I. RETARDATION CORRECTIONS IN THE HELIUM ATOM

II. SELF ENERGY OF AN ELECTRON IN A MAGNETIC FIELD

Thesis by

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I wish to express my gratitude to Professor R. P. Feynman, who proposed these problems and suggested the lines along which they should be attacked. I am indebted to Professor R. F. Christy for valuable counsel.
PART I.

ABSTRACT

An analysis is made of the errors which arise for absolute levels and fine structure separations from the use, for the interaction of two electrons, of the nonretarded Breit expression instead of the true retarded interaction. The effects on the fine structure turn out to be too small for observation. The correction to the ground state energy may become observable if the experimental accuracy is increased. This is probably not the case for any excited level.
PART II.

ABSTRACT

The exact propagation function for a Dirac electron in an arbitrarily intense magnetic field is derived in closed form as a parametric integral. Using the exact relativistic wave functions and the exact propagation function, the energy corrections of order $e^2$, due to the emission and reabsorption of one virtual photon, are calculated and exhibited in closed form as double parametric integrals. These integrals are shown to possess an asymptotic expansion in the small parameter $H/m^2$. This expansion is not a pure power series but involves also terms of the form $H^{n} m/(m^2)$. The terms of order $H$ agree with the known correction to the magnetic moment. The terms of order $H^2$ and $H^3 m/(m^2)$ are exhibited and discussed.
<table>
<thead>
<tr>
<th>Part</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>II. NATURE OF BOUND STATES</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>III. NONRELATIVISTIC APPROXIMATION</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>IV. RELATIVISTIC CORRECTIONS</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>REFERENCES</td>
<td>23</td>
</tr>
<tr>
<td>II.</td>
<td>I. INTRODUCTION</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>II. PROPAGATION FUNCTIONS</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>III. M IN PARAMETRIC FORM</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>IV. THE WAVE FUNCTIONS</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>V. THE ASYMPTOTIC EXPANSION</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>VI. INTERPRETATION AND DISCUSSION</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>REFERENCES</td>
<td>40</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

If the nucleus is considered as an infinitely massive structureless point charge, the problem of the helium atom is a perfectly definite one from the viewpoint of field theory. It is the problem of finding all stationary states, of total charge minus two, of the Dirac field in interaction with the nuclear potential and the radiation field. There is however at present no satisfactory way of finding these states or their energies, and all treatments of such systems proceed by at first constructing an accurate solution of the corresponding nonrelativistic problem, for which powerful methods exist, and then adding various relativistic corrections as perturbations. These corrections fall into four classes, giving the effects of

(1) The well known spin-orbit interaction and other effects which appear in single-electron problems and carry over unchanged when more electrons are present.
(2) Interaction with externally applied fields. (Stark and Zeeman effects.)

(3) Relativistic corrections to the inter-electron interaction.

(4) Action of the electrons on themselves by the emission and reabsorption of virtual photons. (Lamb shift, etc.)

This thesis is primarily concerned with the third item above, but it will turn out to be so intimately related to the fourth item that it will be convenient to treat the two together. The relation arises from the fact that the inter-electron interaction, aside from its Coulomb part, is due to the emission of virtual photons by one electron and their absorption by the other.

II. NATURE OF BOUND STATES

If the electromagnetic field is eliminated by replacing it by retarded interactions, the atom can be pictured in terms of an indefinitely extended Feynman diagram, with the two electrons repeatedly interacting with themselves and each other, and occasionally doubling back their trajectories to form pairs. The effect of the central potential can be considered either as additional
interactions or as included in the propagation functions for the electrons. The figure shows a typical interval in the atom's history, adopting the second alternative and not showing interactions with the central field. This picture does not directly give a particle model of the atom because of the possible presence of pairs. However, if pairs are "usually" absent the state is describable by a $\Psi(x_a,x_b)$ which is the amplitude for electron $a$ to be at point $\vec{r}_a$ at time $t_a$ and electron $b$ to be at $\vec{r}_b$ at time $t_b$ with no photons and no pairs present. By this is meant that a surface can be drawn which is crossed by no photon lines, which intersects the world line of each electron exactly once (at points $a,b$) and which extends in a spacelike manner to infinity. The surface is not required to be spacelike at all points, and the separation of points $a,b$ need not be spacelike. Once $\Psi$ is known, it is possible to derive amplitudes for the presence of one or more photons or pairs. In principle, $\Psi$ can be determined
as the solution of an integral equation, whose kernel is the sum over all irreducible diagrams of the amplitude for propagation from \(a,b\) to \(a',b'\). A diagram is irreducible if it cannot be separated into two parts by such a surface as just described. In actuality no such solutions have yet been obtained. This description is useless when the interaction is so strong that a photon is usually present, for then very long chains of interlaced interactions make important contributions to the kernel.

For an eigenstate, \(\Psi\) involves the mean time \(\frac{1}{2}(t_a+t_b)\) only through an exponential factor, so

\[
\Psi = e^{-i\frac{E}{\hbar}(t_a-t_b)} \phi(p_a, p_b, t_a-t_b)
\]

where \(E\) is the energy of the state. In momentum space \(\Psi\) becomes

\[
\delta(k_a^0+k_b^0-E) \mathcal{N}(k_a^0, p_b, k_b^0-k_a^0)
\]

so that there is a distribution of energies for each electron, with the total energy remaining definite. Nothing is known of the dependence of the wave function on relative time or relative energy in any particular case. Intuitively it would seem probable that there should be only small amplitude for \(|k_a-k_b|\) to be much larger than the binding energy, and in cases where the electron interaction energy is small compared...
to the total binding the amplitude should be large only when the individual energy of each electron approximates the energy of some single-electron state. Yet this intuition is directly contradicted by another, for when the binding is weak and the velocities small, the atom is well described by a nonrelativistic model in which the forces are not retarded and only a single time need be considered. The nonrelativistic $\Psi$ involves $t_a - t_b$ only through a factor $\delta(t_a - t_b)$ so that the wave function is, in momentum space, independent of $k_a - k_b$, corresponding to an infinite spread in relative energy rather than a small spread comparable to the small binding. Evidently some care is needed in considering a single time description as the limit of a many time one.

The retarded interaction between two electrons, which in coordinate space is $e^2 \delta(s_a s_b)$, is in momentum space

$$\gamma \varepsilon e^2 \frac{\partial^2 \partial^2}{k \omega} = -\gamma \varepsilon e^2 \frac{\partial^2 \partial^2 - \frac{1}{4}(\hat{\partial}^2 \partial^2)(\hat{\partial}^2 \partial^2)}{k^2 - \omega^2} + \gamma \varepsilon e^2 \frac{\hat{\partial}^2 \partial^2 - \frac{1}{4}(\hat{\partial}^2 \partial^2)(\hat{\partial}^2 \partial^2)}{k^2 - \omega^2}$$

(1)

The second term is the retarded transverse interaction. The first term, after a gauge transformation, becomes $-\gamma \varepsilon e^2 \partial^2 \partial^2$; the nonretarded Coulomb potential. $\gamma_a$ and $\gamma_b$ are Dirac matrices
referring respectively to electrons a and b, and $\mathbf{k}_\sigma$ is the momentum of the transferred photon. If in the transverse interaction the photon energy $\omega$ is neglected in comparison with $\mathbf{k}$, thereby neglecting the effects of retardation, the result is the Breit–interaction energy, whose expectation value is a principal contributor to the helium fine structure level shifts.

Since $\omega$ vanishes in the coordinate system in which the center of gravity of the two electrons is at rest, the error in using the Breit interaction instead of the true retarded interaction may be expected to be of the order $(\text{Breit energy})(\nabla^2/c^2)$ where $\nabla^2$ is a mean square velocity of the center of gravity. In the excited states of helium the outer electron moves slowly compared to the $1s$ electron, so that $\nabla^2_{\text{cg}} \frac{1}{4} \nabla^2_{1s} \approx \frac{1}{2m}$(binding energy of $1s$ electron). Hence $\nabla^2/c^2 \approx 1/(157)^2$. This estimate predicts an error which would be unobservably small. However this estimate is an unreliable one for several reasons:

(1) The next approximation involves $\frac{\omega}{\mathbf{k}^2}$ which is singular at $\mathbf{k}=0$. This singularity must be investigated before an expansion in $\frac{\omega}{\mathbf{k}^2}$ is justified.
Although normally the energy of each of the two electrons is $\pm m$, their interaction sometimes scatters them into a state where one of them has a negative energy and the energy of the other greatly increased in compensation. For momentum conserved, energy must fail of conservation by $\approx 2m$, so that the life of this abnormal situation is correspondingly brief. Taken between this state and the normal one, the matrices give factors of about unity, so that the Breit interaction gives a matrix element of order $e^2/\lambda^2$ while the true retarded interaction is more like $e^2/m^2$. It is for this reason that the Breit interaction is not suitable for use beyond the first order in perturbation theory, and even in first order Breit found that for agreement with experiment it must be used in connection with a particular method of reduction to large components which amounts to the use of a projection eliminating the negative energies. Since the non-relativistic variational wave functions do not lend themselves to an accurate determination of the magnitude of the negative energy parts, their contribution to the energy is best determined by
choosing the initial wave functions smooth so that momenta as large as \( m \) are practically absent and the state includes no negative energies, and then considering the second order perturbation due to the electrons being scattered into states of very high momentum or negative energy and then back.

(3) The effect of the action of the Coulomb potentials while the electrons are exchanging a transverse photon ought to be of the same magnitude as the Lamb shift. Though very small, this is still large compared to the original error estimate, and should be investigated.

III. NONRELATIVISTIC APPROXIMATION

The Coulomb part of the electron interaction is accurately accounted for by the use of variational principles to find accurate energies. The transverse part may be considered as a perturbation due to the emission of a virtual photon by one electron and its absorption by the other. So long as we limit the photon momenta to be \( \ll m \) by a suitable cutoff, the calculation of the energy shift can be done using a nonrelativistic model and ordinary perturbation theory. This gives
\[ \Delta E = \frac{\alpha^2}{4 \pi \hbar m_e} \sum \sum \int \frac{d^3 k}{(k + \omega_n - \omega_k)} \left( \langle s | \tilde{p}_a \tilde{\epsilon}_e e^{-i \mathbf{k} \cdot \mathbf{r}} | n \rangle \langle n | \tilde{p}_b \tilde{\epsilon}_e e^{i \mathbf{k} \cdot \mathbf{r}} | s \rangle \right) \]

+ similar term with a and b interchanged \hspace{1cm} (2)

If the Breit interaction had been used instead, the only change in the result would have been the substitution of \( k \) for \( k + \omega_n - \omega_k \) in the energy denominator. The difference is

\[ \Delta E - \Delta E_{BR} = \frac{\alpha^2}{4 \pi \hbar m_e} \sum \sum \int \frac{(\omega_n - \omega_k) d^3 k}{k^2 (k + \omega_n - \omega_k)} \text{(same matrix elements)} \] \hspace{1cm} (3)

This is in the same form as Bethe's expression for the nonrelativistic part of the Lamb shift. The only difference is that the operators for emission and absorption of photons with momentum \( k \) and polarization \( \mathbf{e} \) refer to different electrons. If the Lamb shift terms, which are due to emission and reabsorption, are included, the matrix elements become

\[ \langle s | \tilde{p}_a \tilde{\epsilon}_e e^{-i \mathbf{k} \cdot \mathbf{r}} + \tilde{p}_b \tilde{\epsilon}_e e^{-i \mathbf{k} \cdot \mathbf{r}} | n \rangle \langle n | \tilde{p}_b \tilde{\epsilon}_e e^{i \mathbf{k} \cdot \mathbf{r}} + \tilde{p}_a \tilde{\epsilon}_e e^{i \mathbf{k} \cdot \mathbf{r}} | s \rangle \] \hspace{1cm} (4)

Following Bethe, the factor \( \omega_n - \omega_k \) can be included in the matrix elements by commuting the emission operator with the Hamiltonian. Where it occurs in the energy denominator it can be replaced by a mean excitation energy. Then the sum over states is
easily performed by an appeal to the completeness property of the wave functions. With

\[ H = \frac{1}{2m}(p_a^2 + p_b^2) + V_a + V_b + V_{ab} \]  

(5)

the result is the expectation of

\[ \frac{\hbar^2}{3m}(p_a^2 + p_b^2) + \frac{1}{2} \left( \frac{V_a^2}{r_a} + \frac{V_b^2}{r_b} + (1 - \cos \alpha) \frac{(V_{p,0})^2}{K^2} \right) V_{ab} \]  

(6)

which has been summed over polarizations. The first three terms are single-electron effects of no interest here. The third term, for \( V_{ab} = e\gamma \), reduces to

\[ \frac{e^2}{2\gamma^2} \int \frac{1}{r_a} \cos kr \rho \, d\rho \]  

(7)

where an averaging over angles has been performed. The fact that it vanishes as \( kr \to 0 \) has a very simple interpretation. The forced motion of a pair of electrons pushed around by the vacuum fields can be resolved into a relative motion and a motion of the center of gravity. The last does not alter the mean value of the mutual potential energy, and first does not arise in dipole approximation.

Till now the momentum cutoff \( K \) has been left unspecified except that it is \( \ll m \) and \( \gg (\omega_d - \omega_i) \) so
that \((\omega_n - \omega_s)/K\) can be neglected. Under this approximation and dropping the single-electron terms, the energy shift is given by the expectation of

\[
Q = \frac{2e^4}{\pi m^2 r^3} \left( \frac{\sin Kr}{K^3 r^3} - \frac{\cos Kr}{K^2 r^2} - \frac{1}{3} \right)
\]  

(8)

\(Q\) behaves like \(-e^4K^215\pi m^2r\) for \(Kr\) and like \(-2e^4/3\pi m^2r^3\) for \(Kr\) large. Let us choose \(K\) as large as permissible so as to include in the non-relativistic calculation as much as possible of the total effect. \(Q\) is an inverse cubic potential cut off to behave like \(1/r\) inside a radius \(r_0 \cong 3/K\). Since \(K \ll m\), \(r_0\) must be large compared to the Compton wavelength, but it can still be small compared to the size of the atom, so that \(Ka_0 \gg 1\), where \(a_0\) is the helium Bohr radius \(1/2me^2\).

The form of \(Q\) in momentum space will be of importance for any attempt to join this nonrelativistic calculation on to a relativistic calculation which has a low-energy cutoff. Let \(S=\beta K\) be the magnitude of the momentum vector. \(Q(S)\) has a rather complex behavior for \(\beta\) medium or
large, but this depends on the details of the cutoff procedure and is of no interest. Systematically dropping terms of order $\beta$ or smaller, \[
Q(S) = 4\pi \int_{\infty}^{\infty} \frac{\sin Sr}{Sr} Q(r) r^2 dr
\]
\[
\approx \frac{8e^4}{3m^2} \int_{\infty}^{\infty} \frac{\sin \beta z}{\beta z} \left(\frac{\sin z}{z} - 1\right) dz
\]
\[
\approx \frac{8e^4}{3m^2} \left(1 + \ln S/K\right)
\]
(9)

A rough estimate of the effect of $Q$ is easily worked out if the logarithm term is replaced by some mean value. Let us assume that the effect of a relativistic calculation would be to replace $K$ by something of the order of $m$, so the mean value should be about $-\ln(a_0m)\approx -4.22$. Then in coordinate space $Q$ becomes a delta function. For triplet states this would give zero if the approximation were good for such states. It is not, but it is legitimate to conclude that compared to the ground state the effect is quite small both for triplet states and for those excited singlet states where the wavefunctions do not greatly overlap.

For the ground state a calculation gives minus \[(8/3)(5.22)(0.17)\approx -2.36 \text{ cm}^{-1},\] using hydrogenic
wave functions. This is about half the probable error in the experimental determination of the ground state energy. Also, as will be explained in a later section, there is reason to believe the logarithmic part is cancelled by another effect not yet discussed.

IV. RELATIVISTIC CORRECTIONS

The nonrelativistic calculation must be joined on to a relativistic calculation which takes account of the effects of virtual photons with momenta above the cutoff. For these high momenta it is permissible to treat the electrons as lightly bound: the coulomb potential is considered only as a perturbation which acts at most once. The first term to be considered is of order $e^2$ and gives the effect of the transverse interaction acting only once and the coulomb potential not at all.

The major part of this term is the Breit interaction, reduced to Pauli form by assuming the small components to be $\frac{e}{\hbar} / 2m$ times the large components. This reduction is correct to order $v^2/c^2$ except that it acts as a projection eliminating any negative energy states present. These will be accounted for separately as part of a second term of order $e^4$ by considering them to exist only in an intermediate
state, being created by one interaction and destroyed by a second.

Aside from these negative energy states, the Breit interaction has errors of order $v^4/c^4$ from the reduction process, and an error of this same order from the difference $M$ which expresses the effects of retardation.

$$M = 4\pi e^2 \left( \alpha_\nu \right)_\nu \left( \alpha_\omega \right)_\omega \left( \delta_{\mu \nu} - \frac{1}{k^2} \hat{k}_\mu \hat{k}_\nu \right) \left( \frac{1}{k^2} \omega^2 - \frac{1}{k^2} \right)$$

$$\vec{K} = \vec{p}_3 - \vec{p}_\nu = \vec{p}_2 - \vec{p}_\nu \quad \omega = \frac{1}{2} k \cdot \left( \vec{p}_3 + \vec{p}_2 \right)$$

The electrons are actually virtual, but to assume them free should be a good approximation. When reduced to Pauli form, $M$ becomes (for triplet states and ignoring all terms independent of the total spin vector $S$)

$$\frac{m^2 v^4}{2 \pi e^2} M = \left( \frac{5 \cdot k}{k' v} \right)^3 \left( \frac{5 \cdot k^2}{k' v} \right)^2 i \hat{S} \hat{S} \left( \frac{5 \cdot k^2}{k' v} \right) \left( \frac{5 \cdot k^2}{k' v} \right)^2 \left( \frac{5 \cdot k^2}{k' v} \right)^3$$

$$2 \hat{L} = \hat{p}_3 + \hat{p}_2 \quad 2 \hat{d} = \hat{p}_3 - \hat{p}_2 \quad 2 \hat{s} = \hat{p}_3 - \hat{p}_2 \quad \hat{K} = \hat{p}_3 - \hat{p}_2$$

An order of magnitude estimate shows that this is far too small to affect the triplet splittings. The spin independent part is also
completely negligible. Actually $M$ is even smaller than a quick estimate would indicate, for the level splittings it causes are due to anisotropy in momentum space and the largest part of the momentum differences is the momentum of the $ls$ electron which is nearly isotropic. The form of $M$ is of some interest. The terms quadratic in $S$ have effects proportional to $(L \cdot S)^2$ and are interpretable as arising from something very like a velocity dependent tensor force in momentum space. The other terms have effects proportional to $L \cdot S$ and are less easy to visualize.

The classification of effects as being of various orders in $v/c$ is justified by the very rapid decline of the wave functions for large momenta. (Like $p^{-4}$ as can be seen from the non-relativistic integral equation.) This rapid decline shows that small velocities are really predominant and the mean of $v^2$ is really of the same order as the square of the mean of $v^2$. It also justifies the smoothing of the wave functions so that momenta comparable to $m$ are absent.
Next to be considered must be the term of order $e^4$ which gives the effect of the transverse interaction acting once and the coulomb potential also once. But here it is necessary to be careful, for since high velocities will be important in the intermediate states, it is not permissible to consider the transverse interaction small compared to the coulomb potential. The double transverse interaction due to exchanging two transverse photons is no smaller than the above, since $v/c$ is not to be considered small except in the initial and final states. Consequently the full $e^4$ interaction must be used: all Feynman diagrams with two photon lines. The two principal diagrams are a and b. The vacuum polarization is given by c, while d and a number of others like it are best understood as radiative corrections to the scattering by the (retarded) interelectron interaction. The total effect of

\[ a \quad b \quad c \quad d \]
the diagrams like d reduces to

\[ \frac{g_e}{3} \frac{e^v}{m^2} \gamma \beta \mu (\beta \mu, \beta \nu) (e^v (\gamma_\lambda \gamma_\mu) - 3) \]  

(12)

plus a term which will be ignored which is simply

the interaction of the anomalous part of one
electron's magnetic moment with the electromagnetic
fields generated by the charge of the other electron.

q is the transferred momentum and lambda is a photon
rest mass introduced as a lower cutoff. A similar
cutoff must be used for a and b. No cutoff is
needed for d (with proper renormalization) which
reduces to

\[ \frac{e^v (\gamma_\lambda \gamma_\mu) (e^v (\gamma_\lambda \gamma_\mu) - 3)}{15 \pi} - \frac{e^v}{60 \pi} (s \cdot g)^2 \]  

(13)

The procedure here is a model to be followed
in all cases: considering all the initial and final
momenta as small quantities, retain only the lowest
order term, which will be of order \( e^4 / m^2 \), and the
lowest order term capable of contributing to triplet
splittings, which will be of order \( e^4 q^2 / m^4 \). Reduce
this to Pauli form and express in terms of the total
spin vector \( \vec{s} \). The cutoff lambda is to be assigned
a magnitude of approximately \( (2 \alpha)^{\frac{1}{2}} m \), so that \( q^2 / \lambda^2 \)
and $\lambda^2/m^2$ are both small numbers. For diagram a there is a possibility of both electrons being only slightly virtual, corresponding to a longlived intermediate state. This is not properly a part of the $e^4$ correction: from a plus b must be subtracted terms representing the repeated action of a potential which is Coulomb plus projected Breit. (The projection is accomplished by deleting that pole of the electron propagation factor which refers to negative states.) Actually the positive energy part of the Breit interaction makes little difference, its contribution being much smaller than that of the negative states.

For the term of lowest order any spin factors are to be evaluated as for a singlet state. The lowest order term necessarily vanished for triplet states (antisymmetry).

The rationale of this procedure requires some explanation. The presence of the photon rest mass gives to the interaction a finite range which in momentum space implies a behavior regular about the origin, so that an expansion in powers of the initial and final momenta (and hence in powers of $q^2/m^2$) is justified if these momenta are $\ll \lambda$.

This regularity seems surprising since this calculation is to be fitted to a nonrelativistic
one which involves the log of q. The answer to this discrepancy is that there is another effect which should have been included in the nonrelativistic calculation: the effect of the exchange of two transverse photons. This is seemingly of a higher order in v/c, but only seemingly, since what should have been the dominant part of the calculated effect, the dipole approximation, in fact vanished identically, leaving only terms of a higher order in Kr.

No attempt has been made to compute this \( Q'(S) \), but it is easy to show that it also is of the form \( e^4/m^2 \) times a function of \( S/K \), and a reasonable conclusion is that \( Q \) and \( Q' \) are actually of the same form with the logarithm terms cancelling, so the sum \( Q + Q' \) is just a constant times \( e^4/m^2 \). Strictly this is right only for momenta less than the cutoff \( K \) (which is to be roughly identified with lambda) but if \( K \) is large compared to the momenta present in the initial and final states this qualification can be dropped. The result, in coordinate space, is a delta function interaction.

Since the total nonrelativistic part does not involve the log of the cutoff, the same must be true of the relativistic part (at least in the limit of zero momenta) and in fact the lowest order term
must approach a constant as lambda approaches zero, which justifies the earlier statement that it is of order $\frac{4}{m^2}$ rather than say $\frac{e^4}{\lambda^2}$. A similar argument holds for the fine structure term, making use of the fact that the nonrelativistic part is independent of the orientation of the spin vector. (There are terms of order $\frac{e^4 q^2}{m^2 \lambda^2}$ which blow up as lambda goes to zero, but they are the same for all lines of a multiplet. They represent the failure of the expansion in powers of $q$ when $q > \lambda$.)

Since the terms involving $q^2$ are too small to make their effect on absolute levels appreciable, it is only necessary to retain that part which contributes to multiplet splittings, and for this there is no trouble.

Since the major contribution to it comes from negative energy states (strictly, states with pairs present) the $e^4$ term must represent an interaction of short range. Since the intermediate states differ from the initial and final by about $2m$, the interaction should have a range of about a Compton wavelength, which makes logical an expansion in delta functions and their derivatives, or, in momentum space, in ascending powers, as stated in the
beginning. Since observations on level splittings are more precise than those on absolute levels, not only the lowest order term must be retained, but also the lowest order term which depends on the spin orientation.

In the previous section it was calculated that in the most favorable circumstances (ground state) a term $e^4/m^2$ causes a level shift of only $0.17 \text{ cm}^{-1}$. This is only about 3\% of the probable error of the experimental ground state energy, and the calculational accuracy is probably no greater. The best calculated value is $198319 \text{ cm}^{-1}$ while a 1942 experimental value is $198314 \pm 5 \text{ cm}^{-1}$. For excited states the effect is less and the experimental and calculational uncertainties much greater. So no observable effect can be expected unless very generous assumptions are made as to the probable coefficient of the $e^4/m^2$ term.

The possibility of an appreciable effect on the fine structure is far more remote. The lowest order term which can possibly contribute is $e^4q^2/m^4$, which is smaller than the above by a factor of roughly $4/(137)^2$ or $2 \cdot 10^{-4}$. 

This is small compared to the accuracy with which known effects can be calculated, and quite beyond the possibility of experimental verification. Hence, the only effect of any importance is the shift of the lower singlet levels by the delta function interaction. The coefficient of this interaction has not been computed exactly, but according to a rough estimate it should not be greater than about ten at most. This would still require a refinement of the experimental accuracy to make it observable.
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I. INTRODUCTION

The purpose of this investigation is to find the energy, due to self action, of an electron in a uniform magnetic field. The quantity we wish to find is the value per unit time of

\[ M = e^2 \int \int \psi^*(z) \xi_{\mu} \{ K_{+}(z, 1) \xi_{\nu} \} \psi(z) \frac{d^2 z}{d^2 t} \]

where \( \psi \) is some one of the stationary states of an electron in a magnetic field, and \( K_{+} \) is Feynman's propagation function for an electron in such a field. Both are expressed in a form correct for arbitrarily intense fields. \( M \) expresses the effect of the virtual emission and reabsorption of a single photon in the presence of the field, and gives the energy increment correct to order \( e^2 \) in the electron charge, but to all orders in the field intensity. The technique used is to express \( K_{+} \) and \( f_{+} \) as parametric integrals, with the integrands simple functions of space and time, so that all integrations other than those over the parameters are easily performed. \( M \) is then in exact closed form as a double integral, for which an expansion in \( H \), the magnetic field intensity, can be found. This expansion is not a pure power series, but contains terms of the form \( H^k \ln H \).

Since the extra generality takes little trouble,
the preliminary development is done for the case of uniform electric and magnetic fields both present. Without loss of generality, \( E \) and \( H \) are both taken in the \( z \) direction with \( H = 0 \), since any other case can be reduced to this by a suitable Lorentz transformation. Later we specialize to \( E = 0 \). The potential is chosen as

\[
A_\nu = \frac{1}{2} \chi_\mu F_{\mu \nu} \quad F_{\mu \nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \quad F_{1z} = -H \quad F_{34} = +E
\]

\[
A_x = -\frac{1}{2} \gamma H \quad A_z = -\frac{1}{2} \gamma E \quad A_y = +\frac{1}{2} \times H \quad A_t = -\frac{1}{2} \times E
\]

Here and henceforth, the relativistic notation of Feynman\(^1\) is used:

\[
\chi_\mu = x, y, z, t = \vec{p}, \vec{t} \quad A_\mu = \vec{A}, \vec{A}_t \quad \gamma_\mu = \beta \delta_\mu, \beta \quad \gamma = \gamma_\mu a_\mu \\
\frac{\partial}{\partial x^\mu} = -\text{grad} a_\mu \quad \gamma^\nu = \gamma_\mu \frac{\partial}{\partial x^\mu} \\
a_\mu b_\nu = a_\nu b_\mu - \vec{a} \cdot \vec{b} = a \cdot b \\
\delta_{\nu \nu} = 1, \delta_{\mu \mu} = \delta_{zz} = \delta_{33} = -1 \\
\delta_{\mu \nu} a_\nu = a_\mu \quad \phi^+ = \phi^* \beta
\]

II. PROPAGATION FUNCTIONS

Below, \( 0 \) denotes the origin, while \( 1 \) and \( 2 \) denote arbitrary spacetime points. \( f_+ (s^2) \) is the photon propagation function, given by

\[
f_+ (s^2) = \frac{1}{4\pi} \int e^{-i\vec{x}_{\mu 2} \cdot \vec{x}_{\mu 1}} \cdot q (\alpha) d\alpha \quad (x_{\mu 1} x_{\mu 2} = s^2)
\]

The convergence factor \( g(\alpha) \) is introduced to make
the mass correction finite. When the mass correction has been subtracted \( g(\omega) \) will be replaced by 1, so that \( f_4 \) becomes \( g(c^2 \omega) \) which is the true propagation function without cutoff.

\( K_+ \) is defined to be that solution of the inhomogeneous Dirac equation

\[
(i \gamma^2 - \kappa - m)K_+(2,1) = i \delta(2,1)
\]

which shows the same behaviour as does the field free \( K_+ \) when \( 2 \rightarrow 1 \), and also for \( \kappa \rightarrow \) zero \( K_+ \) must reduce to the field free \( K_+ \).

Because of the presence of the potentials,

\( K_+(2,1) \neq K_+(2-1,0) \). Instead

\[
K_+(2,1) = e^{i \beta(z,1)} K_+(2-1,0)
\]

where \( \beta(z,1) = \frac{1}{2} X_\mu(z)X_\nu(1)F_{\mu\nu} \)

This is evident since \( \beta(z,1) \) is zero and

\[
-\frac{\partial}{\partial x_{\mu z}} \beta(z,1) = -\frac{1}{2} X_\nu(1)F_{\mu\nu}
\]

\[
(i \gamma^2 - \kappa(z)) e^{i \beta(z,1)} K_+(2-1,0) = e^{i \beta(z,1)} (i \gamma^2 - \kappa(z)) K_+(2-1,0)
\]

Explicitly,

\[
\beta(z,1) = \frac{1}{2} H (X_1 Y_2 - X_2 Y_1) + \frac{1}{2} E (Z_1 T_2 - Z_2 T_1)
\]

\( K_+ \) can be expressed in terms of another function \( I \) by

\[
K_+(2,1) = (i \gamma^2 - \kappa + m) I(2,1)
\]

\( I \) satisfies

\[
i \delta(2,1) = \left[ \gamma_\mu (i \frac{\partial}{\partial x_{\mu z}} - \kappa_\mu) - m \right] \left[ \gamma_\nu (i \frac{\partial}{\partial x_{\nu z}} - \kappa_\nu) + m \right] I(2,1)
\]

\[
= [-m^2 + (i \frac{\partial}{\partial x_{\mu z}} - \kappa_\mu)^2 - \frac{1}{2} \gamma_\mu \gamma_\nu F_{\mu\nu} ] I(2,1)
\]
The following equations will be useful later:

\[
(i \gamma_2 - A_\tau - m)K_+(2,1) = i \delta(2,1) = -K_+(2,1)(i \gamma_1 + A_\mu + m)
\]

\[
-\mathbb{I}(2,1)(i \gamma_1 + A_\mu + m) = K_+(2,1) = (i \gamma_2 - A_\tau + m)\mathbb{I}(2,1)
\]

\[
(-i \frac{\partial}{\partial x_\mu} - A_\mu) \left\{ K_+(2,1) \right\} = (i \frac{\partial}{\partial x_\mu} - A_\mu) \left\{ K_+(2,1) \right\}
\]

Below, \(I(1)\) means \(I(1,0)\).

\[
I(2,1) = e^{i\phi(2,1)}I(2,1)
\]

\[
i \delta(1) = \left[ -m^2 + \left( i \frac{\partial}{\partial x_\nu} - A_\nu \right)^2 - i \frac{\partial}{\partial x_\nu} \gamma_\nu F_{\mu\nu} \right] I(1)
\]

This is satisfied by

\[
I(1) = \int_{-\infty}^{\infty} e^{-i m^2 u} \varphi(1,u) e^{i \frac{\partial}{\partial x_\nu} \gamma_\nu F_{\mu\nu}} du
\]

provided

\[
i \delta(1) \delta(2) = \left[ i \frac{\partial}{\partial u} + \left( i \frac{\partial}{\partial x_\nu} - A_\nu \right)^2 \right] \varphi(1,u)
\]

\[
\frac{\partial}{\partial x_\mu} \gamma_\nu F_{\mu\nu} = \gamma_2 X_2 H + \gamma_3 y_4 E.
\]

Note that these two terms commute. Assume for \(\varphi\) a form \(\varphi = e^{a + b(x^2 + y^2) + c(z^2 + \xi)}\)

\[
i \delta(1) \delta(u) = \left[ \frac{a^2}{x^2} + \frac{b^2}{y^2} - \frac{H^2}{q} (x^2 + y^2) + iH \left( \frac{a}{x} - \frac{b}{y} \right) \right] \varphi
\]

\[+ \left[ \frac{a^2}{z^2} - \frac{b^2}{\xi^2} + \frac{E^2}{q} (z^2 + \xi^2) + iE \left( \frac{a}{z} + \frac{b}{\xi} \right) \right] \varphi
\]

Then

\[-i \frac{da}{du} = 4(\alpha + c), \quad -i \frac{db}{du} = 4 \beta^2 - \frac{H^2}{q}, \quad -i \frac{dc}{du} = 4 \lambda^2 + \frac{E^2}{q}\]

Choose solutions singular at the origin:
To get a solution of the inhomogeneous equation, take \( \Phi = 0 \) for \( u < 0 \) and choose the constant such that

\[
1 = \int_{u=0+}^{t} \phi \ dx \ dy \ dz \ dt
\]

So finally

\[
\Phi = \frac{EH}{16\pi^2 \sinh Hu \sinh Eu} e^{i \frac{H}{4} \cosh Hu (x^2 + y^2) + i \frac{E}{4} \coth Eu (z^2 - t^2)}
\]

\[
I(2,1) = \int_{0}^{\infty} e^{-i m u} e^{\frac{1}{2} \sqrt{g} \lambda \sqrt{f} \mu \nu \epsilon} e^{i \lambda(2,1)} \Phi(2-1, u) \ du
\]

Note that \( I \) is even in space and time, as it should be.

III. \( M \) IN PARAMETRIC FORM

\( M \) can be simplified by substituting for \( K \) its expression in terms of \( I \) and then reducing the result by partwise integration and the use of various identities.

\[
\frac{M}{e^2} = \iint \psi^*(x) \gamma_{\mu} \left( (\gamma_{\nu} - \delta_{\nu}^{\mu} + m) I(2,1) \right) \gamma_{\nu} \psi(x) f_{\mu} (s_{\nu}) \ d2d1
\]

\[
= 2m \iiint \psi^* \psi - \iint \psi^* \gamma_{\nu} \gamma_{\mu} \gamma_{\nu} \psi \left( \frac{\partial}{\partial \gamma_{\nu}} f_{\mu} \right) \psi
\]
Alternatively
\[ M/e^2 = -\iint \psi^+(2) \gamma_\mu (I(2,1)(\mathbf{p} + \mathbf{h} - m)) \gamma_\mu f_+ \psi(1) \, d2 \, d1 \]
\[ = 2m \iint \psi^+ I f_+ \psi - \iint \psi^+ \gamma_\mu I \gamma_\mu \psi(1) \frac{\delta}{\delta \nu_2} f_+ \]

The backward pointing arrow means that the differential operator operates back on the function \( I \) before it. In performing the reduction, use has been made of the fact that the initial and final wave functions satisfy the Dirac equation. Examination of the difference between the two expressions for \( M \) shows that it vanishes, but half the sum is simpler to use than either.

Specializing to the case \( E = 0 \) and inserting the parametric expressions for \( I \) and \( f_+ \) gives, after some manipulation
\[ M = \frac{e^2}{16 \pi^3} \int_0^\infty g(d \omega) d\omega \int_0^\infty H e^{-i \frac{m^2}{\omega}} \left( J_1 + J_2 + J_3 \right) d\omega \]
\[ J_1 = 2m \iint \psi^+(2) e^{-\gamma_5} \gamma_\mu H \gamma_\mu \psi(1) e^{\gamma_0 \gamma_4} \, d1 \, d2 \]
\[ J_2 = -4 \alpha \cos H \iint \psi^+(2) (\gamma_5 x_2 + \gamma_5 y_2) \psi(1) e^{\gamma_0 \gamma_4} \, d1 \, d2 \]
\[ J_3 = 4 \alpha \iint \psi^+(2) e^{i \gamma_5} \gamma_\mu H \gamma_\mu \gamma_5 \gamma_4 \psi(1) e^{\gamma_0 \gamma_4} \, d1 \, d2 \]
\[ G_0 + G_1 = \frac{\zeta H}{2} (x_2 y_2 - x_1 y_1) + \frac{\zeta H}{2} (\cos H + \sin H)(x_2^2 + y_2^2) + \zeta (\kappa + \frac{1}{\omega}) (x_2^2 + y_2^2) \]

Terms involving \( \gamma_5 \) have been omitted, since in the case \( E = 0 \) it is possible without loss of generality to consider only states with zero momentum along the z axis, and for such states these
terms vanish.

IV. THE WAVE FUNCTIONS

Since electron states in a uniform magnetic field are highly degenerate, there is considerable freedom of choice in selecting a set of states which will include each physically different case just once, with no redundant inclusion of states which differ only by having different distributions of the centers of the circular orbits. The states chosen are those (of zero momentum along the z axis) which have the centers localized as much as possible at the origin \(x=y=0\). There are exactly two such states, of opposite spin orientation, for each non-negative value of \(n\), where minus \(n\) is the z component of orbital angular momentum. The energy of these states depends only on the total angular momentum, so that state \(n\), spin up is degenerate with state \(n-1\), spin down. All states outside the set and with this same angular momentum have higher energies corresponding to greater kinetic angular momentum partly compensated by the vector potential.

Because of conservation of energy and angular momentum, the matrix element \(M\) vanishes if taken between a state in the set and an outside state.
Within the set the only off-diagonal elements that do not automatically vanish from conservation are those that connect the two states of a degenerate pair. These also vanish when the states are properly chosen, so that there remain only the diagonal elements which are energy corrections.

Introducing $x-iy=W$ and $r^2=x^2+y^2$ and ignoring normalization factors, the wave functions are:

For spin up: $k^2=m^2\gamma_2 n \hbar$

$$\psi = \begin{pmatrix} (k+m) W \sin \theta \\ 0 \\ 0 \\ -2i n W \sin \theta \end{pmatrix} e^{-\frac{i}{2} r^2 - ikt}$$

For spin down: $k^2=m^2\gamma_2 (n+1) \hbar$

$$\psi = \begin{pmatrix} 0 \\ (k+m) W \\ i H W \sin \theta \\ 0 \end{pmatrix} e^{-\frac{i}{2} r^2 - ikt}$$

These wave functions are solutions of $(i \partial - \gamma - m) \psi = 0$

given by $\psi = (i \partial - \gamma + m) \phi$

$$[m^2 - (i \partial - \gamma - m)^2] \phi = i \gamma_5 \partial_5 H \phi = (\sigma_5 \sigma_5) \phi = \pm \hbar \phi$$

It is easy to verify that $1, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8$, and $\gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}$, which are the only Dirac matrices appearing in $M$, all give zero when taken between an up and a down state.
When the expressions for $\Psi$ and $\Psi^*$ are inserted into $J_1, J_2,$ and $J_3,$ the resulting integrals, though rather involved, are essentially of Gaussian form. Each breaks into a $t,$ a $z,$ and an $xy$ factor. As an example, $J_1$ for spin up is

$$J_1 = 2mR \int \left[ e^{iH' \cdot (k \cdot m)} W_1^m W_2^n - 4 m^2 W_1^{m^2} W_2^{-n^2} \right] e^{i t} dx_1 dy_1 dx_2 dy_2$$

where

$$R = \left( \int_{-\infty}^{\infty} e^{i t} dt_1 \right) \left( \int_{-\infty}^{\infty} e^{i \alpha t_2} dt_2 \right)$$

$$G = \frac{iH}{2} (x_1 x_1' - x_2 x_2') + \frac{iH}{4} (\cot \theta n + \frac{\theta}{H}) (x_1 x_1' + x_2 x_2') - \frac{i}{4} (r_1 - r_2)^2$$

The exponential factors are the same for each $J$ and for both spin cases, though the other factors vary. By choosing new variables in a suitable manner, $G$ can be written as a sum of squares, and the integrals evaluated in a straightforward way. After considerable labor and a variety of substitutions, the results are

$$M_{up} = \frac{e^x}{4H} \int_0^1 \int_0^1 e^{-i m_2 s} e^{-2 i m_2 z} \frac{(1 - 5\beta_1)^{m_1}}{(1 - 5\beta_2)^{m_2}} A_1$$

$$M_{down} = \frac{e^x}{4H} \int_0^1 \int_0^1 e^{-i m_2 s} e^{-2 i m_2 z} \frac{(1 - 5\beta_1)^{m_1}}{(1 - 5\beta_2)^{m_2}} A_2$$

$$\beta_1 = 1 - e^{i\theta} \frac{\tan \theta}{\theta}$$

$$\beta_2 = 1 - e^{-i\theta} \frac{\tan \theta}{\theta}$$

$$\theta = \frac{\nu u}{1 + \nu u}$$
\[ A_1 = (h+m) \left( \frac{m}{h} + Se^{-2i\beta} \right) \frac{1 - S\beta_1}{1 - S\beta_2} + (2-m) \left( - \frac{m}{h} + Se^{2i\beta} \right) \]
\[-2(k^2-m^2) \frac{S \cos \phi}{k^2} \frac{\cos \frac{2\pi}{\lambda} \frac{m^2}{h}}{1 - S\beta_2} \]

\[ A_2 = (h+m) \left( \frac{m}{h} + Se^{2i\beta} \right) + (2-m) \left( - \frac{m}{h} + Se^{-2i\beta} \right) \frac{1 - S\beta_1}{1 - S\beta_2} \]
\[-2(k^2-m^2) \frac{S \cos \phi}{k^2} \frac{\cos \frac{2\pi}{\lambda} \frac{m^2}{h}}{1 - S\beta_2} \]

In the limiting case \( H \to 0 \) keeping \( k^2m^2 \) constant, \( A_1 \) and \( A_2 \) both become \( 2\frac{m^2}{h} (1+S) \) and \( M \) becomes
\[ M_o = \frac{e^2 m^2}{2\pi} \int_0^\infty \int_0^{\infty} \frac{e^{i m^2 \frac{y^2}{4} u^2}}{u} \left( 1 + \frac{y^2}{4} u^2 \right) \left( 1 + \frac{y^2 u^2}{4} \right) \left( 1 + \frac{y^2 u^2}{4} \right) \]

\( M_o \) represents the energy correction for a particle of unchanged kinetic energy in a vanishing field, and is of the form of a change in the rest mass.

Introducing \( \psi(\alpha) = 1 - e^{-i\phi \alpha} \), \( \frac{1}{nu} = (\beta-1) \alpha \)

\[ A M_o = \frac{e^2 m^2}{2\pi} \int_0^\infty \frac{\psi(\alpha) d\alpha}{\alpha} \int_0^{\infty} \frac{e^{-i m^2 \frac{y^2}{4} u^2}}{u} \left( 1 + \frac{y^2}{4} u^2 \right) \left( 1 + \frac{y^2 u^2}{4} \right) d\beta \]
\[ = \frac{e^2 m^2}{2\pi} \int_1^{\infty} \left( \frac{1}{\beta^2} + \frac{1}{\beta^2} \right) \ln \left( 1 + \frac{k^2}{m^2} \phi (\alpha) \right) d\beta \]

For \( K \gg m \) this is \( \approx \frac{2e^2 m^2 h^2 k + 2e^2 m^2}{8\pi} \) in agreement with known results.
Once the mass correction has been subtracted, the convergence factor $g$ can be replaced by unity. Introducing $\lambda = \frac{m^3}{n^2}$ the final result is

\[
\Delta E_{\text{up}} = \frac{e^{x^2}}{n^2} \int_0^\infty dx \int_0^\infty dz e^{-\lambda z^2} \left[ - \frac{2m^3}{n^2} (1 + x) + e^{-2i\pi m A_2} \left( \frac{(1 - 5B_2)^m - 1}{(1 - 5B_2)^m - 1} A_1 \right) \right]
\]

\[
\Delta E_{\text{down}} = \frac{e^{x^2}}{n^2} \int_0^\infty dx \int_0^\infty dz e^{-\lambda z^2} \left[ - \frac{2m^3}{n^2} (1 + x) + e^{-2i\pi m A_2} \left( \frac{(1 - 5B_2)^m - 1}{(1 - 5B_2)^m - 1} A_1 \right) \right]
\]

V. THE ASYMPTOTIC EXPANSION

The usual technique for asymptotic expansion cannot be directly applied because the coefficient $\lambda S$ in the exponent is not uniformly large. However, if the integrals are rewritten as

\[
\Delta E_{\text{up}} = \frac{e^{x^2}}{n^2} \int_0^\infty dx \int_0^\infty dz \left( e^{\lambda z^2} - e^{-\lambda z^2} \right) + \frac{e^{x^2}}{n^2} \int_0^\infty dx \int_0^\infty dz e^{-\lambda z^2} \left[ - \frac{2m^3}{n^2} (1 + x) + e^{-2i\pi m A_2} \left( \frac{(1 - 5B_2)^m - 1}{(1 - 5B_2)^m - 1} A_1 \right) \right]
\]

\[
\Delta E_{\text{down}} = \frac{e^{x^2}}{n^2} \int_0^\infty dx \int_0^\infty dz \left( e^{\lambda z^2} - e^{-\lambda z^2} \right) + \frac{e^{x^2}}{n^2} \int_0^\infty dx \int_0^\infty dz e^{-\lambda z^2} \left[ - \frac{2m^3}{n^2} (1 + x) + e^{-2i\pi m A_2} \left( \frac{(1 - 5B_2)^m - 1}{(1 - 5B_2)^m - 1} A_1 \right) \right]
\]

where $\lambda' = \frac{\lambda}{x^2}$, the first term is a known form and the bracket in the second term can be conveniently expanded in powers of $S$:

\[
\int_0^\infty dx \int_0^\infty dz e^{-\lambda z^2} \left[ S^2 F_r(c) \right] = \frac{1}{\lambda x^{3/2}} \int_0^\infty dx \int_0^\infty dz e^{-\lambda z^2} F_r(c) S^2
\]

\[
= \sum_{r=1}^{\infty} \frac{1}{\lambda x^{3/2}} \left( C_r + \int_0^\infty dx \int_0^\infty dz e^{-\lambda z^2} \frac{1}{\lambda x^{3/2}} F_r^{(r+1)}(0) \right)
\]

\[
= \sum_{r=1}^{\infty} \left( \frac{1}{\lambda x^2} + \frac{C_r + \lambda x^{3/2} F_r^{(r+1)}(x)}{\lambda x^{3/2}} + \frac{1}{\lambda x^{3/2}} F_r^{(r+2)}(x^2) + \ldots \right)
\]
In this development the asymptotic expansion of the z integral is made possible by replacing the definite S integral by an indefinite integral and introducing the constants $C_r$ which must be separately evaluated. $F_0$ is zero and $F_r$ is $O(z^r)$. Furthermore $F_r$ is of the form

$$F_r(cz) = \sum_{\nu=0}^{r} \beta_{\nu} e^{-i\alpha\nu \frac{z}{(cz)}}$$

where the $\alpha$ are integers which are all positive for the ground state but otherwise include some negative values. The coefficients $\beta$ obey sufficient conditions to keep the integrand finite. By evaluating

$$(-1)^{\nu+1} \int_0^\infty \frac{dz}{2\pi} e^{-i\frac{z}{cz}} F_r(cz) = i\int_0^\infty dz e^{-i\frac{z}{cz}} F_r(cz)$$

and then integrating $(r+1)$ times from $S$ to $\infty$,

$$\int_0^{s'} ds' \int_0^\infty \frac{dz}{2\pi} e^{-i\frac{z}{cz}} F_r(cz) = \int_0^{s'} (\sum_\nu \beta_{\nu} (s) \lambda_n(s+\alpha_\nu)) ds$$

$$= \sum_\nu \lambda_n(s+\alpha_\nu) Q_{1\nu}(s) + Q_{2\nu}(s')$$

Where the $P, Q$ are polynomials. In doing the final $S$ integral it is necessary to circle around the points $S+\alpha_\nu=0$. This gives rise to an imaginary contribution to $\Delta E$ that is interpretable in terms of a decay probability for the state. There is no
such imaginary contribution for the ground state. In the final result the logarithms can be expanded and everything rearranged in inverse powers of $\lambda$. This second method of calculation is much more laborious but has the advantage that it fixes the values of the $C_r$.

Although it is possible in principle to carry out the expansions to any order in $H$, in practice the labor becomes excessive at about the third order. The calculation to second order is quite simple if both methods of calculation are used, the second method being used only for those terms which are capable of contributing to $C_1$ in zeroeth order. The result is

$$\Delta E_{\text{up}} = \frac{\epsilon m}{\gamma \hbar} \left[ -\frac{1}{\lambda} + \frac{8}{3\lambda^2} \ln \frac{1}{2} - \frac{1}{\lambda^2} \left( \frac{13}{9} + \frac{2}{3} \pi m \right) \right]$$

$$\Delta E_{\text{down}} = \frac{\epsilon m}{\gamma \hbar} \left[ +\frac{1}{\lambda} + \frac{8}{3\lambda^2} \ln \frac{1}{2} - \frac{1}{\lambda^2} \left( \frac{13}{9} + \frac{2}{3} \pi m i + \pi i \right) \right]$$

In the case of the ground state it is possible to investigate the convergence of the expansion. In this case (only) the path of $z$ integration can be displaced to the negative imaginary axis so that $\Delta E$ can be expressed in the purely real form
Convergence of the sum is guaranteed since an upper bound can be found for the sum from $N$ to $\infty$. Since $1 + S e^{-\frac{Z}{2}}$

$$0 \leq \beta \leq \min(1, \frac{\pi}{2})$$

$$0 \leq \frac{1}{1 - \beta} \leq \min(\frac{\pi}{2}, 1 + \frac{Z}{2})$$

$$\left| 2 \int_0^s dS \int_0^{\infty} e^{-\frac{s z}{2}} \frac{\beta^\nu}{1 - S \beta} \right|$$

is bounded by $2 \int_0^s \min(A, B) dS$

$$A_s = \int_0^{\infty} e^{-\frac{s z}{2}} \frac{S^N}{1 - S} \frac{(\frac{Z}{2})^\nu}{\nu!} = \frac{(N-1)!}{\lambda^N} \frac{1}{1-S}$$

$$B_s = \int_0^{\infty} e^{-\frac{s z}{2}} \frac{(1+S)(\frac{Z}{2})^N}{\nu!} = \frac{(N-1)!}{\lambda^N} \frac{2^N}{1 + \frac{Z}{2}}$$

Hence the sum is less than

$$\frac{2(N-1)!}{\lambda^N} \left( 2 + \frac{N}{\lambda} \int \frac{A}{N} \right)$$

Furthermore, examination of the exact form of the first $N$ terms shows that their expansion in powers plus (powers)(ln$\lambda$) converges if $(N+1) < \frac{\lambda}{2}$

VI. INTERPRETATION AND DISCUSSION

Since in electromagnetic units $\frac{1}{\lambda} \approx H \cdot 10^{-14}$, these second and higher terms are of little practical interest. They are however of some
theoretical interest. The occurrence of logarithm factors shows that any attempt to arrive at these results by treating the magnetic field as a perturbation would encounter an infrared catastrophe. For a bound electron the binding furnishes a cut-off, but for the free electron it would be necessary to use a cutoff depending on H, which would require care to avoid confusing even the form of the answer. For the bound case the form of the $H^2$ correction is readily found. Bethe's expression for the Lamb shift, modified for the presence of a vector potential, is

$$\Delta E = \frac{2}{3} \frac{e^2}{\hbar} \frac{m}{c^2} \int \psi^* \vec{v} \cdot [\vec{H}, \vec{v}] \psi$$

The first term gives $\frac{1}{m^2} \vec{v}^2 \vec{V}$, the usual Lamb shift. The second is $\frac{H^2}{m^3}$. There is another term $\frac{A \cdot J}{m^3}$ which is dropped because J, the current density creating the field A, presumably vanishes inside the atom. The final answer is

$$\Delta E = \frac{2}{3} \frac{e^2}{\hbar} \frac{H^2}{m^3} \frac{m}{c^2}$$

The coefficient is the same as for the free electron.
Twice the imaginary part of $\Delta \mathcal{E}$ is the probability per unit time of radiative decay to a lower state. Since

$$\frac{1}{\omega} \frac{2}{3} e^2 a^2 = \frac{2}{3} \frac{e}{m} e^2 \frac{m^2}{\omega^2} = \frac{2}{3} e^2 m n \left( \frac{e}{m} \right)^2$$

the term proportional to $n$ agrees with the classical acceleration radiation. It also checks with a quantum mechanical calculation using the dipole approximation. The fact that the states extend indefinitely in the z direction can be ignored in using this approximation since the only effect of this extension is to force the electron to take up the recoil momentum. The extra term which appears only for states with spin down is a measure of the probability of decay by turning over the spin.
References

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