A TYPE OF PSEUDO-NORM

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Introduction and Summary

In mathematical literature, the term pseudo-norm has no one specific definition but is used for functionals satisfying some but not all of the postulates for a norm. The notion of such functionals or "pseudo-norms" is common in the study of linear topological spaces,[1] which, from one point of view, may be regarded as generalizations of normed linear spaces. The particular type of pseudo-norm considered in this thesis is the triangular norm of Menger's "generalized vector space".[2] Menger noticed that only the triangle property of the norm was necessary in order to obtain certain results in the calculus of variations, and thought that a linear space with a generalized triangular "distance" might prove to be a fruitful concept.

We first consider spaces (type K, see text) which are more specialized than those treated by Menger. In this thesis, spaces of the latter type are termed "spaces of type G". Apart from the intrinsic interest of type K spaces, certain aspects of their theory are applied in Chapter IV to the treatment of spaces of type G.

In Chapter I a space of type K is defined, the independence of the pseudo-norm postulates is established, and the question of the continuity of the pseudo-norm is treated. In Chapter II the notion of equivalence classes leads to a vector space of type K/Z, the existence of which depends only on the presence of a pseudo-norm in K. The more general spaces of type G are then introduced. A metric topology defined in terms of the pseudo-norm is discussed in Chapter III and functionals linear with respect to this topology are considered.

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1). See, for example, Hyers, Ref. 7; LaSalle, Ref. 9; von Neumann, Ref. 16; Wehausen, Ref. 19.
2). Menger, Ref. 13, p 96.

The question of the Gâteaux differentiability of the pseudo-norm is taken up in Chapter IV and a connection is established between this property and the existence of functionals linear in the topology of the pseudo-norm.

Chapter V investigates connections between the pseudo-norm and ordering relations in a real vector space. Conditions are found under which a partial ordering can be defined in terms of a given pseudo-norm, and conversely.

CONTENTS

CHAPTER I

Spaces of Type K

1.1. Definition. A space of type K is a set of undefined elements x, y, z, ..., satisfying the following postulates:

1.1.1. Postulates for a linear space. K forms a real linear space (vector space). This means

1.1.1(a). The elements of K form a commutative group under an operation termed addition and denoted by +. The group inverse of an element x is denoted by $-x$, the identity element by θ, and the operation $x + (-y)$ by $x-y$.

1.1.1(b). There is defined in K an associative and doubly-distributive multiplication of elements of K by real numbers a, b, c, ... This operation is called scalar multiplication and is denoted by a dot, thus . . It is subject to the following explicit rules:

$$a \text{ real}, x \in K, \text{ then } a.x \in K;$$

$$a \text{ real}, x, y, \in K, \text{ then } a.(x+y) = a.x + a.y;$$

$$a, b, \text{ real}, x \in K, \text{ then } (a+b).x = a.x + b.x;$$

$$a.(b.x) = (ab).x;$$

$$1.x = x.$$

1.1.2. Postulates for the strong[1)] norm x. There exists a function defined for all $x \in K$ denoted by $\|x\|$ and called the (strong) norm of x such that

1.1.2(a). $\|x\|$ is real;

1.1.2(b). $\|x + y\| \leq \|x\| + \|y\|$;

1.1.2(c). a real, $\|a.x\| = |a|\ \|x\|$;

1.1.2(d). $x \neq \theta \Rightarrow \|x\| > 0$.

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1). The adjective is used merely for convenience in distinguishing this norm from the pseudo-norm of 1.1.3.

1.1.3. Postulates for the pseudo-norm or P-norm, $P(x)$. There exists a function defined for all $x \in K$ denoted by $P(x)$ and called the pseudo-norm of x such that

1.1.3(a). $P(x)$ is real and is not identically zero;

1.1.3(b). $P(x + y) \leq P(x) + P(y)$;

1.1.3(c). $a > 0$, $P(a.x) = a\,P(x)$.

1.1.4. The pseudo-norm is bounded from above, i.e., there exists a constant M such that $P(x) \leq M\,\|x\|$ for all $x \in K$. We shall suppose M to be the infimum of all such possible numbers.

Remarks on the Postulates.

The strong norm of 1.1.2 is the usual positive definite norm of normed linear spaces. The pseudo-norm of 1.1.3 is much more general. It can take on negative values and $P(x)$ can be zero without x being the zero element. The pseudo-norm may be described verbally as a sub-additive positively-homogeneous functional. It will shortly be proved (theorem 1.3.10) that postulate 1.1.4 is equivalent to the continuity of the P-norm with respect to the metric topology of the strong norm.

1.2. Some Immediate Consequences of the Postulates.

We assume for the moment that mathematical entities exist which satisfy 1.1.1 to 1.1.4 (examples will be produced later), and proceed to write down for purposes of reference some known simple consequences of the postulates.

Consequences of Vector Space Postulates.

1.2.1. $0.x = \theta$

Proof: $a.x = (a + 0).x = a.x + 0.x$ by 1.1.1(b).

$\therefore 0.x = \theta$.

1.2.2. $a.\theta = \theta$.

Proof. $a.x = a.(x + \theta) = a.x + a.\theta$ by 1.1.1(b).

$\therefore\ a.\theta = \theta$.

1.2.3. $(-1).x = -x$.

Proof. $\theta = 0.x$ by 1.2.1, $= (-1 + 1).x = (-1).x + 1.x = (-1).x + x$ by 1.1.1(b). Hence $(-1).x$ is the inverse of x, i.e., $(-1).x = -x$.

Consequences of Strong Norm Postulates.

1.2.4. $\|\theta\| = 0$.

Proof. $\|\theta\| = \|2.\theta\|$ by 1.2.2, $= 2\ \|\theta\|$ by 1.1.2(c).

$\therefore\ \|\theta\| = 0$.

1.2.5. $\|x\| = 0$ if and only if $x = \theta$.

Proof. Sufficiency: 1.2.4.

Necessity: Suppose $\|x\| = 0$ but $x \neq \theta$. Then, by 1.1.2(d), $\|x\| > 0$, which is contrary to hypothesis.

1.2.6. $\|x\| = \|-x\|$.

Proof. $\|-x\| = \|-1.x\|$ by 1.2.3, $= |-1|\ \|x\|$ by 1.1.2(c), $= \|x\|$.

1.2.7. $\big|\ \|x\| - \|y\|\ \big| \leqq \|x + y\|$.

Proof. $\|x + y - y\| \leqq \|x + y\| + \|-y\|$ by 1.1.2(b), $= \|x + y\| + \|y\|$ by 1.2.6,

i.e., $\|x\| \leqq \|x + y\| + \|y\|$, $\|x\| - \|y\| \leqq \|x + y\|$.

Interchange x and y. $\|y\| - \|x\| \leqq \|x + y\|$.

Hence $\big|\ \|x\| - \|y\|\ \big| \leqq \|x + y\|$.

1.2.8. $\big|\ \|x\| - \|y\|\ \big| \leqq \|x - y\|$.

Proof. Put $-y$ for y in 1.2.7 and use 1.2.6.

$\big|\ \|x\| - \|y\|\ \big| = \big|\ \|x\| - \|-y\|\ \big| \leqq \|x + (-y)\| = \|x - y\|$.

Consequences of Pseudo-Norm Postulates.

1.2.9. $P(\theta) = 0$.

Proof. $P(\theta) = P(2\theta) = 2\,P(\theta)$ by 1.1.3(c).

$\therefore P(\theta) = 0$.

1.2.10. $P(x) - P(-y) \leq P(x + y)$ and $P(x) - P(y) \leq P(x - y)$.

Proof. By 1.1.3(b), $P(x + y + -y) \leq P(x + y) + P(-y)$,

i.e., $P(x) \leq P(x + y) + P(-y)$, $P(x) - P(-y) \leq P(x + y)$.

The second part now follows if y is replaced by -y.

1.2.11. $P(x) \geq -P(-x)$ and $P(-x) \geq -P(x)$.

Proof. Put $x = \theta$ in the first part of 1.2.10.

$P(\theta) - P(-y) \leq P(\theta + y)$, hence

$-P(-y) \leq P(y)$, which is the first part.

By replacing y by -y, we get $P(-y) \geq -P(y)$.

1.2.12. For all real a, $P(a.x) \geq a\,P(x)$.

Proof. If $a > 0$, $P(a.x) = a\,P(x)$ by 1.1.3(c).

If $a = 0$, $P(a.x) = P(\theta) = 0 = a\,P(x)$.

If $a < 0$, $P(x) \geq -P(-x)$ gives $a\,P(x) \leq -a\,P(-x) = P(-a.-x)$ by

1.1.3(c), i.e., $a\,P(x) \leq P(-a.-x) = P(a.x)$, hence $P(a.x) \geq a\,P(x)$.

1.3. Strong Continuity of the P-Norm.

Mazur[1)] states without proof that condition 1.1.4, $P(x) \leq M\|x\|$, is equivalent to the continuity of $P(x)$ with respect to the strong norm topology. We first give precise definitions and then prove this result.

1.3.1. Definition of Strong Continuity.

A real-valued function $f(x)$ defined on a subset $E \subset K$ is strongly con-

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1). Mazur, Ref. 11, p 130.

tinuous at x_0 if, corresponding to any $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

1.3.2. Definition of Strong Upper Semi-Continuity.

A real-valued function $f(x)$ is strongly upper semi-continuous at x_0 if, corresponding to any $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta$ implies $f(x) < f(x_0) + \varepsilon$.

1.3.3. Definition of Strong Lower Semi-Continuity.

A real-valued function $f(x)$ is strongly lower semi-continuous at x_0 if, corresponding to any $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta$ implies $f(x) > f(x_0) - \varepsilon$.

1.3.4. Lemma. The pseudo-norm $P(x)$ is strongly upper semi-continuous at any point x_0.

Proof. The constant M of $P(x) \leqq M\|x\|$ is positive since $P(x)$ is not identically zero, and, if $P(x) < 0$, then by 1.2.11 $P(-x) > -P(x) > 0$.

$$P(x) - P(x_0) \leqq P(x - x_0) \text{ by 1.2.10}$$
$$\leqq M\,\|x - x_0\| \text{ by 1.1.4}$$
$$< \varepsilon \text{ when } \|x - x_0\| < \varepsilon/M.$$

Choose $\delta = \varepsilon/M$. Then $P(x) < P(x_0) + \varepsilon$ when $\|x - x_0\| < \delta$, that is $P(x)$ is strongly upper semi-continuous at x_0 by the definition 1.3.2.

1.3.5. Lemma. The pseudo-norm $P(x)$ is strongly lower semi-continuous at any point x_0.

Proof. $P(x_0) - P(x) \leqq P(x_0 - x)$ by 1.2.10

$$\leqq M\,\|x_0 - x\| = M\,\|x - x_0\| < \varepsilon \text{ when } \|x - x_0\| < \varepsilon/M.$$

$$\therefore P(x) > P(x_0) - \varepsilon \text{ when } \|x - x_0\| < \delta = \varepsilon/M,$$

i.e., $P(x)$ is strongly lower semi-continuous at x_0.

1.3.6. Theorem. $P(x)$ is strongly continuous at any point $x_0 \in K$.

Proof. By the preceding two lemmas we have $P(x) - P(x_0) < \varepsilon$ and $P(x_0) - P(x) < \varepsilon$ when $\|x - x_0\| < \delta = \varepsilon/M$. This is the definition of strong continuity at x_0.

1.3.7. Definition of Functional. A functional is a mapping of an abstract space into the set of real numbers.

1.3.8. Definition of Positively-Homogeneous Functional. A positively-homogeneous functional is a functional f such that $f(a.x) = a\, f(x)$ for all $a > 0$.

1.3.9. Theorem. For a positively-homogeneous functional f defined on a space of type K to be strongly continuous at $x = \theta$, it is both necessary and sufficient that there exist a constant A (independent of x) such that $|f(x)| \leq A\, \|x\|$ for all $x \in K$. (This theorem is a slight generalization of a result of S. Banach.[1])

Proof. Necessity: If there is no such A, there exists a sequence of points $\{x_n\}$, $x_n \in K$, and a sequence of positive numbers A_n such that $A_n \to \infty$ and $|f(x_n)| > A_n \|x_n\|$.

Define $y_n = x_n/A_n\|x_n\|$. Then $\|y_n\| = 1/A_n$, hence $\lim_{n\to\infty} \|y_n\| = 0$. $|f(y_n)| = |f(x_n/A_n\|x_n\|)| = 1/A_n\|x_n\| \, |f(x_n)|$ since f is positively-homogeneous. Therefore $|f(y_n)| > 1$. Also $f(\theta) = f(2.\theta) = 2\, f(\theta)$, whence $f(\theta) = 0$. So $|f(y_n) - f(\theta)| = |f(y_n)| > 1$ for all y_n. But this is impossible since, by strong continuity, $|f(y_n) - f(\theta)| < \varepsilon$ when $\|y_n - \theta\| = \|y_n\|$ is sufficiently small.

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1). Banach, Ref. 2, p 54.

Sufficiency: $\left|f(x) - f(\theta)\right| = \left|f(x)\right|$ since $f(\theta) = 0$, $\leq A\|x\| < \varepsilon$ for $\|x\| < \varepsilon/A$. This means that f(x) is strongly continuous at θ (see section 1.3.1).

1.3.10. Theorem. In the space K the following relations are equivalent to one another:

(a) There exists a constant M such that $P(x) \leq M\|x\|$ for all $x \in K$;

(b) The P-norm is strongly continuous at $x = \theta$;

(c) There exists a constant A such that $\left|P(x)\right| \leq A\|x\|$ for all $x \in K$.

Proof. We show that (a) implies (b), (b) implies (c), and (c) implies (a).

That (a) implies (b) is stated in theorem 1.3.6.

That (b) implies (c) follows immediately from theorem 1.3.9 since P(x) is a positively-homogeneous functional.

That (c) implies (a) is clear.

We point out that the equivalence of (a) and (b) proves the remark of Mazur noted above.

Effect of Boundedness from Below (Only) on Semi-Continuity of P(x).

As a matter of interest we next inquire whether P(x) necessarily remains either upper or lower semi-continuous if postulate 1.1.4 is changed to

1.1.4*. There exists a constant N such that $P(x) \geq N\|x\|$ for all $x \in K$.

The answer is no in both cases. This is shown by the following counter example. Let K_1 consist of the set of all absolutely convergent series of real numbers under the ordinary rules of term-by-term addition and scalar multiplication. Every element x of K_1 is an enumerable set of real numbers u_n such that $\sum_{n=1}^{\infty}|u_n|$ exists as a finite number. For this we use the notation $x \equiv \{u_n\}$, $\sum_{n=1}^{\infty}|u_n| < \infty$. Define $\|x\| \equiv \sup_n |u_n|$, and $P(x) \equiv \sum_{n=1}^{\infty}|u_n|$. Then K_1 satisfies postulates 1.1.1, 1.1.2, 1.1.3, and 1.1.4*.

Verification of 1.1.1. K_1 is a vector space since absolutely convergent series may be added, subtracted, and multiplied by real numbers, term by term, and remain absolutely convergent. In this space the identity element θ is $\{0, 0, 0, \ldots\}$.

Verification of 1.1.2. $\|x\|$ as defined above for K_1 is a strong norm. It is readily seen to satisfy 1.1.2(a), 1.1.2(c), and 1.1.2(d). We show that 1.1.2(b) is satisfied as follows:

Let $x \equiv \{u_n\}$, $y \equiv \{v_n\}$.

For all n, $|u_n| + |v_n| \leq \sup_n |u_n| + \sup_n |v_n|$ and

$|u_n + v_n| \leq |u_n| + |v_n| \leq \sup_n |u_n| + \sup_n |v_n|$

$\therefore \sup_n |u_n + v_n| \leq \sup_n |u_n| + \sup_n |v_n|$

$\therefore \|x + y\| \leq \|x\| + \|y\|$. This is 1.1.2(b).

Verification of 1.1.3. P(x) as defined in K_1 is a pseudo-norm. 1.1.3(a) and 1.1.3(c) are clearly satisfied, and for 1.1.3(b) we have

$$P(x + y) = \sum_1^\infty |u_n + v_n| \leq \sum_1^\infty |u_n| + \sum_1^\infty |v_n| = P(x) + P(y).$$

Verification of 1.1.4*. We have $P(x) \geq \|x\|$ since $\sum_{n=1}^\infty |u_n| \geq \sup_n |u_n|$. This shows that 1.1.4* holds with $N = 1$.

Denial of Strong Upper Semi-Continuity. We have to show that for some $\varepsilon > 0$, there exists no $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $P(x) < P(x_0) + \varepsilon$. We choose $\varepsilon = 1$, suppose there does exist such a δ, and establish a contradiction.

Let $x_0 = \{1/n^2\}$. Then x_0 is in K_1 since $\sum 1/n^2$ converges. $P(x_0) = \sum_1^\infty 1/n^2$. Now consider the element $x = \{u_n\} \in K_1$ with $u_n = 1/n^2 + \delta/2$ for $n \leq n_0 \equiv [2/\delta + 1]$ where $[2/\delta + 1]$ denotes the greatest integer in $2/\delta + 1$, and $u_n = 1/n^2$ for $n > n_0$.

Another way of writing x is

$$x = \left\{1 + \frac{\delta}{2}, \frac{1}{2^2} + \frac{\delta}{2}, \frac{1}{3^2} + \frac{\delta}{2}, \ldots, \frac{1}{n_0^2} + \frac{\delta}{2}, \frac{1}{(n_0+1)^2}, \frac{1}{(n_0+2)^2}, \ldots\right\}$$

Then $\|x - x_0\| = \sup_n |u_n - 1/n^2| = \delta/2 < \delta.$

However $P(x) = \sum_{n=1}^{\infty} |u_n| = \sum_{1}^{\infty} 1/n^2 + n_0\,\delta/2$

$= P(x_0) + n_0\,\delta/2 > P(x_0) + 1$ since

$n_0 = [2/\delta + 1]$ implies that $n_0 > 2/\delta$, hence $n_0\,\delta/2 > 1$. Thus $P(x) > P(x_0) + 1$ even though $\|x - x_0\| < \delta$. This is the contradiction sought.

Denial of Lower Semi-Continuity of P(x).

We have to show that for some $\varepsilon > 0$ there exists no $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $P(x) > P(x_0) - \varepsilon$. The demonstration parallels the case immediately above.

Take $x_0 = \{1/n^2\}$, $\varepsilon = 1$, but put

$$x = \{u_n\} = \left\{1 - \frac{\delta}{2}, \frac{1}{2^2} - \frac{\delta}{2}, \ldots, \frac{1}{n_0^2} - \frac{\delta}{2}, \frac{1}{(n_0+1)^2}, \frac{1}{(n_0+2)^2}, \ldots\right\}$$

where as before $n_0 = [2/\delta + 1]$.

Then $P(x) = \sum 1/n^2 - n_0\,\delta/2 = P(x_0) - n_0\,\delta/2 < P(x_0) - 1$ since $n_0\,\delta/2 > 1$. However $\|x - x_0\| = \delta/2 < \delta$. Thus $P(x)$ cannot be strongly lower semi-continuous at x_0.

1.4. Examples of Spaces of Type K.

Example I. A Finite Dimensional K-Space.

Let K be the set of ordered n-uples of n real numbers. An element x of K is represented as $x = \{x_1, x_2, \ldots, x_n\}$. Then K forms a linear space with addition and scalar multiplication defined componentwise. Define $\|x\| \equiv (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ where p is an arbitrary fixed positive integer, and $P(x) \equiv \max_i x_i$. It is readily seen that $\|x\|$ satisfies postulates 1.1.2(a), 1.1.2(c), and 1.1.2(d). To verify 1.1.2(b) we make use of the inequality of Minkowski,

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p},$$

i.e., $\|x + y\| \leq \|x\| + \|y\|$.

Clearly also $P(x)$ satisfies postulates 1.1.3(a) and 1.1.3(c). 1.1.3(b) holds since $\max_i (x_i + y_i) \leq \max_i x_i + \max_i y_i$. Finally we have $\max_i x_i \leq (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ so that $P(x) \leq \|x\|$. This is postulate 1.1.4.

Example II. An Infinite Dimensional K-Space.

Consider the Banach space (m) consisting of the set of all bounded sequences of complex numbers[1)] with scalar multiplication restricted to real numbers. (m) being a Banach space, postulates 1.1.1 and 1.1.2 are valid, the strong norm being defined as $\|x\| = \sup_n |\xi_n|$ where x is the sequence $\{\xi_n\}$. The space is made into a K-space by defining the pseudo-norm as $P(x) \equiv \sup_n (\text{Rl}\ \xi_n)$. We proceed to verify that 1.1.3 and 1.1.4 hold.

1.1.3(a): $P(x)$ is certainly real.

1.1.3(b): Let $x = \{\xi_n\}$, $y = \{\eta_n\}$ be elements of (m).

For all n, $\text{Rl}\ \xi_n \leq \sup_n (\text{Rl}\ \xi_n)$ and $\text{Rl}\ \eta_n \leq \sup_n (\text{Rl}\ \xi_n)$.

Hence $\text{Rl}\ (\xi_n + \eta_n) \leq \sup_n (\text{Rl}\ \xi_n) + \sup_n (\text{Rl}\ \eta_n)$.

$\therefore \sup_n \text{Rl}\ (\xi_n + \eta_n) \leq \sup_n \text{Rl}\ \xi_n + \sup_n (\text{Rl}\ \eta_n)$, i.e.,

$P(x + y) \leq P(x) + P(y)$.

1.1.3(c): For $a > 0$, $P(ax) = \sup_n \text{Rl}\ (a\,\xi_n) = a \sup_n (\text{Rl}\ \xi_n) = aP(x)$.

1.1.4: $P(x) \leq \|x\|$ since $\text{Rl}\ \xi_n \leq |\xi_n|$. This verifies that the space is of type K.

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1). Banach, Ref. 2, pp 11, 53.

Example III.

This example makes evident the convexity properties implicit in the positive homogeneity and subadditivity of the pseudo-norm. Let K_1 be any strongly normed linear space, that is, one satisfying 1.1.1 and 1.1.2, and let B be a convex body[1)] containing the zero of K_1 as an interior point. Define P(x) as the Minkowski functional of B, i.e., $P(x) \equiv \inf_{\substack{h>0 \\ x/h \in B}} h$ for all $x \in K_1$. Then K_1 is a space of type K.

Proof. From Ascoli[2)] it follows that $P(x) \geqq 0$; $P(x+y) \leqq P(x) + P(y)$; $P(tx) = t\,P(x)$ for $t > 0$; $P(x) \leqq M\|x\|$.

Hence all the postulates 1.1.1 to 1.1.4 are valid. But in this case the pseudo-norm cannot assume negative values. Further examples are given at a later stage (Chapter III), when some of the deeper properties of these spaces are investigated.

1.5. Independence of Pseudo-Norm Postulates.

In view of 1.1.3(b) and 1.1.4 it is meaningless to try to obtain a system in which 1.1.3(b) and 1.1.4 hold but P(x) is not real. In this section it is established by means of examples that postulates 1.1.3(b), 1.1.3(c), and 1.1.4 are independent of each other and of the remaining postulates. For instance, in order to show that postulate 1.1.3(b) is independent of all the remaining postulates we describe a system in which all the postulates hold except 1.1.3(b).

System Showing the Independence of 1.1.3(b).

Consider the Banach space of bounded sequences of real numbers. An element x of this space is of the form $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ with

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1). Mazur, Ref. 10, p 72. A convex body is a convex set closed with respect to the metric topology of the strong norm in K_1 and containing interior points.

2). Ascoli, Ref. 1, pp 48-50.

$\|x\| = \sup_n |\xi_n|$, where the ξ_n are real numbers. Define $P(x) = \inf_n \xi_n$. This system is known to be a Banach space, which means that postulates 1.1.1 and 1.1.2 are satisfied.

1.1.3(a) holds since $P(x)$ is real.

1.1.3(c) holds since, if $a > 0$, $P(ax) = \inf_n a\,\xi_n = a \inf_n \xi_n = aP(x)$.

1.1.4 holds since $P(x) = \inf_n \xi_n \leq \sup_n |\xi_n| = \|x\|$.

But 1.1.3(b) is not satisfied for we have $P(x + y) = \inf_n (\xi_n + \eta_n)$ $\geq \inf_n \xi_n + \inf_n \eta_n \geq P(x) + P(y)$, where $y = \{\eta_n\}$.

System Showing the Independence of 1.1.3(c).

Let K be the set of real numbers and define $\|x\| = |x|$.

Define the pseudo-norm $P(x)$ as follows: $P(x) = \sqrt{|x|}$ when $|x| \geq 1$; $P(x) = |x|$ when $|x| < 1$. Then it is readily seen that postulates 1.1.1 and 1.1.2 hold.

1.1.3(a) is immediate.

The proof of 1.1.3(b) requires consideration of several cases.

Case 1. $|x|, |y|, |x + y|, \geq 1$.

Then $P(x + y) = \sqrt{|x + y|} \leq \sqrt{|x| + |y|} \leq \sqrt{|x|} + \sqrt{|y|} = P(x) + P(y)$.

Case 2. $|x|, |y|, \geq 1$; $|x + y| < 1$.

Then $P(x + y) = |x + y| < 1$.

$P(x) + P(y) = \sqrt{|x|} + \sqrt{|y|} \geq 2$ since $|x|, |y| \geq 1$.

$\therefore P(x + y) \leq P(x) + P(y)$.

Case 3. $|x| \geq 1$, $|y| < 1$, $|x + y| \geq 1$.

Then $P(x + y) = \sqrt{|x + y|} \leq \sqrt{|x| + |y|}$.

Now $|y| + 2\sqrt{|x|} > 1$ since $|x| \geq 1$. Hence $y^2 + 2|y|\sqrt{|x|} > |y|$ on multiplying by $|y|$. Therefore $y^2 + 2|y|\sqrt{|x|} + |x| > |y| + |x|$, i.e., $(|y| + \sqrt{|x|})^2 > |y| + |x|$. Hence $|y| + \sqrt{|x|} > \sqrt{|y| + |x|}$ $\geq \sqrt{|x + y|}$ by the first line of this proof, i.e., $P(y) + P(x) > P(x + y)$.

Case 4. $|x| \geq 1$, $|y| < 1$, $|x + y| < 1$. Then $P(x + y) = |x + y|$ $< 1 < \sqrt{|x|} + |y|$ since $|x| > 1$,

$< P(x) + P(y)$.

Case 5. $|x| < 1$, $|y| < 1$, $|x + y| \geq 1$.
Then $P(x + y) = \sqrt{|x + y|} \leq |x + y|$ since $|x + y| \geq 1$, $\leq |x| + |y|$ $= P(x) + P(y)$.

Case 6. $|x| < 1$, $|y| < 1$, $|x + y| < 1$.
Then $P(x + y) = |x + y| \leq |x| + |y| = P(x) + P(y)$.

The above six cases establish that 1.1.3(b) is satisfied.

But 1.1.3(c) does not hold. This can be seen by considering the case where $a > 1$ and $x > 1$.

Then $P(ax) = \sqrt{ax} = \sqrt{a}\sqrt{x} = \sqrt{a}\; P(x) \neq a\, P(x)$.

Postulate 1.1.4 holds since $\|x\| = |x|$ and $P(x) = \sqrt{|x|}$ for $|x| \geq 1$,
$= |x|$ for $|x| < 1$.

Now $\sqrt{|x|} \leq |x|$ for $|x| \geq 1$ so that, for all x, $P(x) \leq \|x\|$. Thus the constant M of 1.1.4 has the value 1. This shows that 1.1.3(c) is independent of the other postulates.

System Showing the Independence of Postulate 1.1.4.

We again make use of the example on page 7, there used to show that continuity of the P-norm does not hold if it is merely bounded from below. K is the set of all absolutely convergent series of real numbers. We recall that $\|x\| = \sup_n |u_n|$, $P(x) = \sum_1^\infty |u_n|$, where $x = \{u_n\}$. We have seen (pages 7 and 8) that these definitions satisfy postulates 1.1.1, 1.1.2, and 1.1.3. But 1.1.4 does not hold; i.e., there is no number M independent of x such that $P(x) \leq M\|x\|$. For any given M, no matter how large, it is always possible to construct a convergent series with a sufficient number of terms each equal to $\sup_n |u_n|$ so that the sum is greater than $M \sup_n |u_n|$, i.e., greater than $M\|x\|$.

Otherwise thus: we have seen that a space satisfying postulate 1.1.4 has a strongly continuous pseudo-norm. But we have also seen (page 8) that, in the space of the present example, P(x) is not strongly continuous. Hence postulate 1.1.4 cannot be valid, and so it is independent of the other postulates.

CHAPTER II

The Factor Group K/Z as a Pseudo-Normed Vector Space

The results of this chapter do not depend on the existence of the strong norm in K but only on that of the pseudo-norm.

2.1. Equivalence Relations based on the Pseudo-Norm.

We proceed to discuss two methods of obtaining equivalence relations in a space of type K by means of P(x).

2.1.1. Method 1.

Define $x \approx y$ to mean $P(x) = P(y)$.

Proof that the relation $\approx$ is an equivalence.

1.) $\approx$ is well-defined, since for any elements x, y either $P(x) = P(y)$ or $P(x) \neq P(y)$.

2.) $\approx$ is reflexive: $x \approx x$, since $P(x) = P(x)$.

3.) $\approx$ is symmetric: if $P(x) = P(y)$, then $P(y) = P(x)$. Hence, if $x \approx y$, then $y \approx x$.

4.) $\approx$ is transitive: if $P(x) = P(y)$ and $P(y) = P(z)$, then $P(x) = P(z)$. Hence $x \approx y$ and $y \approx z$ implies $x \approx z$.

So $\approx$ is an equivalence relation.

2.1.2. Lemma. $P(x) = 0$ if and only if $x \approx \theta$.

Proof. Sufficiency:

If $x \approx \theta$ then by definition $P(x) = P(\theta) = 0$.

Necessity:

If $P(x) = 0$, then $P(x) = P(\theta)$ since $P(\theta) = 0$.

Hence $x \approx \theta$.

The lemma shows that under the relation $\approx$ the pseudo-norm has property 1.2.5 of the strong norm, namely $\|x\| = 0$ if and only if $x = \theta$.

We now regard two elements of K as "identical under $\approx$" if they have equal pseudo-norms. K can then be split into equivalence classes such that all elements in any one class are identical under $\approx$. However, elements in the same equivalence class do not have equal strong norms. It is easy to get examples from the space K of bounded sequences to show that $x \approx y$ does not imply that $\|x\| = \|y\|$.

2.1.3. Method 2.

This is an essentially deeper method than 2.1.1 and consists in obtaining a subgroup of elements of K and establishing an algebraic homomorphism by the method of residue classes (cosets). In this we follow the procedure of Morse and Transue[1] but by virtue of our pseudo-norm being more general than the one treated in their investigation we require more stringent conditions in order to obtain as starting point a subgroup of the set K.

We define Z to be the set of elements x of K such that

$P(x) = P(-x) = 0$.[2] In the usual notation

$$Z = \{x \in K \mid P(x) = P(-x) = 0\}.$$

2.1.4. Theorem. The set $Z = \{x \in K \mid P(x) = P(-x) = 0\}$ is a subgroup of K.

Proof. Z is non-empty for $P(\theta) = P(-\theta) = 0 \Rightarrow \theta \in Z$.

The necessary and sufficient conditions for Z to be a subgroup of K are[3]

1.) Closure, i.e., $x \in Z$, $y \in Z$ implies $x + y \in Z$;

2.) Existence of inverse, i.e., $x \in Z$ implies $-x \in Z$.

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1). Morse and Transue, Ref. 15, p 779.

2). The condition that $P(x) = 0$ is not sufficient to make Z a subgroup, since all that this implies is that $P(-x) \geqq 0$ under our postulates on P. This is in contrast to the case treated by Morse and Transue, loc. cit.

3). Van der Waerden, Ref. 18, p 21.

Proof of closure: Let x, y be elements of Z. By the definition of Z, $P(x) = P(-x) = P(y) = P(-y) = 0$.

We have to show that $P(x + y) = P\big(-(x + y)\big) = 0$.

(1) $P(x + y) \leq P(x) + P(y) = 0$.

$P(x + y) = P\big(x - (-y)\big) \geq P(x) - P(-y)$ by 1.2.10, $\geq 0 - 0 = 0$.

(2) From (1), it follows that $P(x + y) = 0$.

Now $P\big(-(x + y)\big) = P(-x - y) \geq P(-x) - P(y)$ by 1.2.10, $\geq 0 - 0 = 0$.

Also $P\big(-(x + y)\big) \leq P(-x) + P(-y)$ by postulate 1.1.3(b), ≤ 0.

Hence $P\big(-(x + y)\big) = 0$.

From (2), $P(x + y) = P\big(-(x + y)\big) = 0$. Therefore $x + y \in Z$.

Proof of existence of inverse: This is immediate from the definition of Z. If we write out the proof, we have that $x \in Z$ implies $P(x) = P(-x) = 0$. Hence $P(-x) = P\big(-(-x)\big) = 0$, which means that $-x \in Z$. This concludes the proof of the theorem.

The question arises: is such a subgroup Z confined solely to the zero of K? If so, then any theory based on Z will be essentially trivial. In other words, are there any elements of K other than θ for which $P(x) = P(-x) = 0$? The following example shows that such elements exist in the most general case.

Let K be the space (m) of bounded sequences $x = \{\xi_n\}$ of complex numbers with $\|x\|$ defined as $\sup_n |\xi_n|$ and P(x) defined as $\sup_n (\mathrm{Rl}\ \xi_n)$. We have previously shown (1.4, Example II) that this is a space of type K. Then $P(x) = P(-x) = 0$ means that the element x is a sequence of pure imaginaries and the subgroup Z consists of all bounded sequences of pure imaginary numbers. Thus our considerations are non-trivial, and we may return to the general theory.

Let Q be the group of residue classes (cosets) of K with respect to Z. Q is called the residue class group (factor group) of K with respect to Z and is denoted by K/Z. An algebraic homomorphism exists between the elements of K and the elements of Q. This mapping divides K into equivalence classes which are the residue classes of Z and constitute the elements of Q. A necessary and sufficient condition that two elements, x, y of K belong to the same residue class is that x - y is in Z.[1] This is written $x \equiv y \pmod{Z}$.

<u>2.1.5. Lemma.</u> For real $a \neq 0$, $aZ = Z$, where $Z = \{x \in K \mid P(x) = P(-x) = 0\}$ and aZ denotes the set of elements ax[2] of K such that x is in Z.

<u>Proof.</u> Case 1. $a > 0$, $x \in Z$.

$P(ax) = a\,P(x) = 0 = a\,P(-x) = P(a.-x) = P(-ax)$. Hence $ax \in Z$ for $a > 0$.

Case 2. $a < 0$, $x \in Z$.

$P(ax) = P(-a.-x) = (-a)\,P(-x)$ by 1.1.3(c), $= 0 = (-a)\,P(x) = P(-ax)$. Hence $ax \in Z$ for $a < 0$.

Cases 1 and 2 show that $aZ \subset Z$. Conversely if $x \in Z$, $1/a\,x \in Z$ by the preceding. So $1/a\,x = y \in Z$, $x = ay \in aZ$. Thus $Z \subset aZ$. Together with the result that $aZ \subset Z$, this establishes the lemma.

<u>2.1.6. Theorem.</u> The residue class group $Q = K/Z$, which has as elements the residue classes of K (mod Z), where $Z = \{x \in K \mid P(x) = P(-x) = 0\}$, forms a linear space.

<u>Proof.</u> We have to show that Q forms a commutative group under addition, and

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1). For the fundamental ideas of coset and factor group, see van der Waerden, Ref. 18, pp 25-34.

2). Here and in future, where convenient, we omit the dot hitherto used to denote scalar multiplication (1.1.1(b)).

that under a suitable definition of scalar multiplication Q satisfies postulates 1.1.1(a) and 1.1.1(b).

In what follows, elements of Q are denoted by $Q_1, Q_2, \ldots$, but Q is not necessarily denumerable.

2.1.7. Definition of Addition of Elements of Q = K/Z.

$$Q_1 + Q_2 \equiv \left\{ q_1 + q_2 \;\middle|\; q_1 \in K \cap Q_1,\ q_2 \in K \cap Q_2 \right\}.$$

2.1.8. Definition of Scalar Multiplication of Elements of Q = K/Z.

For real a, $Q_1 \in Q$, define $aQ_1 \equiv \left\{ aq_1 \;\middle|\; q_1 \in K \cap Q_1 \right\}$ for $a \neq 0$,

$\equiv$ the subgroup Z for $a = 0$.

The proof of theorem 2.1.6 depends on a series of lemmas.

2.1.9. Lemma. Under addition, Q = K/Z forms a commutative group with Z as identity.

Proof. This is a classical algebraic result.

2.1.10. Lemma. If a is real, $Q_1 \in Q$, then $aQ_1 \in Q$.

Proof. Q_1 is a coset of the subgroup $Z = \left\{ x \in K \;\middle|\; P(x) = P(-x) = 0 \right\}$.

Hence $Q_1 = q_1 + Z$ where $q_1 \in K$.

Then aQ_1 = the set of elements $a(q_1 + Z)$

= the set $aq_1 + aZ$

$= aq_1 + Z$ by lemma 2.1.5 and definition 2.1.8

= a coset $Q' \in Q$.

Hence Q = K/Z is closed under multiplication by real numbers.

2.1.11. Lemma. $a(Q_1 + Q_2) = aQ_1 + aQ_2$, where a is real and Q_1, Q_2 are elements of Q.

Proof. If $x \in K \cap a(Q_1 + Q_2)$, $x = a(q_1 + q_2)$ where $q_1 \in K \cap Q_1$ and $q_2 \in K \cap Q_2$. Thus $x = aq_1 + aq_2 \in aQ_1 + aQ_2$.

Hence $a(Q_1 + Q_2) \subset aQ_1 + aQ_2$. Conversely, if $x \in K \cap (aQ_1 + aQ_2)$,

$x = aq_1 + aq_2$ where $q_1 \in K \cap Q_1$, $q_2 \in K \cap Q_2$

$= a(q_1 + q_2) \in a(Q_1 + Q_2)$. Therefore $aQ_1 + aQ_2 \subset a(Q_1 + Q_2)$.

Hence $a(Q_1 + Q_2) = aQ_1 + aQ_2$.

If $a = 0$, $a(Q_1 + Q_2) = Z$ by definition 2.1.8 and $aQ_1 + aQ_2 = Z + Z$ $= Z$ since Z is the group identity. So the lemma is valid for all real a.

<u>2.1.12. Lemma.</u> $(a + b)Q_1 = aQ_1 + bQ_1$ where a, b are real numbers, and $Q_1 \in Q$.

<u>Proof.</u> If $a = b = 0$, $(a + b)Q_1 = Z = Z + Z = aQ_1 + bQ_1$.

If $a \neq 0$, $b = 0$, then $(a + b)Q_1 = aQ_1 = aQ_1 + Z = aQ_1 + bQ_1$ by 2.1.8.

If $a \neq 0$, $b \neq 0$, suppose $x \in K \cap (a + b)Q_1$ and $Q_1 = q_1 + Z$, Q_1 being a coset of Z.

Then $x = (a + b)\ (q_1 + q_0)$ where $q_0 \in Z$

$= a(q_1 + q_0) + b(q_1 + q_0) \in a(q_1 + Z) + b(q_1 + Z) \subset aQ_1 + bQ_1$

$\therefore (a + b)Q_1 \subset aQ_1 + bQ_1$.

Conversely, if $y \in K \cap (aQ_1 + bQ_1)$,

$y = a(q_1 + q_0') + b(q_1 + q_0'')$ where q_0', q_0'' are elements of Z

$= (a + b)q_1 + aq_0' + bq_0''$

$= (a + b)q_1 + z_1 + z_2$ by lemma 2.1.5, where z_1, z_2, are in Z

$= (a + b)q_1 + z_3$ where z_3 is an element of the subgroup Z

$= (a + b)q_1 + (a + b)q_0$ by lemma 2.1.5, where q_0 is in Z

$= (a + b)(q_1 + q_0) \in (a + b)Q_1$

Hence $aQ_1 + bQ_1 \subset (a + b)Q_1$.

This result together with the one above proves the lemma.

2.1.13. Lemma. $a(bQ_1) = (ab)Q_1$ where a, b are real, $Q_1 \in Q$.

Proof. $a(bq_1) = (ab)q_1$ for each $q_1 \in Q_1$.

2.1.14. Lemma. $1.Q_1 = Q_1$ for $Q_1 \in Q$.

Proof. Clear.

Theorem 2.1.6 now follows from lemmas 2.1.9 to 2.1.14 inclusive.

2.2. Definition of a Pseudo-Norm in the Linear Space $Q = K/Z$ where $Z = \{x \in K \mid P(x) = P(-x) = 0\}$.

2.2.1. We shall call a space satisfying postulates 1.1.1 and 1.1.3 a pseudo-normed vector space.[1)]

For $Q_1 \in K/Z$, define $P_z(Q_1) = P(x)$ where x is any element of K belonging to Q_1. Before proving that $P_z(Q_1)$ satisfies the pseudo-norm postulates 1.1.3 we justify this definition by proving

2.2.2. Lemma. All elements x in the same coset Q_1 have equal P-norms.

Proof. Let x, y be elements of $K \cap Q_1$, where Q_1 is the coset $q_1 + Z$, $q_1 \in K$. Then $x = q_1 + z_1$, $y = q_1 + z_2$ where z_1, z_2 are in the subgroup Z.

$\therefore$ $x - y = z_1 - z_2 \in Z$ since Z is a subgroup.

$\therefore$ $P(x - y) = P(y - x) = 0$ by definition of Z.

By 1.2.10, $P(x - y) \geqq P(x) - P(y)$ and $P(y - x) \geqq P(y) - P(x)$.

Hence $0 \geqq P(x) - P(y)$ and $0 \geqq P(y) - P(x)$, so $P(x) - P(y) = 0$ and $P(x) = P(y)$. This proves the lemma.

2.2.3. Theorem. If Q_1 is any element of the linear space K/Z (theorem 2.1.6) and if a functional P_z is defined on K/Z so that $P_z(Q_1) = P(x)$ where

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1). Menger uses the term "generalized vector space". (Ref. 13, p 96.)

$x \in K \cap Q_1$, then $P_z(Q_1)$ satisfies the pseudo-norm postulates 1.1.3(a), 1.1.3(b), and 1.1.3(c).

Proof. 1.1.3(a): $P_z(Q_1)$ is real since $P(x)$ is real.

1.1.3(b): $P_z(Q_1 + Q_2) = P(q_1 + q_2)$ where $q_1 \in Q_1$ and $q_2 \in Q_2$

$$\leqq P(q_1) + P(q_2) = P_z(Q_1) + P_z(Q_2).$$

1.1.3(c): $a > 0$, $P_z(aQ_1) = P(aq_1) = aP(q_1) = aP_z(Q_1)$.

2.2.4. Corollary. The linear space K/Z, P-normed as above, is a pseudo-normed vector space (2.2.1).

2.3. Summary:

The cosets of K (mod Z) where $Z = \{x \in K \mid P(x) = P(-x) = 0\}$ form a linear space K/Z onto which K is mapped homomorphically. Group addition, scalar multiplication, and pseudo-norm relationships are preserved by this homomorphism. The last follows from the fact that if $x \approx Q_1$ and $y \approx Q_2$ where $\approx$ denotes the homomorphic mapping, then $P(x) = P(y)$ implies $P_z(Q_1) = P_z(Q_2)$, and conversely (by lemma 2.2.2). We may note that $P(x) = P(y)$ does not imply that x and y are in the same coset Q_1. This is made evident from the following illustrative example.

2.3.1. The Subgroup Z and the Space K/Z when the Space K is the Space of Bounded Sequences of 1.4, Example II.

We recall that $x \in K$ is a complex sequence $\{\xi_n\}$ with $\|x\| = \sup_n |\xi_n|$ and $P(x) = \sup_n \text{Rl}\, \xi_n$. $P(x) = P(-x) = 0$ implies that $\text{Rl}\, \xi_n = 0$ for all n. Hence x is a sequence of pure imaginary numbers and the subgroup Z consists of the set of all bounded sequences consisting of pure imaginaries. All elements in a particular coset of Z consist of sequences which, term for term, have equal real parts.

To show that $P(x) = P(y)$ does not imply that x and y are in the same coset, consider

$$x = \{0, -1, -1, -1, -1, \ldots\}$$

$$y = \{i, i, i, i, i, \ldots\}, \text{ where } i = \sqrt{-1}.$$

Then $P(x) = 0$, $P(y) = 0$, $P(-x) = 1$, $P(-y) = 0$.

Hence, y is in Z but x is not.

CHAPTER III

The P-Topology and P-Linear Functionals

In this chapter we introduce notions of limit and continuity based on the P-norm.

3.1. The P-Topology in Spaces of Type G/Z.

3.1.1. Definition of Space of Type G, or Pseudo-Normed Vector Space.

A space of type G, or a pseudo-normed vector space, is a real linear space satisfying postulates 1.1.1 and 1.1.3, i.e., a space G differs from a space K in not having a given strong norm (see also 2.2.1).

3.1.2. Theorem. If (i) x is an element of a pseudo-normed vector space G,
(ii) $P(x) = P(-x) = 0$ implies $x = \theta$, the zero of G,
then $B(x) \equiv \max \{P(x), P(-x)\}$ has the properties 1.1.2(a) to 1.1.2(d) of a strong norm.

Proof. The theorem is a direct consequence of the following lemmas.

3.1.3. Lemma. $B(x) \geq 0$ where $B(x) = \max \{P(x), P(-x)\}$.

Proof. By 1.2.11, $P(x) \geq -P(-x)$ and $P(-x) \geq -P(x)$. Hence, if either $P(x)$ or $P(-x)$ is negative, $B(x) = \max \{P(x), P(-x)\} > 0$. So, in general, $B(x) \geq 0$.

3.1.4. Lemma. $B(x) = 0$ implies $x = \theta$ under the hypotheses of theorem 3.1.2.

Proof. If $B(x) = 0$, it is clear from the proof of lemma 3.1.3 that neither $P(x)$ nor $P(-x)$ can be negative. Hence $P(x) = P(-x) = 0$. Then by hypothesis (ii) of theorem 3.1.2, $x = \theta$.

3.1.5. Lemma. $B(x + y) \leq B(x) + B(y)$.

Proof. $P(x + y) \leq P(x) + P(y)$; $P(-\overline{x+y}) \leq P(-x) + P(-y)$.

Hence $\max\{P(x + y), P(-\overline{x+y})\} \leq \max\{P(x), P(-x)\} + \max\{P(y), P(-y)\}$, i.e., $B(x + y) \leq B(x) + B(y)$.

3.1.6. Lemma. $B(ax) = |a|\, B(x)$ for real a.

Proof.
$$\begin{aligned} B(ax) = \max\{P(ax), P(-ax)\} &= \max\{aP(x), aP(-x)\} \text{ if } a > 0 \\ &= \max\{-aP(-x), -aP(x)\} \text{ if } a < 0. \\ &= \max\{|a|\,P(x), |a|\,P(-x)\} \text{ for any real } a. \\ &= |a| \max\{P(x), P(-x)\} = |a|\, B(x). \end{aligned}$$

Lemmas 3.1.3 to 3.1.6 prove that the strong norm postulates 1.1.2 hold. This establishes theorem 3.1.2.

3.1.7. Corollary. If G is a pseudo-normed vector space in which hypothesis (ii) of theorem 3.1.2 does not hold, the functional $B(x) = \max\{P(x), P(-x)\}$ is a positive semi-definite norm, i.e., B(x) satisfies 1.1.2(a), 1.1.2(b), 1.1.2(c), but not 1.1.2(d).

Proof. Lemma 3.1.4 is the only one the proof of which makes use of hypothesis (ii) of theorem 3.1.2.

3.1.8. Lemma. If (i) x is an element of a pseudo-normed vector space G,

(ii) $\max\{P(x), P(-x)\} < \eta$,

then $|P(x)| < \eta$ and $|P(-x)| < \eta$.

Proof. By hypothesis (ii), $P(-x) < \eta$, hence $-P(-x) > -\eta$.

By 1.2.11, $P(x) \geq -P(-x) > -\eta$.

Again, by (ii), $P(x) < \eta$, hence $|P(x)| < \eta$.

Similarly $|P(-x)| < \eta$.

The Space G/Z.

We have previously noted (page 15) that the existence of the pseudo-normed vector space K/Z of corollary 2.2.4 depends only on the pseudo-norm P(x). Consequently for any pseudo-normed vector space G we obtain in similar fashion the factor group space G/Z where Z is the subgroup $\{x \in G \mid P(x) = P(-x) = 0\}$. Now in spaces of type K or G it is not generally true that $P(x) = P(-x) = 0$ implies $x = \theta$ (see example 2.3.1). However, this property holds for spaces K/Z and G/Z, as is shown by the next theorem.

3.1.9. Theorem. If (i) G is a pseudo-normed vector space, (see 3.1.1),

(ii) Z is the subgroup $= \{x \in G \mid P(x) = P(-x) = 0\}$,

(iii) Q is any element of the space G/Z, (see immediately above),

(iv) $B(Q) \equiv \max \{P(Q), P(-Q)\}$,[1)]

then the space G/Z is a strongly normed linear space under the definition $\|Q\| \equiv B(Q)$.

Proof. By corollary 2.2.4, G/Z is a pseudo-normed vector space and hence satisfies hypothesis (i) of theorem 3.1.2.

If $P(Q) = P(-Q) = 0$, all the elements x of G present in the coset Q satisfy the relation $P(x) = P(-x) = 0$, since all elements in the same coset have equal pseudo-norms (definition 2.2 and lemma 2.2.2). Therefore $Q = Z$, the zero of G/Z, and G/Z satisfies the remaining hypothesis (ii) of theorem 3.1.2. Accordingly, B(Q) has the properties of a strong norm in the space G/Z.

3.1.10. Definition of P-Topology. The (strong) metric topology of G/Z, based on the strong norm $B(Q) = \max \{P(Q), P(-Q)\}$, $Q \in G/Z$, will be termed the P-topology.

The usual concepts of the theory of normed linear spaces can now be applied to G/Z with its P-topology. In particular, we have the notions of limit, continuity, linear functional, and sequential completeness, all in

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1). Here and in future we omit the subscript z hitherto used for the P-norm in the factor group space K/Z.

terms of the strong norm $B(Q) = \max\{P(Q), P(-Q)\}$. In order to distinguish this topology from the strong norm topology of 1.1.2 in the space K we use the prefix P and speak of P-limit, P-continuity, etc.

3.2. P-Linear Functionals in Spaces of Type G/Z.

3.2.1. Definition of P-Linear Functional in G/Z. A P-linear functional in G/Z is a functional defined over G/Z that is additive, homogeneous,[1] and P-continuous. This means that, if $F(Q)$ is a P-linear functional, $F(aQ_1 + bQ_2) = aF(Q_1) + bF(Q_2)$, a, b, real, and for any point Q_0 and any $\varepsilon > 0$, there exists a $\delta(Q_0, \varepsilon) > 0$ such that $|F(Q - Q_0)| < \varepsilon$ when $B(Q - Q_0) < \delta$.

The existence of linear functionals on a strongly-normed linear space is a classical result. Before using it, we treat a simple case directly in terms of the pseudo-norm.

3.2.2. Lemma. If Q_0 is an element of G/Z and t is a real variable, then the functional F defined over the vector subspace $S = \{tQ_0\}$ by $F(tQ_0) \equiv \text{sign } t \left[P(tQ_0) + P(-tQ_0)\right]$ is P-linear over S (definition 3.2.1).

Proof. Homogeneity:

$t > 0$. $F(tQ_0) = tP(Q_0) + tP(-Q_0) = tF(Q_0)$

$$
\begin{aligned}
t < 0.\quad F(tQ_0) &= -\left[P(tQ_0) + P(-tQ_0)\right] \\
&= -\left[-tP(-Q_0) - tP(Q_0)\right] \text{ since } -t > 0, \\
&= t\left[P(-Q_0) + P(Q_0)\right] = tF(Q_0).
\end{aligned}
$$

Additivity:

Since we are restricted to elements of $S = \{tQ_0\}$,

$F(Q_1 + Q_2) = F(\alpha Q_0 + \beta Q_0) = F(\overline{\alpha + \beta}\, Q_0)$ where $Q_1 = \alpha Q_0$, $Q_2 = \beta Q_0$,

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1). It is sufficient to define a linear functional on a normed linear space as one which is additive and continuous. Banach, Ref. 2, p 36, proves that such a functional is also homogeneous.

i.e., $F(Q_1+Q_2) = (\alpha + \beta) F(Q_0)$ by homogeneity,

$= \alpha F(Q_0) + \beta F(Q_0)$

$= F(\alpha Q_0) + F(\beta Q_0)$

$= F(Q_1) + F(Q_2)$.

P-Continuity:

Let $Q_1 = \alpha Q_0$ be an arbitrary fixed element of $S = \{tQ_0\}$, and $Q = tQ_0$ be an element of S such that $Q \neq Q_1$.

$|F(Q) - F(Q_1)| = |F(Q - Q_1)|$ by homogeneity and additivity

$= |F(\overline{t-\alpha}\, Q_0)|$

$= |\text{sign}\,(t - \alpha)\,[P(\overline{t-\alpha}\, Q_0) + P(-\,\overline{t-\alpha}\, Q_0)]$

$\leq 2 \max \{P(\overline{t-\alpha}\, Q_0), P(-\,\overline{t-\alpha}\, Q_0)\}$ since

$\max \{P(x), P(-x)\} \geq 0$ (lemma 3.1.3),

$\leq 2\, B(\overline{t-\alpha}\, Q_0) = 2\, B(Q - Q_1) < \varepsilon$ for $B(Q - Q_1) < \varepsilon/2$.

Thus F(Q) is continuous at Q_1 in terms of the strong norm B(Q).

The lemma now follows from the definition (3.2.1) of P-linear functional.

3.2.3. Functional Modulus.

We denote the functional modulus (or norm) of a P-linear functional F defined on G/Z by $\boxed{F}$. That is, $\boxed{F}$ is the smallest number k such that $|F(Q)| \leq k\, B(Q)$ for all $Q \in G/Z$.[1)]

The existence of P-linear functionals defined over the full space G/Z can be shown by applying the Hahn-Banach extension theorem to lemma 3.2.2. However, we go directly to the following result of Banach.[2)]

3.2.4. Theorem. For every $Q_0 \in G/Z$ there exists a P-linear functional F(Q)

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1). See Banach, Ref. 2, pp 54-55, and also theorem 1.3.9 of this thesis.
2). Banach, loc. cit., p 55, theorem 3.

defined on G/Z such that $F(Q_0) = B(Q_0)$ and $\boxed{F} = 1$, where $B(Q_0) = \max\{P(Q_0), P(-Q_0)\}$.

3.2.5. Corollary. If Q_0 is an element of G/Z different from the zero element Z, then there exists a P-linear functional F(Q) defined on G/Z such that $F(Q_0) = P(Q_0) + P(-Q_0)$, and

$$\boxed{F} = 1 + \frac{\min\{P(Q_0), P(-Q_0)\}}{\max\{P(Q_0), P(-Q_0)\}}$$

Proof. By 3.2.4 there exists a P-linear functional L such that

$L(Q_0) = B(Q_0)$ and $\boxed{L} = 1$.

Define $F(Q) = \left[1 + \frac{\min\{P(Q_0), P(-Q_0)\}}{\max\{P(Q_0), P(-Q_0)\}}\right] L(Q)$.

Then $F(Q_0) = \max\{P(Q_0), P(-Q_0)\} + \min\{P(Q_0), P(-Q_0)\} = P(Q_0) + P(-Q_0)$, and $|F(Q)| \leq \left(1 + \frac{\min\{P(Q_0), P(-Q_0)\}}{\max\{P(Q_0), P(-Q_0)\}}\right) B(Q)$,

since $|L(Q)| \leq B(Q)$.

Hence $\boxed{F} \leq 1 + \frac{\min\{P(Q_0), P(-Q_0)\}}{B(Q_0)}$.

But the possibility of less in this inequality is excluded, since

$F(Q_0) = \left[1 + \frac{\min\{P(Q_0), P(-Q_0)\}}{B(Q_0)}\right] B(Q_0)$.

This establishes the corollary.

3.3. Functionals P-Linear over the Original Space G.

In this section we generalize a result of Morse and Transue[1)] and obtain the existence of non-zero functionals defined on the original pseudo-

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1). M. Morse and W. Transue, Ref. 15, p 779. These writers consider the case of a positive semi-definite "pseudo-norm" $\|x\|$ which satisfies the conditions $\|x\| \geq 0$, $\|x + y\| \leq \|x\| + \|y\|$, $\|\alpha x\| = |\alpha| \cdot \|x\|$, but which can be zero without x being zero.

normed vector space G which are P-linear in a sense to be made clear. Elements of the space G will be denoted by small latin letters x, y, z.

3.3.1. Definition of P-Limit of a Sequence $\{x_n\}$, $x_n \in G$.

A sequence $\{x_n\}$, $x_n \in G$, is said to P-converge to a limit $x_0 \in G$ if, for all $n > n_0(\mathcal{E})$, $B(x_n - x_0) < \mathcal{E}$, where $B(x) = \max\left\{P(x), P(-x)\right\}$. This is written $x_n \xrightarrow{P} x_0$ and x_0 is called the P-limit of the sequence $\{x_n\}$.

The P-limit of a sequence of elements of G is not necessarily unique. This circumstance is a direct consequence of the fact that B(x) can be zero without x being zero. It is made evident by the following lemma.

3.3.2. Lemma. If (i) G is a pseudo-normed vector space,

(ii) $x_0 \in G$ is a P-limit of a sequence $\{x_n\}$, $x_n \in G$ for all n,

(iii) Z is the subgroup $= \left\{x \in G \mid P(x) = P(-x) = 0\right\}$,

(iv) y is any element in the same coset of Z as x_0,

then y is also a P-limit of the sequence $\{x_n\}$.

Proof. By definition, $x_n \xrightarrow{P} x_0 \Rightarrow B(x_n - x_0) < \mathcal{E}$ for $n > n_0(\mathcal{E})$.

y, x_0 in the same coset $\Rightarrow y - x_0 \in Z \Rightarrow P(y - x_0) = P(x_0 - y) = 0$.

$$P(x_n - y) = P(x_n - x_0 + x_0 - y) \leqq P(x_n - x_0) + P(x_0 - y)$$
$$\leqq P(x_n - x_0) < \mathcal{E} \text{ for } n > n_0.$$

Similarly $P(y - x_n) < \mathcal{E}$ for $n > n_0$.

$\therefore B(x_n - y) < \mathcal{E}$ for $n > n_0(\mathcal{E})$. Hence $x_n \xrightarrow{P} y$ and y is a P-limit of $\{x_n\}$.

Definition of P-Continuity of a Functional Defined over G. We give two definitions and then prove they are equivalent.

3.3.3. A functional f is P-continuous with respect to G at $x_0 \in G$ if to any $\mathcal{E} > 0$ there corresponds a $\delta\ (x_0, \mathcal{E}) > 0$ such that $B(x - x_0) < \delta$ implies

$\left|f(x) - f(x_0)\right| < \varepsilon$.

3.3.4. A functional f is P-continuous with respect to G at $x_0 \in G$ if, for any $\varepsilon > 0$ and any sequence $\{x_n\}$, $x_n \in G$, that P-converges to x_0, there exists a positive integer $n_0(\varepsilon)$ such that $\left|f(x_n) - f(x_0)\right| < \varepsilon$ for $n > n_0(\varepsilon)$.

3.3.5. Lemma. Definitions 3.3.3 and 3.3.4 are equivalent.

Proof. We first show that 3.3.3 implies 3.3.4.

By definition of P-convergence, $x_n \xrightarrow{P} x_0 \Rightarrow$ there exists a $n_0(\delta)$ such that $B(x_n - x_0) < \delta$ for $n > n_0(\delta)$. Hence, by 3.3.3, $\left|f(x) - f(x_0)\right| < \varepsilon$. It remains to prove that 3.3.4 implies 3.3.3.

Deny 3.3.3. Then for some $\varepsilon > 0$ there exists a sequence of positive numbers $\{\delta_n\}$ and a sequence $\{x_n\}$ of elements of G such that $\delta_n \longrightarrow 0$, $B(x_n - x_0) < \delta_n$ and $\left|f(x_n) - f(x_0)\right| \geqq \varepsilon$. Since $\delta_n \longrightarrow 0$, $B(x_n - x_0) < \delta_n$ means that $x_n \xrightarrow{P} x_0$. So definition 3.3.4 applies and $\left|f(x_n) - f(x_0)\right| < \varepsilon$ for all n sufficiently large. This establishes the contradiction sought.

3.3.6. Definition of Functional P-Linear over a Vector Subspace E of a Pseudo-Normed Vector Space G.

A functional f defined on $E \subset G$ will be called P-linear with respect to G if it is additive[1)] and P-continuous with respect to G at each point of E.

3.3.7. Theorem. If f is a functional P-linear with respect to G on a pseudo-normed linear space G in the sense of 3.3.6, then $f(ax) = a\,f(x)$ for all real a, that is, f is homogeneous.

Proof. The proof is similar, mutatis mutandis, to that of Theorem 2, page 36, of Banach's book (Ref. 2). First, the additivity of f(x) is sufficient

---

1). "Additive" means it satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in G$.

to establish by purely algebraic means that $f(rx) = r\, f(x)$ for any rational number r. For any real number a, let $\{r_n\}$ be a sequence of rational numbers converging to a. Then for arbitrary fixed $x \in G$, $B(r_n x - ax) = |r_n - a|\, B(x) < \varepsilon_1$ for $|r_n - a| < \varepsilon_1/B(x)$, i.e., for $n > n_0(\varepsilon_1)$. Hence $r_n x \xrightarrow{P} ax$ by 3.3.1. By continuity, $|f(r_n x) - f(ax)| < \varepsilon_2$ for $n > n_0(\varepsilon_2)$. Hence $f(ax) = \lim_{n\to\infty} f(r_n x) = \lim_{n\to\infty} r_n f(x)$ since r_n is rational, $= f(x) \lim_{n\to\infty} r_n = a\, f(x)$. This proves the theorem.

3.4. A Functional P-Linear over G/Z Induces a Functional P-Linear with Respect to G Defined over G, and Conversely.

3.4.1. Theorem. If (i) G is a pseudo-normed vector space, and Z denotes the subgroup $\{x \in G \mid P(x) = P(-x) = 0\}$,

(ii) F is a functional P-linear over the pseudo-normed space G/Z of cosets of Z,

then there exists a functional f defined over G which is P-linear with respect to G.

Proof. For $x \in G$, define $f(x) = F(Q_x)$ where Q_x is the coset of G/Z containing x.

Additivity of f:

$$\begin{aligned} f(x + y) &= F(Q_{x+y}) \text{ by definition of } f \\ &= F(Q_x + Q_y) \text{ by the algebraic homomorphism between } G \text{ and } G/Z \\ &= F(Q_x) + F(Q_y) \text{ since } F \text{ is additive} \\ &= f(x) + f(y). \end{aligned}$$

Hence f is additive.

P-continuity of f:

For any sequence $\{x_n\}$, $x_n \in G$, that P-converges to $x_0 \in G$, the corresponding sequence $\{Q_{x_n}\}$ of elements of G/Z converges to Q_{x_0} in the strong topology (3.1.10) of B(Q). This is so since $B(x_n - x_0) = B(Q_{x_n} - Q_{x_0})$ by section 2.2, where $B(x) = \max\{P(x), P(-x)\}$ for $x \in G$, and $B(Q) = \max\{P(Q), P(-Q)\}$ for $Q \in G/Z$. Hence, by P-continuity of F, $F(Q_{x_n}) \to F(Q_{x_0})$ and so, by definition of f, $f(x_n) \to f(x_0)$. This means f is P-continuous at the arbitrary point x_0. Thus f, being both additive and P-continuous with respect to G, is P-linear with respect to G.

<u>3.4.2. Theorem.</u> (Converse of theorem 3.4.1) If

(i) G and G/Z are the spaces of the preceding theorem,

(ii) f is a functional defined over G which is P-linear with respect to G,[1)] then there exists a P-linear functional F defined over G/Z such that $F(Q) = f(x_Q)$ where $Q \in G/Z$ and x_Q is any element of G in the coset Q.

<u>Proof.</u> We have seen that if a sequence $\{x_n\}$, $x_n \in G$, P-converges to an element $x_0 \in G$, then $\{x_n\}$ P-converges to every element in the same coset as x_0 (lemma 3.3.2). Since f is P-continuous with respect to G, the definition of 3.3.4 implies that $f(x_n) \to f(x_0)$ and also that $f(x_n) \to f(y)$ where y is any element in the same coset as x_0. Hence f(x) is constant for all x in the same coset. This justifies our defining $F(Q) \equiv f(x_Q)$ where x_Q is any element in the coset Q.

Additivity of F:

$$
\begin{aligned}
F(Q_1 + Q_2) &= f(x_{Q_1+Q_2}) \text{ by definition,} \\
&= f(x_{Q_1} + x_{Q_2}) \text{ by the algebraic homomorphism,} \\
&= f(x_{Q_1}) + f(x_{Q_2}) \text{ since f is additive,} \\
&= F(Q_1) + F(Q_2).
\end{aligned}
$$

P-continuity of F:

For $Q, Q_0, \in G/Z$, $|F(Q) - F(Q_0)| = |f(x_Q) - f(x_{Q_0})| < \varepsilon$ for $B(x_Q - x_{Q_0}) < \delta(\varepsilon)$, since f is P-continuous. As in the proof of theorem 3.4.1,

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1). See 3.3.6 for the definition.

$B(x_Q - x_{Q_0}) = B(Q - Q_0)$. Hence $|F(Q) - F(Q_0)| < \varepsilon$ when $B(Q - Q_0) < \delta(\varepsilon)$, i.e., F is continuous at arbitrary $Q_0 \in G/Z$. Then F, being additive and P-continuous in G/Z, is P-linear (see footnote, p 27).

3.4.3. Theorem. If f(x) is an additive functional defined on a pseudo-normed vector space G, and B(x) denotes max $\{P(x), P(-x)\}$, then, for f(x) to be P-linear with respect to G,[1] it is both necessary and sufficient that there exist a number C such that $|f(x)| \leq C\,B(x)$ for all $x \in G$.

Proof. Necessity:

If f(x) is P-linear over G, then by theorem 3.4.2 there exists a corresponding P-linear functional F(Q) in the strongly normed topology of G/Z such that $f(x) = F(Q_x)$ where Q_x is the coset containing x. Using the fact that in strongly-normed linear spaces linear functionals are bounded,[2] we have $|F(Q)| \leq C\,B(Q)$ for all $Q \in G/Z$. Hence $|f(x)| \leq C\,B(x)$ for all $x \in G$, since $B(x) = B(Q_x)$ by 2.2.

Sufficiency:

Since f(x) is additive, $f(2\theta) = f(\theta + \theta) = f(\theta) + f(\theta)$, i.e., $f(\theta) = 2\,f(\theta)$, hence $f(\theta) = 0$. Therefore $0 = f(\theta) = f(x + -x) = f(x) + f(-x)$ by additivity. Hence $f(-x) = -f(x)$. Using this result, $|f(x) - f(x_0)| = |f(x) + f(-x_0)| = |f(x - x_0)|$ by additivity, $\leq C\,B(x - x_0)$ by hypothesis, $< \varepsilon$ for $B(x - x_0) < \varepsilon/C$. This means f(x) is continuous at any point x_0.

3.4.4. Corollary. If f(x) is P-linear with respect to G over G, there exists a smallest number $\boxed{f}$ such that $|f(x)| \leq \boxed{f}\,B(x)$ for all $x \in G$. Moreover, $\boxed{f} = \boxed{F}$, the modulus of the functional F in G/Z induced (theorem 3.4.2) by f. (For definition of $\boxed{F}$, see 3.2.3).

Proof. From the properties of linear functionals in strongly normed linear

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1). See 3.3.6.
2). Banach, Ref. 2, p 54.

spaces, we know that $\boxed{F}$ is the smallest number such that $|F(Q)| \leq \boxed{F}\, B(Q)$ for all $Q \in K/Z$. Since the range of values of $f(x)$ is the same as that of $F(Q)$ and since $B(Q) = B(x_Q)$, it follows that a smallest number $\boxed{f}$ exists and $\boxed{f} = \boxed{F}$.

Theorems 3.4.1 and 3.4.2 show that a given functional P-linear over the quotient space G/Z induces a P-linear functional on the original space G, and conversely. This result is similar to that obtained by Morse and Transue[1)] but is more general inasmuch as their "pseudo-norm" is restricted to non-negative values.

Existence of Non-Zero P-Linear Functionals on the Space G. From theorems 3.2.4, 3.4.1, and corollary 3.4.4, we obtain

3.4.5. Theorem. Corresponding to any point $x_0 \in G$ such that $B(x_0) \equiv \max \{P(x_0), P(-x_0)\} \neq 0$, there exists a non-zero functional $f(x)$, P-linear with respect to G, such that $f(x_0) = B(x_0)$ and $|f(x)| \leq B(x)$ for all $x \in G$.

3.5. Completeness in Pseudo-Normed Vector Spaces G and G/Z.

In the space G/Z where $B(Q) \equiv \max \{P(Q), P(-Q)\}$ is a strong norm (theorem 3.1.9), completeness is the usual notion that every Cauchy sequence has a limit.[2)] But the space G is not a metric space and for it we need the following definitions.

3.5.1. Definition of P-Fundamental Sequence of Elements of G.

A sequence $\{x_n\}$, $x_n \in G$, is P-fundamental if, for any $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that $B(x_m - x_p) < \varepsilon$ for all $m, p, > n_0$ where $B(x) = \max \{P(x), P(-x)\}$.

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1). Morse and Transue, Ref. 15, pp 779-780.
2). See, for example, Hille, Ref. 6, p 5.

3.5.2. Definition of a P-Complete Space G. A pseudo-normed space G is P-complete if, for any P-fundamental sequence $\{x_n\}$, $x_n \in G$, there exists an element $x_0 \in G$ such that the sequence $\{x_n\}$ P-converges to x_0.[1)]

3.5.3. Theorem. If a pseudo-normed vector space G is P-complete, so is the pseudo-normed vector space G/Z, and conversely.

Proof. Let $\{Q_n\}$, $Q_n \in G/Z$, be a Cauchy sequence, i.e., $B(Q_m - Q_p) < \varepsilon$ for all m, p, $> n_0(\varepsilon)$. Let x_n be an element of G in the coset Q_n, n = 1, 2, 3, ..., . Then the sequence $\{x_n\}$ is a fundamental sequence in G since $B(x_m - x_p) = B(Q_m - Q_p)$.[2)] Hence, by hypothesis, there exists $x_0 \in G$ such that $x_n \xrightarrow{P} x_0$. Then if Q_0 is the coset containing x_0, $Q_n \rightarrow Q_0$ in the strong topology of G/Z. The existence of Q_0 proves that G/Z is P-complete.

Conversely, if $\{x_n\}$ is a fundamental sequence in G, $B(x_m - x_p) < \varepsilon$ for m, p, $> n_0(\varepsilon)$. Hence $B(Q_m - Q_p) < \varepsilon$ where Q_n is the coset containing x_n, n = 1, 2, 3, ..., so $\{Q_n\}$ is a Cauchy sequence in G/Z. By hypothesis there exists $Q_0 \in G/Z$ such that $\lim_{n\to\infty} B(Q_n - Q_0) = 0$. Hence, for any x_0 in the coset Q_0, $\lim_{n\to\infty} B(x_n - x_0) = 0$, so $x_n \xrightarrow{P} x_0$ and G is P-complete. This completes the proof.

3.6. Examples of Spaces of Type K.

We now consider further examples of spaces of type K and give a more detailed discussion than in 1.4. It is to be carefully noted that in spaces of type K we have the metric topology of the given strong norm $\|x\|$ and we have defined the notion of P-limit and P-continuity in terms of the pseudo-norm P(x). Also, in the vector space K/Z we have a normed topology based on P(Q), $Q \in K/Z$. In spaces where the subgroup $Z = \{x \in K \mid P(x) = P(-x) = 0\}$ reduces to the single element θ, the spaces K/Z and K are identical and we

1). See 3.3.1 for definition of P-convergence in G.
2). See definition 2.2.

then have two normed topologies in K, the $\| \ \|$ topology and the P-topology. This case arises in some of the examples discussed below. It will appear that the two metric topologies are not necessarily equivalent.

3.6.1. The Two-Dimensional Real Vector Space.

Let K be the set of ordered pairs of real numbers with addition and scalar multiplication defined componentwise in the usual way. An element x of K is represented as (x_1, x_2). Define $\|x\| = \sqrt{x_1^2 + x_2^2}$, $P(x) = \max (x_1, x_2)$. It is readily verified that $\|x\|$ and $P(x)$ satisfy the postulates. The normal subgroup Z consists of only one element, the zero $\theta = (0, 0)$. It is easy to show that the metric topology based on P(x) is equivalent to the strong topology of $\|x\|$ (use the Hausdorff equivalence theorem). Similar results hold for n-dimensional real vector space with $\|x\| = (\sum_{i=1}^{n} |\xi_i|^p)^{1/p}$, p any fixed positive integer, and $P(x) = \max_i \xi_i$ where the element x is represented by $(\xi_1, \xi_2, \dots, \xi_n)$.

3.6.2. The Space of Bounded Sequences of Complex Numbers with Scalar Multiplication Restricted to Real Numbers.

If the element x is the sequence $\{\xi_n\}$, we define $\|x\| = \sup_i |\xi_i|$ and $P(x) = \sup_i (\mathrm{Rl}\ \xi_i)$. We have seen[1)] that the subgroup Z consists of sequences of pure imaginaries and a coset (element of K/Z) consists of all sequences having identical real parts in corresponding terms. But the strong norms of elements in the same coset take on a wide range of values so that in K/Z there is only one metric topology, the P-topology. For example, $P(Z) = 0$ but there is no restriction except finiteness on the imaginary parts of sequences belonging to the coset Z, so that Z contains elements of K whose strong norms take every non-negative value.

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1). Discussed in 1.4, example II, again on p 17, and in 2.3.1.

3.6.3. Theorem. The space K of example 3.6.2 (bounded sequences) is P-complete in the sense of 3.5.1.

Proof. We prove the space K/Z is complete in its metric P-topology. The result then follows by theorem 3.5.3.

The elements of K/Z consist of classes of sequences having the same real parts. We take as representative of each class (coset) a sequence of real numbers only. Then the norm is $B(x) = \max (\sup_i \xi_i, \sup_i -\xi_i)$ where $x = \{\xi_n\}$, $= \sup_i |\xi_i|$ since the ξ_i are real. Thus B(x) becomes the classical norm in the Banach space (m)[1] which is known to be complete. Application of theorem 3.5.3 gives the result.

3.6.4. The Space of Convergent Sequences of Complex Numbers with Scalar Multiplication Restricted to the Reals and Definitions of Strong Norm and Pseudo-Norm as in the Previous Example.

This space of convergent sequences is clearly a subspace of the space of bounded sequences. From the discussion in 3.6.2 it follows that there exists a one-to-one norm-preserving correspondence between the K/Z space of this example 3.6.4 and the Banach space (c)[2] of convergent sequences.

P-Linear Functionals in the Space of Convergent Sequences.

The remark immediately preceding allows us to prove the following result very readily.

3.6.5. Theorem. Every functional f(x), P-linear with respect to the space K of convergent sequences of 3.6.4, and defined on K, is of the form $f(x) = C \lim_{n\to\infty} \xi_n + \sum_{n=1}^{\infty} C_n \xi_n$, where the sequence $\{\xi_n\}$ is the sequence of

---

1). Banach, Ref. 2, pp 11 and 53-54.
2). Banach, ibid.

real parts of the terms of the sequence x, and where $|C| + \sum_{n=1}^{\infty} |C_n| = \boxed{f}$.[1]

Proof. It is known that any real linear functional F(q), where $q = \{\xi_n\}$, defined over the real Banach space (c) of convergent sequences is of the form

$$F(q) = C \lim_{n\to\infty} \xi_n + \sum_{n=1}^{\infty} C_n \xi_n$$

where $|C| + \sum_{n=1}^{\infty} |C_n| = |F|$.[2]

From 3.6.2 and 3.6.4 it follows that in the space K/Z, where K is the space of convergent sequences of 3.6.4, any P-linear functional F(q) is of the form

$$F(q) = C \lim_{n\to\infty} \xi_n + \sum_{n=1}^{\infty} C_n \xi_n$$

where $|C| + \sum_{n=1}^{\infty} |C_n| = \boxed{F}$ and where $\{\xi_n\}$ is a real sequence representative of the element $q \in K/Z$. By theorem 3.4.1, F(q) induces in the space K a functional f(x) P-linear with respect to K such that f(x) = F(q) where q is the coset containing x. Conversely, by theorem 3.4.2, to any functional f(x) P-linear with respect to K there corresponds a P-linear functional F(q) on K/Z. The proofs of theorems 3.4.1 and 3.4.2 show this correspondence to be biunique. Moreover, by corollary 3.4.4, $\boxed{f}_K = \boxed{F}_{K/Z}$. These facts establish the theorem.

3.6.6. Let K be the Space (C) of Real-Valued Continuous Functions x(t) Defined on $0 \leq t \leq 1$, Normed by $\|x\| = \max_{0\leq t\leq 1} |x(t)|$, and Pseudo-Normed by $P(x) = \max_{0\leq t\leq 1} x(t)$.

The subgroup Z reduces to the single element θ and the elements of the space K/Z are the same as those of K. The P-topology is the same as the topology of the strong norm, since $B(x) = \max\{P(x), P(-x)\} = \max_{0\leq t\leq 1} |x(t)| = \|x\|$.

---

1). For $\boxed{f}$, see 3.2.3 and corollary 3.4.4.
2). See Banach, loc. cit., pp 65-67.

Consequently the same class of functionals is linear with respect to both topologies.

3.6.7. Let K be the Space (C) of Real-Valued Continuous Functions Defined on $0 \leq t \leq 1$, Normed by $\|x\| \equiv \max_{0 \leq t \leq 1} |x(t)|$, and Pseudo-Normed by $P(x) \equiv \sup_{0 \leq t \leq 1} \int_0^t x(s)\, ds$.

It is easy to verify that the pseudo-norm postulates are satisfied. We note that $P(x) \geqq 0$ but $P(x)$ may be zero without x being zero.

The Subgroup Z Consists of only One Element, i.e., $P(x) = P(-x) = 0 \Rightarrow x = \theta$.

To see that this is so, suppose that $x(t) \neq 0$ for all t, $0 \leq t \leq 1$. Then $x(t)$, being continuous, is greater (say) than zero over some interval of length $\delta > 0$. Then $\int_\delta x(t)\, dt > 0$. As t increases from 0 to 1, such an interval δ cannot occur before a value of t where $x(t) < 0$, otherwise $\sup_t \int_0^t x(s)\, ds > 0$. But, similarly, $x(t)$ cannot take a value < 0 before $x(t) > 0$ since, by hypothesis, $\sup_t \int_0^t -x(s)\, ds = 0$. Hence $x(t) = 0$ identically. Hence $K \equiv K/Z$ and in the space K we have two normed topologies, one based on $\|x\|$ and the other on $P(x)$. We next prove that

The Topologies of the Strong Norm and the Pseudo-Norm in the Space K of 3.6.7 are Not Equivalent.

Proof. It is sufficient to show that a neighborhood in the $\|x\|$ topology does not contain any neighborhood of the $P(x)$ topology. Let $N_s = \{x \mid \|x\| < \varepsilon\}$ be a given neighborhood of the origin in the $\|x\|$ topology. Any neighborhood of the $P(x)$ topology is of the form $N_p = \{x \mid B(x) < \varepsilon_1\}$ where $B(x) \equiv \max \{P(x), P(-x)\}$.

Define the function y(t) as follows (also see Figure 1):

$$
\begin{aligned}
y(t) &= 8\,\varepsilon^2\, t/\varepsilon_1 && \text{for } 0 \leq t \leq \varepsilon_1/4\varepsilon \\
&= -8\,\varepsilon^2\, t/\varepsilon_1 + 4\,\varepsilon && \text{for } \varepsilon_1/4\varepsilon \leq t \leq \varepsilon_1/2\varepsilon \\
&= 0 && \text{for } \varepsilon_1/2\varepsilon \leq t \leq 1
\end{aligned}
$$

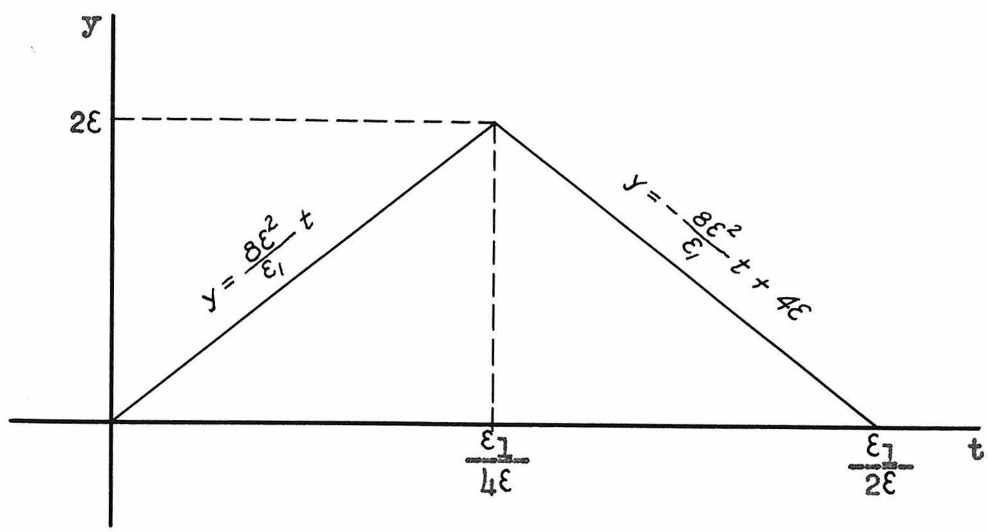

Figure 1.

y(t) is continuous, so $y \in K$.

$$\sup_{0\leq t\leq 1} \int_0^t y = \varepsilon_1/2 < \varepsilon_1, \text{ and } \sup_{0\leq t\leq 1} \int_0^t -y = 0 < \varepsilon_1.$$

Hence, $B(y) < \varepsilon_1$ so $y \in N_p$. But $\max_{0\leq t\leq 1} |y(t)| = 2\varepsilon > \varepsilon$. Thus y is not in N_s. This means that there is no P-neighborhood contained in a given $\|x\|$ neighborhood and so the topologies are not equivalent.

<u>Completeness.</u> It is known that this space of continuous functions is complete with respect to the topology of the strong norm ((C) is a Banach space). This is no longer true in the P-topology. We prove the following result:

The Space of Continuous Functions of 3.6.7, Pseudo-Normed by $P(x) = \sup_{0 \leq t \leq 1} \int_0^t x(s)\, ds$, is not P-Complete.[1)]

Proof. Consider the sequence of continuous functions $x_n(t)$ defined as follows:

$$x_n(t) = 0 \quad \text{for } 0 \leq t \leq \tfrac{1}{4}$$
$$= n(t-\tfrac{1}{4}) \quad \text{for } \tfrac{1}{4} \leq t \leq \tfrac{1}{4} + 1/n$$
$$= 1 \quad \text{for } \tfrac{1}{4} + 1/n \leq t \leq 1.$$

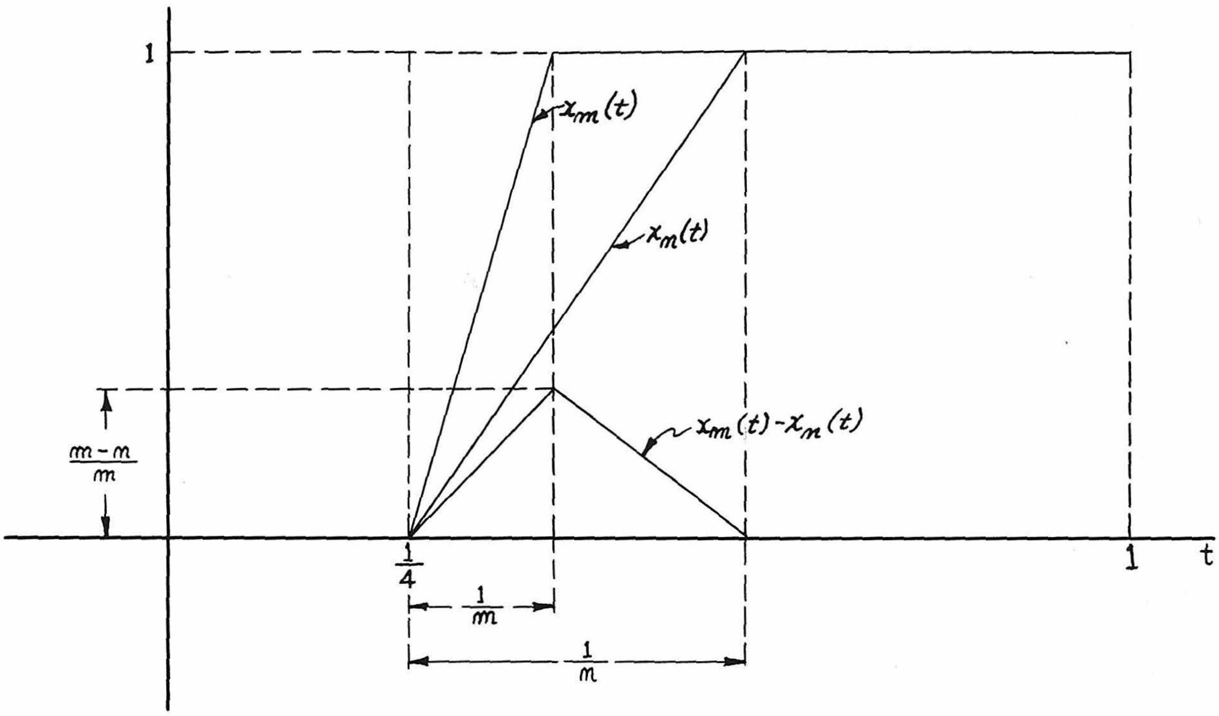

Figure 2.

Then for $m > n \geq 2$,

$$x_m(t) - x_n(t) = 0 \quad \text{for } 0 \leq t \leq \tfrac{1}{4}$$
$$= (m-n)(t-\tfrac{1}{4}) \quad \text{for } \tfrac{1}{4} \leq t \leq \tfrac{1}{4} + 1/m$$
$$= -n(t-\tfrac{1}{4}-1/n) \quad \text{for } \tfrac{1}{4} + 1/m \leq t \leq \tfrac{1}{4} + 1/n$$
$$= 0 \quad \text{for } \tfrac{1}{4} + 1/n \leq t \leq 1.$$

This computation is more easily followed by referring to Figure 2.

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1). For the definition, see 3.5.1, 3.5.2.

It is readily seen that $P(x_m - x_n) \equiv \sup_{0 \leq t \leq 1} \int_0^t (x_m - x_n) = (m - n)/2mn$, and that $P(x_n - x_m) \equiv 0$. $\therefore B(x_m - x_n) \equiv \frac{m-n}{2mn} = \frac{1}{2n} - \frac{1}{2m} < \varepsilon$ for all positive integers $m, n, > 1/2\varepsilon$. Hence $\{x_n(t)\}$ is a P-fundamental sequence. From Figure 2 it is clear that the step function with value 0 for $0 \leq t \leq \frac{1}{4}$ and value 1 for $\frac{1}{4} < t \leq 1$ is the P-limit function of the sequence $\{x_n(t)\}$ but it is not continuous and so is not an element of the space K/Z (here the same as K). Hence the space is not P-complete.

<u>P-Linear Functionals in the Space 3.6.7 of Real-Valued Continuous Functions $x(t)$ on $0 \leq t \leq 1$, Pseudo-Normed by $P(x) \equiv \sup_{0 \leq t \leq 1} \int_0^t x(s)\, ds$.</u>

We prove the following analogue of the theorem of F. Riesz on linear functionals in the Banach space (C) of continuous functions.

<u>3.6.8. Theorem.</u> Every P-linear functional $f(x)$ defined on the space (C) of real-valued continuous functions $x(t)$ on $0 \leq t \leq 1$, pseudo-normed by $P(x) \equiv \sup_{0 \leq t \leq 1} \int_0^t x(s)\, ds$, is of the form $f(x) \equiv \int_0^1 x(t)\, dg$ where $g(t)$ is a function of bounded variation independent of x.

<u>Proof.</u> The method of proof is to approximate continuous functions by step functions. We first define a P-normed linear space containing both classes of functions. Let (M) be the set of measurable functions bounded in the interval $[0, 1]$. From a result of measure theory to the effect that fg is integrable if f is integrable and g is essentially bounded and measurable,[1)] it follows that any function $x(t) \in (M)$ is integrable on $[0, 1]$ and on any subinterval of $[0, 1]$. (Define $f \equiv 1$ on the subinterval in question, 0 elsewhere, and take for g the function $x(t)$). Hence we may define $P(x) \equiv \sup_{0 \leq t \leq 1} \int_0^t x$ for $x \in (M)$. Then $P(x)$ is a pseudo-norm, and if in (M) we adopt a common procedure and regard functions differing on sets of measure zero

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1). See, e.g., Halmos, Ref. 4*, p 113, Theorem D.

as identical, then the subgroup Z in (M) reduces to the single element θ as in the space of continuous functions (C) of 3.6.7. This space (C) becomes a linear subspace of (M) with the same definition of pseudo-norm in both (C) and (M).

By the Hahn-Banach extension theorem, if f(x) is a P-linear functional on (C), there exists a P-linear functional F on (M) such that F(x) = f(x) for $x \in (C)$ and $\boxed{F}_{(M)} = \boxed{f}_{(C)}$, where $\boxed{F}_{(M)}$, $\boxed{f}_{(C)}$, are the respective functional moduli (see 3.2.3).

Define the step function $\xi_t \equiv \xi_t(u) \equiv 1$ for $0 \leq u \leq t$

$\equiv 0$ for $t < u \leq 1$

and place $F(\xi_t) = g(t)$. Note that ξ_t is in (M). We next show, following Banach, that g(t) is a function of bounded variation. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ and $\varepsilon_i = \text{sign}\left[g(t_i) - g(t_{i-1})\right]$ for i = 1, 2, ..., n. Then

$$\sum_{i=1}^{n} \left| g(t_i) - g(t_{i-1})\right| = \sum_{i=1}^{n} \left\{g(t_i) - g(t_{i-1})\right\} \varepsilon_i$$

$$= F\left(\sum_{i=1}^{n}(\xi_{t_i} - \xi_{t_{i-1}})\varepsilon_i\right) \text{ by linearity of } F,$$

$$= \boxed{F}_{(M)}\, B\left\{\sum_{i=1}^{n} (\xi_{t_i} - \xi_{t_{i-1}})\varepsilon_i\right\}.^{1)}$$

Now $B\left\{\sum_{i=1}^{n}(\xi_{t_i} - \xi_{t_{i-1}})\varepsilon_i\right\}$ = max of the pseudo-norms of $\pm \sum_{i=1}^{n}(\xi_{t_i}-\xi_{t_{i-1}})\varepsilon_i$ and so will have its greatest possible value when all the ε_i are positive. Hence $B\left\{\sum_{i=1}^{n}(\xi_{t_i} - \xi_{t_{i-1}})\varepsilon_i\right\} \leq 1$ by the definition of the pseudo-norm.

$\therefore \sum_{i=1}^{n} \left| g(t_i) - g(t_{i-1})\right| \leq \boxed{F}_{(M)}$, which is independent of n. $\therefore$ total $\underset{0 \leq t \leq 1}{\text{variation}}$ of $g(t) = \boxed{F}_{(M)} = \boxed{f}_{(C)}$ by the Hahn-Banach theorem, i.e., g is a function of bounded variation.

The next step consists in approximating a given continuous function x(t) by the following sequence of step functions $\left\{z_n\right\}$, n = 1, 2, ...,

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1). See 3.2.3.

where $z_n \equiv z_n(u) \equiv \sum_{r=1}^{n} x(r/n)\left\{\xi_{r/n}(u) - \xi_{(r-1)/n}(u)\right\}$. From this, $B(x - z_n) \to 0$ as $n \to \infty$, i.e., $z_n \xrightarrow{P} x$. From the P-continuity of F we have $\lim_{n\to\infty} F(z_n) = F(P\text{-}\lim_{n\to\infty} z_n) = F(x) = f(x)$ since $x \in (C)$. From the additivity and homogeneity of F, it follows that $F(z_n) = \sum_{r=1}^{n} x(r/n)\left\{g(r/n) - g(\frac{r-1}{n})\right\}$. Since $x(t)$ is continuous and $g(t)$ is a function of bounded variation, $\lim_{n\to\infty} F(z_n)$ is the Riemann-Stieltjes integral, $\int_0^1 x(t)\, dg(t)$. Hence $f(x) = \int_0^1 x(t)\, dg$ for all $x \in (C)$. This proves the theorem.

We remark that this investigation does not yield an exact value for the functional modulus (contrast the classical result of F. Riesz). The best result this proof allows is that $\boxed{f} \geqq$ total $\underset{0\leqq t\leqq 1}{\text{variation}}$ of $g(t)$.

3.6.9. The Space (C) of Real-Valued Continuous Functions Defined on $0 \leqq t \leqq 1$, Normed by $\|x\| = \max_{0\leqq t\leqq 1} |x(t)|$, and Pseudo-Normed by $P(x) = \sup_{\frac{1}{2}\leqq t\leqq 1} \int_0^t x(s)\, ds$.

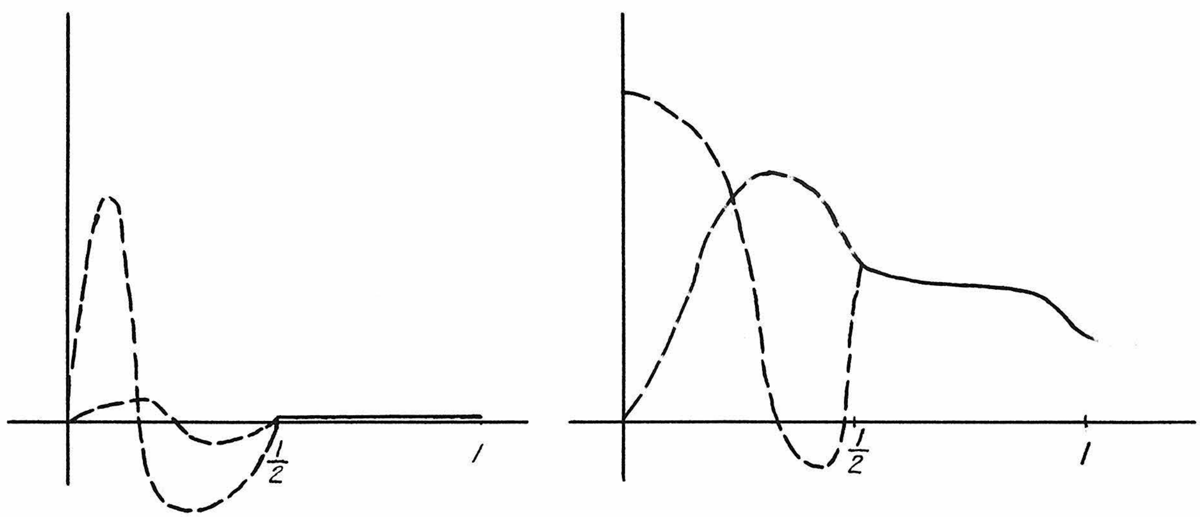
This space is more general than that of 3.6.7. Here P(x) may take negative values. We prove that P satisfies the triangular inequality. If x, y are elements of the space,

$$\begin{aligned} P(x + y) &= \sup_{\frac{1}{2}\leqq t\leqq 1} \int_0^t (x + y) = \int_0^{\frac{1}{2}} (x + y) + \sup_{\frac{1}{2}\leqq t\leqq 1} \int_{\frac{1}{2}}^t (x + y) \\ &\leqq \int_0^{\frac{1}{2}} x + \int_0^{\frac{1}{2}} y + \sup_{\frac{1}{2}\leqq t\leqq 1} \int_{\frac{1}{2}}^t x + \sup_{\frac{1}{2}\leqq t\leqq 1} \int_{\frac{1}{2}}^t y \\ &\leqq \sup_{\frac{1}{2}\leqq t\leqq 1} \int_0^t x + \sup_{\frac{1}{2}\leqq t\leqq 1} \int_0^t y \\ &\leqq P(x) + P(y). \end{aligned}$$

The other postulates are readily seen to be satisfied.

Nature of the Subgroup $Z = \{x \mid P(x) = P(-x) = 0\}$.

$P(x) = 0 \Rightarrow \int_0^{\frac{1}{2}} x(s)ds \leqq 0$ since $\sup_{\frac{1}{2}\leqq t\leqq 1} \int_{\frac{1}{2}}^t x(s)\, ds \geqq 0$. Similarly $P(-x) = 0 \Rightarrow \int_0^{\frac{1}{2}} -x(s)\, ds \leqq 0$. Therefore $\int_0^{\frac{1}{2}} x(s)\, ds = 0$ for $x \in Z$, and consequently $\sup_{\frac{1}{2}\leq t\leq 1}\int_{\frac{1}{2}}^{t} x(s)\,ds$ and $\sup_{\frac{1}{2}\leq t\leq 1}\int_{\frac{1}{2}}^{t} -x(s)\,ds$ are zero for $x \in Z$. As in 3.6.7, the latter result implies that $x(t) = 0$ for $\frac{1}{2} \leq t \leq 1$. Hence the functions $x(t)$ of Z are zero over $\frac{1}{2} \leq t \leq 1$ and their integrals over the interval $\left[0, \frac{1}{2}\right]$ are zero. Two functions $x(t)$, $y(t)$ are in the same coset if $x(t) = y(t)$ for $\frac{1}{2} \leq t \leq 1$, and if $\int_0^{\frac{1}{2}} \left(x(t) - y(t)\right)dt = 0$. Figure 3 helps to explain the situation.



Functions in the Subgroup Z   Functions in the Same Coset

Figure 3.

The pseudo-norm of 3.6.9 is more general than that of 3.6.7. Hence the space K/Z of 3.6.9 is not P-complete, else the space of 3.6.7 would also be complete.

CHAPTER IV

The Pseudo-Norm as a Functional

In this chapter we shall confine ourselves to pseudo-normed vector spaces of type G[1] and G/Z, where $Z = \{x \in G \mid P(x) = P(-x) = 0\}$.

4.1. Continuity of P(x).

We have seen (theorem 3.1.9) that a space G/Z can be made into a strongly normed linear space under $B(Q) = \max\{P(Q), P(-Q)\}$, $Q \in K/Z$, and that for $x \in G$, $B(x) = \max\{P(x), P(-x)\}$ is a positive semi-definite norm in the space G. Also theorems 3.4.1 and 3.4.2 show that there exists a bi-unique correspondence between P-continuous functionals defined on G/Z and P-continuous functionals defined on G. This being so, it is immaterial in dealing with questions of continuity with respect to the norm B whether we work in a space of type G or in the more special space G/Z.

4.1.1. Theorem. The pseudo-norm is P-continuous, i.e., continuous with respect to the topology of the norm B(x).

Proof. We have just pointed out that we may work either with a pseudo-normed vector space of type G or one of type G/Z. Select the latter. For $Q \in G/Z$, $P(Q) \leq B(Q)$, where $B(Q) = \max\{P(Q), P(-Q)\}$ is a strong norm in G/Z. This means that G/Z is a space of type K[2] in which $\|x\|$ is replaced by B(Q) and $P(x) \leq M\|x\|$ is replaced by $P(Q) \leq 1\, B(Q)$. Theorem 1.3.6 now applies and completes the proof.

4.1.2. Definition of Convex Function.

A real-valued function f(u) of a real variable u is convex if $f(q_1u_1 + q_2u_2) \leq q_1f(u_1) + q_2f(u_2)$ for any u_1, u_2 and any q_1, q_2 such that

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1). See 3.1.1.
2). See pp 1, 2 for the definition of a space of type K.

$q_1 \geqq 0$, $q_2 \geqq 0$, $q_1 + q_2 = 1$.

4.1.3. Theorem. For arbitrary but fixed x, y, ϵ G, and a real variable u, P(x + uy) is a convex and continuous function of u.

Proof. Denote P(x + uy) by f(u).

Convexity:

$$f(q_1u_1 + q_2u_2) = P(x + q_1u_1y + q_2u_2y)$$
$$= P(q_1\overline{x + u_1y} + q_2\overline{x + u_2y}) \text{ since } q_1 + q_2 = 1,$$
$$\leqq q_1 \, P(x + u_1y) + q_2 \, P(x + u_2y)$$
$$\leqq q_1 \, f(u_1) + q_2 \, f(u_2).$$

Continuity:

$$f(u + \Delta u) - f(u) = P(x + uy + y\Delta u) - P(x + uy)$$
$$\leqq P(y\Delta u) \text{ by 1.2.10, } \leqq |\Delta u| \max \{P(y), P(-y)\} \text{ by}$$

1.1.3(c) and lemma 3.1.3.

A similar result holds for f(u) - f(u + Δu). Hence $|f(u + \Delta u) - f(u)| \leqq |\Delta u| \, B(y) < \varepsilon$ for $|\Delta u| < \varepsilon/B(y)$ when $B(y) \neq 0$, and $= 0$ when $B(y) = 0$. Thus f(u) is continuous. This completes the proof of theorem 4.1.3.

By applying certain results[1)] of the theory of convex continuous functions to the above function P(x + uy), we immediately obtain the following theorem.

4.1.4. Theorem. For arbitrary fixed elements x, y of a pseudo-normed vector space G, the real-valued function f(u) = P(x + uy) of the real variable u possesses a right derivative and a left derivative at each point u; moreover f(u) has a derivative except perhaps at an enumerable set of values of u.[2)]

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1). Hardy, Littlewood, and Polya, Ref. 5, pp 91, 94.
2). Using distinct methods, Mazur, Ref. 10, page 75, and James, Ref. 8, Theorem 6.6, prove the first part of this theorem, and Mazur, ibid, page 77, then applies a result of Siérpinski to obtain the second part.

4.2. Gâteaux Differentiability of the Pseudo-Norm.

4.2.1. Definition of Gâteaux Differential. The functional P(x) is said to be Gâteaux-differentiable at the point x if $\lim_{h\to 0} \frac{P(x+hy)-P(x)}{h}$ exists for all y in the space G. The value of this limit, if it exists, will be denoted by $P_g(x; y)$ and called the Gâteaux differential of P(x) at x with increment y.

4.2.2. Definition of Right and Left Differentials of P(x). Theorem 4.1.4 gives the existence of both right and left derivatives of P(x + uy) with respect to u. For u = 0, this means that $\lim_{h\to 0+} \frac{P(x+hy)-P(x)}{h}$ and $\lim_{h\to 0-} \frac{P(x+hy)-P(x)}{h}$ exist for all x, y, ∈ G. (But these limits are not necessarily equal.) We set $P_+(x; y) = \lim_{h\to 0+} \frac{P(x+hy)-P(x)}{h}$ and $P_-(x; y) = \lim_{h\to 0-} \frac{P(x+hy)-P(x)}{h}$ and term these the right differential and the left differential respectively of P(x) at the point x with increment y.

4.2.3. Theorem.[1] For fixed x ∈ G, the right differential $P_+(x; y)$ of P(x) has the following properties:

(i) $t > 0$, $P_+(x; ty) = t\, P_+(x; y)$,

(ii) $P_+(x; y+z) \leq P_+(x; y) + P_+(x; z)$,

(iii) $P_+(x; y) \leq P(y)$.

Proof. (i) $t > 0$, $P_+(x; ty) = \lim_{h\to 0+} \frac{P(x+hty)-P(x)}{h} = \lim_{h\to 0+} \frac{t[P(x+hty)-P(x)]}{ht}$

$$= t \lim_{h\to 0+} \frac{P(x+hty)-P(x)}{ht}$$

$$= t \lim_{ht\to 0+} \frac{P(x+hty)-P(x)}{ht} = t\, P_+(x; y).$$

(ii) $\frac{P(x+h\overline{y+z})-P(x)}{h} = \frac{P(2x+2hy+2hz)-P(x)-P(x)}{2h}$

$$\leq \frac{P(x+2hy)-P(x)}{2h} + \frac{P(x+2hz)-P(x)}{2h} \text{ for } h > 0.$$

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1). Proved also by Mazur, Ref. 10, p 75.

Now $\lim_{h \to 0+} \frac{P(x+2hy) - P(x)}{2h}$ exists and is in fact $P_+(x; y)$. Similarly $\lim_{h \to 0+} \frac{P(x+2hz) - P(x)}{2h} = P_+(x; z)$. Therefore, as $h \to 0+$, the above inequality gives $P_+(x; y+z) \leq P_+(x; y) + P_+(x; z)$.

(iii) For $h > 0$, $\frac{P(x + hy) - P(x)}{h} \leq \frac{P(hy)}{h} = P(y)$. Hence

$$\lim_{h \to 0+} \frac{P(x + hy) - P(x)}{h} \leq P(y), \text{ i.e., } P_+(x; y) \leq P(y).$$

4.2.4. Corollary. For fixed x, the right differential[1] $P_+(x; y)$, regarded as a function of the increment y, is also a pseudo-norm in the vector space G.

Proof. Properties (i), (ii) of theorem 4.2.3, are postulates 1.1.3(c) and 1.1.3(b) which a pseudo-norm satisfies in a space of type G.[2] We can assure that $P_+(x; y)$ is not identically zero in y (postulate 1.1.3(a) for a pseudo-norm) by selecting the fixed x so that $P(x) \neq 0$. Since $P_+(x; x) = P(x)$, $P_+(x; y) \not\equiv 0$. This establishes the corollary.

We can now apply theorem 4.1.4 and definition 4.2.2 to the right differential of P(x) and obtain the result that the right differential at y with increment z of the right differential of P(x) at x with increment y exists. If we denote this second right differential by $P_{++}(x; y; z_y)$, and apply 4.2.3 we get the following result.

4.2.5. Corollary. (i) $t > 0$, $P_{++}(x; y; tz_y) = tP_{++}(x; y; z_y)$,

(ii) $P_{++}(x; y; (z+w)_y) \leq P_{++}(x; y; z_y) + P_{++}(x; y; w_y)$,

(iii) $P_{++}(x; y; z_y) \leq P_+(x; y)$.

4.2.6. Lemma. $P_-(x; y) = -P_+(x; -y)$ where $P_-(x; y)$ is the left differential of P(x) with increment y, and $P_+(x; -y)$ is the right differential of P(x) with increment -y.

---

1). For definition of $P_+(x; y)$, see 4.2.2.
2). See 3.1.1 for definition of a space of type G.

Proof. By definition, $P_-(x; y) = \lim_{h\to 0-} \frac{P(x+hy) - P(x)}{h}$

$$= \lim_{h\to 0-} - \frac{P(x + |h|(-y)) - P(x)}{|h|}$$

$$= - \lim_{h\to 0+} \frac{P(x + h(-y)) - P(x)}{h}$$

$$= - P_+(x; -y) \text{ by definition.}$$

4.2.7. Lemma. A necessary and sufficient condition for the existence of the Gâteaux differential of the pseudo-norm P(x) at $x \in G$ is that $P_+(x; y) \equiv \lim_{h\to 0+} \frac{P(x+hy) - P(x)}{h}$ and $-P_+(x; -y) \equiv - \lim_{h\to 0+} \frac{P(x-hy) - P(y)}{h}$ be equal for all $y \in G$.

Proof. This follows readily from the definition 4.2.1 of Gâteaux differential and lemma 4.2.6. In detail, the existence of the Gâteaux differential $P_g(x; y) \equiv \lim_{h\to 0} \frac{P(x+hy) - P(x)}{h}$ is equivalent to $\lim_{h\to 0+} \frac{P(x+hy) - P(x)}{h} = P_g(x; y) = \lim_{h\to 0-} \frac{P(x+hy) - P(x)}{h}$, i.e., $P_+(x; y) = P_-(x; y)$, both limits existing in virtue of 4.2.2. Then by lemma 4.2.6, $P_+(x; y) = -P_+(x; -y)$.

4.2.8. Theorem. If the Gâteaux differential of the pseudo-norm exists at $x \in G$, it is P-linear with respect to G[1] in the increment y, i.e., additive, homogeneous, and continuous in the P-topology.[2]

Proof: Homogeneity:

For $t > 0$, theorem 4.2.3 gives $P_+(x; ty) = tP_+(x; y)$ and for $t < 0$, $P_+(x; ty) = P_+(x; (-t)(-y)) = -tP_+(x; -y)$. Since $P_g(x; y)$ exists, $P_g(x; y) = P_+(x; y)$. For $t > 0$, $P_g(x; ty) = P_+(x; ty) = tP_+(x; y) = tP_g(x; y)$.

For $t < 0$, $P_g(x; ty) = P_+(x; ty) = -tP_+(x; -y)$ by the first part of the proof. Since $P_g(x; y)$ exists, lemma 4.2.7 gives $P_g(x; -y) = P_+(x; -y) = -P_+(x; y)$. Using this and the above, we get $P_+(x; ty) = -tP_+(x; -y) = (-t)(-P_+(x; y)) = tP_+(x; y)$. This completes the proof of homogeneity.

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1). See Chapter III, in particular definition 3.3.6 and theorem 3.3.7.
2). An equivalent result is stated without proof by Mazur, Ref. 11, p 130.

Additivity:

Since $P_g(x; y)$ is homogeneous in y, $P_g(x; -\overline{y+z}) = -P_g(x; y+z)$. Now $P_g(x; -\overline{y+z}) = P_+(x; -y-z) \leqq P_+(x; -y) + P_+(x; -z)$ by theorem 4.2.3, $\leqq P_g(x; -y) + P_g(x; -z) \leqq - P_g(x; y) - P_g(x; z)$ since $P_g(x; y)$ is homogeneous in y by the first part of this proof. Thus $-P_g(x; y+z) \leqq -P_g(x; y) - P_g(x; z)$, hence $P_g(x; y+z) \geqq P_g(x;y) + P_g(x; z)$. But $P_g(x; y+z) = P_+(x; y+z) \leqq P_+(x; y) + P_+(x; z)$ by theorem 4.2.3. Therefore $P_g(x; y+z) = P_g(x; y) + P_g(x; z)$.

P-continuity:

By theorem 4.2.3, $P_g(x; y) \leqq P(y)$; and $P(y) \leqq B(y)$, where $B(y) \equiv \max\left\{P(y), P(-y)\right\}$ is the norm for the P-topology. The method of theorem 4.1.1 then shows immediately that $P_g(x; y)$ is P-continuous in y. This completes the proof of the theorem.

<u>Remark on Fréchet differential:</u> Theorem 4.2.8 does not prove that the existence of the Gâteaux differential of P(x) implies the existence of the Fréchet differential. For the latter it is necessary in addition that the approach to the limit be uniform with respect to y in some y-neighborhood of the origin.[1] In fact, Fréchet[2] has shown that in the space (C) of real-valued continuous functions of 3.6.6, pseudo-normed by $P(x) = \max_{0 \leqq t \leqq 1} x(t)$, the pseudo-norm nowhere possesses a Fréchet differential although there are points where its Gâteaux differential exists.

<u>4.2.9 Theorem.</u> A necessary and sufficient condition that the pseudo-norm be Gâteaux-differentiable at $x \in G$ is that $\lim_{u \to \infty} \left[P(ux+y+z) - P(ux+y)\right] = \lim_{u \to \infty} \left[P(ux+z) - P(ux)\right]$ for any y, z, in the space G.[3]

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1). See Mazur, Ref. 11, p 130, and, under much more general conditions, Michal, Ref. 14, p 412.
2). Fréchet, Ref. 4, pp 245-249.
3). This theorem was suggested by a similar result of James, Ref. 8, p 28 concerning the differentiability of a strong norm.

Proof. Necessity:

$$\lim_{u\to\infty}\left[P(ux+y+z)-P(ux+y)\right] = \lim_{h\to 0+} P(\tfrac{1}{h}x+y+z) - P(\tfrac{1}{h}x+y)$$

$$= \lim_{h\to 0+}\frac{P(x+h\,\overline{y+z}) - P(x+hy)}{h}$$

$$= \lim_{h\to 0+}\frac{P(x+h\,\overline{y+z}) - P(x) - \{P(x+hy)-P(x)\}}{h}$$

$= P_g(x;\ y+z) - P_g(x;\ y) = P_g(x;\ z)$ by linearity (theorem 4.2.8),

$$= \lim_{h\to 0+}\frac{P(x+hz)-P(x)}{h} = \lim_{u\to\infty} u\{P(x+z/u)-P(x)\}$$

$= \lim_{u\to\infty}\left[P(ux+z)-P(ux)\right]$. This proves the necessity.

Sufficiency:

Both limits appearing in the enunciation always exist since, just as in the proof of necessity,

$$\lim_{u\to\infty}\left[P(ux+y+z)-P(ux+y)\right]$$

$$= \lim_{h\to 0+}\left[\frac{P(x+h\,\overline{y+z})-P(x)}{h} - \frac{P(x+hy)-P(x)}{h}\right]$$

$= P_+(x;\ y+z) - P_+(x;\ y)$. Similarly,

$\lim_{u\to\infty}\left[P(ux+z)-P(ux)\right] = P_+(x;\ z)$. Hence the hypothesis becomes

$P_+(x;\ y+z) - P_+(x;\ y) = P_+(x;\ z)$; and this holds for all $y, z, \in G$.

Putting $z = -y$ we get

$P_+(x;\ \theta) - P_+(x;\ y) = P_+(x;\ -y)$, i.e., $P_+(x;\ y) = -P_+(x;\ -y)$. By lemma 4.2.7, this last relation is equivalent to the existence of the Gâteaux differential of the pseudo-norm at x. This proves the theorem.

Gâteaux Differential of P(x) at Zero.

It is easy to see that a strong norm cannot be Gâteaux differentiable at $x = \theta$, for

$$\lim_{h \to 0+} \frac{\|\theta + hy\| - \|\theta\|}{h} = \|y\| \text{ while}$$

$$\lim_{h \to 0-} \frac{\|\theta + hy\| - \|\theta\|}{h} = \lim_{h \to 0-} \frac{|h| \ \|y\|}{h} = -\|y\| \neq \|y\| \text{ for all } y.$$

However, the notion of pseudo-norm applies to a wider class of functionals than that of a strong norm, and under special conditions a pseudo-norm is differentiable at zero.

4.2.10. Theorem. $P(x)$ possesses a Gâteaux differential at θ if and only if $P(-y) = -P(y)$ for all y in the pseudo-normed vector space G.

Proof. Necessity:

$$P_g(\theta; y) = \lim_{h \to 0+} \frac{P(\theta + hy) - P(\theta)}{h} = P(y) \text{ and}$$

$$P_g(\theta; y) = \lim_{h \to 0-} \frac{P(\theta + hy) - P(\theta)}{h} = \lim_{h \to 0-} \frac{P\{(-h)(-y)\}}{h}$$

$$= \lim_{h \to 0-} \frac{-h\, P(-y)}{h} = -P(-y)$$

Thus $P(y) = -P(-y)$.

Sufficiency:

$P(-y) = -P(y) \Rightarrow P(-hy) = -P(hy)$ for $h > 0$. Hence

$\dfrac{P(-hy)}{-h} = \dfrac{-P(hy)}{-h} = \dfrac{P(hy)}{h}$ for $h > 0$. This implies that

$\dfrac{P(\theta - hy) - P(\theta)}{-h} = \dfrac{P(\theta + hy) - P(\theta)}{h}$, $h > 0$, therefore

$-\lim_{h \to 0+} \dfrac{P(\theta - hy) - P(\theta)}{h} = \lim_{h \to 0+} \dfrac{P(\theta + hy) - P(\theta)}{h}$, i.e.,

$-P_+(\theta; -y) = P_+(\theta; y)$, and this, by lemma 4.2.7, implies the existence of the Gâteaux differential at θ.

4.2.11. Theorem. If, in a pseudo-normed vector space G, the Gâteaux differential of $P(x)$ exists at zero, then it exists at all non-zero points and

$P_g(x; y) = P(y)$. Thus $P_g(x; y)$ is independent of x, and in particular $P_g(x; y) = P_g(-x; y)$.

Proof. By theorem 4.2.10, $P(-y) = -P(y)$.

For $h > 0$, $\frac{P(x + hy) - P(x)}{h} \leq \frac{P(hy)}{h} = P(y)$ and

$\frac{P(x + hy) - P(x)}{h} \geq \frac{P(x) - P(-hy) - P(x)}{h}$ by 1.2.10,

$\geq \frac{-P(-hy)}{h} = \frac{-hP(-y)}{h} = -P(-y) = P(y)$.

Hence $\lim_{h\to 0+} \frac{P(x + hy) - P(x)}{h} = P(y)$. Similarly

$\lim_{h\to 0+} \frac{P(x-hy) - P(x)}{h} = P(-y) = -P(y)$. The former result is

$P_+(x; y) = P(y)$ and the latter is $P_+(x; -y) = -P(y)$, i.e., $-P_+(x; -y) = P(y)$. Thus $P_+(x; y) = -P_+(x; -y) = P(y)$ and this implies, by lemma 4.2.7, that the Gâteaux differential $P_g(x; y) = P(y)$. This completes the proof.

4.2.12. Theorem. (A converse of Theorem 4.2.11.) If the Gâteaux differential of the pseudo-norm exists at all non-zero points x of a vector space G, and is such that $P_g(x; y) = P_g(-x; y)$ for all increments y, then the pseudo-norm is Gâteaux-differentiable at zero.

Proof. By the existence of the Gâteaux differential, $P_+(x; y) = P_g(x; y) = P_g(-x; y) = P_+(-x; y)$. Putting $y = x$, we get $P_+(x; x) = P_+(-x; x)$, i.e.,

$$\lim_{h\to 0+} \frac{P(x + hx) - P(x)}{h} = \lim_{h\to 0+} \frac{P(-x + hx) - P(-x)}{h}, \text{ i.e.,}$$

$$\lim_{h\to 0+} \frac{(1 + h)\, P(x) - P(x)}{h} = \lim_{h\to 0+} \frac{(1 - h)\, P(-x) - P(-x)}{h} \text{ since } 1 - h > 0.$$

This yields $P(x) = -P(-x)$. Since this holds for all $x \in G$, we obtain the existence of the Gâteaux differential at Θ by means of theorem 4.2.10. This concludes the proof.

4.2.13. Examples of $P_+(x; y)$ and $P_g(x; y)$ in Simple Cases.

4.2.13(a). Let G be the two dimensional vector space of real number couples, $x = (x_1, x_2)$ for $x \in G$, with $P(x) = \max(x_1, x_2)$.

The Gâteaux differential of P(x) does not exist at zero since $P(y) \neq -P(-y)$ for all $y \in G$ (theorem 4.2.10). Neither does $P_g(x; y)$ exist at $x = (1, 1)$. To see this, put $y = (y_1, y_2)$. Then

$$\lim_{h\to 0+} \frac{P(x+hy) - P(x)}{h} = \lim_{h\to 0+} \frac{1 + h\max(y_1, y_2) - 1}{h} = \max(y_1, y_2);$$

$$\text{but } \lim_{h\to 0-} \frac{P(x+hy) - P(x)}{h} = \lim_{h\to 0-} \frac{\max\{1+hy_1, 1+hy_2\} - 1}{h}$$

$$= \lim_{h\to 0-} \frac{\max\{hy_1, hy_2\}}{h} = \lim_{h\to 0-} \frac{h\min\{y_1, y_2\}}{h} = \min(y_1, y_2).$$

Since $\max(y_1, y_2) \neq \min(y_1, y_2)$ for all y, $P_g(x; y)$ does not exist at $x = (1, 1)$. In fact it is easily seen that it exists nowhere.

4.2.13(b). Let G be the two-dimensional vector space of real number couples with the almost trivial pseudo-norm $P(x) = x_2$ where $x = (x_1, x_2)$. In this case $P(-x) = -P(x)$ for all $x \in G$, so by theorems 4.2.10 and 4.2.11 the Gâteaux differential of P(x) exists everywhere and, in fact, $P_g(x; y) = P(y) = y_2$, where y is the element (y_1, y_2).

4.2.13(c). Let G be the vector space of real-valued functions of the form $x(t) = a(t^2 - t)$ defined on $0 \leq t \leq 1$. Define $P(x) = \max_{0\leq t\leq 1} a(t^2 - t)$.

$$\text{Then } P(x) = -a/4 \text{ if } a < 0,$$
$$= 0 \text{ if } a \geq 0.$$

Lemma. The Gâteaux differential of P(x) does not exist at zero.

Proof. Let $y = a(t^2 - t)$, $a > 0$. Then $P(\theta + hy) - P(\theta) = P\big(ha(t^2 - t)\big)$. Hence $P_+(\theta; y) = \lim_{h\to 0+} \frac{P\big(ha(t^2 - t)\big)}{h} = 0$ since $a > 0$. On the other hand, $P_-(\theta; y) = \lim_{h\to 0-} \frac{P\big(ha(t^2 - t)\big)}{h} = -a/4 \neq 0 = P_+(\theta; y)$. This establishes the lemma.

Lemma. The P-norm is Gâteaux-differentiable at $x = -(t^2 - t)$.

Proof. Let y be any element of G. Then $y = a(t^2 - t)$.

$$\frac{P(x+hy) - P(x)}{h} = \frac{P[(1-ha)(t - t^2)] - P[-(t^2 - t)]}{h}$$

$$= \frac{(1-ha)\,P(t-t^2) - \frac{1}{4}}{h} \quad \text{since } 1 - ha > 0 \text{ for } |h|$$

sufficiently small, $= \dfrac{(1-ha)\frac{1}{4} - \frac{1}{4}}{h} = -a/4.$

Therefore $\lim\limits_{h\to 0} \dfrac{P(x+hy) - P(x)}{h} = -a/4$ and exists for all increments y. This means that P(x) is Gâteaux-differentiable at $x = -(t^2 - t)$.

4.3. A Connection Between Gâteaux-Differentiability of the Pseudo-Norm and P-Linear Functionals.

4.3.1. Theorem. Let G be a pseudo-normed vector space and x_0 an element of G such that $P(x_0) > P(-x_0)$. Then, in order that the Gâteaux differential of the pseudo-norm should exist at x_0, it is necessary and sufficient that there exist one and only one P-linear functional F such that $F(x_0) = P(x_0)$ and $\boxed{F} = 1$ where $\boxed{F}$ is the functional modulus of F.[1)]

Proof. By 1.2.11, $P(x_0) > P(-x_0)$ implies $P(x_0) > 0$. Hence $B(x_0)$ $\left(\text{i.e., } \max\{P(x_0), P(-x_0)\}\right) = P(x_0)$.

Necessity:

Define $F(y) = P_g(x_0; y)$, the Gâteaux differential of P(x) at x_0 with increment y. Then F(y) is a P-linear functional of y by theorem 4.2.8, and $F(x_0) = P_g(x_0; x_0) = \lim\limits_{h\to 0} \dfrac{P(x_0 + hx_0) - P(x_0)}{h} = P(x_0)$. Moreover, by theorem 4.2.3, $F(y) = P_+(x_0; y) \leq P(y)$; hence $F(y) \leq \max\{P(y), P(-y)\} = B(y)$, so that $\boxed{F} \leq 1$. Since $F(x_0) = P(x_0) = B(x_0)$, $\boxed{F} = 1$. This establishes the

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1). See definition 3.3.6 and corollary 3.4.4 for explanations of "P-linear functional" and "functional modulus".
A similar theorem concerning the Gâteaux differential of a strong norm is stated by Mazur, Ref. 11, p 130.

existence of one such functional F. To prove its uniqueness, we need the following lemma which is the analogue of a result obtained by Mazur, James, and McShane,[1] for the case of a strong norm.

<u>Lemma.</u> If $P(x_0) > P(-x_0)$, and $F(x)$ is a P-linear functional on a pseudo-normed vector space G of functional modulus $\boxed{F} = 1$ such that $F(x_0) = P(x_0)$, then $-P_+(x_0; -y) \leq F(y) \leq P_+(x_0; y)$.

<u>Proof.</u> Our method is an adaptation of that used by McShane.[2] For any real number a and any $y \in G$, the homogeneity and additivity of F give $F\left\{x_0 + a\left(y - \frac{F(y)}{P(x_0)} x_0\right)\right\} = F(x_0) + aF(y) - \frac{aF(y)}{P(x_0)} F(x_0) = F(x_0)$, since $F(x_0) = P(x_0)$. Since F is P-linear,

$$F\left\{x_0 + a\left(y - \frac{F(y)}{P(x_0)} x_0\right)\right\} \leq \boxed{F}\, B\left\{x_0 + a\left(y - \frac{F(y)}{P(x_0)} x_0\right)\right\}.$$

Since $\boxed{F} = 1$, $B\left\{x_0 + a\left(y - \frac{F(y)}{P(x_0)} x_0\right)\right\} \geq P(x_0)$ for all real a.

For $a = \dfrac{t}{1 + t\dfrac{F(y)}{P(x_0)}}$, t real, this becomes

$$B\left\{x_0 + \frac{t}{1 + t\frac{F(y)}{P(x_0)}}\left(y - \frac{F(y)}{P(x_0)} x_0\right)\right\} \geq P(x_0).$$

For $|t|$ sufficiently small, $1 + t\frac{F(y)}{P(x_0)} > 0$, hence the identity

$$x_0 + ty = \left(1 + t\frac{F(y)}{P(x_0)}\right)\left\{x_0 + \frac{t}{1 + t\frac{F(y)}{P(x_0)}}\left(y - \frac{F(y)}{P(x_0)} x_0\right)\right\} \text{ yields}$$

$$B(x_0 + ty) = \left(1 + t\frac{F(y)}{P(x_0)}\right) B\left\{x_0 + \frac{t}{1 + t\frac{F(y)}{P(x_0)}}\left(y - \frac{F(y)}{P(x_0)} x_0\right)\right\}$$

$$\geq \left(1 + t\frac{F(y)}{P(x_0)}\right) P(x_0) = P(x_0) + t\,F(y).$$

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1). Mazur, Ref. 10, p 75; James, Ref. 8, p 102; McShane, Ref. 12, p 402.
2). McShane, ibid.

Since $B(x_0 + ty) = \max\{P(x_0 + ty), P(-x_0 - ty)\}$, since $P(x_0) > P(-x_0)$, and since $P(x_0 + ty)$ is a continuous function of t (theorem 4.1.3), it follows that, for $|t|$ sufficiently small, $B(x_0 + ty) = P(x_0 + ty)$. Under this condition the above relation becomes $P(x_0 + ty) - P(x_0) \geqq t\, F(y)$.

Hence, $F(y) \leqq \lim_{t \to 0+} \dfrac{P(x_0 + ty) - P(x_0)}{t} = P_+(x_0;\, y)$

and $F(y) \geqq \lim_{t \to 0-} \dfrac{P(x_0 + ty) - P(x_0)}{t} = -P_+(x_0;\, -y)$ by lemma 4.2.6.

Thus $-P_+(x_0;\, -y) \leqq F(y) \leqq P_+(x_0;\, y)$, which proves the lemma.

Uniqueness of $F(y)$ when $P_g(x_0;\, y)$ Exists.

By the above lemma, any P-linear functional F with $F(x_0) = P(x_0)$ and $\boxed{F} = 1$ must satisfy the inequality $-P_+(x_0;\, -y) \leqq F(y) \leqq P_+(x_0;\, y)$. The existence of $P_g(x_0;\, y)$ implies that these three members are all equal to $P_g(x_0;\, y)$. Hence F is unique. This completes the proof of necessity for theorem 4.3.1.

Sufficiency: We use the following result of Mazur.[1)] If x_0 and z are distinct elements of a linear space G pseudo-normed by $P(x)$, and a is a real number such that $-P_+(x_0;\, -z) \leqq a \leqq P_+(x_0;\, z)$, there exists in G an additive and homogeneous functional $F(x)$ such that $F(x_0) = P(x_0)$, $F(z) = a$, and $F(x) \leqq P(x)$ for $x \in G$.

It is to be noted that no question of the continuity of $F(x)$ is involved explicitly in this result. Suppose $P(x)$ is not Gâteaux-differentiable at x_0. Then there exists at least one element $y_0 \in G$ such that $-P_+(x_0;\, -y_0) \neq P_+(x_0;\, y_0)$. Corollary 4.2.4 justifies the application of consequence 1.2.11 to $P_+(x_0;\, y)$. This yields the inequality $-P_+(x_0;\, -y_0) < P_+(x_0;\, y_0)$. Hence there exist distinct real numbers a, b such that $-P_+(x_0;\, -y_0) \leqq a < b \leqq P_+(x_0;\, y_0)$. We now apply the above result of Mazur with z replaced by y_0

1). Mazur, Ref. 10, p 75.

and obtain the existence of two additive homogeneous functionals in G, F(x) and H(x), such that $F(x_0) = H(x_0) = P(x_0)$; $F(y_0) = a$, $H(y_0) = b$; $F(x) \leq P(x)$ and $H(x) \leq P(x)$ for all $x \in G$. Moreover this last implies that both F(x) and H(x) are P-continuous, for $F(x) \leq P(x) \leq B(x) = M\,B(x)$ where $M = 1$. Since an additive and homogeneous functional F(x) satisfies the pseudo-norm postulates 1.1.3(a), 1.1.3(b), 1.1.3(c), and B(x) is the norm for the P-topology, the methods of theorems 1.3.6 and 4.1.1 show that F(x) is P-continuous. Also $\boxed{F} = 1$ since $F(x_0) = P(x_0) = B(x_0)$. As a similar result applies to H(x), we have two P-linear functionals F(x) and H(x) such that $F(x_0) = H(x_0) = P(x_0)$, and $\boxed{F} = \boxed{H} = 1$. This is contrary to the hypothesis of only one such functional. Hence P(x) is Gâteaux-differentiable at x_0. This completes the proof of sufficiency and, consequently, of the theorem.

CHAPTER V

The Pseudo-Norm and Ordering Relations

5.1. Ordering Relations Based on a Given Pseudo-Norm.

5.1.1. Definition of Quasi-Ordering Relation.[1)]

Let X be a set of elements between certain pairs of which there is defined a relation $\leq$. $\leq$ is termed a quasi-ordering relation if it satisfies

O1: For all $x \in X$, $x \leq x$. (reflexive)

O2: If $x \in X$, $y \in X$, $z \in X$, $x \leq y$, $y \leq z$, then $x \leq z$. (transitive)

5.1.2. Theorem. If G is a real vector space in which is defined a pseudo-norm $P(x)$, $x \in G$, there exists in G a quasi-ordering relation $\leq$ with the following properties:

1. $x \in G$, $y \in G$, $x \leq y$, $a > 0$, $\Rightarrow ax \leq ay$.
2. $x \in G$, $y \in G$, $x \leq y$, $\Rightarrow -y \leq -x$.
3. $x_1 \in G$, $x_2 \in G$, $y_1 \in G$, $y_2 \in G$, $x_1 \leq y_1$, $x_2 \leq y_2$, $\Rightarrow x_1 + x_2 \leq y_1 + y_2$.
4. $x \in G$, θ = the zero of G, $x \leq \theta$, a and b real, $a > b$, $\Rightarrow ax \leq bx$.
5. $x \leq \theta$, $a > 1$, $\Rightarrow ax \leq x$.
6. If y_0 is an element of G such that $P(-y_0) < 0$, then for any $x \in G$ there exists a real number a depending on x, $a(x)$, such that $x \leq ay_0$.
7. Archimedean Property. If $P(y) \geq 0$ and $x \leq \varepsilon y$ for all $\varepsilon > 0$, then $x \leq \theta$, the zero of G.

Proof. We define $x \leq y$ to mean $P(x - y) \leq 0$. Then O1 is satisfied since $P(x - x) = P(\theta) = 0 \leq 0$ for all $x \in G$. To prove O2, let $x \leq y$ and $y \leq z$, i.e., $P(x - y) \leq 0$ and $P(y - z) \leq 0$. Then $P(x - z) = P(x - y + y - z)$

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1). See Birkhoff, Ref. 3, p 4.

$\leq P(x - y) + P(y - z) \leq 0$. Hence $x \leq z$. This proves 02, so by definition 5.1.1 G is quasi-ordered.

Proof of Properties 1 - 7.

1. $P(ax - ay) = aP(x - y)$ since $a > 0$, ≤ 0 since $x \leq y$. Therefore $ax \leq ay$.

2. $P\big(-y - (-x)\big) = P(x - y) \leq 0$ since $x \leq y$. Therefore $-y \leq -x$.

3. $P(\overline{x_1 + x_2} - \overline{y_1 + y_2}) \leq P(x_1 - y_1) + P(x_2 - y_2) \leq 0$ since $x_1 \leq y_1$ and $x_2 \leq y_2$. Therefore $x_1 + x_2 \leq y_1 + y_2$.

4. $P(ax - bx) = (a - b)\, P(x)$ since $a > b$, ≤ 0 since $x \leq \theta$. Therefore $ax \leq bx$.

5. $P(ax - x) = (a - 1)\, P(x)$ since $a - 1 > 0$, ≤ 0 since $x = \theta$, i.e., $ax \leq x$.

6. There are two cases to be considered.

   First Case: $P(x) > 0$.

   Then $P(x - ay_0) \leq P(x) + a\, P(-y_0) \leq 0$ for $a \geq \dfrac{P(x)}{|P(-y_0)|}$,

   Therefore $x \leq ay_0$.

   Second Case: $P(x) \leq 0$.

   Then $P(x - ay_0) \leq P(x) + a\, P(-y_0) \leq 0$ for any $a \geq 0$,

   i.e., $x \leq ay_0$.

7. Archimedean property. $x \leq \varepsilon y$ for all $\varepsilon > 0 \Rightarrow P(x - \varepsilon y) \leq 0$ for all $\varepsilon > 0$, $\Rightarrow 0 \geq P(x) - \varepsilon P(y)$ by consequence 1.2.10. Hence, for any $\eta > 0$, $P(x) \leq \varepsilon P(y) < \eta$ by suitable choice of ε. Therefore $P(x) \leq 0$, $P(x - \theta) \leq 0$, and $x \leq \theta$.

5.1.3. Definition of Partial Ordering.

A set X is said to be partially ordered if there exists a binary relation $\leq$ defined for certain elements of X that satisfies the following postulates.

O1: For all $x \in X$, $x \leq x$. (reflexive)

O2: $x \in X$, $y \in X$, $z \in X$, $x \leq y$, $y \leq z$, $\Rightarrow x \leq z$. (transitive)

O3: $x, y, \in X$, $x \leq y$, $y \leq x$, $\Rightarrow x = y$. (antisymmetric)

That is, a partial ordering is a quasi-ordering which satisfies the additional postulate O3.

5.1.4. Theorem. The pseudo-normed vector space G/Z,[1] whose elements are the cosets of the subgroup $Z = \{x \in G \mid P(x) = P(-x) = 0\}$, is partially ordered by the defining relation $Q_1 \leq Q_2 \equiv P(Q_1 - Q_2) \leq 0$ where Q_1, Q_2 are elements of G/Z. Also this partial ordering satisfies properties 1 - 7 of theorem 5.1.2.

Proof. Since G/Z is a pseudo-normed vector space, O1, O2, and properties 1 - 7 of theorem 5.1.2 are satisfied. It remains to prove O3. So suppose $Q_1 \leq Q_2$ and $Q_2 \leq Q_1$. Then $P(Q_1 - Q_2) \leq 0$ and $P(Q_2 - Q_1) \leq 0$. By 1.2.11, $P(Q_1 - Q_2) \geq -P(Q_2 - Q_1) \geq 0$. Therefore $0 \leq P(Q_1 - Q_2) \leq 0$, whence $P(Q_1 - Q_2) = 0$. Similarly $P(Q_2 - Q_1) = 0$. Hence $Q_1 - Q_2 = Z$, the zero element of G/Z. Therefore $Q_1 = Q_2$. This completes the proof.

We note without proof that the partial ordering of theorem 5.1.4 can be obtained from the quasi-ordering of theorem 5.12 as a special case of a result of Schröder quoted by Birkhoff in his book "Lattice Theory".[2] Birkhoff shows that any quasi-ordering can be made into a partial ordering by means of an equivalence relation.

In future cases where a partial ordering is called for we shall drop the notation G/Z and speak of a pseudo-normed vector space G in which $P(x) = P(-x) = 0$ implies that $x = \theta$.

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1). For the definition of G/Z, see, inter alia, sections 2.2 and 3.1.
2). Birkhoff, Ref. 3, p 4.

The Moore-Smith Property.

5.1.5. Definition of Moore-Smith Property.[1)]

A set X with a (quasi- or partial) ordering relation $\leq$ possesses the Moore-Smith property if, for any $x, y, \in X$, there exists an element $z \in X$ such that $x \leq z$ and $y \leq z$.

5.1.6. Theorem. Let G be a pseudo-normed vector space in which $x \leq y$ is defined to mean $P(x-y) \leq 0$ as in theorem 5.1.2. If in G there exists an element y_o such that $P(-y_o) < 0$, then the set G under the relation $\leq$ possesses the Moore-Smith property.

Proof. By property 6 of theorem 5.1.2 there exist positive numbers a, b such that $x \leq ay_o$ and $y \leq by_o$ for any $x, y, \in G$. Put $c = \max(a, b)$. Since $\theta \leq y_o$, properties 1 and 3 of theorem 5.1.2 give $ay_o \leq ay_o + (c-a)y_o \leq cy_o$, and similarly $by_o \leq cy_o$. Hence $z \equiv cy_o$ is such that $x \leq z$ and $y \leq z$. This proves the theorem.

5.2. Definition of a Pseudo-Norm in Terms of a Given Ordering Relation.

5.2.1. Theorem. Let E be a real vector space with a given partial ordering relation (01, 02, 03, of 5.1.3) denoted by $\leq$ and satisfying the following conditions:

(i) $x, y, \in E$, $a > 0$, $x \leq y$, $\Rightarrow ax \leq ay$;

(ii) $x_1 \leq y_1$, $x_2 \leq y_2$, $\Rightarrow x_1 + x_2 \leq y_1 + y_2$;

(iii) There exists in E an element $e \neq \theta$ such that $\theta \leq e$ and such that for any $x \in E$ there exists a positive real number a depending on x such that $x \leq a.e.$

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1). See Birkhoff, Ref. 3, p xi. This concept is due to E. H. Moore, Proc. Nat. Acad. Sci. (1915), pp 628-632.

Then a functional $\pi(x)$ can be defined on the space E with the properties 1.1.3(a), 1.1.3(b), 1.1.3(c) of a pseudo-norm.

Proof. The proof makes use of the following lemma.

5.2.2. Lemma. Under the hypotheses of theorem 5.2.1, $x \leq y \Rightarrow -y \leq -x$.

Proof. By O1, $-x - y \leq -x - y$. Since $x \leq y$, hypothesis (ii) gives $x - x - y \leq y - x - y$, i.e., $-y \leq -x$. This establishes the lemma.

To prove the theorem, define $\pi(x) = \inf_{x \leq s.e} s$, $x \in E$.

Proof of 1.1.3(a) that $\pi(x)$ is a real number[1] not identically zero.

First, $\pi(e) \neq 0$ for, if $\pi(e) = 0$, by the definition of inf there would exist, for any $\mathcal{E} > 0$, an $\mathcal{E}_1$ such that $0 \leq \mathcal{E}_1 < \mathcal{E}$ and $e \leq \mathcal{E}_1 e$. Now $-\mathcal{E}_1 e \leq -\mathcal{E}_1 e$, hence, by hypothesis (ii), $e - \mathcal{E}_1 e \leq \theta$, that is, $(1-\mathcal{E}_1)e \leq \theta$, hence $e \leq \theta$ by (i), on multiplying by $1/(1-\mathcal{E}_1)$, which is positive for $\mathcal{E} < 1$. But $e \leq \theta$ is contrary to hypothesis, so that $\pi(e) \neq 0$.[2]

The possibility that $\pi(x) = +\infty$ for some x is excluded by hypothesis (iii). Finally we proceed to show that $\pi(x) \neq -\infty$. Suppose x is such that $\inf_{x \leq se} s = -\infty$. This means that for any $A > 0$, there exists $A' > A$ such that $x \leq -A'e$. Hence, by lemma 5.2.2, $A'e \leq -x$. Now choose A as the positive number of hypothesis (iii) applied to $-x$. Then $-x \leq Ae$. By (i), $\theta \leq (A'-A)e$, hence $-x + \theta \leq Ae + (A'-A)e$ by (ii), i.e., $-x \leq A'e$. O3 applied to the results $A'e \leq -x$ and $-x \leq A'e$ gives $x = -A'e$. From this it follows that $\pi(x)$ cannot be less than $-A'$ for, if it were, there would exist $b > 0$ such that $x \leq (-A'-b)e$ and so $A'e + b.e \leq -x$ by lemma 5.2.2. The above result that $-A'e = x$ then yields $b.e \leq \theta$ by (ii). This is contrary to hypothesis (iii) since $\theta \leq e$ and $b > 0$. Hence there is no $x \in E$ for which $\pi(x) = -\infty$.

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1). We are excluding ∞ and $-\infty$ as real numbers.
2). The method of proof can easily be adapted to show that, in fact, $\pi(e) = 1$.

Proof of 1.1.3(c) that $\pi(tx) = t\pi(x)$ for $t > 0$.

$$\pi(tx) = \inf_{tx \leq s.e} s = \inf_{x \leq \frac{s}{t}e} s \text{ by hypothesis (i), } = \inf_{x \leq r.e} tr$$

$$= t \inf_{x \leq r.e} r = t\pi(x).$$

Proof of 1.1.3(b) that $\pi(x+y) \leq \pi(x) + \pi(y)$.

Let $\pi(x) \equiv \inf_{x \leq s.e} s = b$, $\pi(y) \equiv \inf_{y \leq s.e} s = c$. By the definition of π, for any $\varepsilon > 0$ there exist numbers b', c' such that $b \leq b' < b + \varepsilon$, $c \leq c' < c + \varepsilon$, and such that $x \leq b'.e$, $y \leq c'.e$. Then $x + y \leq (b' + c')e$ by hypothesis (ii). Therefore $\inf_{x+y \leq s.e} s \leq b' + c' < b + c + 2\varepsilon$. Since ε is an arbitrary positive number, this implies $\inf_{x+y \leq s.e} s \leq b + c$, i.e., $\pi(x+y) \leq \pi(x) + \pi(y)$. This completes the proof of theorem 5.2.1.

5.2.3. Corollary. If, in addition to the hypotheses of theorem 5.2.1, the partial ordering in E possesses the Archimedean property that $x \leq \varepsilon e$ for all $\varepsilon > 0$ implies $x = \theta$, then the pseudo-norm $\pi(x)$ is such that $\pi(x) = \pi(-x) = 0$ implies $x = \theta$.

Proof. If $\pi(x) = 0$, $\inf_{x \leq s.e} s = 0$. Hence for any $\varepsilon > 0$ there exists ε_1 such that $0 \leq \varepsilon_1 < \varepsilon$ and $x \leq \varepsilon_1 e$. Since $\theta \leq (\varepsilon - \varepsilon_1)e$, this gives $x \leq \varepsilon_1 e + (\varepsilon - \varepsilon_1)e$ by hypothesis (ii), i.e., $x \leq \varepsilon e$. This being true for all $\varepsilon > 0$, the Archimedean property implies $x \leq \theta$. Similarly $\pi(-x) = 0$ leads to $-x \leq \theta$, and hence $\theta \leq x$ by lemma 5.2.2. Therefore $x = \theta$ by O3. This proves the corollary.

5.2.4. Corollary. Let E be a real vector space with a partial ordering relation $\leq$ under the hypotheses (i), (ii), and (iii) of theorem 5.2.1. Then the set E under the ordering $\leq$ possesses the Moore-Smith property (5.1.5).

Proof. The proof proceeds exactly as in the case of theorem 5.1.6, the element e of hypothesis (iii) of theorem 5.2.1 being used in place of y_o.

5.3. Connection Between Set-Ups of Sections 5.1 and 5.2.

Let us summarize certain aspects of theorems 5.1.4 and 5.2.1. In theorem 5.1.4 a given pseudo-norm in a vector space leads to a partial ordering, while in theorem 5.2.1 a given partial ordering in a vector space leads to the existence of a pseudo-norm. Thus we may start with a given pseudo-norm $P(x)$ and obtain a partial ordering $\leq$ in terms of which we can define another pseudo-norm $\pi(x)$ as in theorem 5.2.1. The question arises how the P-topology (Chapter III) based on $P(x)$ will compare with the P-topology based on $\pi(x)$. Conversely, being given a partial ordering $\leq$, we can define a pseudo-norm $\pi(x)$ as in theorem 5.2.1, then define a new partial ordering $\underset{\sim}{\leq}$ based on $\pi(x)$ as in theorem 5.1.4. What connection, if any, will exist between the two partial orderings?[1] The rest of this chapter attempts to answer these questions.

5.3.1. Lemma. Enunciation of Lemma 5.3.1.

Let G be a pseudo-normed vector space in which $P(x) = P(-x) = 0$ implies $x = \theta$ and in which there exists an element y_0 such that $P(-y_0) < 0$. Define $x \leq y$ to mean $P(x-y) \leq 0$. By theorem 5.1.4, $x \leq y$ $(x, y, \in G)$ is a partial ordering in G satisfying hypotheses (i), (ii), (iii), of theorem 5.2.1, and the hypothesis of corollary 5.2.3, the element y_0 taking the place of the element e. Hence, by theorem 5.2.1 and corollary 5.2.3, $\pi(x) = \inf_{x \leq sy_0} s$ is a functional defined on G with the properties of a pseudo-norm such that $\pi(x) = \pi(-x) = 0$ implies $x = \theta$. So $\max\{\pi(x), \pi(-x)\}$ is a strong norm in G (theorem 3.1.9).

Then, under these hypotheses, the following results hold:

(1) If $\pi(x) = 0$, $P(x) = 0$.

(2) If $\pi(x) > 0$, $P(x)/P(y_0) \leq \pi(x) \leq P(x)/-P(-y_0)$.

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1). These ideas are largely adapted from a paper by M. H. Stone, Ref. 17.

(3) If $\pi(x) < 0$, $P(x)/-P(-y_o) \leq \pi(x) \leq P(x)/P(y_o)$.

This completes the enunciation of the lemma.

<u>Proof of Lemma 5.3.1.</u>

Since $\pi(x) = \inf_{x \leq sy_o} s$, for arbitrary $\varepsilon > 0$ there exists a real number a such that $\pi(x) \leq a < \pi(x) + \varepsilon$ and $x \leq ay_o$.

(1) <u>$\pi(x) = 0$.</u> Then $0 \leq a < \varepsilon$ and $x \leq ay_o$. This means that $P(x - ay_o) \leq 0$, hence $0 \geq P(x) - aP(y_o)$, i.e., $P(x) \leq aP(y_o)$. Since ε is arbitrary and $0 \leq a < \varepsilon$, and $P(y_o) \geq -P(-y_o) > 0$, it follows that $P(x) \leq 0$. Also $\inf_{x \leq sy_o} s = 0 \Rightarrow x \not\leq -\varepsilon y_o \Rightarrow P(x + \varepsilon y_o) \not\leq 0 \Rightarrow P(x + \varepsilon y_o) > 0$. Therefore $0 < P(x) + \varepsilon P(y_o)$, $P(x) > -\varepsilon P(y_o)$, hence $P(x) \geq 0$ since ε is arbitrary and $P(y_o) > 0$. Then $P(x) = 0$ by the above opposite inequality.

(2) <u>$\pi(x) > 0$.</u> As above, $\pi(x) \leq a < \pi(x) + \varepsilon$ and $x \leq ay_o$. Hence $0 \geq P(x - ay_o) \geq P(x) - aP(y_o)$ since $a > 0$. Therefore $a \geq P(x)/P(y_o)$. Since $\pi(x) \leq a < \pi(x) + \varepsilon$ for arbitrary positive ε, this last result implies that $\pi(x) \geq P(x)/P(y_o)$. In order to prove the remaining half of the inequality, we note that $\pi(x) \leq a < \pi(x) + \varepsilon$ implies $a - \varepsilon < \pi(x)$. Choose ε so that $0 < \varepsilon < \pi(x)$. Then $a - \varepsilon > 0$. Now $a - \varepsilon < \pi(x) \Rightarrow x \not\leq (a-\varepsilon)y_o \Rightarrow P(x - \overline{a-\varepsilon}\ y_o) \not\leq 0 \Rightarrow 0 < P(x - \overline{a-\varepsilon}\ y_o) < P(x) + (a-\varepsilon)P(-y_o)$ since $a - \varepsilon > 0$. Hence $-(a-\varepsilon)P(-y_o) < P(x)$ and therefore $a - \varepsilon < P(x)/-P(-y_o)$ since, by hypothesis, $P(-y_o) < 0$. Since the inequalities $a - \varepsilon < P(x)/-P(-y_o)$ and $a - \varepsilon < \pi(x) \leq a < \pi(x) + \varepsilon$ hold for arbitrary small ε, $\pi(x) \leq P(x)/-P(-y_o)$. This is the second half of the required relation, and completes the proof of case (2) of lemma 5.3.1.

(3) <u>$\pi(x) < 0$.</u> Choose positive ε so that $\pi(x) \leq a < \pi(x) + \varepsilon < 0$,

hence $a < 0$. As in the previous cases, $0 \geqq P(x - ay_o) \geqq P(x) - P(ay_o) \geqq P(x) + aP(-y_o)$ since $a < 0$. Hence $-aP(-y_o) \geqq P(x)$, $a \geqq P(x)/-P(-y_o)$. Since this is true for arbitrary small positive ε, $\pi(x) \geqq P(x)/-P(-y_o)$. For the second part, we have $a - \varepsilon < \pi(x) < 0$. Hence $x \not\leq (a - \varepsilon)y_o$. Therefore $P(x - \overline{a-\varepsilon}y_o) > 0$, $0 < P(x) + P(-\overline{a-\varepsilon}y_o) = P(x) - (a-\varepsilon)P(y_o)$, i.e., $(a-\varepsilon)P(y_o) < P(x)$, $a - \varepsilon < P(x)/P(y_o)$. Since ε can be arbitrarily small, this gives $\pi(x) \leqq P(x)/P(y_o)$. This completes the proof of lemma 5.3.1.

5.3.2. Theorem. Under the hypotheses of lemma 5.3.1, the (strong) P-topology induced by $\pi(x)$ is equivalent to the (strong) P-topology induced by the original pseudo-norm, $P(x)$.

Proof. We use the Hausdorff criterion and show that, if $N_P(x)$ is a given neighborhood in the topology of $P(x)$, there exists a neighborhood $N_\pi(x)$ of the $\pi(x)$ topology contained in $N_P(x)$, and conversely.

Let $N_P(x) = \{x \mid \max [P(x), P(-x)]\} < \varepsilon$.

Define $N_\pi(x) = \{x \mid \max [\pi(x), \pi(-x)] < \varepsilon/P(y_o)$.

To prove $N_\pi(x) \subset N_P(x)$: From lemma 5.3.1 (2), $\pi(x) > 0$ implies $P(x) > 0$ and $P(x) \leqq \pi(x)P(y_o)$. Also lemma 5.3.1 (3) shows that $\pi(x) < 0$ implies $P(x) < 0$ and $P(x) \geqq \pi(x)P(y_o)$, i.e., $|P(x)| = -P(x) \leqq -\pi(x)P(y_o) = |\pi(x)|P(y_o)$. Also $\pi(x) = 0 \Rightarrow P(x) = 0$ by 5.3.1 (1), hence, for all x, $|P(x)| \leqq |\pi(x)| P(y_o)$.

By lemma 3.1.8, $\max \{\pi(x), \pi(-x)\} < \eta$ implies $|\pi(x)| < \eta$ and $|\pi(-x)| < \eta$. If $x \in N_\pi(x)$, $|P(x)| \leqq |\pi(x)|P(y_o) < \varepsilon$ since $\eta = \varepsilon/P(y_o)$. Similarly $|P(-x)| < \varepsilon$. Therefore $\max \{P(x), P(-x)\} < \varepsilon$. Hence $x \in N_P(x)$ and $N_\pi(x) \subset N_P(x)$.

Conversely, let $N_{\pi}(x) = \{x \mid \max(\pi(x), \pi(-x)) < \varepsilon\}$. Define $N_P(x) = \{x \mid \max(P(x), P(-x)) < -\varepsilon P(-y_0)\}$.

To prove that $N_P(x) \subset N_{\pi}(x)$.

Again we use lemma 5.3.1. When $\pi(x) > 0$, $\pi(x) \leq P(x)/-P(-y_0)$; for $\pi(x) < 0$, $-\pi(x) \leq -P(x)/-P(-y_0)$; for $\pi(x) = 0$, $P(x) = 0$. So in all cases $|\pi(x)| \leq |P(x)|/-P(-y_0)$. If $x \in N_P(x)$, $|P(x)| < -\varepsilon P(-y_0)$ and $|P(-x)| < -\varepsilon P(-y_0)$ by lemma 3.1.8, hence $|\pi(x)| < \varepsilon$ and $|\pi(-x)| < \varepsilon$. Therefore $\max\{\pi(x), \pi(-x)\} < \varepsilon$, which means that $x \in N_{\pi}(x)$. Thus $N_P(x) \subset N_{\pi}(x)$. This completes the proof of theorem 5.3.2.

We now consider the following set-up. A given partially ordered vector space E allows a pseudo-norm $\pi(x)$ to be defined in it (theorem 5.2.1). Then a second ordering relation $\leq$ is defined in terms of $\pi(x)$. We seek to compare the two ordering relations in E.

5.3.3. Lemma. Let E be a real vector space partially ordered by means of the relation $\leq$ (01, 02, 03 of 5.1.3) which satisfies the hypotheses of theorem 5.2.1. By theorem 5.2.1, $\pi(x) \equiv \inf_{x \leq s \cdot e} s$ is a pseudo-norm in E. Then $x \leq y$ implies $\pi(x) \leq \pi(y)$.

Proof. Suppose $\pi(x) > \pi(y)$. Choose $\varepsilon > 0$ such that $\pi(y) + \varepsilon < \pi(x)$. By the definition of π, there exist real numbers a, b such that $\pi(y) \leq b < \pi(y) + \varepsilon < \pi(x) \leq a < \pi(x) + \varepsilon$, $y \leq b.e$, $x \not\leq b.e$. But $x \leq y$ and $y \leq b.e$ implies $x \leq b.e$, a contradiction. Hence $\pi(x) \leq \pi(y)$.

5.3.4. Corollary. Under the hypotheses of lemma 5.3.3, $x \leq y$ implies $\pi(x - y) \leq 0$.

Proof. $x \leq y \Rightarrow x - y \leq \theta$ by hypothesis (ii) of theorem 5.2.1, since $-y \leq -y$ by 01. Hence $\pi(x - y) \leq \pi(\theta)$ by lemma 5.3.3. Since π is a pseudo-norm, $\pi(\theta) = 0$, hence $\pi(x - y) \leq 0$.

5.3.5. Lemma. Let E be a real vector space with a partial ordering relation $\leqq$ satisfying the hypotheses of theorem 5.2.1, and let $\pi(x) \equiv \inf_{x \leqq s.e} s$ be the pseudo-norm of theorem 5.2.1. Then $\pi(x) \leqq 0 \Rightarrow x \leqq \mathcal{E}e$ for all $\mathcal{E} > 0$.

Proof. From the definition of $\pi(x) = \inf_{x \leqq s.e} s$, $\pi(x) \leqq 0$ implies that for any $\mathcal{E} > 0$ there exists a real number a such that $\pi(x) \leqq a < \mathcal{E}$ and $x \leqq a.e$. Since $\theta \leqq e$ and $\mathcal{E} - a > 0$, hypothesis (i) of theorem 5.2.1 gives $\theta \leqq (\mathcal{E} - a)e$, whence hypothesis (ii) of theorem 5.2.1 gives $x \leqq \mathcal{E}e$ since $x \leqq a.e$. This proves the lemma.

5.3.6. Theorem: Let $\leqq$ be a given partial ordering relation in a real vector space E such that the hypotheses of theorem 5.2.1 are satisfied, and let $\pi(x)$, $x \in E$, be the consequent pseudo-norm of 5.2.1 and lemma 5.3.3. Suppose, in addition, that the given partial ordering $\leqq$ satisfies the Archimedean property of corollary 5.2.3. Define a new binary relation $\precsim$ in E as follows: $x \precsim y$ if and only if $\pi(x-y) \leqq 0$ for x, y, $\in$ E. Then, under the above hypotheses, $x \leqq y$ if and only if $x \precsim y$, where x, y, $\in$ E.

Proof. Necessity: $x \leqq y \Rightarrow x \precsim y$.

By corollary 5.3.4, $x \leqq y \Rightarrow \pi(x-y) \leqq 0 \Rightarrow x \precsim y$ by definition.

Sufficiency: $x \precsim y \Rightarrow x \leqq y$.

$x \precsim y \Rightarrow \pi(x-y) \leqq 0$ by definition. Therefore $x - y \leqq \mathcal{E}e$ for all $\mathcal{E} > 0$, by lemma 5.3.5. Then, by the Archimedean property, $x - y \leqq \theta$, i.e., $x \leqq y$. This completes the proof.

5.3.7. Corollary. Under the hypotheses of theorem 5.3.6, the binary relation $\precsim$ is a partial ordering.

Proof. The relation $\leqq$ is a partial ordering, and theorem 5.3.6 states that the relation $\precsim$ is logically identical with $\leqq$.

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