A RATIONAL DESIGN PROCEDURE FOR MACHINE FOUNDATIONS

Thesis by

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To my wife

Alice

in appreciation for her

understanding, love, and patience.

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ABSTRACT

The problems involved in the design of machine foundations are discussed, followed by a short review of the literature of the subject. The general theory of vibration for single and multiple degree of freedom systems is briefly reviewed, with special emphasis on its application to machine foundation design. A procedure for the analysis of machine foundations is then developed on the basis of a simplified equivalent system. Procedures for determining the elastic coefficients and the inertia parameters of the soil are next considered.

The purpose and extent of the experimental investigations are discussed, followed by a description of the instrumentation used and the nature and accuracy of the data obtained. The data is then analyzed and checked against the theory presented. The data required for the design of machine foundations is discussed, and a procedure for design and analysis is recommended. In conclusion recommendations for further study and research are made.

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INTRODUCTION

The need for a satisfactory rational method of designing machine foundations has existed for a long time. Although the cost of the foundation usually represents only a small fraction of the total cost of an installation, the behavior of the foundation is a major consideration in insuring satisfactory performance of the machine. All too often has the difficulty and importance of designing a suitable machine foundation been underestimated.

That knowledge in this field has lagged behind other branches of technology is partly due to the fact that the responsibility for a satisfactory installation is divided between two branches of engineering. The machine designer's responsibility generally does not extend beyond the design of the machine; the foundation engineer must design a foundation for it -- a problem much more complex than the design of a foundation which supports only static loads. This is because additional dynamic forces are involved which alter the behavior of both the foundation and the soil.

In order to arrive at a rational design procedure for machine foundations, the author's research was directed at an investigation of the behavior of foundations subject to periodic dynamic forces. The problem was resolved into three phases:

1. A review of the literature of the subject.
2. A study of the problem as a problem in the application of the theory of vibrations.
3. An experimental investigation of foundations subject to periodic dynamic forces.

The review of the literature revealed that many of the usual design practices are contradictory and that much of the available data on foundation behavior is incomplete and vague. The experimental investigation was therefore designed to furnish the necessary information for checking theoretical behavior. It was established that reasonable predictions can be made on the basis of the theory of vibrations, provided that the effect of the elastic and inertia properties of the soil are incorporated. A method of computing the necessary coefficients was developed and the procedure for analyzing a foundation outlined.

To permit mathematical treatment, several simplifying assumptions were made in developing the theory. Exact numerical results are therefore not to be expected; nevertheless, results obtained by its use are in good agreement with experimental observations and should be sufficiently accurate for most design problems arising in practice. The theory also explains why many of the empirical design practices which have been established by experience constitute sound engineering practice and are desirable for dependable and economic machine foundations.

CHAPTER I

REVIEW OF THE LITERATURE ON MACHINE FOUNDATION DESIGN

Introduction

The first attempts at devising design procedures were almost entirely empirical. The next attempt was a rationalization of the problem by an application of the theory of vibrations in which the spring constants required were determined from static deflection. About 1933 German engineers developed several vibration machines and used them to determine the dynamic soil constants. This was followed by experimentation in other countries, principally in Great Britain, and led to the theory of ground "self-frequency". The latter theory was further expanded into the so-called "pressure bulb" theory.

Empirical Design Methods

Manufacturers of industrial machines generally furnish "certified" foundation drawings which merely show the location of holes for anchor bolts and give the overall dimensions required to accomodate the machine. Recommended values of minimum yardage of concrete in the foundation are sometimes given. The remainder of the design is left to the draftsman. The yardage of the foundation is generally based on the assumption that the foundation will rest on hard firm subsoil, and is justified on the basis of the theory of mass damping.(1) If P is the unbalanced inertia force, A is the allowable amplitude of displacement, and M is the mass of the foundation, then

$$P = M\,a = M\,\omega^2 A$$

where a is the acceleration, and ω is the operating frequency of the machine in radians per second. Solving for M, we have

$$M = \frac{P}{\omega^2 A}$$

An arbitrary constant is sometimes introduced to allow for the apparent mass of the soil which moves with the foundation block.[2] This theory ignores the effect of damping, the action of the soil as a spring, and the phase difference between the force P and the displacement A.

Some of the rules of thumb based on this theory which have found favor are:

a. Weight of foundation block should be equal to not less than ten times the weight of the moving parts of the machine.

b. Weight of foundation block should be equal to some multiple of the total weight of the machine.*

c. Weight of foundation block should be equal to some constant times the rated horsepower output divided by the number of cylinders.**

For the guidance of the draftsman the following rules and specifications have been laid down by various authors:

1. The base of the machine foundation should not be higher than the base of adjoining foundations.

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* Cozens tabulated recommended ratios of foundation weight to engine weight for several types of engines.[3] The recommended ratio for steam engines ranges from 4:1 to 3:1 ; for gas engines, from 3:1 for single cylinder to 2:1 for eight cylinder; for Diesel engines, from 2.75:1 for 2 cylinder to 1.9:1 for 8 cylinders.

** Larkin[1] tabulated the yardage per horsepower recommended by various manufacturers for 85 different gas engines. The engines included were 3 to 8 cylinder engines with rated output ranging from 75 to 360 HP. All the engines were of medium speed, ranging from 200 to 400 rpm. The average yardage recommended per horsepower may be expressed by the relation

$$V = 0.06\ (1 + 4/n) \quad Yd.^3/HP$$

where n is the number of cylinders.

2. For reciprocating engines the depth of the foundation should be at least five times the piston stroke, with a minimum of five feet.

3. The width of the base should not be less than the height; if necessary, a cantilevered footing slab may be used.

4. Cork isolation pads are recommended for high speed engines, or if noise abatement is required.

5. To prevent cracking, faces of the foundation should be reinforced with 5/8 in. deformed bars, spaced 12 in. on center, both horizontally and vertically. Low shrinkage and low water-cement ratio concrete should be used.

6. Whenever possible symmetrical arrangement should be used, and the center of gravity of the base contact area should coincide with the center of gravity of the machine or of the action line of the resultant dynamic forces.

7. Where soil conditions are unsatisfactory, piles should be used to consolidate the soil and to transfer the load to a stronger stratum.

8. Conservative allowable soil bearing values should be used in determining the base contact area.

9. Pockets or other suitable spaces may be left so that additional mass may be added should the natural frequency of the foundation coincide with the operating frequency of the machine.(4)

10. Cantilevered projections should be eliminated wherever possible, or when unavoidable, should be stiffened with brackets.(5)

11. Where foundations project thru a floor they should be separated to prevent "moment" continuity.(5)

12. Preloading of the foundation is desirable to prevent misalignment due to settlement.(6)

Semi-rational Design Methods

During the early thirties, German engineers began advocating a more rational analysis of machine foundations based on the theory of vibrations. Much work was done on the problem of isolation by Rausch, Von Schlippe, Steinbach, Geiger and others. Their most notable contribution was the development of spring supported foundations, which satisfactorily solved the problem for medium to high frequency machines. By the use of springs the problem was made determinate, since for low natural frequencies of the suspended system, the dynamic forces transmitted to the ground can be kept very small.

Rausch[7] classified machine foundations into three groups on the basis of operating frequency.

A. Low to medium frequencies	0 - 500 rpm
B. Medium to high frequencies	300 - 1000 rpm
C. High frequencies	Greater than 1000 rpm

Group A consists of large reciprocating engines, compressors, blowers etc. Reciprocating engines operate at frequencies from 50 to 250 rpm but have considerable second harmonic content, so that sizable dynamic forces up to 500 rpm must be withstood. Foundations of the block type with large soil contact surface are recommended for this group. The natural frequency of these foundations is generally higher than the operating frequency of the machine. An exception to this case is a foundation on piles which may have a very low frequency for the horizontal mode.

The second group consists of foundations for medium size reciprocating engines such as diesel and gas engines, as well as blowers and other rotating machinery. In this group, the natural frequency of a foundation which rests directly on the ground is apt to coincide with the operating frequency. A spring supported foundation is therefore recommended in order to make the natural frequency of the foundation block much lower than the operating frequency. This type of foundation is readily designed by application of the theory of vibration and will therefore not be discussed in further detail in this thesis.

The third group consists principally of high speed internal combustion engines, electric motors and steam turbines. In these installations the operating frequency of the machine is generally well above the natural frequency of the foundation. Massive foundations with small contact area are recommended and cork isolation pads may be used to increase the damping. These pads also tend to reduce the natural frequency of the foundation by reducing the effective spring constant. Framed foundations, such as are required for turbines, are treated separately. They are generally much more complex since the columns supporting the foundation table act as springs, thereby introducing additional degrees of freedom to the system.

For cast-in-place block foundations, Rausch determines the natural modes of oscillation from static soil constants. Damping is neglected both in determining resonant frequency and amplitude of oscillation. Allowable values for amplitude are given. Formulae for determining size and type of springs required for type B foundations are given. This subject is also treated by De Gruben.[8]

In general these authors ignore the mass effect of the soil which vibrates with the foundation. Static measurements or calculations are depended upon for the determination of the soil spring constants. To account for the dynamic augment, soil pressures are arbitrarily increased by a factor of five.

Determination of Dynamic Soil Constants.

About the time of the above development much interest was shown in the design and theory of vibration machines.[9] Several machines were constructed and were used by the Degebo (Deutsche Gesellschaft fur Bodenmechanik) in a series of extensive tests on various types of soil.[10] The classic work in this field was that of Lorenz.[11] He proposed that the natural frequency of a vibrator resting on soil be expressed by the relation

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k'Ag}{W_s + W_v}}$$

where f_n is the frequency in cycles per second, k' the spring constant, A the surface area, g the acceleration of gravity, W_s the effective weight of the soil moving with the vibrator, and W_v the weight of the vibrator. The unit spring constant k' is generally called the "coefficient of dynamic subgrade reaction".[12] It has been found that this coefficient is consistently larger than the coefficient of subgrade reaction determined from static tests. Lorenz in his analysis assumed that k' was constant for a given soil, and hence was able to determine the apparent mass of soil, W_s. He concluded that W_s is not constant, but varies with frequency, contact pressure and dynamic force. An independent set of experiments were made by Barkan[13] on cohesive soils. Barkan assumed that the apparent mass, W_s, is constant, and therefore concluded that k' varies with intensity of contact pressures, size of contact area and with the frequency of load application. Attempts have been made by other investigators to correlate these different approaches. Lorenz' equation is based on the tacit assumption that the weight, W_s, of the vibrating soil constitutes part of the weight of the rigid vibrator, and that the seat of the forces of elastic restitution has no weight. Actually, the boundary of the zone which vibrates under the influence of the impulse is not sharply defined and depends on the physical properties of the subgrade. Several attempts have been made to obtain a more accurate conception of the interaction which occurs between the vibrator and the soil. The assumption made in Reisner's analysis is that the vibrator rests on the horizontal surface of a semi-infinite elastic isotropic mass.[14] This problem has also been treated by Quinlan.[15] The results of these analyses have not been too useful when applied to cohesionless soils, since for these soils the modulus of elasticity varies with depth. Tchebotarioff[16] has proposed a method of normalizing the data on the basis of calculating the natural frequency for unit contact pressure, thus:

$$f_n = \sqrt{\frac{A}{W_v}} \times \frac{1}{2\pi}\sqrt{\frac{k'g}{1 + W_s/W_v}}$$

$$= f_{nr}/\sqrt{p}$$

where

$$p = \frac{W_v}{A}$$

and f_{nr} is the reduced natural frequency. On this basis he finds that there appears to be some correlation of all published data to date, which may be expressed approximately by the relation

$$f_{nr} = 1500/\sqrt[4]{A}$$

Ground Self-frequencies.

As early as 1885, it was noted by Rayleigh, that under certain conditions the ground appears to have a natural or "self-frequency", which depends on the physical properties of the soil. This phenomenon was also studied by Lamb in 1904, and by the Degebo in 1934-36, and by Sezawa and Kanai in 1937. In 1946 Bergstrom and Linderholm in Sweden showed that a correlation exists between the self-frequency and the bearing capacity of the ground, and proposed making use of this phenomenon as a practical method of determining bearing values for ordinary building sites. Andrews and Crockett(17) in 1944 made a number of vibrograph studies in which the self-frequency was excited by dropping heavy weights on the ground; damping factors were determined by measuring the decay of the oscillations. The frequencies obtained in this manner appear to be somewhat lower than those obtained by the Degebo with continuous excitation. This is shown graphically in Fig. 1.1. On the basis of these self-frequencies and in an attempt to account for the mass of soil moving with the foundation, Crockett and Hammond have advanced the "pressure bulb" theory.(18)

The Pressure Bulb Theory.

Most investigators have realized that a portion of the soil moves with the foundation and effectively behaves as an additional or apparent mass. Lorenz(11) attempted to measure this mass but had to assume that the dynamic subgrade reaction remained constant. In 1948, Crockett and Hammond(17) proposed that the pressure bulb be used as a measure of the apparent mass. If the soil is assumed to behave elastically, the pressure at any point may be determined from the Boussinesq equations. If points of equal pressure are then considered, they will be found to lie on a bulb-shaped surface. Crockett and Hammond assume that the mass of the soil within this envelope or bulb can be taken as the apparent mass. The size of the bulb, of course, depends on the pressure intensity selected. Presumably this value may be determined experimentally.

Summary.

Progress in the field of machine foundation design has been very slow. Only in the last two decades has a rational approach to the problem been

GROUND "SELF FREQUENCY"

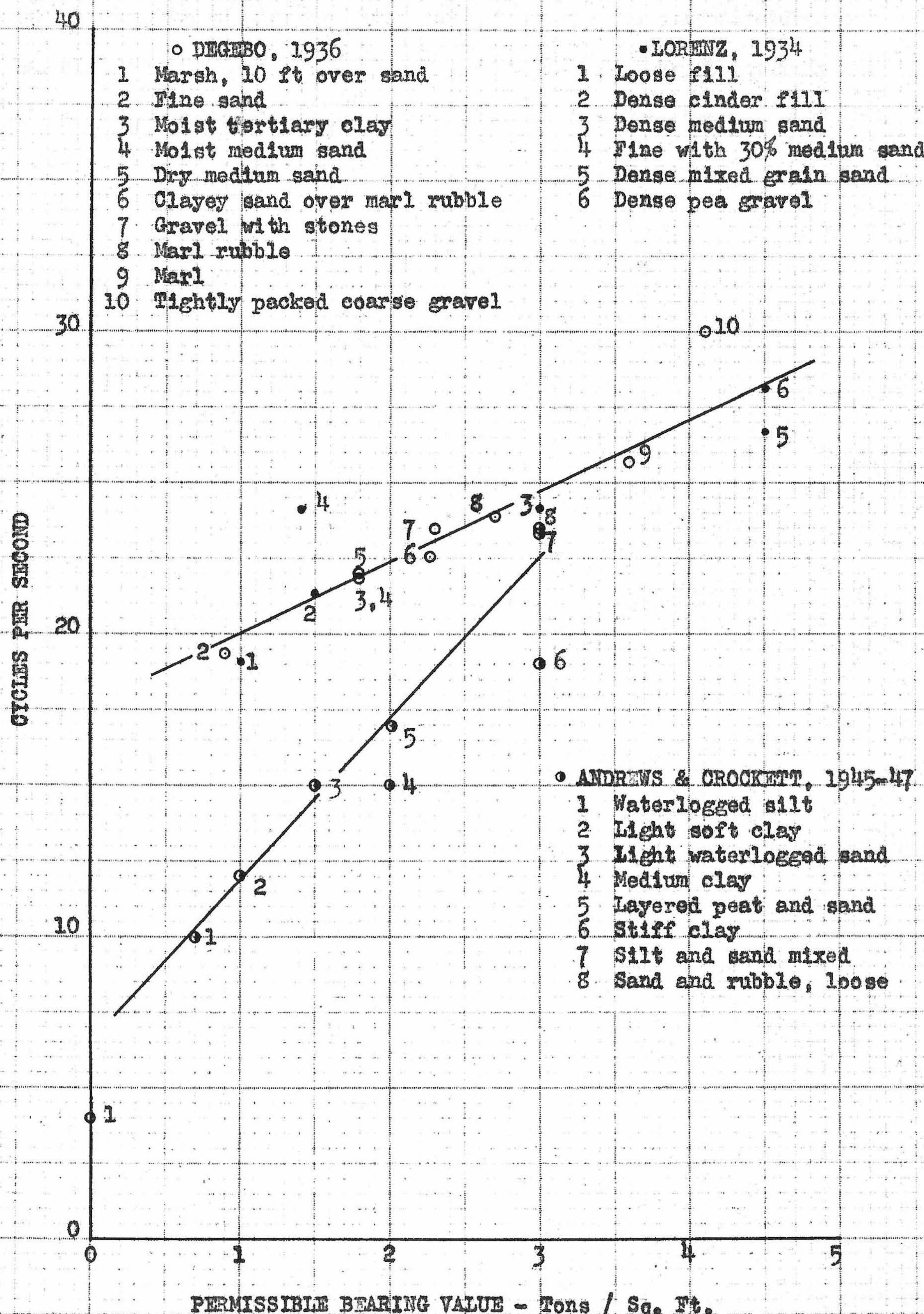


Figure 1.1

attempted. The methods of analysis advanced to date have either over-simplified the problem or have been limited in their application to special cases. As a result, engineers still rely almost entirely on judgement and experience in designing machine foundations.

CHAPTER II

THEORY OF VIBRATION OF ONE DEGREE OF FREEDOM SYSTEMS AND ITS APPLICATION TO MACHINE FOUNDATION DESIGN

Mechanical oscillation is caused by the interaction of inertia and restoring forces when a body is disturbed from its position of static equilibrium. The requirements for oscillation are the presence of a restoring force and the kinetic energy of the mass. In the case of machine foundations, the latter is furnished by the mass of the machine and its foundation, plus a portion of the soil which moves with it; the restoring force is furnished by the elasticity of the soil. The mass of the system may be expressed by m = W/g and the exciting force by F(t). In general, restoring forces for small displacements may be considered linear, for if we express the restoring force as

$$f(x) = k_1 x + k_2 x^2 + k_3 x^3 + \dots \tag{2.1}$$

then for sufficiently small values of x we can write

$$f(x) \approx k x \tag{2.2}$$

Since (in the design of machine foundations) we are interested in keeping the displacements very small, the above simplification is justified. In (2.2) k is called the spring constant.

Unless external energy is supplied, a disturbed system will gradually come to rest because of energy losses. The sources of these energy losses are dissipation and damping. In the case at hand, dissipation is due to radiation of energy into the soil mass on which the foundation rests, and damping is furnished by the friction between soil particles. The combined effect is most conveniently expressed as a damping force, $\beta\dot{x}$, where it is assumed that the damping force is viscous, i.e., proportional to the velocity. If the damping force does not follow the viscous law the problem is non-linear and serious complications in the mathematical treatment of the problem ensue. It is therefore customary to treat the damping force on the basis of an equivalent viscous damping factor, β_e, which causes the same energy loss in the system. It is one of the purposes of this thesis to determine the seriousness of this simplification in predicting the behavior of machine foundations.

The Equivalent System.

If the elastic restraints of a system are of such a nature that the system can only vibrate parallel to, or in a plane about, a fixed axis, it is said to have one degree of freedom. Otherwise, the degree of freedom is equal to the number of coordinates required to define the displacement. In the most general case, the movement of a rigid system such as a block foundation can be resolved into three translatory and three rotational components, and such a system therefore can have, at most, six degrees of freedom.

Consider the simplest case, a system with one degree of freedom. The machine and its foundation plus the equivalent effective mass of the soil moving with it is considered as a mass resting on a damped spring.

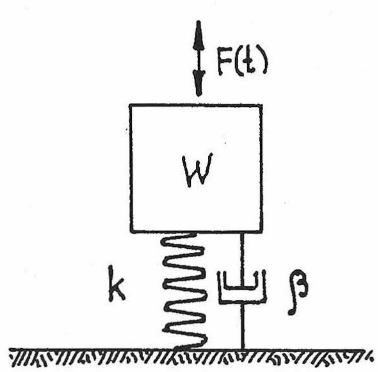


Figure 2.1

The differential equation for this system is

$$m\ddot{x} = -kx - \beta\dot{x} + F(t) \quad . \tag{2.3}$$

<u>Undamped Free Oscillation</u>.

For the case of zero damping and no exciting force, equation (2.3) reduces to

$$m\ddot{x} + kx = 0 \quad . \tag{2.4}$$

It can readily be verified that a solution of equation (2.4) is

$$x = A \cos\sqrt{\frac{k}{m}}\, t \quad , \tag{2.5}$$

where A is the amplitude of oscillation, and the period in seconds is given by

$$\tau = \frac{2\pi}{\sqrt{\frac{k}{m}}} \quad . \tag{2.6}$$

The frequency of oscillation, in cycles per second, is

$$f = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{k}{W}} = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{1}{\delta_{st}}} \quad , \tag{2.7}$$

since $m = W/g$ and the static deflection is given by

$$\delta_{st} = \frac{W}{k} \quad . \tag{2.8}$$

The simplicity of these relationships is appealing and has led many investigators[7] into trying to determine the effective spring constant of a foundation by determining the static deflection of the foundation, either by direct measurement or by computation. The difficulty in this approach to the problem is that these deformations are normally very small and hence cannot be determined very accurately. Furthermore, the deformation rate is dependent on the load intensity since soils are not elastic materials. The equivalent static deformation, therefore, is not a constant, but is a function of the intensity of load and the rate of load application.

Damped Oscillation.

If damping is considered, equation (2.3) may be rewritten as

$$m\ddot{x} + \beta\dot{x} + kx = 0 \tag{2.9}$$

The general solution of equation (2.9) is of the form

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{2.10}$$

where

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m} \tag{2.11}$$

The condition for oscillatory motion is that $\beta^2 < 4km$. When $\beta^2 = 4km$, the system is said to be critically damped. Critical damping is therefore defined as

$$\beta_c = 2\sqrt{km} \tag{2.12}$$

For small damping, $\beta < \beta_c$, equation (2.10) may be written

$$x = e^{-\frac{\beta}{2m}t}\left(C_1' \sin\sqrt{\frac{k}{m} - \left(\frac{\beta}{2m}\right)^2}\,t + C_2' \cos\sqrt{\frac{k}{m} - \left(\frac{\beta}{2m}\right)^2}\,t\right) \tag{2.13}$$

For the case $C_1' = 0$ and $\beta = 0$ equation (2.13) reduces to equation (2.5). Inspection of equation (2.13) reveals that damping decreases both the amplitude and the frequency of the vibration. The period for this case is given by the relation

$$\tau = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{\beta}{2m}\right)^2}} \tag{2.14}$$

Damping is conveniently expressed by the ratio:

$$c = \frac{\beta}{\beta_c} \tag{2.15}$$

For most soil conditions "c" has been found to range from 0.05 to 0.15. The effect of damping on the frequency of the vibrating system is of interest. Rewriting equation (2.14), we have

$$\tau = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{\beta^2}{\beta_c^2}\frac{k}{m}}} = \frac{2\pi}{\sqrt{\frac{k}{m}}\sqrt{1-c^2}} \qquad (2.16)$$

Expanding by the binomial theorem,

$$\tau = \frac{2\pi}{\sqrt{\frac{k}{m}}}\left(1 + \frac{1}{2}c^2 + \frac{3}{8}c^4 + \ldots\right) \qquad (2.17)$$

$$\simeq \frac{2\pi}{\sqrt{\frac{k}{m}}}\left(1 + \frac{c^2}{2}\right) \quad , \quad \text{for } c \ll 1$$

It is seen therefore that for $c = 0.10$ the error introduced in the computation of the resonant frequency by neglecting damping is only about one half of one per cent. For practically all design problems this degree of accuracy is more than sufficient.

Damping, in a freely vibrating system, may be measured by the ratio between successive peaks. Thus, if

$$x_1 = e^{-\frac{\beta}{2m}t_1}$$

and

$$x_2 = e^{-\frac{\beta}{2m}(t_1+\tau)}$$

then

$$\frac{x_1}{x_2} = \left(e^{-\frac{\beta}{2m}t_1}\right)\left(e^{+\frac{\beta}{2m}(t_1+\tau)}\right) = e^{\frac{\beta}{2m}\tau}$$

The logarithm of this ratio is generally called the logarithmic decrement δ; therefore

$$\delta = \log\frac{x_1}{x_2} = \frac{\beta}{2m}\tau \qquad (2.18)$$

For small damping ratios where $c = \frac{\beta}{\beta_c} \ll 1$

$$\tau = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

and therefore,

$$\delta = 2\pi \frac{\beta}{\beta_c} = 2\pi c \tag{2.19}$$

Forced Oscillation.

We will next consider the motion of a system with one degree of freedom under the action of a periodic external force. The equation of motion (2.13) can be written as

$$m\ddot{x} + \beta\dot{x} + kx = F_o \sin \omega t \tag{2.20}$$

In machine foundations we are only interested in the steady state oscillation which is given by the particular solution of the above equation. Thus,

$$x = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}} \sin(\omega t - \phi) \tag{2.21a}$$

where

$$\phi = \tan^{-1} \frac{\beta\omega}{k - m\omega^2} \tag{2.21b}$$

If we define the frequency of the undamped free oscillation of the system by

$$\omega_n = \sqrt{\frac{k}{m}}, \qquad \text{radians per second} \tag{2.22}$$

and recall that the critical damping is given by

$$\beta_c = 2\sqrt{km}$$

equation (2.21) may be rewritten in the form

$$x = \frac{\frac{F_o}{k}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2c\frac{\omega}{\omega_n}\right)^2}} \sin(\omega t - \phi) \tag{2.23a}$$

where

$$\phi = \tan^{-1} \frac{2c\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{2.23b}$$

The maximum amplitude, A, occurs when $\sin(\omega t - \phi) = 1$

Defining the static deflection for the exciting force F_o, by

$$\delta'_{st} = \frac{F_o}{k} \tag{2.24}$$

we obtain from equation (2.23)

$$\frac{A}{\delta'_{st}} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2c\frac{\omega}{\omega_n}\right)^2}} \tag{2.25}$$

This ratio is called the dynamic amplification factor. A plot of this factor is shown in Fig. 2.2 for several damping ratios. These curves may be normalized by dividing the amplitude by the amplitude for $\frac{\omega}{\omega_n} = 1$. The resulting curves (Fig. 2.3) are useful in determining the damping factor of a system by comparing them with a normalized curve of the measured displacements. Inspection of equation (2.25) reveals that the most important factor determining the amplitude is the frequency ratio, ω/ω_n. If equation (2.25) is rewritten in the form

$$A = \frac{F_o}{k\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2c\frac{\omega}{\omega_n}\right)^2}} \tag{2.26}$$

it can be seen that when the frequency of the external force is in synchronism with the frequency of the free undamped system, (i.e. $\omega = \omega_n$,)

$$A = \frac{F_o}{k\left(\frac{2\beta}{2\sqrt{km}}\right)\left(\frac{\omega}{\sqrt{k/m}}\right)} = \frac{F_o}{\beta\omega} \tag{2.27}$$

For low frequency ratios, ($\omega/\omega_n \simeq 0$,)

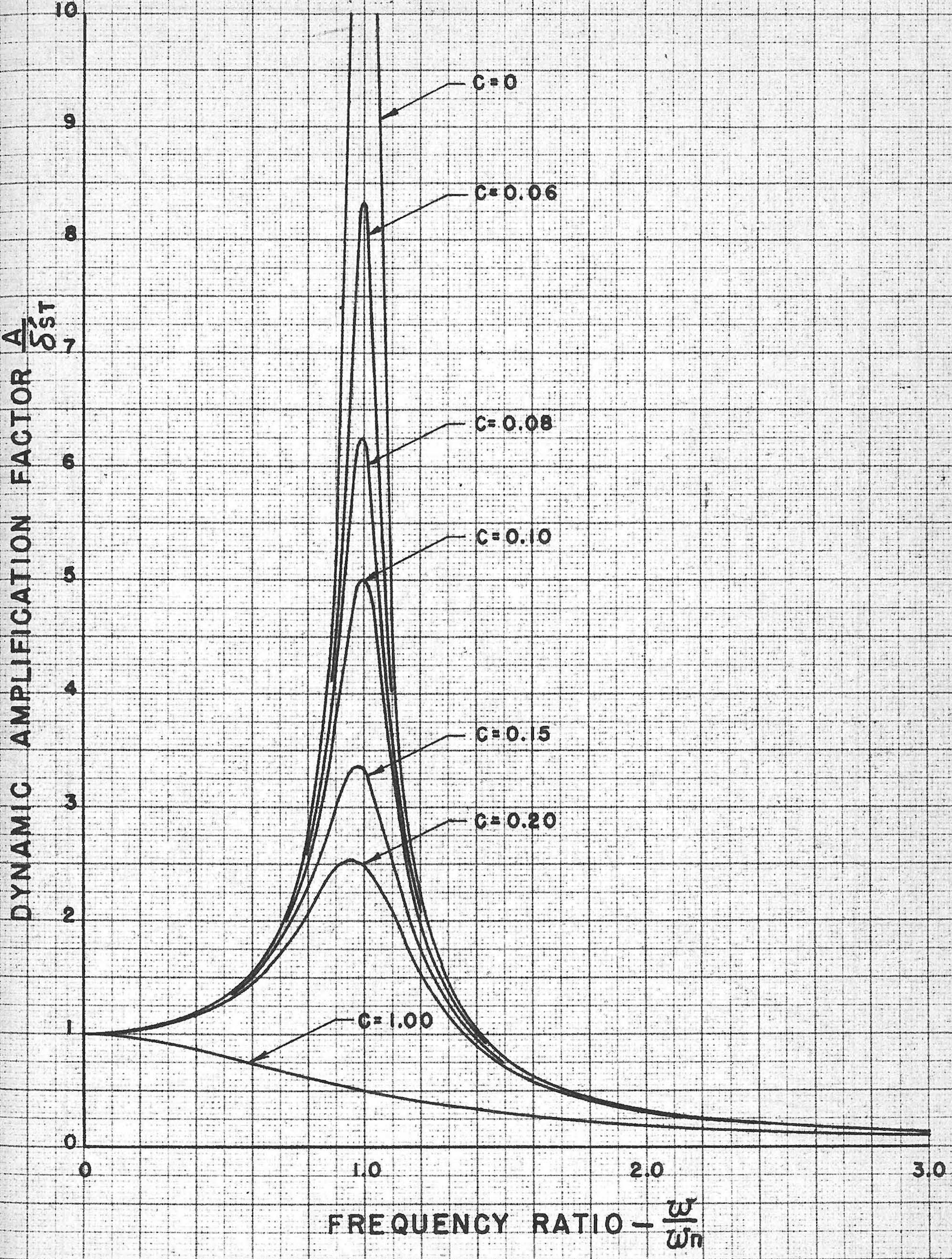
$$A \approx \frac{F_o}{k} \tag{2.28}$$

For high frequency ratios and small damping, ($\omega/\omega_n \gg 1$, $c \ll 1$,)

$$\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 \simeq \left(\frac{\omega}{\omega_n}\right)^4 \quad \text{and} \quad c^2 \simeq 0$$

hence

$$A \approx \frac{F_o}{k\left(\frac{\omega}{\omega_n}\right)^2} = \frac{F_o}{m\omega^2} \tag{2.29}$$



DYNAMIC AMPLIFICATION FACTOR $\frac{A}{\delta_{ST}}$
FREQUENCY RATIO — $\frac{\omega}{\omega_n}$
C=0
C=0.06
C=0.08
C=0.10
C=0.15
C=0.20
C=1.00
10
9
8
7
6
5
4
3
2
1
0
0
1.0
2.0
3.0

Figure 2.2

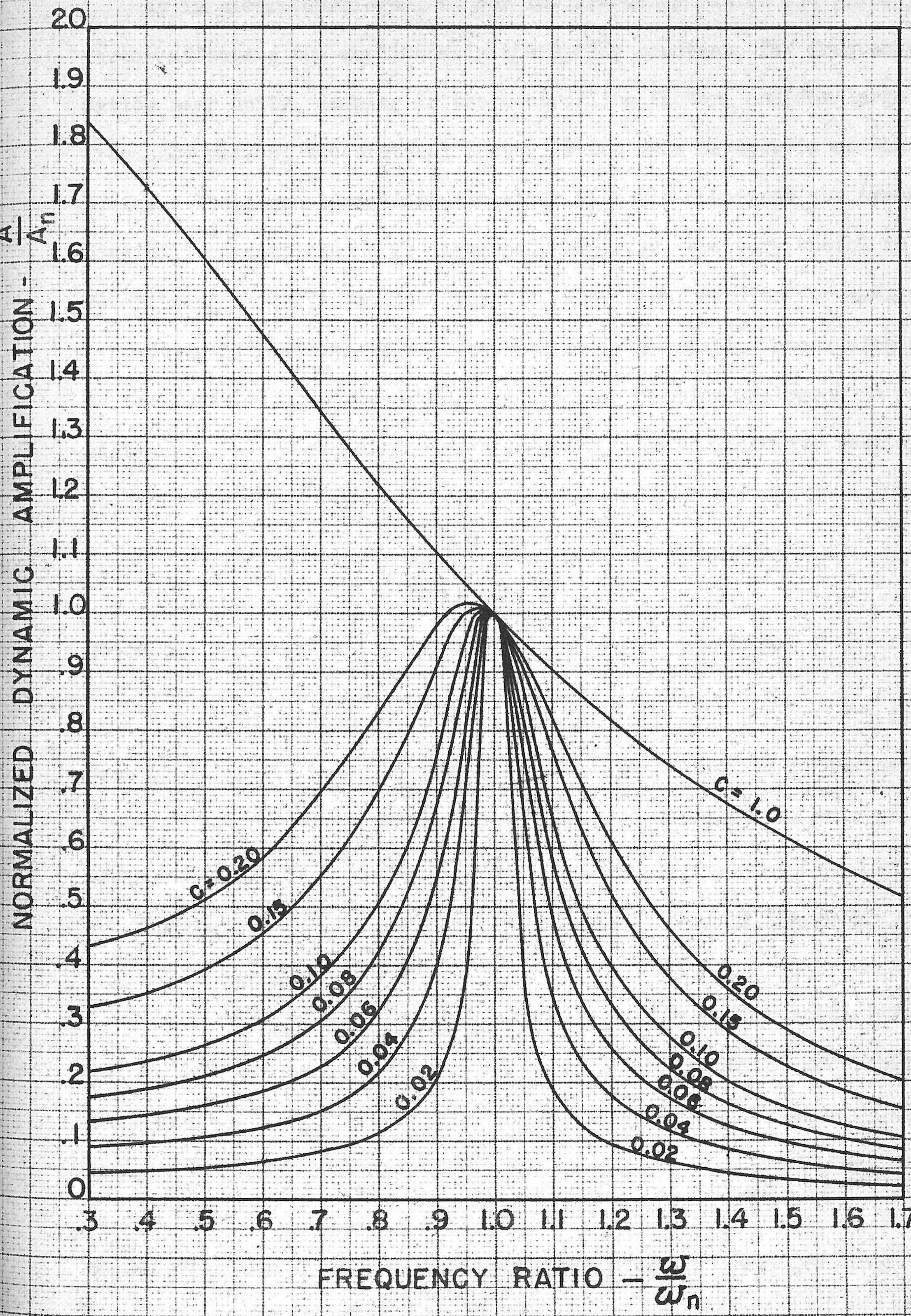

NORMALIZED DYNAMIC AMPLIFICATION - $\frac{A}{A_n}$
FREQUENCY RATIO - $\frac{\omega}{\omega_n}$
C = 1.0
C = 0.20
0.15
0.10
0.08
0.06
0.04
0.02
0.20
0.15
0.10
0.08
0.06
0.04
0.02
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7

Figure 2.3

It is clear therefore that for low frequency ratios the principal factor affecting the amplitude is the spring constant; for frequency ratios near unity, damping is the controlling factor; and for large frequency ratios, the critical factor is the mass or inertia of the system. These considerations, together with the power and the transmissibility relations to be developed presently, are very useful in determining criteria for the design of economical machine foundations.

It is interesting to note that these relationships are analogous to the familiar electric network equations; β being analogous to resistance, $\frac{k}{\omega}$ to capacitive reactance and ωm to inductive reactance. This becomes quite clear if we rewrite (2.26) in the form

$$A = \frac{F_0/\omega}{\sqrt{\left(\frac{k}{\omega} - \omega m\right)^2 + \beta^2}} \tag{2.30}$$

where the denominator represents the mechanical impedance of the system

$$Z = \sqrt{\left(\frac{k}{\omega} - \omega m\right)^2 + \beta^2} \tag{2.31}$$

From (2.31) it is immediately evident that the impedance is dominated by k for low frequency ratios, by β for frequency ratios near unity, and by m for large frequency ratios.

Forced Oscillation; Exciting Force Proportional to the Square of the Frequency.

In the design of machine foundations, the periodic exciting force which must be considered is normally due to an unbalance, and the force will therefore be proportional to the square of the frequency. Thus

$$F(t) = m'e\,\omega^2 \sin \omega t = F_0'\,\omega^2 \sin \omega t \tag{2.32}$$

where m' is the unbalanced mass and e its eccentricity. Defining

$$r = m'e/m = W'e/W \tag{2.33}$$

equation (2.25) takes the form

$$\frac{A}{r} = \frac{(\omega/\omega_n)^2}{\sqrt{(1 - \omega/\omega_n)^2 + (2c\,\frac{\omega}{\omega_n})^2}} \tag{2.34}$$

This ratio is called the dynamic amplification ratio(4) and has been plotted for several damping ratios in Fig. 2.4.

It is of interest that for large frequency ratios the right hand side of equation (2.34) approaches unity. We may therefore write

$$W = \frac{W'e}{A}\,C \tag{2.35}$$

where C approaches unity for large frequency ratios if the damping is small. For example, when

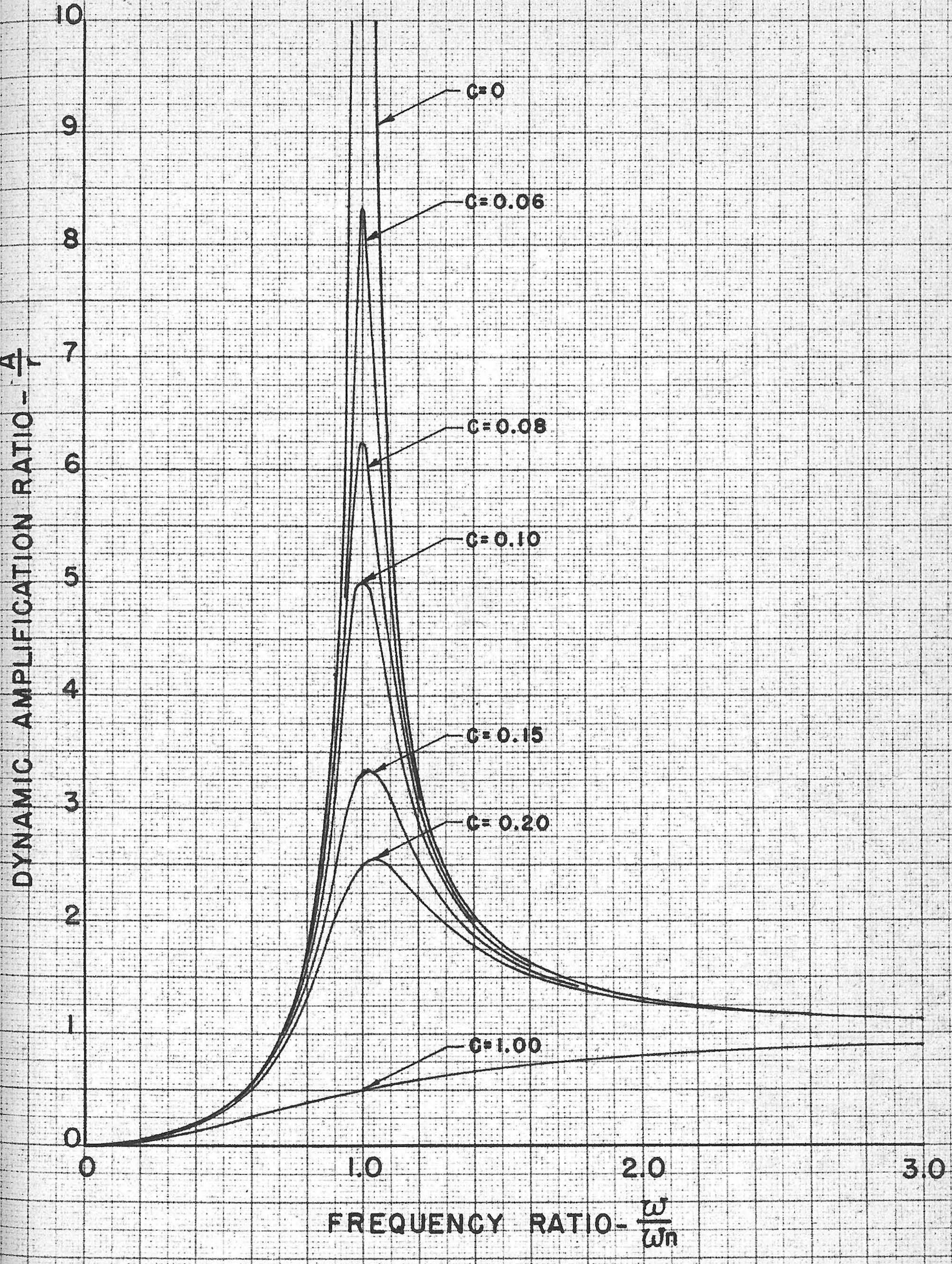
$$\frac{\omega}{\omega_n} = 2 \quad \text{and} \quad c = 0, \qquad C = 1.33$$

and when

$$\frac{\omega}{\omega_n} = 2 \quad \text{and} \quad c = 0.10, \qquad C = 1.29$$

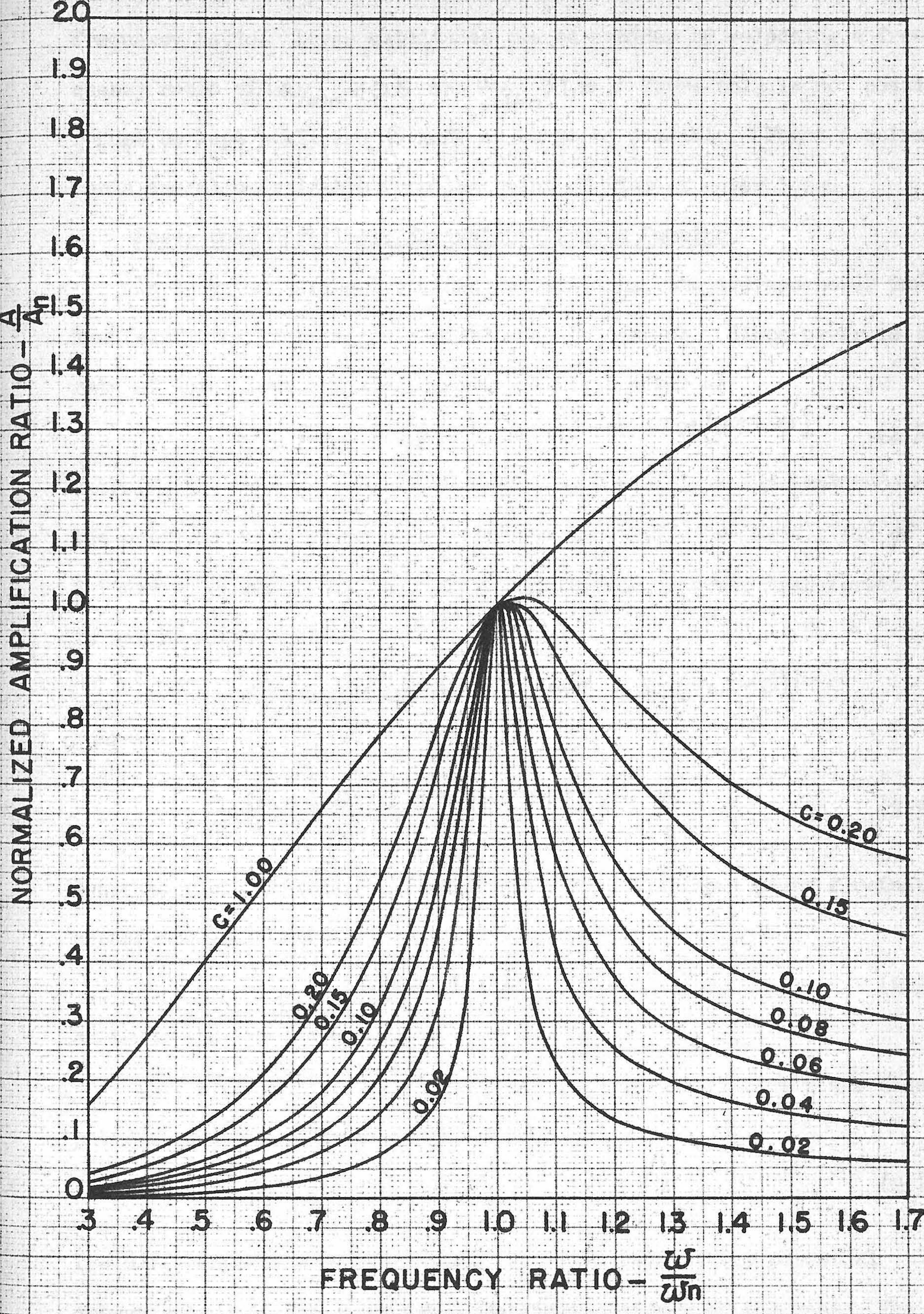
Since c can be estimated closely from the frequency response curve, equation (2.35) furnishes a useful relation for experimentally determining the effective mass of a system(9). The effective spring factor can then be determined from equation (2.22). Some caution, however, should be exercised in applying this procedure to determining the apparent mass of soil moving with a foundation. There is considerable evidence that the apparent mass of the soil depends to a large extent on the amplitude and the phase relationships of the vibration and is therefore not constant.

The dynamic amplification <u>ratio</u> (equation (2.34), Fig. 2.4) may be normalized in the same manner as the dynamic amplification <u>factor</u>. The normalized curves, Fig. 2.5, may be used to determine damping factors when the exciting force varies as the square of the frequency.



10
9
8
7
6
5
4
3
2
1
0
DYNAMIC AMPLIFICATION RATIO - $\frac{A}{r}$
C=0
C=0.06
C=0.08
C=0.10
C=0.15
C=0.20
C=1.00
0
1.0
2.0
3.0
FREQUENCY RATIO - $\frac{\omega}{\omega_n}$

Figure 2.4



NORMALIZED AMPLIFICATION RATIO – $\frac{A}{A_n}$
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
C=1.00
0.20
0.15
0.10
0.02
C=0.20
0.15
0.10
0.08
0.06
0.04
0.02
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
FREQUENCY RATIO – $\frac{\omega}{\omega_n}$

Figure 2.5

Inspection of Fig. 2.2 and Fig. 2.4 reveals the importance of the frequency ratio. Large amplitudes can be avoided by selecting a frequency ratio of $\omega/\omega_n < 0.5$ or $\omega/\omega_n > 1.5$. Where this is not possible r must be kept small and β made as large as possible. These requirements are also desirable from the point of view of power loss.

Power Considerations: Equivalent Viscous Damping.

In a system having sustained oscillations, the average power input must equal the average power dissipated in damping. Since power is the rate of doing work, we can express the input power as

$$P_i = Fv = F\frac{dx}{dt} = F_o \sin\omega t \cdot A\omega\cos(\omega t - \phi) \tag{2.36}$$

where F_o is the magnitude of the exciting force, A is the amplitude of the oscillation as given by either equation (2.25) or (2.34), and ϕ is defined by equation (2.23b). By a simple trigonometric transformation it can be shown that:

$$\sin\omega t \cos(\omega t - \phi) = \frac{1}{2}\left(\sin\phi + \sin(2\omega t - \phi)\right)$$

Therefore

$$P_i = \frac{F_o A\omega}{2}\left[\sin\phi + \sin(2\omega t - \phi)\right] \tag{2.37}$$

Inspection of the above equations shows that the input power fluctuates around the mean level

$$P_{i_{ave}} = \frac{F_o A\omega}{2}\sin\phi \tag{2.38}$$

at twice the input frequency.

This average input power must be equal to the average power dissipated by damping. The power dissipated is given by

$$P_d = F_d\, v = \beta A^2\omega^2\cos^2(\omega t - \phi) \tag{2.39}$$

The difference between (2.39) and (2.37) represents the potential energy stored in the spring and the kinetic energy of the mass. The average value of the power dissipated is

$$P_{d_{ave.}} = \frac{\beta A^2 \omega^2}{2} \tag{2.40}$$

since the average value of $\cos^2(\omega t - \phi)$ is 1/2. Equating (2.38) and (2.40), we obtain

$$\beta = \frac{F_o \sin\phi}{\omega A} \tag{2.41}$$

At resonance, $\sin\phi = 1$, and therefore

$$\beta = \frac{F_o}{\omega A_n} \tag{2.42}$$

So far it has been assumed that the damping is viscous, (i.e., the damping force is proportional to the velocity). In the problem under consideration, this is not entirely the case. In a machine foundation energy is dissipated by radiation and also by friction between the soil particles. This friction or Coulomb damping is not proportional to the velocity. A good approximation, however, can be made by determining an equivalent viscous damping factor from energy considerations. This equivalent damping factor may be defined from equation (2.40), thus

$$\beta_e = \frac{2 P_{d_{ave.}}}{A^2 \omega^2} \tag{2.43}$$

Determination of the Damping Factor.

We have already discussed how the damping factor may be determined from the normalized amplification curves. Another convenient method which can be used for small damping, is to determine the damping ratio from the width of the resonance curve at a point where the amplitude is equal to $\sqrt{2}\ A_{max.}$ by means of the relation

$$c = \beta/\beta_c \simeq \frac{\Delta f}{2f} \tag{2.44}$$

for, from equation (2.25), (and since $A_{max} \simeq A_n$)

$$\frac{\sqrt{2}\,A_n}{2\,\delta_{st}} = \frac{\sqrt{2}}{4c} = \frac{1}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2c\,\omega/\omega_n)^2}}$$

Solving for $(\omega/\omega_n)^2$, we find

$$(\omega/\omega_n)^2 = (1-2c^2) \pm 2c\sqrt{c^2+1}$$

but,

$$\frac{\Delta\omega}{\omega} \simeq \frac{\omega_1^2-\omega_2^2}{\omega_1^2+\omega_2^2} = \frac{2c\sqrt{c^2+1}}{1-2c^2} = 2c\left(1+\frac{3c^2}{2}+\frac{19c^4}{4}+\dots\right)$$

$$\simeq 2c$$

For $c = 0.10$ the error introduced in determining damping by this method is about 2.5 per cent, and for $c = 0.20$ the error is about 10 per cent.

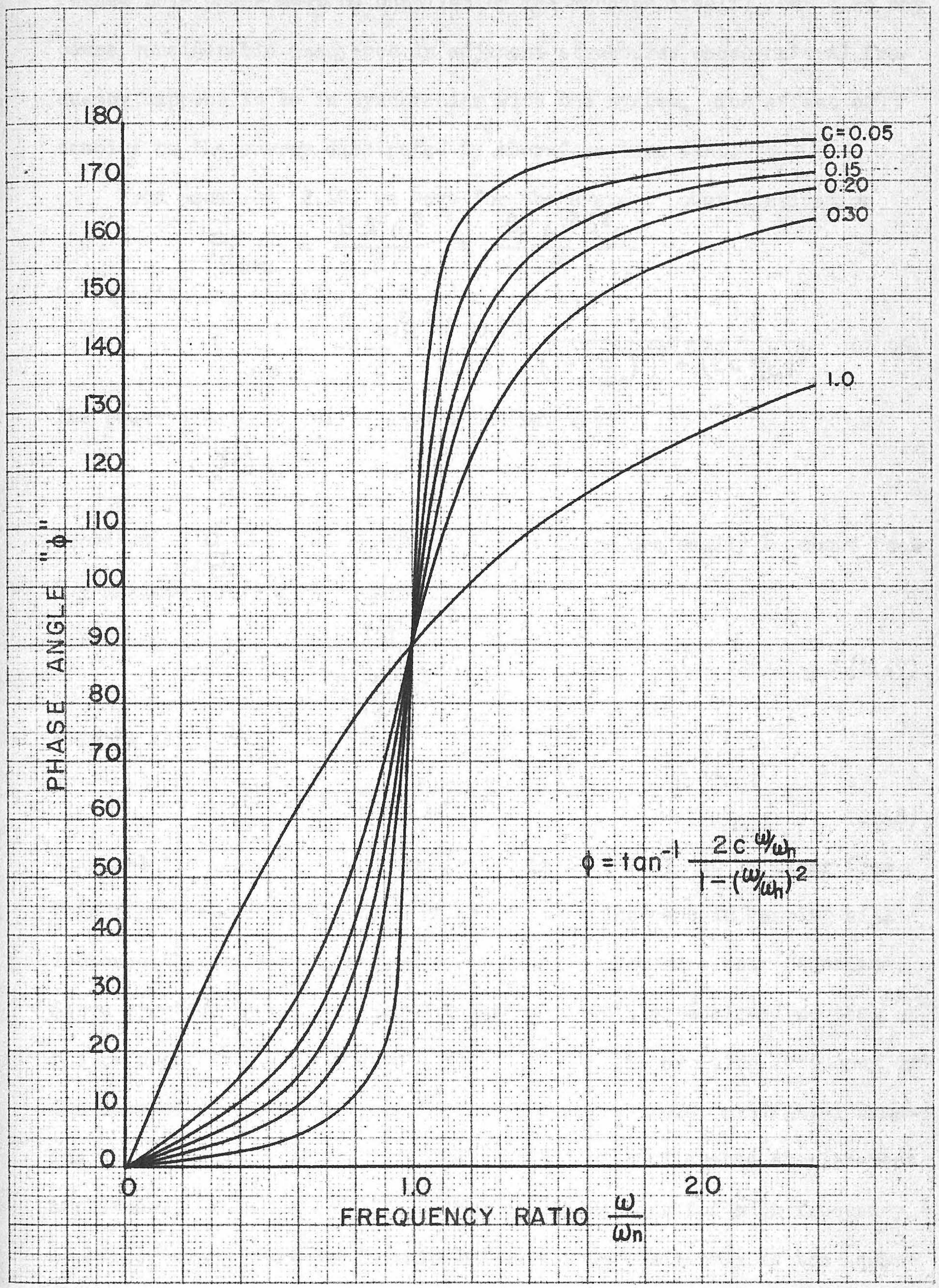
For flat response curves the damping ratio may be determined from phase shift measurements. From equation (2.23b) it is seen that for a frequency ratio of unity (i.e. $\omega/\omega_n = 1$), the phase angle between the external force and the resulting displacement is exactly ninety degrees for all damping ratios. Therefore the natural frequency (i.e. the frequency of undamped free oscillation) is also readily determined by phase shift measurements. Solving equation (2.23b) for c, we have

$$c = \frac{1-\left(\frac{\omega}{\omega_n}\right)^2}{2\,\frac{\omega}{\omega_n}}\tan\phi \qquad (2.45)$$

The damping ratio can be calculated from the above equation by measuring ϕ for some frequency ratio different from unity. The relationship between ϕ and ω/ω_n is shown graphically in Fig. 2.6 for several damping ratios.

<u>Effect of Damping on the Dissipated Power.</u>

Next to the amplitude of oscillation, the most important criterion in machine foundation design is the energy dissipated. Dissipation of large amounts of energy not only constitutes a decrease in efficiency



180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
PHASE ANGLE "ϕ"
C=0.05
0.10
0.15
0.20
0.30
1.0
$\phi = \tan^{-1} \frac{2c\, \omega/\omega_n}{1-(\omega/\omega_n)^2}$
0
1.0
2.0
FREQUENCY RATIO $\frac{\omega}{\omega_n}$

Figure 2.6

(since this power must be supplied by the machine itself), but also may cause considerable annoyance in adjacent structures whose natural frequency happens to be in synchronism with the system. The effect of damping on the energy dissipated is therefore important.

From equation (2.40) we have for the average power dissipation

$$P_{d_{ave.}} = \frac{\beta A^2 \omega^2}{2} = \frac{\beta_c\, \omega^2 A^2 c}{2}$$

On substituting equation (2.34) for A we have

$$P_{d_{ave.}} = \frac{\beta_c\, \omega^2 r^2}{2} \left[\frac{\omega}{\omega_n}\right]^4 \frac{c}{\left(1 - (\frac{\omega}{\omega_n})^2\right)^2 + \left(2c\,\frac{\omega}{\omega_n}\right)^2} \qquad (2.46)$$

The power dissipated will reach a maximum when

$$\frac{dP_d}{dc} = 0$$

provided $\frac{d^2 P_d}{dc^2}$ is negative. It can therefore easily be shown that for maximum power dissipation

$$c^2 = \frac{\left(1 - (\frac{\omega}{\omega_n})^2\right)^2}{4\left(\frac{\omega}{\omega_n}\right)^2} \qquad (2.47)$$

solving for ω/ω_n, we find

$$\frac{\omega}{\omega_n} = \pm c + \sqrt{1 + c^2} \qquad (2.48)$$

This relationship is shown graphically in Fig. 2.7. It is clear from this figure and from equation (2.47) that an increase in damping will increase the power dissipation except in a narrow zone near resonance. Since practically all machine foundations should be designed so that the resonant frequency does not coincide with the exciting frequency or its harmonics, it can be seen that, as a general rule, artificial damping should be kept to a minimum, compatible with allowable displacement amplitudes. This is especially true of foundations for high frequency machines, since the power dissipated varies approximately as the square of the exciting frequency.

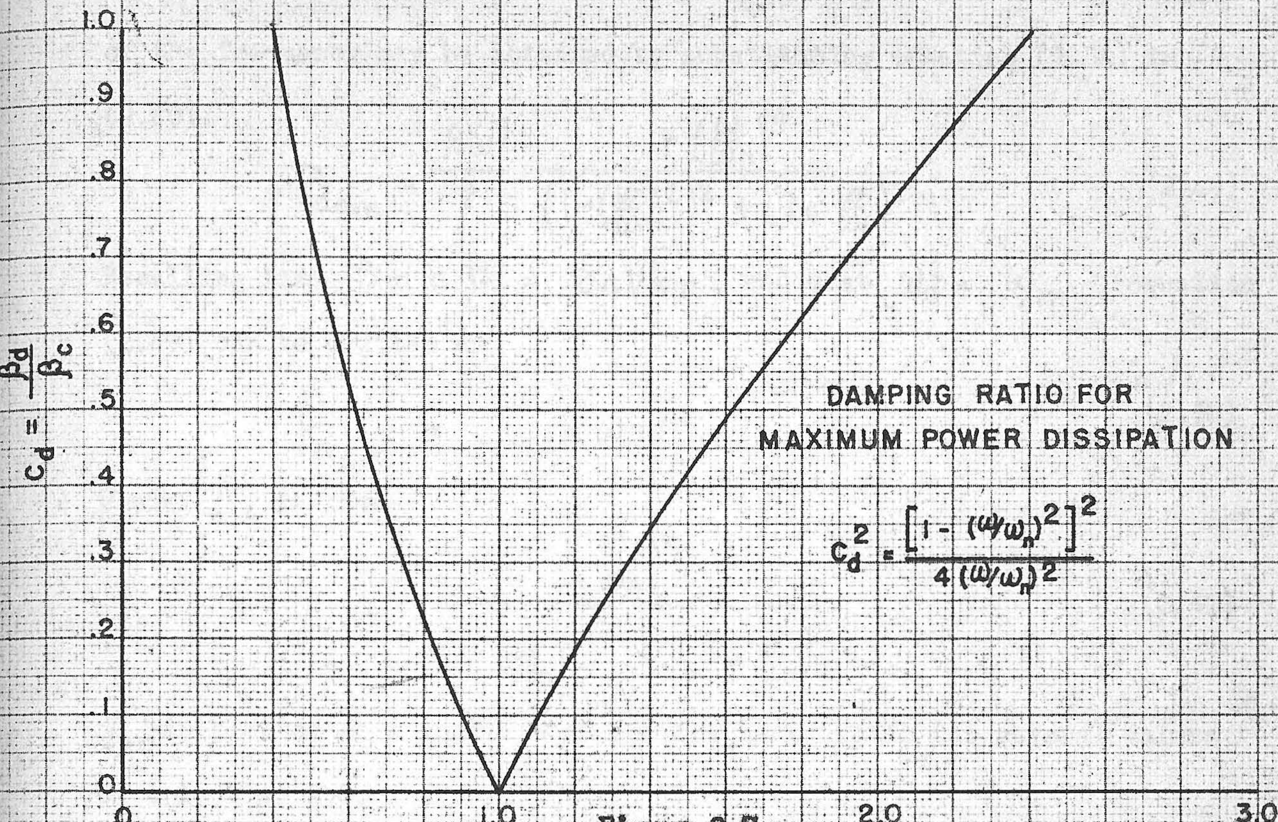


Figure 2.7

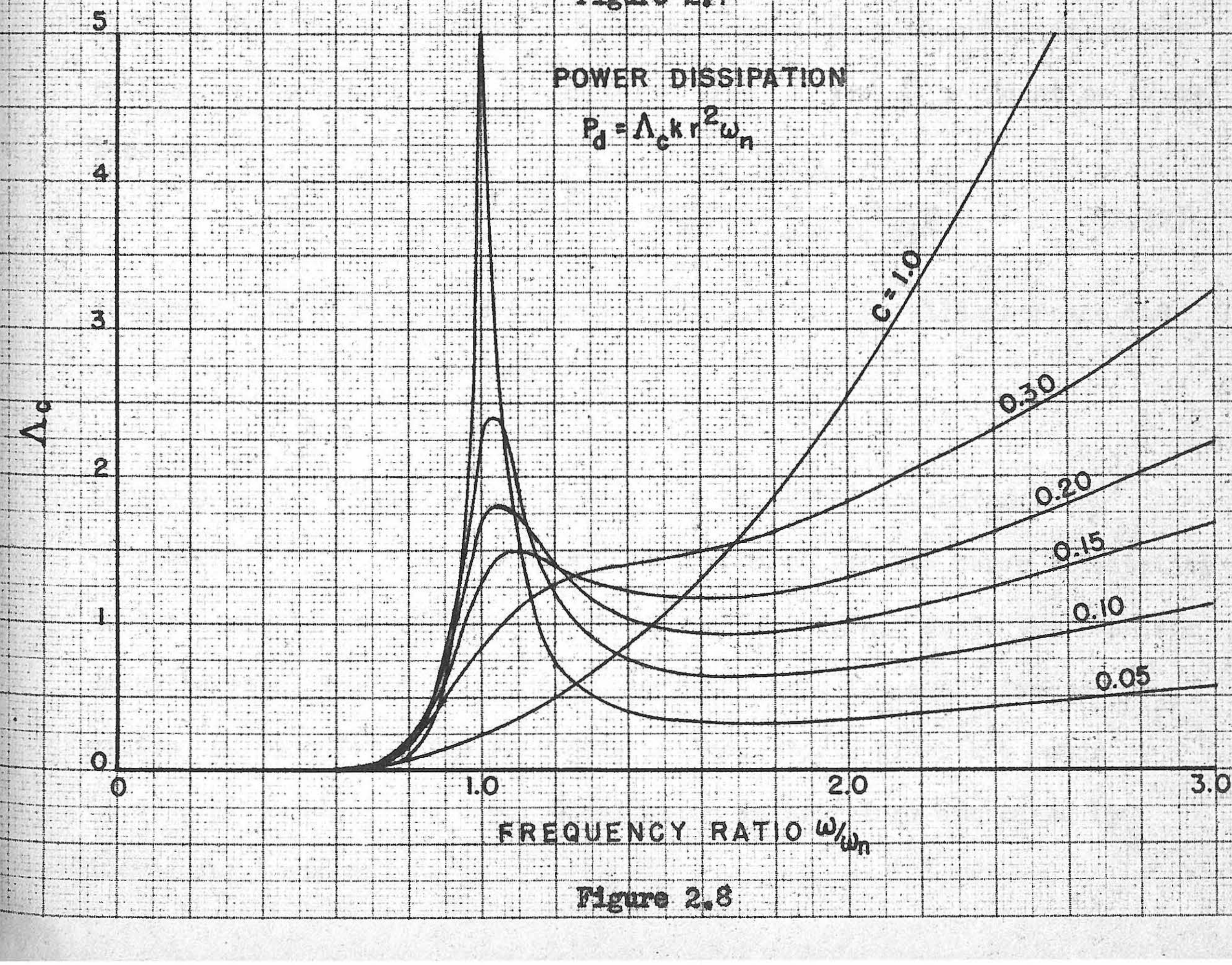


Figure 2.8

The power dissipated when the exciting force varies as the square of the frequency may be obtained by substituting equation (2.34) in (2.40).

$$P_{d\,ave.} = \frac{\beta\omega^2}{2} \frac{r^2\left(\frac{\omega}{\omega_n}\right)^4}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2c\frac{\omega}{\omega_n}\right)^2} \tag{2.49}$$

Recalling that $\beta = c\,\beta_c = 2c\sqrt{km}$, and that $\omega_n^2 = k/m$, equation (2.49) can be written as

$$P_{d\,ave} = \Lambda_c\, k\,\omega_n r^2 \tag{2.50}$$

where Λ_c is given by

$$\Lambda_c = \frac{\left(\frac{\omega}{\omega_n}\right)^6 c}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2c\frac{\omega}{\omega_n}\right)^2} \tag{2.51}$$

Λ_c is plotted as a function of the frequency ratio for several values of c in Fig. 2.8.

To show the dependence of P_d on k, m, and β for various frequency ratios, we can rewrite (2.49) in terms of k, m, and β. Equation (2.49) becomes

$$P_{d\,ave.} = \frac{1}{2}\left[\frac{W'e}{g}\right]^2 \frac{\omega^4\beta}{\left(\frac{k}{\omega} - \omega m\right)^2 + \beta^2} \tag{2.52}$$

From equation (2.52) it is clear that increasing β will increase the power dissipation except when

$$\beta \geqslant \frac{k}{\omega} - m\omega$$

for only in that case is the denominator of (2.52) dominated by β^2. This means that only in the zone near resonance can the power dissipation be decreased by increasing β, for then $k/\omega \simeq \omega m$. As in the case for displacements, the denominator is dominated by k for small frequency ratios and by m for large frequency ratios. For a given k/m ratio (i.e. a given ω_n), the power dissipation is therefore decreased by increasing k and m.

Transmissibility.

The third factor which must be considered in the design of a machine foundation is the magnitude of the forces transmitted to the soil. In order to discuss this phase it is convenient to visualize the soil as consisting of two zones; (a) the zone adjacent to the foundation in which the major portion of the distortion of the soil takes place, and (b) an outer or rigid zone. In the first zone the damping is principally due to intergranular friction while in the rigid zone it is essentially due to radiation of the transmitted forces. Since the damping due to radiation may be quite small, it is desirable to keep the transmitted forces small in order to prevent disturbance of adjacent structures whose natural frequencies coincide with the exciting frequency.

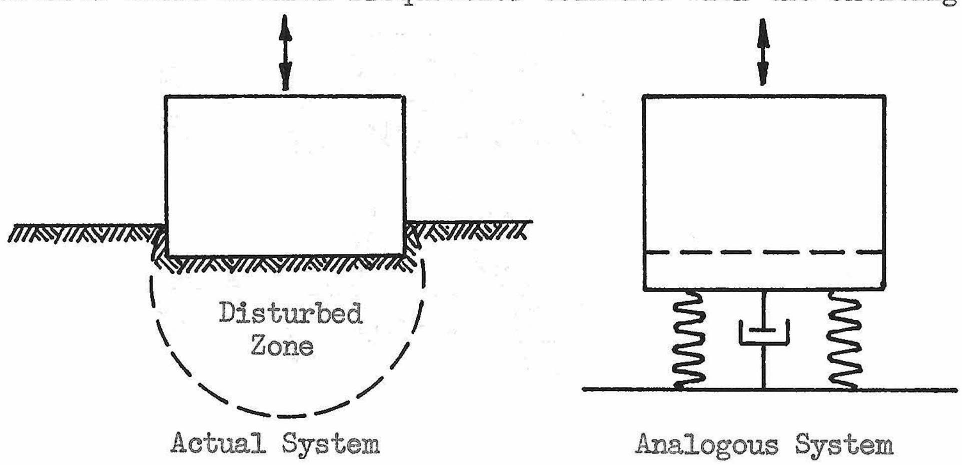


Figure 2.9

To study this problem, we replace the actual system with an analogous system. The force transmitted by the springs is given by

$$F_a = kx + \beta\dot{x}$$
$$= kA\sin(\omega t-\phi) + \beta\omega A\cos(\omega t-\phi) \qquad (2.53)$$

where A is defined by equation (2.25).

The amplitude of the transmitted force is therefore

$$F_{a_{max}} = kA\sqrt{1 + \left(\frac{\beta\omega}{k}\right)^2} \tag{2.54}$$

Transmissibility is defined as the ratio of the transmitted force divided by the exciting force, and is given by

$$\frac{F_a}{F_o} = \frac{\sqrt{1 + \left(\frac{\beta\omega}{k}\right)^2}}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\frac{\beta\omega}{\beta_c\omega_n}\right)^2}} \tag{2.55}$$

If we define

$$N = \frac{\omega}{\omega_n}$$

and since $\omega_n = \sqrt{\frac{k}{m}}$, $\beta_c = 2\sqrt{km}$, and $c = \frac{\beta}{\beta_c}$, equation (2.55) can be written in the following alternative forms:

$$\frac{F_a}{F_o} = \frac{\sqrt{1 + (2cN)^2}}{\sqrt{(1-N^2)^2 + (2cN)^2}} \tag{2.56}$$

$$\frac{F_a}{F_o} = \frac{\sqrt{1 + \frac{N^2\beta^2}{km}}}{\sqrt{(1-N^2)^2 + \frac{N^2\beta^2}{km}}} \tag{2.57}$$

Transmissibility as a function of the frequency ratio is plotted for several damping ratios in Fig. 2.9.

From equations (2.56) and (2.57) we can draw the following conclusions:

1. For $N < \sqrt{2}$ the transmissibility is greater than unity.
2. For $N < \sqrt{2}$ the transmissibility is reduced by increasing β and decreasing m and k.
3. For $N > \sqrt{2}$ the transmissibility is reduced by decreasing β and increasing m and k.

The effect of β, k, and m on displacement, power dissipation and transmissibility are summarized in table 2.1.

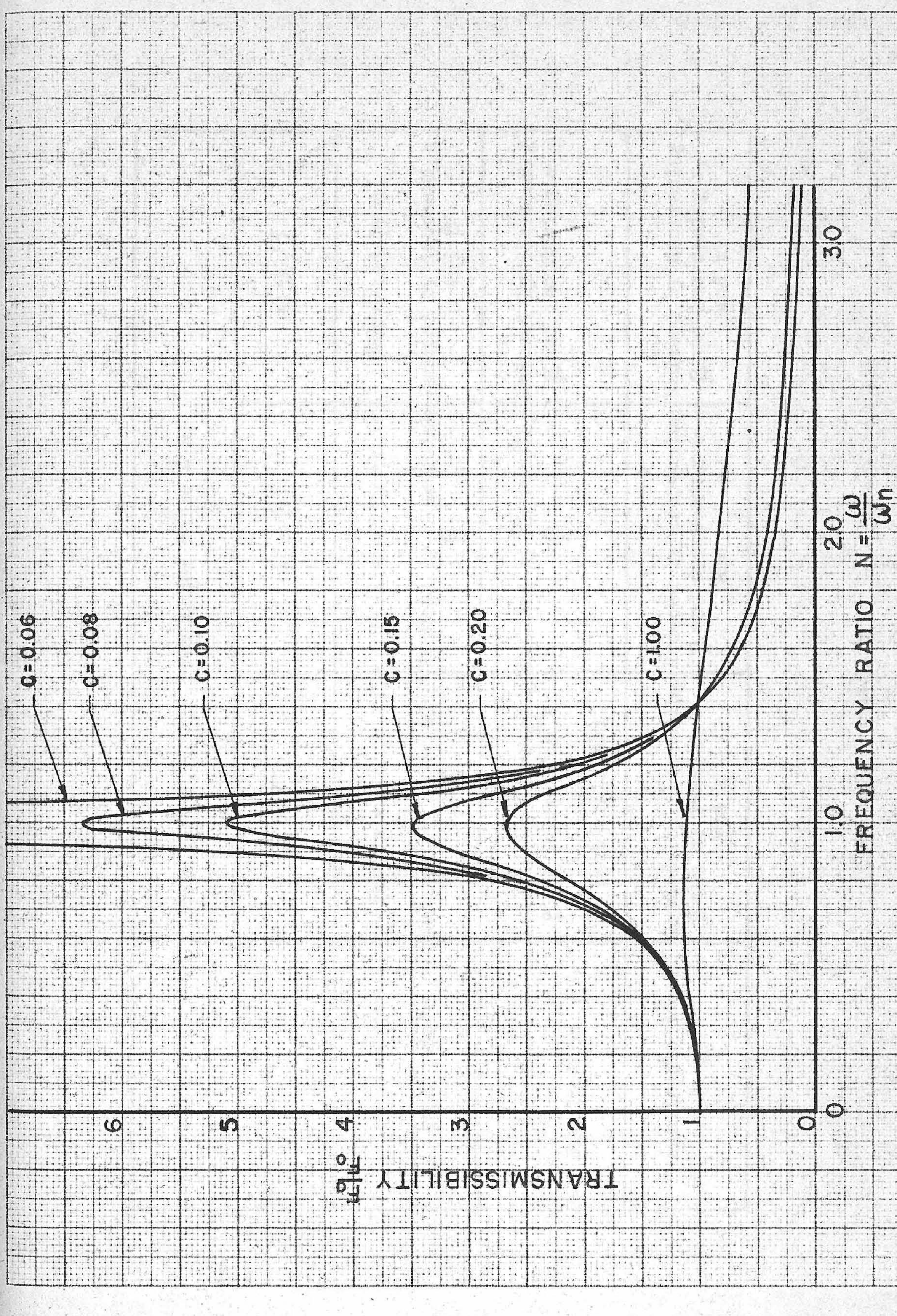

C=0.06
C=0.08
C=0.10
C=0.15
C=0.20
C=1.00
TRANSMISSIBILITY $\frac{F_o}{F}$
0
1
2
3
4
5
6
0
1.0
2.0
3.0
FREQUENCY RATIO $N=\frac{\omega}{\omega_n}$

Figure 2.9

	FREQUENCY RATIO - N		
	$N < \sqrt{2}$	$N \simeq 1$	$N > \sqrt{2}$
Displacement Amplitude	Dominated by k Reduced by increasing k and β	Dominated by β Reduced by increasing β	Dominated by m Reduced by increasing m and β
Power Dissipation	Reduced by decreasing β and increasing k (not critical)	Reduced by increasing β	Reduced by decreasing β and increasing m
Transmissibility	Reduced by increasing β and decreasing m and k	Reduced by increasing β and decreasing m and k	Reduced by decreasing β and increasing m and k

Table 2.1

From table 2.1 it can be seen that the requirements for minimum displacement, power dissipation and transmissibility are contradictory and hence a compromise must be made in an actual design. The problem is further complicated by the interdependence of k, m, and β for most soil conditions. For machines operating at low frequencies, good soil conditions are required. Excessive vibration and settlement may be expected if this type of equipment is founded on loose or compressible soils. This type of soil is characterized by a low elastic modulus, resulting in a low effective spring constant and large damping factor. In many cases soil conditions can be improved by grouting, driving piles, or by some other method of stabilization. These methods of stabilization have also proved effective for correcting existing foundations which vibrated excessively.[18,19,20]

CHAPTER III

MULTIPLE DEGREE OF FREEDOM SYSTEMS

EQUATIONS OF MOTION FOR A BLOCK FOUNDATION

The discussion in Chapter II was limited to systems having one degree of freedom. In practice, foundations can oscillate simultaneously about a number of axes. Specifically, a block foundation (that is a foundation in which the machine and its foundation can be treated as a rigid unit) will have six degrees of freedom, namely three translations and three rotations. In order to examine the possible modes of oscillation resulting from these six degrees of freedom, it is convenient to apply Lagrange's method for deriving the equations of motion.[21,22]

In the discussion of single degree of freedom systems, it was noted that damping causes only a small shift in the resonant frequency. From figures 2.2 and 2.4 it can further be seen that small damping has only a minor effect on the amplitude ratio for frequency ratios of less than 0.5 and greater than 1.5. In considering problems with several degrees of freedom it is convenient to neglect damping in determining the natural frequencies of the principal modes. The problem is thereby considerably simplified without much sacrifice of accuracy.

<u>Lagrange's Equations</u>.

In more complicated systems, the problem of how to determine the differential equations for the system must first be solved. Starting from Newton's law we have:

$$\bar{F} = m\ddot{\bar{r}} \tag{3.1}$$

Converting into energy terms, we find

$$\begin{aligned} \bar{F}\cdot d\bar{r} &= m\ddot{\bar{r}}\cdot d\bar{r} = m\ddot{\bar{r}}\cdot\frac{d\bar{r}}{dt}dt = \frac{d}{dt}\left(\frac{m}{2}\dot{\bar{r}}^2\right)dt \\ dW &= dT \end{aligned} \tag{3.2}$$

where W is work and T is energy. The advantage of relationship (3.2) is that it involves scalar quantities (i.e. velocities) instead of the vector quantities of (3.1). In order to rewrite (3.2) in a form leading to the differential equation, a system of generalized coordinates is defined. In general, for any system there exists a set of generalized coordinates $q_1, q_2, q_3, \ldots,$ such that the coordinates are independent and correspond to the number of degrees of freedom. Let us assume that the system consists of "n" mass points and that the configuration of these points is given by "r" independent parameters. Let us further assume that the Cartesian coordinates of the n mass points can be expressed in terms of the r coordinates by equations of the form

$$x_i = x_i(q_1, q_2, q_3, \ldots\ldots q_r)$$
$$y_i = y_i(q_1, q_2, q_3, \ldots\ldots q_r)$$
$$z_i = z_i(q_1, q_2, q_3, \ldots\ldots q_r)$$

where $i = 1, 2, 3, \ldots, n$.

Since the coordinates $q_1, q_2, q_3, \ldots, q_r$ are independent, the increment of work resulting from a small variation of q_k is

$$dW = \frac{\partial W}{\partial q_k} dq_k = Q_k \, dq_k \tag{3.3}$$

where $Q_k = \dfrac{\partial W}{\partial q_k}$ is called the generalized force.

The generalized force has the dimensions of a force or a moment, and the dimension may be determined by the rule that $Q_k \cdot q_k$ has the dimension of work. The kinetic energy may be expressed in Cartesian coordinates by the relation:

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right) \tag{3.4}$$

Since

$$\dot{x}_i = \frac{dx}{dt} = \frac{\partial x_i}{\partial q_k}\frac{dq_k}{dt} = \frac{\partial x_i}{\partial q_k}\dot{q}_k$$

$$\dot{y}_i = \frac{\partial y_i}{\partial q_k}\dot{q}_k$$

$$\dot{z}_i = \frac{\partial z_i}{\partial q_k}\dot{q}_k ,$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\left[\left(\frac{\partial x_i}{\partial q_k}\dot{q}_k\right)^2 + \left(\frac{\partial y_i}{\partial q_k}\dot{q}_k\right)^2 + \left(\frac{\partial z_i}{\partial q_k}\dot{q}_k\right)^2\right]$$

$$= T(q_k, \dot{q}_k) \qquad (3.5)$$

and

$$dT = \frac{dT}{dt}dt = \left[\frac{\partial T}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial T}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right]dt$$

T is a homogeneous quadratic function of the $\dot{q}$'s ; therefore

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\dot{q}_k\right) = \frac{\partial T}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \dot{q}_k\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)$$

and

$$\frac{\partial T}{\partial \dot{q}_k}\dot{q}_k = 2T$$

Therefore,

$$dT = \left[\frac{\partial T}{\partial q_k}\frac{dq_k}{dt} + \frac{d}{dt}(2T) - \dot{q}_k\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\right]dt$$

$$= \frac{\partial T}{\partial q_k}dq_k + 2\,dT - dq_k\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)$$

Transposing,

$$dT = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]dq_k = dW = Q_k\,dq_k$$

therefore,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q} = Q_k \tag{3.6}$$

When the system is conservative,

$$Q_k = -\frac{\partial U}{\partial q_k}$$

where U is the potential energy. Equation (3.6) can then be written in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T-U)}{\partial q_k} = 0 \tag{3.7}$$

Equations (3.6) and (3.7) are called Lagrange's equations.

<u>Small Oscillations of a Conservative System about an Equilibrium Point.</u>

The coordinates may be selected in such a way that in the equilibrium position $q_1 = q_2 = q_3 = \ldots = q_n = 0$, and the level of the potential energy is zero. Expanding the potential energy in the neighborhood of the equilibrium position by means of a Taylor series, we have

$$\begin{aligned} U = {} & \left[\frac{\partial U}{\partial q_1}\right]_0 q_1 + \left[\frac{\partial U}{\partial q_2}\right]_0 q_2 + \cdots + \frac{1}{2}\left[\frac{\partial^2 U}{\partial q_1^2}\right]_0 q_1^2 + \cdots \\ & + \left[\frac{\partial^2 U}{\partial q_1 \partial q_2}\right]_0 q_1 q_2 + \left[\frac{\partial^2 U}{\partial q_1 \partial q_3}\right]_0 q_1 q_3 + \cdots \\ & + \text{higher order terms} \end{aligned} \tag{3.8}$$

Since the expansion is about the equilibrium position (U = 0)

$$-\frac{\partial U}{\partial q_1} = Q_1 = 0, \tag{3.9a}$$

and

$$-\frac{\partial U}{\partial q_2} = Q_2 = 0, \ldots \tag{3.9b}$$

If the motion of the system is restricted to small oscillations, all terms of order higher than the second may be neglected. Equation (3.8) can then be written in the form:

$$U = \frac{1}{2} \sum_i \sum_j \left[\frac{\partial^2 U}{\partial q_i \, \partial q_j}\right]_0 q_i q_j$$

$$= \frac{1}{2} \sum_i \sum_j k_{ij} q_i q_j \tag{3.10}$$

where the k_{ij}'s are called the elastic coefficients, and are equal to

$$k_{ij} = \left[\frac{\partial^2 U}{\partial q_i \, \partial q_j}\right]_0 \tag{3.11}$$

For stable systems the potential energy, U, has a minimum value at the equilibrium position. Hence for $q_1 = q_2 = q_3 = \ldots = q_r = 0$, $U = 0$, and is also a minimum; it follows that U must be positive everywhere else. U is therfore a positive definite quadratic function.

The kinetic energy may be expressed in a similar manner. To transform equation (3.4) into general coordinates, we recall that

$$\begin{aligned} x_i &= f_1(q_1, q_2, q_3, \ldots, q_r) \\ y_i &= f_2(q_1, q_2, q_3, \ldots, q_r) \\ z_i &= f_3(q_1, q_2, q_3, \ldots, q_r) \end{aligned} \tag{3.12}$$

where $q_1, q_2, q_3, \ldots q_r$ are independent coordinates, and that therefore

$$\begin{aligned} \dot{x}_i &= \frac{\partial f_1}{\partial q_1}\dot{q}_1 + \frac{\partial f_1}{\partial q_2}\dot{q}_2 + \cdots &&= \sum_{i=1}^{r} \frac{\partial f_1}{\partial q_i}\dot{q}_i \\ \dot{y}_i &= &&= \sum_{i=1}^{r} \frac{\partial f_2}{\partial q_i}\dot{q}_i \\ \dot{z}_i &= &&= \sum_{i=1}^{r} \frac{\partial f_3}{\partial q_i}\dot{q}_i \end{aligned} \tag{3.13}$$

Substituting in equation (3.4) we obtain

$$T=\frac{1}{2}\sum m_i\left\{\left(\frac{\partial f_1}{\partial q_1}\right)^2\dot{q}_1^2+\left(\frac{\partial f_1}{\partial q_2}\right)^2\dot{q}_2^2+\cdots+2\left(\frac{\partial f_1}{\partial q_1}\right)\left(\frac{\partial f_1}{\partial q_2}\right)\dot{q}_1\dot{q}_2+\cdots\right.$$
$$\left.+\left(\frac{\partial f_2}{\partial q_1}\right)^2\dot{q}_1^2+\left(\frac{\partial f_2}{\partial q_2}\right)^2\dot{q}_2^2+\cdots+2\left(\frac{\partial f_2}{\partial q_1}\right)\left(\frac{\partial f_2}{\partial q_2}\right)\dot{q}_1\dot{q}_2+\cdots\right\} \tag{3.14}$$

The coefficients of $\dot{q}_1^2, \dot{q}_2^2, \dot{q}_1\dot{q}_2$, etc. are called the inertia parameters and may be denoted by m_{ij}. Expanding in a Taylor series about the equilibrium point,

$$m_{ij}=\left[(m_{ij})_0+\left(\frac{\partial m_{ij}}{\partial q_1}\right)_0 q_1+\left(\frac{\partial m_{ij}}{\partial q_2}\right)_0 q_2+\cdots\right] \tag{3.15}$$

All terms except $(m_{ij})_0$ contribute third or higher order terms to the kinetic energy T, and therefore may be neglected for small oscillations. The expression for kinetic energy therefore reduces to

$$T=\frac{1}{2}\sum_i\sum_j (m_{ij})_0\,\dot{q}_i\dot{q}_j \tag{3.16}$$

Since the kinetic energy can not be negative by definition, T is a positive definite form of the velocities. T does not depend on the coordinates, therefore

$$\frac{\partial T}{\partial q_i}=0$$

and Lagrange's equation reduces to the simple form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial U}{\partial q_i}=0 \tag{3.17}$$

If these equations are written explicitly for r=3, the following equations are obtained upon substitution of equations (3.10) and (3.16) in equation (3.17):

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + m_{13}\ddot{q}_3 = -(k_{11}q_1 + k_{12}q_2 + k_{13}q_3)$$
$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{23}\ddot{q}_3 = -(k_{21}q_1 + k_{22}q_2 + k_{23}q_3)$$
$$m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + m_{33}\ddot{q}_3 = -(k_{31}q_1 + k_{32}q_2 + k_{33}q_3)$$

The terms $k_{12} = k_{21}$, $k_{23} = k_{32}$, and $k_{31} = k_{13}$ are called static coupling terms, whereas the terms $m_{12} = m_{21}$, $m_{13} = m_{31}$, and $m_{23} = m_{32}$ are called the dynamic coupling terms. The systems in which we are interested, namely block foundations, contain only static coupling terms.

Finally, if the system is subjected to an external forcing function, $F_i \sin(\omega t + \psi)$, the relationship becomes:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = F_i \sin(\omega t + \psi) \tag{3.18}$$

Since in our application there is no dynamic coupling, the kinetic energy is given by:

$$T = \frac{1}{2}\sum_i^r m_i \dot{q}_i^2,$$

and when expanded, equation (3.18) assumes the form

$$m_i \ddot{q}_i + k_{i1}q_1 + k_{i2}q_2 + k_{i3}q_3 + \dots + k_{ir}q_r = F_i \sin(\omega t + \psi) \tag{3.19}$$

For a single degree of freedom system, equation (3.19) reduces to the familiar equation

$$m\ddot{q} + kq = F\sin(\omega t + \psi) \tag{3.20}$$

For a single degree of freedom system the frequency of free oscillation may be found by assuming $q = A \sin \omega t$ to be a solution, and substituting in the reduced equation

$$m\ddot{q} + kq = 0$$

Then

$$-\omega^2 m + k = 0$$

from which the frequency of free oscillation is determined to be

$$\omega_n^2 = k/m$$

The same procedure may be extended to a multiple degree of freedom system. Assuming the solutions to be of the form

$$q_i = A_i \sin(\omega t + \psi) \tag{3.21}$$

we obtain

$$-m_i\omega^2 A_i + \sum_{j=1}^{n} k_{ij} A_j = 0 \tag{3.22}$$

For a rational solution, the determinant of the coefficients of A_{ij} must vanish, therefore

$$\begin{vmatrix} (k_{11}-m_1\omega^2) & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & (k_{22}-m_2\omega^2) & k_{23} & \cdots & \cdots \\ k_{31} & k_{32} & (k_{33}-m_3\omega^2) & \cdots & \\ \cdots & & & & \\ k_{n1} & k_{n2} & & & (k_{nn}-m_n\omega^2) \end{vmatrix} = 0 \tag{3.23}$$

The determinant (3.23) may be solved for the n roots of ω. The most general solution, therefore, is

$$q_i = \sum_{r=1}^{n} A_i^{(r)} \sin(\omega_r t + \psi_r) \tag{3.24}$$

where $A_i^{(r)}$ is the coefficient corresponding to the frequency for the i th coordinate.

Orthogonality Relations.

If the result of (3.24) is substituted back into equation (3.22) then for the r th mode

$$m_i \omega_r^2 A_i^{(r)} = \sum_{j=1}^{n} k_{ij} A_j^{(r)} \tag{3.25}$$

and similarly for the s th mode

$$m_i \omega_s^2 A_i^{(s)} = \sum_{j=1}^{n} k_{ij} A_j^{(s)} \tag{3.26}$$

Multiplying (3.25) by $A_i^{(s)}$ and (3.26) by $A_i^{(r)}$ and summing over i we obtain:

$$\sum_{i=1}^{n} m_i \omega_r^2 A_i^{(r)} A_i^{(s)} = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} A_j^{(r)} A_i^{(s)}$$

$$\sum_{i=1}^{n} m_i \omega_s^2 A_i^{(s)} A_i^{(r)} = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} A_j^{(s)} A_i^{(r)} = \sum_{j=1}^{n}\sum_{i=1}^{n} k_{ji} A_i^{(s)} A_j^{(r)}$$

But, $k_{ij} = k_{ji}$, hence

$$(\omega_r^2 - \omega_s^2) \sum_{i=1}^{n} m_i A_i^{(r)} A_i^{(s)} = 0$$

Since ω_r and ω_s are different roots, $(\omega_r^2 - \omega_s^2) \neq 0$, and

$$\sum_{i=1}^{n} m_i A_i^{(r)} A_j^{(s)} = 0 \tag{3.27}$$

This relationship is called the orthogonality condition.

Forced Oscillation Amplitudes.

Equation (3.24) gives the relative amplitude for the various modes, but the absolute amplitude is unknown because the coefficients $A_i^{(r)}$ are unknown. To determine the amplitude of oscillation, it is again convenient to recall the procedure followed for single degree of freedom systems. If we substitute

$$q = A \sin(\omega t + \psi)$$

in equation (3.20), we obtain

$$-A\omega^2 m + kq = F$$

Solving for A,

$$A = \frac{F}{(k - m\omega^2)} = \frac{F/k}{1 - (\omega/\omega_n)^2}$$

and hence

$$q = \frac{F/k}{1 - (\omega/\omega_n)^2} \sin(\omega t + \psi)$$

We again recall that the above expression is valid only for the case of zero damping. Even for appreciable values of damping, however, the expression for A is reasonably accurate for frequency ratios less than 0.5 or greater than 1.5.

Accordingly, for multiple degree of freedom systems we try the substitution

$$q_i = \sum_r c_r \phi_i^{(r)} \sin(\omega t + \psi) \tag{3.28}$$

The orthogonality condition (3.27) then becomes,

$$\sum_{i=1}^{n} m_i c_r \phi_i^{(r)} c_r \phi_i^{(s)} = 0$$

Then, since $c_r \neq 0$

$$\sum_{i=1}^{n} m_i \phi_i^{(r)} \phi_i^{(s)} = 0 \tag{3.29}$$

Substituting the assumed solution (3.28) in equation (3.19) we have

$$-m_i \omega^2 \sum_{r=1}^{n} c_r \phi_i^{(r)} + k_{i1} \sum_{r=1}^{n} c_r \phi_1^{(r)} + k_{i2} \sum_{r=1}^{n} c_r \phi_2^{(r)} + \dots = F_i \tag{3.30}$$

but since $A_i^{(r)}$ in (3.25) is replaced by $c_r \phi_i^{(r)}$,

$$\begin{aligned} k_{i1} \sum_{r=1}^{n} c_r \phi_1^{(r)} + k_{i2} \sum_{r=1}^{n} c_r \phi_2^{(r)} + \dots &= \sum_{j=1}^{n} k_{ij} \sum_{r=1}^{n} c_r \phi_j^{(r)} \\ &= \sum_{j=1}^{n} k_{ij} (c_1 \phi_j^{(1)} + c_2 \phi_j^{(2)} + \dots) \\ &= \sum_{r=1}^{n} c_r \sum_{j=1}^{n} k_{ij} \phi_j^{(r)} \\ &= \sum_{r=1}^{n} c_r m_i \omega_r^2 \phi_i^{(r)} \end{aligned} \tag{3.31}$$

Substituting the result (3.31) in equation (3.30), we have

$$\sum_{r=1}^{n} c_r m_i \phi_i^{(r)} (\omega_r^2 - \omega^2) = F_i \tag{3.32}$$

If we now write

$$F_i = \sum_{r=1}^{n} f_r m_i \phi_i^{(r)} \tag{3.33}$$

then

$$\sum_{r=1}^{n} c_r m_i \phi_i^{(r)} (\omega_r^2 - \omega^2) = \sum_{r=1}^{n} f_r m_i \phi_i^{(r)}$$

and hence

$$c_r (\omega_r^2 - \omega^2) = f_r$$

Solving for c_r,

$$c_r = \frac{f_r}{\omega_r^2 - \omega^2} \tag{3.34}$$

The desired solution is therefore

$$q_i = \sum_{r=1}^{n} \frac{f_r}{(\omega_r^2 - \omega^2)} \phi_i^{(r)} \sin(\omega t + \psi) \tag{3.35}$$

The problem of determining the oscillation amplitudes is solved provided we can determine the coefficients f_r from the given forces F_1, F_2, F_3 ... F_n. From equation (3.33) we have

$$F_j = \sum_{r=1}^{n} f_r m_j \phi_j^{(r)} = f_1 m_j \phi_j^{(1)} + f_2 m_j \phi_j^{(2)} + \dots + f_n m_j \phi_j^{(n)}$$

Then

$$F_j \phi_j^{(r)} = f_1 m_j \phi_j^{(1)} \phi_j^{(r)} + f_2 m_j \phi_j^{(2)} \phi_j^{(r)} + \dots + f_n m_j \phi_j^{(n)} \phi_j^{(r)}$$

Summing over j we have, owing to the orthogonality condition (3.29),

$$\sum_{j=1}^{n} F_j \phi_j^{(r)} = \sum_{j=1}^{n} f_r m_j \phi_j^{(r)2}.$$

And hence

$$f_r = \frac{\sum_{j=1}^{n} F_j \phi_j^{(r)}}{\sum_{j=1}^{n} m_j [\phi_j^{(r)}]^2} \tag{3.36}$$

It follows, then, that

$$q_i = \sum_{r=1}^{n} \frac{\sum_{j=1}^{n} F_j \phi_j^{(r)}}{\sum_{j=1}^{n} m_j [\phi_j^{(r)}]^2} \; \frac{\phi_i^{(r)}}{(\omega_r^2 - \omega^2)} \sin(\omega t + \gamma) \tag{3.37}$$

Equations of Motion for a Block Foundation.

Let us consider a block foundation resting directly on the soil.

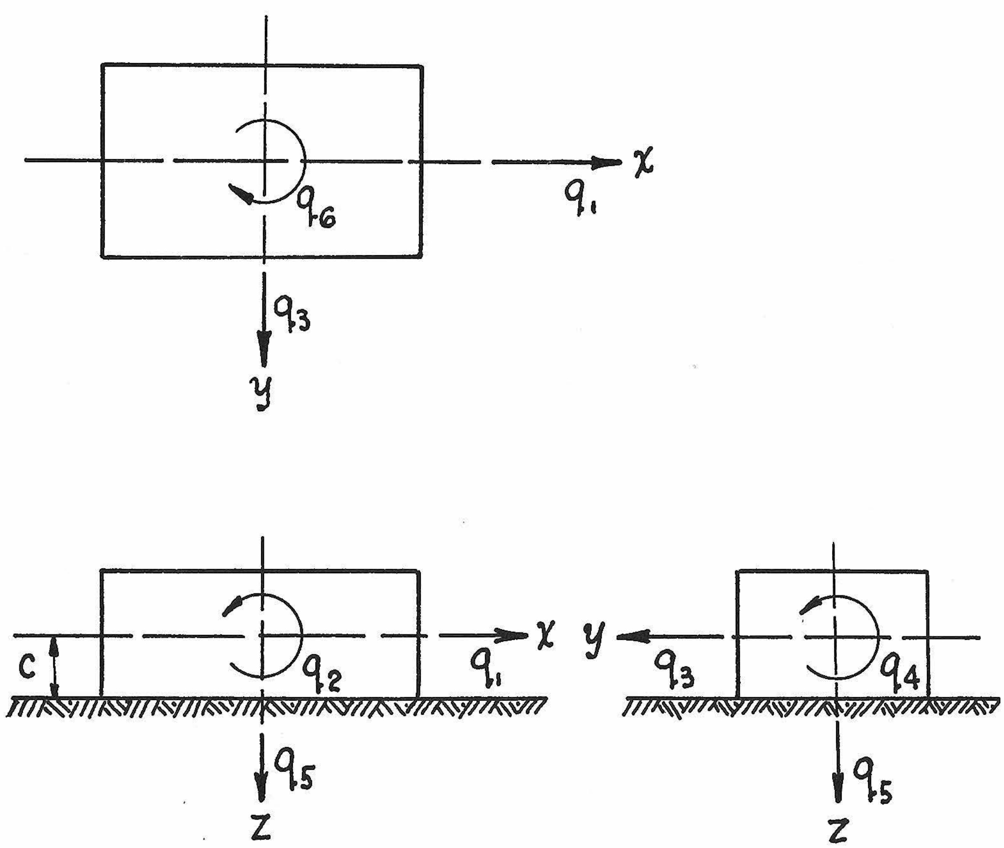


Figure 3.1

Let the q_i's represent the displacements or rotations as shown in Fig. 3.1. The center of coordinates, i.e. the equilibrium point, is the center of the combined mass of the foundation block, the machine and the apparent mass of soil moving with it.

If the block is depressed a unit distance into the soil, the soil will exert a force on the block which we may define by k_z. This force is called the vertical spring constant of the soil. Similarly, for unit displacements in the x and y directions, the spring constants are k_x and k_y, and for rotations about the x, y, and z axes thru the centroid of the contact surface, the spring constants are k_{yz}, k_{xz} and k_{xy} respectively.

The inertia parameters are:

$$\begin{aligned} m_1 &= m_3 = m_5 = m + m' \\ m_2 &= I_y + I'_y \\ m_4 &= I_x + I'_x \\ m_6 &= I_z + I'_z \end{aligned} \tag{3.38}$$

where m is the mass of the block and machine, m' the apparent mass of the soil, I is the mass moment of inertia of the block and machine, and I' the mass moment of inertia of the soil moving with the block. The subscripts refer to the axes about which the moments of inertia are determined.

Since for this case there is no dynamic coupling, the kinetic energy of the system is given by:

$$T = \frac{1}{2}\sum_{i=1}^{6} m_i \dot{q}_i^2 \tag{3.39}$$

The potential energy may be computed from the spring factors, thus

$$U = \frac{1}{2}\Big[k_x(q_1 + cq_2)^2 + k_y(q_3 - cq_4)^2 + k_z q_5^2 + k_{xz} q_2^2 + k_{yz} q_4^2 + k_{xy} q_6^2\Big] \tag{3.40}$$

The elastic coefficients may be determined from equation (3.11)

$$k_{ij} = \left[\frac{\partial^2 U}{\partial q_i \partial q_j}\right]_0 \tag{3.11}$$

Therefore,

$$\begin{aligned}
k_{11} &= k_x & k_{12} &= c k_x = k_{21} \\
k_{22} &= c^2 k_x + k_{xz} \\
k_{33} &= k_y & k_{34} &= -c k_y = k_{43} \\
k_{44} &= c^2 k_y + k_{yz} \\
k_{55} &= k_z \\
k_{66} &= k_{yz}
\end{aligned} \tag{3.41}$$

Assume that the foundation is under the action of a periodic external force, F sin($\omega t + \psi$). This force can be resolved into six components corresponding to the coordinates q_1, q_2, ..., q_6. Therefore

$$F \sin(\omega t + \psi) = \sum_{i=1}^{6} F_i \sin(\omega t + \psi) \tag{3.42}$$

Substituting the results of (3.39), (3.40), and (3.42) in equation (3.18) the equations of motion for the system are obtained.

$$
\begin{aligned}
m_1 \ddot{q}_1 + k_x (q_1 + c q_2) &= F_1 \sin(\omega t + \psi) \\
m_2 \ddot{q}_2 + c k_x (q_1 + c q_2) + k_{xz} q_2 &= F_2 \sin(\omega t + \psi) \\
m_3 \ddot{q}_3 + k_y (q_3 - c q_4) &= F_3 \sin(\omega t + \psi) \\
m_4 \ddot{q}_4 - c k_y (q_3 - c q_4) + k_{yz} q_4 &= F_4 \sin(\omega t + \psi) \\
m_5 \ddot{q}_5 + k_z q_5 &= F_5 \sin(\omega t + \psi) \\
m_6 \ddot{q}_6 + k_{xy} q_6 &= F_6 \sin(\omega t + \psi)
\end{aligned}
\tag{3.43}
$$

Using the elastic coefficients defined by (3.41), equations (3.43) can be rewritten in the following form:

$$
\begin{aligned}
m_1 \ddot{q}_1 + k_{11} q_1 + k_{12} q_2 &= F_1 \sin(\omega t + \psi) \\
m_2 \ddot{q}_2 + k_{21} q_1 + k_{22} q_2 &= F_2 \sin(\omega t + \psi) \\
m_3 \ddot{q}_3 + k_{33} q_3 + k_{34} q_4 &= F_3 \sin(\omega t + \psi) \\
m_4 \ddot{q}_4 + k_{43} q_3 + k_{44} q_4 &= F_4 \sin(\omega t + \psi) \\
m_5 \ddot{q}_5 + k_{55} q_5 &= F_5 \sin(\omega t + \psi) \\
m_6 \ddot{q}_6 + k_{66} q_6 &= F_6 \sin(\omega t + \psi)
\end{aligned}
\tag{3.44}
$$

From equation (3.43) and (3.44) it is clear that there are coupled oscillations consisting of horizontal translation and rotation in the xz and yz planes.

Recalling (3.23) it is seen that the natural frequencies of the system must satisfy the determinant

$$\begin{vmatrix} k_{11}-m_1\omega^2 & k_{12} & 0 & 0 & 0 & 0 \\ k_{21} & k_{22}-m_2\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33}-m_3\omega^2 & k_{34} & 0 & 0 \\ 0 & 0 & k_{43} & k_{44}-m_4\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{55}-m_5\omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{66}-m_6\omega^2 \end{vmatrix} = 0$$

Solving for ω we have,

$$m_1 m_2 \omega^4 - (k_{22}m_1 + k_{11}m_2)\omega^2 + k_{11}k_{22} - k_{12}k_{21} = 0$$

$$\omega^2 = \frac{\dfrac{k_{11}}{m_1} + \dfrac{k_{22}}{m_2} \pm \sqrt{\dfrac{k_{11}^2}{m_1^2} - \dfrac{2k_{11}k_{22}}{m_1 m_2} + \dfrac{k_{22}^2}{m_2^2} + \dfrac{4k_{12}k_{21}}{m_1 m_2}}}{2}$$

$$\omega_{1,2}^2 = \frac{\dfrac{k_{11}}{m_1} + \dfrac{k_{22}}{m_2} \pm \sqrt{\left(\dfrac{k_{11}}{m_1} - \dfrac{k_{22}}{m_2}\right)^2 + \dfrac{4\,k_{12}\,k_{21}}{m_1\,m_2}}}{2} \qquad (3.45)$$

Similarly,

$$\omega_{3,4}^2 = \frac{\dfrac{k_{33}}{m_3} + \dfrac{k_{44}}{m_4} \pm \sqrt{\left(\dfrac{k_{33}}{m_3} - \dfrac{k_{44}}{m_4}\right)^2 + \dfrac{4\,k_{34}\,k_{43}}{m_3\,m_4}}}{2} \qquad (3.46)$$

$$\omega_5^2 = \frac{k_{55}}{m_5} \qquad (3.47)$$

$$\omega_6^2 = \frac{k_{66}}{m_6} \qquad (3.48)$$

The relative amplitudes of the coupled modes are found by substituting back into the reduced equations corresponding to (3.44). Thus, with

$$q_i = c_r\,\phi_i^{(r)} \sin(\omega_r t + \psi)$$

we have

$$-\omega_1^2 m_1 c_1 \phi_1^{(1)} + k_{11} c_1 \phi_1^{(1)} + k_{12} c_1 \phi_2^{(1)} = 0$$

from which

$$\phi_2^{(1)} = \frac{\omega_1^2 m_1 - k_{11}}{k_{12}} \phi_1^{(1)} \tag{3.49}$$

Similarly,

$$\phi_2^{(2)} = \frac{\omega_2^2 m_1 - k_{11}}{k_{12}} \phi_1^{(2)} \tag{3.50}$$

$$\phi_4^{(3)} = \frac{\omega_3^2 m_3 - k_{33}}{k_{34}} \phi_3^{(3)} \tag{3.51}$$

$$\phi_4^{(4)} = \frac{\omega_4^2 m_3 - k_{33}}{k_{34}} \phi_3^{(4)} \tag{3.52}$$

From equation (3.36) it can be seen that the coefficients f_r are given by

$$f_1 = \frac{F_1 \phi_1^{(1)} + F_2 \phi_2^{(1)}}{m_1[\phi_1^{(1)}]^2 + m_2[\phi_2^{(1)}]^2} \tag{3.53}$$

$$f_2 = \frac{F_1 \phi_1^{(2)} + F_2 \phi_2^{(2)}}{m_1[\phi_1^{(2)}]^2 + m_2[\phi_2^{(2)}]^2} \tag{3.54}$$

$$f_3 = \frac{F_3 \phi_3^{(3)} + F_4 \phi_4^{(3)}}{m_3[\phi_3^{(3)}]^2 + m_4[\phi_4^{(3)}]^2} \tag{3.55}$$

$$f_4 = \frac{F_3 \phi_3^{(4)} + F_4 \phi_4^{(4)}}{m_3[\phi_3^{(4)}]^2 + m_4[\phi_4^{(4)}]^2} \tag{3.56}$$

$$f_5 = \frac{F_5}{m_5 \phi_5^{(5)}} \tag{3.57}$$

$$f_6 = \frac{F_6}{m_6 \phi_6^{(6)}} \tag{3.58}$$

$$q_1 = \left[\frac{f_1}{(\omega_1^2-\omega^2)}\phi_1^{(1)} + \frac{f_2}{(\omega_2^2-\omega^2)}\phi_1^{(2)}\right]\sin(\omega t+\psi)$$

$$q_2 = \left[\frac{f_1}{(\omega_1^2-\omega^2)}\phi_2^{(1)} + \frac{f_2}{(\omega_2^2-\omega^2)}\phi_2^{(2)}\right]\sin(\omega t+\psi)$$

$$q_3 = \left[\frac{f_3}{(\omega_3^2-\omega^2)}\phi_3^{(3)} + \frac{f_4}{(\omega_4^2-\omega^2)}\phi_3^{(4)}\right]\sin(\omega t+\psi) \qquad (3.59)$$

$$q_4 = \left[\frac{f_3}{(\omega_3^2-\omega^2)}\phi_4^{(3)} + \frac{f_4}{(\omega_4^2-\omega^2)}\phi_4^{(4)}\right]\sin(\omega t+\psi)$$

$$q_5 = \frac{f_5}{(\omega_5^2-\omega^2)}\phi_5^{(5)}\sin(\omega t+\psi)$$

$$q_6 = \frac{f_6}{(\omega_6^2-\omega^2)}\phi_6^{(6)}\sin(\omega t+\psi)$$

Application of the above results to a numerical problem will be shown in Chapter VI.

The case of a foundation partially embedded in the soil may be treated in a similar manner. This case will be somewhat more complicated due to the introduction of additional static coupling terms.

CHAPTER IV

EQUIVALENT SOIL SPRING CONSTANTS

The methods developed in Chapter III for determining the modes and amplitudes of oscillation of a block foundation depend on the availability of suitable dynamic spring constants. It is the purpose of this chapter to develop relationships which may be used to calculate these constants. Unfortunately, from a mathematical standpoint, soil is not a homogeneous, isotropic, elastic material, and hence does not lend itself to rigorous mathematical treatment. Some simplifying assumptions are therefore made which, although they introduce some error into the final results, permit an approximate evaluation of the required soil constants under many different boundary conditions.

The bearing capacity of soils is determined by its shear strength. The shear strength is given by Coulomb's law which states that

$$S = C + N \tan \phi \tag{4.1}$$

where S is the maximum shearing resistance, C is the cohesion, N is the normal load and ϕ is the angle of internal friction. For cohesionless soils such as sands, C is relatively small and may be neglected. Since the normal load is proportional to the depth, the shearing strength also tends to be proportional to the depth. For very cohesive soils, such as clays, ϕ tends to be very small and the shearing strength is essentially constant. Since the effective soil modulus is approximately proportional to the shearing strength, the following assumptions are made in determining the effective spring constants:

a. For cohesionless soils the modulus of elasticity is proportional to the effective depth.

b. For cohesive soils the modulus of elasticity is constant. For intermediate soils the spring constants may be estimated by interpolation on the basis of Coulomb's law.

A further simplification is made in determining the stress distribution due to the dynamic load. In a homogeneous, isotropic, elastic material, the stress distribution may be determined by the well known Boussinesq equations. These equations are based on an assumed pressure distribution on the soil surface; they are also approximate since the exact pressure distribution is not known. In the following development of expressions for the effective spring factors, it is assumed that only a truncated cone or pyramid of soil directly under the foundation is effective in distributing the load, and that the stress distribution over any horizontal section is uniform.

The spring factor may be defined as the force exerted on the system when it is displaced a unit distance from the equilibrium position, or the moment, when rotated thru a unit angle. The dimensions of the spring factor are such that the product of the spring factor and the displacement has the dimension of work. Since a foundation in general has six degrees of freedom, there are six spring factors to be determined for each surface in contact with the soil.

Spring Factors for Horizontal Contact Surface - Cohesionless Soils.

Vertical Displacement (k_z^{xy})

Consider an elemental cube of dimensions dz subjected to a vertical load dP which causes a distortion $d\delta$.

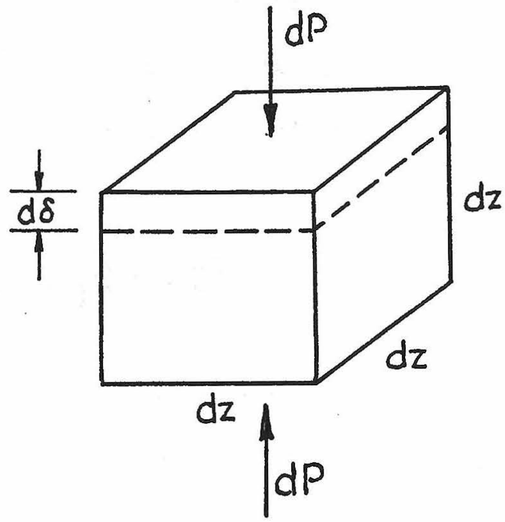


Figure 4.1

The elastic modulus is defined as

$$E = \frac{dP/(dz)^2}{d\delta/dz} = \frac{dP}{d\delta\, dz} \tag{4.2}$$

Consider next a rectangular area, of length a and width b, loaded with a uniform load, q.

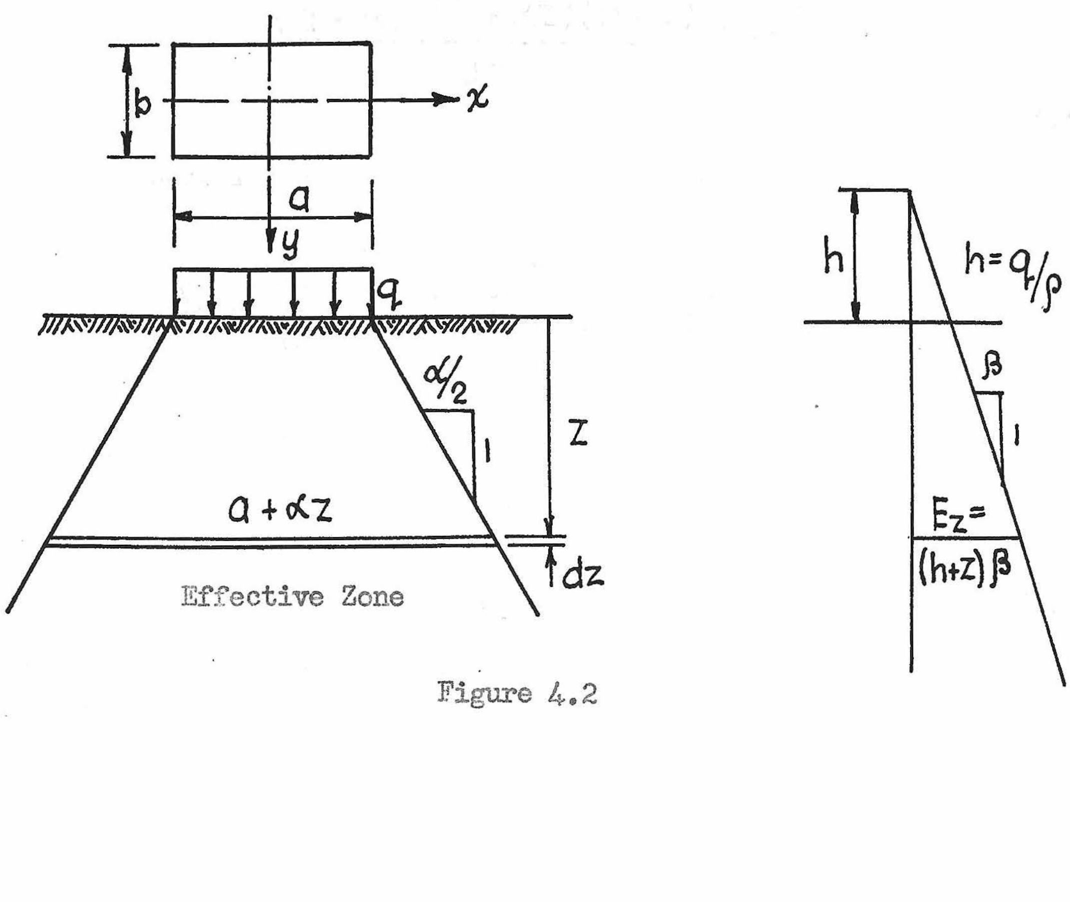


Figure 4.2

Let the effective zone be determined by the surface area ab and the planes sloping at an angle $\tan^{-1}\alpha/2$. Since the load acts as an equivalent surcharge, the effective soil modulus for any depth is given by the relation:

$$E(z) = \beta(h+z) \tag{4.3}$$

where β is the rate at which the modulus increases with depth, and h is the equivalent surcharge which is given by

$$h = \frac{q}{\rho} \tag{4.4}$$

where ρ is the unit weight of the soil. The total pressure on any horizontal section is then

$$P_z = \frac{(a+\alpha z)(b+\alpha z)}{dz^2} E\, dz\, d\delta$$

$$= \frac{(a+\alpha z)(b+\alpha z)(h+z)}{dz} \beta\, d\delta \tag{4.5}$$

The total surface deformation is then given by

$$\delta_z = \frac{P_z}{\beta} \int_0^\infty \frac{dz}{(a+\alpha z)(b+\alpha z)(h+z)} \tag{4.6}$$

Defining,

$$r = a/b, \qquad a \geqslant b$$

and

$$s = \alpha h/b \tag{4.7}$$

equation (4.6) can be rewritten in the form

$$\delta_z = \frac{P_z}{\beta b^2} \int_0^\infty \frac{dz}{(r+z)(1+z)(s+z)} \tag{4.8}$$

By definition the spring factor k_z^{xy} is given by the relation

$$k_z^{xy} = \frac{P_z}{\delta_z} \tag{4.9}$$

therefore

$$k_z^{xy} = \beta b^2 \gamma_z^{xy} \tag{4.10}$$

where

$$\frac{1}{\gamma_z^{xy}} = \int_0^\infty \frac{dz}{(r+z)(1+z)(s+z)} \tag{4.11}$$

Equation (4.11) must be evaluated for the following five cases:

$r \neq s \neq 1$; $r = 1$, $s \neq 1$; $r \neq 1$, $s = 1$; $r = s \neq 1$; $r = s = 1$

The evaluation of the integral (4.11) is given in Appendix A and the following results are obtained.

$$\gamma_z^{xy} = \frac{r-s}{\frac{\log s}{s-1} - \frac{\log r}{r-1}}, \quad r \neq s \neq 1$$

$$\gamma_z^{xy} = \frac{s-1}{1 - \frac{\log s}{s-1}}, \quad r=1,\ s \neq 1$$

$$\gamma_z^{xy} = \frac{r-1}{1 - \frac{\log r}{r-1}}, \quad s=1,\ r \neq 1$$

$$\gamma_z^{xy} = \frac{s(s-1)}{\frac{s \log s}{s-1} - 1}, \quad r = s \neq 1$$

$$\gamma_z^{xy} = 2, \quad r = s = 1 \tag{4.12}$$

Values of γ_z^{xy}/r as a function of s are plotted in Fig. 4.3 for several ratios of r.

It is also of some interest to consider the case of a long narrow foundation for which equation (4.5) becomes:

$$P = abq = a\beta \frac{(b+\alpha z)(h+z)}{dz} d\delta \tag{4.13}$$

The surface deformation for this case is then given by:

$$\delta = \frac{P}{\beta r b^2} \int_0^\infty \frac{dz}{(1+z)(s+z)} \tag{4.14}$$

For this case the value of γ_z^{xy} is

$$\frac{1}{\gamma_z^{xy}} = \frac{1}{r} \int_0^\infty \frac{dz}{(1+z)(s+z)} \tag{4.15}$$

The evaluation of this integral is also given in Appendix A, and the following results are obtained:

$$\gamma_z^{xy} = \frac{r}{\frac{\log s}{s-1}}, \qquad s \neq 1$$

$$\gamma_z^{xy} = r, \qquad s = 1 \tag{4.16}$$

The plot of γ_z^{xy}/r as r tends to infinity, is also shown in Fig. 4.3. In some applications circular footings are used. For this case equation (4.5) takes the form

$$P = \frac{\pi}{4} dq = \frac{\pi}{4} \frac{(d+\alpha z)^2 (h+z)}{dz} \beta d\delta \tag{4.17}$$

where d is the diameter of the foundation. It is readily seen that this equation reduces to the same form as (4.8) for the case r = 1, except for the constant $\pi/4$. For this case, therefore

$$k_z^{xy} = \frac{\pi}{4} \beta d^2 \gamma_z^{xy} \tag{4.18}$$

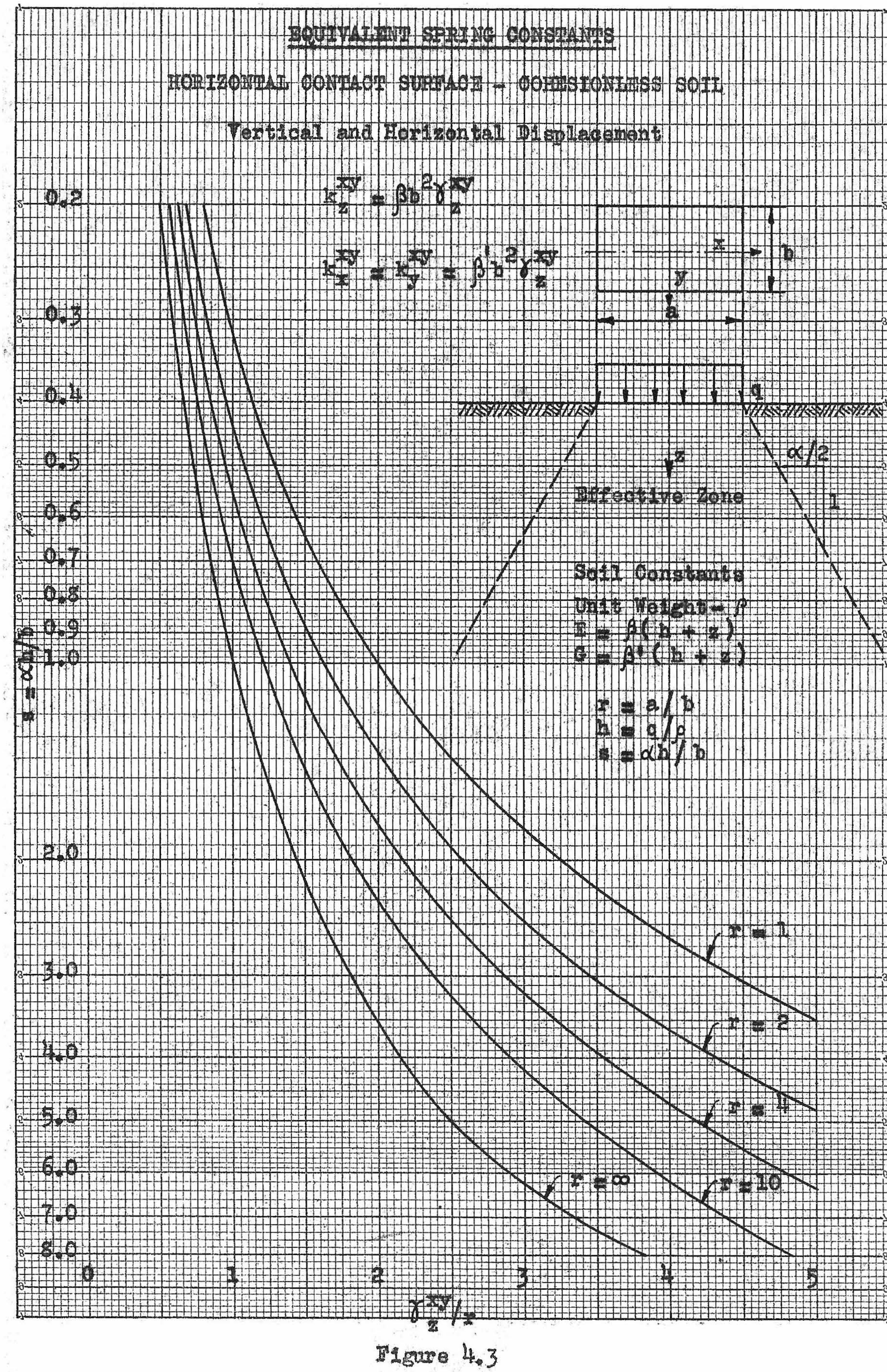

EQUIVALENT SPRING CONSTANTS
HORIZONTAL CONTACT SURFACE - COHESIONLESS SOIL
Vertical and Horizontal Displacement
$k_z^{xy} = \beta b^2 \gamma_z^{xy}$
$k_x^{xy} = k_y^{xy} = \beta' b^2 \gamma_z^{xy}$
x
y
b
a
q
z
$\alpha/2$
1
Effective Zone
Soil Constants
Unit Weight = ρ
$E = \beta(h + z)$
$G = \beta'(h + z)$
$r = a/b$
$h = q/\rho$
$s = \alpha h/b$
$s = \alpha h/b$
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
0
1
2
3
4
5
γ_z^{xy}/r
r = 1
r = 2
r = 4
r = 10
r = ∞

Figure 4.3

where γ_z^{xy} has the same values as obtained from (4.12) for the case r = 1.

Horizontal Displacement (k_x^{xy}, k_y^{xy})

Consider an elemental cube of dimensions dz subjected to a shearing force dF causing a distortion $d\tau$.

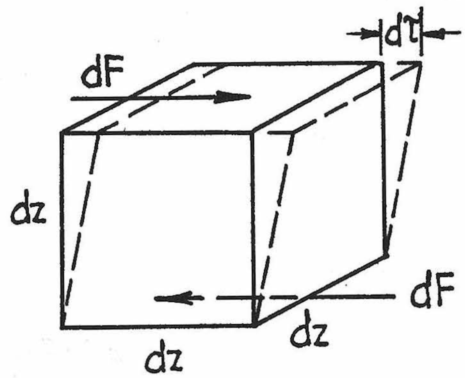


Figure 4.4

Here

$$G = \frac{dF/(dz)^2}{d\tau/dz} = \frac{dF}{dz\,d\tau} \tag{4.19}$$

where G is the shearing modulus and may be determined from the well known relation:[28]

$$G = \frac{E}{2(1+\mu)} \tag{4.20}$$

For sand, Poisson's ratio μ may be taken as 0.35.[15]

If G(z) is then defined as

$$G(z) = \beta'(h+z) \tag{4.21}$$

it is evident that β' is given by

$$\beta' = \frac{\beta}{2(1+\mu)} \tag{4.22}$$

The spring factor may therefore be calculated from the relation

$$k_x^{xy} = k_y^{xy} = \beta' b^2 \gamma_x^{xy} \tag{4.23}$$

where γ_x^{xy} is equal to the γ_z^{xy} of the previous section.

<u>Rotation about x-axis</u> (k_{yz}^{xy})

The case under consideration is that of rotation about a horizontal centroidal axis parallel to the long dimension a. (Cf. Fig. 4.2) It is assumed that horizontal planes are not distorted but remain plane after rotation.

The moment on any horizontal section is given by

$$M_x = \int_{\frac{-b-\alpha z}{2}}^{\frac{b+\alpha z}{2}} y^2 d\theta \left[\beta(h+z)\,dz \right] \frac{dy(a+\alpha z)}{(dz)^2}$$

$$= \beta \frac{(h+z)(a+\alpha z)}{dz} d\theta \int_{\frac{-b-\alpha z}{2}}^{\frac{b+\alpha z}{2}} y^2 d\theta$$

$$= \frac{\beta}{12} \frac{(h+z)(a+\alpha z)(b+\alpha z)^3}{dz} d\theta \tag{4.24}$$

The rotation of the contact surface is then

$$\theta_x = \frac{12 M_x}{\beta} \int_0^\infty \frac{dz}{(a+\alpha z)(h+z)(b+\alpha)^3} \tag{4.25}$$

Recalling that $r = a/b$ and $s = \alpha h/b$, equation (4.25) may be written in the form

$$\theta_x = \frac{12 M_x}{\beta b^4} \int_0^\infty \frac{dz}{(r+z)(s+z)(1+z)^3} \tag{4.26}$$

By definition, the spring factor is

$$k_{yz}^{xy} = \frac{M_x}{\theta_x} \tag{4.27}$$

therefore,

$$k_{yz}^{xy} = \beta b^4 \gamma_{YZ}^{XY} \tag{4.28}$$

where γ_{YZ}^{XY} is defined by the relation

$$\frac{1}{\gamma_{YZ}^{XY}} = 12\int_0^\infty \frac{dz}{(r+z)(s+z)(1+z)^3} \tag{4.29}$$

This integral is evaluated in Appendix B. The following expressions may be used for calculating the value of γ_{YZ}^{XY} :

$$\gamma_{YZ}^{XY} = \frac{1}{12}\left[\frac{(r-s)(s-1)(r-1)}{\frac{(r-1)}{(s-1)^2}\log s - \frac{(s-1)}{(r-1)^2}\log r + (r-s)\left(\frac{1}{2} - \frac{1}{r-1} - \frac{1}{s-1}\right)}\right], \qquad r \neq s \neq 1$$

$$\gamma_{YZ}^{XY} = \frac{1}{12}\left[\frac{(s-1)^3}{1 - \frac{s-1}{2} + \frac{(s-1)^2}{3} - \frac{\log s}{s-1}}\right], \qquad r = 1,\ s \neq 1$$

$$\gamma_{YZ}^{XY} = \frac{1}{12}\left[\frac{(r-1)^3}{1 - \frac{r-1}{2} + \frac{(r-1)^2}{3} - \frac{\log r}{r-1}}\right], \qquad r \neq 1,\ s = 1$$

$$\gamma_{YZ}^{XY} = \frac{1}{12}\left[\frac{(r-1)^3}{\frac{3\log r}{r-1} - \left(\frac{1}{r} + 2 - \frac{r-1}{2}\right)}\right], \qquad r = s \neq 1$$

$$\gamma_{YZ}^{XY} = \frac{1}{3}, \qquad r = s = 1 \tag{4.30}$$

Values of γ^{xy}_{yz}/r as a function of s are plotted in Fig. 4.5 for various ratios of r.

The case for a long narrow strip is also of interest. Equation (4.24) reduces to

$$M_x = \frac{a\beta}{12} \frac{(h+z)(b+\alpha z)^3}{dz} d\theta \tag{4.31}$$

and equation (4.26) reduces to

$$\theta_x = \frac{12 M_x}{r \beta b^4} \int_0^\infty \frac{dz}{(s+z)(1+z)^3} \tag{4.32}$$

This integral is also evaluated in Appendix B. The spring factor for this case is again determined by equation (4.28), where γ^{xy}_{yz} is given by

$$\gamma^{xy}_{yz} = \frac{1}{12}\left[\frac{r(s-1)^2}{\frac{\log s}{s-1} - \frac{3-s}{2}}\right], \qquad s \neq 1$$

$$\gamma^{xy}_{yz} = \frac{r}{4}, \qquad s = 1 . \tag{4.33}$$

γ^{xy}_{yz}/r for this case is plotted in Fig. 4.5.

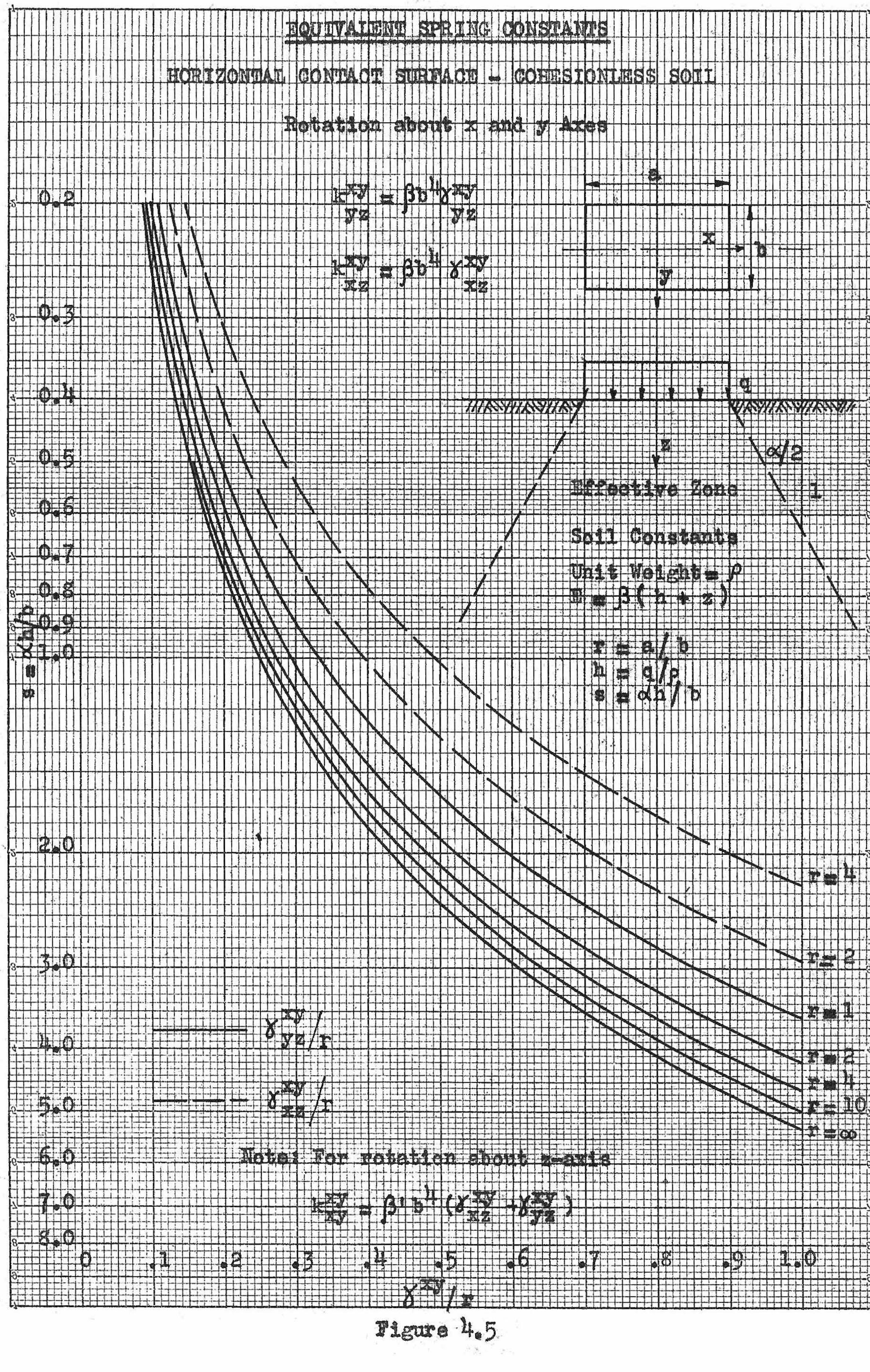

EQUIVALENT SPRING CONSTANTS
HORIZONTAL CONTACT SURFACE - COHESIONLESS SOIL
Rotation about x and y Axes
$k^{xy}_{yz} = \beta b^4 \gamma^{xy}_{yz}$
$k^{xy}_{xz} = \beta b^4 \gamma^{xy}_{xz}$
a
b
x
y
q
z
α/2
1
Effective Zone
Soil Constants
Unit Weight = ρ
E = β(h + z)
r = a/b
h = q/ρ
s = αh/b
s = αh/b
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
r = 4
r = 2
r = 1
r = 2
r = 4
r = 10
r = ∞
γ^{xy}_{yz}/r
γ^{xy}_{xz}/r
Note: For rotation about z-axis
$k^{xy}_{xy} = \beta' b^4 (\gamma^{xy}_{xz} + \gamma^{xy}_{yz})$
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
γ^{xy}/r

Figure 4.5

The spring factor for a circular plate is readily calculated.

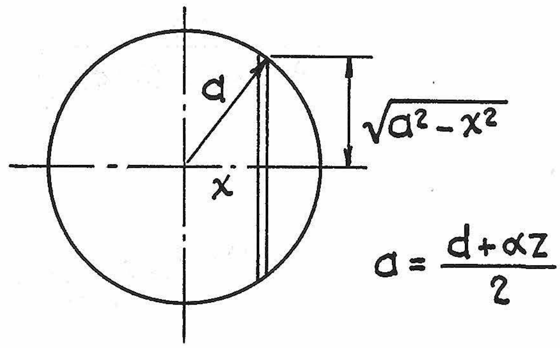


Figure 4.6

Referring to Fig. 4.6 it is seen that the moment is

$$M = \int_{-a}^{+a} x^2 \, d\theta \, \beta (h+z) \, dz \; \frac{2\sqrt{a^2-x^2}}{(dz)^2} \, dx$$

$$= 2\beta \frac{(h+z)}{dz} d\theta \int_{-a}^{+a} x^2 (a^2-x^2)^{\frac{1}{2}} \, dx$$

$$= 2\beta \frac{(h+z)}{dz} d\theta \left[-\frac{x t^3}{4} + \frac{a^2 x t}{8} + \frac{a^4}{8} \sin^{-1}\frac{x}{a} \right]_{-a}^{+a} \qquad (4.34)$$

$$= \frac{\pi}{4} \beta \frac{(h+z)}{dz} d\theta \, a^4$$

But, $$a = \frac{d+\alpha z}{2}$$

So

$$M = \frac{\pi}{64} \beta \, d\theta (h+z)(d+\alpha z)^4 .$$

Now if we define

$$s = \alpha h/d$$

then

$$\theta = \frac{64M}{\pi\beta d^4}\int_0^\infty \frac{dz}{(s+z)(1+\alpha z)^4} \tag{4.35}$$

Recalling equations (4.25) and (4.29) it is readily seen that

$$k_{yz}^{xy} = k_{xz}^{xy} = \frac{3\pi}{16}\beta b^4 \gamma_{YZ}^{XY} \tag{4.36}$$

where γ_{YZ}^{XY} is defined by equation (4.30) for the case r = 1.

<u>Rotation about y-axis</u> (k_{xz}^{xy})

For this case the moment equation is

$$M_y = \int_{\frac{-a-\alpha}{2}}^{\frac{a+\alpha}{2}} x^2\, d\theta\, \beta\,(h+z)\,dz\; \frac{dx\,(b+\alpha z)}{(dz)^2}$$

$$= \frac{\beta}{12}\,\frac{(h+z)(b+\alpha z)(a+\alpha z)^3}{dz}\,d\theta \tag{4.37}$$

With $r = a/b$, and $s = \alpha h/b$, the equation for θ becomes

$$\theta_y = \frac{12 M_y}{b^4}\int_0^\infty \frac{dz}{(s+z)(1+z)(r+z)^3}$$

For this case the spring factor is therefore calculated from

$$k_{xz}^{yz} = \beta b^4 \gamma_{xz}^{xy} \tag{4.38}$$

where γ_{xz}^{xy} is defined by the relation

$$\frac{1}{\gamma_{xz}^{xy}} = 12\int_0^\infty \frac{dz}{(s+z)(1+z)(r+z)^3} \tag{4.39}$$

Integral (4.39) is evaluated in Appendix B. The following relations may be used to calculate γ_{xz}^{xy}

$$\gamma_{xz}^{xy} = \frac{1}{12}\left[\frac{(r-s)(s-1)(r-1)}{\frac{r-1}{(r-s)^2}\log\frac{s}{r} + \frac{r-s}{(r-1)^2}\log r + \frac{s-1}{r}\left(\frac{1}{2} + \frac{1}{r-s} + \frac{1}{r-1}\right)}\right], \quad r \neq s \neq 1$$

$$\gamma_{xz}^{xy} = \frac{1}{12}\left[\frac{(s-1)^3}{\left(1 - \frac{s-1}{2} + \frac{(s-1)^2}{3}\right) - \frac{\log s}{s-1}}\right], \quad r=1,\ s\neq 1$$

$$\gamma_{xz}^{xy} = \frac{1}{12}\left[\frac{(r-1)^3}{-\frac{3}{r-1}\log r - \left(1 - \frac{2}{r} + \frac{r-1}{2r^2}\right)}\right], \quad r\neq 1,\ s=1$$

$$\gamma_{xz}^{xy} = \frac{1}{12}\left[\frac{(r-1)^3}{\frac{\log r}{r-1} - \left(\frac{1}{r} + \frac{r-1}{2r^2} + \frac{(r-1)^2}{3r^2}\right)}\right], \quad r = s \neq 1$$

$$\gamma_{xz}^{xy} = \frac{1}{3}, \quad r = s = 1 \tag{4.40}$$

In Fig. 4.5 γ_{xz}^{xy}/r for several values of r is plotted.

<u>Rotation about z-axis</u> (k_{xy}^{xy})

Recalling equation (4.21) for the shearing modulus, the moment on any horizontal section can be shown to be:

$$M_z = \frac{\beta'(h+z)}{dz}\left[2\int_0^{\frac{a+\alpha z}{2}}(b+\alpha z)\,x^2\,d\theta\,dx + 2\int_0^{\frac{b+\alpha z}{2}}(a+\alpha z)\,y^2\,d\theta\,dy\right]$$

$$= \frac{1}{12}\,\frac{\beta'(h+z)}{dz}\,d\theta\left[(b+\alpha z)(a+\alpha z)^3 + (a+\alpha z)(b+\alpha z)^3\right] \tag{4.41}$$

Equation (4.41) may be written

$$\left.\begin{aligned} M_z &= M_z' + M_z'' \\ \text{where} \quad M_z' &= \frac{1}{12}\beta' \frac{(h+z)(b+\alpha z)(a+\alpha z)^3}{dz} d\theta \\ \text{and} \quad M_z'' &= \frac{1}{12}\beta' \frac{(h+z)(a+\alpha z)(b+\alpha z)^3}{dz} d\theta \end{aligned}\right\} \tag{4.42}$$

The spring factor is given by:

$$k_{xy}^{xy} = \frac{M_z}{\theta_z} = \frac{M_z'}{\theta_z} + \frac{M_z''}{\theta_z} \tag{4.43}$$

$$k_{xy}^{xy} = \beta' b^4 (\gamma' + \gamma'') = \beta' b^4 \gamma_{xy}^{xy} \tag{4.44}$$

where

$$\frac{1}{\gamma'} = 12 \int_0^\infty \frac{dz}{(s+z)(1+z)(r+z)^3} \tag{4.45}$$

and

$$\frac{1}{\gamma''} = 12 \int_0^\infty \frac{dz}{(s+z)(r+z)(1+z)^3} \tag{4.46}$$

It is readily verified that integral (4.45) is identical to (4.39), and integral (4.46) to (4.29). It follows therefore, that

$$\gamma_{xy}^{xy} = \gamma' + \gamma'' = \gamma_{xz}^{xy} + \gamma_{yz}^{xy} \tag{4.47}$$

where γ_{xz}^{xy} is given by equations (4.40) and γ_{yz}^{xy} by equations (4.30). The ratios γ_{xz}^{xy}/r and γ_{yz}^{xy}/r are plotted for several values of r in Fig. 4.5.

Spring Factors for Horizontal Contact Surface - Cohesive Soils.

Vertical Displacement (k_z^{xy})

The derivation of the spring factors for cohesive soils is similar to the derivation for cohesionless soils, except that the modulus is assumed to be constant instead of increasing with depth. Equation (4.5) is replaced by

$$P = abq = \frac{E(a+\alpha z)(b+\alpha z)}{dz} d\delta \tag{4.48}$$

The total surface deformation is then

$$\delta = \frac{P}{E}\int_0^\infty \frac{dz}{(a+\alpha z)(b+\alpha z)}$$

$$= \frac{P}{\alpha E b}\int_0^\infty \frac{dz}{(r+z)(1+z)} \tag{4.49}$$

where r is again defined as a/b. Recalling equation (4.9), the spring factor is given by

$$k_z^{xy} = E\alpha b\, \gamma_z^{xy} \tag{4.50}$$

where γ_z^{xy} is defined by the relation

$$\frac{1}{\gamma_z^{xy}} = \int_0^\infty \frac{dz}{(r+z)(1+z)} \tag{4.51}$$

The evaluation of (4.51) is similar to the evaluation of the integral (4.15) which is given in Appendix A. Therefore

$$\begin{aligned} \gamma_z^{xy} &= \frac{r-1}{\log r}, \quad r \neq 1 \\ \gamma_z^{xy} &= 1, \quad r = 1 \end{aligned} \tag{4.52}$$

Horizontal Displacement (k_x^{xy}, k_y^{xy})

As in the case for cohesionless soil, the only change is the substitution of G for E, where G is defined by equation (4.20). Data for Poisson's ratio for cohesive soils is not available; for saturated cohesive soils, $\mu = 0.50$ would be a reasonable assumption. The spring factor is calculated from:

$$k_x^{xy} = k_y^{xy} = G \alpha b \gamma_z^{xy} \tag{4.53}$$

where γ_z^{xy} is given by equation (4.52).

Rotation about x-axis (k_{yz}^{xy})

The moment on any horizontal section is

$$M_x = \int_{\frac{-b-\alpha z}{2}}^{\frac{b+\alpha z}{2}} E\, y^2 \, d\theta \, dz \; \frac{dy(a+\alpha z)}{(dz)^2}$$

$$= \left[E \frac{(a+\alpha z)}{dz} d\theta \frac{x^3}{3} \right]_{\frac{-b-\alpha z}{2}}^{\frac{b+\alpha z}{2}}$$

$$= \frac{E}{12} \frac{(a+\alpha z)(b+\alpha z)^3}{dz} d\theta \tag{4.54}$$

The rotation of the contact surface is therefore

$$\theta_x = \frac{12 M_x}{E} \int_0^\infty \frac{dz}{(a+\alpha z)(b+\alpha z)^3}$$

$$= \frac{12 M_x}{\alpha E b^3} \int_0^\infty \frac{dz}{(r+z)(1+z)^3} \tag{4.55}$$

The spring factor is

$$k_{yz}^{xy} = \frac{M_x}{\theta_x} = \alpha E b^3 \gamma_{yz}^{xy} \tag{4.56}$$

where γ_{yz}^{xy} is defined by the integral

$$\frac{1}{\gamma_{yz}^{xy}} = 12\int_0^\infty \frac{dz}{(r+z)(1+z)^3} \tag{4.57}$$

This integral is of the same form as the integral in equation (4.32) which is evaluated in Appendix B. γ_{yz}^{xy} may be calculated from the equations

$$\gamma_{yz}^{xy} = \frac{1}{12}\left[\frac{(r-1)^2}{\frac{\log r}{r-1} - \frac{3-r}{2}}\right], \quad r \neq 1$$

$$\gamma_{yz}^{xy} = \frac{1}{4}, \quad r = 1 \tag{4.58}$$

Rotation about y-axis (k_{yz}^{xy})

For this case, the moment on any horizontal section is

$$M_y = \frac{E}{12}\,\frac{(b+\alpha z)(d+\alpha z)^3}{dz}\,d\theta \tag{4.59}$$

and the rotation of the contact surface is

$$\theta_y = \frac{12\,M_y}{\alpha E b^3}\int_0^\infty \frac{dz}{(1+z)(r+z)^3} \tag{4.60}$$

The spring factor is

$$k_{xz}^{xy} = \frac{M_y}{\theta_y} = \alpha E b^3 \gamma_{xz}^{xy} \tag{4.61}$$

where γ_{xz}^{xy} is defined by the integral

$$\frac{1}{\gamma_{xz}^{xy}} = 12\int_0^\infty \frac{dz}{(r+z)^3(1+z)} \tag{4.62}$$

This integral is evaluated in Appendix C and yields the following expressions for γ_{xz}^{xy}:

$$\gamma_{xz}^{xy} = \frac{1}{12}\left[\frac{(r-1)^2}{\frac{\log r}{r-1} - \frac{3r-1}{2r^2}}\right], \quad r \neq 1$$

$$= \frac{1}{4}, \quad r = 1 \tag{4.63}$$

<u>Rotation about z-axis</u> (k_{xy}^{xy})

For this case the moment on any horizontal plane is given by:

$$M_z = G\,dz \int_{\frac{-a-\alpha z}{2}}^{\frac{a+\alpha z}{2}} \int_{\frac{-b-\alpha z}{2}}^{\frac{b+\alpha z}{2}} (x^2+y^2)\, d\theta \, \frac{dy\,dx}{(dz)^2}$$

$$= \frac{4}{3}\frac{d\theta}{dz} G\left[x^3y + xy^3\right]_0^{\frac{a+\alpha z}{2},\ \frac{b+\alpha z}{2}}$$

$$= \frac{G}{12}\frac{d\theta}{dz}\left\{(b+\alpha z)(a+\alpha z)^3 + (a+\alpha z)(b+\alpha z)^3\right\} \tag{4.64}$$

$$M_z = M_z' + M_z''$$

where

$$M_z' = \frac{G}{12}\frac{(b+\alpha z)(a+\alpha z)^3}{dz}d\theta$$

$$M_z'' = \frac{G}{12}\frac{(a+\alpha z)(b+\alpha z)^3}{dz}d\theta \tag{4.65}$$

Therefore,

$$k_{xy}^{xy} = \frac{M_z}{\theta_z} = \frac{M_z'}{\theta_z} + \frac{M_z''}{\theta_z}$$

$$= \alpha G b^3(\gamma' + \gamma'') \tag{4.66}$$

where

$$\frac{1}{\gamma'} = 12\int_0^\infty \frac{dz}{(1+z)(r+z)^3} \tag{4.67}$$

and

$$\frac{1}{\gamma''} = 12\int_0^\infty \frac{dz}{(r+z)(1+z)^3} \tag{4.68}$$

Equation (4.67) is seen to be identical to equation (4.62) and equation (4.68) is identical to equation (4.57). Therefore $\gamma' = \gamma_{xz}^{xy}$, and may be evaluated from equation (4.63); $\gamma'' = \gamma_{yz}^{xy}$, and is given by equation (4.58). γ/r for cohesive soil is plotted in Fig. 4.7.

<u>Spring Factors for Vertical Contact Surface - Cohesionless Soils.</u>

<u>Horizontal Displacement Normal to Contact Surface</u> (k_y^{xz})

The case of a contact surface parallel to the xz plane will be considered. The equations for a contact surface parallel to the yz plane are similar except that y replaces x and b replaces a wherever they appear in the equations.

Consider the lateral pressure on a foundation block. Let the

EQUIVALENT SPRING CONSTANTS

HORIZONTAL CONTACT SURFACE - COHESIVE SOIL

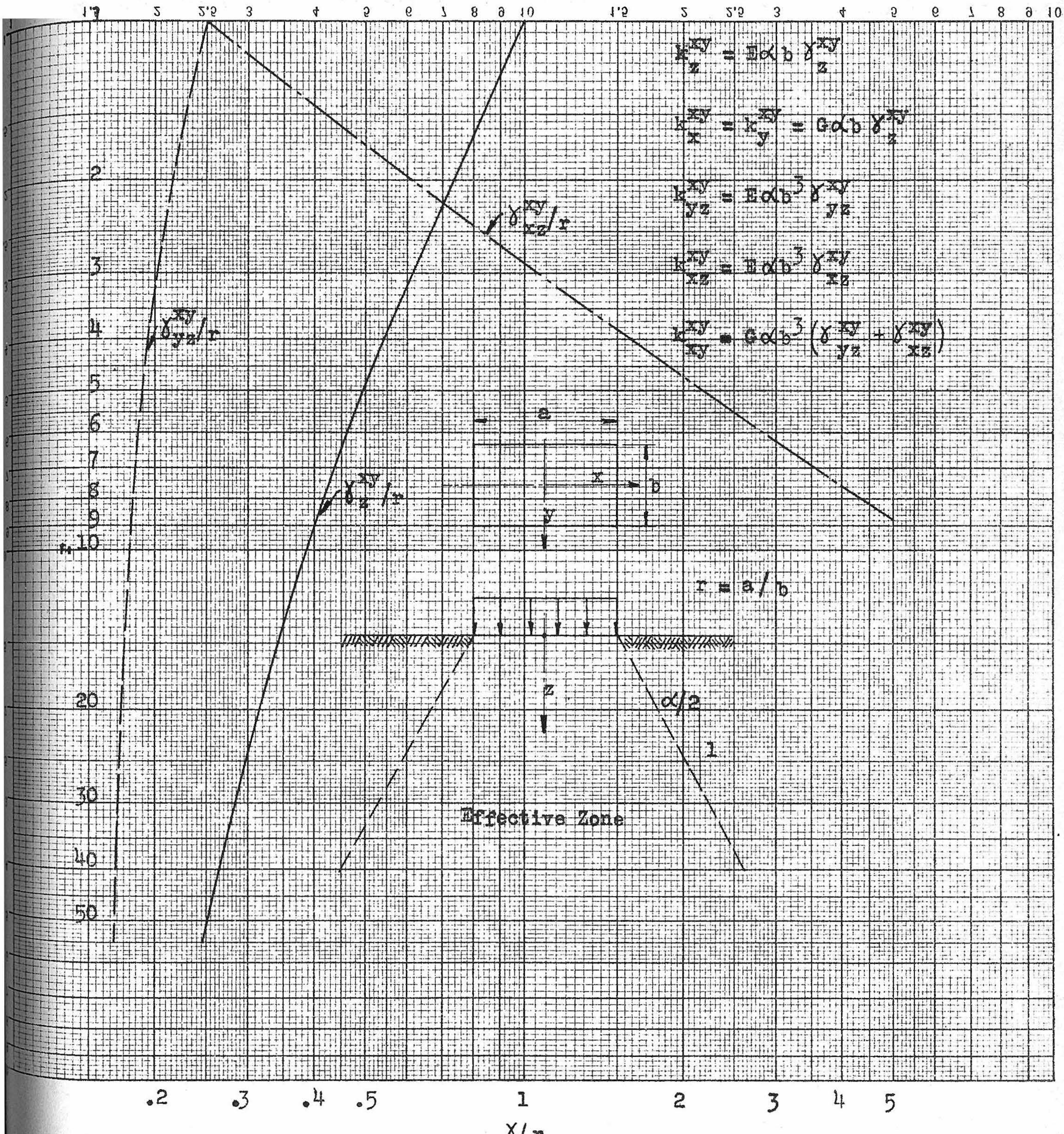


Figure 4.7

depth of embedment be D_f and the length of the contact surface be a. The soil modulus is given by

$$E = \beta z \tag{4.69}$$

where z is measured from the surface of the soil.

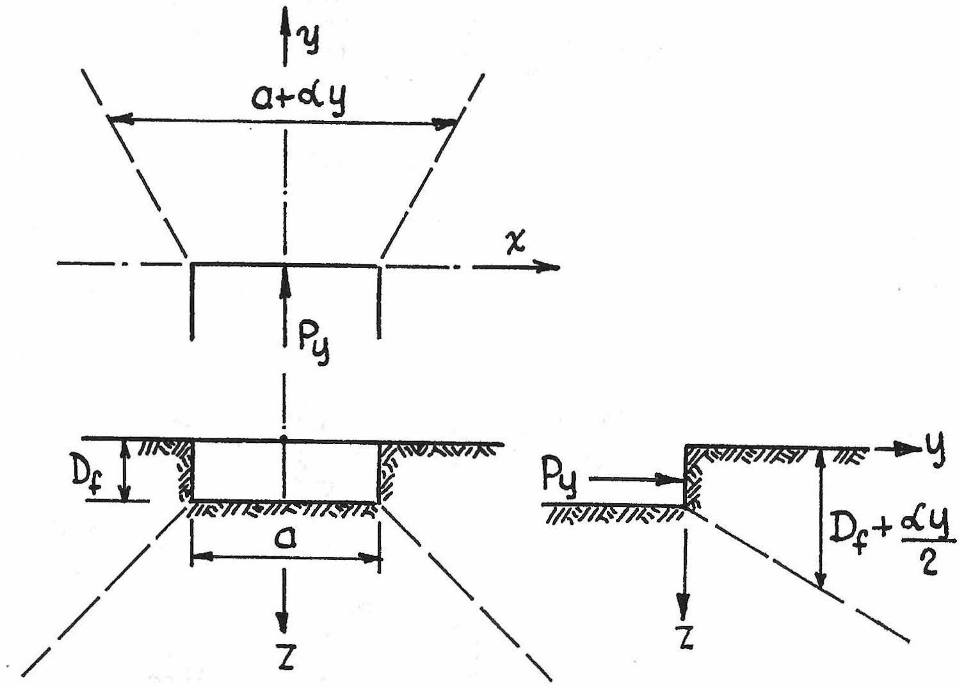


Figure 4.8

Recalling equation (4.2),

$$dP = E\,dz\,d\delta = \beta z\,dz\,d\delta$$

the normal pressure on any section of the effective zone parallel to the contact surface is

$$P_y = \frac{a+\alpha y}{dy}\int_0^{\frac{2D+\alpha y}{2}} \beta\, d\delta\, z\, dz$$

$$= \beta\,\frac{a+\alpha y}{dy}\,d\delta\left[\frac{(2D_f+\alpha y)^2}{8}\right]$$

$$= \frac{\beta}{8}\,\frac{(a+\alpha y)(2D_f+\alpha y)^2 d\delta}{dy} \tag{4.70}$$

Defining

$$t = \frac{2D_f}{a} \tag{4.71}$$

we obtain for the displacement of the contact surface

$$\delta_y = \frac{8 P_y}{\alpha \beta a^2} \int_0^\infty \frac{dy}{(1+y)(t+y)^2} \tag{4.72}$$

Note that a uniform horizontal displacement is assumed. The resultant pressure, P_y, will therefore act at a distance, $D_f/3$, from the base of the foundation, since E is a linear function of z. The spring factor is given by

$$k_y^{xz} = \frac{P_y}{\delta_y} = \alpha \beta a^2 \gamma_y^{xz} \tag{4.73}$$

where γ_y^{xz} is defined by

$$\frac{1}{\gamma_y^{xz}} = 8 \int_0^\infty \frac{dy}{(1+y)(t+y)^2} \tag{4.74}$$

This integral is of similar form to the integral evaluated in Appendix A. (Cf. equation (A.7)). γ_y^{xz} is therefore equal to

$$\gamma_y^{xz} = \frac{1}{8} \left[\frac{t(t-1)}{\frac{t \log t}{t-1} - 1} \right], \qquad t \neq 1$$

$$\gamma_y^{xz} = \frac{1}{4}, \qquad t = 1 \tag{4.75}$$

A plot of γ_y^{xz} is shown in Fig. 4.9.

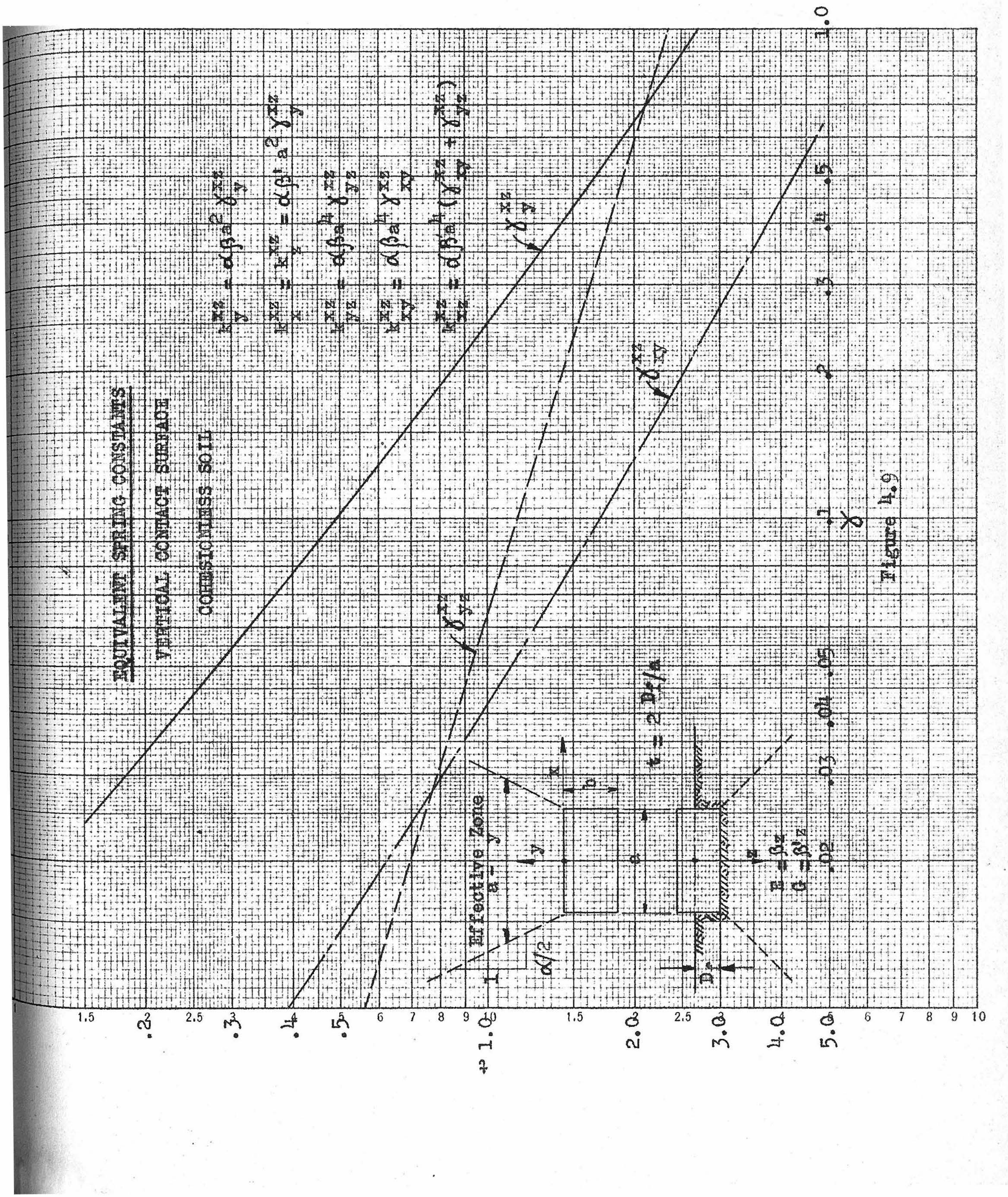

EQUIVALENT SPRING CONSTANTS
VERTICAL CONTACT SURFACE
COHESIONLESS SOIL
$k_y^{xz} = \alpha\beta a^2 \gamma_y^{xz}$
$k_x^{xz} = k_y^{xz} = \alpha\beta' a^2 \gamma_y^{xz}$
$k_{yz}^{xz} = \alpha\beta a^4 \gamma_{yz}^{xz}$
$k_{xy}^{xz} = \alpha\beta a^4 \gamma_{xy}^{xz}$
$k_{xz}^{xz} = \alpha\beta' a^4 (\gamma_{xy}^{xz} + \gamma_{yz}^{xz})$
γ_y^{xz}
γ_{xy}^{xz}
γ_{yz}^{xz}
$t = 2 D_f/a$
Effective Zone
$\alpha/2$
$E = \beta z$
$G = \beta' z$
D_f
γ
t

Figure 4.9

Lateral Displacement (k_x^{xz} , k_z^{xz})

The only change in determining the spring factor for this case is that G is substituted for E. The spring factor is therefore

$$k_x^{xz} = k_z^{xz} = \alpha \beta' a^2 \gamma_y^{xz} \tag{4.76}$$

where β' is

$$\beta' = \frac{\beta}{2(1+\mu)} \tag{4.22}$$

and γ_y^{xz} is given by equations (4.75).

Rotation about x-axis (k_{yz}^{xz})

The moment on any vertical section of the effective zone is:

$$M_x = \frac{a+\alpha y}{dy} \int_0^{\frac{2D_f+\alpha y}{2}} z^2 \, d\theta \, (\beta z \, dz)$$

$$= \frac{\beta}{64} \frac{(a+\alpha y)(2D_f+\alpha y)^4}{dy} d\theta \tag{4.77}$$

The rotation of the contact surface is

$$\theta_x = \frac{64 \, M_x}{\alpha \beta a^4} \int_0^\infty \frac{dy}{(1+y)(t+y)^4} \tag{4.78}$$

hence the spring factor is

$$k_{yz}^{xz} = \frac{M_x}{\theta_x} = \alpha \beta a^4 \gamma_{yz}^{xz} \tag{4.79}$$

where γ_{yz}^{xz} is determined from the relation

$$\frac{1}{\gamma_{yz}^{xz}} = 64 \int_0^\infty \frac{dy}{(1+y)(t+y)^4} \tag{4.80}$$

Referring to (B.19) and (B.20) of Appendix B it is readily seen that:

$$\gamma_{yz}^{xz} = 64 \left[\frac{(t-1)^3}{\frac{\log t}{t-1} - \left\{ \frac{1}{t} + \frac{(t-1)}{2t^2} + \frac{(t-1)^2}{3t^3} \right\}} \right] , \quad t \neq 1$$

$$= \frac{1}{16} , \qquad t = 1 \tag{4.81}$$

See Fig. 4.9 for a plot of γ_{yz}^{xz} .

<u>Rotation about z-axis</u> (k_{xy}^{xz})

$$M_z = \frac{2}{dy} \int_0^{\frac{a+\alpha y}{2}} x^2 \, d\theta \, dx \int_0^{\frac{2D_f+\alpha y}{2}} \beta z \, dz$$

$$= \frac{\beta d\theta}{96} \frac{(a+\alpha y)^3 (2D_f+\alpha y)^2}{dy} \tag{4.82}$$

The rotation of the contact plane is

$$\theta_z = \frac{96 M_z}{\beta} \int_0^\infty \frac{dy}{(a+\alpha y)^3 (2D_f+\alpha y)^2}$$

$$= \frac{96 M_z}{\alpha \beta a^4} \int_0^\infty \frac{dy}{(1+y)^3 (t+y)^2} \tag{4.83}$$

The spring factor is

$$k_{xy}^{xz} = \frac{M_z}{\Theta_z} = \alpha\beta a^4 \gamma_{xy}^{xz} \tag{4.84}$$

where γ_{xy}^{xz} is defined by

$$\frac{1}{\gamma_{xy}^{xz}} = 96\int_0^\infty \frac{dy}{(1+y)^3(t+y)^2} \tag{4.85}$$

This integral is evaluated in Appendix B. (Cf. (B.10), (B.7)).

Therefore

$$\gamma_{xy}^{xz} = \frac{1}{96}\left[\frac{(t-1)^3}{\frac{3\log t}{t-1} - \left\{\frac{1}{t} + 2 - \frac{(t-1)}{2}\right\}}\right], \quad t \neq 1$$

$$\gamma_{xy}^{xz} = \frac{1}{24}, \qquad t = 1 \tag{4.86}$$

γ_{xy}^{xz} is plotted in Fig. 4.9.

<u>Rotation about y-axis</u> (k_{xz}^{xz})

Recalling that $G = \beta' z\, dz$, the moment on a section of the effective zone parallel to the contact plane is

$$M_y = \frac{2}{dy}\int_0^{\frac{a+\alpha y}{2}}\int_0^{\frac{2D_f+\alpha y}{2}} (x^2+z^2)\, d\theta\, \beta' z\, dz\, dx$$

$$= \frac{\beta' d\theta}{dy}\left[\frac{(a+\alpha y)^3(2D_f+\alpha y)^2}{96} + \frac{(a+\alpha y)(2D_f+\alpha y)^4}{}\right] \tag{4.87}$$

Equation (4.87) may also be written

$$M_y = M_y' + M_y''$$

where

$$M_y' = \frac{\beta' d\theta}{96} \frac{(a+\alpha y)^3 (2D_f + \alpha y)^2}{dy}$$

$$M_y'' = \frac{\beta' d\theta}{64} \frac{(a+\alpha y)(2D_f + \alpha y)^4}{dy} \tag{4.88}$$

The spring factor is

$$k_{xz}^{xz} = \frac{M_y}{\theta_y} = \frac{M_y'}{\theta_y} + \frac{M_y''}{\theta_y} \tag{4.89}$$

Therefore

$$k_{xz}^{xz} = \alpha \beta' a^4 (\gamma' + \gamma'') = \alpha \beta' a^4 \gamma_{xz}^{xz} \tag{4.90}$$

where γ' and γ'' are defined by the equations

$$\frac{1}{\gamma'} = 96 \int_0^\infty \frac{dy}{(1+y)^3 (t+y)^2} \tag{4.91}$$

and

$$\frac{1}{\gamma''} = 64 \int_0^\infty \frac{dy}{(1+y)(t+y)^4} \tag{4.92}$$

Equations (4.91) and (4.92) are identical to equations (4.85) and (4.80) respectively. Therefore $\gamma' = \gamma_{XY}^{XZ}$, and may be calculated from equation (4.86); $\gamma'' = \gamma_{YZ}^{XZ}$, and may be calculated from equation (4.81).

Spring Factors for Vertical Contact Surface - Cohesive Soils.

Displacement Normal to Contact Surface (k_y^{xz})

For this case the soil modulus is assumed constant; hence

$$P_y = \frac{(a+\alpha y)(2D_f+\alpha y)}{2\,dy}\,E\,d\delta \tag{4.93}$$

Recalling that $t = 2\,D_f/a$,

$$\delta_y = \frac{2P_y}{\alpha E a}\int_0^\infty \frac{dy}{(1+y)(t+y)} \tag{4.94}$$

The spring factor is

$$k_y^{xz} = \frac{P_y}{\delta_y} = \alpha E a\,\gamma_y^{xz} \tag{4.95}$$

where γ_y^{xy} is defined by

$$\frac{1}{\gamma_y^{xz}} = 2\int_0^\infty \frac{dy}{(1+y)(t+y)} \tag{4.96}$$

Referring to (A.11), it may be shown that

$$\gamma_y^{xz} = \frac{1}{2}\left[\frac{t-1}{\log t}\right], \quad t \neq 1$$

$$\gamma_y^{xz} = \frac{1}{2}, \quad t = 1 \tag{4.97}$$

γ_y^{xz} is plotted in Fig. 4.10.

Lateral Displacement (k_x^{xz}, k_z^{xz})

The only change for this case is that G replaces E in equation

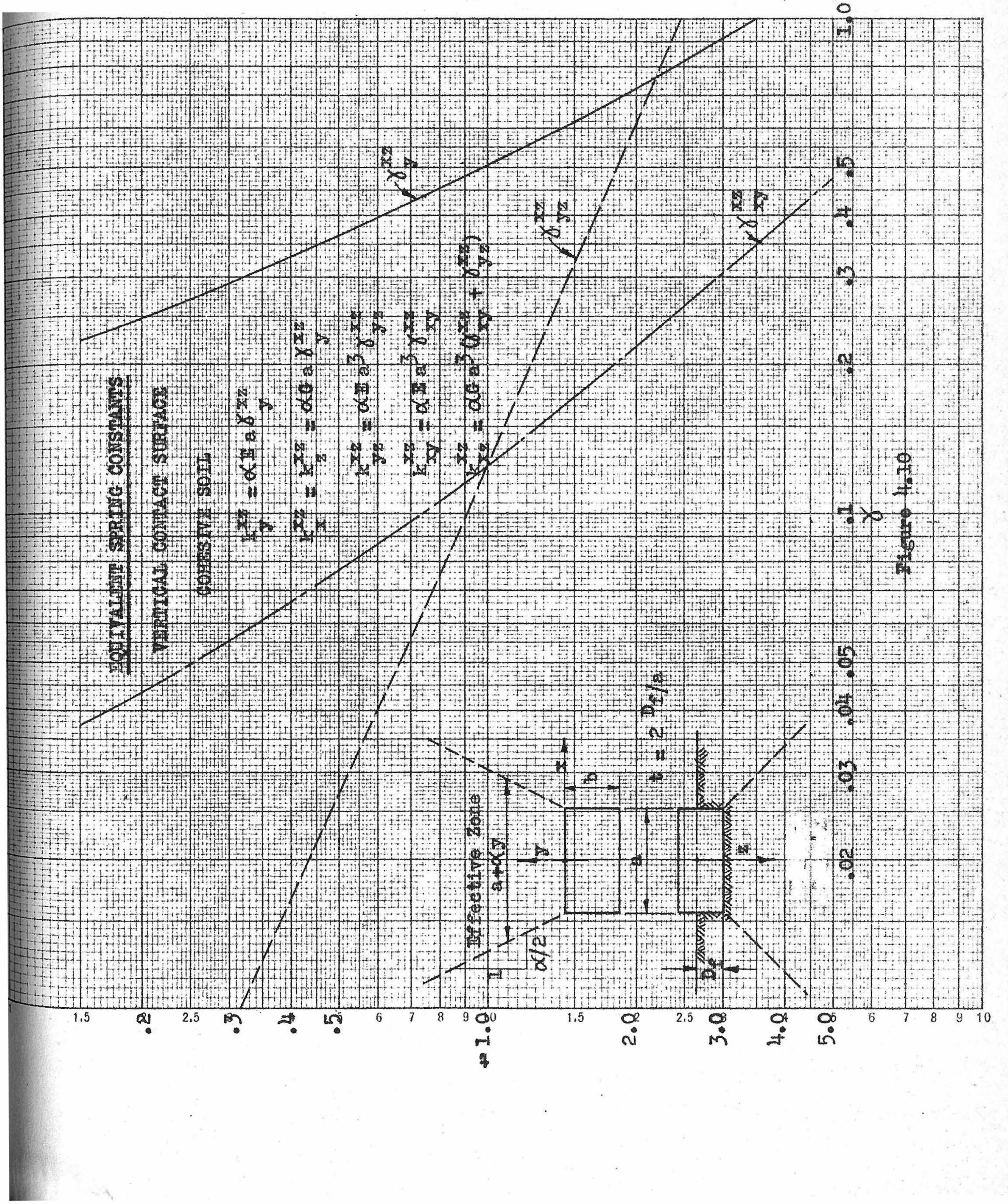

EQUIVALENT SPRING CONSTANTS
VERTICAL CONTACT SURFACE
COHESIVE SOIL
$k_y^{xz} = \alpha E a \gamma_y^{xz}$
$k_x^{xz} = k_z^{xz} = \alpha G a \gamma_y^{xz}$
$k_{yz}^{xz} = \alpha E a^3 \gamma_{yz}^{xz}$
$k_{xy}^{xz} = \alpha E a^3 \gamma_{xy}^{xz}$
$k_{xz}^{xz} = \alpha G a^3 (\gamma_{xy}^{xz} + \gamma_{yz}^{xz})$
γ_y^{xz}
γ_{yz}^{xz}
γ_{xy}^{xz}
Effective Zone
$a + \alpha y$
$\alpha/2$
$t = 2\ D_f/a$
D_f
γ

Figure 4.10

(4.95), thus

$$k_x^{xz} = k_z^{xz} = \alpha\, G\, a\, \gamma_y^{xz} \tag{4.98}$$

<u>Rotation about x-axis</u> (k_{yz}^{xz})

The moment for any vertical section of the effective zone parallel to the contact surface is

$$M_x = \frac{a+\alpha y}{dy} \int_0^{\frac{2D_f+\alpha y}{2}} z^2\, d\theta\, E\, dz$$

$$= \frac{(a+\alpha y)(2D_f+\alpha y)^3}{24\, dy} E\, d\theta \tag{4.99}$$

The rotation of the contact surface is therefore

$$\theta_x = \frac{24 M_x}{\alpha E a^3} \int_0^{\infty} \frac{dy}{(1+y)(t+y)^3} \tag{4.100}$$

The spring factor is

$$k_{yz}^{xz} = \frac{M_x}{\theta_x} = \alpha E\, a^3\, \gamma_{yz}^{xz} \tag{4.101}$$

where γ_{yz}^{xz} is defined by

$$\frac{1}{\gamma_{yz}^{xz}} = 24 \int_0^{\infty} \frac{dy}{(1+y)(t+y)^3} \tag{4.102}$$

Now (4.102) is of identical form to (4.62): therefore

$$\gamma_{yz}^{xz} = \frac{1}{24}\left[\frac{(t-1)^2}{\frac{\log t}{(t-1)} - \frac{(3t-1)}{2t^2}}\right], \quad t \neq 1 \qquad (4.103)$$

$$= \frac{1}{8}, \quad t = 1$$

γ_{yz}^{xz} is plotted in Fig. 4.10.

Rotation about z-axis (k_{xy}^{xz})

The moment is

$$M_z = \frac{2}{dy}\int_0^{\frac{a+\alpha y}{2}} x^2\, d\theta \; \frac{2D_f + \alpha y}{2}\, E\, dx$$

$$= \frac{(2D_f + \alpha y)(a+\alpha y)^3}{24\, dy}\, E\, d\theta \qquad (4.104)$$

and the surface rotation is

$$\theta_z = \frac{24 M_z}{\alpha E a^3}\int_0^{\infty} \frac{dy}{(t+y)(1+y)^3} \qquad (4.105)$$

The spring factor is

$$k_{xy}^{xz} = \alpha E a^3\, \gamma_{xy}^{xz} \qquad (4.106)$$

where γ_{xy}^{xz} is defined by

$$\frac{1}{\gamma_{xy}^{xz}} = 24\int_0^{\infty} \frac{dy}{(t+y)(1+y)^3} \tag{4.107}$$

Equation (4.107) is of identical form to equation (4.57); it follows that

$$\gamma_{xy}^{xz} = \frac{1}{24}\left[\frac{(t-1)^2}{\frac{\log t}{(t-1)} - \frac{(3-t)}{2}}\right], \quad t \neq 1$$

$$\gamma_{xy}^{xz} = \frac{1}{8}, \qquad t = 1 \tag{4.108}$$

γ_{xy}^{xz} is plotted in Fig. 4.10.

<u>Rotation about y-axis</u> (k_{xz}^{xz})

The twisting moment on any vertical section of the effective zone parallel to the contact plane is

$$M_y = \frac{2}{dy}\int_0^{\frac{a+\alpha y}{2}}\int_0^{\frac{2D_f+\alpha y}{2}} (x^2+z^2)\, d\theta\, G\, dz\, dx$$

$$= \frac{G\, d\theta}{24\, dy}\left[(a+\alpha y)^3(2D_f+\alpha y)+(a+\alpha y)(2D_f+\alpha y)^3\right] \tag{4.109}$$

$$M_y = M_y' + M_y''$$

where

$$M_y' = \frac{G}{24}\left[\frac{(a+\alpha y)^3(2D_f+\alpha y)}{dy}\right] d\theta \tag{4.110}$$

and

$$M_y'' = \frac{G}{24}\left[\frac{(a+\alpha y)(2D_f+\alpha y)^3}{dy}\right] d\theta$$

(4.110a)

The spring constant therefore is

$$k_{xz}^{xz} = \frac{M_y}{\theta_y} = \frac{M_y'}{\theta_y} + \frac{M_y''}{\theta_y}$$

$$= \alpha G a^3(\gamma' + \gamma'') = \alpha G a^3 \gamma_{xz}^{xz}$$

(4.111)

It may be shown that $\gamma' = \gamma_{xy}^{xz}$, and $\gamma'' = \gamma_{yz}^{xz}$, and therefore can be calculated from equations (4.108) and (4.103) respectively.

<u>General Remarks on the Use of Equivalent Spring Factors.</u>

In the above development it has been assumed that negative soil stresses with respect to the normal stress state can occur. Cohesionless soils cannot take tensile stress. The results are therefore not valid if these negative stresses exceed the initial dead load compressive stresses in the soil. For this case the k values are not constant but will decrease with increased displacement amplitudes and the resulting oscillation will be non-linear. This phenomenon is especially likely to occur in foundations subject to horizontal dynamic forces. Figure 4.11 shows the cracks which developed in the soil adjacent to a test block when large periodic horizontal forces were applied. Under normal operating conditions, however, only small displacements can be tolerated and therefore k may be assumed constant.

Figure 4.11

If the soil is subject to shrinkage, shrinkage cracks can be expected to develop. The effective depth of the surcharge is reduced with a corresponding change in the k values. The behavior of foundations on this type of soil can be expected to vary with seasonal fluctuations of soil moisture.

The values of β, α, μ, G, and E must be obtained experimentally, preferably by the use of dynamic tests. It must be remembered that the spring factors calculated by the expressions developed in this chapter are based on an approximate theory and therefore extreme accuracy cannot be expected in predicting the behavior of a foundation under all conditions. The principal problem in analyzing a foundation is the determination of the critical frequencies. In general one attempts to design a foundation in such a manner that the frequency components of the exciting forces do not coincide with the critical frequencies of the foundation. In many design problems an approximate determination of the critical frequencies is sufficient.

By considering parallel or series combinations of equivalent springs, this theory can be used to calculate effective spring factors for odd shaped foundations or for those which rest on non-uniform soils. In this respect the theory is quite flexible in its application to problems encountered in practice.

CHAPTER V

APPARENT MASS

In order to use the results of Chapters II and III, a knowledge of the equivalent spring factors is not sufficient; it is also necessary to estimate the effect of the mass of the body of soil which participates in the vibrations of the system. Recalling equation (2.7)

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

it can be seen that in the equivalent system, m consists of the mass of the machine and the foundation block, plus some unknown mass representing the effect of the soil. The foundation-soil system may be considered analogous to a foundation of mass m oscillating on damped springs of mass m_s. However, for purposes of analysis it is convenient to replace these springs by weightless springs and an apparent mass, m', which is added to the mass of the foundation. We may then rewrite the frequency equation as

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m+m'}} \tag{5.1}$$

A method of determining the apparent mass experimentally will be discussed in Chapter VII.

Horizontal Contact Surface - Cohesionless Soil.

Apparent Mass

An estimate of the apparent mass m' can be made under certain conditions, by equating the kinetic energy of the apparent mass to the kinetic energy of the soil in the effective zone as defined in the previous chapter. Therefore:

$$T' = \frac{m'\dot{\delta}_z^2}{2} = \frac{m'\omega^2\delta_z^2}{2} = \frac{1}{2}\frac{\rho\omega^2}{g}\int_0^\infty A(z)\left[\delta(z)\right]^2 dz$$

Solving for m' we have:

$$m' = \frac{\rho}{g}\int_0^\infty \frac{A(z)\left[\delta(z)\right]^2 dz}{\delta_z^2} \tag{5.2}$$

where A(z) is the area of the effective zone at depth z and is given by

$$A(z) = (a+\alpha z)(b+\alpha z) \tag{5.3}$$

and δ (z) is the displacement of a section of the zone at depth z.

Recalling equation (4.5) and solving for $\delta(z)$ we have:

$$\delta(z) = \frac{P_z}{\beta}\int_z^\infty \frac{dz}{(a+\alpha z)(b+\alpha z)(h+z)} \tag{5.4}$$

Again defining r = a/b, s = α h/b and with x = α z/b, we obtain on substitution of equations (5.3) and (5.4) in equation (5.2)

$$m' = \frac{\rho}{g}\frac{b^3}{\alpha}\;\frac{\displaystyle\int_0^\infty (r+x)(1+x)\left[\int_x^\infty \frac{dx}{(r+x)(1+x)(s+x)}\right]^2 dx}{\left[\displaystyle\int_0^\infty \frac{dx}{(r+x)(1+x)(s+x)}\right]^2} \tag{5.5}$$

The general solution of equation (5.5) has not been obtained, but the special case, r = s = 1, is readily solved. For this case equation (5.5) reduces to

$$m' = \frac{\rho}{g}\frac{b^3}{\alpha}\int_0^\infty \frac{dx}{(1+x)^2}$$

(5.6)

$$= \frac{\rho}{g}\frac{b^3}{\alpha}\ , \quad r = s = 1$$

(5.7)

Approximate solutions of equation (5.5) may be obtained for finite values of r and s by numerical or graphical integration to a finite limit. It is convenient to select a limit, z, as some multiple of α/b, the magnitude of the limit depending on the rate of convergence.

For the case of an infinitely long strip the infinite integral for the apparent mass per unit length does not converge on a finite value. For this case the expression for the apparent mass per unit length is:

$$m' = \frac{\rho}{g}\frac{\int_0^\infty (b+\alpha z)\left[\int_z^\infty \frac{dz}{(b+\alpha z)(h+z)}\right]^2 dz}{\left[\int_0^\infty \frac{dz}{(b+\alpha z)(h+z)}\right]^2}$$

$$= \frac{\rho}{g}\frac{b^2}{\alpha}\frac{\int_0^\infty (1+x)\left[\int_x^\infty \frac{dx}{(1+x)(s+x)}\right]^2 dx}{\left[\int_0^\infty \frac{dx}{(1+x)(s+x)}\right]^2}$$

(5.8)

For the special case, s = 1, equation (5.8) reduces to

$$m' = \frac{\rho}{g}\,\frac{b^2}{\alpha}\int_0^{\infty}\frac{dx}{(1+x)}$$

$$= \frac{\rho}{g}\,\frac{b^2}{\alpha}\log(1+x)\Big]_{x=0}^{x=\infty} \tag{5.9}$$

<u>Apparent Mass Moment of Inertia</u>

The apparent mass moment of inertia may be estimated in a similar manner. For this case the kinetic energy is given by

$$T' = \frac{I'}{2}\dot{\theta}_z^2 = \frac{I'\omega^2\theta_z^2}{2} = \frac{1}{2}\,\frac{\rho}{g}\,\omega^2\int_0^{\infty} I(z)\left[\theta(z)\right]^2 dz$$

Solving for I' we have

$$I' = \frac{\rho}{g}\,\frac{\int_0^{\infty} I(z)\left[\theta(z)\right]^2 dz}{\theta_z^2} \tag{5.10}$$

where I(z) is the moment of inertia of a horizontal section of the effective zone at depth z and is given by

$$I_x(z) = \frac{(a+\alpha z)(b+\alpha z)^3}{12}$$

or (5.11)

$$I_y(z) = \frac{(a+\alpha z)^3(b+\alpha z)}{12}$$

and $\theta(z)$ is the rotation of a section of the effective zone at depth z. Recalling equations (4.24) and (4.37), we have

$$\theta_x(z) = \frac{12\,M_x}{\beta}\int_z^\infty \frac{dz}{(a+\alpha z)(h+z)(b+\alpha z)^3}$$

$$\theta_y(z) = \frac{12\,M_y}{\beta}\int_z^\infty \frac{dz}{(b+\alpha z)(h+z)(a+\alpha z)^3} \tag{5.12}$$

Recalling that r = a/b, and s = α h/b, and with x = α z/b, we obtain on substitution of (5.11) and (5.12) in equation (5.10)

$$I_x' = \frac{\rho b^5}{12 g\alpha}\,\frac{\displaystyle\int_0^\infty (r+x)(1+x)^3\left[\int_x^\infty \frac{dx}{(r+x)(s+x)(1+x)^3}\right]^2 dx}{\left[\displaystyle\int_0^\infty \frac{dx}{(r+x)(s+x)(1+x)^3}\right]^2} \tag{5.13}$$

$$I_y' = \frac{\rho b^5}{12 g\alpha}\,\frac{\displaystyle\int_0^\infty (1+x)(r+x)^3\left[\int_x^\infty \frac{dx}{(r+x)^3(s+x)(1+x)}\right]^2 dx}{\left[\displaystyle\int_0^\infty \frac{dx}{(r+x)^3(s+x)(1+x)}\right]^2} \tag{5.14}$$

Equations (5.13) and (5.14) can again be readily solved for the special case, r = s = 1, for which case

$$I_x' = I_y' = \frac{\rho b^5}{12 g\alpha}\int_0^\infty \frac{dx}{(1+x)^4}$$

$$= \frac{1}{36}\,\frac{\rho b^5}{g\alpha}\,, \qquad r = s = 1 \tag{5.15}$$

For other values of r and s, equations (5.13) and (5.14) may again be solved by numerical integration to a finite limit. It should be noted that convergence is much more rapid for this case.

Horizontal Contact Surface- Cohesive Soil

Apparent Mass

Since, for cohesive soils, the modulus of elasticity of the soil has been assumed constant, the velocity of propagation of stress is constant and does not increase with depth as in cohesionless soils. Consequently, wave propagation theory rather than static displacements must be used for the calculation of apparent mass.* For, if the procedure of the previous section is applied to the case of cohesive soils, it is found that the resulting infinite integral does not converge on a finite value. Recalling equation (4.48) we have

$$\delta(z) = \frac{P}{E}\int_z^{\infty} \frac{dz}{(a+\alpha z)(b+\alpha z)}$$

which on substitution in equation (5.2) results in

$$m' = \frac{\rho b^3}{g\alpha} \frac{\int_0^{\infty}(r+x)(1+x)\left[\int_x^{\infty}\frac{dx}{(r+x)(1+x)}\right]^2 dx}{\left[\int_0^{\infty}\frac{dx}{(r+x)(1+x)}\right]^2} \tag{5.16}$$

For the case r = 1, equation (5.16) reduces to

$$m' = \frac{\rho b^3}{g\alpha}\int_0^{\infty} dx \tag{5.17}$$

---

* Solutions based on this theory have been obtained for a few special cases by E. Reissner (14) and by P. Quinlan (15).

hence

$$m' = \frac{\rho b^3}{g\alpha} \left. x \right]_0^\infty , \quad r = 1 \tag{5.18}$$

Very little data is available for cohesive soils; there is, however, some indication that the apparent mass may be very large.[17]

Mass Moment of Inertia.

Referring to equations (4.54) and (4.59), it may be verified that for cohesive soils

$$\theta_x(z) = \frac{12 M_x}{E} \int_z^\infty \frac{dz}{(a+\alpha z)(b+\alpha z)^3}$$

and (5.19)

$$\theta_y(z) = \frac{12 M_y}{E} \int_z^\infty \frac{dz}{(a+\alpha z)^3(b+\alpha z)}$$

Substituting in equation (5.10) we have

$$I_x' = \frac{\rho b^5}{12 g\alpha} \, \frac{\int_0^\infty (r+x)(1+x)^3 \left[\int_x^\infty \frac{dx}{(r+x)\,(1+x)^3} \right]^2 dx}{\left[\int_0^\infty \frac{dx}{(r+x)\,(1+x)^3} \right]^2} \tag{5.20}$$

and

$$I_y' = \frac{\rho b^5}{12 g\alpha} \, \frac{\int_0^\infty (r+x)^3(1+x) \left[\int_x^\infty \frac{dx}{(r+x)^3\,(1+x)} \right]^2 dx}{\left[\int_0^\infty \frac{dx}{(r+x)^3\,(1+x)} \right]^2} \tag{5.21}$$

For the case r = 1, equations (5.20) and (5.21) reduce to

$$I_x' = I_y' = \frac{\rho b^5}{12 g \alpha} \int_0^\infty \frac{dx}{(1+x)^2}$$

$$= \frac{\rho b^5}{12 g \alpha} \quad , \quad r=1 \tag{5.22}$$

Vertical Contact Surface - Cohesionless Soil

Apparent Mass

When the foundation is partially or entirely embedded in the soil, the apparent mass and the mass moment of inertia are increased due to the motion of the soil adjacent to the foundation. Consider first the apparent mass due to a vertical contact surface in the xz-plane, of height D_f and width a. The expression for apparent mass for this case is

$$m' = \frac{\rho}{2g} \frac{\int_0^\infty (2D_f + \alpha y)(a + \alpha y)\left[\delta(y)\right]^2 dy}{\left[\delta_y\right]^2} \tag{5.23}$$

Referring to equation (4.70) we have for δ (y)

$$\delta(y) = \frac{8 P_y}{\beta} \int_y^\infty (a + \alpha y)(2D_f + \alpha y)^2 dy \tag{5.24}$$

Defining $t = 2D_f/a$, and with $x = \alpha y/a$, we have on substitution of (5.24) in (5.23)

$$m' = \frac{\rho a^3}{2g\alpha} \frac{\int_0^\infty (t+x)(1+x)\left[\int_x^\infty \frac{dx}{(1+x)(t+x)^2}\right]^2 dx}{\left[\int_0^\infty \frac{dx}{(1+x)(t+x)^2}\right]^2} \tag{5.25}$$

For the special case t = 1, equation (5.25) reduces to

$$m' = \frac{\rho a^3}{2g\alpha}\int_0^\infty \frac{dx}{(1+x)^2}$$

$$= \frac{\rho a^3}{2g\alpha}\,, \qquad t=1 \tag{5.26}$$

Moment of Inertia.

For a vertical contact surface, equation (5.13) takes the form

$$I' = \frac{\rho}{g}\,\frac{\int_0^\infty I(y)\,[\theta(y)]^2\,dy}{\theta^2} \tag{5.27}$$

where I(y) is given by

$$I_z(y) = \frac{(a+\alpha y)^3(2D_f+\alpha y)}{24}$$

or (5.28)

$$I_x(y) = \frac{(a+\alpha y)(2D_f+\alpha y)^3}{24}$$

Referring to equation (4.82) and (4.77) we have for θ,

$$\theta_z(y) = \frac{96M_z}{\beta}\int_y^\infty \frac{dy}{(a+\alpha y)^3(2D_f+\alpha y)^2}$$

and (5.29)

$$\theta_x(y) = \frac{64M_x}{\beta}\int_y^\infty \frac{dy}{(a+\alpha y)(2D_f+\alpha y)^4}$$

Recalling that $t = 2D_f/a$, and with $x = \alpha y/a$, we obtain on substitution of (5.28) and (5.29) in equation (5.27)

$$I_z' = \frac{\rho a^5}{24 g \alpha} \frac{\int_0^\infty (t+x)(1+x)^3 \left[\int_x^\infty \frac{dx}{(1+x)^3 (t+x)^2} \right]^2 dx}{\left[\int_0^\infty \frac{dx}{(1+x)^3 + (t+x)^2} \right]^2} \tag{5.30}$$

and

$$I_x' = \frac{\rho a^5}{24 g \alpha} \frac{\int_0^\infty (t+x)^3 (1+x) \left[\int_x^\infty \frac{dx}{(1+x)(t+x)^4} \right]^2 dx}{\left[\int_0^\infty \frac{dx}{(1+x)(t+x)^4} \right]^2} \tag{5.31}$$

For the special case $t = 1$

$$I_z' = I_x' = \frac{\rho a^5}{24 g \alpha} \int_0^\infty \frac{dx}{(1+x)^4}$$

$$= \frac{\rho a^5}{72 g \alpha} \quad , \quad t = 1 \tag{5.32}$$

<u>Vertical Contact Surface - Cohesive Soil</u>

<u>Apparent Mass</u>.

Referring to equation (4.93) we have for $\delta(z)$

$$\delta(z) = \frac{2P_y}{E} \int_z^\infty \frac{dy}{(a + \alpha y)(2D_f + \alpha y)} \tag{5.33}$$

Substituting into equation (5.23) and recalling that $t = 2D_f/a$ and $x = \alpha y/a$, we obtain

$$m' = \frac{\rho a^3}{2g\alpha} \frac{\int_0^\infty (t+x)(1+x)\left[\int_x^\infty \frac{dx}{(1+x)(t+x)}\right]^2 dx}{\left[\int_0^\infty \frac{dx}{(1+x)(t+x)}\right]^2} \tag{5.34}$$

As in the case of cohesive soil and a horizontal contact surface, the infinite integral does not converge, for with $t = 1$, we have

$$m' = \frac{\rho a^3}{2g\alpha}\int_0^\infty dx = \frac{\rho a^3}{2g\alpha}\, x\Big]_0^\infty \tag{5.35}$$

Moment of Inertia.

For cohesive soil, we have from equations (4.104) and (4.99)

$$\Theta_z(y) = \frac{24 M_z}{E}\int_y^\infty \frac{dy}{(2D_f+\alpha y)(a+\alpha y)^3}$$

and (5.36)

$$\Theta_x(y) = \frac{24 M_x}{E}\int_y^\infty \frac{dy}{(a+\alpha y)(2D_f+\alpha y)^3}$$

On substitution in equation (5.27) we obtain

$$I_z' = \frac{\rho a^5}{24 g\alpha} \frac{\int_0^\infty (1+x)^3(t+x)\left[\int_x^\infty \frac{dx}{(1+x)^3(t+x)}\right]^2 dx}{\left[\int_0^\infty \frac{dx}{(1+x)^3(t+x)}\right]^2} \tag{5.37}$$

and

$$I_x' = \frac{\rho a^5}{24 g\alpha} \frac{\int_0^\infty (1+x)(t+x)^3\left[\int_x^\infty \frac{dx}{(1+x)(t+x)^3}\right]^2 dx}{\left[\int_0^\infty \frac{dx}{(1+x)(t+x)^3}\right]^2} \tag{5.38}$$

For t = 1, (5.37) and (5.38) reduce to

$$I_z' = I_x' = \frac{\rho a^5}{24 g \alpha} \int_0^\infty \frac{dx}{(1+x)^2}$$

$$I_z' = I_x' = \frac{\rho a^5}{24 g \alpha}\,, \qquad t = 1 \tag{5.39}$$

<u>General Remarks on the Use of Apparent Mass and Mass Moment of Inertia Values</u>.

The restrictions imposed by soil and load conditions which apply to the determination of the spring factors, also apply to the apparent mass terms. Indeed, from the derivation it is clear that the spring factor and the apparent mass of a foundation are intimately connected. It must be kept in mind that the theory only approximates the actual stress conditions in the soil and that significant errors may be introduced in calculating the apparent mass terms. Moreover, the accuracy of the calculation depends on the rate of convergence of the infinite integral. It has been pointed out that for cohesive soils with constant modulus of elasticity, the velocity of propagation of stress is constant. For this case, the infinite integrals obtained by the use of static displacements do not converge, and wave propagation theory must be used in the calculations. Unfortunately, very little data is available for cohesive soils at the present time and it is impossible to check the accuracy of the theory for this case.

Soils whose modulus is constant with depth are seldom encountered in nature. Even in normally loaded clays, there is some increase of modulus with depth due to the consolidation of the lower layers by the weight of the overburden. For these cases the expressions for cohesionless soil may be used by assuming an imaginary value of h, such that the soil modulus is given by the equation

$$E = \beta(h + z) \tag{5.40}$$

The integrals which arise in determining the apparent mass terms are much more difficult to evaluate than the ones for the equivalent spring factors. For those cases where convergence is sufficiently rapid, numerical or graphical integration to a finite limit may be used. An example of this procedure will be given in Chapter VII. Special problems arising from non-uniform soil conditions or odd foundation shapes can also be treated by this method.

CHAPTER VI

EXPERIMENTAL INVESTIGATION

Purpose and Scope.

The purpose of this investigation was to determine the effect of several parameters on the behavior of a foundation under the action of a periodic external force. The parameters investigated were:

1. Size and shape of contact area.
2. Relative magnitude of the external force.
3. Direction of the force.
4. Weight of the foundation.
5. Depth of embedment in the ground.

Since variation of the above parameters resulted in a large number of tests, it was decided to investigate the foundation behavior for only one type of soil, namely a clean, well-graded sand. The reason for this choice is that this type of soil has proven to be most susceptible to vibration problems in actual installations. The behavior of the foundation was determined by measuring amplitudes of vertical and horizontal oscillation, and by measuring the dynamic soil reaction on the base of the foundation.

From these measurements the following information was obtained.

1. Critical frequency of the excited modes of oscillation.
2. Amplitude of oscillation as a function of the frequency ratio.
3. Estimate of the dynamic soil constants.
    a. Effective spring factor
    b. Equivalent damping factor
    c. Apparent mass of the soil

4. Dynamic soil pressure.

   a. Pressure distribution and magnitude

   b. Relationship between pressure and displacement amplitudes.

5. A measure of the non-linearity of the system.

The data obtained was used to test the accuracy of the theoretical development.

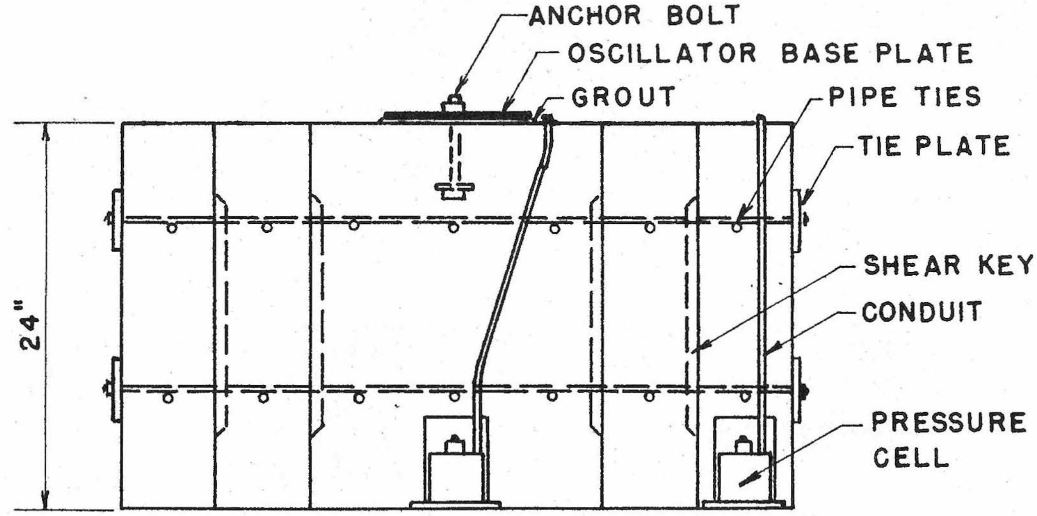
Description of the Test Site.

The test site consisted of a pit ten feet square by six feet deep and filled with washed concrete sand. This pit had previously been used for an experimental investigation of soil compaction and the soil characteristics were therefore known.[(15)] A large steel tripod and a two ton hoist were used to handle the heavy vibrator unit and to lift the foundation block.

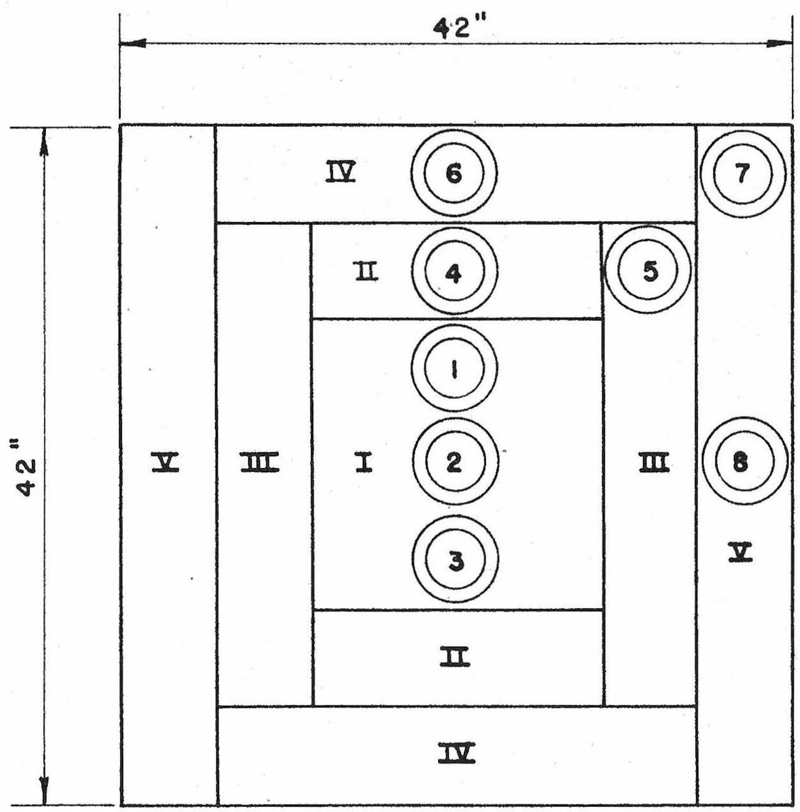
Test Block.

The concrete foundation block was cast in five increments, so that tests could be made after the addition of each increment. The basic size was 18 inches square by 24 inches high. This block was then increased in size by the addition of 6 inch sections cast on each side. The casting sequence is shown in Fig. 6.1. The foundation was reinforced with 1/2 inch diameter steel pipe. Couplings were used to extend the pipe when additional sections were cast on. Extensible forms were constructed, so that only one set of forms was required for all pours. (Fig. 6.2). To insure that the block would act as a unit, shear keys were provided, and tie rods were inserted thru the pipe and bolted to tie plates. A base plate for the oscillator was fastened to the foundation by means of two 3/4 inch anchor bolts. A thin layer of grout

TEST BLOCK

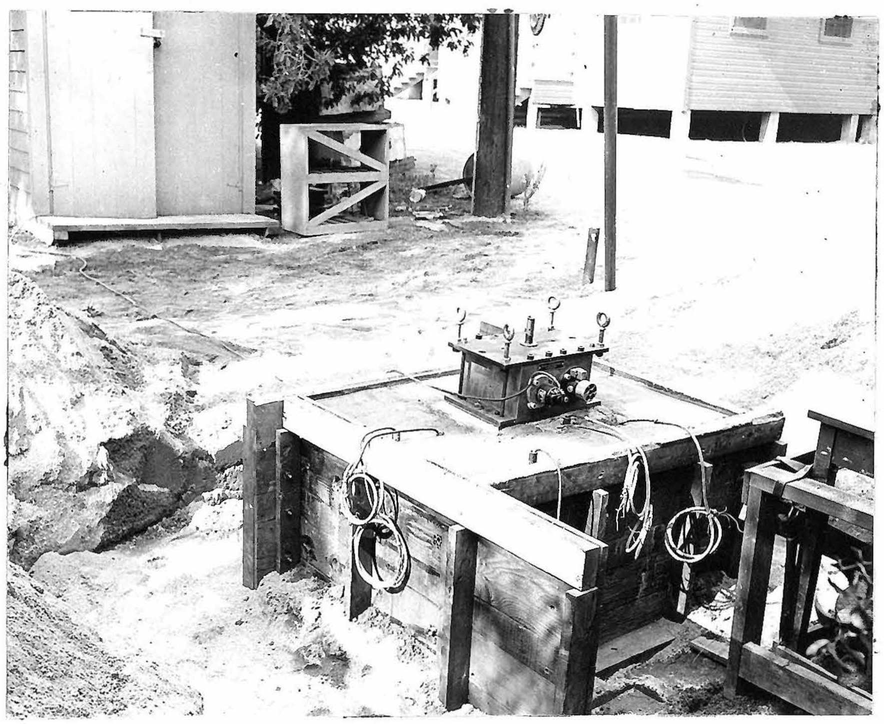


SECTION



CASTING SEQUENCE
AND PRESSURE CELL LOCATION

Figure 6.1



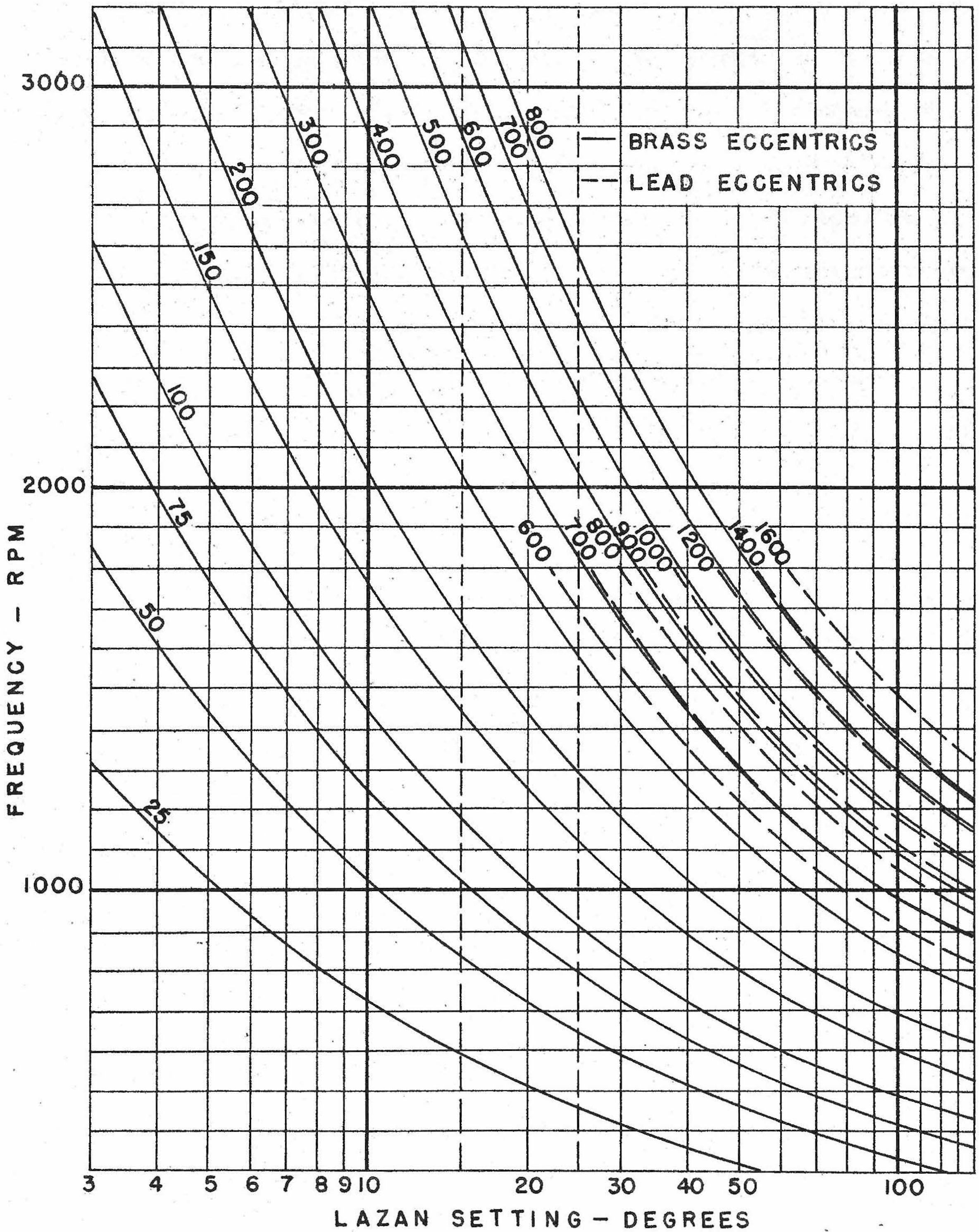
Pour IV - Forms in Place

Figure 6.2

was placed under this plate to insure uniform pressure distribution. Eight pressure cells were installed in the base of the block in the locations indicated in Fig. 6.1. The cells were installed in such a manner that they could readily be removed at the end of the test. The electrical lead-in wires to the cells were brought in through the top of the block by means of pipe conduit.

Oscillator.

A Lazan oscillator (model LA -1), manufactured by the Baldwin Locomotive Works, was used to produce the periodic force. This oscillator weighed approximately 61 lbs. and its overall dimensions were 12.5 x 11.25 x 6.5 inches. The periodic force generated was the resultant of centrifugal forces produced by brass or lead eccentric weights fastened to two parallel counter-rotating shafts. The oscillator was so designed that the unbalanced forces added in one direction and mutually cancelled in all other directions. The magnitude of the periodic force could be adjusted while the oscillator was running or at rest, by means of an external force control knob, which controlled the relative position of the eccentrics on the two parallel shafts. The relative position of the eccentrics was indicated on a counter located adjacent to the control knob. The dynamic force generated was determined from a force rating chart. (Fig. 6.3) The maximum allowable force was ± 1600 pounds. This output could be obtained at 1800 rpm with the brass eccentrics, and at 1300 rpm with the lead eccentrics. The maximum allowable frequency was 3450 rpm for the brass eccentrics, and 1700 rpm for the lead eccentrics. The oscillator was mounted on the base plate of the foundation, by means of a heavy steel cage. Provision was made on the top plate of the cage for securing additional static load weights.



DYNAMIC FORCE ±LB.
LAZAN OSCILLATOR

Figure 6.3

The oscillator was driven by a one horsepower direct current motor thru a six foot flexible shaft. Frequencies from 170 to 3450 rpm could be obtained by means of a thyratron speed control unit. This unit was regulated by the feedback from a small D.C. generator connected directly to one of the oscillator shafts. The output from this generator was also used as a tachometer to measure the oscillator frequency.

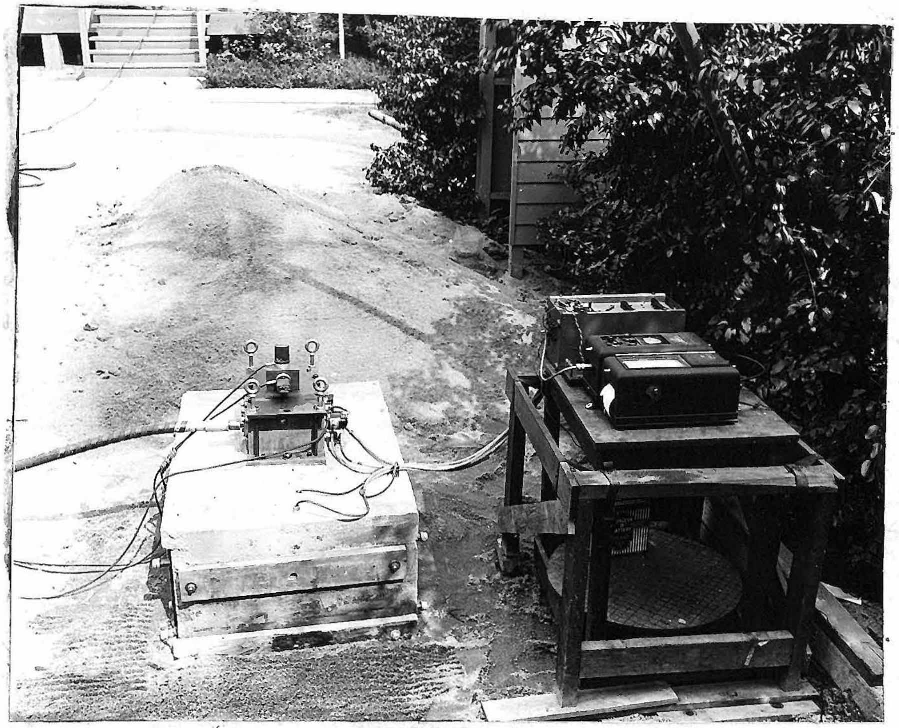
To produce a horizontal dynamic force, the oscillator was mounted on its side, and the mounting cage was adapted by means of longer bolts and special base clamping plates. The mounting position of the oscillator for vertical and horizontal force is shown in Figs. 6.4 and 6.5 respectively.

Vibration Pickup.

The displacements were measured by means of two velocity pickups. (Consolidated Engineering Corp. Type 4 - 102 A) The output from these pickups was read on a vibration meter. (Consolidated Engineering Corp. Type 1 -110B) This meter contained an integrating circuit so that peak to peak displacements could be read directly. The two pickups were mounted on an angle bracket attached to the center of the top plate of the oscillator cage as shown in Figs. 6.4 and 6.5.

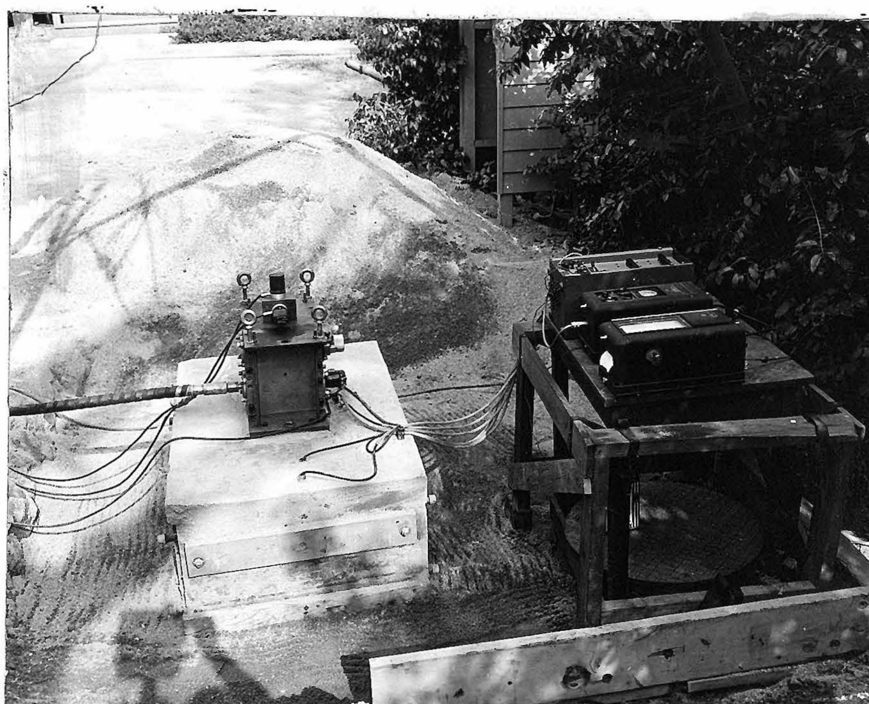
Pressure Cells.

In order to determine the dynamic pressure distribution on the base of the foundation, eight pressure cells were constructed and installed. A section of one of these cells is shown in Fig. 6.6. The principle of operation of these cells is as follows: changes in pressure are transmitted from the pressure plate to the case thru two concentric shells, arranged in such a manner that when one shell is in compression, the other shell is in tension. The active elements in the cell are four



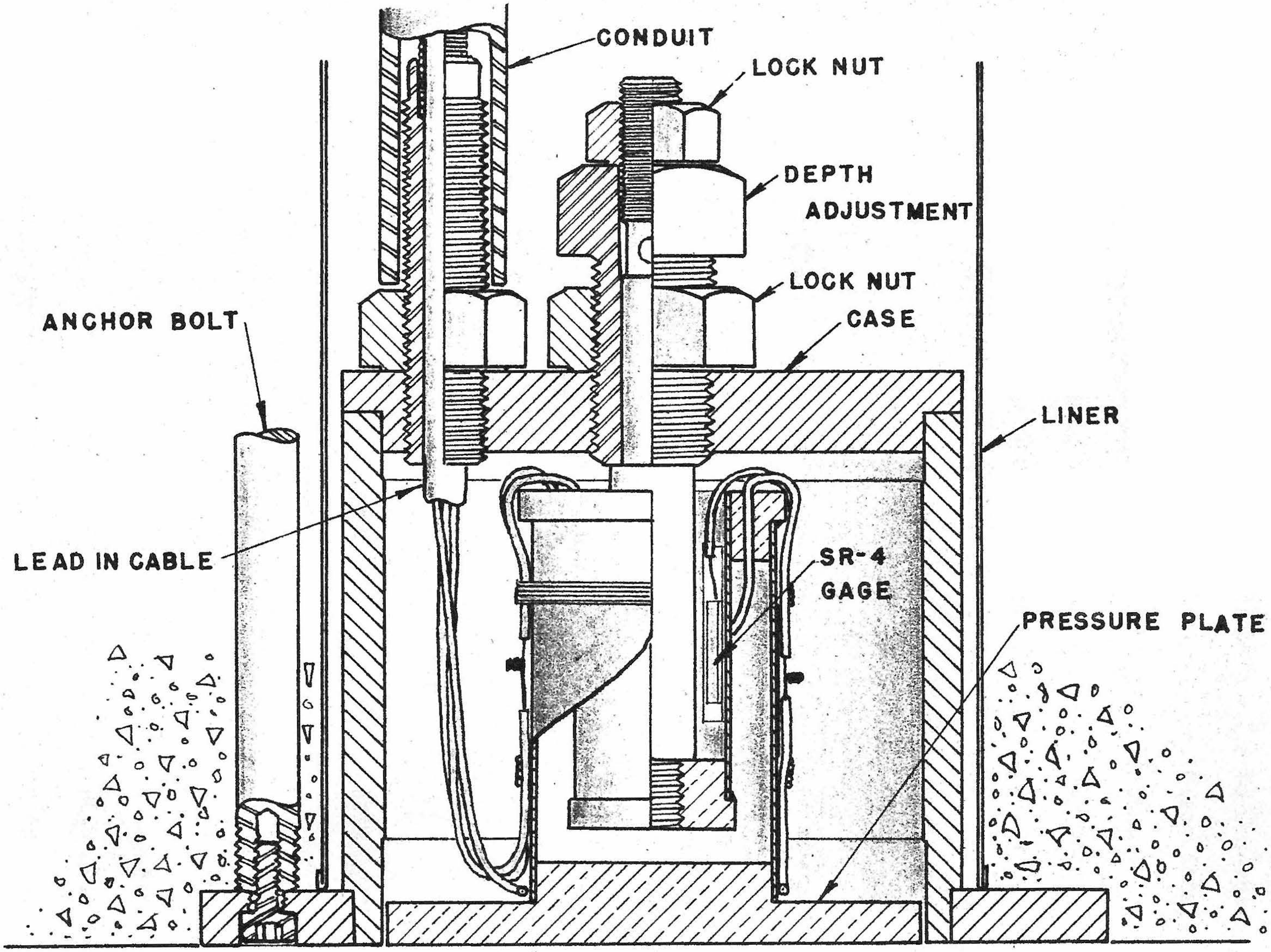
Oscillator Mounted for Vertical Force

Figure 6.4



Oscillator Mounted for Horizontal Force

Figure 6.5



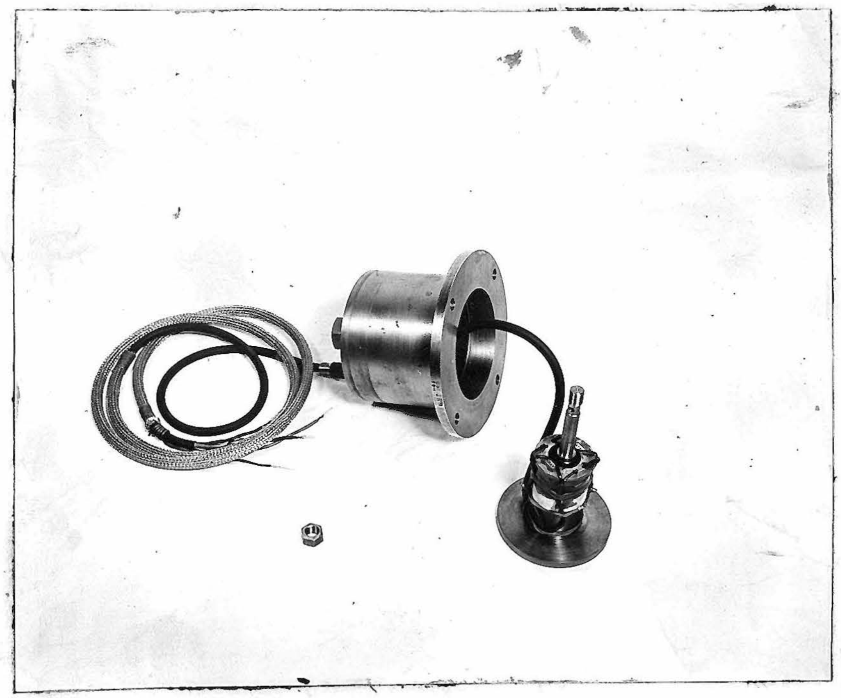
PRESSURE CELL

Figure 6.6

type C-5, bonded wire SR - 4 strain gages, two mounted on each shell. The gages are mounted on diametrically opposite sides of the shells so as to cancel out the effect of bending strains due to non-uniform load distribution on the pressure plate. The gages are electrically connected in a balanced bridge circuit, and the leads from the gages are brought out thru a watertight fitting in the top of the case. Provision is made to adjust the position of the pressure plate flush with the bottom mounting flange of the case. The cell is installed inside a thin protective metal liner, and is anchored to the foundation by four anchor bolts. The cell may be removed after the four socket head screws in the case flange are unscrewed. Details of the cell are shown in Figs. 6.7 and 6.8. The cells were calibrated for static loads by applying measured loads to the pressure plate by means of a platform scale and a loading yoke. Corresponding strains were measured by measuring the unbalance produced in the strain gage bridge circuit with a Brown Instrument Co. millivoltmeter. This calibration was then correlated to the unbalance obtained by shunting one leg of the bridge with a fixed resistor.

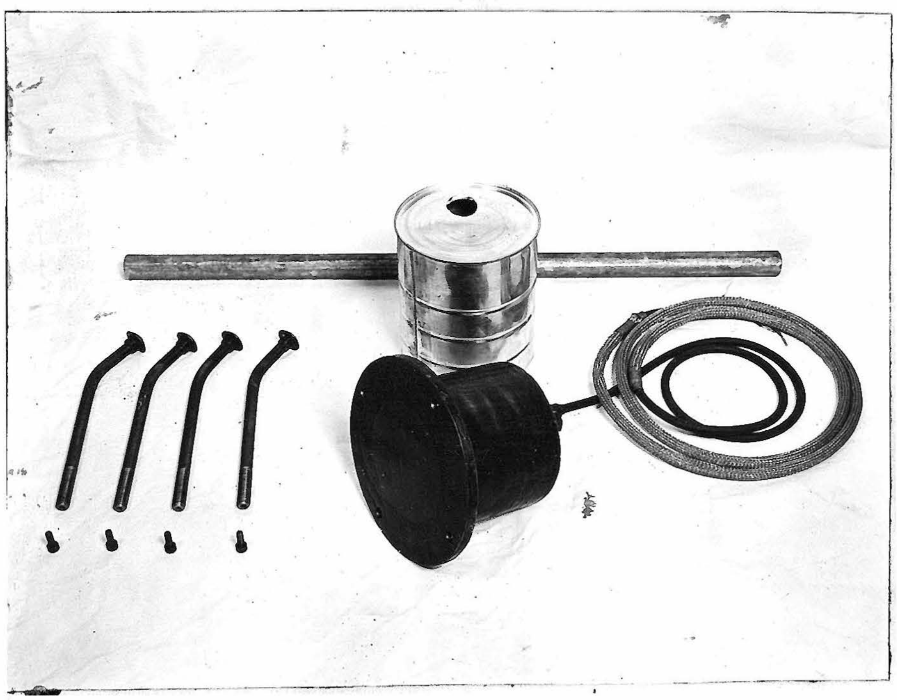
The static load calibration was found to be essentially the same for all cells, and it was therefore possible to use a single calibration constant. A typical static pressure calibration curve is shown in Fig. 6.9.

The periodic dynamic pressure change was measured by means of a Brush Development Co. type BL - 905 Amplifier and a BL - 202 Direct Inking Oscillograph. Pressures were calibrated by using the signal produced by unbalancing the strain gage bridge with the fixed resistor which had been correlated with the static pressure calibration. A



Pressure Cell

Figure 6.7



Pressure Cell Assembly
and Installation Accessories

Figure 6.8

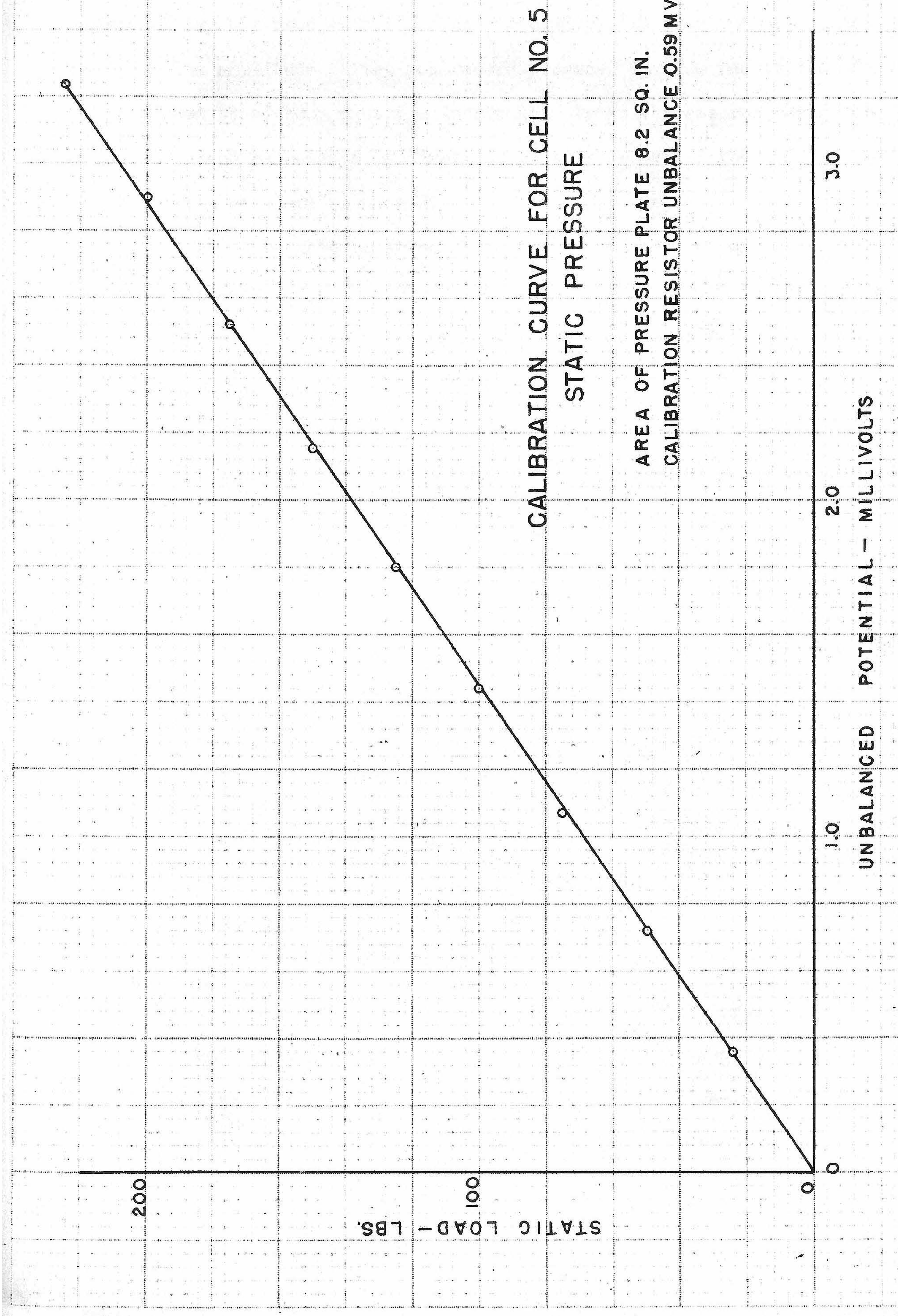

CALIBRATION CURVE FOR CELL NO. 5
STATIC PRESSURE
AREA OF PRESSURE PLATE 8.2 SQ. IN.
CALIBRATION RESISTOR UNBALANCE 0.59 MV
STATIC LOAD – LBS.
200
100
0
0
1.0
2.0
3.0
UNBALANCED POTENTIAL – MILLIVOLTS

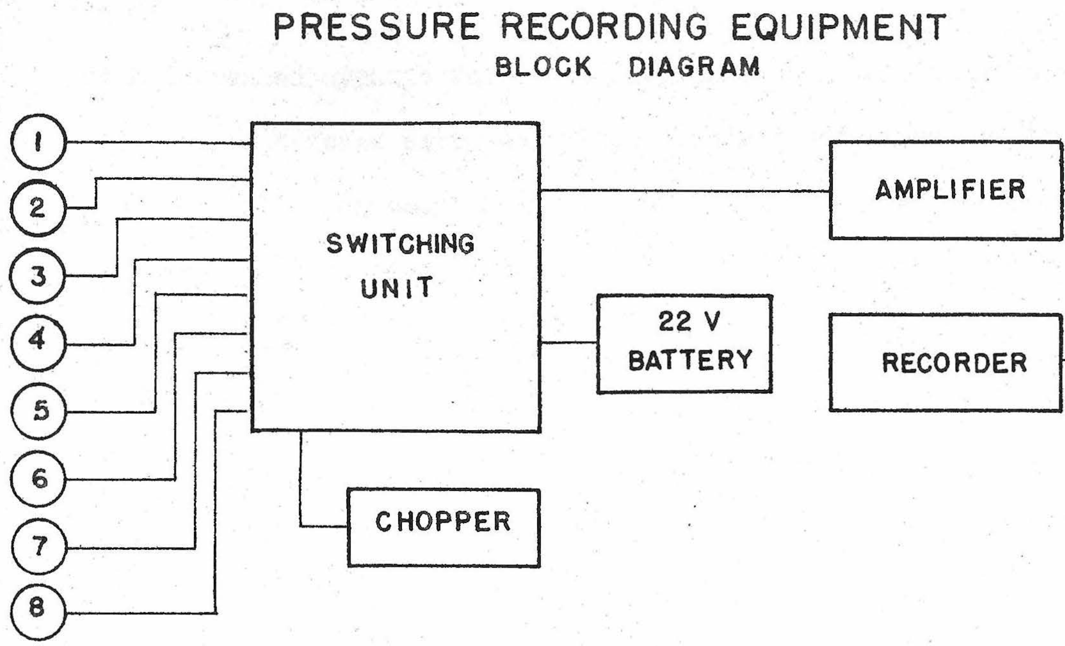
Figure 6.9

telegraph key was used as a "chopper" to produce the intermittent signal needed. The amplitude of the pen record produced in this manner therefore corresponded to a known pressure increment. Dynamic pressures could then be measured and calculated by direct proportion, provided the gain of the amplifier was left unchanged.

In order to be able to select the signal from any one of the eight pressure cells, a switching unit was designed and constructed. Provision was made to permit the output from the pressure cells to be switched to either the Brown Millivoltmeter, for static pressure measurement, or to the Brush for dynamic measurement. A D.C. voltmeter was incorporated to check the voltage applied to the bridge circuit. Power for the bridge circuit was supplied by a 6 volt battery for static measurements and by an externally connected 22 volt battery for dynamic pressure measurements. Balancing potentiometers were provided to zero-set the output from each cell on the Brown Millivoltmeter. The calibration resistor was also installed in this unit, and provision made to permit its use as a shunt across any one of the four legs of the bridge. The arrangement of this equipment is shown in Fig. 6.10. The recording equipment was mounted on a spring supported table in order to isolate it from the vibrations transmitted thru the ground. (Figs. 6.4, 6.5)

Test Procedure.

Displacement and pressure measurements were made for each test block size and for both vertical and horizontal external force. The effect of embedment was studied by building up the surcharge in three eight inch lifts. The effect of additional static load was studied only for vertical force on the 18 inch square block. The additional load was obtained by attaching steel plates to the top of the oscillator cage. For these tests the velocity pickups were mounted on the side of the cage. Since non-linear effects tended to become more pronounced


PRESSURE RECORDING EQUIPMENT
BLOCK DIAGRAM
1
2
3
4
5
6
7
8
SWITCHING UNIT
AMPLIFIER
22 V BATTERY
RECORDER
CHOPPER

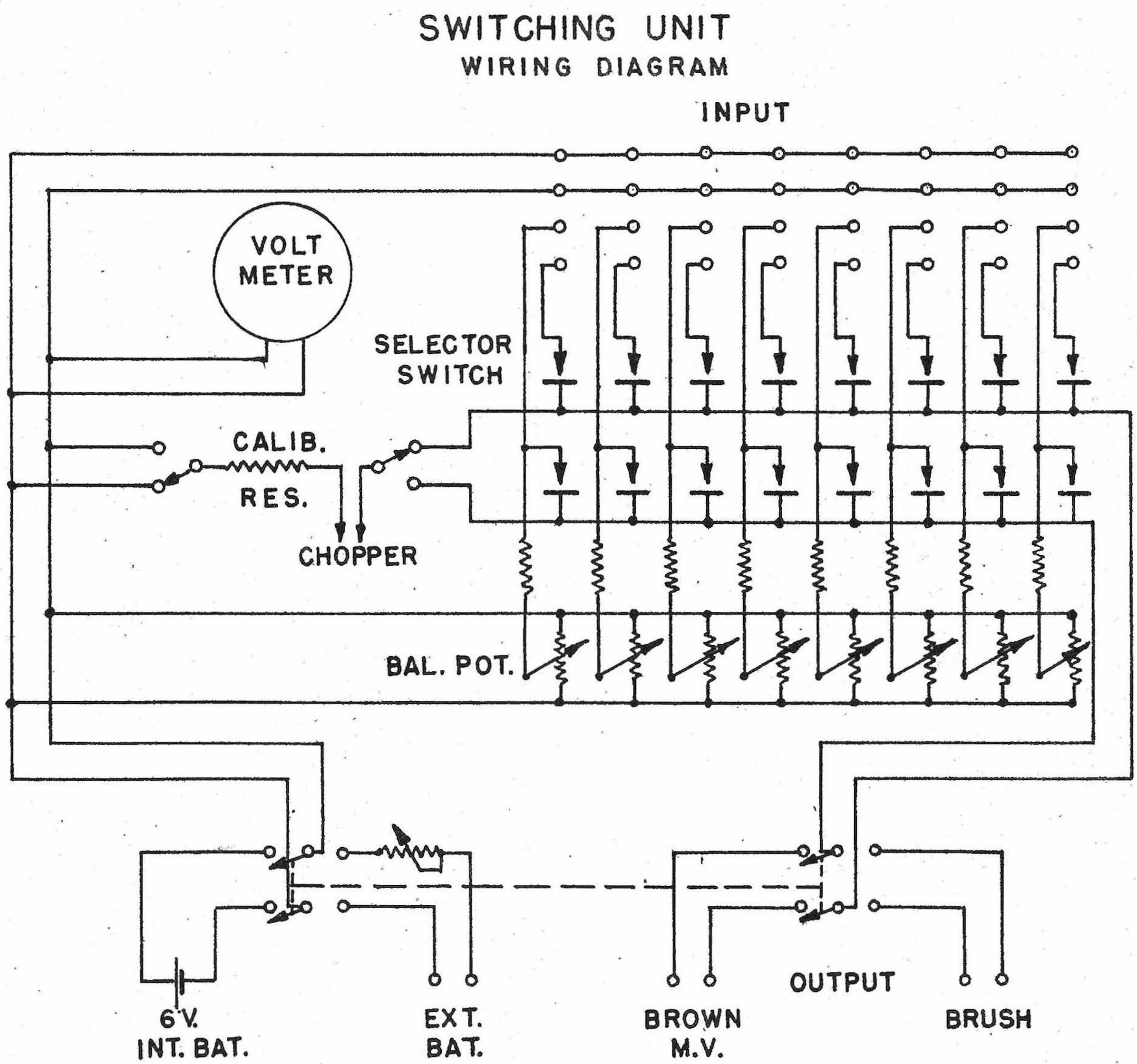

SWITCHING UNIT
WIRING DIAGRAM
INPUT
VOLT METER
SELECTOR SWITCH
CALIB. RES.
CHOPPER
BAL. POT.
OUTPUT
6 V. INT. BAT.
EXT. BAT.
BROWN M.V.
BRUSH

Figure 6.10

with increased dynamic force and amplitude, all test runs were made with constant force settings. This required adjustment of the control knob for each frequency at which pressure and displacement readings were made; all readings were made after stable conditions were obtained.

Test Data.

The displacements measured with the vertical and horizontal pickups were plotted and are shown in Appendix D. Pressure data was obtained by measuring peak to peak amplitudes on the oscillograph record. Pressures were then calculated in the following manner.

From the static calibration tests, it was determined that the effect of the shunt resistor was equivalent to 4.85 psi pressure. The dynamic pressure change is therefore

$$\Delta p = 700 \frac{A}{A_c} \text{ psf}$$

where A is the peak to peak amplitude of the pressure signal, and A_c is the peak to peak amplitude of the calibration signal. The dynamic pressure amplitude is therefore

$$p_d = 350 \frac{A}{A_c} \text{ psf}$$

Pressure data obtained in this manner was also plotted and is shown in Appendix D.

Accuracy of Test Results.

The frequency response of the pickup is flat ± 5% from 8 to 700 cycles, for double amplitudes up to 0.250 in.[24] In the significant range, 15 - 30 cps, the sensitivity is about 96% of the nominal sensitivity. The sensitivity decreases somewhat with amplitude (down approximately 5% at 0.4 in. double amplitude), and increases approximately 1% for an increase in temperature of 10° F. The vibration meter is essentially a vacuum tube voltmeter with a ± 2% flat response range of 5 to 5000 cps.[25] When properly calibrated, the system can be expected to measure displacements with an error not exceeding ± 8%. Due to the variation of soil properties with moisture and temperature conditions which could not be controlled, it was felt that the accuracy obtained was sufficient for all practical purposes.

The degree of accuracy of the pressure readings is somewhat harder to determine and varies widely with the magnitude of the pressure. The Brush amplifier and oscillograph has been designed to give an essentially flat response up to 120 cps. With maximum gain the sensitivity of the system was about 1200 psf per inch deflection. Assuming that the amplitude can be measured to the nearest 0.01 inch, errors of ± 12 psf may be expected. For small pressure readings the error is somewhat larger, due to pen drag on the paper. For large pressures, the wave form tended to be distorted. This was partly due to the fact that under these conditions the static pressure was exceeded. It is estimated that the pressure readings are accurate up to 15% or ± 15 psf, whichever is greater.

A further difficulty encountered in obtaining consistent pressure readings was the development of "hard spots". This problem was especially serious when rocking modes were excited. Under these conditions, the sand tended to pack more along the edges and the corners of the foundation block, resulting in larger pressure readings than were obtained under normal conditions.

CHAPTER VII

EVALUATION AND INTERPRETATION OF TEST DATA

Experimental Determination of Damping Ratio, Spring Factor and Apparent Mass.

Damping ratios, spring factors and apparent mass may be determined by the methods outlined in Chapter II. Since the soil is assumed to be linearly elastic, the base pressures must be proportional to the displacements. That this is indeed the case may be seen by comparing the amplitude and pressure response curves in Appendix D. Minor variations in the shape of corresponding response curves may be explained by the fact that the rocking modes are accentuated in the pressure measurements (due to the position of pressure cells). It follows, therefore, that either the pressure or the amplitude response curves may be used for calculating the dynamic soil constants. Because of the experimental difficulties encountered in making accurate pressure measurements, the amplitude response curves were selected. For the calculation of spring constants and apparent mass, the data used was further restricted to those tests in which only the vertical mode was excited.

Damping ratios were determined by plotting normalized vertical displacement curves for a viscously damped system (Fig. 2.3). A plot of normalized data curves superimposed on the theoretical curves is shown in Figs. (7.1), (7.2) and (7.3). It may be noted that the equivalent damping varies somewhat with amplitude. Since much of the damping is due to intergranular friction it is to be expected that the equivalent viscous damping factor is less for large amplitudes than for small amplitudes of oscillation. It was noted that the spring factor also tends to decrease with amplitude and with frequency of oscillation.

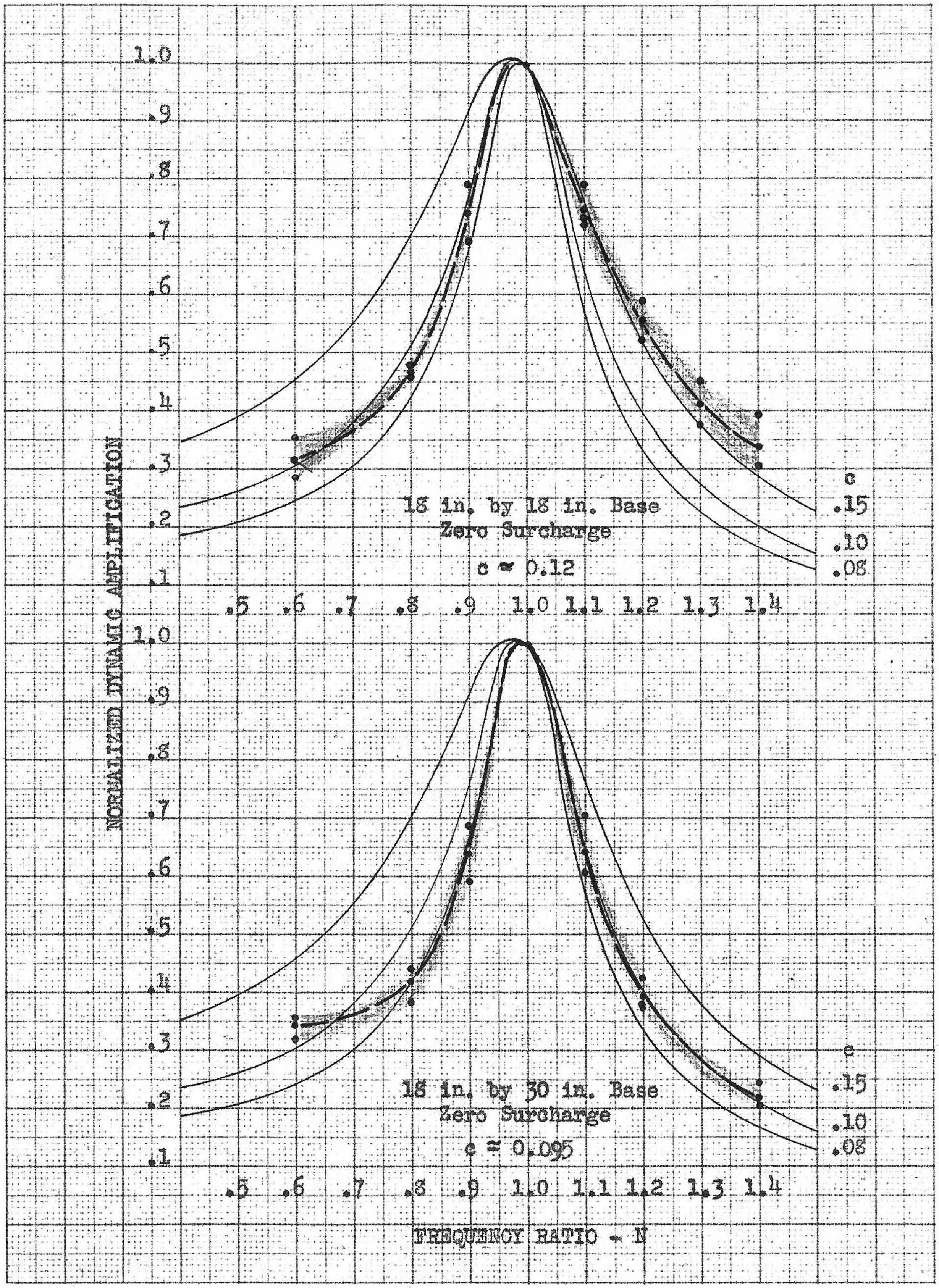

NORMALIZED DYNAMIC AMPLIFICATION
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
18 in. by 18 in. Base
Zero Surcharge
c ≈ 0.12
c
.15
.10
.08
.5 .6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4
18 in. by 30 in. Base
Zero Surcharge
c ≈ 0.095
FREQUENCY RATIO - N

Figure 7.1

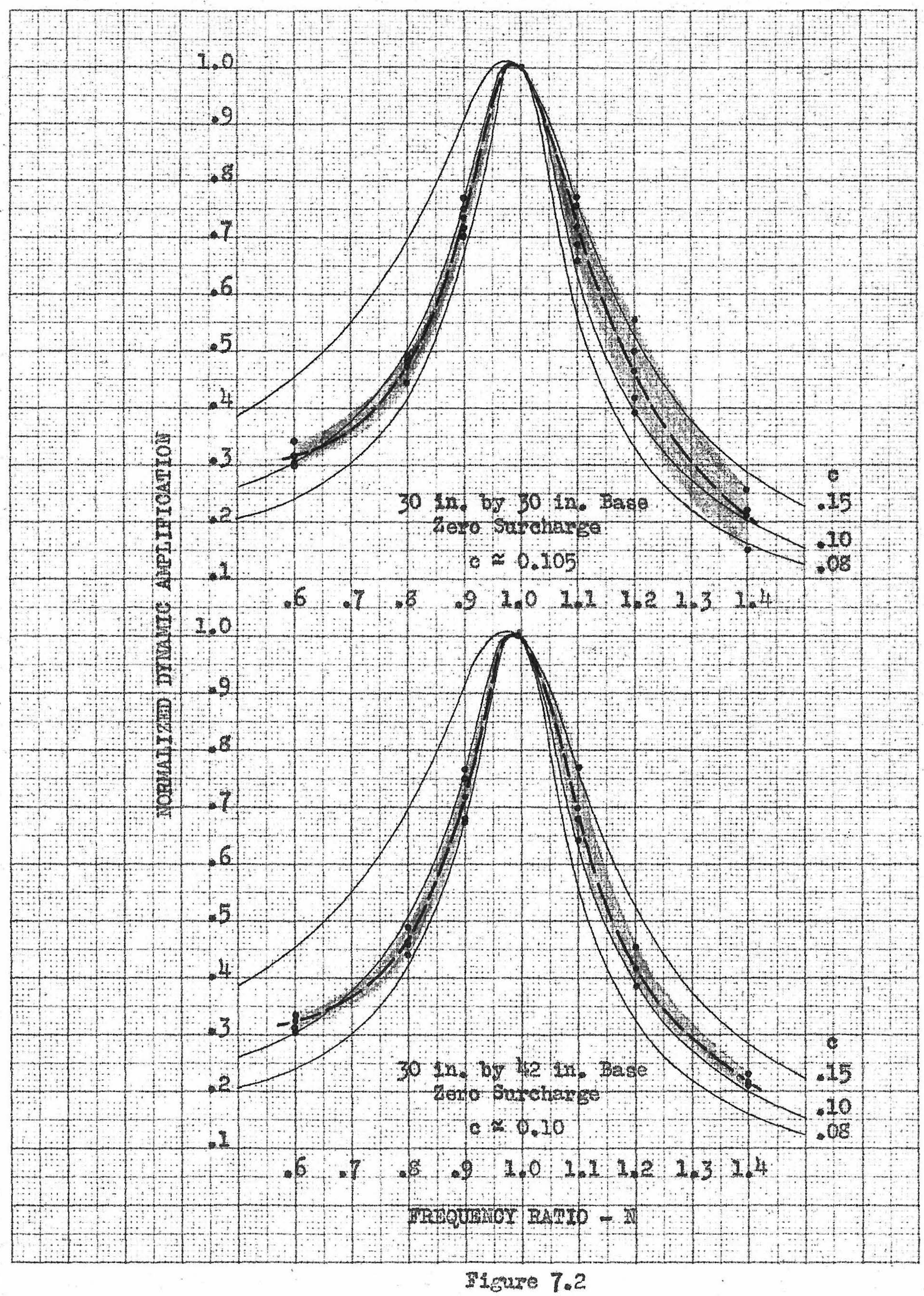

NORMALIZED DYNAMIC AMPLIFICATION
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
c
.15
.10
.08
30 in. by 30 in. Base
Zero Surcharge
c ≈ 0.105
.6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
c
.15
.10
.08
30 in. by 42 in. Base
Zero Surcharge
c ≈ 0.10
.6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4
FREQUENCY RATIO - N

Figure 7.2

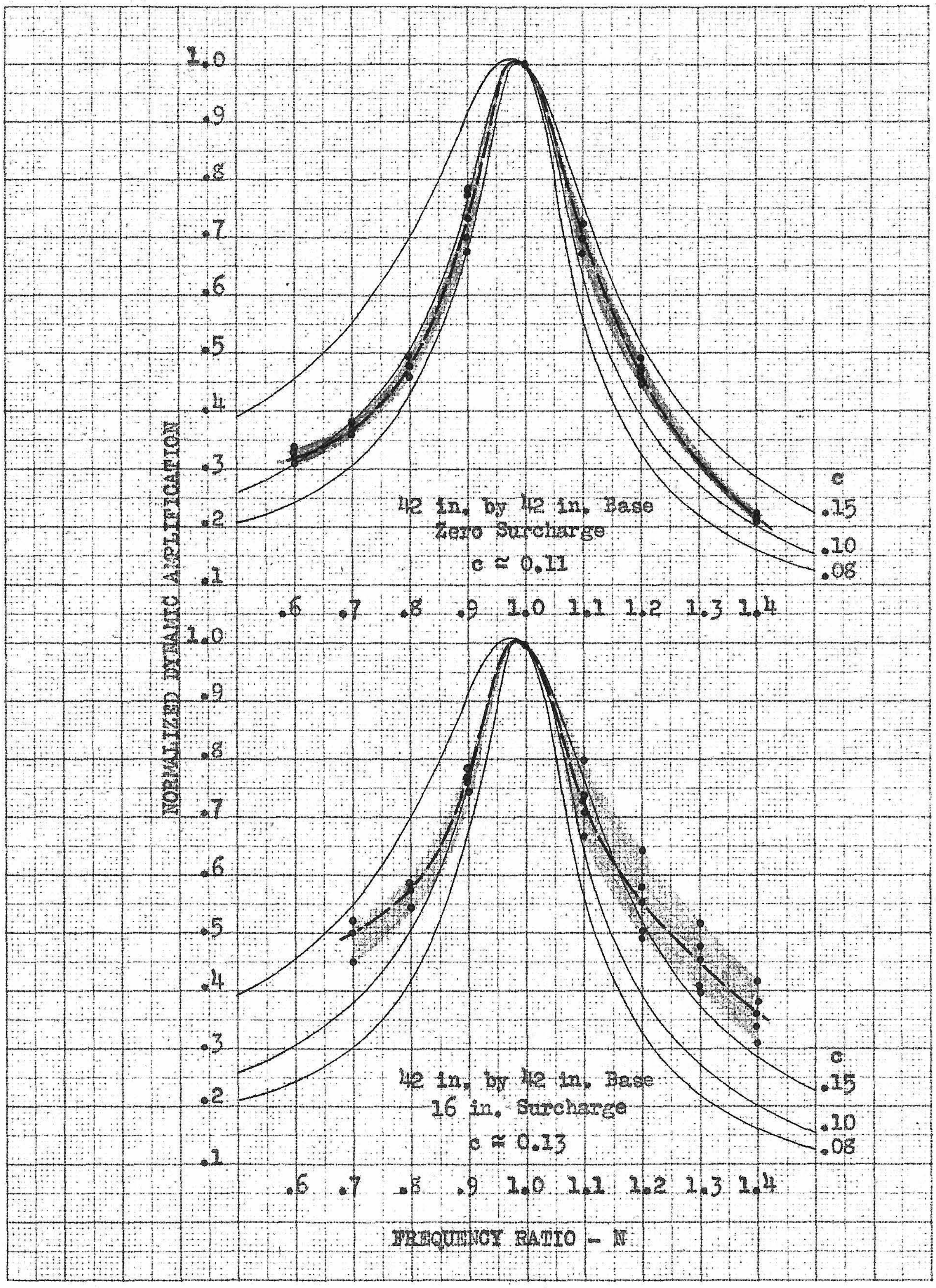

NORMALIZED DYNAMIC AMPLIFICATION
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
42 in. by 42 in. Base
Zero Surcharge
c ≈ 0.11
c
.15
.10
.08
.6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
42 in. by 42 in. Base
16 in. Surcharge
c ≈ 0.13
c
.15
.10
.08
.6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4
FREQUENCY RATIO - N

Figure 7.3

This phenomenon causes a tilting to the left of the response curve which is characteristic of a system with a soft spring.[27] The effective equivalent viscous damping therefore varied somewhat with the magnitude of the exciting force as is shown by the shaded areas in the figures. The mean value is shown by the dashed lines; the damping ratios indicated were estimated on the basis of these curves. The effective equivalent viscous damping ratio was found to range from 0.08 to 0.15 for the vertical mode, and from 0.06 to 0.10 for the coupled horizontal displacement and rotation mode.

It was noted in Chapter II that for small frequency ratios the displacement amplitude is dominated by the spring factor and for large frequency ratios, by the mass of the oscillating system. Using the dynamic amplification factors plotted in Fig. 2.2, it was possible to estimate the "static deflection", δ'_{st}, of the exciting force both for frequency ratios smaller than and greater than unity. The spring factors were then calculated from the δ'_{st} values for the small frequency ratios by the relation

$$k = \frac{F_o}{\delta'_{st}} \tag{7.1}$$

and the total effective mass of the system from the δ'_{st} values for the large frequency ratios by the relation

$$m = \frac{F_o}{\delta'_{st}\ \omega_n^2} \tag{7.2}$$

where ω_n, the frequency of the undamped free system was estimated from the frequency at which the amplitude was a maximum and from the damping factor. The values of k, m, and c obtained for the cases investigated are summarized in table (7.1).

Base Size	Mass of foundation m lb. sec²/ft.	Dynamic Force F_o lb.	Damping Ratio c	Spring Factor k kips/ft.	Total Mass m + m' lb. sec²/ft.	Apparent Mass m' lb. sec²/ft.
18 x 18	32	100	.12	3640	66	34
		200	.12	2880	77	45
		300	.11	2490	80	48
		Average	.12	2880	74	42
18 x 30	36	200	.08	3400	124	88
		300	.08	2720	120	84
		400	.095	2920	109	73
		500	.12	2830	111	75
		Average	.095	2960	116	80
30 x 30	59	300	.09	3460	158	99
		400	.09	3600	169	110
		500	.11	3440	175	116
		600	.12	3660	152	93
		Average	.105	3540	164	105
30 x 42	81	200	.09	4860	177	96
		300	.095	4200	175	94
		400	.11	4000	176	95
		500	.10	3640	178	97
		Average	.10	4180	176	95
42 x 42	112	300	.11	5080	225	113
		400	.11	5030	232	120
		500	.10	4700	230	118
		600	.105	4780	239	127
		Average	.11	4900	232	120

Table 7.1a

Zero Surcharge

Base Size	Mass of Foundation m $\frac{lb.\ sec^2}{ft}$	Dynamic Force F_o lb.	Damping Ratio c	Spring Factor k $\frac{kips}{ft}$	Total Mass m + m' $\frac{lb.sec^2}{ft}$	Apparent Mass m' $\frac{lb.\ sec^2}{ft}$
18 x 18	27	200 400		3400 2800	66 60	39 33
		Average		3100	63	36
18 x 30	36	300 400 500		3120 2620 2690	93 103 109	57 67 73
		Average		2810	102	66
30 x 30	59	400 500 600 700		4100 3860 4340 4030	129 147 156 165	70 88 97 106
		Average		4080	149	90
30 x 42	81	300 400 500 600		6600 6480 5880 6090	179 187 185 202	98 106 104 121
		Average		6260	188	107
42 x 42	112	400 500 600 700	.11 .125 .105 .095	8160 7920 6910 7200	270 269 266 264	159 157 154 152
		Average	.11	7550	267	155

Table 7.1b
8 in. Surcharge

<u>Check of Theoretical Spring Constants.</u>

The experimental spring constants tabulated in table (7.1a) and (7.1b) were used to check the accuracy of the theoretical development in Chapter IV. Solving equation (4.10) for β,

$$\beta = \frac{k_z^{xy}}{b^2 \gamma_z^{xy}} \tag{7.3}$$

For a given soil, β (the rate of increase of the modulus with depth) is constant, therefore the value of β computed by equation (7.3) for the various foundation sizes should be constant. The computations for the case of zero surcharge are shown in table (7.2). Values of γ_z^{xy} were obtained from Fig. (4.3) with α equal to unity and ρ equal to 110 lbs. per cu. ft. The maximum deviation of the value of β from the average value (β = 273,000 lbs. per cu. ft.) was 13 per cent. An improved fit could have been obtained by increasing the value of α; it should be noted, however, that the scatter of the experimental data is of the same order of magnitude as the change in the computed value of β. It should also be noted from tables (7.1a) and (7.1b) that the spring factors tend to decrease with increased dynamic force. For design purposes the value of β should be estimated for the maximum permissible displacement amplitude.

It was noted in the tests that there was a marked decrease in oscillation amplitude for even a small surcharge. This result is due to the fact that for zero surcharge the sand tended to ooze away from the edges, thereby decreasing the effective spring factor. The computed values of β for the case of 8 inch surcharge are tabulated in table (7.4). The effect of the surcharge is computed by assuming Poisson's ratio for sand to be 0.35 and using Fig. 4.9. These computations are shown in table (7.3).

Base Size	r	h $\frac{W}{ab\rho}$	s $\frac{\alpha h}{b}$	γ_z^{xy}/r	γ_z^{xy}	$\gamma_z^{xy} b^2$ ft^2	k $\frac{kips}{ft}$	β $\frac{kips}{ft^3}$
18 x 18	1.00	4.14	2.76	4.15	4.15	9.35	2880	308
18 x 30	1.67	2.79	1.86	2.65	4.41	9.93	2960	298
30 x 30	1.00	2.76	1.10	2.12	2.12	13.25	3540	267
30 x 42	1.40	2.70	1.08	1.90	2.66	16.62	4180	251
42 x 42	1.00	2.68	.76	1.67	1.67	20.45	4900	240

Table 7.2

a	t $2D_f/a$	γ_y^{xz}	$\gamma_y^{xz} a^2 \frac{\beta'}{\beta}$
18	.89	.21	.175
30	.53	.11	.254
42	.38	.07	.326

Table 7.3

Base Size	$\gamma_z^{xy} b^2$ ft^2	Side Correction ft^2	k/β ft^2	k $\frac{kips}{ft}$	β $\frac{kips}{ft^3}$
18 x 18	8.21	.70	8.91	3100	346
18 x 30	9.93	.86	10.79	2810	260
30 x 30	13.25	1.02	14.27	4080	286
30 x 42	16.62	1.16	17.78	6260	350
42 x 42	20.45	1.30	21.75	7550	346

Table 7.4

The average value of β in table (7.4) is 318,000 lbs. per cu. ft., which is a twenty per cent increase over the average value for the tests with zero surcharge. As a further check the spring factors for full embedment have been computed and compared with the experimental values. The average value of β determined in table (7.4) was used to calculate the spring constants. Results of these computations are shown in tables (7.5) and (7.6). It should be noted that the predicted values are in excellent agreement with the values determined experimentally.

Apparent Mass.

The apparent mass was calculated for the 42 x 42 inch base by numerical integration. For this calculation the density was assumed to be 110 lbs. per cu. ft. and s was taken to be 0.75. For this case r = 1 and equation (5.5) reduces to

$$m' = \frac{\rho}{g} \frac{b^3}{\alpha} C_m \tag{7.4}$$

where

$$C_m = \frac{\int_0^\infty (1+x)^2 \left[\frac{(s-1)}{(1+x)} - \log \frac{s+x}{1+x} \right]^2 dx}{\left[(s-1) - \log s \right]^2} \tag{7.5}$$

The numerical integration is shown in Appendix E. The calculated value of m' is therefore

$$m' = \frac{110}{32.2} \frac{(3.5)^3}{1} (0.835)$$

$$= 122 \ \text{lb sec}^2/\text{ft.}$$

The average of the experimental values is shown in table (7.1a) and is 120 lb. sec^2/ft.

a	t $2\,D_f/a$	γ_y^{xz}	$\gamma_y^{xz} a^2 \frac{\beta'}{\beta}$
18	2.66	.98	.83
30	1.60	.47	1.09
42	1.14	.30	1.36

Table 7.5

Base Size	$\gamma_z^{xy} b^2$ ft^2	Side Correction ft^2	k/β ft^2	Computed k $\frac{kips}{ft}$	Experimental k $\frac{kips}{ft}$
18x 18	9.35	3.32	12.67	4030	-
18x 30	9.93	3.84	13.77	4380	5200
30x 30	13.25	4.37	17.62	5600	5550
30x 42	16.62	4.90	21.52	6850	6900
42x 42	20.45	5.44	25.89	8240	8640

Table 7.6

Apparent Mass Moment of Inertia.

The apparent mass moment of inertia was calculated for the 42 x 42 inch base. Since for this case $r = 1$, equation (5.14) reduces to

$$I' = \frac{\rho b^5}{12 g} C_i \tag{7.6}$$

where

$$C_i = \frac{\int_0^\infty (1+x)^4 \left[\frac{S-1}{1+x} - \frac{1}{2}\left(\frac{S-1}{1+x}\right)^2 + \frac{1}{3}\left(\frac{S-1}{1+x}\right)^3 - \log \frac{S+x}{1+x} \right]^2 dx}{\left[(s-1) - \frac{(s-1)^2}{2} + \frac{(s-1)^3}{3} - \log s \right]^2} \tag{7.7}$$

The numerical integration of equation (7.7) is shown in Appendix F. The mass moment of inertia is therefore

$$I' = \frac{(110)(3.5)^5}{(12)(32.2)} (0.30)$$
$$= 44.9 \text{ lb. ft. sec}^2$$

Coupled Modes.

As a final check on the theory, the frequencies of the equivalent undamped system were computed for the case of the 42 x 42 inch base with zero surcharge. Using the value for apparent mass determined above, the total effective mass of the foundation and soil is

$$m + m' = 112 + 122 = 234 \text{ lb. sec}^2/\text{ft.}$$

The center of mass of the system is

$$C = \frac{(112)(1)}{234} = 0.48 \text{ ft.}$$

The mass moment of inertia of the foundation about the y-axis in the contact plane is

$$I_y'' = \frac{\rho b^2 d}{3}\left[\frac{b^2}{4} + d^2\right] = \frac{m}{3}\left[\frac{b^2}{4} + d^2\right]$$

$$= \frac{112}{3}\left[\frac{12.25}{4} + 4\right]$$

$$= 263.6 \text{ lb. ft. sec}^2$$

The mass moment of inertia about the center of mass is therefore

$$I_y + I_y' = I_y'' + I' - (m+m')c^2$$

$$= 263.6 + 44.9 - 234(0.48)^2$$

$$= 255 \text{ lb. ft. sec.}^2$$

Referring to equation (3.38), the inertia parameters are

$$m_1 = m_3 = m_5 = 234 \text{ lb. sec.}^2/\text{ft.}$$

$$m_2 = m_4 = 255 \text{ lb. sec}^2/\text{ft}$$

For r = 1 and s = 0.75 we have (from Fig. (4.3))

$$\gamma_z = 1.67$$

and from Fig. (4.5)

$$\gamma_{yz} = \gamma_{xz} = 0.265$$

Using the average value of β obtained from the experiments with zero surcharge, we have from table (7.2)

$$\beta = 273 \text{ kips/ft}^3$$

and with μ = 0.35

$$\beta' = \frac{273}{2(1.35)} = 101 \text{ kips/ft}^3$$

The spring factors are

$$k_z = \beta b^2 \gamma_z = 273\,(3.5)^2(1.67) \qquad = 5{,}590 \text{ kips/ft.}$$

$$k_x = k_y = \beta' b^2 \gamma_z = 101\,(3.5)^2(1.67) \qquad = 2{,}070 \text{ kips/ft.}$$

$$k_{xz} = k_{yz} = \beta b^4 \gamma_{xz} = 273\,(3.5)^4(0.265) = 10{,}850 \text{ kip ft.}$$

Referring to equation (3.11) we have

$$k_{11} = k_x \qquad = 2{,}070 \text{ kips/ft.}$$

$$k_{12} = k_{21} = c k_x = (0.48)(2070) \qquad = 994 \text{ kips}$$

$$k_{22} = c^2 k_x + k_{xz} = (0.48)^2(2070) + 10{,}850 = 11{,}327 \text{ kip ft.}$$

$$k_{55} = k_z \qquad = 5{,}590 \text{ kips/ft.}$$

Substituting in equation (3.45)

$$\omega^2_{1,2} = \frac{\frac{k_{11}}{m_1} + \frac{k_{22}}{m_2} \pm \sqrt{\left(\frac{k_{11}}{m_1} - \frac{k_{22}}{m_2}\right)^2 + 4\,\frac{k_{12}\,k_{21}}{m_1\,m_2}}}{2}$$

$$= 26{,}675 \pm \sqrt{334{,}600{,}000}$$

$$= 8{,}385\,;\ 44{,}965$$

Therefore

$$\omega_1 = 91.5 \text{ rad./sec.}$$

$$= 91.5\,\frac{60}{2\pi} = 875 \text{ rpm}$$

and

$$\omega_2 = 212 \text{ rad./sec.}$$

$$= 212\,\frac{60}{2\pi} = 2020 \text{ rpm}$$

Since the base is square, the frequencies for the coupled modes in the yz-plane are identical to the above values. The observed frequencies for maximum amplitude for this case ranged from 810 rpm for a 150 pound dynamic force to 940 rpm for a 50 pound horizontal dynamic force. (Appendix D.). The higher mode was not excited in this case.

The frequency of the vertical mode of the equivalent free undamped system may be computed from equation (3.47)

$$\omega_5^2 = \frac{k_{55}}{m_5}$$

Therefore

$$\omega_5 = \sqrt{\frac{5,590,000}{234}} = 154.5 \ \text{rad./sec.}$$

$$= 154.5 \frac{60}{2\pi} = 1475 \ \text{rpm}$$

The observed frequencies for maximum vertical displacement amplitude ranged from 1390 rpm for 600 pound vertical dynamic force to 1420 rpm for a 300 pound dynamic force. (Appendix D.)

<u>Equations of Motion for the Coupled Mode.</u>

Referring to equation (3.49) we have

$$\phi_2^{(1)} = \frac{(8.385)(234) - 2070}{994} \phi_1^{(1)}$$

$$= -0.109 \phi_1^{(1)}$$

Similarly, upon substituting in equation (3.50), we obtain

$$\phi_2^{(2)} = \frac{(44.965)(234) - 2070}{994} \phi_1^{(2)}$$

$$= 8.50 \phi_1^{(2)}$$

The horizontal dynamic force F was applied about 6.5 inches above the top of the foundation; therefore

$$F_1 = F$$

$$F_2 = 2.05 F$$

$$F_3 = F_4 = F_5 = F_6 = 0$$

Substituting in equation (3.53) we have

$$f_1 = \frac{1 + (2.05)(-0.109)}{234 + 255(-0.109)^2} \frac{F}{\phi_1^{(1)}} = 0.003,28 \frac{F}{\phi_1^{(1)}}$$

and in equation (3.54)

$$f_2 = \frac{1 + (2.05)(8.5)}{234 + 255(8.5)^2} \frac{F}{\phi_1^{(2)}} = 0.000,988 \frac{F}{\phi_1^{(2)}}$$

Finally, upon substituting in equations (3.59) we obtain for the equations of motion

$$q_1 = \left[\frac{3.28}{\omega_1^2 - \omega^2} + \frac{0.988}{\omega_2^2 - \omega^2} \right] \frac{F}{1000} \sin(\omega t + \psi)$$

and

$$q_2 = \left[\frac{-0.336}{\omega_1^2 - \omega^2} + \frac{8.4}{\omega_2^2 - \omega^2} \right] \frac{F}{1000} \sin(\omega t + \psi)$$

The dimension of q_1 is in feet and of q_2 in radians when F is in pounds and ω is in radians per second. It must be remembered that damping has been neglected in the above equations and therefore they cannot be used to calculate the displacement amplitudes near the resonant frequencies.

Pressure Distribution.

The pressure cell readings were somewhat more erratic than the displacement readings. This was partly due to the development of "hard spots" in the sand as the sand was consolidated by the vibration of the foundation. It was difficult to obtain consistent pressure recordings. The pressure response curves (Appendix D) are very similar to the ones obtained for displacement. The pressure distribution was found to be affected by the frequency; pressures near the edges of the foundation increase with increase in frequency. Contours of equal dynamic pressure are shown in Fig. (7.4) for the 30 x 42 inch base and for the 42 x 42 inch base. These contours were obtained by averaging the pressure readings for several dynamic loads. It should be noted that the maximum dynamic pressure may be several times the magnitude of the average pressure. Account of this fact should be taken when selecting safe bearing values.

Conclusions.

Dynamic soil constants can best be determined from displacement measurements. Care should be taken that only a single mode is excited and that the test base or plate is in contact with the soil during the complete cycle of oscillation.

Correlation between the computed and experimental values for the equivalent spring factor was very good. (Cf. tables (7.2), (7.4) and (7.6)) The theoretical value for apparent mass was calculated only for one case; for this case the correlation with the measured value was excellent. The predicted critical frequencies for the rocking modes based on computed spring factors and apparent mass and mass moment of inertia values were also in good agreement with observed frequencies. It may be concluded therefore that the behavior of machine foundations on cohesionless soils may be accurately predicted by the theory presented.

CONTOURS OF EQUAL PRESSURE

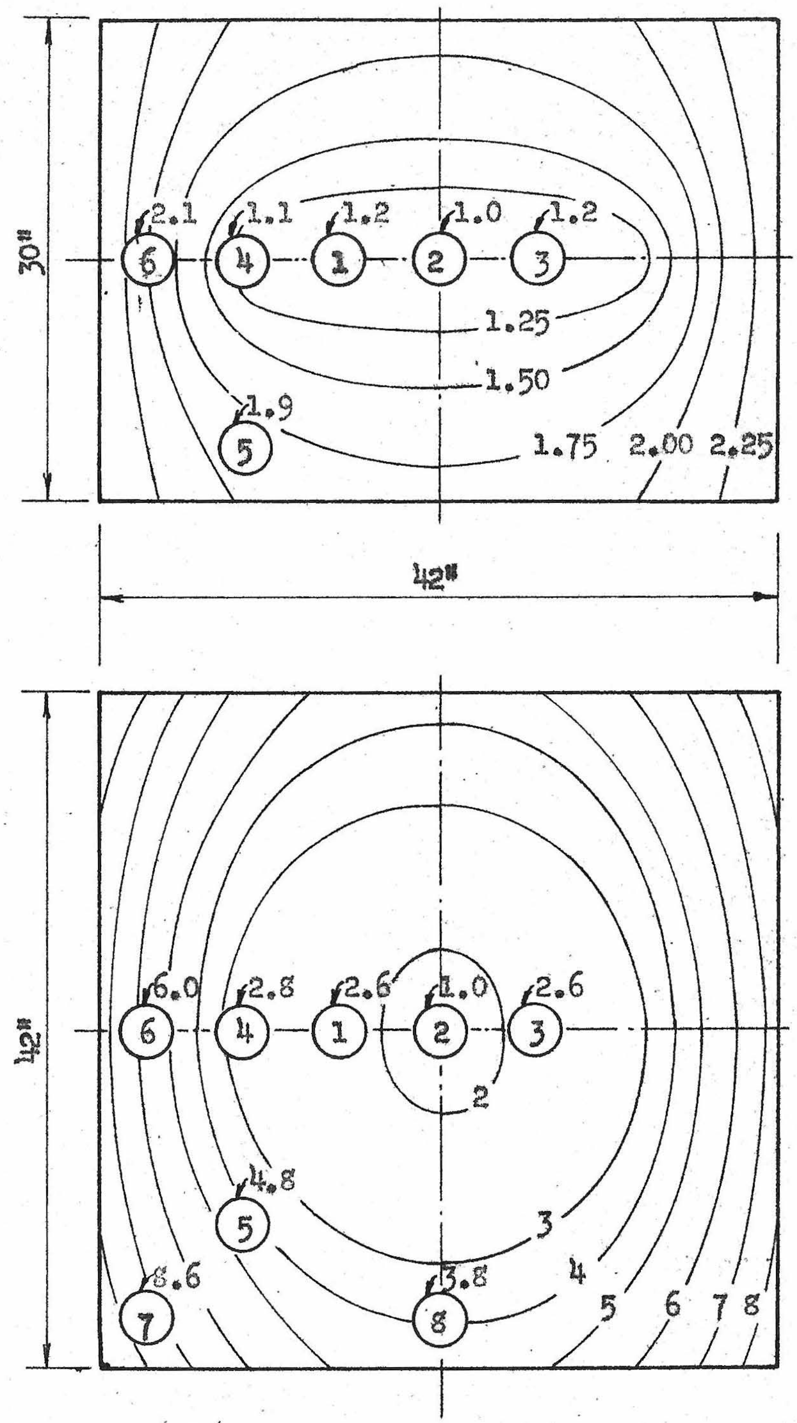


Figure 7.4

CHAPTER VIII

DESIGN PROCEDURES

RECOMMENDATIONS FOR FURTHER STUDY AND RESEARCH

Design Data Required.

Before proceeding with a machine foundation design the following data must be obtained:

1. Type of machine and magnitude, direction and frequency of the dynamic forces to be resisted. This information should generally be furnished by the manufacturer. Where this information is not furnished, a conservative estimate must be made on the basis of the type and the design features of the machine. In some cases it may be possible to measure the unbalanced forces by strain gage techniques, or by the use of dynamic load measuring devices.

2. For major installations a soils investigation should be made. Type and characteristics of the soil should be determined and where possible dynamic field tests should be made. Depth of the stratum should be investigated and test holes made to a depth not less than three times the maximum overall dimension of the foundation, or, in the case of long narrow foundations, to a depth at least five times the width of the base. From these tests and from undisturbed samples removed from the test holes, the following design data should be obtained:

a. The dynamic modulus of the soil.

b. The effective damping factor.

c. The maximum permissible design load.

Preliminary Design.

Owing to the complexity of the problem, and since in most instances certain functional and space requirements will have to be satisfied, the trial design method is recommended. In general the critical mode of oscillation will be determined by the type of machine and the proposed method of installation. The frequency ratio can then be estimated from a knowledge of soil conditions. The next problem is to determine the maximum allowable amplitude of oscillation. Very little data is available on this factor. The allowable amplitude depends on

1. The type of machine and its operating frequency;
2. The location of the installation and the type of activity carried on in its environment;
3. The type of soil on which the machine is founded.

The first item is of importance since some machines may be damaged by excessive vibration. Generally smaller amplitudes are permissible for the higher frequency machines. Amplitudes may have to be restricted in some installations to prevent damage to adjacent structures or annoyance to workers in the vicinity. In some cases noise and vibration isolation may have to be provided. The type of soil is an important factor since the damping factor for some soils, notably waterlogged soils, may be very small and hence disturbances may be transmitted for long distances.

Rausch[7] recommends that the maximum amplitude of oscillation should not exceed

$$A = 9.54/f \text{ in.}$$

where f is in revolutions per minute, for frequencies less than 1800 rpm, or

$$A = 17{,}600/f^2 \text{ in.},$$

for frequencies greater than 1800 rpm. Studies of the physiological effect of vibrations have been made by the automotive industry. The results, however, are not applicable to foundation problems, since the annoyance level in moving vehicles is generally much higher than can be tolerated in a building or factory.[22] A study by the Liberty Mutual Life Insurance Co. of Boston shows that amplitudes in excess of

$$A = 0.36/f \text{ in.}$$

are easily noticeable to persons, and amplitudes in excess of

$$A = 30/ f^{1.3} \text{ in.}$$

are troublesome.[28] It was further shown that the amplitude level causing structural damage is much higher than the annoyance level.

From the allowable amplitude, and the amplitude ratio (Figs. 2.3, 2.5) the "static" deflection, δ'_{st}, can be determined. The minimum values of the spring factor k and the total mass (m + m') can then be calculated from the equations

$$k = \frac{F}{\delta'_{st}}$$

and

$$m + m' = \frac{F}{\delta'_{st}\,\omega_n^2}$$

where ω_n is the estimated resonant frequency in radians per second. The base size is then determined by trial and error using the graphs in Chapter IV, and the weight of the foundation is calculated by estimating the apparent mass and subtracting from the total mass required. The preliminary design can then be completed to satisfy functional and space requirements. The above procedure may be summarized by the following steps:

1. Determine the probable critical mode.
2. Estimate the frequency ratio and determine the amplitude ratio.
3. Determine the allowable amplitude.

4. Calculate k and (m + m')

5. Determine the overall foundation dimensions using the graphs in Chapter IV.

6. Complete the preliminary design to satisfy other requirements.

Design Analysis.

After the preliminary design is completed it should be carefully analyzed. The following procedure is recommended.

1. Determine the spring factors. (Graphs, Chapter IV)

2. Calculate the inertia terms using the procedures outlined in Chapter V.

3. Calculate the modes of oscillation using the equations developed in Chapter III.

4. Calculate the maximum amplitude of oscillation.

5. Check the maximum amplitude with permissible limits.*

6. Determine the transmissibility ratio (Fig. 2.9) and check whether the maximum soil pressure developed is within design limits.

7. For frequency ratios greater than 0.7 check whether power dissipation is excessive.

8. Check special features, functional requirements and structural requirements.

9. In cases where the operating frequency or a critical harmonic falls close to one of the resonant frequencies of the foundation, provision should be made for remedial measures should they be required. Remedial measures are of three main types:

   a. Dynamic balancing devices - suitable only for constant speed machines.

   b. Provision for addition of mass.

   c. Stiffening or structural modification.

---

* Revise the design if necessary and repeat steps 1-5.

Recommendations for Further Study and Research.

Reliable information on the behavior of machine foundations on cohesive soil is not available. An extension to other types of soil of the experiments performed on sand would furnish the data required to check the theory developed. An analysis should be made of both satisfactory and unsatisfactory existing installations in order to determine reliable criteria for maximum amplitudes. This information should eventually be presented in code form for the guidance of the foundation designer.

Refinement of the theory presented may be possible and desirable. More data is required for determining the maximum soil pressures developed. The effect of special foundation shapes and partial contact should be investigated; also the use of cast-in-place piers tied to mat foundations.

Much work yet remains to be done, both in theoretical development and experimental verification. Nevertheless, it is the author's belief that by the use of the theory and procedures developed in this thesis the design and analysis of machine foundations may be put on a rational basis.

APPENDIX A

1. Evaluation of the integral

$$\frac{1}{\gamma} = \int_0^\infty \frac{dz}{(r+z)(s+z)(1+z)} \tag{A.1}$$

a. For $r \neq s \neq 1$. Separating into partial fractions

$$\frac{1}{\gamma} = \int_0^\infty \left[\frac{1}{(s-r)(1-r)} \frac{dz}{(r+z)} + \frac{1}{(r-s)(1-s)} \frac{dz}{(s+z)} + \frac{1}{(r-1)(s-1)} \frac{dz}{(1+z)} \right]$$

$$= \frac{1}{(r-s)(s-1)(r-1)} \left[s \log \frac{r+z}{1+z} + \log \frac{s+z}{r+z} + r \log \frac{1+z}{s+z} \right]_0^\infty$$

$$= \frac{1}{(r-s)(s-1)(r-1)} \left[(r-1) \log s - (s-1) \log r \right]$$

$$= \frac{1}{(r-s)} \left[\frac{\log s}{(s-1)} - \frac{\log r}{(r-1)} \right]$$

Therefore

$$\gamma = \frac{(r-s)}{\dfrac{\log s}{(s-1)} - \dfrac{\log r}{(r-1)}} \tag{A.2}$$

b. For $r = 1$, $s \neq 1$. (A.1) reduces to

$$\frac{1}{\gamma} = \int_0^\infty \frac{dz}{(s+z)(1+z)^2} \tag{A.3}$$

Separating into partial fractions

$$\frac{1}{\gamma} = \int_0^\infty \left[\frac{1}{(s-1)^2} \frac{dz}{(s+z)} + \frac{-1}{(s-1)^2} \frac{dz}{(1+z)} + \frac{1}{(s-1)} \frac{dz}{(1+z)^2} \right]$$

$$= \frac{1}{(s-1)^2} \left[\log \frac{(s+z)}{(1+z)} - \frac{(s-1)}{(1+z)} \right]_0^\infty$$

$$= \frac{1}{(s-1)^2} \left[(s-1) - \log s \right]$$

Thus

$$\gamma = \frac{(s-1)}{1 - \dfrac{\log s}{(s-1)}} \tag{A.4}$$

c. For $s = 1$, $r \neq 1$. (A.1) reduces to

$$\frac{1}{\gamma} = \int_0^\infty \frac{dz}{(r+z)(1+z)^2} \tag{A.5}$$

which is if the same form as (A.3). Consequently

$$\gamma = \frac{(r-1)}{1 - \frac{\log r}{r-1}} \tag{A.6}$$

d. For $r = s \neq 1$, (A.1) reduces to

$$\frac{1}{\gamma} = \int_0^\infty \frac{dz}{(s+z)^2(1+z)} \tag{A.7}$$

Separating into partial fractions

$$\frac{1}{\gamma} = \int_0^\infty \left[\frac{1}{(s-1)^2}\frac{dz}{(1+z)} - \frac{1}{(1-s)^2}\frac{dz}{(s+z)} + \frac{1}{(1-s)}\frac{dz}{(s+z)^2}\right]$$

$$= \frac{1}{(s-1)^2}\left[\log\frac{1+z}{s+z} + \frac{(s-1)}{(s+z)}\right]_0^\infty$$

$$= \frac{1}{(s-1)^2}\left[\log s - \frac{(s-1)}{s}\right]$$

Hence

$$\gamma = \frac{s(s-1)}{\frac{s\log s}{s-1} - 1} \tag{A.8}$$

e. For $r = s = 1$, (A.1) reduces to

$$\frac{1}{\gamma} = \int_0^\infty \frac{dz}{(1+z)^3} \tag{A.9}$$

$$= \left[-\frac{1}{2}\frac{1}{(1+z)^2}\right]_0^\infty = \frac{1}{2}$$

Therefore

$$\gamma = 2 \tag{A.10}$$

2. Evaluation of the integral

$$\frac{1}{\gamma} = \int_0^\infty \frac{dz}{(s+z)(1+z)} \tag{A.11}$$

a. For $s \neq 1$. Separating into partial fractions

$$\frac{1}{\gamma} = \int_0^\infty \left[\frac{1}{(s-1)} \frac{dz}{(1+z)} - \frac{1}{(s-1)} \frac{dz}{(s+z)} \right]$$

$$= \frac{1}{(s-1)} \left[\log \frac{1+z}{s+z} \right]_0^\infty = \frac{1}{(s-1)} \log s$$

Therefore

$$\gamma = \frac{s-1}{\log s} \tag{A.12}$$

b. For $s = 1$. (A.11) reduces to

$$\frac{1}{\gamma} = \int_0^\infty \frac{dz}{(1+z)^2} = \left[-\frac{1}{1+z} \right]_0^\infty = 1 \tag{A.13}$$

hence

$$\gamma = 1 \tag{A.14}$$

APPENDIX B

1. Evaluation of the integral

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(r+z)(s+z)(1-z)^3} \tag{B.1}$$

a. For $r \neq s \neq 1$. Separating into partial fractions

$$\frac{1}{\gamma} = 12\int_0^\infty \Bigg[\frac{1}{(s-r)(1-r)^3}\,\frac{dz}{(r+z)} + \frac{1}{(r-s)(1-s)^3}\,\frac{dz}{(s+z)}$$

$$+\left(\frac{1}{(r-1)(s-1)^3} + \frac{1}{(r-1)^2(s-1)^2} + \frac{1}{(r-1)^3(s-1)}\right)\frac{dz}{(1+z)}$$

$$+\left(\frac{-1}{(r-1)(s-1)^2} + \frac{-1}{(r-1)^2(s-1)}\right)\frac{dz}{(1+z)^2} + \frac{1}{(r-1)(s-1)}\,\frac{dz}{(1+z)^3}\Bigg]$$

$$= \Bigg[\frac{12}{(r-s)(r-1)^3(s-1)^3}\Bigg\{(s^3-3s^2+3s)\log\frac{r+z}{1+z}$$

$$-(r^3-3r^2+3r)\log\frac{s+z}{1+z} + \log\frac{r+z}{s+z}\Bigg\}$$

$$+\frac{12}{(r-1)^2(s-1)^2}\left\{\frac{(r+s-2)}{(1+z)} - \frac{(r-1)(s-1)}{2(1+z)^2}\right\}\Bigg]_0^\infty$$

$$= \frac{12}{(r-s)(s-1)(r-1)}\Bigg[\frac{(r-1)}{(s-1)^2}\log s - \frac{(s-1)}{(r-1)^2}\log r$$

$$+(r-s)\left\{\frac{1}{2} - \frac{1}{(r-1)} - \frac{1}{(s-1)}\right\}\Bigg]$$

Therefore

$$\gamma = \frac{\frac{1}{12}(r-s)(s-1)(r-1)}{\frac{(r-1)}{(s-1)^2}\log s - \frac{(s-1)}{(r-1)^2}\log r + (r-s)\left(\frac{1}{2} - \frac{1}{(r-1)} - \frac{1}{(s-1)}\right)} \tag{B.2}$$

b. For $r=1$, $s\neq 1$, the integral reduces to

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(s+z)(1+z)^4} \tag{B.3}$$

Separating into partial fractions

$$\frac{1}{\gamma} = 12\int_0^\infty \Big[\frac{1}{(s-1)^4}\frac{dz}{(s+z)} + \frac{-1}{(s-1)^4}\frac{dz}{(1+z)} + \frac{1}{(s-1)^3}\frac{dz}{(1+z)^2} + \frac{-1}{(s-1)^2}\frac{dz}{(1+z)^3} + \frac{1}{(s-1)}\frac{dz}{(1+z)^4}\Big]$$

$$= \frac{12}{(s-1)^4}\left[\log\frac{s+z}{1+z} - \frac{(s-1)}{(1+z)} + \frac{(s-1)^2}{2(1+z)^2} - \frac{(s-1)^3}{3(1+z)^3}\right]_0^\infty$$

$$= \frac{12}{(s-1)^3}\left[-\frac{\log s}{s-1} + (1 - \frac{(s-1)}{2} + \frac{(s-1)^2}{3}\right]$$

Therefore

$$\gamma = \frac{\frac{1}{12}(s-1)^3}{\left(1-\frac{(s-1)}{2}+\frac{(s-1)^2}{3}\right) - \frac{\log s}{(s-1)}} \tag{B.4}$$

c. $s=1$, $r\neq 1$. For this case the integral reduces to

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(r+z)(1+z)^4} \tag{B.5}$$

This integral is similar to (B.3), hence

$$\gamma = \frac{\frac{1}{12}(r-1)^3}{\left(1-\frac{(r-1)}{2}+\frac{(r-1)^2}{3}\right) - \frac{\log r}{(r-1)}} \tag{B.6}$$

d. $r=s\neq 1$. (B.1) reduces to

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(r+z)^2(1+z)^3} \tag{B.7}$$

Separating into partial fractions

$$\frac{1}{\gamma} = 12\int_0^\infty \Big[\frac{-3}{(r-1)^4}\frac{dz}{(r+z)} + \frac{-1}{(r-1)^3}\frac{dz}{(r+z)^2} + \frac{3}{(r-1)^4}\frac{dz}{(1+z)} + \frac{-2}{(r-1)^3}\frac{dz}{(1+z)^2} + \frac{1}{(r-1)^2}\frac{dz}{(1+z)^3}\Big]$$

$$\frac{1}{\gamma} = \frac{12}{(r-1)^3}\left[\frac{-3}{(r-1)}\log\frac{r+z}{1+z} + \frac{1}{(r+z)} + \frac{2}{(1+z)} - \frac{(r-1)}{2(1+z)^2}\right]_0^\infty$$

$$= \frac{12}{(r-1)^3}\left[\frac{3\log r}{(r-1)} - \left\{\frac{1}{r} + 2 - \frac{(r-1)}{2}\right\}\right]$$

hence

$$\gamma = \frac{\frac{1}{12}(r-1)^3}{\frac{3\log r}{(r-1)} - \left\{\frac{1}{r} + 2 - \frac{(r-1)}{2}\right\}} \tag{B.8}$$

e. $r = s = 1$. The integral reduces to the form

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(1+z)^5} \tag{B.9}$$

$$= -\frac{12}{4}\left.\frac{1}{(1+z)^4}\right]_0^\infty = 3$$

therefore

$$\gamma = \frac{1}{3} \tag{B.10}$$

2. Evaluation of the integral

$$\frac{1}{\gamma} = \frac{12}{r}\int_0^\infty \frac{dz}{(s+z)(1+z)^3} \tag{B.11}$$

a. $s \neq 1$. Separating into partial fractions

$$\frac{1}{\gamma} = \frac{12}{r}\int_0^\infty\left[\frac{-1}{(s-1)^3}\frac{dz}{(s+z)} + \frac{1}{(s-1)^3}\frac{dz}{(1+z)} + \frac{-1}{(s-1)^2}\frac{dz}{(1+z)^2} + \frac{1}{(s-1)}\frac{dz}{(1+z)^3}\right]$$

$$\frac{1}{\gamma} = \frac{12}{r(s-1)^2}\left[\frac{-1}{(s-1)}\log\frac{s+z}{1+z} + \frac{1}{1+z} - \frac{s-1}{2(1+z)^2}\right]_0^\infty$$

$$= \frac{12}{r(s-1)^2}\left[\frac{\log s}{(s-1)} - \left\{1 - \frac{(s-1)}{2}\right\}\right]$$

Therefore

$$\gamma = \frac{r}{12}\left[\frac{(s-1)^2}{\frac{\log s}{(s-1)} - \frac{(3-s)}{2}}\right] \tag{B.12}$$

b. $s = 1$. The integral reduces to

$$\frac{1}{\gamma} = \frac{12}{r}\int_0^\infty \frac{dz}{(1+z)^4} \tag{B.13}$$

$$= -\frac{12}{3r}\left.\frac{1}{(1-z)^3}\right]_0^\infty = \frac{4}{r}$$

hence

$$\gamma = \frac{r}{4} \tag{B.14}$$

3. Evaluation of the integral

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(s+z)(1+z)(r+z)^3} \tag{B.15}$$

a. For $r \neq s \neq 1$. Separating into partial fractions

$$\frac{1}{\gamma} = 12\int_0^\infty \left[\frac{-1}{(s-1)(r-s)^3}\,\frac{dz}{(s+z)} + \frac{1}{(s-1)(r-1)^3}\,\frac{dz}{(1+z)}\right.$$

$$+ \left\{\frac{1}{(r-s)(r-1)^3} + \frac{1}{(r-s)^2(r-1)^2} + \frac{1}{(r-s)^3(r-1)}\right\}\frac{dz}{(r+z)}$$

$$\left.+ \left\{\frac{1}{(r-s)(r-1)^2} + \frac{1}{(r-s)^2(r-1)}\right\}\frac{dz}{(r+z)^2} + \frac{1}{(r-s)(r-1)}\,\frac{dz}{(r+z)^3}\right]$$

$$\frac{1}{\gamma} = \left[\frac{12}{(s-1)(r-s)^3(r-1)^3}\left\{-r^3\log\frac{s+z}{1+z} + (3r^2-3r+1)\log\frac{s+z}{r+z}\right.\right.$$

$$\left. + (3r^2s - 3rs^2 + s^3)\log\frac{r+z}{1+z}\right\}$$

$$\left. - \frac{12}{(r-s)^2(r-1)^2}\left\{\frac{(r-s)+(r-1)}{(r+z)} + \frac{(r-s)(r-1)}{2(r+z)^2}\right\}\right]_0^\infty$$

$$= \frac{12}{(r-s)(s-1)(r-1)}\left[\frac{(r-1)}{(r-s)^2}\log\frac{s}{r} + \frac{(r-s)}{(r-1)^2}\log r\right.$$

$$\left. + \frac{s-1}{r}\left\{\frac{1}{2} + \frac{1}{(r-s)} + \frac{1}{(r-1)}\right\}\right]$$

hence

$$\gamma = \frac{\dfrac{(r-s)(s-1)(r-1)}{12}}{\left[\dfrac{(r-1)}{(r-s)^2}\log\dfrac{s}{r} + \dfrac{(r-s)}{(r-1)^2}\log r + \dfrac{s-1}{r}\left\{\dfrac{1}{2} + \dfrac{1}{(r-s)} + \dfrac{1}{(r-1)}\right\}\right]} \tag{B.16}$$

b. $r = 1$, $s \neq 1$. Integral (B.15) reduces to

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(s+z)(1+z)^4}$$

which is identical to (B.3). The value is therefore given by (B.4)

c. $s = 1$, $r \neq 1$. Integral (B.15) reduces to

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(1+z)^2(r+z)^3} \tag{B.17}$$

Separating into partial fractions

$$\frac{1}{\gamma} = 12\int_0^\infty \left[\frac{-3}{(r-1)^4}\frac{dz}{(1+z)} + \frac{1}{(r-1)^3}\frac{dz}{(1+z)^2} + \frac{3}{(r-1)^4}\frac{dz}{(r+z)}\right.$$

$$\left. + \frac{2}{(r-1)^3}\frac{dz}{(r+z)^2} + \frac{1}{(r-1)^2}\frac{dz}{(r+z)^3}\right]$$

$$\frac{1}{\gamma} = \frac{12}{(r-1)^3}\left[-\frac{3}{(r-1)}\log r + \left\{1 + \frac{2}{r} + \frac{(r-1)}{2r^2}\right\}\right] \tag{B.18}$$

d. $r = s \neq 1$. Integral (B.15) reduces to

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(1+z)(r+z)^4} \tag{B.19}$$

Separating into partial fractions

$$\frac{1}{\gamma} = 12\int_0^\infty \left[\frac{1}{(r-1)^4}\,\frac{dz}{(1+z)} + \frac{-1}{(r-1)^4}\,\frac{dz}{(r+z)} + \frac{-1}{(r-1)^3}\,\frac{dz}{(r+z)^2} + \frac{-1}{(r-1)^2}\,\frac{dz}{(r+z)^3} + \frac{-1}{(r-1)}\,\frac{dz}{(r+z)^4}\right]$$

$$= \frac{12}{(r-1)^3}\left[\frac{-1}{(r-1)}\log\frac{r+z}{1+z} + \left\{\frac{1}{(r+z)} + \frac{(r-1)}{2(r+z)^2} + \frac{(r-1)^2}{3(r+z)^3}\right\}\right]_0^\infty$$

$$= \frac{12}{(r-1)^3}\left[\frac{\log r}{(r-1)} - \left\{\frac{1}{r} + \frac{(r-1)}{2r^2} + \frac{(r-1)^2}{3r^3}\right\}\right] \tag{B.20}$$

e. $r = s = 1$. The integral reduces to (B.9), therefore

$$\gamma = \frac{1}{3}$$

APPENDIX C

Evaluation of the integral

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(1+z)(r+z)^3} \tag{C.1}$$

a. For $r \neq 1$. Separating into partial fractions

$$\frac{1}{\gamma} = 12\int_0^\infty \left[\frac{1}{(r-1)^3}\,\frac{dz}{(1+z)} + \frac{-1}{(r-1)^3}\,\frac{dz}{(r+z)} + \frac{-1}{(r-1)^2}\,\frac{dz}{(r+z)^2} + \frac{-1}{(r-1)}\,\frac{dz}{(r+z)^3}\right]$$

$$= \frac{12}{(r-1)^3}\left[-\log\frac{r+z}{1+z} + (r-1)\left\{\frac{1}{(r+z)} + \frac{(r-1)}{2(r+z)^2}\right\}\right]_0^\infty$$

$$= \frac{12}{(r-1)^2}\left[\frac{\log r}{(r-1)} - \left\{\frac{1}{r} + \frac{(r-1)}{2r^2}\right\}\right]$$

Hence

$$\gamma = \frac{\dfrac{(r-1)^2}{12}}{\dfrac{\log r}{(r-1)} - \left\{\dfrac{1}{r} + \dfrac{(r-1)}{2r^2}\right\}} \tag{C.2}$$

b. For $r = 1$. (C.1) reduces to

$$\frac{1}{\gamma} = 12\int_0^\infty \frac{dz}{(1+z)^4} \tag{C.3}$$

$$= \left[\frac{-4}{(1-z)^3}\right]_0^\infty = 4$$

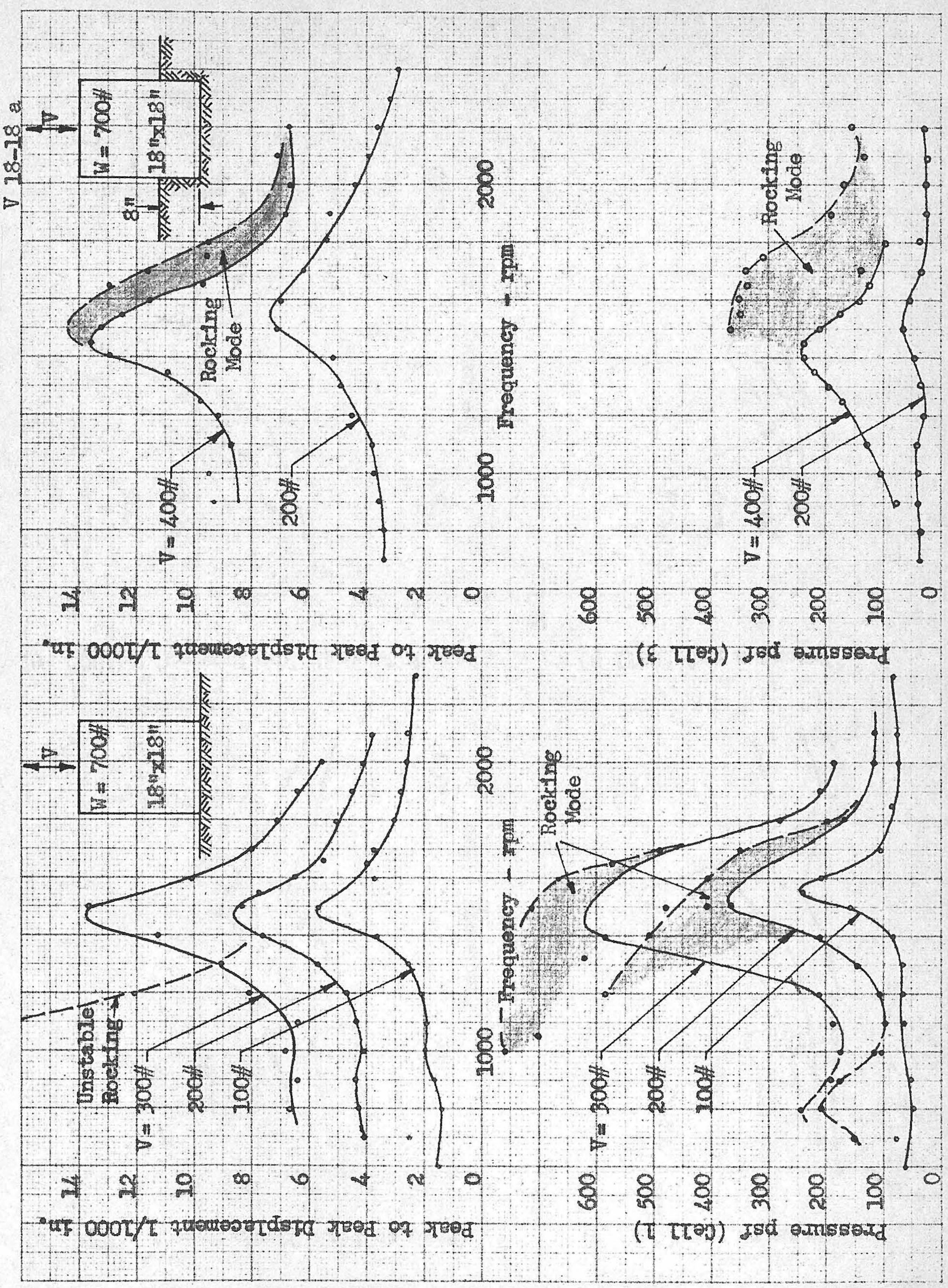
Therefore

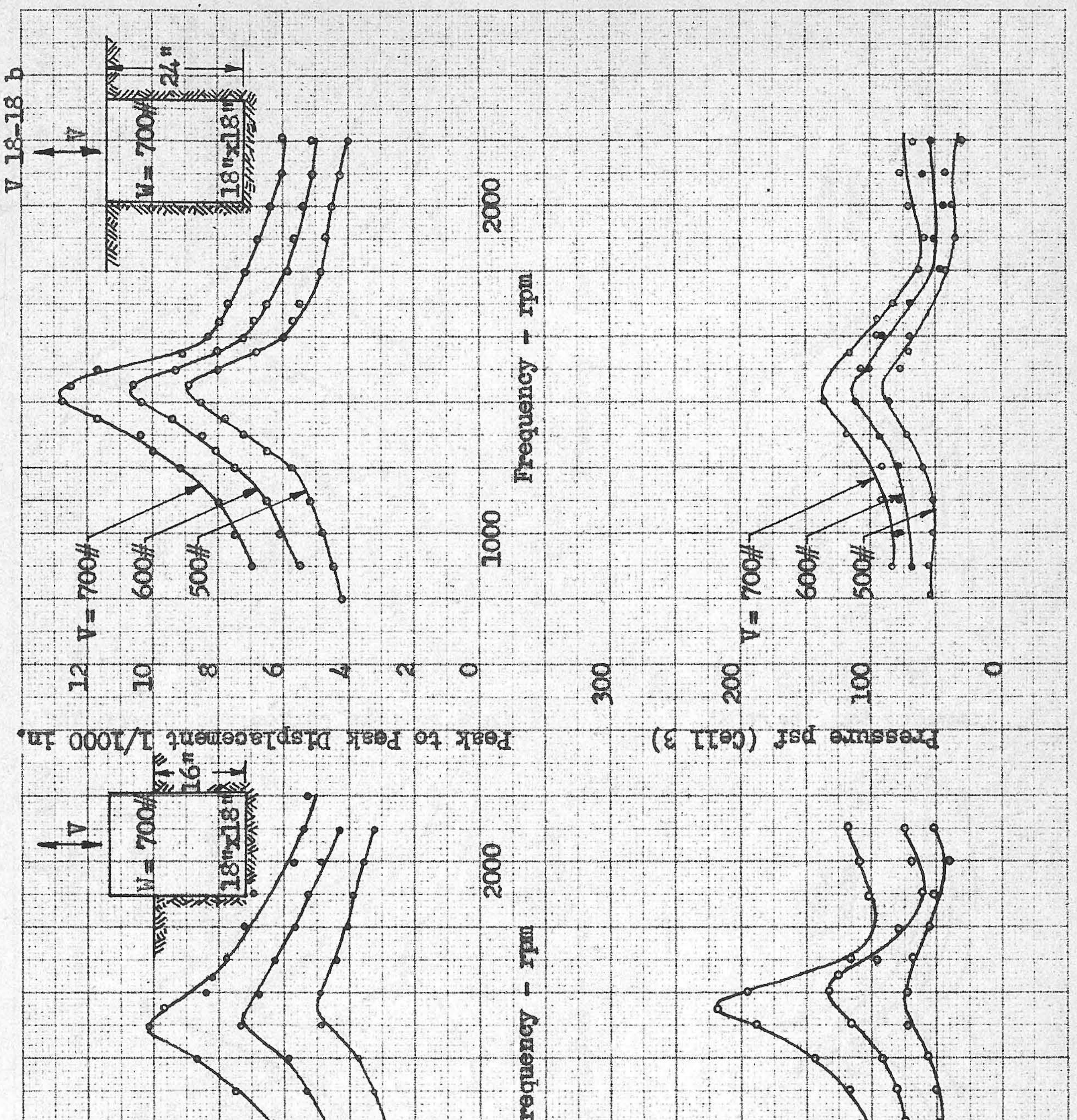
$$\gamma = \frac{1}{4} \tag{C.4}$$

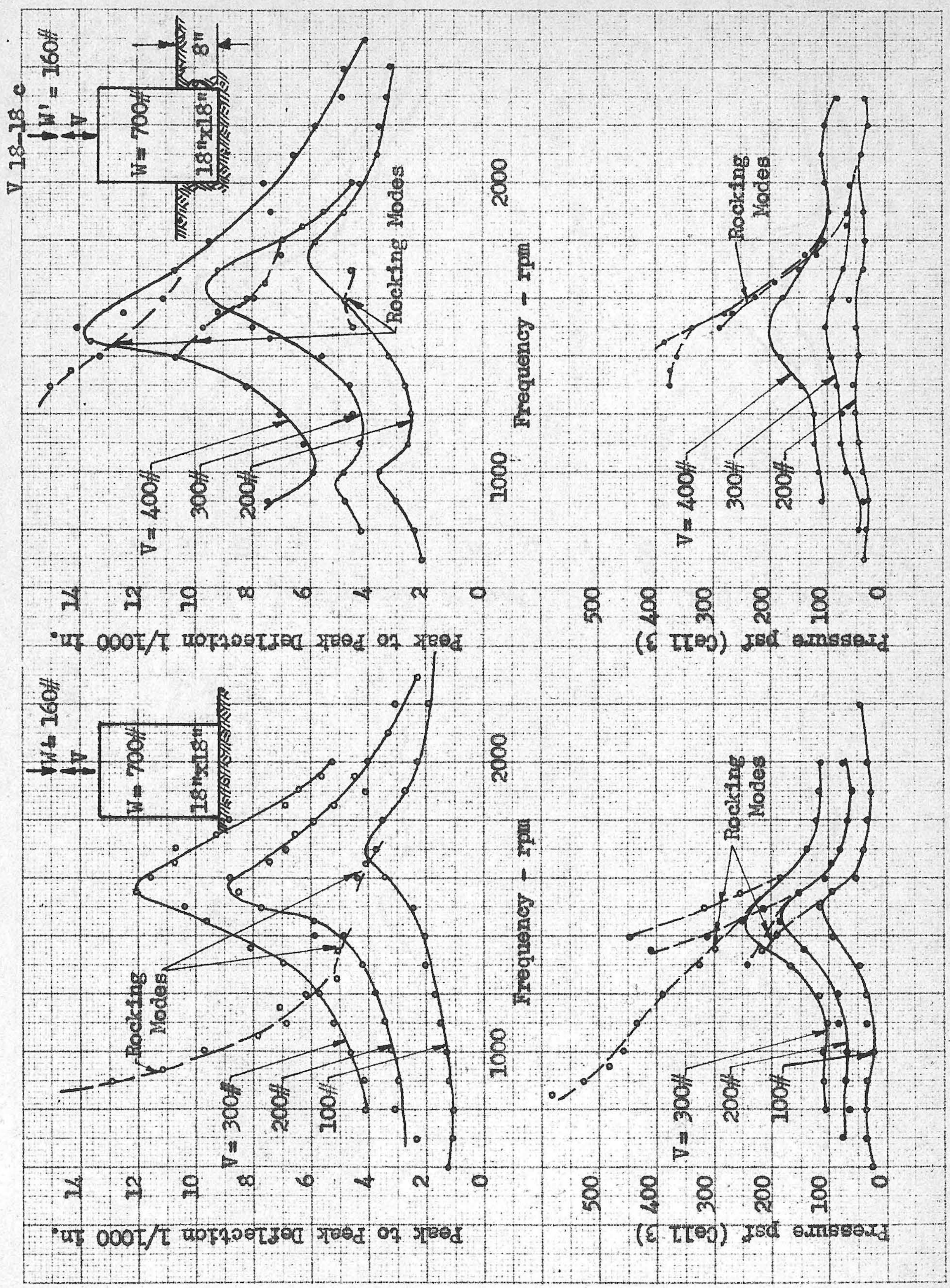
APPENDIX D

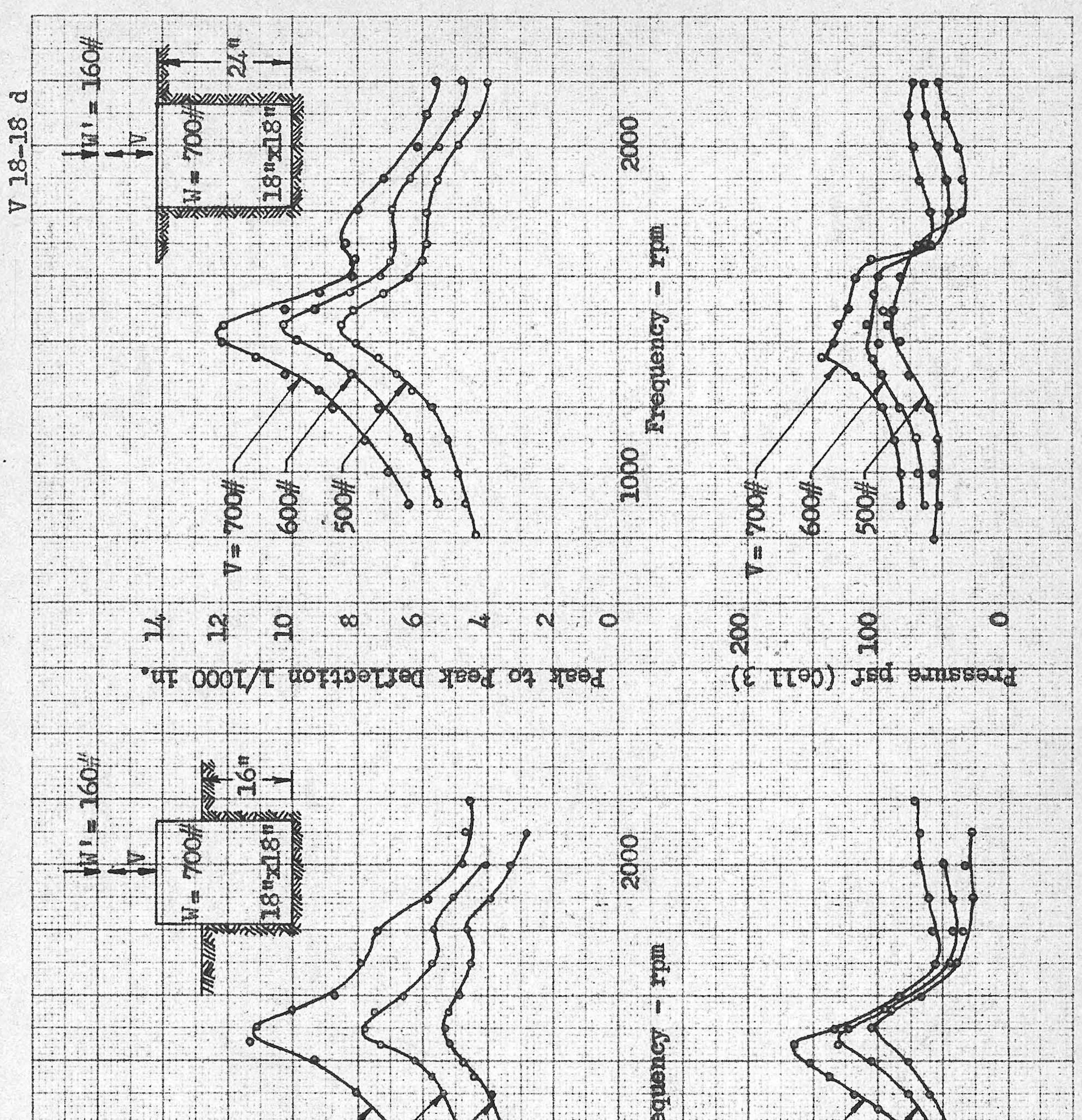
Experimental Data

Displacement and Pressure Response Curves

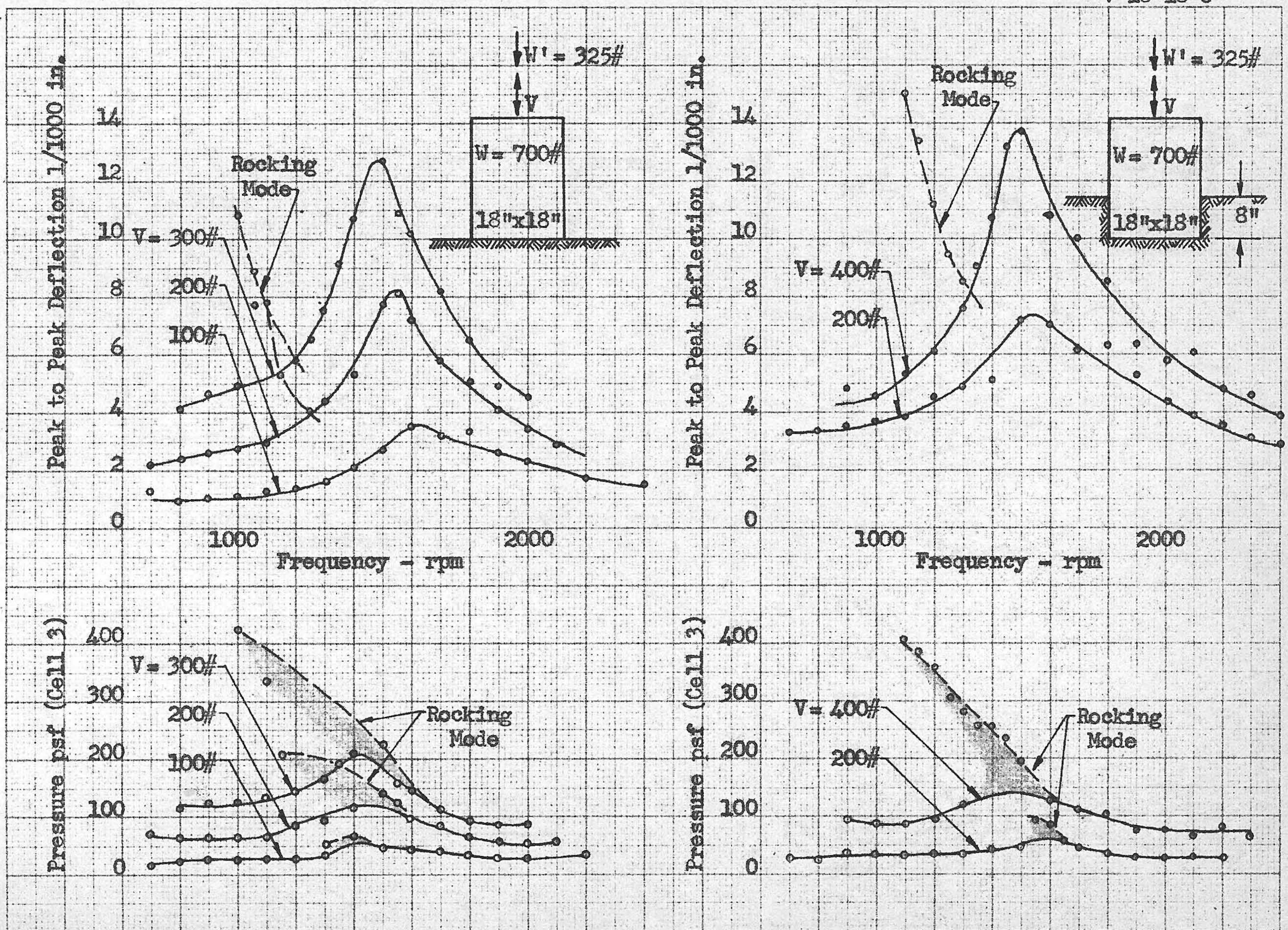

V 18-18 a
W = 700#
18"x18"
8"
Unstable Rocking
Rocking Mode
V = 300#
200#
100#
V = 400#
200#
Frequency - rpm
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 1)
Pressure psf (Cell 3)

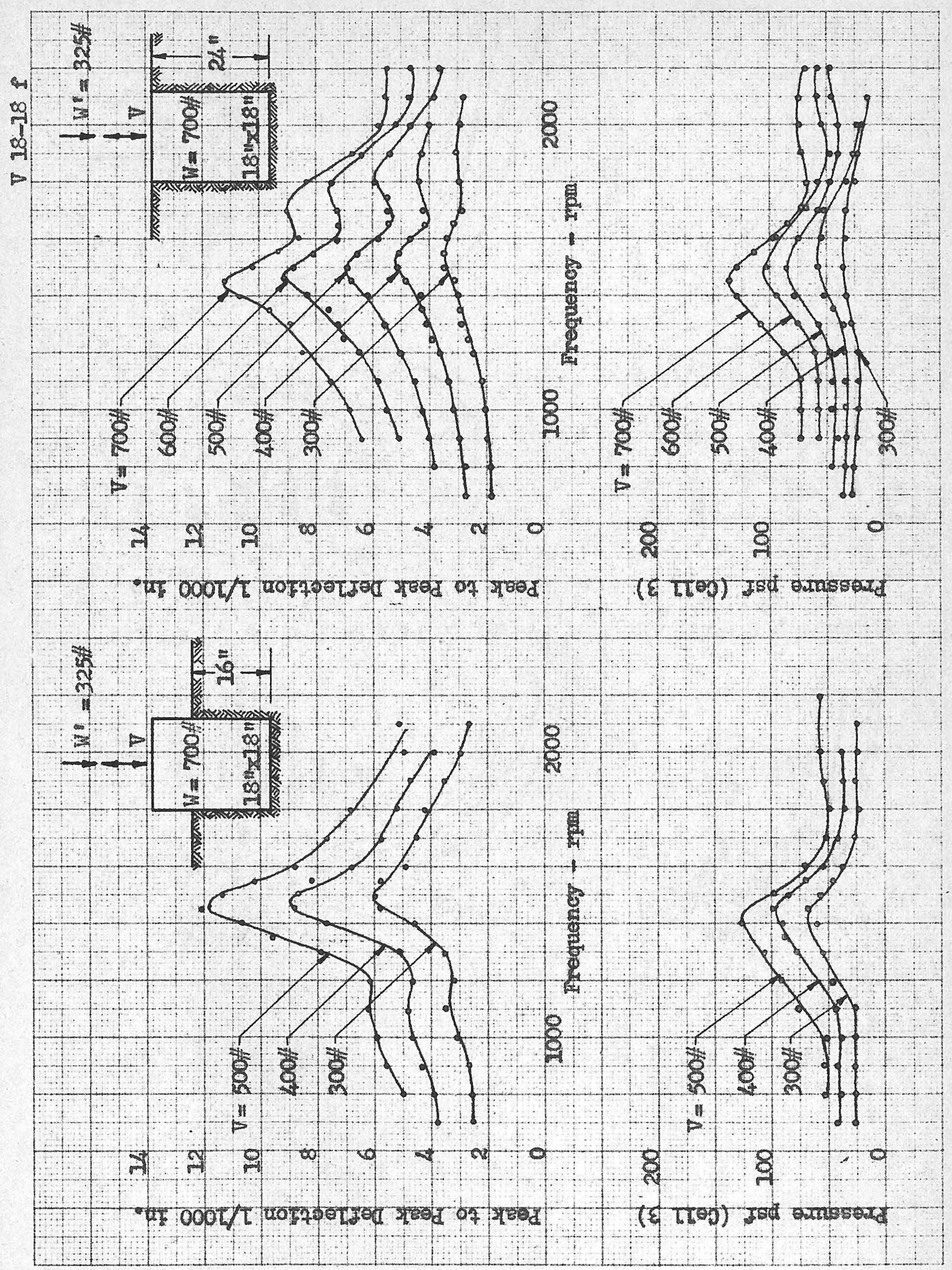

V 18-18 b
V
W = 700#
18"x18"
24"
16"
V = 700#
600#
500#
V = 500#
400#
300#
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm
12
10
8
6
4
2
0
300
200
100
1000
2000


V 18-18-c
W' = 160#
V
W = 700#
18"x18"
8"
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm
Rocking Modes
V = 400#
300#
200#
W'= 160#
V = 300#
100#
0
1000
2000
2
4
6
8
10
12
14
100
200
300
400
500


V 18-18 d
W' = 160#
W = 700#
18"x18"
24"
V = 700#
600#
500#
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm
W' = 160#
W = 700#
18"x18"
16"
V = 500#
400#
300#
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm

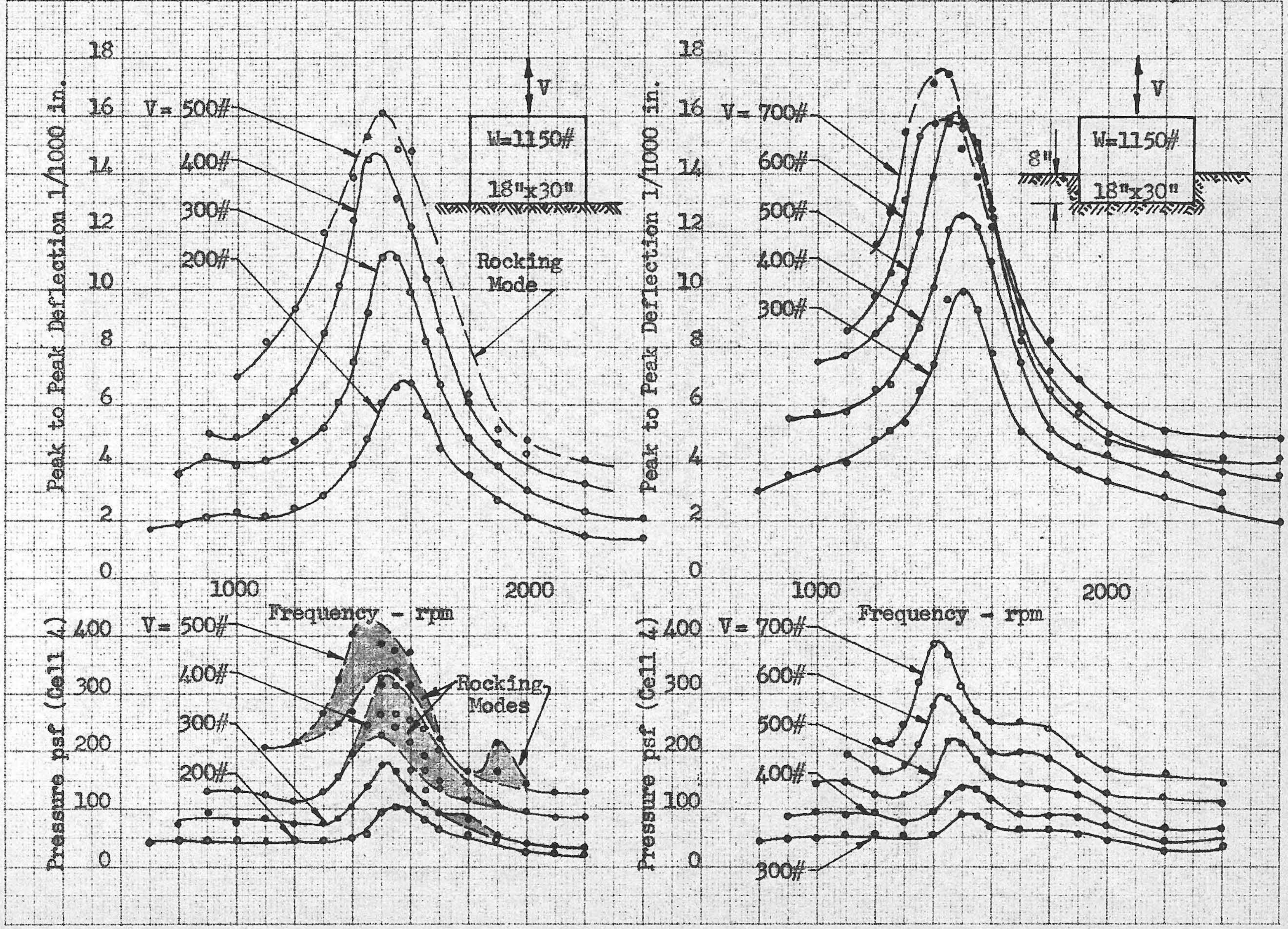
V 18-18 e


W' = 325#
V
W = 700#
18"x18"
8"
Rocking Mode
V = 300#
200#
100#
V = 400#
200#
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm
0
2
4
6
8
10
12
14
100
200
300
400
1000
2000

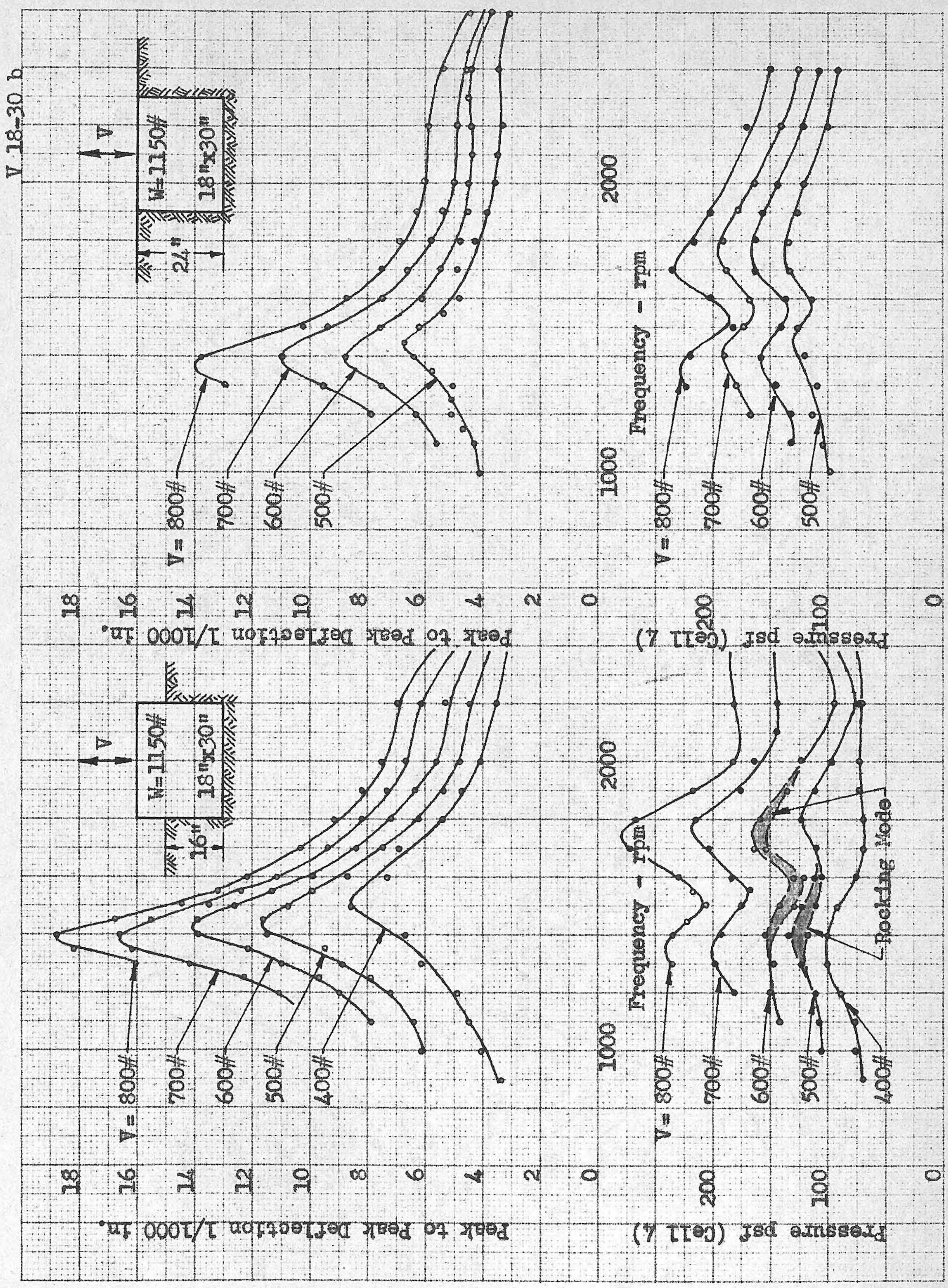


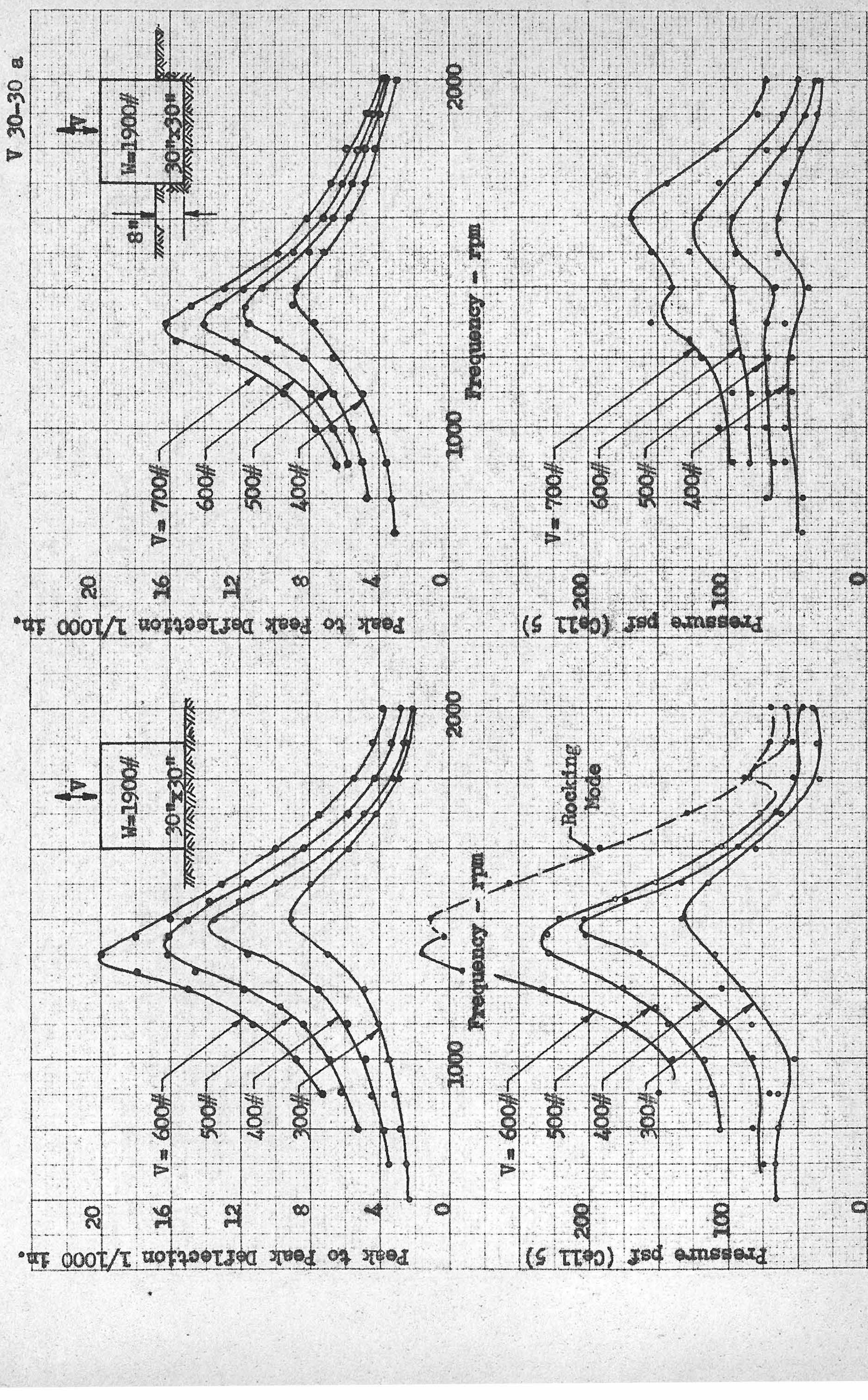
V 18-18 f
W' = 325#
V
W = 700#
18"x18"
24"
V = 700#
600#
500#
400#
300#
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm
W' = 325#
V
W = 700#
18"x18"
16"
V = 500#
400#
300#
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm

V 18-30 a

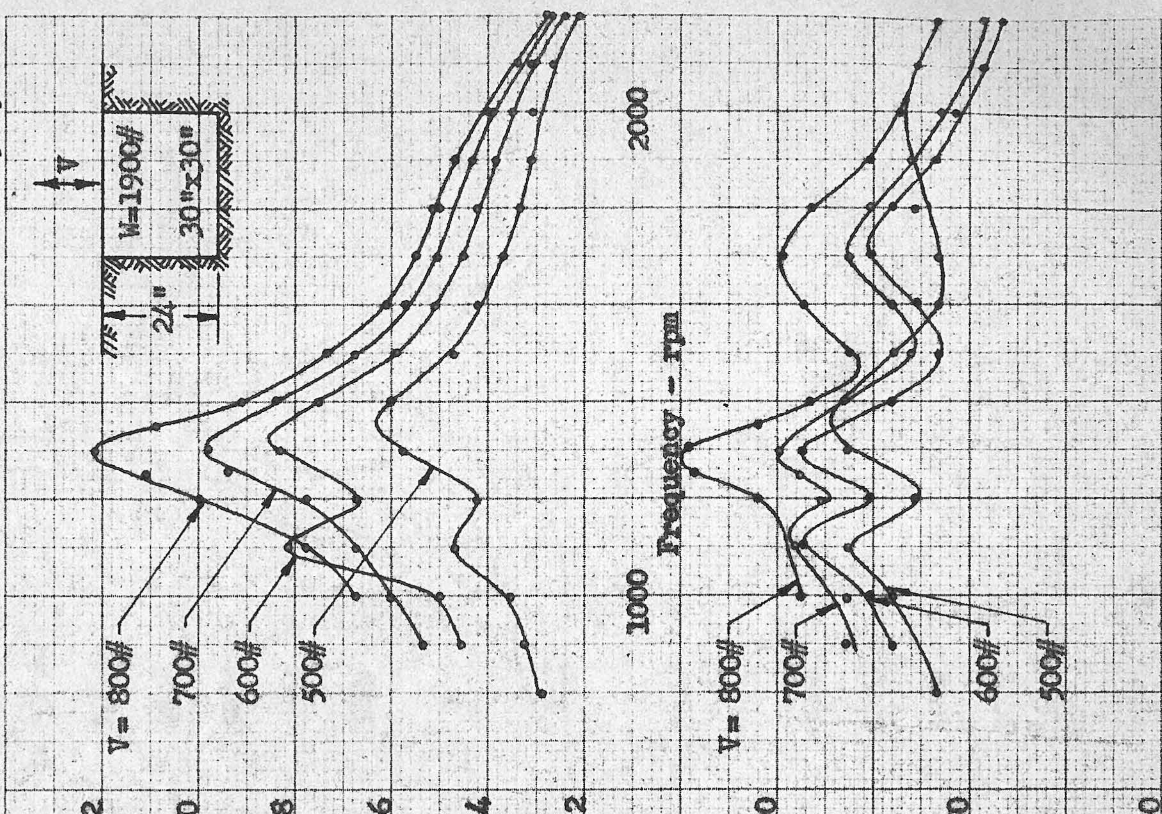


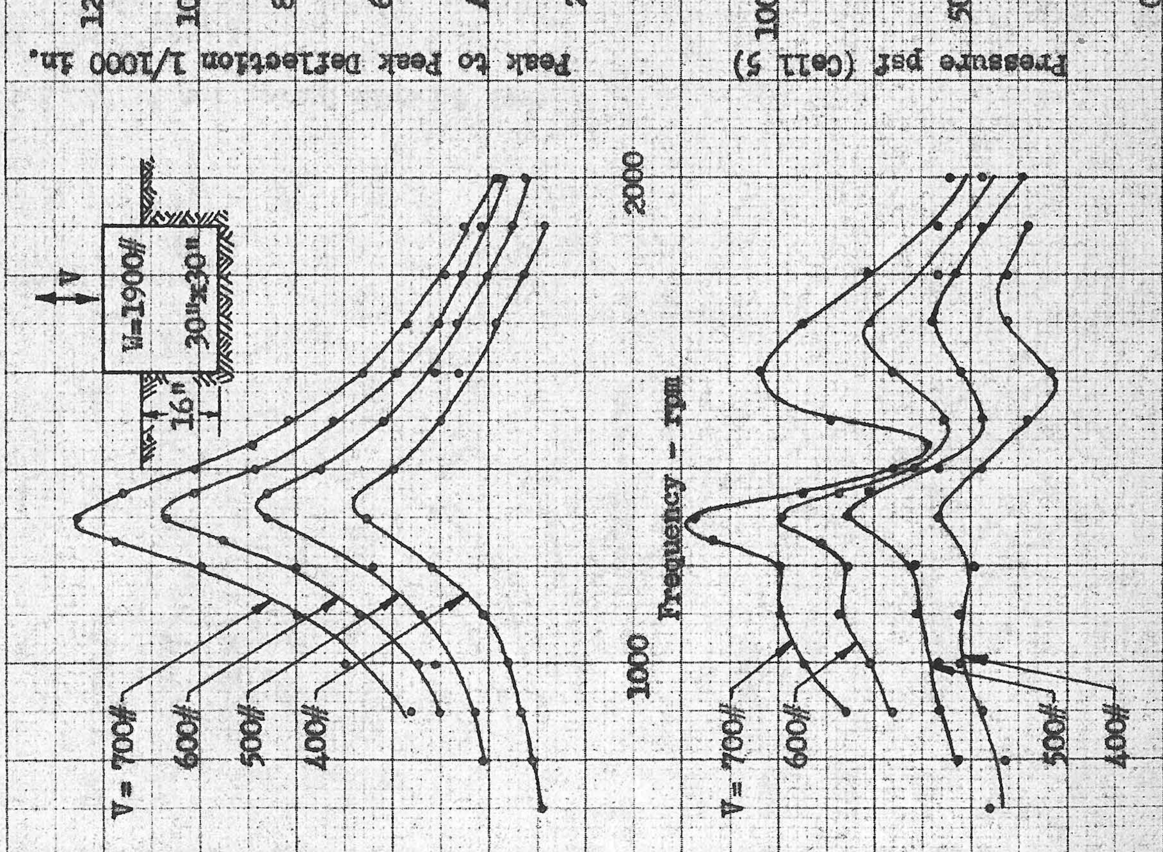
Peak to Peak Deflection 1/1000 in.
18
16
14
12
10
8
6
4
2
0
V= 500#
400#
300#
200#
V
W=1150#
18"x30"
Rocking
Mode
1000
2000
Frequency - rpm
V= 700#
600#
500#
400#
300#
8"
Pressure psf (Cell 4)
400
300
200
100
0
Rocking
Modes


V 18-30 b
W=1150#
18"x30"
24"
V
V = 800#
700#
600#
500#
Peak to Peak Deflection 1/1000 in.
Frequency - rpm
Pressure psf (Cell 4)
W=1150#
18"x30"
16"
V
V = 800#
700#
600#
500#
400#
Rocking Mode
Peak to Peak Deflection 1/1000 in.
Frequency - rpm
Pressure psf (Cell 4)

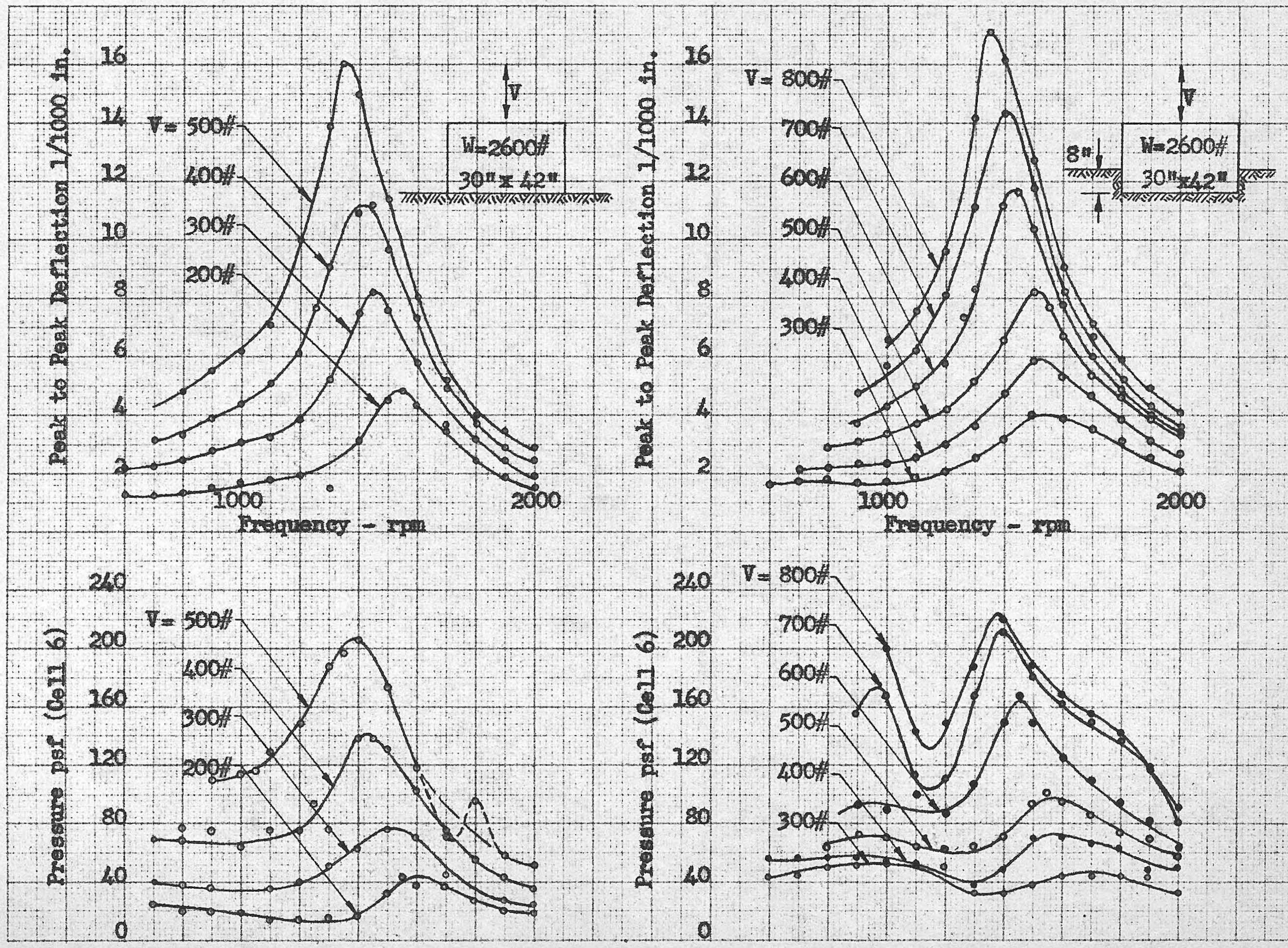

V 30-30 a
W=1900#
30"x30"
8"
V
V = 600#
500#
400#
300#
V = 700#
600#
500#
400#
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 5)
Frequency - rpm
Rocking Mode
0
4
8
12
16
20
100
200
1000
2000

V 30-30 b

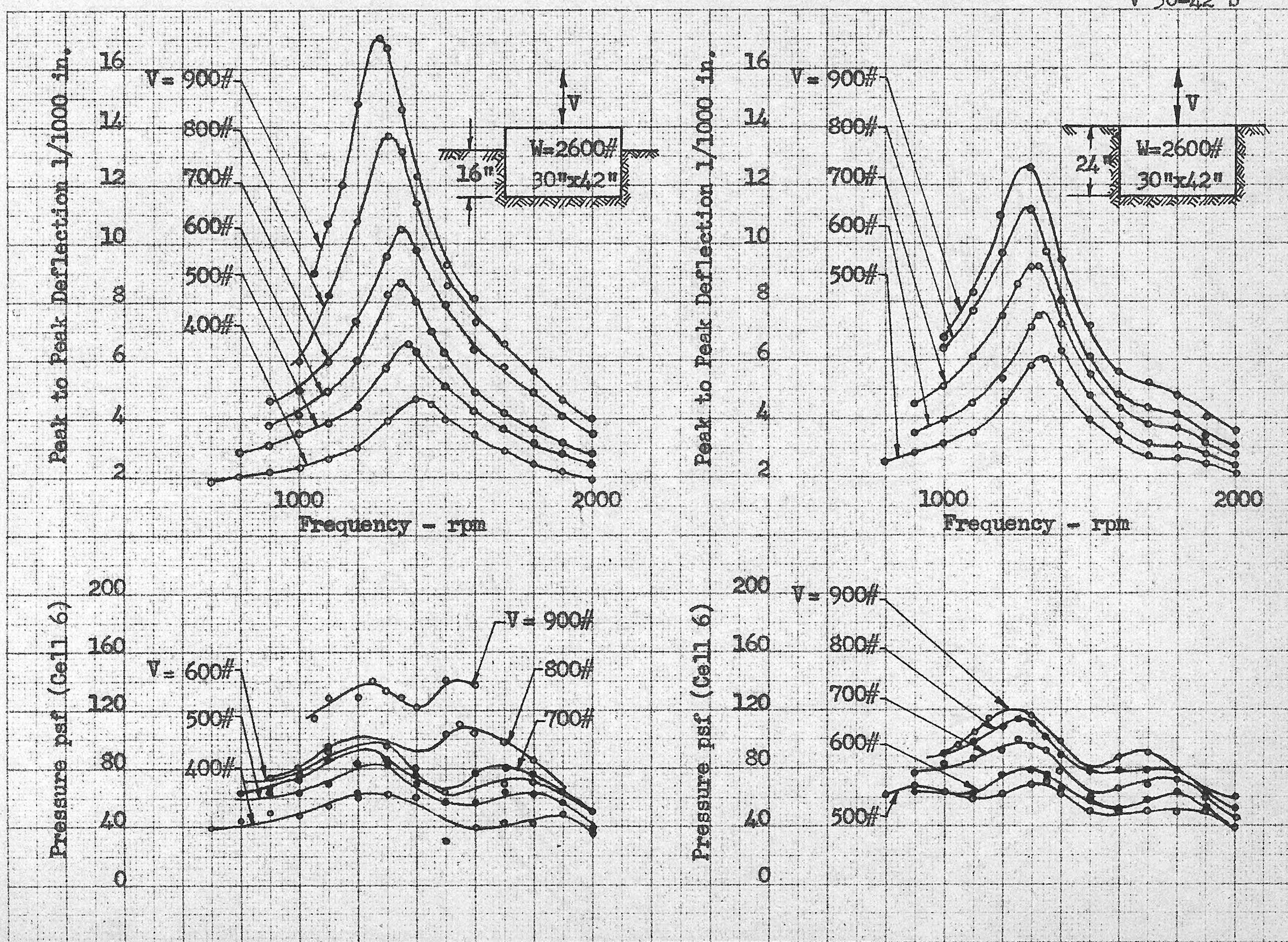

Peak to Peak Deflection 1/1000 in.
12
10
8
6
4
2
V = 800#
700#
600#
500#
1000
2000
Frequency - rpm
W=1900#
30"x30"
24"
V
Pressure psf (Cell 5)
100
50
0
V = 800#
700#
600#
500#


Peak to Peak Deflection 1/1000 in.
12
10
8
6
4
2
V = 700#
600#
500#
400#
1000
2000
Frequency - rpm
W=1900#
30"x30"
16"
V
Pressure psf (Cell 5)
100
50
0
V = 700#
600#
500#
400#

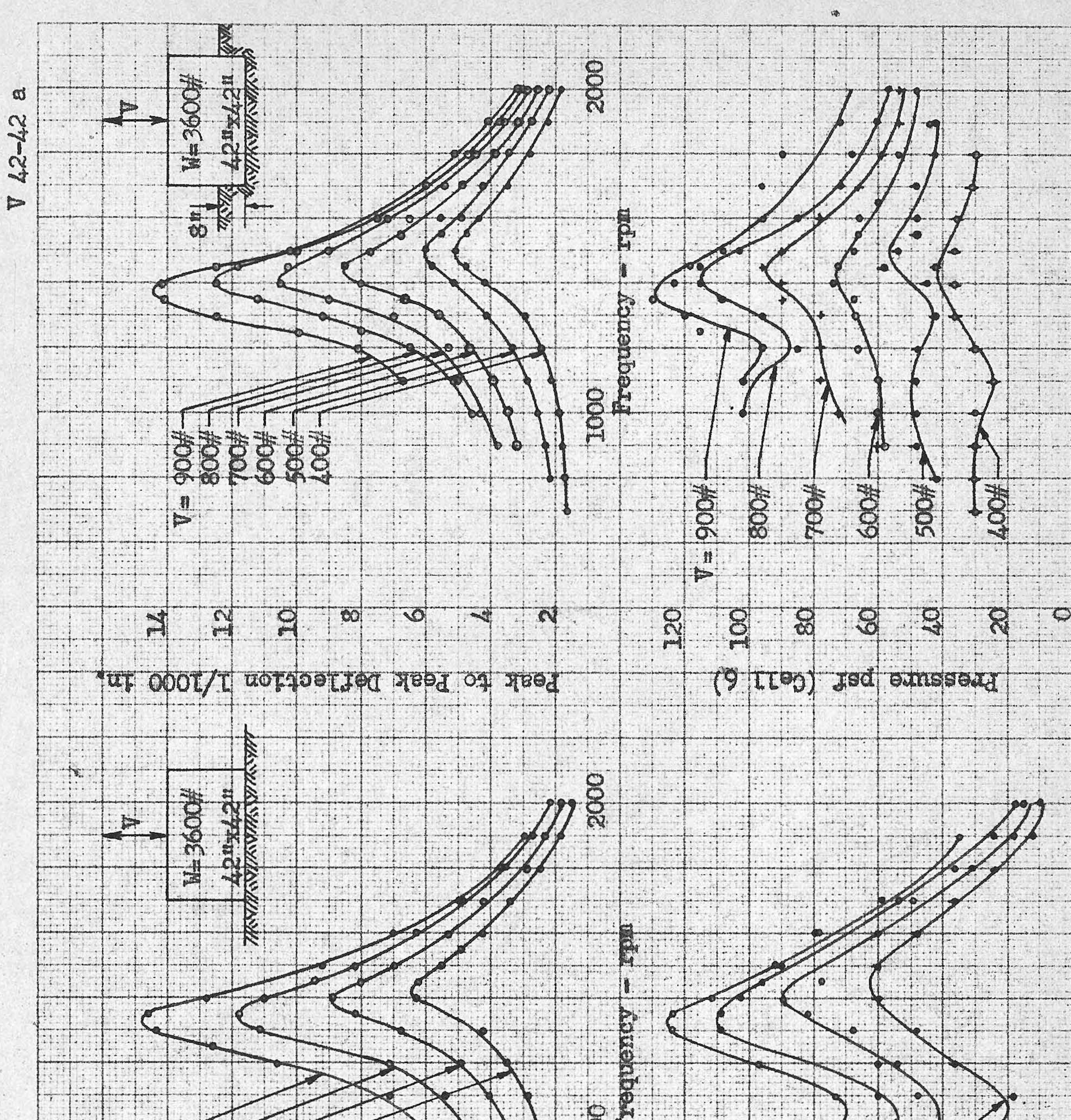
V 30-42 a


Peak to Peak Deflection 1/1000 in.
16
14
12
10
8
6
4
2
V= 500#
400#
300#
200#
W=2600#
30" x 42"
V
1000
2000
Frequency - rpm
Peak to Peak Deflection 1/1000 in.
V= 800#
700#
600#
500#
400#
300#
W=2600#
30"x42"
8"
V
Frequency - rpm
Pressure psf (Cell 6)
240
200
160
120
80
40
0
V= 500#
400#
300#
200#
Pressure psf (Cell 6)
V= 800#
700#
600#
500#
400#
300#

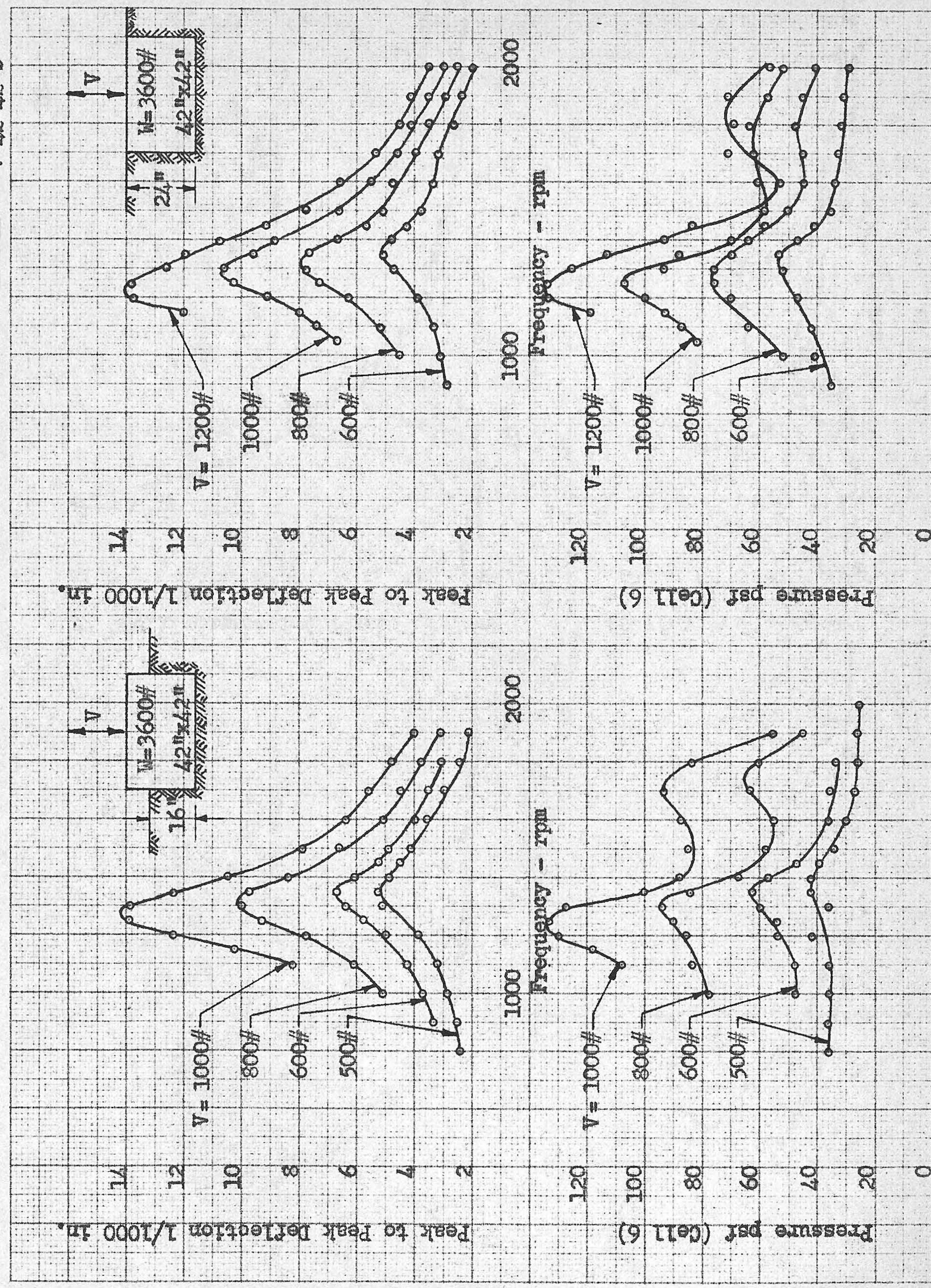
V 30-42 b



Peak to Peak Deflection 1/1000 in.
16
14
12
10
8
6
4
2
V = 900#
800#
700#
600#
500#
400#
W=2600#
30"x42"
16"
V
1000
2000
Frequency - rpm
Pressure psf (Cell 6)
200
160
120
80
40
0
V = 600#
500#
400#
V = 900#
800#
700#
Peak to Peak Deflection 1/1000 in.
V = 900#
800#
700#
600#
500#
W=2600#
30"x42"
24"
V
1000
2000
Frequency - rpm
Pressure psf (Cell 6)
V = 900#
800#
700#
600#
500#

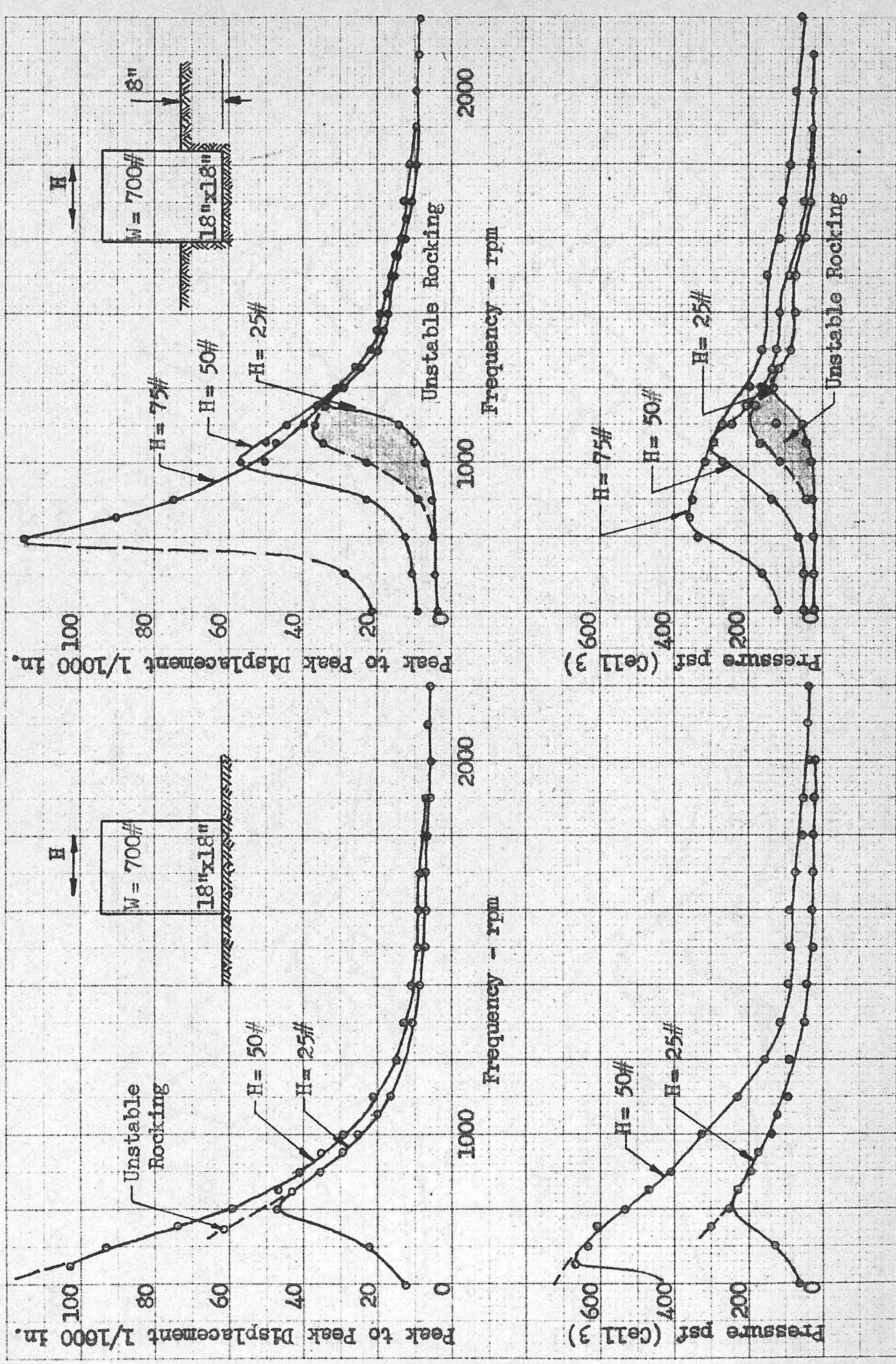

V 42-42 a
V
W=3600#
42"x42"
8"
V = 900#
800#
700#
600#
500#
400#
1000
2000
Frequency - rpm
Peak to Peak Deflection 1/1000 in.
14
12
10
8
6
4
2
120
100
80
60
40
20
0
Pressure psf (Cell 6)
V = 600#
500#
400#
300#

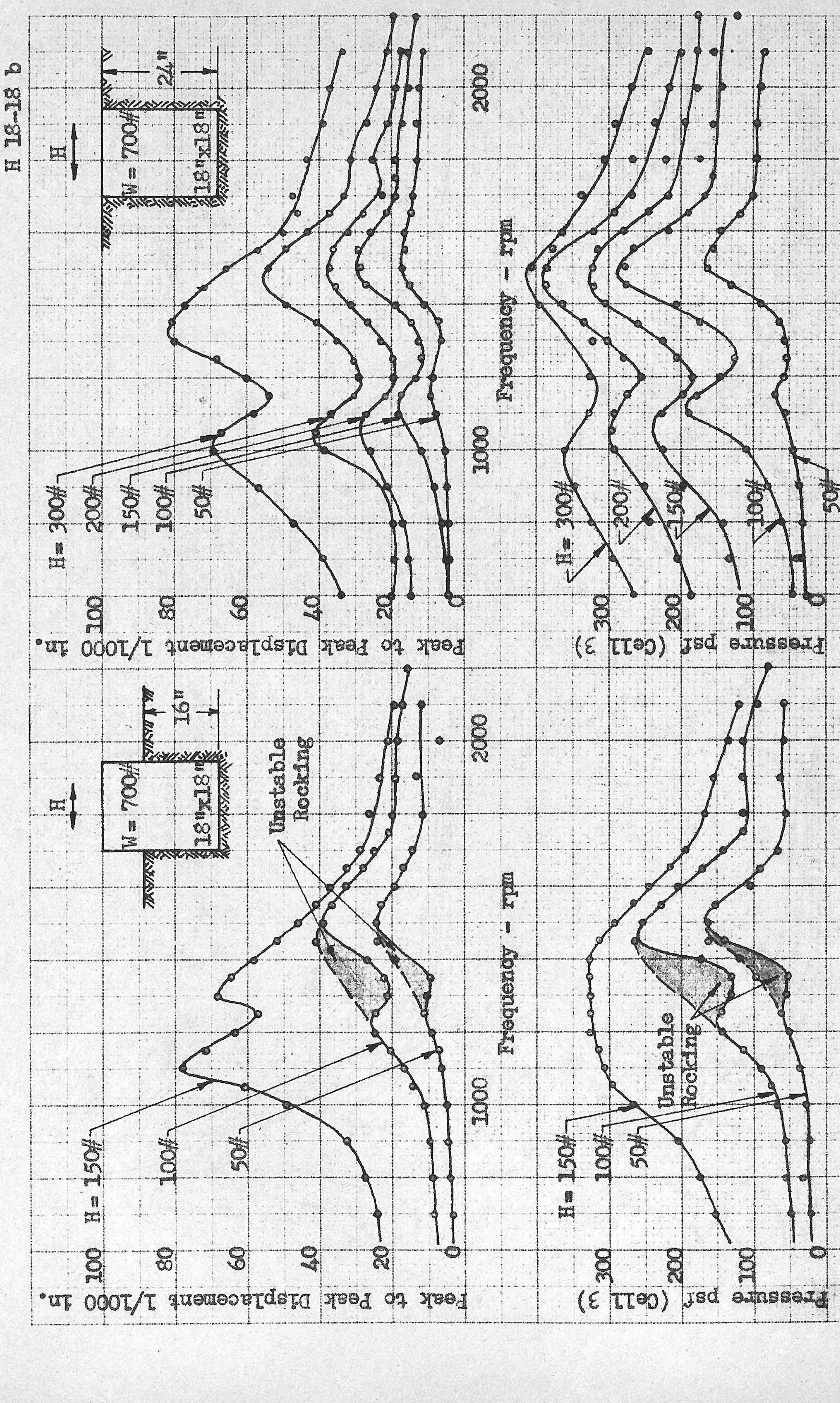
V 42-42 b



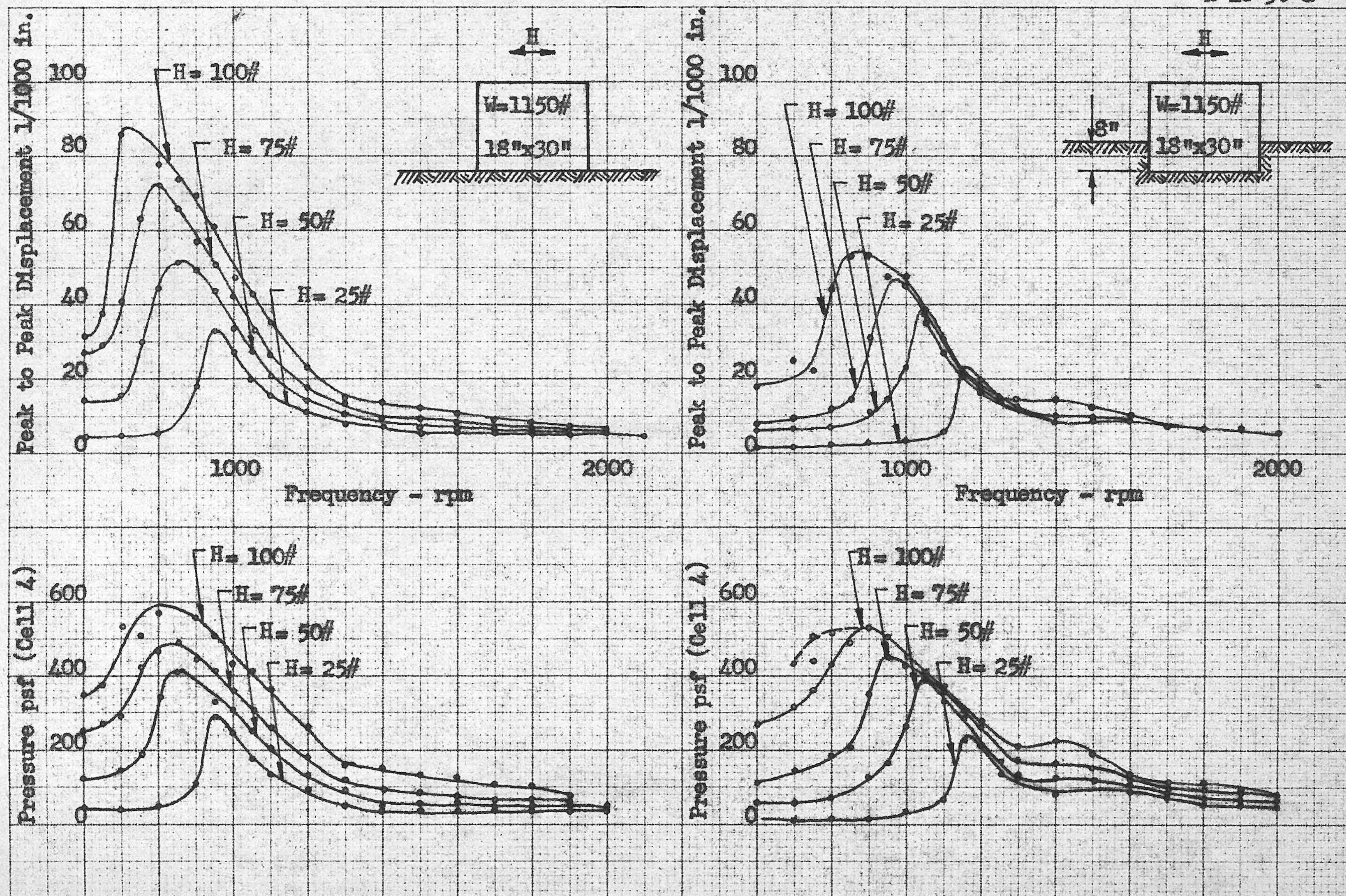
Peak to Peak Deflection 1/1000 in.
Pressure psf (Cell 6)
Frequency - rpm
W=3600#
42"x42"
16"
24"
V
V = 1000#
800#
600#
500#
V = 1200#
1000#
800#
600#
1000
2000

H 18-18 a

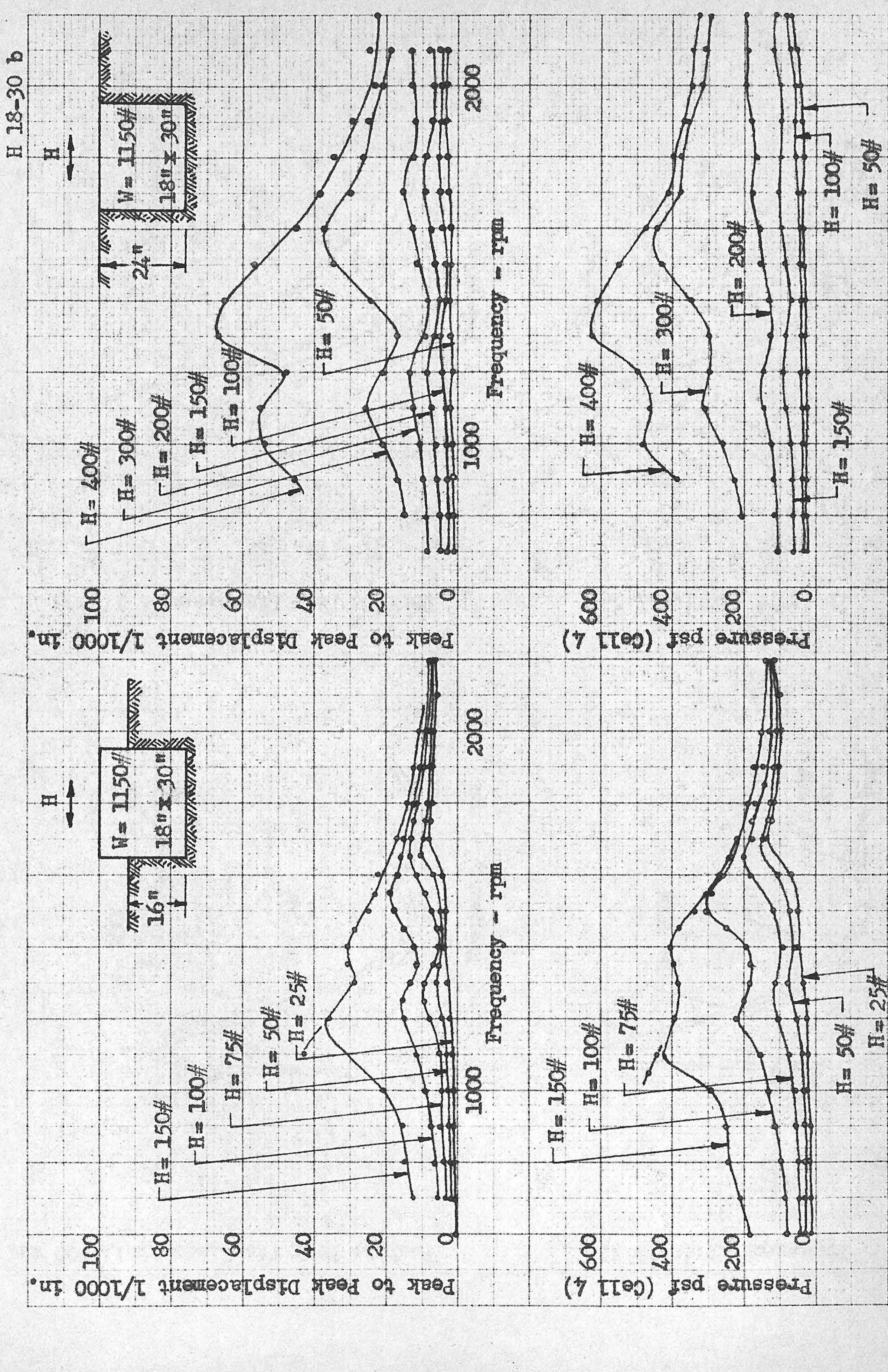

H
W = 700#
18"x18"
8"
Unstable Rocking
H = 75#
H = 50#
H = 25#
Frequency - rpm
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 3)
Unstable Rocking
H = 75#
H = 50#
H = 25#
Frequency - rpm
Pressure psf (Cell 3)
H
W = 700#
18"x18"
Unstable Rocking
H = 50#
H = 25#
Frequency - rpm
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 3)
H = 50#
H = 25#
Frequency - rpm
Pressure psf (Cell 3)

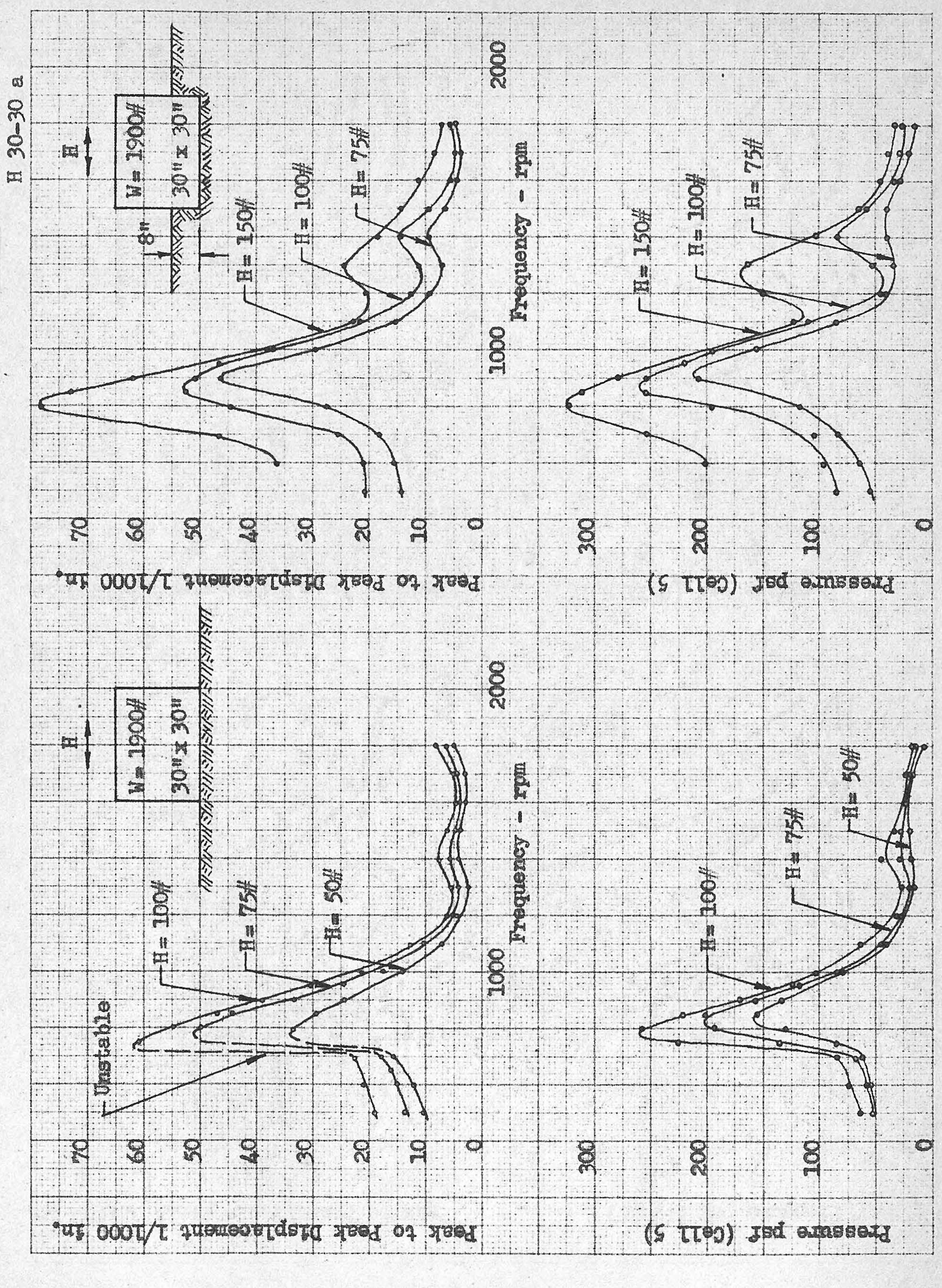

H 18-18 b
H
W = 700#
18"x18"
24"
16"
H = 300#
200#
150#
100#
50#
H = 150#
100#
50#
Unstable Rocking
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 3)
Frequency - rpm
0
20
40
60
80
100
200
300
1000
2000

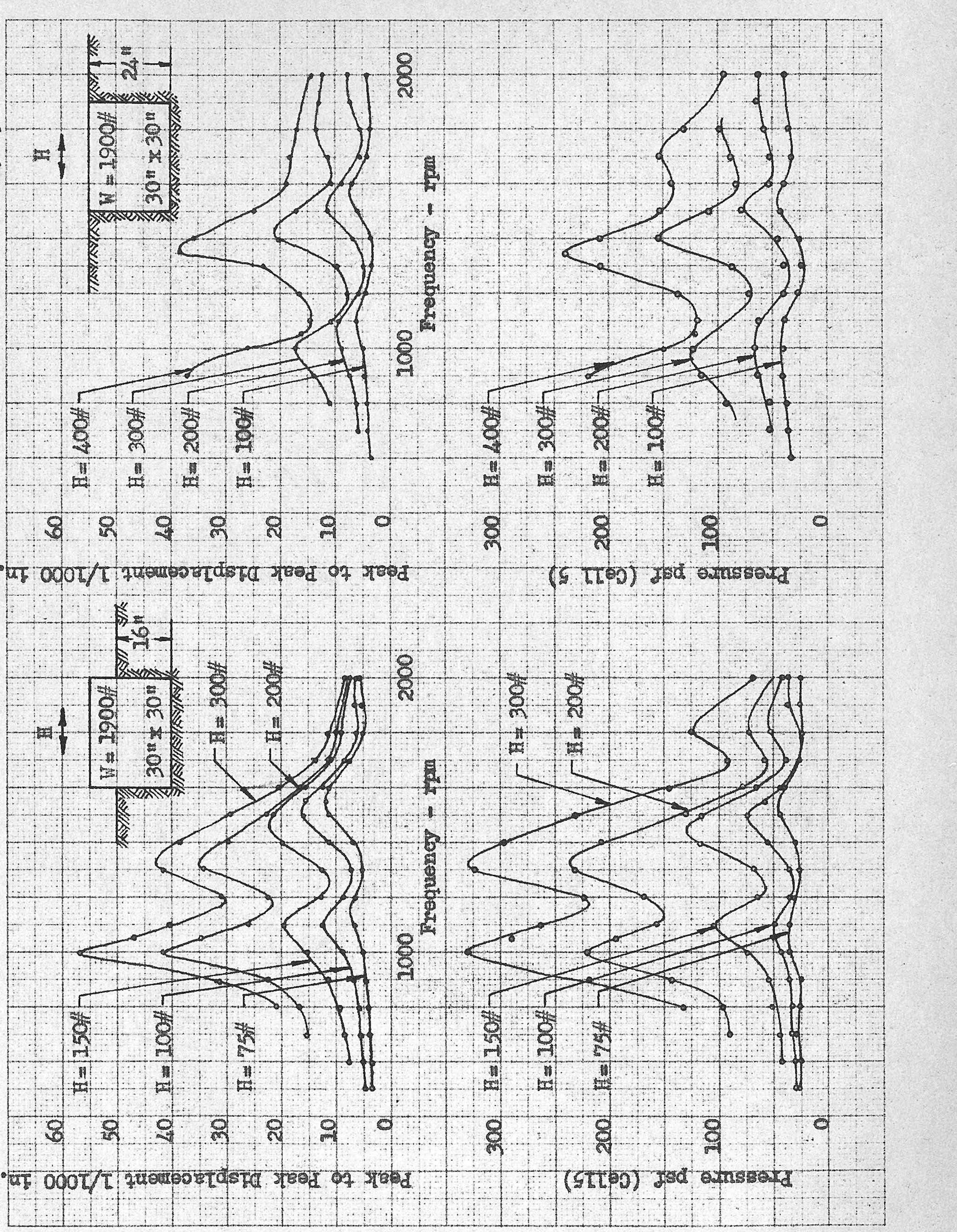
H 18-30 a



W=1150#
18"x30"
H
8"
H= 100#
H= 75#
H= 50#
H= 25#
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 4)
Frequency - rpm
0
20
40
60
80
100
200
400
600
1000
2000

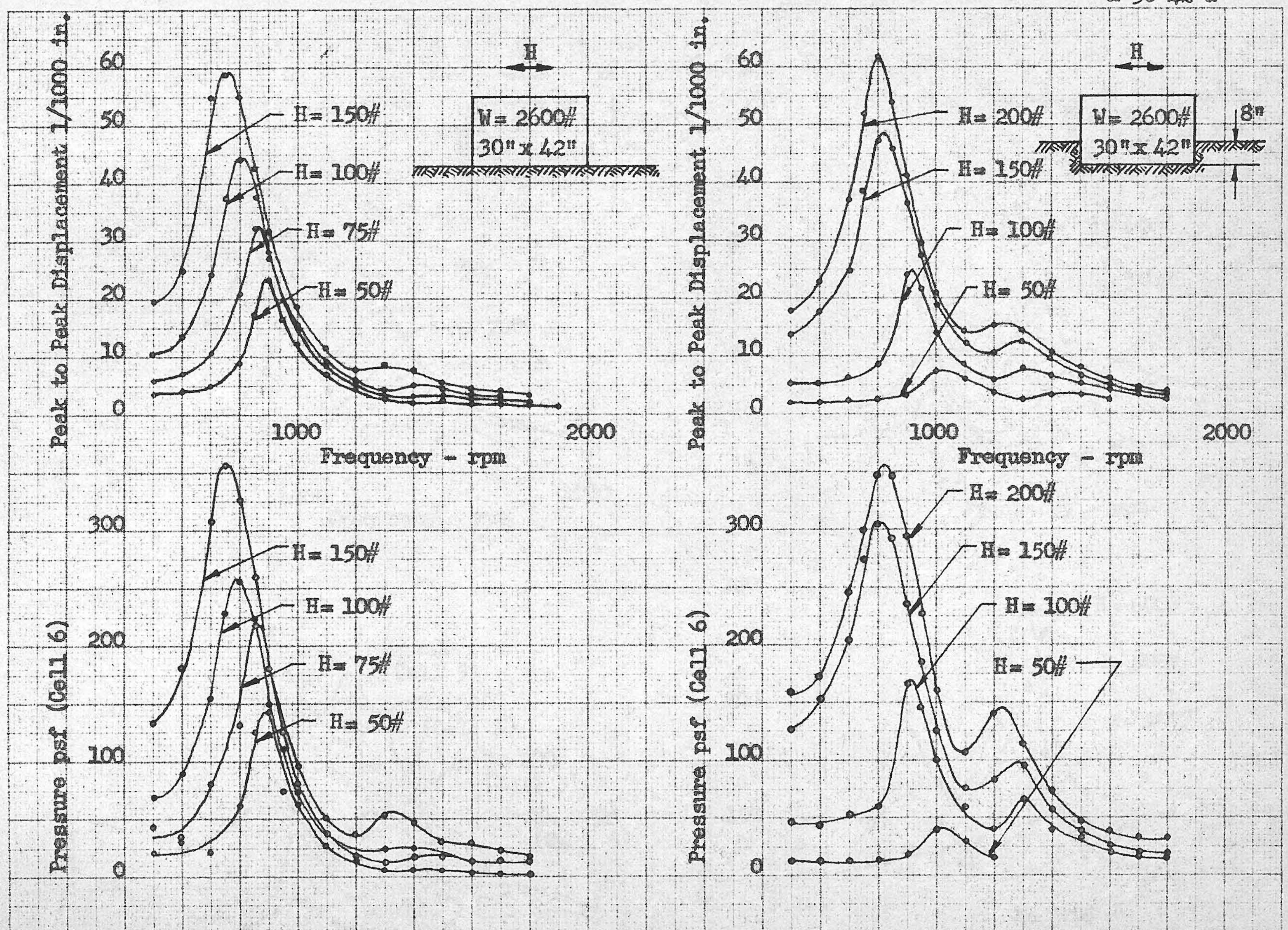

H 18-30 b
W = 1150#
18" x 30"
24"
16"
H
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 4)
Frequency - rpm
1000
2000
0
20
40
60
80
100
200
400
600
H = 400#
H = 300#
H = 200#
H = 150#
H = 100#
H = 75#
H = 50#
H = 25#


H 30-30 a
W = 1900#
30" x 30"
H
8"
H = 150#
H = 100#
H = 75#
H = 50#
Unstable
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 5)
Frequency - rpm
0
10
20
30
40
50
60
70
100
200
300
1000
2000

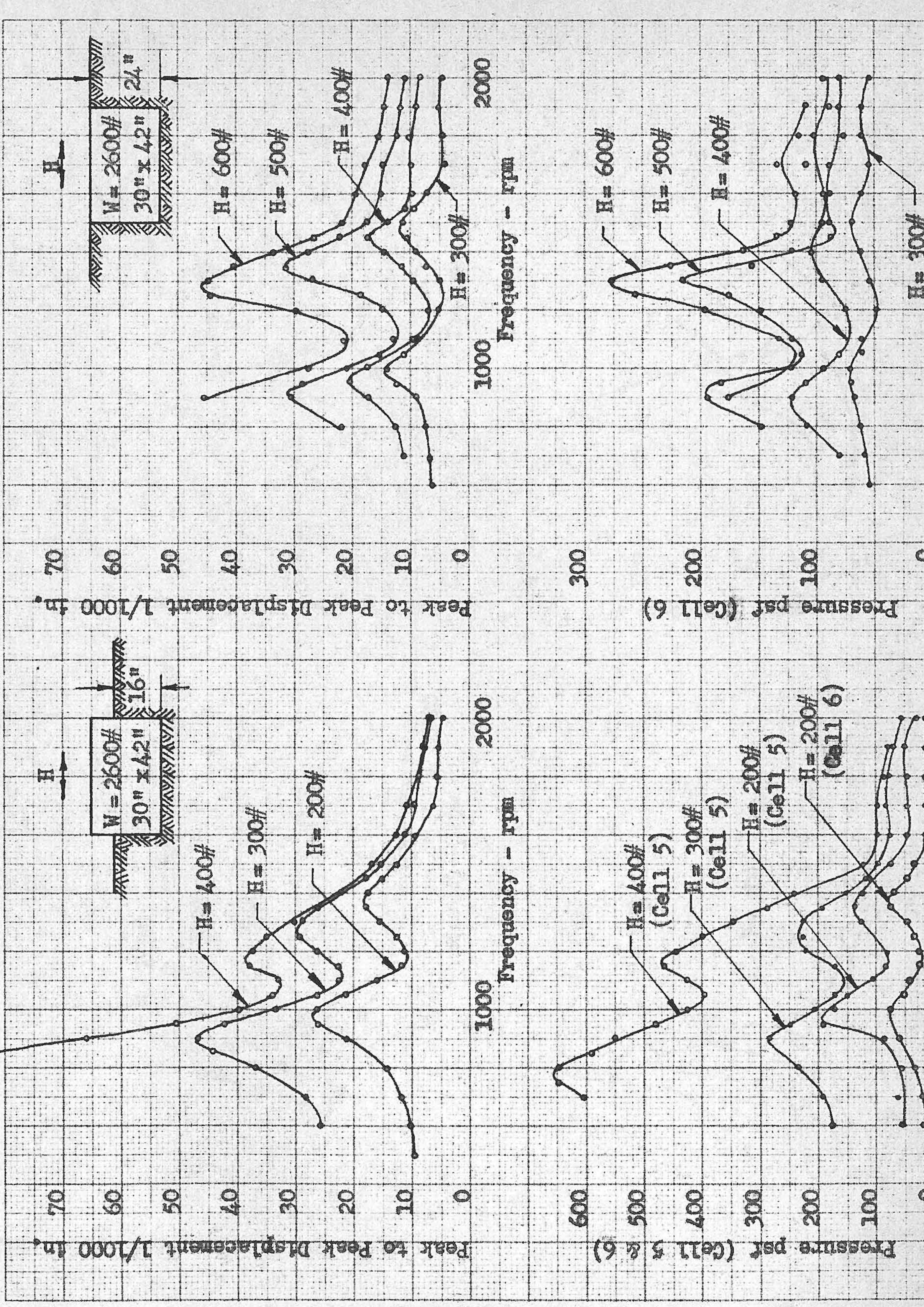


H 30-30 b
H
W = 1900#
30" x 30"
24"
H = 400#
H = 300#
H = 200#
H = 100#
Peak to Peak Displacement 1/1000 in.
Frequency - rpm
Pressure psf (Cell 5)
16"
H = 150#
H = 100#
H = 75#
Pressure psf (Cell5)

H 30-42 a

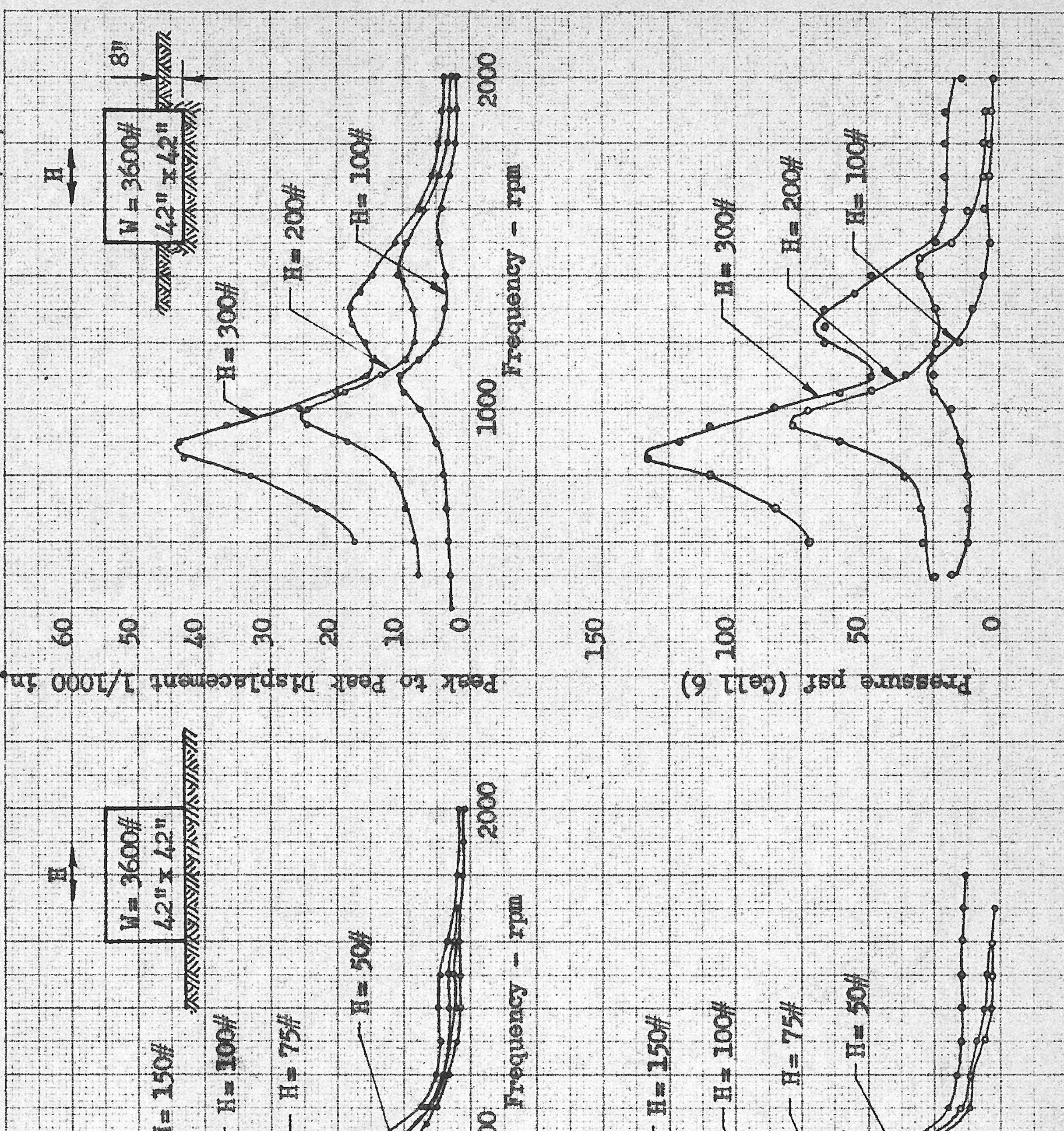

Peak to Peak Displacement 1/1000 in.
60
50
40
30
20
10
0
1000
2000
Frequency - rpm
H = 150#
H = 100#
H = 75#
H = 50#
H
W = 2600#
30" x 42"
H = 200#
H = 150#
H = 100#
H = 50#
8"
Pressure psf (Cell 6)
300
200
100
0

H 30-42 b

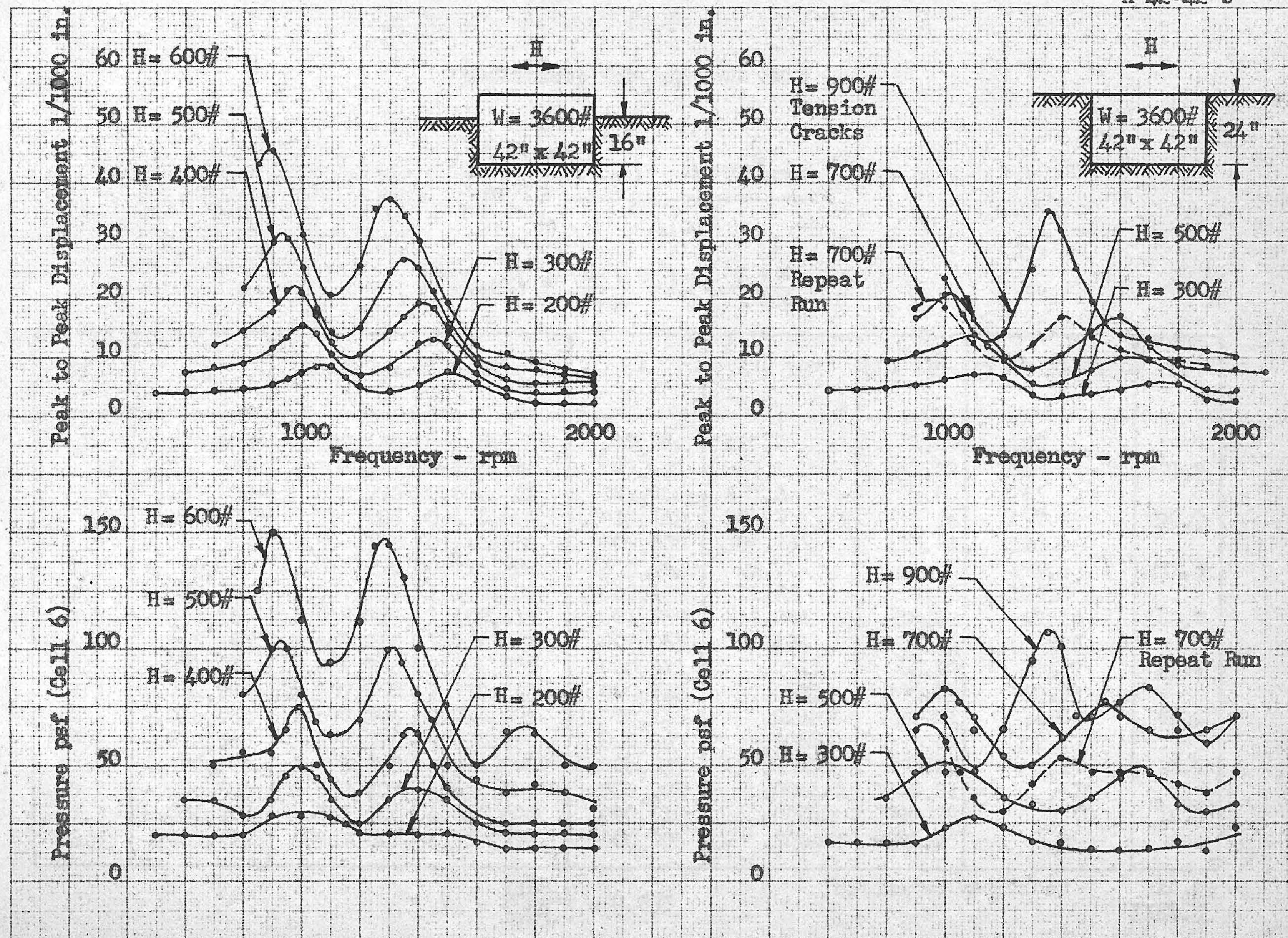


W = 2600#
30" x 42"
16"
24"
H
H = 400#
H = 300#
H = 200#
H = 600#
H = 500#
H = 300#
H = 400# (Cell 5)
H = 300# (Cell 5)
H = 200# (Cell 5)
H = 200# (Cell 6)
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 5 & 6)
Pressure psf (Cell 6)
Frequency - rpm
1000
2000

H 42-42 s


W = 3600#
42" x 42"
8"
H
H = 300#
H = 200#
H = 100#
H = 150#
H = 100#
H = 75#
H = 50#
Peak to Peak Displacement 1/1000 in.
Pressure psf (Cell 6)
Frequency - rpm
60
50
40
30
20
10
0
150
100
50
1000
2000

H 42-42 b


Peak to Peak Displacement 1/1000 in.
60
50
40
30
20
10
0
H = 600#
H = 500#
H = 400#
H = 300#
H = 200#
H
W = 3600#
42" x 42"
16"
1000
2000
Frequency - rpm
H = 900#
Tension
Cracks
H = 700#
H = 700#
Repeat
Run
H = 500#
H = 300#
W = 3600#
42" x 42"
24"
Pressure psf (Cell 6)
150
100
50
H = 700#
Repeat Run
H = 900#

APPENDIX E

Coefficient for Apparent Mass

$r = 1,\ s = 0.75$

$$C_m = \int_0^\infty \frac{(1+x)^2\ A(x)^2\,dx}{A(0)^2}$$

$$A(x) = \frac{s-1}{1+x} - \log\frac{s+x}{1+x}$$

x	$\frac{s-1}{1+x}$	log(s + x)	log(1 + x)	A	A x 10^3	$(1+x)^2$	$(1+x)^2A^2 \times 10^6$	Δ
0	-.250,000	-.287,682	0	.037,682	1420	1.00	1420	
.15	-.217,391	-.105,361	0.139,762	.027,732	769.1	1.32	1018	321.9
.3	-.192,308	.048,790	0.262,364	.021,266	452.2	1.69	765	
.6	-.156,250	.300,105	0.470,004	.013,649	186.3	2.56	477	299.9
.9	-.131,579	.500,775	0.541,854	.009,500	90.25	3.61	326	
1.5	-.100,000	.810,930	0.916,291	.005,361	28.74	6.25	179.4	231.5
2.1	-.080,645	1.047,319	1.131,402	.003,438	11.82	9.61	113.7	
3.3	-.058,140	1.398,717	1.458,615	.001,758	3.091	18.49	57.15	150.7
4.5	-.045,455	1.658,228	1.704,748	.001,065	1.134	30.25	34.35	
6.9	-.031,646	2.034,706	2.066,863	.000,511	.2611	62.40	16.29	87.9
9.3	-.024,272	2.307,560	2.332,144	.000,312	.0973	106.09	10.33	
14.1	-.016,556	2.697,994	2.714,695	.000,145	.02102	228.01	4.80	51.6
18.9	-.012,563	2.978,074	2.990,720	.000,083	.00689	396.01	2.73	
28.5	-.008,475	3.375,878	3.384,390	.000,037	.001369	870.25	1.19	26.1
38.1	-.006,394	3.659,707	3.666,122	.000,021	.000441	1528.81	.674	
57.3	-.004,288	4.061,304	4.065,602	.000,010	.000100	3398.89	.340	14.0
76.5	-.003,226	4.347,047	4.350,278	.000,005	.000025	6006.25	.150	

Estimated Remainder 14.0

1188.6

$$C_m = \frac{1188.6}{1420.0} = 0.835$$

APPENDIX F

Coefficient for Mass Moment of Inertia

r = 1, s = 0.75

$$C_i = \frac{\int_0^\infty (1+x)^4 \, B(x)^2 \, dx}{B(0)^2}$$

$$B(x) = \frac{s-1}{1+x} - \frac{1}{2}\left(\frac{s-1}{1+x}\right)^2 + \frac{1}{3}\left(\frac{s-1}{1+x}\right)^3 + \log\frac{s+x}{1+x}$$

x	$\frac{s-1}{1+x}$	$\frac{1}{2}\left(\frac{s-1}{1+x}\right)^2$	$\frac{1}{3}\left(\frac{s-1}{1+x}\right)^3$	$\log\frac{s+x}{1+x}$	B	$B^2 \times 10^6$	$(1+x)^4$	$(1+x)^4 B^2 \times 10^6$	Δ
0	-.250,000	.031,250	-.005,208	-.287,682	.001,224	1.488,2	1.	1.488	
.15	-.217,391	.023,629	-.003,425	-.245,123	.006,678	.459,7	1.749	.804	.2581
.3	-.192,308	.018,491	-.002,374	-.213,574	.000,401	.160,8	2.856	.459	
.6	-.156,250	.012,207	-.001,272	-.169,899	.000,170	.028,9	6.554	.1895	.1309
.9	-.131,579	.008,657	-.000,759	-.141,079	.000,084	.007,056	13.03	.0920	
1.5	-.100,000	.005,000	-.000,333	-.105,361	.000,028	.000,784	39.06	.0306	.0451
2.1	-.080,645	.003,252	-.000,175	-.034,083	.000,011	.000,121	92.35	.0112	
3.3	-.058,140	.001,690	-.000,065	-.059,898	.000,003	.000,009	341.9	.0031	.0098
4.5	-.045,455	.001,033	-.000,031	-.046,520	.000,001	.000,001	951.1	.0009	
								Estimated Remainder	.0030
									.447

$$C_i = \frac{0.447}{1.488} = \underline{\underline{0.30}}$$

REFERENCES

1. Larkin, K.H. "Design of Diesel Engine Foundations" American Society of Mechanical Engineers. Transactions, v. 64, p. 341-348; discussion p. 348-350, (1942)

2. Hool, G.A. and Kinne, W.S. "Foundations, Abutments and Footings", 2nd Edition. McGraw-Hill (1943) p. 315-320.

3. Cozens, W.J. "Machinery Foundations", Institution of Electrical Engineers, Journal, v. 82, p. 327-328 (1938).

4. Newcomb, W.K. "Principles of Foundation Design for Engines and Compressors", American Society of Mechanical Engineers, Transactions v. 73, April 1951, p. 307-319.

5. Vesselowsky, S. "Experimental and Theoretical Investigation of Turbine Foundations", American Society of Mechanical Engineers, Transactions, v.62, p. A63-A70, (1940); discussion v. 63, p. A141-143 (1941).

6. "Turbine-generator Foundation Design", Engineering News-Record, v. 137, p. 316-320, September 5, 1946.

7. Rausch, E. "Maschinenfundamente und Andere Dynamische Bauaufgaben", 2nd Edition, VDI Verlag.Berlin, 3 vols. (1943).

8. De Gruben, K. "Natural Frequency of Oscillation of Spring Mounted Machines", Verein Deutscher Ingenieure. Zeitschrift, v. 86, p. 633-637, October 17, 1942.

9. Spath, W. "Theory and Practice of Vibration Testing Machines", Translation No. 51, U.S. Experimental Model Basin. (1938).

10. Hertwig, A., Fruh, G. and Lorenz, H. "Determining the Most Important Soil Properties for Buildings by Means of Forced Vibrations", Verein Deutscher Ingenieure, Zeitschrift, v. 77, p. 550. (1933).

11. Lorenz, H. "Neue Ergebnisse der Dynamische Baugrundforschungen", Verein Deutscher Ingenieure, Zeitschrift, v. 78, p. 379-385. March 24, 1934.

12. Terzaghi, K. "Theoretical Soil Mechanics", John Wiley and Sons, Inc. (1943)

13. Barkan, D.D. "Experimental Studies of Vibrations of Mat Foundations Resting on Cohesive Saturated Soils", Transactions for Engineering Foundations Research (Vios); Moscow U.S.S.R. (1934)

14. Reissner, Erich, "Stationare, Axialsymmetrische, durch ein schuttelnde Masse erregte Schwingungen eines homogenen elastischen Halbraumes", Ing. Archiv. Vol. 7, (1936)

15. "An Investigation of the Compaction of Soil by Vibration" Research Report, California Institute of Technology, March 1950.

16. Tschebotarioff, G.P. and Ward, E.R. "Resonance of Machine Foundations and the Soil Coefficients which affect it". Second International Conference on Soil Mechanics and Foundation Engineering, Proceedings, Rotterdam, v. 1, p. 309-313, (1948).

17. Crockett, H.A. and Hammond, R.E.R. "Natural Oscillation of Ground and Industrial Foundations". Second International Conference on Soil Mechanics and Foundation Engineering, Proceedings, Rotterdam, v. 3, p. 88-93 (1948).

18. "Grouting Stops Compressor Sway". Engineering and Mining Journal, v. 149, p. 93-94. July 1948.

19. "Grouting Checks Foundation Vibrations". Engineering News Record, v. 129, p. 561-562. October 22, 1942.

20. "Chemical Injections Solidify Soil to Stop Foundation Settlement". Construction Methods, v. 27, p. 78-79, 156, 158, 160, 162. Dec. 1945.

21. Karman, T. von, and Biot, M.A. "Mathematical Methods in Engineering". McGraw-Hill (1940).

22. Thornton, D.L. "Mechanics Applied to Vibration and Balancing". John Wiley and Sons, Inc. (1940).

23. Timoshenko, S. "Strength of Materials" v. 1, D. Van Nostrand and Co., (1947).

24. Operation and Maintenance Manual - Velocity Pickup Type 4-102A Consolidated Engineering Corp., Pasadena, Calif.

25. Operation and Maintenance Manual - Vibration Meter Type 1-110B Consolidated Engineering Corp., Pasadena, Calif.

26. Stoker, J.J. "Non-linear Vibrations", Interscience Publishers, Inc., New York (1950)

27. "Will My Blasting Damage Property?", Engineering News Record, May 4, 1950, p. 36.

The author wishes to draw attention to the following bibliography.

Bibliography On Machinery Foundations;

Design, Construction, Vibration Elimination.

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