AN ANALYSIS OF ERRORS
IN THE ELECTRIC-ANALOG COMPUTER

Thesis by

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ABSTRACT

This thesis is the result of work done in connection with the California Institute of Technology Electric-Analog Computer. Several methods are developed for determining the accuracy of the solutions of various types of problems by electric circuit analogies. These are used to obtain expressions for the errors involved in the solutions of specific examples.

The first part deals with the error involved in the solution of problems with continuously distributed physical properties by means of circuit analogies of lumped parameters. The errors of mode frequencies of several mechanical vibration problems are given in the form of asymptotic series.

In the second part, investigation is made of the effect of the statistical deviation of the actual values of the computer elements from their nominal values. This effect is computed for circuits for some of the problems considered in the first section.

The third part describes the analog computer solution of the transient stresses in a model airplane wing under landing impact. This solution is compared with another computed solution and with an experimental test of the same model wing.

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PREFACE

The statement of a problem which is to be solved with the electric-analog computer consists of three distinct parts. The first part is a description of the system. This may be in the form of a set of mathematical equations, or it may be a description of the subject of the problem, consisting of physical measurements and properties. The second part establishes the excitation of the system. The third part states what answers are desired; that is, prescribes the quantities to be investigated.

The solution of the problem follows the same steps. First an electric circuit is constructed which is an analog of the physical or mathematical system. The second step is the application of electrical excitation to various parts of the circuit. And finally the answers are obtained by observation of the electrical quantities present in the circuit.

Errors in the solution may be traced back to one of these three processes. The analogy between the electric circuit and the physical or mathematical system may be faulty. The excitation of the electric circuit may not correspond to the forces to which the physical system is subjected. Observation of the electric analog by means of meters may introduce further error. This thesis will treat in detail only errors derived from the first source, imperfect analogy.

The other sources do contribute to errors, and in many classes of problems cannot be ignored. When the excitation consists of an arbitrary function of time to be applied to the circuit, the combatting of error from this source lies chiefly in the design and use of special equipment to generate the desired functions. (1)* Errors due to metering may be caused by the disturbing of the analog-circuit when the meters are connected, or by transmitting to the operator information at variance with that which exists in the circuit. The elimination of these errors lies in metering-system design, in which the chief limitations are the quality of ferromagnetic materials and parasitic impedances.

The analogy between the physical or mathematical system and its electric analog may be inaccurate for a number of reasons. One is that the differential equations of the original system may be represented in the circuit by difference equations. This is most often the result of using lumped electrical parameters as an analog of distributed physical properties which are functions of a continuous space variable. Errors belonging to this class are considered in Part I of this thesis.

Another reason is the imperfection of the individual

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* Raised numbers in parentheses designate the references listed at the end of the text.

elements of the computer, including inherent parasitic inductance, capacitance, and resistance; deviation of the actual value from the nominal value; and non-linearities in the operating characteristics. Part II deals with the limitation of precision caused by the deviation of actual values from nominal values.

A third reason is that it may be impossible to reduce the original physical system to a set of equations without introducing several qualifying assumptions which affect the validity of the answers obtained.

In discussing the error and precision involved in the solution of a problem by means of a certain circuit it is necessary to specify what quantity constitutes the answer in order to define the error and precision. It would be difficult to define and ascertain the error of an answer which consists, for example, of the response of a system to an arbitrary transient excitation. In the case of vibration problems, the frequency of normal mode vibration is an important answer, and is a quantity which is more readily analyzed. In this thesis, most of the problems treated are vibration problems, and the expressions which are derived are in terms of mode frequencies.

I ERRORS DUE TO THE LUMPING OF CONTINUOUS PHYSICAL PROPERTIES

1.1 Introduction

The theory underlying the construction of electric circuits of discrete elements to represent continuous physical systems has been considerably developed. (2,3) The most certain method of determining the error introduced by the lumping of parameters is to calculate the solution to the physical problem, calculate the solution of the circuit analogy, and compare the results. This cannot be done in general, for the usual problem to which the computer is applied cannot be solved by exact analytical methods. However the process of checking any computer involves using it to get answers to problems which can be solved by other means. It is then hoped that the errors present in the general solutions can be estimated from a knowledge of the errors which exist in the test problems.

Certain eigenvalue problems are a good test for errors introduced by lumping parameters, for solutions may be obtained for the problem with continuous properties, as well as for the electric circuit which represents it. The electrical analogy for this type of problem is a passive, non-dissipative circuit, for which the normal modes of oscillation are determined. For the purpose of defining errors the mode frequencies are considered to be the answer.*

In this part are considered eigenvalue problems associated with linear second and fourth order partial differential equations with constant coefficients. For the continuous physical system the eigenvalue solution is obtained from a differential equation by standard methods. For the lumped circuit the mode frequencies are obtained from the solution of a difference equation or difference equations. For many of the cases considered the mode frequencies cannot be expressed explicitly from the difference equation solution, but are developed from it in the form of series.

1.2 The Fourth Order Eigenvalue Problem

The equation considered here is that describing the lateral motion of a uniform beam bending in one plane.

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (1)$$

y = lateral deflection

x = longitudinal dimension

ρ = mass per unit length

EI = stiffness to bending in x-y plane

t = time

Effects of rotary inertia, finite shearing strain,

* If the physical system under consideration is non-conservative, but has linear dissipation, the same method may be used. The circuit will contain resistance, and the mode frequencies will be complex numbers, representing rate of decay as well as rate of oscillation.

and damping are neglected. To reduce this to an eigenvalue problem* sinusoidal oscillations are assumed to exist at a frequency of $\omega/2\pi$. $y(x,t)$ is replaced by $Y(x)e^{j\omega t}$, and $\frac{\partial^2 y}{\partial t^2}$ is replaced by $-\omega^2 Y e^{j\omega t}$. Equation (1) is now written

$$\frac{d^4 Y}{dx^4} - k^4 Y = 0, \tag{2}$$

where $k = \left(\frac{\rho\omega^2}{EI}\right)^{1/4}$.

k is a positive, real number. In general Y is a complex number. However, in normal mode vibrations, all portions of the system vibrate in the same phase, so that Y may be considered real without loss of generality. y is determined by taking the real part of $Y e^{j\omega t}$. The general solution of equation (2) is

$$Y = A \cosh kx + B \sinh kx + C \cos kx + D \sin kx. \tag{3}$$

For comparison with solutions by finite-difference methods, two sets of boundary conditions are applied, giving the following standard solutions.(4)

(a) Simply supported beam.

At $x = 0$, $Y = 0$ and $\frac{d^2 Y}{dx^2} = 0$.

At $x = 1$, $Y = 0$ and $\frac{d^2 Y}{dx^2} = 0$.

$$Y = A_m \sin k_m x \tag{4}$$

$$k_m = m\pi \tag{5}$$

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*The term "eigenvalue problem" refers to a differential equation resulting from the separation of variables of a partial differential equation, and a set of boundary conditions. In order that the eigenvalue problem have a solution, it will be found that a term in the differential equation, initially undetermined, must take on one of a set of discrete values, known as the eigenvalues. For the vibration problems considered in this thesis, the variable which is eliminated by separation is time, t. The eigenvalue and the frequency of oscillation are related in a simple manner.

$m = 1, 2, 3, \ldots$

(b) Cantilever beam.

At $x = 0$, $Y = 0$ and $\frac{dY}{dx} = 0$. (Clamped end.)

At $x = 1$, $\frac{d^2Y}{dx^2} = 0$ and $\frac{d^3Y}{dx^3} = 0$. (Free end.)

$$Y = A_m(\cosh k_m x - \cos k_m x) + B_m(\sinh k_m x - \sin k_m x), \qquad (6)$$

where k_m is a solution of the equation

$$1 + \cosh k \cos k = 0, \qquad (7)$$

and

$$\frac{B_m}{A_m} = - \frac{\cosh k_m + \cos k_m}{\sinh k_m + \sin k_m} = (-1)^m \left[\tan \tfrac{1}{2}k_m\right]^{(-1)^m}$$

The first few eigenvalues for the cantilever beam are listed in Table 1.

The simplest type of electric circuit analogy for the beam in bending is given in Fig. 1*, page 8, together with

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Table 1 Eigenvalues of the Uniform, Continuous, Cantilever Beam

mode number, m	k_m
1	1.875
2	4.694
3	7.855
4	10.996
5	14.137
..	...

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* The transformers of Fig. 1 are assumed to be "ideal". That is, all the turns are linked by all the flux, the sum of the ampere-turns is zero, and no energy is dissipated. T represents the ratio of the number of turns in the primary (y) circuit to the number of turns in the secondary (ϕ) circuit.

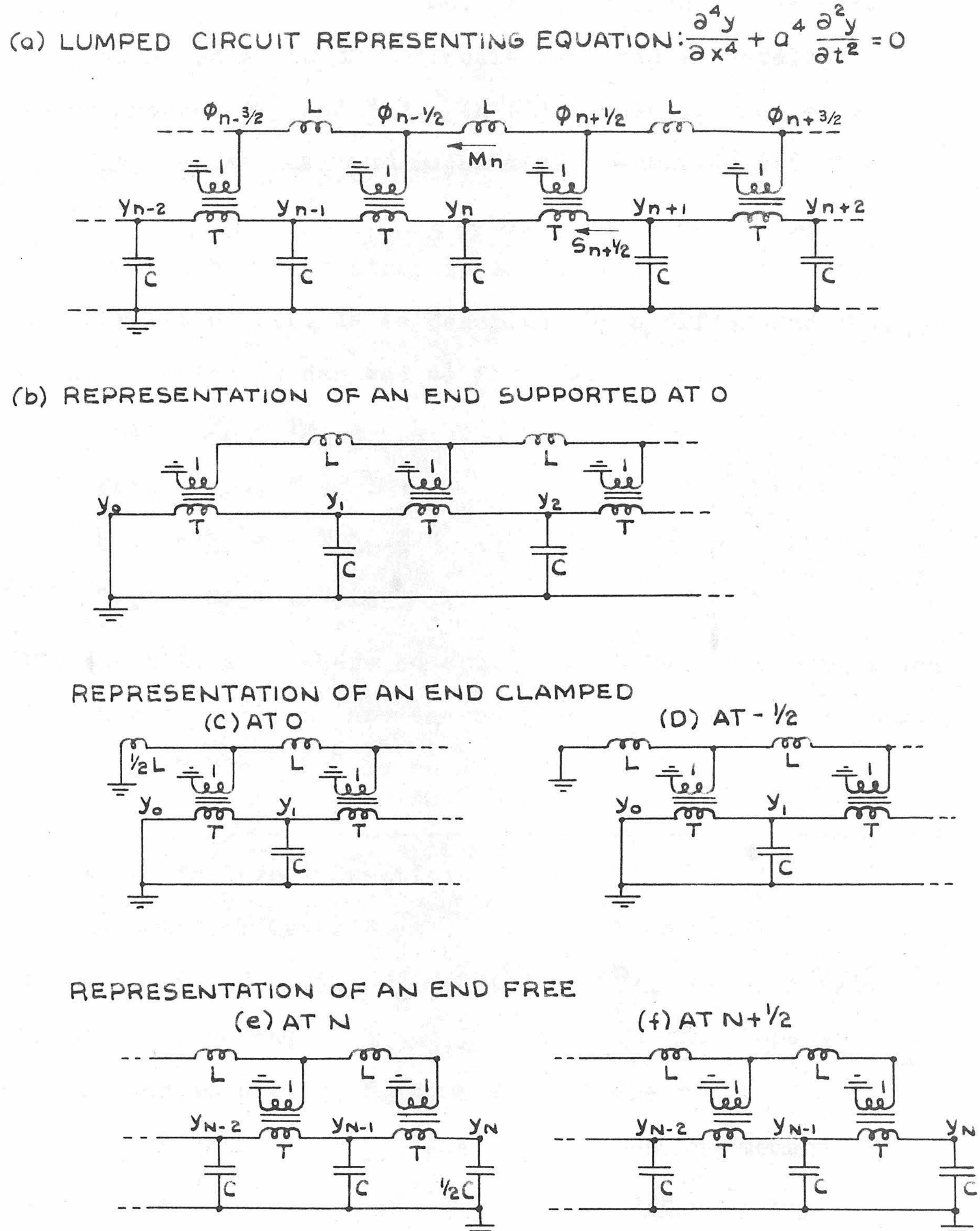

FIGURE 1 ELECTRIC NETWORK FOR THE SOLUTION OF THE UNIFORM BEAM IN BENDING.

the representation of several types of boundary conditions. The theory on which this circuit is based is developed in references (2) and (5). In this analogy, the corresponding electrical and mechanical quantities are given in Table 2.

When subject to sinusoidal electric oscillations, the circuit of Fig. 1a is described by a difference equation which is derived as follows.

$$Y_{n+1} - Y_n = T\underline{\phi}_{n+\frac{1}{2}}$$

$$\underline{\phi}_{n+\frac{1}{2}} - \underline{\phi}_{n-\frac{1}{2}} = j\omega L\underline{M}_n$$

$$\underline{M}_{n+1} - \underline{M}_n = - T\ \underline{S}_{n-\frac{1}{2}}$$

$$\underline{S}_{n+\frac{1}{2}} - \underline{S}_{n-\frac{1}{2}} = j\omega CY_n$$

The quantities in these equations which have subscripts are functions of only n, and are related to the electrical quantities of Table 2 by the relations

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Table 2. Analogous Quantities in the Circuit of Fig. 1a

Electrical Quantities	Mechanical Quantities
Voltages at nodes, y_n, y_{n+1}, etc.	$\frac{\partial y}{\partial t}$, Bending velocity
Voltages at nodes, $\phi_{n-\frac{1}{2}}$, $\phi_{n+\frac{1}{2}}$, etc.	$\frac{\partial \phi}{\partial t} = \frac{\partial^2 y}{\partial x \partial t}$, Slope velocity
Branch currents $S_{n-\frac{1}{2}}$, $S_{n+\frac{1}{2}}$, etc.	Shear
Branch currents M_n, M_{n+1}, etc.	Bending moment
Capacitors C	Mass
Inductors L	Bending flexibility

$$y_n(t) = \Re(Y_n e^{j\omega t})$$

$$\phi_{n+\frac{1}{2}}(t) = \Re(\underline{\phi}_{n+\frac{1}{2}} e^{j\omega t})$$

$$M_n(t) = \Re(\underline{M}_n e^{j\omega t})$$

$$S_{n+\frac{1}{2}}(t) = \Re(\underline{S}_{n+\frac{1}{2}} e^{j\omega t}).$$

$\Re$ means "the real part of." The four difference equations may be reduced to one by substituting for the terms on the left side of each equation according to the preceding equation. The result of this process is

$$Y_{n+2} - 4Y_{n+1} + (6-z^4)\,Y_n - 4Y_{n-1} + Y_{n-2} = 0 \qquad (8)$$

where

$$z = \sqrt{T\omega\sqrt{LC}},$$

and is real and positive. The general solution of this difference equation is

$$Y_n = A \cosh n\theta_1 + B \sinh n\theta_1 + C \cos n\theta_2 + D \sin n\theta_2 \qquad (9)$$

in which θ_1 and θ_2 are determined from the equation

$$2 \sinh \tfrac{1}{2}\theta_1 = 2 \sin \tfrac{1}{2}\theta_2 = z = \sqrt{T\omega\sqrt{LC}}. \qquad (10)$$

The details of this solution are given in Appendix 1. Equations (8) and (9) are valid for $2 \leqq n \leqq N - 2$ when the end conditions are those given in Figs. 1b to 1f. By writing out the equations of these terminating circuits and defining Y_n for $n < 2$ and $N - 2 < n$ by equation (8) or (9) a convenient mathematical statement of the boundary conditions is arrived at:

Fig. 1b. End simply supported at 0 $\qquad Y_o = 0, \quad Y_1 + Y_{-1} = 0 \qquad (11)$

Fig. 1c. End clamped at 0 $\qquad Y_o = 0, \quad Y_1 - Y_{-1} = 0 \qquad (12)$

Fig. 1d. End clamped at $-\frac{1}{2}$ $$Y_o = 0 \quad (13)$$
$$Y_{-1} = 0$$

Fig. 1e. End free at N $$Y_{N+1} - 2Y_N + Y_{N-1} = 0 \quad (14)$$
$$Y_{N+2} - 2Y_{N+1} + 2Y_{N-1} - Y_{N-2} = 0$$

Fig. 1f. End free at $N + \frac{1}{2}$. $$Y_{N+1} - 2Y_N + Y_{N-1} = 0 \quad (15)$$
$$Y_{N+2} - 2Y_{N+1} + Y_N = 0$$

1.3 Solution by Wave Propagation Characteristics. Simply Supported Beam.

From equations (3) and (9) it is observed that the solutions of both the differential and difference equations are composed of two parts, one of which is a sinusoidal function of longitudinal distance. If the entire solution is sinusoidal (in equations (3) and (9) $A = B = 0$) the relation of the frequency of oscillations to the wavelength may be found directly from the differential equation of a uniform medium, or from the difference equation of a periodic structure - in the latter case by a method due to Brillouin.[7] This method involves substituting in the difference equation an assumed solution of sinusoidal form. Applied to equation (8) it consists of carrying out the work of Appendix 1, but considering only those values of θ which are pure imaginaries. This method has been applied to eight different electric-circuit analogies for the beam in bending.*

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* In "An Improved Electrical Analogy for the Analysis of Beams in Bending" by W. T. Russell and R. H. MacNeal. (To be published.)

Referring to the differential equation (2), and its solution, equation (3), and setting $A = B = 0$, wavelength λ and oscillatory frequency are related as follows.

$$\frac{2\pi}{\lambda} = k = \left(\frac{\rho\omega^2}{EI}\right)^{1/4} \tag{16}$$

Similarly, from the difference equation (8) and its solution, equation (9), results the relation

$$\theta_2 = 2 \sin^{-1} \tfrac{1}{2}z = 2 \sin^{-1} \left[\tfrac{1}{2} (T^2\omega^2 LC)^{1/4}\right]. \tag{17}$$

θ_2 is the finite-difference equivalent of k, the wave number.

From equation (4), the simply supported, continuous beam has sinusoidal mode shapes. The equivalent circuit of N cells analogous to this will have the end condition of Fig. 1b at nodes 0 and N. It is apparent from equations (11) that the end conditions will be satisfied by a sinusoidal solution:

$$Y_n = A \sin \frac{mn\pi}{N} \qquad m = 1, 2, 3, \ldots, N - 1 \tag{18}$$

so that the frequency may be obtained from

$$z_m = 2 \sin \tfrac{1}{2}\theta_2 = 2 \sin \frac{m\pi}{2N} = 2 \sin \frac{k_m}{2N} = \frac{k_m}{N} - \frac{k_m^3}{24N^3} + \frac{k_m^5}{1920N^5} - \cdots \tag{19}$$

Except for an over-all factor relating time measurements in the analogous systems, the circuit is constructed to satisfy the equation

$$N^4T^2LC = \frac{\rho}{EI} \tag{20}$$

This allows comparison of the mode frequencies of the two systems

$$\frac{\Delta\omega_m}{\omega_{mo}} = \frac{\omega_m}{\omega_{mo}} - 1 = \frac{z_m^2}{T\sqrt{LC}} \cdot \frac{\sqrt{\rho}}{k_m^2\sqrt{EI}} - 1 = \frac{N^2 z_m^2}{k_m^2} - 1$$

$$\frac{\Delta\omega_m}{\omega_{mo}} = \frac{\sin^2 \frac{k_m}{2N}}{\left(\frac{k_m}{2N}\right)^2} - 1$$

$$\frac{\Delta\omega_m}{\omega_{mo}} = -\frac{k_m^2}{12N^2} + \frac{k_m^4}{360N^4} - \cdots \qquad (21)$$

$$k_m = m\pi \qquad (5)$$

As in the case of the clamped and free ends illustrated in Fig. 1, an analogy can be constructed for a support midway between nodes. The equations for this type of analogy are satisfied by a sinusoidal mode shape. Thus the preceding development is applicable, and equation (21) gives the correct error, provided N represents the effective length in cells.

1.4 Solution by Asymptotic Series. Cantilever Beam.

Unless the boundary conditions permit a sinusoidal solution the method of wave propagation characteristics cannot be used to determine the mode frequencies. For the uniform beam, only the beam with both ends simply supported will have a sinusoidal mode shape. Problems involving other end conditions must be attacked by different methods. The cantilever beam is an important object of study on the electric analog computer. A method of analysis suggested by R. H. MacNeal is employed here to determine the error in mode frequencies of circuit-analogies of the cantilever beam based on Fig. 1, and to compare it with the error in mode frequencies of the simply supported beam of the preceding section.

There are four possible combinations of end conditions given in Fig. 1. These will be designated (ce), (cf), (de), (df), the (ce) beam having the end condition of Figs. 1c and 1e, etc. These are summarized in Table 3, page 15. The first step is to reduce the general solution to the difference equation,

$$Y_n = A \cosh n\theta_1 + B \sinh n\theta_1 + C \cos n\theta_2 + D \sin n\theta_2, \quad (9)$$

and the boundary conditions to a form analogous to equation (7) of the continuous beam. This is essentially a process of browbeating hyperbolic and trigonometric functions. For the first two beams these details are recorded in Appendix 2. For all four beams the resulting equation, giving an implicit expression of the frequency, is obtained and given in Table 4, page 16. The length of the circuit-analogy, measured in cells, is designated M. It will be noticed that the two beams (cf) and (de) have identical equations, and hence will have identical mode frequencies.

It is seen that both members of the equations in Table 4 are functions of z, and hence of ω. The solutions could be found by plotting these functions on a graph and noting the points at which the functions are equal. This method is of limited value as it would have to be repeated for each value of M.

The equations of Table 4 may be put in a form which will make them susceptible to the theory of implicit functions. First, permit M to be continuously variable,

Table 3 Summary of Cantilever Beams Considered

Beam Designation	Clamped at	Free at	M, Length in Cells	Boundary Conditions
(ce)	0	N	N	$Y_0 = 0$ $Y_1 - Y_{-1} = 0$ $Y_{N+1} - 2Y_N + Y_{N-1} = 0$ $Y_{N+2} - 2Y_{N+1} + 2Y_{N-1} - Y_{N-2} = 0$
(cf)	0	$N + \frac{1}{2}$	$N + \frac{1}{2}$	$Y_0 = 0$ $Y_1 - Y_{-1} = 0$ $Y_{N+1} - 2Y_N + Y_{N-1} = 0$ $Y_{N+2} - 2Y_{N+1} + Y_N = 0$
(de)	$-\frac{1}{2}$	N	$N + \frac{1}{2}$	$Y_0 = 0$ $Y_{-1} = 0$ $Y_{N+1} - 2Y_N + Y_{N-1} = 0$ $Y_{N+2} - 2Y_{N+1} + 2Y_{N-1} - Y_{N-2} = 0$
(df)	$-\frac{1}{2}$	$N + \frac{1}{2}$	$N + 1$	$Y_0 = 0$ $Y_{-1} = 0$ $Y_{N+1} - 2Y_N + Y_{N-1} = 0$ $Y_{N+2} - 2Y_{N+1} + Y_N = 0$

Table 4 Implicit Solution of Finite-Difference Cantilever Beam Mode Frequencies

$$2 \sinh \tfrac{1}{2}\theta_1 = 2 \sin \tfrac{1}{2}\theta_2 = z = \sqrt{T\omega\sqrt{LC}}$$

Beam	Implicit Solution
(ce)	$1 + \cosh M\theta_1 \cos M\theta_2 = \sinh M\theta_1 \sin M\theta_2 \tanh \tfrac{1}{2}\theta_1 \tan \tfrac{1}{2}\theta_2$
	$= \sinh M\theta_1 \sin M\theta_2 \dfrac{z^2}{4\sqrt{1 - \frac{z^4}{16}}}$
(cf) and (de)	$1 + \cosh M\theta_1 \cos M\theta_2 = 1 - \cosh \tfrac{1}{2}\theta_1 \cos \tfrac{1}{2}\theta_2$
	$= 1 - \sqrt{1 - \frac{z^4}{16}}$
(df)	$1 + \cosh M\theta_1 \cos M\theta_2 = - \sinh M\theta_1 \sin M\theta_2 \tanh \tfrac{1}{2}\theta_1 \tan \tfrac{1}{2}\theta_2$
	$= - \sinh M\theta_1 \sin M\theta_2 \dfrac{z^2}{4\sqrt{1 - \frac{z^4}{16}}}$
continuous beam	$1 + \cosh k \cos k = 0$

and not restricted to discrete values. Now, as the expansion desired will be for large values of M, replace M by $w^{-\frac{1}{2}}$. Also, for convenience, replace z by $v\sqrt{w}$. By these means, any of the equations of Table 4 may be put in the form

$$f \equiv f(v,w) = 0. \tag{22}$$

Furthermore, each of these is satisfied by

$$f(k_m, 0) = 0, \tag{23}$$

in which k_m is any solution of equation (7).
It is assumed that a solution exists of the form $v = v(w)$. Designating $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$ by subscripts v and w, respectively, and $\frac{dv}{dw}$ and $\frac{d^2v}{dw^2}$ by v' and v'', respectively, successive differentiation of equation (22) with respect to w results in

$$\frac{df}{dw} = f_v v' + f_w = 0 \tag{24}$$

$$\frac{d^2f}{dw^2} = f_{vv}v'^2 + 2f_{vw}v' + f_v v'' + f_{ww} = 0 \tag{25}$$

These equations may be solved for v', v'', etc.

$$v' = -\frac{f_w}{f_v} \tag{26}$$

$$v'' = -\frac{f_{vv}\, v'^2}{f_v} - \frac{2f_{vw}v'}{f_v} - \frac{f_{ww}}{f_v} \tag{27}$$

$$= -\frac{f_{vv}\, f_w^2}{f_v^3} + \frac{2f_{vw}f_w}{f_v^2} - \frac{f_{ww}}{f_v}$$

Then the function v(w) may be written

$$v(w) = k_m + \frac{wv'}{1!} + \frac{w^2v''}{2!} + \cdots \tag{28}$$

v', v'', etc., are to be calculated from equations (27), (28), etc., in which the partial derivatives of f are all evaluated at $(k_m, 0)$. According to the theory of implicit functions [8], this development of $v(w)$ is valid provided $f_v(k_m, 0)$ is different from zero, and f and all its derivatives used in equations (26), (27), etc. exist and are continuous in the neighborhood of $(k_m, 0)$.

For example, applying the transformations stated at the beginning of the preceding paragraph to the second equation of Table 4 results in

$$f(v,w) \equiv \cosh\left(\frac{2}{\sqrt{w}} \sinh^{-1} \frac{v\sqrt{w}}{2}\right) \cos\left(\frac{2}{\sqrt{w}} \sin^{-1} \frac{v\sqrt{w}}{2}\right) + \sqrt{1 - \frac{v^4 w^2}{16}} \qquad (29)$$

It is necessary to restrict this definition of f to values of w other than zero. When $w = 0$,

$$f(v,w) = f(v,0) \equiv \cosh v \cos v + 1 \qquad (30)$$

It is not immediately apparent that this function satisfies the provisions stated at the end of the preceding paragraph, but they may be verified by recourse to the fundamental definition of partial differentiation, or by expansion of the expressions in parentheses in equation (29) in power series in $(v\, w)$. (The existence of such series is of interest, as it shows that the partial derivatives in equations (26) and (27) will exist; that v may be expanded in powers of w, and hence in the even powers of $\frac{1}{M}$; and that z may be expanded in odd powers of $\frac{1}{M}$.)

The solution of a problem by the method of implicit functions will be given later, in Appendix 4, in which a problem from section 1.6 is solved. The present problem, the solution of the equations of Table 4 for mode frequencies, is obtained by a somewhat different, but related method, as follows.

From the above development, and by comparison with the simply supported beam, it is expected that the difference between Mz_m and k_m, defined as

$$u \equiv \frac{Mz-k}{k}, \tag{31}$$

will have an expansion of the type

$$u = \frac{a_1}{M^2} + \frac{a_2}{M^4} + \cdots \tag{32}$$

so that

$$z = \frac{k}{M}(1+u) = \frac{k}{M}\left(1 + \frac{a_1}{M^2} + \frac{a_2}{M^4} + \cdots\right). \tag{33}$$

(Subscripts m, designating mode number, are understood.) The equations of Table 4 are attacked by substituting appropriate double power series in u and $\frac{1}{M^2}$ for the terms z, θ_1, θ_2, etc. The amount of calculation is kept within reason by maintaining a balance among the various orders of infinitesimals consistent with the number of terms a_1, a_2, etc., desired. The zero order terms always cancel by virtue of equation (7). Terms in u, $\frac{1}{M^2}$ constitute the first order infinitesimals. Terms in u^2, $\frac{u}{M^2}$, $\frac{1}{M^4}$ constitute the second order infinitesimals, and so forth. This calculation is carried out for the (ce) beam in Appendix 3, with the first two terms in the expression of error determined.

The results for all the cantilever beams considered are given symbolically in Table 5, page 20, and numerically in Table 6, page 21. Figs. 2 and 3, pages 22 and 23, present comparisons of the calculated errors in cantilever

Table 5 Finite-Difference Cantilever Beam Mode Frequencies

$$\sqrt{T\omega\sqrt{LC}} = z = \frac{k}{M}(1+u) = \frac{k}{M}\left(1 + \frac{a_1}{M^2} + \frac{a_2}{M^4} + \cdots\right)$$

Beam	a_1	a_2
(ce)	$-\frac{k}{24}\cdot\frac{k\alpha - 6\gamma}{\beta}$	$\frac{k^2}{576}\left[\frac{33k^2}{10} - \frac{k^3\gamma}{\beta} + \frac{(2k^2+9k\alpha-12\gamma)(k\alpha-6\gamma)}{\beta^2} + \frac{k\gamma(k\alpha-6\gamma)^2}{\beta^3}\right]$
(cf) and (de)	$-\frac{k^2\alpha}{24\beta}$	$\frac{k^3}{576}\left[-\frac{27k}{10} + \frac{18-k^2\gamma}{\beta} + \frac{k(2k+3\alpha)\alpha}{\beta^2} + \frac{k^2\gamma\alpha^2}{\beta^3}\right]$
(df)	$-\frac{k}{24}\cdot\frac{k\alpha+6\gamma}{\beta}$	$\frac{k^2}{576}\left[-\frac{87k^2}{10} - \frac{k^3\gamma}{\beta} + \frac{(2k^2-3k\alpha+12\gamma)(k\alpha+6\gamma)}{\beta^2} + \frac{k\gamma(k\alpha+6\gamma)^2}{\beta^3}\right]$

α = tan k + tanh k

β = tan k - tanh k

γ = sinh k sin k

k = eigenvalue of continuous beam (see Table 1)

M = length in cells

Table 6 Finite-Difference Cantilever Beam Mode Frequencies

$$z = \frac{k}{M}\left(1 + \frac{a_1}{M^2} + \frac{a_2}{M^4} + \dots\right). \qquad \frac{\Delta\omega}{\omega_0} = \frac{b_1}{M^2} + \frac{b_2}{M^4} + \dots \qquad b_1 = 2a_1$$

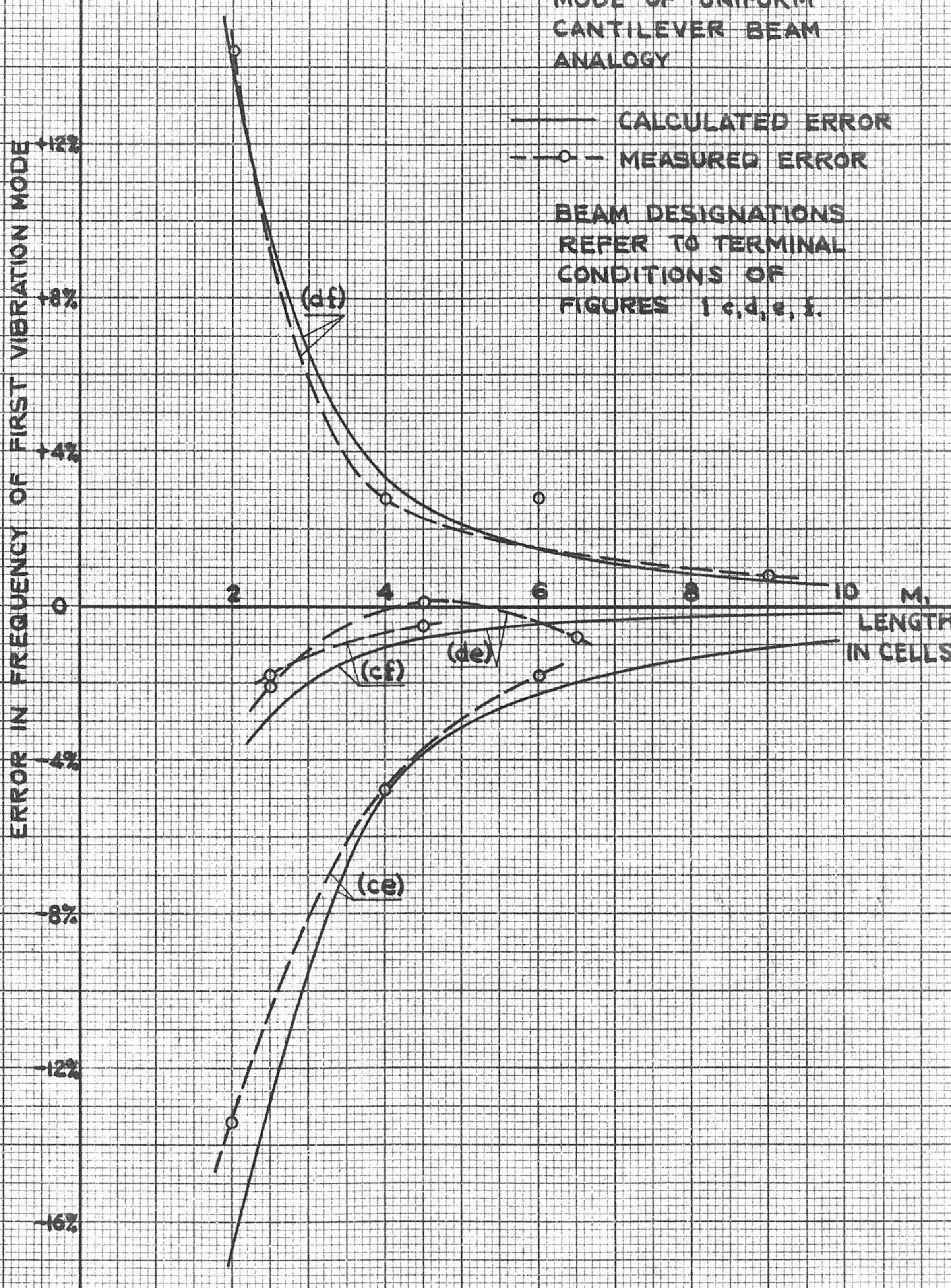
$$b_2 = a_1^2 + 2a_2$$

Beam	Mode	a_1	a_2	b_1	b_2
(ce)	1	-.423	.224	-.846	.628
(ce)	2	-2.15	8.64	-4.35	21.9
(cf) and (de)	1	-.0789	-.0757	-.158	-.145
(cf) and (de)	2	-.952	.359	-1.90	1.62
(df)	1	.265	.0254	.530	.121
(df)	2	.243	-2.35	.486	-4.64

Table 7 Limiting Values of the Terms Giving the Higher Mode Frequencies of Finite-Difference Cantilever Beams

Beam	a_1	a_2
(ce)	$-\frac{k(k+6)}{24}$	$\frac{k^2(k^2+100k+240)}{1920}$
(cf) and (de)	$-\frac{k^2}{24}$	$\frac{k^4}{1920}$
(df)	$-\frac{k(k-6)}{24}$	$\frac{k^2(k^2-100k+240)}{1920}$

FIGURE 2 ERRORS IN FREQUENCY OF FIRST VIBRATION MODE OF UNIFORM CANTILEVER BEAM ANALOGY



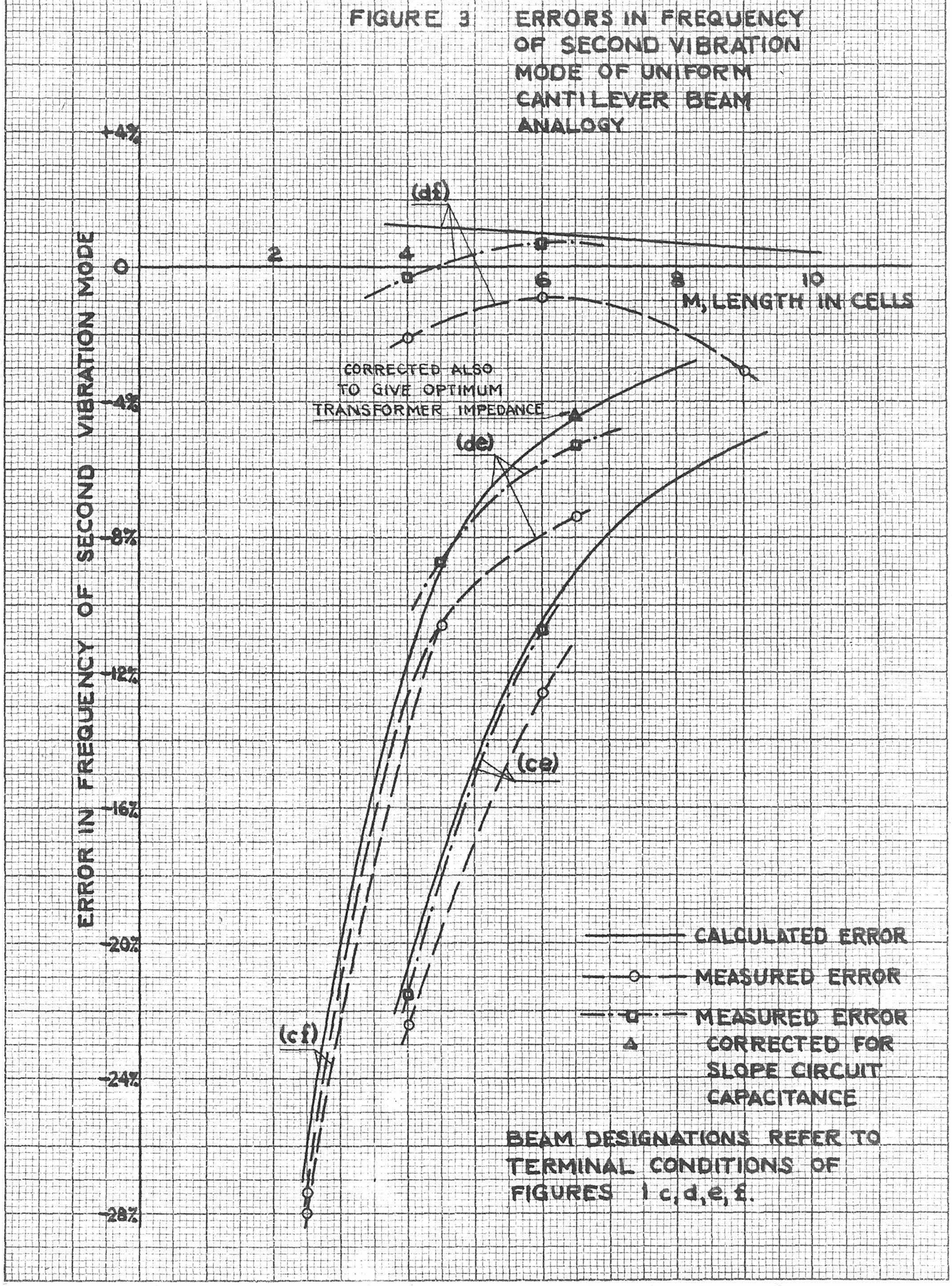


FIGURE 3 ERRORS IN FREQUENCY OF SECOND VIBRATION MODE OF UNIFORM CANTILEVER BEAM ANALOGY
ERROR IN FREQUENCY OF SECOND VIBRATION MODE
+4%
0
-4%
-8%
-12%
-16%
-20%
-24%
-28%
2
4
6
8
10
M, LENGTH IN CELLS
(df)
(de)
(ce)
(cf)
CORRECTED ALSO TO GIVE OPTIMUM TRANSFORMER IMPEDANCE
CALCULATED ERROR
MEASURED ERROR
MEASURED ERROR CORRECTED FOR SLOPE CIRCUIT CAPACITANCE
BEAM DESIGNATIONS REFER TO TERMINAL CONDITIONS OF FIGURES 1 c, d, e, f.

beam mode frequencies and experimental errors determined by C. H. Wilts, for the first and second modes, respectively. The experimental data of Fig. 3 were obtained at a computer frequency higher than optimum, at which the effect of parasitic capacitance and inductance was appreciable. Consequently points were also obtained in which a correction was made to account for this effect.

A comparison of the error of mode frequencies of the cantilever beam with that of the simply supported beam is made by comparing the coefficients of $(k/M)^2$ in the expansions of either u or $(\Delta\omega/\omega_0)$. The coefficients of $(k/M)^2$ rather than of $(1/M)^2$ are considered because the former gives the error in terms of the number of cells per wavelength of the sinusoidal part of the solution. On this basis, it is found that, for the first mode, only the (cf) and (de) cantilever beams have less error than the simply supported beam, and for the second mode, only the (df) cantilever beam has less error than the simply supported beam.

In the higher bending modes of the uniform cantilever beam, the sinusoidal part of the solution predominates in the mode shape. Hence it is expected that the errors in the higher mode frequencies will approximate those in the simply supported beam. To determine if this is true, it is noticed that as k increases, some of the terms of the expressions in Table 4 increase exponentially, so that the remaining terms become insignificant by comparison. Specifically

$$\beta^{-1} \rightarrow (-1)^m 2e^{-k}$$

$$\frac{\alpha}{\beta} \rightarrow 1$$

$$\frac{\gamma}{\beta} \rightarrow -1$$

$$k_m \rightarrow (2m-1)\tfrac{1}{2}\pi$$

In this manner, the limiting values of a_1 and a_2 are determined. These are given in Table 7, page 21. Because of the exponential nature of the terms involved, the convergence to these forms is very rapid. By comparison with equation (19), it is seen that the error of the higher mode frequencies of the cantilever beam does approach that of the simply supported beam, although less rapidly for (ce) and (df) beams than for the (cf) and (de) beams.

The method of reducing the difference equations of a circuit-analogy to a single equation analogous to the eigenvalue equation of the continuous system, and the solution of this equation in a series form, have been applied in this section to one set of end conditions of one beam amalogy. It may also be applied to other sets of end conditions, and to other circuit-analogies of the uniform beam. (See reference[3] and the footnote, p. 11). Another application is the analysis of the effect of unequal lumping, an example of which is presented later in Part I.

It is questionable whether one number alone can be a fair index of the accuracy of a particular type of analogy. For example, by the method of wave propagation characteristics, the accuracy of the simply supported beam is determined. But when the end conditions are changed to make the circuit analogous to a cantilever beam, the accuracy is made significantly better or worse, depending on the particular end analogies used. Thus it appears that the accuracy of the analogy of the beam exclusive of end conditions is best expressed by the entire range of values it may take as the various combinations of end conditions are applied.

It is evident that there is some correlation between the error in mode frequencies and the form of the circuit termination. Although it is not possible, for a single circuit, to separate the part of the error arising from lumping from the part due to end conditions, it is possible to determine the contribution to the error caused by a change in circuit termination.

For example, in the case of the cantilever beam studied, reference to Table 5 shows that replacing the representation of the clamped end (c) by (d) results in a change in u of $-\frac{\gamma k}{4\beta}\cdot\frac{1}{M^2}$, regardless of whether the free end is represented by (e) or (f). (Only the first term in the expansion of u is considered.) Similarly replacing (e) by (f) results in a change in u of $-\frac{\gamma k}{4\beta}\cdot\frac{1}{M^2}$, regardless of whether the clamped end analogy is (c) or (d). Since the change due to varying the representation of one end is independent of the other end, the effects of changing the analogies at the two ends may be superimposed.

1.5 The Second Order Eigenvalue Problem.

The equation considered here is the wave equation in one dimension,

$$\frac{\partial^2 y}{\partial x^2} - a^2 \frac{\partial^2 y}{\partial t^2} = 0 \qquad (34)$$

This equation describes numerous physical phenomena, including the vibration of a uniform string, longitudinal and torsional wave motion in an elastic prismatic bar, and motion of electric waves along a transmission line. As in the fourth order problem, steady state oscillatory motion is assumed. Again $y(x,t)$ is replaced by $Y(x)e^{j\omega t}$, and $\frac{\partial^2 y}{\partial t^2}$ by $-\omega^2 Y e^{j\omega t}$. Equation (34) becomes

$$\frac{d^2 Y}{dx^2} + a^2 \omega^2 Y = 0 \qquad (35)$$

The general solution is

$$Y = A \cos a\omega x + B \sin a\omega x. \qquad (36)$$

Two sets of boundary conditions will be considered, giving solutions as follows.

(a) $Y = 0$ at $x = 0$. $Y = 0$ at $x = 1$.

$$Y = A_m \sin m\pi x \qquad (37)$$

$$\omega_{mo} = \frac{k_m}{a}, \; k_m = m\pi, \; m = 1, 2, 3, \ldots \qquad (38)$$

(b) $Y = 0$ at $x = 0$. $\frac{dY}{dx} = 0$ at $x = 1$.

$$Y = A_m \sin (2m-1) \tfrac{1}{2}\pi x \qquad (39)$$

$$\omega_{mo} = \frac{k_m}{a}, \; k_m = (2m-1)\tfrac{1}{2}\pi, \quad m = 1, 2, 3, \ldots \qquad (40)$$

m is the mode number.

Several circuits are developed to represent equation (34), approximating differentiation with respect to x by finite differences in accordance with the principles set forth in Part I of reference (2). These are given in Fig. 4, page 29. In these circuits, the node voltages are analogous to $\frac{\partial y}{\partial t}$. The circuit embodying uniform cell size (Fig. 4a) may be solved by standard methods.

The difference equations are, from the summation of currents leaving node n,

$$- Y_{n+1} + (2-z^2) Y_n - Y_{n-1} = 0 \qquad (41)$$

in which

$$z = \omega\sqrt{LC}.$$

The solution is

$$Y_n = A \cos n\theta + B \sin n\theta, \qquad (42)$$

where

$$z = 2 \sin \tfrac{1}{2}\theta. \qquad (43)$$

With boundary conditions as given in Fig. 4d, equation (42) is valid for $1 \leqq n \leqq N-1$. By writing out the equations of the circuits of Fig. 4d and defining Y_0, Y_{-1}, Y_N, and Y_{N+1} by equation (42), this mathematical statement of the boundary equations is obtained:

$$\left.\begin{array}{ll} Y = 0 \text{ at } 0 & Y_0 = 0 \\ Y = 0 \text{ at } -\tfrac{1}{2} & Y_0 + Y_{-1} = 0 \\ \frac{dY}{dx} = 0 \text{ at } N & Y_{N+1} - Y_{N-1} = 0 \\ \frac{dY}{dx} = 0 \text{ at } N+\tfrac{1}{2} & Y_{N+1} - Y_N = 0 \end{array}\right\} \qquad (44)$$

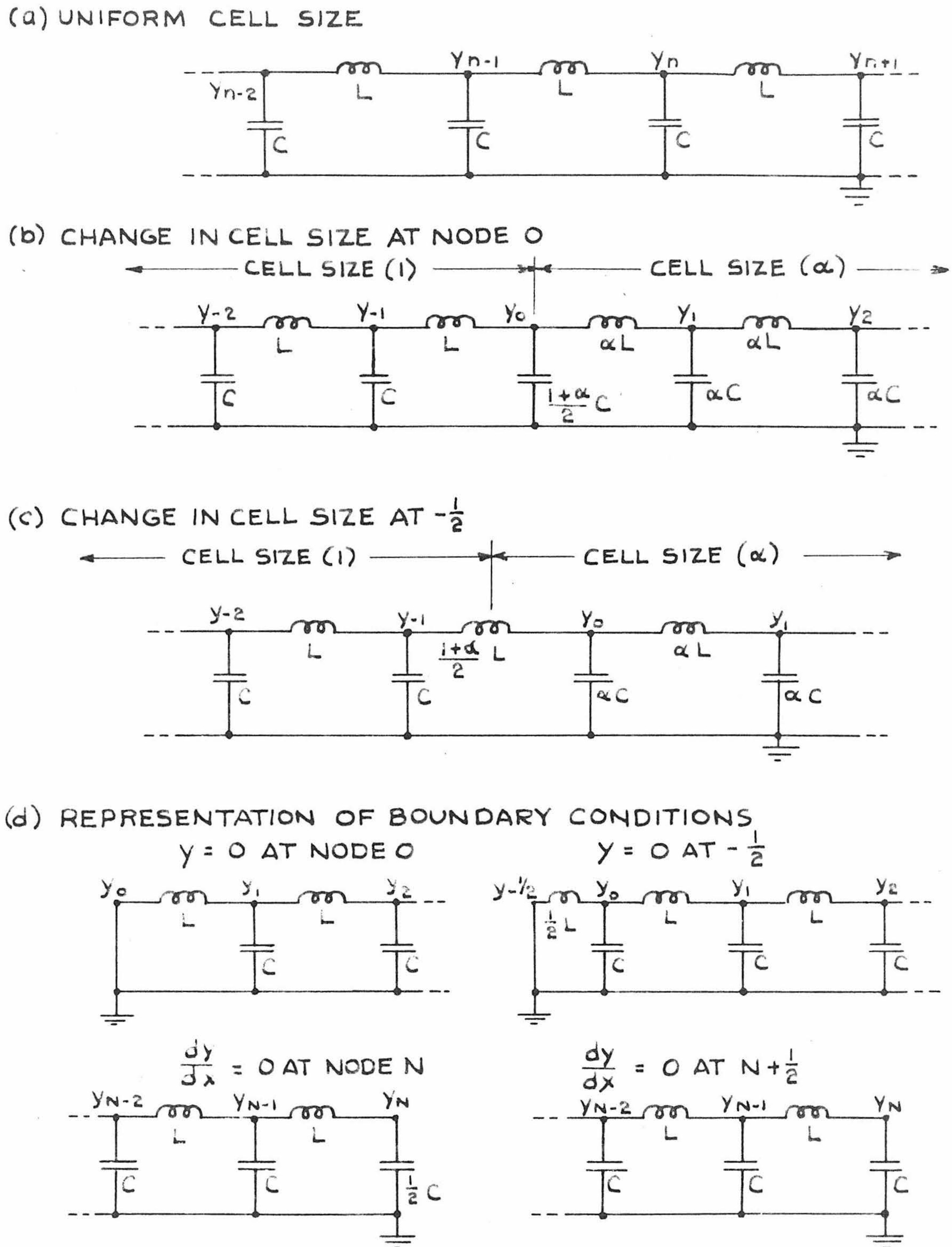


FIGURE 4 ELECTRICAL NETWORKS FOR THE SOLUTION OF THE EQUATION $\frac{\partial^2 y}{\partial x^2} - a^2 \frac{\partial^2 y}{\partial t^2} = 0$

Considering the same boundary conditions as in the case of the continuous beam, the solutions are

(a) $Y = 0$ at 0. $Y = 0$ at N.

$$Y_n = A_m \sin \frac{mn\pi}{N} \qquad m = 1, 2, 3, .., N-1 \qquad (45)$$

$$\omega_m \sqrt{LC} = z_m = 2 \sin \frac{m\pi}{2N} \qquad (46)$$

(b) $Y = 0$ at 0. $\frac{dY}{dx} = 0$ at N.

$$Y_n = A_m \sin \frac{(2m-1)n\pi}{2N} \qquad m = 1, 2, 3, .., N \qquad (47)$$

$$\omega_m \sqrt{LC} = 2 \sin \frac{(2m-1)\pi}{4N} \qquad (48)$$

Except for over-all factors relating time and impedance measurements of the analogous systems,

$$N^2 LC = a^2 \qquad (49)$$

Using this relation, the mode frequencies may be compared with those of the continuous system, and the error due to lumping is determined.

(a) $$\omega_m = \frac{2N}{a} \sin \frac{m\pi}{2N}$$

$$\frac{\Delta\omega_m}{\omega_{mo}} = \frac{\omega_m}{\omega_{mo}} - 1 = \frac{\sin \frac{m\pi}{2N}}{\frac{m\pi}{2N}} - 1 = \frac{\sin \frac{k_m}{2N}}{\frac{k_m}{2N}} - 1$$

$$= -\frac{k_m^2}{24N^2} + \frac{k_m^4}{1920N^4} - \cdots \qquad (50)$$

(b) $$\omega_m = \frac{2N}{a} \sin \frac{(2m-1)\pi}{4N}$$

$$\frac{\Delta\omega_m}{\omega_{mo}} = \frac{\omega_m}{\omega_{mo}} - 1 = \frac{\sin \frac{(2m-1)\pi}{4N}}{\frac{(2m-1)\pi}{4N}} - 1 = \frac{\sin \frac{k_m}{2N}}{\frac{k_m}{2N}} - 1$$

$$= -\frac{k_m^2}{24N^2} + \frac{k_m^4}{1920N^4} - \cdots \qquad (51)$$

1.6 Unequal Lumping.

Figs. 4b and 4c give circuit analogies of the system of equation (34) in which the cell size is not uniform.

Each cell on the right side of the circuit represents a segment of the physical system which is α times as long as that represented by a cell on the left side. The method of section 1.4 can be used to determine the mode frequencies of circuits of this type.

The circuits of Figs. 4b and 4c are described by two difference equations. To the left of the change in cell size

$$- Y_{n-1} + (2-z^2) Y_n - Y_{n+1} = 0 \tag{52}$$

$$Y_n = A \cos n\theta_1 + B \sin n\theta_1 \tag{53}$$

where

$$2 \sin \tfrac{1}{2} \theta_1 = z = \omega\sqrt{LC}. \tag{54}$$

To the right of the change in cell size

$$-Y_{n-1} + (2-\alpha^2 z^2) Y_n - Y_{n+1} = 0 \tag{55}$$

$$Y_n = C \cos n\theta_2 + D \sin n\theta_2 \tag{56}$$

where

$$2 \sin \tfrac{1}{2}\theta_2 = \alpha z = \alpha\omega\sqrt{LC}. \tag{57}$$

Two circuits are considered which have unequal cell size, and which are analogs of problem (a) of the preceding section. Both are constructed from the circuits of Fig. 4, as follows:

circuit (i)

$Y = 0$	at $-N_1$
Relative cell size 1	
Change in cell size	at 0
Relative cell size α	
$Y = 0$	at N_2

circuit (ii)

$Y = 0$	at $-N_1 - \frac{1}{2}$

Relative cell size 1
Change in cell size at $-\frac{1}{2}$
Relative cell size α
$Y = 0$ at $N_2 - \frac{1}{2}$

Two circuits are also considered which are analogs of problem (b) of the preceding section, and are constructed as follows:

circuit (iii)

$Y = 0$ at $-N_1$
Relative cell size 1
Change in cell size at 0
Relative cell size α
$\frac{dY}{dx} = 0$ at N_2

circuit (iv)

$Y = 0$ at $-N_1 - \frac{1}{2}$
Relative cell size 1
Change in cell size at $-\frac{1}{2}$
Relative cell size α
$\frac{dY}{dx} = 0$ at $N_2 - \frac{1}{2}$

The end conditions are determined from equations (44). The mathematical description of the change in cell size is determined by summing current flow from each node not satisfied by equations (52) or (55). In circuits (i) and (iii), employing Fig. 4b, this is done for node 0:

$$-Y_{-1} + \left(\frac{1+\alpha}{\alpha} - \frac{1+\alpha}{2} z^2\right) Y_0 - \frac{1}{\alpha} Y_1 = 0 \qquad (58)$$

where Y_0 satisfies both of equations (53) and (56). In circuits (ii) and (iv), employing Fig. 4c, equations are written for nodes -1 and 0;

$$-Y_{-2} + \left(\frac{3+\alpha}{1+\alpha} - z^2\right) Y_{-1} - \frac{2}{1+\alpha} Y_0 = 0 \qquad (59)$$

$$-\frac{2\alpha}{1+\alpha} Y_{-1} + \left(\frac{1+3\alpha}{1+\alpha} - z^2\right) Y_0 - Y_1 = 0 \qquad (60)$$

where Y_{-1} is defined by equation (53) and Y_0 by equation

(56). By making use of equations (44) and either equation (58) or equations (59) and (60), difference-equation solutions, which are implicit solutions for mode frequencies, are obtained as given in Table 8, page 34. In the case of circuit (i) the details of this procedure are carried out in Appendix 4. The methods of section 1.4 are used to determine the first term in the expansion of the error in mode frequencies, the results of which are given in Table 9, page 34. Again, in the case of circuit (i), this procedure is recorded in Appendix 4.

It is not proposed to analyze these results here to determine optimum circuitry, but merely to list them as an example of the method of solution for mode frequency error by series expansion.

Table 8 Difference Equation Solutions. Unequal Lumping.

Circuit	Implicit Solution
(i)	$\sin(N_1\theta_1 + N_2\theta_2) + \nu \sin N_1\theta_1 \cos N_2\theta_2 = 0$
(ii)	$\sin(N_1\theta_1 + N_2\theta_2) + \nu \cos N_1\theta_1 \sin N_2\theta_2 = 0$
(iii)	$\cos(N_1\theta_1 + N_2\theta_2) - \nu \sin N_1\theta_1 \sin N_2\theta_2 = 0$
(iv)	$\cos(N_1\theta_1 + N_2\theta_2) + \nu \cos N_1\theta_1 \cos N_2\theta_2 = 0$

$$\omega\sqrt{LC} = z = 2 \sin \tfrac{1}{2}\theta_1 = \frac{2}{\alpha} \sin \tfrac{1}{2}\theta_2$$

$$\nu = \frac{\cos \frac{1}{2}\theta_2}{\cos \frac{1}{2}\theta_1} - 1 = (1-\alpha^2)\left(\frac{1}{8} z^2 + \frac{3+\alpha^2}{128} z^4 + \cdots\right)$$

Table 9 Mode Frequencies of Circuits with Unequal Lumping

$$\omega\sqrt{LC} = z = \frac{k}{M}(1+u) = \frac{k}{M}\left(1 + \frac{a_1}{M^2} + \frac{a_2}{M^4} + \cdots\right)$$

Circuit	a_1
(i)	$-\frac{k}{24}\left\{\left[\gamma_1(1-\alpha^2) + \alpha^2\right] k + \frac{3}{2}(1-\alpha^2) \sin 2\gamma_1 k\right\}$
(ii)	$-\frac{k}{24}\left\{\left[\gamma_1(1-\alpha^2) + \alpha^2\right] k - \frac{3}{2}(1-\alpha^2) \sin 2\gamma_1 k\right\}$
(iii)	$-\frac{k}{24}\left\{\left[\gamma_1(1-\alpha^2) + \alpha^2\right] k + \frac{3}{2}(1-\alpha^2) \sin 2\gamma_1 k\right\}$
(iv)	$-\frac{k}{24}\left\{\left[\gamma_1(1-\alpha^2) + \alpha^2\right] k - \frac{3}{2}(1-\alpha^2) \sin 2\gamma_1 k\right\}$

$M = N_1 + \alpha N_2$

$\gamma_1 = \frac{N_1}{M}$

(i) and (ii): $k_m = m\pi \qquad m = 1, 2, 3, \ldots, N_1 + N_2$

(iii) and (iv): $k_m = \frac{2m-1}{2}\pi \qquad m = 1, 2, 3, \ldots, N_1 + N_2$

II THE EFFECT OF STATISTICAL DEVIATION OF THE VALUES OF THE ELEMENTS

2.1 General

The analyses of Part I are all concerned with the accuracy of solutions. Given a circuit-analogy of a physical or mathematical system, and a statement of what constitutes the answer, they determine what error will exist. It is assumed that any time the circuit is constructed in the computer, each element will be exactly what the circuit diagram prescribes, and the answer, and hence the error, will be the same each time.

Part II is concerned with the precision of solutions. Considering that the values of the elements are not exactly their nominal values, and that the elements are not identical, it is apparent that when a circuit is constructed in the computer by selecting computer elements at random, successive constructions of the same circuit will not give identical answers. Even if the elements are set to assigned values with the aid of a meter, their actual values will be subject to a deviation depending on the precision of the meter, and may be treated as if coming from an original population with this precision.

The precision with which an answer is determined depends on the precision of the value of each element in the circuit and on the extent to which changes in the value of the element affect the answer. Specifically, if an answer f depends on various circuit parameters x_1, x_2, ... , x_N according to the relation

$$f = f\,(x_1,\ x_2,\ \ldots\ ,\ x_N), \tag{1}$$

and the circuit parameters are selected from populations with uncorrelated distributions having r.m.s. deviations $\sigma_1, \sigma_2, \ldots, \sigma_N$ about means $\overline{x}_1$, $\overline{x}_2$, ..., $\overline{x}_N$, respectively, then f is characterized by a mean

$$\overline{f} = f\,(\overline{x}_1,\ \overline{x}_2,\ \ldots,\ \overline{x}_N) \tag{2}$$

and an r.m.s. deviation

$$\sigma_f = \left[\sum_{n=1}^{N} \left(\sigma_n \frac{\partial f}{\partial x_n} \right)^2 \right]^{\frac{1}{2}} \tag{3}$$

in which the partial derivatives are evaluated at ($\overline{x}_1$, $\overline{x}_2$, ..., $\overline{x}_N$).[9] It is assumed that σ_n are small so that the Taylor series expansion of f about $\overline{f}$ is sufficiently accurate with only the linear terms present.

Determination of the precision of answers by equation (3) does not directly require knowledge of the answer, but it does require knowledge of the partial derivatives of the answer with respect to the parameters. Once again eigenvalue problems are convenient for the determination of precision. Regarding the mode frequencies as answers, the partial derivatives can be obtained by a method due to Rayleigh.[10] This method relates increments in mode frequencies to increments in the potential and kinetic energies associated with given mode shapes.

When applied to a conservative electric circuit, electrostatic and electromagnetic energies are considered.

Assuming that a particular mode shape, consisting of node voltages V_1, V_2, ..., V_i, ..., V_N, is known for a given electrical system, then the maximum instantaneous electrostatic energy associated with this mode is

$$W_e = \frac{1}{2}\sum C_j V_j^2 \tag{4}$$

and the maximum instantaneous magnetic energy is

$$W_m = \frac{1}{2}\sum L_j I_j^2 = \frac{1}{2\omega^2}\sum \frac{V_j^2}{L_j} \tag{5}$$

The terms V_j are the voltages across the elements C_j, L_j, and are determined from the known voltages V_i. For the purposes of this development, any energy associated with mutual inductances is included in equation (5) by replacing them in the circuit by equivalent self-inductance. In normal mode oscillation of an electric circuit, the energy is at times entirely magnetic, and at other times entirely electrostatic. Thus the normal mode frequency may be determined by equating W_e and W_m.

$$\omega^2 = \frac{\sum \frac{V_j^2}{L_j}}{\sum C_j V_j^2} \tag{6}$$

To determine the partial derivatives of ω with respect to the circuit parameters L_j, C_j, it is assumed for the moment that the voltages defining the mode shape, V_i, and hence V_j, remain unchanged.

$$\frac{\partial\omega}{\partial C_r} = \frac{1}{2\omega}\frac{\partial(\omega^2)}{\partial C_r} = -\frac{V_r^2 \sum \frac{V_j^2}{L_j}}{2\omega\left(\sum C_j V_j^2\right)^2} = -\frac{\omega}{2C_r}\frac{C_r V_r^2}{\sum C_j V_j^2} \tag{7}$$

$$\frac{\partial \omega}{\partial L_r} = \frac{1}{2\omega}\frac{\partial(\omega^2)}{\partial L_r} = -\frac{1}{2\omega L_r^2}\frac{V_r^2}{\sum C_j V_j^2} = -\frac{\omega}{2L_r}\frac{\frac{V_r^2}{L_r}}{\sum \frac{V_j^2}{L_j}} \tag{8}$$

The expressions of equations (7) and (8) may be used directly in equation (3).

The assumption of unchanged mode shape bears investigation. Following Rayleigh's development, (10) let the normal coordinates of the given system be ϕ_1, ϕ_2, ..., ϕ_N. While considering one mode, the q^{th}, for example, change one of the elements, C_r, for example, by a small amount μC_r. Then it is shown that the shape of the new normal vibration mode may be expressed as

$$\phi_q' = \mu_1\phi_1 + \mu_2\phi_2 + \cdots + \phi_q + \cdots + \mu_N\phi_N \tag{9}$$

in which the μ_i's are of the same order as μ. However, it is shown also that ω^2 as calculated by equation (6) has a stationary value when the mode shape is that of a normal mode. By equation (9) the mode shape is changed by terms of order μ, so that ω^2 will experience change of order μ^2. Thus, in differentiating equation (6) to obtain equations (7) and (8), taking the limit $\Delta C_r \rightarrow 0$ or $\Delta L_r \rightarrow 0$ insures that all change in ω is directly due to the change in the value of the element, and not to the change in mode shape.

The method developed here is used to determine the precision with which normal vibration frequencies may be determined by the use of some of the circuits considered in Part I.

2.2 The Second Order Eigenvalue Problem

The example considered here is the circuit of Fig. 4a with $V = 0$ at nodes 0 and N. The normal mode solutions are obtained from equations (45) and (46), section 1.5.

$$V_n = \sin \frac{mn\pi}{N} \qquad m = 1, 2, 3, \ldots, N-1 \tag{10}$$

$$\omega_m \sqrt{LC} = z_m = 2 \sin \frac{m\pi}{2N} \tag{11}$$

Considering $L_{n+\frac{1}{2}}$ to be the inductor connecting nodes n and n+1, and $V_{n+\frac{1}{2}}$ the voltage across $L_{n+\frac{1}{2}}$,

$$V_{n+\frac{1}{2}} = V_{n+1} - V_n = \sin \frac{m(n+1)\pi}{N} - \sin \frac{mn\pi}{N}$$

$$V_{n+\frac{1}{2}} = 2 \sin \frac{m\pi}{2N} \cos \frac{m(n+\frac{1}{2})\pi}{N} \tag{12}$$

It is necessary to determine the summations employed in equations (7) and (8)

$$\sum C_j V_j^2 = C \sum_{n=1}^{N-1} \sin^2 \frac{mn\pi}{N} = \frac{CN}{2} \tag{13}$$

$$\sum \frac{V_j^2}{L_j} = \frac{4}{L} \sin^2 \frac{m\pi}{2N} \sum_{n=0}^{N-1} \cos^2 \frac{m(n+\frac{1}{2})\pi}{N} = \frac{2N}{L} \sin^2 \frac{m\pi}{2N} \tag{14}$$

The details of the summations are given in Appendix 5. Substituting in equations (7) and (8),

$$\frac{\partial \omega}{\partial C_r} = -\frac{\omega}{NC} \sin^2 \frac{mr\pi}{N} \tag{15}$$

$$\frac{\partial \omega}{\partial L_{r+\frac{1}{2}}} = -\frac{\omega}{NL} \cos^2 \frac{m(r+\frac{1}{2})\pi}{N} \tag{16}$$

It is now assumed that each of the capacitors C_r is chosen from a large population having a mean capacitance C and r.m.s. deviation $\sigma_c = \epsilon_c C$. Similarly, the inductors are distributed about a mean inductance L with r.m.s. deviation

$\sigma_L = \epsilon_L L$. Equations (15) and (16) may now be substituted in equation (3).

$$\sigma_\omega^2 = \epsilon_c^2 C^2 \frac{\omega^2}{N^2C^2} \sum_{r=1}^{N-1} \sin^4 \frac{mr\pi}{N} + \epsilon_L^2 L^2 \frac{\omega^2}{N^2L^2} \sum_{r=0}^{N-1} \cos^4 \frac{m(r+\frac{1}{2})\pi}{N}$$

$$\sigma_\omega^2 = \omega^2 \frac{3}{8N} (\epsilon_c^2 + \epsilon_L^2) \qquad m = 1, 2, \ldots, N-1; \; m \neq \frac{N}{2} \qquad (17)$$

$$\sigma_\omega^2 = \omega^2 \frac{1}{N}\left(\frac{1}{2}\epsilon_c^2 + \frac{1}{4}\epsilon_L^2\right) \qquad m = \frac{N}{2} \qquad (18)$$

The details of these summations are also given in Appendix 5. If $\epsilon_c = \epsilon_L = \epsilon$, then

$$\sigma_\omega = \omega\sqrt{\frac{3\epsilon^2}{4N}} = \omega \frac{\epsilon}{2}\sqrt{\frac{3}{N}} \qquad m = 1, 2, \ldots, N-1 \qquad (19)$$

The result confirms expectation that the deviation of the mode frequency is proportional to the mode frequency and the per unit deviation of the elements, and inversely proportional to the square root of the number of cells used.

From equation (50), section 1.5, it is seen that the error due to lumping diminishes as $\frac{1}{N^2}$, much faster than the statistical deviation. The two effects will be equal when N is given by

$$N_o = \left(\frac{m^4 \pi^4}{432\,\epsilon^2}\right)^{\frac{1}{3}} \qquad (20)$$

In the California Institute computer, ϵ will be approximately .01, if pains are not taken to select special elements. In this case $N_o = 13.1$ for the first mode. For this number of cells,

$$\frac{\sigma_\omega}{\omega} = \frac{\Delta\omega_1}{\omega_1} = .0024 \qquad (21)$$

It is possible that, because of aging, parasitic effects, or other reasons, the mean of the values of the elements C_r may be $C + \delta C$ instead of C, and of the elements

L_r, $L + \delta L$. Assuming that δC and δL are small compared with C and L, the expressions determined for σ_ω will still apply. However, the mean frequency, $\omega_m + \delta\omega$, will differ from the calculated frequency for the circuit, ω_m.

$$\omega_m + \delta\omega = \frac{2 \sin \frac{m\pi}{2N}}{\sqrt{(L+\delta L)(C+\delta C)}}$$

$$\frac{\delta\omega}{\omega_m} \cong - \frac{\delta L}{2L} - \frac{\delta C}{2C} \qquad (22)$$

2.3 <u>The Fourth-Order Eigenvalue Problem</u>

The problem is that of the simply supported beam, taken up in section 1.3. The circuit is that of Figs. 1a and 1b, with supports at nodes 0 and N. The node voltages, V_n, given in equation (18) of section 1.3, are the same as those in the problem just considered. v_n, the voltages across the elements L_n, are determined from Fig. 1a.

$$V_n = \sin \frac{mn\pi}{N} \qquad m = 1, 2, \ldots, N-1 \qquad (23)$$

$$v_n = \phi_{n+\frac{1}{2}} - \phi_{n-\frac{1}{2}} = \frac{1}{T} (V_{n+1} - 2V_n + V_{n-1})$$

$$v_n = -\frac{1}{T} z^2 \sin \frac{mn\pi}{N} = - \omega\sqrt{LC} \sin \frac{mn\pi}{N} \qquad (24)$$

The last relation is from equation (11) of Appendix 2.

$$\sum C_j V_j^2 = C \sum_{n=1}^{N-1} \sin^2 \frac{mn\pi}{N} = \frac{CN}{2} \qquad (25)$$

$$\sum \frac{v_j^2}{L_j} = \omega^2 C \sum_{n=1}^{N-1} \sin^2 \frac{mn\pi}{N} = \omega^2 C \frac{N}{2} \qquad (26)$$

$$\frac{\partial\omega}{\partial C_r} = - \frac{\omega}{NC} \sin^2 \frac{mr\pi}{N} \qquad (27)$$

$$\frac{\partial\omega}{\partial L_r} = - \frac{\omega}{NL} \sin^2 \frac{mr\pi}{N} \qquad (28)$$

Again it is assumed that $\sigma_c = \epsilon_c C$, $\sigma_L = \epsilon_L L$. From equation (3) and Appendix 5,

$$\sigma_\omega^2 = \frac{\omega^2}{N^2C^2}\epsilon_c^2 C^2 \sum_{r=1}^{N-1} \sin^4\frac{mr\pi}{N} + \frac{\omega^2}{N^2L^2}\epsilon_L^2 L^2 \sum_{r=1}^{N-1} \sin^4\frac{mr\pi}{N}$$

$$\sigma_\omega^2 = \frac{\omega^2}{N}\frac{3}{8}(\epsilon_c^2 + \epsilon_L^2) \qquad m = 1, 2, \ldots, N-1;\ m \neq \frac{N}{2} \qquad (29)$$

$$\sigma_\omega^2 = \frac{\omega^2}{N}\frac{1}{2}(\epsilon_c^2 + \epsilon_L^2) \qquad m = \frac{N}{2} \qquad (30)$$

If $\epsilon_c = \epsilon_L = \epsilon$

$$\sigma_\omega = \omega\frac{\epsilon}{2}\sqrt{\frac{3}{N}} \qquad m = 1, 2, \ldots, N-1;\ m \neq \frac{N}{2} \qquad (31)$$

$$\sigma_\omega = \frac{\omega\epsilon}{\sqrt{N}} \qquad m = \frac{N}{2} \qquad (32)$$

Except for the possible mode $m = \frac{N}{2}$, the results are identical with those for the second order eigenvalue problem.

From equation (21), section 1.3, it is seen that the error due to lumping is increased, compared with the second order problem. Here, the deviation is less than the error due to lumping unless N is greater than N_o, where

$$N_o = \left(\frac{m^4\pi^4}{108\epsilon^2}\right)^{\frac{1}{3}} \qquad (33)$$

Again taking $m = 1$, $\epsilon = .01$,

$$N_o = 20.8 \qquad (34)$$

$$\frac{\sigma_\omega}{\omega} = \frac{\Delta\omega_1}{\omega_1} = .0019 \qquad (35)$$

It is concluded from the examples of these two sections that the expected deviation of the answer resulting from the distribution of the elements about their means will be negligible compared with limitations imposed by the accuracy of the solution. If the number of cells is less than N_o, the error due to lumping overshadows the effect of deviation of the element values. If the number of cells is greater than N_o, the expected deviation is well within the specifications of the computer operation.

However, the effect of mean error of the values of the elements, from equation (22), may not be negligible, as it does not decrease as N is increased.

2.4 Distribution of the Elements of the California Institute Computer.

The elements of the California Institute computer have been measured accurately for purposes of calibration, and some of the results of these measurements are given in Table 10, page 44. The capacitors and inductors were measured with an a.c. impedance bridge at 1000 cycles, and the resistors with a d.c. bridge. Each capacitor measurement is of a single unit, and is independent of the other measurements. For use in the computer, each element is made up of several units of different nominal values, and the proper capacitance is formed by connecting units whose capacitance have the correct sum.

The computer inductor elements each consist of three coils wound on three separate cores. One coil has a large inductance, one intermediate, and one small. Each coil has several taps. The coils are connected in series, so that by various settings of the taps, a wide range of inductance values is available. The data of Table 10(b) are for the various tap settings of the larger inductors only. The uniformity of the r.m.s. deviations is explained by the fact that each row represents measurements on the same coils and cores, with only the tap settings changed. It should be

Table 10. Distribution of the Values of Elements, California Institute of Technology Electric-Analog Computer

(a) Capacitors

Quantity Measured	Nominal Value, Microfarads	In Per Cent of Nominal Value: Mean Error	R.m.s. Deviation
80	.01	2.20%	1.00%
160	.02	.82	.88
80	.05	.19	.60
79	.1	.32	.93
159	.2	-.27	1.26
80	.5	.04	1.08
32	1.	3.10	1.12
32	2.	3.20	.82

(b) 20 Inductors

Nominal Value, Henrys	In Per Cent of Nominal Value: Mean Error	R.m.s. Deviation
.06	-.30%	1.30%
.12	-.52	1.24
.18	-.17	1.18
.24	.09	1.18
.30	.25	1.22
.36	.07	1.16
.42	.86	1.19
.48	.85	1.45
.54	1.03	1.19
.60	.90	1.19
.66	1.29	1.20
.72	1.24	1.22
.78	1.25	1.20
.84	1.63	1.24
.90	1.95	1.23

(Continued on p. 45)

Table 10. (Continued)

(c) 78 Resistors

Nominal Value, Ohms	In Per Cent of Nominal Values: Mean Error	R.m.s. Deviation
2	6.75%	3.62%
4	2.82	1.82
6	1.63	1.23
8	1.01	.90
10	2.43	1.00
20	1.66	.66
30	1.39	.54
40	1.28	.52
50	1.23	.50
60	1.14	.42
70	1.10	.41
80	1.08	.39
90	1.06	.37
100	.97	.37
100	.191	.58
200	.092	.43
300	.020	.36
400	.029	.31
500	.017	.29
600	.012	.29
700	.024	.28
800	.018	.26
900	.013	.27
1000	-.100	.24
1000	.47	.30
2000	.46	.32
3000	.49	.23
4000	.49	.20
5000	.51	.18
6000	.51	.13

mentioned that the values of mean error are influenced by the frequency of the metering bridge, somewhat higher than the range of optimum operation of the inductors.

The resistances tabulated in Table 10(c) are divided into four groups. In any group, the resistance of the nth row is actually made up n resistors, each with the nominal value of the first row of the group. This construction explains the variations of the r.m.s. deviations within each group. The large errors of the low-resistance measurements are caused chiefly by the contact resistance of metering relays in series with each element.

The data of several of the rows of Table 10 were tested to determine whether the hypothesis that the distribution of values is a normal distribution might be supported. The results were not conclusive, but did not disprove the hypothesis. The method employed was the chi-squared test for goodness of fit.[11] For the various rows, probabilities that values selected from the assumed normal distribution would form a better fit with this distribution than the values actually measured were, successively, 0.65, 0.07, 0.83, 0.21, 0.69.

Depending upon the requirements of the problem, and the actual elements used, the settings of the elements are arrived at in some cases from their nominal values, and in other cases by reference to accurate measurements of their values. The applications of Part II are chiefly with regard to circuits in which the elements are selected at random from a group, the values of which are distributed in a close range about the nominal value.

III LANDING TEST OF A MODEL AIRPLANE WING

3.1 Experimental Test.

In the past few years experimental tests have been made at the National Bureau of Standards for the purpose of verifying the accuracy of several methods of calculating the response of an airplane wing to transient forces imposed by landing. A detailed description of these tests is given in references (12) and (13).

The tests were performed on a model airplane wing about 11 feet long. The model wing was made of sheet and angles of aluminum alloy, with "landing struts" projecting downwards. The model was dropped in a condition as free from strain as possible, alighting on the struts. The forces transmitted to the wing through the landing struts and the resulting stresses at several points in the wing were measured by means of wire strain gauges fastened to the model.

The tests were conducted so that the motion of the wing consisted wholly of bending in the vertical plane, vertical translation, and roll. The character of the landing could be changed by varying the nature of the surfaces which were struck by the landing struts. In the test for which a solution was obtained on the electric-analog computer the wing was subjected to a "soft", unsymmetrical, two-point landing impact. By a "soft" impact, it is meant that the time of contact of the landing strut with the landing surface is comparable with the longest period of free bending vibration. The asymmetry was provided by allowing one of the two landing struts to alight before the other.

3.2 Analytical Methods of Computation.

Several analytical methods of computation were employed to obtain solutions of the problem, to which the electric-analog computer was applied. Some of these were included in the report of the experimental tests, reference (13), and others were published later. These methods were considered not only for accuracy, but also with regard to the length and difficulty of the computations involved. These methods are outlined here briefly.

The method of normal modes, and a few variations of this method, are considered in reference (13). The basic normal modes solution is described in reference (14). The entire motion of the structure is considered as the sum of motions, each of which maintains the shape of one of the normal modes of vibration. Similarly, the external forces are broken into the generalized forces corresponding to each mode. Because of the orthogonality of the normal coordinates, the variation of each normal coordinate with time depends on the generalized force forthat mode alone. These variations are determined independently, and their sum gives the entire motion of the structure.

By separating the statical response from the vibrational response in each mode due to exciting forces of fixed spacial distribution, Williams [15] has obtained a form which converges more rapidly. Part of the solution is obtained by assuming the external forces act statically, and the remainder of the solution, due to the inertia forces of each mode, depends chiefly on the lower frequency modes.

Another method outlined in reference (13), due to Levy, makes use of modes which are no longer normal. Each mode shape is calculated from all the lower modes. This has the advantage that it is not necessary to have all the normal modes calculated. There is a difficulty in estimating the accuracy of this form of solution -- consideration of the first neglected term involves recalculation of all the preceding terms, because of the non-orthogonality of the modes.

Of these three solutions, the second two proved somewhat superior in the case of "soft" landings. All three were unsatisfactory for "hard" impact, because of the number of modes which had to be considered.

A "statistical" method, developed in reference (14), was applied to the problem.[16] In this method, answers are obtained by summing the maximum responses in each mode, regardless of the time of occurrence of the maximum. Thus the answers are not given as functions of time. The method includes the further simplification of using a forcing function taken from a table of standard forms as an approximation to the actual excitations in the problem. The results of this "statistical" method were regarded as too conservative for practical value.

The same problem was solved also by a step-by-step method of calculation at the Bureau of Standards.(17) This solution is based on a method first applied to the study of aircraft vibration problems by Houbolt.(18) In this method, the partial differential equation is replaced by finite-difference equations in both variables. The forces on a given mass segment at any time are equated to the mass times the acceleration which is expressed in terms of the deflections of the segment at immediately preceding time increments. It is not necessary to have the normal modes for the step-by-step calculations. Although there is a large amount of numerical work involved, it is chiefly adaptable to handling by means of punched cards. This method was regarded as superior to the other solutions considered, and is compared here with the results of the experimental test and with the analog computer solution.

3.3 Analog Computer Solution

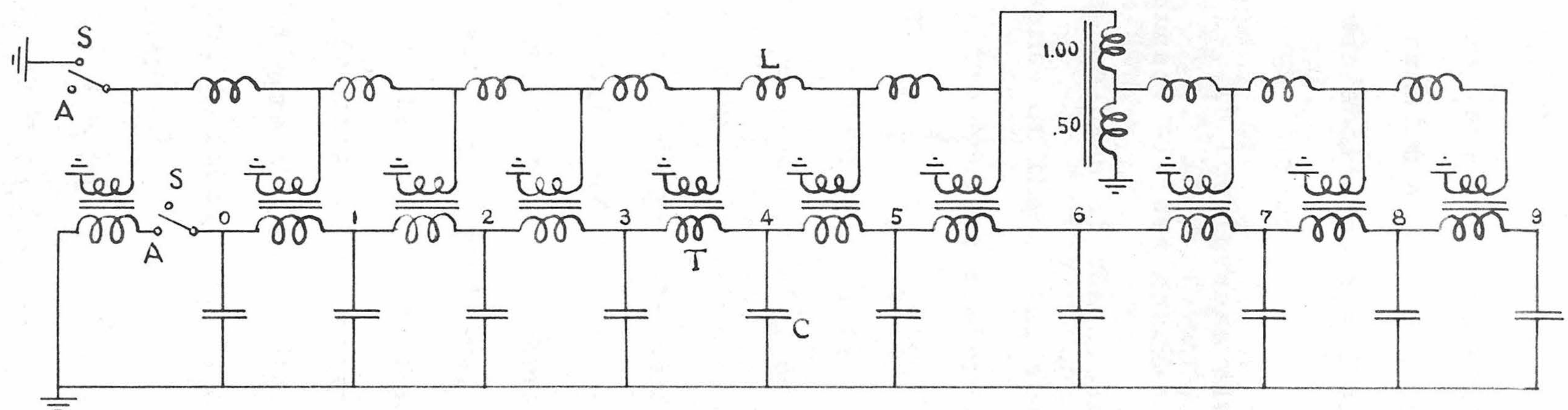
The model wing subjected to the experimental test was symmetric, and about 11 feet long. The two landing struts were located at stations 16.5 and -16.5 (measured in inches from the centerline). The data describing the wing, as obtained from the Bureau of Standards, consisted of values of masses already lumped at ten stations, x_1, x_2, ..., x_{10}, along the half-wing, and a curve of EI versus distance along the wing, x.

Two circuits (Fig. 5, page 53) were used to obtain two computer solutions to the problem. They are constructed according to the method developed in reference (5). In one case the half-wing was divided into ten cells, and in the other, five cells. In the ten-cell analog, the nodes corresponded to the ten stations of the original data, and the capacitances were determined directly from the lumped masses. To determine the values of inductance to be used, a curve of $\frac{1}{EI}$ versus x was plotted. The flexibility associated with each node was calculated by graphically performing the integration

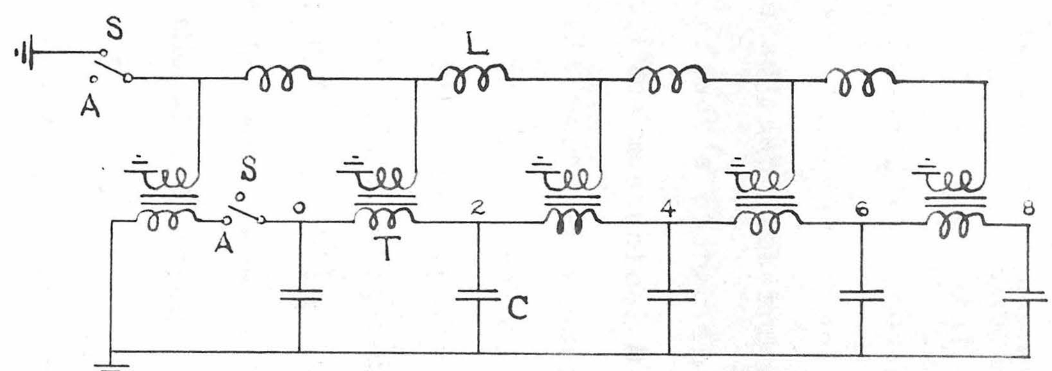
$$\frac{1}{K_n} = \int_{x_{n-\frac{1}{2}}}^{x_{n+\frac{1}{2}}} \frac{1}{EI}\, dx. \qquad (1)$$

$x_{n-\frac{1}{2}}$ and $x_{n+\frac{1}{2}}$ are the points obtained by bisecting the internodal intervals adjacent to node n. The inductance L_n is calculated from K_n, the constant of the lumped spring associated with node n. Values of T, the transformer turns ratios, were made proportional to the distances between stations. The single transformer in the slope circuit of Fig. 5a was introduced to make a change in the impedance base along the wing, for the range of values, K_n, from root to tip was greater than the range of

FIGURE 5 CIRCUITS ANALOGOUS TO HALF THE MODEL AIRPLANE WING



(a) TEN-CELL CIRCUIT



(b) FIVE-CELL CIRCUIT

inductances, L_n, that it was practical to employ.

The five-cell analog (Fig. 5b) was constructed in the same way, with successive nodes falling at alternate stations of the original data. The mass at each station that was abolished was divided among the adjacent nodes as though it were statically supported at those nodes, that is, in the inverse ratio of the adjacent internodal distances. It was not necessary to effect a change in the impedance base of the slope circuit. The mechanical constants of the system and the circuit parameters used in the two analogs are presented in Table 11, page 55.

The excitation of the system was specified by giving time histories of the forces applied at the landing struts. These are given in Fig. 6a, page 56, $f_+(t)$ being applied at the strut at station +16.5, and $f_-(t)$ being applied at -16.5. The results desired were the wing bending moments $M_+(t)$ at station +17.5 and $M_-(t)$ at -17.5.

The solution was carried out in two steps. In the first step the circuit represented half of a wing in symmetric bending vibration, and in the second, half of a wing in anti-symmetric vibration. The excitations applied were the symmetric and anti-symmetric parts of the landing forces, $f_s(t)$ and $f_a(t)$, respectively (Fig. 6b), where

$$f_s(t) = \tfrac{1}{2}\left[f_+(t) + f_-(t)\right] \qquad (2)$$

$$f_a(t) = \tfrac{1}{2}\left[f_+(t) - f_-(t)\right]. \qquad (3)$$

Table 11. Mechanical and Electrical Parameters. Analog Computer Solution of Wing Landing Problem.

(a) Ten-cell Circuit

Node	Station in.	Δx in.	Flexibility rad/lb·in x10^{-6}	Mass $\frac{\text{lb sec}^2}{\text{in.}}$	Inductance hy.	Capacitance μfd.	Transformer Turns Ratio T
-	0		-	-	-	-	
		3.0					.300
0	3.0		.334	.03142	.0356	7.380	
		6.5					.650
1	9.5		.825	.00161	.0879	.378	
		7.0					.700
2	16.5		1.25	.00850	.1331	1.996	
		8.0					.800
3	24.5		2.27	.00205	.2418	.481	
		9.5					.950
4	34.0		5.37	.00899	.572	2.110	
		10.0					1.000
5	44.0		9.46	.00106	1.008	.2489	
		8.0					.800
6	52.0		14.5	.000728	.1717	.1710	
		4.5					1.350
7	56.5		30.5	.000510	.3606	.1198	
		5.5					1.650
8	62.0		175.	.000295	2.070	.0693	
		5.0					1.500
9	67.0		300	.000129	-	.0303	

(b) Five-cell Circuit

Node	Station in.	Δx in.	Flexibility rad/lb·in x10^{-6}	Mass $\frac{\text{lb sec}^2}{\text{in.}}$	Inductance hy.	Capacitance μfd.	Transformer Turns Ratio T
-	0		-	-	-	-	
		3.0					.250
0	3.0		.739	.03210	.0970	8.826	
		13.5					1.123
2	16.5		2.75	.01054	.3611	2.900	
		17.5					1.457
4	34.0		9.85	.01040	1.291	2.860	
		18.0					1.498
6	52.0		32.4	.001598	4.244	.4394	
		10.0					.832
8	62.0		494.	.000654	-	.1798	

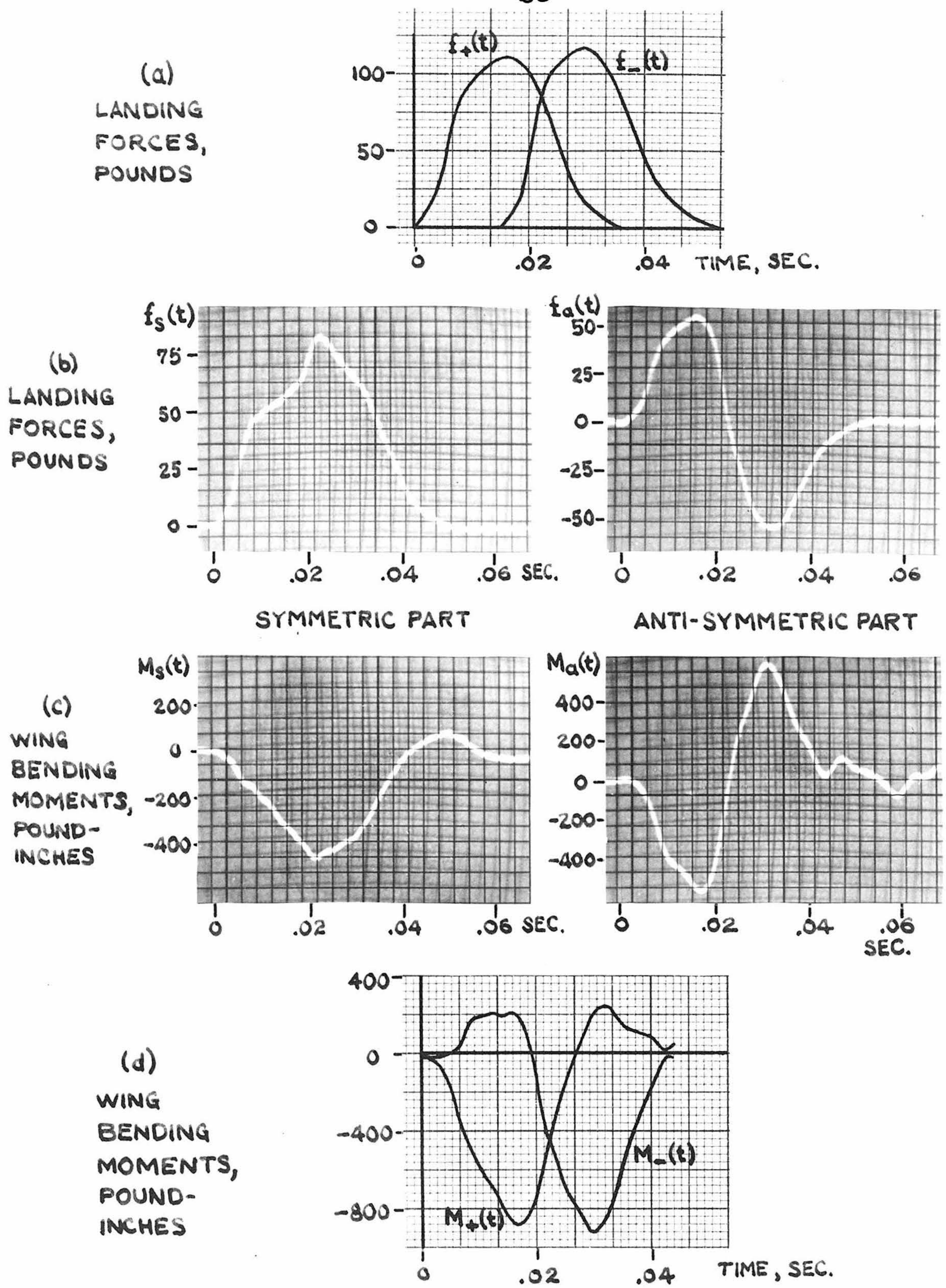


FIGURE 6 STEPS IN THE ANALOG COMPUTER SOLUTION FOR TRANSIENT BENDING MOMENT IN MODEL WING

The exciting forces, $f_s(t)$ and $f_a(t)$, were introduced into the circuit as currents at node 2. The forces were first established electrically as time-varying voltages by an arbitrary function generator operating on the photoelectric principle (reference (1), p. 957). By the use of current feedback, a dc amplifier was employed as a current generator to provide the desired excitation for the circuit.

The bending moments were determined from

$$M_+(t) = M_s(t) + M_a(t) \quad (4)$$

$$M_-(t) = M_s(t) - M_a(t), \quad (5)$$

The quantities $M_s(t)$ and $M_a(t)$ being obtained from measurements of currents in the inductors in the symmetric and anti-symmetric circuits, respectively. Resolving the solution into symmetric and anti-symmetric components results in an economy of computer elements and a simplification in the application of the exciting forces, but requires more measurements.

As the currents in the inductors are analogous to bending moments at the nodes, there was no current analogous to the moment at station 17.5. This moment had to be interpolated from measurements at 16.5 and 24.5 in the ten-cell wing, and at 16.5 and 34.0 in the five-cell wing. Fig. 6c presents the moments obtained from measurements at station 16.5 in the symmetric and anti-symmetric ten-cell circuit. Fig. 6d gives the moments at stations

+17.5, -17.5, determined by equations (4) and (5).

A comparison of the results obtained by several means is given in Figs. 7 and 8, pages 59 and 60. The bending moment at station 17.5 is plotted in Fig. 7 and the moment at station -17.5 in Fig. 8. The four curves are the solutions by circuit-analogs of ten and five cells, by experimental test with the model wing, and the solution of reference (17) by a step-by-step method based on Houbolt's method. In the step-by-step solution the mass of the half-wing was lumped at four stations. The time interval of the step calculations was .001 seconds.

It is difficult to make an accurate estimate of the error involved in each of these solutions. A comparison of the results of the various methods is perhaps as significant as any other method. The results of Part I are not directly applicable to the analog computer solution, as the wing is not a uniform beam, and as the lumped masses are not entirely derived from the distributed mass, but, at some stations, consist partly of discrete masses mounted on the wing. However, the accuracies of certain processes of the analog computer solution of this problem may be given.

The error in producing the forcing functions of Fig. 6b was determined by recording the actual currents introduced in the circuit and comparing with the desired functions. The difference between the two functions was

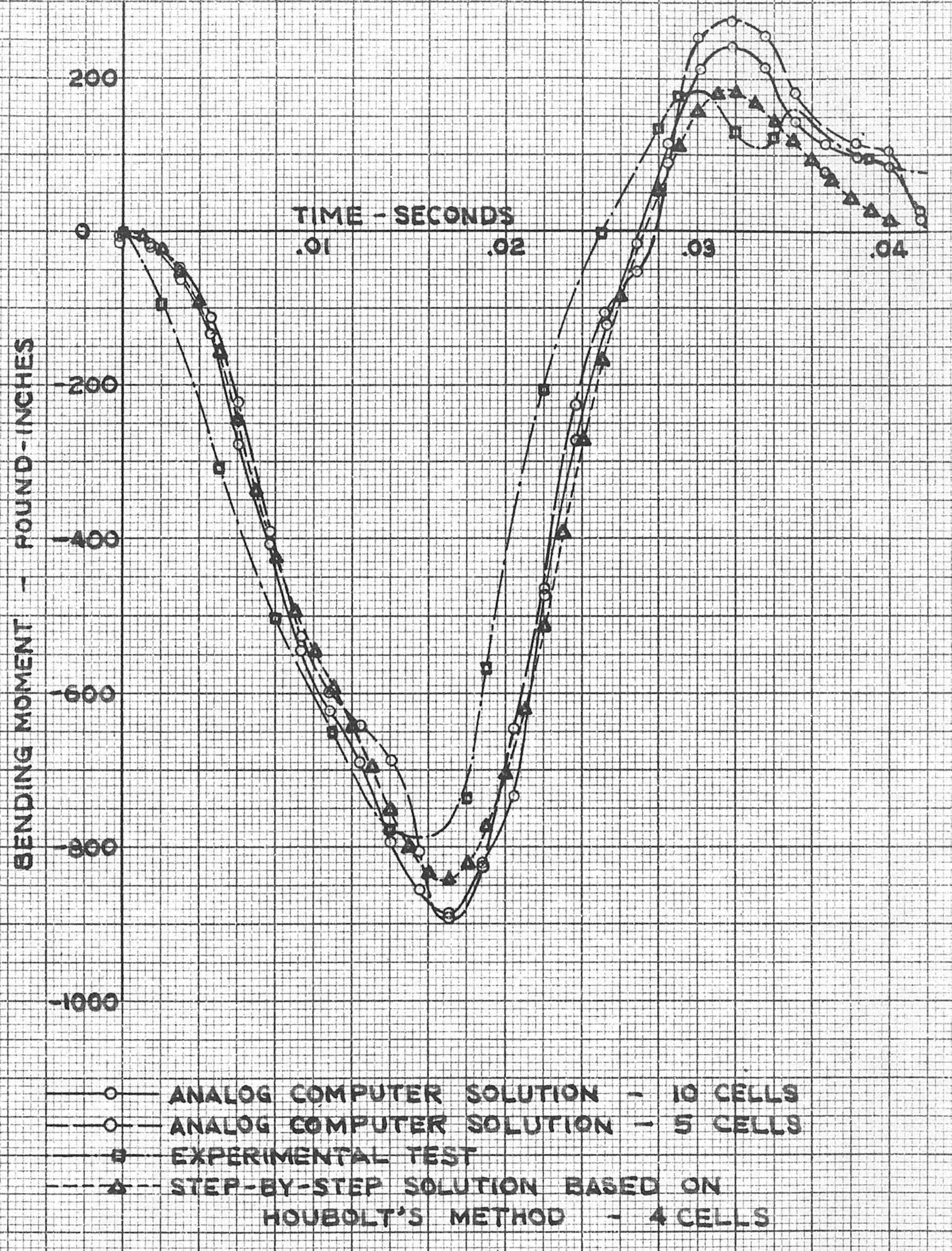


FIGURE 7 $M_+(t)$ BENDING MOMENT AT STATION +17.5

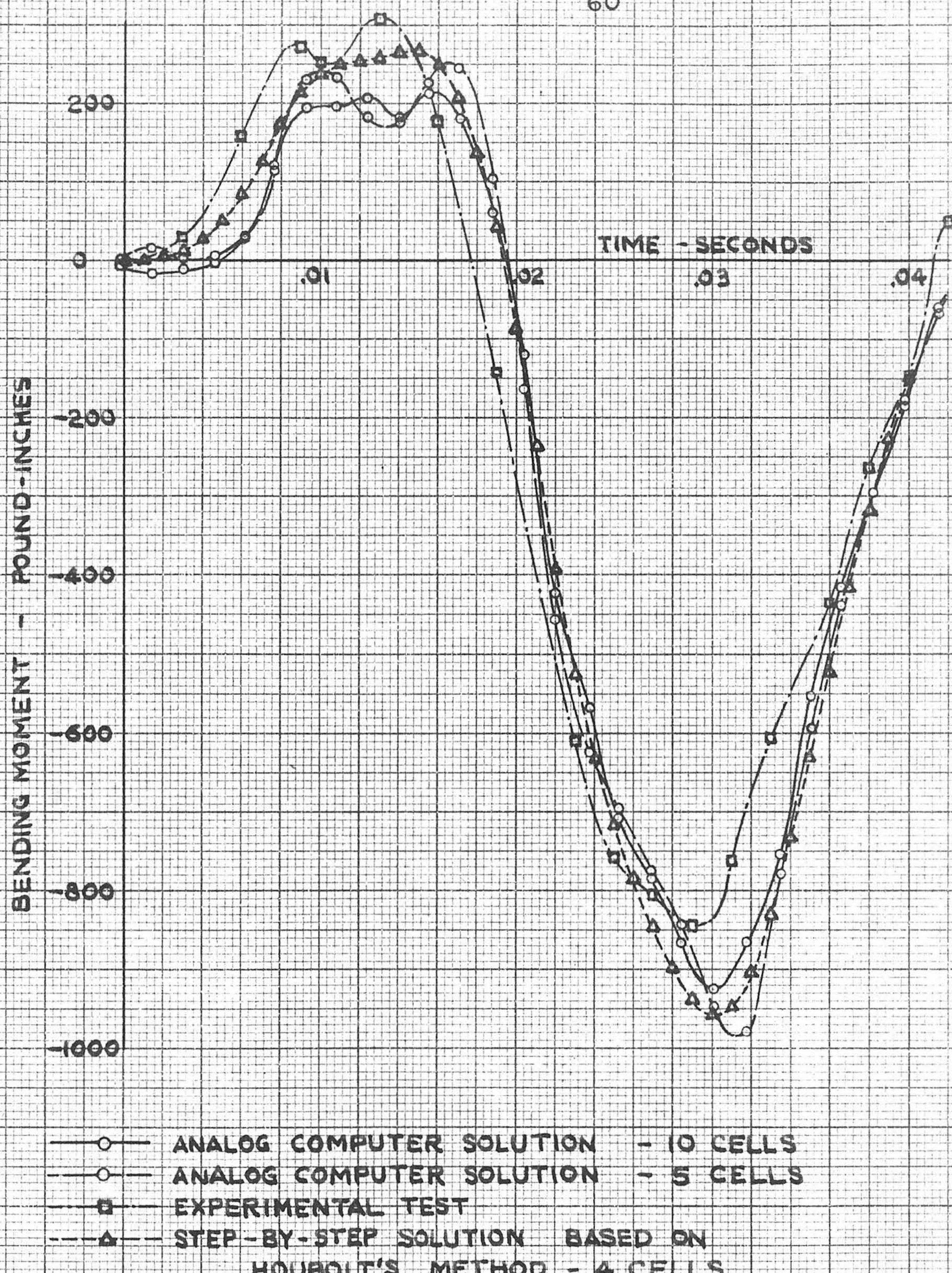


FIGURE 8 $M_{-}(t)$ BENDING MOMENT AT STATION -17.5

at most four per cent of the maximum value of the function, and, in absolute value, averaged two per cent of the maximum value. The elements of the circuits were all measured, and determined to be within one per cent of the desired values. The process of measuring the quantities by photographing the oscillograph screen was considered reproducible with a precision of two per cent.

There were no data included in the report of the experimental test which could be used to estimate the accuracy of that solution.

The step-by-step solution involves two approximations, one in using lumped physical parameters, the other in taking finite increments of time. The concentrating of the masses at four points along the half-wing, in this solution, is somewhat more coarse than the five-cell analog computer solution. The accuracy involved in taking finite time increments may be estimated in the light of the results obtained.

In the step-by-step solution, the acceleration of a point, $\frac{d^2y}{dt^2}$, was obtained from the equation

$$\left[\ddot{y}(t)\right] = \frac{1}{\epsilon^2}\left[2y(t) - 5y(t-\epsilon) + 4y(t-2\epsilon) - y(t-3\epsilon)\right], \qquad (6)$$

in which $y(t-\epsilon)$, $y(t-2\epsilon)$, etc., are values of the displacement, y, at preceding time increments, and in which a value of .001 seconds was used for ϵ. The error of this expression may be expressed in terms of the higher derivatives of y by writing the Taylor series expansion of each term on the right side of the equation, and adding:

$$\left[\ddot{y}(t)\right] = \frac{d^2y}{dt^2} + \frac{4}{3}\epsilon^2\frac{d^4y}{dt^4} + \epsilon^3\frac{d^5y}{dt^5} + \cdots \tag{7}$$

The derivatives are evaluated at time t.

Because of the tabular form of the results, it is easier to find the error in terms of differences of y than in terms of its derivatives. Defining

$$\Delta^2 y(t) \cdot\equiv\cdot\ y(t+\epsilon) - 2y(t) + y(t-\epsilon) \tag{8}$$

$$\Delta^4 y(t) \cdot\equiv\cdot\ \Delta^2\left[\Delta^2 y(t)\right], \tag{9}$$

the following relation is obtained, again by Taylor series expansion:

$$\frac{1}{\epsilon^2}\left[\Delta^2 y(t) - \frac{1}{12}\Delta^4 y(t)\right] = \frac{d^2y}{dt^2} - \frac{1}{90}\epsilon^4\frac{d^6y}{dt^6} + \cdots \tag{10}$$

The error in evaluating $\frac{d^2y}{dt^2}$ by equation (10) is very small compared with the error of equation (6). The results in reference (17) may be used to compare $\left[\ddot{y}(t)\right]$ with $\frac{d^2y}{dt^2}$ as given by the more accurate method of equation (10). This comparison was made for a number of values of $\left[\ddot{y}(t)\right]$. The greatest difference found in this comparison was 13 per cent of the maximum value of $\frac{d^2y}{dt^2}$, and, in absolute value, the difference averaged about four per cent.

It is difficult to predict what the effect of an error in one equation used in the step-by-step process will have on the result of the entire process.

For a problem of this type, having no exact analytical expression of the answer, the best estimate of the accuracy of any one method of solution may be a comparison of solutions by several methods, as in Figs. 7 and 8.

References

1. Electronic Techniques Applied to Analog Methods of Computation
G. D. McCann, C. H. Wilts, and B. N. Locanthi
Proc. IRE. Vol. 37. No. 8. Aug. 1949. p. 954.

2. The Solution of Partial Differential Equations by Means of Electrical Networks
R. H. MacNeal
C.I.T. Ph.D. Thesis (1949).

3. Lumped Parameter Analogies for Continuous Mechanical Systems
W. T. Russell
C.I.T. Ph.D. Thesis (1950).

4. Vibration Problems in Engineering
S. Timoshenko
D. Van Nostrand. New York. 1937.

5. Beam Vibration Analysis with the Electric Analog Computer
G. D. McCann and R. H. MacNeal
Journal of Applied Mechanics. March, 1940.

6. Calculus of Finite Differences. Chapter 11
G. Boole
MacMillan and Co. London. 1880.

7. Wave Propagation in Periodic Structures
L. Brillouin
McGraw-Hill. New York. 1946.

8. Differential and Integral Calculus. Volume II. Chapter III
R. Courant
Interscience Publishers. New York. 1948.

9. Treatment of Experimental Data
A. G. Worthing and J. Geffner
Wiley and Sons. New York. 1943.

10. The Theory of Sound. Volume I. pp 109-115
Rayleigh
MacMillan. London. Second Edition. 1894.

11. Introduction to Mathematical Statistics. p. 186
P. G. Hoel
Wiley & Sons. New York. 1947

12. Experimental Verification of Theory of Landing Impact
W. Ramberg and A. E. McPherson
Jour. of Research NBS. RP1936. Vol. 41. Nov, 1948

13. Transient Vibration in an Airplane Wing Obtained by Several Methods
W. Ramberg
Jour. of Research NBS. RP1984. Vol. 42. May, 1949

14. Dynamic Loads on Airplane Structures During Landing
M. A. Biot and R. H. Bisplinghoff
NACA Wartime Report W-92. Oct, 1944

15. Dynamic Loads in Aeroplanes under Given Impulsive Loads with Particular Reference to Landing and Gust Loads on a Large Flying Boat
D. Williams and R. P. N. Jones
British Aeronautical Research Council Technical Report. R&M No. 2221. 1948

16. "Statistical" Computation of Bending Response Due to Unsymmetric Impact of an Airplane Model
J. B. Woodson
NBS Lab. No. 6.4/1-181 PR5. BuAer TED NBS 2410
Aug. 24, 1949

17. A Step-by-Step Method of Determining the Dynamic Response of Aircraft in Landing
W. D. Kroll and S. Levy
NBS Lab. No. 6.4/1-181 PR6. BuAer TED NBS 2410
Sept. 21, 1949

18. A Recurrence Matrix Solution for the Dynamic Response of Aircraft in Gusts
J. D. Houbolt
Technical Note No. 2060 NACA. March, 1950

List of Appendixes

Appendix 1 General Solution of the Difference Equation of the Circuit-Analogy of the Beam in Bending.

This solution is carried out by standard means, an exposition of which is given, for example, in Reference (6). The difference equation describing the circuit of Fig. 1a is eq. (8); (Part I),

$$Y_{n+2} - 4Y_{n+1} + (6 - z^4)Y_n - 4Y_{n-1} + Y_{n-2} = 0$$

where z is a positive real number. Let $Y_n = e^{n\theta}$

Then

$$z^4 e^{n\theta} = Y_{n+2} - 4Y_{n+1} + 6Y_n - 4Y_{n-1} + Y_{n-2}$$

$$= e^{n\theta}(e^{2\theta} - 4e^{\theta} + 6 - 4e^{-\theta} + e^{-2\theta})$$

$$= e^{n\theta}(e^{\frac{1}{2}\theta} - e^{-\frac{1}{2}\theta})^4$$

$$= e^{n\theta}\, 16 \sinh^4 \tfrac{1}{2}\theta$$

$$2 \sinh \tfrac{1}{2}\theta = \pm z, \ \pm jz.$$

The four possible values of θ, then, are $\pm\theta_1$, $\pm j\theta_2$*, where

$$2 \sinh \tfrac{1}{2}\theta_1 = z$$

$$2 \sin \tfrac{1}{2}\theta_2 = z.$$

The general solution is made up of the individual solutions.

$$Y_n = A_1 e^{n\theta_1} + A_2 e^{-n\theta_1} + A_3 e^{jn\theta_2} + A_4 e^{-jn\theta_2}$$

or,

$$Y_n = A \cosh n\theta_1 + B \sinh n\theta_1 + C \cos n\theta_2 + D \sin n\theta_2$$

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* θ_1 and θ_2 are real when $z \leq 2$. This includes all the practical cases of eigenvalue solution by the circuit of Fig. 1a. However for $z > 2$, θ_2 is no longer real, and the possible values of θ are

$$\theta = \pm 2 \sinh^{-1} \tfrac{1}{2}z$$

$$\theta = \pm 2 \cosh^{-1} \tfrac{1}{2}z + j\pi$$

To any of these solutions may be added $jm\pi$, where m is any integer.

Appendix 2 Determination of Implicit Solution for Mode Frequencies of Cantilever Beams (ce) and (cf).

1. Beam (ce). Clamped at 0. Free at N.

The general solution is given by equation (9), Part I.

$$Y_n = A\cosh n\theta_1 + B\sinh n\theta_1 + C\cos n\theta_2 + D\sin n\theta_2 \tag{1}$$

The boundary conditions are taken from Table 3, remembering that M, the length in cells, equals N, the free node number.

$$Y_0 = 0 \tag{2}$$

$$Y_1 - Y_{-1} = 0 \tag{3}$$

$$Y_{M+1} - 2Y_M + Y_{M-1} = 0 \tag{4}$$

$$Y_{M+2} - 2Y_{M+1} + 2Y_{M-1} - Y_{M-2} = 0 \tag{5}$$

Substituting equation (1) into equations (2) and (3), respectively,

$$A + C = 0$$

$$2B\sinh\theta_1 + 2D\sin\theta_2 = 0$$

Incorporating these relations, equation (1) may be rewritten

$$\begin{aligned} Y_n = {} & A(\cosh n\theta_1 - \cos n\theta_2) \\ & + E(\sin\theta_2 \sinh n\theta_1 - \sinh\theta_1 \sin n\theta_2) \end{aligned} \tag{6}$$

The following relations will be useful.

$$z^2 = 4\sinh^2 \tfrac{1}{2}\theta_1 = 2(\cosh\theta_1 - 1) = 4\sin^2 \tfrac{1}{2}\theta_2 = 2(1 - \cos\theta_2) \tag{7}$$

$$\begin{aligned} & \cosh(n+1)\theta_1 - 2\cosh n\theta_1 + \cosh(n-1)\theta_1 \\ & \quad = 2(\cosh\theta_1 - 1)\cosh n\theta_1 = z^2 \cosh n\theta_1 \end{aligned} \tag{8}$$

$$\begin{aligned} & \sinh(n+1)\theta_1 - 2\sinh n\theta_1 + \sinh(n-1)\theta_1 \\ & \quad = 2(\cosh\theta_1 - 1)\sinh n\theta_1 = z^2 \sinh n\theta_1 \end{aligned} \tag{9}$$

$$\begin{aligned} & \cos(n+1)\theta_2 - 2\cos n\theta_2 + \cos(n-1)\theta_2 \\ & \quad = 2(\cos\theta_2 - 1)\cos n\theta_2 = -z^2 \cos n\theta_2 \end{aligned} \tag{10}$$

$$\sin(n+1)\theta_2 - 2\sin n\theta_2 + \sin(n-1)\theta_2$$

$$= 2(\cos\theta_2 - 1)\sin n\theta_2 = -z^2 \sin n\theta_2 \tag{11}$$

Substituting equation (6) into equation (4), making use of equations (8) through (11), results in

$$A(\cosh M\theta_1 + \cos M\theta_2)$$

$$+ E(\sin\theta_2 \sinh M\theta_1 + \sinh\theta_1 \sin M\theta_2) = 0 \tag{12}$$

Substituting equation (6) into equation (5),

$$(Y_{M+2} - 2Y_{M+1} + Y_M) - (Y_M - 2Y_{M-1} + Y_{M-2}) = 0$$

$$A[\cosh(M+1)\theta_1 + \cos(M+1)\theta_2]$$

$$+E[\sin\theta_2 \sinh(M+1)\theta_1 + \sinh\theta_1 \sin(M+1)\theta_2]$$

$$-A[\cosh(M-1)\theta_1 + \cos(M-1)\theta_2]$$

$$-E[\sin\theta_2 \sinh(M-1)\theta_1 + \sinh\theta_1 \sin(M-1)\theta_2] = 0$$

$$2A[\sinh M\theta_1 \sinh\theta_1 - \sin M\theta_2 \sin\theta_2]$$

$$+2E[\sin\theta_2 \cosh M\theta_1 \sinh\theta_1 + \sinh\theta_1 \cos M\theta_2 \sin\theta_2] = 0 \tag{13}$$

Considering A and E as variables in the equations (12) and (13), the condition for the existence of normal modes of vibration is determined by setting the coefficient determinant to zero:

$$\sinh\theta_1 \sin\theta_2 (\cosh M\theta_1 + \cos M\theta_2)^2$$

$$= (\sin\theta_2 \sinh M\theta_1 + \sinh\theta_1 \sin M\theta_2)(\sinh M\theta_1 \sinh\theta_1 - \sin M\theta_2 \sin\theta_2),$$

$$\sinh\theta_1 \sin\theta_2 (\cosh^2 M\theta_1 + 2\cosh M\theta_1 \cos M\theta_2 + \cos^2 M\theta_2)$$

$$= \sinh^2 M\theta_1 \sinh\theta_1 \sin\theta_2 - \sinh M\theta_1 \sin M\theta_2 \sin^2\theta_2$$

$$+\sinh M\theta_1 \sinh^2\theta_1 \sin M\theta_2 - \sinh\theta_1 \sin^2 M\theta_2 \sin\theta_2$$

This substitution is made:

$$\cosh^2 M\theta_1 + \cos^2 M\theta_2 = 2 + \sinh^2 M\theta_1 - \sin^2 M\theta_2$$

resulting in

$$2 \sinh \theta_1 \sin \theta_2 \left(1 + \cosh M\theta_1 \cos M\theta_2\right)$$

$$= \sinh M\theta_1 \sin M\theta_2 \left(\sinh^2 \theta_1 - \sin^2 \theta_2\right)$$

$$1 + \cosh M\theta_1 \cos M\theta_2 = \sinh M\theta_1 \sin M\theta_2 \frac{\sinh^2 \theta_1 - \sin^2 \theta_2}{2 \sinh \theta_1 \sin \theta_2}. \quad (14)$$

This answer may be given in several forms:

$$1 + \cosh M\theta_1 \cos M\theta_2 = \sinh M\theta_1 \sin M\theta_2 \tanh \tfrac{1}{2}\theta_1 \tan \tfrac{1}{2}\theta_2 \quad (15)$$

$$1 + \cosh M\theta_1 \cos M\theta_2 = \sinh M\theta_1 \sin M\theta_2 \frac{z^2}{4\sqrt{1 - \frac{z^4}{16}}} \quad (16)$$

2. Beam (cf). Clamped at 0. Free at $N + \frac{1}{2}$.

This differs from the preceding beam in that the free end conditions are, from Table 3,

$$Y_{N+1} - 2Y_N + Y_{N-1} = 0 \quad (17)$$

$$Y_{N+2} - 2Y_{N+1} + Y_N = 0 \quad (18)$$

Equations (1), (2), (3), and (6) apply to this beam. Substituting equation (6) into equations (17) and (18), respectively, gives, by analogy with equation (12),

$$A\left[\cosh N\theta_1 + \cos N\theta_2\right]$$

$$+ E\left[\sinh N\theta_1 \sin \theta_2 + \sinh \theta_1 \sin N\theta_2\right] = 0 \quad (19)$$

$$A\left[\cosh (N+1)\theta_1 + \cos (N+1)\theta_2\right]$$

$$+ E\left[\sinh (N+1)\theta_1 \sin \theta_2 + \sinh \theta_1 \sin (N+1)\theta_2\right] = 0 \quad (20)$$

Substituting $M - \frac{1}{2}$ for N and $M + \frac{1}{2}$ for $N + 1$ results in a symmetric form

$$A\left[\cosh (M-\tfrac{1}{2})\theta_1 + \cos (M-\tfrac{1}{2})\theta_2\right]$$

$$+ E\left[\sinh (M-\tfrac{1}{2})\theta_1 \sin \theta_2 + \sinh \theta_1 \sin (M-\tfrac{1}{2})\theta_2\right] = 0 \quad (21)$$

$$A\left[\cosh (M+\tfrac{1}{2})\theta_1 + \cos (M+\tfrac{1}{2})\theta_2\right]$$

$$+ E\left[\sinh (M+\tfrac{1}{2})\theta_1 \sin \theta_2 + \sinh \theta_1 \sin (M+\tfrac{1}{2})\theta_2\right] = 0 \quad (22)$$

It is more convenient to use the following forms, which are $\frac{1}{2}$ [(22)+(21)] and $\frac{1}{2}$ [(22)-(21)], respectively

$$A\left[\cosh M\theta_1 \cosh \tfrac{1}{2}\theta_1 + \cos M\theta_2 \cos \tfrac{1}{2}\theta_2\right]$$
$$+E\left[\sinh M\theta_1 \cosh \tfrac{1}{2}\theta_1 \sin\theta_2 + \sinh\theta_1 \sin M\theta_2 \cos\tfrac{1}{2}\theta_2\right] = 0 \qquad (23)$$
$$A\left[\sinh M\theta_1 \sinh \tfrac{1}{2}\theta_1 - \sin M\theta_2 \sin \tfrac{1}{2}\theta_2\right]$$
$$+E\left[\cosh M\theta_1 \sinh \tfrac{1}{2}\theta_1 \sin\theta_2 + \sinh\theta_1 \cos M\theta_2 \sin\tfrac{1}{2}\theta_2\right] = 0 \qquad (24)$$

The condition for the existence of normal modes is stated independent of A and E by setting the coefficient determinant of equations (23) and (24) to zero.

$$(\cosh^2 M\theta_1 - \sinh^2 M\theta_1) \cosh\tfrac{1}{2}\theta_1 \sinh\tfrac{1}{2}\theta_1 \sin\theta_2$$
$$+ (\cos^2 M\theta_2 + \sin^2 M\theta_2) \sinh\theta_1 \cos\tfrac{1}{2}\theta_2 \sin\tfrac{1}{2}\theta_2$$
$$+ \cosh M\theta_1 \cos M\theta_2 \left(\sinh\theta_1 \cosh\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2\right.$$
$$\left. + \sinh\tfrac{1}{2}\theta_1 \sin\theta_2 \cos\tfrac{1}{2}\theta_2\right)$$
$$- \sinh M\theta_1 \sin M\theta_2 \left(\sinh\theta_1 \sinh\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2\right.$$
$$\left. - \cosh\tfrac{1}{2}\theta_1 \sin\theta_2 \sin\tfrac{1}{2}\theta_2\right) = 0 \qquad (25)$$

Now $\sinh\theta_1$ is replaced by $2\sinh\frac{1}{2}\theta_1 \cosh\frac{1}{2}\theta_1$, $\sin\theta_2$ by $2\sin\frac{1}{2}\theta_2 \cos\frac{1}{2}\theta_2$, $\sinh\frac{1}{2}\theta_1$ by $\sin\frac{1}{2}\theta_2$ (by equation (7)), and the result is divided by $2\sin^2\frac{1}{2}\theta_2$.

$$\cosh\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 + \cosh\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2$$
$$+ \cosh M\theta_1 \cos M\theta_2 \left(\cosh^2\tfrac{1}{2}\theta_1 + \cos^2\tfrac{1}{2}\theta_2\right)$$
$$- \sinh M\theta_1 \sin M\theta_2 \left(\cosh\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 - \cosh\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2\right) = 0 \qquad (26)$$

From equation (7),

$$\cosh^2\tfrac{1}{2}\theta_1 + \cos^2\tfrac{1}{2}\theta_2 = 2 + \sinh^2\tfrac{1}{2}\theta_1 - \sin^2\tfrac{1}{2}\theta_2 = 2$$

Thus (26) becomes

$$\cosh\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 + \cosh M\theta_1 \cos M\theta_2 = 0$$

This may be stated variously:

$$1 + \cosh M\Theta_1 \cos M\Theta_2 = 1 - \cosh \tfrac{1}{2}\Theta_1 \cos \tfrac{1}{2}\Theta_2 \qquad (27)$$

$$1 + \cosh M\Theta_1 \cos M\Theta_2 = 1 - \sqrt{1 - \frac{z^4}{16}} \qquad (28)$$

Appendix 3 Determination of Mode Frequencies of the Cantilever Beam (ce).

The equation to be solved is given in Table 4 and Appendix 2.

$$1+\cosh M\theta_1 \cos M\theta_2 = \sinh M\theta_1 \sin M\theta_2 \frac{z^2}{4\sqrt{1-\frac{z^4}{16}}} \tag{1}$$

The definition of u, equation (31) of Part I, gives

$$z = \frac{k}{M}(1+u) \tag{2}$$

θ_1 and θ_2 are expanded in series from their definitions, equation (10)Part I.

$$\theta_1 = 2\sinh^{-1}\frac{z}{2} = 2\sinh^{-1}\frac{k}{2M}(1+u) = 2\left[\frac{k(1+u)}{2M} - \frac{1}{6}\frac{k^3(1+u)^3}{8M^3} + \frac{3}{40}\frac{k^5(1+u)^5}{32M^5} - \cdots\right]$$

$$\theta_1 = \frac{k}{M} + \left(\frac{ku}{M} - \frac{k^3}{24M^3}\right) + \left(-\frac{k^3u}{8M^3} + \frac{3k^5}{640M^5}\right) + \cdots \quad \cdot\equiv\cdot \quad \frac{k}{M} + \frac{\delta_1}{M} \tag{3}$$

$$\theta_2 = 2\sin^{-1}\frac{z}{2} = 2\sin^{-1}\frac{k}{2M}(1+u) = 2\left[\frac{k(1+u)}{2M} + \frac{1}{6}\frac{k^3(1+u)^3}{8M^3} + \frac{3}{40}\frac{k^5(1+u)^5}{32M^5} + \cdots\right]$$

$$\theta_2 = \frac{k}{M} + \left(\frac{ku}{M} + \frac{k^3}{24M^3}\right) + \left(\frac{k^3u}{8M^3} + \frac{3k^5}{640M^5}\right) + \cdots \quad \cdot\equiv\cdot \quad \frac{k}{M} + \frac{\delta_2}{M} \tag{4}$$

The grouping of terms results from assuming that u and $\frac{1}{M^2}$ are of comparable order, and both small compared with 1. Substituting in the terms of equation (1),

$$\begin{aligned} 1+\cosh M\theta_1 \cos M\theta_2 &= 1+\cosh(k+\delta_1)\cos(k+\delta_2) \\ &= 1 + \cosh k \cos k \cosh\delta_1 \cos\delta_2 \\ &\quad - \cosh k \sin k \cosh\delta_1 \sin\delta_2 \\ &\quad + \sinh k \cos k \sinh\delta_1 \cos\delta_2 \\ &\quad - \sinh k \sin k \sinh\delta_1 \sin\delta_2 \end{aligned}$$

Similarly,

$$\begin{aligned} \sinh M\theta_1 \sin M\theta_2 &= \sinh k \sin k \cosh\delta_1 \cos\delta_2 \\ &\quad + \sinh k \cos k \cosh\delta_1 \sin\delta_1 + \end{aligned}$$

$$+ \cosh k \sin k \sinh \delta_1 \cos \delta_2$$

$$+ \cosh k \cos k \sinh \delta_1 \sin \delta_2 .$$

From equations (3) and (4),

$$\cosh \delta_1 \cos \delta_2 = (1 + \tfrac{1}{2}\delta_1^2 + \cdots)(1 - \tfrac{1}{2}\delta_2^2 + \cdots)$$

$$= 1 - \frac{k^4 u}{12 M^2} + \cdots$$

$$\cosh \delta_1 \sin \delta_2 = (1 + \tfrac{1}{2}\delta_1^2 + \cdots)(\delta_2 - \cdots)$$

$$= \left(ku + \frac{k^3}{24M^2}\right) + \left(\frac{k^3 u}{8M^2} + \frac{3k^5}{640M^4}\right) + \cdots$$

$$\sinh \delta_1 \cos \delta_2 = (\delta_1 + \cdots)(1 - \tfrac{1}{2}\delta_2^2 + \cdots)$$

$$= \left(ku - \frac{k^3}{24M^2}\right) + \left(-\frac{k^3 u}{8M^2} + \frac{3k^5}{640M^4}\right) + \cdots$$

$$\sinh \delta_1 \sin \delta_2 = (\delta_1 + \cdots)(\delta_2 - \cdots)$$

$$= k^2 u^2 - \frac{k^6}{576 M^4} + \cdots$$

Also

$$\frac{z^2}{4\sqrt{1 - \frac{z^4}{16}}} = \frac{z^2}{4} + \cdots = \frac{k^2}{4M^2} + \frac{k^2 u}{2M^2} + \cdots$$

Equation (1) now becomes

$$1 + \left(1 - \frac{k^4 u}{12M^2}\right)\cosh k \cos k - \left[\left(ku + \frac{k^3}{24M^2}\right) + \left(\frac{k^3 u}{8M^2} + \frac{3k^5}{640M^4}\right)\right]\cosh k \sin k$$

$$+ \left[\left(ku - \frac{k^3}{24M^2}\right) + \left(-\frac{k^3 u}{8M^2} + \frac{3k^5}{640M^4}\right)\right]\sinh k \cos k$$

$$- \left(k^2 u^2 - \frac{k^6}{576M^4}\right)\sinh k \sin k$$

$$= \left(\frac{k^2}{4M^2} + \frac{k^2 u}{2M^2}\right)\sinh k \sin k + \left(\frac{k^3 u}{4M^2} + \frac{k^5}{96M^4}\right)\sinh k \cos k$$

$$+ \left(\frac{k^3 u}{4M^2} - \frac{k^5}{96M^4}\right)\cosh k \sin k$$

From equation (7), Part I, $\cosh k \cos k$ can be replaced by -1, $\cosh k \sin k$ by $-\tan k$, and $\sinh k \cos k$ by $-\tanh k$.

$$\frac{k^4 u}{12M^2} + \left(ku + \frac{k^3}{24M^2} + \frac{3k^3 u}{8M^2} - \frac{11k^5}{1920\,M^4}\right)\tan k$$
$$+ \left(-ku + \frac{k^3}{24M^2} + \frac{3k^3 u}{8M^2} + \frac{11k^5}{1920\,M^4}\right)\tanh k$$
$$+ \left(-\frac{k^2}{4M^2} - \frac{k^2 u}{2M^2} - k^2u^2 + \frac{k^6}{576\,M^4}\right)\sinh k \sin k = 0 \qquad (5)$$

Equation (5) may be written

$$c_1 u^2 + \left(c_2 + \frac{c_3}{M^2}\right)u + \frac{c_4}{M^2} + \frac{c_5}{M^4} = 0 \qquad (6)$$

in which

$$c_1 = -k^2 \sinh k \sin k$$
$$c_2 = k\,(\tan k - \tanh k)$$
$$c_3 = \frac{k^4}{12} + \frac{3k^3}{8}(\tan k + \tanh k) - \frac{k^2}{2} \sinh k \sin k$$
$$c_4 = \frac{k^3}{24}(\tan k + \tanh k) - \frac{k^2}{4} \sinh k \sin k$$
$$c_5 = \frac{11k^5}{1920}(-\tan k + \tanh k) + \frac{k^6}{576} \sinh k \sin k$$

The solution of (6) is

$$u = \frac{a_1}{M^2} + \frac{a_2}{M^4} + \cdots$$
$$a_1 = -\frac{c_4}{c_2}$$
$$a_2 = -\frac{c_5}{c_2} + \frac{c_3 c_4}{c_2^2} - \frac{c_1 c_4^2}{c_2^3}$$
$$a_1 = -\frac{k}{24}\,\frac{k(\tan k + \tanh k) - 6 \sinh k \sin k}{\tan k - \tanh k}$$
$$a_2 = \frac{k^2}{576}\left\{\frac{33k^2}{10} - \frac{k^3 \sinh k \sin k}{\tan k - \tanh k}\right.$$
$$+ \frac{[2k^2 + 9k(\tan k + \tanh k) - 12 \sinh k \sin k]}{(\tan k - \tanh k)^2} \cdot$$
$$\cdot [k(\tan k + \tanh k) - 6 \sinh k \sin k]$$
$$\left. + \frac{k \sinh k \sin k\,[k(\tan k + \tanh k) - 6 \sinh k \sin k]^2}{(\tan k - \tanh k)^3}\right\}$$

Appendix 4 Solution of Circuit (i), Section 1.6

Circuit (i) is constructed from the circuits of Figs. 4b and 4d with these boundary and end conditions:

$Y = 0$	at $-N_1$
Relative cell size 1	
Change in cell size	at 0
Relative cell size α	
$Y = 0$	at N_2

From equations (44), (56), and (58), Part I,

$$Y_{-N_1} = 0 \tag{1}$$

$$Y_n = A\cos n\theta_1 + B\sin n\theta_1 \qquad -N_1 \leqq n \leqq 0 \tag{2}$$

$$Y_n = C\cos n\theta_2 + D\sin n\theta_2 \qquad 0 \leqq n \leqq N_2 \tag{3}$$

$$Y_{N_2} = 0 \tag{4}$$

$$-Y_{-1} + \left(\frac{1+\alpha}{\alpha} - \frac{1+\alpha}{2}z^2\right)Y_0 - \frac{1}{\alpha}Y_1 = 0 \tag{5}$$

$$\omega\sqrt{LC} = z = 2\sin\tfrac{1}{2}\theta_1 = \frac{2}{\alpha}\sin\tfrac{1}{2}\theta_2 \tag{6}$$

By inspection, a form satisfying equations (1) to (4) may be written for Y_n.

$$Y_n = A\sin(N_1+n)\theta_1 \qquad -N_1 \leqq n \leqq 0 \tag{7}$$

$$Y_n = A\frac{\sin N_1\theta_1}{\sin N_2\theta_2}\sin(N_2-n)\theta_2 \qquad 0 \leqq n \leqq N_2 \tag{8}$$

Substituting in equation (5),

$$-\sin(N_1-1)\theta_1 + \left(\frac{1+\alpha}{\alpha} - \frac{1+\alpha}{2}z^2\right)\sin N_1\theta_1 - \frac{\sin N_1\theta_1}{\alpha\sin N_2\theta_2}\sin(N_2-1)\theta_2 = 0 \tag{9}$$

Substituting trigonometric expressions and rearranging results in

$$\cos N_1\theta_1\sin\theta_1 + \frac{\sin N_1\theta_1\cos N_2\theta_2\sin\theta_2}{\alpha\sin N_2\theta_2} + \left(-\cos\theta_1 + \frac{1+\alpha}{\alpha} - \frac{1+\alpha}{2}z^2 - \frac{\cos\theta_2}{\alpha}\right)\sin N_1\theta_1 = 0 \tag{10}$$

The term in parentheses may be reduced by application of equation (6)

$$\left(-\cos\theta_1 + \frac{1+\alpha}{\alpha} - \frac{1+\alpha}{2}z^2 - \frac{1}{\alpha}\cos\theta_2\right)$$

$$= 2\sin^2\tfrac{1}{2}\theta_1 - 1 + \frac{1}{\alpha} + 1 - \frac{1+\alpha}{2}z^2 + \frac{2}{\alpha}\sin^2\tfrac{1}{2}\theta_2 - \frac{1}{\alpha}$$

$$= \frac{z^2}{2} - \frac{1+\alpha}{2}z^2 + \alpha\frac{z^2}{2}$$

$$= 0$$

Equation (10) may be rewritten

$$\cos N_1\theta_1 \sin N_2\theta_2 + \sin N_1\theta_1 \cos N_2\theta_2 \frac{\sin\theta_2}{\alpha\sin\theta_1} = 0$$

$$\cos N_1\theta_1 \sin N_2\theta_2 + \sin N_1\theta_1 \cos N_2\theta_2 \frac{\cos\frac{1}{2}\theta_2}{\cos\frac{1}{2}\theta_1} = 0$$

$$\sin(N_1\theta_1 + N_2\theta_2) + \sin N_1\theta_1 \cos N_2\theta_2 \left(\frac{\cos\frac{1}{2}\theta_2}{\cos\frac{1}{2}\theta_1} - 1\right) = 0 \qquad (11)$$

By equation (6) the last term may be written in terms of z.

$$\frac{\cos\frac{1}{2}\theta_2}{\cos\frac{1}{2}\theta_1} - 1 = \sqrt{\frac{4-\alpha^2 z^2}{4-z^2}} - 1$$

$$= (1-\alpha^2)\left(\frac{1}{8}z^2 + \frac{3+\alpha^2}{128}z^4 + \cdots\right) \qquad (12)$$

The series form of the solution for mode frequencies is based on the implicit function theory outlined in section 1.4. First the following definitions and substitutions are made:

Length in cells $= N_1 + \alpha N_2 \equiv M \equiv \frac{1}{\sqrt{w}}$

$\gamma_1 \equiv N_1\sqrt{w}$

$\gamma_2 \equiv N_2\sqrt{w}$

$$z = \frac{k}{M}(1+u) = k\sqrt{w}\,(1+u)$$

$$g(u,w) = N_1\theta_1 = 2N_1 \sin^{-1}\frac{z}{2} = 2\frac{\gamma_1}{\sqrt{w}} \sin^{-1}\frac{k\sqrt{w}}{2}(1+u)$$

$$g(u,w) \cdot\equiv\cdot \gamma_1 k(1+u) + \frac{\gamma_1 k^3}{24} w(1+u)^3 + \cdots$$

$$h(u,w) = N_2\theta_2 = 2N_2 \sin^{-1}\frac{\alpha z}{2} = 2\frac{\gamma_2}{\sqrt{w}} \sin^{-1}\frac{k\alpha\sqrt{w}}{2}(1+u)$$

$$h(u,w) \cdot\equiv\cdot \gamma_2 \alpha k(1+u) + \frac{\gamma_2 \alpha^3 k^3}{24} w(1+u)^3 + \cdots$$

$$\gamma_1 + \alpha\gamma_2 = 1 \qquad (13)$$

g and h are defined in terms of the series to make them and their derivatives continuous at $w = 0$. Equation (11) may now be written

$$f(u,w) = 0 \qquad (14)$$

in which

$$f(u,w) \cdot\equiv\cdot \sin(g+h) + \sin g \cos h \,(1-\alpha^2)\left[\frac{k^2 w}{8}(1+u)^2 + \cdots\right] \qquad (15)$$

From equation (38), section 1.6,

$$k_m = m\pi \qquad m = 1, 2, 3, \ldots$$

$$\sin k_m = 0$$

$$\cos k_m = (-1)^m$$

$$g(0,0) = \gamma_1 k$$

$$h(0,0) = \alpha\gamma_2 k$$

$$f(0,0) = \sin(\gamma_1 + \alpha\gamma_2)k = \sin k = 0$$

To determine the implicit function $u(w)$, which is expanded about the point (0,0) some of the partial derivatives must be determined. These are designated by subscripts.

$$g_u(0,0) = \gamma_1 k$$

$$g_w(0,0) = \frac{\gamma_1 k^3}{24}$$

$$h_u(0,0) = \alpha \gamma_2 k$$

$$h_w(0,0) = \frac{\alpha^3 \gamma_2 k^3}{24}$$

$$f_u(0,0) = (g_u + h_u)\cos(g+h) = k \cos k = (-1)^m k$$

$$f_w(0,0) = (g_w + h_w)\cos(g+h) + k^2 \frac{1}{8}(1-\alpha^2)\sin\gamma_1 k \cos\alpha\gamma_2 k$$

$$= (-1)^m(\gamma_1 + \alpha^3\gamma_2)\frac{k^3}{24} + \frac{k^2}{8}(1-\alpha^2)\sin\gamma_1 k \cos(k-\gamma_1 k)$$

$$= (-1)^m[\gamma_1(1-\alpha^2)+\alpha^2]\frac{k^3}{24} + \frac{k^2}{16}(1-\alpha^2)(-1)^m \sin 2\gamma_1 k$$

From equation (26), section 1.4,

$$u'(0,0) = -\frac{f_w(0,0)}{f_u(0,0)} = -\frac{k}{24}\left\{[\gamma_1(1-\alpha^2)+\alpha^2]k + \frac{3}{2}(1-\alpha^2)\sin 2\gamma_1 k\right\}$$

$$u = a_1 w + a_2 w^2 + \cdots$$

$$u = \frac{a_1}{M^2} + \frac{a_2}{M^4} + \cdots$$

$$a_1 = -\frac{k}{24}\left\{[\gamma_1(1-\alpha^2)+\alpha^2]k + \frac{3}{2}(1-\alpha^2)\sin 2\gamma_1 k\right\}$$

Appendix 5 Summation of the Trigonometric Series of Sections 2.2 and 2.3

$$\sum_{n=1}^{N-1} \sin^2 \frac{mn\pi}{N} = \sum_{n=0}^{N-1} \sin^2 \frac{mn\pi}{N}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2mn\pi}{N}\right)$$

$$= \frac{N}{2} - \frac{1}{2}\mathcal{R}\sum_{n=0}^{N-1} e^{j\frac{2mn\pi}{N}}$$

$$= \frac{N}{2} - \frac{1}{2}\mathcal{R}\left[\frac{1-e^{j2m\pi}}{1-e^{j\frac{2m\pi}{N}}}\right]$$

$$\sum_{n=1}^{N-1} \sin^2 \frac{mn\pi}{N} = \frac{N}{2} \qquad m = 1, 2, \ldots, N-1 \qquad (1)$$

$$\sum_{n=0}^{N-1} \cos^2 \frac{m(n+\frac{1}{2})\pi}{N} = \sum_{n=0}^{N-1}\left(\frac{1}{2} + \frac{1}{2}\cos\frac{m(2n+1)\pi}{N}\right)$$

$$= \frac{N}{2} + \frac{1}{2}\mathcal{R}\left[e^{j\frac{m\pi}{N}}\sum_{n=0}^{N-1} e^{j\frac{2mn\pi}{N}}\right]$$

$$= \frac{N}{2} + \frac{1}{2}\mathcal{R}\left[e^{j\frac{m\pi}{N}}\frac{1-e^{j2m\pi}}{1-e^{j\frac{2m\pi}{N}}}\right]$$

$$\sum_{n=0}^{N-1} \cos^2 \frac{m(n+\frac{1}{2})\pi}{N} = \frac{N}{2} \qquad m = 1, 2, \ldots, N-1 \qquad (2)$$

$$\sum_{n=1}^{N-1} \sin^4 \frac{mn\pi}{N} = \sum_{n=0}^{N-1} \sin^4 \frac{mn\pi}{N}$$

$$= \sum_{n=0}^{N-1}\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2mn\pi}{N}\right)^2$$

$$= \sum_{n=0}^{N-1}\left(\frac{1}{4} - \frac{1}{2}\cos\frac{2mn\pi}{N} + \frac{1}{8} + \frac{1}{8}\cos\frac{4mn\pi}{N}\right)$$

$$= \frac{3}{8}N - \frac{1}{2}\mathcal{R}\left[\frac{1-e^{j2m\pi}}{1-e^{j\frac{2m\pi}{N}}}\right] + \frac{1}{8}\mathcal{R}\left[\frac{1-e^{j4m\pi}}{1-e^{j\frac{4m\pi}{N}}}\right]$$

$$\sum_{n=1}^{N-1} \sin^4 \frac{mn\pi}{N} = \frac{3}{8}N \qquad m = 1, 2, \cdots, N-1; \quad m \neq \frac{N}{2} \tag{3}$$

$$\sum_{n=1}^{N-1} \sin^4 \frac{mn\pi}{N} = \frac{N}{2} \qquad m = \frac{N}{2} \tag{4}$$

$$\sum_{n=0}^{N-1} \cos^4 \frac{m(n+\frac{1}{2})\pi}{N} = \sum_{n=0}^{N-1} \left[\frac{1}{2} + \frac{1}{2}\cos\frac{m(2n+1)\pi}{N}\right]^2$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{4} + \frac{1}{2}\cos\frac{m(2n+1)\pi}{N} + \frac{1}{8} + \frac{1}{8}\cos\frac{2m(2n+1)\pi}{N}\right]$$

$$= \frac{3}{8}N + \frac{1}{2}\mathcal{R}\left[e^{j\frac{m\pi}{N}}\frac{1-e^{j2m\pi}}{1-e^{j\frac{2m\pi}{N}}}\right] + \frac{1}{8}\mathcal{R}\left[e^{j\frac{2m\pi}{N}}\frac{1-e^{j4m\pi}}{1-e^{j\frac{4m\pi}{N}}}\right]$$

$$\sum_{n=0}^{N-1} \cos^4 \frac{m(n+\frac{1}{2})\pi}{N} = \frac{3}{8}N \qquad m = 1, 2, \ldots, N-1; \quad m \neq \frac{N}{2} \tag{5}$$

$$\sum_{n=0}^{N-1} \cos^4 \frac{m(n+\frac{1}{2})\pi}{N} = \frac{N}{4} \qquad m = \frac{N}{2} \tag{6}$$

List of Symbols

a	constant in differential equation
a_1, a_2, b_1, b_2, A, B, C, D	constants
C	capacitance
e	base of natural logarithms
EI	bending stiffness of beam
f, g, h	functions
i, j	indexes of summation
I	branch currents
j	$\sqrt{-1}$
k	constant in differential equation
k, k_m	eigenvalue (m^{th} mode) of continuous system
K	spring constant
L	inductance
m	mode number
M	circuit length in cells
M	bending moment
n	node number
N	number of cells
r	node number of variable element
$\Re$	real part of
S	shear
t	independent variable, time
T	transformer turns ratio
u	$\frac{Mz - k}{k}$

v, w	variables
v_n, V_n	node voltages
w	M^{-2}
W_e	electrostatic energy
W_m	electromagnetic energy
x	independent variable
$\overline{x}$	mean of x
y	dependent variable
Y	amplitude of oxcillations of y
y_n	node voltages
Y_n	amplitude of node voltage oscillations
z	$\omega\sqrt{LC}$ in 2nd order lumped system
z	$\sqrt{T\omega\sqrt{LC}}$ in 4th order lumped system
α	change in cell size
α, β, γ	constants
δ, Δ	increments
ϵ	per unit r.m.s. deviation
ϵ	increment of time
θ, θ_1, θ_2	parameters used in solutions of difference equations
λ	wavelength
μ	small number
ν	see Table 8, p. 34
ρ	mass per unit length
σ	r.m.s. deviation
ϕ, ϕ_n	slope
ϕ_1, ϕ_2, ...	normal coordinates

ω	frequency of oscillations
ω, ω_m	frequency of (m^{th}) normal mode
$\equiv$	is defined as