PRODUCTION AND ANNIHILATION
OF ANTIPROTONS

Thesis by

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ABSTRACT

If protons and neutrons are Dirac particles, as one usually assumes, the corresponding antiparticles should exist, but these have never been observed. The coupling of nucleons with the meson field offers processes by which these particles could be created in energetic collisions between nucleons and mesons or other nucleons. In this thesis the pseudoscalar meson theory is used to calculate cross-sections for the production of antiprotons in such collision processes. These are applied to estimate the numbers of antiprotons to be expected from the interaction of cosmic-ray particles with the nucleons of the atmosphere. It is found that meson production is about 60 times more frequent than antinucleon production for the complete primary cosmic-ray spectrum, but that antinucleon production is of comparable probability with meson production for energies greater than about 10^{11} e.v. Cross-sections are also calculated for the annihilation of antiprotons in collisions with protons and neutrons, with emission of mesons.

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INTRODUCTION

The theory of the electron proposed by Dirac in 1928 was remarkably successful in explaining the properties of that particle as known at the time[1]. For instance, it yielded the Sommerfeld expression for the fine-structure of the spectral lines of hydrogen. In addition, the fact that there was a spin angular momentum of $\frac{1}{2}\hbar$ associated with the electron came automatically from the theory. However, there was an important difficulty with the theory, namely that it offered the existence of states of negative energy for the electron. By interaction with the radiation field, transitions from states of positive to states of negative energy were possible, so that it appeared that the electron could lose an infinite amount of energy, gradually coasting down through states of more and more negative energy. It was not possible to disregard these states, for their presence was necessary in order that the set of solutions of the Dirac equation be "complete" in the mathematical sense. I.e., the expansion of an arbitrary wave-function in terms of the eigensolutions of the Dirac equation would in general have to include some of those associated with states of negative energy.

This difficulty of negative-energy states was circumvented by the assumption that in a vacuum all the states of negative energy are filled with electrons[2]. Then transitions to them are forbidden by the Pauli exclusion principle, which makes it impossible for two particles (fermions) to occupy the same state. But now it appeared that if sufficient energy were supplied -- namely, more than $2mc^2$, where m is the mass of the electron -- an electron in such a state of negative energy could be raised to a state of positive energy. The resultant hole in the "sea" of negative-energy electrons would behave like a particle of electronic mass but of opposite charge. At first, Dirac associated these positive particles with protons, but later the discovery by Anderson and Neddermeyer of the positron provided a physical realization of these hypothetical "anti-particles". The processes by which these are produced always involves the creation of a <u>pair</u> of particles, a positive and a negative electron, at the same time.

Many experiments, together with considerations of nuclear and molecular structure, have demonstrated that the proton and the neutron also have a spin angular momentum of $\frac{1}{2}\hbar$ and obey Fermi statistics. The only theory known which provides such properties for an elementary particle is the Dirac theory. Protons and neutrons, or "nucleons", as they are generically called, have about them the short-range nuclear force field, in addition to the electromagnetic field of the proton. To explain this, one assumes coupling of the nucleons with the field of mesons, so that nucleons can act as sources or sinks of mesons as well as of photons. These theories of the nucleon have as yet given only qualitative agreement with experiment. The lack of exact quantitative agreement may be due to the inadequacy of the methods of calculation. For instance, all meson theories predict that the nucleons will have an additional magnetic moment due to their interaction with the meson field (emission and absorption of virtual charged mesons), but none of the theories has thus far yielded the experimental values of the anomalous moments of the proton and the neutron.

A major difficulty with the interpretation of the proton as a Dirac particle is that the corresponding anti-particle, or "antiproton", should exist. This antiproton would have the same mass as the proton, but a negative charge and a magnetic moment of equal magnitude but of opposite sign. Such a particle has never been observed. It is the purpose of this thesis to investigate some of the properties predicted by current theory for the antiproton, to determine whether the fact that it has not been observed can be explained in this way.

The production of antiprotons through the electromagnetic interaction of protons is possible by the same type of processes whereby electron-positron pairs are made, provided sufficient energy is available. Since cross-sections for such reactions are proportional to the square of the Compton wavelength of the particle produced, it is seen that those for making proton-antiproton pairs would be smaller by a factor of $(1/1836)^2 = 3 \times 10^{-7}$ than the cross-sections for producing positron-electron pairs at corresponding energies. Thus the probability of such reactions would be unobservably small.

It is also through the coupling of nucleons with the meson field that we can expect reactions to occur in which nucleon pairs are produced. The coupling constant $g^2/\hbar c$ which defines the strength of the interaction is on the order of unity, as compared with the electromagnetic coupling constant $e^2/\hbar c = 1/137$. Since the coupling constant occurs to the third or fourth power in the cross-sections (computed in lowest order) for the various production processes, this difference in the sizes of the coupling constants may offset the above noted factor of 3×10^{-7}.

This theory requires that both the proton and the neutron be Dirac particles, so that there are two "seas", one of negative-energy protons, the other of negative-energy neutrons; and there are both antiprotons and antineutrons, each represented by "holes" in the respective seas. A neutron in a negative-energy state could give up a negative meson to become a proton which fills the hole which represents an antiproton. This corresponds to the virtual emission of negative mesons by antiprotons: $P^- \rightarrow \pi^- + N^*$, where N^* represents an antineutron. In this work we shall be primarily concerned with the antiproton, since the antineutron would be much more difficult to observe.

Nearly all of the calculations made by meson theory of processes in which nucleons occur in intermediate states, such as meson production (cf. Section VII below), assume the Dirac theory and hence the possible existence of the antiproton. For instance, the observed decay of the neutral meson into two gamma rays is usually explained by saying that the neutral meson turns into a virtual proton-antiproton pair, which annihilate by their coupling with the electromagnetic field into two photons.

The antiproton was considered in certain attempts to explain the origin of cosmic radiation, such as that of O. Klein[3], who assumed the existence of whole galaxies of "reversed matter". In these, nuclei were composed of antiprotons and antineutrons, and surrounded by positrons to form atoms. If such a galaxy of reversed matter were to drift through one composed of ordinary matter, the nuclei would annihilate into energetic mesons or gamma-rays, and the mesons would ultimately decay into electrons and neutrinos. These electrons were supposed to constitute the cosmic radiation. Since it is now reasonably certain that the primary radiation consists of protons and heavier nuclei, and that the content of electrons and gamma-rays is negligible, such theories as that of Klein are no longer considered.

On the other hand, N. Arley[4] assumed the primary radiation striking the atmosphere to consist of nearly equal numbers of protons and antiprotons. The antiprotons were to annihilate with protons of the air nuclei to form energetic gamma-rays, which in turn initiate the cascade showers observed as a part of the soft component of cosmic rays. In order to fit observations of the soft component, annihilation cross-sections on the order of 10^{-25} cm^2 were needed. This could not be obtained from the Dirac theory, which yields values of less than $10^{-33} cm^2$. At present, showers are believed to be initiated by the gamma-rays produced in the decay of the neutral mesons, which are formed in the high-energy collisions between the primary particles and the air nuclei.

One process by which antiprotons might be produced in the atmosphere is the collision of an energetic pimeson with a nucleon; e.g., $\pi^+ + N \rightarrow P^- + 2P$, or $\pi^- + P \rightarrow P^- + P + N$. The first attempt to estimate the cross-section for such a reaction was made by McConnell[5]. He used the Weizsäcker-Williams method (cf. Section VI below) in the following way. First he calculated, using the Heitler theory of radiation damping, the cross-section for the reaction $\pi^+ + \pi^- \rightarrow P^- + P$. Then in the collision of an energetic meson with a nucleon, he transformed to a Lorentz system in which the meson was at rest, with the nucleon moving at high velocity. The field of the nucleon was represented by mesons, which reacted with the resting meson to make a nucleon-antinucleon pair according to the above process. Now, at the time, it was thought necessary to use the theory of radiation damping in order to avoid certain divergences and other anomalies in the meson theory, but since then it has been shown that this method is unnecessary and even gives unreasonable results in certain cases[6]. Besides that, the method is so difficult to apply that results can be obtained only in the limits of very low or very high energies. Hence we have calculated the cross-section for a typical meson-nucleon process using the Feynman methods of perturbation theory (cf. Section IV). It is assumed that, despite the difficulties of the meson theory, this will give a rough description of the behavior of the cross-section at different energies, as well as an order-of-magnitude estimate of its size.

Antiprotons could also be produced in the upper atmosphere in the collisions of energetic primary cosmic-ray particles with nucleons of the air. We have attempted to estimate the probability of such processes by using the Weizsäcker-Williams method (Section VI). For collisions at extremely high energies, greater than about 10^{12} e.v., it is known from studies using photographic emulsions that large-scale disruptions of the nucleus occur, with several mesons and nucleons thrown off. For such occurrences, it is best to use a statistical method such as that of Fermi[7] to estimate the comparative numbers of mesons and antinucleons formed.

Once antiprotons are formed, what becomes of them? Clearly annihilation into two gamma-rays by collisions with a proton would be an improbable event, for the same reasons as those noted above for the unlikelihood of production through the electromagnetic coupling. On the other hand, annihilation with either a proton or a neutron to produce mesons could occur. The probabilities of such events are calculated in Section III below.*

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* The annihilation of antiprotons and protons to produce mesons was first considered by McConnell[5], using the theory of radiation damping. Because of the unreliability of this method, the cross-sections have been recalculated using ordinary perturbation theory. Independent work by Ashkin, Auerbach, and Marshak has already been published[8]. They give details concerning the distinctive appearance of annihilation events in photographic emulsions.

I. KINEMATICS OF THREE-PARTICLE PROCESSES.

For later reference we wish to develop notations and formulas for the kinematical relations involved in processes in which three particles are produced. The treatment will be relativistic except where otherwise noted, so that the term "energy of a particle" always refers to the total energy, including rest mass, and not to the kinetic energy alone. Masses and momenta are measured in energy units, being multiplied by c^2 and c respectively, where c is the velocity of light.

In the center-of-mass (c.m.) system, the energies of the three particles produced are E_1, E_2, and E_3 respectively, and the ordinary momentum vectors are $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$. These are combined in the four-vectors $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$, so that

(1.01) $$\hat{p}_1 = (E_1, \bar{p}_1), \text{ etc.}$$

Then if T is the total energy available in the c.m. system, and if $\hat{P}$ denotes the four-vector $(T,0)$, we have

(1.02) $$\hat{P} = \hat{p}_1 + \hat{p}_2 + \hat{p}_3$$

by the usual conservation laws of energy and momentum.

We introduce the scalar product of two four-vectors as follows:

(1.03) $$\hat{p}_1 \cdot \hat{p}_2 = E_1E_2 - \bar{p}_1 \cdot \bar{p}_2 = E_1E_2 - p_1p_2 \cos\theta$$

where p_1, p_2 are the magnitudes of $\bar{p}_1$, $\bar{p}_2$, and θ is the angle between these three-vectors. Then if m_1, m_2, m_3 are the masses of these three particles,

(1.04) $$\hat{p}_1 \cdot \hat{p}_1 = \hat{p}_1^{\,2} = m_1^{\,2} = E_1^{\,2} - p_1^{\,2}, \text{ etc.}$$

Thus we can use (1.02) to obtain

$$(1.05)\qquad \hat{p}_3^{\,2} = m_3^{\,2} = (\hat{P} - \hat{p}_1 - \hat{p}_2)^2 =$$
$$\hat{P}^2 + \hat{p}_1^{\,2} + \hat{p}_2^{\,2} - 2\hat{P}\cdot\hat{p}_1 - 2\hat{P}\cdot\hat{p}_2 + 2\hat{p}_1\cdot\hat{p}_2 =$$
$$T^2 + m_1^{\,2} + m_2^{\,2} - 2TE_1 - 2TE_2 + 2(E_1E_2 - p_1p_2\cos\theta)$$

or

$$(1.06)\qquad p_1p_2\cos\theta = \tfrac{1}{2}(T^2 + m_1^{\,2} + m_2^{\,2} - m_3^{\,2})$$
$$- TE_1 - TE_2 + E_1E_2 = \tfrac{1}{2}F^2 - TE_2 + E_1E_2$$

where

$$(1.07)\qquad F^2 = T^2 + m_1^{\,2} + m_2^{\,2} - m_3^{\,2} - 2TE_1 = G^2 + m_2^{\,2} - m_3^{\,2}.$$

Thus for a given total energy T, and a fixed momentum $\bar{p}_1$ for particle 1, the end of the momentum vector $\bar{p}_2$ lies on a surface of revolution about $\bar{p}_1$ given by (1.06). We shall need the extreme values of E_2, p_2, which occur for $\theta = 0$ and $\theta = \pi$. Let

$$(1.08)\qquad E_2 = E_2', \quad p_2 = p_2' \qquad \text{for} \quad \theta = 0$$

$$(1.09)\qquad E_2 = E_2'', \quad p_2 = p_2'' \qquad \text{for} \quad \theta = \pi.$$

From (1.06) we see that these are the roots of the equation

$$(1.10)\qquad p_1^{\,2}\, p_2^{\,2} = p_1^{\,2}\,(E_2^{\,2} - m_2^{\,2}) = \left[\tfrac{1}{2}F^2 - (T - E_1)E_2\right]^2$$

or

$$(1.11)\qquad 4G^2E_2^{\,2} - 4F^2(T - E_1)E_2 + F^4 + 4m_2^{\,2}p_1^{\,2} = 0$$

Then we must have

$$(1.12)\qquad E_2'' + E_2' = F^2G^{-2}(T - E_1)$$

$$(1.13)\qquad E_2'' - E_2' = p_1G^{-2}\left[G^2 - (m_2 + m_3)^2\right]^{\frac{1}{2}}\left[G^2 - (m_2 - m_3)^2\right]^{\frac{1}{2}}$$

From these it can be shown that

$$(1.14)\qquad p_2'' + p_2' = (T - E_1)G^{-2}\left[G^2 - (m_2 + m_3)^2\right]^{\frac{1}{2}}\left[G^2 - (m_2 - m_3)^2\right]^{\frac{1}{2}}$$

$$p_2'' - p_2' = F^2 G^{-2} p_1 \tag{1.15}$$

Now $p_2'' + p_2'$ is the longest diameter of the $\bar{p}_2$-surface. As the energy E_1 increases, this diameter decreases, and it vanishes for the maximum value E_1^o. This occurs when $G^2 - (m_2 + m_3)^2 = 0$ so that

$$E_1^o = \left[T^2 + m_1^2 - (m_2 + m_3)^2\right] / 2T \tag{1.16}$$

When $E_1 = E_1^o$,

$$E_2' = E_2'' = \frac{m_2}{m_2 + m_3}\left[\frac{T^2 - m_1^2 + (m_2 + m_3)^2}{2T}\right] \tag{1.17}$$

In this case the $\bar{p}_2$-surface has shrunk to a point, and both $\bar{p}_2$ and $\bar{p}_3$ are in the direction opposite to $\bar{p}_1$. The other extreme is for $E_1 = m_1$, $\bar{p}_1 = 0$, in which case symmetry shows the $\bar{p}_2$-surface to be a sphere, with

$$E_2 = \frac{F^2}{2(T - m_1)} = \frac{(T - m_1)^2 + m_2^2 - m_3^2}{2(T - m_1)} \tag{1.18}$$

We now wish to derive an expression for the differential volume of momentum space

$$d\tau = \frac{d^3\bar{p}_1\, d^3\bar{p}_2}{dT} = \frac{p_1 E_1 dE_1 d\Omega_1\, p_2 E_2 dE_2\, dx\, d\alpha}{dT} \tag{1.19}$$

where $d\Omega_1$ is an element of solid angle in the space of $\bar{p}_1$, and α is a polar angle which, with θ, locates the position of the vector $\bar{p}_2$. $x = \cos\theta$. Using (1.06),

$$dx = dT\left(\frac{\partial x}{\partial T}\right)_{E_2} = \frac{E_3}{p_1 p_2}\, dT$$

so that

$$d\tau = E_1 E_2 E_3\, dE_1\, d\Omega_1\, dE_2\, d\alpha \tag{1.20}$$

In integrations, α runs from 0 to 2π, while $E_2' \leqslant E_2 \leqslant E_2''$. The integration over E_1 is carried out last. The factor $E_1E_2E_3$ will always be cancelled out by other terms, and it is clear that any pair of particle momenta can be used in $d\tau$.

We now calculate the total volume of momentum space available to the three particles produced in the reaction, in the non-relativistic limit where

(1.21) $\quad U = T - (m_1 + m_2 + m_3) \ll m_1$ or m_2 or m_3

and none of the masses vanishes. In this case we can set $E_1 = m_1$, etc., in (1.17). The integration over α gives 2π, that over E_2 gives $E_2'' - E_2'$, and that over Ω_1 gives 4π, so that, using (1.13),

$$(1.22)\quad \int d\tau = 8\pi^2 m_1 m_2 m_3 \cdot \int_{m_1}^{E_1^0} \frac{p_1\, dE_1}{G^2}\left[G^2-(m_2+m_3)^2\right]^{\frac{1}{2}}\left[G^2-(m_2-m_3)^2\right]^{\frac{1}{2}}$$

In all these terms except the first bracket, we can assume $E_1 = E_1^0 = m_1$, so that $G = m_2 + m_3$. In the first bracket, however, we put, by (1.07),

$$(1.23)\quad G^2 - (m_2 + m_3)^2 = T^2 - 2TE_1 + m_1^2 - (m_2 + m_3)^2 = U(T - m_1 + m_2 + m_3) - 2T(E_1 - m_1)$$

Non-relativistically, $E_1 - m_1 = p_1^2/2m_1$ so that

$$(1.24)\quad G^2 - (m_2 + m_3)^2 = 2U(m_2 + m_3) - \frac{Tp_1^2}{m_1} = \frac{T}{m_1}(p_0^2 - p_1^2)$$

where p_0 is the maximum value attained by p_1, approximately. Thus (1.22) becomes

$$(1.25)\quad \int d\tau = \frac{16\pi^2 (m_2 m_3)^{3/2}}{(m_2+m_3)^2}\sqrt{\frac{T}{m_1}}\int_0^{p_0} p^2\sqrt{p_0^2 - p^2}\, dp = \frac{4\pi^3 (m_1 m_2 m_3)^{3/2}}{(m_1+m_2+m_3)^{3/2}}\, U^2$$

We shall need also the volume of available momentum space in the case of four particles of equal mass M produced in a reaction with total energy T in the c.m. system. To calculate this we use a method due to Fermi[7]. Again let $U = T - 4M$ be the excess energy, i.e., the total kinetic energy of the final particles. Then, non-relativistically,

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 = 2MU \tag{1.26}$$

$$\bar{p}_1 + \bar{p}_2 + \bar{p}_3 + \bar{p}_4 = 0 \tag{1.27}$$

Introduce the new vectors $\bar{q}$, $\bar{r}$, $\bar{s}$ given by

$$\begin{aligned} \bar{p}_1 &= \bar{q} + \bar{r} + \bar{s}, \qquad & \bar{p}_2 &= \bar{q} - \bar{r} - \bar{s} \\ \bar{p}_3 &= -\bar{q} + \bar{r} - \bar{s}, \qquad & \bar{p}_4 &= -\bar{q} - \bar{r} + \bar{s} \end{aligned} \tag{1.28}$$

Then

$$q^2 + r^2 + s^2 = \tfrac{1}{2}MU \tag{1.29}$$

where $q^2 = q_x^2 + q_y^2 + q_z^2$, etc. It is seen that

$$d^3\bar{p}_1\, d^3\bar{p}_2\, d^3\bar{p}_3 = J^3\, d^3\bar{q}\, d^3\bar{r}\, d^3\bar{s} \tag{1.30}$$

where $J = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & +1 \end{vmatrix} = 4$ is the Jacobian of the transformation (1.28). Thus

$$\int d\tau = \int \frac{d^3\bar{p}_1\, d^3\bar{p}_2\, d^3\bar{p}_3}{dT} = 2^6 \frac{d}{dT}\int d^3\bar{q}\, d^3\bar{r}\, d^3\bar{s} \tag{1.31}$$

where the latter integral is the volume of the nine-dimensional sphere (1.29), the radius of which is $R = \sqrt{\tfrac{1}{2}MU}$. Using the well-known formula for the volume of an n-dimensional sphere,

$$V_n = \frac{2}{n}\frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n)} R^n \tag{1.32}$$

the integral becomes

$$(1.33) \quad \int d\tau = 2^6 \frac{d}{dU} \frac{2^4}{105\sqrt{\pi}} \frac{2}{9} \pi^{9/2} \left(\tfrac{1}{2} M U\right)^{9/2}$$

$$= \frac{2^{11/2} \pi^4}{105} M^{9/2} U^{7/2}$$

II. APPLICATION OF THE FEYNMAN METHOD

In these calculations we shall use the methods of R.P. Feynman[9], and our notation will be the same, except where otherwise noted. In particular we use his convention as to the Dirac matrices γ_μ, with the definitions

(2.01) $$\gamma_4 = \beta, \quad \gamma_1 = \beta\alpha_x, \quad etc., \quad \bar{\gamma} = \beta\bar{\alpha}.$$

We use the notation

(2.02) $$\gamma_\mu p_\mu = \tilde{p} = \beta[p_4 - \bar{\alpha}\cdot\bar{p}], \quad \hat{p} = (p_4, \bar{p}).$$

Considering only the charged meson theory at present, we state the following prescription whereby the matrix element of the collision operator H in momentum space is written down from the Feynman diagram for the given process: For each virtual-nucleon line there is a term $(\tilde{p} - M)^{-1}$ where $\hat{p}$ is the four-momentum of the nucleon. For each virtual-meson line we put $(\hat{q}^2 - \mu^2)^{-1}$ where $\hat{q}$ is the four-momentum of the meson and μ is the mass of the meson. At each intersection where a virtual meson is emitted or absorbed, there is an interaction term of the form

$$\sqrt{4\pi}\, g\hbar c\, \bar{\psi}(p_f)\, O\, \psi(p_i)$$

where g is the meson coupling constant, and $\psi(p_i)$ and $\psi(p_f)$ are four-component plane-wave Dirac wave-functions for the initial and final nucleons.* O is an operator depending upon the

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* The factor of $\sqrt{4\pi}$ occurs because we are using unrationalized units.

type of coupling. For scalar coupling it is 1, for pseudoscalar γ_5, and for pseudovector $\gamma_5 \tilde{q}/\mu$. $\bar{\psi}(p) = \psi^*(p)\beta$ is the adjoint of $\psi(p)$. At an intersection involving a real meson of energy E, the interaction term above is multiplied by $(2E)^{-\frac{1}{2}}$.

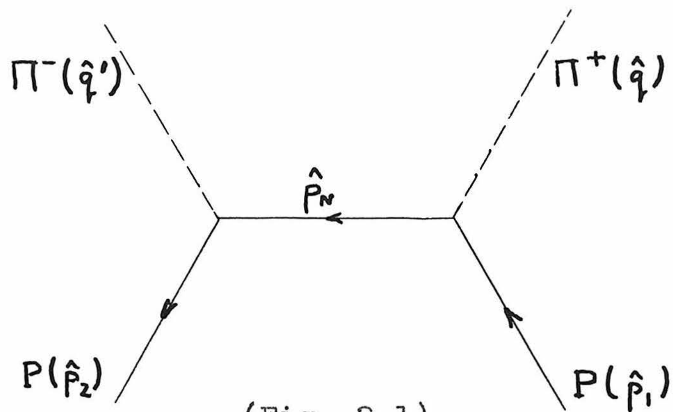
In determining the cross-section for the given process, the matrix element thus written down will be multiplied by its adjoint and summed over the various spin states of the initial and final nucleons by the spur technique presented by Feynman[9]. As is there pointed out, it is necessary to multiply by a factor $\left(\bar{\psi}\psi/\psi^*\psi\right) = M/E$ for each real nucleon involved, in order correctly to adjust the normalization, where M is the mass and E the energy of the nucleon.

The differential cross-section for a process resulting from the collision of two particles of relative velocity v is then

(2.03) $$d\sigma = \frac{2\pi}{\hbar v} |H|^2_{av} \rho_F$$

where $|H|^2_{av}$ is the absolute square of the matrix element of H summed over the spins of the final particles and averaged over the spins of the initial particles. ρ_F is the usual density of final states in momentum space.

As an illustration of the above prescription, we shall work out the cross-section for the annihilation of a proton and an antiproton into two charged mesons. In the c.m. system the proton is represented by the four-vector $\hat{p}_1 = (E, \bar{p})$, while the antiproton has $(E, -\bar{p})$. The process, represented by the



(Fig. 2.1)

diagram in Fig. 2.1, consists of the proton emitting two mesons $\hat{q} = (E, \bar{k})$ and $\hat{q}' = (E, -\bar{k})$ and going into the negative-energy state $\hat{p}_2 = (-E, \bar{p})$. The intermediate state is a neutron of momentum

(2.04) $$\hat{p}_N = (0, \bar{p} - \bar{k}) = \hat{p}_1 - \hat{q} = \hat{p}_2 + \hat{q}'$$

Then the pertinent matrix element is

(2.05) $$H = \frac{2\pi g^2 \hbar^2 c^2}{E} \bar{\psi}(p_2) O_2 (\tilde{p}_N - M)^{-1} O_1 \psi(p_1).$$

The relative velocity of the initial nucleons is

(2.06) $$v = (2p/E)c$$

The density of final states is

(2.07) $$\rho_F = \frac{d^3 \bar{k}}{(2\pi \hbar c)^3 dT} = \frac{kE\,d\Omega}{2(2\pi\hbar c)^3}$$

where $T = 2E$ is the total energy, and $d\Omega = \sin\theta\, d\theta\, d\varphi$ is the element of solid-angle for the meson direction. We get an additional factor of $\frac{1}{4}$ from the average over the spins of the initial nucleons, so that the differential cross-section is

(2.08) $$d\sigma = \frac{2\pi}{\hbar c}\left(\frac{E}{2p}\right)\frac{1}{4}\sum |H|^2 \frac{kE\,d\Omega}{2(2\pi\hbar c)^3}$$

where $\sum$ indicates that $|H|^2$ is summed over all possible initial and final states. $\psi(p_1)$ satisfies the Dirac equation

(2.09) $$\tilde{p}_1 \psi(p_1) = M \psi(p_1).$$

In the case of pseudoscalar coupling, $O = \gamma_5$, and we can simplify H by writing

(2.10) $$\bar{\psi}(p_2)\gamma_5 (\tilde{p}_N - M)^{-1} \gamma_5 \psi(p_1) =$$

$$\bar{\psi}(p_2)\gamma_5 (\tilde{p}_N + M) \gamma_5 \psi(p_1) (\hat{p}_N^2 - M^2)^{-1} =$$

$$\frac{\bar{\psi}(p_2)(\tilde{p}_N - M)\psi(p_1)}{\hat{p}_N^2 - M^2} = \frac{\bar{\psi}(p_2)(\tilde{p}_1 - \tilde{q} - M)\psi(p_1)}{-(\bar{p}-\bar{k})^2 - M^2} = \frac{\bar{\psi}(p_2)\tilde{q}\psi(p_1)}{(\bar{p}-\bar{k})^2 + M^2}.$$

(2.11) $$\therefore H = \frac{2\pi g^2 \hbar^2 c^2}{E[(\bar{p}-\bar{k})^2 + M^2]} \bar{\psi}(p_2)\tilde{q}\psi(p_1)$$

Taking the absolute square and summing,

$$\sum |\bar{\psi}(p_2)\tilde{q}\psi(p_1)|^2 = (2M)^{-2} Sp(\tilde{p}_2 + M)\tilde{q}(\tilde{p}_1 + M)\tilde{q}$$

(2.12) $$= M^{-2}[2(\hat{p}_2\cdot\hat{q})(\hat{p}_1\cdot\hat{q}) + \tfrac{1}{2}\mu^2(\hat{p}_1 - \hat{p}_2)^2] =$$

$$-2k^2(E^2 - p^2\cos^2\theta)M^{-2}.$$

This must be multiplied by $-M^2/E^2$, as noted above, so that

(2.13) $$\sum |H|^2 = \frac{(2\pi g^2\hbar^2c^2)^2 \cdot 2k^2(E^2 - p^2\cos^2\theta)}{E^4[(\bar{p}-\bar{k})^2 + M^2]^2}.$$

Hence the differential cross-section is

(2.14) $$d\sigma = \frac{g^4 k^3 (E^2 - p^2\cos^2\theta)\, d\Omega}{8pE^2[(\bar{p}-\bar{k})^2 + M^2]^2}$$

in the c.m. system, with

$$(\bar{p} - \bar{k})^2 + M^2 = E^2 + k^2 - 2pk\cos\theta = 2E(E - p\cos\theta) - \mu^2$$

It is of interest to derive our formula (2.05) from the perturbation theory of Feynman in this simple case. We note that we desire the transition amplitude A(2,1) from an initial plane-wave state

(2.15) $f(1) = (2\pi)^{-3/2}\,\psi(p_1)\,e^{-i\hat{p}_1\cdot\hat{x}_1}\,, \quad \hat{p}_1 = (E_1, \bar{p}_1)$

to a final plane-wave state

(2.16) $g(2) = (2\pi)^{-3/2}\,\psi(p_2)\,e^{-i\hat{p}_2\cdot\hat{x}_2}\,, \quad \hat{p}_2 = (-E_2, -\bar{p}_2)$

where E_1, E_2 are the energies and $\bar{p}_1$, $\bar{p}_2$ the momenta of the initial proton and antiproton respectively. This transition amplitude is given by

(2.17) $A(2,1) = -\iint \bar{g}(2)\,\beta\,K^{(2)}(2,1)\,\beta\,f(1)\,d^3\bar{x}_1\,d^3\bar{x}_2$

where

(2.18) $K^{(2)}(2,1) = -\iint K_{+2}(2,4)\,V(4)\,K_{+n}(4,3)\,V(3)\,K_{+1}(3,1)\,d\tau_3\,d\tau_4$

Here K_{+1} is the free-particle kernel for the initial proton, given by

(2.19) $K_{+1}(2,1) = \frac{i}{(2\pi)^4}\int (\tilde{p}_1 - M)^{-1}\exp(-i\,\hat{p}_1\cdot\hat{x}_{21})\,d^4 p_1$

where $\hat{x}_{21} = \hat{x}_2 - \hat{x}_1 = (t_2 - t_1, \bar{x}_2 - \bar{x}_1)$ and $d^4p_1 = dE_1 d^3\bar{p}_1$. (We take $\hbar = c = 1$ in this discussion.) Similar expressions hold for K_{+n} and K_{+2}. $d\tau_3 = dt_3 d^3\bar{x}_3$. Combining these expressions and using

(2.20) $f(3) = \int K_{+1}(3,1)\,\beta\,f(1)\,d^3\bar{x}_1$

with a similar expression for $\bar{g}(4)$, we find

(2.21)
$$A(2,1) = i(2\pi)^{-4}\iint \bar{g}(4)\,V(4)\,(\tilde{p}_N - M)^{-1}\cdot$$
$$\exp(-i\,\hat{p}_N\cdot\hat{x}_{43})\,V(3)\,f(3)\,d\tau_3\,d\tau_4\,d^4 p_N\,.$$

Now at the space-time point 3, the potential acting is given by charged-meson theory to be

$$V(3) = g\sqrt{\frac{2\pi}{Q_1}}\; O_1 \; \tau_{PN} \; e^{i\hat{q}_1\cdot\hat{x}_3} \tag{2.22}$$

where $\hat{q}_1 = (Q_1, \bar{q}_1)$, $q_2 = (Q_2, \bar{q}_2)$ are the four-momenta of the emitted mesons. O is the coupling operator used above, and τ_{PN} is an operator which changes a proton into a neutron. Similarly,

$$V(4) = g\sqrt{\frac{2\pi}{Q_2}}\; O_2 \; \tau_{NP} \; e^{i\hat{q}_2\cdot\hat{x}_4} \tag{2.23}$$

Since $\tau_{PN}\tau_{NP} = 1$ we get

$$\begin{aligned} A(2,1) &= i(2\pi)^{-7}\iiint d^4p_N\, d\tau_3\, d\tau_4 \,\cdot \\ &\exp[i(\hat{p}_2 - \hat{p}_N + \hat{q}_2)\cdot\hat{x}_4]\,\exp[i(\hat{q}_1 - \hat{p}_1 + \hat{p}_N)\cdot\hat{x}_3]\,\cdot \\ &\frac{2\pi g^2}{\sqrt{Q_1Q_2}}\;\bar{\psi}(p_2)\,O_2\,(\tilde{p}_N - M)^{-1}\,O_1\,\psi(p_1)\,. \end{aligned} \tag{2.24}$$

The last set of terms is merely H of (2.05) in this more general reference system. Now integrating over the space parts of $\hat{x}_3$ and $\hat{x}_4$, and then over the space part of $\hat{p}_N$,

$$\begin{aligned} A(2,1) &= i(2\pi)^{-1}\iiint d^4p_N\, dt_3\, dt_4\; \delta(\bar{q}_2 - \bar{p}_2 - \bar{p}_N)\,\cdot \\ &\delta(\bar{q}_1 - \bar{p}_1 + \bar{p}_N)\, e^{i(Q_2 - E_2 - E_N)t_4}\, e^{i(Q_1 - E_1 + E_N)t_3}\, H = \\ &i(2\pi)^{-1}\iiint dE_N\, dt_3\, dt_4\; \delta(\bar{q}_1 + \bar{q}_2 - \bar{p}_1 - \bar{p}_2)\,\cdot \\ &e^{i(Q_2 - E_2 - E_N)t_4}\, e^{i(Q_1 - E_1 + E_N)t_3}\, H\,. \end{aligned} \tag{2.25}$$

Let $E_1 + E_2 = E_i$ = initial energy, $Q_1 + Q_2 = E_f$ = final energy; $\bar{p}_1 + \bar{p}_2 = \bar{p}_f$ = initial momentum, $\bar{q}_1 + \bar{q}_2 = \bar{p}_f$ = final momentum. H can be written $\left(|E_N| = \sqrt{p_N^2 + M^2}\right)$,

$$H = \frac{2\pi g^2}{\sqrt{Q_1 Q_2}} \frac{\bar{\psi}(p_2)\, O_2 (\tilde{p}_N + M)\, O_1\, \psi(p_1)}{\hat{p}_N^2 - M^2} = \tag{2.26}$$

$$\frac{2\pi g^2}{\sqrt{Q_1 Q_2}} \frac{\bar{\psi}(p_2)\, O_2 (E_N \gamma_4 - \bar{\gamma}\cdot\bar{p}_N + M)\, O_1\, \psi(p_1)}{(E_N - |E_N|)(E_N + |E_N|)}$$

Now in integrating over E_N, we take $|E_N|$ to have an infinitesimal negative imaginary part, δ .[9] The integrand contains the factor $\exp\left[-iE_N(t_4 - t_3)\right]$. Thus for $t_3 < t_4$ we can complete the contour in the lower half of the E_N-plane and evaluate the integral by taking the residue at $E_N = +\sqrt{p_N^2 + M^2} - i\delta$ $= |E_N| - i\delta$, while for $t_3 > t_4$ we must complete the contour in the upper half of the plane and take the residue at $E_N =$ $- |E_N| + i\delta$. ($\delta \doteq 0$). Thus we get for $t_3 < t_4$, using the Cauchy residue formula,

$$\frac{1}{2\pi}\int dE_N\, e^{-iE_N(t_4 - t_3)}\, H = -\frac{i}{2}\, e^{-i|E_N|(t_4 - t_3)} \cdot \tag{2.27}$$

$$\frac{2\pi g^2}{\sqrt{Q_1 Q_2}} \frac{\bar{\psi}(p_2)\, O_2 (|E_N|\gamma_4 - \bar{p}_N\cdot\bar{\gamma} + M)\, O_1\, \psi(p_1)}{|E_N|}$$

while for $t_3 > t_4$

$$\frac{1}{2\pi}\int dE_N\, e^{-iE_N(t_4 - t_3)}\, H = +\frac{i}{2}\, e^{+i|E_N|(t_4 - t_3)} \cdot \tag{2.28}$$

$$\frac{2\pi g^2}{\sqrt{Q_1 Q_2}} \frac{\bar{\psi}(p_2)\, O_2 (-|E_N|\gamma_4 - \bar{p}_N\cdot\bar{\gamma} + M)\, O_1\, \psi(p_1)}{-|E_N|}$$

Combining, we ultimately obtain

$$\begin{aligned} A(2,1) = {} & \frac{2\pi g^2}{\sqrt{Q_1 Q_2}} \frac{1}{2|E_N|} \int_0^T dt_4 \Big\{ \int_0^{t_4} dt_3\, e^{i(Q_2 - E_2 - |E_N|)t_4} \cdot \\ & e^{i(Q_1 - E_1 + |E_N|)t_3} \left[\bar{\psi}(p_2)\, O_2 (|E_N|\gamma_4 - \bar{p}_N\cdot\bar{\gamma} + M)\, O_1\, \psi(p_1)\right] \\ & + \int_{t_4}^T dt_3\, e^{i(Q_2 - E_2 + |E_N|)t_4}\, e^{i(Q_1 - E_1 - |E_N|)t_3} \cdot \\ & \left[\bar{\psi}(p_2)\, O_2 (-|E_N|\gamma_4 - \bar{p}_N\cdot\bar{\gamma} + M)\, O_1\, \psi(p_1)\right]\Big\}\, \delta(\bar{p}_f - \bar{p}_i), \end{aligned} \tag{2.29}$$

where the integrals over t_3, t_4 are extended over the time of observation of the system, 0 to T. Carrying out the integration on t_3, we get

$$A(2,1) = \frac{2\pi g^2}{\sqrt{Q_1 Q_2}} \frac{1}{2|E_N|} \delta(\bar{p}_f - \bar{p}_i) \cdot$$

$$(2.30) \quad \int_0^T dt_4 \Big\{ \frac{e^{i(E_f - E_i)t_4} - e^{i(Q_2 - E_2 - |E_N|)t_4}}{i(Q_1 - E_1 + |E_N|)} \cdot$$
$$[\bar{\psi}(p_2) O_2 (|E_N|\gamma_4 - \bar{p}_N \cdot \bar{\gamma} + M) O_1 \psi(p_1)] +$$
$$\frac{e^{i(Q_1 - E_1 - |E_N|)T} e^{i(Q_2 - E_2 + |E_N|)t_4} - e^{i(E_f - E_i)t_4}}{i(Q_1 - E_1 - |E_N|)} \cdot$$
$$[\bar{\psi}(p_2) O_2 (-|E_N|\gamma_4 - \bar{p}_N \cdot \bar{\gamma} + M) O_1 \psi(p_1)] \Big\}$$

Now considering times T long compared to $\hbar$ over the various energies involved, only the terms in exp $i(E_f - E_i)t_4$ will contribute to the integral over t_4, the others averaging out to zero. Thus we get

$$(2.31) \quad A(2,1) = -\frac{2\pi g^2}{\sqrt{Q_1 Q_2}} \frac{e^{i(E_f - E_i)T} - 1}{E_f - E_i} \delta(\bar{p}_f - \bar{p}_i) \cdot$$
$$\Big\{ \frac{\bar{\psi}(p_2) O_2 (|E_N|\gamma_4 - \bar{\gamma}\cdot\bar{p}_N + M) O_1 \psi(p_1)}{|E_N| (Q_1 - E_1 + |E_N|)} +$$
$$\frac{\bar{\psi}(p_2) O_2 (-|E_N|\gamma_4 - \bar{\gamma}\cdot\bar{p}_N + M) O_1 \psi(p_1)}{-|E_N| (Q_1 - E_1 - |E_N|)} \Big\}.$$

Note that the terms in the brace are just those one would obtain in writing down the matrix element for such a process using the methods of the Dirac hole theory. They can be recombined to give

$$(2.32) \quad A(2,1) = \frac{e^{i(E_f - E_i)T} - 1}{E_f - E_i} H \, \delta(\bar{p}_f - \bar{p}_i)$$

The δ-function merely expresses conservation of momentum in the process, while the exponential term gives conservation of energy for long times of observation T. This expression (32) is then handled in the usual way to obtain the cross-section formula (2.03), with H given by (2.05), replacing E by $\sqrt{Q_1 Q_2}$.

III. ANNIHILATION OF ANTIPROTONS.

(a) Annihilation into Charged Mesons.

As shown in the previous section, the differential cross-section for the process

(3.01) $\quad P(\bar{p}) + P^{-}(-\bar{p}) \longrightarrow \Pi^{+}(\bar{q}) + \Pi^{-}(-\bar{q})$

with the pseudoscalar coupling is, by (2.14),

$$d\sigma = \frac{2\pi g^4 q^3 (E^2 - p^2\cos^2\theta)\sin\theta\, d\theta}{8pE^2(E^2+q^2-2pq\cos\theta)^2} \tag{3.02}$$

To get the total cross-section, we integrate over the angle θ, so that

$$\begin{aligned}\sigma &= \frac{\pi g^4 q}{8E^2 p}\left[-1+\frac{E^2+q^2}{2pq}\log\frac{M^2+(p+q)^2}{M^2+(p-q)^2} - \frac{\mu^4}{\mu^4+4M^2q^2}\right] \\ &= \frac{\pi g^4 q}{8E^2 p}\left[-1+\frac{E}{p}\log\frac{E+p}{E-p} + O\left(\frac{\mu^4}{M^4}\right)\right].\end{aligned} \tag{3.03}$$

Let E_o, p_o be the energy and momentum of the incident antiproton in the system in which the proton is initially at rest (laboratory system). Then

$$E^2 = \tfrac{1}{2}M(E_o + M),\quad p^2 = \tfrac{1}{2}M(E_o - M),\quad \frac{E+p}{E-p} = \frac{E_o+p_o}{M} \tag{3.04}$$

These give

$$\sigma = \frac{\pi}{4}\left(\frac{g^2}{M}\right)^2 \frac{M}{p_o}\left[1-\frac{\mu^2}{M(E_o+M)}\right]\left[-1+\frac{p_o}{E_o-M}\log\frac{E_o+p_o}{M}\right] + O\left(\frac{\mu^4}{M^4}\right) \tag{3.05}$$

as the total cross-section for annihilation in the laboratory system. Non-relativistically this is

$$\sigma = \frac{1}{4}\pi r_o^2 \frac{c}{v}\left[1-\frac{\mu^2}{2M^2}+O\left(\frac{\mu^4}{M^4}\right)\right] \tag{3.06}$$

where $r_o = g^2/M = 2.1 \times 10^{-14}(g^2/\hbar c)$ cm., and v is the relative velocity of proton and antiproton. For very high energies,

(3.07) $$\sigma \doteq \frac{1}{4}\pi r_0^2 \frac{M}{E_0}\left(\log\frac{2E_0}{M}-1\right).$$

The scalar theory gives, with O = 1,

(3.08) $$d\sigma = \frac{g^4 q\, d\Omega}{8pE^2}\left[\frac{q^2(E^2-p^2\cos^2\theta)+M^2 p(p-4q\cos\theta)}{(E^2+q^2-2pq\cos\theta)^2}\right]$$

which becomes, upon integration,

(3.09) $$\sigma = \frac{\pi g^4 q}{8E^2 p}\left[-1+\frac{4M^2+2E^2-\mu^2}{2pq}\log\frac{M^2+(p+q)^2}{M^2+(p-q)^2}-\frac{(4M^2-\mu^2)^2}{\mu^4+4M^2q^2}\right].$$

This reduces to (3.06) and (3.07) in the limits of low and high energies, respectively, so that we need not consider the scalar case further.

Another interesting case is that of pseudoscalar mesons with gradient coupling. Here we have $O = \gamma_5\tilde{q}/\mu$, so that H becomes

(3.10) $$H = \frac{2\pi g^2}{E\mu^2}\bar{\Psi}(p_2)\gamma_5\tilde{q}_2(\tilde{p}_N-M)^{-1}\gamma_5\tilde{q}_1\psi(p_1) =$$

$$-\frac{2\pi g^2}{E\mu^2}\frac{\bar{\Psi}(p_2)\tilde{q}_2(\tilde{p}_N-M)\tilde{q}_1\psi(p_1)}{\hat{p}_N^2-M^2} = \frac{2\pi g^2}{E\mu^2}\frac{\bar{\Psi}(p_2)(\tilde{p}_N-M)^3\psi(p_1)}{\hat{p}_N^2-M^2}$$

since

(3.11) $$\tilde{q}_1\,\psi(p_1) = (\tilde{p}_1-\tilde{p}_N)\,\psi(p_1) = -(\tilde{p}_N-M)\,\psi(p_1)$$

(3.12) $$\bar{\Psi}(p_2)\tilde{q}_2 = \bar{\Psi}(p_2)(\tilde{p}_N-\tilde{p}_2) = \bar{\Psi}(p_2)(\tilde{p}_N-M).$$

We can write (3.10) as

(3.13) $$H = \frac{2\pi g^2}{E\mu^2}\bar{\Psi}(p_2)\left[2M+\left(\frac{\hat{p}_N^2+3M^2}{\hat{p}_N^2-M^2}\right)\tilde{q}_1\right]\psi(p_1)$$

Squaring and averaging by the usual spur technique give

$$(3.14)\quad d\sigma = \frac{\pi g^4 q \sin\theta\, d\theta}{4E^2\mu^4 p}\left\{ q^2(E^2 - p^2\cos^2\theta) + 2M^2\mu^2 + \frac{2M^2\mu^4}{E^2+q^2-2pq\cos\theta} - \frac{4M^4\mu^4}{(E^2+q^2-2pq\cos\theta)^2}\right\}$$

If we integrate this expression over θ and neglect terms of order μ^2/M^2, we find at low energies

$$(3.15)\qquad \sigma \doteq \frac{\pi g^4 M^2}{\mu^4}\frac{c}{v}.$$

At very high energies, on the other hand, we get

$$(3.16)\qquad \sigma \doteq \frac{\pi g^4 M E_0}{6\mu^4}.$$

Thus the cross-section for annihilation increases with the energy E_0 of the incident antiproton. It was this anomaly which led McConnell[5] to apply the Heitler theory of radiation damping to this calculation. But as we have seen, the pseudoscalar direct coupling yields the expression (3.07), which drops off at high energies in a reasonable manner. When McConnell wrote his papers, the pseudoscalar direct coupling was in disrepute because of a misconception as to the nature of the nuclear potential it yields. This difficulty has since then been cleared up.

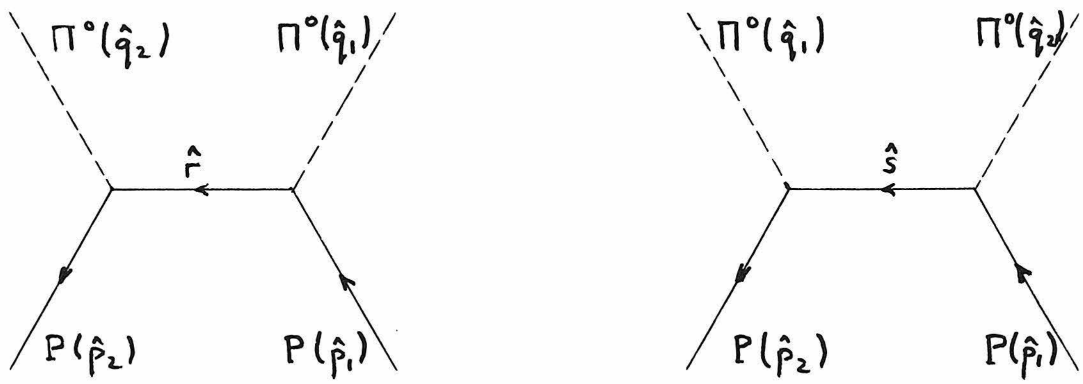
(b) Annihilation into Neutral Mesons.

In the following work, wherever the effects of neutral mesons are included, we use the form of meson theory which assumes coupling through the isotopic-spin operator τ_3, so that there is a difference in sign depending upon whether the neutral meson is emitted by a proton or by a neutron. For a virtual neutral meson, the sign of the term is positive when emission and absorption are by the same type of nucleon, and it is negative when the meson is emitted by one type and absorbed by the other type of nucleon. The coupling constant will be g_0. For the symmetric theory of Kemmer, take $g_0 = g/\sqrt{2}$.

We shall now study the following process:

$$P^{+}(\bar{p}) + P^{-}(-\bar{p}) \longrightarrow \Pi^{0}(\bar{q}) + \Pi^{0}(-\bar{q}). \tag{3.17}$$

In this case there are two possible Feynman diagrams, depending upon which of the neutral mesons is emitted first. These are given in Fig. 3.1.



(Fig. 3.1)

$$\hat{r} = \hat{p}_1 - \hat{q}_1 = \hat{p}_2 + \hat{q}_2, \quad \hat{s} = \hat{p}_1 - \hat{q}_2 = \hat{p}_2 + \hat{q}_1 \tag{3.18}$$

where $\hat{r}$, $\hat{s}$ are the four-momenta of the intermediate protons. Then the operator H, in the pseudoscalar theory, becomes

$$H = 2\pi g_0^2 E^{-1}\left[\bar{\psi}(p_2)\gamma_5(\tilde{r}-M)^{-1}\gamma_5\psi(p_1) + \bar{\psi}(p_2)\gamma_5(\tilde{s}-M)^{-1}\gamma_5\psi(p_1)\right] = \tag{3.19}$$

$$\frac{2\pi g_0^2}{E}\left(\frac{1}{\hat{r}^2-M^2} - \frac{1}{\hat{s}^2-M^2}\right)\bar{\psi}(p_2)\tilde{q}_1\psi(p_1)$$

since $\quad \bar{\psi}(p_2)\tilde{q}_2\psi(p_1) = -\bar{\psi}(p_2)\tilde{q}_1\psi(p_1).$

This gives, in the center-of-mass system,

$$(3.20)\quad d\sigma = \frac{g_0^4 q^3 p \cos^2\theta\,[(E^2+q^2)^2 - 4p^2q^2\cos^2\theta - \mu^4]\sin\theta\, d\theta\, d\varphi}{2E^2\,[(E^2+q^2)^2 - 4p^2q^2\cos^2\theta]^2}$$

The integration on φ runs only from 0 to π, since the two neutral mesons are indistinguishable. Integrating on φ and θ,

$$(3.21)\quad \sigma = \frac{\pi g_0^4 q}{4E^2 p}\left\{ -1 + \frac{1}{4pq}\left(E^2+q^2+\frac{\mu^4}{2(E^2+q^2)}\right)\log\frac{M^2+(p+q)^2}{M^2+(p-q)^2} - \frac{\mu^4}{2(\mu^4+4M^2q^2)}\right\}.$$

Transforming to the laboratory system by use of (3.04), and neglecting terms of order μ^4/M^4,

$$(3.22)\quad \sigma = \frac{\pi g_0^4}{4M(E_0-M)}\left\{\log\frac{E_0+p_0}{M} - \frac{2p_0}{E_0+M}\right\}\left(1-\frac{\mu^2}{M(E_0+M)}\right).$$

Non-relativistically, with v the relative velocity of proton and antiproton,

$$(3.23)\quad \sigma \doteq \frac{\pi}{24}\left(\frac{g_0^2}{M}\right)^2 \frac{v}{c}\left(1-\frac{\mu^2}{2M^2}\right)$$

while for very high energies,

$$(3.24)\quad \sigma \doteq \frac{\pi}{4}\left(\frac{g_0^2}{M}\right)^2 \frac{M}{E_0}\left(\log\frac{2E_0}{M} - 2\right)$$

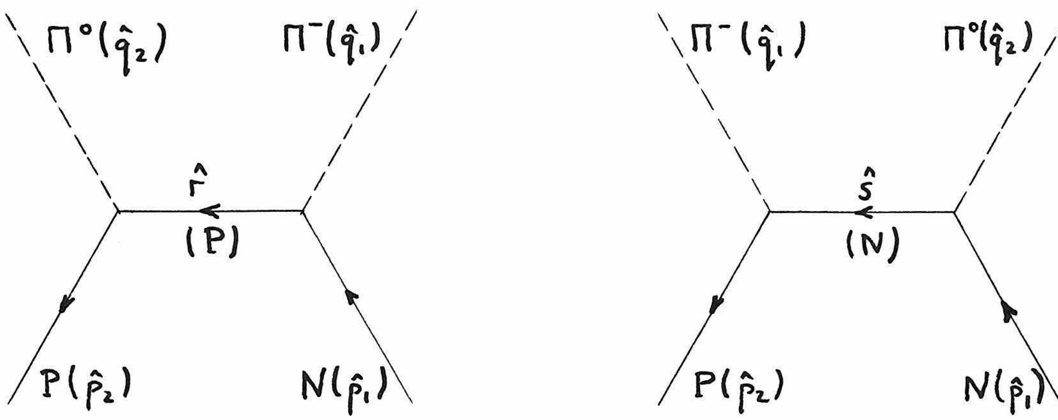
Thus we see that annihilation at low energies is predominantly into charged mesons. This also follows from the fact that for very low energies, the initial nucleons form an S-state with J = 0 or J = 1. The parity of such a state with a particle and an antiparticle is odd. The neutral mesons, having spin zero and Bose statistics, can be produced only in states of even parity and even J. Thus the annihilation into neutral mesons is forbidden in the limit of zero relative velocity, because of the necessity for the conservation of angular momentum and parity. (This argument is due to Mr. M. Ruderman.)

(c) Annihilation with a Neutron.

Since our theory treats protons and neutrons on an equal basis, we must expect that an antiproton could react with a neutron, producing a negative meson and a neutral meson:

$$P^{-}(-\bar{p}) + N(\bar{p}) \longrightarrow \Pi^{-}(\bar{q}) + \Pi^{0}(-\bar{q}) \tag{3.25}$$

Here again there are two possible processes, illustrated by the accompanying diagrams (Fig. 3.2). In the former case, the



(Fig. 3.2)

intermediate nucleon is a proton; in the latter, a neutron. As noted above, the symmetric theory then requires a difference in sign of the corresponding terms of H, which we can write down as follows:

$$\begin{aligned} H = 2\pi g g_0 E^{-1} \{ \bar{\Psi}(p_2)\gamma_5(\tilde{r}-M)^{-1}\gamma_5\psi(p_1) - \\ \bar{\Psi}(p_2)\gamma_5(\tilde{s}-M)^{-1}\gamma_5\psi(p_1) \} = \\ 2\pi g g_0 E^{-1}\left[(\hat{r}^2-M^2)^{-1}+(\hat{s}^2-M^2)^{-1}\right]\bar{\Psi}(p_2)\,\tilde{q}_1\,\psi(p_1). \end{aligned} \tag{3.26}$$

This gives the differential cross-section in the center-of-mass system:

$$d\sigma = \frac{\pi g^2 g_0^2 q}{4E^2 p}\left[(E^2+q^2)^2 - 4p^2q^2\cos^2\theta - \mu^4\right]\cdot \tag{3.27}$$
$$\left[(E^2+q^2)^2 - 4p^2q^2\cos^2\theta\right]^{-2}(E^2+q^2)^2 \sin\theta\, d\theta .$$

The total cross-section is

$$\sigma = \frac{\pi g^2 g_0^2 (E^2+q^2)}{8E^2p^2}\left\{\left[1-\frac{\mu^4}{2(E^2+q^2)^2}\right]\log\frac{M^2+(p+q)^2}{M^2+(p-q)^2} \right. \tag{3.28}$$
$$\left. - \frac{2pq\mu^4}{(\mu^4+4M^2q^2)(E^2+q^2)}\right\}$$

In the laboratory system this is, to terms of order μ^4/M^4,

$$\sigma = \frac{\pi g^2 g_0^2}{2M^2}\cdot\left[1-\frac{\mu^2}{M(E_0+M)}\right]\frac{M}{E_0-M}\log\frac{E_0+p_0}{M}. \tag{3.29}$$

At low velocities,

$$\sigma \doteq \frac{\pi g^2 g_0^2}{M^2}\frac{c}{v}\left(1-\frac{\mu^2}{2M^2}\right) \tag{3.30}$$

while at high energies,

$$\sigma \doteq \frac{\pi g^2 g_0^2}{2M^2}\frac{M}{E_0}\log\frac{2E_0}{M} \tag{3.31}$$

Thus the cross-section for annihilation of an antiproton with a neutron is twice that for annihilation with a proton, non-relativistically, if we put $g_0^2 = \frac{1}{2}g^2$ as in the Kemmer theory.

(d) Annihilation with a Bound Neutron.

It is of interest to consider the possibility that a slowly moving antiproton will annihilate with a neutron bound in a nucleus, with only one meson being emitted, the nucleus taking up the extra momentum. In this case, we can write the wave-function of the neutron as a unit Dirac spinor for a state of positive energy times the Schrödinger wave-function $\psi(r)$ of the neutron, supposed bound in some sort of potential well which represents the rest of the nucleus. The neutron will emit a negative meson and become a proton of negative energy, filling the hole which corresponds to the antiproton. Thus the final wave-function will be a unit Dirac spinor for a state of negative energy (and proper spin) times a factor exp $\left[-i\bar{p}'\cdot\bar{r}/\hbar c\right]$, where $\bar{p}'$ is the momentum of the antiproton. With pseudoscalar coupling, the Hamiltonian operator will be

$$(3.32)\quad H = -ig\hbar c\sqrt{\frac{2\pi}{E_\pi}}\int d^3\bar{r}\,(\psi_P{}^*\beta\gamma_5\psi_N)\,e^{-i\bar{q}\cdot\bar{r}/\hbar c}$$

where $\bar{q}$ is the momentum of the emitted meson. $E_\pi = \sqrt{q^2+\mu^2}$ = energy of emitted meson $\cong$ 2M. The spinor parts of the wave-functions, combined with $\beta\gamma_5$ will give unity, provided the neutron is of spin opposite to that of the antiproton, and they give zero otherwise. Thus annihilation will occur with about half of the neutrons in the nucleus, namely those of proper spin. Thus (3.32) becomes

$$(3.33)\quad H = -ig\hbar c\sqrt{\frac{2\pi}{E_\pi}}\int d^3\bar{r}\,\psi(\bar{r})\,e^{i(\bar{p}'-\bar{q})\cdot\bar{r}/\hbar c}$$

the integration being taken over all space. Now $|\bar{q}|$ is of the order of 2M, while $|\bar{p}'|$ is small compared to M, in the non-relativistic limit we are considering. Hence (3.33) represents the Fourier component of the bound-neutron wave-function corresponding to a momentum of about 2 Bev. Let $\bar{p} = -\bar{p}' + \bar{q}$. Then

$$(3.34)\quad H = -ig\frac{(2\pi)^2}{\sqrt{E_\pi}}\,\varphi(\bar{p})$$

where we have taken $\hbar = c = 1$. Also

$$(3.35)\quad \varphi(\bar{p}) = (2\pi)^{-\frac{3}{2}}\int\psi(\bar{r})\,e^{-i\bar{p}\cdot\bar{r}}\,d^3\bar{r}$$

is the momentum representation of the wave-function of the bound neutron. The cross-section for annihilation is then

(3.36) $$\sigma = \frac{2\pi}{\hbar v} \frac{1}{2} |H|^2 \rho_F$$

where v is the velocity of the incident antiproton, and ρ_F is the density of final states, given by

(3.37) $$\rho_F = \frac{q E_\pi}{(2\pi)^3} \int d\Omega_1 = \frac{4\pi q E_\pi}{(2\pi)^3}$$

(3.38) $$\therefore \sigma \cong 8\pi^3 g^2 v^{-1} q \, |\varphi(\bar{p})|^2$$

with $q \cong \sqrt{4M^2 - \mu^2} \cong 2M$.

To compute $\varphi(p)$, we write the Schrödinger equation

(3.39) $$-\frac{\hbar^2}{2M} \nabla^2 \psi + V\psi = E\psi$$

where V is the potential binding the neutron. Now substitute the inverse of (3.35):

(3.40) $$\psi(\bar{r}) = (2\pi)^{-\frac{3}{2}} \int \varphi(\bar{p}) e^{i\bar{p}\cdot\bar{r}} d^3\bar{p}$$

so that

(3.41) $$(2\pi)^{-\frac{3}{2}} \int \left(E - \frac{p^2}{2M}\right) \varphi(\bar{p}) e^{i\bar{p}\cdot\bar{r}} d^3\bar{p} = V(\bar{r})\psi(\bar{r}).$$

Take the inverse Fourier transform of this equation:

(3.42) $$\varphi(\bar{p}) = (2\pi)^{-\frac{3}{2}} \left(E - \frac{p^2}{2M}\right)^{-1} \int V(\bar{r}) \psi(\bar{r}) e^{-i\bar{p}\cdot\bar{r}} d^3\bar{r}$$

Now for $|\bar{p}| \cong 2M$, the exponential varies rapidly, while $\psi(\bar{r})$, being the ground-state wave-function, varies relatively slowly over the nucleus, where V(r) is different from zero. Hence we can replace $\psi(\bar{r})$ by its average ovdr the nucleus, ψ_a, and write, since $p^2/2M \gg E$,

$$(3.43)\quad \varphi(\bar{p}) \cong -2M\,p^{-2}(2\pi)^{-3/2}\psi_a \int V(\bar{r})\,e^{-i\bar{p}\cdot\bar{r}}\,d^3\bar{r}$$
$$= -2M\,p^{-2}(2\pi)^{-\frac{3}{2}}\psi_a\, v(\bar{p})$$

where $v(\bar{p})$ is the Fourier component of the potential for large momentum transfer $\bar{p}$. This can be estimated from high-energy neutron-proton scattering, for the differential cross-section of which the Born approximation gives the formula

$$(3.44)\quad d\sigma_{sc} = \frac{M^2}{4\pi^2}\,|v(\bar{p})|^2\,d\Omega\,.$$

Replacing $d\Omega$ by 4π to estimate the total scattering cross-section, σ_{sc} , we can combine (3.38), (3.43-.44) to obtain

$$(3.45)\quad \sigma \cong \frac{\pi}{2}\left(\frac{g^2}{\hbar c}\right)\frac{c}{v}\,|\psi_a|^2\,M^{-3}\,\sigma_{sc}$$

We can estimate $|\psi_a|^2$ by the reciprocal of the nuclear volume per nucleon, which is approximately $|\psi_a|^2 \sim a^{-3}$, where $a \cong 1.5 \times 10^{-13}$ cm., so that $|\psi_a|^2 M^{-3} \sim (M_a)^{-3}$; $Ma \cong 0.14$.

$$(3.46)\quad \therefore\ \sigma \cong \frac{\pi}{2}\left(\frac{g^2}{\hbar c}\right)\frac{c}{v}(Ma)^{-3}\,\sigma_{sc}\,.$$

At the highest energy available, about 270 Mev, the total n-p scattering cross-section[10] is $\sigma_{sc} \cong 0.038 \times 10^{-24}$ cm^2 = $86(\hbar c/M)^2$. This corresponds to a momentum of about 0.38M in the center-of-mass system, or an average momentum transfer of about $\frac{1}{2}$M, so that our result will be somewhat too high. We get for this one-meson annihilation cross-section

$$(3.47)\quad \sigma \lesssim \frac{\pi}{2}\,\frac{g^2}{\hbar c}\left(\frac{\hbar c}{M}\right)^2\frac{c}{v}\,(0.2)$$

This is seen to be about 1/5 or less of the cross-section (3.30) for annihilation into two mesons.

(e) Annihilation in Matter.

When an antiproton passes through matter, it may annihilate by any one of the processes (3.01), (3.17), and (3.25). For energies of more than 1 Bev. or so, the annihilation cross-sections are small, and the antiproton may be expected to lose energy by the same processes as does an ordinary proton. The ionization energy-loss would be approximately the same for an antiproton as for a proton at all energies. For kinetic energies less than 1 Bev. or so, the cross-sections σ_1 for annihilation with a proton into two charged mesons, and σ_2 for annihilation with a neutron as in (3.25), will become large, and the antiprotons will disappear in this manner. We can neglect the cross-section (3.22) for annihilation into two neutral mesons, for this is smaller than the others by a factor of nearly 1/100 at these low energies.

Let n be the number of atoms per cm^3., each atom containing Z protons and N neutrons. Then the probability that the antiproton is annihilated in going a distance dx is

$$dY = n(Z\sigma_1 + N\sigma_2)\,dx \tag{3.48}$$

where σ_1, σ_2 are given in (3.05) and (3.29) respectively. In a distance dx, the antiproton loses an energy dE = (dE/dx)dx. Thus in slowing down from an energy E_2 to energy E_1, the probability of annihilation is

$$Y(E_2, E_1) = n\int_{E_1}^{E_2} (Z\sigma_1 + N\sigma_2)\left(\frac{dE}{dx}\right)^{-1} dE. \tag{3.49}$$

Values of the energy loss (dE/dx) and the range of protons in S.T.P. air and in aluminum hawe been calculated by Smith[11]. We use his results and evaluate (3.49) by numerical integration, taking $g_0 = g/\sqrt{2}$ and $g^2/\hbar c = 1$. (For antiprotons in air of kinetic energies less than 10 Mev, we use the energy-loss curve for protons given by Bethe[12].) Thus we obtain the following tables, where E_1, E_2 are kinetic energies in Mev, and ΔR is the range traversed by the antiproton in slowing down from energy E_2 to energy E_1.

<u>Air</u>

E_2 (Mev)	E_1 (Mev)	Y	ΔR (cm)	ΔR ($\frac{gm}{cm^2}$)
2000	1000	1.89×10^{-1}	4.36×10^{5}	564
1000	400	1.45	2.23	289
400	300	2.58×10^{-2}	2.82×10^{4}	36.5
300	200	2.54	2.34	30.3
200	100	2.37	1.67	21.6
100	45	1.09	5.40×10^{3}	6.99
45	25	3.23×10^{-3}	1.11	1.44
25	15	1.22	0.352	0.455
10	6	4.05×10^{-4}	68	88×10^{-3}
6	2	5.42	40	52
2	1	1.29	4.85	6.3
1	0.6	1.02	1.30	1.7
0.6	0.2	0.88	0.76	1.0

Aluminum*

E_2 (Mev)	E_1 (Mev)	Y	ΔR (cm)
2000	1000	9.93×10^{-2}	218
1000	400	7.69	112
400	300	1.38	14.2
300	200	1.36	11.8
200	100	1.27	8.51
100	45	5.92×10^{-3}	2.77
45	25	1.77	0.574
25	12	9.56×10^{-4}	0.226
12	3	5.11	0.077

It is seen that an antiproton of a few Bev. energy has a good chance of annihilating in flight, with a probability of about 0.35 $(g^2/\hbar c)^2$ for a kinetic energy of 2 Bev. in air.

If it escapes annihilation in flight, the antiproton will be slowed down and eventually stop, being captured by a positively charged nucleus. Wightman[14] has calculated the moderation times of antiprotons in hydrogen gas and finds times on the order of 10^{-10} sec. for slowing down from 10 Mev. to capture by an H_2 molecule. In heavy substances the times would be expected to be somewhat shorter than this. We note that the antiproton would arrive in the lowest Bohr orbit about one of the nuclei by radiating photons in a time τ_R given roughly by the ordinary radiation formula

$$\tau_R^{-1} \sim \frac{e^2 v^3 a^2}{\hbar c^3} \tag{3.52}$$

* Note that a standard photographic emulsion such as is used in nuclear investigations has a stopping power and average Z comparable to that of aluminum[13]. Hence we can use these results for aluminum to get the annihilation probability in an emulsion by multiplying the Y of the table by the ratio of the densities, which is about 1.45.

where a is the Bohr radius $\hbar^2/ZMe^2$, while $h\nu$ would be on the order of the ground-state energy, $(Ze^2/\hbar c)^2 Mc^2$. Thus we get $\tau_R^{-1} \sim Z^4 \left(\frac{e^2}{\hbar c}\right)^5 \frac{M}{\hbar}$ or $\tau_R \sim Z^{-4} \cdot 3 \cdot 10^{-14}$ sec.

The lifetime τ_a of an antiproton in the ground state about a nucleus can be estimated by noting that it is given approximately by the cross-section $Z\sigma_1 + N\sigma_2 \cong \frac{\pi}{4} r_0^2 (Z+2N) \frac{c}{v}$ multiplied by the relative velocity v, times the density $|\psi(0)|^2$ of antiprotons at the nucleus. Here $\psi(0)$ is the value at the origin of $\psi(r)$, the wave-function of the antiproton in the lowest Bohr orbit. Taking this as the hydrogen-like wave-function

(3.53) $$\psi(r) = \left(Z^3/\pi a_0^3 \right)^{\frac{1}{2}} e^{-Zr/a_0}$$

where $a_0 = \dfrac{\hbar^2 c^2}{M' e^2}$, $M' = \dfrac{AM}{A+1}$, $A = Z + N$

we find

(3.54) $$|\psi(0)|^2 = \frac{1}{\pi} \left(\frac{Ze^2}{\hbar c} \right)^3 \left(\frac{M}{\hbar c} \right)^3 (A^{-1} + 1)^{-3}$$

(3.55) $$\therefore \tau_a^{-1} \cong \frac{1}{8} \left(\frac{g^2}{\hbar c} \right)^2 \frac{Z+2N}{(1+A^{-1})^3} \left(\frac{Ze^2}{\hbar c} \right)^3 \frac{M}{\hbar}$$

(We have divided by two to account for the fact that the antiproton will annihilate only with nucleons of opposite spin, in the non-relativistic limit.) This gives

(3.56) $$\tau_a \cong \left(\frac{g^2}{\hbar c} \right)^{-2} \frac{(1+A^{-1})^3}{Z^3 (Z+2N)} \times 1.44 \times 10^{-17} \text{ sec.}$$

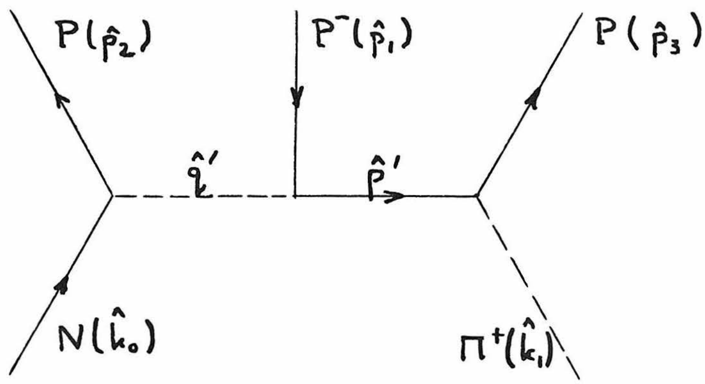
Of course for most nuclei, the ground-state radius is of the same order of magnitude as the nuclear radius, so that the antiproton will spend most of its time within the nucleus, and the annihilation time τ_a will be much shorter than the estimate (3.56). (For the special case of the antiproton bound to a proton, see Appendix I.) This time is very short compared with the time taken by the antiproton to slow down and be captured by the atomic nucleus.

IV. PRODUCTION OF ANTIPROTONS IN MESON-NUCLEON COLLISIONS.

The first production process we shall consider will be the result of the collision of a positive meson and a neutron:

$$\Pi^+(\hat{k}_1) + N(\hat{k}_o) \rightarrow P^-(\hat{p}_1) + P(\hat{p}_2) + P(\hat{p}_3) \quad (4.01)$$

This is described by the diagram of Fig. 4.1.



(Fig. 4.1)

Here we have, in the center-of-mass system,

$$\hat{k}_o = (E_o, \bar{k}), \ \hat{k}_1 = (\varepsilon, -\bar{k}) \quad (4.02)$$

$$\hat{p}_1 = (E_1, \bar{p}_1), \ \hat{p}_2 = (E_2, \bar{p}_2), \ \hat{p}_3 = (E_3, \bar{p}_3) \quad (4.03)$$

The antiproton is considered as a proton going backward in time, and the associated spinor $\psi(p_1)$ satisfies

$$(-\tilde{p}_1 - M) \ \psi(p_1) = 0 \quad (4.04)$$

For the other protons,

$$(\tilde{p}_2 - M) \psi(p_2) = 0, \ (\tilde{k}_o - M) \psi(k_o) = 0, \text{ etc.} \quad (4.05)$$

The intermediate neutron has the four-momentum

$$\hat{p}' = \hat{p}_3 - \hat{k}_1 = \hat{p}_1 + \hat{q}' \quad (4.06)$$

where $\hat{q}'$ is the four-momentum of the intermediate meson:

$$\hat{q}' = \hat{k}_o - \hat{p}_2 = \hat{p}' - \hat{p}_1 \quad (4.07)$$

Since the emergent protons are identical particles, we must consider also the process obtained by interchanging $\hat{p}_2$ and $\hat{p}_3$ in the diagram. The momenta of the intermediate particles are then designated by a double prime:

(4.08) $$\hat{p}'' = \hat{p}_2 - \hat{k}_1, \quad \hat{q}'' = \hat{k}_0 - \hat{p}_3$$

We note that if T is the total energy in the c.m. system,

(4.09) $$2TE_0 = T^2 + M^2 - \mu^2, \; 2T\varepsilon = T^2 - M^2 + \mu^2, \; T = E_0 + \varepsilon.$$

Dividing by two to average over spins of the incident neutron, the differential cross-section for the process (4.01) is given by

(4.10) $$d\sigma = \frac{1}{2}\,\frac{2\pi}{\hbar v}\sum |H|^2 \rho_F$$

The relative velocity of neutron and meson is v, given by

(4.11) $$\frac{v}{c} = \frac{k}{E_0} + \frac{k}{\varepsilon} = \frac{kT}{\varepsilon E_0}$$

The density of final states ρ_F is

(4.12) $$\rho_F = \frac{d^3\bar{p}_1\, d^3\bar{p}_2}{(2\pi\hbar c)^6\, dT} = (2\pi\hbar c)^{-6}\, d\tau$$

where $d\tau$ is given by (1.20). $\sum |H|^2$ is the sum over the states of the initial and final nucleons of the proper transition matrix element in momentum space. Assuming pseudoscalar charged mesons, this can be written down from the diagram of Fig. 4.1, according to our prescription of Section II. We thus obtain

(4.13) $$H = (4\pi)^{3/2} g^3 \hbar^3 c^3 (2\varepsilon)^{-\frac{1}{2}} \cdot 2^{-\frac{1}{2}} (O_1 - O_2)$$

where

(4.14) $$O_1 = \bar{\psi}(p_2)\gamma_5\psi(k_0)(\hat{q}'^2 - \mu^2)^{-1}\cdot$$
$$\bar{\psi}(p_3)\gamma_5(\tilde{p}' - M)^{-1}\gamma_5\psi(p_1)$$

and O_2 is obtained by exchanging $\hat{p}_2$ and $\hat{p}_3$ everywhere, changing primes to double primes. Requirements of anti-symmetry between the final proton states cause us to write $2^{-\frac{1}{2}}(O_1 - O_2)$ in place of O_1 alone. Using (4.05) and (4.06), we can simplify (4.14) to obtain

$$O_1 = -\frac{\bar{\psi}(p_2)\gamma_5\psi(k_0)\bar{\psi}(p_3)\tilde{k}_1\psi(p_1)}{(\hat{p}'^2 - M^2)(\hat{q}'^2 - \mu^2)} \tag{4.15}$$

(cf. (2.10)). Combining (4.10-.12),

$$d\sigma = \frac{g^6 E_0\, d\tau}{4\pi^2 \hbar c k T}\sum |O_1 - O_2|^2 \tag{4.16}$$

Using the spur technique, and noting (4.04), we have

$$\sum |O_1|^2 = \frac{S_p(\tilde{p}_2 + M)\gamma_5(\tilde{k}_0 + M)\gamma_5 \cdot S_p(\tilde{p}_3 + M)\tilde{k}_1(-\tilde{p}_1 + M)\tilde{k}_1}{(2M)^4 (\hat{p}'^2 - M^2)^2 (\hat{q}'^2 - \mu^2)^2} \tag{4.17}$$

$$S_p(\tilde{p}_2 + M)\gamma_5(\tilde{k}_0 + M)\gamma_5 = -4(M^2 - \hat{p}_2\cdot\hat{k}_0) = -2\hat{q}'^2 \tag{4.18}$$

$$\begin{aligned} S_p(M + \tilde{p}_3)\tilde{k}_1(M - \tilde{p}_1)\tilde{k}_1 &= \\ S_p[M^2\tilde{k}_1^2 - \tilde{p}_3\tilde{k}_1\tilde{p}_1\tilde{k}_1] &= \\ -2[4(\hat{p}_3\cdot\hat{k}_1)(\hat{p}_1\cdot\hat{k}_1) - \mu^2(\hat{p}_1 + \hat{p}_3)^2] \end{aligned} \tag{4.19}$$

$$\sum |O_1|^2 = \frac{\hat{q}'^2[4(\hat{p}_3\cdot\hat{k}_1)(\hat{p}_1\cdot\hat{k}_1) - \mu^2(\hat{p}_1 + \hat{p}_3)^2]}{4M^4(\hat{q}'^2 - \mu^2)^2(\hat{p}'^2 - M^2)^2} \tag{4.20}$$

$$\bar{O}_1 O_2 = \frac{\bar{\psi}(p_3)\gamma_5\psi(k_0)\bar{\psi}(k_0)\gamma_5\psi(p_2)\bar{\psi}(p_2)\tilde{k}_1\psi(p_1)\bar{\psi}(p_1)\tilde{k}_1\psi(p_3)}{(\hat{q}'^2 - \mu^2)(\hat{q}''^2 - \mu^2)(\hat{p}'^2 - M^2)(\hat{p}''^2 - M^2)} \tag{4.21}$$

$$\sum \bar{O}_1 O_2 = \frac{S_p(M + \tilde{p}_3)\gamma_5(M + \tilde{k}_0)\gamma_5(M + \tilde{p}_2)\tilde{k}_1(M - \tilde{p}_1)\tilde{k}_1}{(2M)^4(\hat{q}'^2 - \mu^2)(\hat{q}''^2 - \mu^2)(\hat{p}'^2 - M^2)(\hat{p}''^2 - M^2)} \tag{4.22}$$

A somewhat tedious application of spur methods yields

$$\begin{aligned} &S_p(M + \tilde{p}_3)\gamma_5(M + \tilde{k}_0)\gamma_5(M + \tilde{p}_2)\tilde{k}_1(M - \tilde{p}_1)\tilde{k}_1 = \\ &4(\hat{p}_1\cdot\hat{k}_1)[(\hat{p}_3\cdot\hat{k}_1)(\hat{k}_0 - \hat{p}_2)^2 + (\hat{p}_2\cdot\hat{k}_1)(\hat{k}_0 - \hat{p}_3)^2 - (\hat{k}_0\cdot\hat{k}_1)(\hat{p}_3 - \hat{p}_2)^2] \\ &- \mu^2[(\hat{p}_3 + \hat{p}_1)^2(\hat{k}_0 - \hat{p}_2)^2 + (\hat{p}_1 + \hat{p}_2)^2(\hat{k}_0 - \hat{p}_3)^2 - (\hat{k}_0 + \hat{p}_1)^2(\hat{p}_3 - \hat{p}_2)^2]. \end{aligned} \tag{4.23}$$

Then we can write

$$(4.24)\quad M^4\sum \bar{O}_1 O_2 = -\frac{(\hat{p}_2-\hat{p}_3)^2[4(\hat{k}_0\cdot\hat{k}_1)(\hat{p}_1\cdot\hat{k}_1)-\mu^2(\hat{k}_0+\hat{p}_1)^2]}{16(\hat{q}'^2-\mu^2)(\hat{q}''^2-\mu^2)(\hat{p}'^2-M^2)(\hat{p}''^2-M^2)}$$

$$+\frac{1}{4}M^4\sum|O_1|^2\frac{(\hat{q}'^2-\mu^2)(\hat{p}'^2-M^2)}{(\hat{q}''^2-\mu^2)(\hat{p}''^2-M^2)}+\frac{1}{4}M^4\sum|O_2|^2\frac{(\hat{q}''^2-\mu^2)(\hat{p}''^2-M^2)}{(\hat{q}'^2-\mu^2)(\hat{p}'^2-M^2)}.$$

These sums $\sum$ must be multiplied by $-M^4/E_0E_1E_2E_3$ to correct the normalization.

It is easiest to complete the evaluation of the total cross-section in the case in which the total energy T is just above the threshold energy 3M. Then the produced particles are non-relativistic. We then have

$$(4.25)\qquad T \doteq 3M,\ E_0 \doteq \tfrac{5}{3}M - \mu^2/6M,\ \varepsilon \doteq \tfrac{4}{3}M + \mu^2/6M$$

$$(4.26)\quad \sum|O_1-O_2|^2 = \sum|O_1|^2 = \frac{3}{16}\left(M^2-\frac{1}{16}\mu^2\right)\left(M^2+\frac{1}{2}\mu^2\right)^{-2}.$$

Using (1.25) we have

$$(4.27)\qquad \int d\tau = \frac{4\pi^3M^3}{3\sqrt{3}}\,U^2.$$

We note that if E_π is the energy of the incident π-meson in the laboratory system (neutron at rest), and if ΔE_π is the energy in excess of the threshold energy of $4M - \mu^2/2M$,

$$(4.28)\qquad U \doteq \frac{1}{3}\Delta E_\pi = \frac{1}{3}\left(E_\pi - 4M + \mu^2/2M\right)$$

Combining these results and substituting into (4.16),

$$(4.29)\quad \sigma \doteq 2^{-6}\,3^{-\frac{5}{2}}\,\pi\left(\frac{g^2}{\hbar c}\right)^3\left(\frac{\hbar c}{M}\right)^2\left(\frac{\Delta E_\pi}{M}\right)^2\omega$$

where

$$(4.30)\quad \omega = \left(1+\frac{\mu^2}{2M^2}\right)^{-2}\left(1-\frac{\mu^2}{16M^2}\right)^{\frac{1}{2}}\left(1-\frac{\mu^2}{4M^2}\right)^{-\frac{1}{2}} \cong 1-\frac{29\mu^2}{32M^2}$$

$$(4.31) \qquad \therefore \sigma \doteq 1.36 \times 10^{-30} \left(\frac{g^2}{\hbar c}\right)^3 \left(\frac{\Delta E_\pi}{M}\right)^2 \ cm^2.$$

For energies much larger than the threshold, it is necessary to take into account the angular dependence of the expressions (4.20) and (4.24), and these must be integrated over the momentum space of the final particles. This integration is feasible only if certain approximations are made. The first of these is to set the mass of the meson $\mu = 0$. The error induced by this will be on the order of μ^2/M^2. The second will be to neglect the first term in (4.24), which is an exchange term proportional to $(\hat{p}_2 - \hat{p}_3)^2$, and which vanishes in the non-relativistic limit. (4.20) now becomes, after some simplification, using (4.06) and (4.07),

$$(4.32) \qquad -M^4 \sum |O_1|^2 = -\frac{\hat{p}_1 \cdot \hat{k}_1}{4\left[(\hat{k}_0 - \hat{p}_2)^2 - \mu^2\right](\hat{p}_3 \cdot \hat{k}_1)}$$

while (4.24) can be written

$$(4.33) \qquad \sum \bar{O}_1 O_2 = \frac{1}{4} \sum |O_1|^2 + \frac{1}{4} \sum |O_2|^2$$

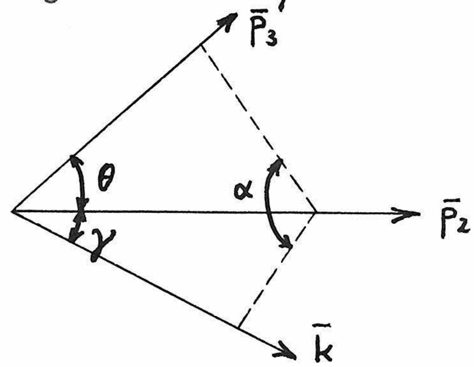
Since O_1 and O_2 differ only in the interchange of $\hat{p}_2$ and $\hat{p}_3$, they give the same result after the integration over momentum space, so that only one need be considered, and we obtain for the total cross-section

$$(4.34) \quad \sigma = -\iint \frac{g^6 \, dE_2 \, d\Omega_2 \, dE_3 \, d\alpha \, (\varepsilon E_1 + \bar{p}_1 \cdot \bar{k})}{16\pi^2 \hbar c k T \left[(\hat{k}_0 - \hat{p}_2)^2 - \mu^2\right](\varepsilon E_3 + \bar{p}_3 \cdot \bar{k})}$$

since $\hat{p}_1 \cdot \hat{k}_1 = \varepsilon E_1 + \bar{p}_1 \cdot \bar{k}$ etc. α is a polar angle measuring the location of the vector $\bar{p}_3$ with respect to a plane taken for convenience through $\bar{p}_2$ and $\bar{k}$. (v. Fig. 4.2). All the results of section I can be used by replacing $\hat{p}_1$ by $\hat{p}_2$ and $\hat{p}_2$ by

$\hat{p}_3$. The angle between $\bar{p}_2$ and $\bar{p}_3$ is θ, while that between $\bar{p}_2$ and $\bar{k}$ is γ. Then we have, by spherical trigonometry,

(4.35) $$\bar{p}_3 \cdot \bar{k} = p_3 k(\cos\theta \cos\gamma + \sin\theta \sin\gamma \cos\alpha)$$



(Fig. 4.2)

(4.36) $$\int_0^{2\pi} \frac{\varepsilon E_1 + \bar{p}_1 \cdot \bar{k}}{\varepsilon E_3 + \bar{p}_3 \cdot \bar{k}}\, d\alpha = \int_0^{2\pi} \left[\frac{\varepsilon (T - E_2) - \bar{p}_2 \cdot \bar{k}}{\varepsilon E_3 + \bar{p}_3 \cdot \bar{k}} - 1 \right] d\alpha =$$

$$2\pi \left[\frac{\varepsilon (T - E_2) - p_2 k \cos\gamma}{\sqrt{(\varepsilon E_3 + p_3 k \cos\theta \cos\gamma)^2 - (p_3 k \sin\theta \sin\gamma)^2}} - 1 \right] =$$

$$2\pi \left[\frac{T - E_2 - p_2 \cos\gamma}{\sqrt{(E_3 \cos\gamma + p_3 \cos\theta)^2 + M^2 \sin^2\gamma}} - 1 \right]$$

where we have used $\mu = 0$ to replace ε by $k = \sqrt{\varepsilon^2 - \mu^2}$. This expression is now to be integrated on $E_3' \leq E_3 \leq E_3''$, where we have, by (1.06), (1.16-.15), with $m_1 = m_2 = m_3 = M$,

(4.37) $$p_2 p_3 \cos\theta = \tfrac{1}{2}(T^2 + M^2) - TE_2 - (T - E_2) E_3$$

and

(4.38) $$E_3'' + E_3' = T - E_2, \quad E_3'' - E_3' = p_2 R$$

(4.39) $$p_3'' + p_3' = (T - E_2) R, \quad p_3'' - p_3' = p_2$$

where by (1.07), (1.16),

(4.40) $$R = \frac{\sqrt{G^2 - 4M^2}}{G}, \quad G^2 = F^2 = T^2 - 2TE_2 + M^2, \quad E_2^0 = \frac{T^2 - 3M^2}{2T}$$

Let

(4.41) $$X = E_3 \cos\gamma + p_3 \cos\theta$$

Then it can be shown that

(4.42) $$dE_3 = \frac{-p_2\ dx}{T - E_2 - p_2 \cos\gamma}$$

with the limits of integration $x'' \leq x \leq x'$, where

(4.43) $$x' = E_3' \cos\gamma + p_3', \quad x'' = E_3'' \cos\gamma - p_3''$$

(4.44) $$\sqrt{x'^2 + M^2 \sin^2\gamma} = E_3' + p_3' \cos\gamma$$
$$\sqrt{x''^2 + M^2 \sin^2\gamma} = E_3'' - p_3'' \cos\gamma$$

(4.45) $$\int_{x''}^{x'} \frac{dx}{\sqrt{x^2 + M^2 \sin^2\gamma}} = \log \frac{E_3' + p_3'}{E_3'' - p_3''} = \log \frac{1+R}{1-R} = 2 \tanh^{-1} R$$

where

(4.46) $$R^2 = \frac{E_2^0 - E_2}{E_0 - E_2}, \qquad E_0 = \frac{T^2 + M^2}{2T}$$

by (4.09) with $\mu = 0$. Now put these results into (4.34), using

(4.47) $$(\hat{k}_0 - \hat{p}_2)^2 - \mu^2 = -2(E_0 E_2 - M^2 + \tfrac{1}{2}\mu^2 - p_2 k \cos\gamma)$$

and $d\Omega = 2\pi \sin\gamma\, d\gamma$.

(4.48) $$\therefore \sigma = \frac{g^6}{8\hbar c k T} \int_M^{E_2^0} dE_2 \int_0^{\pi} \frac{p_2 (2 \tanh^{-1} R - R) \sin\gamma\, d\gamma}{E_0 E_2 - M^2 + \frac{1}{2}\mu^2 - p_2 k \cos\gamma}.$$

Before integrating over γ, we do a partial integration on E_2, to obtain

(4.49) $$\sigma = \frac{g^6}{4\hbar c k T} \int_M^{E_2^0} \frac{dE_2}{p_2} (\tanh^{-1} R - R)(E_0 - E_2) \cdot \left[(E_0 - E_2) M^2 + \tfrac{1}{2}\mu^2 E_2\right] \int_0^{\pi} \frac{\sin\gamma\, d\gamma}{(E_0 E_2 - M^2 + \frac{1}{2}\mu^2 - p_2 k \cos\gamma)^2}$$

Integrating over γ, we get

$$(4.50)\quad \sigma = \frac{g^6}{2\hbar ckT}\int_M^{E_2^\circ}\frac{dE_2}{p_2}(E_0-E_2)\left[M^2(E_0-E_2)+\tfrac{1}{2}\mu^2E_2\right]\cdot$$

$$(\tanh^{-1}R-R)\left[M^2(E_0-E_2)^2+\mu^2(E_0E_2-M^2)+\tfrac{1}{4}\mu^4\right]^{-1}.$$

Inspection shows that at $E_2 = M$ the integrand become infinite with $1/p_2$, so that for very high energies we can approximate (4.50) by putting $E_0 = M$ in the integrand:

$$(4.51)\quad \sigma \doteq \frac{g^6}{2\hbar ckT}\,\frac{M(E_0-M)}{M(E_0-M)+\frac{1}{2}\mu^2}\cdot$$

$$\left(\cosh^{-1}\sqrt{\frac{E_0-M}{E_0-E_2^\circ}}-\sqrt{\frac{E_2^\circ-M}{E_0-M}}\right)\int_M^{E_2^\circ}\frac{dE_2}{p_2}.$$

Then using $\int_M^{E_2^\circ}(E_2^2-M^2)^{-\frac{1}{2}}dE_2=\cosh^{-1}\frac{E_2^\circ}{M}$ and $\cos h^{-1}x \sim \log 2x$

for large x, we can estimate σ by

$$(4.52)\quad \sigma \sim \frac{g^6}{\hbar c(T^2-M^2)}\left[1+\frac{\mu^2}{2M(E_0-M)}\right]\left(\log\frac{T}{M}-1\right)\log\frac{T}{M}\sim$$

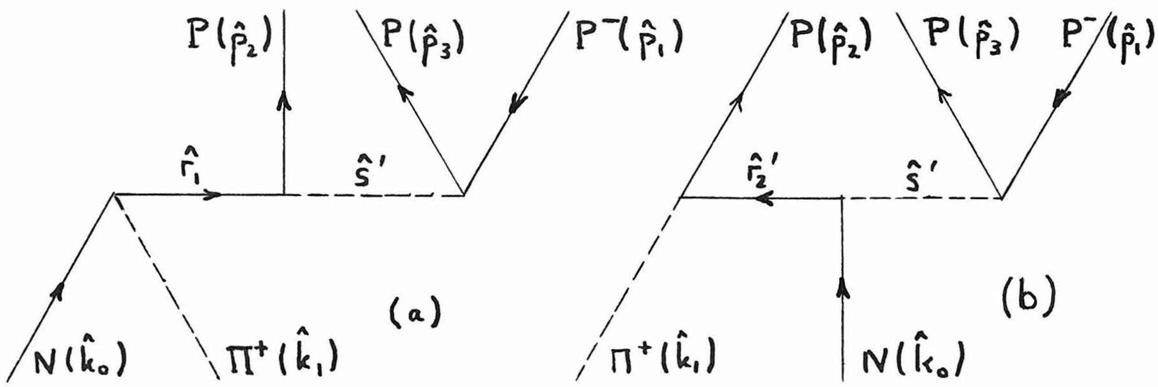
$$\frac{g^6}{8\hbar cME_\pi}\log\frac{2E_\pi}{M}\left(\log\frac{2E_\pi}{M}-2\right)$$

using the transformation formula to obtain the energy E_π of the incident meson in the laboratory system:

$$(4.53)\qquad 2ME_\pi = T^2 - M^2 - \mu^2$$

The graph, Fig. 4.3 at the end of the text, gives values of σ obtained by numerical integration.

It is of interest to ascertain the effect of the inclusion of virtual neutral mesons on the result of such a calculation. If we do this, we have two more diagrams to consider. They are given in Fig. 4.4 below.



(Fig. 4.4)

Each diagram represents two processes, the second being obtained from the one shown by interchanging $\hat{p}_2$ and $\hat{p}_3$ everywhere, replacing primes by double primes.

(4.54) $$\hat{r}_1 = \hat{k}_o + \hat{k}_1, \quad \hat{r}_2' = \hat{p}_2 - \hat{k}_1, \quad \hat{r}_2'' = \hat{p}_3 - \hat{k}_1$$

(4.55) $$\hat{s}' = \hat{p}_1 + \hat{p}_3, \quad \hat{s}'' = \hat{p}_1 + \hat{p}_2$$

Remembering our rule about signs in the symmetric theory, we get for the matrix element of H in this case

(4.56) $$H_N = (4\pi)^{3/2}\, g g_o^{\,2} (2\varepsilon)^{-\frac{1}{2}} \cdot 2^{-\frac{1}{2}} (O_3 - O_4)$$

where

(4.57) $$O_3 = \bar{\psi}(p_3)\gamma_5\psi(p_1)(\hat{s}'^2-\mu^2)^{-1} \cdot \left[\bar{\psi}(p_2)\gamma_5(\tilde{r}_1-M)^{-1}\gamma_5\psi(k_o) - \bar{\psi}(p_2)\gamma_5(\tilde{r}_2'-M)^{-1}\gamma_5\psi(k_o)\right] = \bar{\psi}(p_3)\gamma_5\psi(p_1)\bar{\psi}(p_2)\tilde{k}_1\psi(k_o) \cdot (\hat{s}'^2-\mu^2)^{-1}\left[(\hat{r}_1^{\,2}-M^2)^{-1} + (\hat{r}_2'^2-M^2)^{-1}\right],$$

and where O_4 is obtained from O_3 by interchanging $\hat{p}_2$ and $\hat{p}_3$ in all terms. Then the cross-section is proportional to

$$(4.58)\quad |H|^2 = |H_c + H_N|^2 = 2\varepsilon^{-1}(2\pi)^3(\hbar c)^6 g^2 \left| g^2(O_1 - O_2) + g_0^2(O_3 - O_4)\right|^2$$

where H_c is the H of (4.13). Working this out in the non-relativistic limit and substituting into (4.10), we obtain for the total cross-section, neglecting $\mu^2 \ll M^2$,

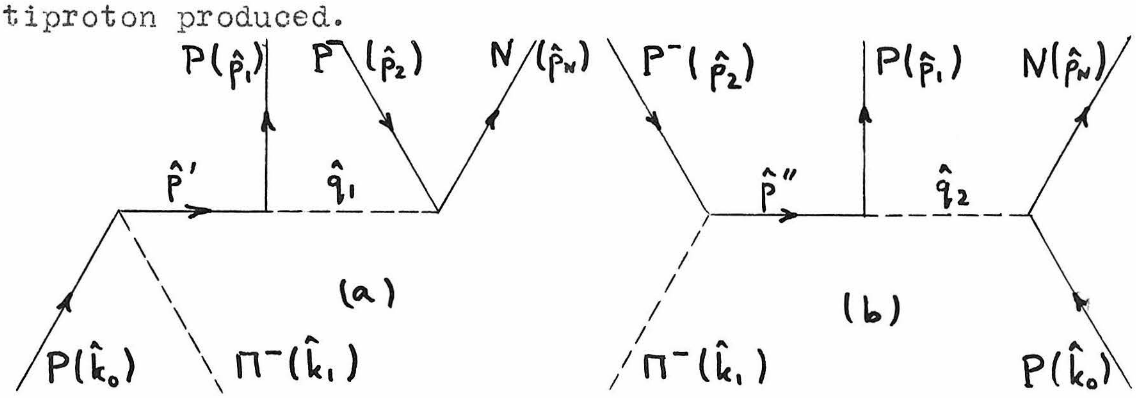
$$(4.59)\quad \sigma = \frac{\pi(3g^2 + 2g_0^2)^2 g^2}{2^6\, 3^{9/2}\, (\hbar c)^3} \left(\frac{\hbar c}{M}\right)^2 \left(\frac{\Delta E_\pi}{M}\right)^2$$

so that in the Kemmer theory, with $g_0^2 = \frac{1}{2}g^2$, the cross-section is 16/9 the value obtained by including only charged virtual mesons, in the non-relativistic limit.

We should consider also the effect of negative mesons colliding with protons, so that antiprotons might be produced in the reaction

$$(4.60)\quad P(\hat{k}_0) + \Pi^-(\hat{k}_1) \rightarrow P(\hat{p}_1) + P^-(\hat{p}_2) + N(\hat{p}_N)$$

Again considering only charged mesons in intermediate states, we now have two diagrams, which correspond to the interchange of the two protons entering the reaction-- the initial proton and the proton going backward in time which represents the antiproton produced.



(Fig. 4.5)

(4.61) $\hat{p}' = \hat{k}_o + \hat{k}_1, \; \hat{p}'' = -\hat{p}_2 + \hat{k}_1$

(4.62) $\hat{q}_1 = \hat{p}_2 + \hat{p}_N, \; \hat{q}_2 = -\hat{p}_N + \hat{k}_o$

(4.63) $\tilde{p}_1 \, \psi(p_1) = M \, \psi(p_1), \; \tilde{p}_2 \, \psi(p_2) = -M \, \psi(p_2)$

$\tilde{p}_N \, \psi(p_N) = M \, \psi(p_N), \; \tilde{k}_o \, \psi(k_o) = M \, \psi(k_o)$

Then the matrix element can be written as

(4.64) $$H = (4\pi)^{3/2} g^3 \hbar^3 c^3 (2\varepsilon)^{-\frac{1}{2}} (O_1 - O_2)$$

where

(4.65) $$O_1 = \bar{\psi}(p_N) \gamma_5 \psi(p_2) (\hat{q}_1^2 - \mu^2)^{-1} \cdot \bar{\psi}(p_1) \gamma_5 (\tilde{p}' - M)^{-1} \gamma_5 \psi(k_o) = \frac{\bar{\psi}(p_N) \gamma_5 \psi(p_2) \bar{\psi}(p_1) \tilde{k}_1 \psi(k_o)}{(\hat{q}_1^2 - \mu^2)(\hat{p}'^2 - M^2)}$$

(4.66) $$O_2 = \bar{\psi}(p_N) \gamma_5 \psi(k_o) (\hat{q}_2^2 - \mu^2)^{-1} \cdot \bar{\psi}(p_1) \gamma_5 (\tilde{p}'' - M)^{-1} \gamma_5 \psi(p_2) = \frac{\bar{\psi}(p_N) \gamma_5 \psi(k_o) \bar{\psi}(p_1) \tilde{k}_1 \psi(p_2)}{(\hat{q}_2^2 - \mu^2)(\hat{p}''^2 - M^2)}$$

(4.66) is seen to be the same type of term as (4.15), while (4.65) is new. Here we must omit the factor $2^{-\frac{1}{2}}$ which appeared in (4.13) because the final integration over momentum space in this latter case includes only one of the "identical" protons, the antiproton observed, while in the former case, both the protons are observed as produced in the reaction. Then the cross-section is given by (4.16), with an additional factor of two, where now

(4.67) $$\sum |O_1|^2 = \frac{Sp\,(M + \tilde{p}_N) \gamma_5 (M - \tilde{p}_2) \gamma_5 \cdot Sp\,(M + \tilde{p}_1) \tilde{k}_1 (M + \tilde{k}_o) \tilde{k}_1}{(2M)^4 (\hat{q}_1^2 - \mu^2)^2 (\hat{p}'^2 - M^2)^2}$$

(4.68) $$\therefore \sum |O_1|^2 = \frac{\hat{q}_1^2 [4 (\hat{k}_o \cdot \hat{k}_1)(\hat{p}_1 \cdot \hat{k}_1) + \mu^2 (\hat{k}_o - \hat{p}_1)^2]}{4M^4 (\hat{q}_1^2 - \mu^2)^2 (\hat{p}'^2 - M^2)^2} .$$

As in (4.20),

$$(4.69)\quad \sum |O_2|^2 = \frac{\hat{q}_2^{\,2}\left[4(\hat{p}_2\cdot\hat{k}_1)(\hat{p}_1\cdot\hat{k}_1) - \mu^2(\hat{p}_1+\hat{p}_2)^2\right]}{4M^4(\hat{q}_2^{\,2}-\mu^2)^2(\hat{p}''^2-M^2)^2}$$

A long spur calculation shows

$$(4.70)\quad M^4\sum \bar{O}_1 O_2 = \frac{(\hat{p}_2+\hat{k}_0)^2\left[4(\hat{p}_1\cdot\hat{k}_1)(\hat{p}_N\cdot\hat{k}_1) + \mu^2(\hat{p}_1-\hat{p}_N)^2\right]}{4(\hat{q}_1^{\,2}-\mu^2)(\hat{q}_2^{\,2}-\mu^2)(\hat{p}'^2-M^2)(\hat{p}''^2-M^2)}$$

$$+ M^4\sum|O_1|^2\frac{(\hat{q}_1^{\,2}-\mu^2)(\hat{p}'^2-M^2)}{(\hat{q}_2^{\,2}-\mu^2)(\hat{p}''^2-M^2)} + M^4\sum|O_2|^2\frac{(\hat{q}_2^{\,2}-\mu^2)(\hat{p}''^2-M^2)}{(\hat{q}_1^{\,2}-\mu^2)(\hat{p}'^2-M^2)}$$

Proceeding as before, in the non-relativistic limit, we find for the total cross-section, neglecting $\mu^2 \ll M^2$

$$(4.71)\qquad \sigma \doteq \frac{7\pi}{2^5\cdot 3^{9/2}}\left(\frac{g^2}{\hbar c}\right)^3\left(\frac{\hbar c}{M}\right)^2\left(\frac{\Delta E_\pi}{M}\right)^2$$

This is 14/9 the cross-section (4.29) for the process (4.01).

For higher energies it is again not possible to carry out the integration over momentum space except by setting $\mu = 0$ and neglecting the first term of (4.70). This latter approximation introduces a more serious error here, however, for even in the non-relativistic limit, the term neglected contributed 2/7 of the total cross-section (4.71). Hence our final result can be expected to be in error by about this amount. As in (4.33), the cross-section reduces to

$$(4.72)\qquad \sigma = \frac{g^6 E_0\, d\tau}{4\pi^2\hbar c k T}\left(\sum|O_1|^2 + \sum|O_2|^2\right)$$

The second term of this will give the cross-section (4.34) or (4.50). The first term is smaller by a factor of 1/10 or less. If we call the contribution of this term σ_1, we easily obtain

$$(4.73)\qquad \sigma_1 = \frac{g^6}{8\hbar c k T^3}\int_M^{E_1^0}\frac{p_1 E_1\, dE_1}{E_0-E_1}\sqrt{\frac{E_1^0-E_1}{E_0-E_1}}$$

Numerical integration then gives the following values, tabulated against the energy E_π of the incident meson in the laboratory system:

E_π / M	6.1	12.0	34.5	52.4	100.5
$\sigma_1 \cdot \left(\frac{g^2}{\hbar c}\right)^{-3} \left(\frac{\hbar c}{M}\right)^{-2} \cdot 10^4$	2.75	7.80	7.09	6.22	5.69

We can thus conclude that the total cross-section for the production of antiprotons in the collision of negative mesons and protons according to (4.60) is of the same order of magnitude as the cross-section for the positive-meson-neutron process (4.01), and in later work we shall take them equal, using the more precisely known cross-section (4.50) for both.

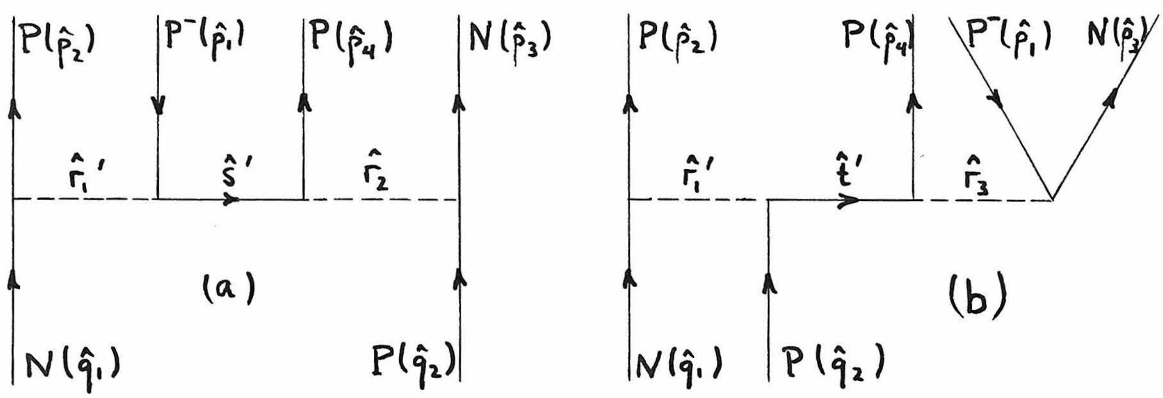
Inspection of the integrand of (4.34) shows that it is largest when $\bar{p}_2$ is in the direction of $\bar{k}$. Hence the protons are emitted mostly in that direction, and the antiprotons mostly in the direction of the incident meson.

V. PRODUCTION OF ANTIPROTONS IN NUCLEON-NUCLEON COLLISIONS -- PERTURBATION METHOD (NON-RELATIVISTIC)

We shall now calculate the cross-section for the process by which a proton and a neutron collide to produce a pair:

(5.01) $N(\hat{q}_1) + P(\hat{q}_2) \longrightarrow P^-(\hat{p}_1) + P(\hat{p}_2) + N(\hat{p}_3) + P(\hat{p}_4)$

We assume for simplicity that there are only charged mesons in intermediate states, so that there are two possible diagrams (Fig. 5.1), the one being obtained from the other by the interchange of the two protons entering the reaction-- the original proton and the proton moving backward in time which corresponds to the antiproton produced.



(Fig. 5.1)

In the center-of- mass system, we have

(5.02) $\hat{q}_1 = (E_o, \bar{q})$, $\hat{q}_2 = (E_o, -\bar{q})$, $\hat{p}_1 = (E_1, \bar{p}_1)$, etc.

with the Dirac equations

(5.03) $\tilde{q}_1 \ \psi(q_1) = M \ \psi(q_1)$, $\tilde{q}_2 \ \psi(q_2) = M \ \psi(q_2)$,

$\tilde{p}_1 \ \psi(p_1) = -M \ \psi(p_1)$, $\tilde{p}_i \ \psi(p_i) = M \ \psi(p_i)$, $i = 2,3,4.$

T is the total energy in the c.m. system, and T = 4M is the threshold for the reaction in that system. Let $\bar{\bar{E}}$ be the energy of one of the incident nucleons in the laboratory system, i. e., the system in which the other initial nucleon is at rest. Then the Lorentz transformation gives

(5.04) $$T^2 = (\bar{E} + M)^2 - \bar{p}^2 = 2M(\bar{E} + M)$$

and we find that $\bar{E} = 7M$ is the threshold energy in the laboratory system.

Because of the number of variables involved and their complicated relationships, we have not been able to evaluate the total cross-section for this process by integration over the momentum space of the four particles produced, except in the limit in which these particles have kinetic energies small compared to M. Then the energy $\bar{E}$ of the incident nucleon is only slightly greater than 7M. The absolute square of the collision matrix element H reduces to a constant, and the total cross-section is given by

(5.05) $$\sigma = \frac{2\pi}{\hbar v} \cdot \frac{1}{4} \cdot \sum |H|^2 (2\pi\hbar c)^{-9} \int d\tau$$

where $\int d\tau$ is given by (1.33), and can be written

(5.06) $$\int d\tau = \frac{\pi^4}{105 \cdot 2\sqrt{2}} M^8 \left(\frac{\bar{E} - 7M}{M}\right)^{7/2}$$

using $T - 4M = U \doteq \frac{1}{4}(\bar{E} - 7M)$ for $U \ll 4M$. In (5.05), v is the relative velocity of the initial nucleons in the c.m. system, and it is given by

(5.07) $$v/c = 2q/E_0$$

The factor $\frac{1}{4}$ in (5.05) comes from the average over the spins of the initial nucleons.

The matrix element can be written down from the diagrams of Fig. 5.1 in the usual manner, and we obtain

$$(5.08)\quad H = (4\pi g^2\hbar^2c^2)^2\left[\frac{1}{\sqrt{2}}(O_1'-O_1'') - \frac{1}{\sqrt{2}}(O_2'-O_2'')\right]$$

$$= \frac{1}{\sqrt{2}}(4\pi g^2\hbar^2c^2)^2\,(O_1'-O_1''-O_2'+O_2''),$$

where from Fig. 5.1 (a),

$$(5.09)\quad O_1' = \bar{\psi}(p_2)\,\gamma_5\,\psi(q_1)\,(\hat{r}_1'^2-\mu^2)^{-1}\cdot$$

$$\bar{\psi}(p_4)\,\gamma_5\,(\tilde{s}'-M)^{-1}\gamma_5\,\psi(p_1)\,(\hat{r}_2^2-\mu^2)^{-1}\bar{\psi}(p_3)\,\gamma_5\,\psi(q_2)$$

From diagram (b) of the same figure,

$$(5.10)\quad O_2' = \bar{\psi}(p_2)\,\gamma_5\,\psi(q_1)\,(\hat{r}_1'^2-\mu^2)^{-1}\cdot$$

$$\bar{\psi}(p_4)\,\gamma_5\,(\tilde{t}'-M)^{-1}\gamma_5\,\psi(q_2)\,(\hat{r}_3^2-\mu^2)^{-1}\bar{\psi}(p_3)\,\gamma_5\,\psi(p_1)$$

O_1'', O_2'' are obtained from O_1', O_2' by interchanging $\hat{p}_2$ and $\hat{p}_4$ in all terms. From the diagrams we also obtain the relations between the four-momenta of the particles involved:

$$(5.11)\quad \begin{aligned} &\hat{r}_1' = \hat{q}_1 - \hat{p}_2,\ \hat{r}_2 = \hat{p}_3 - \hat{q}_2,\ \hat{r}_3 = \hat{p}_1 + \hat{p}_3 \\ &\hat{s}' = -\hat{p}_1 + \hat{r}_1' = \hat{p}_4 + \hat{r}_2 = -\hat{p}_1 - \hat{p}_2 + \hat{q}_1 \\ &\hat{t}' = \hat{r}_1' + \hat{q}_2 = \hat{r}_3 + \hat{p}_4 = \hat{q}_1 + \hat{q}_2 - \hat{p}_2 = \hat{p}_1 + \hat{p}_3 + \hat{p}_4 \end{aligned}$$

$$(5.12)\quad \therefore\ O_1' = \frac{\bar{\psi}(p_2)\,\gamma_5\,\psi(q_1)\,\bar{\psi}(p_4)\,\tilde{r}_1'\,\psi(p_1)\,\bar{\psi}(p_3)\,\gamma_5\,\psi(q_2)}{(\hat{r}_1'^2-\mu^2)\,(\hat{s}'^2-M^2)\,(\hat{r}_2^2-\mu^2)}$$

$$(5.13)\quad O_2' = \frac{\bar{\psi}(p_2)\,\gamma_5\,\psi(q_1)\,\bar{\psi}(p_4)\,\tilde{r}_1'\,\psi(q_2)\,\bar{\psi}(p_3)\,\gamma_5\,\psi(p_1)}{(\hat{r}_1'^2-\mu^2)\,(\hat{t}'^2-M^2)\,(\hat{r}_3^2-\mu^2)}$$

after some simplification. In the non-relativistic limit, the Dirac equations for the final particles become simply

$$(5.14)\quad \gamma_4\,\psi(p_1) = -\psi(p_1),\ \gamma_4\,\psi(p_i) = +\psi(p_i),\ i=2,3,4,$$

while for the initial nucleons

(5.15) $$E_0 \doteq 2M, \quad q \doteq M\sqrt{3}$$

(5.16) $$(2M\gamma_4 - q\gamma_1)\psi(q_1) = M\psi(q_1)$$
$$(2M\gamma_4 + q\gamma_1)\psi(q_2) = M\psi(q_2)$$

if we take the x-axis in the direction of the incident nucleons.

In addition we have

(5.17) $$\tilde{r}_1' = M\gamma_4 - q\gamma_1$$

(5.18) $$\hat{r}_1'^2 - \mu^2 = \hat{r}_2^2 - \mu^2 = -(2M^2 + \mu^2)$$
$$\hat{r}_3^2 - \mu^2 = 4M^2 - \mu^2$$
$$\hat{s}'^2 - M^2 = -4M^2, \quad \hat{t}'^2 - M^2 = 8M^2$$

(5.19) $$\bar{\psi}(p_4)\tilde{r}_1'\psi(p_1) = \bar{\psi}(p_4)(M\gamma_4 - q\gamma_1)\psi(p_1)$$
$$= -q\,\bar{\psi}(p_4)\gamma_1\psi(p_1)$$

since by (5.14) $\bar{\psi}(p_4)\gamma_4\psi(p_1) = \bar{\psi}(p_4)\psi(p_1) = -\bar{\psi}(p_4)\psi(p_1) = 0$

(5.20) $$\bar{\psi}(p_4)\tilde{r}_1'\psi(q_2) = \bar{\psi}(p_4)(M\gamma_4 - q\gamma_1)\psi(q_2) =$$
$$\bar{\psi}(p_4)(3M\gamma_4 - M)\psi(q_2) = 2M\,\bar{\psi}(p_4)\psi(q_2)$$

since by (5.16), $-q\gamma_1\psi(q_2) = (2M\gamma_4 - M)\psi(q_2)$

and by (5.15) $\bar{\psi}(p_4)\gamma_4 = \bar{\psi}(p_4)$

(5.21) $$\therefore\ O_1' = q\,\frac{\bar{\psi}(p_2)\gamma_5\psi(q_1)\,\bar{\psi}(p_4)\gamma_1\psi(p_1)\,\bar{\psi}(p_3)\gamma_5\psi(q_2)}{4M^2(2M^2+\mu^2)^2}$$

(5.22) $$O_2' = -\frac{\bar{\psi}(p_2)\gamma_5\psi(q_1)\,\bar{\psi}(p_4)\psi(q_2)\,\bar{\psi}(p_3)\gamma_5\psi(p_1)}{4M(4M^2-\mu^2)(2M^2+\mu^2)}$$

(5.23) $$\sum|\bar{\psi}(p_2)\gamma_5\psi(q_1)|^2 = \frac{1}{4M^2}\,Sp\,(\tilde{p}_2 + M)\gamma_5(\tilde{q}_1 + M)\gamma_5$$
$$= -\frac{1}{2M^2}(\hat{q}_1 - \hat{p}_2)^2 = +1 = \sum|\bar{\psi}(p_3)\gamma_5\psi(q_2)|^2$$

$$(5.24)\quad \sum |\bar{\Psi}(p_4)\gamma_1\psi(p_1)|^2 = \frac{1}{4} S_p(1+\gamma_4)\gamma_1(1-\gamma_4)\gamma_1$$
$$= -\frac{1}{4} S_p(1+\gamma_4)^2 = -2.$$

$$(5.25)\quad \therefore \sum |O_1'|^2 = -\frac{3}{8M^2(2M^2+\mu^2)^4} = \sum |O_1''|^2$$

$$(5.26)\quad \sum |\bar{\Psi}(p_4)\psi(q_2)|^2 = (4M^2)^{-1} S_p(\tilde{p}_4+M)(\tilde{q}_2+M)$$
$$= (2M^2)^{-1}(\hat{p}_4+\hat{q}_2)^2 = 3$$

$$(5.27)\quad \sum |\bar{\Psi}(p_3)\gamma_5\psi(p_1)|^2 = -\frac{(\hat{p}_1+\hat{p}_3)^2}{2M^2} = -2$$

$$(5.28)\quad \therefore \sum |O_2'|^2 = \sum |O_2''|^2 = -\frac{3}{8M^2(2M^2+\mu^2)^2(4M^2-\mu^2)^2}$$

Similarly the cross-product terms can be evaluated:

$$(5.29)\quad \sum \bar{O}_1' O_1'' = \frac{1}{2}\sum |O_1'|^2$$

$$(5.30)\quad \sum \bar{O}_1' O_2' = \sum \bar{O}_1'' O_2'' = -3\left[16M^2(2M^2+\mu^2)^3(4M^2-\mu^2)\right]^{-1}$$

$$(5.31)\quad \sum \bar{O}_2' O_2'' = \frac{1}{2}\sum |O_2'|^2$$

$$(5.32)\quad \sum \bar{O}_1' O_2'' = \sum \bar{O}_2' O_1'' = 0$$

Combining all these and multiplying by $M^4/E_o(-E_1)E_2E_3 = -\frac{1}{2}$,

$$(5.33)\quad \sigma_{ps} = \frac{1}{35}\sqrt{3}\cdot 2^{-\frac{29}{2}}\left(\frac{g^2}{\hbar c}\right)^4\left(\frac{\hbar c}{M}\right)^2\left(\frac{\bar{E}-7M}{M}\right)^{7/2}\omega =$$
$$= 9.40\cdot 10^{-34}\left(\frac{g^2}{\hbar c}\right)^4\left(\frac{\bar{E}-7M}{M}\right)^{7/2}\ \mathrm{cm}^2.$$

where

$$(5.34)\quad \omega = (1-\tfrac{1}{2}x+\tfrac{1}{4}x^2)(1+\tfrac{1}{2}x)^{-4}(1-\tfrac{1}{4}x)^{-2},\quad x = \mu^2/M^2.$$

The case of scalar coupling is obtained by replacing γ_5 by $\underline{1}$ in (5.09) and (5.10). Then in (5.10) the factor $\bar{\Psi}(p_3)\psi(p_1)$

will give zero because of the orthogonality of states of positive and of negative energy in the limit of zero momentum. Thus only diagram (a) above contributes to the cross-section non-relativistically. Working out the spurs in the same manner as above, we find

$$\sigma_{sc} = \frac{3}{35}\sqrt{3}\; 2^{-\frac{25}{2}} \left(\frac{g^2}{\hbar c}\right)^4 \left(\frac{\hbar c}{M}\right)^2 \left(\frac{\bar{E}-7M}{M}\right)^{7/2} \left(1+\frac{\mu^2}{2M^2}\right)^{-4} \tag{5.34}$$

which is about 12 times the pseudoscalar result if the same coupling constant g is used in each.

In the case of pseudoscalar mesons with gradient coupling, (5.09) becomes

$$\begin{aligned} \mu^4 O_1' = {} & \bar{\psi}(p_2)\gamma_5 \tilde{r}_1' \psi(q_1) (\hat{r}_1'^2 - \mu^2)^{-1} \cdot \\ & \bar{\psi}(p_4)\gamma_5 \tilde{r}_2 (\tilde{s}' - M)^{-1} \gamma_5 \tilde{r}_1' \psi(p_1) \cdot \\ & (\hat{r}_2^2 - \mu^2)^{-1} \bar{\psi}(p_3)\gamma_5 \tilde{r}_2 \psi(q_2). \end{aligned} \tag{5.35}$$

Using (5.11), we can rewrite the middle term as follows:

$$\begin{aligned} & \bar{\psi}(p_4)\gamma_5 \tilde{r}_2 (\tilde{s}' + M)\gamma_5 \tilde{r}_1' \psi(p_1) (\hat{s}'^2 - M^2)^{-1} = \\ & \bar{\psi}(p_4) \tilde{r}_2 (\tilde{s}' - M) \tilde{r}_1' \psi(p_1) (\hat{s}'^2 - M^2)^{-1} = \\ & \bar{\psi}(p_4) (\tilde{s}' - \tilde{p}_4)(\tilde{s}' - M)(\tilde{s}' + \tilde{p}_1) \psi(p_1) (\hat{s}'^2 - M^2)^{-1} = \\ & \bar{\psi}(p_4) (\tilde{s}' - M)^3 \psi(p_1) (\hat{s}'^2 - M^2)^{-1} = \\ & \bar{\psi}(p_4) \left[(\hat{s}'^2 + 3M^2)\tilde{s}' - M(3\hat{s}'^2 + M^2) \right] \psi(p_1) (\hat{s}'^2 - M^2)^{-1} \end{aligned} \tag{5.36}$$

Non-relativistically, by (5.18), $\hat{s}'^2 + 3M^2 = 0$ and $\bar{\psi}(p_4)\psi(p_1) = 0$ (orthogonality of positive- and negative- energy states), so that $O_1' = 0$, and only diagram (b) of Fig. 5.1 contributes in this case. This gives

(5.37) $\mu^4 O_2' = \bar{\psi}(p_2)\gamma_5 \tilde{r}_1' \psi(q_1) (\hat{r}_1'^2 - \mu^2)^{-1} \cdot$

$\bar{\psi}(p_4)\gamma_5 \tilde{r}_3 (\tilde{t}' - M)^{-1} \gamma_5 \tilde{r}_1' \psi(q_2) \cdot$

$(\hat{r}_3^2 - \mu^2)^{-1} \bar{\psi}(p_3)\gamma_5 \tilde{r}_3 \psi(p_1) =$

$-(2M)^2 \bar{\psi}(p_2)\gamma_5 \psi(q_1) \bar{\psi}(p_4)(\tilde{t}' - M)^3 \psi(q_2) \cdot$

$\bar{\psi}(p_3)\gamma_5 \psi(p_1) (\hat{r}_1'^2 - \mu^2)^{-1} (\hat{t}'^2 - M^2)^{-1} (\hat{r}_3^2 - \mu^2)^{-1}$

where we have used $\bar{\psi}(p_2)\gamma_5 \tilde{r}_1' \psi(q_1) =$

$\bar{\psi}(p_2)\gamma_5 (\tilde{q}_1 - \tilde{p}_2)\psi(q_1) = \bar{\psi}(p_2)\gamma_5 (M - \tilde{p}_2)\psi(q_1) =$

$\bar{\psi}(p_2)(M + \tilde{p}_2)\gamma_5 \psi(q_1) = 2M \bar{\psi}(p_2)\gamma_5 \psi(q_1)$, etc.

Working out the cross-section as before, we obtain

(5.38) $\sigma_{psv} = \frac{1}{35\sqrt{3}} \cdot 2^{-\frac{31}{2}} \left(\frac{g^2}{\hbar c}\right)^4 \left(\frac{2M}{\mu}\right)^8 \cdot$

$$\left(\frac{\hbar c}{M}\right)^2 \left(\frac{\bar{E} - 7M}{M}\right)^{\frac{7}{2}} \left(1 + \frac{\mu^2}{2M^2}\right)^{-2} \left(1 - \frac{\mu^2}{4M^2}\right)^{-2}.$$

VI. APPLICATION OF THE WEIZSACKER-WILLIAMS METHOD.

The Weizsäcker-Williams (W.-W.) method was first applied to problems involving the electromagnetic interaction of fast moving electrons[15,16]. If an electron is moving with nearly the speed of light, its electromagnetic field is compressed in the direction of motion because of the Lorentz contraction. To an observer this field appears as a pulse of very short duration, and such a pulse can be represented by a certain distribution of quanta moving in the same direction as the electron. Thus the interaction of the moving electron with a given system, say an atom, can be calculated from the previously known interaction cross-section for high-energy photons with the system. Many examples are given in the paper of Williams[16].

The W.-W. method was first applied to meson-nucleon interactions by Heitler and Peng[17] in calculating meson production. We shall follow their treatment in developing the necessary equations for the case of pseudoscalar mesons, in order that we might understand the approximations involved. For pseudoscalar mesons, the source-free field equations are ($\hbar = c = 1$)

(6.01) $$\partial \Psi / \partial t = \mu \Phi$$

(6.02) $$\mu \bar{\Gamma} = -\nabla \Psi$$

(6.03) $$-\partial \Phi / \partial t = \mu \Psi + \nabla \cdot \bar{\Gamma}$$

where μ is the mass of the meson in these units. These can be combined to give the Klein-Gordon equation

(6.04) $$\frac{\partial^2 \Psi}{\partial t^2} = \nabla^2 \Psi - \mu^2 \Psi$$

with a corresponding equation for Φ and for each component of $\bar{\Gamma}$. These have plane-wave solutions of the form

(6.05) $\Psi = A \exp i(\bar{p}\cdot\bar{x} - Et)$, $\bar{\Gamma} = -i\bar{p}\mu^{-1}\Psi$, $\Phi = -iE\mu^{-1}\Psi$

where

(6.06) $$E^2 = p^2 + \mu^2$$

The density ρ and current $\bar{S}$ of mesons are given by

(6.07) $$\rho = (4\pi)^{-1}(\bar{\Gamma}^*\cdot\bar{\Gamma} + \Phi^*\Phi + \Psi^*\Psi)$$
$$\bar{S} = (4\pi)^{-1}(\Phi^*\bar{\Gamma} + \Phi\bar{\Gamma}^*)$$

satisfying the continuity equation

(6.08) $$\nabla\cdot\bar{S} + \frac{\partial\rho}{\partial t} = 0$$

The meson field of a resting nucleon, which is treated as a point source of mesons, is given by

(6.09) $\Psi' = f(\bar{\sigma}\cdot\nabla')\varphi(r')$, $\varphi(r') = \dfrac{e^{-\mu r'}}{r'}$,

$f = g\mu/2M$, where g is the usual coupling constant. $\bar{\sigma}$ is the spin vector of the source nucleon. (Quantities referred to the system in which the nucleon is at rest will be denoted by primes.) By (6.01) $\Phi' = 0$.

Now consider a system in which the nucleon is moving in a straight line with a speed v. Since $\bar{\Gamma}$ and Φ together form a four-vector, we have

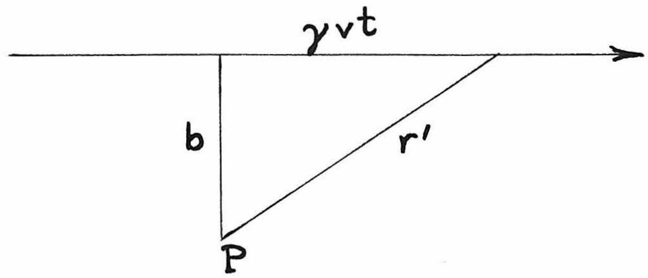
(6.10) $\Gamma_x = \gamma(\Gamma_x' + v\Phi')$, $\Phi = \gamma(\Phi' + v\Gamma_x')$

or, since $\Phi' = 0$,

(6.11) $\Gamma_x = \gamma\Gamma_x'$, $\Phi = \gamma v\Gamma_x' = v\Gamma_x$

where $\gamma = (1 - v^2)^{-\frac{1}{2}}$.

We seek the field at time t at a point P a distance b from the nucleon path. (v. Fig. 6.1) If the nucleon passes



(Fig. 6.1)

P at time t = 0, then by the Lorentz contraction rule,

(6.12) $$r' = \sqrt{x'^2 + b^2} = \sqrt{\gamma^2 v^2 t^2 + b^2}$$

since $x' = \gamma x = \gamma vt$. By (6.03), (6.09), (6.11),

(6.13) $$\Gamma_x = -\frac{\gamma}{\mu}\frac{\partial \Psi'}{\partial x'} = -\frac{1}{\mu v}\frac{\partial \Psi'}{\partial t} =$$

$$-\frac{f}{\mu v}\frac{\partial}{\partial t}\left[\sigma_x \frac{\partial}{\partial x'} + \sigma_z \frac{\partial}{\partial b}\right]\varphi(r') =$$

$$-\frac{f}{\mu v}\frac{\partial}{\partial t}\left[\frac{\sigma_x}{\mu v}\frac{\partial}{\partial t} + \sigma_z \frac{\partial}{\partial b}\right]\varphi(r')$$

where r' is expressed by (6.12). The meson current in the x-direction passing the point P is given by

(6.14) $$S_x = \frac{1}{4\pi}\left(\phi^* \Gamma_x + \phi\, \Gamma_x^*\right) = \frac{1}{2\pi} v\, |\Gamma_x|^2$$

Now we make a Fourier analysis of the meson field observed at P. Let

(6.15) $$\varphi(r') = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} e^{i\omega t} h(\omega)\, d\omega$$

where

(6.16) $$h(\omega) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{i\omega t} \frac{\exp[-\mu\sqrt{\gamma^2 v^2 t^2 + b^2}]}{\sqrt{\gamma^2 v^2 t^2 + b^2}} dt$$

$$= \frac{1}{\gamma v}\sqrt{\frac{2}{\pi}}\, K_0(\mu b \sec\alpha), \quad \text{with } \tan\alpha = \frac{\omega}{\gamma\mu v}.$$

To prove this last formula, make the substitution

(6.17) $$\gamma v t = b \sinh(\varphi - i\alpha), \quad r' = b \cosh(\varphi - i\alpha),$$

$$dt = \frac{b}{\gamma v} \cosh(\varphi - i\alpha)\, d\varphi$$

Then we get

(6.18) $$h(\omega) = (2\pi)^{-\frac{1}{2}} \frac{b}{\gamma v} \int_{-\infty + i\alpha}^{+\infty + i\alpha} \exp\left[-i \frac{\omega b}{\gamma v} \sinh(\varphi - i\alpha) - \mu b \cosh(\varphi - i\alpha)\right] d\varphi$$

$$= \frac{1}{\sqrt{2\pi}} \frac{b}{\gamma v} \int_{-\infty + i\alpha}^{+\infty + i\alpha} \exp[-\mu b \sec\alpha \cosh\varphi]\, d\varphi$$

if we pick $\tan\alpha = \omega/\mu\gamma v$ so that the terms in $\sinh\varphi$ in the exponent vanish. Deforming the path of integration to lie along the real φ-axis, we get the standard integral form

(6.19) $$K_0(z) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-z\cosh\varphi}\, d\varphi$$

This gives (6.16).

Substituting from (6.16) into (6.13),

(6.20) $$\Gamma_x = \frac{f}{\pi\gamma v^2 \mu} \left\{ \frac{\sigma_x}{\gamma v} \int_{-\infty}^{+\infty} \omega^2 e^{i\omega t} K_0(\mu b \sec\alpha)\, d\omega \right.$$

$$\left. - i\sigma_z \int_{-\infty}^{+\infty} \omega\mu \sec\alpha\, e^{i\omega t} K_0'(\mu b \sec\alpha)\, d\omega \right.$$

The integrated current in the x-direction past the point P is then given by

$$J_x = \int_{-\infty}^{+\infty} S_x \, dt = \frac{v}{2\pi}\int_{-\infty}^{+\infty} |\Gamma_x|^2 \, dt = \tag{6.21}$$

$$\frac{2v}{c}\left(\frac{f^2}{\pi^2\gamma^2 v^4 \mu^2}\right)\Big\{\gamma^{-2}v^{-2}\int_0^\infty \omega^4 [K_0(\mu b \sec\alpha)]^2 d\omega$$

$$+ \mu^2 \int_0^\infty \omega^2 \sec^2\alpha \, [K_0'(\mu b \sec\alpha)]^2 d\omega\Big\},$$

where we have used the theorem that if $g(\omega)$ is the Fourier transform of $f(t)$,

$$\int_{-\infty}^{+\infty} |f(t)|^2 \, dt = 2\pi \int_{-\infty}^{+\infty} |g(\omega)|^2 d\omega. \tag{6.22}$$

If we now integrate over all impact parameters from $b = b_{min}$ to $b = \infty$, where b_{min} is some minimum impact parameter to be determined later, we find that the total momentum in the pulse is

$$I_x = \int_{b_{min}}^\infty J_x \,(2\pi b \, db) = \tag{6.23}$$

$$\frac{4f^2}{\pi\gamma^2 v^3 \mu^2}\int_{b_{min}}^\infty \Big\{\gamma^{-2}v^{-2}\int_0^\infty \omega^4 [K_0(\mu b \sec\alpha)]^2 d\omega +$$

$$\mu^2\int_0^\infty \omega^2 \sec^2\alpha \, [K_0'(\mu b \sec\alpha)]^2 d\omega\Big\} \, b \, db.$$

Let $z = \mu b \sec\alpha$, $\bar{z} = \mu b_{min} \sec\alpha$.

$$I_x = \frac{4f^2}{\pi\gamma^2 v^3 \mu^2}\int_0^\infty \frac{\omega^2 \, d\omega}{\mu^2 \sec^2\alpha} \cdot \tag{6.24}$$

$$\Big\{\frac{\omega^2}{\gamma^2 v^2}\int_{\bar{z}}^\infty z K_0^2 \, dz + \mu^2 \sec^2\alpha \int_{\bar{z}}^\infty z K_0'^2 \, dz\Big\}$$

Now using

$$\int z[K_0(z)]^2 dz = -\frac{1}{2} z^2 \{[K_1(z)]^2 - [K_0(z)]^2\} \tag{6.25}$$

and

(6.26) $$\int z[K_0'(z)]^2 dz = -zK_0(z)K_1(z) + \frac{1}{2}z^2\{[K_1(z)]^2 - [K_0(z)]^2\},$$

and $\tan\alpha = \omega/\gamma v\mu$ we get

(6.27) $$I_x = \frac{2f^2}{\pi\gamma^2 v^3\mu^2}\int_0^\infty \omega^2 d\omega \cdot \left\{2\bar{z}K_0(\bar{z})K_1(\bar{z}) - \bar{z}^2\cos^2\alpha\left([K_1(\bar{z})]^2 - [K_0(\bar{z})]^2\right)\right\}$$

Now assume that this pulse can be represented by a distribution of mesons such that there are $q(\omega)d\omega$ mesons of energy between ω and $\omega + d\omega$. If we assume the energy $E_n = \gamma M$ of the nucleon is very large, then we can take $\omega \gg \mu$, so that each meson carries a momentum ω in the x-direction. Then the total momentum transfer is approximately

(6.28) $$I_x \doteq \int_0^\infty \omega q(\omega) d\omega .$$

Comparing with (6.27) we see that we can take

(6.29) $$q(\omega) = \frac{2f^2\omega}{\pi\gamma^2\mu^2}\left[2\bar{z}K_0K_1 - \bar{z}^2\cos^2\alpha\,(K_1^2 - K_0^2)\right],$$

where we have put $v = c = 1$. Using $f = g\mu/2M$ and $E_n = \gamma M$, we can write this as

(6.30) $$q(\omega)d\omega = \frac{1}{2\pi}\left(\frac{g^2}{\hbar c}\right)\frac{\omega d\omega}{E_n^2} \cdot \left[2zK_0K_1 - \mu^2b^2(K_1^2 - K_0^2)\right]$$

putting

(6.31) $$b = b_{min}, \quad z = \mu b \sec\alpha = \mu b\left[1 + \left(\frac{M}{\mu}\right)^2\left(\frac{\omega}{E_n}\right)^2\right]^{\frac{1}{2}}.$$

Now if a high-energy nucleon interacts with a given system, the cross-section can be found by first calculating the cross-section for interaction of a meson of energy ω with the system. This is then to be multiplied by $q(\omega)d\omega$ and integrated over ω, to get the nucleon interaction cross-section. In this it must be assumed that the nucleon is very little deflected during the collision, since we had to take the path of the nucleon as a straight line in the above derivation. Hence the method is applicable only for nucleon energies $E_n \gg M$, the nucleon mass. The spectrum $q(\omega)d\omega$ holds only for $\omega \gg \mu$ since we had to assume the meson momentum $\sqrt{\omega^2-\mu^2} \cong \omega$ in setting up (6.28). On the other hand, we do not expect to have mesons in the spectrum of energy comparable to or greater than the nucleon energy, and indeed, the interaction of such mesons would cause such a large momentum transfer to the nucleon that the path could no longer be considered straight. Hence we must cut off the spectrum at a meson energy of $\omega = \beta E_n$, where β is a fraction less than 1. We shall follow Heitler and Peng in taking $\beta \sim \frac{1}{2}$.

It is also necessary to pick a reasonable value of the minimum impact parameter b. For a collision with a nucleon at rest, take $b \approx \hbar c/M$, the Compton wave-length of the nucleon. The momentum transfer must be much less than M in order that the path of the moving nucleon be considered straight relative to the position of the other nucleon, so that we then see from the Uncertainty Principle that $b \sim \Delta x \gtrsim \hbar c/M$. Since then we have $b\mu \ll 1$, the quantity in brackets in (6.30) can be shown to be roughly $2 \log 2/Cz$, where C = Euler's constant = 0.5772. Thus $q(\omega)$ depends on the impact parameter b only logarithmically.

Of course, with all these approximations and uncertainties in the various quantities b, β , etc. which occur in the expression (6.30) for the meson distribution $q(\omega)$, it is clear that we can expect the method to give no more than an order-of-magnitude estimate of the cross-sections calculated therewith. Our results may be in error by as much as a factor of five or ten. It should be noted also that we are using the theory of a neutral meson field and neglecting any effects which may be due to the change of isotopic spin of the nucleon. We are forced to assume that the distribution of positive mesons in the field of a moving proton, or of negative mesons in the field of a neutron, is given also by (6.30).

We shall now apply this method to the production of nucleon pairs in the collision of a fast moving nucleon of energy E with another nucleon at rest. It is assumed that the mesons in the field of the moving nucleon beget nucleon pairs on the resting nucleon through processes like (4.01) and (4.60), and the cross-sections for such processes are taken as that calculated in the first part of Section IV and expressed in (4.50). Call this cross-section, as a function of the incident meson energy $E_\pi = \omega$, $\sigma'(\omega)$. Then the total cross-section for the production of a nucleon pair in the collision is

$$\sigma(E) = 2\int_{4M}^{\beta E} \sigma'(\omega)\, q(\omega)\, d\omega \tag{6.32}$$

The upper limit of integration will be taken as $\beta E \sim \frac{1}{2}E$ in accordance with the above arguments, while the lower limit is $\omega = 4M$, the threshold of the meson-nucleon process. The factor of two occurs because we must consider the two Lorentz systems in which first the one nucleon and then the other is taken as initially at rest. Write $\sigma'(\omega)$ as

$$\sigma'(\omega) = \left(\frac{g^2}{\hbar c}\right)^3 \left(\frac{\hbar c}{M}\right)^2 f(\omega) \tag{6.33}$$

where $f(\omega)$ is the quantity plotted in the graph of Fig. 4.3. Put (6.30) in the form

$$q(\omega)\, d\omega = \frac{1}{2\pi}\left(\frac{g^2}{\hbar c}\right) \frac{\omega\, d\omega}{E^2}\, G(z) \tag{6.34}$$

where

$$G(z) = 2z\, K_0(z)\, K_1(z) - \mu^2 b^2 \left\{ [K_1(z)]^2 - [K_0(z)]^2 \right\}. \tag{6.35}$$

$$z = \sqrt{\left(\frac{\mu}{M}\right)^2 + \left(\frac{\omega}{E}\right)^2} = \sqrt{a^2 + x^2} \tag{6.36}$$

with $a = \mu/M = 0.156$, $x = \omega/E$.

$$\therefore \sigma(E) = \frac{1}{\pi}\left(\frac{g^2}{\hbar c}\right)^4 \left(\frac{\hbar c}{M}\right)^2 \int_{\frac{4M}{E}}^{1/2} x\, f(Ex)\, G(z)\, dx . \tag{6.37}$$

$G(z)$ is evaluated from tables of modified Bessel functions, and it is tabulated below.

x	0	0.05	0.10	0.15	0.20	0.25	0.30
G(z)	3.017	2.993	2.930	2.783	2.582	2.362	2.142
x	0.35	0.40	0.45	0.50			
G(z)	1.933	1.741	1.566	1.408			

The integration in (6.37) is carried out numerically and the results are tabulated below.

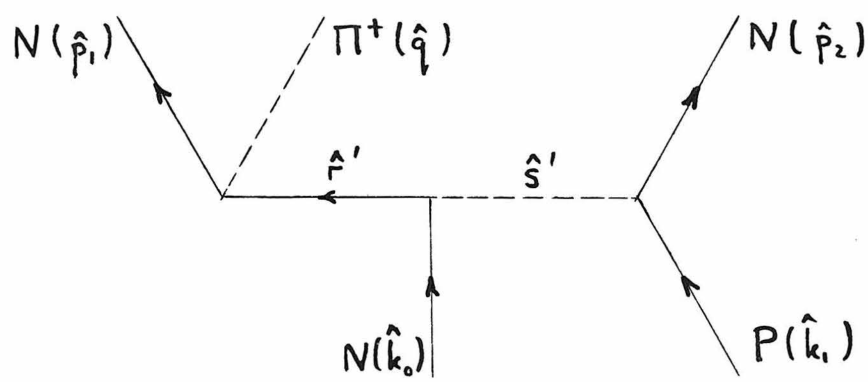
E/M	13.3	16	20	26.7	40	80	160
$\sigma\left(\frac{g^2}{\hbar c}\right)^{-4}\left(\frac{\hbar c}{M}\right)^2\cdot 10^3$	0.14	0.29	0.53	0.87	1.32	1.76	1.73

VII. PRODUCTION OF MESONS IN NUCLEON-NUCLEON COLLISIONS

For sake of comparison it is of value to estimate the cross-section for the production of mesons in nucleon-nucleon collisions. We shall consider the process

(7.01) $\quad N(\hat{k}_0) + P(\hat{k}_1) \longrightarrow N(\hat{p}_1) + N(\hat{p}_2) + \pi^+(\hat{q})$

This calculation has been carried out by C. Morette[18], but her method of momentum-space integration is so complicated and difficult to understand that we shall repeat it using the methods and formulas developed above. The pertinent diagram is given in Fig. 7.1 below.



(Fig. 7.1)

From this we can write down the transition matrix element

(7.02) $$H = (\sqrt{4\pi}\, g \hbar c)^3 (2\varepsilon)^{-\frac{1}{2}}\, 2^{-\frac{1}{2}} (O' - O'')$$

where

(7.03) $$O' = \bar{\psi}(p_1)\, \gamma_5 (\tilde{r}' - M)^{-1} \gamma_5\, \psi(k_0) \cdot (\hat{s}'^2 - \mu^2)^{-1}\, \bar{\psi}(p_2)\, \gamma_5\, \psi(k_1)$$

and O" is got by interchanging $\hat{p}_1$ and $\hat{p}_2$ everywhere.

(7.04) $$\hat{r}' = \hat{p}_1 + \hat{q}, \quad \hat{s}' = \hat{p}_2 - \hat{k}_1$$

In the center-of-mass system let

(7.05) $\hat{k}_0 = (E_0, \bar{k})$, $\hat{k}_1 = (E_0, -\bar{k})$, $\hat{p}_1 = (E_1, \bar{p}_1)$,

$\hat{p}_2 = (E_2, \bar{p}_2)$, $\hat{q} = (\varepsilon, \bar{q})$, $T = 2E_0$

We can simplify (7.03) to obtain

(7.06) $$O' = \frac{\bar{\psi}(p_2)\,\gamma_5\,\psi(k_1)\,\bar{\psi}(p_1)\,\tilde{q}\,\psi(k_0)}{(\hat{s}'^2 - \mu^2)(\hat{r}'^2 - M^2)}$$

The differential cross-section for the process is given by

(7.07) $$d\sigma = \frac{2\pi}{\hbar v}\,\frac{1}{4}\sum |H|^2 \rho_F$$

where we have a factor $\frac{1}{4}$ to average over the spins of the initial nucleons. The relative velocity v is given by

(7.08) $$v/c = 2k/E_0$$

while the density of final states is, by (1.20),

(7.09) $$\rho_F = (2\pi\hbar c)^{-6}\,E_1 E_2\,\varepsilon\,dE_2\,d\Omega_2\,d\varepsilon\,d\alpha$$

where α is a polar angle locating the vector $\bar{q}$ and measured from the plane through $\bar{p}_2$ and $\bar{k}$. Combining (7.02) and (7.07-.09), we get

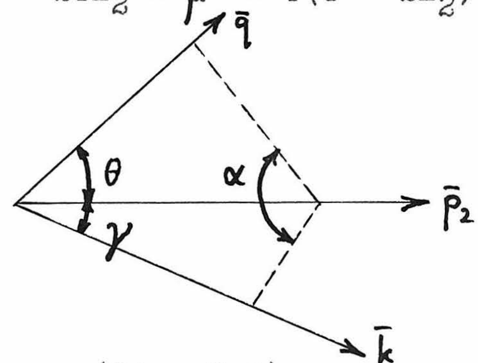
(7.10) $$d\sigma = \frac{g^6\,dE_2\,d\Omega_2\,d\varepsilon\,d\alpha}{16\pi^2\hbar c\,k\,E_0}\,M^4 \sum |O' - O''|^2$$

where we have already multiplied by the factor $M^4/E_0^2 E_1 E_2$ which corrects the spurs for normalization. Evaluating the sum of the absolute squares of H by the same methods as previously, we again obtain from $\sum \bar{O}'O''$ an exchange term proportional to $(\hat{p}_1 - \hat{p}_2)^2$ which we are obliged to neglect in order to carry out the integration. Since we are interested only in high-energy collisions, we may also neglect the meson mass μ. Then we find for the total cross-section

(7.11) $$\sigma = \frac{g^6}{16\pi^2\hbar c k T}\int \frac{(\hat{k}_0\cdot\hat{q})\,dE_2\,d\Omega_2\,d\varepsilon\,d\alpha}{(\hat{r}'^2 - M^2)(\hat{s}'^2-\mu^2)} =$$
$$\frac{g^6}{32\pi^2\hbar c k T^2}\int \frac{(E_0\varepsilon - \bar{q}\cdot\bar{k})\,dE_2\,d\Omega_2\,d\varepsilon\,d\alpha}{(T-2E_2)(E_2E_0+\bar{p}_2\cdot\bar{k}-M^2+\frac{1}{2}\mu^2)}$$

The first integration is over all positions of $\vec{q}$, which lies on a surface given by (cf. (1.06))

(7.12) $$\bar{p}_2\cdot\bar{q} = p_2 q\cos\theta = \tfrac{1}{2}F^2 - (T - E_2)\varepsilon\,,$$
$$F^2 = T^2 - 2TE_2 - \mu^2 \doteq T(T - 2E_2) = G^2 - M^2$$



(Fig. 7.2)

Let the angle between $\bar{p}_2$ and $\bar{k}$ be γ. Then Fig. 7.2 gives

(7.13) $$\bar{q}\cdot\bar{k} = qk\,(\cos\theta\cos\gamma + \sin\theta\sin\gamma\cos\alpha\,)$$

Thus the integration over α is easily carried out, after which we integrate over ε, and use the relations

(7.14) $$\varepsilon'' + \varepsilon' = F^2G^{-2}(T - E_2),\quad \varepsilon'' - \varepsilon' = p_2F^2G^{-2}$$

(cf. (1.12-.13)). Integrating then over γ, we obtain

(7.15) $$\sigma = \frac{g^6}{16\hbar c k^2}\int_M^{\frac{1}{2}T} \frac{dE'(T-2E')}{(T^2-2TE'+M^2)^2}\,\cdot$$
$$\left\{(T^2-2TE'+2M^2)\ln\frac{E_0E'-M^2+kp'}{M(E_0-E')} + 2kp'\right\},\ p'^2 = E'^2 - M^2,$$

since the maximum value of nucleon energy is given by (1.16) as

(7.16) $E'^0 = [T^2 + M^2 - (M + \mu)^2]/2T = [T^2 - 2M\mu + \mu^2]/2T \cong \frac{1}{2}T$

for $T - 2M \gg \mu$.

We note that if E_n is the energy of one of the nucleons in the Lorentz system in which the other is at rest,

(7.17) $T^2 = 2M(E_n + M)$, $k^2 = \frac{1}{2}M(E_n - M)$

(7.15) is integrated numerically, and it gives the following typical values:

E_n/M	5.5	12.5	19.5	41.3	80.9	161
$\sigma\cdot\left(\frac{g^2}{\hbar c}\right)^{-3}\left(\frac{\hbar c}{M}\right)^{-2}\cdot 10^2$	1.37	0.97	0.80	0.47	0.30	0.18

(7.15) is quite well represented at high energies by

(7.18) $$\sigma \doteq \frac{g^6}{16\hbar c M E_n}\left(\log\frac{2E_n}{M} - 1\right)$$

It appears that meson-production and pair-production become of comparable probability at very high energies.

VIII. PRODUCTION OF ANTIPROTONS IN COSMIC RAYS.

Using our calculated cross-sections we can now estimate the numbers of antiprotons produced in the atmosphere by cosmic radiation. Consider first the interaction of pi-mesons with nucleons to produce pairs by processes like (4.01). Since the energy required is so high, we can neglect the effect of the nuclear binding, and we can use the cross-section expressed by (4.50). This will be written as

$$\sigma(E) = \left(\frac{g^2}{\hbar c}\right)^3 \left(\frac{\hbar c}{M}\right)^2 f(E) \tag{8.01}$$

where f(E) is plotted in Fig. 4.3, and E is the energy of the incident pi-meson in units of M, the nucleon mass.

According to the work of Sands[19], mu-mesons are produced in the upper atmosphere with a spectrum of the form $E^{-\gamma}\,dE$, with $\gamma \cong 2.5$, for high energies, and we can take this form also for the spectrum of pi-mesons in the atmosphere. Thus the probability that a pi-meson will have energy between E and E + dE is given by

$$p(E)dE = (\gamma - 1)(E_c)^{\gamma - 1}E^{-\gamma}\,dE, \quad E_c < E < \infty \tag{8.02}$$

where E_c is a cut-off energy on the order of $\frac{1}{2}M$.

We shall consider loss of pi-mesons only by decay, neglecting nuclear interactions. The fraction of mesons which survive a distance x in the atmosphere is given by $\exp(-x/c\tau') = \exp(-x\mu/c\tau E)$, where $\tau' = (E/\mu)\tau$ is the lifetime of the mesons corrected for relativistic time-dilation. τ is the lifetime of the meson at rest, approximately 2.6×10^{-8} sec.[20] If

$\rho(x)$ is the density of nucleons at distance x, the probability that a meson of energy E will interact before reaching sea-level or before decaying is

(8.03) $$g(E) = \int_0^H \rho(x) e^{-x\mu/c\tau E} \sigma(E)\, dx.$$

H is the distance of production of pi-mesons above sea-level, and $\rho(x) = \rho_0 \exp[-\alpha(H - x)]$, where $\rho_0 = 7.8 \times 10^{20}$ cm^{-3} is the density of nucleons at sea level. $\alpha \cong 1.4 \times 10^{-6}$ cm^{-1}. If we assume mesons produced at about 100 gm/cm^2 below the top of the atmosphere, $e^{-\alpha H} = 0.1$, and H = 16.4 KM. (8.03) becomes

(8.04) $$g(E) = \rho_0 \sigma(E) e^{-\alpha H} \left(1 - e^{-\delta H}\right) \delta^{-1},$$

$$\delta = \frac{\mu}{c\tau E} - \alpha = \left(199 \frac{M}{E} - 1.4\right) \cdot 10^{-6} \text{ cm}^{-1}$$

Thus the probability that a pi-meson will make an antiproton in the atmosphere is given by

(8.05) $$Y = \tfrac{1}{2} \int_{4M}^{\infty} g(E) p(E)\, dE = \tfrac{1}{2} \rho_0 e^{-\alpha H} \left(\frac{g^2}{\hbar c}\right)^3 \left(\frac{\hbar c}{M}\right)^2 \cdot$$
$$(\gamma - 1) E_c^{\gamma - 1} \int_{4M}^{\infty} E^{-\gamma} f(E) \left(1 - e^{-\delta H}\right) \delta^{-1}\, dE,$$

where the factor $\frac{1}{2}$ occurs because a positive meson produces an antiproton only in interacting with a neutron, a negative meson only with a proton. Numerical integration with $\gamma = 2.5$ gives a value of 92.1 for the integral, so that

(8.06) $$Y = 2.4 \times 10^{-6} (E_c/M)^{1.5} (g^2/\hbar c)^3$$

or about $10^{-6}(g^2/\hbar c)^3$.*

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* We leave all results expressed in terms of the coupling constant, since its value will probably be revised when current calculations of the nuclear interaction potential taking into account exchanges of two mesons are completed (R.P. Feynman, class lectures). It should be remembered that we use unrationalized units, in which the Yukawa potential is of the form $g^2 \exp(-\mu r)/r$ for the scalar theory. In pseudoscalar theory, $g^2/\hbar c$ is of the order of unity.

We can also calculate the production of antiprotons by the primary cosmic rays. Since the energies are so high, collisions can be considered as between individual nucleons only, neglecting nuclear binding, and we can use the cross-sections estimated in Section VI.

The spectrum of primaries can be taken as that in (8.02), with a cut-off energy determined by the latitude. For a latitude of 41^{o}N., $E_c \cong 4.7$ Bev. or 4.4M. In penetrating a distance equivalent to d gm/cm^2 of atmosphere, the number of antinucleons produced per incident primary nucleon is

$$Y = N_a d \int_{7M}^{\infty} \sigma(E) p(E) dE \tag{8.07}$$

where $N_a = 6.02 \times 10^{23}$ is the Avogadro number, and $\sigma(E)$ is the cross-section calculated in Section VI, as a function of incident nucleon energy. Numerical integration is used to obtain, with $\gamma = 2.5$,

$$Y = 5.3 \times 10^{-9} d(E_c/M)^{1.5}(g^2/\hbar c)^4 = 4.9 \times 10^{-8} d(g^2/\hbar c)^4 \tag{8.08}$$

Thus, considering that a primary nucleon penetrates to an average depth of 100 gm/cm^2 before losing most of its energy by meson production and other processes, we see that we can expect only about $5 \times 10^{-6}(g^2/\hbar c)^4$ antiprotons per primary cosmic-ray nucleon.

It is instructive to compare the effectiveness of different parts of the primary cosmic-ray spectrum for antinucleon and for meson production, by evaluating (8.07) for different cut-off energies, using the cross-sections calculated in Sections VI and VII. Carrying out the numerical integrations, we obtain for the yields Y_n of anti-nucleons and Y_π of mesons, per primary nucleon, the following values. ($g^2/\hbar c = 1$ and $d = 100$ gm/cm^2 here.)

E_c/M	4.4	100	1000
Y_n	4.9×10^{-6}	4.3×10^{-5}	1.9×10^{-5}
Y_π	2.9×10^{-4}	4.7×10^{-5}	7.0×10^{-6}
$Y_n/Y_\pi \quad (g^2/\hbar c)^{-1}$	1/60	1/1.1	2.8/1

Thus it is seen that meson production exceeds anti-nucleon production by a factor of the order of 60 for the entire primary spectrum, but that the two are comparable for primaries of energies $\gtrsim 10^{11}$ or 10^{12} e.v.

IX. CONCLUSION.

In the preceding sections we have used standard perturbation methods and the pseudoscalar meson theory to compute the cross-sections for the production of antiprotons in meson-nucleon and in nucleon-nucleon collisions. These were then applied to determine the numbers of antiprotons to be expected due to these processes in the interaction of energetic cosmic-ray particles with the nucleons of the atmosphere. These numbers were found to be very small, on the order of $10^{-6}(g^2/\hbar c)^4$ per primary particle, so that it is not too surprising, on this basis, that antiprotons have not yet been observed in cosmic radiation. However, it is of value to compare antinucleon production with meson production, as was done at the end of the last section. There it was found that meson production exceeds antinucleon production by a large factor of about 60 for the entire primary spectrum. (Note that the meson production calculated is itself too small to account for the observed numbers of mesons in cosmic rays.) But for primaries of extreme energies, the cross-sections become comparable, and one would expect about as many antinucleons as mesons to be produced. Of course, there is evidence that at such energies multiple production occurs, and one is inclined to doubt that the simple perturbation treatment used here is adequate.

Fermi[7] has gone to the opposite extreme and used a statistical treatment to estimate the numbers of pi-mesons and antinucleons to be expected in energetic collisions. He assumes that the entire energy of the collision is concentrated in a small volume which is a sphere of radius $\hbar c/\mu$ but contracted in the direction of motion by a Lorentz factor $2M/W$, where W is the total energy in the center-of-mass system. This energy is assumed to be distributed among states containing different numbers of pi-mesons and nucleon pairs, according to their statistical weights. By so doing he finds reasonable numbers for the multiplicity of meson production. At low energies of about 10 to 100 Bev. for the incident nucleon, only about 0.002 times as many antinucleons as mesons are produced. (The cross-sections calculated by perturbation methods are in a ratio of about 0.1 $g^2/\hbar c$ at these energies.) At extreme energies, greater than about 10^3 Bev., the number of mesons to be expected is given as about 0.33 $(E_n/M)^{\frac{1}{4}}$, with the number of antinucleons as 0.76 $(E_n/M)^{\frac{1}{4}}$. Here E_n is the energy of the incident nucleon in the laboratory system. Both these numbers are on the same order of magnitude, in vague agreement with our results as to the cross-sections. Fermi assumes a total cross-section for all processes of about $\pi(\hbar c/\mu)^2 = 129(\hbar c/M)^2 = 5.7 \times 10^{-26}$ cm^2. This is much larger than any of the cross-sections obtained here, unless one takes a very large value of the coupling constant. The maximum of our cross-section for the production of antinucleons in nucleon-nucleon collisions, from the table on p. 66, is about $1.8 \times 10^{-3}(\hbar c/M)^2(g^2/\hbar c)^4 = 8.0 \times 10^{-31}$ $(g^2/\hbar c)^4$ cm^2.

While both methods give comparable values for the ratio of the numbers of pi-mesons to antinucleons produced in collisions at extreme energies, Fermi's statistical treatment would seem to predict a smaller relative total production of antinucleons than the perturbation method. Since antiprotons have not yet been observed, this may indicate that his method and viewpoint are somewhat closer to reality than the conventional theory used here.

Antinucleons produced in the very energetic collisions observed in photographic emulsions would have such a high energy that they would not be expected to be annihilated in the same emulsion, and they would thus be indistinguishable from ordinary nucleons. One could hope to observe antiprotons only lower down in the atmosphere after they had lost most of their energy, and it seems indicated that their total number would be very much smaller than the numbers of mesons or nucleons in cosmic radiation.

APPENDIX I. ANNIHILATION OF ANTIPROTONS IN BOUND STATES.

An antiproton in hydrogen will ultimately be captured into an S-state orbit about a proton. It is of interest to determine the lifetime against annihilation in such a state. First, it can be shown that annihilation is forbidden if the state is a singlet ($^1S_0$). The parity of such an S-state with a nucleon and an antinucleon is odd. (This is the same as the case of positronium, cf. reference 21). The two mesons coming off can then have only odd values of total J, i.e., J = 1, 3, etc., since they have zero spin. Hence annihilation into two mesons can occur only from the triplet ($^3S_1$) state, and not from the singlet state, since total angular momentum must be conserved.*

It is instructive to obtain this conclusion using previous methods. Using pseudoscalar coupling, it is shown in Section II that the matrix element for the transition is proportional to $\bar{\psi}_f \tilde{q} \psi_i = \psi_f^* (\varepsilon - \bar{\alpha} \cdot \bar{q}) \psi_i$, where ψ_i is the Dirac wave-function for the initial proton and ψ_f that for the proton in a negative-energy state which fills the hole which represented the antiproton. $\hat{q} = (\varepsilon , \bar{q})$ is the four-momentum of one of the emitted mesons. In the non-relativistic limit, $\psi_f^* \psi_i = 0$ because of the orthogonality of states

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* In the singlet state, annihilation would be most likely into two charged mesons and a neutral meson, each taking on the average 1/3 of the available energy.

of positive and negative energy. Hence our result depends only on the term $\psi_f^* \bar{\alpha} \cdot \bar{q}\, \psi_i$ or $\psi_f^* \alpha_z \psi_i$, if we take the z-axis along the direction $\bar{q}$ of the meson. (This is permissible in a singlet S-state, which has spherical symmetry and no preferred axes.) The same result would follow from the scalar coupling, in this limit.

We must, however, take into account that the initial singlet state must be anti-symmetric in the spin parts of the wave-functions of the initial particles (proton and antiproton). To convert the matrix element into a form in which the wave-function of the antiproton appears, we use the charge-conjugation operator C, which has the property that if ψ represents a particle of positive charge, and satisfies the Dirac equation, then $\varphi^* = C\psi$ represents a particle of negative charge which is really a hole in the sea of negative-energy states of the former positive particle[22]. Then our matrix element can be written down, putting $\alpha_z = O$, $\psi_f = C^{-1}\varphi^*$, $\psi_f^* = \varphi C^{-1*}$,

$$M = \psi_f^* O \psi_i = \sum_{\alpha\beta} \psi_{f\alpha}^* O_{\alpha\beta} \psi_{i\beta} = \tag{I.01}$$

$$\sum_{\alpha\beta} (\varphi C^{-1*})_\alpha O_{\alpha\beta} \psi_\beta = \sum_{\alpha\beta\gamma} \varphi_\gamma C^{-1*}_{\gamma\alpha} O_{\alpha\beta} \psi_\beta = \sum_{\beta\gamma} N_{\gamma\beta} \varphi_\gamma \psi_\beta ,$$

where $N = C^{-1*}O$. Now let superscripts (1) and (2) refer to the two spin states (up or down) of the initial particles. For the singlet state (total spin zero), we must use, instead of (I.01),

$$M = \frac{1}{\sqrt{2}} \sum_{\beta\gamma} N_{\gamma\beta} \left(\varphi_\gamma^{(1)} \psi_\beta^{(2)} - \varphi_\gamma^{(2)} \psi_\beta^{(1)} \right). \tag{I.02}$$

In the usual system of $\vec{\alpha}, \beta$ matrices, in which they are Hermitian, a suitable charge-conjugation operator C is given by

$$C = -i\beta\alpha_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{I.03}$$

Then $C^{-1*} = C$, and

$$N = C\alpha_z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{I.04}$$

Non-relativistically, $\varphi^{(1)} = \psi^{(1)} = (0\ 0\ 1\ 0)$, $\varphi^{(2)} = \psi^{(2)} = (0\ 0\ 0\ 1)$. Putting these into (I.02), we see that M = 0 so that the transition is forbidden in the singlet state.*

The annihilation into two charged mesons for an S-state of the bound proton and antiproton must therefore go only in the triplet state. The lifetime τ is simply given by

$$\tau^{-1} = 4v\,\sigma(0)\,|\Phi(0)|^2 = \pi r_0^2 c\,|\Phi(0)|^2 \tag{I.05}$$

where $|\Phi(0)|^2$ is the density of nucleons at the origin, and v is the relative velocity of the proton and antiproton. $\sigma(0)$ is the zero-energy limit of the annihilation cross-section, which is given by (3.06). We multiply by a factor of four since we must use the sum over initial spin states, and

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* In the annihilation of a positron and an electron, in the ground-state of positronium, into two gamma-rays, it turns out that one obtains for M, $\psi_f^* \gamma_5 \psi_i$, so that $O = \gamma_5$, and

$$N = C^{-1*}O = C\gamma_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

so that $M \neq 0$, and the process is allowed in the singlet state. It is forbidden in the triplet state from other considerations[21].

not the average, which is used in determining the cross-section. $\Phi(r)$ is the wave-function of the ground-state of the proton-antiproton system, given by the usual formula

$$\Phi(r) = (\pi a^3)^{-\frac{1}{2}} e^{-r/a} \tag{I.06}$$

where $a = 2\hbar^2c^2/Me^2$ is the Bohr radius, using the reduced mass of $\frac{1}{2}M$. Then (I.05) becomes

$$\tau^{-1} = \frac{1}{8}\left(\frac{g^2}{\hbar c}\right)^2\left(\frac{e^2}{\hbar c}\right)^3 \frac{M}{\hbar} \tag{I.07}$$

or $\tau = 1.44 \times 10^{-17} (g^2/\hbar c)^{-2}$ sec.

APPENDIX II. MU-MESON DECAY SPECTRUM.

As a further application of the formulas of section I for momentum-space integration, we shall work out the energy-spectrum of the electrons from the decay of the mu-meson. Such spectra have been worked out by Tiomno, Wheeler, and Rau[23] for the five usual beta-interactions. We shall use the completely anti-symmetric interaction of Wigner and Critchfield[24]. Then the matrix element for the interaction can be written

(II.01)
$$H = G \begin{vmatrix} \Psi_4^* & \Phi_1 & \varphi_1 & \psi_1 \\ -\Psi_3^* & \Phi_2 & \varphi_2 & \psi_2 \\ -\Psi_2^* & \Phi_3 & \varphi_3 & \psi_3 \\ \Psi_1^* & \Phi_4 & \varphi_4 & \psi_4 \end{vmatrix}$$

where Ψ is the plane-wave Dirac spinor for the decaying meson, and Φ, φ, ψ are spinors for the particles produced, which have momenta $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$, energies E_1, E_2, E_3, and masses m_1, m_2, m_3, respectively. M is the mass of the mu-meson. The components of the Dirac plane-wave spinors are tabulated in Heitler[15], p. 86, and we can use them to evaluate the sum of the absolute squares of H over the spin states of the final particles. In the c.m. system, the state of the initial (resting) meson is given by $\Psi = (0\ 0\ 0\ 1)$. Then we get, using also conservation of energy and momentum as embodied in (1.06),

(II.02)
$$\Sigma |H|^2 = \sum_{i=1}^{3} |H_i|^2$$

where

(II.03) $|H_1|^2 = G^2(E_1 + m_1)\left[M^2 + m_1^2 - (m_2 + m_3)^2 - 2ME_1\right]/8E_1E_2E_3$,

and $|H_2|^2$, $|H_3|^2$ are obtained by cyclic permutation.

Assuming the decay to produce an electron of mass $m_1 = m$ and two neutrinos of zero mass, we obtain

(II.04) $\sum |H|^2 = G^2 \left[(E_1 + m)(M^2 + m^2 - 2ME_1) + E_2(M^2 - m^2 - 2ME_2) + E_3\, (M^2 - m^2 - 2ME_3) \right] / 8E_1E_2E_3.$

The decay probability per second is given by the usual formula

(II.05) $$dw = \frac{2\pi}{\hbar} \sum |H|^2 \rho_F$$

where ρ_F is the density of final states, given according to (1.20) as

(II.06) $\rho_F = 2(2\pi)^2(2\pi\hbar c)^{-6}\; E_1E_2E_3dE_1dE_2$

where we have already integrated over angles, which do not appear in $\sum |H|^2$. In the integration over neutrino energies E_2, E_3, the last two terms of (II.04) give an equal contribution. Using (1.12-.13), we obtain for this case,

(II.07) $$\int dE_2 = p_1, \quad \int E_2 dE_2 = \tfrac{1}{2}p_1\, (M - E_1),$$
$$\int E_2^{\,2} dE_2 = \frac{1}{12}\, p_1 \left[3(M - E_1)^2 + p_1^{\,2} \right]$$

where $p_1^{\,2} = E_1^{\,2} - m^2$. Hence the probability that an electron is emitted with an energy between E_1 and $E_1 + dE_1$ is given by

(II.08) $$dw = \frac{G^2 p_1}{4(2\pi)^3 \hbar^7 c^6} \left\{ m\left(M^2 - \tfrac{2}{3} mM + m^2\right) + 2E_1\left(M^2 - mM + m^2\right) - \tfrac{10}{3} ME_1^{\,2} \right\} dE_1, \quad m < E_1 < \bar{E}_1,$$

with $\bar{E}_1 = (M^2 + m^2)/2M$, by (1.16).

Neglecting $m \ll M$, this simplifies to

(II.09) $$dw = \frac{G^2 M}{2(2\pi)^3 \hbar^7 c^6} p_1 E_1 \left(M - \tfrac{5}{3} E_1\right) dE_1, \quad m < E_1 < \tfrac{1}{2} M$$

Then the mean-life of the mu-meson, τ , is got by integrating over E_1:

$$\frac{1}{\tau} = \int dw \doteq \frac{G^2 M^5}{2^7 (2\pi)^3 \hbar^7 c^6} \tag{II.10}$$

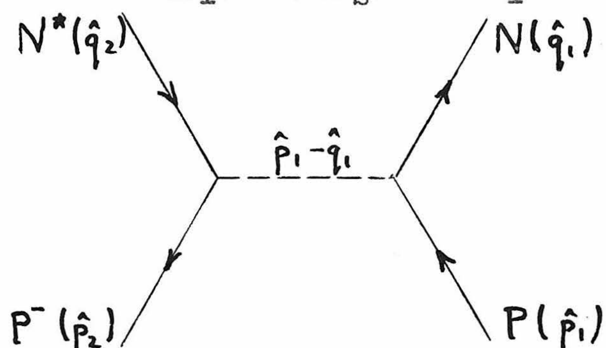
Taking the meson mass as $M = 210m$, and $\tau = 2.15 \times 10^{-6}$ sec., we find $G = 3.22 \times 10^{-49}$ erg-cm^3.

APPENDIX III. ADDITIONAL PROCESSES WITH ANTIPROTONS.

(a) Charge Exchange.

A proton colliding with an antiproton could exchange a charged meson with it to form a neutron-antineutron pair:

(III.01) $$P(\hat{p}_1) + \bar{P}(\hat{p}_2) \longrightarrow N(\hat{q}_1) + N^*(\hat{q}_2)$$



(Fig. III.1)

This is represented by the diagram of Fig. III.1, and it is seen to be analogous to proton-neutron scattering, except that one of the nucleons is moving backwards in time. In fact, the cross-section in lowest order will have the same form as that for scattering with one meson exchanged. The matrix element is given by

(III.02) $$H = 4\pi g^2 \hbar^2 c^2\, \bar{\psi}(p_2)\, \gamma_5\, \psi(q_2) \cdot \left[(\hat{p}_1 - \hat{q}_1)^2 - \mu^2 \right]^{-1} \bar{\psi}(q_1)\, \gamma_5\, \psi(p_1),$$

with

(III.03) $$\tilde{p}_1 \psi(p_1) = M\psi(p_1), \quad \tilde{p}_2 \psi(p_2) = -M\psi(p_2)$$
$$\tilde{q}_1 \psi(q_1) = M\psi(q_1), \quad \tilde{q}_2 \psi(q_2) = -M\psi(q_2)$$

The cross-section is then

(III.04) $$d\sigma = \frac{2\pi}{\hbar v} \frac{1}{4} \sum |H|^2 \rho_F$$

with

(III.05) $\quad v = (2p/E)c \quad$ and $\quad \rho_F = \dfrac{pE\,d\Omega}{2(2\pi\hbar c)^3}$

in the c.m. system. Thus the differential cross-section is

$$d\sigma = \frac{g^4}{16E^2} \frac{Q^4}{(Q^2+\mu^2)^2}\, d\Omega \tag{III.06}$$

where $Q = |\bar{p} - \bar{q}|$ is the momentum transfer in the c.m. system. For large energies the total cross-section is

$$\sigma = \frac{\pi g^4}{4E^2} = \frac{\pi g^4}{2M(E_0+M)}, \tag{III.07}$$

where E_0 is the energy of the incident antiproton in the laboratory system. At low energies, if v is the relative velocity of the pair,

$$\sigma \doteq \frac{\pi g^4}{12M^2}\left(\frac{M}{\mu}\right)^4\left(\frac{v}{c}\right)^4 \tag{III.08}$$

Thus this charge-exchange reaction has small probability in comparison with annihilation of the pair into two mesons, for the pseudoscalar coupling.

In the scalar theory, however, with $\gamma_5 = 1$ in (III.02), the cross-section is larger, and (III.06) becomes

$$d\sigma = \frac{g^4}{16E^2} \frac{(4M^2+Q^2)^2}{(Q^2+\mu^2)^2}\, d\Omega \tag{III.09}$$

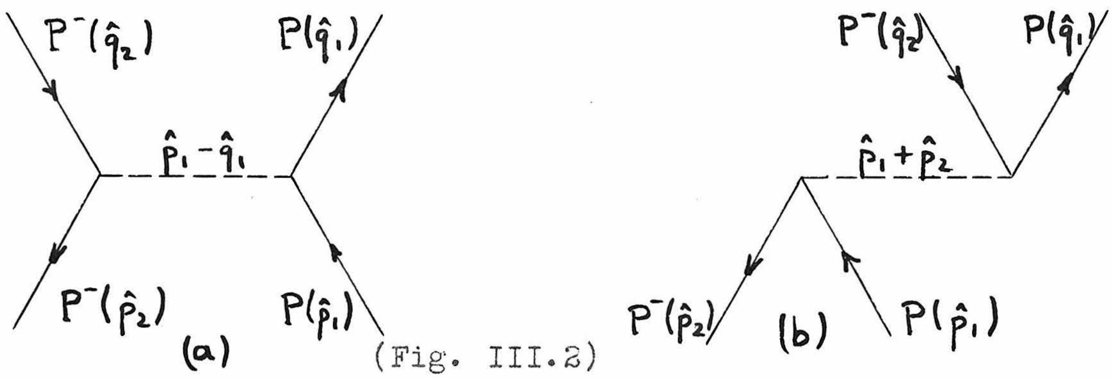
so that at low velocities

$$\sigma \doteq \pi\left(\frac{g^2}{M}\right)^2\left(\frac{M}{\mu}\right)^4 \tag{III.10}$$

This is larger than the annihilation cross-section, except for energies less than about 1 e.v. Of course, if an antineutron is heavier than an antiproton, just as an ordinary neutron is heavier than a proton, this mass difference would act to prevent the exchange of charge at low energies (less than about 1 Mev.). But on the scalar theory, an antiproton slowing down from about 100 Mev. would be seen to disappear and reappear as it lost and recovered its negative charge in collisions.

(b) Scattering of Antiprotons.

In lowest order, the scattering of an antiproton with a proton would take place through the exchange of a neutral meson in accordance with the diagrams of Fig. III.2.



(Fig. III.2)

The notation is the same as in III(a). We can write down the matrix element immediately:

(III.11) $$H = 4\pi g_0^2 \hbar^2 c^2 \Big\{ \bar{\psi}(p_2)\gamma_5\psi(q_2)\left[(\hat{p}_1-\hat{q}_1)^2-\mu^2\right]^{-1} \cdot \bar{\psi}(q_1)\gamma_5\psi(p_1) - \bar{\psi}(q_1)\gamma_5\psi(q_2)\left[(\hat{p}_1+\hat{p}_2)^2-\mu^2\right]^{-1}\bar{\psi}(p_2)\gamma_5\psi(p_1)\Big\}.$$

Working out the cross-section in the usual way, we obtain

(III.12) $$d\sigma = \frac{g_0^4}{16E^2}\left[\left(1-\frac{\mu^2}{4E^2}\right)^{-2} + \left(1-\frac{\mu^2}{4E^2}\right)^{-1}\left(1+\frac{\mu^2}{Q^2}\right)^{-1} + \left(1+\frac{\mu^2}{Q^2}\right)^{-2}\right] d\Omega$$

where E is the energy and $Q = |\bar{p} - \bar{q}|$ is the momentum transfer in the center-of-mass system. The non-relativistic cross-section is a constant:

(III.13) $$\sigma \doteq \frac{\pi g_0^4}{4M^2}\left(1-\frac{\mu^2}{4M^2}\right)^{-2}$$

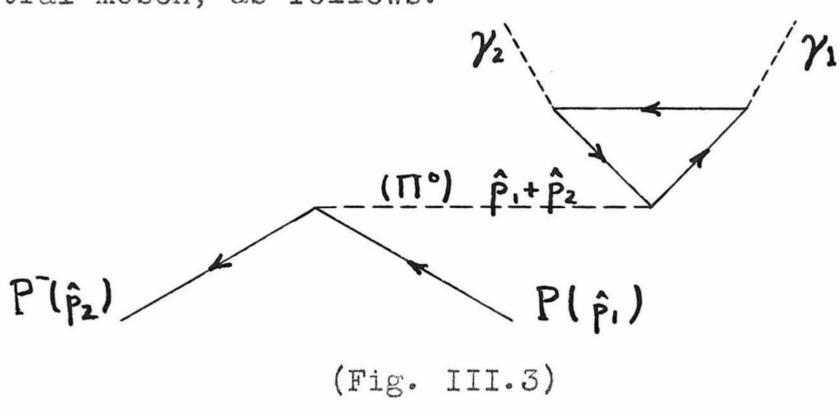
The scattering of antiprotons with neutrons takes place through formation of a virtual charged meson. The diagram is similar to Fig. III.2(B). The cross-section is just the first term of (III.12), so that the total cross-section in the c.m. system is

(III.14) $$\sigma = \frac{\pi g^4}{4E^2}\left(1-\frac{\mu^2}{4E^2}\right)^{-2}$$

Non-relativistically this reduces to (III.13), replacing g_0 by g. Thus scattering is seen to be of smaller probability than annihilation at all energies, in the pseudoscalar theory.

(c) Annihilation into Gamma Rays.

The second diagram of Fig. III.2 above leads one to ask of the possible importance of a process in which the anti-proton combines with a proton to form a virtual neutral meson, which then decays into two gamma rays as neutral mesons have been observed to do. The cross-section for such a reaction can be worked out in terms of the mean-life τ of the neutral meson, as follows.



(Fig. III.3)

The mean-life is given by

(III.15) $$\frac{1}{\tau} = \frac{2\pi}{\hbar}|H_1|^2 \rho'$$

where H_1 is the matrix element for the decay, and ρ' is the volume of momentum space, given by

(III.16) $$\rho' = \frac{4\pi q^2}{2(2\pi\hbar c)^3} = \frac{\pi\mu^2}{2(2\pi\hbar c)^3}$$

where $q = \frac{1}{2}\mu$ is the energy of each gamma ray in the c.m. system. On the other hand, the annihilation cross-section is

(III.17) $$\sigma = \frac{2\pi}{\hbar v}|H|^2_{av}\,\rho,$$

where $v/c = 2p/E$ is the relative velocity in the c.m. system, with $\hat{p}_1 = (E, \bar{p})$, $\hat{p}_2 = (E, -\bar{p})$ the four-momenta of the incident proton and antiproton. Then also

(III.18) $$\rho = \frac{4\pi E^2}{2(2\pi\hbar c)^3}$$

and $|H|^2_{av}$ is the average over spins of the square of the matrix-element, given by

(III.19) $$H = \sqrt{4\pi}\, g_0 \hbar c\, \bar{\psi}(p_2)\gamma_5\psi(p_1)\cdot \left[(\hat{p}_1 + \hat{p}_2)^2 - \mu^2\right]^{-1}\sqrt{2\mu}\, H_1,$$

where the factor $\sqrt{2\mu}$ corrects for a factor included in H_1 because the initial meson was there free. Thus we get, using (III.15),

(III.20) $$c\sigma\tau = \frac{8\pi g_0^2\hbar^2c^2E^3}{\mu p(4E^2-\mu^2)^2} \doteq \frac{\pi g_0^2\hbar^2c^2}{2\mu p E}$$

Non-relativistically, with $E \doteq M$, $p = \frac{1}{2}Mv/c$,

(III.21) $$\sigma = \pi\left(\frac{g_0^2}{\hbar c}\right)\left(\frac{\hbar c}{M}\right)^2\frac{c}{v}\left(\frac{\hbar}{\mu\tau}\right)$$

The factor $h/\mu\tau \sim 10^{-8}$ for $\tau \sim 10^{-15}$ sec. Thus this process would have a cross-section even smaller than the direct annihilation into two photons, the cross-section for which is given by[15]

$$\sigma = \pi \left(\frac{e^2}{M} \right)^2 \frac{c}{v} . \tag{III.22}$$

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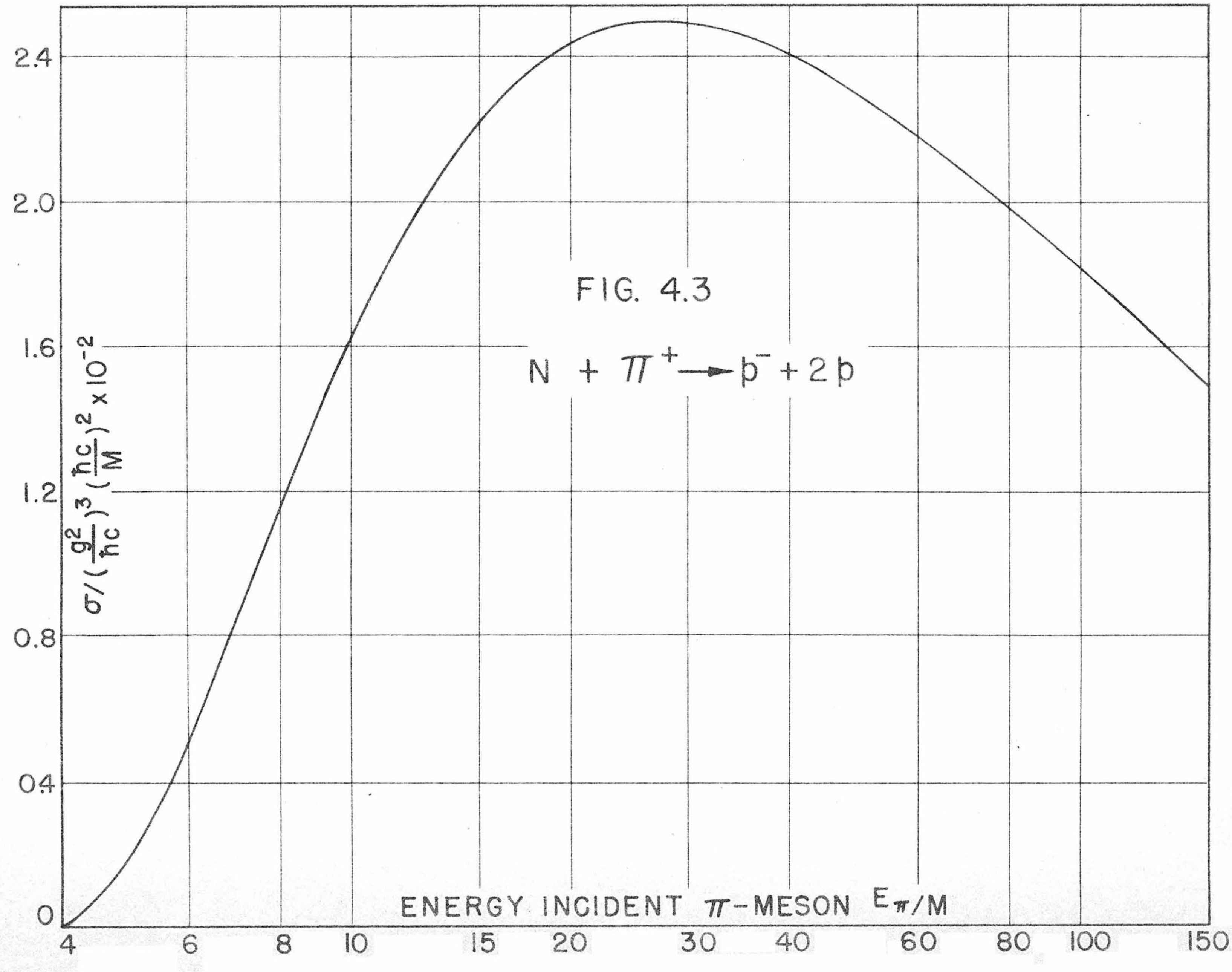

FIG. 4.3

$N + \pi^+ \longrightarrow p^- + 2p$