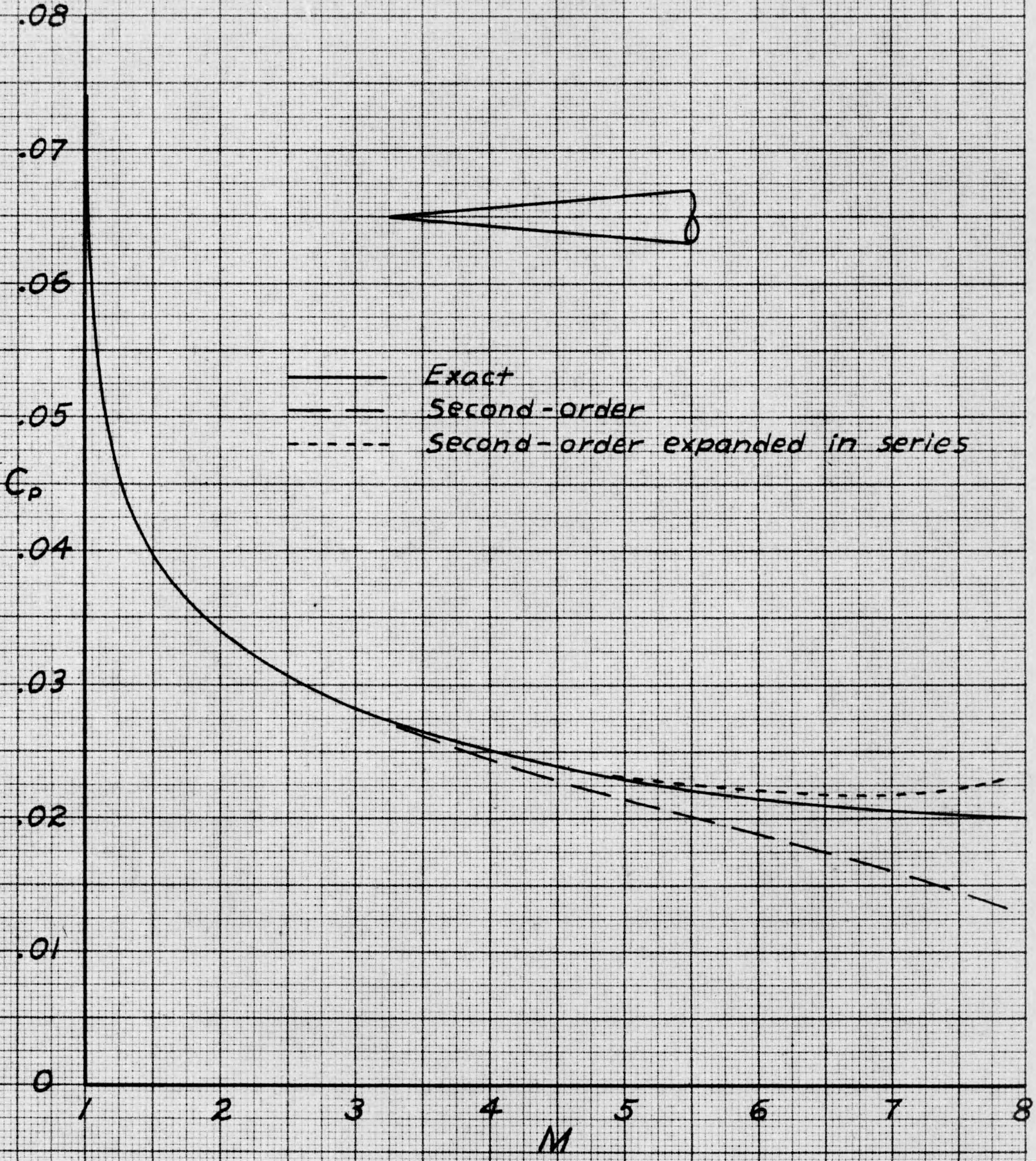

Fig. 2.8. Effect of Expanding in Series upon Second-order Pressure on a Cone.

(a) 5° Semi-vertex Angle

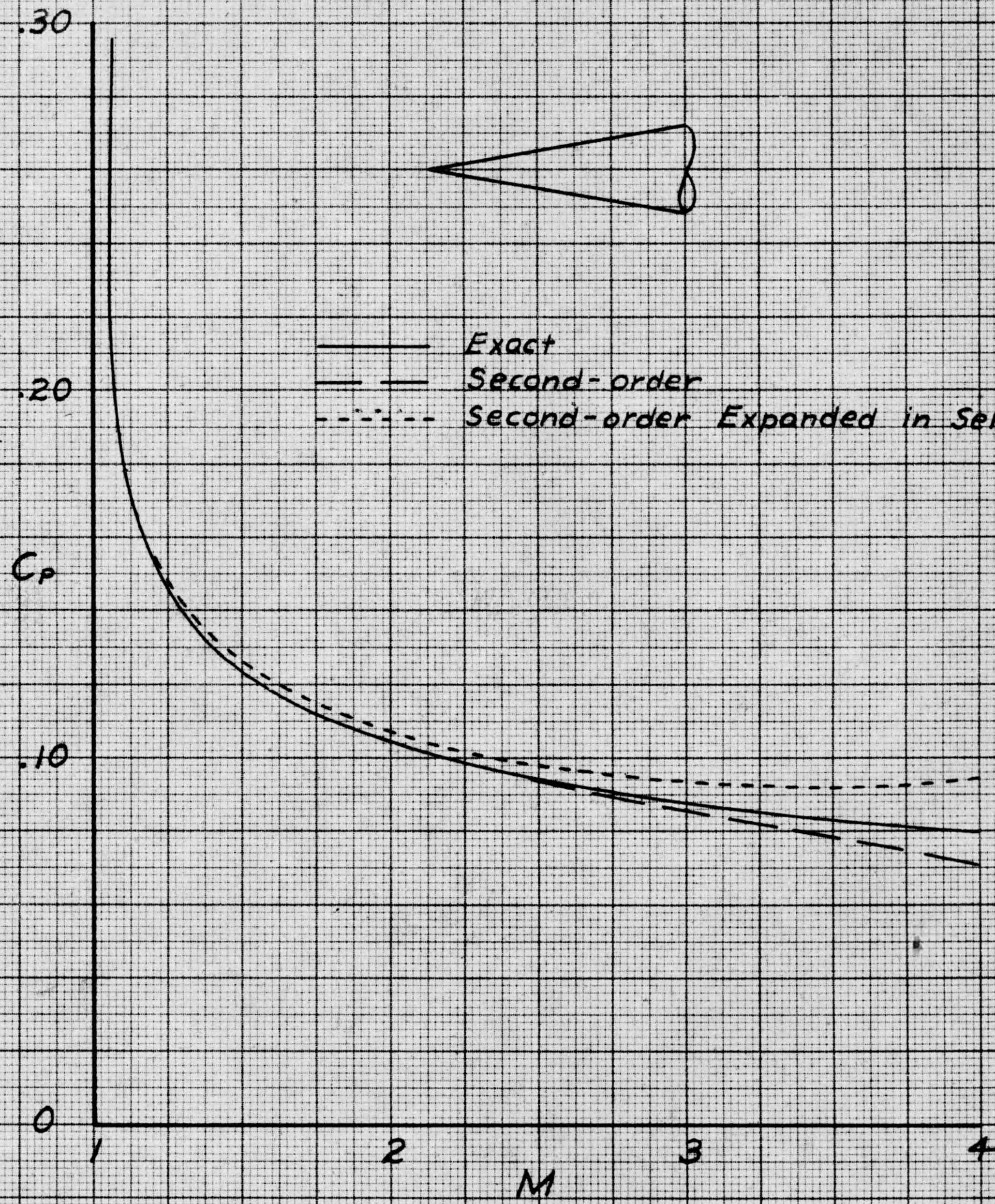

Fig. 2.8. Continued.

(b) 10° Semi-vertex Angle

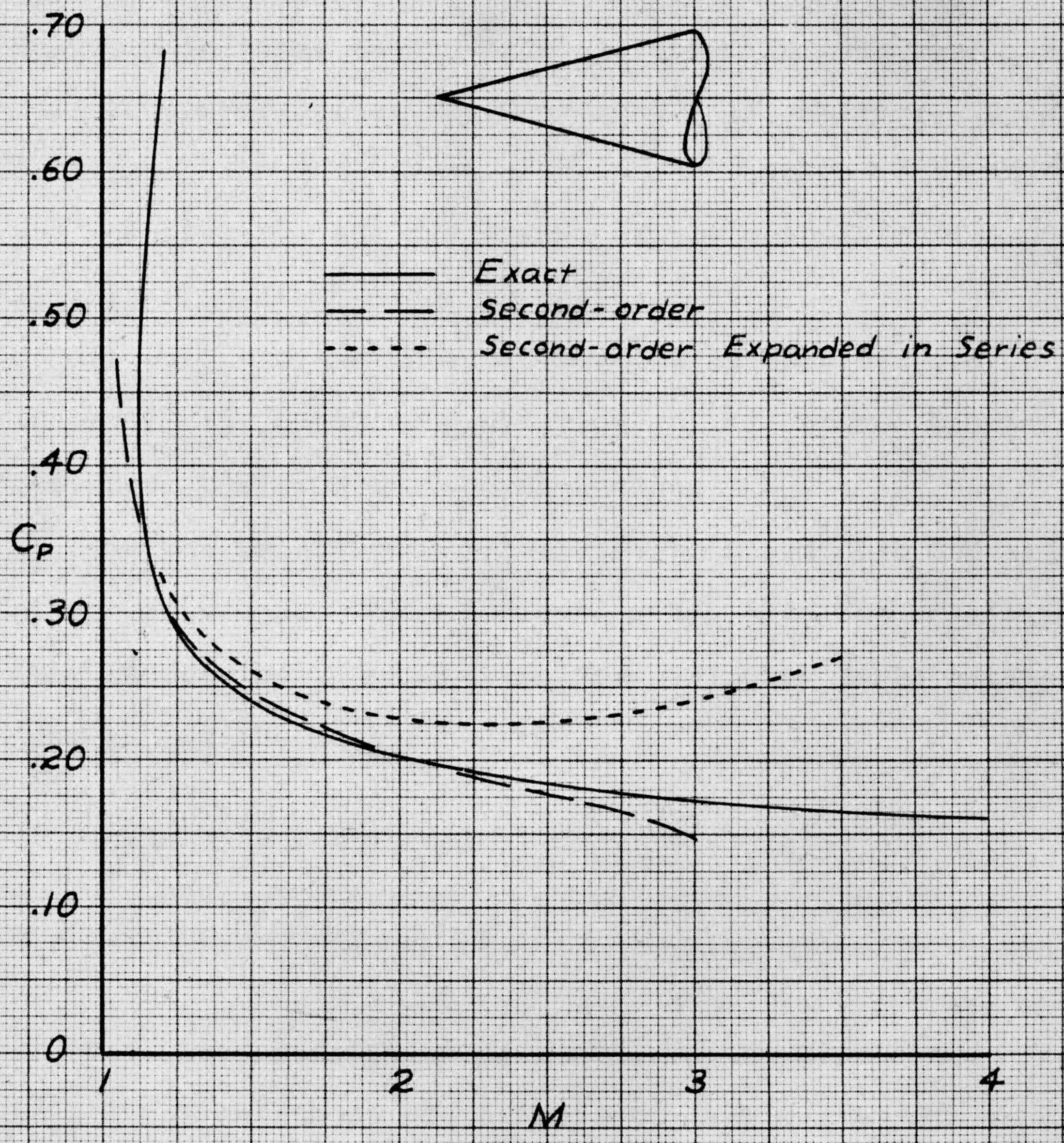

Fig. 2.8. Concluded

(c) 15° Semi-vertex Angle

unnecessary loss of accuracy.

15. The Shock Wave Angle

The solution for plane flow past a corner (Section 11) suggests that the second-order solution for the cone may fail to converge near the Mach cone. If, however, it does converge there, the first-order shock wave position and consequently the entropy change can be calculated from the fact that to first order an oblique shock bisects the Mach directions. It was noted in Section 13 that first-order theory is incapable of predicting any difference between the shock position and the Mach cone.

Assume provisionally that the solution does converge at the Mach cone, while indicating by (?) the possibility that it does not. From (2.24) the velocity perturbations just behind the Mach cone are

$$\begin{aligned} \left(\frac{u}{U}\right)_{t=1} &\overset{(?)}{=} -2M^2N\epsilon^4 \\ \left(\frac{v}{U}\right)_{t=1} &\overset{(?)}{=} 2\beta M^2N\epsilon^4 \end{aligned} \tag{2.29}$$

so that the perturbation is normal to the Mach cone. From (2.11) the cotangent of the revised Mach angle just behind the cone is found to be

$$\beta^{(1)} \equiv \sqrt{M^{(1)2}-1} \overset{(?)}{=} \beta\left[1-2M^4N(N-1)\epsilon^4\right] \tag{2.30}$$

The upward stream inclination there is $\left(\frac{v}{U}\right)_{t=1}$, so that the

Mach lines just behind the Mach cone have the slope

$$\frac{dr}{dx} \overset{(?)}{=} \frac{1}{\beta}\left[1 + 2M^4N^2\epsilon^4\right] \tag{2.31}$$

Therefore the slope of the shock wave differs from that of the original Mach cone by

$$\tan\theta_{sw} - \frac{1}{\beta} \overset{(?)}{=} \beta M^4N^2\epsilon^4 = \frac{(\gamma+1)^2M^8}{4\beta^5}\epsilon^4 \tag{2.32a}$$

This problem has been treated rigorously in an ingenious manner by Lighthill (Ref. 15) and also by Broderick (Ref. 16), who find that actually

$$\tan\theta_{sw} - \frac{1}{\beta} = \frac{3}{8}\frac{(\gamma+1)^2M^8}{\beta^5}\epsilon^4 \tag{2.32b}$$

which is $1\frac{1}{2}$ times the result of (2.32a). The discrepancy means that the second-order solution does not converge near the Mach cone. The question of convergence for bodies of revolution in general will be considered further in Section 23.

It seems remarkable that the solution developed above is in error only to the extent of a constant factor. The possibility that this is true more generally will be considered in Section 28.

It is well known that the entropy increase through a weak oblique shock wave is proportional to the cube of its deflection from the Mach direction. Consequently, the entropy increase through the shock wave from a cone is $O(\epsilon^{12})$, as noted by Lighthill (Ref. 15).

III. General Solutions for Plane and Axially-Symmetric Flow

16. The Role of a Particular Solution

In the preceding chapter several simple solutions of the second-order iteration equation were found by direct methods. It will now be shown that for plane and axially-symmetric flows a particular solution of the equation can be written down at once in terms of the first-order solution. This essentially solves the problem, because the complete solution consists of a particular integral plus a solution of the homogeneous equation, and the latter can be obtained by existing methods. That is,

$$\varphi = \psi + \chi \tag{3.1}$$

where

ψ = any particular solution of the non-homogeneous iteration equation

χ = a correction potential which is a solution of the corresponding homogeneous equation $\Box \Phi = 0$ and which serves to satisfy the boundary conditions.

and the problem for χ is identical with the usual first-order problem, whose solution is assumed to be available.

The role of the particular solution is to transfer the non-homogeneity in the problem from the equation, where it is troublesome, to the boundary conditions, where it can be handled by existing theory. For linear differential equations it is always possible in principle to transfer non-homogeneities in this way from the equation to the boundary conditions and vice-versa, by adding a suitable function to the dependent variable (see Ref. 6, vol. I, p. 236).

Since the particular solution ψ will be found in terms of the first-order solution, it will vanish upstream of the plane $x = 0$. Then the correction potential must also vanish there, so that

$$\chi(0, y, z) = \chi_x(0, y, z) = 0 \tag{3.2}$$

The tangency condition for χ is given by (1.18b):

$$\chi_c = (1+\phi_x)[\text{Slope}] - \phi_c - \psi_c \quad \text{on the surface} \tag{3.3}$$

or, in the case of planar systems, from (1.19b):

$$[\chi_c]_{plane} = [1+\phi_x]_{\substack{\text{plane or}\\ \text{surface}}} [\text{Slope}]_{surface} - [\phi_c]_{surface} - [\psi_c]_{surface} \tag{3.4}$$

17. The General Solution for Plane Flow

For plane flow, the first-order solution is

$$\phi = H(x-\beta y) + K(x+\beta y) \tag{3.5}$$

where H and K are functions chosen so as to satisfy the boundary conditions. In the iteration equation, all triple products can be neglected, and (1.12) becomes

$$\varphi_{yy} - \beta^2\varphi_{xx} = 2M^2\left[(N-1)\beta^2\phi_{xx}\phi_x + \phi_{xy}\phi_y\right] \qquad (3.6a)$$

It can readily be verified that a particular solution of this equation is given by

$$\psi = M^2\phi_x\left[(1-\tfrac{N}{2})\phi + \tfrac{N}{2}y\phi_y\right] \qquad (3.7a)$$

To this must be added a solution χ of the homogeneous equation, which is given by

$$\chi = h(x-\beta y) + k(x+\beta y) \qquad (3.8)$$

where h and k are functions determined by the second-order boundary conditions (3.2) and (3.3) or (3.4).

For flow past a single boundary (such as one surface of an airfoil) the first-order potential (3.5) contains only one or the other of the functions H and K. In this case $\phi_{xy}\phi_y = \beta^2\phi_{xx}\phi_x$, so that the iteration equation reduces to

$$\varphi_{yy} - \beta^2\varphi_{xx} = 2\beta^2M^2N\phi_{xx}\phi_x = \beta^2M^2N(\phi_x^2)_x \qquad (3.6b)$$

The particular solution may then be simplified to

$$\psi = M^2 \frac{N}{2} y \Phi_x \Phi_y \tag{3.7b}$$

and the correction potential (3.8) contains only h or only k, according as (3.5) contains only H or K.

For example, for the flow past a slightly curved wall which was treated in Section 9, equations (3.6b) and (3.7b) give the additional second-order potential as

$$\varphi = \psi + \chi = -\frac{M^2 N}{2\beta} \epsilon^2 y \{g'(x-\beta y)\}^2 + h(x-\beta y)$$

Imposing the tangency condition (3.4)

$$h'(x) = \epsilon^2 \left[\frac{2-M^2N}{2\beta^2} \{g'(x)\}^2 - g(x) g''(x) \right]$$

so that

$$h(x) = -\epsilon^2 \left[g(x) g'(x) + \frac{M^2(N-2)}{2\beta^2} \int_0^x \{g'(\xi)\}^2 d\xi \right]$$

and

$$\varphi = -\epsilon^2 g(x-\beta y)\, g'(x-\beta y) - \frac{M^2 N}{2\beta} \epsilon^2 y \{g'(x-\beta y)\}^2 + \frac{M^2(N-2)}{2\beta^2} \epsilon^2 \int_0^{x-\beta y} \{g'(\xi)\}^2 d\xi$$

which is the same as the previous result (2.7).

18. The Particular Solution for Axially-Symmetric Flow

Consider flow past an axially-symmetric body, which will be assumed to be either a slender pointed body with nose at the origin, or one which extends indefinitely upstream with constant radius a for $x \leq 0$ (Fig. 3.1). With suitable modification, the subsequent development can be applied to

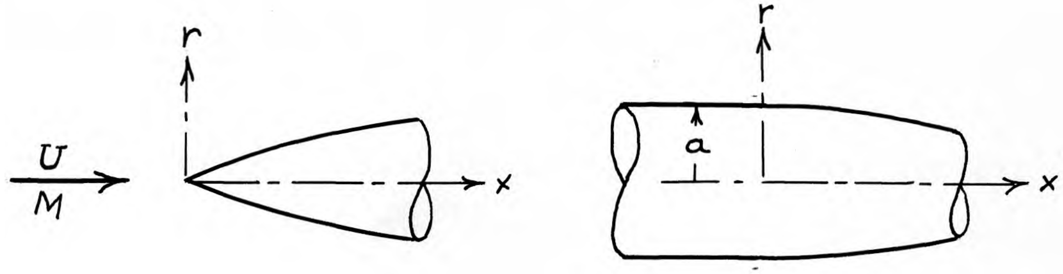

Fig. 3.1. Flow Past Bodies of Revolution

other shapes, such as annular bodies. The meridian curve can be represented in the first case by

$$r = R = \epsilon \rho(x) \qquad x \geq 0 \tag{3.9a}$$

and in the second by

$$r = R = \begin{cases} a & x \leq 0 \\ a + \epsilon \rho(x) & x \geq 0 \end{cases} \tag{3.9b}$$

Here ϵ is again a parameter small compared with unity, and $\rho(x)$ is a function vanishing at $x = 0$ and possessing such conditions of continuity as may be found necessary to insure convergence of the iteration process.

The first-order problem is

$$\phi_{rr} + \frac{\phi_r}{r} - \beta^2 \phi_{xx} = 0 \tag{3.10}$$

with the usual conditions of tangency and unperturbed upstream flow. The solution is well known to be (Ref. 17)

$$\phi(x,r) = -\int_{c}^{x-\beta r} \frac{F(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} = -\int_{0}^{\cosh^{-1}\frac{x-c}{\beta r}} F(x-\beta r\cosh u)\,du \tag{3.11}$$

The second form is useful for carrying out differentiation, after which the first form can be restored. The derivatives which will be required are

$$\begin{aligned} \phi_x &= -\int_{0}^{\cosh^{-1}\frac{x-c}{\beta r}} F'(x-\beta r\cosh u)\,du = -\int_{c}^{x-\beta r} \frac{F'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \\ \phi_r &= \beta\int_{0}^{\cosh^{-1}\frac{x-c}{\beta r}} F'(x-\beta r\cosh u)\cosh u\,du = \frac{1}{r}\int_{c}^{x-\beta r} \frac{(x-\xi)F'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \end{aligned} \tag{3.12}$$

With coordinates as shown in Fig. 3.1, the lower limit of integration c is zero for the pointed body and $-\beta a$ for the semi-infinite body. F(x) may be regarded as the strength of a supersonic line source along the x-axis. F is determined by the tangency condition, which gives an integral equation of Volterra type for F':

$$\int_{c}^{x-\beta R(x)} \frac{(x-\xi)F'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 R^2(x)}} = R(x)\,R'(x) \tag{3.13}$$

From (1.13) the iteration equation is found to be

$$\varphi_{rr} + \frac{\varphi_r}{r} - \beta^2\varphi_{xx} = M^2\left[\begin{array}{l} 2(N-1)\beta^2\phi_{xx}\phi_x + 2\phi_{xr}\phi_r \\ +\ \phi_{rr}\phi_r^2 \\ +\ \ldots\ O(\phi_{xx}\phi_x^2, \phi_{xx}\phi_r^2, \phi_{xr}\phi_x\phi_r) \end{array}\right. \tag{3.14}$$

and the solution for the cone suggests that the terms indicated in the last line are negligible.

It will now be shown that a particular solution of this equation is given by

$$\psi = M^2\phi_x(\phi + Nr\phi_r) - \frac{M^2}{4} r(\phi_r)^3 \tag{3.15}$$

The first group of terms contributes the first line in (3.14), as can be verified by direct substitution. The last term in (3.15) accounts for the term $\phi_{rr}\phi_r^2$ as follows:

$$\begin{aligned}
\Box\left(-\frac{M^2}{4} r\phi_r^3\right) &= -M^2\phi_r\Big[\frac{9}{4}\phi_r\phi_{rr} + \frac{3}{2}r\phi_{rr}^2 + \frac{3}{4}r\phi_r\phi_{rrr} + \frac{1}{4}\phi_r^2 \\
&\qquad - \frac{3}{2}\beta^2 r\phi_{xr}^2 - \frac{3}{4}\beta^2 r\phi_r\phi_{xxr}\Big] \\
&= M^2\phi_r\Big[\phi_{rr}\phi_r - \frac{3}{4}r\phi_r\left(\phi_{rrr} + \frac{\phi_{rr}}{r} - \frac{\phi_r}{r^2} - \beta^2\phi_{xxr}\right) \\
&\qquad - \left(\phi_r + \frac{3}{2}r\phi_{rr}\right)\left(\phi_{rr} + \frac{\phi_r}{r} - \beta^2\phi_{xx}\right) - \beta^2\Big(\frac{3}{2}r\phi_{xx}\phi_{rr} + \phi_{xx}\phi_r \\
&\qquad + \frac{3}{2}r\phi_{xr}^2\Big)\Big] \\
&= M^2\phi_{rr}\phi_r^2 - \beta^2 M^2\phi_r\left(\frac{3}{2}r\phi_{xx}\phi_{rr} + \phi_{xx}\phi_r + \frac{3}{2}r\phi_{xr}^2\right)
\end{aligned}$$

where repeated use is made of the fact that ϕ satisfies (3.10). The last group of terms consists of triple products involving x-derivatives, which have already been neglected in (3.14), so that the result is proved.

The correction potential χ is a solution of (3.10) and can be written as

$$\chi = -\int_c^{x-\beta r} \frac{f(\xi)\,d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} = -\int_0^{\cosh^{-1}\frac{x-c}{\beta r}} f(x - \beta r\cosh u)\,du \tag{3.16}$$

Using (3.12) the second-order tangency condition (3.3) is

$$\int_{c}^{x-\beta R(x)} \frac{(x-\xi) f'(\xi) d\xi}{\sqrt{(x-\xi)^2 - \beta^2 R^2(x)}} = R(x)\Big[R'(x)\{1+\phi_x(x,R)\} - \phi_r(x,R) - \psi_r(x,R)\Big] \qquad (3.17)$$

which is again a Volterra integral equation.

19. Method of Solution for Analytic Bodies

Discovery of a particular integral for bodies of revolution reduces the second-order problem to the same form as the first-order problem -- namely, the solution of a Volterra integral equation. Various methods of attacking this problem are listed by Hayes (Ref. 10, p. 140). Karman and Moore first solved the integral equation using a step-by-step method, which will be discussed in Section 22. However, another procedure seems preferable if the meridian curve is analytic, for example, if it is given by a polynomial in x. It might be supposed that any shape encountered in practice could be approximated sufficiently well by a polynomial, but it will be seen that this is not practical if the body has discontinuities in slope or curvature.

It will be assumed that the unknown source strength $F(x)$ appearing in the expression (3.11) for the first-order potential can be represented by a few terms of a polynomial, of which three terms will be retained here:

$$F(x) = Ax + Bx^2 + Cx^3 + \ldots \qquad (3.18)$$

The unknown coefficients A, B, C . . . are of order ϵ^2 for pointed shapes, but for quasi-cylindrical bodies may be either $O(\epsilon)$ or $O(\epsilon^2)$. In the case of a pointed body, $A = \epsilon^2$ where ϵ is the tangent of the semi-vertex angle, because the first term alone gives the conical solution discussed in Section 13.

Carrying out the integration in (3.11) and introducing the more convenient conical coordinates (1.14), the first-order perturbation potential becomes

$$\begin{aligned}\phi = &-Ax\left[\operatorname{sech}^{-1}t - \sqrt{1-t^2}\right] - Bx^2\left[(1+\tfrac{1}{2}t^2)\operatorname{sech}^{-1}t - \tfrac{3}{2}\sqrt{1-t^2}\right] \\ &- Cx^3\left[(1+\tfrac{3}{2}t^2)\operatorname{sech}^{-1}t - (\tfrac{11}{6}+\tfrac{2}{3}t^2)\sqrt{1-t^2}\right] - \ldots\end{aligned} \tag{3.19}$$

The functions of t alone which occur inside the brackets are the functions $T_{mo}(t)$, m = 1, 2, 3, . . . introduced in a more formal manner by Hayes (Ref. 10, p. 38), who has discussed their properties in detail. Using equations (1.16), the derivatives of the potential are found to be

$$\begin{aligned}
\phi_x &= -A\operatorname{sech}^{-1}t - 2Bx(\operatorname{sech}^{-1}t - \sqrt{1-t^2}) - 3Cx^2\left[(1+\tfrac{1}{2}t^2)\operatorname{sech}^{-1}t - \tfrac{3}{2}\sqrt{1-t^2}\right] - \ldots \\
\phi_r &= \beta A\frac{\sqrt{1-t^2}}{t} + \beta Bx\left(\frac{\sqrt{1-t^2}}{t} - t\operatorname{sech}^{-1}t\right) + \beta Cx^2\left[(1+2t^2)\frac{\sqrt{1-t^2}}{t} - 3t\operatorname{sech}^{-1}t\right] \\
\phi_{xx} &= -\frac{A}{x}\frac{1}{\sqrt{1-t^2}} - 2B\operatorname{sech}^{-1}t - 6Cx(\operatorname{sech}^{-1}t - \sqrt{1-t^2}) - \ldots \\
\phi_{xr} &= \frac{\beta A}{x}\frac{1}{t\sqrt{1-t^2}} + 2\beta B\frac{\sqrt{1-t^2}}{t} + 3\beta Cx\left(\frac{\sqrt{1-t^2}}{t} - t\operatorname{sech}^{-1}t\right) + \ldots \\
\phi_{rr} &= -\frac{\beta^2 A}{x}\frac{1}{t^2\sqrt{1-t^2}} - \beta^2 B\left(\frac{\sqrt{1-t^2}}{t} + \operatorname{sech}^{-1}t\right) - \beta^2 Cx\left[(1-4t^2)\frac{\sqrt{1-t^2}}{t^2} + 3\operatorname{sech}^{-1}t\right] - \ldots
\end{aligned} \tag{3.20}$$

The constants A, B, C . . . are now determined by imposing the first-order tangency condition (1.18a)

$$\phi_r = [Slope] \tag{3.21}$$

at a corresponding number of points on the surface of the body. The deviation from the tangency at intermediate points can, if desired, be calculated as a check on the approximation. If it is unsatisfactory, additional terms in (3.19) must be used, or the method of Section 22 adopted.

The particular solution for the second approximation is given by (3.15), which has derivatives

$$\begin{aligned} \psi_x &= M^2\left[\phi_{xx}(\phi+Nr\phi_r) + \phi_x(\phi_x+Nr\phi_{xr}) - \tfrac{3}{4}r\phi_{xr}\phi_r^2\right] \\ \psi_r &= M^2\left[\phi_{xr}(\phi+Nr\phi_r) + \phi_x\{(N+1)\phi_r+Nr\phi_{rr}\} - \tfrac{1}{4}\phi_r^2(\phi_r+3r\phi_{rr})\right] \end{aligned} \tag{3.22}$$

The second-order correction potential χ can be represented by a series of the form (3.19), with new constants a, b, c . . . which are $O(\epsilon^4)$ for pointed bodies and $O(\epsilon^2)$ or $O(\epsilon^4)$ for quasi-cylindrical shapes. The first derivatives of χ can be calculated using equations (3.20). Then, just as before, the constants a, b, c . . . are determined by imposing the second-order tangency condition (3.3)

$$\chi_r = (1+\phi_x)[Slope] - \phi_r - \psi_r \tag{3.23}$$

at the points on the surface of the body. Finally, the second-order perturbation velocities are given by

$$\begin{aligned} \frac{u}{U} &= \Phi^{(2)}_x = \phi_x + \psi_x + \chi_x \\ \frac{v}{U} &= \Phi^{(2)}_r = \phi_r + \psi_r + \chi_r \end{aligned} \qquad (3.24)$$

and the pressure coefficient can be calculated from (1.21). Summarizing, the computing procedure is as follows:

1. Choose a suitable number of terms in (3.19), and from (3.20) calculate ϕ_r at a corresponding number of points on the surface in terms of the unknown constants A, B, C . . .

2. Determine the constants so that (3.21) is satisfied at the chosen points. (Note that for a pointed body $A = \epsilon^2$.)

3. At the points on the surface calculate the values of ϕ (3.19) and its six derivatives (3.20).

4. Calculate ψ_x and ψ_r at those points from (3.22).

5. Determine new constants a, b, c . . . for χ given by (3.19) such that (3.23) is satisfied at the surface points.

6. Calculate the velocities from (3.24) and the pressure coefficient from (1.21).

This method of calculating the second-order solution is seen to involve approximately twice as much labor as a careful first-order solution.

20. Use of the Slender-Body Approximation

It was shown by Karman (Ref. 17) that for slender bodies the source strength F(x) appearing in equations (3.11) and

(3.12) is approximately equal to the rate of change of cross-sectional area:

$$F(x) \sim \frac{1}{2\pi}\frac{dS}{dx} = R(x)\,R'(x) \qquad (3.25)$$

Lighthill has shown (Ref. 11) that if R(x) and its first two derivatives are of order ϵ and R' is continuous, then this asymptotic determination of F(x) is correct to the order of terms retained in the first approximation.

For the second approximation, F(x) may be determined in this way only if the body has continuous curvature. This is clear from the example of a semi-infinite body (Fig.

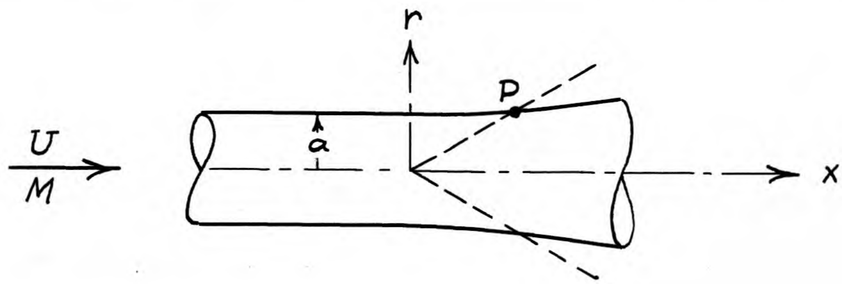

Fig. 3.2. Body with Curvature Discontinuity

3.2), which exhibits the essential features of the limitation. Suppose that the body is represented, according to (3.9b), by

$$r = R(x) = \begin{cases} a & x \le 0 \\ a + \epsilon\rho(x) = a + \epsilon\frac{x^2}{2} & x \ge 0 \end{cases} \qquad (3.26)$$

so that it has a discontinuity in curvature at x = 0.

Then the slender-body approximation (3.25) gives

$$F(x) \sim \begin{cases} 0 & x \leq 0 \\ (a + \epsilon \frac{x^2}{2})\epsilon x \quad = Ax + Cx^3 & x \geq 0 \end{cases} \qquad (3.27)$$

As noted above, the term Ax occuring here yields the conical solution discussed in Section 13. According to equation (2.29), this solution involves a velocity jump across the Mach line from the origin in the second approximation. Clearly no such jump actually occurs at point P in Fig. 3.2. Hence in this case the approximation involved in the slender-body method is too gross for purposes of a second-order solution. If, however, the body has continuous curvature, with discontinuities in R''', only terms with coefficients B and higher appear in (3.27). In this event it can be shown that the second-order pressure distribution remains smooth.

Under these restrictions, the slender-body approximation may prove useful if the meridian curve can be represented (or approximated) by a simple analytic expression. Probably the only practical case is that of a polynomial representation for R(x), in which event the slender-body result can replace step (2) of the procedure outlined in the previous section.

The source strength f(x) for the second-order correction potential χ (3.16) may likewise be determined by the slender-body method under proper restrictions. The result corresponding to (3.25) is (cf. equation 3.17)

$$f(x) \sim R(x)\left[R'(x)\{1+\phi_x(x,R)\} - \phi_r(x,R) - \psi_r(x,R)\right] \tag{3.28}$$

and is applicable if the bracket has continuous second derivative. Again the method is useful only if the right-hand side can be approximated by a polynomial, in which case it replaces step (5) of the procedure outlined above.

21. Series Expansion

If the meridian curve can be represented by an analytic function, the second-order solution for any body of revolution can be expanded in a series of terms of the form

$$t^{2m}\left(\log \frac{2}{t}\right)^2, \qquad t^{2m}\log\frac{2}{t}, \qquad t^{2m} \qquad\qquad m = 0, 1, 2 \ldots$$

with coefficients depending on x. This expansion was carried out for the cone in Section 13. The resultant series converges within the Mach cone. It may be noted that the logarithmic terms arise from the expansion (2.25) of terms in (3.19) which contain $\operatorname{sech}^{-1} t$.

Broderick (Ref. 14) has chosen such an expansion as the starting point for a second-order solution for slender pointed bodies of revolution. The analysis is rather lengthy since the simplification resulting from the discovery of a particular integral does not appear. The results are definitely limited to shapes for which the cross-sectional area is given by an analytic function, or at least possesses continuous derivatives up to a considerable order. Powers of t are retained only up to t^4, so that the solution is accurate to $O(\epsilon^4)$ only near the surface of the body, where $t = O(\epsilon)$. Even on the surface, the results of Section 14 indicate that for bodies of reasonable thickness much of the advantage of a second approximation has been lost by expanding in series.

22. Method of Solution for Non-Analytic Bodies

The previous method of solution is not suitable for bodies of revolution having discontinuities in slope or curvature. The reason for this is that the corresponding pressure distributions have jumps and corners, as indicated in Fig. 3.3. The analytic source strength used in the

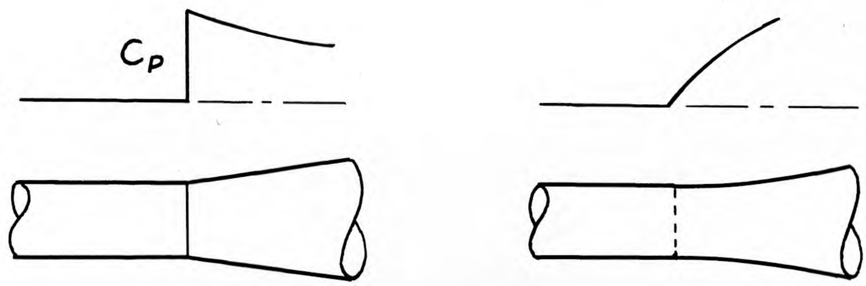

Fig. 3.3. Pressure Distributions Near Discontinuities in Slope and Curvature

previous method yields an analytic pressure distribution. Consequently an impractical number of terms would be required to give a reasonable approximation to such discontinuities.

For these cases, and for complicated shapes for which a few terms of the expansion (3.19) are insufficient, the integral equations (3.13) and (3.17) must be solved numerically using a step-by-step procedure. In first-order theory the usual method, introduced by Karman and Moore (Ref. 18) is to assume that the source distribution can be approximated by a polygonal graph. This is equivalent to superimposing a number of conical source lines of different strengths, each shifted downstream with respect to its predecessor, as indicated in Fig. 3.4. The latter viewpoint is more conven-

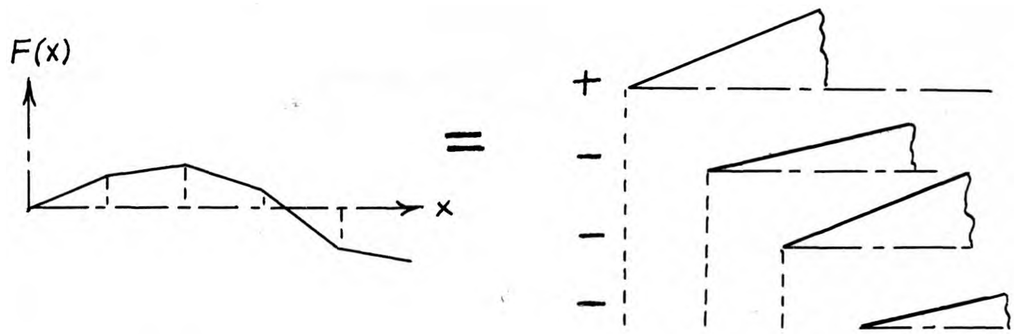

Fig. 3.4. Equivalence of Polygonal Source Strength and Sum of Conical Sources

ient for computation. The strengths of the source lines are determined in succession by satisfying the tangency condition at a series of points on the surface of the body. The details of this procedure are clearly explained in Ref. 1.

For purposes of a second approximation, this procedure must be modified in one respect. Conical source lines alone cannot be superimposed, since it was shown in Section 20 that within a region of continuous curvature they would produce false pressure jumps along their Mach cones. However, the procedure can be carried out using in addition source lines of quadratic strength. These correspond to the term in (3.19) having coefficient B, and it was observed in Section 20 that the corresponding pressure distribution is continuous.

A single source line of this type represents the flow past a slender pointed body with cusped nose (Fig. 3.5),

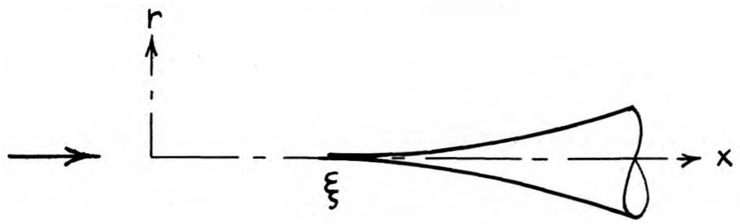

Fig. 3.5. Body Formed by Source Line of Quadratic Strength

as is clear from the slender-body approximation (3.25). If the source line begins at $x = \xi$, the potential and its derivatives are, according to (3.19) and (3.20)

$$
\begin{aligned}
\phi &= -B\,\xi^2\left[(1+\tfrac{1}{2}\tau^2)\,\mathrm{sech}^{-1}\tau - \tfrac{3}{2}\sqrt{1-\tau^2}\right] \\
\phi_x &= -2B\,\xi\,(\mathrm{sech}^{-1}\tau - \sqrt{1-\tau^2}) \\
\phi_r &= \beta B\,\xi\left(\frac{\sqrt{1-\tau^2}}{\tau} - \tau\,\mathrm{sech}^{-1}\tau\right) \\
\phi_{xx} &= -2B\ \mathrm{sech}^{-1}\tau \\
\phi_{xr} &= 2\beta B\,\frac{\sqrt{1-\tau^2}}{\tau} \\
\phi_{rr} &= -\beta B\left(\frac{\sqrt{1-\tau^2}}{\tau} + \mathrm{sech}^{-1}\tau\right)
\end{aligned}
\qquad \text{where } \tau = \frac{\beta r}{x-\xi} \qquad (3.29)
$$

The method of solution is indicated in Fig. 3.6 for bodies having continuous slope. Points ξ_n are chosen

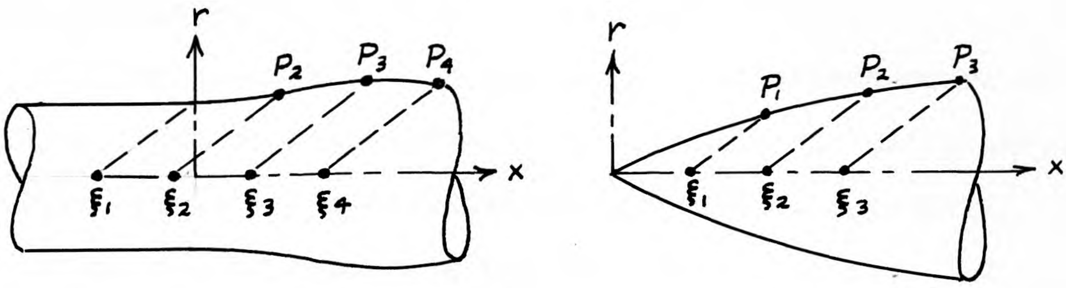

Fig. 3.6. Method of Solution for Non-Analytic Bodies

along the axis, at each of which a quadratic source line is to begin. Calling the potential due to the nth such line ϕ_n , its strength B_n is found by imposing the tangency condition at the point P_{n+1} on the surface, which lies on the Mach line from ξ_{n+1} . For this purpose, the tangency condition (3.21) can be written

$$(\phi_n)_r = [Slope] - \sum_{i=1}^{n-1} (\phi_i)_r \qquad at \quad P_{n+1} \tag{3.30}$$

from which each of the B_n can be found in turn. For a pointed body with finite vertex angle the solution should start with a conical source line, following which the procedure is the same. For a conical source line starting at the origin

$$\begin{aligned} \phi &= -Ax(\operatorname{sech}^{-1}t - \sqrt{1-t^2}) & \phi_{xx} &= -\frac{A}{x}\frac{1}{\sqrt{1-t^2}} \\ \phi_x &= -A \operatorname{sech}^{-1} t & \phi_{xr} &= \frac{\beta A}{x}\frac{1}{t\sqrt{1-t^2}} \\ \phi_r &= \beta A \frac{\sqrt{1-t^2}}{t} & \phi_{rr} &= -\frac{\beta^2 A}{x}\frac{1}{t^2\sqrt{1-t^2}} \end{aligned} \tag{3.31}$$

The strength A can be taken to be ϵ^2, where the semi-vertex angle is $\tan^{-1}\epsilon$, or can be determined from the tangency condition at point P_1 (Fig. 3.6).

The velocities due to the particular second-order solution can then be calculated at the points P_n using equation (3.22). Finally the second-order correction potential χ is determined by repeating the procedure used for ϕ, finding constants b_n such that the second-order tangency condition

$$(\chi_n)_r = (1+\phi_x)[\text{Slope}] - \phi_r - \psi_r - \sum_{i=1}^{n-1}(\chi_i)_r \tag{3.32}$$

is satisfied at the points P_{n+1}. Summarizing, the procedure is the following:

0. If the body has a sharp nose of finite angle, choose a conical potential ϕ_0 given by (3.31), with A determined by the tangency condition near the nose.

1. Divide the axis into intervals by points ξ_n, and locate the points P_n (Fig. 3.6). At each such point calculate $\tau_{mn} = \dfrac{\beta R_n}{x_n - \xi_m}$ and the various functions of τ_{mn} appearing in (3.29).

2. Determine constants B_n in succession so that the tangency condition (3.30) is satisfied.

3. Calculate the contributions of all the components ϕ_n (including ϕ_0 if required) to the potential and its first and second derivatives at the points P_n. Add to obtain the total first-order values.

4. From (3.22) calculate ψ_x and ψ_r at the points P_n.

5. Repeat steps (0) and (2) for the second-order correction potentials χ_o (for a pointed body) and χ_n, involving constants a and b_n which are determined so that (3.32) is satisfied.

6. Calculate the first derivatives of χ_o and of the χ_n at the points P_n. Add to obtain the total χ_x and χ_r.

7. Calculate the second-order velocities from (3.24) and the pressure coefficient from (1.21).

The question of whether this procedure can be applied to bodies having slope discontinuities will be considered in the following sections.

23. Solution Behind Discontinuity in Slope or Higher Derivative

Thus far the second approximation for axially-symmetric flow has been tacitly assumed to be part of some process that converges, so that it gives an improved representation of the non-linear solution. Whether this is actually the case will now be investigated near a discontinuity in the slope or in some higher derivative.

Consider the semi-infinite body which has constant radius a upstream, and is formed by a first-order source distribution along the axis of strength

$$F(x) = C x^k \qquad x \geq 0 \tag{3.33}$$

Clearly the body will depart from a cylinder downstream of the point $x = \beta a$, as indicated in Fig. 3.7.

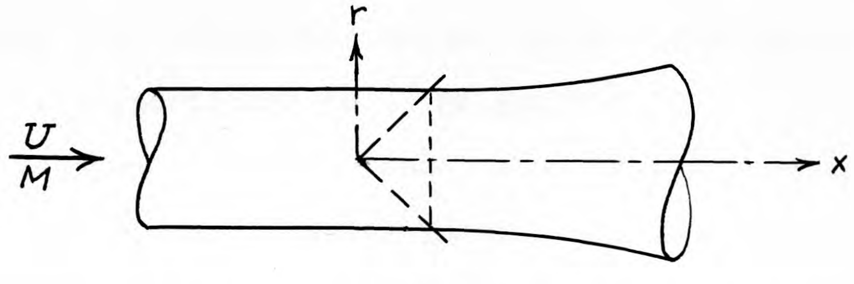

Fig. 3.7. Body with Discontinuity in Some Derivative

According to (3.11), the first-order potential is

$$
\begin{aligned}
\phi(x,r) &= -C\int_0^{x-\beta r}\frac{\xi^k\,d\xi}{\sqrt{(x-\xi)^2-\beta^2 r^2}} \\
&= -C\,\frac{(x-\beta r)^{k+\frac{1}{2}}}{\sqrt{2\beta r}}\int_0^1\frac{(1-s)^k\,ds}{\sqrt{s\left(1+\frac{x-\beta r}{2\beta r}s\right)}}
\end{aligned}
\qquad (3.34)
$$

This integral converges within and on the downstream Mach cone and represents, except for a multiplicative constant, the analytical continuation of the hypergeometric function

$$
F\left(\tfrac{1}{2},\tfrac{1}{2};\,k+\tfrac{3}{2};\,-\frac{x-\beta r}{2\beta r}\right)
$$

(Ref. 19, p. 248). Consequently, for $x < 3\beta r$

$$
\phi(x,r) = -C\sqrt{\frac{\pi}{2\beta}}\,\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}\,\frac{(x-\beta r)^{k+\frac{1}{2}}}{\sqrt{r}}\,F\left(\tfrac{1}{2},\tfrac{1}{2};k+\tfrac{3}{2};-\frac{x-\beta r}{2\beta r}\right) \qquad (3.35)
$$

Differentiating and expanding in series gives

$$
\phi_r(x,r) = C\sqrt{\frac{\pi\beta}{2}}\,\frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})}\,\frac{(x-\beta r)^{k-\frac{1}{2}}}{\sqrt{r}}\left[1+\frac{3}{2(2k+1)}\,\frac{x-\beta r}{2\beta r}-\dots\right] \qquad (3.36)
$$

so that just behind the point $x = \beta a$ the radius of the body is proportional to $(x-\beta a)^{k-\frac{1}{2}}$.

Setting $k = \frac{1}{2}$ and $C = \frac{2\epsilon}{\pi}\sqrt{\frac{2a}{\beta}}$ yields a body which has a discontinuity in slope of magnitude ϵ. Physically it is clear that the flow immediately behind this corner should be identical with that for the plane case (Section 11). From (3.35) the first-order potential and its derivatives are found to be

$$
\begin{aligned}
\phi &= -\frac{\epsilon}{\beta}(x-\beta r)\sqrt{\frac{a}{r}}\left[1 - \frac{1}{8}\,\frac{x-\beta r}{2\beta r} + \ldots\right] \\
\phi_x &= -\frac{\epsilon}{\beta}\sqrt{\frac{a}{r}}\left[1 - \frac{1}{4}\,\frac{x-\beta r}{2\beta r} + \ldots\right] \\
\phi_r &= \epsilon\sqrt{\frac{a}{r}}\left[1 + \frac{3}{4}\,\frac{x-\beta r}{2\beta r} - \ldots\right] \\
\phi_{xx} &= \frac{\epsilon}{8\beta^2 r}\sqrt{\frac{a}{r}}\left[1 - \frac{9}{8}\,\frac{x-\beta r}{2\beta r} + \ldots\right] \\
\phi_{xr} &= \frac{3\epsilon}{8\beta r}\sqrt{\frac{a}{r}}\left[1 - \frac{5}{8}\,\frac{x-\beta r}{2\beta r} + \ldots\right] \\
\phi_{rr} &= -\frac{7}{8}\frac{\epsilon}{r}\sqrt{\frac{a}{r}}\left[1 + \frac{57}{56}\,\frac{x-\beta r}{2\beta r} - \ldots\right]
\end{aligned}
\qquad (3.37)
$$

At $r = a$ as $(x-\beta r)$ approaches zero, the first-order velocities ϕ_x and ϕ_r approach the values for plane flow past a corner.

The velocities due to the particular second-order solution are calculated from (3.22), which gives

$$
\begin{aligned}
\psi_x &= \left(1-\frac{N}{4}\right)M^2\frac{\epsilon^2}{\beta^2} + \ldots \\
\psi_r &= -\left(1-\frac{N}{4}\right)M^2\frac{\epsilon^2}{\beta} + \ldots
\end{aligned}
\qquad (3.38)
$$

The second of these is $-\beta$ times the first, which means that the velocity perturbation due to ψ is normal to the original Mach wave. The important consequence of this fact is that imposing the second-order tangency condition causes all terms containing N to cancel, since from (3.37) the velocity due to the second-order correction potential χ is also normal to the Mach wave just behind the corner. Hence the result is incorrect; the second-order solution breaks down immediately behind a corner.

Exactly the same result is found in the same way for a discontinuity in curvature or in any higher derivative. Terms involving N drop out, so that the solution is incorrect just behind the discontinuity. Mathematically, the iteration can converge only for an analytic body. Whether the solution is actually useless for other shapes will be considered in the next section.

24. Comparison with Numerical Solutions

It has been seen that the iteration process for bodies of revolution fails immediately behind a discontinuity in a derivative of any order. Yet for practical purposes it may be that the solution thereafter approaches the proper form so rapidly that the local failure is unimportant. Whether this is the case can apparently be determined only by comparison with exact solutions.

Fortunately, several numerical solutions are available which are ideal for this purpose. A number of cases of axially-symmetric flow have been solved by the method of characteristics at the Douglas Aircraft Company, Santa Monica, Calif. (Ref. 20). The solutions were carried out in unusual detail, and the effect upon the accuracy of varying the lattice size was investigated. The bodies considered are sufficiently slender that good agreement with perturbation methods may be expected.

The first solution is that for a slender ogive to which a conical tip has been affixed, so that a discontinuity in curvature occurs at the point of tangency. Fig. 3.8 shows the shape of the body, the pressure distribution obtained by the method of characteristics, and the second-order result calculated by the method of Section 22. The source lines used for the calculation are indicated by drawing the Mach wave from the front of each. The first-order solution is also shown for comparison.

Immediately behind the discontinuity in curvature, excellent agreement is found between the second approximation and the numerical solution. Evidently the local failure of the iteration process which was discussed in Section 23 is of no practical importance in this case.

Farther back, the second-order pressure distribution lies below the numerical result. Liepmann and Lapin (Ref. 20) have pointed out that the characteristics solution approaches the correct solution from one direction only as the lattice size is reduced. In this case the jump in pres-

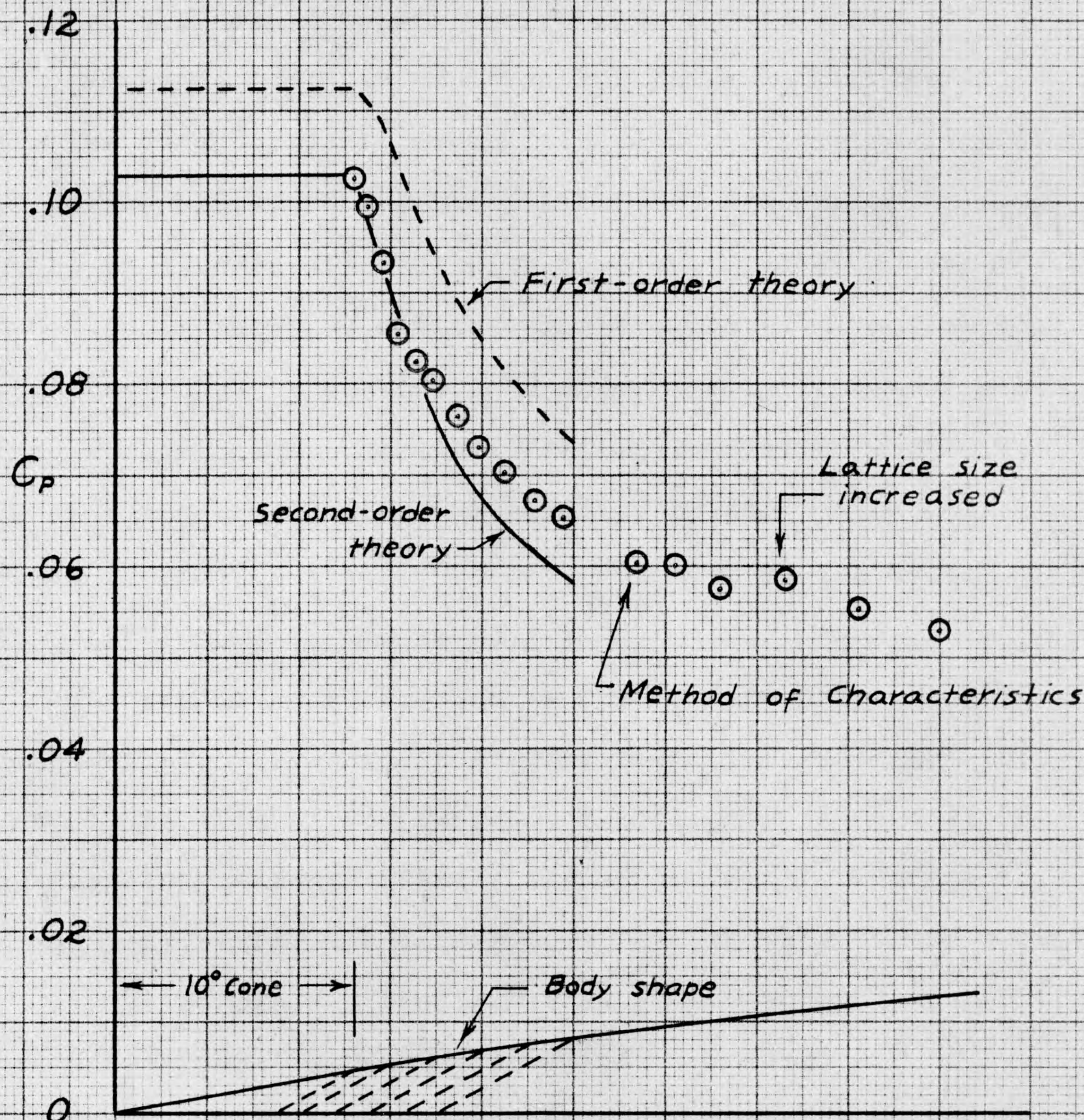

Fig. 3.8. Pressure Distribution on Body of Revolution with Discontinuity in Curvature, M = 2.075

sure at the point where the lattice size was increased (Fig. 3.8) indicates that the true solution lies below. Consequently, the second approximation agrees more closely with the exact solution than would appear from the figure.

The second body to be considered consists of a cone of ten degree semi-vertex angle followed by a circular cylinder, and so involves a discontinuity in slope. Fig. 3.9 shows the shape of the body and the pressure distributions obtained from first-order theory, second-order theory, and the method of characteristics. For the first approximation the solution for the cone (2.20) was modified by adding the solution (3.34) to produce a sharp corner, followed by the usual superposition of solutions (3.29).

The second approximation is seen to lie nearer the characteristics solution than does the first-order result. However, in view of the results of Section 23 this must be regarded as accidental, since the second-order solution fails immediately behind the corner. Any superiority of the second approximation lies in the fact that it subsequently runs more nearly parallel to the numerical solution.

This observation suggests that the second approximation may perhaps be used for bodies with corners provided that the resulting pressure distribution is shifted vertically after each corner to give the two-dimensional jump. This suggestion is quite tentative, and must be investigated more carefully before it can be considered sound.

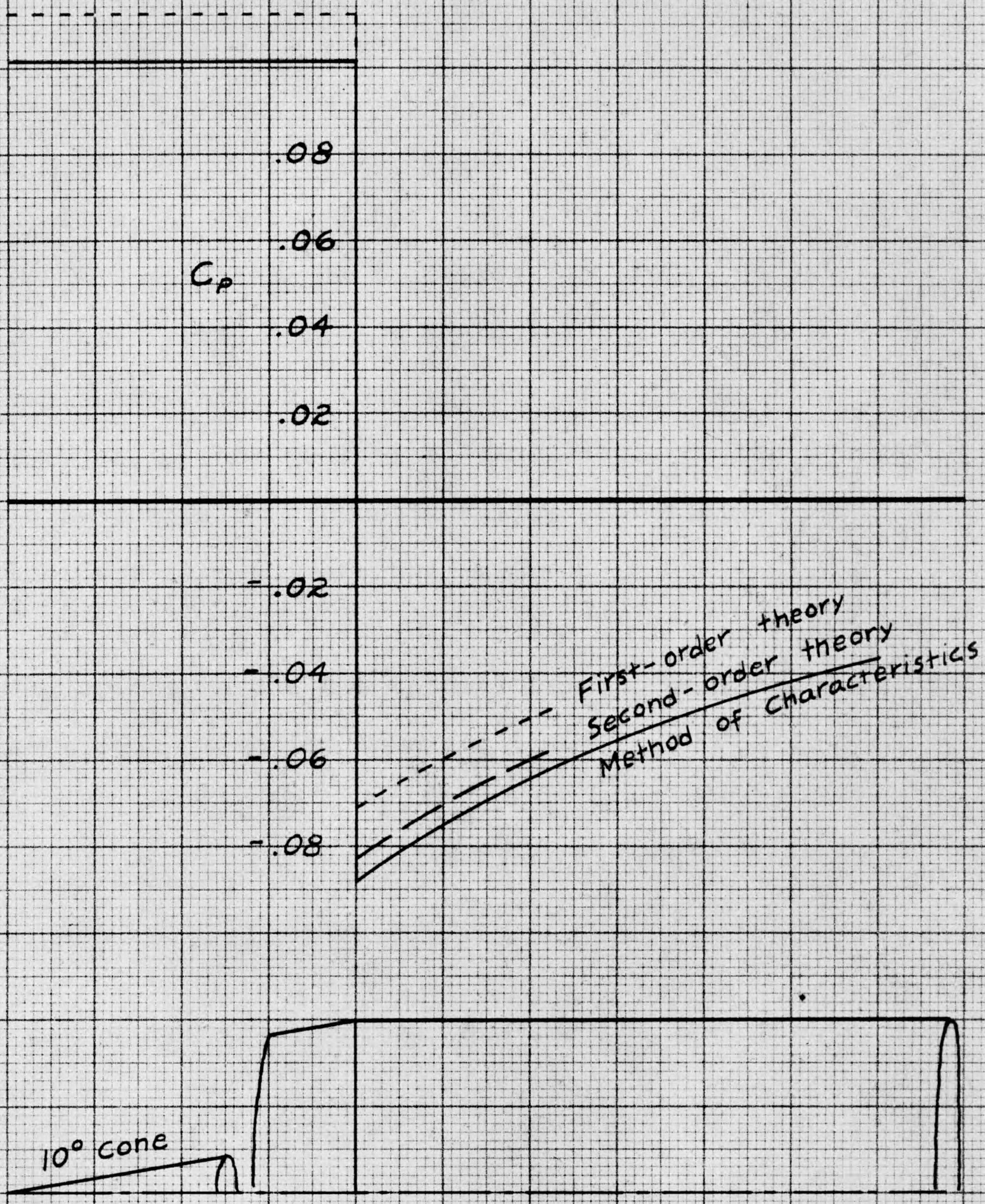

Fig. 3.9. Pressure Distribution on Cone-Cylinder Combination, M = 2.075.

IV. Three-Dimensional Problems

25. A Partial Particular Solution

It might be hoped that a particular solution, which so greatly simplifies the iteration for plane and axially-symmetric flows, could be found for the general three-dimensional case. The various methods of existing first-order theory could then be applied immediately to the problems of second-order flow past such shapes as bodies at angle of attack and three-dimensional wings.

A part of the particular solution is found at once, being common to the two special cases. Consider the three-dimensional iteration equation (1.12)

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} 2(N-1)\beta^2\Phi_{xx}\Phi_x + 2\Phi_{xy}\Phi_y + 2\Phi_{xz}\Phi_z \\ + \Phi_{yy}\Phi_y^2 + 2\Phi_{yz}\Phi_y\Phi_z + \Phi_{zz}\Phi_z^2 \end{bmatrix} \qquad (4.1)$$

It can be readily verified that for the last two terms in the first line, which do not involve N, a particular solution is given by

$$\psi = M^2\Phi\,\Phi_x \qquad (4.2)$$

which appears in both (3.7a) and (3.15).

The triple products in the last line of (4.1) are negligible in certain problems, and otherwise could probably be handled approximately, which is all that is necessary. In any event, however, an additional particular solution must be found for

$$\varphi_{yy} + \varphi_{zz} - \beta^2 \varphi_{xx} = 2M^2\beta^2 N \, \phi_{xx}\phi_x \qquad (4.3)$$

It has not been possible to find a particular solution of this equation in terms of the first-order potential. The solutions for plane and axially-symmetric flow do not appear to suggest a generalization. On the other hand, there is no assurance that such a solution cannot be found, so that one is tempted to search further.

The right-hand side of (4.3) vanishes if $\gamma = -1$, so that N = 0. However, investigation of the previous solutions indicates that the idea of here taking $\gamma = -1$ is not legitimate.

In the absence of a complete particular integral, the remaining non-homogeneous equation must be attacked by more conventional methods. In principle, it is always possible to find a particular solution of a linear non-homogeneous equation with the aid of the fundamental solution associated with the differential operator. For the three-dimensional wave operator which occurs here, the fundamental solution is

$$\frac{1}{\sqrt{(x-\xi)^2 - \beta^2[(y-\eta)^2 - (z-\zeta)^2]}} \qquad (4.4)$$

which can be interpreted as the potential at any point (x,y,z) lying inside the downstream Mach cone from a unit supersonic source at (ξ, η, ζ). With the aid of Green's formula, it can be shown that a particular solution of

$$\varphi_{yy} + \varphi_{zz} - \beta^2 \varphi_{xx} = F(x, y, z)$$

is given by

$$\psi(x,y,z) = \iiint \frac{F(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 - \beta^2\left[(y-\eta)^2 + (z-\zeta)^2\right]}} \qquad (4.5)$$

where the integration extends throughout that portion of the forward Mach cone from the point (x,y,z) within which F is defined.

In practice, the integration of (4.5) is generally not feasible. For example, even the simplification of axial symmetry reduces (4.5) only to a double integral of F multiplied by an elliptic integral of complicated argument. Avoiding such integrals by discovery of the particular solution clearly represents a great simplification in this case.*

In the following sections, one example of a three-dimensional solution will be given, and the possibility of treating other shapes will be discussed.

---

*Comparing the two methods would lead to the evaluation of definite integrals involving complete elliptic integrals, which might be of some interest.

26. The Cone at an Angle

The problem of a cone at an angle of attack illustrates the use of separation of variables to reduce the iteration equation to tractable form.

Two alternative coordinate systems are suitable for bodies of revolution at an angle. In wind axes the body is inclined, while in body axes the stream impinges obliquely. The latter system is simpler for first-order problems, and is probably better for the second approximation also. However, wind axes will be used here, since otherwise the iteration equations of Chapter I must be re-derived.

To facilitate imposing the tangency condition, it is convenient to apply an oblique transformation (see, for example, Ref. 21, p. 18). This effectively unyaws the axis of the body (but distorts the surface) while leaving the wave

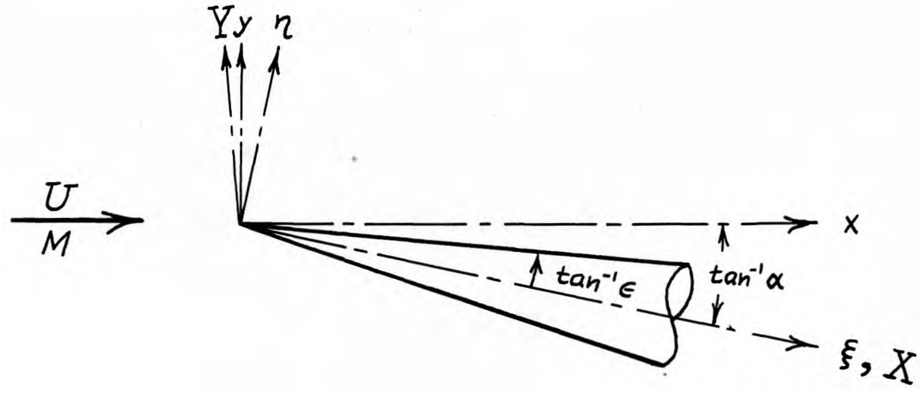

Fig. 4.1. Coordinate Systems for Cone at an Angle

operator unchanged. Thus three different coordinate systems are required:

Wind Axes: x, y, z

Body Axes: ξ, η, ζ $\qquad \xi, \rho, \theta$ $\qquad \xi, \tau, \theta$

Oblique Axes: X, Y, Z $\qquad X, R, \Theta$ $\qquad X, T, \Theta$

the latter two being used also in cylindrical and conical form. The three systems are related according to the following table

$$z = \zeta = Z$$

$\xi = \dfrac{x-\alpha y}{\sqrt{1+\alpha^2}}$	$x = \dfrac{\xi+\alpha\eta}{\sqrt{1+\alpha^2}}$	$X = \dfrac{x+\beta^2\alpha y}{\sqrt{1-\beta^2\alpha^2}}$	$x = \dfrac{X-\beta^2\alpha Y}{\sqrt{1-\beta^2\alpha^2}}$	$\xi = \dfrac{(1+\alpha^2)X-\alpha M^2 Y}{\sqrt{(1+\alpha^2)(1-\beta^2\alpha^2)}}$	$X = \dfrac{(1-\beta^2\alpha^2)\xi+\alpha M^2\eta}{\sqrt{(1+\alpha^2)(1-\beta^2\alpha^2)}}$
$\eta = \dfrac{y+\alpha x}{\sqrt{1+\alpha^2}}$	$y = \dfrac{\eta-\alpha\xi}{\sqrt{1+\alpha^2}}$	$Y = \dfrac{y+\alpha x}{\sqrt{1-\beta^2\alpha^2}}$	$y = \dfrac{Y-\alpha X}{\sqrt{1-\beta^2\alpha^2}}$	$\eta = \sqrt{\dfrac{1-\beta^2\alpha^2}{1+\alpha^2}}\, Y$	$Y = \sqrt{\dfrac{1+\alpha^2}{1-\beta^2\alpha^2}}\, \eta$

Now to simplify the solution, it will be assumed that the angle of attack is so small that the square of α can be neglected. This will give a solution non-linear in the body thickness but linear in α, and will therefore yield the correct initial slope of the lift curve. Non-linear terms in α can be retained at the expense of algebraic complication. The above table reduces to

$$z = \zeta = Z$$

$\xi = x-\alpha y$	$x = \xi+\alpha\eta$	$X = x+\beta^2\alpha y$	$x = X-\beta^2\alpha Y$	$\xi = X-\alpha M^2 Y$	$X = \xi+\alpha M^2\eta$
$\eta = y+\alpha x$	$y = \eta-\alpha\xi$	$Y = y+\alpha x$	$y = Y-\alpha X$	$\eta = Y$	$Y = \eta$

To this approximation

$$
\begin{aligned}
R &= \rho \\
T &= \tau\left(1 - \alpha \frac{M^2}{\beta} \tau \cos\theta\right) \\
\Theta &= \theta
\end{aligned}
\tag{4.6}
$$

the surface of the cone is

$$
\begin{aligned}
\tau &= \beta\epsilon \\
T &= \beta\epsilon - \beta M^2 \epsilon^2 \alpha \cos\theta
\end{aligned}
\tag{4.7}
$$

and the velocity components are related by

$$
\begin{aligned}
\phi_\xi &= \bar{\Phi} - T\bar{\Phi}_T \\
\phi_\rho &= \beta\bar{\Phi}_T + \alpha M^2(\bar{\Phi} - T\bar{\Phi}_T)\cos\theta \\
\frac{\phi_\theta}{\rho} &= \frac{\bar{\Phi}_\theta}{T} - \beta\alpha M^2(\bar{\Phi} - T\bar{\Phi}_T)\sin\theta
\end{aligned}
\tag{4.8}
$$

where, as in (1.15), the conical potential is introduced by

$$
\Phi(X, R, \Theta) = X\,\bar{\Phi}(T, \Theta) \qquad \text{where} \quad T = \frac{\beta R}{X}
\tag{4.9}
$$

The first-order problem, referred to oblique coordinates, is found to be

$$
\begin{aligned}
(1-T^2)\Phi_{TT} + \frac{\Phi_T}{T} + \frac{\Phi_{\Theta\Theta}}{T^2} &= 0 \\
\beta\,\Phi_T(\beta\epsilon, \Theta) &= (\epsilon - \alpha\cos\Theta) \\
\Phi(\infty, \Theta) = \Phi_T(\infty, \Theta) &= 0
\end{aligned}
\tag{4.10}
$$

where the bars denoting conical potentials have been dropped. The solution is the sum of potentials for a conical line source and dipole (Ref. 1, p. 74):

$$\phi(T,\Theta) = -\epsilon^2\left[\left(\operatorname{sech}^{-1}T - \sqrt{1-T^2}\right) - \beta\alpha\cos\Theta\left(\frac{\sqrt{1-T^2}}{T} - T\operatorname{sech}^{-1}T\right)\right] \tag{4.11}$$

After considerable transformation of coordinates, starting for example from equation (1.12), the iteration equation in oblique coordinates can be shown to be

$$(1-T^2)\varphi_{TT} + \frac{\varphi_T}{T} + \frac{\varphi_{\Theta\Theta}}{T^2} = M^2\left[\begin{aligned}&2(N-1)\left[T^2\phi_{TT}(\phi - T\phi_T) + \beta\alpha\cos\Theta\, T\phi_{TT}(3T\phi_T - 2\phi)\right]\\ &-2\left[T\phi_{TT}\phi_T - \beta\alpha\cos\Theta\,\phi_{TT}\left\{(1+T^2)\phi_T - T(\phi - T\phi_T)\right\}\right]\\ &+\beta^2\phi_{TT}\phi_T^2 + \ldots\end{aligned}\right. \tag{4.12}$$

Substituting (4.11) into the right-hand side gives

$$(1-T^2)\varphi_{TT} + \frac{\varphi_T}{T} + \frac{\varphi_{\Theta\Theta}}{T^2} = \epsilon^4 M^2\left[\begin{aligned}&2(N-1)\frac{\operatorname{sech}^{-1}T}{\sqrt{1-T^2}} + 2\frac{1}{T^2} - \beta^2\epsilon^2\frac{\sqrt{1-T^2}}{T^4}\\ &-\beta\alpha\cos\Theta\left[8\frac{1}{T^3} + 2(3N-2)\frac{1}{T} + \right.\\ &\left.\qquad + 4(2N-1)\frac{\operatorname{sech}^{-1}T}{T\sqrt{1-T^2}} - 4\beta^2\epsilon^2\frac{\sqrt{1-T^2}}{T^5}\right]\end{aligned}\right. \tag{4.13}$$

This is reduced to two total differential equations by setting

$$\varphi(T,\Theta) = \varphi^{I}(T) + \alpha\cos\Theta\,\varphi^{II}(T) \tag{4.14}$$

The equation for φ^{I} is identical with that solved previously for the cone at zero angle (2.21a), so that φ^{I} is given by (2.22). The equation for φ^{II} is

$$(1-T^2)\varphi^{II}_{TT} + \frac{\varphi^{II}_T}{T} - \frac{\varphi^{II}}{T^2} = -\epsilon^4\beta M^2\left[\begin{array}{l} 4(2N-1)\frac{\operatorname{sech}^{-1}T}{T\sqrt{1-T^2}} + 8\frac{1}{T^3} \\ +2(3N-2)\frac{1}{T} - 4\beta^2\epsilon^2\frac{\sqrt{1-T^2}}{T^5}\end{array}\right. \tag{4.15}$$

Setting

$$\varphi^{II}(T) = T\,\omega(T) \tag{4.16}$$

reduces this to a linear first-order equation in ω_T, which can be integrated as (2.21a) was to give

$$\varphi^{II} = \epsilon^4\beta M^2\left[\begin{array}{l} D\left(\frac{\sqrt{1-T^2}}{T} - T\operatorname{sech}^{-1}T\right) + (5N+3)\frac{1}{T} - 4\,\frac{\sqrt{1-T^2}\operatorname{sech}^{-1}T}{T} \\ +(2N+1)\,T(\operatorname{sech}^{-1}T)^2 + \frac{1}{2}\beta^2\epsilon^2\,\frac{\sqrt{1-T^2}}{T^3}\end{array}\right. \tag{4.17}$$

The constants C in (2.22) and D in (4.17) are determined by the tangency condition that on the surface of the cone

$$\Phi^{(2)}_\rho = \epsilon(1+\phi_\xi) - \alpha\cos\theta \tag{4.18}$$

Using (4.6) and (4.11), and expressing values of functions on the cone in terms of their values at $t=\beta\epsilon$ by means of Taylor expansions, this can be reduced to the two conditions

$$\varphi_T^{I}(\beta\epsilon) = \frac{\epsilon}{\beta}(1-\epsilon^2 \operatorname{sech}^{-1}\beta\epsilon) - \frac{\epsilon}{\beta}\sqrt{1-\beta^2\epsilon^2}$$

$$\varphi_T^{II}(\beta\epsilon) = \frac{1}{\beta}\left[\sqrt{1-\beta^2\epsilon^2} - 1 - M^2\frac{\epsilon^2}{\sqrt{1-\beta^2\epsilon^2}} - \epsilon^2 \operatorname{sech}^{-1}\beta\epsilon\right] \tag{4.19}$$

The first of these is the same as (2.21b) for the cone at zero angle, so that φ^{I} is completely independent of α to first order. The constant D can be determined from the second of (4.19).

27. Series for Surface Pressure Coefficient

The solution can be expanded in powers of t and $\log\frac{2}{t}$, as was done for the unpitched cone in Section 14. The constant D is then found to be

$$D = \frac{2M^2+1}{M^2}\log\frac{2}{\beta\epsilon} - \left(5N-1+\frac{5}{2M^2}\right) + \ldots \tag{4.20}$$

Then calculating the velocity components from (4.8) and the surface pressure coefficient from (1.23) gives

$$C_p = C_{p_0} - 4\epsilon\alpha\cos\theta\left[1-\epsilon^2\left(M^2\log\frac{2}{\beta\epsilon} - \frac{3}{2}M^2+1\right)\right] \tag{4.21}$$

where C_{p_0} is the value for the cone at zero angle of attack, given by (2.28). Integrating gives the normal force coefficient, based on cross-sectional area:

$$C_N \equiv \frac{\text{Normal Force}}{\frac{1}{2}\rho_0 U^2\,(\text{Area})} = 2\alpha\left[1-\epsilon^2\left(M^2\log\frac{2}{\beta\epsilon} - \frac{3}{2}M^2+1\right)\right] \tag{4.22}$$

This result has also been obtained by Lighthill (Ref. 22), who extended Broderick's method of solution by series (see Section 21) to bodies of revolution at an angle of attack. He also retains terms in α^3.

Stone (Ref. 23) has developed a theory for cones at an angle which is linearized with respect to α, but otherwise exact. Kopal (Ref. 24) has published tables of the numerical results of this theory. A comparison of equation (4.22) with this exact theory and with the first-order solution (Ref. 25) is shown in Fig. 4.2 for five and ten degree cones. The results of Section 14 suggest that the agreement might improve if the solution were not expanded in series.

28. The Shock Wave Position

If the solution were valid at the Mach cone, the velocity components there would be, from (2.22) and (4.17)

$$-\left(\frac{u}{U}\right) = \frac{1}{\beta}\left(\frac{v}{U}\right) \overset{(?)}{=} \epsilon^4 M^2\left[2N-3(3N-1)\beta\alpha\cos\theta\right] \tag{4.23}$$

Comparing with equations (2.29) and (2.32a), it is found that the deflection of the shock wave away from the Mach cone would be

$$\theta_{sw} - \sin^{-1}\frac{1}{M} \overset{(?)}{=} \frac{(\gamma+1)^2 M^6}{4\beta^3}\epsilon^4 - \frac{3}{4}(\gamma+1)(3N-1)M^4\epsilon^4\alpha\cos\theta \tag{4.24}$$

Hence the ratio of the angular rotation of the shock wave to that of the cone would be

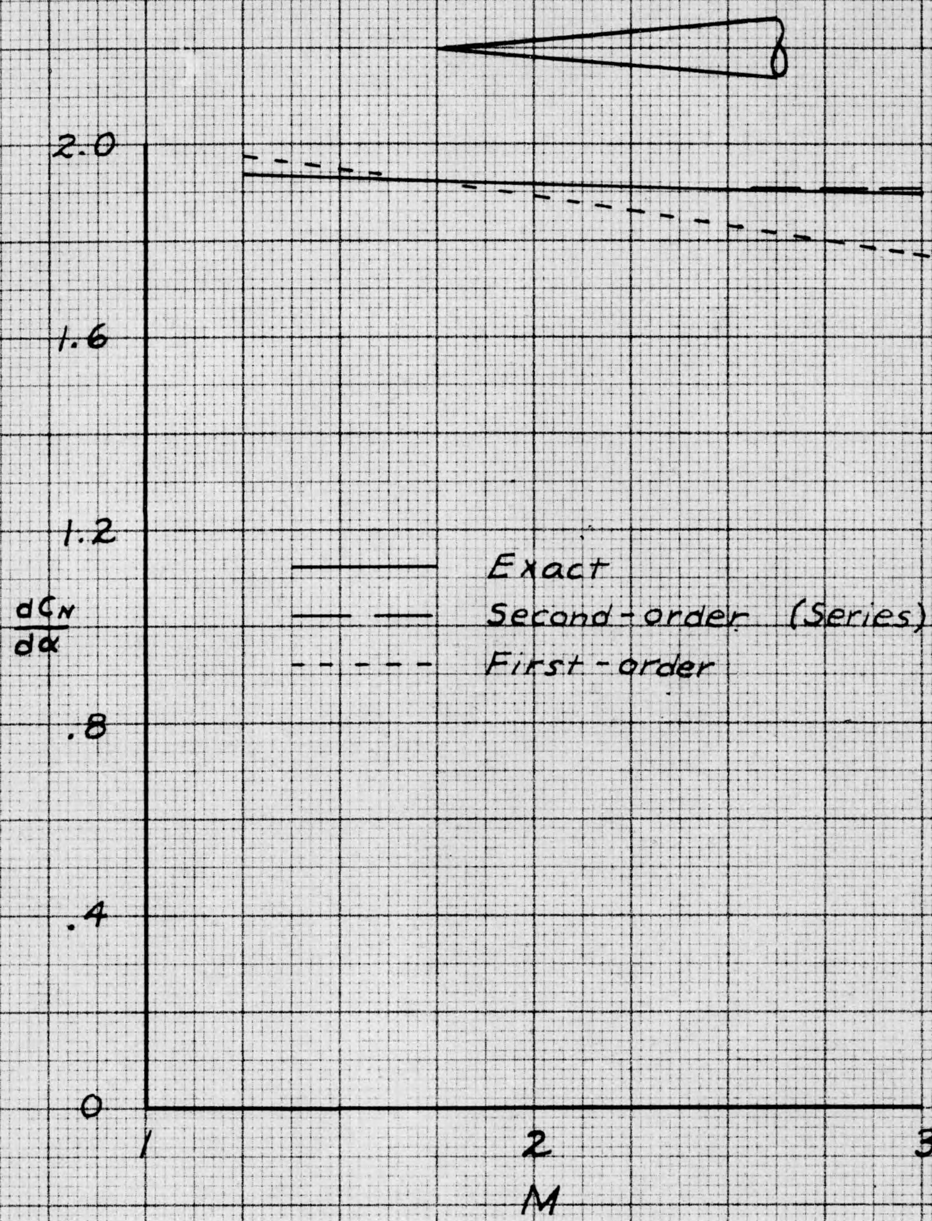

Fig. 4.2. Normal Force Slope for Cone.

(a) 5° Semi-vertex Angle

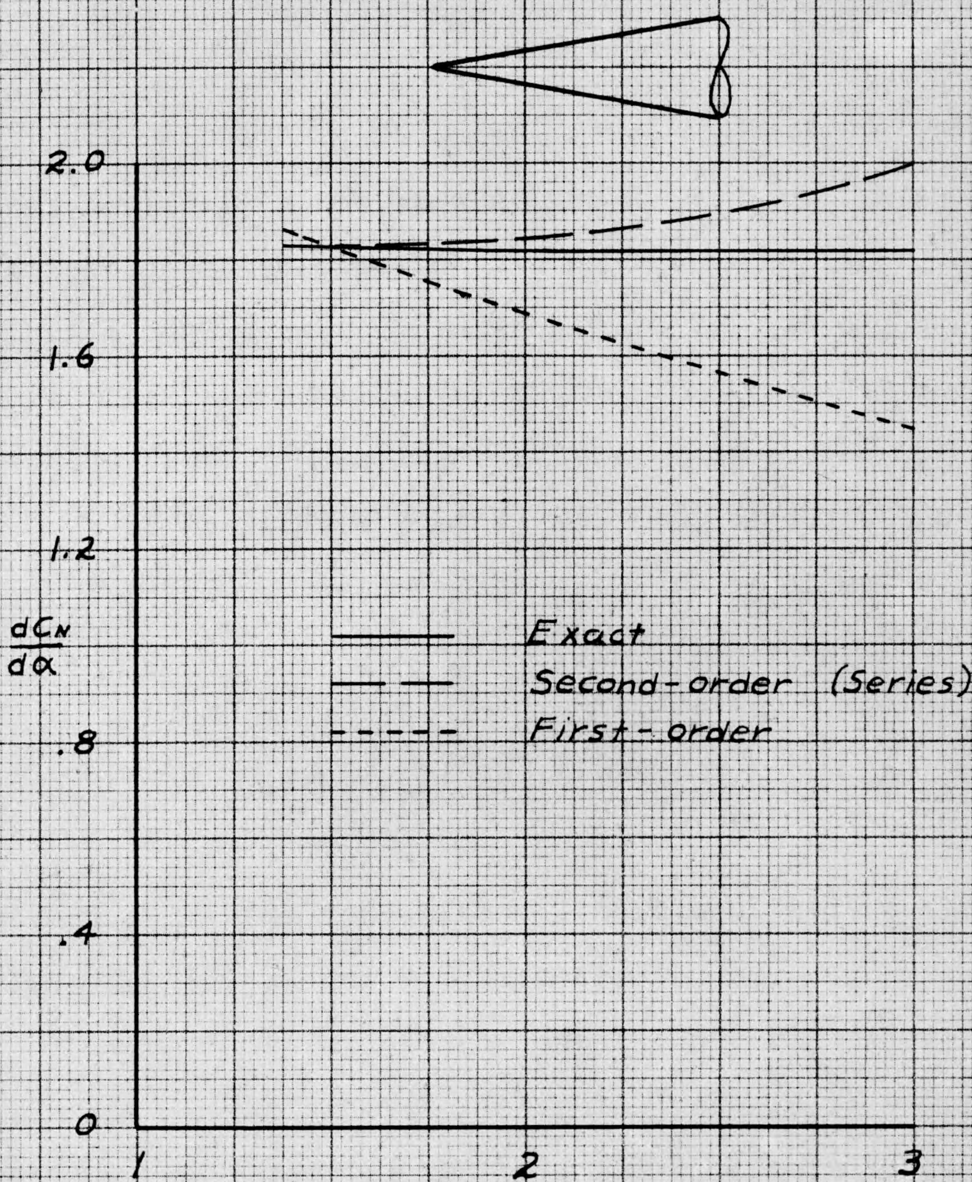

Fig. 4.2. Concluded.

(b) 10° Semi-vertex Angle

$$\frac{\delta}{\alpha} \overset{(?)}{=} \frac{3}{2}\beta^2 M^2 N(3N-1)\,\epsilon^4 \qquad (4.25a)$$

It was seen in Section 14 that although the solution does not in fact converge at the Mach cone, the shock wave deflection calculated in this way is correct for the unpitched cone except for a factor of $1\frac{1}{2}$. It might be supposed that the same factor would correct the second term in (4.24). Kopal (Ref. 24) tabulates values of $\frac{\delta}{\alpha}$ calculated from Stone's theory, and from these it appears that a factor of 3, rather than $1\frac{1}{2}$, is required, so that actually

$$\frac{\delta}{\alpha} = \frac{9}{2}\beta^2 M^2 N\,(3N-1)\,\epsilon^4 \qquad (4.25b)$$

Fig. 4.3 shows a comparison of this modified result with the exact values for a five degree cone.

It must be emphasized that (4.25b) represents nothing more than a conjecture. It could probably be verified, however, by extending the solution of Lighthill (Ref. 15) or Broderick (Ref. 16) to the case of angle of attack.

29. Possible Treatment of Wings

Undoubtedly the most useful application of first-order theory is to thin flat wings. No attempt has so far been made to find the second-order solution for a wing. It seems

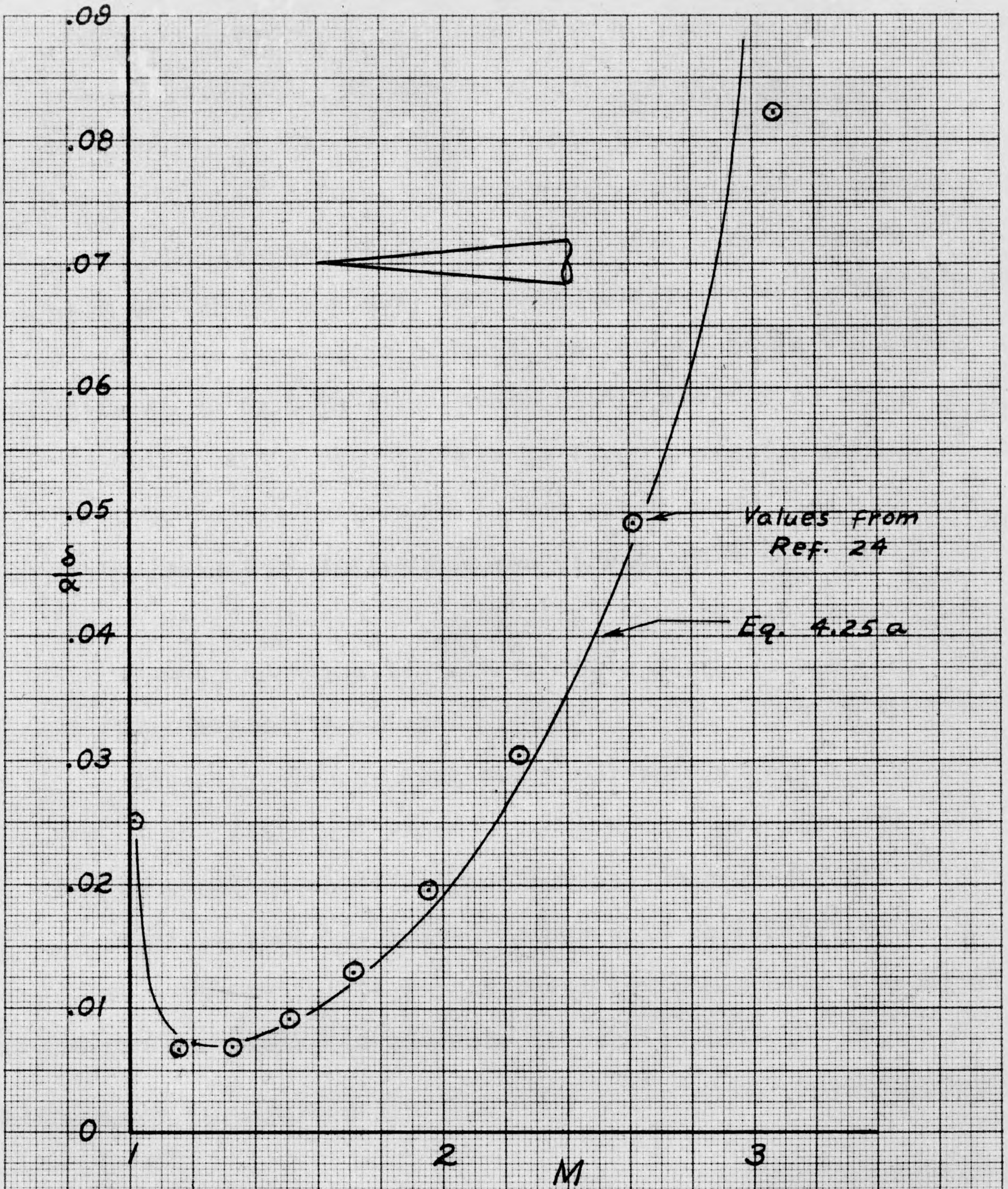

Fig. 4.3. Ratio of Shock Wave Rotation to Angle of Attack for 5° Cone.

likely, however, that solutions can be found at least for conical problems. In this case the iteration equation can be reduced, by the standard conical theory (Ref. 8, 21) to the problem of solving Poisson's equation inside a circle.

Two difficulties can be anticipated. First, if the wing has subsonic edges, infinite velocities arise there, so that the assumption of small perturbations is violated. It is known that in first-order theory this is no essential objection, since the pressure is found correctly except in the immediate neighborhood of the singularity, and the integrated values of lift and moment are correct to first order. Kaplan has indicated (Ref. 5c) that this result extends to the second approximation for subsonic flow, so that probably no real difficulty exists.

Secondly, if the wing has supersonic edges, the failure of the iteration process along Mach lines from the apex can be expected to affect the surface pressures. Again it is possible that integrated values will be correct to second order. Otherwise, it may be possible to adjust the solution in those regions, as was done in Section 11.

V. Concluding Remarks

30. Future Investigation

Two large classes of problems which have only been touched upon deserve further study. One of these is wings, the other bodies of revolution at an angle of attack. The example of the cone at an angle (Section 26) was undoubtedly made awkward by the use of wind coordinates. The iteration equation should be re-derived in body coordinates, and the solution extended to general bodies of revolution. It is possible that in this form a particular integral could be discovered. That there is good possibility of success with this problem is suggested by the fact that Lighthill was able to obtain a general solution by assuming a series expansion (Ref. 22).

The possibility of discovering particular integrals of the iteration equation might be investigated more systematically. If none can be found for general three-dimensional flow, special cases such as conical flow should be studied.

31. Higher Approximations

It seems unlikely that a third or higher approximation would be justified. Other neglected factors, chiefly viscosity and heat conduction, should certainly be considered first. However, the Busemann second-order result has been extended to fourth order (Ref. 26), and various writers have considered the third approximation for plane subsonic flow (Ref. 3b, 4b, 5). If a third approximation should be considered worthwhile, the iteration could be repeated. Again the cases of flow past a curved wall and a cone would serve as helpful examples.

32. Application to Subsonic Flow

The iteration equation and the particular integrals are in no way restricted to supersonic flow. The particular integral for plane flow might profitably be compared with the subsonic solutions of Refs. 3, 4, and 5.

The particular solution for axially-symmetric flow makes possible a second-order solution for bodies of revolution at subsonic speed. In this case, the integral equation cannot be solved step by step, but can be treated by the methods used for the airship problem.

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