LAPLACE TRANSFORM IN
COMMUTATIVE BANACH ALGEBRAS

Thesis by

Robert Y. Dean

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1952

ACKNOWLEDGMENT

The author wishes to express his sincerest thanks to Professor H. F. Bohnenblust for his most valuable suggestions and criticisms, for his patience and understanding, and for the generous assistance he has given during the development of this thesis.

ABSTRACT

After a brief review of the general theory of commutative complex Banach algebras in Section I, Section II introduces and discusses some important facts about the generalized Laplace-Stieltjes integral. Section III consists of an investigation of the regions of ordinary and absolute convergence of the Laplace-Stieltjes integral, and is followed by specializations to Dirichlet and power series in Section IV. Section V contains a consideration of the analyticity of functions defined by Laplace-Stieltjes integrals, while Section VI concludes the thesis with some remarks on the existence and distribution of singularities of such functions.

TABLE OF CONTENTS

Section I

INTRODUCTION

During the past decade and a half increasingly more and more mathematical attention has been concentrated on the theory of those mathematical systems which have become commonly known as normed linear rings or Banach algebras. As early as 1934, in a generalization of Fredholm integral equation theory, Michal and Martin [1] laid down the postulates and gave several infinite dimensional examples of what they called "a special linear vector space S . . . having additional properties abstracted from those of a space of linear transformations". Their postulates are identical with those now used for a Banach algebra with unit element. Later, in 1936, Nagumo [2] studied the properties of the group of regular elements and the generalized exponential function in an abstract system which he called a linear metric ring.

The major impetus to the study of this new discipline appears however to have begun with Gelfand in Russia who in the period from 1939 to 1941 published a series of abstracts on the subject culminating with his now famous papers [3] and [4] on Normierte Ringe.

Since that time the theory has flourished under the contributions of a host of authors writing on the algebraic, topological, and function-theoretic aspects of such systems. The bibliography contained herein is by no means complete or exhaustive. A fairly comprehensive list of references may be found in [5] and [6].

The properties of Banach algebras are such that they lend themselves most readily to generalizations of techniques and theories of classical analysis. Among these generalizations, has been the extension of the scope of analytic function theory. Here the development of the theory possesses remarkable similarities to that taken by the classical course. Nevertheless, there arise new and interesting phenomena which in the classical case either do not exist or are trivial. One of these peculiarities occurs in connection with power series, where in general the region of convergence is no longer a sphere, and as a consequence the distribution of singularities of the function which the series represents appears to be of an extremely complex nature.

It was the latter problem -- that of determining the existence and distribution of singularities on the boundary of the region of convergence of a power series -- that led to the present study.

In the case of the complex plane it has long been known that many statements concerning the occurrence and location of singularities on the circle of convergence of a power series are specializations of broader results known for Dirichlet series, or more generally, Laplace-Stieltjes integrals. Such statements are embodied in a class of results usually referred to as gap and density theorems. Motivated by this knowledge, it seemed appropriate to attack the same question for commutative Banach algebras in an analogous manner. Section II of this work therefore begins with the definition of the generalized Laplace-Stieltjes integral and concludes with several results of a general nature concerning such integrals.

Prerequisite to studying the occurrence of singularities on the boundary of the region of convergence of a Laplace-Stieltjes integral, however, is the necessity of having an adequate description of the region itself. Here a marked deviation from the classical situation arises. The familiar half-plane of convergence is replaced by a set considerably more complex in nature. In fact, as yet a complete description of this set is not known. Nevertheless, a partial description can be given, and for several classes of Banach algebras a full treatment is possible. It is to these matters that we devote Section III.

Section IV consists of specializations of the results of Sections II and III to generalized Dirichlet and Power series.

In Section V we consider functions defined by the Laplace-Stieltjes integral and show that they are analytic in a sense to be defined at that time.

The thesis concludes with Section VI, in which some remarks are made concerning the occurrence and location of singularities on the boundary of the region of convergence. A special case is treated and a suggested plan of attack for further research along this line is given.

In the interest of more or less completeness, the remaining paragraphs of this section will consist of the definition, examples, and a brief résumé of some of the well-known and fundamental results on Banach algebras to which it will be necessary to refer in the main part of this thesis.

Although not at all times essential, it will be convenient to restrict our considerations to a special class of Banach algebras — those characterized by the fact that they are commutative, complex, and possess a unit element. It is to this special class that all ensuing comments apply.

DEFINITION 1.1. A set of elements $\mathcal{B}$ (always denoted hereafter by capital letters S, T, ..., X, Y, Z) is said to be a commutative complex Banach algebra with unit if

(a) $\mathcal{B}$ is a Banach Space [7].

(b) $\mathcal{B}$ is a commutative algebra over the field of complex numbers, the elements of which will always be denoted by small case letters a, b, c, ..., x, y, z. In short, this means that a multiplication is defined in $\mathcal{B}$ which satisfies the properties:

(i) $XY = YX$

(ii) $X(YZ) = (XY)Z$

(iii) $(aX)Y = a(XY)$

(iv) $X(aY + bZ) = aXY + bXZ$.

(c) $\mathcal{B}$ has a unit element I satisfying $IX = X$.

(d) $\|XY\| \leq \|X\| \; \|Y\|$, $\|I\| = 1$.

For illustrative purposes we include the following set of examples which have been taken from the literature, and are among those most frequently encountered by the analyst.

EXAMPLES

1. The field of complex numbers with the norm taken as the absolute value.

2. The set of all n by n diagonal matrices of complex numbers (a_{ij}), with the norm taken as $\max_{1 \leq i,j \leq n} |a_{ij}|$.

3. The set of all n by n matrices of complex numbers (a_{ij}) such that $a_{ij} = 0$ for $j < i$, and $a_{i-1\ j-1} = a_{ij}$ for $j \geqslant i$. The norm may be taken as in example 2.

4. The ring of complex-valued functions f(x), continuous on a compact space $\mathcal{S}$, with $\|f\| = \sup_{x \in \mathcal{S}} |f(x)|$.

5. The ring of complex-valued functions f(x) of bounded variation on the interval $a \leq x \leq b$, with multiplication defined point-wise and $\|f\| = \sup_{a \leq x \leq b} |f(x)| + V_{[a,b]} f(x)$.

6. The ring of functions analytic over a bounded domain D in the complex plane and continuous over the closure $\overline{D}$ of D, with $\|f\| = \sup_{\overline{D}} |f(z)|$.

7. The ring of absolutely convergent Fourier series $f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$, with $\|f\| = \sum_{-\infty}^{\infty} |a_n|$.

8. The ring generated by any bounded linear operator T defined over an arbitrary Banach space B.

9. The ring generated by a one-parameter group or semi-group of linear transformations over a Banach space.

10. The ring of complex-valued functions f(x) defined over the interval $a \leq x \leq b$, possessing n continuous derivatives, with $\|f\| = \sum_{m=0}^{n} \sup_{a \leq x \leq b} |f^{(m)}(x)| / m!$.

For the following list of definitions and theorems we refer the reader to [5] and [6] where proofs and references to the original papers are to be found.

DEFINITION 1.2. An element $X \varepsilon \mathcal{B}$ is said to be regular if there is an element X^{-1} called the inverse of X such that $XX^{-1} = X^{-1}X = I$. A non-regular element is called singular. The resolvent set $\rho(X)$ is the set of all complex numbers z for which $(zI - X)$ is regular. The spectrum $\sigma(X)$ of X is the complement in the complex plane of $\rho(X)$, and is a closed non-vacuous point set.

THEOREM 1.1. The set of regular elements in $\mathcal{B}$ form an open (not necessarily connected) set $\mathcal{O}$, and X^{-1} is continuous on $\mathcal{O}$.

DEFINITION 1.3. The set $\mathcal{J} \subset \mathcal{B}$ is called a non-trivial ideal if

(a) $X, Y \varepsilon \mathcal{J}$, $A, B \varepsilon \mathcal{B}$ imply $AX + BY \varepsilon \mathcal{J}$.

(b) $\mathcal{J} \neq \mathcal{B}$.

An ideal is said to be maximal if it is not a proper subset of another (non-trivial) ideal.

THEOREM 1.2.

(a) An ideal contains no regular elements.

(b) The closure of an ideal is an ideal.

(c) A maximal ideal is closed.

(d) Every ideal is contained in a maximal ideal.

(e) An element X is contained in a maximal ideal if and only if it has no inverse.

DEFINITION 1.4. The quotient algebra. Let $\mathcal{I}$ be an ideal in $\mathcal{B}$ and $X - \mathcal{I}$ the set of all elements in $\mathcal{B}$ of the form $X - Y$ with $Y \varepsilon \mathcal{I}$. The classes $X - \mathcal{I}$ form an algebra denoted by $\mathcal{B}/\mathcal{I}$ according to the definitions

(i) $a(X - \mathcal{I}) = aX - \mathcal{I}$.

(ii) $(X - \mathcal{I}) + (Y - \mathcal{I}) = X + Y - \mathcal{I}$

(iii) $(X - \mathcal{I})(Y - \mathcal{I}) = XY - \mathcal{I}$

THEOREM 1.3. If $\mathcal{I}$ is a closed ideal in $\mathcal{B}$ then $\mathcal{B}/\mathcal{I}$ is a commutative Banach algebra under the norm $|||X - \mathcal{I}||| = \inf_{Y \varepsilon \mathcal{I}} \|X - Y\|$.

THEOREM 1.4. If a commutative complex Banach algebra is a field then it is isometrically isomorphic to the field of complex numbers.

THEOREM 1.5. If $\mathcal{B}$ is a commutative complex Banach algebra with unit element and if the norm satisfies the condition $\|XY\| = \|X\| \|Y\|$ for all $X, Y \varepsilon \mathcal{B}$, then $\mathcal{B}$ is isomorphic to the complex field.

THEOREM 1.6. Let $\mathcal{I}$ be a closed ideal in $\mathcal{B}$. Then $\mathcal{B}/\mathcal{I}$ is the complex number system if and only if $\mathcal{I}$ is maximal.

DEFINITION 1.5. Let $\mathfrak{M}$ be the set of maximal ideals in $\mathcal{B}$, then for every $X \varepsilon \mathcal{B}$ and $\mathcal{M} \varepsilon \mathfrak{M}$ there is a uniquely determined complex number z such that

$$X - \mathcal{M} = zI - \mathcal{M}.$$

The function $X(\mathcal{M})$ on $\mathfrak{M}$ is defined by $X(\mathcal{M}) = z$.

THEOREM 1.7. The function $X(\mathcal{M})$ has the properties:

(a) $(X + Y)(\mathfrak{M}) = X(\mathfrak{M}) + Y(\mathfrak{M})$

(b) $(XY)(\mathfrak{M}) = X(\mathfrak{M})Y(\mathfrak{M})$

(c) $(I)(\mathfrak{M}) = 1$

(d) $(aX)(\mathfrak{M}) = aX(\mathfrak{M})$

(e) $|X(\mathfrak{M})| \leq \|X\|$

(f) $X(\mathfrak{M}) = 0$ if and only if $X \in \mathfrak{M}$

(g) If $\mathfrak{M}_1 \neq \mathfrak{M}_2$ there exists an X such that $X(\mathfrak{M}_1) \neq X(\mathfrak{M}_2)$

(h) $X(\mathcal{M}) = \sigma(X)$

(i) If $z \in \sigma(X)$ there exists and $\mathfrak{M} \in \mathcal{M}$ such that $X(\mathfrak{M}) = z$

(j) If for given X and $\mathfrak{M}$, $X(\mathfrak{M}) \neq 0$, then there exists a $Y \in \mathcal{B}$ such that $Y(\mathfrak{M}) = 1/X(\mathfrak{M})$.

<u>THEOREM 1.8.</u> For a fixed maximal ideal $\mathfrak{M} \in \mathcal{M}$ and X ranging over $\mathcal{B}$, the values $X(\mathfrak{M})$ define a linear bounded multiplicative functional $F^*(X;\mathfrak{M})$ on $\mathcal{B}$. Conversely, if $F^*(X) \neq 0$ is such a functional on $\mathcal{B}$ and if $\mathfrak{N}$ is the set of points in $\mathcal{B}$ where $X(\mathfrak{M}) = 0$, then $\mathfrak{N} \in \mathcal{M}$ and $F^*(X) = X(\mathfrak{N})$ for all X.

Henceforth we shall employ the symbol M* to denote either a maximal ideal or its corresponding multiplicative functional. No confusion will arise since the context will always make it clear which interpretation is to be used.

<u>DEFINITION 1.6.</u> The <u>spectral</u> <u>radius</u> of X is the least upper bound of $|z|$ as z varies over the spectrum $\sigma(X)$ of X. It will be denoted by $|\sigma(X)|$.

THEOREM 1.9. $|\sigma(X)| = \sup_{M^* \varepsilon \mathfrak{M}} |M^*(X)| = \lim_{n \to \infty} \|X^n\|^{1/n}$.

DEFINITION 1.7. A generalized nilpotent is an element $X \varepsilon \mathcal{B}$ such that $\lim_{n \to \infty} \|X^n\|^{1/n} = |\sigma(X)| = 0$. The set of all generalized nilpotents is called the radical of $\mathcal{B}$.

THEOREM 1.10. The radical of a commutative Banach algebra is the intersection of all maximal ideals in $\mathcal{B}$.

THEOREM 1.11. Let $M_o^* \varepsilon \mathfrak{M}$, $\epsilon > 0$ and $X_i \varepsilon \mathcal{B}$ $(i = 1,2,\ldots,n)$. The set of elements

$$\mathcal{U}(M_o^*; \epsilon ; X_1, X_2, \ldots, X_n) = \left\{ M^* \varepsilon \mathfrak{M} \Big| \; |M^*(X_i) - M_o^*(X_i)| < \epsilon , \; i = 1,2,\ldots,n \right\}$$

is called a neighborhood of M_o^*. With such a neighborhood system $\mathfrak{M}$ becomes a compact Hausdorff space and the functions $M^*(X)$ are continuous for all $M^* \varepsilon \mathfrak{M}$.

THEOREM 1.12. Let $\mathcal{B}$ be a commutative Banach algebra and $\mathfrak{M}$ the compact Hausdorff space of maximal ideals in $\mathcal{B}$. Let $\mathcal{C}(\mathfrak{M})$ be the Banach algebra of continuous functions on $\mathfrak{M}$. Then the mapping $X \to M^*(X)$ is a homomorphic mapping of $\mathcal{B}$ into $\mathcal{C}(\mathfrak{M})$. It will be an isomorphic mapping if and only if $\mathcal{B}$ has no radical.

Section II

THE LAPLACE-STIELTJES INTEGRAL

We begin this section with a collection of fundamental definitions and results which we shall need later for the study of the Laplace-Stieltjes integral

(1) $$F(S) = \int_0^\infty \exp(-tS)dA(t).$$

DEFINITION 2.1. A function $A(t)$ of the real variable t defined on the closed interval $[a,b]$ to the Banach algebra $\mathcal{B}$ is said to be of strongly bounded variation in that interval if

$$\sup \sum \|A(t_i) - A(t_{i-1})\| < \infty,$$

where all possible partitions of $[a,b]$ are allowed. The supremum will be called the strong total variation [5, page 39].

DEFINITION 2.2. Let $F(t)$ and $A(t)$ be functions defined over the closed interval $[a,b]$ to the Banach algebra $\mathcal{B}$. Let Δ be a subdivision of the interval $[a,b]$ by the points

$$a = t_0 < t_1 < \dots < t_n = b,$$

and let $\delta = \max (t_{i+1} - t_i)$. If the limit (in the normed topology of $\mathcal{B}$)

$$\lim_{\delta \to 0} \sum_{i=0}^{n-1} F(u_i) \left[A(t_{i+1}) - A(t_i) \right]$$

where

$$t_i \leq u_i \leq t_{i+1} \qquad (i = 0,1,\dots,n-1)$$

exists independently of the manner of subdivision and of the choice of the numbers u_i, then the limit is called the Riemann-Stieltjes integral of F(t) with respect to A(t) from a to b and will be denoted by

$$(2) \qquad \int_a^b F(t)dA(t).$$

THEOREM 2.1. If F(t) is strongly continuous (i.e., continuous in the normed topology of $\mathcal{B}$) and A(t) is of strongly bounded variation, then the Riemann-Stieltjes integral of F(t) with respect to A(t) from a to b exists. Further, if F* is an arbitrary bounded linear functional belonging to the conjugate space $\mathcal{B}^*$, then

$$(3) \qquad F^*\left[\int_a^b F(t)dA(t)\right] = \int_a^b F^*\left[F(t)dA(t)\right] .$$

The proof follows standard patterns, examples of which can be found in [5, page 51] and [8, theorem 11] .

DEFINITION 2.3. If A(t) is of strongly bounded variation in $a \leq t \leq b$, it is said to be normalized there if

$$A(a) = 0$$

$$A(t) = \frac{A(t+) + A(t-)}{2} \qquad (a < t < b).$$

THEOREM 2.2. A function of strongly bounded variation has right and left hand limits everywhere and is strongly continuous except for a countable set of discontinuities of the first kind, and hence may be normalized. See [5, page 203] .

As in the classical case [9, page 14] if A(t) is not normalized, we may do so by defining the function

$$(4)\qquad \begin{cases} B(a) = 0 \\ B(t) = \dfrac{A(t+) + A(t-)}{2} - A(a) & (a < t < b) \\ B(b) = A(b) - A(a). \end{cases}$$

Since the replacement of A(t) in (2) by the function B(t) defined in (4) leaves the value of the integral unchanged, we shall assume henceforth that our functions of strongly bounded variation are always normalized.

<u>DEFINITION 2.4.</u> Let A(t) be a function on $[0,\infty)$ to $\mathcal{B}$, and let A(t) be of strongly bounded variation over every finite interval $[0,b]$. Then since exp(-tS) is strongly continuous the integral

$$(5)\qquad F(S;b) = \int_0^b \exp(-tS)\,dA(t)$$

exists for finite positive values of b. If, for a particular $S \in \mathcal{B}$, $\lim_{b\to\infty} F(S;b)$ exists in the normed topology of $\mathcal{B}$, we will denote the limit by

$$(1)\qquad F(S) = \int_0^\infty \exp(-tS)\,dA(t),$$

and will call F(S) the <u>Laplace-Stieltjes transform</u> of A(t). Furthermore, the set of $S \in \mathcal{B}$ for which (1) converges will be denoted by $\mathcal{C}$, and will be called the <u>set of convergence</u> of (1). The open interior of the set $\mathcal{C}$ will be denoted by the symbol $\mathcal{O}_c$, and will be called the <u>region of convergence</u> of (1).

In the classical case $\mathcal{C} < \overline{\mathcal{O}}_c$. This statement is in general no longer valid for Banach algebras. Example 3.2 discussed below can easily be specialized to give an example where $\mathcal{O}_c$ is void, and $\mathcal{C}$ is not void.

In preparation for a discussion of the region $\mathcal{O}_c$ in Section II we now include some results of a general nature concerning the integral (1).

For a given integral it will soon become apparent that it is the spectral properties of the element S rather than its norm which plays the dominant role in determining whether or not it belongs to the region of convergence. For this reason it will frequently be necessary to make use of

<u>THEOREM 2.3.</u> Let $\mathcal{B}$ be a Banach algebra with unit element and let S be an arbitrary element belonging to $\mathcal{B}$, then the expression

$$(6) \qquad R1(S) \equiv -\lim_{t \to \infty} \frac{\ln \|\exp(-tS)\|}{t}$$

exists and is equal to

$$(7) \qquad \min_{z \in \sigma(S)} R1(z) = \min_{M^* \in \mathfrak{M}} R1 M^*(S) .$$

<u>Proof:</u> Let S be a fixed element belonging to $\mathcal{B}$ and consider the real-valued function of t defined by

$$(8) \qquad h(t) = \ln \|\exp(-tS)\| \qquad (0 \leq t < \infty).$$

Since $\|\exp[-(t_1 + t_2)S]\| \leq \|\exp(-t_1 S)\| \, \|\exp(-t_2 S)\|$, we have

$$h(t_1 + t_2) \leq h(t_1) + h(t_2). \tag{9}$$

Equation (9) together with the fact that for arbitrary t $\|\exp(-tS)\| > 0$ shows that h(t) is a finite-subadditive function of t [5, pages 135, 136] over any interval $0 \leq t < \infty$ and hence

$$\lim_{t \to \infty} f(t)/t = \lim_{t \to \infty} \frac{\ln \|\exp(-tS)\|}{t} \quad \text{exists.} \tag{10}$$

Letting $t \to \infty$ through integral values and comparing with theorem 1.9 we have

$$\lim_{n \to \infty} \frac{\ln \|\exp(-nS)\|}{n} = \lim_{n \to \infty} \frac{\ln \|[\exp(-S)]^n\|}{n} = \ln \left\{ \max_{M^* \varepsilon \mathfrak{M}} |M^*[\exp(-S)]| \right\}$$

$$= \ln \left\{ \max_{M^* \varepsilon \mathfrak{M}} | e^{-M^*(S)}| \right\} = \ln \left\{ \max_{M^* \varepsilon \mathfrak{M}} e^{-Rl M^*(S)} \right\}$$

$$= \ln \left\{ e^{-\min_{M^* \varepsilon \mathfrak{M}} Rl M^*(S)} \right\} = -\min_{M^* \varepsilon \mathfrak{M}} Rl M^*(S) ,$$

from which the conclusion of the theorem is evident.

The importance of theorem 2.3 lies in the fact that for large values of t, $\|\exp(-tS)\|$ is governed by Rl(S). In this respect we have the following

<u>COROLLARY:</u> Let Rl(S) = a, then for arbitrary b > a there exists a t_o such that

$$\|\exp(-tS)\| < e^{-bt} \quad \text{for all } t > t_o. \tag{11}$$

<u>THEOREM 2.4.</u> If the integral

$$\int_0^\infty \exp(-tS)dA(t) \tag{1}$$

converges for $S = S_o$ (i.e., $S_o \varepsilon \mathcal{C}$) then it converges for any S such that

(12) $$\mathrm{RlM}^*(S) > \mathrm{RlM}^*(S_o) \quad \text{for all } M^* \varepsilon \mathfrak{M} .$$

Proof:

$$\int_p^q \exp(-tS)dA(t) = \int_p^q \exp\left[-t(S-S_o+S_o)\right] dA(t) = \int_p^q \exp\left[-t(S-S_o)\right] dB(t),$$

where

$$B(t) = \int_0^t \exp(-uS_o)dA(u).$$

Note that $B(0) = 0$, $B(\infty)$ exists and since $B(t)$ is strongly continuous there exists a constant $m > 0$ such that $\| B(t)\| < m$ for all t. An integration by parts yields

$$\int_p^q \exp(-tS)dA(t) = B(q)\exp\left[-q(S-S_o)\right] - B(p)\exp\left[-p(S-S_o)\right]$$
$$+ (S-S_o) \int_p^q \exp(-tS)B(t)dt.$$

By hypothesis $\mathrm{RlM}^*(S-S_o) > 0$ for all $M^* \varepsilon \mathfrak{M}$, hence by the Corollary to Theorem 2.3 there exists an $r > 0$ and a t_o such that

$$\| \exp\left[-t(S-S_o)\right]\| < e^{-rt} \qquad (t > t_o).$$

Then

$$\left\| \int_p^q \exp(-tS)dA(t)\right\| \leqslant 2me^{-rp} + \|S-S_o\| \, m \int_p^q e^{-rt}dt \qquad (t_o < p < q),$$

and thus given arbitrary $\epsilon > 0$, a simple computation shows that if

$$q > p > \max \left\{ t_o, \frac{1}{r} \ln \frac{\epsilon r}{2m \| S-S_o \|}, \frac{1}{r} \ln \frac{\epsilon}{4m} \right\}$$

then

$$\left\| \int_p^q \exp(-tS)dA(t) \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{and (1) converges.}$$

COROLLARY: If $S_o \varepsilon \mathcal{O}_c$ and S_1 is any element such that

(13) $\quad RlM^*(S_1) \geqslant RlM^*(S_o)$ for every $M^* \varepsilon \mathfrak{M}$, then $S_1 \varepsilon \mathcal{O}_c$.

Proof: This follows from the fact that the spectrum of an element is a continuous function of the norm. (See Theorem 1.7(e).) $S_o \varepsilon \mathcal{O}_c$ implies there exists an $r > 0$ such that an open sphere of radius r about S_o also belongs to $\mathcal{O}_c$. In particular the element $S_2 = S_o - 2r/3\ I \ \varepsilon \ \mathcal{O}_c$, and

(14) $\quad RlM^*(S_2) = RlM^*(S_o) - 2r/3 \quad$ for every $M^* \varepsilon \mathfrak{M}$.

Now let S be any element contained in an open sphere of radius $r/3$ about S_1. Then for all $M^* \varepsilon \mathfrak{M}$

$$RlM^*(S) > RlM^*(S_1) - r/3 \geqslant RlM^*(S_o) - r/3 = RlM^*(S_2) + r/3,$$

and therefore

(15) $\quad RlM^*(S) > RlM^*(S_2)$ for all $M^* \varepsilon \mathfrak{M}$.

(15) implies an entire sphere about S_1 belongs to the region $\mathcal{C}$, and hence $S_1 \varepsilon \mathcal{O}_c$.

THEOREM 2.5. Let $\mathcal{O}_c$ be the region of convergence of the integral

(1) $$\int_0^\infty \exp(-tS)dA(t).$$

Then if $S_o \varepsilon \mathcal{O}_c$ and R is an arbitrary element belonging to the radical of $\mathcal{B}$, $(S_o + R) \varepsilon \mathcal{O}_c$.

Proof: This follows immediately from the corollary to theorem 2.4 and the fact that for all $M^* \varepsilon \mathfrak{M}$

$$M^*(S_o + R) = M^*(S_o).$$

In close analogy to a standard result of classical Laplace transform theory [10] we have

THEOREM 2.6. If the integral

$$(1) \qquad \int_0^\infty \exp(-tS)dA(t)$$

converges for $S = S_o$, then it converges uniformly with respect to z for all S of the form

$$S = S_o + zI, \quad \text{where} \quad |\arg(z)| \leq \theta < \pi/2.$$

Proof: Define

$$B(t) = \int_t^\infty \exp(-uS_o)dA(u).$$

By hypothesis, for any $\epsilon > 0$ there exists a $t_o = t_o(\epsilon)$ such that

$$\|B(t)\| < \epsilon \quad \text{for } t > t_o.$$

Now

$$\int_{t_1}^{t_2} \exp\left[-t(S_o + zI)\right] dA(t) = \int_{t_1}^{t_2} e^{-tz} \exp(-tS_o)dA(t) = -\int_{t_1}^{t_2} e^{-tz} dB(t),$$

and an integration by parts gives

$$\int_{t_1}^{t_2} \exp\left[-t(S_o + zI)\right] dA(t) = -e^{-tz}B(t)\Big|_{t_1}^{t_2} - z \int_{t_1}^{t_2} e^{-tz}B(t)dt.$$

Consequently

$$\left\| \int_{t_1}^{t_2} \exp\left[-t(S_o + zI)\right] dA(t)\right\| \leq e^{-t_2 Rl(z)} \|B(t_2)\| + e^{-t_1 Rl(z)} \|B(t_1)\| + |z| \int_{t_1}^{t_2} e^{-tRl(z)} \|B(t)\| \, dt.$$

Then if $t_2 > t_1 > t_o$,

$$\left\| \int_{t_1}^{t_2} \exp\left[-t(S_o + zI)\right] dA(t)\right\| < \epsilon e^{-t_2 Rl(z)} + \epsilon e^{-t_1 Rl(z)} + \epsilon |z| \int_{t_1}^{\infty} e^{-tRl(z)} dt$$

$$= \epsilon \left\{ e^{-t_2 Rl(z)} + e^{-t_1 Rl(z)} + \frac{|z|}{Rl(z)} e^{-t_1 Rl(z)} \right\}.$$

But for $|\arg(z)| \leq \theta < \pi/2$, $Rl(z) > 0$, and hence $e^{-t_1 Rl(z)} \leq 1$, $e^{-t_2 Rl(z)} \leq 1$, and $\dfrac{|z|}{Rl(z)} \leq \dfrac{1}{\cos(\theta)}$, from which we get

$$\left\| \int_{t_1}^{t_2} \exp\left[-t(S_o + zI)\right] dA(t)\right\| < \epsilon \left(2 + \frac{1}{\cos(\theta)}\right).$$ The conclusion of the theorem follows immediately from this inequality.

Section III

REGIONS OF CONVERGENCE OF THE LAPLACE-STIELTJES INTEGRAL

It is well known in the classical case where $\mathcal{B}$ is the complex plane, that if the function A(t) is prescribed in advance for the integral (1), then it is possible to give specific criteria [9] with which to decide whether or not a given S lies in the region of convergence $\mathcal{O}_c$. Furthermore, the criteria provide a description of $\mathcal{O}_c$ (if it exists) as an open half-plane. Since it will be necessary to refer to these criteria we include

THEOREM 3.1. For the integral

$$\int_0^\infty e^{-ts}da(t) \tag{16}$$

there exists two real numbers c and a such that the integral is convergent for $Rl(s) > c$, but not for any s with $Rl(s) < c$, and it is absolutely convergent for $Rl(s) > a$, but not for any s with $Rl(s) < a$. We have

$$-\infty \leq c \leq a \leq +\infty,$$

and

$$c = \overline{\lim_{t \to \infty}} \frac{\ln |a(t) - a(\infty)|}{t}, \tag{17}$$

$$a = \overline{\lim_{t \to \infty}} \frac{\ln |v(t) - v(\infty)|}{t}, \tag{18}$$

where $a(\infty) = \lim_{t \to \infty} a(t)$ or 0 according as the limit exists or not, and $v(\infty)$ is defined similarly for the function v(t) which denotes the total variation of a(t) in the interval $[0,t]$.

Attempts to find analogous criteria for an arbitrary Banach algebra have not as yet been completely successful. Several special cases can be treated in full however, and for certain classes of Banach algebras some interesting results concerning the relation between the general shape of the region $\mathcal{O}_c$ and functions over the space $\mathfrak{M}$ of maximal ideals can be given.

As a preliminary to discussing the region of convergence of (1) we make the following definition.

DEFINITION 3.1. Let $F^* \neq 0$ be any bounded linear functional belonging to $\mathcal{B}^*$, the conjugate space of $\mathcal{B}$, and let c be any real number. The set

(19) $$\mathcal{H}(F^*;c) = \left\{ S \in \mathcal{B} \;\middle|\; \mathrm{Rl}F^*(S) > c \right\}$$

will be called the half-space determined by F^* and c. In particular if the functional is multiplicative, i.e., one of those functionals associated with some maximal ideal $M^* \varepsilon \mathfrak{M}$, then $\mathcal{H}(M^*;c)$ will be called a distinguished half-space.

Because of the continuity and linearity of F^*, $\mathcal{H}(F^*;c)$ is clearly an open convex set in $\mathcal{B}$. In fact it is easily verified that the set

(20) $$\mathcal{P}(F^*;c) = \left\{ S \in \mathcal{B} \;\middle|\; \mathrm{Rl}F^*(S) = c \right\}$$

defines a "hyperplane" which divides $\mathcal{B}$ into two disjoint convex sets.

Judging from the situation in the complex plane it might be supposed at this point that the region of convergence of (1) would consist of a distinguished half-space. That this is not the case is adequately demonstrated by a simple example.

EXAMPLE 3.1. Let $\mathcal{B}$ be the set of all complex diagonal matrices of order two. One finds that there are but two maximal ideals in $\mathcal{B}$. Letting

$$S = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}, \qquad A(t) = \begin{pmatrix} u(t) & 0 \\ 0 & v(t) \end{pmatrix},$$

where u(t) and v(t) are complex-valued functions of bounded variation, then $M_1^*(S) = z$, $M_2^*(S) = w$, and

$$(21) \qquad \int_0^\infty \exp(-tS)dA(t) = \begin{pmatrix} \int_0^\infty e^{-zt}du(t) & 0 \\ 0 & \int_0^\infty e^{-wt}dv(t) \end{pmatrix}.$$

It is apparent that if (21) is to converge it is necessary and sufficient that

$$(22) \qquad \begin{aligned} M_1^*\left[\int_0^\infty \exp(-tS)dA(t)\right] &= \int_0^\infty e^{-zt}du(t) \quad \text{and} \\ M_2^*\left[\int_0^\infty \exp(-tS)dA(t)\right] &= \int_0^\infty e^{-wt}dv(t) \end{aligned}$$

converge in the ordinary sense. This implies that

$$(23) \qquad \begin{cases} Rl(z) > c\,[u(t)] & \text{and} \\ Rl(w) > c[v(t)]. \end{cases}$$

Stating conditions (23) in the style of definition 3.1 we have

$$(24) \qquad RlM_i^*(S) > c\left[M_i^*(A(t))\right] \qquad (i = 1,2).$$

This shows that the region of convergence consists of the intersection of two distinguished half-spaces.

Motivated by this example we make the following definition.

DEFINITION 3.2. To every bounded, linear, multiplicative functional $M^* \varepsilon \mathfrak{M}$ let there correspond a real number $c(M^*)$. The set

$$(25) \qquad \mathcal{Q}_{c(M^*)} = \bigcap_{M^* \varepsilon \mathfrak{M}} \mathcal{H}\left[M^*; c(M^*)\right] = \left\{ S \,\varepsilon\, \mathcal{B} \;\middle|\; \mathrm{Rl}M^*(S) > c(M^*), \text{ all } M^* \varepsilon \mathfrak{M} \right\}$$

will be called a cylindrical quadrant. Furthermore, if the numbers $c(M^*)$ are all equal we shall say the cylindrical quadrant is diagonal.

Remark: It is to be noted that the numbers $c(M^*)$ in the above definition define a function over the space $\mathfrak{M}$ of maximal ideals, and consequently the decision as to whether a given S_0 belongs to the cylindrical quadrant or not depends upon the ordering of the two functions $\mathrm{Rl}M^*(S_0)$ and $c(M^*)$. This does not mean however, that at all times that the determining function $c(M^*)$ for a given quadrant is unique. Examples of Banach algebras can be given for which the function $c(M^*)$ can be altered by decreasing its value at one point to form a new function $d(M^*) \neq c(M^*)$, and yet $\mathcal{Q}_{c(M^*)} = \mathcal{Q}_{d(M^*)}$. This follows from the fact that for a given S the function $\mathrm{Rl}M^*(S)$ is continuous over $\mathfrak{M}$.

Several important properties of cylindrical quadrants are given by

THEOREM 3.2. Let $\mathcal{Q}$ be a cylindrical quadrant contained in $\mathcal{B}$, then

(i) $\mathcal{Q}$ is a convex set.

(ii) If $S_0 \epsilon \mathcal{Q}$ and S is any element belonging to $\mathcal{B}$ such that for every $M^* \epsilon \mathcal{M}$ $\mathrm{Rl}M^*(S) \geqslant \mathrm{Rl}M^*(S_0)$, then $S \epsilon \mathcal{Q}$.

(iii) If $S_0 \epsilon \mathcal{Q}$ and R is an arbitrary element belonging to the radical of $\mathcal{B}$, then $S_0 + R \epsilon \mathcal{Q}$.

Proof. (i) is obvious since $\mathcal{Q}$ is the intersection of convex distinguished half-spaces. (ii). Let $\mathcal{Q} = \left\{ S \middle| \mathrm{Rl}M^*(S) > c(M^*), \text{ all } M^* \epsilon \mathcal{M} \right\}$, $S_0 \epsilon \mathcal{Q}$, and S be an element satisfying (ii) of the theorem. Then for all $M^* \epsilon \mathcal{M}$ $\mathrm{Rl}M^*(S) \geqslant \mathrm{Rl}M^*(S_0) > c(M^*)$, and hence $S \epsilon \mathcal{Q}$. (iii) follows from the fact that for all $M^* \epsilon \mathcal{M}$ and R belonging to the radical $\mathrm{Rl}M^*(S_0 + R) = \mathrm{Rl}M^*(S_0) > c(M^*)$.

Example 3.1 is somewhat misleading in that it may have given the impression that (1) will converge if for each $M^* \epsilon \mathcal{M}$ the integral

$$(26) \qquad M^* \left[\int_0^\infty \exp(-tS)dA(t) \right] = \int_0^\infty e^{-tM^*(S)} dM^*[A(t)]$$

converges in the ordinary sense. At the risk of belaboring a point the following example is cited to illustrate that this is not the case.

<u>EXAMPLE 3.2.</u> Let $\mathcal{B}$ be the set of two by two complex matrices of the form

$$S = \begin{pmatrix} z & w \\ 0 & z \end{pmatrix}.$$

Here there is but one maximal ideal and $M^*(S) = z$. Letting

$$A(t) = \begin{pmatrix} u(t) & v(t) \\ 0 & u(t) \end{pmatrix}$$

we have

$$(27) \qquad \int_0^\infty \exp(-tS)dA(t) = \begin{pmatrix} \int_0^\infty e^{-tz}du(t) & \int_0^\infty e^{-tz}dv(t) + w\int_0^\infty (-t)e^{-tz}du(t) \\ 0 & \int_0^\infty e^{-tz}du(t) \end{pmatrix} .$$

Here

$$M^* \left[\int_0^\infty \exp(-tS)dA(t) \right] = \int_0^\infty e^{-tz}du(t) \quad ,$$

and hence for convergence it is necessary that

$$Rl(z) = RlM^*(S) > c\left[u(t)\right] = c\left[M^*(A(t))\right] .$$

A glance at (27) shows however that it is also necessary to have

$$(28) \qquad Rl(z) = RlM^*(S) > c\left[v(t)\right] ,$$

a condition that is apparently not detectable by the use of M^*. The region of convergence can now be described as

$$(29) \qquad \mathcal{O}_c = \left\{ S \in \mathcal{B} \;\middle|\; RlM^*(S) > \max\left[c(u(t)), \quad c(v(t))\right] \right.$$

which is plainly a distinguished half-space.

By choosing the functions u(t) and v(t) in (27) properly it is easily seen that it is possible for

$$M^* \left[\int_0^\infty \exp(-tS)dA(t) \right]$$

to converge for all $S \in \mathcal{B}$ and yet have (27) convergent for no $S \in \mathcal{B}$.

Despite the fact that the second condition (28) of the last example was not predicted by use of the multiplicative functional, it is significant that $\mathcal{O}_c$ is nevertheless a cylindrical quadrant. There is considerable evidence to support the view that this is the general situation. Although it appears impossible to give a specific formula that will define the region $\mathcal{O}_c$ for an arbitrary Banach algebra, it is believed that it is always a cylindrical quadrant, and as such is defined by an upper semi-continuous function over the space of maximal ideals.

We state the above opinion in the form of a

CONJECTURE. Denote by $\mathcal{C}$ the set of $S \epsilon \mathcal{B}$ for which the integral (1) converges, and by $\mathcal{O}_c$ the interior of $\mathcal{C}$. Then $\mathcal{O}_c$ is a cylindrical quadrant in the sense of definition 3.2.

The following paragraphs will be devoted to the discussion of a special case of the integral (1) for which it is possible to give a precise description of the region $\mathcal{O}_c$. This is followed by examples of several classes of Banach algebras for which the conjecture can be verified. In any case however, it follows from the corollary to theorem 2.4 and theorem 2.5, that $\mathcal{O}_c$ possesses properties (ii) and (iii) given in theorem 3.2 for a cylindrical quadrant.

Before proceeding to the special cases we state

THEOREM 3.3. If for any particular Banach algebra the region of convergence $\mathcal{O}_c$ is a cylindrical quadrant, then it can be described as

$$(30) \qquad \mathcal{Q}_{c(M^*)} = \left\{ S \,\epsilon\, \mathcal{B} \;\middle|\; RlM^*(S) > c(M^*) = \inf_{S \,\epsilon\, \mathcal{O}_c} RlM^*(S), \text{ all } M^* \epsilon \mathfrak{M} \right\} .$$

Proof: Let

$$\mathcal{O}_c = \mathcal{Q}_{d(M^*)} = \left\{ S \in \mathcal{B} \mid \mathrm{Rl}M^*(S) > d(M^*), \text{ all } M^* \in \mathfrak{M} \right\}$$

and define

$$\mathcal{Q}_{c(M^*)} = \left\{ S \in \mathcal{B} \mid \mathrm{Rl}M^*(S) > c(M^*) = \inf_{S \in \mathcal{O}_c} \mathrm{Rl}M^*(S), \text{ all } M^* \in \mathfrak{M} \right\}.$$

We show that $\mathcal{Q}_{c(M^*)} = \mathcal{Q}_{d(M^*)}$.

(i) Let $S_o \in \mathcal{Q}_{c(M^*)}$, then

$\mathrm{Rl}M^*(S_o) > c(M^*) \geqslant d(M^*)$, hence $S_o \in \mathcal{O}_c = \mathcal{Q}_{d(M^*)}$ and $\mathcal{Q}_{c(M^*)} \subset \mathcal{Q}_{d(M^*)}$.

(ii) Let $S_o \in \mathcal{O}_c = \mathcal{Q}_{d(M^*)}$. Since $\mathcal{O}_c$ is open there exists a $\delta > 0$ such that $S_o + S \in \mathcal{O}_c$ for all S for which $\|S\| < \delta$. In particular, if $|z| < \delta$, then $S_o + zI \in \mathcal{O}_c$. Choose z such that $-\delta < \mathrm{Rl}(z) < 0$, then for all $M^* \in \mathfrak{M}$

(31) $\mathrm{Rl}M^*(S_o) > \mathrm{Rl}M^*(S_o + zI) = \mathrm{Rl}M^*(S_o) + \mathrm{Rl}(z) > d(M^*)$. But $S_o + zI \in \mathcal{O}_c$ and hence

(32) $\mathrm{Rl}M^*(S_o + zI) \geqslant \inf \mathrm{Rl}M^*(S) = c(M^*)$.

The inequalities (31) and (32) show that for every $M^* \in \mathfrak{M}$

$\mathrm{Rl}M^*(S_o) > c(M^*)$, hence $S_o \in \mathcal{Q}_{c(M^*)}$ and $\mathcal{Q}_{d(M^*)} \subset \mathcal{Q}_{c(M^*)}$.

We now discuss three special cases when the region of convergence $\mathcal{O}_c$ can be identified as a cylindrical quadrant.

Case 1: For an arbitrary commutative Banach algebra with unit element if the function A(t) is a multiple of the identity I,

(33) $A(t) = a(t)I$,

where a(t) is a complex-valued function of bounded variation, then the integral (1) reduces to the form

$$(34) \qquad \int_0^\infty \exp(-tS)\,da(t).$$

Under these conditions the region of convergence of (34) is a diagonal cylindrical quadrant, and is described in

THEOREM 3.4. For the integral

$$(34) \qquad \int_0^\infty \exp(-tS)\,da(t)$$

there exists a real number $-\infty \leq c \leq \infty$ given by the expression

$$(17) \qquad c = \overline{\lim_{t \to \infty}} \frac{\ln|a(t) - a(\infty)|}{t},$$

where $a(\infty) = \lim_{t \to \infty} a(t)$ or 0 according as the limit exists or not, such that (34) is convergent for $\mathrm{Rl}(S) > c$, and divergent for $\mathrm{Rl}(S) < c$. This describes σ_c as the diagonal cylindrical quadrant

$$\sigma_c = \bigcap_{M^* \varepsilon \mathfrak{M}} \mathcal{H}(M^*;c) = \left\{ S \varepsilon \mathcal{B} \;\middle|\; \mathrm{Rl}M^*(S) > c, \text{ all } M^* \varepsilon \mathfrak{M} \right\}.$$

Proof: Bearing in mind the convention established for the meaning of the term $a(\infty)$, let

$$(35) \qquad \overline{\lim_{t \to \infty}} \frac{\ln|a(t) - a(\infty)|}{t} = c.$$

Equation (35) implies that for arbitrary $\delta > 0$, there exists a $t_0(\delta)$ such that

$$(36) \qquad |a(t) - a(\infty)| < e^{(c+\delta)t} \qquad \text{for } t > t_0.$$

Secondly, if $\mathrm{Rl}(S) = d > b > c$, then by the corollary to theorem 2.3

there exists a t_1 such that

(37) $$\| \exp(-tS) \| < e^{-bt} \quad \text{for } t > t_1.$$

Now

$$\int_p^q \exp(-tS)da(t) = \int_p^q \exp(-tS)d\left[a(t) - a(\infty)\right] ,$$

hence on integrating by parts and employing the bounds (36) and (37), we have for arbitrary $\epsilon > 0$

$$\left\| \int_p^q \exp(-tS)da(t) \right\| < \epsilon \text{ , provided}$$

$$q > p > \max\left\{ t_o, t_1, \frac{1}{(c+\delta-b)} \ln\left(\frac{\epsilon}{4}\right), \frac{1}{(c+\delta-b)} \ln\left[\frac{\epsilon(c+\delta-b)}{\|S\|}\right]\right\}.$$

This shows the convergence of (34) for $Rl(S) > c$.

On the other hand, assume that the integral (34) converges for some S_o such that $Rl(S_o) = h < c$. By theorem 1.7(i) there exists at least one $M_o^* \varepsilon \mathfrak{M}$ such that

$$RlM_o^*(S_o) = h < c.$$

By theorem 2.1 the strong convergence of (34) implies weak convergence, hence for every bounded linear functional $F^* \varepsilon \mathfrak{B}^*$ the integral

(38) $$\int_0^\infty F^*\left[\exp(-tS_o)\right] da(t)$$

is convergent. In particular for M_o^*

$$\int_0^\infty M_o^*\left[\exp(-tS_o)\right] da(t) = \int_0^\infty e^{-tM_o^*(S_o)} da(t)$$

converges, which by theorem 3.1 is impossible since $RlM_o^*(S_o) = h < c$.

This completes the proof of the theorem.

Note that in theorem 3.4, as in the classical case no information is obtained about the convergence of the integral for those S for which Rl(S) = c.

In case 1 it was a specialization of the form of the integral for which the conjecture was verified. We now consider two cases in which it is the Banach algebra itself which is specialized.

Case 2: If the Banach algebra is finite dimensional it is possible to verify the conjecture. The proof of this fact depends upon the following lemma.

LEMMA 3.1. If $\mathcal{B}$ is finite dimensional then there exist X_j satisfying the conditions

(39)
$$\text{(i)}\quad M_i^*(X_j) = \delta_{ij}$$
$$\text{(ii)}\quad X_i X_j = 0 .$$

Proof: There are only a finite number of maximal ideals M_i^*, and since they are linearly independent there exist U_j such that

(40)
$$M_i^*(U_j) = \delta_{ij} .$$

Clearly the elements $U_i U_j$ $(i \neq j)$ belong to the radical because $M^*(U_i U_j) = 0$ $(i \neq j)$ for all M^*. But $U_i U_j$ $(i \neq j)$ in the radical implies there exists an integer n_{ij} such that

$$(U_i U_j)^{n_{ij}} = 0 \qquad (i \neq j).$$

Let $n_o = \max\limits_{i,j}(n_{ij})$ $(i \neq j)$, then

(41) $$(U_i U_j)^{n_o} = 0 \qquad (i \neq j).$$

Setting

(42) $$X_j = U_j^{n_o}$$

we have

$$\text{(i)} \quad M_i^*(X_j) = M_i^*(U_j^{n_o}) = \left[M_i^*(U_j)\right]^{n_o} = (\delta_{ij})^{n_o} = \delta_{ij},$$

$$\text{(ii)} \quad X_i X_j = U_i^{n_o} U_j^{n_o} = (U_i U_j)^{n_o} = 0,$$

as was to be shown.

THEOREM 3.5. Let $\mathcal{B}$ be finite dimensional. Then for the integral

(1) the region $\mathcal{O}_c$ is the cylindrical quadrant

(43) $$Q_{c(M^*)} = \left\{ S \in \mathcal{B} \;\middle|\; \mathrm{Rl}M^*(S) > c(M^*) = \inf_{S \in \mathcal{O}_c} \mathrm{Rl}M^*(S), \text{ all } M^* \in \mathfrak{M} \right\}.$$

Proof: Let $S_o \in Q_{c(M^*)}$, then there exists a finite number of elements $S_1, S_2, \ldots, S_n$ such that

(44) $$\text{(i)} \quad S_i \in \mathcal{O}_c \qquad i = 1, 2, \ldots, n$$
$$\text{(ii)} \quad \mathrm{Rl}M^*(S_o) > \min_{1 \le i \le n} \mathrm{Rl}M^*(S_i) \text{ all } M^* \in \mathfrak{M}.$$

Since there are but a finite number of maximal ideal M_k, $(k = 1, 2, \ldots, p)$ the S_i may be relabelled in such a manner that we may write

(45) $$\mathrm{Rl}M_k^*(S_o) > \mathrm{Rl}M_k^*(S_k), \qquad k = 1, 2, \ldots, p.$$

Now by lemma 3.1 determine X_j $(j = 1, 2, \ldots, p)$ such that

(46a) $$M_k^*(X_j) = \delta_{kj}$$

(46b) $$X_i X_j = 0 \quad (i \neq j).$$

Next choose an a such that for all $i, j = 1, 2, \ldots, p$

(47) $$\mathrm{RlM}_i(S_j) + a < 0,$$

and construct the elements

(48) $$Z_k = X_k(S_k + aI) - aI \qquad k = 1,2,\ldots,p.$$

A simple computation and the use of (47) shows

(49) $$\mathrm{RlM}_i(Z_k) \geqslant \mathrm{RlM}_i(S_k) \qquad i = 1,2,\ldots,p,$$

then by the corollary to theorem 2.4 $Z_k \in \mathcal{O}_c$, $k = 1,2,\ldots,p$.

Finally define

(50) $$Z = \sum_{k=1}^{p} \left[X_k(S_k + aI) \right] - aI.$$

By property (46b) for the X_k's an easy computation yields

(51) $$\int_a^b \exp(-tZ)dA(t) = \sum_{k=1}^{p} \int_a^b \exp(-tZ_k)dA(t)$$

and consequently

(52) $$\left\| \int_a^b \exp(-tZ)dA(t) \right\| \leqslant \sum_{k=1}^{p} \left\| \int_a^b \exp(-tZ_k)dA(t) \right\| .$$

But the Z_k's $\in \mathcal{O}_c$, and hence by choosing a and b sufficiently large the left side of (52) can be made less than arbitrary $\epsilon > 0$. This at least implies that $Z \in \mathcal{C}$. Since the Z_k's $\in \mathcal{O}_c$, a standard argument based on the fact that $\mathcal{O}_c$ is open allows one to conclude $Z \in \mathcal{O}_c$.

From (50) we see that

(53) $$\mathrm{RlM}_i^*(Z) = \mathrm{Rl} \sum_{k=1}^{p} \left\{ M_i(X_k) \left[M(S_k) + a \right] \right\} - a,$$

and using (46a) and (47) it follows that

(54) $$\mathrm{RlM}_i^*(Z) \geqslant \mathrm{RlM}_i^*(S_i), \qquad i = 1,2,\ldots,p.$$

But (54) together with (45) imply

$$(55) \qquad \mathrm{RlM}_i^*(S_o) \geqslant \mathrm{RlM}_i^*(Z) \qquad i = 1,2,\dots,p$$

and the corollary to theorem 2.4 again implies that $S_o \in \mathcal{O}_c$ as was to be shown.

On the other hand, let $S_o \in \mathcal{O}_c$. Then there exists an $r > 0$ such that $S_o - rI \in \mathcal{O}_c$. Hence

$$\mathrm{RlM}^*(S_o) > \mathrm{RlM}^*(S_o - rI) \geqslant c(M^*).$$

Case 3: To discuss the second class of specialized Banach algebras for which it is possible to verify the conjecture we make the following

DEFINITION 3.3. Two elements $X, Y \in \mathcal{B}$ will be said to be *equivalent at* M_o^* (written $X \sim Y$ at M_o^*) if $M^*(X) = M^*(Y)$ for all M^* in a neighborhood of M_o^*. (See theorem 1.11 in this respect.)

We assume now that the algebra $\mathcal{B}$ satisfies the following two properties.

Property 1: If $S, S_1, S_2, \dots, S_n \in \mathcal{B}$, and if for each $M^* \in \mathfrak{M}$ there exists an i such that $S \sim S_i$ at M^*, then

$$(56) \qquad \|S\| \leq \sum_{i=1}^{n} \|S_i\| .$$

Property 2: If $S_o \in \mathcal{B}$ and if for some $M_o^* \in \mathfrak{M}$ $\mathrm{RlM}_o^*(S_o) > 0$, then there exists an $S_1 \in \mathcal{B}$ such that

$$(57) \qquad \begin{cases} \text{(i)} & \mathrm{RlM}^*(S_1) > 0 \text{ for all } M^* \in \mathfrak{M}, \\ \text{(ii)} & S_1 \sim S_o \text{ at } M_o^* . \end{cases}$$

It is easily verified that those Banach algebras given in examples

4, 5, 6, and 10 of Section I satisfy the above two properties.

We proceed to verify that the conjecture is true for algebras satisfying properties 1 and 2.

LEMMA 3.2. If for every $M^* \varepsilon \mathfrak{M}$,

$$R1M^*(S) > \min_{1 \leq i \leq n} R1M^*(S_i) \qquad i = 1,2,\ldots,n$$

then there exist elements $T_1, T_2, \ldots, T_m$ with the properties

(i) for each $k = 1,2,\ldots,m$, there exists an $i = 1,2,\ldots,n$ such that for every $M^* \varepsilon \mathfrak{M}$

$$R1M^*(T_k) > R1M^*(S_i),$$

(ii) for each $M^* \varepsilon \mathfrak{M}$, there exists k such that

$$S \sim T_k \text{ at } M^*.$$

Proof: Choose an $M_o^* \varepsilon \mathfrak{M}$, then there exists an i such that

$$R1M_o^*(S) > R1M_o^*(S_i) \text{ or}$$

$$R1M_o^*(S - S_i) > 0.$$

By property 2, there exists an element $U \varepsilon \mathfrak{B}$ such that for all $M^* \varepsilon \mathfrak{M}$ $R1M^*(U) > 0$ and $U \sim (S - S_i)$ at M_o^*. Set $T_{M_o^*} = U + S_i$, then $T_{M_o^*} \sim S$ at M_o^* and for all $M^* \varepsilon \mathfrak{M}$, $R1M^*(T_{M_o^*}) > R1M^*(S_i)$.

By this construction a collection $\{T_{M^*}\}$ is obtained such that

(i) given M^*, there exists an i for which

$$R1M^*(T_{M^*}) > R1M^*(S_i) \text{ all } M^*,$$

(ii) given M^*, $T_{M^*} \sim S$ at M^*.

Since $\mathfrak{M}$ is compact, the collection can be reduced to a finite collection which satisfies the conditions of the lemma.

LEMMA 3.3. If for all $M^* \varepsilon \mathfrak{M}$ $R1M^*(S) > \min_{1 \leq i \leq n} R1M^*(S_i)$ and

$S_i \in \mathcal{O}_c$ then $S \in \mathcal{O}_c$.

Proof: Determine the finite collection $T_1, T_2, \ldots, T_m$ by lemma 3.2. Then since for each $k = 1,2,\ldots,m$ and all $M^* \in \mathfrak{M}$, $\mathrm{RlM}^*(T_k) > \mathrm{RlM}^*(S_i)$ for some i, it follows from theorem 2.4 that the $T_k \in \mathcal{O}_c$ since the $S_i \in \mathcal{O}_c$. Furthermore if $S \sim T_k$ at M^* then

$$\int_p^q \exp(-tS)dA(t) \sim \int_p^q \exp(-tT_k)dA(t) \qquad \text{at } M^*.$$

By property 1

$$\left\| \int_p^q \exp(-tS)dA(t) \right\| \leq \sum_{k=1}^m \left\| \int_p^q \exp(-tT_k)dA(t) \right\| \quad ,$$

and since $T_k \in \mathcal{O}_c$, $S \in \mathcal{O}_c$.

THEOREM 3.6. For each $M^* \in \mathfrak{M}$ let

$$c(M^*) = \inf_{S \in \mathcal{O}_c} \mathrm{RlM}^*(S),$$

and let S_o be such that

$$\mathrm{RlM}^*(S_o) > c(M^*),$$

then $S_o \in \mathcal{O}_c$.

Proof: We first note that $c(M^*)$ is an upper semi-continuous function over $\mathfrak{M}$ since it is the infimum of continuous functions. Consequently, at each point $M^* \in \mathfrak{M}$ there exists an $S_{M^*} \in \mathcal{O}_c$ such that $\mathrm{RlM}^*(S_o) > \mathrm{RlM}^*(S_{M^*})$. But $\mathfrak{M}$ is compact and hence there exist $S_1, S_2, \ldots, S_n \in \mathcal{O}_c$ such that for all $M^* \in \mathfrak{M}$

$$\mathrm{RlM}^*(S_o) > \min_{1 \leq i \leq n} \mathrm{RlM}^*(S_i),$$

and an application of lemma 3.3 shows that $S_o \in \mathcal{O}_c$.

THEOREM 3.7. Let $c(M^*) = \inf \mathrm{Rl}M^*(S)$, then $S \varepsilon \sigma_c$ implies $\mathrm{Rl}M^*(S) > c(M^*)$.

Proof: $S \varepsilon \sigma_c$ implies there exists an $r > 0$ such that if $|z| < r$, then $S + zI \varepsilon \sigma_c$ and $\mathrm{Rl}M^*(S + zI) = \mathrm{Rl}M^*(S) + \mathrm{Rl}(z) \geqslant c(M^*)$, hence $\mathrm{Rl}M^*(S) > c(M^*)$.

THEOREM 3.8. If a Banach algebra satisfies properties 1 and 2 then σ_c is a cylindrical quadrant.

Proof: From theorems 3.6 and 3.7 it follows that the statement $\mathrm{Rl}M^*(S) > c(M^*)$ all $M^* \varepsilon \mathfrak{M}$ is equivalent to the statement $S \varepsilon \sigma_c$ and thus σ_c is the cylindrical quadrant

$$Q_{c(M^*)} = \left\{ S \varepsilon \mathcal{B} \;\middle|\; \mathrm{Rl}M^*(S) > c(M^*) = \inf_{S \varepsilon \sigma_c} \mathrm{Rl}M^*(S), \text{ all } M^* \varepsilon \mathfrak{M} \right\}.$$

The discussion in the previous paragraphs has concerned itself with the question of ordinary convergence of the integral (1) and a partial description of the region of convergence σ_c.

The question now arises, as to what we shall mean by absolute convergence. It would be desirable of course to formulate a definition that would reduce to the already existing notion of absolute convergence when $\mathcal{B}$ is chosen as the complex field. It is immediately apparent that this is possible in several ways. One might say that the integral (1) converges absolutely for a given S if any one of the following three integrals converges:

$$(58) \qquad \int_0^\infty \| \exp(-tS)dA(t) \| \quad ,$$

(59) $$\int_0^\infty \|\exp(-tS)\| \quad \|dA(t)\| ,$$

(60) $$\int_0^\infty \exp(-tS) \ \|dA(t)\| .$$

In the light of theorem 1.5 it is a priori evident that in general the regions of convergence of (58) and (59) will not coincide, and as a consequence there are several notions of absolute convergence.

Before discussing integrals (58), (59), and (60) we introduce some notation. Let

$\mathcal{C} = \{ S \mid \text{integral (1) converges} \}$

σ_c = the interior of $\mathcal{C}$,

$\mathcal{A} = \{ S \mid \text{integral (58) converges} \}$

σ_a = the interior of $\mathcal{A}$,

$\mathcal{A}_s = \{ S \mid \text{integral (59) converges} \}$

σ_{sa} = the interior of $\mathcal{A}_s$,

$\mathcal{V} = \{ S \mid \text{integral (60) converges} \}$

σ_v = the interior of $\mathcal{V}$.

From the properties of the norm it is obvious that $\mathcal{C} > \mathcal{A} > \mathcal{A}_s$ and hence $\sigma_c > \sigma_a > \sigma_{sa}$.

As in the case of integral (1) no definite criteria have been established with which one may describe the region σ_a. We state without proof however

<u>THEOREM 3.9</u>: If $S_o \varepsilon \sigma_a$ and R is an arbitrary element belonging to the radical of $\mathcal{B}$, then $(S_o + R) \varepsilon \sigma_a$.

As a first step toward establishing criteria for the description of the region σ_{sa} we quote [5, pg. 204] without proof the following theorem which is analogous to theorem 3.1.

THEOREM 3.10. For the integral

$$(61) \qquad \int_0^\infty e^{-tz} dA(t)$$

there exist two real numbers c and a such that the integral is convergent for $Rl(z) > c$, but not for any z with $Rl(z) < c$, and it is absolutely convergent for $Rl(z) > a$, but not for any z with $Rl(z) < a$. We have

$$(62) \qquad -\infty \leq c \leq a \leq \infty,$$

and

$$(63) \qquad c = \overline{\lim} \frac{\ln \|A(\infty) - A(t)\|}{t},$$

$$(64) \qquad a = \overline{\lim} \frac{\ln |V(\infty) - V(t)|}{t},$$

where $A(\infty) = \lim_{t \to \infty} A(t)$ or 0 according as the limit exists or not and $V(\infty)$ is similarly defined for the function $V(t)$ which denotes the strong total variation of $A(t)$.

We can now prove

THEOREM 3.11. For the integral

$$(59) \qquad \int_0^\infty \|\exp(-tS)\| \ \|dA(t)\|$$

there exists a real number $-\infty \leq a \leq \infty$ given by the expression

$$(64) \qquad a = \overline{\lim_{t \to \infty}} \frac{\ln |V(t) - V(\infty)|}{t},$$

where $V(t)$ denotes the strong variation of $A(t)$ in the interval $[0,t]$, and $V(\infty) = \lim_{t \to \infty} V(t)$ or 0 according as the limit exists or not, such that (59) is convergent for $Rl(S) > a$, and is divergent for $Rl(S) < a$. This establishes the region σ_{sa} as the diagonal cylindrical quadrant

$$\sigma_{sa} = \bigcap_{M^* \epsilon \mathfrak{M}} \mathcal{H}(M^*;a) = \left\{ S \epsilon \mathcal{B} \;\middle|\; RlM^*(S) > a, \text{ all } M^* \epsilon \mathfrak{M} \right\}.$$

Proof: This is an immediate consequence of theorem 3.10 and the corollary to theorem 2.3 concerning the relationship between $Rl(S)$ and the rate of growth of $\|\exp(-tS)\|$.

In the particular case where $\mathcal{B}$ is the complex field $\sigma_a = \sigma_{sa}$. That this is not true in general is demonstrated by the following example which shows that for a given $A(t)$ the integral (59) may diverge for a certain S and yet the integral (58) converges in an entire neighborhood of S.

EXAMPLE: Let $\mathcal{B}$ be the Banach algebra of complex-valued continuous functions $F(x)$, $0 \leq x \leq 1$, with the usual norm

(65) $$\|F(x)\| = \sup_{0 \leq x \leq 1} |F(x)| .$$

Integral (1) then assumes the form

(66) $$\int_0^\infty e^{-tS(x)} d_t A(x,t) .$$

Let

(67) $$A(x,t) = -2a(x)e^{-t/2}, \text{ where}$$

(68) $$a(x) = \begin{cases} 1 & 0 \leq x \leq 1/4 \\ 2-4x & 1/4 \leq x \leq 1/2 \\ 0 & 1/2 \leq x \leq 1 \end{cases} .$$

Then

(69) $$d_t A(x,t) = a(x)e^{-t/2}\,dt.$$

Choose

(70) $$S_o(x) = \begin{cases} -1/4 & 0 \le x \le 1/2 \\ 5/4 - 3x & 1/2 \le x \le 3/4 \\ -1 & 3/4 \le x \le 1 \end{cases} .$$

Computation shows

(71) $$\| e^{-tS_o(x)}\, d_t A(x,t) \| = e^{-t/4}\,dt,$$

(72) $$\| d_t A(x,t) \| = e^{-t/2}\,dt, \text{ and}$$

(73) $$\| e^{-tS_o(x)} \| \; \| d_t A(x,t) \| = e^{t/2}\,dt.$$

From (71) and (73) it follows that integral (58) converges while integral (59) diverges for $S_o(x)$. Now define a neighborhood of $S_o(x)$ by

(74) $$\mathcal{N}(S_o(x)) = \left\{ S(x) \in \mathcal{B} \;\middle|\; \sup_x |S(x) - S_o(x)| < \frac{1}{8} \right\}.$$

For any $S(x) \in \mathcal{N}(S_o(x))$

(75) $$\| e^{-tS(x)} d_t A(x,t) \| \le e^{-t/8}\,dt,$$

and hence integral (58) converges.

It may be pointed out however, that it is not possible for (58) to converge and (59) to diverge for every $S(x) \in \mathcal{B}$, for the two integrals must agree at least for all $S(x)$ which are a multiple of the identity I.

Finally we come to the case of integral (60) for which we have

THEOREM 3.12. For the integral

(60) $$\int_0^\infty \exp(-tS)\, \| dA(t) \|$$

there exists a real number $-\infty \leq c \leq \infty$ given by the expression

$$c = \overline{\lim_{t \to \infty}} \frac{\ln |V(t) - V(\infty)|}{t}, \tag{64}$$

where $V(t)$ denotes the strong variation of $A(t)$ in the interval $[0,t]$, and $V(\infty) = \lim_{t \to \infty} V(t)$ or 0 according as the limit exists or not, such that (60) is convergent for $Rl(S) > c$, and is divergent for $Rl(S) < c$.

Proof: This follows immediately from theorem 3.4 since $\|dA(t)\|$ is merely a multiple of the identity, and shows that $\sigma_v = \sigma_{sa} = \bigcap_{M^* \varepsilon \mathfrak{M}} \mathcal{H}(M^*;c)$.

Section IV

DIRICHLET AND POWER SERIES

As in the case of the complex plane, a special case of the Laplace-Stieltjes integral is the generalized Dirichlet series

$$(76) \qquad f(S) = \sum_{n=1}^{\infty} A_n \exp(-t_n S)$$

$$0 \leq t_1 < t_2 < t_3 \ldots \qquad \lim_{n \to \infty} t_n = \infty.$$

For if $A(t)$ is defined by the equations

$$(77) \qquad \begin{cases} A(t) = A_1 + A_2 + \ldots + A_n & (t_n < t < t_{n-1}) \\ A(0) = 0 \\ A(t) = \dfrac{A(t+) + A(t-)}{2} & (t > 0) \end{cases}$$

we have

$$(78) \qquad \int_0^{\infty} \exp(-tS)dA(t) = \sum_{n=1}^{\infty} A_n \exp(-t_n S)$$

whenever the integral or the series converges.

In particular if $t_n = n$, and we set

$$\exp(-nS) = \left[\exp(-S)\right]^n = Z^n$$

then (78) reduces to

$$(79) \qquad \int_0^{\infty} \exp(-tS)dA(t) = \sum_{n=1}^{\infty} A_n Z^n.$$

It must be pointed out however, that in general a power series cannot be represented as a Laplace-Stieltjes integral. This follows from the fact that the equation $\exp(-S) = Z$ has a solution for a given

Z if and only if it belongs to that component of the regular elements of $\mathcal{B}$ which contains the identity I [5, page 451]. Lorch [11,12] has shown that in a commutative Banach algebra the set of regular elements either has one or infinitely many components. In fact the number of components may well be non-denumerable.

The failure of power series to become completely a special case of the Laplace-Stieltjes integral does not alter the previous theorems, however. As is to be expected it is no longer Rl(S), but the spectral radius

$$(80) \qquad |\sigma(Z)| = \lim_{n \to \infty} \|Z^n\|^{1/n} = \inf_{M^* \varepsilon \mathfrak{M}} |M^*(Z)|$$

that now plays the essential role in the convergence theory. Since only a slight revision of techniques is required, we state without proofs the following theorems.

<u>THEOREM 4.1.</u> If the series $\sum_{0}^{\infty} A_n Z^n$ converges for $Z = Z_o$, then it converges for any Z such that $|\sigma(Z)| < |\sigma(Z_o)|$.

<u>THEOREM 4.2.</u> Let σ_c denote the interior of the set of elements ζ for which the series $\sum_{0}^{\infty} A_n Z^n$ converges. Then if $Z_o \varepsilon \sigma_c$ and R is an arbitrary element belonging to the radical of $\mathcal{B}$, $(Z_o + R) \varepsilon \sigma_c$.

<u>THEOREM 4.3.</u> For the series $\sum_{n=0}^{\infty} \|A_n\| \, \|Z^n\|$, there exists a real number $0 \leq a \leq +\infty$ given by the expression

$$a = 1/\overline{\lim_{n \to \infty}} \|A_n\|^{1/n}$$

such that the series converges for all Z for which $|\sigma(Z)| < a$, and diverges for all Z for which $|\sigma(Z)| > a$.

Finally, as a special case of a theorem which may be found in [5, page 88] we have

THEOREM 4.4. For the power series

$$\sum_{n=0}^{\infty} A_n Z^n \tag{81}$$

there exists a real number a given by the expression

$$a = \frac{1}{\overline{\lim\limits_{n \to \infty}} \|A_n\|^{1/n}},$$

such that (81) converges absolutely for $\|Z\| < a$. On every spherical surface $\|Z\| = r > a$ there are points where the series diverges. Moreover the series is uniformly convergent for $\|Z\| < (1- \epsilon)a$, $\epsilon > 0$, and fails to converge uniformly on any spherical surface $\|Z\| = r < a$.

The form of the conjecture stated in Section III undergoes obvious appropriate changes and can be verified for the same classes of Banach algebras that were treated previously.

Section V

THE LAPLACE-STIELTJES INTEGRAL AS AN ANALYTIC FUNCTION

The problem of extending analytic function theory has been considered in one form or another by many authors.* We refer the reader to [5, chapters 3,4, chapter 4, paragraphs 5.15-5.17, and chapter 22, paragraph 22.9] where numerous references are to be found.

Particularly applicable to the case of a commutative Banach algebra with unit element, where one desires to study functions on the algebra to itself, is the following definition of differentiability and analyticity given by Lorch [11] .

DEFINITION 5.1. Let $\mathcal{B}$ be a commutative complex Banach algebra with a unit element. A single-valued function F(Z) whose domain $\mathcal{D}$ and range $\mathcal{R}$ are in $\mathcal{B}$ is said to have a derivative $F'(Z_0)$ at $Z = Z_0$ if for each $\epsilon > 0$ a $\delta > 0$ can be found such that for all H in $\mathcal{B}$ with $\|H\| < \delta$

$$\|F(Z_0 + H) - F(Z_0) - HF'(Z_0)\| < \epsilon \|H\| . \tag{82}$$

If F(Z) has a derivative everywhere in $\mathcal{D}$, then it is said to be Lorch-analytic in $\mathcal{D}$.

---

* The theory of analytic function in real and complex normed linear spaces was initiated by A. D. Michal and R. S. Martin in a seminar at the California Institute of Technology during the years 1931-1932. See footnote 1, page 2 of [13] .

The theory of Lorch-analytic functions closely parallels the classical course both in its methods and principal identities. In order to develop a Cauchy theorem Lorch defined the following analogue of the Riemann integral.

DEFINITION 5.2: Let $\mathcal{D}$ be an open connected subset of $\mathcal{B}$. Let $W = F(Z)$ be a function on $\mathcal{B}$ to itself with domain $\mathcal{D}$. Let $\mathcal{P}$ be a rectifiable arc in $\mathcal{D}$. By this it is meant that $\mathcal{P}$ is given by an equation $Z = Z(t)$, $0 \leq t \leq 1$, where $Z(t)$ is continuous and of strongly bounded variation. We then define

$$(83) \qquad \int_{\mathcal{P}} F(Z)dZ = \lim_{n \to \infty} \sum_{k=1}^{n} F\left[Z(u_{n,k})\right] \left\{ Z(t_{n,k}) - Z(t_{n,k-1}) \right\} .$$

where $\max_k (t_{n,k} - t_{n,k-1}) \to 0$.

If $F(Z)$ is continuous in $\mathcal{D}$, the existence of the integral is established in the usual manner and it has the properties of linearity and boundedness which are to be expected.

In particular we have the following

THEOREM 5.1:

$$(84) \qquad \left\| \int_{\mathcal{P}} F(Z)dZ \right\| \leq \max_{\mathcal{P}} \|F(Z)\| \; \ell(\mathcal{P}),$$

where $\ell(\mathcal{P})$ is the length of $\mathcal{P}$, that is, the strong total variation of $Z(t)$ in the interval $[0,1]$.

A function $F(Z)$ Lorch-analytic in a region $\mathcal{D}$ can be shown to be continuous and Fréchet differentiable in $\mathcal{D}$. (See [5, page 72] for terminology and references to the literature). As such, it is but a matter of computation to show that $F(Z)$ possesses derivatives of all orders in $\mathcal{D}$ which are themselves Lorch-analytic functions. Furthermore if A denotes an interior point of $\mathcal{D}$ one can establish the formulas

$$F^{(n)}(A) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(A + zI)dz}{z^{n+1}} \qquad (n = 0,1,2,\ldots), \tag{85}$$

where z describes any curve γ surrounding the origin in the complex plane once in the positive sense and in such a manner that $|z|$ remains small enough to assure that $(A + zI)$ remains in the domain of definition of $F(Z)$. For the justification of the above statements, and further details we refer to [5, Chapter 4, especially pages 113-114].

With the aid of (85) it is now possible to represent $F(Z)$ and its derivatives in a form more closely analogous to the classical formulas. In fact, we have

THEOREM 5.2. Let $F(Z)$ be Lorch-analytic in an open connected region $\mathcal{D} < \mathcal{B}$, then the formulas

$$F^{(n)}(A) = \frac{1}{2\pi i} \int_{\Gamma} F(Z)(Z-A)^{-(n+1)}dZ \quad (n = 0,1,2,\ldots) \tag{86}$$

are valid for every path Γ of the form $Z = A + zI$, where z again describes any simple, closed, positively oriented path γ surrounding the origin once in the complex plane, provided that $|z|$ remains small enough to assure that $(A + zI)$ remains in the domain of definition of $F(Z)$.

Proof: Letting $Z = A + zI$ clearly reduces (86) to the form (85).

Should the domain $\mathcal{D}$ include that portion of the subspace zI (z varying over the complex numbers) which includes the spectrum of A considered as elements of the algebra, then an argument on the deformation of paths will allow a restatement of the above theorem. The path $Z = zI$ may now be chosen, where z surrounds the spectrum of A in the complex plane once in the positive sense.

In the latter form, formula (86) has been used by H. Poincaré [14] , F. Riesz [15] , L. Fantappiè [16] , and more recently by Dunford [17,18] , Lorch [11] and A. E. Taylor [19] .

To be brief, the usual pattern of fundamental theorems may be developed, culminating with the establishment of a Taylor expansion for a Lorch-analytic function into a power series which converges in the largest sphere with center at the point of development in which the function is analytic. As has already been pointed out however, the series may converge for points outside of this sphere -- for example, points in the radical of $\mathcal{B}$. Conversely, a power series of the form

$$F(Z) = \sum_{n=0}^{\infty} A_n(Z - A)^n$$

defines a Lorch-analytic function in the interior of its set of convergence.

As a first step toward establishing the Lorch-analyticity of the function

$$(1) \qquad F(S) = \int_0^{\infty} \exp(-tS)dA(t)$$

within the region $\mathcal{O}_c$, we prove the following analogue of the Vitali theorem.

<u>THEOREM 5.3.</u> Let $\{F_n(S)\}$ be a sequence of function such that

(i) $F_n(S)$ is Lorch-analytic in an open region $\mathcal{O}$, $n = 1,2,\ldots$,

(ii) $F_n(S) \to F(S)$ for each $S \in \mathcal{O}$, and

(iii) for each $S \in \mathcal{O}$, there exists a closed sphere $\mathcal{S}(S)$ and a constant $m(S)$ such that $\|F_n(S)\| \leq m(S)$, $n = 1,2,\ldots$, then the limit function $F(S)$ is Lorch-analytic within $\mathcal{O}$.

<u>Proof:</u> It will be sufficient to verify that $F(S)$ is Lorch-analytic in a neighborhood of each point $S \in \mathcal{O}$.

Let S_o be an arbitrary point belonging to $\mathcal{O}$. By (iii) there exists a sphere $\mathcal{S}(S_o;r_o)$ of radius $r_o > 0$ about S_o contained entirely in $\mathcal{O}$, and a positive constant m, such that

$$\|F_n(S)\| \leq m, \quad S \in \mathcal{S}(S_o;r_o) \qquad n = 1,2,3,\ldots \ .$$

Choose an $r_1 < r_o$ and form the sphere

$$\mathcal{S}(S_o;r_1) = \left\{ S \mid \|S - S_o\| \leq r_1 \right\}.$$

By (i) and theorem 5.2, for S_1 and $S_2 \in \mathcal{S}(S_o;r_1)$

$$F_n(S_1) - F_n(S_2) = \frac{1}{2\pi i} \int_\Gamma F_n(W)(W-S_1)^{-1}\, dW - \frac{1}{2\pi i} \int_\Gamma F_n(W)(W-S_2)^{-1}\, dW$$

$$= \frac{1}{2\pi i} (S_1 - S_2) \int_\Gamma F_n(W)(W-S_1)^{-1}(W-S_2)^{-1}\, dW ,$$

where Γ may be chosen as

$$W = S_o + r_o e^{i\theta} I \qquad (0 \leq \theta \leq 2\pi).$$

We then have

$\|F_n(S_1) - F_n(S_2)\| \leq \frac{\|S_1 - S_2\| m r_o}{d^2}$, where $d > 0$ is the minimum distance from Γ to $\mathscr{S}(S_o;r_1)$. Thus given $\epsilon > 0$, if $\|S_1 - S_2\| < \frac{d^2 \epsilon}{r_o m} = \delta(\epsilon)$, then

$$\|F_n(S_1) - F_n(S_2)\| < \epsilon \qquad n = 1,2,\ldots \quad .$$

This states that the $F_n(S)$ are equi-uniformly continuous in $\mathscr{S}(S_o;r_1)$.

Now choose $r_2 < r_1$ and define the path Γ_1 as

$$W = S_o + re^{i\theta}I, \quad 0 \leq \theta \leq 2\pi \quad , r_2 < r < r_1.$$

By the choice of r, Γ_1 belongs to the sphere $\mathscr{S}(S_o;r_1)$ and hence by the preceding argument, given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\|F_n(W_1) - F_n(W_2)\| < \epsilon \qquad n = 1,2,\ldots$$

for any W_1, $W_2 \in \Gamma_1$ for which $\|W_1 - W_2\| < \delta(\epsilon)$. By (ii) $F_n(W) \to F(W)$, and hence

$$\|F(W_1) - F(W_2)\| < \epsilon \qquad \text{for } \|W_1 - W_2\| < \delta(\epsilon).$$

Since Γ_1 is a compact set it is possible to choose a finite number of points W_i $(i = 1,2,\ldots,p)$ along Γ_1 such that

$$\|W_i - W_{i+1}\| < \delta(\epsilon) \quad (i = 1,2,\ldots,p) \; (W_{p+1} = W_1),$$

and for arbitrary $W \in \Gamma_1$ there exists at least one W_j such that

$$\|W - W_j\| < \delta(\epsilon).$$

Again by (ii) since $F_n(W) \to F(W)$, we have the existence of p integers $n_i(W_i)$ $i = 1,2,\ldots,p$, such that, given $\epsilon > 0$

$$\|F_n(W_i) - F_m(W_i)\| < \epsilon \qquad \text{for } n,\, m > n_i(W_i) \quad i = 1,2,\ldots,p.$$

Let

$$n_o = \max_{1 \leq i \leq p} \left[n_i(W_i) \right] \quad ,$$

then for arbitrary $W \varepsilon \Gamma_1$

$$\|F_n(W) - F_m(W)\| \leq \|F_n(W) - F_n(W_i)\| + \|F_n(W_i) - F_m(W_i)\| + \|F_m(W_i) - F_m(W)\|$$

$$< 3\epsilon \qquad \text{provided } n,m > n_o \text{ and}$$

W_i is chosen so that $\|W - W_i\| < \delta(\epsilon)$.

Thus the $F_n(W)$ converge uniformly to $F(W)$ for $W \varepsilon \Gamma_1$.

Now restricting S to the sphere

$$\mathcal{S}(S_o;r_2) = \left\{ S \;\middle|\; \|S - S_o\| \leq r_2 \right\} ,$$

$$F_n(S) - F_m(S) = \frac{1}{2\pi i} \int_{\Gamma_1} \left\{ F_n(W) - F_m(W) \right\} (W-S)^{-1} dW,$$

from which it follows

$$\| F_n(S) - F_m(S) \| \leq \frac{\epsilon r}{d} , \; n > n_o.$$

But n_o is independent of $S \varepsilon \mathcal{S}(S_o, r_2)$, and hence the $F_n(S)$ converge uniformly to $F(S)$ in this sphere and consequently to a Lorch analytic function. This completes the proof of the theorem.

We next establish two lemmas.

<u>LEMMA 5.1.</u> If $A(t)$ is a function of strongly bounded variation in every finite interval $[0,b]$ then the sequence of functions defined by

$$(87) \qquad F_n(S) = \int_0^n \exp(-tS)dA(t) \qquad n = 1,2,\ldots$$

are Lorch analytic throughout $\mathcal{B}$ and their k-th derivatives are given by the formula

(88) $$F_n^{(k)}(S) = \int_0^n (-t)^k \exp(-tS)dA(t) \qquad n = 1,2,\ldots \quad .$$

Proof: The proof will be demonstrated for the case k = 1 only. From (87) and (88) we have

$$\|F_n(S+H)-F_n(S)- HF'(S)\| = \left\| \int_0^n \exp(-tS) \left\{ \exp(-tH) - I + tH\right\} dA(t)\right\|$$

$$\leq \|H\| \left\{ \|H\| \int_0^n t^2 e^{t\,\|H\|} \|\exp(-tS)\| \;\|dA(t)\| \right\} = \|H\| \left\{ \|H\| \; m(n)\right\} \quad .$$

Then given $\epsilon > 0$, for all H such that $\|H\| < \frac{\epsilon}{m(n)}$ we have $\|F_n(S + H) - F_n(S) - HF'(S)\| < \epsilon \|H\|$. Similar proofs may be devised for higher derivatives.

LEMMA 5.2. Let $\mathcal{O}_c$ denote the interior of the region of convergence of the integral

(1) $$\int_0^\infty \exp(-tS)dA(t) \; ,$$

and let

(87) $$F_n(S) = \int_0^n \exp(-tS)dA(t) \qquad (n = 1,2,\ldots).$$

Then if S_o is an arbitrary element belonging to the region $\mathcal{O}_c$, there exists a real number $r(S_o) > 0$, and a constant $m(S_o)$, such that within the sphere $\mathcal{S}(r;S_o) = \|S-S_o\| < r(S_o)$

$$\|F_n(S)\| \leq m(S_o) \qquad (n = 1,2,\ldots).$$

Proof: Let S_o be an arbitrary point belonging to $\mathcal{O}_c$. Since $\mathcal{O}_c$ is open, there exists an open sphere of radius $\rho(S_o)$ about S_o which also lies in $\mathcal{O}_c$. In particular

$$S_1 = S_o - \rho/_2 \, I \quad \text{is in} \quad \mathcal{O}_c.$$

Now we restrict S to lie in the sphere $\|S - S_o\| \leq \rho/_3$.

We have

$$(89) \qquad F_n(S) = \int_0^n \exp(-tS)dA(t) = \int_0^n \exp\left[-t(S-S_1+S_1)\right] dA(t)$$

$$= \int_0^n \exp\left[-t(S-S_1)\right] dB(t) \, ,$$

where

$$(90) \qquad B(t) = \int_0^t \exp(-uS)dA(u).$$

An integration by parts yields

$$f_n(S) = \exp\left[-n(S-S_1)\right] B(n) + (S-S_1) \int_0^n \exp\left[-t(S-S_1)\right] B(t)dt,$$

and hence

$$(91) \quad \|f_n(S)\| \leq \left\|\exp\left[-n(S-S_1)\right]\right\| \, \|B(n)\| + \|S-S_1\| \int_0^n \left\|\exp\left[-t(S-S_1)\right]\right\| \, \|B(t)\| \, dt.$$

For each finite n, the right side of (91) exists giving a sequence of constants: $m_1, m_2, \ldots, m_n, \ldots$.

The restriction $\|S - S_o\| \leq \rho/_3$ implies that $Rl(S-S_1) \geqslant \frac{\rho}{6} > \frac{\rho}{7} > 0$, hence there exists an n_o such that

$$(92) \qquad \| \exp[-n(S-S_1)] \| < e^{-n\rho/7} \qquad (n > n_o).$$

From (90), B(t) is a continuous function of t, and since S_1 lies in $\mathcal{O}_c$ $B(\infty)$ exists, and therefore for some positive constant c

$$(93) \qquad \| B(n) \| < c \quad \text{for all } n.$$

Using (92) and (93) in (91) yields

$$(94) \qquad \| F_n(S) \| < e^{-n_o\rho/7} + 5c\rho/6 \int_0^{n_o} \| \exp[-t(S-S_1)] \| \, dt - 35c/6 \, e^{-n_o\rho/7}$$

$$= m \qquad (n > n_o).$$

Setting

$$m(S_o) = \max \{ m, m_1, m_2, \ldots, m_{n_o} \} ,$$

we have for S belonging to $\mathcal{S}(S_o) = \{ S \mid \|S-S_o\| \leq \rho/3 = r(S_o) \}$

$$\| F_n(S) \| < m(S_o) \qquad (n = 1,2,3,\ldots.)$$

as was to be shown.

From the last two lemmas we see that the functions

$$(87) \qquad F_n(S) = \int_0^n \exp(-tS)dA(t) \qquad n = 1,2,\ldots$$

satisfy the conditions of theorem 5.3 and hence we have established

<u>THEOREM 5.4.</u> The function defined by the Laplace-Stieltjes integral

$$(1) \qquad F(S) = \int_0^\infty \exp(-tS)dA(t)$$

represents a Lorch-analytic function within the region $\mathcal{O}_c$.

A more general theory of analytic functions on one Banach Space to another may be found summarized in [5, Chapter 4] where adequate references to an extensive literature on the subject is available. When couched in the language of Banach algebras the definition of an analytic function assumes the following form.

DEFINITION 5.3. A function F(Z) on $\mathcal{B}$ to $\mathcal{B}$, defined in the domain $\mathcal{D}$ is said to be analytic in $\mathcal{D}$ if it is single-valued, locally bounded, and Gateaux-differentiable [5, page 72] in $\mathcal{D}$.

The question arises if perhaps it is not possible for a Lorch-analytic function to be analytic in the sense of the above definition in a more extensive domain than that in which it is Lorch-analytic. That this is not possible is shown in

THEOREM 5.5. Let F(Z) be analytic (in the sense of definition 5.3) in an open connected set $\mathcal{O} < \mathcal{B}$. Let $\mathcal{S}_L$ be a sphere, center at Z_0, contained interior to $\mathcal{O}$ in which F(Z) is Lorch-analytic. Then F(Z) is a Lorch-analytic function throughout $\mathcal{O}$.

Proof: Let W be an arbitrary point in $\mathcal{O}$. By hypothesis we may join Z_0 to W by a path Γ lying entirely within $\mathcal{O}$. To each Z_Γ belonging to Γ there exists a sphere, center at Z_Γ, with a non-zero radius $\rho(Z_\Gamma)$ in which F(Z) may be represented by an absolutely convergent F-power series

$$(95) \qquad F(Z) = \sum_{n=0}^{\infty} \frac{\delta^n F(Z_\Gamma ; Z - Z_\Gamma)}{n!} \qquad \| Z - Z_\Gamma \| < \rho(Z_\Gamma).$$

By standard arguments $\rho(Z_\Gamma)$ can be shown to be a continuous non-zero function of Z_Γ on the compact set Γ and hence attains its lower bound $\rho_o > 0$.

By the compactness of Γ we may choose a finite number of points $Z_o, Z_1, \ldots, Z_p = W$ along Γ such that $\| Z_{i+1} - Z_i \| < \rho_o/2$ $i = 0,1,\ldots,p-1$ and cover Γ with the p spheres $\mathcal{S}_i = \{ Z \mid \| Z - Z_i \| < \rho_o \}$ $i = 0,1,\ldots,p$. Furthermore, the $\mathcal{S}_i$ have the property that the center Z_{i+1} of $\mathcal{S}_{i+1}$ is contained in $\mathcal{S}_i$, $i = 0,1,\ldots,p-1$.

Let Z' be the first one of the $Z_i \in \Gamma$ contained in $\mathcal{S}_L$ and for which the sphere $\| Z - Z' \| < \rho_o$ includes a portion of $\mathcal{O} - \mathcal{S}_L$. Since Z' is interior to $\mathcal{S}_L$, F(Z) may be expanded in a Lorch-power series

$$(96) \qquad F(Z) = \sum_{0}^{\infty} \frac{F^{(n)}(Z')\,(Z-Z')^n}{n!} \qquad \| Z-Z' \| < \rho(Z').$$

Simultaneously we have the F-power series

$$(97) \qquad F(Z) = \sum_{n=0}^{\infty} \frac{\delta^n F(Z';Z-Z')}{n!} \qquad \| Z-Z' \| < \rho_o.$$

By hypothesis, (96) and (97) agree in $\| Z-Z' \| < \rho(Z')$ and for such Z it may be shown [5, page 72] that

$$(98) \qquad \delta^n F(Z';Z-Z') = F^{(n)}(Z')(Z-Z') \qquad \| Z-Z' \| < \rho(Z').$$

Now let $Z \neq Z'$ be an arbitrary point in $\| Z-Z' \| < \rho_o$, and set

$$(99) \qquad Z'' = Z + \frac{\rho(Z')}{2} \frac{(Z-Z')}{\| Z-Z' \|} .$$

Obviously

$$\|Z'' - Z'\| = \frac{\rho(Z')}{2} < \rho(Z')$$

and hence $(Z'' - Z')$ satisfies (98), that is

$$\delta^n F(Z';Z''-Z') = F^{(n)}(Z')(Z''-Z'). \tag{100}$$

Substituting from (99) into (100) and making use of the linearity and homogeneity of the G-differential we have

$$\delta^n F(Z';Z-Z') = F^{(n)}(Z')(Z-Z')^n \qquad \|Z-Z'\| < \rho_0. \tag{101}$$

This requires that

$$F(Z) = \sum_{0}^{\infty}{}' \frac{\delta^n F(Z';Z-Z')}{n!} = \frac{F^{(n)}(Z')(Z-Z')^n}{n!} \tag{102}$$

be absolutely convergent for all Z such that $\|Z-Z'\| < \rho_0$. Since the right side is an absolutely convergent series of Lorch-analytic functions for $\|Z-Z'\| < \rho_0$ and converges to F(Z) in that domain; then F(Z) is a Lorch analytic function outside of $\mathcal{S}_L$. Proceeding in an analogous manner we thus see it is possible to reach an arbitrary point $W \in \mathcal{O}$ in a finite number of steps, at each step establishing the Lorch-analyticity of F(Z).

Section VI

SINGULARITIES ON THE BOUNDARY OF THE REGION σ_c

When $\mathcal{B}$ is the complex plane it is well known that a function defined by a Laplace-Stieltjes integral may fail to have any singularities on the abscissa of convergence of the integral. Indeed, it is not difficult to construct examples for which the abscissa of convergence is finite and yet the function represented is entire.

In the remaining paragraphs of this section the analogous situation is considered for an arbitrary Banach algebra. A special case is discussed for which it can be asserted that there exists singularities on the boundary of the region σ_c. When the term singularity is used it will be in the following sense.

DEFINITION 6.1. A point S_0 on the boundary of the region σ_c will be called a singularity of the function defined by

$$(1) \qquad F(S) = \int_0^\infty \exp(-tS)dA(t)$$

if, given any neighborhood $\mathcal{N}(S_0)$ of S_0, it is impossible to extend the definition of F(S) into $\mathcal{N}(S_0)$ in such a manner that it remains analytic and agrees with F(S) in $\mathcal{N}(S_0) \cap \sigma_c$.

It has already been pointed out that a function F(S) Lorch-analytic in a region σ may be expanded in a power series about any point S_0 in σ . Furthermore, the series converges not only in the largest sphere contained in the domain of analyticity, but for all points of that sphere translated in the direction of the radical of $\mathcal{B}$. This serves to extend the domain of analyticity into a cylindrical set parallel to the radical. Because of this fact we have

THEOREM 6.1. If the Lorch-analytic function F(S) defined by (1) has a singularity at the point S_0 on the boundary of the region $\mathcal{O}_c$, and if R is an arbitrary element belonging to the radical of $\mathcal{B}$, then $S_0 + R$ is a singularity of F(S).

Proof: We first recall from theorem 2.5 that if $S \in \mathcal{O}_c$ then $S + R \in \mathcal{O}_c$ for arbitrary R in the radical, so that F(S) is defined and by theorem 5.4 is analytic in a cylinder in the direction of the radical. It also follows that if S_0 is a boundary point of $\mathcal{O}_c$ then so is $S_0 + R$. Now if $S_0 + R$ were not a singular point of F(S), it could then be included within the interior of the set of convergence of some power series representing F(S) and agreeing with F(S) in $\mathcal{O}_c$. But by the remarks made previous to the theorem this would imply that F(S) could be analytically extended back to include the point S_0. This is a contradiction.

We now consider a case when the existence of singularities on the boundary of $\mathcal{O}_c$ can be assured.

THEOREM 6.2. Let the integral defining the function

$$f(s) = \int_0^\infty e^{-ts} da(t) \tag{16}$$

have abscissa of convergence $Rl(s) = c$, and let s_0 be a singularity of f(s) on $Rl(s) = c$. Then the function defined by the integral

(34) $$F(S) = \int_0^\infty \exp(-tS)da(t)$$

possesses a singularity at any point S_o on the boundary of $\mathcal{O}_c$ for which there exists an $M_o^* \varepsilon \mathfrak{M}$ such that $M_o^*(S_o) = s_o$.

Proof. Let S_o be such a point. Assuming S_o is not a singular point of (34), then for any $r > 0$ the point $S_1 = S_o + rI \varepsilon \mathcal{O}_c$, $Rl(S_1) > c$, and F(S) may be expanded in a power series

(103) $$F(S) = \sum_{n=0}^{\infty} A_n(S - S_1)^n,$$

where

(104) $$A_n = \frac{1}{n!} \int_0^\infty (-t)^n \exp(-tS_1)da(t) \qquad n = 0,1,\ldots .$$

The series (103) will include the point S_o within its sphere of convergence. This implies however that

(105) $$M_o^*\left[F(S)\right] = \sum_{n=0}^{\infty} M_o^*(A_n)\left[M_o^*(S) - M_o^*(S_1)\right]^n$$

is a convergent power series which includes the point $s_o = M_o^*(S_o)$ within its circle of convergence. Since

(106) $$M_o^*(A_n) = \int_0^\infty (-t)^n e^{-tM_o^*(S_1)}da(t) \qquad n = 0,1,\ldots ,$$

we see that the power series (105) represent (16) at s_o. But this is a contradiction to the assumption that s_o was a singularity of (16).

Theorem 6.2 implies that any of the known results concerning singularities of the function represented by (16) apply to integral (34) in the sense stated above. In particular we refer to [9, page 58], [20, pages 89-92], and [21, Chapter 4] where many of these results are discussed.

In conclusion we sketch the proof of a generalization of a theorem used in the theory of Dirichlet series by Cramér [22] and Ostrowski [23]. To do so we first need

LEMMA 6.1. Let $P(z)$ be a $\mathcal{B}$-valued entire function of the complex variable z of exponential type, i.e., there exists a non-negative constant k such that for arbitrary $\epsilon > 0$

$$(107) \qquad \|P(z)\| < e^{(k+\epsilon)|z|} \quad \text{for } |z| \text{ sufficiently large.}$$

Let

$$(108) \qquad P(z) = \sum_{n=0}^{\infty} C_n z^n$$

be the power series expansion of $P(z)$ and define the function

$$(109) \qquad Q(z) = \sum_{n=0}^{\infty} n!\, C_n / z^{n+1}$$

constructed by using the same C_n as in (108). Then $P(z)$ and $Q(z)$ are related by the following formulas:

$$(110) \qquad Q(w) = \int_0^{\infty} e^{-wz} P(z)\,dz$$

$$(111) \qquad P(z) = \frac{1}{2\pi i} \int_{|w| = k+\epsilon} e^{zw} Q(w)\,dw.$$

The proof follows mutatis mutandis that of the classical case and may be found in [10, pages 61-65].

Now let σ_c be the open region of convergence of the function defined by

$$(1)\qquad F(S) = \int_0^\infty \exp(-tS)\,dA(t),$$

and let P(z) be an entire function of the complex variable z satisfying the conditions of lemma 6.1. Introduce the new function C(S) defined by the integral

$$(112)\qquad C(S) = \int_0^\infty \exp(-tS)P(t)\,dA(t).$$

By use of (107) it can be verified that the integral (112) has an open region of convergence which at least includes the set

$$(113)\qquad \mathcal{O}_k = \left\{ S \in \mathcal{B} \mid S = S_o + kI, \text{ some } S_o \in \mathcal{O}_c \right\}.$$

From theorem 2.4 it follows that $\mathcal{O}_k < \mathcal{O}_c$.

Now let $\mathcal{E}$ be an open set contained in $\mathcal{B}$ and such that $\mathcal{E} \cap \mathcal{O}_k$ is non-void. Further, let G(S) be an analytic function defined in $\mathcal{E}$ and such that

$$(114)\qquad G(S) = F(S) \quad \text{for } S \in \mathcal{E} \cap \mathcal{O}_c \supset \mathcal{E} \cap \mathcal{O}_k.$$

Define the region

$$(115)\qquad \mathcal{H} = \mathcal{E} \cup \mathcal{O}_c.$$

Since $\mathcal{O}_c$ and $\mathcal{O}_k$ are open sets it is easily seen that $S \in \mathcal{O}_k$ implies that $S + \zeta I \in \mathcal{O}_c$ for all ζ such that $|\zeta| \leq k$. Moreover, it follows that $S \in \mathcal{E} \cap \mathcal{O}_k$ implies $S + \zeta I \in \mathcal{H}$ for $|\zeta| \leq k$.

Construct within $\mathcal{H}$ the set

$$(116)\qquad \mathcal{H}_k = \left\{ S \in \mathcal{H} \mid S + \zeta I \in \mathcal{H},\ |\zeta| \leq k \right\}.$$

Certainly $\mathcal{H}_k$ is non-void since it contains at least $\xi \cap \mathcal{O}_k$. Furthermore, $\mathcal{H}_k$ is an open set. This follows from the fact that the minimum distance from the compact disk $S + \xi I$, $(|\xi| \leq k)$ to the complement of $\mathcal{H}$ is definitely positive.

Now let $\widetilde{\mathcal{H}}_k$ be that component of $\mathcal{H}_k$ which is connected with $\mathcal{O}_k$. Again, $\widetilde{\mathcal{H}}_k$ is non-void since $\xi \cap \mathcal{O}_k < \mathcal{H}_k \cap \mathcal{O}_k < \widetilde{\mathcal{H}}_k$.

Define the function

$$(117) \qquad J(S) = \frac{1}{2\pi i} \int_{|w| = k + \epsilon} G(S-wI)\, Q(w)dw,$$

where $Q(w)$ represents the analytic continuation of the function defined by (110). If S is restricted to remain in $\widetilde{\mathcal{H}}_k$, then ϵ can be chosen sufficiently small, so that as w traverses the path $|w| = k + \epsilon$, $S - wI$ remains in the domain of analyticity of G(S), and hence by a slight modification of a standard theorem of complex variable theory [24, page 99] it can be shown that J(S) defines a Lorch analytic function of S in $\widetilde{\mathcal{H}}_k$.

Moreover, since S is restricted to lie in $\widetilde{\mathcal{H}}_k$, then by (114) J(S) may be written as

$$(118) \qquad J(S) = \frac{1}{2\pi i} \int_{|w| = k + \epsilon} F(S-wI)\, Q(w)dw.$$

If S is even further restricted to lie in $\widetilde{\mathcal{H}}_k \cap \mathcal{O}_k$ we may use equation (1) as a representation of F(S). Substituting from (1) into (118) we have

$$(119) \qquad J(S) = \frac{1}{2\pi i} \int_{|w| = k + \epsilon} \left(\int_0^\infty \exp\left[-t(S-wI)\right] dA(t) \right) Q(w)dw.$$

The restriction $S \in \widetilde{\mathcal{N}}_k \cap \mathcal{O}_k$ and the choice of ϵ sufficiently small makes the inner integral in (119) uniformly convergent with respect to w (see theorem 2.6), and an interchange of order of integration is justified. Carrying out this process yields

$$(120) \qquad J(S) = \int_0^\infty \exp(-tS) \left(\frac{1}{2\pi i} \int_{|w| = k + \epsilon} e^{tw} Q(w)\, dw \right) dA(t),$$

which by (111) becomes

$$(121) \qquad J(S) = \int_0^\infty \exp(-tS) P(t)\, dA(t) = C(S).$$

We thus have

<u>THEOREM 6.3.</u> Let F(S), P(z), and G(S) be the functions described above, and define the function C(S) as in (112). Then J(S) as defined in (117) is an analytic extension of C(S) into $\widetilde{\mathcal{N}}_k$.

It is to be noted that J(S) need not be equal to C(S) at all points where both are defined, since this region may not be simply connected. This multi-valuedness of the extension can even be realized in the classical case where $\mathcal{B}$ is the complex plane.

The importance of the last theorem lies in the fact that in the classical case it plays an essential role in gap and density theorems. The function P(z) is an entire function which is associated with the sequence of exponents of a Dirichlet series, and aids in a discussion of the density of that sequence. Its function is essentially that of modifying the given Dirichlet series to one with all positive coefficients. Once the coefficients are positive the existence of singularities on the abscissa of convergence can then be assured. It appears reasonable that a similar type of analysis could be extended to those special Banach algebras called Banach algebras in which there exists a notion of positiveness. It is the intention of the author to carry out such an investigation in the near future.

REFERENCES

1. Michal, A. D., and Martin, R. S., Some expansions in vector space. Journal de Mathématiques Pures et Appliquées (1934), 13, 69-91.
2. Nagumo, M., Einige analytische Untersuchungen in linearen metrischen Ringen. Japanese Journal of Mathematics (1936), 13, 61-80.
3. Gelfand, I., Normierte Ringe. Rec. Math. N. S. (1941), 9, 3-24.
4. Gelfand, I., Ideale und Primärideale in Normierte Ringe. Rec. Math. N. S. (1941), 9, 41-48.
5. Hille, E., Functional Analysis and Semi-Groups. Coll. Pub. Amer. Math. Soc., (1948), 31.
6. Proceedings of the Symposium on Spectral Theory and Differential Problems. Math. Dept., Oklahoma Agriculture and Mechanical College, Stillwater, Oklahoma. (1951).
7. Banach, S., Théorie des Opérations Linéaires. Warszawa (1932).
8. Dunford, N., Uniformity in linear spaces. Trans. Amer. Math. Soc. (1938), 44, 305-356.
9. Widder, D. V., The Laplace Transform. Princeton Univ. Press (1946).
10. Doetsch, G., Theorie und Anwendung der Laplace-Transformation. J. Springer, Berlin (1937).
11. Lorch, E. R., The theory of analytic functions in normed abelian vector rings. Trans. Amer. Math. Soc. (1943), 54, 414-425.
12. Lorch, E. R., The structure of normed abelian rings. Bull. Amer. Math. Soc., (1944), 50, 447-463.

13. Michal, A. D., On a non-linear total differential equation in normed linear spaces. Acta Mathematica (1948), 80, 1-21.

14. Poincaré, H., Sur les groupes continus. Trans. Cambridge Phil. Soc. (1900), 18, 182-185.

15. Riesz, R., Les systems d'équations linéaires à une infinité d'inconnues. Gauthier-Villars, Paris (1913).

16. Fantappiè, L., Überblick über die Theorie der analytischen Funktionale und ihre Anwendungen. Jahresber. D.M.V., (1933), 43, 1-25.

17. Dunford, N., Spectral theory. Bull. Amer. Math. Soc., (1943), 49, 637-651.

18. Dunford, N., Spectral theory I, Convergence to projections. Trans. Amer. Math. Soc. (1943), 49, 185-217.

19. Taylor, A. E., Analysis in complex Banach Spaces. Bull. Amer. Math. Soc. (1943), 49, 652-669.

20. Levinson, N., Gap and Density Theorems. Coll. Pub. Amer. Math. Soc. (1940), 26.

21. Bernstein, V., Séries de Dirichlet. Paris (1933).

22. Cramér, H., Un théorème sur les séries de Dirichlet et son application. Arkiv for Math., Astr. och Fys. (1918), 13.

23. Ostrowski, A., Einige Bemerkungen über Singularitäten Taylorscher und Dirichletscher Reihen. Berlin Sitz. (1921).

24. Titchmarsh, E. C., The Theory of Functions. Oxford Univ. Press (1932).