A STUDY OF SOME TWO-DIMENSIONAL FIELD THEORY MODELS

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ABSTRACT

Recently it has been pointed out[(6)] that quantum electrodynamics with a massless fermion in one-space, one-time dimension exhibits behavior which can be interpreted as being analogous to scaling and confinement. To learn more about the occurrence of such behavior, we study several other two-dimensional quantum field theories. After reviewing the Thirring model, we construct operator solutions of scale-invariant generalizations with internal degrees of freedom. We find that the physics of these models can be equally well described in terms of a field theory of fermions or of bosons. Because the physical excitations are massless in these models, the physical states can also be described, in general, as many-fermion or many-boson states.

We also generalize Lowenstein and Swieca's operator solution of quantum electrodynamics in two-dimensions to a model with more than one massless, charged fermion. Although the resulting physical states are all electrically neutral, we find that the fermions may not be completely confined in the sense that some of their quantum numbers may be represented by particles in the spectrum. Once again, we discover that the physics can be represented by a fermion field theory or by a boson theory.

The fact that the same physics can be represented by interacting fermions or interacting bosons can be understood by considering the polarization fields associated with the fermion charge and axial charge. This general physical picture allows us to extend the fermion-boson duality to theories with non-trivial scattering. Although these models cannot be solved, we can use formal operator structures to test, in a semi-classical approximation, for the possible existence of charged fermion states in the spectrum. Because the fermion states are equivalent to coherent states of bosons, their physics is approximated by solutions to the appropriate classical boson theory. Therefore, in two dimensions, confinement may be a phenomenon associated with the classical behavior of a theory.

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I. INTRODUCTION

Although quantum field theory has been an accepted and much-used tool in the study of the physics of elementary particles for several decades now, the detailed structure of exact solutions of quantum theories with interacting fields continues to elude the many onslaughts of both mathematicians and physicists. The most phenomenologically accurate attempts to probe such structure have involved the use of perturbation theory, particularly in the study of quantum electrodynamics. The essence of the perturbation approach is the iterative solution of the equations of motions for the interacting quantum fields by expansion in terms of a set of free (non-interacting) fields. If the numerical values of the parameters involved in the expansion are sufficiently small, we can obtain a rather precise approximation to the exact solutions by truncating the series at an appropriate term. It is obvious that perturbation theory is of great value in physical situations where the exact solution bears a close resemblence to a collection of free fields. Thus, in quantum electrodynamics, where we have experimentally observed that an electron "acts like" a free electron and a photon "acts like" a non-interacting quantum at low energies, the numerical accuracy of low order perturbation theory is remarkable.

On the other hand, the experimental study of hadrons, strongly-interacting particles, has not encouraged the use of perturbation theory in theoretical calculations. Only in a few limited situations do these hadrons propagate or interact in a fashion which can be modelled by small perturbations of free field theories. In fact, as the energies involved in accelerator experiments have increased, there has been an accumulation of evidence that the observed hadrons are most likely composite in nature. Their very stability (if we can use the term for objects with lifetimes as short as 10^{-24} sec.) and existence may be a product of the forces which also govern their mutual interactions. Clearly, the inherent non-linearities of such theories can not be well approximated by free fields. As a result, particle physicists have attempted to study interacting field theories by methods which take these non-linearities into account. Much progress has been made by these approaches which usually are based on the fact that the interacting field theories possess a certain symmetry or mathematical structure which must be reflected in the exact solutions. A recent example of such an analysis is the discovery by renormalization group methods that non-abelian gauge theories are "asymptotically free"; that is, as the energies involved are increased the particles described by such theories look more and more like free particles, even though at low energies the theory may be dominated by non-linear effects.

An alternative plan of attack in the study of relativistic quantum field theories is to analyze simple models, which exhibit some phenomenologically interesting behavior, but are not physically realistic, in order to explore the mathematic structure of the solutions which are naturally more accessible than the solutions of full-fledged theories. The hope is that the detailed structures of these simpler theories will suggest possible analogous structures in realistic models. That eventuality is our basic motivation for studying two dimensional (one-space, one-time) field theories.

The particular phenomenological picture which has suggested the possible relevance of two-dimensional models was first developed to explain the data from the deep inelastic electron-proton scattering experiments.[24] These results displayed "scaling". In the simplest terms, the proton seemed to be composed of constituents which participated in the high energy collisions as if they were not bound. At the same time, these apparent constituents are so tightly confined within the physically observed hadrons that up to the present, they have never been observed as independent entities produced in a collision of hadrons. Thus, the paradoxical situation indicated that at high energies the data could possibly be explained by a field theory whose basic fields correspond to the weakly interacting constituents, while the low energy spectral data displayed little direct trace of these fields (except possibly for explicit symmetries of the theory).

Casher, Kogut, and Susskind[6] noted that two-dimensional quantum electrodynamics with massless fermions is a field-theoretic model exhibiting properties which can be interpreted as analogues of scaling and confinement. Two-dimensional quantum electro-dynamics is a non-interacting theory (the S-matrix = 1), but if we couple the scalar fermion current to an external field, the effect of the external sources is to produce (or scatter) pairs of non-interacting fermions which move away from each other, and then crumble into clouds of massive bosons. When the external source has very high frequency components, the excitations resemble free fermion pairs over appreciable times, so that their production (or scattering) is a model for the scaling found in the deep inelastic electroproduction. Yet, because the fermions are not stable, they are never to be found in the spectral content of the final states which are completely composed of massive bosons. Bound by electromagnetism, the fermion-antifermion pairs are totally confined within the bosons.

In an attempt to understand what aspects of this model may be relevant and generalizable to other theories and more physically realistic situations, this thesis studies several two-dimensional quantum field theories. We begin by considering some soluble models, in which we find a correspondence between theories formulated in terms of fermion fields and theories formulated with boson fields. Some of these models have been solved previously by others, and others are generalizations for which we present operator solutions by extending the original constructions and arguments. The physics in these models can be equally well represented in terms of interacting fermion fields or interacting boson fields. We show this explicitly by constructing the operator correspondence and arguing that it is physically reasonable.

Of course, the fact that a physical situation can be modelled through fields of given statistics and Lorentz transformation properties does not imply that there are physical particles with corresponding characteristics. We find, in fact, that some of these exactly soluble theories have only boson states, whereas others have both bosons and fermions. Furthermore, in the latter case, our fermion-boson correspondence allows us to represent bosons as fermion-anti-fermion pairs and fermions as coherent states of bosons. This ambiguity arises because the fields involved are massless.

The physical argument which leads to the duality between fermion and boson representations of the same physics is certainly not dependent on whether the model is exactly soluble. Thus, we are led to extending the correspondence to theories with interactions we cannot analyze completely.

Although we are unable to calculate the particle spectrum in detail for such theories, the duality allows us to represent some fermion models in terms of boson field theories which are more amenable to semi-classical approximations than their fermion counterparts. We consider several examples of such models which might provide interesting dynamical analogues of the scaling-confinement phenomena if we could solve them exactly.

In presenting the above ideas we discuss the following specific topics. In Chapter II, we study the Thirring model, the theory of a massless spinor field coupled to its own current. This model was the first interacting fermion field theory to be solved. Its history has been reviewed elsewhere(28). By applying the arguments of Dell'Antonio, Frishman, and Zwanziger(11) to Klaiber's operator solution(17), we develop an operator solution to the Thirring model in which the fermion operator is simply a coherent state operator involving only free, massless bosons. Yet these bosons can be viewed as kinematically bound states of a free, massless fermion-anti-fermion pair. We present a physical picture which explains why this fermion-boson duality is reasonable.

We also consider a technical point, the definition of normal products of several fields at the same point. We argue that Lowenstein's prescription(19) should be modified to give a more physically motivated result. The modification is trivial in the Thirring model, but leads to differences in our later generalizations.

In Chapter III, we construct operator solutions of a class of scale-invariant abelian and U(n) non-abelian generalizations of the Thirring model. Since these exactly soluble models involve only free, massless quanta, the physics involved is not qualitatively different from the Thirring situation. Included among these models are two sets of theories which were investigated through an algebraic approach by Dashen and Frishman(9). We construct the corresponding operator solutions and show that the dynamics of the two different sets of theories is not distinct. One set of solutions is simply a relabelling of the other set. We also comment on the qualitative difference between the generalization of Lowenstein's normal product and the generalization of our product to these theories.

The operator solution of Lowenstein and Swieca[20] for two-dimensional quantum electrodynamics with massless fermions is summarized in Chapter IV. Although the spectrum consists of only free, massive bosons (no fermion states), in contrast to the Thirring model, we emphasize the mathematical and physical similarity of the operator solutions. In particular, the QED theory is also representable in terms of either fermion fields or boson fields. Generalization of the case to more than one species of fermion is made, and we interpret the resulting spectrum as only partial confinement of the fermions.

After noting the interchangeability of fermion and boson representations in all the soluble theories, in Chapter V we extend this duality through formal operator structures to general interacting theories. Since these theories have non-trivial scattering, there is, as yet, no method of exact solution known. Semi-classical arguments are applied to several particular models to discuss whether or not we expect fermion states in such theories.

Finally, in Chapter VI we summarize the general physical picture we have developed and discuss what approaches may make these results more relevant to the actual four-dimensional phenomena we wish to model. For clarity, Appendix F contains a table of the theories we have studied and lists some of the properties of each.

To keep the main text as readable as possible, we have only sketched the various solutions of the models. More complete mathematical details are included in the appendices. Hopefully, the resulting presentation is clear enough to allow the interested reader to follow the logic, without becoming ensnarled in the tedious task of decoding equations.

We should note that while this work was being done and later while we revised this text, many other authors have published similar or related investigation. We have listed some of this work in the bibliography, but in particular we should mention the papers by Mandelstam(21) and Coleman(8).

II. THE THIRRING MODEL

The Thirring model[11,15,16,17,28] is the theory of a two-component spinor coupled locally to its own current in two space-time dimensions. Denoting the spinor field as $\psi(x)$, we can write the formal Lagrangian density:

$$\mathcal{L} = \bar{\psi}\, i\not{\partial}\psi - \frac{g}{2} J_\mu J^\mu,$$

$$\text{where} \quad J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x). \tag{II.1}$$

Our γ matrices are the two-dimensional analogues of the four-dimensional Dirac matrices. They are explicitly defined in Appendix A, along with our other conventions. The formal equation of motion resulting from (II.1) is:

$$i\not{\partial}\psi = g J_\mu \gamma^\mu \psi. \tag{II.2}$$

Klaiber[17] has given a complete operator solution for the quantum field theories corresponding to (II.1) (until ψ's Lorentz properties are given, (II.1) and (II.2) do not uniquely determine a quantum field theory). Since the operator structures we will be discussing are very closely related to Klaiber's solution, we will now review his construction. The simplest way to introduce the quantities involved in the solution is to consider the classical solution to (II.2) first.

Classical Thirring Solution

We begin with a free, massless Dirac spinor χ, which satisfies the equation:

$$i\partial\!\!\!/ \chi = 0. \tag{II.3}$$

In two dimensions, (II.3) implies that both the current and axial current associated with χ are conserved:

$$\partial_\mu(\bar{\chi}\gamma^\mu\chi) \equiv \partial_\mu j^\mu = 0,$$

$$\partial_\mu(\bar{\chi}\gamma^\mu\gamma^5\chi) \equiv \partial_\mu j^{5\mu} = 0. \tag{II.4}$$

Furthermore, the γ-matrix identity,

$$\gamma_\mu\gamma^5 = \epsilon_{\mu\nu}\gamma^\nu, \tag{II.5}$$

relates the current and axial current:

$$j^5_\mu = \epsilon_{\mu\nu} j^\nu. \tag{II.6}$$

If we limit ourselves to fields χ and j_μ which are bounded in spatial extent, we can define spatially-integrated fields:

$$\phi(x) = -\frac{\sqrt{\pi}}{2}\int_{-\infty}^{\infty} dz\ \epsilon(x^1 - z)\, j_0(z, x^0),$$

$$\tilde{\phi}(x) = -\frac{\sqrt{\pi}}{2}\int_{-\infty}^{\infty} dz\ \epsilon(x^1 - z)\, j_0^5(z, x^0)$$

$$= \frac{\sqrt{\pi}}{2}\int_{-\infty}^{\infty} dz\ \epsilon(x^1 - z)\, j_1(z, x^0),$$

where

$$\epsilon(x^1 - z) = \begin{cases} 1 & x^1 > z \\ 0 & x^1 = z \\ -1 & x^1 < z \end{cases} \quad . \tag{II.7}$$

The second form for $\tilde{\phi}(x)$ follows from (II.6).

ϕ and $\tilde{\phi}$ are polarization or dipole density fields:

$$\frac{1}{\sqrt{\pi}}\,\epsilon_{\mu\nu}\,\partial^\nu\phi = j_\mu,$$

$$\frac{1}{\sqrt{\pi}}\,\epsilon_{\mu\nu}\,\partial^\nu\tilde{\phi} = j_\mu^5 . \tag{II.8}$$

To understand why the fields ϕ and $\tilde{\phi}$ might play an important role in the Thirring model, we can consider the polarization field in classical electromagnetism. When an external electric field is applied across the boundary between two different dielectrics, a surface charge is induced. On a macroscopic scale, where the size of an individual dipole in the dielectric is negligible, no individual particles carry the surface charge which is a collective property of the dipoles. The induced charge density corresponds to a change in the polarization field in going from one dielectric to the other.

If we apply our considerations to two dimensions, we note that a surface charge is point-like. Thus, the dynamics of point charges is equivalent to the dynamics of polarization fields in two dimensions. This equivalence extends beyond electromagnetism to any theory involving a charge whether this charge interacts in the same way as electric charge or not. In the Thirring model, there are two conserved charges corresponding to the current, $\bar{\psi}\gamma_\mu\psi$, and the axial currents, $\bar{\psi}\gamma_\mu\gamma^5\psi$. We will find that the related polarizations, which turn out to be ϕ and $\tilde{\phi}$, will be prominent in both the classical and quantum theories.

Using χ, ϕ, and $\tilde{\phi}$, we can write a family of classical solutions to (II.2):

$$\psi = e^{i[(\alpha+\sqrt{\pi})\tilde{\phi} + (\beta+\sqrt{\pi})\gamma^5\phi]}\chi,$$

$$J_\mu \equiv \overline{\psi}\gamma_\mu\psi = \overline{\chi}\gamma_\mu\chi \equiv j_\mu,$$

$$\text{where} \quad \beta - \alpha = g. \tag{II.9}$$

To check that (II.9) is a solution of (II.2), we simply note that the differentiation of ψ simply brings down derivatives of ϕ and $\tilde{\phi}$ from the exponential. However, these derivatives are, by construction, the current components j^μ. Since, $J^\mu = j^\mu$, (II.2) follows. The classical Thirring model solution consists of a free Dirac wave function modified by a position-dependent phase. Note that even if we choose $g = 0$, (II.9) still represents a one-parameter family of distinct solutions for various values of $\alpha(=\beta)$. Furthermore, if we apply an arbitrary external current, so that:

$$J_\mu = \overline{\psi}\gamma_\mu\psi + J_\mu^{\text{external}}, \tag{II.10}$$

the solution to (II.2) can still be written as in (II.9), with the polarization fields modified to include the external contributions:

$$\phi(x) = -\frac{\sqrt{\pi}}{2}\int_{-\infty}^{\infty} dz \;\; \epsilon(x^1 - z)\;\left\{ j_0 + J_0^{\text{external}} \right\},$$

$$\tilde{\phi}(x) = \frac{\sqrt{\pi}}{2}\int_{-\infty}^{\infty} dz \;\; \epsilon(x^1 - z)\left\{ j_1 + J_1^{\text{external}} \right\}. \qquad \text{(II.11)}$$

Quantum Thirring Model

Since the classical Thirring model solution can be constructed from a free Dirac field, even when an arbitrary external perturbation is involved, it is not surprising that the solution of the corresponding quantum field theory problem also can be constructed from free fields. Beginning with the formal expression (II.1), the vacuum expectation values of the quantized field operators (or Wightman functions) can be calculated by various standard techniques such as perturbation theory, functional techniques, or implementation of Ward identities(16). Klaiber's operator solution duplicates these results in terms of free field annihilation and creation operators and canonical algebra. The construction begins with a quantized, free, massless spinor field which we will denote as χ. As in the classical case, the associated current, j_μ, and axial current j^5_μ, are conserved. The relation (II.6) between these currents still holds. As a result, we may once again define polarization fields $\phi(x)$ and $\tilde{\phi}(x)$ completely in analogy to (II.7). In addition, the conservation of the currents implies that j_μ, j_μ^5, ϕ and $\tilde{\phi}$ all satisfy the Klein-Gordon equation for a free, massless field:

$$\partial^2 j_\mu(x) = 0 = \partial^2 j_\mu^5(x),$$

$$\partial^2 \phi(x) = 0 = \partial^2 \tilde{\phi}(x). \qquad \text{(II.12)}$$

Therefore, an invariant decomposition can be made of all the relevant fields into creation parts, $\chi^{(+)}$, $j_\mu^{(+)}$, $j_\mu^{5(+)}$, $\phi^{(+)}$, and $\tilde{\phi}^{(+)}$, and annihilation parts, $\chi^{(-)}$, $j_\mu^{(-)}$, $j_\mu^{5(-)}$, $\phi^{(-)}$, and $\tilde{\phi}^{(-)}$. The quantized Thirring field can be written:

$$\psi = e^{i[(\alpha+\sqrt{\pi})\tilde{\phi}^{(+)} + (\beta+\sqrt{\pi})\gamma^5\phi^{(+)}]}\,\chi\cdot \cdot e^{i[(\alpha+\sqrt{\pi})\tilde{\phi}^{(-)} + (\beta+\sqrt{\pi})\gamma^5\phi^{(-)}]}, \qquad \text{(II.13)}$$

where the γ^5 matrices are all understood to act on χ. Clearly, (II.13) is simply a Wick-ordered, quantized version of the classical field (II.9).

Before we verify that ψ satisfies the Thirring equation of motion, we should note that the details of the construction are complicated by the infrared structure of the operators. In two dimensions, a free, massless, quantized scalar field (like ϕ and $\tilde{\phi}$) is not well-defined because of infrared divergences. These divergences arise from infrared contributions to the scalar field which prevent it from being a true local operator. Some definition in terms of limits, called "regularization" is needed to be precise. The regularization may be accomplished in a positive-definite, non-covariant manner, or in a covariant framework with ghosts. In either approach, the actual matrix elements or Wightman functions are completely covariant, and the Lorentz-invariant properties of the theory are maintained. The details of the regularization do not appear in the final result. However, the infrared problem is not a mere technicality. In our later discussion, it will become clear that the infrared behavior of massless scalars is tied to the fermion-boson correspondence in two dimensions. At this point, we can proceed by ignoring such difficulties; our formulae will contain no essential errors. Appendix B outlines Klaiber's regularization.

To continue with Klaiber's construction, we can explicitly calculate the Poincaré properties (properties under relativistic transformation) of the field ψ defined in (II.13), since the Poincaré properties of the free fields, χ, j_μ, j_μ^5, ϕ, and $\tilde{\phi}$ are all known. In fact, the stress-energy tensor which generates the Poincaré transformation can be written in three equivalent forms (the equivalence can be shown explicitly by using the definition j_μ and j_μ^5 in terms of χ as in (7)):

$$\begin{aligned} \Theta_{\mu\nu} &= \frac{i}{4} :\left\{ \bar{\chi}\gamma_\mu \partial_\nu \chi + \bar{\chi}\gamma_\nu \partial_\mu \chi - (\partial_\mu \bar{\chi})\gamma_\nu \chi - (\partial_\nu \bar{\chi})\gamma_\mu \chi \right\}: \\ &= \frac{\pi}{2} :\left\{ j_\mu j_\nu + j_\mu^5 j_\nu^5 \right\}: \\ &= :\left\{ \partial_\mu \phi \, \partial_\nu \phi - \frac{g_{\mu\nu}}{2} \partial_\alpha \phi \, \partial^\alpha \phi \right\}: . \end{aligned} \tag{II.14}$$

Here, the : : symbol indicates that the operators must be Wick-ordered to eliminate the usual zero-point infinities. Commuting $\theta_{\mu\nu}$ and ψ yields the Poincaré structure for ψ. The time and spatial derivatives are:

$$\begin{aligned} -i\partial_0 \psi &= [P_0, \psi] = \left[\int dz \, \Theta_{00}(z, x^0), \psi \right] \\ &= \sqrt{\pi}\, \gamma_0 \left\{ \alpha \left(j_0^{(+)} \gamma^0 \psi + \gamma^0 \psi j_0^{(-)} \right) + \right. \\ &\qquad \left. - \beta \left(j_1^{(+)} \gamma^1 \psi + \gamma^1 \psi j_1^{(-)} \right) \right\}, \end{aligned} \tag{II.15}$$

$$-i\partial_1\psi = [P_1, \psi] = \left[\int dz\ \Theta_{01}(z, x^0), \psi\right]$$

$$= \sqrt{\pi}\,\gamma_1\left\{\alpha\left(j_1^{(+)}\gamma^1\psi + \gamma^1\psi j_1^{(-)}\right) + \right.$$
$$\left. -\beta\left(j_0^{(+)}\gamma^0\psi^{\cdot} + \gamma^0\psi j_0^{(-)}\right)\right\}. \qquad \text{(II.16)}$$

(II.15) and (II.16), the right-hand sides of which can be obtained by direct differentiation or by commutation, are the appropriately ordered analogues of derivatives of the classical solution. If we can identify $J_\mu(=\bar{\psi}\gamma_\mu\psi)$ with $j_\mu(=\bar{\chi}\gamma_\mu\chi)$, ψ will represent an operator solution of (II.2). To discuss J_μ, we must consider normal products of several spinor fields at one point. Before exploring the details of these products, we can briefly mention some further properties of ψ.

In two space-time dimensions the only Lorentz transformations are boosts generated by

$$M_{01} = \int dz\ (x_0\Theta_{10} - z\Theta_{00}). \qquad \text{(II.17)}$$

From (II.14) we can calculate

$$[M_{01}, \psi] = -i(x_0\partial_1 - x_1\partial_0)\psi - \frac{i}{2\pi}\alpha\beta\gamma^5\psi. \tag{II.18}$$

The second term on the right-hand side of (II.18) can be interpreted as arising from the "spin", $s(= \frac{\alpha\beta}{2\pi})$, of ψ. It is the part of ψ's transformation properties which does not arise from the change of coordinate systems when a boost is performed. An analogous term appears in four dimensions if a massless particle with helicity $\pm s$ is boosted along its direction of motion. Of course, the lack of true rotations in two dimensions means that s does not correspond to a true angular momentum and that s is not quantized by physical rotations by an angle of 2π. Thus, the "spin" of a field in two dimensions may take any real value. For a canonical free spinor field ($\alpha = \sqrt{\pi} = \beta$), $s = 1/2$. It should be noted that the existence of fields with arbitrary spin does not necessarily require the existence of particles with corresponding Lorentz properties, since the field operator may only produce states with many particles. In fact, the spin-statistics connection (see Appendix B) precludes the existence of particles with s neither integer nor half-integer.

We can also calculate the properties of ψ under scale changes ($x \to \lambda x$) by commuting with

$$D = \int dz\, (x^0\, \Theta_{00} + z\,\Theta_{10}) \qquad \text{(II.19)}$$

The results are:

$$[D, \psi] = -i x \cdot \partial \psi - \frac{i}{4\pi} \{\alpha^2 + \beta^2\} \psi, \qquad \text{(II.20)}$$

$$\left[\frac{dD}{dt}, \psi\right] = 0 . \qquad \text{(II.21)}$$

Equation (II.21) tells us that the theory is scale invariant, i.e. if $x \to \lambda x$, the equations and matrix elements remain the same if the fields transform like $\psi \to \lambda^{+d}\psi$, where d is the dimension of ψ. d is given by $\frac{1}{4\pi}$ $\{\alpha^2 + \beta^2\}$. For a canonical free spinor d = 1/2. From (II.20) we see that ψ has, in general, anomalous dimensions ($d \neq 1/2$). One consequence of anomalous dimensions is non-canonical equal-time singularity structure. In other words, the properties of ψ under a scale transformation, $x \to \lambda x$, is not compatible with the usual value of the anticommutator at equal times:

$$\left\{ \psi_\alpha(x), \psi^\dagger_\beta(0) \right\} \Big|_{x^0=0} = \delta_{\alpha\beta}\ \delta(x^1). \tag{II.22}$$

Therefore, for $d \neq 1/2$, the singular function on the right-hand side of (II.22) must have some other form. From the viewpoint of perturbation theory, the anomalous dimensions and non-canonical singularities arise from a modification in the propagator of ψ coming from the sum of diagrams to all orders in g, the coupling constant. Physically, the non-canonical structure implies that the field ψ need not have any single particle component, but may correspond to multiparticle excitations only. Later we shall see that the latter is true.

From the Poincaré results above, it is clear that choosing a value for any two of the five parameters, α, β, g, s, and d, determines a unique solution from the class given by (II.13). Further, if $s = 1/2$, a given value of g determines a unique solution. We also note that the canonical current commutation relations are preserved (if $J_\mu = j_\mu$, as we will see) but the charge and axial charge associated with ψ are renormalized:

$$[j_\mu, j_\nu] = -\frac{i}{\pi}\partial_\mu\partial_\nu\Delta_0 \,, \tag{II.23}$$

$$[j_\mu, \psi] = \left\{\frac{-\alpha}{\sqrt{\pi}} g_{\mu\nu} + \frac{-\beta}{\sqrt{\pi}}\gamma^5\epsilon_{\mu\nu}\right\}\partial^\nu\Delta_0\,\psi. \tag{II.24}$$

Other details of the models easily studied are the general Wightman functions, the spin-statistics connection, the breaking of discrete symmetries, and the cluster decomposition properties of the solutions (see Appendix B).

Before proceeding to the discussion of normal products, we should make an observation about the dynamics of the model which will be important in both the normal products and our later study of the generalized Thirring models. It is instructive to rewrite (II.15) and (II.16) in light cone coordinates (defined in Appendix A):

$$
\begin{aligned}
[P_L, \psi_R] &= \sqrt{\pi}(\alpha-\beta)\left\{j_L^{(+)}\psi_R + \psi_R j_L^{(-)}\right\},\\
[P_R, \psi_R] &= \sqrt{\pi}(\alpha+\beta)\left\{j_R^{(+)}\psi_R + \psi_R j_R^{(-)}\right\},\\
[P_L, \psi_L] &= \sqrt{\pi}(\alpha+\beta)\left\{j_L^{(+)}\psi_L + \psi_L j_L^{(-)}\right\},\\
[P_R, \psi_L] &= \sqrt{\pi}(\alpha-\beta)\left\{j_R^{(+)}\psi_L + \psi_L j_R^{(-)}\right\}.
\end{aligned} \tag{II.25}
$$

In this form, we find a hidden symmetry of the model. There are essentially two coupling constants, $g(= -\sqrt{\pi}(\alpha-\beta))$ and $-\sqrt{\pi}(\alpha+\beta)$. If the values of these two couplings are interchanged, we retain the same dynamics except that the roles of ψ_R and ψ_L are also interchanged. In particular, free fermion dynamics ($\alpha = -\sqrt{\pi}=\beta$) is reproduced by a $g = 2\pi$ theory ($\alpha = -\sqrt{\pi}$, $\beta = \sqrt{\pi}$). This dynamical redundancy should be kept in mind when we consider all our Thirring structures, since the physics of the interchanged theory is not different from its parent.

Normal Products

The equations (II.25) also point out the fact that the dynamics of a given chiral component of ψ are different in the two different light-like directions. Thus, it will not be a surprise to find that the limit of products of fields as their coordinates approach each other depends on the direction of approach. With this idea in mind, we can now consider the structure of normal products of ψ's and ψ^{+}'s. The most important of these products is the current J_μ associated with ψ. Only after we can identify J_μ with j_μ, the free fermion current, can we say that ψ is an operator solution of (II.2).

There is an ambiguity in the definition of normal products which amounts to only a trivial overall normalization factor in the Thirring model proper. However, in the generalized models to be studied later, the ambiguity becomes more serious. To settle this question, we will need to discuss the products in some detail.

Lowenstein[19] has defined a multilocal normal product which is covariant and which has a unique local field as its limit as the field coordinates approach each other. For instance, if we denote the product obtained via his prescription by L{...}, the current is given simply by (except for infrared correction terms which are not relevant here):

$$J_\mu(x) = \lim_{y \to x} L\left\{ \overline{\Psi}(y)\, \gamma_\mu \Psi(x) \right\}. \tag{II.26}$$

Note that the limit is independent of both the direction of approach and the parameters α and β. If we look in detail at Lowenstein's prescription (see Appendix B), we find that the regularization involves an exponential operator which, at short distances, effectively cancels the exponential of polarization fields ϕ and $\tilde{\phi}$ multiplying the free fermion operator χ in (II.13). This definition of the normal product essentially treats the polarization fields in ψ as an operator gauge transformation of χ. With this interpretation, the Thirring model becomes the theory of a free fermion and its currents, subject to gauge transformations which leave the current invariant.

Our alternative prescription attempts to define the normal products of coincident fields in a way which reflects the dynamics more directly. We derive the currents from a short-distance expansion of bilocal operator products, in a manner similar to that employed by Dell'Antonio, Frishman, and Zwanziger[(11)]. $R\{\psi(x)\psi^+(y)\}$ is defined as a regularized product (Appendix B) with the multiplicative coordinate singularities and the vacuum expectation value of the ordinary product $\psi(x)\psi^+(y)$ removed. $R\{\psi(x)\psi^+(y)\}$ can be easily expanded for short distances to obtain our bilinear normal products. We discover that the limits of the regularized scalar and pseudoscalar densities, $R\{\bar{\psi}(y)\psi(x)\}$ and $R\{\bar{\psi}(y)\gamma^5\psi(x)\}$, as $x \rightarrow y$, are unique local operators (equal, in fact, to

Lowenstein's corresponding products). These limits are our normal products for $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$.

The short distance expansion for the current densities (ignoring infrared terms) can be expressed in light cone coordinates:

$$x_L R\{\psi^\dagger(x)\frac{1+\gamma^5}{\sqrt{2}}\psi(0)\} \xrightarrow[x\to 0]{} \frac{1}{\sqrt{\pi}}[-(\alpha-\beta)x_R j_L - (\alpha+\beta)x_L j_R + \mathcal{O}(x^2)+\ldots],$$

$$x_R R\{\psi^\dagger(x)\frac{1-\gamma^5}{\sqrt{2}}\psi(0)\} \xrightarrow[x\to 0]{} \frac{1}{\sqrt{\pi}}[-(\alpha+\beta)x_R j_L - (\alpha-\beta)x_L j_R + \ldots].$$

(II.27)

For arbitrary values of α and β, the limits of the bilocals seem to be non-covariant, depending on the direction of approach. However, in two dimensions, there are two invariant bilinear vector products, scalar and pseudoscalar. We can re-express (II.27):

$$x^\mu R\{\bar{\psi}(x)\gamma_\mu\psi(0)\} \to -\frac{1}{\sqrt{\pi}}[\alpha x^\mu j_\mu + \ldots],$$

$$x^\mu \epsilon_{\mu\nu} R\{\bar{\psi}(x)\gamma^\nu\psi(0)\} \to -\frac{1}{\sqrt{\pi}}[\beta x^\mu \epsilon_{\mu\nu} j^\nu + \ldots]. \qquad \text{(II.28)}$$

Thus, the R-products have covariant expansions, but there are two unequally normalized terms, so that the behavior is different along each of the light-cone directions. As we said earlier, equations (II.25) indicate that such a result is likely, since the dynamics of ψ are governed by different coupling constants along the two directions. The physical picture associated with the two different terms will be clearer after we discuss the coherent state operator forms for ψ, a little later.

Using the parity properties of the terms in (II.27), we can now define two different normal products from the expansion of $R\{\bar{\psi}(y)\psi(x)\}$ which yield a current and an axial current:

$$N(\bar{\Psi}\gamma_\mu\Psi) = -\frac{\alpha}{\sqrt{\pi}}\, j_\mu ,$$

$$N_5(\bar{\Psi}\gamma_\mu\gamma^5\Psi) = -\frac{\beta}{\sqrt{\pi}}\, \epsilon_{\mu\nu} j^\nu . \qquad \text{(II.29)}$$

Clearly, these products differ from Lowenstein's products only in overall normalization, a difference which we eliminate in any case, by insisting that the currents (and axial currents) satisfy the canonical commutation relations, (II.23). However, when the Thirring model is generalized to include internal degrees of freedom, a generalization of L{...} leads only to the currents of the original free fermions, while the short distance expansions of the bilocals will allow mixing of the currents by the interaction. There is already an indication of the qualitative difference between the two approaches in the results in (II.29). If we consider the structure of the γ matrices, we note that the vector current J_μ should correspond to the sum of the currents associated with the two chiral projections of the field ψ, ψ_R and ψ_L. J_μ^5, the axial current, corresponds to the difference between the ψ_R and ψ_L currents. Thus, J_μ and J_μ^5 are independent quantities, which are related only coincidentally in our construction. From the commutation relation (II.24), the form of ψ (II.13), and the definition of ϕ and $\tilde{\phi}$ in (II.7), we find that ψ_R carries a charge $\frac{\alpha}{\sqrt{\pi}}$ and an axial charge $\frac{\beta}{\sqrt{\pi}}$, while the values for ψ_L are $\frac{\alpha}{\sqrt{\pi}}$ and $-\frac{\beta}{\sqrt{\pi}}$, respectively. The factors in (II.28) exactly reflect these charges. Lowenstein's prescription eliminates the dynamical effects which yield the renormalized values of the charges and the currents are given by the free current structure.

With the currents J_μ and J_μ^5 derived from (II.29) shown to be identical to j_μ and j_μ^5, equations (II.15) and (II.16) become the Thirring equations of motion. Using the given operator structure for J_μ, and ψ, we can now calculate all the Wightman functions of the theory. Normal products corresponding to field products other than the ones we have discussed are similarly defined by means of short distance expansions (Appendix B). In general, there will be several different normal products for a single field product, each derived from a different approach to the limit of coincident points as in the case of J_μ and J_μ^5. As we have said, the difference in these limits is not surprising since the dynamics of the Thirring model differs in the two light-like directions.

The Thirring Field as a Boson Coherent State Operator

A reasonable question at this point is why we can express $\theta_{\mu\nu}$, the stress-energy tensor, in (II.14) as a bilinear in j_μ (or $\partial_\mu\phi$) as well as a bilinear in χ. It turns out that even ψ can be constructed from j_μ alone (with the proper infrared regularization). Led by the algebraic arguments of Dell'Antonio, Frishman, and Zwanziger[(11)]. we extend Klaiber's explicit construction to write:

$$\psi = :e^{i[\alpha\tilde{\phi} + \beta\gamma^5\phi]}: \left[\frac{\mathcal{m}}{2\pi}\right]^{1/2} \mathfrak{C}. \tag{II.30}$$

The : : indicate the operators must be Wick-ordered. The $\mathcal{m}$ factor gives ψ the correct physical dimensions, with $\mathcal{m}$ a mass involved in the infrared regularization. $\mathfrak{C}$ is a coordinate-independent, unitary, spinor operator which commutes with $j\mu$. $\mathfrak{C}$ essentially plays the role of a placeholder allowing ψ to have two components (the γ^5 acts on $\mathfrak{C}$). Also, since each ψ field carries a $\mathfrak{C}$ and each ψ^+ a $\mathfrak{C}^+$, the states in the different charge sectors are labelled by the number of $\mathfrak{C}$'s and $\mathfrak{C}^+$'s used in their construction.

To understand how the form (II.30) might be relevant, we first discuss coherent states in two dimensions in general. Suppose $\phi(x)$ is a free scalar field with mass μ and canonical commutation relations:

$$[\phi(x), \phi(y)] = i\Delta_{\mu^2}(x-y). \tag{II.31}$$

We can construct coherent states from ϕ:

$$|g,h,t\rangle = :e^{-i\int_{-\infty}^{\infty} dz\,[g(z)\partial^0\phi(z,t) + h(z)\partial^1\phi(z,t)]}: |0\rangle. \tag{II.32}$$

$|0\rangle$ denotes the Fock vacuum. These states are eigenstates of the annihilation part of the field operator and its canonically conjugate momentum.

$$\phi^{(-)}(x)|g,h,t\rangle = -\int_{-\infty}^{\infty} dz\,\left[g(z)\,\partial^0\Delta^{(+)}_{\mu^2}(x^0-t,x^1-z) + \right.$$
$$\left. + h(z)\,\partial^1\Delta^{(+)}_{\mu^2}(x^0-t,x^1-z)\right]|g,h,t\rangle,$$
$$\xrightarrow[x^0\to t]{} g(x^1)\,|g,h,t\rangle,$$

$$\partial^0\phi^{(-)}(x)|g,h,t\rangle \xrightarrow[x^0\to t]{} \frac{\partial h}{\partial z}(x^1)\,|g,h,t\rangle. \tag{II.33}$$

Thus, the expectation value of ϕ in the coherent state is a classical field which satisfies the same equation of motion as ϕ and has as boundary conditions,

$$\langle g,h,t|\ \phi(x^1,t)\,|g,h,t\rangle = g(x^1)\ \langle g,h,t|g,h,t\rangle,$$

$$\langle g,h,t|\ \partial_0\phi(x^1,t)\,|g,h,t\rangle = \tag{II.34}$$
$$= \frac{\partial h}{\partial z}(x^1)\ \langle g,h,t|g,h,t\rangle.$$

These coherent states correspond to collective excitations with an indeterminate number of quanta. As we shall see later (II.39 and II.40), the field operator ϕ, when it is applied to such states can be thought of as having a c-number piece as well as the usual q-number or operator piece. In fact, when we deal with coherent states in which the c-number contribution to matrix elements of interest is much larger than the q-number contribution, we have a physical situation where the quantum fluctuations are comparatively small. The quantum field ϕ can then be approximated by a classical field satisfying the same equations of motion, and we are in a regime where classical physics may be a satisfactory approximation to the quantum theory.

Of course, all our formal definitions like (II.36) are valid only if the expressions are mathematically well-defined. The norm of $|g,h,t>$ is divergent unless (for g,h real):

$$\int_{-\infty}^{\infty} dK \frac{1}{[K^2+\mu^2]^{1/2}} \; |G(K)|^2 < \infty,$$

$$\int_{-\infty}^{\infty} dK \frac{K^2}{[K^2+\mu^2]^{3/2}} \; |H(K)|^2 < \infty, \quad \text{where}$$

$$G(K) \equiv [K^2 + \mu^2]^{1/2} \int_{-\infty}^{\infty} dz \; e^{-iKz} g(z),$$

$$H(K) \equiv [K^2 + \mu^2]^{1/2} \int_{-\infty}^{\infty} dz \; e^{-iKz} h(z). \qquad \text{(II.35)}$$

In order to insure that the expectation values of the energy and momentum are finite, we have the more stringent conditions:

$$\int_{-\infty}^{\infty} dK \, |G(K)|^2 < \infty,$$

$$\int_{-\infty}^{\infty} dK \frac{K^2}{K^2+\mu^2} |H(K)|^2 < \infty. \qquad \text{(II.36)}$$

Physically, the infrared integrability condition corresponds to requiring that the ϕ field is damped at $\pm\infty$, so that the total energy of the state is finite, while the ultraviolet integrability condition corresponds to requiring that the quantum mechanical state is sufficiently smeared out to allow a finite total energy.

An alternative approach to coherent states involves the unitary operator:

$$\mathcal{U}(g,h,t) = e^{i\int_{-\infty}^{\infty} dz\,[g(z)\partial^0\phi(z,t) + h(z)\partial^1\phi(z,t)]} \quad . \tag{II.37}$$

Obviously, we can construct the coherent states with U:

$$|g,h,t\rangle = \sqrt{\langle g,h,t\,|\,g,h,t\rangle}\;\;\mathcal{U}^{-1}\,|0\rangle. \tag{II.38}$$

U induces a transformation on the field ϕ , in co-ordinate space:

$$\mathcal{U}\,\phi(x)\,\mathcal{U}^{-1} = \phi(x) + \int_{-\infty}^{\infty} dz\,[g(z)\,\partial^0 + h(z)\,\partial^1]\,\Delta_{\mu^2}(x^0 - t, x^1 - z), \tag{II.39}$$

and in momentum space (d(K) is the free field annihilation operator defined in Appendix A):

$$\mathcal{U}\,d(K)\,\mathcal{U}^{-1} = d(K) - G(K) - \frac{K}{\sqrt{K^2+\mu^2}}\,H(K). \tag{II.40}$$

On the other hand,

$$\langle g,h,t|\,\phi(x)\,|g,h,t\rangle = \langle 0|\,\mathcal{U}\phi(x)\mathcal{U}^{-1}|0\rangle\;\;\langle g,h,t\,|\,g,h,t\rangle. \tag{II.41}$$

Thus, the vacuum matrix elements of the transformed field are the same as the matrix elements of ϕ in the coherent state (up to a normalization).

Mathematically, U is a well-defined unitary operator only if the conditions (II.34) are satisfied. In addition, to insure finite energy states, we must satisfy (II.36). In such cases, the coherent state matrix elements yield a mathematically equivalent representation of the free field operator algebra. The physics is the same in the coherent state and Fock representations, even though the corresponding matrix elements differ by c-number background fields (as in II.39)).

If we now consider coherent states of a massless scalar, we find a qualitative difference in the structure. (Strictly speaking, a free massless scalar field cannot be defined in two dimensions because of the infrared problem encountered, for instance, in defining the two-point function. To be rigorous, we would consider a vector field instead, or impose a cut-off like Klaiber's. To emphasize the analogy with a massive scalar, however, we will continue as if the massless scalar were well-defined.) When $\mu^2 = 0$, (II.36) can be satisfied, without requiring (II.35) which could be violated by infrared divergentes. In this case, although neither $|g, h, t>$ nor U could be constructed in a well-defined manner as before, the energy of the state $|g, h, t>$ if it were constructed would be finite.

Physically, this situation corresponds to a state with an infinite number of infrared quanta whose total energy is finite. The number operator has an infinite value in such a state, and the matrix elements of the field operator are infrared divergent. An analogous problem arises in four-dimensional quantum electrodynamics involving "soft" photons.

Suppose we ignore the mathematical difficulties in defining U, and simply consider the formal transformation (II.39) or (II.40). The matrix elements of the transformed field constitute a mathematically inequivalent representation of the free field algebra. The mathematical inequivalence is simply due to the fact U does not exist in a mathematical sense. The physical picture we drew above would lead us to believe there is no inequivalence in the physics. These coherent states differ only in having an infinite number of infrared quanta. In fact, this type of state has been studied by Borchers, Haag, and Schroer (4). They find that these representations are physically equivalent to the Fock representation in the sense that a distinction can be made only by a global measurement over an infinite volume which is not possible in an actual physical situation. In our particular states, the global measurements required would be the limiting values of ϕ and $\int dx^1\, \partial_0 \phi$ at $z \to = \infty$.

Let us now turn in particular to the Thirring model. The expression for ψ in (II.13) has the form of a coherent state operator for the field ϕ multiplying a free spinor χ (with the appropriate ordering). Comparing with (II.32) we see that

$$
\begin{aligned}
g(z) &= -\frac{\sqrt{\pi}}{2}(\alpha+\sqrt{\pi})\,\epsilon(z-x^1), \\
h(z) &= -\frac{\sqrt{\pi}}{2}(\beta+\sqrt{\pi})\,\epsilon(z-x^1)\,\gamma^5,
\end{aligned}
\qquad \text{(II.42)}
$$

for this coherent state. Thus, the operator is improper for two reasons. It violates conditions (II.35) in the infrared and conditions (II.36) in the ultraviolet. The latter is easily remedied if we remember that ψ is constructed to be a local field. Since a completely localized quantum state has a divergent energy, the actual physical states are formed by smearing the local operator as is generally required in quantum field theory. The effect on g(z) and h(z) is to round the corners of the step function. The corresponding G(K) and H(K) are damped at high momenta and (II.36) is satisfied.

The infrared structure in (II.42) is exactly the type we have described as inducing mathematically inequivalent, but physically equivalent representations of the field algebra. Furthermore, we noted that the different inequivalent representations are catalogued by the asymptotic values of ϕ and $\tilde{\phi}$. On the other hand, the identification of the gradients of ϕ and $\tilde{\phi}$ with the Thirring currents allows us to express the charge and axial charge:

$$Q = \int dz\, j_0 = \frac{1}{\sqrt{\pi}} \int dz \frac{d\phi}{dz} = \frac{1}{\sqrt{\pi}} (\phi(\infty) - \phi(-\infty)),$$

$$\tilde{Q} = \int dz\, j_0^5 = \frac{1}{\sqrt{\pi}} (\tilde{\phi}(\infty) - \tilde{\phi}(-\infty)). \qquad \text{(II.43)}.$$

The different inequivalent representations are simply the different charge and axial charge sectors of the theory. The physical equivalence of the representations implies that within a given sector of the theory the physics is the physics of the Thirring currents which are free fields. The ψ field is the operator which intertwines the different inequivalent representations, i.e. it is the operator which accomplishes the transformations like (II.39) and (II.40).

Klaiber's infrared regularization scheme, which we have been implicitly employing, is a means of implementing the transformations between the inequivalent representations, while maintaining the coherent state operator form in the definition of ψ. The infrared divergence which invalidated the operator is eliminated by a cut-off in the momentum allowed in ϕ and $\tilde{\phi}$. Condition (II.35) is satisfied by the resulting operator. The associated coherent state is damped out at large spatial co-ordinates. Since the coherent "cloud" which surrounds the free fermion (represented by χ) does not extend to infinity, it no longer carries any charge to renormalize the free charge and axial charge to the interacting values $\frac{\alpha}{\sqrt{\pi}}$ and $\frac{\beta}{\sqrt{\pi}}$,respectively. The current must then be corrected by terms proportional to the total charge operators which "count" each ψ field and renormalize its charge. (For further details, see Appendix B).

Although ψ acts as an intertwining operator between different charge sectors, we see that charge carried by ψ is divided between χ and the coherent state operator. Studying the commutation relations for ψ and j^{μ} , we find this separation is artificial. The free spinor can be eliminated entirely in favor of an additional coherent state operator. The resulting form for ψ is (II.30), in which the entire charge and current densities are represented by the boson operators. The matrix elements and commutation relations all remain the same as before. The normal products can be obtained once again from short-distance expansions.

In fact, since the boson fields are free, the expansions will simply be expansions in terms of Wick products and derivatives of Wick products.

The Physical Picture Presented by the Thirring Model

We have already pointed out the equivalence between point charges and discontinuities in polarization fields in two dimensions in our consideration of the classical Thirring solutions. The physics should be independent of which framework we chose to analyze the model. We can now see this duality is manifested in the quantum solutions. In the form of (II.13), ψ can be interpreted as a free fermion field clothed in a cloud of fermion-anti-fermion pairs created by the interaction. In the form (II.30), ψ represents a coherent state of polarization quanta. The duality holds even for the free case ($\alpha = -\sqrt{\pi} = \beta$). Thus, the state created by applying a free fermion field $\psi(x)$ on the vacuum can be understood equally well as a state with a point charge at x^1 or a state with a constant dipole density filling all space, but changing sign at x^1.

For arbitrary values of α and β, if we use the form (II.30), the structure of the state created by $\psi_R(x)$ on the vacuum is a polarization wave of the form $-\frac{\sqrt{\pi}(\alpha + \beta)}{2}\epsilon(z - x^1)$

moving to the right at the speed of light and a polarization wave of the form $-\frac{\sqrt{\pi}(\alpha - \beta)}{2}\,\varepsilon(z - x^1)$ moving to the left at the speed of light. The state associated with $\psi_L(x)$ is analogous. Since the polarization quanta are massless, the wave fronts maintain their shape as they travel off to infinity. This lack of dispersion of the polarization waves leads us to the question of the particle content of the model.

Our explicit constructions make it clear that the Thirring model can be viewed as a theory of free, massless fermions. The ψ field is then a composite operator constructed from the fermion fields. Similarly, the currents J_μ and J_μ^5 are bilinears in ψ and ψ^+. The massless pole in the current two-point functions arises from the fact that the fermion anti-fermion pairs moving in the same direction are kinematically bound in two dimensions into massless current quanta.

Alternatively, the model can be viewed as a free, massless boson theory. The ψ field is a coherent state operator. The current quanta are the bosons. In either of these equally valid approaches to the spectrum of the model, ψ does not correspond to a particle, in general. We must now consider whether there are any further interpretations of the spectrum, especially ones in which ψ corresponds to a particle.

We see immediately that unless $\alpha = \pm\,\beta$, ψ cannot be a single particle operator since it creates two massless excitations that move off in opposite directions. From the explicit structure of ψ in (II.30) we see that for $\alpha \neq \pm\,\beta$, ψ can be written as the product of two coherent state operators, one of which produces a single right-moving excitation, the other a left-moving excitation.

$$\psi = \underbrace{:e^{i[\frac{\alpha+\beta\gamma^5}{2}][\tilde{\Phi}+\phi]}:}_{\text{right-moving excitation}}\ \underbrace{:e^{i[\frac{\alpha-\beta\gamma^5}{2}][\tilde{\Phi}-\phi]}:}_{\text{left-moving excitation}}\ \left[\frac{m}{2\pi}\right]^{1/2}\mathbb{C}. \qquad \text{(II.44)}$$

Thus, the problem is reduced to the cases $\alpha = \pm\,\beta$, since the general case simply involves the production of two of the $(\alpha = \pm\,\beta)$-excitations. Actually, we already know that for $\alpha = -\sqrt{\pi} = \beta$, the boson coherent state is equivalent to a free fermion. For all α and β , the "spin" of the boson coherent state is $s = \frac{\alpha\beta}{2\pi}$. We find that unless s is integral or half-integral, ψ and ψ^+ neither commute nor anti-commute at spacelike separations (see Appendix B). Taking Fourier transforms of ψ and ψ^+, we find that the momentum space operators, which would be the single particle creation and annihilation operators if ψ corresponded to a particle, do not commute (or anti-commute) for different momenta. As a result, single particles corresponding to ψ cannot be isolated and defined.

However, for $s = \frac{n}{2}$, with n integer, ψ and ψ^+ commute or anticommute properly. We can write a momentum space decomposition for ψ :

$$\psi_{\left\{ {R \atop L} \right\}} = \int \frac{dK}{4\pi|K|} (|K| \pm K)^s \left\{ a(K)e^{-ik\cdot x} + b^{+}(K)e^{ik\cdot x} \right\}. \tag{II.45}$$

The a and b are canonical:

$$a(K)a^{+}(K') - (-1)^n a^{+}(K')a(K) = 4\pi|K|\,\delta(K-K'),$$

$$a(K)a(K') - (-1)^n a(K')a(K) = 0,$$

etc. (II.46)

We find then that boson coherent states corresponding to integral "spin" are equivalent to single bosons. These corresponding to half-integral "spin" are equivalent to single fermions.

In summary, an arbitrary boson coherent excitation created by an operator of the form (II.30) can be interpreted as a single particle of any "spin" (half-integral or integral) surrounded by a coherent cloud of bosons which carries the remaining "spin". The particle part of ψ could be written as in (II.45) and the boson cloud would be represented by an additional coherent state operator. For the special cases $|\alpha| = \sqrt{\pi n} = |\beta|$, n integer, the Thirring model can be viewed as the theory of free, massless spin n/2 particles.

We conclude our discussion of the Thirring model with a comment on the normal products defined earlier. The preceding analysis shows that ψ_R is associated with right-moving bosons of the type $(\alpha + \beta)\, \partial_R\phi$ and left-moving bosons $(\alpha - \beta)\, \partial_L\phi$. There is an analogous composition of ψ_L. There are therefore two different currents associated with ψ, $(\alpha + \beta) j_\mu = (\alpha + \beta)\varepsilon_{\mu\nu}\partial^\nu\phi$ and $(\alpha - \beta) j_\mu = (\alpha - \beta)\varepsilon_{\mu\nu}\partial^\nu\phi$. The fact that the currents are proportional is an artifact of our construction. We could equally well choose them to be two independent massless, free vectors. The normal products $N\{\ldots\}$ and $N_5\{\ldots\}$ we defined reflect this possibility, since the independence of the two currents is maintained. Lowenstein's prescription does not allow for such a freedom.

III. THIRRING GENERALIZATIONS

A natural step to take after studying the Thirring Model is to generalize the model by incorporating internal degrees of freedom in the ψ field. Of course, we can only explicitly write down models if they are constructed from free fields like the Thirring Model. We limit ourselves to such soluble models here. Later, we will discuss the more general case.

Dashen and Frishman(9) have recently investigated a model in which ψ has a U(n) internal symmetry. ψ is then a 2n-component field. If we denote the standard n-by-n U(n) matrices (see Appendix A) by λ^i, i = 0,..., n^2-1, the formal currents are:

$$J_\mu \equiv J_\mu^0 = \bar{\Psi}\gamma_\mu\frac{\lambda^0}{\sqrt{2}}\Psi = \frac{1}{\sqrt{n}}\bar{\Psi}\gamma_\mu\Psi,$$

$$J_\mu^a = \bar{\Psi}\gamma_\mu\frac{\lambda^a}{\sqrt{2}}\Psi, \qquad a = 1,\ldots,n^2-1. \qquad \text{(III.1).}$$

The model considered by Dashen and Frishman has formal Lagrangian density:

$$\mathcal{L} = \bar{\Psi}\, i\partial\!\!\!/\,\Psi - \frac{G^2}{2}J_\mu J^\mu - \frac{g^2}{2}J_\mu^a J^{\mu a}. \qquad \text{(III.2)}$$

Dashen and Frishman attempted to find solutions corresponding to (III.2) which exhibit scale invariance and the canonical (free) equal-time commutation relations for the currents, J_μ and J_μ^a. Using the algebraic structure of the operators, the U(n) symmetry, and the restriction of scale invariance, they find that there are two classes of solutions. One class corresponds to $g^2 = 0$, so that the only interaction is the U(1) term, $\frac{G^2}{2} J_\mu J^\mu$. This class is clearly a simple generalization of the normal Thirring model. The other class involves $g^2 = \frac{2\pi}{n+1}$. However, as we will show in our explicit operator construction, this second class of solutions is physically equivalent to the $g^2 = 0$ solutions. The two classes are related by the same symmetry that we explicitly displayed in the Thirring model proper via equations (II. 24). Thus, Dashen and Frishman's original conjecture that they had found a set of solutions dynamically independent from the $g^2 = 0$ case is not completely accurate (This remark has also been made by several other authors e.g. (2)).

Construction of ψ

To study the model associated with (III.2) and other scale invariant generalizations of the Thirring model, we will first generalize our construction of ψ in (III.30). Then, by varying the parameters of the construction, we can satisfy several different equations of motion for ψ . To begin with, we single out the n diagonal matrices in the standard representation $U(n)$. We denote these diagonal matrices by τ^i, $i = 0, \ldots, n-1$. Since we are constructing n fermion degrees of freedom, we employ n free abelian currents, with zero divergence and curl, (analogous to j_μ in Chapter II) j^i_μ, $i = 0, \ldots, n-1$. Each of the j^i_μ has free communtation relations:

$$[j_\mu{}^i(x), j_\nu{}^k(y)] = -\frac{i}{\pi}\delta^{ik}\partial_\mu\partial_\nu\Delta_0(x-y). \tag{III.3}$$

Following our previous construction, the polarization fields ϕ^i and $\tilde{\phi}^i$ are defined (Klaiber's regularization scheme is implicitly assumed) so that

$$j_\mu{}^i = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi^i = \frac{1}{\sqrt{\pi}}\partial_\mu\tilde{\phi}^i = \epsilon_{\mu\nu}j^{5\nu i}. \tag{III.4}$$

Using ϕ^i and $\tilde{\phi}^i$,

$$\psi = :e^{i\frac{\tau^j}{2}[\alpha_{jk}\tilde{\phi}^k + \beta_{jk}\gamma^5\phi^k]}:\left[\frac{m}{2\pi}\right]^{1/2}\mathfrak{C}. \tag{III.5}$$

Here, α and β are real n-by-n matrices, left-multiplied by τ^j and right-multiplied by ϕ^k and $\tilde{\phi}^k$. $\mathfrak{C}$ is a 2n-component spinor operator which is co-ordinate independent, unitary, and commutes with all the j^i_μ. m is once again a mass involved in the infrared regularization.

Clearly, the construction of ψ is completely analogous to the Thirring construction. ψ is an improper coherent state operator which yields different mathematically inequivalent representations of the current algebra, (III.3).

Definition and Normalization of the "Diagonal" Currents

In order to define the current operators associated with ψ , we need to construct normal products of ψ and ψ^+. If we limit ourselves to the "diagonal" currents formally defined by

$$K_\mu^i = \overline{\Psi}\gamma_\mu \frac{\tau^i}{\sqrt{2}}\Psi,$$

$$K_\mu^{5i} = \overline{\Psi}\gamma_\mu\gamma^5 \frac{\tau^i}{\sqrt{2}}\Psi, \qquad i = 0,\ldots,n-1, \qquad \text{(III.6)}$$

we can define the corresponding normal products by short-distance expansion in exactly the same way as in the Thirring model. Using our previous notation for the two different products, the results analogous to (II.29) are:

$$N(\overline{\Psi}\gamma_\mu \frac{\tau^i}{\sqrt{2}}\Psi) = -\frac{\alpha^{im}}{\sqrt{\pi}}\, j^m_\mu,$$

$$N_5(\overline{\Psi}\gamma_\mu\gamma^5 \frac{\tau^i}{\sqrt{2}}\Psi) = -\frac{\beta^{im}}{\sqrt{\pi}}\, \epsilon_{\mu\nu} j^{m\nu}. \qquad \text{(III.7)}$$

We normalize the currents by insisting on the commutation relations:

$$[K_\mu^m, K_\nu^m] = -\frac{i}{\pi}\partial_\mu\partial_\nu\Delta_0, \qquad \text{(no sum on m)}$$

$$[K_\mu^{5m}, K_\nu^{5m}] = -\frac{i}{\pi}\partial_\mu\partial_\nu\Delta_0. \qquad \text{(no sum on m)} \qquad \text{(III.8)}$$

We can express K_μ^m and K_μ^{5m}:

$$K_\mu^m = -\frac{\tilde{\alpha}^{m\ell}}{\sqrt{\pi}}\, j_\mu^\ell,$$

$$K_\mu^{5m} = -\frac{\tilde{\beta}^{m\ell}}{\sqrt{\pi}}\,\epsilon_{\mu\nu}\, j^{\ell\nu}, \qquad \text{(III.9)}$$

by defining the renormalized matrices:

$$\tilde{\alpha}^{m\ell} = \frac{1}{A^m}\,\alpha^{m\ell}, \qquad \text{(no sum on m)}$$

$$A^m = \sqrt{\frac{\alpha^{m\ell}\alpha^{m\ell}}{\pi}}, \qquad \text{(no sum on m)}$$

$$\tilde{\beta}^{m\ell} = \frac{1}{B^m}\,\beta^{m\ell}, \qquad \text{(no sum on m)}$$

$$B^m = \sqrt{\frac{\beta^{m\ell}\beta^{m\ell}}{\pi}}. \qquad \text{(no sum on m)} \qquad \text{(III.10)}$$

From (III.9), we see that ψ carries renormalized charges whose magnitudes depend on α and β .

$$[K^m_\mu, \psi] = \frac{1}{\sqrt{\pi}} \tilde{\alpha}^{m\ell} \left\{ \alpha^{k\ell} g_{\mu\nu} + \beta^{k\ell} \gamma^5 \epsilon_{\mu\nu} \right\} \frac{\tau^k}{\sqrt{2}} \partial^\nu \Delta_0 \psi,$$

$$[K^{5m}_\mu, \psi] = \frac{1}{\sqrt{\pi}} \tilde{\beta}^{m\ell} \left\{ \alpha^{k\ell} \epsilon_{\mu\nu} + \beta^{k\ell} \gamma^5 g_{\mu\nu} \right\} \frac{\tau^k}{\sqrt{2}} \partial^\nu \Delta_0 \psi. \qquad \text{(III.11).}$$

In general, $K^{5m}_\mu \neq \varepsilon_{\mu\nu} K^{5m\nu}$, and K^m_μ and K^ℓ_ν do not commute for $m \neq \ell$. It should be noted here that a generalization of Lowenstein's normal product would have produced the results $K^m_\mu = j^m_\mu = \varepsilon_{\mu\nu} K^{5m\nu}$. Clearly, (III.9) better reflects the physical currents associated with ψ.

Equation of Motion and other Properties of ψ

Since ψ is constructed from the j^m_μ, we can write the stress-energy tensor:

$$\Theta_{\mu\nu} = \frac{\pi}{2} : \left\{ j^m_\mu j^m_\nu + j^{5m}_\mu j^{5m}_\nu \right\} :$$

$$= : \left\{ \partial_\mu \phi^m \partial_\nu \phi^m - \frac{g_{\mu\nu}}{2} \partial_\alpha \phi^m \partial^\alpha \phi^m \right\} : . \qquad \text{(III.12)}$$

Commuting $\theta_{\mu\nu}$ with ψ gives the transformation properties of ψ. In particular,

$$i\not\partial \psi = -\frac{\pi}{\sqrt{2}} : \left\{ A^{im} K^m_\mu + B^{im} K^{5m}_\mu \gamma^5 \right\} \gamma^\mu \tau^i \psi : ,$$

where $A^{im} = A^i \delta^{im}$, (no sum on i)

$B^{im} = B^i \delta^{im}$. (no sum on i) (III.13)

If the matrix $\tilde{\alpha}$ is non-singular, we can use its inverse $\tilde{\alpha}^{-1}$ to write:

$$i\not{\partial}\psi = -\frac{\pi}{\sqrt{2}} :\left\{A^{im} - B^{i\ell}\tilde{\beta}^{\ell n}(\tilde{\alpha}^{-1})^{nm}\right\}K^{m}{}_{\mu}\gamma^{\mu}\tau^{i}\psi: . \tag{III.14}$$

Similarly, for non-singular $\tilde{\alpha}$, the other linear combination of derivatives is

$$i\not{\partial}\gamma^{0}\psi = -\frac{\pi}{\sqrt{2}} :\left\{A^{im} + B^{i\ell}\tilde{\beta}^{\ell n}(\tilde{\alpha}^{-1})^{nm}\right\}K^{m}_{\mu}\gamma^{\mu}\gamma^{0}\tau^{i}\psi: . \tag{III.15}$$

We have not placed any constraints on the matrices α and β , and as a result the various components of ψ do not all have the same "spin" or dimension:

$$[M_{01}, \psi] = -i(x_0\partial_1 - x_1\partial_0)\psi - \frac{i}{4\pi}\alpha^{i\ell}\beta^{m\ell}\tau^{i}\tau^{m}\gamma^{5}\psi,$$

$$[D, \psi] = -ix\cdot\partial\psi - \frac{i}{8\pi}\left\{\alpha^{i\ell}\alpha^{m\ell} + \beta^{i\ell}\beta^{m\ell}\right\}\tau^{i}\tau^{m}\psi,$$

$$\left[\frac{dD}{dt}, \psi\right] = 0. \tag{III.16}$$

Thus, we have constructed a field ψ which satisfies a very general equation of motion involving the "diagonal" currents K^{i}_{μ} and K_{μ}^{5i}. Of course, usually we are interested in a more symmetrical case where, for instance, all components of ψ transform the same way under boosts and scale changes. Requiring such symmetry is equivalent to restricting the form of α and β .

To point out the fact that our 2n-component field still has the symmetry of the Thirring model displayed in (II.25), we note the effect of changing the sign of β. The forms of the interactions in equations (III.14) and (III.15) are interchanged. The transformation properties of $(1 + \gamma^5)\psi$ and $(1 - \gamma^5)\psi$ under Lorentz transformations are also exchanged. We see once again that the same physics can be associated with two different equations of motion.

There is nothing new in the physical picture generated by our new models. ψ once more produces waves of polarization moving at the speed of light. The difference is that there are n different types of dipoles involved.

Introduction of All the U(n) Currents

Up till now we have only dealt with n of the U(n) currents and n of the U(n) axial currents. Thus, the structures described above are equivalent to the theory of n Thirring spinor "stacked on top of each other". The only additional property we have added is the possibility of current-current interactions between the different Thirring spinors. As we have seen, the only effect of these interactions is to mix the currents with each other.

In order to study the model based on (III.2) and similar models, we must consider the "non-diagonal" currents formally defined (along with the "diagonal" ones) in (III.1). In addition, there are also the axial currents:

$$J_\mu^5 = \overline{\Psi}\gamma_\mu\gamma^5 \frac{\lambda^0}{\sqrt{2}}\Psi = \frac{1}{\sqrt{n}}\overline{\Psi}\gamma_\mu\gamma^5\Psi, \tag{III.17}$$

$$J_\mu^{5a} = \overline{\Psi}\gamma_\mu\gamma^5 \frac{\lambda^a}{\sqrt{2}}\Psi, \qquad a = 1,\ldots,n^2-1.$$

It would appear that we are greatly increasing the complexity of our models. In fact, if we let χ be a free, massless fermion with a U(n) internal symmetry in two dimensions, then all its associatd currents are conserved and curlless:

$$\partial_\mu :\overline{\chi}\gamma^\mu \frac{\lambda^0}{\sqrt{2}}\chi: \; = 0 = \; \epsilon_{\mu\nu}\partial^\mu :\overline{\chi}\gamma^\nu\frac{\lambda^0}{\sqrt{2}}\chi:,$$

$$\partial_\mu :\overline{\chi}\gamma^\mu \frac{\lambda^a}{\sqrt{2}}\chi: \; = 0 = \epsilon_{\mu\nu}\partial^\mu :\overline{\chi}\gamma^\nu\frac{\lambda^a}{\sqrt{2}}\chi:, \qquad a = 1,\ldots, n^2-1$$

$$:\overline{\chi}\gamma^\mu\gamma^5\frac{\lambda^0}{\sqrt{2}}\chi: \; = \; \epsilon^{\mu\nu} :\overline{\chi}\gamma_\nu\frac{\lambda^0}{\sqrt{2}}\chi:,$$

$$:\overline{\chi}\gamma^\mu\gamma^5\frac{\lambda^a}{\sqrt{2}}\chi: \; = \epsilon^{\mu\nu} :\overline{\chi}\gamma_\nu\frac{\lambda^a}{\sqrt{2}}\chi:, \qquad a = 1,..,n^2-1. \tag{III.18}$$

As a result, each of the currents satisfies a free, massless Klein-Gordon equation:

$$\partial^2 j_\mu^i = 0, \qquad i = 0, \ldots, n^2-1,$$
$$\text{where} \quad j_\mu^i = :\bar{\chi}\gamma_\mu \frac{\lambda^i}{\sqrt{2}}\chi: . \tag{III.19}$$

Further, there is a pole at zero mass-squared in the current commutator matrix elements:

$$\langle 0| [j_\mu^\ell(x), j_\nu^k(y)] |0\rangle = -\frac{i}{\pi}\delta^{\ell k}\partial_\mu\partial_\nu\Delta_0(x-y). \tag{III.20}$$

Thus, there seems to be n^2 bosons in the theory of a free, massless fermion, rather than the n bosons we would have used to construct the field (as in (III.5) with $\alpha^{\ell m} = \sqrt{\pi}\,\delta^{\ell m} = \beta^{\ell m}$).

On the other hand, the n current bosons we used to construct (III.5) can be thought of as a massless fermion-anti-fermion pair moving in the same direction. The pair is bound by the kinematics of massless particles in two dimensions into a massless boson. Clearly, the additional n^2-n bosons associated with the "non-diagonal" currents of a free fermion are likewise kinematically bound fermion-anti-fermion pairs. Thus, all n^2 bosons are obtained from the n fermion degrees of freedom.

Translating back into the picture of fermions as coherent state operators involving only the n bosons, ϕ^i, we see that the "diagonal" currents are formed by superimposing two polarization waves of opposite polarity. The result in the limit of coincident waves is a single dipole excitation, a linear combination of the $\partial_\mu\phi^i$'s (as in III.7). The "non-diagonal" currents are formed by superimposing two polarization waves composed of different types of dipoles. The result is a polarization wave involving the ϕ^i's which in the free fermion case ($\alpha^{\ell m} = \sqrt{\pi} = \beta^{\ell m}$) behaves like a "spin" one particle. These polarization wave bosons are the other n^2-n bosons in the free theory. Therefore, all the n^2 bosons in the free fermion theory can be described in terms of the n bosons originally used in constructing ψ.

The same situation holds in the case of general matrices α and β. We will see that the "non-diagonal" currents will be coherent state operators constructed from bilinear products of ψ^+ and ψ. Since there are only n fermion degrees of freedom, (III.5) is the most general field we can build out of free, massless currents, even after including the full U(n) structure. In particular, the same fields ψ which satisfy the equation of motion (III.14) involving only "diagonal" currents may also satisfy equations of motion involving all n^2 U(n) currents (and axial currents).

Definition of "Non-Diagonal" Currents

We define the "non-diagonal" current operators in terms of the limit of a regularized bilocal product in a manner similar to (II.26). Unless we are dealing with a particular case in which some of the n internal degrees of freedom of ψ coincide (such can be handled by the use of products like (II. 27)), the bilocals will not have vacuum expectation values, and the regularization will be simply Wick-ordering. The limits of the bilocals are unique operators:

$$R\left\{\psi^{\dagger}(x)\, \frac{1 \pm \gamma^5}{2}\, \lambda^i\, \psi(0)\right\} \xrightarrow[x \to 0]{} \frac{m}{2\pi} : \mathbb{C}^{\dagger} e^{-i\frac{\tau^j}{\sqrt{2}}[\alpha_{jk}\tilde{\phi}^k + \beta_{jk}\gamma^5\phi^k]} \cdot$$

$$\cdot\, \frac{1 \pm \gamma^5}{2}\lambda^i\, e^{i\frac{\tau^l}{\sqrt{2}}[\alpha_{lm}\tilde{\phi}^m + \beta_{lm}\gamma^5\phi^m]}\, \mathbb{C} : . \qquad \text{(III.21)}$$

However, in general, $R\{\psi^{+}(0)\, \frac{1 \pm \gamma^5}{2}\, \lambda^i\, \psi(0)\}$ is not suitable as a current because it does not transform like the component of a vector under Lorentz boosts. By commuting with $\theta_{\mu\nu}$, we can calculate the "spin" of the R-product which is a complex function of the α and β matrices. Thus, we find that a general interaction of the form in

(III.14), which obviously breaks the original U(n) symmetry of the free field theory, also destroys the normal Lorentz properties of the "non-diagonal" currents. In such models, these currents play roles, like the ψ field, as improper coherent state operators which transform one representation of the "diagonal" current algebra to an mathematically inequivalent one.

To discuss theories with n^2 vector currents, we must therefore confine ourselves to situations where the expression in (III.21) is composed only of terms with "spin" = ± 1. For the $(1 + \gamma^5)$ - component of (III.21), the terms with spin + 1 are analogous to the term proportional to $(\alpha + \beta)$ in the upper equation of (II.27), while the spin -1 terms are analogous to the $(\alpha - \beta)$ terms. The reverse is true for the $(1 - \gamma^5)$ - component. Proceeding analogously to (II.29), the currents and axial currents are defined as linear combinations of the terms in (III.21). For instance, the "non-diagonal" current components, J^i_R, are defined to be the sum of the spin + 1 components of $R\{ \psi^+(0) \frac{1 + \gamma^5}{2} \lambda^i \psi(0)\}$ and $R\{\psi^+(0) \frac{1 - \gamma^5}{2} \lambda^i (0)\}$, while the axial current components, J^{5i}_R, are the difference of the spin + 1 components. (The non-covariant appearance of equation (III.21) involving terms with different Lorentz properties on the right-hand side is due to the non-covariant definition for the regularized product $R\{..\}$ in such cases. This definition is notationally easier to handle. Of course, if we picked our currents out of the short distance expansion for the full product, $\psi^{+}(x)\,\frac{1 \pm \gamma 5}{2}\,\lambda^{i}\,(0)$, the Lorentz covariance of our prescription would be clear.)

Although the "non-diagonal" currents whose construction we have outlined are chosen to behave as vectors under Lorentz transformations, we have not specified whether they have canonical scale dimensions. They need not satisfy the canonical equal-time current commutation relations, and in general, are not conserved.

Transforming "Diagonal" Currents into "Non-Diagonal" Currents

We are now in a position to see whether the ψ field constructed in (III.5) does indeed satisfy equations of motion involving the "non-diagonal" currents. Since we already know that ψ satisfied an equation of motion with only "diagonal" currents, we are interested in converting products with "diagonal" currents into products with all the $U(n)$ currents. The easiest way to illustrate such relationships is to employ the properties of completeness and orthogonality of the $U(n)$ and Dirac matrices under the trace operation to derive a generalized

Fierz-type identity. For arbitrary matrices P and Q:

$$\mathrm{tr}[PQ] = \frac{1}{4}\left\{\mathrm{tr}[P\gamma_\mu\lambda^i]\,\mathrm{tr}[Q\gamma^\mu\lambda^i] + \mathrm{tr}[P\gamma^0\gamma^\mu\lambda^i]\,\mathrm{tr}[Q\gamma_\mu\gamma_0\lambda^i]\right\},$$

where i is summed from 0 to $n^2 - 1$. (III.22)

If we let $\tau(x)$ be an anti-commuting c-number 2n-component spinor, we can chose the specific matrices P and Q:

$$P_{\ell m} = \left\{\frac{1+\gamma^5}{2}\,\tau(x_1)\right\}_\ell \left\{\tau^\dagger(y_1)\,\frac{1+\gamma^5}{2}\right\}_m ,$$

$$Q_{\ell m} = \left\{\frac{1+\gamma^5}{2}\,\tau(x_2)\right\}_\ell \left\{\tau^\dagger(y_2)\,\frac{1+\gamma^5}{2}\right\}_m . \qquad \text{(III.23)}$$

Substituting into (III.22):

$$\left[\tau^\dagger(y_1)\,\frac{(1+\gamma^5)\lambda^0}{2}\,\tau(x_2)\right]\left[\tau^\dagger(y_2)\,\frac{(1+\gamma^5)\lambda^0}{2}\,\tau(x_1)\right] =$$

$$= -\frac{1}{n}\left[\tau^\dagger(y_1)\,\frac{(1+\gamma^5)\lambda^a}{2}\,\tau(x_1)\right]\left[\tau^\dagger(y_2)\,\frac{(1+\gamma^5)\lambda^a}{2}\,\tau(x_2)\right],$$

where a is summed from 0 to $n^2 - 1$. (III.24)

If we set $x_1 = x_2 = y_1 = y_2$, (III.24) becomes an identity which relates a product involving only the U(1) current, J^0_μ, to a product involving the n^2-1 currents, J^a_μ. We can derive identities for other current products by choosing other appropriate forms for the matrices P and Q.

To apply (III.24) and the other identities to operator products involving ψ and ψ^+, we begin with the point-split form ($x_1 \neq x_2$, etc.). The identity between multilocal operators obtained by substituting ψ and ψ^+ for τ and τ^+ can be converted to an identity for normal products by making the proper regularizations and short distance expansions on both sides of the equation. As a result, expressions involving only "diagonal" currents can be re-expressed in terms of all the currents.

A simple example (which has been discussed in a different context by Coleman, Gross, and Jackiw[(7)]) is the free fermion case. The free currents, j^i_μ, are defined in (III.19) and the axial currents are $j^{5i}_\mu = \varepsilon_{\mu\nu}\, j^{i\nu}$. The "diagonal" currents among the j^i_μ and j^{5i}_μ are denoted K^i_μ and K^{5i}_μ as in (III.6). (III.24) and the corresponding equation for the $(1 - \gamma^5)$ - components allow us to calculate the stress energy tensor:

$$\theta_{\mu\nu} = \frac{\pi}{2} :\left\{ K^i_\mu K^i_\nu + K^{5i}_\mu K^{5i}_\nu \right\}: ,$$

$$= \frac{\pi}{2} :\left\{ j^0_\mu j^0_\nu + j^{50}_\mu j^{50}_\nu + \frac{1}{n+1}\left[j^a_\mu j^a_\nu + j^{5a}_\mu j^{5a}_\nu \right]\right\}: ,$$

where a is summed from 1 to $n^2 - 1$. (III.25)

The first form of $\theta_{\mu\nu}$ is expressed only in terms of the "diagonal" currents, while the second form includes all n^2 currents. In deriving (III.25) it is important to do the short distance expansion carefully, since naive application of (III.24) would imply that the second form of $\theta_{\mu\nu}$ was zero. In particular, it should be noted that $\bar{\psi}\gamma_\mu \frac{\lambda^0}{\sqrt{2}} \psi$ has a vacuum expectation value which must be subtracted from the operator product while $\bar{\psi}\gamma_\mu \frac{\lambda^a}{\sqrt{2}} \psi$ does not, for a = 1,..., $n^2 - 1$ (see Appendix C).

Explicit Examples of U(n)-Type Models

We now consider four explicit examples of the types of structures we have outlined in general. The first two models are the two families of scale invariant solutions corresponding to (III.2), found by other methods by Dashen and Frishman. Along with scale invariance, they insisted that the currents satisfy the canonical equal-time algebra:

$$\left[J_\mu^l(x), J_\nu^m(y)\right]\Big|_{x^0=y^0} = \frac{i}{\pi}\,\delta^{lm}\,(g_\mu{}^0 g_\nu{}^1 + g_\mu{}^1 g_\nu{}^0)\,\delta'(x^1-y^1) +$$
$$+ \sqrt{2}\, i\, f^{lmn}\,(J_\mu^n(x)\, g_\nu{}^0 + J_\nu^n(x)\, g_\mu{}^0)\,\delta(x^1-y^1). \qquad \text{(III.26)}$$

The first class of solutions is a straightforward generalization of the normal Thirring model because $g^2 = 0$ in (III.2). The only coupling is the U(1) term. This class is specified by the matrices:

$$\alpha^{\ell m} = \sqrt{\pi}\left\{\delta^{\ell m} + (A^0 - 1)\delta^{\ell 0}\delta^{m0}\right\},$$

$$\beta^{\ell m} = \sqrt{\pi}\left\{\delta^{\ell m} + (B^0 - 1)\delta^{\ell 0}\delta^{m0}\right\}. \qquad \text{(III.27)}$$

A^0 and B^0 may take on any real values. The field ψ carries well-defined charges:

$$[J_\mu, \psi] = \left\{A^0 g_{\mu\nu} + B^0\gamma^5\epsilon_{\mu\nu}\right\}\partial^\nu\Delta_0\frac{\lambda^0}{\sqrt{2}}\psi,$$

$$[J^a_\mu, \psi] = \left\{g_{\mu\nu} + \gamma^5\epsilon_{\mu\nu}\right\}\partial^\nu\Delta_0\frac{\lambda^a}{\sqrt{2}}\psi,$$
$$a = 1,\ldots, n^2 - 1. \qquad \text{(III.28)}$$

The Poincaré properties are summarized:

$$i\partial\!\!\!/\psi = -\pi(A^0 - B^0) : J_\mu\gamma^\mu\frac{\lambda^0}{\sqrt{2}}\psi: ,$$

$$i\partial\!\!\!/\gamma^0\psi = -\pi\left\{(A^0 + B^0 - 2) : J_\mu\gamma^\mu\gamma^0\frac{\lambda^0}{\sqrt{2}}\psi: + 2 : K^i_\mu\gamma^\mu\gamma^0\frac{\tau^i}{\sqrt{2}}\psi:\right\},$$

$$= -\pi\left\{(A^0 + B^0) : J_\mu\gamma^\mu\gamma^0\frac{\lambda^0}{\sqrt{2}}\psi: + \frac{2}{n+1} : J^a_\mu\gamma^\mu\gamma^0\frac{\lambda^a}{\sqrt{2}}\psi:\right\},$$

$$[D, \psi] = -ix\cdot\partial\psi - \frac{i}{2n}\left\{n - 1 + \frac{(A^0)^2 + (B^0)^2}{2}\right\}\psi,$$

$$[M_{01}, \psi] = -i(x_0\partial_1 - x_1\partial_0)\psi - \frac{i}{2n}\left\{n - 1 + A^0B^0\right\}\gamma^5\psi. \qquad \text{(III.29).}$$

If we want ψ to have spin 1/2, $A^0B^0 = 1$. Also, the U(1) coupling constant is $G = -\pi(A^0 - B^0)$. Although ψ has an anomalous dimension in general, the currents are simply the free currents. Thus, this class of solutions is essentially a free spinor with an overall U(1) interaction which renormalizes the U(1) charges. The physical picture of the state created by operating with ψ^{+} on the vacuum is a free fermion surrounded by a boson coherent state of J_{μ} -type dipoles extending to infinity.

By using the symmetry of the Thirring constructions we have emphasised before, we can reach the second class of solutions singled out by Dashen and Frishman from our first class. The transformation is performed by changing the sign of β. The matrices specifying the solution are:

$$\alpha^{\ell m} = \sqrt{\pi}\left\{\delta^{\ell m} + (A^{o}-1)\delta^{\ell o}\delta^{m o}\right\},$$

$$\beta^{\ell m} = \sqrt{\pi}\left\{-\delta^{\ell m} + (1-B^{o})\delta^{\ell o}\delta^{m o}\right\}. \qquad \text{(III.30)}$$

The axial charges are now found to have changed sign:

$$[J_{\mu}, \psi] = \left\{A^{o} g_{\mu\nu} - B^{o}\gamma^{5}\epsilon_{\mu\nu}\right\}\partial^{\nu}\Delta_{o}\frac{\lambda^{o}}{\sqrt{2}}\psi,$$

$$[J_{\mu}^{a}, \psi] = \left\{g_{\mu\nu} - \gamma^{5}\epsilon_{\mu\nu}\right\}\partial^{\nu}\Delta_{o}\frac{\lambda^{a}}{\sqrt{2}}\psi. \qquad \text{(III.31)}$$

Finally, the Poincaré properties become:

$$i\partial\!\!\!/\psi = -\pi\left\{(A^0+B^0-2):J_\mu\gamma^\mu\frac{\lambda^0}{\sqrt{2}}\psi: + 2:K^i_\mu\gamma^\mu\frac{\tau^i}{\sqrt{2}}\psi:\right\},$$

$$= -\pi\left\{(A^0+B^0):J_\mu\gamma^\mu\frac{\lambda^0}{\sqrt{2}}\psi: + \frac{2}{n+1}:J^a_\mu\gamma^\mu\frac{\lambda^a}{\sqrt{2}}\psi:\right\},$$

$$i\partial\!\!\!/\gamma^0\psi = -\pi\left\{(A^0-B^0):J_\mu\gamma^\mu\frac{\lambda^0}{\sqrt{2}}\psi:\right\},$$

$$[D,\psi] = -ix\cdot\partial\psi - \frac{i}{2n}\left\{n-1+\frac{(A^0)^2+(B^0)^2}{2}\right\}\psi,$$

$$[M_{01},\psi] = -i(x_0\partial_1-x_1\partial_0)\psi - \frac{i}{2n}\left\{-n+1-A^0B^0\right\}\gamma^5\psi. \qquad \text{(III.32)}$$

In this class of solutions, ψ has spin 1/2 if $A^0B^0 = -2n+1$. The coupling constants are $G = -\pi(A^0+B^0)$ and $g = -\frac{2\pi}{n+1}$. Note that the minimum dimension for ψ, if we choose the spin to be 1/2, is $\frac{3}{2} - \frac{1}{n}$. Thus, ψ never has a single fermion state associated with it in this class of solutions. This result is hardly surprising if we consider the explicit construction. As we indicated, the first class of solutions consists of a field which is free with respect to the SU(n) currents, with only a U(1) interaction. Changing the sign of β exchanges the left and right chiral components of ψ, leaving the free field dynamics intact, but reversing the spin. Modifying the U(1) interaction allows us to choose spin 1/2 again. In other words, a right chiral component of ψ in this second class of solutions is a free left chiral component of a free field plus a coherent cloud of right-moving U(1)-type bosons. Thus, the dynamics of these models is essentially the same as the first class of solutions.

If we choose ψ to have spin 1/2 and canonical commutation relations for the currents, in our scale invariant models, the only other acceptable forms for α and β are unitary transformations of the two classes described above. Such solutions would differ from each other only in the redefinition of the original fields ϕ^i and $\tilde{\phi}^i$ (or j_μ^i) used to construct ψ . The uniqueness of our solutions can be verified by looking at the details of the construction. The essence of the arugment is that in order to maintain the canonical scale of the currents and their equal-time commutation relations, the only change from the free field case allowed in the structure of ψ is a U(1)-type of coherent state operator which drops out of all the SU(n) currents.

We now turn to other possible models involving all n^2 currents. There is a choice of maintaining the SU(n) symmetry but abandoning scale invariance, or maintaining scale invariance with anomalous current dimensions but breaking the symmetry. The former are not exactly soluble and we will discuss some of the possibilities in Chapter V. The latter are easily constructed from our general structures. Our other two classes of solutions are simple examples.

Let the matrices α and β be:

$$\alpha^{lm} = \sqrt{\pi}\left\{\delta^{lm} + (C-1)\delta^{l1}\delta^{m1} + (A^0-1)\delta^{l0}\delta^{m0}\right\},$$

$$\beta^{lm} = \sqrt{\pi}\left\{\delta^{lm} + (\tfrac{1}{C}-1)\delta^{l1}\delta^{m1} + (B^0-1)\delta^{l0}\delta^{m0}\right\}. \qquad \text{(III.33)}$$

A^0 and B^0 are real parameters as before, and C is any non-zero real number. The solution constructed using (III.33) obviously has the SU(n) symmetry broken along the direction corresponding to the K^1_μ current. The Poincaré structure is given:

$$i\partial\!\!\!/\psi = -\pi :\left\{(A^0-B^0)\, J_\mu\gamma^\mu\frac{\lambda^0}{\sqrt{2}}\psi + (C-\tfrac{1}{C})\, K^1_\mu\gamma^\mu\frac{\tau^1}{\sqrt{2}}\psi\right\}:,$$

$$i\partial\!\!\!/\gamma^0\psi = -\pi:\left\{(A^0+B^0-2)J_\mu\gamma^\mu\gamma^0\frac{\lambda^0}{\sqrt{2}}\psi + 2K^i_\mu\gamma^\mu\gamma^0\frac{\tau^i}{\sqrt{2}}\psi + \right.$$
$$\left. + (C+\tfrac{1}{C}-2)K^1_\mu\gamma^\mu\gamma^0\frac{\tau^1}{\sqrt{2}}\psi\right\}:$$
$$= -\pi:\left\{(A^0+B^0)\, J_\mu\gamma^\mu\gamma^0\frac{\lambda^0}{\sqrt{2}}\psi + \frac{2}{n+1}J^a_\mu\gamma^\mu\gamma^0\frac{\lambda^a}{\sqrt{2}}\psi + \right.$$
$$\left. + (C+\tfrac{1}{C}-2)\frac{1}{n+1}K^1_\mu\gamma^\mu\gamma^0\frac{\tau^1}{\sqrt{2}}\psi\right\}:,$$

$$[D,\psi] = -ix\cdot\partial\psi - \frac{i}{2n}\left\{n-1+\frac{(A^0)^2+(B^0)^2}{2} + n(\tau^1)^2\frac{(C-1)^2}{2C}\right\}\psi,$$

$$[M_{01},\psi] = -i(x_0\partial_1 - x_1\partial_0)\psi - \frac{i}{2n}\left\{n-1+A^0B^0\right\}\gamma^5\psi. \qquad \text{(III.34)}$$

The currents J^a_μ which are orthogonal to the direction of the breaking, i.e. for which $[\lambda^i, \tau^1] = 0$, have canonical structure. The remaining currents have pieces with anomalous dimensions whose value depends on C. These pieces lead to non-conservation of the currents:

$$\partial_\mu J^{a\mu} = (c - \tfrac{1}{c})\, f^{ab} :K^1_\mu J^{b\mu}: ,$$

$$\partial_\mu J^{5a\mu} = (c - \tfrac{1}{c})\, f^{ab} :K^1_\mu J^{5b\mu}: ,$$

$$\text{where } f^{ab} = -\frac{i}{\sqrt{8}}\, \mathrm{tr}\left\{[\lambda^a, \tau^1]\lambda^b\right\}. \qquad \text{(III.35)}$$

The physical reason for this non-conservation can be found by considering the explicit form of the currents. In the previous examples where the currents were all conserved, their structure was the same as in the free case. The components J^a_R could be interpreted as consisting of a massless fermion-anti-fermion pair moving to the right at the spead of light. Therefore, $\partial_L J^a_R = 0$. The analogous behavior for J^a_L gives us:

$$\partial_\mu J^{a\mu} = \partial_L J^a_R + \partial_R J^a_L = 0. \qquad \text{(III.36).}$$

However, for the currents in (III.35), the K^1_μ interaction has changed some of the components ψ_R from pure right-moving coherent state operators to the product of right- and left-moving coherent state operators as in (II.44). Thus, the J^a_R now depends on x_L as well as x_R, and the non-conservation of the currents is due to the fact that the charged excitations separate along opposite lightlike directions.

The final example is the $\beta \to -\beta$ counterpart of the above example. Once again the current structure is maintained but the equations of motion are reversed:

$$\alpha^{\ell m} = \sqrt{\pi}\left\{\delta^{\ell m} + (C-1)\delta^{\ell 1}\delta^{m1} + (A^0-1)\delta^{\ell 0}\delta^{m0}\right\},$$

$$\beta^{\ell m} = \sqrt{\pi}\left\{-\delta^{\ell m} - (\tfrac{1}{C}-1)\delta^{\ell 1}\delta^{m1} - (B^0-1)\delta^{\ell 0}\delta^{m0}\right\},$$

$$i\partial\!\!\!/\psi = -\pi :\left\{(A^0+B^0)J_\mu\gamma^\mu\frac{\lambda^0}{\sqrt{2}}\psi + \frac{2}{n+1}J_\mu^a\gamma^\mu\frac{\lambda^a}{\sqrt{2}}\psi + \right.$$
$$\left. + (C+\tfrac{1}{C}-2)\frac{1}{n+1}K_\mu^1\gamma^\mu\frac{\tau^1}{\sqrt{2}}\psi\right\}:,$$

$$i\partial\!\!\!/\gamma^0\psi = -\pi :\left\{(A^0-B^0)J_\mu\gamma^\mu\gamma^0\frac{\lambda^0}{\sqrt{2}}\psi + (C-\tfrac{1}{C})K_\mu^1\gamma^\mu\gamma^0\frac{\tau^1}{\sqrt{2}}\psi\right\}:,$$

$$[D,\psi] = -ix\cdot\partial\psi - \frac{i}{2n}\left\{n-1+\frac{(A^0)^2+(B^0)^2}{2} + n(\tau^1)^2\frac{(C-1)^2}{2C}\right\}\psi,$$

$$[M_{01},\psi] = -i(x_0\partial_1 - x_1\partial_0)\psi - \frac{i}{2n}\left\{1-n-A^0B^0\right\}\gamma^5\psi. \quad \text{(III.37)}.$$

Even though the U(n) symmetry is explicitly broken, we find we can write the equations of motion in terms of all the n^2 U(n) currents.

<u>The Physical Picture of the Generalized Models</u>

As we pointed out before, the physical picture associated with all these structures is a simple extension of the picture we developed for the Thirring model. The only new piece of the models is the inclusion of the "non-diagonal" currents which are additional coherent state operators similar to the components of ψ . If the symmetry of the interaction is sufficiently great, some of these additional currents behave like single particle operators in the same way that the "diagonal" currents, K_μ^i , do. We conclude by remarking that our choice of "diagonal" and "non-diagonal" is, of course, not unique. We could have constructed ψ from any set of n of the U(n) matrices which all commute, rather than our particular choice, the τ^i.

IV. TWO-DIMENSIONAL QED AND GENERALIZATIONS

In Chapters II and III, we considered the Thirring model and some exactly soluble generalizations of the model with internal degrees of freedom. One common characteristic of these models is that the interactions do not change the physical spectrum from that of the theory of free fermions. It is true that the fields ψ which are defined by (II.30) and (III.5), and which satisfy the equations of motion associated with the various formal Lagrangians are different in structure from the free fermion field, in general. For instance, the two-point function, $\langle 0 \mid \psi(x)\bar{\psi}(y) \mid 0\rangle$, differs from the free two-point function indicating the ψ^+ acting on the vacuum does not create the same states as the free field acting on the vacuum. However, by examining the explicit structure of ψ, we found that the states created by ψ^+ acting on the vacuum are, in general, actually multiparticle states in which the individual particles are free quanta from the free theory. Thus, the physical spectrum of the Thirring model and its scale invariant generalizations is the same as the spectrum of free theories.

On the other hand, quantum electrodynamics with massless fermions has also been solved exactly in two space-time dimensions,[5,13,20,22] and the physical spectrum does not contain any fermions. Instead, as we shall see explicitly, the only particles in the theory are free, massive bosons which represent dynamically bound fermion-anti-fermion pairs. Casher, Kogut, and Susskind[6] noted that this model exhibits properties which can be interpreted as analogues of scaling and confinement, phenomena which we discussed in the introduction. Two-dimensional quantum electrodynamics is a theory with only free particles, so we cannot analyze any dynamical situations within the theory itself. However, Casher, Kogut, and Susskind pointed out that if the scalar density, $\bar{\psi}\psi$, is coupled to an external field, the effect of the external sources is to produce (or scatter) pairs of excitations which would be fermions, if the electromagnetic interaction were turned off. With the interaction present, the excitations crumble into clouds of massive bosons. Still, when the external source has very high frequency components, i.e. the source is pumping a lot of energy into the system, the excitations resemble free fermions over appreciable times, providing a model for the scaling which is found in deep inelastic electroproduction. The fact that the spectrum consists of bosons only models the confinement of quarks within hadrons. Casher, Kogut, and Susskind went on to elaborate on the analogy between the two-dimensional theory and real scattering in four dimensions.

It is difficult to extract a model for detailed dynamics from two-dimensional quantum electrodynamics, since the theory involves only free massive bosons. Furthermore, there is an ambiguity concerning the type of confinement at work in the model. A qualitative distinction can be made between two different sorts of confinement. The first, which we can denote as "shielding", involves the neutralization of a particular property of the interacting particles. In the example of QED, this property is the electric charge. Thus, if shielding is at work, the electric charge of all the particles in the theory is confined or neutralized. No electrically charged particles can be isolated. However, other properties such as the statistics or other varieties of charge associated with the original particles may still be carried by the electrically shielded particles.

In contrast, the confinement may be "total", in which none of the quantum numbers associated with the original particles can be isolated once the interactions are present. The properties of the original particles are reflected in the final physical spectrum only as contributions to the composite properties of the bound states which make up that spectrum. Total confinement, not shielding, is the mechanism which is relevant to the Yang-Mills type of models that are, at present, favored candidates as descriptions of the dynamics of hadrons. In these models, the quarks (fermionic constituents of the hadrons) carry Yang-Mills or "color" charge. Since color charged states have never been isolated, a confinement of the color charge is expected in the dynamics. Total confinement of the quarks is necessary, in most of the models, because the quarks also carry fractional electric charge. If the color charge were simply shielded, physical particles with fractional electric charge could be isolated.

Two-dimensional QED is not complex enough a theory to distinguish between shielding and total confinement. The charged fermion can only be neutralized by its antiparticle. The resulting neutral states cannot carry any quantum numbers because the antifermion's quantum numbers cancel the fermion's contribution exactly. Feynman[29] suggested that studying generalizations of two-dimensional QED in which there is more than one massless, charged fermion, or in which there are massive fermions could possibly yield a better model for the dynamics and clarify the question of the type of confinement present in the models. The latter models with massive fermions are not exactly soluble, and we will discuss them in Chapter V. The former models are exactly soluble. In this chapter, we will present a solution which is a generalization of the operator solution of Lowenstein and Swieca[20] to two-dimensional QED with a single fermion. Using the physical picture of the Thirring model we have developed, we discuss the type of confinement appearing in the model. Our conclusions agree with those reached by Feynman through a perturbative approach.

Operator Solution of QED with a Single Massless Fermion

We begin by reviewing the operator solution of Lowenstein and Swieca to QED with a single massless fermion, emphasizing the similarity to the Thirring model constructions. The formal mathematical details are thoroughly covered in their paper.

The formal Lagrangian with which we begin is:

$$\mathcal{L} = \overline{\Psi}(i\not{\partial} - e\not{A})\Psi + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}F_{\mu\nu}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}). \qquad \text{(IV.1)}$$

The resulting equations of motion are:

$$i\not{\partial}\Psi = e\not{A}\Psi,$$

$$\partial^{\mu}F_{\mu\nu} = \partial^{2}A_{\nu} - \partial_{\nu}\partial^{\mu}A_{\mu} = -e\overline{\Psi}\gamma_{\nu}\Psi = -eJ_{\nu}. \qquad \text{(IV.2)}$$

The latter equations yield the formal equations for J_{μ} and J_{μ}^{5}, the axial current:

$$\partial^{\mu}J_{\mu} = 0,$$

$$\partial^{\mu}J_{\mu}^{5} = \partial^{\mu}\epsilon_{\mu\nu}J^{\nu} = -e\epsilon_{\mu\nu}\partial^{2}\partial^{\mu}A^{\nu}. \qquad \text{(IV.3)}$$

In comparison, the analogous Thirring model equations are:

$$i\not{\partial}\Psi = -\sqrt{\pi}(\alpha - \beta):\not{J}\Psi:,$$

$$\partial_{\mu}J^{\mu} = 0 = \partial_{\mu}J^{5\mu}. \qquad \text{(IV.4)}$$

The crucial point in the Thirring construction is that we were able to express ψ in terms of dipole fields ϕ and $\tilde{\phi}$, whose gradients and curls produce the current and axial current. The conservation of both of the currents implied that ϕ (and $\tilde{\phi}$) were free, massless fields. Equations (IV.3) imply that the polarization density in QED, whose derivatives yield the electromagnetic current, satisfies a more complex equation. It is easy to see from Lowenstein and Swieca's construction that the field we identify as the polarization density will be a free, massive boson, with mass-squared $\frac{e^2}{\pi}$. Denoting the density as σ, we find:

$$(\partial^2 + \frac{e^2}{\pi})\,\sigma = 0,$$

$$[\sigma(x), \sigma(y)] = i\Delta_{e^2/\pi}(x-y). \qquad \text{(IV.5)}$$

We can make a canonical momentum space decomposition of σ in terms of creation and annihilation operators:

$$\sigma(x) = \int \frac{dK/4\pi}{\sqrt{K^2 + \frac{e^2}{\pi}}} \left\{ a(K)\, e^{-ik\cdot x} + a^{\dagger}(K)\, e^{ik\cdot x} \right\}. \qquad \text{(IV.6)}$$

In analogy to the Thirring model, we formally define another polarization field corresponding to $\tilde{\phi}$:

$$\tilde{\sigma}(x) = \int \frac{dK}{4\pi K} \left\{ a(K)\, e^{-ik\cdot x} + a^{\dagger}(K)\, e^{ik\cdot x} \right\}. \tag{IV.7}$$

At this point, we would expect to follow the same route as in the Thirring model. The spinor field, carrying a charge and axial charge, should be expressible as a coherent state operator of the dipole fields. However, as in the Thirring case, there are infrared difficulties. $\tilde{\sigma}$ is not a well-defined field. For instance, it is easy to calculate the two-point function, $<0 \mid \tilde{\sigma}(x)\tilde{\sigma}(0) \mid 0>$. The result is infrared divergent. The problem is more serious than in the Thirring construction, where we found that the infrared divergences came from infinite numbers of quanta with zero energy. Physically, there was no real difficulty since the energy and momenta all remained finite. In QED, the polarization quanta are massive, and states with infinite numbers of quanta must involve infinite energies.

We can see the same problems from the coherent state viewpoint if we try to construct ψ in analogy to the Thirring spinor:

$$\psi = :e^{i[\alpha\tilde{\sigma} + \beta\gamma^5\sigma]}\left[\frac{e^2}{4\pi^3}\right]^{1/4}\mathbb{C}: . \tag{IV.8}$$

Once again, : : indicates Wick-ordering of the operators, and $\mathbb{C}$ is the "place-holding" operator. If ψ is a well-defined operator, we can follow the arguments that led to (II.43) to conclude that it carries an electric charge $\frac{-\alpha e}{\sqrt{\pi}}$, since

$$eQ = \frac{e}{\sqrt{\pi}}(\sigma(\infty) - \sigma(-\infty)). \tag{IV.9}$$

(Recalling our heuristic discussion of the significance of the polarization fields in Chapter II, we should note that in QED the polarization field is proportional to the electric field: $E(x) = e\sigma(x)$. As a result, (IV.9) is a two-dimensional version of Gauss' Law.) To see whether ψ is a proper operator, we refer back to our discussion of coherent states of massive, free fields. ψ has the form (II.37) with the functions g and h given by (II.42). Thus, the necessary conditions for ψ to be well-defined mathematically, (II.35) and (II.36), are violated. The ultraviolet divergences can be avoided, as in the case of the Thirring model, by smearing the local field operator. However, in contrast to the Thirring model, both (II.35) and (II.36) are violated by an infrared divergence. Thus, ψ is associated with not only infinite numbers of quanta but also infinite energies.

Physically, the problem is clear, since the state created by acting with ψ on the vacuum has a non-zero expectation value for the electric field (or massive polarization field) at $x^1 = \pm\infty$ (for $\alpha \neq 0$). Such states require infinite energies to construct, since the energy density involves the electric field term, $\frac{E^2}{2\pi}$ (or the mass term for the polarization, $\frac{1}{2}\frac{e^2}{\pi}\sigma^2$). On the other hand, it is exactly the behavior at spatial infinity which allows ψ to carry an electric charge, as we saw using (IV.9). Thus, we find that physical states with finite energies must have no net electric charge. We see that there is a qualitative difference between the infrared structure in the Thirring model and QED. In the former case, ψ connects states with finite numbers of quanta and energy to states with infinite numbers of quanta and finite energy. In the latter, states with finite energy are connected to states with infinite energy. Therefore, in QED, although ψ is the field operator in the equations of motion, actual physical states can be described only by acting on the vacuum with electrically neutral operators like $\bar{\psi}(x)\psi(y)$. As we shall see later, this requirement is natural since the invariance of QED under gauge transformations implies that all the physics must be contained in the gauge invariant operators of the theory which cannot carry a net charge. Physically, the infinite energy of the state associated with ψ implies that there are no states in the spectrum which we can identify as single fermion states. The elementary particle we would have identified as the fermion of the theory is confined within the bound states which make up the spectrum.

To continue with the construction, ψ can be made a well-defined operator by employing a non-invariant cut-off in a positive-definite Hilbert space or by introducing ghosts which cancel the infrared divergences.[20] Even with such regularizations, Lowenstein and Swieca point out that (in some gauges) the gauge-dependent matrix elements will show pathological behavior. In contrast, gauge-invariant quantities all behave properly. As in the Thirring model, we will ignore the infrared corrections (detailed by Lowenstein and Swieca) since the physics is properly represented by our mathematically improper operators. It will even turn out that there is a set of gauges in which the infrared problem vanishes entirely.

If we return to (IV.7), we find that, unlike $\tilde{\phi}$ in the Thirring model, $\tilde{\sigma}$ is non-covariant. In addition,

$$\partial_\mu \tilde{\sigma} = \epsilon_{\mu\nu}\partial^\nu \sigma - \frac{ie^2}{\pi} g_\mu{}^0 \int \frac{dK}{4\pi K\sqrt{K^2+\frac{e^2}{\pi}}} \left\{a(K)e^{-ik\cdot x} - a^\dagger(K)e^{ik\cdot x}\right\}. \quad \text{(IV.10)}$$

The explicit non-covariance of $\tilde{\sigma}$ implies that ψ in (IV.8) is in general non-covariant. However, this Lorentz structure is acceptable if we are in a gauge which is not covariant. We shall see that for $\alpha \neq 0$ that is the case.

Keeping all the preceding mathematical discussion and our Thirring construction in mind, we now can write down the QED fields in terms of σ and $\tilde{\sigma}$:

$$\psi = :e^{i[\alpha\tilde{\sigma} - \sqrt{\pi}\gamma^5\sigma]}\left[\frac{e^2}{4\pi^3}\right]^{1/4} \mathbb{C}:,$$

$$J_\mu = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\sigma,$$

$$A_\mu = -\frac{1}{e}(\alpha\partial_\mu\tilde{\sigma} + \sqrt{\pi}\,\epsilon_{\mu\nu}\partial^\nu\sigma),$$

$$F_{\mu\nu} = \epsilon_{\mu\nu}\frac{e}{\sqrt{\pi}}\sigma. \tag{IV.11}$$

These fields then satisfy the QED equations of motion:

$$i\not{\partial}\psi = :e\not{A}\psi:,$$

$$\partial^\mu F_{\mu\nu} = -eJ_\nu. \tag{IV.12}$$

β has been set to $\sqrt{\pi}$ by the requirement that J_μ satisfies the canonical equal-time commutation relations and that ψ carries charge e (in the Coulomb gauge discussed below, or in the indefinite metric solution in Appendix D). Looking at the equation for A_μ, we see that the separation into $\partial_\mu\tilde{\sigma}$ and $\epsilon_{\mu\nu}\partial^\nu\sigma$ is a unique (up to free, massless contributions) decomposition into curlless and divergenceless components, respectively. Furthermore, the $\tilde{\sigma}$ factor in ψ can be thought of as co-ordinate dependent overall phase multiplying the rest of the spinor. As a result, the choice of α corresponds to the choice of a type of gauge. In particular, $\alpha = 0$ is a Lorentz gauge ($\partial_\mu A^\mu = 0$), while $\alpha = -\sqrt{\pi}$ is a Coulomb gauge ($\partial_1 A^1 = 0$). As we indicated before, only $\alpha = 0$ is a covariant gauge. Moreover, in this gauge the infrared problems associated with $\tilde{\sigma}$ are absent.

Normal Products and Gauge Invariance

To complete the operator solution, we must verify that J_μ is indeed the current associated with ψ. We are led once more to the question of normal products. Actually, the construction of normal products of several fields at the same point is entirely analogous to the constructions in the Thirring model. Once again, all the operators can be decomposed invariantly into annihilation and creation parts. Thus, we can make short-distance expansions of multilocal products in terms of Wick-ordered products of the free σ field. After the vacuum expectation terms have been eliminated, the leading operator with the correct Lorentz transformation properties in the short-distance expansion of the multilocal product is the normal product. For finite co-ordinate separations in the multilocal product, the explicit expansion is complicated because the singular function $\Delta_{e^2/\pi}$ is not a simple logarithm like Δ_0, which appeared in the Thirring model. However, as the limit of coincident points is approached, $\Delta_{e^2/\pi} \to \Delta_0$, except that e^2/π replaces the cut-off mass-squared, $\mathcal{m}^2$, used to define Δ_0. Thus, the normal product structure in QED exactly parallels the structure in the Thirring model.

If we proceed to construct the current from the bilocals $\{\psi^+(x) \frac{1+\gamma^5}{\sqrt{2}} \psi(y)\}$, we find, following (II.26):

$$x_L R\{\psi^\dagger(x) \tfrac{1+\gamma^5}{\sqrt{2}} \psi(0)\} \xrightarrow[x\to 0]{} \tfrac{1}{\pi}\left[-\alpha x_\mu \partial^\mu \tilde{\sigma} - \sqrt{\pi}\, x_\mu \epsilon^{\mu\nu}\partial_\nu \sigma + \ldots\right],$$

$$x_R R\{\psi^\dagger(x) \tfrac{1-\gamma^5}{\sqrt{2}} \psi(0)\} \to \tfrac{1}{\pi}\left[-\alpha x_\mu \partial^\mu \tilde{\sigma} + \sqrt{\pi}\, x_\mu \epsilon^{\mu\nu}\partial_\nu \sigma + \ldots\right].$$

(IV.13)

The currents formed from (IV.13) would involve $\partial^\mu \tilde{\sigma}$. They are not only non-covariant, but they disagree with J^μ in (IV.11). This disturbing result occurs because we have not implemented gauge invariance. Formally, the gauge invariant current in QED is defined as the limit of the fermion bilocal with a gauge integral inserted. Since we are dealing with a free field in our construction, this gauge invariant bilocal,

$\psi_\alpha^+(x) e^{ie\int_x^y A_\mu(\xi) d\xi^\mu} \psi_\beta(y)$, can be explicitly evaluated. The gauge integral is path-dependent. The simplest convention is to choose the path to be always taken along the straight line between the two points. Using this prescription, we find that the non-covariant piece of $\partial^\mu \tilde{\sigma}$ is cancelled and the α dependence is removed from (IV.13):

$$x^\mu R\{\overline{\Psi}(x)\gamma_\mu e^{ie\int_x^0 A_\nu d\xi^\nu}\psi(0)\} \to \frac{1}{\sqrt{\pi}}[x^\mu \epsilon_{\mu\nu}\partial^\nu\sigma + \ldots],$$

$$x^\mu \epsilon_{\mu\nu} R\{\overline{\Psi}(x)\gamma^\nu e^{ie\int_x^0 A_\rho d\xi^\rho}\psi(0)\} \to \frac{1}{\sqrt{\pi}}[x^\mu \partial_\mu\sigma + \ldots]. \tag{IV.14}$$

Thus, we can define

$$J_\mu = \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu}\partial^\nu\sigma,$$

$$J_\mu^5 = \frac{1}{\sqrt{\pi}} \partial_\mu\sigma. \tag{IV.15}$$

and (IV.11) does constitute a set of operator solutions of QED.

At this point we see that the Coulomb gauge is a more physical gauge, since the gauge integral connecting a $\bar{\psi}$ - ψ pair in a gauge-invariant bilocal at equal times is unnecessary in this gauge. A gauge-invariant bilocal at equal times, like $\bar{\psi}(x)e^{ie\int_x^y A_\mu d\xi^\mu}\psi(y)$, represents a charge -e at x, a charge e at y, and an electric field extending between the two. In the Coulomb gauge ψ already carries the electric field with it. At the other extreme, in the Lorentz gauge, ψ has no electric field ($\dot{\alpha} = 0$) and the gauge integral is needed to set up the field between x and y. On the other hand, it is precisely the electric field in the Coulomb gauge which caused all the mathematical difficulties we discussed earlier. The Lorentz gauge avoids all the problems associated with $\tilde{\sigma}$. Furthermore, we can effectively demonstrate the gauge freedom available in the Lorentz gauges by employing our Thirring constructions. If $\tilde{\phi}$ and j^μ are defined as in (II.8), we can make an entire class of operator gauge transformations:

$$\psi(x) \longrightarrow :e^{i\lambda\tilde{\phi}(x)}\psi(x):,$$

$$eA_\mu(x) \longrightarrow eA_\mu(x) - \lambda j_\mu(x). \qquad \text{(IV.16)}$$

With the gauge freedom in (IV.16) we have reproduced Lowenstein and Swieca's class of positive-definite Hilbert space solutions, except that our Thirring construction eliminates the free spinor χ implicit in their construction. Clearly, gauge operators like ψ and A_μ can be made to include any sort of Thirring contribution, but the gauge-invariant operators like J_μ, $F_{\mu\nu}$, etc., which contain all the physics, are unaffected.

The Poincaré transformation properties of the QED operators are determined by the stress-energy tensor:

$$\Theta_{\mu\nu} = :\left\{\partial_\mu\sigma\,\partial_\nu\sigma - g_{\mu\nu}\left[\partial^\alpha\sigma\,\partial_\alpha\sigma - \frac{e^2}{\pi}\sigma^2\right]\right\}. \qquad \text{(IV.17)}$$

Of course, a Thirring component like (II.14) should be added if we make a gauge transformation like (IV.16). Since only the Lorentz gauge is covariant, that is the only gauge whose Poincaré properties are interesting. We find that the spin, as defined in the Thirring models, of ψ is zero. In fact, ψ commutes with $\bar{\psi}$ at space-like distances. This odd behavior is due to the fact that the operator solution has been constructed in a positive definite Hilbert space. If we embed our operator structure in an indefinite metric space, we encounter Gupta-Bleuler type solutions in which the space of physical states is only a subspace of the full Hilbert space. In such solutions, ψ, J_μ, and A_μ can have canonical spin and equal-time commutation relations. Perturbation theory and functional methods[13,22] lead to these indefinite metric solutions. In Appendix D, we outline Lowenstein and Swieca's construction of these operator solutions.

The Physical Picture of Two-Dimensional QED with a Single Massless Fermion

Once again, we can consider QED as a theory of point charges or a theory of polarization fields. The connection between these two alternate frameworks is the fact that we have constructed the field corresponding to the point charges, ψ, as an improper coherent state operator of the polarization bosons. As we have seen, if we start from the polarization field framework, the model becomes extremely simple. The dipoles turn out to be free, massive bosons. In contrast to the Thirring model, multiparticle boson states disperse because the mass allows the velocities to range from zero to the speed of light. The coherent states of the dipoles are not kinematically bound as in the massless case. Thus, the physical spectrum consists of massive dipoles alone, without any of the collective excitations which in the massless case allows us to construct massless particles with any spin. Even if such collective states did exist, we find they would have finite energy only if they were electrically neutral, i.e., only if the dipole density were to go to zero at $x^1 = \pm\infty$. Thus, in particular, we cannot produce a charged spinor state. Of course, as long as a state has no net charge, we can locally approximate a point charge to any arbitrary accuracy by creating a sharp change in the dipole density. However, because the distribution is artifically created out of free particles, the dipole gradient will simply spread out and then cancel against gradients of the other sign.

If we view the same physics from the charged fermion framework, the picture appears to be much more colorful. Rather than the kinematical binding in the Thirring model which turns a massless fermion-anti-fermion pair into a current dipole, the fermion-anit-fermion pairs are bound dynamically by the electric interaction. Since, in two dimensions, any state with a net electric charge must have non-zero electric fields extending to infinity, only neutral states can be found in the finite energy spectrum.

This fact is reflected immediately by the field ψ which (in the Coulomb gauge) carries charge e. It only connects the vacuum to infinite energy states. In order to return to a physical state, we must also apply a ψ^+ field, so that the state is neutral overall.

The bound state created from the fermion-anti-fermion pair can be thought of as the point limit of a dipole of charges. As the charges are brought together, the energy in the electric field goes down as the distance between the charges. However, in the limit, a finite contribution from this energy remains, becoming the mass of the dipole quantum. This limit is a quantum effect analogous to the four-dimensional problem of two infinite capacitor plates separated by a vanishing distance (3). Since the bound state began with a massless fermion-anti-fermion pair, and the interaction is attractive, it might seem strange that the boson has a positive mass-squared. Actually, it is clear that the separated point charges have infinite masses because of their electric fields. The finite positive mass of the dipole is simply the difference between this electric mass and the potential energy gained by bringing the charges together.

Because of the quantum phenomenon of pair production and the existence of the finite energy bound state, the charged fermion state is not simply promoted to infinite energy, but it is completely eliminated. Even if we ignore the energy involved, we cannot create a stable charged fermion. When we try to create such a state by operating ψ on the vacuum, we find at a later time that the infinite energy is carried by an infinite number of the free dipole quanta filling more and more of space. The energy in the electric field of the original charge allows pairs to be produced in the vacuum. The original charge is neutralized by binding with an antiparticle and its partner carries off the net charge. As this process is repeated the net charge is spread out till its expectation is fluctuating throughout space. In the wake of these fluctuations dipoles are formed everywhere. The same vacuum polarization occurs for neutral states which consist initially of several localized charge distributions. In these situations, however, the fermions and antifermions can eventually pair off entirely into free neutral dipoles which disperse. Thus, we find that the fermions are confined within the composite bosons.

Single Fermions with Both QED and Thirring Interactions

We briefly note here that by considering a more general coherent state operator, with $\beta \neq -\sqrt{\pi}$, as in (IV.8), and by making the identifications:

$$J_\mu = \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \sigma,$$

$$A_\mu = \frac{\sqrt{\pi}}{e} (\partial_\mu \tilde{\sigma} - \epsilon_{\mu\nu} \partial^\nu \sigma),$$

$$F_{\mu\nu} = \epsilon_{\mu\nu} \frac{e}{\sqrt{\pi}} \sigma, \tag{IV.18}$$

we can solve a more generalized model, in the Coulomb gauge:

$$i\partial\!\!\!/ \psi = :\left\{ \sqrt{\pi} (\beta - \alpha) J\!\!\!/ - \frac{\alpha e}{\sqrt{\pi}} A\!\!\!/ \right\} \psi:,$$

$$\partial_\mu F^{\mu\nu} = -eJ^\nu. \tag{IV.19}$$

Of course, such equations of motion can be derived from a Lagrangian only if $\alpha = -\sqrt{\pi}$. The details of such constructions including definition of normal products and implementation of the gauge symmetries are obvious analogues of our previous discussions. The physical picture is a simple extension of the QED analysis.

QED With Two Massless Fermions

We now proceed to the theory of one gauge field coupled to two independent massless fermions. As we noted before, this model allows us to consider the question of confinement more closely. We wish to find out whether both spinor fields are bound into neutral dipoles, or some neutralized spinor remains in the spectrum.

The formal Lagrangian involved is:

$$\mathcal{L} = \overline{\Psi}_1(i\not{\partial} - e_1\not{A})\Psi_1 + \overline{\Psi}_2(i\not{\partial} - e_2\not{A})\Psi_2 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \\ - \frac{1}{2}F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{IV.20}$$

The resulting equations of motion are:

$$i\not{\partial}\Psi_1 = :e_1\not{A}\Psi_1:,$$

$$i\not{\partial}\Psi_2 = :e_2\not{A}\Psi_2:,$$

$$\partial_\mu F^{\mu\nu} = -e_1 J_1^\nu - e_2 J_2^\nu. \tag{IV.21}$$

In attempting to construct ψ_1 and ψ_2 as coherent state operators we run into the same mathematical difficulties as in the single fermion cases. Since we have discussed these difficulties in detail, we will simply proceed here by dropping the infrared corrections which are needed to regularize our operators. Once again, the physics is correctly represented.

If we let $e = \sqrt{e_1^2 + e_2^2}$, we now define σ to be a free, massive boson field with mass-squared $\frac{e^2}{\pi}$. $\tilde{\sigma}$ can be constructed as in (IV.7). In addition, we employ a set of Thirring fields, j_μ, ϕ, and $\tilde{\phi}$ related by (II.8). ψ_1 and ψ_2 can then be constructed (in a positive-definite Hilbert space):

$$\psi_1 = :e^{i\theta_1[\alpha\tilde{\sigma} - \sqrt{\pi}\gamma^5\sigma + \lambda\tilde{\phi}]}\, e^{-i\theta_2\sqrt{\pi}[\tilde{\phi} + \gamma^5\phi]} \left[\frac{\mathfrak{M}^{2\theta_2} e^{2\theta_1}}{4\pi^{(2+\theta_1)}}\right]^{1/4} \mathfrak{C}_1:,$$

$$\psi_2 = :e^{i\theta_2[\alpha\tilde{\sigma} - \sqrt{\pi}\gamma^5\sigma + \lambda\tilde{\phi}]}\, e^{i\theta_1\sqrt{\pi}[\tilde{\phi} + \gamma^5\phi]} \left[\frac{\mathfrak{M}^{2\theta_1} e^{2\theta_2}}{4\pi^{(2+\theta_2)}}\right]^{1/4} \mathfrak{C}_2:,$$

$$\text{where } \theta_1 = \frac{e_1}{e}, \quad \theta_2 = \frac{e_2}{e}. \qquad \text{(IV.22)}$$

The $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are "placeholders" analogous to $\mathfrak{C}$ in our other solutions, and $\mathfrak{M}$ is the cut-off mass used to define the Thirring fields. We have implemented some of the operator gauge invariance by letting the two gauge parameters α and λ be arbitrary. The other fields are given by;

$$A_\mu = -\frac{1}{e}(\alpha\partial_\mu\tilde{\sigma} + \lambda\sqrt{\pi}\,\epsilon_{\mu\nu}j^\nu + \sqrt{\pi}\,\epsilon_{\mu\nu}\partial^\nu\sigma),$$

$$F_{\mu\nu} = \epsilon_{\mu\nu}\frac{e}{\sqrt{\pi}}\sigma,$$

$$J_1^\mu = \frac{\theta_1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\sigma + \theta_2 j^\mu,$$

$$J_2^\mu = \frac{\theta_2}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\sigma - \theta_1 j^\mu. \qquad \text{(IV.23)}$$

By following our previous discussions of normal products, we could show that J_1^μ and J_2^μ are indeed the gauge-invariant currents associated with ψ_1 and ψ_2. As in the single fermion case, ψ_1 and ψ_2 do not transform like spin 1/2 fields. By embedding the Lorentz gauge solution in an indefinite metric space, we generate Gupta-Bleuler type solutions which transform properly (see Appendix D).

The Physical Picture of QED with Two Massless Fermions

In terms of the polarization fields, the generalized model is obviously the theory of two free bosons, one massless and one massive. Since they are independent, the physical spectrum must simply be the superposition of a Thirring spectrum and a QED spectrum. In particular, there should be a state which is a collective excitation of Thirring dipoles that we have identified as a Thirring spinor carrying a charge associated with $j_\mu = \frac{1}{e}(e_2 J_{1\mu} - e_1 J_{2\mu})$. Thus it appears that this model involves only shielding of the electric charge, and not total confinement of the fermions. This behavior will be clearer from the fermion framework. Of course, only electrically neutral states have finite energies. This restriction implies that all the Thirring fields carry no charge, which is easily seen since they all commute with the electric current, $e_1 J_{1\mu} + e_2 J_{2\mu}$.

From the viewpoint of a fermion theory, we find two fermion-anti-fermion bound states, one kinematically bound and one dynamically bound. Because the two types of fermions can turn into each other through a virtual photon, the bound states are linear combinations of both types of fermions. The dynamically bound boson can be thought of as a state involving the superposition of two dipoles, one of the $\psi_1^\dagger\psi_1$ -type, one of the $\psi_2^\dagger\psi_2$ -type, such that the expectation of the electric field has a non-zero limit. The kinematically bound boson can be thought of as a similar superposition, this time with the two different types of dipoles oriented so that the electric field expectation is zero, allowing the state to be massless.

If we consider charged operators, such as the fermion fields (in the Coulomb gauge), we find that they only connect infinite energy states to the vacuum. However, if we supply an infinite energy and produce a state like $\psi_1(x) \mid 0>$, the result is qualitatively different from the single fermion model. As in the latter case, the point charge polarizes the vacuum. The resulting fermions and anti-fermions bind into massive pairs which gradually fill the entire space. In the two-fermion model, these bound states consist of both fermion species, and although the pair production spreads the net charge out, the Thirring charge remains localized. Its electric charge is shielded, but the Thirring spinor, surrounded by a cloud of dipoles, is not totally confined.

Returning to physical states with finite energy, we see that unless $\frac{e_1}{e_2}$ is a rational fraction, the neutrality requirement implies that the number of 1-type fermions must equal the number of 1-type anti-fermions. Similarly, the 2-type charge must be zero. Thus, the net Thirring charge must also be zero. However, within such a state, once the Thirring spinors have been electrically neutralized, they move freely through space, confined in no way. If $\frac{e_1}{e_2}$ is rational, states with non-zero Thirring charge are also physical.

We conclude our description by noting that the masslessness of the spinors seems crucial to the fact that the confinement is not total. In order to shield the spinors electrically, it is necessary to produce dipoles out of the vacuum. For instance, a ψ_1 excitation can be neutralized without being totally confined by pair production of 2-type dipoles. The number of such dipoles must be infinite to cancel the original electric charge. Because the fermions are massless we found that the $(e_2 J_{1\mu} - e_1 J_{2\mu})$ dipoles were massless. An infinite number of these dipoles shield the ψ_1 electric charge without requiring an infinite energy. However, if the fermions were massive, the $(e_2 J_{1\mu} - e_1 J_{2\mu})$ dipoles would presumably be massive. Shielding could not occur without infinite energies, and total confinement would be required. We discuss the theory with massive fermions, which has not been solved, in the next chapter.

It should be clear that further generalizations of QED to cases with any number of massless fermions, which may also interact via generalized Thirring interactions of the type discussed in Chapter IV, can be made easily.

V. FORMAL OPERATOR STRUCTURE IN THEORIES WITH NON-TRIVIAL SCATTERING

Thus far, we have discussed only exactly soluble models. Not surprisingly, the operator solutions of these models involve only free fields, so that there is no physical scattering in the final analysis. The complex energy-momentum content of the Wightman functions (vacuum matrix elements) in the fermion theories is simply due to the fact that the fermion field corresponds to multiparticle states, rather than some single fermion. In fact, by considering these fermion theories in terms of the physics of the polarization fields associated with the fermion charges, ϕ and $\tilde{\phi}$ in the Thirring model, σ and $\tilde{\sigma}$ in QED, we discovered that the simplicity of the physics in the models made explicit. ϕ is a free, massless scalar field, and σ is a free, massive scalar field. The states in QED are nothing but free massive boson states. The states in the Thirring model are nothing but free, massless boson states including the coherent states with infinite numbers of zero energy bosons.

The heuristic argument we used to explain the correspondence between the fermion theories and the boson theories involved the fact that in two dimensions point charges and surface polarization charges (which are also point-like) cannot be distinguished. If we know the dynamics of the fermions, the corresponding polarization field can be described in terms of the positions and momenta of the fermions. On the other hand, if we know the dynamics of the polarization field, the fermion currents are simply the gradients of the polarization field. There is nothing in this picture which depends on the detailed dynamics. Thus, we would expect to be able to extend the correspondence between fermion theories and boson theories to models with non-trivial scattering. Of course, we do not expect, at present, to find explicit operator solutions in such models whether we consider the fermion framework or the boson framework, since we do not know of any exact solutions of theories with non-trivial scattering. Still, we will exhibit the correspondence by manipulating the formal operator structures in analogy with our previous operator constructions. While this thesis was being written, Mandelstam published a similar construction connecting the Thirring model and Sine-Gordon theory [21]. Coleman has demonstrated the same connection perturbatively [8].

Before we establish the correspondence, we should briefly mention some reasons why the relation between fermion theories and boson theories is of interest to us. Firstly, there is simply the intrinsic interest in connecting two field theory models which, at first sight, do not have anything in common. Secondly, as we have indicated previously, one of the prominent dynamical mechanisms which currently is conjectured to be important in the physics of hadrons is the total confinement of the fermion constituents of the hadrons. QED with a massless fermion provides an oversimplified model of confinement, and, as a result, it may prove interesting to study similar two-dimensional models, such as QED with several massive fermions. In studying massless QED, we found the equivalent boson theory much simpler (free!) than the fermion formulation. We expect that this relative simplicity of the boson viewpoint will persist in the more complex two-dimensional models of confinement. Because the boson framework more accurately approximates the physical spectrum, the dynamics is simpler than in the fermion framework where the mechanism of the confinement must be explicitly represented. For instance, a low order perturbative approximation in the boson approach may give a reasonable picture of scattering in certain kinematic regions, while such an approximation in the fermion approach will not even give a good picture of the spectrum. Thus, the fermion-boson equivalence may make models of confinement easier to study.

A third possible application of our correspondence is in semi-classical approximations to the two-dimensional fermion quantum theories. Whether confinement is present or not, the classical boson theory may provide a better approximation than the classical fermion theory. (Although we will show a correspondence between boson and fermion quantum theories, the associated classical theories need not describe the same physics. QED with a single massless fermion is an example. From Chapter IV, we see that the classical boson theory is that of a free, massive boson. On the other hand, the classical fermion theory associated with the Lagrangian (IV.1) can be solved in terms of free, massless classical fermion fields in a manner analogous to the classical Thirring solution, (II.9) [(27)]) Several factors favor the boson viewpoint. In the classical fermion theory, the solutions of the equation of motion will, in general, not resemble any of the matrix elements of the quantum fermion theory. Since the quantum fermion field creates or destroys fermion quanta, the equation of motion is not satisfied by a particular matrix element in the quantum theory, but relates different matrix elements. Of course, in the classical boson theory, the matrix elements will not satisfy the boson equations of motion for the same reasons. However, as we saw explicitly in the Thirring and QED constructions, the states in the quantum boson theory corresponding to one or more fermions are coherent states of bosons. These are the states in a quantum theory whose matrix elements are best approximated by classical fields. Roughly speaking, the matrix elements of the boson field between coherent states satisfy the classical equation momentarily until the interactions between the boson quanta break up the coherence. Thus, if we are interested in a semi-classical approach to a quantum theory, the classical boson theory may produce accurate results more easily.

In addition, if we are interested in bound states of several fermions or other multiparticle phenomena, the details may be more accessible in the classical boson theory. Such states simply correspond to classical boson field configurations in which the gradients of the field (corresponding to the fermion currents) are non-zero in several localized regions of space. From the classical fermion viewpoint, such multifermion states cannot be represented accurately in terms of a single fermion field. Thus, the best description of the multiparticle systems within the classical fermion framework is a Hartree-Fock type of solution in which the classical fermion field represents an average of some sort of the individual wavefunctions of the fermions. Since this average does not represent the wavefunction of any individual particle in the system, it is not as detailed a description as provided by the boson solutions.

Finally, the classical fermion theory is hampered by the inability to describe fermions and anti-fermions simultaneously. For instance, the charge density is positive definite in the classical fermion theory. Thus, a state such as a bound fermion-anti-fermion pair can only be considered by introducing a second classical field of opposite charge. Within the classical boson theory, fermions and antifermions are only distinguished by the sign of the change in the polarization (boson) field. The fermion currents which are the derivatives of the boson field are not positive definite in this approach. Once again, the classical boson theory would seem to better approximate the quantum theory.

The Construction of the Fermion Operator in a Boson Theory

Our construction will simply be a generalization of the previous constructions in the soluble models. We begin with a scalar field theory associated with the formal Lagrangian:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - F(\phi). \tag{V.1}$$

F (ϕ) is a functional of ϕ , but not of its derivatives. Since we are dealing with theories we cannot explicitly solve, we are limited in our knowledge of the operators in the theory. However, the theories associated with (V.1) are all super-renormalizable in two dimensions (in perturbation theory, only the single loop diagrams with a single vertex can be divergent). Thus, we can assume that the scalar field and its canonically conjugate momentum, π, satisfy canonical equal-time commutation relations:

$$[\pi(t,x^1), \phi(t,y^1)] = -i\delta(x^1-y^1). \tag{V.2}$$

The natural framework for our construction is the Schrödinger picture in which the explicit time dependence of the single-particle annihilation and creation operators, which we cannot calculate, is replaced by the formal time translation operator,

$$\mathcal{U}(t) = e^{it\int dz\,\mathcal{H}(z)}. \tag{V.3}$$

$\mathcal{H}$ is the Hamiltonian density of the theory. U(t) operating on a state in the Schrödinger picture translates the state in time from $t = 0$ to time t. The Schrödinger field operators are time-independent. We make a canonical momentum space decomposition in terms of annihilation and creation operators, $a(K)$ and $a^+(K)$:

$$\phi(x^1) = \int \frac{dK}{4\pi|K|} \left\{ a(K)\, e^{iKx^1} + a^+(K)\, e^{-iKx^1} \right\},$$

$$\pi(x^1) = -i \int \frac{dK}{4\pi} \left\{ a(K)\, e^{iKx^1} - a^+(K)\, e^{-iKx^1} \right\}. \tag{V.4}$$

(To make the analogy with the Thirring model clear, we have chosen ϕ to be a massless scalar, i.e. any mass term has been swept into $F(\phi)$. Consequently, the operators in (V.4) need to be regularized as in the Thirring construction. Once again, we will not make the regularization explicit since our naive expressions will contain no essential errors.)

In the Schrödinger picture, the stress-energy components are:

$$\theta_{00} = :\frac{1}{2}\left\{ \pi^2 + (\partial_1\phi)^2 \right\} + F(\phi):,$$

$$\theta_{01} = \theta_{10} = :\pi \partial_1 \phi:,$$

$$\theta_{11} = :\frac{1}{2}\left\{ \pi^2 + (\partial_1\phi)^2 \right\} - F(\phi):. \tag{V.5}$$

In two dimensions, the Wick-ordering with respect to $a(K)$ and $a^+(K)$ makes $\theta_{\mu\nu}$ a finite, well-defined operator. Thus, (V.5) specifies a unique theory with a given set of coupling constants. It should be noted that, in practice, the infrared regularization procedure must be explicitly defined before (V.5) describes a unique theory, since the regularization involves a mass which sets the scale for the coupling constants. Of course, commutation of ϕ with $\theta_{\mu\nu}$ yields the equation of motion:

$$[P_\mu,[P^\mu,\phi]] = :\frac{\delta F(\phi)}{\delta\phi}:. \qquad (V.6).$$

We now construct a field out of the scalar operators which we will show is the solution of a spinor theory. Using $\mathcal{M}$ and $\mathfrak{C}$ as defined in Chapter II,

$$\psi = :e^{i[\alpha\tilde{\phi} + \beta\gamma^5\phi]}\left[\frac{\mathcal{M}}{2\pi}\right]^{1/2}\mathfrak{C}:,$$

$$\tilde{\phi}(x^1) = -\frac{1}{2}\int_{-\infty}^{\infty} dz\; \epsilon(x^1 - z)\,\pi(z). \qquad (V.7).$$

Before we can calculate the equations of motion for ψ, we need to define the normal products of ψ and ψ^+. We simply follow the same procedure of making short-distance expansions as in our previous constructions. The only difference is that the time-dependence must be explicitly inserted in the bilocal products which must be expanded; for instance, the currents are derived from products like $U(x^0)\ \psi^+(x)^1\ U^{-1}(x^0)\ U\ (y^0)\ \psi\ (y^1)\ U^{-1}(y^0)$. The other consideration in calculating the current is implementation of gauge invariance in those theories which possess the symmetry. The insertion of a gauge integral is required exactly as in (IV.14) in QED. The final result of the expansions is simply a straightforward generalization of our previous results. For the $\psi^+\psi$ bilinears:

$$:\bar{\psi}\psi: = \frac{m}{2\pi}\left\{\mathbb{C}^+\gamma^0 \frac{1+\gamma^5}{2}\mathbb{C} :e^{2i\beta\phi}: + \mathbb{C}^+\gamma^0\frac{1-\gamma^5}{2}\mathbb{C} :e^{-2i\beta\phi}:\right\},$$

$$:\bar{\psi}\gamma^5\psi: = \frac{m}{2\pi}\left\{\mathbb{C}^+\gamma^0 \frac{1+\gamma^5}{2}\mathbb{C} :e^{2i\beta\phi}: - \mathbb{C}^+\gamma^0\frac{1-\gamma^5}{2}\mathbb{C} :e^{-2i\beta\phi}:\right\},$$

$$J_0 = -\frac{1}{\sqrt{\pi}}\,\partial_1\phi = -J_1^5,$$

$$J_1 = -\frac{1}{\sqrt{\pi}}\,\pi = -J_0^5. \qquad \text{(V.8)}$$

As in our previous constructions, J_μ and J^5_μ have been normalized by choosing canonical equal-time relations for them.

We now calculate the Schrödinger form of the equations of motion for ψ:

$$[P_\mu, \gamma^\mu \psi] = :\left\{[(\alpha-\beta)\partial^1\phi + \alpha B]\gamma^0 - [(\alpha-\beta)\pi]\gamma^1\right\}\psi:,$$

$$[P_\mu, \gamma^\mu \gamma^0 \psi] = :\left\{[(\alpha+\beta)\partial^1\phi + \alpha B]\gamma^0 - [(\alpha+\beta)\pi]\gamma^1\right\}\gamma^0\psi:,$$

$$\text{where} \quad \partial_1^2 B = -\partial_1 : \frac{\delta F(\phi)}{\delta\phi} : . \tag{V.9}$$

Equations (V.9) are still partially expressed in terms of the boson operators. Depending on the exact structure of F(ϕ), the substitution of fermion operators into (V.9) can give different results, as we shall see in examples later. For the general case we can rewrite (V.9):

$$[P_\mu, \gamma^\mu \psi] = :\{g \not{J} + \not{A}\}\psi:,$$

$$[P_\mu, \gamma^\mu \gamma^0 \psi] = :\{\bar{g} \not{J} + \not{A}\}\psi:,$$

$$\begin{aligned} \text{with} \quad A_\mu &= (\alpha - \tfrac{g+\bar{g}}{2}) J_\mu + \alpha B g_\mu{}^0, \\ \partial_1^2 B &= i\partial_1 [P_\mu, J^{5\mu}], \\ g - \bar{g} &= -2\beta. \end{aligned} \tag{V.10}$$

In general, (V.10) cannot be derived from a formal Lagrangian involving only ψ unless it is non-local like QED in the Coulomb gauge. Thus, relatively simple boson theories lead to fairly strange fermion theories. We can also calculate the Lorentz properties of ψ:

$$[M_{01}, \psi] = -i(x_0\partial_1 - x_1\partial_0)\psi - \frac{i}{2\pi}\alpha\beta\gamma^5\psi. \qquad \text{(V.11).}$$

As in the Thirring model, ψ transforms like a spinor only if $\alpha\beta = \pi$.

Our general result for the fermion theory equivalent to the boson theory associated with (V.1) is expressed rather clumsily in (V.10). In practice, we will usually start with a given fermion theory and find the equivalent boson theory, using the relations in (V.7) and (V.8) to express the fermion operators as boson operators. As we shall see the resulting boson theories are more conventional in appearance.

The Possibility of Fermion States in a Boson Theory

The existence of a fermion operator in a boson theory (or vice-versa) does not imply the existence of a corresponding particle in the spectrum of the theory, as we have seen clearly in QED. (In fact, our formal demonstration does not guarantee the existence of the quantum theories. For instance, the boson theory specified by $:F(\phi):=:\dfrac{\lambda\phi^3}{3!}:$

has no lower bound for the energy density, since $:\phi^3:$ is not positive definite. The corresponding fermion theory must suffer from the same affliction.) Since we cannot calculate the exact matrix elements of the non-trival theory, we cannot really analyze the particle content of the theory to determine whether there are fermions. Still we can consider whether the existence of fermions is possible in a theory by recalling how the fermion states arose in the soluble models. We found out in Chapter II that the different fermion sectors corresponded to mathematically inequivalent, but physically equivalent representations of the equal-time current commutation relations. More physically, since the fermion current J_μ was identified with $\frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu}\partial^\nu\phi$ (or $\frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu}\partial^\nu\sigma$), we used (II.43) (or(IV.9)) to see that states with non-zero charge have different limiting expectation value for ϕ (or σ) at $z = \pm \infty$. In order to exhibit this kind of behavior, the states must include infinite numbers of infrared (zero momentum) boson quanta.

When the total energy of the infrared quanta is finite, charged states (including fermion states) are possible. If the energy required by the infrared quanta is infinite, the physical states must be neutral. In theories with one species of fermion which we are considering here, the neutrality of the spectrum would indicate that there are no fermion states. In those models in which charged states are allowed, the definite presence or absence of fermions can only be established by considering the Lorentz properties of those charged states. This structure is not accessible to us in the non-trivial theories, so we can only determine whether the models have non-zero charge sectors. The states making up these charged sectors could be fermion states, or states of a fermion surrounded by a cloud of polarization which cannot be removed without infinite energy, or some loosely bound, condensed state of bosons (like a liquid phase) which is not a single particle state at all. In order to distinguish between these possibilities in a specific model, much more detailed work would be necessary.

We noted above that the existence of charged sectors requires states in which the expectation of ϕ can go to various different limits at spatial infinity. We can denote the possible limits of ϕ as φ_i, $i = 1, 2, \ldots$. Since we require that the states have finite energy, the field configuration as it approaches φ_i must contribute zero additional energy to the state. In other words, each value of φ_i must correspond to a space-time independent quantum state with zero energy density everywhere and φ_i as the expectation value of ϕ. Clearly, the normal vacuum is such a state with $\varphi_i = 0$. The other such states are other degenerate vacua with $\varphi_i \neq 0$. Thus, the problem of the presence of charged sectors is the problem of finding degenerate vacua. The charged states are those which approach different vacua at $z = \pm \infty$. Looking back at the Thirring model, we see that the free, massless scalar field theory has a vacuum associated with any real value of φ_i. Therefore, not only are there an infinite number of sectors for each value of α and β, there are the different Thirring theories with different α and β. In QED, the massive, free scalar theory has only the normal vacuum, and no charged sectors exist.

Turning to the general boson theory corresponding to (V.1), if we were looking for the degenerate vacua of the classical theory, they would be given by $\phi = \varphi_i$, where φ_i are the absolute minima of $F(\phi)$. In the quantum theory, the expectation values φ_i which allow $:F(\phi):$ to be minimized are changed from the classical values by quantum corrections. There is one exception to this quantum difficulty. If $F(\phi)$ is periodic in ϕ, then the existence of a single minimum requires periodically spaced degenerate minima. For other forms of $F(\phi)$, we cannot calculate the quantum corrections without making detailed approximations and perturbative calculations. Thus, in discussing the explicit examples with such $F(\phi)$, we will limit ourselves to weak coupling limits where the classical energy associated with a value of $\phi = \varphi_i$ is expected to be a good estimate of the minimum energy of a quantum state with expectation value of ϕ equal to φ_i.

Before leaving the subject of quantum corrections, we should note that we can estimate the stability of the classical vacua against quantum fluctuations of some kinds of extending an analysis done by Coleman for the sine-Gordon model(8). He pointed out that for $F(\phi) = M^2 \cos b\phi$, where M is a mass and b is a numerical parameter, it is possible to find states for which the expectation of $:F(\phi):$ is arbitrarily large and negative, if $b^2 > 8\pi$. His trial states are generalized coherent states, similar to those used in the theory of superfluidity, which have no expectation for ϕ, but shift $:\phi^2:$ by a constant. Thus, for such values of b^2, although the charged sectors of the theory can be reached from the classical vacuum with a finite energy, there are neutral quantum states, in which space is filled with dipole excitations (oriented in both directions so that the total charge is zero), with an arbitrarily large negative energy density. The theory no longer is well-defined.

Generalizing Coleman's approach we can calculate the expectation of $:F(\phi):$ in a state constructed to have an constant expectation value φ for ϕ , and an independent expectation, η^2 for $:\phi^2:$ (8). If the minimum energy (expectation of $:F(\phi):$) achieved for this set of states is not in or near the classical vacuum, we must assume that quantum fluctuations entirely invalidate any conclusion drawn from the classical energies. We merely summarize the results here (Appendix E goes into slightly more detail). For most of the models we will explicitly mention in this chapter (listed in Appendix F), there is a term or terms in $F(\phi)$ which are sinusoidal functions of ϕ . In these cases, the behavior is essentially the same as in the sine-Gordon theory. The classical vacua remain the true minima until the coefficient of ϕ in the sinusoidal function is too large in which case the theory no longer makes sense. In the last model we consider, $F(\phi) = - \frac{\mu^2 \phi^2}{2!} + \frac{\lambda\phi^4}{4!}$. Classically, there are two degenerate minima at $\phi = \pm \frac{\sqrt{6\mu^2}}{\lambda}$. Using the trial states described above, we find that for small coupling λ (relative to μ^2 and $\mathfrak{m}^2$, the infrared cut-off), the classical vacua are slightly shifted but remain minima, but for large coupling there is a single absolute minimum with expectation of $\phi = 0$, expectation of $:\phi^2:$ non-zero. Thus, for small λ there appear to be different charge sectors, but for large coupling these disappear, leaving only neutral states. With these quantum effects in mind, we now can look at some specific examples.

Massive Fermion Examples of the Correspondence between Fermion and Boson Quantum Theories

The simplest and most obvious generalization which can be made to all our soluble models is the inclusion of a mass for the fermions. Let us begin by adding a mass for the fermion in the Thirring model. Instead of referring to the general equations (V.10), the easiest way to deduce the correct form for $F(\phi)$ to generate a mass term in the equation of motion for ψ is to write the mass term $: m\,\bar{\psi}\psi :$ in terms of bosons using (V.8). We eliminate $\mathfrak{C}$ and $\mathfrak{C}^+$ by proper choice of the vacuum in the same way Lowenstein and Swieca do in QED. (See Appendix E. All our equivalent boson theories will be written in terms of a parity-symmetric vacuum.) After subtracting the vacuum expectation we find:

$$:F(\phi): = :m\overline{\Psi}\Psi: = -\frac{m\mathcal{m}}{\pi} :\{\cos 2\beta\phi - 1\}: . \qquad \text{(V.12)}$$

Thus, we find that the quantum fermion theory, known as the massive Thirring model, corresponding to

$$\mathcal{L} = \overline{\Psi}(i\not{\partial} - m)\Psi - \frac{\sqrt{\pi}\,(\beta - \frac{\pi}{\beta})}{2} J_\mu J^\mu , \qquad \text{(V.13)}$$

is equivalent to the boson theory, known as the sine-Gordon model, with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m\mathcal{m}}{\pi} \{\cos 2\beta\phi - 1\}, \qquad \text{(V.14)}$$

where ψ is given in terms of ϕ by (V.7). We have set $\alpha = \frac{\pi}{\beta}$, since the equations of motion derived from (V.13) are covariant only if ψ is a spin 1/2 field. Using the boson stress-energy tensor (V.5) and (V.7) we can explicitly check that:

$$[P_\mu, \gamma^\mu\Psi] = :\{\sqrt{\pi}\,(\frac{\pi}{\beta} - \beta)\not{J} + m\}\Psi: . \qquad \text{(V.15)}$$

As we noted before, the boson theory seems to break down for $\beta^2 > 2\pi$. For other β, we have periodic degenerate vacua for expectation value of $\phi = \varphi_n = \frac{n\pi}{\beta}$. Consequently, we have charge sectors with the charge quantized in units of $\frac{\sqrt{\pi}}{\beta}$. To check our consistency, we see that ψ, as defined in (V.7), carries charge $\frac{\alpha}{\sqrt{\pi}} = \frac{\sqrt{\pi}}{\beta}$, and it does indeed map states from one charge sector to the next. (Actually, because of the anti-symmetric boundary conditions we used to define $\tilde{\phi}$, ψ must be supplemented by a coherent state operator which shifts ϕ by $-\frac{\pi}{2\beta}$ throughout all space before it takes you from one minimum of $F(\phi)$ to the next.)

The correspondence between the theories (V.13) and (V.14) is particularly interesting because if we choose $\beta = -\sqrt{\pi}$, ψ is a free, massive fermion field. Just as a free, massless fermion can be interpreted as a collective excitation of free massless bosons, a free, massive fermion is a collective excitation of interacting sine-Gordon bosons. Thus, we may be encouraged that even for $\beta \neq -\sqrt{\pi}$, the different charge sectors may indeed correspond to states with fermions, perhaps "dressed" with a cloud of bosons.

For most theories with non-trivial scattering, we have no further information on the spectrum or dynamics. In the case of the sine-Gordon equation, though, Fadeev and Takhatjan[(12)] have solved the classical equation exactly, and Dashen, Hasslacher, and Neveu[(10)] have quantized the model in a semi-classical approximation. Very briefly, the localized classical solutions with finite energy fall into two classes. One set of solutions are the solitons and antisolitons -- waves which propagate without changing shape and for which the field ϕ tends to different limiting values at $z = \pm\infty$. The solitons and antisolitons all have the same mass which determines their energy-momentum relation. These solutions are presumably the classical analogues of the excitations created by acting on the vacuum with our fermion field ψ. To reinforce this connection, Orfanidis and the author have shown that localized, finite energy solutions of the classical massive Thirring model, (V.13), can be constructed from the soliton and antisoliton solutions of the sine-Gordon equation [30].

The other class of localized sine-Gordon solutions are bound soliton-antisoliton pairs which have a continuous mass spectrum going from zero to twice the soliton mass. Scattering solutions can also be constructed which appear to be the linear superposition of solitons, antisolitons, and pairs when these excitations are initially infinitely separated. After the nonlinear scattering as the excitations approach and pass through each other, the original solitons, antisolitons, and pairs appear with the original velocities and shapes as the separations become large once again.

The quantized spectrum derived by semi-classical quantization is exactly the same as the classical spectrum except that the pair spectrum is discrete. Also, the approximation falls apart for $\beta^2 > 2\pi$. Dashen, Hasslacher, and Neveu argue that the approximate quantization may provide the exact spectrum in a manner similar to the Bohr-Sommerfeld prescription for the hydrogen atom. As evidence, they show that their results agree with perturbation results for $\beta << 1$ (boson perturbation) and for $\beta \approx -\sqrt{\pi}$ (fermion perturbation) and with n-body analysis in two dimensions. Although it has not been determined whether the quantized soliton and antisoliton states are fermion states, obviously, these detailed quantum results are consistent with our naive analysis which is encouraging.

A second example of the boson-fermion correspondence involves QED with a massive fermion:

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m - eA\!\!\!/)\Psi + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu). \qquad (V.16).$$

Following our previous argument concerning the introduction of a mass term, we find that the corresponding boson theory is defined by:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{e^2\phi^2}{2\pi} + \frac{m\mathcal{M}}{\pi}\left\{\cos 2\sqrt{\pi}\phi - 1\right\}. \qquad (V.17).$$

We see that the Coulomb interaction, which is responsible for the $\frac{e^2}{\pi}\phi^2$ term, prevents the existence of charged sectors and fermion states in the same way as in the massless fermion case. If the electromagnetic coupling is small, $e^2 \ll m\mathcal{M}$, the fermion-anti-fermion bound state spectrum should be very similar to the discrete pair spectrum (for $\beta \leq 2\sqrt{\pi}$) in the sine-Gordon quantum theory, at least for the several lowest states.

The situation is more complex if we consider the massive counterpart of the two fermion generalization of QED we studied in Chapter IV:

$$\mathcal{L} = \overline{\Psi}_1(i\partial\!\!\!/ - m_1 - e_1 A\!\!\!/)\Psi_1 + \overline{\Psi}_2(i\partial\!\!\!/ - m_2 - e_2 A\!\!\!/)\Psi_2 + \\ + \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2} F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{V.18}$$

The equivalent boson Lagrangian, involving two boson fields this time, is:

$$\mathcal{L} = \tfrac{1}{2}\left\{(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 - \tfrac{1}{\pi}(e_1\phi_1 + e_2\phi_2)^2\right\} + \\ + \frac{m_1\mathcal{M}}{\pi}\left\{\cos 2\sqrt{\pi}\phi_1 - 1\right\} + \\ + \frac{m_2\mathcal{M}}{\pi}\left\{\cos 2\sqrt{\pi}\phi_2 - 1\right\}. \tag{V.19}$$

In this model, unless $\frac{e_1}{e_2}$ is a rational fraction, there is only one classical vacuum, occuring for $\phi_1 = 0 = \phi_2$. Thus, no charged sectors or fermions will be present. If $\frac{e_1}{e_2}$ is a rational fraction, it is possible to find non-zero constant ϕ_1 and ϕ_2 which simultaneously minimize the mass contributions and the electromagnetic term. These values of ϕ_1 and ϕ_2 allow different Thirring charge sectors (the electric charge is obviously zero). However, unlike the massless model discussed in Chapter IV, we do not expect single fermion states in the charged sectors. It is easy to see from the structure of (V.19) that the operators which connect the normal vacuum to the charged sectors will be products of ψ_1 and ψ_2^+ (or vice-versa) which are electrically neutral. Thus, we expect the charged states to be composites of the 1-and 2-type elementary particles. The confinement in the model is total and not simply shielding.

U(n) Thirring Models

We pointed out in Chapter III that the soluble U(n)-symmetric Thirring models were scale invariant with the currents satisfying the canonical equal-time commutation relations. As Dashen and Frishman[(9)] noted, scale invariance and canonical equal-time structure for the currents requires that all the U(n) currents and axial currents be conserved. (This result can be demonstrated in two dimensions by considering the constraints placed on matrix elements of the currents by scale invariance and canonical dimensions.) However, if the coupling constant, g, is non-zero in the Lagrangian (III.2), the formal equations of motion imply that the axial currents are not conserved:

$$\partial^{\mu} J_{\mu}^{5a} = \sqrt{2}\, g f^{abc} J_{\mu}^{5b} J^{c\mu} . \qquad (V.20).$$

If (V.20) is to hold in the quantum theory, we see that the scale invariance must be broken, and the theory will not be exactly soluble. We can use our correspondence prescription to find the boson theory which corresponds to (III.2). Hopefully, we can get a better idea of the structure of these non-trivial U(n) models.

We begin by defining n two-component spinors, each constructed as in (V.7):

$$\psi_{\ell} = : e^{-i\sqrt{\pi}[\tilde{\phi}_{\ell} + \gamma^5 \phi_{\ell}]} \left(\frac{m}{2\pi}\right)^{1/2} \mathbb{C}_{\ell} : , \quad \ell = 1, \ldots, n. \qquad (V.21).$$

Each ϕ_{ℓ} is a Schrödinger scalar field as in (V.4), and the $\tilde{\phi}_{\ell}$ are constructed following (V.7). The α and β appearing in (V.7) have all been set to $-\sqrt{\pi}$ to guarantee canonical equal-time commutation relations for all the ψ and all the currents. Note that (V.21) is not expressed in the explicit U(n) notation and matrices used in Chapter III (for instance, in (III.5)).

To obtain a simpler form for the interaction, we use the transformation (III.22). It is straightforward to derive (analogous to (III.25)):

$$:J_\mu^0 J^{0\mu} + J_\mu^a J^{a\mu}: \; = - :\left\{(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma^5\psi)^2\right\}: \; . \qquad \text{(V.22)}$$

As in Chapter III, the J_μ^a include all n^2-1 SU(n) currents, while the K_μ^i will denote only the n "diagonal" currents. We have seen in Chapters II and III that interactions of the Thirring type which involve the "diagonal" currents give rise to non-canonical equal-time singularities for the ψ commutation relations ($\alpha \neq -\sqrt{\pi} \neq \beta$ in (II.30), for instance). This will be the case in these models also. Therefore, to maintain the canonical equal-time behavior at first, we will explicitly subtract out the "diagonal" contribution in (V.22). The modified interaction term has the form:

$$\frac{g^2}{2} :\left\{J_\mu^0 J^{0\mu} + J_\mu^a J^{a\mu} - K_\mu^i K^{i\mu}\right\}: \; = - \frac{g^2}{2} :\left\{(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma^5\psi)^2 + K_\mu^i K^{i\mu}\right\}: \; .$$

(V.23).

Evaluating the right hand side of (V.23) in terms of the boson fields, we obtain:

$$- \frac{g^2 \mathcal{M}^2}{2\pi^2} : \left(\sum_l \cos 2\sqrt{\pi}\,\phi_l\right)^2 + \left(\sum_l \sin 2\sqrt{\pi}\,\phi_l\right)^2 - n : =$$

$$= - \frac{g^2 \mathcal{M}^2}{2\pi^2} : \sum_{l>m} \cos\left\{2\sqrt{\pi}\,(\phi_l - \phi_m)\right\} : \quad . \tag{V.24}$$

We have an equivalence between the quantum theories associated with the two Lagrangians:

$$\mathcal{L} = \overline{\Psi}(i\not{\partial})\Psi - \frac{g^2}{2}\left\{J^0_\mu J^{0\mu} + J^a_\mu J^{a\mu} - K^i_\mu K^{i\mu}\right\}, \tag{V.25}$$

$$\text{and} \quad \mathcal{L} = \sum_l \tfrac{1}{2}(\partial_\mu \phi_l)^2 + \frac{g^2 \mathcal{M}^2}{2\pi^2} \sum_{l>m} \cos\left\{2\sqrt{\pi}\,(\phi_l - \phi_m)\right\}. \tag{V.26}$$

We see that the breaking of scale invariance, which we had expected because of formal equations obtained from the fermion Lagrangian, appears explicitly in the boson Lagrangian since the coupling $\dfrac{g^2 \mathcal{M}^2}{2\pi^2}$ has dimensions of mass-squared. In fact, the individual interaction terms in (V.26) resemble the mass terms we found in our previous examples, except that the argument of the cosine is a linear combination of the boson fields.

Clearly, when we ask about the possibility of charged sectors, there are infinite families of degenerate minima of the classical energy. Thus fermion states may appear in these models. We should note that if we use Coleman's type of trial state to test the stability of the vacuum, the model, (V.26), developes the same disease of having no ground state as the sine-Gordon model, when $g^2 > \frac{\pi}{2\,(n-1)}$ (Appendix E).

Finally, if we want to reinstate the coupling of the "diagonal" currents in our model, we must allow the ψ_ℓ to become:

$$\psi_\ell = :e^{-i\left[\frac{\pi}{\beta}\tilde{\phi}_\ell + \beta\gamma^5\phi_\ell\right]}\left[\frac{m}{2\pi}\right]^{1/2}\mathbb{C}_\ell: \quad . \tag{V.27}$$

The equivalence can be worked out again for (V.27), but the equal-time commutation relations are modified in the same way as in the Thirring model discussed in Chapter II.

<u>Fermions in a Boson Theory with a Finite-Order Polynomial Interaction</u>

As a final example, we consider the boson theory associated with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\mu^2\phi^2}{2} - \frac{\lambda\phi^4}{4!} \quad . \tag{V.28}$$

Goldstone and Jackiw[14] have recently used a semi-classical calculation to argue that a fermion state exists in this theory. However, they find only one charged sector. No multiply-charged states exist. All these results are suppose to hold for weak coupling where the quantum fluctuations we discussed earlier do not change the classical picture drastically.

Since the classical energy is minimum at $\phi = \pm\sqrt{\frac{6\mu^2}{\lambda}}$, our naive considerations would agree with the charge sector structure given above. We can use our equivalence to ask what kind of fermion theory would exhibit such behavior. Using the correspondences in (V.7) and (V.8), we can develop the equations:

$$i\partial\!\!\!/\psi = :A\!\!\!/\psi:,$$

$$A_\mu = g_\mu^0 A_0$$

$$\partial_1^2 A_0(x) =: \sqrt{\pi}\, J_0(x) \Big\{ 2\mu^2 - \sqrt{6\mu^2\pi\lambda} \int_{x^1}^{\infty} dz\; J_0(z, x^0) + \frac{\lambda\pi}{2}\Big[\int_{x^1}^{\infty} dz\; J_0(z, x^0)\Big]^2 \Big\}, \qquad \text{(V.29).}$$

where J_μ is the current associated with the field ψ. These bizarre equations describe a Coulomb gauge formulation of a modified electrodynamics in which the magnitude of the effective charge associated with a fermion depends on the number of fermions and anti-fermions and their positions in the rest of space. Thus, the single fermion sector is allowed, but if we attempt to insert further numbers of fermions the Coulomb repulsion gives the resulting state an infinite energy. Clearly, the boson theory (V.28) is much easier to study.

VI. DISCUSSION

The major idea that evolves from our study of two-dimensional field theories is that the physics of interacting fermions can be conveniently represented in terms of a field theory of bosons instead of a fermion field theory. Furthermore, within the boson framework, the fermion field operator is a boson coherent state operator. Thus, we can get a certain amount of information by considering solutions of the classical boson theory which carry fermion number (i.e., satisfy the appropriate boundary conditions to correspond to the fermion states in the quantum theory). In some cases, the behavior of the classical solutions may be a good approximation for matrix elements of the fermion theory, because the coherent states involve filling all space with boson excitations and therefore the effect of quantum fluctuations may be relatively small.

How does this duality between fermion and boson representations of a physical model relate to the analogues of scaling and confinement that originally interested us in the two-dimensional theories? In order for a theory to exhibit approximate scaling at large momenta, the fields must behave like canonical free fields at short distances. We see that in our models that amounts to avoiding Thirring-type interactions which lead to anomalous short distance singularities. This question is totally independent of whether or not asymptotic fermions exist. In the equivalent boson framework, we find that we need only require that the interaction be superrenormalizable.

Confinement, on the other hand, depends on the infrared or long distance behavior of the theories. Here the duality becomes extremely useful. As we have noted, we can find a boson theory corresponding to any given fermion theory we would like to study. The states in the boson theory which are equivalent to fermion states involve coherent boson excitations. We argued in Chapter II that the boson field represents a polarization density. Consequently, a localized fermion is represented by a polarization gradient localized in space. The boson field, however, will not be localized in such a state. Thus, a fermion state, viewed in terms of the boson field, is a macroscopic state that fills all space. Such states may perhaps be well approximated by classical fields since the quantum fluctuations may be small relative to the total field strengths. If Dashen, Hasslacher, and Neveu's [10] results are exact, as they conjecture, then the states in the sine-Gordon theory are indeed well approximated by the classical solutions. The quantum effects only cause the bound state spectrum to be discrete rather than continuous.

In these terms, the existence of fermion sectors in a theory can be investigated, at least semi-classically, by looking for classical solutions of the boson equation of motion which behave at infinity like a fermion state matrix element. We found in the theories we considered that the electromagnetic interactions were equivalent to massive boson theories which do not allow the extended solutions corresponding to fermions. Thus, these theories displayed a lack of non-zero charge sectors. It is important to note, as we did in Chapter IV, that this lack is not equivalent to total confinement. If the electric charge can be shielded by massless excitations, the asymptotic states may still carry some of the quantum numbers of the original fermion. We can visualize the physical states as a bare charged fermion, surrounded by a cloud of massless polarization dipoles extending to infinity. On the other hand, if the only excitations available to shield the fermion are massive, it would require an infinity energy to build the cloud. As a result, such a situation would lead to total confinement.

Finally, can we apply any of these ideas we have found in two dimensions to the physically relevant case of four dimensional field theories? There are two approaches which suggest themselves as possibly useful applications of the two-dimensional results. The first avenue is the isolation of physical situations where the number of degrees of freedom is limited to two. Obviously, the two-dimensional results may be relevant in such instances. The most interesting cases would be models where the boundary conditions in the constrained degrees of freedom determined the coupling parameters in the residual two-dimensional problem. As a heuristic example, suppose some scattering process at high energies is dominated by physics along the direction of the incident particles in the center-of-mass. Further, let this physics be represented by the massive Thirring model whose dynamics are given by (10), where β is determined by an external variable - say the total energy.

The results of a series of scatterings at different β will be qualitatively different. At low β, the final state will dissolve immediately into pieces corresponding to the Thirring boson states. For $\beta \lesssim \sqrt{\pi}$, the Thirring fermion state is only slightly above the lowest boson state in the spectrum, and the final state will keep the form of a coherent state for a longer time. Long range correlations and clustering of the final state particles would most likely develop. Using this approach we could build a phenomenology in which a range of two-dimensional models corresponds to a range of four-dimensional kinematics.

The other research program that holds some promise is the implementation of some coherent state - point particle duality in four dimensions. Such a correspondence would allow simpler analysis of certain problems, in exactly the same way as we found in two dimensions. It would be foolish to apply such an approach to low energy QED where a free fermion is a good approximation, but in models like the SU(n) Yang-Mills theory where the spinor field may not be represented in the physical spectrum, the coherent states may provide a better approximation. (In fact, while this thesis was being revised, extensive work has been done on the possibility of classical soliton solutions and their quantized counterparts in the SU(n) and other similar theories.)

Regardless of whether the two suggestions described above prove to be feasible methods of attack, the understanding of quantum field theories in one-space, one-time dimension should provide insights to the structure of more realistic models. Hopefully, the reader agrees.

APPENDIX A: CONVENTIONS AND NOTATION

1. Gamma Matrices and Tensors:

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \gamma^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1,$$

$\varepsilon_{\mu\nu}$ is the antisymmetric symbol with $\varepsilon_{01} = 1$,

$g_{\mu\nu}$ is the symmetric metric tensor, $g_{00} = 1$, $g_{11} = -1$,

$$\gamma^\mu\gamma^5 = \varepsilon^\mu{}_\nu\gamma^\mu,$$

2. Light Cone Co-ordinates:

$$x_R = \frac{1}{\sqrt{2}}(x^0 + x^1), \qquad x_L = \frac{1}{\sqrt{2}}(x^0 - x^1),$$

$$\partial_R = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1), \qquad \partial_L = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1).$$

For an arbitrary vector V_μ:

$$V_R = \frac{1}{\sqrt{2}}(V^0 + V^1),$$

$$V_L = \frac{1}{\sqrt{2}}(V^0 - V^1).$$

For an arbitrary spinor ψ:

$$\psi_R = \frac{1}{2}(1 + \gamma^5)\psi,$$

$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi.$$

3. Singular Functions:

For masses $\mu^2 > 0$

$$\Delta^{\pm}_{\mu^2}(x-y) = -i\int Dk\, e^{\mp ik\cdot(x-y)},$$

$$\text{where } Dk = \frac{d^2k}{2\pi}\delta(k^2-\mu^2)\theta(k^0),$$

$$\Delta_{\mu^2}(x-y) = \Delta^{+}_{\mu^2} - \Delta^{-}_{\mu^2}.$$

For mass $\mu^2 = 0$, the cut-off singular functions with cut-off parameter $\kappa\,(K=k^1)$:

$$\Delta^{\pm}_{0}(x-y) = -i\int Dk\left\{e^{\mp ik\cdot(x-y)} - \theta(\kappa-|K|)\right\},$$

$$\Delta^{+}_{0}(x) = \frac{i}{4\pi}\ln\left[-\eta^2x^2 + i\epsilon x^0\right],\quad \eta = \kappa e^{-\Gamma'(1)},$$

$$\tilde{\Delta}^{\pm}_{0}(x-y) = -i\int Dk\,\epsilon(K)\left\{e^{\mp ik\cdot(x-y)} - \theta(\kappa-|K|)\right\},$$

$$\tilde{\Delta}^{+}_{0}(x) = \frac{i}{4\pi}\ln\left|\frac{x^0-x^1}{x^0+x^1}\right| + \frac{1}{4}\epsilon(x^1)\theta(-x^2),$$

$$\Delta^{\pm}_{\kappa}(x) = -i\int Dk\left\{e^{\mp ik\cdot x} - 1\right\}\theta(\kappa-|K|),$$

$$\tilde{\Delta}^{\pm}_{\kappa}(x) = -i\int Dk\,\epsilon(K)\left\{e^{\mp ik\cdot x} - 1\right\}\theta(\kappa-|K|),$$

$$D^{\pm}_{\kappa}(x,y) = -i\int Dk\left\{e^{\mp ik\cdot x} - \theta(\kappa-|K|)\right\}e^{\pm ik\cdot y},$$

$$\tilde{D}^{\pm}_{\kappa}(x,y) = -i\int Dk\ \epsilon(K)\left\{e^{\mp ik\cdot x} - \theta(\kappa - |K|)\right\}e^{\pm ik\cdot y},$$

$$D^{\pm}_{\kappa\kappa}(x,y) = -i\int Dk\ \left\{e^{\mp ik\cdot x} - \theta(\kappa - |K|)\right\}\left\{e^{\pm ik\cdot y} - \theta(\kappa - |K|)\right\},$$

$$\tilde{D}^{\pm}_{\kappa\kappa}(x,y) = -i\int Dk\ \epsilon(K)\left\{e^{\mp ik\cdot x} - \theta(\kappa - |K|)\right\}\left\{e^{\pm ik\cdot y} - \theta(\kappa - |K|)\right\}.$$

4. Free Fields:

Scalar: $$\phi(x) = \int Dk\left\{d(K)e^{-ik\cdot x} + d^{\dagger}(K)e^{ik\cdot x}\right\},$$

$$\phi^{(+)}(x) = \int Dk\ \ d^{\dagger}(K)\,e^{ik\cdot x},$$

$$\phi^{(-)}(x) = \int Dk\ \ d(K)\,e^{-ik\cdot x},$$

$$[\phi(x), \phi(y)] = i\Delta_{\mu^2}(x-y).$$

Spinor: $$\chi(x) = \int Dk\left\{a(K)\,e^{-ik\cdot x}u(K) + b^{\dagger}(K)\,e^{ik\cdot x}v(K)\right\},$$

$$u(K) = \frac{1}{\sqrt{k^0 - m}}\begin{bmatrix} k^1 \\ k^0 - m\end{bmatrix},\quad v(K) = \frac{1}{\sqrt{k^0+m}}\begin{bmatrix} k^1 \\ k^0 + m\end{bmatrix},$$

$$\{\chi(x), \chi(y)\} = i(i\partial\!\!\!/ + m)\,\Delta_{m^2}(x-y).$$

5. U(n) Matrices:

The λ_i, $i = 0, 1, \ldots, n^2 - 1$, are the n^2, n - by - n hermitian matrices associated with the standard U(n) representation,

$$\{\lambda_\ell, \lambda_m\} = 2 d_{\ell m n} \lambda^n ,$$

$$[\lambda_\ell, \lambda_m] = 2 i f_{\ell m n} \lambda^n ,$$

$$\mathrm{tr}\,[\lambda_i \lambda_j] = 2 \delta_{ij} ,$$

$$\mathrm{tr}\,[\lambda_i] = \sqrt{2n}\, \delta_{io} .$$

For convenience, we single out the diagonal λ_1 as the τ_j, $j = 0, 1, \ldots, n - 1$. $\tau_o = \lambda_o$, and the other τ_j are, for example, in U(3), $\tau_1 = \lambda_3$, $\tau_2 = \lambda_8$.

APPENDIX B: THE THIRRING MODEL

In this Appendix, we construct in some detail an operator solution of the Thirring model in a positive definite Hilbert space. The solution follows the one given by Klaiber in his comprehensive article(17). As we mentioned in Chapter II, a massless scalar field is not well-defined in two dimensions. One symptom is the fact that the two point function is infrared divergent. Klaiber resolves this difficulty by brute force. The divergences are eliminated by cutting the momenta off at a value κ (see Appendix A). This procedure is not manifestly Lorentz covariant, but all the Wightman functions in the Thirring model can be shown to be covariant. In Chapter II, we explained that the cut-off implements the inequivalent representations of j_μ, while maintaining the Fock space operator structure. The dimensional constant κ appears in the final Wightman functions as the scale for the momenta. Since the theories are scale invariant, rescaling κ simply rescales the momenta without any observable effect. In the formal constructions discussed in Chapter V, the boson theories are no longer scale invariant, and if we actually could solve the models, the limit $\kappa \to o$ would have to be taken. Presumably the dimensional coupling constants in such theories appear in the exact solution as masses which eliminate the infrared divergences.

Using the free fermion annihilation and creation operators, a and b (Appendix A), we can construct the truncated ϕ and $\tilde{\phi}$ fields:

$$\phi^{(\pm)}(x) = \int Dk \ \epsilon(K) \ d^{\dagger}_{(\cdot)}(K) \left\{ e^{\pm ik\cdot x} - \Theta(\mathcal{K} - |K|) \right\},$$

$$\tilde{\phi}^{(\pm)}(x) = \int Dk \ d^{\dagger}_{(\cdot)}(K) \left\{ e^{\pm ik\cdot x} - \Theta(\mathcal{K} - |K|) \right\},$$

where
$$d^{\dagger}(K) = -i \int \frac{dK'}{2\pi} \left\{ a^{\dagger}(K')\, b^{\dagger}(K-K') \frac{\Theta((K-K')K')}{\sqrt{|K-K'||K'|}} + \right.$$
$$+ a^{\dagger}(K+K')\, a(K') \frac{\Theta(KK')}{\sqrt{|K+K'||K'|}} +$$
$$\left. - b^{\dagger}(K+K')\, b(K') \frac{\Theta(KK')}{\sqrt{|K+K'||K'|}} \right\}. \tag{B.1}$$

The ϕ and $\tilde{\phi}$ can be decomposed invariantly like normal free fields:

$$[d(K), d(K')] = 0 = [d^{+}(K), d^{+}(K')],$$

$$[d^{+}(K), d(K)] = 4|K|\,\delta(K-K'). \qquad \text{(B.2)}$$

To restore covariance and restore the charge operators (zero momentum components of ϕ and $\tilde{\phi}$), we modify the operators:

$$\Phi^{(\pm)} = \phi^{(\pm)} \mp \left(\frac{\alpha}{\sqrt{\pi}} + 1\right) Q\,\tilde{\Delta}^{\mp}_{K} \mp \left(\frac{\beta}{\sqrt{\pi}} + 1\right) \tilde{Q}\,\Delta^{\mp}_{K},$$

$$\tilde{\Phi}^{(\pm)} = \tilde{\phi}^{(\pm)} \mp \left(\frac{\alpha}{\sqrt{\pi}} + 1\right) Q\,\Delta^{\mp}_{K} \mp \left(\frac{\beta}{\sqrt{\pi}} + 1\right) \tilde{Q}\,\tilde{\Delta}^{\mp}_{K},$$

where $$Q = \int_{-\infty}^{\infty} dz\; j^{0}(z,t),$$

$$\tilde{Q} = \int_{-\infty}^{\infty} dz\; j^{1}(z,t). \qquad \text{(B.3)}$$

α and β are the parameters appearing in (II.6). The Thirring solution is then constructed:

$$\psi(x) = e^{i\rho^{+}(x)}\chi(x)\, e^{i\rho^{-}(x)},$$

$$J_\mu = \epsilon_{\mu\nu}\partial^\nu \Phi,$$

$$\rho^{\pm}(x) = \sqrt{\pi}\Big[(\alpha+\sqrt{\pi})\,\tilde{\Phi}^{(\pm)} + (\beta+\sqrt{\pi})\,\gamma^5_x\,\Phi^{(\pm)} + \pm\big\{(\alpha+\sqrt{\pi})Q + (\beta+\sqrt{\pi})\gamma^5_x \dot{\tilde{Q}}\big\}\big\{\Delta^{\pm}_K + \gamma^5_x \tilde{\Delta}^{\pm}_K\big\}\Big]. \tag{B.4}$$

γ^5_x acts on $\chi(x)$. Since all the operator structure is explicitly known, we can now calculate all the Wightman functions. However, (B.4) still involves the free spinor. Imitating Klaiber's method, we can eliminate the spinor. The solution is re-expressed:

$$\psi(x) = e^{i\tilde{\rho}^{+}(x)}\left[\frac{m}{2\pi}\right]^{1/2} \mathbb{C}\, e^{i\tilde{\rho}^{-}(x)},$$

$$J_\mu = \epsilon_{\mu\nu}\partial^\nu \Phi,$$

with $\tilde{\rho}^{\pm} = \sqrt{\pi}\,[\alpha\,\tilde{\Phi}^{(\pm)} + \beta\,\Phi^{(\pm)}\gamma^5_x],$

$$\tilde{\Phi}^{(\pm)} = \tilde{\phi}^{(\pm)} \mp \frac{\alpha}{\sqrt{\pi}} D\Delta^{\mp}_K \mp \frac{\beta}{\sqrt{\pi}} \tilde{D}\tilde{\Delta}^{\mp}_K,$$

$$\Phi^{(\pm)} = \phi^{(\pm)} \mp \frac{\alpha}{\sqrt{\pi}} D\tilde{\Delta}^{\mp}_K \mp \frac{\beta}{\sqrt{\pi}} \tilde{D}\Delta^{\mp}_K. \tag{B.5}$$

The C and D are co-ordinate independent operators. C is a two-component spinor, while D and $\tilde{D}$ are single hermitian operators.

$$[\mathbb{C}_i, \mathbb{C}_j] = 0 = [\mathbb{C}_i, \mathbb{C}_j^\dagger] \qquad (i, j = 1, 2),$$

$$[D, \mathbb{C}] = -\mathbb{C}, \qquad [\tilde{D}, \mathbb{C}] = -\gamma^5 \mathbb{C},$$

$$[D, \tilde{D}] = 0,$$

$$\langle 0 | (\mathbb{C}_1)^m (\mathbb{C}_2)^n (\mathbb{C}_1^\dagger)^m (\mathbb{C}_2^\dagger)^n | 0 \rangle = 1. \tag{B.6}$$

All Wightman functions involving C's are zero unless, the number of C_i and C_i^+ are equal for $i = 1, 2$. D and $\tilde{D}$ have no effect on the matrix elements of the theory. The stress-energy tensor is

$$\theta_{\mu\nu} = \frac{\pi}{2} : \left\{ J_\mu J_\nu + J^5_\mu J^5_\nu \right\} : . \tag{B.7}$$

The Poincaré properties in Chapter II follow from (B.7). For completeness, we display the general Wightman function;

$$\langle 0|\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_1)\cdots\bar{\psi}(y_n)|0\rangle =$$
$$= e^{iF}\langle 0|\chi(x_1)\cdots\chi(x_n)\bar{\chi}(y_1)\cdots\bar{\chi}(y_n)|0\rangle,$$

where
$$\begin{aligned}F = &\sum_{j<k}\Big[(-\alpha^2+\pi)\Delta_0^+(x_j-x_k) + (-\beta^2+\pi)\gamma^5_{x_j}\gamma^5_{x_k}\Delta_0^+(x_j-x_k) + \\ &\qquad + (-\alpha\beta+\pi)(\gamma^5_{x_j}+\gamma^5_{x_k})\tilde{\Delta}_0^+(x_j-x_k)\Big] + \\ &+\sum_{j,k}\Big[(\alpha^2-\pi)\Delta_0^+(x_j-y_k) + (-\beta^2+\pi)\gamma^5_{x_j}\gamma^5_{y_k}\Delta_0^+(x_j-y_k) + \\ &\qquad + (\alpha\beta-\pi)(\gamma^5_{x_j}-\gamma^5_{y_k})\tilde{\Delta}_0^+(x_j-y_k)\Big] + \\ &+\sum_{j<k}\Big[(-\alpha^2+\pi)\Delta_0^+(y_j-y_k) + (-\beta^2+\pi)\gamma^5_{y_j}\gamma^5_{y_k}\Delta_0^+(y_j-y_k) + \\ &\qquad + (\alpha\beta-\pi)(\gamma^5_{y_j}+\gamma^5_{y_k})\tilde{\Delta}_0^+(y_j-y_k)\Big].\end{aligned} \tag{B.8}$$

The two point function in momentum space has a simple spectral content:

$$\int d^2x\, e^{ip\cdot x}\,\frac{\gamma^0+\gamma^5}{\sqrt{2}}\langle 0|\psi(x)\psi^\dagger(0)|0\rangle\frac{\gamma^0+\gamma^5}{\sqrt{2}} =$$

$$= -\frac{\pi}{m}\,\frac{\Theta(p^0)\,\Theta(p^2)}{\Gamma(\frac{a+b}{4}-s)\Gamma(\frac{a+b}{4}+s)}\left[\frac{4m^2}{p^2}\right]^{1+s-\frac{a+b}{4}} \times$$

$$\times\begin{bmatrix}\left(\frac{p^0+p^1}{2m}\right)^{2s} & 0\\ 0 & \left(\frac{p^0-p^1}{2m}\right)^{2s}\end{bmatrix},$$

where $$a = \frac{\alpha^2}{\pi},$$

$$b = \frac{\beta^2}{\pi},$$

$$s = \frac{\alpha\beta}{2\pi}. \tag{B.9}$$

One of the interesting properties of the Thirring solution is the generalized spin-statistics connection. For space-like separation of the points x and y:

$$\psi(x)\psi^{+}(y) = e^{i\pi s\,(\gamma^5_x + \gamma^5_y)\,\epsilon(x^1 - y^1)}\,\psi^{+}(y)\,\psi(x),$$

$$\psi(x)\psi(y) = e^{-i\pi s\,(\gamma^5_x + \gamma^5_y)\,\epsilon(x^1 - y^1)}\,\psi(y)\,\psi(x). \tag{B.10}$$

For integer s, the ψ's commute at space-like distances. For half-integer s, the ψ's anti-commute. For all other s, locality does not hold, even though the theory is micro-causal. We should point out that the commutation structure involving different components of ψ in both the simple Thirring model and the U(n) structures of Chapter III can be changed from commutation to anticommutation (or vice versa) by means of a Klein transformation [24]. Thus, we need only be concerned with the commutation structure of a particular component with itself.

If either $\alpha = o$ or $\beta = o$, the charge or axial charge associated with ψ vanishes. In addition, the theory no longer has the cluster decomposition property. These diseases are symptomatic of the fact that the theory is degenerate for these values of the parameters, leaving only half the physical degrees of freedom.

In our construction we have used the Fock vacuum. Obviously, we could generate an equivalent representation of the matrix elements by choosing a different vacuum:

$$|n_1, n_2\rangle = (\mathbb{C}_1^{\dagger})^{n_1} (\mathbb{C}_2^{\dagger})^{n_2} |0\rangle, \quad n_1, n_2 \text{ integer.} \tag{B.11}$$

We can produce models which break any of the discrete symmetries like parity and fermion number by selecting a physical vacuum built from a coherent combination of the states in (B.11);

$$|v\rangle = \sum_{n=-\infty}^{\infty} e^{in(\theta_1 + \gamma^5 \theta_2)} (\mathbb{C}^{\dagger})^n |0\rangle. \tag{B.12}$$

For instance,

$$\tfrac{1}{2}(1 \pm \gamma^5)\,\mathbb{C}\,|v\rangle = e^{i(\theta_1 \pm \theta_2)}\,|v\rangle,$$

$$\langle v|\,\tfrac{1}{2}(1 \pm \gamma^5)\,\psi(x)\,|v\rangle = \left[\frac{m}{2\pi}\right]^{1/2} e^{-i(\theta_1 \pm \theta_2)}. \tag{B.13}$$

In Chapter V, the interactions may explicitly break some discrete symmetry of the theory. The physical vacuum will then be one of the states given by (B.12).

To complete the discussion of the Thirring solution, we explicitly define the operator products involved. The bilocal field product can be written, using (B.4) and (B.5) ($\hat{=}$ denotes correspondence rather than strict equality):

$$\begin{aligned}
\psi(x)\psi^{+}(y) = {} & e^{i(\alpha\beta-\pi)(\gamma^5_x+\gamma^5_y)\tilde{\Delta}^{+}_{0}(x-y)}\,\cdot \\
& \cdot\, e^{i[(\alpha^2-\pi)+\gamma^5_x\gamma^5_y(\beta^2-\pi)]\Delta^{+}_{0}(x-y)}\,\cdot \\
& \cdot\, e^{i(\rho^{+}(x)-\rho^{+}(y))}\,\chi(x)\chi^{+}(y)\,e^{i(\rho^{-}(x)-\rho^{-}(y))} \\
\hat{=} {} & e^{i\alpha\beta(\gamma^5_x+\gamma^5_y)\tilde{\Delta}^{+}_{0}(x-y)}\,e^{i[\alpha^2+\gamma^5_x\gamma^5_y\beta^2]\Delta^{+}_{0}(x-y)}\,\cdot \\
& \cdot\, e^{i(\tilde{\rho}^{+}(x)-\tilde{\rho}^{+}(y))}\,\mathbb{C}\,\frac{m}{2\pi}\,\mathbb{C}^{\dagger}\,e^{i(\tilde{\rho}^{-}(x)-\tilde{\rho}^{-}(y))}.
\end{aligned} \tag{B.14}$$

Lowenstein's product is best expressed in terms of the first form of (B.14) with the explicit presence of χ.

$$L\left\{\psi(x)\psi^{+}(y)\right\} = e^{i(\rho^{+}(x)-\rho^{+}(y))} : \chi(x)\chi^{+}(y) : e^{i(\rho^{-}(x)-\rho^{-}(y))} +$$

$$+\left\{e^{i(\Omega(x)-\Omega(y))} - 1\right\} e^{i(K^{+}(x)-K^{+}(y))} e^{i(K^{-}(x)-K^{-}(y))} \langle 0|\chi(x)\chi^{+}(y)|0\rangle,$$

with

$$\Omega(x) = -\sqrt{\pi}\left\{(\alpha+\sqrt{\pi})Q + (\beta+\sqrt{\pi})\gamma^{5}_{x}\tilde{Q}\right\}\left\{\Delta_{\chi} + \gamma^{5}_{x}\tilde{\Delta}_{\chi}\right\}.$$

$$K^{\pm}(x) = +\sqrt{\pi}\left\{(\alpha+\sqrt{\pi})\tilde{\Phi}^{(\pm)} + (\beta+\sqrt{\pi})\gamma^{5}_{x}\,\Phi^{(\pm)}\right\}. \tag{B.15}$$

The regularized product we use is given by

$$R\left\{\psi(x)\psi^{+}(y)\right\} = e^{i(\rho^{+}(x)-\rho^{+}(y))}\chi(x)\chi^{+}(y)\,e^{i(\rho^{-}(x)-\rho^{-}(y))} + \\ - \langle 0|\chi(x)\chi^{+}(y)|0\rangle$$

$$\hat{=}\; e^{i\pi(\gamma^{5}_{x}+\gamma^{5}_{y})\tilde{\Delta}^{+}_{0}(x-y)}\, e^{i\pi(1+\gamma^{5}_{x}\gamma^{5}_{y})\Delta^{+}_{0}(x-y)} \cdot \\ \cdot\left\{e^{i(\tilde{\rho}^{+}(x)-\tilde{\rho}^{+}(y))}\,\mathbb{C}\,\frac{m}{2\pi}\,\mathbb{C}^{+}\,e^{i(\tilde{\rho}^{-}(x)-\tilde{\rho}^{-}(y))} - \langle 0|\mathbb{C}\,\frac{m}{2\pi}\,\mathbb{C}^{+}|0\rangle\right\}.$$

(B.16)

The normal products are then derived as in Chapter II. If we utilize the structure of the singular functions and the representation in which the free spinor is eliminated we can see that the normal products are alternatively obtained:

$$N(\bar{\Psi}\gamma_{\mu}\Psi) = -\frac{i}{2\pi}\partial_{\mu}\, \mathrm{tr}\left\{ e^{i(\tilde{\rho}^{+}(o)-\tilde{\rho}^{+}(x))}\, \mathbb{C}\mathbb{C}^{\dagger}\, e^{i(\tilde{\rho}^{-}(o)-\tilde{\rho}^{-}(x))} \right\}\Big|_{x=o} ,$$

$$N_{5}(\bar{\Psi}\gamma_{\mu}\gamma^{5}\Psi) = -\frac{i}{2\pi}\partial_{\mu}\, \mathrm{tr}\left\{ e^{i(\tilde{\rho}^{+}(o)-\tilde{\rho}^{+}(x))}\, \mathbb{C}\mathbb{C}^{\dagger} e^{i(\tilde{\rho}^{-}(o)-\tilde{\rho}^{-}(x))}\gamma^{5} \right\}\Big|_{x=o} . \tag{B.17}$$

We see that when ψ is represented totally as a coherent state operator as in (B.5), the bilinear normal products are either simply the Wick product (with infrared corrections) or, in the cases where the Wick product yields a c-number only, the derivative of the bilocal Wick product. Put more elegantly, the normal products are the leading operators with the correct Lorentz behavior in the short distance expansion of the product $\psi(o)\psi^{+}(x)$.

Extending our consideration to other normal products, we find the generalization to many fields at the same point is fairly straightforward. The simplest description of the procedure is in terms of the pure coherent state picture of (B.5). The general normal product is the leading q-number operator with the correct Lorentz properties in a short distance expansion of the simple field product. The field product must be corrected by subtracting the vacuum expectation values of each subset of the local fields involved in the product. This additional procedure follows the Wick product expansion used for free local fields:

$$
\begin{aligned}
:\phi_1 \cdots \phi_n: \; = \; & \phi_1 \cdots \phi_n - \sum_{j \neq k} :\phi_1 \cdots \underbrace{\phi_j \cdots \phi_k} \cdots \phi_n: + \\
& - \sum_{j \neq k \neq l \neq m} :\phi_1 \cdots \phi_j \cdots \phi_l \cdots \phi_k \cdots \phi_m \cdots \phi_n: + \\
& - \cdots \quad . \qquad \text{(B.18)}
\end{aligned}
$$

The brackets indicate that the pair of fields is replaced in the product by its vacuum expectation value.

Since the simple products of coincident fermion fields like $\{\frac{(1+\gamma^5)}{2}\psi\}\ \{\frac{(1+\gamma^5)}{2}\psi\}$ are zero, the normal products involving such expressions vanish. Thus, the higher order normal products are most conveniently expressed in terms of J_μ and J^5_μ. Obviously, such products are simply Wick-products (with infrared corrections).

APPENDIX C: THIRRING GENERALIZATIONS

The infrared regularization scheme in these generalizations follows Klaiber's approach. Very briefly:

$$\psi(x) = e^{i\tilde{\rho}^{+}(x)} \left[\frac{m}{2\pi}\right]^{1/2} \mathbb{C}\, e^{i\tilde{\rho}^{-}(x)},$$

$$\tilde{\rho}^{\pm} = \sqrt{\frac{\pi}{2}}\, \tau_i \left[\tilde{K}^{i(\pm)} + \gamma^5 K^{i(\pm)}\right]$$

$$K^{i(\pm)} = -\frac{\beta^i{}_\ell}{\sqrt{\pi}} \left\{\phi^{\ell(\pm)} \mp \frac{\alpha^\ell{}_m}{\sqrt{\pi}} D^m \tilde{\Delta}^{\mp}_{\kappa} \mp \frac{\beta^\ell{}_m}{\sqrt{\pi}} \tilde{D}^m \Delta^{\mp}_{\kappa}\right\},$$

$$\tilde{K}^{i(\pm)} = -\frac{\alpha^i{}_\ell}{\sqrt{\pi}} \left\{\tilde{\phi}^{\ell(\pm)} \mp \frac{\alpha^\ell{}_m}{\sqrt{\pi}} D^m \Delta^{\mp}_{\kappa} \mp \frac{\beta^\ell{}_m}{\sqrt{\pi}} \tilde{D}^m \tilde{\Delta}^{\mp}_{\kappa}\right\}. \tag{C.1}$$

The different components of ψ will anti-commute at space-like distances only if we perform a Klein transformation on ψ.

It is obvious that the Wightman functions, spectrum, spin-statistics connection, and cluster decomposition follow directly from (C.1) and our discussion in Appendix B.

As an example of the application of (III.22) to relate "diagonal" current products to "non-diagonal" products, we outline the derivation of an identity for the free fermion case. ψ can be expressed as in (III.5) with $\alpha_{ij} = \beta_{ij} = \sqrt{\pi}\delta_{ij}$, and the currents are:

$$j^b_\mu(x) = \overline{\psi}(x)\gamma_\mu \frac{\lambda^b}{\sqrt{2}}\psi(x), \qquad b = 1,\dots,n^2-1,$$

$$j^0_\mu(x) = \overline{\psi}(x)\gamma_\mu \frac{\lambda^0}{\sqrt{2}}\psi(x) - \langle 0|\overline{\psi}(x)\gamma_\mu\frac{\lambda^0}{\sqrt{2}}\psi(x)|0\rangle. \tag{C.2}$$

Consider,

$$j^b_R(\eta)\, j^b_R(0) = \left[\overline{\psi}(\eta)\gamma_R\frac{\lambda^b}{\sqrt{2}}\psi(\eta)\right]\left[\overline{\psi}(0)\gamma_R\frac{\lambda^b}{\sqrt{2}}\psi(0)\right], \qquad b = 1,\dots,n^2-1.$$

$$\begin{aligned} &= \left[\overline{\psi}(\eta)\gamma_R\frac{\lambda^a}{\sqrt{2}}\psi(\eta)\right]\left[\overline{\psi}(0)\gamma_R\frac{\lambda^a}{\sqrt{2}}\psi(0)\right] + \\ &\quad -\left[\overline{\psi}(\eta)\gamma_R\frac{\lambda^0}{\sqrt{2}}\psi(\eta)\right]\left[\overline{\psi}(0)\gamma_R\frac{\lambda^0}{\sqrt{2}}\psi(0)\right], \\ &\qquad a = 0,\dots,n^2-1. \end{aligned} \tag{C.3}$$

Using (III.22),

$$\begin{aligned} j^b_R(\eta)\, j^b_R(0) = &-n\left[\overline{\psi}(\eta)\gamma_R\frac{\lambda^0}{\sqrt{2}}\psi(0)\right]\left[\overline{\psi}(0)\gamma_R\frac{\lambda^0}{\sqrt{2}}\psi(\eta)\right] + \\ &-\left[\overline{\psi}(\eta)\gamma_R\frac{\lambda^0}{\sqrt{2}}\psi(\eta)\right]\left[\overline{\psi}(0)\gamma_R\frac{\lambda^0}{\sqrt{2}}\psi(0)\right] + \\ &+\left[\overline{\psi}(\eta)\gamma_R\frac{\lambda^0}{\sqrt{2}}\psi(\eta)\right]\langle 0|\, tr\Big(\{\overline{\psi}_\ell(0),\psi_m(0)\}\Big(\gamma_R\frac{\lambda^0}{\sqrt{2}}\Big)_{\ell m}\Big)|0\rangle + \\ &+\left[\overline{\psi}(\eta)\gamma_R\frac{\lambda^a}{\sqrt{2}}\right]_\ell \langle 0|\{\psi_\ell(\eta),\overline{\psi}_m(0)\}|0\rangle\left(\gamma_R\frac{\lambda^a}{\sqrt{2}}\psi(0)\right]_m. \end{aligned}$$

(C.4)

If we now subtract the vacuum expectation, $\langle o \mid j_R^b(\eta) j_R^b(o) \mid o \rangle$, from (C.4) and then take the limit $\eta \to o$ (averaging over η directions), we get two contributions. The first comes from the finite operators in each $[\bar{\psi}\gamma_R \frac{\lambda^o}{\sqrt{\pi}} \psi]$ factor. This yields $-(n+1){:}\, j_R^o(o) j_R^o(o){:}$. The other term comes from the first term in (C.4) resulting from the $\frac{1}{\eta}$ singularity of one factor multiplying the operator part of the other factor which vanishes like η. Using the explicit form (III.5), we find that this yields $(n+1){:}K_R^i(o)K_R^i(o){:}$. The final result is:

$$:j_R^o j_R^o + \frac{1}{n+1} j_R^b j_R^b: \; = \; :K_R^i K_R^i: . \qquad \text{(C.5)}$$

Coleman, Gross, and Jackiw(7) do essentially this calculation in complete detail for the free fermion case.

The Poincaré structure for (C.1) is:

$$\Theta_{\mu\nu} = \frac{\pi^2}{2} : \left\{ (\tilde{\alpha}^{-1})^{\ell}{}_{m} K_\mu^m \, (\tilde{\alpha}^{-1})_{\ell}{}^{n} K_\nu^n + (\tilde{\beta}^{-1})^{\ell}{}_{m} K_\mu^{5m} \, (\tilde{\beta}^{-1})_{\ell}{}^{n} K_\nu^{5n} \right\},$$

$$[P_\mu, \psi] = -\frac{\pi}{\sqrt{2}} \gamma_\mu : \left\{ A^i{}_m K_\mu^m + B^i{}_m \gamma^5 K_\mu^{5m} \right\} \gamma^\mu \tau_i \psi :, \quad \text{(no sum on } \mu\text{)},$$

$$[D, \psi] = -i x \cdot \partial \psi - \frac{i}{8\pi} \left\{ \alpha^i{}_\ell \alpha^{k\ell} + \beta^i{}_\ell \beta^{k\ell} \right\} \tau_i \tau_k \psi,$$

$$[M_{01}, \psi] = -i(x_0 \partial_1 - x_1 \partial_0)\psi - \frac{i}{4\pi} \alpha^i{}_\ell \beta^{k\ell} \tau_i \tau_k \gamma^5 \psi. \qquad \text{(C.6)}$$

If all the U(n) currents are included in the model, the structures (C.1) and (C.6) are unchanged, although we may be able to rewrite the equations of motion in a more symmetrical fashion. The additional currents are defined as a Wick product as we see in (III.21). The Lorentz properties of these currents can be calculated by commutation with M_{01}. The result is similar to the result for ψ in (C.6) since both ψ and the "off-diagonal" currents are improper coherent operators. The condition that the currents are Lorentz vectors is a condition on α and β. Similarly, the dimension of the "off-diagonal" currents is determined by α and β. From (C.6) we can see that the minimum dimension for ψ is the free value if we insist that it transform as a Lorentz spinor. The analogous result is true for the currents, and, as we stated in Chapter III, this constraint leads to free current structure for the solutions with canonical dimensions. If we study models with anomalous dimensions, the form of ψ and J^i_μ will lead to extra operator terms in the equal time commutation structure.

APPENDIX D; QED

The covariant form of Lowenstein and Swieca's solution can be expressed;

$$\psi = :e^{i\sqrt{\pi}[\tilde{\phi} + \gamma^5(\phi+\eta+\sigma)]} \left[\frac{e}{2\pi^{3/2}}\right]^{1/2} \mathcal{C}: ,$$

$$J_\mu = \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu (\sigma+\eta) + j_\mu ,$$

$$A_\mu = \frac{\sqrt{\pi}}{e} \epsilon_{\mu\nu} \partial^\nu (\sigma+\eta),$$

$$F_{\mu\nu} = \epsilon_{\mu\nu} \frac{e}{\sqrt{\pi}} \sigma. \tag{D.1}$$

The ϕ, $\tilde{\phi}$, j_μ have the same structure as in Chapter II. The η field is a free, massless ghost;

$$\partial^2 \eta = 0,$$

$$[\eta(x), \eta(y)] = -i\Delta_0(x-y). \tag{D.2}$$

Using the stress-energy tensor, it is easy to show that ψ has spin 1/2, and canonical dimension. The equations of motion are;

$$i\not\partial\psi = :e\not A\psi:,$$

$$\partial^\mu F_{\mu\nu} = -eJ_\nu + e\mathcal{J}_\nu,$$

$$\mathcal{J}_\mu = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\eta + j_\mu. \tag{D.3}$$

$\mathcal{J}_\mu$ creates states of zero norm since the ghost and Thirring contributions cancel. Thus, the proper equations of motion and a positive definite solution is insured if we limit the physical subspace by the Gupta-Bleuler condition

$$\mathcal{J}_\mu^{(-)}\,|\,\text{physical state}\rangle = 0. \tag{D.4}$$

The form of A_μ in this solution has an interesting interpretation. The free electromagnetic field does not correspond to any real degrees of freedom, since it can be gauged away. In the Lorentz gauge, we could write such a free, massless field as the sum of j_μ and the curl of the ghost η. Turning on the interaction gives a mass to the positive metric contribution to A_μ and leaves η alone.

For the two fermion case, the Gupta-Bleuler solution is a simple generalization of (D.1)

$$\psi_1 = :e^{i\theta_1\sqrt{\pi}\,\gamma^5[\sigma+\eta]}\,e^{i\sqrt{\pi}[\tilde{\phi}_1+\gamma^5\phi_1]}\left[\frac{m}{2\pi}\right]^{1/2}\mathbb{C}_1: ,$$

$$\psi_2 = :e^{i\theta_2\sqrt{\pi}\,\gamma^5[\sigma+\eta]}\,e^{i\sqrt{\pi}[\tilde{\phi}_2+\gamma^5\phi_2]}\left[\frac{m}{2\pi}\right]^{1/2}\mathbb{C}_2: ,$$

$$J_{1\mu} = \frac{\theta_1}{\sqrt{\pi}}\,\epsilon_{\mu\nu}\,\partial^\nu(\sigma+\eta) + j_{1\mu},$$

$$J_{2\mu} = \frac{\theta_2}{\sqrt{\pi}}\,\epsilon_{\mu\nu}\,\partial^\nu(\sigma+\eta) + j_{2\mu},$$

$$A_\mu = \frac{\sqrt{\pi}}{e}\,\epsilon_{\mu\nu}\,\partial^\nu(\sigma+\eta),$$

$$F_{\mu\nu} = \epsilon_{\mu\nu}\frac{e}{\sqrt{\pi}}\,\sigma. \tag{D.5}$$

The equations of motion are:

$$i\partial\!\!\!/\psi_1 = :e_1 A\!\!\!/\,\psi_1: ,$$

$$i\partial\!\!\!/\psi_2 = :e_2 A\!\!\!/\,\psi_2: ,$$

$$\partial^\mu F_{\mu\nu} = e\mathcal{J}_\nu - e_1 J_{1\nu} - e_2 J_{2\nu},$$

$$\mathcal{J}_\mu = \frac{1}{\sqrt{\pi}}\,\epsilon_{\mu\nu}\,\partial^\nu\eta + \theta_1 j_{1\mu} + \theta_2 j_{2\mu}. \tag{D.6}$$

APPENDIX E: FORMAL STRUCTURES IN NON-TRIVIAL THEORIES

Choice of Vacuum

The interaction described by (V.12) is not diagonal in the spaces associated with C_1 and C_2. Thus, the interaction not only modifies the ϕ dynamics, but also maps one $|n_1, n_2>$ into another. We saw in Appendix B that the various $|n_1, n_2>$ are all acceptable vacua. The most convenient representation of the modified theory is, therefore, with a linear combination of the Thirring vacua. For instance,

$$|v> = \sum_{n=-\infty}^{\infty} (\mathbb{C}_1^{\dagger}\mathbb{C}_2)^n |0>. \tag{E.1}$$

With this vacuum, the entire Hamiltonian is diagonal since

$$\mathbb{C}_1^{\dagger}\mathbb{C}_2 |v> = |v>. \tag{E.2}$$

Such a choice of vacuum is not surprising since it corresponds to the breaking of chiral invariance inherent in a mass term.

Quantum Fluctuations

Coleman's argument[8] concerning the existence of a ground state for a given theory involves using a class of trial states to find an upper bound for the energy density of the physical vacuum. If this upper bound can be shown to be arbitrarily large and negative, the theory does not make physical sense. The set of trial states Coleman employs are essentially dilitation transforms of the Fock vacuum:

$$|\lambda\rangle = e^{i\lambda D}|0\rangle, \tag{E.3}$$

where D is given by (II.19).

Coleman phrases the argument differently since he uses a method of infrared cutoff different from Klaiber's, normal ordering at a finite mass. He defines the Hamiltonian for a theory by normal ordering the boson operators at a mass m. For instance, the Hamiltonian density corresponding to (V.26) is (following Coleman's notation[8]):

$$\mathcal{H} = N_m \left\{ \sum_{\ell} \frac{\pi_\ell^2 + (\partial_1 \phi_\ell)^2}{2} - \frac{g^2 m^2}{2\pi^2} \sum_{\ell > k} \cos[2\sqrt{\pi}(\phi_\ell - \phi_k)] \right\}. \tag{E.4}$$

The trial states are then the states which would have been the Fock vacua, if the normal ordering had been done at a different mass μ. These states can be constructed explicitly in two dimensions in terms of ϕ and π. In these states, $:\phi^2:$ acquires an expectation, although ϕ does not:

$$\langle 0,\mu | \phi | 0,\mu \rangle = 0,$$

$$\langle 0,\mu | N_m(\phi^2) | 0,\mu \rangle = \frac{1}{4\pi} \ln \frac{m^2}{\mu^2}. \tag{E.5}$$

Since we can also introduce an independent expectation for ϕ by using a coherent state operator,

$$\begin{aligned} \langle \varphi, 0 | \, \phi \, | \varphi, 0 \rangle &= \langle 0 | e^{i\int dz\, \varphi \pi(z)} \, \phi \, e^{-i\int dz\, \varphi \pi(z)} | 0 \rangle \\ &= \varphi, \end{aligned} \tag{E.6}$$

we can vary the parameters ψ and μ^2 independently to minimize the expectation of $\mathcal{H}$.

For $\mathcal{H}$ in (E.4), we find:

$$\langle 0,\mu|\mathcal{H}|0,\mu\rangle = \frac{n}{8\pi}(\mu^2-m^2) - \frac{g^2\mu^2}{2\pi^2}\frac{n(n-1)}{2}. \qquad \text{(E.7)}$$

This expectation has a minimum with respect to μ^2 only if $\frac{2g^2}{\pi}(n-1) < 1$. Only if this condition is fulfilled does the theory associated with (E.4) make sense. Similar results hold for the other theories in Chapter V.

APPENDIX F: TABLE OF THEORIES INVESTIGATED

Boson Representation	⟷ Fermion Representation	Comments
a single massless free boson $\partial^2\phi = 0$	⟷ the massless Thirring model $i\not\partial\psi = g\not{j}\psi$	1. Exactly soluble (we extend Klaiber's solution by eliminating the free spinor operator in his formalism, Chapter II). 2. In general, the Thirring field operator only produces multiparticle states which can be interpreted as composed of fermions, bosons, or both.
massless, sine-Gordon $\partial^2\phi = -\frac{m\mathfrak{m}}{2\beta\pi}\sin 2\beta\phi$	⟷ massive Thirring model $i\not\partial\psi = \sqrt{\pi}\left(\frac{\pi}{\beta}-\beta\right)\not{j}\psi + m\psi$	1. Not solved, although Dashen et. al.[10] have calculated the spectrum semi-classically (Chapter V). 2. Both "fermions" and bosons appear in semi-classical spectrum. Bosons appear as bound "fermion-antifermion" pair. The "fermion" appears classically as a soliton solution[23], whose statistics have not been checked in the semi-classical calculations. 3. $\beta = \sqrt{\pi}$ is the special case of a free fermion.

n free, massless bosons ↔ $\partial^2\phi_\ell = 0$	scale-invariant, massless Thirring generalizations with 2n-component spinor $i\not{\partial}\psi = -\pi(\not{A}^0+B^0)\not{J}\frac{\lambda^0}{\sqrt{2}}\psi - \frac{2\pi}{n+1}\not{J}^a\frac{\lambda^a}{\sqrt{2}}\psi$, etc.	1. Exactly soluble (Chapter III). 2. Spectrum as in the massless Thirring model.
n massless, generalized sine-Gordon bosons ↔ $\partial^2\phi_\ell = -\frac{g^2\mathcal{m}^2}{8\pi^{5/2}}\sum_{m\neq\ell}\sin\{2\sqrt{\pi}(\phi_\ell-\phi_m)\}$	non-scale invariant, massless U(n) Thirring generalization with canonical equal-time structure $i\not{\partial}\psi = g\not{J}^a\frac{\lambda^a}{\sqrt{2}}\psi$, etc.	1. Not solved (Chapter V). 2. Semi-classically appears to have charged fermion sectors, as well as boson states.
free, massive boson ↔ $\partial^2\phi = -\frac{e^2}{\pi}\phi$	QED with massless fermion $i\not{\partial}\psi = e\not{A}\psi$ $\partial_\mu F^{\mu\nu} = -ej^\nu$	1. Exactly solved by Lowenstein and Swieca[20] and others. 2. Only bosons in the spectrum. No fermion quantum numbers appear.
massive sine-Gordon boson ↔ $\partial^2\phi = -\frac{e^2}{\pi}\phi - \frac{m\mathcal{m}}{2\pi^{3/2}}\sin 2\sqrt{\pi}\phi$	QED with massive fermion $i\not{\partial}\psi = (e\not{A}+m)\psi$ $\partial_\mu F^{\mu\nu} = -ej^\nu$	1. Not solved (Chapter V). 2. Semi-classically appears to have only interacting bosons in the spectrum.

massive and massless free bosons $\longleftrightarrow$ QED with two massless fermions $\partial^2\sigma = -\frac{e_1^2+e_2^2}{\pi}\sigma$ $\quad$ $i\not{\partial}\psi_1 = e_1\not{A}\psi_1$ $\partial^2\phi = 0$ $\quad$ $i\not{\partial}\psi_2 = e_2\not{A}\psi_2$ $\partial_\mu F^{\mu\nu} = -e_1 J_1^\nu - e_2 J_2^\nu$	1. Exactly soluble (Chapter IV). 2. Only electrical neutral states appear. However, Thirring quantum numbers of the original fermions appear in the spectrum. Whether fermions or only bosons are present depends on the values of e_1 and e_2. In general the states associated with ψ_1 and ψ_2 are multiparticle only.
massive, coupled sine-Gordon bosons $\partial^2\phi_1 = -\frac{e_1^2}{\pi}\phi_1 - \frac{e_1e_2}{\pi}\phi_2 - \frac{m_1\mathcal{M}}{2\pi^{3/2}}\sin 2\sqrt{\pi}\phi_1$ $\partial^2\phi_2 = -\frac{e_2^2}{\pi}\phi_2 - \frac{e_1e_2}{\pi}\phi_1 - \frac{m_2\mathcal{M}}{2\pi^{3/2}}\sin 2\sqrt{\pi}\phi_2$ $\longleftrightarrow$ QED with two massive fermions $i\not{\partial}\psi_1 = (e_1\not{A}+m)\psi_1$ $i\not{\partial}\psi_2 = (e_2\not{A}+m)\psi_2$ $\partial_\mu F^{\mu\nu} = -e_1 J_1^\nu - e_2 J_2^\nu$	1. Not solved (Chapter V). 2. Semi-classically neither electrical nor Thirring charge appears in general. Confinement is total.

finite poly-nomial boson interaction $\longleftrightarrow$	non-local fermion interaction	1. Semi-classically investigated by Goldstone and Jackiw[14]. 2. A single "fermion" charge sector exists, but multiply-charged sectors do not.
$\partial^2\phi = \mu^2\phi - \frac{\lambda\phi^3}{3!}$	(Chapter V) (!)	

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