I   RETARDATION CORRECTIONS

IN THE HELIUM ATOM

II   SELF ENERGY OF AN ELECTRON

IN A MAGNETIC FIELD

Thesis by

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PART I.

ABSTRACT

An analysis is made of the errors which arise for absolute levels and fine structure separations from the use, for the interaction of two electrons, of the nonretarded Breit expression instead of the true retarded interaction. The effects on the fine structure turn out to be too small for observation. The correction to the ground state energy may become observable if the experimental accuracy is increased. This is probably not the case for any excited level.

PART II.

ABSTRACT

The exact propagation function for a Dirac electron in an arbitrarily intense magnetic field is derived in closed form as a parametric integral. Using the exact relativistic wave functions and the exact propagation function, the energy corrections of order e^2, due to the emission and reabsorption of one virtual photon, are calculated and exhibited in closed form as double parametric integrals. These integrals are shown to possess an asymptotic expansion in the small parameter H/m^2. This expansion is not a pure power series but involves also terms of the form $H^k \ln \left(\frac{m^2}{H} \right)$. The terms of order H agrees with the known correction to the magnetic moment. The terms of order H^2 and $H^2 \ln \left(\frac{m^2}{H} \right)$ are exhibited and discussed.

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I.   INTRODUCTION

If the nucleus is considered as an infinitely massive structureless point charge, the problem of the helium atom is a perfectly definite one from the viewpoint of field theory. It is the problem of finding all stationary states, of total charge minus two, of the Dirac field in interaction with the nuclear potential and the radiation field. There is however at present no satisfactory way of finding these states or their energies, and all treatments of such systems proceed by at first constructing an accurate solution of the corresponding nonrelativistic problem, for which powerful methods exist, and then adding various relativistic corrections as perturbations. These corrections fall into four classes, giving the effects of

> (1)   The well known spin-orbit inter-
> action and other effects which
> appear in single-electron problems
> and carry over unchanged when more
> electrons are present.

(2) Interaction with externally applied fields. (Stark and Zeeman effects.)

(3) Relativistic corrections to the inter-electron interaction.

(4) Action of the electrons on themselves by the emission and reabsorption of virtual photons. (Lamb shift, etc.)

This thesis is primarily concerned with the third item above, but it will turn out to be so intimately related to the fourth item that it will be convenient to treat the two together. The relation arises from the fact that the inter-electron interaction, aside from its Coulomb part, is due to the emission of virtual photons by one electron and their absorption by the other.

II. NATURE OF BOUND STATES

If the electromagnetic field is eliminated by replacing it by retarded interactions, the atom can be pictured in terms of an indefinitely extended Feynman diagram, with the two electrons repeatedly interacting with themselves and each other, and occasionally doubling back their trajectories to form pairs. The effect of the central potential can be considered either as additional

interactions or as included in the propagation
functions for the electrons. The figure shows
a typical interval in the atom's history, adopting
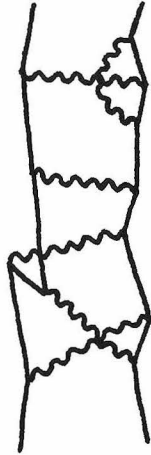
the second alternative and not
showing interactions with the
central field. This picture
does not directly give a part-
icle model of the atom because
of the possible presence of
pairs. However, if pairs are
"usually" absent the state is
describable by a $\psi(x_a, x_b)$ which is the amplitude
for electron a to be at point $\vec{r}_a$ at time t_a and
electron b to be at $\vec{r}_b$ at time t_b with no photons
and no pairs present. By this is meant that a
surface can be drawn which is crossed by no photon
lines, which intersects the world line of each
electron exactly once (at points a,b) and which
extends in a spacelike manner to infinity. The
surface is not required to be spacelike at all
points, and the separation of points a,b need not
be spacelike. Once ψ is known, it is possible to
derive amplitudes for the presence of one or more
photons or pairs. In principle, ψ can be determined

as the solution of an integral equation, whose kernel
is the sum over all irreducible diagrams of the
amplitude for propagation from a,b to a',b'. A
diagram is irreducible if it cannot be separated
into two parts by such a surface as just described.
In actuality no such solutions have yet been ob-
tained. This description is useless when the inter-
action is so strong that a photon is usually present,
for then very long chains of interlaced interactions
make important contributions to the kernel.

For an eigenstate, ψ involves the mean time
$\frac{1}{2}(t_a + t_b)$ only through an exponential factor, so
$\psi = e^{-iE\frac{1}{2}(t_a+t_b)}\phi(\vec{r}_a, \vec{r}_b, t_a-t_b)$ where E is the energy of
the state. In momentum space ψ becomes

$$\delta(k_a^4 + k_b^4 - E)\, n(\vec{k}_a, \vec{k}_b, k_a^4 - k_b^4)$$

so that there is a distribution of energies for
each electron, with the total energy remaining
definite. Nothing is known of the dependence of
the wave function on relative time or relative
energy in any particular case. Intuitively it
would seem probable that there should be only
small amplitude for $|k_a - k_b|$ to be much larger
than the binding energy, and in cases where the
electron interaction energy is small compared

to the total binding the amplitude should be large
only when the individual energy of each electron
approximates the energy of some single-electron
state. Yet this intuition is directly contradicted
by another, for when the binding is weak and the
velocities small, the atom is well described by a
nonrelativistic model in which the forces are not
retarded and only a single time need be considered.
The nonrelativistic ψ involves $t_a - t_b$ only through
a factor $\delta(t_a - t_b)$ so that the wave function is,
in momentum space, independent of $k_a^4 - k_b^4$, correspond-
ing to an infinite spread in relative energy rather
than a small spread comparable to the small binding.
Evidently some care is needed in considering a single
time description as the limit of a many time one.

The retarded interaction between two electrons,
which in coordinate space is $e^2 \delta(s_{ab}^2)$, is in momentum
space

$$4\pi e^2 \frac{\gamma_\mu^a \gamma_\mu^b}{k_\sigma k_\sigma} = -4\pi e^2 \frac{\gamma_t^a \gamma_t^b - \frac{1}{k^2}(\vec{\gamma}^a \cdot \vec{k})(\vec{\gamma}^b \cdot \vec{k})}{k^2 - \omega^2} + 4\pi e^2 \frac{\vec{\gamma}^a \cdot \vec{\gamma}^b - \frac{1}{k^2}(\vec{\gamma}^a \cdot \vec{k})(\vec{\gamma}^b \cdot \vec{k})}{k^2 - \omega^2} \tag{1}$$

The second term is the retarded transverse
interaction. The first term, after a gauge trans-
formation, becomes $-4\pi e^2 \gamma_t^a \gamma_t^b / k^2$, the nonretarded
Coulomb potential. γ_a and γ_b are Dirac matrices

referring respectively to electrons a and b, and k_σ is the momentum of the transferred photon. If in the transverse interaction the photon energy ω is neglected in comparison with k, thereby neglecting the effects of retardation, the result is the Breit[1] interaction energy, whose expectation value is a principal contributor to the helium fine structure level shifts.

Since ω vanishes in the coordinate system in which the center of gravity of the two electrons is at rest, the error in using the Breit interaction instead of the true retarded interaction may be expected to be of the order (Breit energy)$(\overline{v^2}/c^2)$ where $\overline{v^2}$ is a mean square velocity of the center of gravity. In the excited states of helium the outer electron moves slowly compared to the 1s electron, so that $\overline{v^2_{cg}} \approx \frac{1}{4}\overline{v^2_{1s}} \approx \frac{1}{2m}$ (binding energy of 1s electron). Hence $\overline{v}/c^2 \approx 1/(137)^2$. This estimate predicts an error which would be unobservably small. However this estimate is an unreliable one for several reasons:

(1) The next approximation involves $\frac{\omega^2}{k^4}$ which is singular at k=0. This singularity must be investigated before an expansion in $\frac{\omega^2}{k^2}$ is justified.

(2) Although normally the energy of each of
the two electrons is $\approx$ m, their interaction some-
times scatters them into a state where one of them
has a negative energy and the energy of the other
greatly increased in compensation. For momentum
conserved, energy must fail of conservation by
$\approx$ 2m, so that the life of this abnormal situation
is correspondingly brief. Taken between this
state and the normal one, the γ matrices give
factors of about unity, so that the Breit inter-
action gives a matrix element of order e^2/k^2
while the true retarded interaction is more like
e^2/m^2 . It is for this reason that the Breit inter-
action is not suitable for use beyond the first
order in perturbation theory, and even in first
order Breit[2] found that for agreement with exper-
iment it must be used in connection with a part-
icular method of reduction to large components
which amounts to the use of a projection elimin-
ating the negative energies. Since the non-
relativistic variational wave functions do not
lend themselves to an accurate determination of
the magnitude of the negative energy parts, their
contribution to the energy is best determined by

choosing the initial wave functions smooth so that
momenta as large as m are practically absent and the
state includes no negative energies, and then con-
sidering the second order perturbation due to the
electrons being scattered into states of very high
momentum or negative energy and then back.

(3) The effect of the action of the Coulomb
potentials while the electrons are exchanging a
transverse photon ought to be of the same magnitude
as the Lamb shift. Though very small, this is
still large compared to the original error estimate,
and should be investigated.

III. NONRELATIVISTIC APPROXIMATION

The Coulomb part of the electron interaction
is accurately accounted for by the use of variational
principles to find accurate energies. The trans-
verse part may be considered as a perturbation due
to the emission of a virtual photon by one electron
and its absorption by the other. So long as we
limit the photon momenta to be $\ll$ m by a suitable
cutoff, the calculation of the energy shift can be
done using a nonrelativistic model and ordinary
perturbation theory. This gives

$$\Delta E = \frac{-e^2}{4\pi^2 m^2} \sum_n \sum_\sigma \int \frac{d\vec{k}}{k(k+\omega_n-\omega_s)} \left(\langle s | \vec{p}_a \cdot \vec{\epsilon}_\sigma e^{-i\vec{k}\cdot\vec{r}_a} | n \rangle \langle n | \vec{p}_b \cdot \vec{\epsilon}_\sigma e^{i\vec{k}\cdot\vec{r}_b} | s \rangle \right.$$

$$+ \text{similar term with a and b interchanged} \qquad (2)$$

If the Breit interaction had been used instead, the only change in the result would have been the substitution of k for $k + \omega_n - \omega_s$ in the energy denominator. The difference is

$$\Delta E - \Delta E_{BR.} = \frac{e^2}{4\pi^2 m^2} \sum_n \sum_\sigma \int \frac{(\omega_n-\omega_s)d\vec{k}}{k^2(k+\omega_n-\omega_s)} \left(\begin{array}{c} \text{same matrix} \\ \text{elements} \end{array} \right) \qquad (3)$$

This is in the same form as Bethe's expression for the nonrelativistic part of the Lamb shift. The only difference is that the operators for emission and absorption of photons with momentum k and polarization ϵ refer to different electrons. If the Lamb shift terms, which are due to emission and reabsorption, are included, the matrix elements become

$$\langle s | \vec{p}_a \cdot \vec{\epsilon}_\sigma e^{-i\vec{k}\cdot\vec{r}_a} + \vec{p}_b \cdot \vec{\epsilon}_\sigma e^{-i\vec{k}\cdot\vec{r}_b} | n \rangle \langle n | \vec{p}_a \cdot \vec{\epsilon}_\sigma e^{i\vec{k}\cdot\vec{r}_a} + \vec{p}_b \cdot \vec{\epsilon}_\sigma e^{i\vec{k}\cdot\vec{r}_b} | s \rangle$$

$$(4)$$

Following Bethe, the factor $\omega_n - \omega_s$ can be included in the matrix elements by commuting the emission operator with the Hamiltonian. Where it occurs in the energy denominator it can be replaced by a mean excitation energy. Then the sum over states is

easily performed by an appeal to the completeness
property of the wave functions. With

$$H = \frac{1}{2m}(p_a^2 + p_b^2) + V_a + V_b + V_{ab} \tag{5}$$

the result is the expectation of

$$\frac{k^2}{3m}(p_a^2 + p_b^2) + \frac{1}{3}\nabla^2 V_a + \frac{1}{3}\nabla^2 V_b + (1 - \cos k \cdot r)\left(\nabla^2 - \frac{(k \cdot \nabla)^2}{k^2}\right)V_{ab} \tag{6}$$

which has been summed over polarizations. The
first three terms are single-electron effects of
no interest here. The third term, for $V_{ab} = e^2/r$,
reduces to

$$\frac{e^2}{2r^3}\int_{-1}^{1}(3\mu^2 - 1)\cos kr\mu \, d\mu \tag{7}$$

where an averaging over angles has been performed.
The fact that it vanishes as $kr \to 0$ has a very
simple interpretation. The forced motion of a
pair of electrons pushed around by the vacuum
fields can be resolved into a relative motion and
a motion of the center of gravity. The last does
not alter the mean value of the mutual potential
energy, and first does not arise in dipole approx-
imation.

Till now the momentum cutoff K has been left
unspecified except that it is $\ll m$ and $\gg (\omega_a - \omega_s)$ so

that $(\omega_m - \omega_s)/K$ can be neglected. Under this approximation and dropping the single-electron terms, the energy shift is given by the expectation of

$$Q = \frac{2e^4}{\pi m^2 r^3}\left(\frac{\sin Kr}{K^3 r^3} - \frac{\cos Kr}{K^2 r^2} - \frac{1}{3}\right) \qquad (8)$$

Q behaves like $-e^4 K^2/15\pi m^2 r$ for Kr and like $-2e^4/3\pi m^2 r^3$ for Kr large. Let us choose K as large as permissible so as to include in the non-relativistic calculation as much as possible of the total effect. Q is an inverse cubic potential cut off to behave like $1/r$ inside a radius $r_0 \approx 3/K$. Since $K \ll m$, r_0 must be large compared to the Compton wavelength, but it can still be small compared to the size of the atom, so that $Ka_0 \gg 1$, where a_0 is the helium Bohr radius $1/2me^2$.

The form of Q in momentum space will be of importance for any attempt to join this nonrelativistic calculation on to a relativistic calculation which has a low-energy cutoff. Let $S = \beta K$ be the magnitude of the momentum vector. $Q(S)$ has a rather complex behavior for β medium or

large, but this depends on the details of the
cutoff procedure and is of no interest. System-
atically dropping terms of order β or smaller,

$$Q(S) = 4\pi \int_0^\infty \frac{\sin Sr}{Sr} Q(r)r^2 dr$$

$$\approx \frac{8e^4}{3m^2} \int_0^\infty \frac{\sin \beta z}{\beta z} \left(\frac{\sin z}{z} - 1\right) dz$$

$$\approx \frac{8e^4}{3m^2} \left(-1 + \ln S/K\right) \tag{9}$$

A rough estimate of the effect of Q is easily
worked out if the logarithm term is replaced by
some mean value. Let us assume that the effect
of a relativistic calculation would be to replace
K by something of the order of m, so the mean
value should be about $-\ln(a_0 m) \approx -4.22$. Then in
coordinate space Q becomes a delta function. For
triplet states this would give zero if the approx-
imation were good for such states. It is not, but
it is legitimate to conclude that compared to the
ground state the effect is quite small both for
triplet states and for those excited singlet states
where the wavefunctions do not greatly overlap.
For the ground state a calculation gives minus
$(8/3)(5.22)(0.17) \approx -2.36$ cm^{-1}, using hydrogenic

wave functions. This is about half the probable
error in the experimental determination[3] of the
ground state energy. Also, as will be explained
in a later section, there is reason to believe the
logarithmic part is cancelled by another effect
not yet discussed.

IV. RELATIVISTIC CORRECTIONS

The nonrelativistic calculation must be joined
on to a relativistic calculation which takes account
of the effects of virtual photons with momenta above
the cutoff. For these high momenta it is permiss-
able to treat the electrons as lightly bound: the
coulomb potential is considered only as a perturb-
ation which acts at most once. The first term to
be considered is of order e^2 and gives the effect
of the transverse interaction acting only once and
the coulomb potential not at all.

The major part of this term is the Breit inter-
action, reduced to Pauli form by assuming the small
components to be $\vec{p} \cdot \vec{\sigma}/2m$ times the large components.
This reduction is correct to order v^2/c^2 except that
it acts as a projection eliminating any negative
energy states present. These will be accounted for
separately as part of a second term of order e^4 by
considering them to exist only in an intermediate

state, being created by one interaction and de-
-stroyed by a second.

Aside from these negative energy states, the
Breit interaction has errors of order v^4/c^4 from
the reduction process, and an error of this same
order from the difference M which expresses the
effects of retardation.

$$M = 4\pi e^2 {}_3(\alpha_\mu)_{1\upsilon}(\alpha_\nu)_2 \left(\delta_{\mu\nu} - \tfrac{1}{k^2}k_\mu k_\nu\right)\left(\tfrac{1}{k^2-\omega^2} - \tfrac{1}{k^2}\right)$$

$$\vec{k} = \vec{p}_3 - \vec{p}_1 = \vec{p}_2 - \vec{p}_4 \qquad \omega \approx \tfrac{1}{2}\vec{k}\cdot(\vec{p}_1+\vec{p}_2) \qquad (10)$$

The electrons are actually virtual, but to
assume them free should be a good approximation.
When reduced to Pauli form, M becomes (for triplet
states and ignoring all terms independent of the
total spin vector S)

$$\frac{m^4}{2\pi e^2}M = \frac{(S\cdot k)^2(\ell\cdot k)^2}{k^4} - \frac{(S\cdot k')^2(\ell\cdot k')^2}{k'^4} - i\,\vec{S}\cdot(\vec{a}\times\vec{b})\left(\frac{(\ell\cdot k)^2}{k^4} + \frac{(\ell\cdot k)^2}{k'^4}\right)$$

$$2\vec{\ell} = \vec{p}_1 + \vec{p}_2 \qquad 2\vec{a} = \vec{p}_1 - \vec{p}_2 \qquad 2b = \vec{p}_3 - \vec{p}_4 \qquad \vec{k}' = \vec{p}_3 - \vec{p}_2 \qquad (11)$$

An order of magnitude estimate shows that
this is far too small to affect the triplet
splittings. The spin independent part is also

completely negligible. Actually M is even smaller than a quick estimate would indicate, for the level splittings it causes are due to anisotropy in momentum space and the largest part of the momentum differences is the momentum of the ls electron which is nearly isotropic. The form of M is of some interest. The terms quadratic in S have effects proportional to $(L \cdot S)^2$ and are interpretable as arising from something very like a velocity dependent tensor force in momentum space. The other terms have effects proportional to $L \cdot S$ and are less easy to visualize.

The classification of effects as being of various orders in v/c is justified by the very rapid decline of the wave functions for large momenta. (Like p^{-4} as can be seen from the non-relativistic integral equation.) This rapid decline shows that small velocities are really predominant and the mean of v^4 is really of the same order as the square of the mean of v^2. It also justifies the smoothing of the wave functions so that momenta comparable to m are absent.

Next to be considered must be the term of
order e^4 which gives the effect of the transverse
interaction acting once and the coulomb potential
also once. But here it is necessary to be careful,
for since high velocities will be important in the
intermediate states, it is not permissible to con-
sider the transverse interaction small compared
to the coulomb potential. The double transverse
interaction due to exchanging two transverse photons
is no smaller than the above, since v/c is not to
be considered small except in the initial and final
states. Consequently the full e^4 interaction[10] must
be used: all Feynman diagrams with two photon lines.
The two principal diagrams are a and b. The vacuum
polarization is given by c, while d and a number of
others like it are best understood as radiative
corrections to the scattering by the (retarded)
interelectron interaction. The total effect of
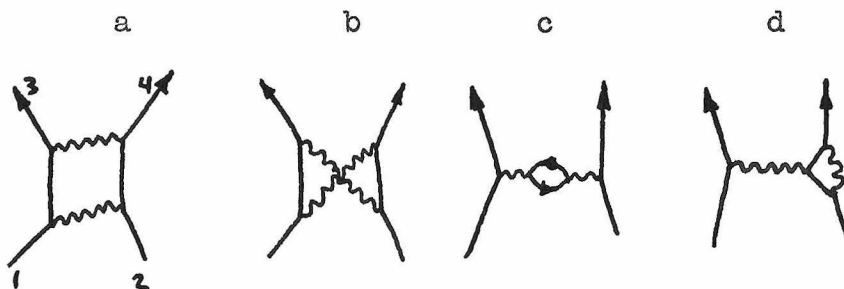
the diagrams like d reduces to

$$\frac{8}{3}\frac{e^4}{m^2}\gamma_\mu \gamma_\mu \frac{q^2}{q^2-\lambda^2}\left(\ln\frac{m}{\lambda}-\frac{3}{8}\right) \tag{12}$$

plus a term which will be ignored which is simply
the interaction of the anomolous part of one
electron's magnetic moment with the electromagnetic
fields generated by the charge of the other electron.
q is the transferred momentum and lambda is a photon
rest mass introduced as a lower cutoff. A similar
cutoff must be used for a and b. No cutoff is
needed for d (with proper renormalization) which
reduces to

$$\frac{e^4}{15\pi}\frac{{}_3(\gamma_\mu)_{1\,4}(\gamma_\nu)_2}{q^2 m^2}\left(q_\mu q_\nu - \delta_{\mu\nu}q^2\right)$$
$$\approx \frac{e^4}{15m^2\pi} - \frac{e^4}{60m^2\pi}(\vec{s}\cdot\vec{q})^2 \tag{13}$$

The procedure here is a model to be followed
in all cases: considering all the initial and final
momenta as small quantities, retain only the lowest
order term, which will be of order e^4/m^2, and the
lowest order term capable of contributing to triplet
splittings, which will be of order $e^4 q^2/m^4$. Reduce
this to Pauli form and express in terms of the total
spin vector $\vec{S}$. The cutoff lambda is to be assigned
a magnitude of approximately $(2\alpha)^{\frac{1}{2}}m$, so that q^2/λ^2

and λ^2/m^2 are both small numbers. For diagram a
there is a possibility of both electrons being only
slightly virtual, corresponding to a longlived inter-
mediate state. This is not properly a part of the
e^4 correction: from a plus b must be subtracted terms
representing the repeated action of a potential which
is Coulomb plus projected Breit. (The projection is
accomplished by deleting that pole of the electron
propagation factor which refers to negative states.)
Actually the positive energy part of the Breit inter-
action makes little difference, its contribution being
much smaller than that of the negative states.

For the term of lowest order any spin factors
are to be evaluated as for a singlet state. The
lowest order term necessarily vanished for triplet
states (antisymmetry).

The rationale of this procedure requires some
explanation. The presence of the photon rest mass
gives to the interaction a finite range which in
momentum space implies a behavior regular about
the origin, so that an expansion in powers of the
initial and final momenta (and hence in powers of
q^2/m^2) is justified if these momenta are $\ll \lambda$.

This regularity seems surprising since this
calculation is to be fitted to a nonrelativistic

one which involves the log of q. The answer to this discrepancy is that there is another effect which should have been included in the nonrelativistic calculation: the effect of the exchange of two transverse photons. This is seemingly of a higher order in v/c, but only seemingly, since what should have been the dominant part of the calculated effect, the dipole approximation, in fact vanished identically, leaving only terms of a higher order in Kr. No attempt has been made to compute this $Q'(S)$, but it is easy to show that it also is of the form e^4/m^2 times a function of S/K, and a reasonable conclusion is that Q and Q' are actually of the same form with the logarithm terms cancelling, so the sum Q plus Q' is just a constant times e^4/m^2. Strictly this is right only for momenta less than the cutoff K (which is to be roughly identified with lambda) but if K is large compared to the momenta present in the initial and final states this qualification can be dropped. The result, in coordinate space, is a delta function interaction.

Since the total nonrelativistic part does not involve the log of the cutoff, the same must be true of the relativistic part (at least in the limit of zero momenta) and in fact the lowest order term

must approach a constant as lambda approaches zero, which justifies the earlier statement that it is of order e^4/m^2 rather than say e^4/λ^2. A similar argument holds for the fine structure term, making use of the fact that the nonrelativistic part is independent of the orientation of the spin vector. (There are terms of order $e^4 q^2/m^2\lambda^2$ which blow up as lambda goes to zero, but they are the same for all lines of a multiplet. They represent the failure of the expansion in powers of q when $q > \lambda$.) Since the terms involving q^2 are too small to make their effect on absolute levels appreciable, it is only necessary to retain that part which contributes to multiplet splittings, and for this there is no trouble.

Since the major contribution to it comes from negative energy states (strictly, states with pairs present) the e^4 term must represent an interaction of short range. Since the intermediate states differ from the intial and final by about 2m, the interaction should have a range of about a Compton wavelength, which makes logical an expansion in delta functions and their derivatives, or, in momentum space, in ascending powers, as stated in the

beginning. Since observations on level splittings are more precise than those on absolute levels, not only the lowest order term must be retained, but also the lowest order term which depends on the spin orientation.

In the previous section it was calculated that in the most favorable circumstances (ground state) a term e^4/m^2 causes a level shift of only 0.17 cm^{-1}. This is only about 3% of the probable error of the experimental ground state energy, and the calculational accuracy is probably no greater. The best calculated value is 198319cm^{-1} while a 1942 experimental value is 198314 ± 5cm^{-1}. For excited states the effect is less and the experimental and calculational uncertainties much greater. So no observable effect can be expected unless very generous assumptions are made as to the probable coefficient of the e^4/m^2 term.

The possibility of an appreciable effect on the fine structure is far more remote. The lowest order term which can possibly contribute is e^4q^2/m^4, which is smaller than the above by a factor of roughly $4/(137)^2$ or $2 \cdot 10^{-4}$.

This is small compared to the accuracy with which known effects can be calculated, and quite beyond the possibility of experimental verification. Hence, the only effect of any importance is the shift of the lower singlet levels by the delta function interaction. The coefficient of this interaction has not been computed exactly, but according to a rough estimate it should not be greater than about ten at most. This would still require a refinement of the experimental accuracy to make it observable.

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I. INTRODUCTION

The purpose of this investigation is to find
the energy, due to self action, of an electron in
a uniform magnetic field. The quantity we wish to
find is the value per unit time of

$$M = e^2 \iint \Psi^+(2) \gamma_\mu K_+(2,1) \gamma_\mu \Psi(1) f_+(S_{12}^2) \, d2 \, d1$$

where Ψ is some one of the stationary states of an
electron in a magnetic field, and K_+ is Feynman's
propagation function for an electron in such a field.
Both are expressed in a form correct for arbitrarily
intense fields. M expresses the effect of the virtual
emission and reabsorption of a single photon in the
presence of the field, and gives the energy increment
correct to order e^2 in the electron charge, but to
all orders in the field intensity. The technique
used is to express K_+ and f_+ as parametric integrals,
with the integrands simple functions of space and
time, so that all integrations other than those
over the parameters are easily performed. M is
then in exact closed form as a double integral, for
which an expansion in H, the magnetic field intensity,
can be found. This expansion is not a pure power
series, but contains terms of the form $H^K \ln H$.

Since the extra generality takes little trouble,

the preliminary developement is done for the case
of uniform electric and magnetic fields both present.
Without loss of generality, E and H are both taken
in the z direction with $H \geqq 0$, since any other case
can be reduced to this by a suitable Lorentz trans-
formation. Later we specialize to $E = 0$. The pot-
ential is chosen as

$$A_\nu = \tfrac{1}{2} X_\mu F_{\mu\nu} \qquad F_{\mu\nu} = \frac{\partial A_\nu}{\partial X^\mu} - \frac{\partial A_\mu}{\partial X^\nu} \qquad F_{12} = -H \qquad F_{34} = +E$$

$$A_x = -\tfrac{1}{2} y H \qquad A_z = -\tfrac{1}{2} t E$$
$$A_y = +\tfrac{1}{2} x H \qquad A_t = -\tfrac{1}{2} z E$$

Here and henceforth, the relativistic notation
of Feynman[1] is used:

$$X_\mu = x, y, z, t = \vec{r}, t \qquad\qquad \not{a} = \gamma_\mu a_\mu$$
$$A_\mu = \vec{A}, A_4 \qquad\qquad \frac{\partial}{\partial X_\mu} = -grad, \frac{\partial}{\partial t}$$
$$\gamma_\mu = \beta\vec{\alpha}, \beta \qquad\qquad \not{\nabla} = \gamma_\mu \frac{\partial}{\partial x}$$
$$a_\mu b_\mu = a_4 b_4 - \vec{a}\cdot\vec{b} = a\cdot b \qquad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$$
$$\delta_{44} = 1, \; \delta_{11} = \delta_{22} = \delta_{33} = -1 \qquad \not{a}\not{b} + \not{b}\not{a} = 2a\cdot b$$
$$\delta_{\mu\nu} a_\nu = a_\mu \qquad\qquad \phi^+ = \phi^* \beta$$

II. PROPAGATION FUNCTIONS

Below, 0 denotes the origin, while 1 and 2
denote arbitrary spacetime points. $f_+(S_{12}^2)$ is the
photon propagation function, given by

$$f_+(S_{21}^2) = \frac{1}{\pi} \int_0^\infty e^{-i\alpha X_{\mu 21} X_{\mu 21}} g(\alpha) \, d\alpha \qquad (X_\mu X_\mu = S^2)$$

The convergence factor $g(\alpha)$ is introduced to make

the mass correction finite. When the mass correction has been subtracted $g^{(\alpha)}$ will be replaced by 1, so that f_+ becomes $\delta_+(S_{12}^2)$ which is the true propagation function without cutoff.

K_+ is defined to be that solution of the inhomogeneous Dirac equation

$$(i \not\nabla_2 - \not A_2 - m)K_+(2,1) = i\,\delta(2,1)$$

which shows the same behaviour as does the field free K_+ when $2 \to 1$, and also for $\not A \to$ zero K_+ must reduce to the field free K_+.

Because of the presence of the potentials, $K_+(2,1) \neq K_+(2-1,0)$. Instead

$$K_+(2,1) = e^{i\beta(2,1)} K_+(2-1,0) \quad \text{where} \quad \beta(2,1) = \tfrac{1}{2} X_\mu(2) X_\nu(1) F_{\mu\nu}$$

This is evident since $\beta(1,1)$ is zero and

$$-\frac{\partial}{\partial X_{\mu 2}}\beta(2,1) = -\tfrac{1}{2} X_\nu(1) F_{\mu\nu}$$

$$\left(i\frac{\partial}{\partial X_{\mu 2}} - A_\mu(2)\right) e^{i\beta(2,1)} K_+(2-1,0) = e^{i\beta(2,1)}\left(i\frac{\partial}{\partial X_{\mu 2}} - A_\mu(2-1)\right) K_+(2-1,0)$$

Explicitly,

$$\beta_+(2,1) = \tfrac{1}{2} H(X_1 Y_2 - X_2 Y_1) + \tfrac{1}{2} E(Z_1 t_2 - Z_2 t_1)$$

K_+ can be expressed in terms of another function I by

$$K_+(2,1) = (i \not\nabla_2 - \not A_2 + m) I(2,1)$$

I satisfies

$$i\,\delta(2,1) = \left[\gamma_\mu\left(i\frac{\partial}{\partial X_{\mu 2}} - A_{\mu 2}\right) - m\right]\left[\gamma_\nu\left(i\frac{\partial}{\partial X_{\nu 2}} - A_{\nu 2}\right) + m\right] I(2,1)$$

$$= \left[-m^2 + \left(i\frac{\partial}{\partial X_{\mu 2}} - A_{\mu 2}\right)^2 - \tfrac{i}{2}\gamma_\mu \gamma_\nu F_{\mu\nu}\right] I(2,1)$$

The following equations will be useful later:

$$(i\not{\nabla}_2 - \not{A}_2 - m)K_+(2,1) = i\delta(2,1) = -K_+(2,1)(i\overleftarrow{\not{\nabla}}_1 + \not{A}_1 + m)$$

$$-I(2,1)(i\overleftarrow{\not{\nabla}}_1 + \not{A}_1 + m) = K_+(2,1) = (i\not{\nabla}_2 - \not{A}_2 + m)I(2,1)$$

$$\left(-i\frac{\partial}{\partial X_{\mu 1}} - A_{\mu 1}\right)\begin{Bmatrix} K_+(2,1) \\ I(2,1) \end{Bmatrix} = \left(+i\frac{\partial}{\partial X_{\mu 2}} - A_{\mu 2}\right)\begin{Bmatrix} K_+(2,1) \\ I(2,1) \end{Bmatrix}$$

Below, I(1) means I(1,0). $I(2,1) = e^{i\beta(2,1)}I(2-1)$

$$i\delta(1) = \left[-m^2 + \left(i\frac{\partial}{\partial X_\nu} - A_\nu\right)^2 - \frac{i}{2}\gamma_\mu\gamma_\nu F_{\mu\nu}\right]I(1)$$

This² is satisfied by

$$I(1) = \int_{-\infty}^{\infty} e^{-im^2 u}\varphi(1,u)e^{\frac{i}{2}\gamma_\mu\gamma_\nu F_{\mu\nu} u}\, du$$

provided

$$i\delta(1)\delta(2) = \left[i\frac{\partial}{\partial u} + \left(i\frac{\partial}{\partial X_\nu} - A_\nu\right)^2\right]\varphi(1,u)$$

$-\frac{1}{2}\gamma_\mu\gamma_\nu F_{\mu\nu} = \gamma_1\gamma_2 H + \gamma_3\gamma_4 E$. Note that these two terms commute. Assume for φ a form $\varphi = e^{a + b(x^2+y^2) + c(z^2-t^2)}$

$$i\delta(1)\delta(u) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{H^2}{4}(x^2+y^2) + iH\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\right]\varphi$$

$$+ \left[\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} + \frac{E^2}{4}(z^2-t^2) + iE\left(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}\right)\right]\varphi$$

$$+ i\frac{\partial\varphi}{\partial u}$$

Then

$$-i\frac{da}{du} = 4(b+c)\,, \qquad -i\frac{db}{du} = 4b^2 - \frac{H^2}{4}\,, \qquad -i\frac{dc}{du} = 4c^2 + \frac{E^2}{4}$$

Choose solutions singular at the origin:

$$b = \frac{iH}{4} \cot Hu$$

$$c = \frac{iE}{4} \coth Eu$$

$$e^a = \frac{EH \cdot \text{CONST}}{\sin Hu \sinh Eu}$$

To get a solution of the <u>inhomogeneous</u> equation, take $\varphi \equiv 0$ for $u \leq 0$ and choose the constant such that

$$1 = \int\limits_{u=0_+} \varphi \, dx \, dy \, dz \, dt$$

$$\frac{1}{\text{CONST}} = \int \frac{1}{u^2} e^{\frac{i}{4u}(x^2+y^2+z^2-t^2)} \, dx \, dy \, dz \, dt = 16\pi^2 i$$

So finally

$$\varphi = \frac{EH}{16\pi^2 i \, \sin Hu \, \sinh Eu} \, e^{i\frac{H}{4}\cot Hu(x^2+y^2) + i\frac{E}{4}\coth Eu(z^2-t^2)}$$

$$I(2,1) = \int_0^\infty e^{-im^2 u} e^{\frac{1}{2}\gamma_\mu \gamma_\nu F_{\mu\nu} u} e^{i\beta(2,1)} \varphi(2-1,u) \, du$$

Note that I is even in space and time, as it should be.

III. M IN PARAMETRIC FORM

M can be simplified by substituting for K its expression in terms of I and then reducing the result by partwise integration and the use of various identities.

$$M/e^2 = \iint \psi^+(2) \gamma_\mu \Big((\nabla_2 - A_2 + m) I(2,1) \Big) \gamma_\mu \psi(1) f_+(s_{12}^2) \, d2 \, d1$$

$$= 2m \iint \psi^+ I f_+ \psi - \iint \psi^+ \gamma_\mu \gamma_\nu I \gamma_\mu (i\frac{\partial}{\partial x_{\nu 2}} f_+) \psi$$

Alternatively

$$M/e^2 = -\iint \psi^+(2)\gamma_\mu \left(I(2,1)(i\overleftarrow{\nabla}_1 + A_1 - m)\right)\gamma_\mu f_+(s_{12}^2)\psi(1)\,d2\,d1$$

$$= 2m \iint \psi^+ I f_+ \psi - \iint \psi^+ \gamma_\mu I \gamma_\nu \gamma_\mu \psi \left(i \frac{\partial}{\partial x_{\nu_2}} f_+\right)$$

The backward pointing arrow means that the diff-
erential operator operates back on the function
I before it. In performing the reduction, use
has been made of the fact that the initial and
final wave functions satisfy the Dirac equation.
Examination of the difference between the two
expressions for M shows that it vanishes, but
half the sum is simpler to use than either.
Specializing to the case E=0 and inserting the
parametric expressions for I and f_+ gives, after
some manipulation

$$M = \frac{e^2}{16\pi^3 i} \int_0^\infty g(\alpha)\,d\alpha \int_0^\infty \frac{H e^{-im^2 u}}{u \sin Hu}(J_1 + J_2 + J_3)\,du$$

$$J_1 = 2m \iint \psi^+(2) e^{-\gamma_1 \gamma_2 Hu} \psi(1) e^{G_0 + G_1}\,d1\,d2$$

$$J_2 = -4\alpha \cos Hu \iint \psi^+(2)(\gamma_1 X_{21} + \gamma_2 Y_{21})\psi(1) e^{G_0 + G_1}\,d1\,d2$$

$$J_3 = 4\alpha \iint \psi^+(2) e^{\gamma_1 \gamma_2 Hu} \gamma_4 t_{21}\psi(1) e^{G_0 + G_1}\,d1\,d2$$

$$G_0 + G_1 = \frac{iH}{2}(x_1 Y_2 - x_2 Y_1) + \frac{iH}{4}\left(\cot Hu + \frac{4\alpha}{H}\right)(x_{21}^2 + Y_{21}^2) + i\left(\alpha + \frac{1}{4u}\right)(z_{21}^2 - t_{21}^2)$$

Terms involving γ_3 have been omitted, since
in the case E = 0 it is possible without loss of
generality to consider only states with zero mom-
entum along the z axis, and for such states these

terms vanish.

IV. THE WAVE FUNCTIONS

Since electron states in a uniform magnetic field are highly degenerate, there is considerable freedom of choice in selecting a set of states which will include each physically different case just once, with no redundant inclusion of states which differ only by having different distributions of the centers of the circular orbits. the states chosen are those (of zero momentum along the z axis) which have the centers localized as much as possible at the origin x=y=0. There are exactly two such states, of opposite spin orientation, for each non-negative value of n, where minus n is the z component of orbital angular momentum. The energy of these states depends only on the total angular momentum, so that state n, spin up is degenerate with state n-1, spin down. All states outside the set and with this same angular momentum have higher energies corresponding to greater kinetic angular momentum partly compensated by the vector potential.

Because of conservation of energy and angular momentum, the matrix element M vanishes if taken between a state in the set and an outside state.

Within the set the only off-diagonal elements that
do not automatically vanish from conservation are
those that connect the two states of a degenerate
pair. These also vanish when the states are prop-
erly chosen, so that there remain only the diagonal
elements which are energy corrections.

Introducing $x - iy = W$ and $r^2 = x^2 + y^2$ and ignoring
normalization factors, the wave functions are:

For spin up: $\qquad k^2 = m^2 + 2nH$

$$\psi = \begin{pmatrix} (k+m) W^m \\ 0 \\ 0 \\ -2inW^{m-1} \end{pmatrix} e^{-\frac{H}{4}r^2 - ikt}$$

For spin down: $\qquad k^2 = m^2 + 2(n+1)H$

$$\psi = \begin{pmatrix} 0 \\ (k+m) W^m \\ iHW^{m+1} \\ 0 \end{pmatrix} e^{-\frac{H}{4}r^2 - ikt}$$

These wave functions are solutions of

$$(i\not{D} - \not{A} - m)\psi = 0$$

given by $\qquad \psi = (i\not{D} - \not{A} + m)\phi$

$$\left[m^2 - \left(i\frac{\partial}{\partial x_\mu} - A_\mu\right)^2 \right]\phi = i\gamma_1\gamma_2 H\phi = \begin{pmatrix} \sigma_z & \\ & \sigma_z \end{pmatrix} H\phi = \pm H\phi$$

It is easy to verify that $1, \gamma_1, \gamma_2, \gamma_4, \gamma_1\gamma_2,$ and $\gamma_1\gamma_2\gamma_4,$
which are the only Dirac matrices appearing in M, all
give zero when taken between an up and a down state.

When the expressions for Ψ and Ψ^+ are inserted into J_1, J_2, and J_3, the resulting integrals, though rather involved, are essentially of Gaussian form. Each breaks into a t, a z, and an xy factor. As an example, J_1 for spin up is

$$J_1 = 2mR \iint [e^{iHu}(k+m)^2 W_1^m \bar{W}_2^m - 4m^2 W_1^{m-1} \bar{W}_2^{m-1}] e^G dx_1 dy_1 dx_2 dy_2$$

where $\quad R = \left(\int_{-\infty}^{\infty} e^{ikt_{21} - i(\alpha + \frac{1}{4u})t_{21}^2} dt_{21} \right) \left(\int_{-\infty}^{\infty} e^{i(\alpha + \frac{1}{4u})z_{21}^2} dz_{21} \right)$

$$G = \frac{iH}{2}(X_1 Y_{21} - X_{21} Y_1) + \frac{iH}{4}(\cot Hu + \frac{4\beta}{H})(X_{21}^2 + Y_{21}^2) - \frac{H}{4}(r_1^2 - r_2^2)$$

The exponential factors are the same for each J and for both spin cases, though the other factors vary. By choosing new variables in a suitable manner, G can be written as a sum of squares, and the integrals evaluated in a straightforward way. After considerable labor and a variety of substitutions, the results are

$$M_{up} = \frac{e^2}{4\pi} \int_0^1 g(\omega) ds \int_0^{\infty} \frac{du}{u} e^{-im^2 su} e^{-2ims z} \frac{(1-S\beta_1)^{m-1}}{(1-S\beta_2)^m} A_1$$

$$M_{down} = \frac{e^2}{4\pi} \int_0^1 g(\omega) ds \int_0^{\infty} \frac{du}{u} e^{-im^2 su} e^{-2i(m+1)s z} \frac{(1-S\beta_1)^m}{(1-S\beta_2)^{m+1}} A_2$$

$$z = Hu \quad S = \frac{4\alpha u}{1 + 4\alpha u}$$

$$\beta_1 = 1 - e^{iz} \frac{\sin z}{z}$$

$$\beta_2 = 1 - e^{-iz} \frac{\sin z}{z}$$

$$A_1 = (k+m)\left(\frac{m}{k} + Se^{-2iZ}\right)\frac{1-S\beta_1}{1-S\beta_2} + (k-m)\left(-\frac{m}{k} + Se^{2iZ}\right)$$

$$-2(k^2-m^2)\frac{S}{kZ}\frac{\cos Z \sin Z}{1-S\beta_2}$$

$$A_2 = (k+m)\left(\frac{m}{k} + Se^{2iZ}\right) + (k-m)\left(-\frac{m}{k} + Se^{-2iZ}\right)\frac{1-S\beta_1}{1-S\beta_2}$$

$$-2(k^2-m^2)\frac{S}{kZ}\frac{\cos Z \sin Z}{1-S\beta_2}$$

In the limiting case $H \to 0$ keeping $k^2 m^2$ constant, A_1 and A_2 both become $2\frac{m^2}{k}(1+S)$ and M becomes

$$M_0 = \frac{e^2}{2\pi}\frac{m^2}{k}\int_0^1 g(\alpha)\,dS \int_0^\infty \frac{du}{u} e^{-im^2 Su}(1+S)$$

$$= \frac{e^2}{2\pi}\frac{m^2}{k}\int_0^\infty g(\alpha)\,d\alpha \int_0^\infty \frac{du}{u} e^{-im^2\frac{4\kappa u^2}{1+4\kappa u}}\left(1 + \frac{4\kappa u}{1+4\kappa u}\right)\frac{4\kappa}{(1+4\kappa u)^2}$$

M_0 represents the energy correction for a particle of unchanged kinetic energy in a vanishing field, and is of the form of a change in the rest mass. Introducing $g(\alpha) = 1 - e^{-iK^2/4\alpha}$, $\frac{1}{4u} = (\beta-1)\alpha$

$$\frac{k}{m}M_0 = \frac{e^2 m}{2\pi}\int_0^\infty \frac{g(\alpha)\,d\alpha}{\alpha}\int_1^\infty e^{-i\frac{m^2}{4\alpha\beta(\beta-1)}}\frac{1+\beta}{\beta^3}\,d\beta$$

$$= \frac{e^2 m}{2\pi}\int_1^\infty \left(\frac{1}{\beta^3} + \frac{1}{\beta^2}\right)\ln\left(1 + \frac{K^2}{m^2}\beta(\beta-1)\right)d\beta$$

For $K \gg m$ this is $\simeq \frac{3}{2\pi}e^2 m \ln\frac{K}{m} + \frac{3e^2 m}{8\pi}$ in agreement with known results.

Once the mass correction has been subtracted, the convergence factor g can be replaced by unity. Introducing $\lambda = \frac{m^2}{H}$ the final result is

$$\Delta E_{up} = \frac{e^2}{4\pi} \int_0^1 ds \int_0^\infty \frac{dz}{z} e^{-i\lambda sz} \left[-\frac{2m^2}{k}(1+s) + e^{-2ims\,z} \frac{(1-s\beta_1)^{m-1}}{(1-s\beta_2)^m} A_1 \right]$$

$$\Delta E_{down} = \frac{e^2}{4\pi} \int_0^1 ds \int_0^\infty \frac{dz}{z} e^{-i\lambda sz} \left[-\frac{2m^2}{k}(1+s) + e^{-2i(m+1)s\,z} \frac{(1-s\beta_1)^m}{(1-s\beta_2)^{m+1}} A_2 \right]$$

V. THE ASYMPTOTIC EXPANSION

The usual technique for asymptotic expansion cannot be directly applied because the coefficient λS in the exponent is not uniformly large. However, if the integrals are rewritten as

$$\Delta E_{up} = \frac{m^2}{k} \frac{e^2}{2\pi} \int_0^1 (1+s)ds \int_0^\infty \frac{dz}{z} (e^{-i\lambda' sz} - e^{-i\lambda sz}) + \frac{e^2}{4\pi} \int_0^1 ds \int_0^\infty \frac{dz}{z} e^{-i\lambda' sz} \left[-\frac{2m^2}{k}(1+s) + \frac{(1-s\beta_1)^{m-1}}{(1-s\beta_2)^m} A_1 \right]$$

$$\Delta E_{down} = \frac{m^2}{k} \frac{e^2}{2\pi} \int_0^1 (1+s)ds \int_0^\infty \frac{dz}{z} (e^{-i\lambda' sz} - e^{-i\lambda sz}) + \frac{e^2}{4\pi} \int_0^1 ds \int_0^\infty \frac{dz}{z} e^{-i\lambda' sz} \left[-\frac{2m^2}{k}(1+s) + \frac{(1-s\beta_1)^m}{(1-s\beta_2)^{m+1}} A_2 \right]$$

where $\lambda' = \frac{k^2}{H}$, the first term is a known form and the bracket in the second term can be conveniently expanded in powers of S:

$$\int_0^1 ds \int_0^\infty \frac{dz}{z} e^{-i\lambda' sz} \left[\sum_r s^r F_r(iz) \right] = \sum_r \frac{1}{\lambda'^{r+1}} \int_0^{\lambda'} ds \int_0^\infty \frac{dz}{z} e^{-isz} F_r(iz) s^r$$

$$= \sum_{r=1}^\infty \frac{1}{\lambda'^{r+1}} \left(C_r + \int^{\lambda'} ds \cdot s^r \sum_{a=0}^\infty \frac{1}{r+a} \frac{1}{s^{r+a}} F_r^{(r+a)}(0) \right)$$

$$= \sum_{r=1}^\infty \left(\frac{1}{r} \frac{F_r^{(r)}}{\lambda'^r} + \frac{C_r + \ln\lambda' F_r^{r+1}/(r+1)}{\lambda'^{r+1}} + \frac{1}{r+2} \frac{F_r^{r+2}}{\lambda'^{r+2}} + \cdots \right)$$

In this developement the asymptotic expansion
of the z integral is made possible by replacing the
definite S integral by an indefinite integral and
introducing the constants C_r which must be separately
evaluated. F_0 is zero and F_r is $O(z^r)$. Furthermore
F_r is of the form

$$F_r(iz) = \sum_{b=0}^{r} \sum_{\nu} \beta_{\nu b} \, e^{-i \alpha_{\nu b} \frac{z}{z}} (iz)^{-b}$$

where the α are integers which are all positive for
the ground state but otherwise include some negative
values. The coefficients β obey sufficient conditions
to keep the integrand finite. By evaluating

$$(-1)^{r+1} \frac{\partial^{r+1}}{\partial S^{r+1}} \int_0^\infty \frac{dz}{z} e^{-iSz} F_r(iz) = i \int_0^\infty dz \, e^{-iSz} (iz)^r F_r(iz)$$

and then integrating $(r+1)$ times from S to ∞ ,

$$\int_0^{\lambda'} dS \cdot S^r \int_0^\infty \frac{dz}{z} e^{-iSz} F_r(iz) = \int_0^{\lambda'} S^r \left(\sum_{\nu} P_{\nu}(S) \ln(S+\alpha_{\nu}) \right) dS$$

$$= \sum_{\nu} \ln(\lambda'+\alpha_{\nu}) Q_{1\nu}(\lambda') + Q_{2\nu}(\lambda')$$

Where the P,Q are polynomials. In doing the final
S integral it is necessary to circle around the
points $S+\alpha_{\nu}=0$. This gives rise to an imaginary
contribution to ΔE that is interpretable in terms
of a decay probability for the state. There is no

such imaginary contribution for the ground state. In the final result the logarithms can be expanded and everything rearranged in inverse powers of λ. This second method of calculation is much more laborious but has the advantage that it fixes the values of the C_r.

Although it is possible in principle to carry out the expansions to any order in H, in practice the labor becomes excessive at about the third order. The calculation to second order is quite simple if both methods of calculation are used, the second method being used only for those terms which are capable of contributing to C_1 in zeroeth order. The result is

$$\Delta E_{up} = \frac{e^2 m}{4\pi}\left[-\frac{1}{\lambda} + \frac{8}{3\lambda^2}\ln\frac{\lambda}{2} - \frac{1}{\lambda^2}\left(\frac{13}{9} + \frac{4}{3}\pi m i\right)\right]$$

$$\Delta E_{down} = \frac{e^2 m}{4\pi}\left[+\frac{1}{\lambda} + \frac{8}{3\lambda^2}\ln\frac{\lambda}{2} - \frac{1}{\lambda^2}\left(\frac{13}{9} + \frac{4}{3}\pi m i + 4\pi i\right)\right]$$

In the case of the ground state it is possible to investigate the convergence of the expansion. In this case (only) the path of z integration can be displaced to the negative imaginary axis so that ΔE can be expressed in the purely real form

$$\frac{2\mathcal{Z}}{e^2 m}\Delta E = \int_0^1 S ds \int_0^\infty \frac{dz}{z} e^{-S\frac{\lambda}{2}z}(e^{-z}-1) + \sum_{r=1}^\infty \int_0^1 S^r ds \int_0^\infty \frac{dz}{z} e^{-S\frac{\lambda}{2}z}(1+Se^{-z})\beta^r$$

$$\beta = 1 - \frac{1-e^{-z}}{z}$$

Convergence of the sum is guaranteed since an upper bound can be found for the sum from N to ∞. Since $1 + Se^{-z} \leq 2$

$$0 \leq \beta \leq \min(1, \frac{z}{2})$$

$$0 \leq \frac{1}{1-S\beta} \leq \min(\frac{1}{1-S}, 1+z)$$

$$\left| 2\int_0^1 S^N ds \int_0^\infty \frac{dz}{z} e^{-\frac{\lambda}{2}Sz} \frac{\beta^N}{1-S\beta}\right| \quad \text{is bounded by} \quad 2\int_0^1 \min(A_S, B_S) ds$$

$$A_S = \int_0^\infty \frac{dz}{z} e^{-\frac{\lambda}{2}Sz} \frac{S^N}{1-S}\left(\frac{z}{2}\right)^N = \frac{(N-1)!}{\lambda^N}\frac{1}{1-S}$$

$$B_S = \int_0^\infty \frac{dz}{z} e^{-\frac{\lambda}{2}Sz}(1+z)\left(\frac{Sz}{2}\right)^N = \frac{(N-1)!}{\lambda^N}\left(\frac{2N}{\lambda S}+1\right)$$

Hence the sum is less than

$$\frac{2(N-1)!}{\lambda^N}\left(2 + \frac{N}{\lambda}\ln\frac{\lambda}{N}\right)$$

Furthermore, examination of the exact form of the first N terms shows that their expansion in powers plus (powers)(ln$H$) converges if $(N+1) \leq \frac{\lambda}{2}$

VI. INTERPRETATION AND DISCUSSION

Since in electromagnetic units $\frac{1}{\lambda} \approx H \cdot 10^{-14}$, these second and higher terms are of little practical interest. They are however of some

theoretical interest. The occurrence of logarithm factors shows that any attempt to arrive at these results by treating the magnetic field as a perturbation would encounter an infrared catastrophe. For a bound electron the binding furnishes a cutoff, but for the free electron it would be necessary to use a cutoff depending on H, which would require care to avoid confusing even the form of the answer. For the bound case the form of the H^2 correction is readily found. Bethe's expression for the Lamb shift, modified for the presence of a vector potential, is

$$\Delta E = \frac{2}{3} \frac{e^2}{\pi} \ln \frac{m}{\langle \omega_n - \omega_s \rangle_{ave}} \int \psi^* \vec{v} \cdot [H, \vec{v}] \psi$$

$$\vec{v} \cdot [H, \vec{v}] = \frac{1}{m^2} \vec{p} \cdot [V, \vec{p}] + \frac{m}{2} \vec{v} [v^2, \vec{v}]$$

The first term gives $\frac{1}{m^2} \nabla^2 V$, the usual Lamb shift. The second is $\frac{H^2}{m^3}$. There is another term $\frac{A \cdot J}{m^3}$ which is dropped because J, the current density creating the field A, presumably vanishes inside the atom. The final answer is

$$\Delta E = \frac{2}{3} \frac{e^2}{\pi} \frac{H^2}{m^3} \ln \frac{m}{\epsilon}$$

The coefficient is the same as for the free electron.

Twice the imaginary part of ΔE is the probability per unit time of radiative decay to a lower state. Since

$$\frac{1}{\omega} \frac{2}{3} e^2 a^2 = \frac{2}{3} \frac{\omega}{H} e^2 \frac{v^2 H^2}{m^2} = \frac{4}{3} e^2 m n \left(\frac{H}{m^2}\right)^2$$

the term proportional to n agrees with the classical acceleration radiation. It also checks with a quantum mechanical calculation using the dipole approximation. The fact that the states extend indefinitely in the z direction can be ignored in using this approximation since the only effect of this extension is to force the electron to take up the recoil momentum. The extra term which appears only for states with spin down is a measure of the probability of decay by turning over the spin.

References

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2. R. P. Feynman, Phys. Rev. 80, 453 (1950)