HYDRODYNAMIC FORCES INDUCED IN FLUID CONTAINERS BY EARTHQUAKES

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Civil Engineer

California Institute of Technology

Pasadena, California

1956

ACKNOWLEDGMENT

The author wishes to thank Professor George W. Housner for his guidance and help which proved invaluable during the course of this investigation. The suggestions of Professor D. E. Hudson are also gratefully acknowledged.

Special thanks are extended to Professor T. K. Caughey for his help in setting up and checking out the analog computer circuit.

ABSTRACT

This thesis describes an experimental method of measuring the hydrodynamic forces induced in fluid containers when the containers are subjected to varying accelerations. Experiments were performed on a model of a rectangular tank and the forces due to the convective pressures were measured and compared to those predicted by a simplified linear theory in which the fluid system is reduced to a system of simple oscillators. An investigation was made to determine the limit of applicability of this simplified theory and to observe the effect of the non-linearity of the fluid motion on the predicted forces. It was found that the natural frequency of vibration diverged significantly from that for small amplitudes when the amplitude to depth ratio exceeded 0.05. The force-amplitude relation remained essentially constant for amplitude to depth ratios up to 0.20.

An electric analog computer was used to compute the dynamic shear forces induced in a typical water tower structure when its base was subjected to the ground accelerations of an actual earthquake. The effect of varying the tower stiffness, on the shear forces induced, was also studied and the results were compared with a modal type of solution. It was found that the modal type of solution gave good results and it is recommended for analyzing earthquake forces on elevated water tanks when an analog computer is not available.

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I. INTRODUCTION

The problem dealt with in this thesis is one arising from the need for a simplified method of determining the nature and the magnitude of the forces exerted by a confined fluid when its container is given horizontal accelerations. It is the concern of the structural engineer to know these forces in order that he may be able to design a structure, such as an elevated water tower, to withstand loads induced by earthquakes. The problem arises in the design of oil tanks, aquariums, and many other structures which confine fluids in regions where earthquakes occur. The importance of the problem is indicated by the case of the elevated water tower, whose failure during an earthquake could imperil a community threatened with fire or a contaminated water supply.

Several solutions have been published for problems concerned with the dynamic forces of confined fluids. These are dealt with in references (1)*, (2), (3) and (4) and they were obtained by the classical hydrodynamical approach of finding a solution to Laplace's equation that satisfies the imposed boundary conditions. While these solutions give very precise measures of the dynamic forces for small oscillations, they prove to be unwieldy when one attempts to apply them in designing a structure.

The purpose of the present study was to investigate the range of validity of a simplified method for determining the dynamic forces on fluid containers proposed by Housner[5]. This method

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*Numbers in parentheses refer to references listed at end of report.

was derived for only small fluid displacements as a result of linear approximations regarding the nature of the oscillations. For sufficiently small oscillations it had been shown that the method over estimates the force of the oscillating fluid by less than 2 percent, but it had not been established at what amplitude of oscillation the non-linear character would become important. To investigate this a series of experiments was performed which measured the forces and frequencies for oscillations of relatively large amplitudes. The objective of the experiments was to determine at what amplitude the non-linear character of the oscillations become observable, to evaluate the non-linear characteristics at large amplitudes, and to determine if at large amplitudes the velocity field of the fluid was sufficiently altered so as to give a force-amplitude relation different from that measured at small amplitudes.

II. THEORY OF FLUID OSCILLATIONS IN TANKS

The following analysis is for a rectangular tank with vertical walls and horizontal bottom containing a non-viscous, incompressible fluid whose top surface is free. The same analysis can be applied to cylindrical tanks as shown in Appendix A. The analysis of the fluid pressures is simplified by distinguishing between the impulsive and the convective fluid pressures. The physical significance of these two types of pressures can be seen by considering the following situation. Let a tank of fluid be at rest and at time t = o let the walls of the tank be given a horizontal acceleration of magnitude $\dot{u}_o$. The acceleration of the walls imparts an acceleration to the fluid so that beginning at t = o a fluid pressure is developed against the walls and the magnitude of the pressure is proportional to $\dot{u}_o$. After the elapse of some time the fluid will have been excited into oscillations and this motion will induce additional pressures against the tank walls. At any time "t" the total fluid pressure will consist of one part that is directly proportional to the instantaneous wall acceleration, $\dot{u}_o$, and one part, associated with the oscillations, that depends upon the sum of all the accelerations, $\dot{u}_o$, that have been imparted to the tank walls prior to time 't'. The first of these is called the 'impulsive pressure' and the second is called the 'convective pressure'.

The analysis of the impulsive pressures is presented first and this will be followed by the analysis of the convective pressures.

Analysis of Impulsive Pressures

Consider a rectangular tank of unit width in which the surface of the fluid is horizontal (fig. 1). If the container is subjected to a sudden horizontal acceleration, $\dot{u}_o$, normal to one wall, pressures will be induced which will impart acceleration to the fluid. The pressures on the walls of the container due to the acceleration, $\dot{u}_o$, will now be determined employing the following notation.

x = Horizontal coordinate measured from the center of the tank

y = Vertical coordinate measured from the free surface of the fluid

u = Horizontal velocity of the fluid

$\dot{u}$ = Horizontal acceleration of the fluid

$\dot{u}_o$ = The initial horizontal acceleration of the container walls

v = Vertical velocity of fluid

$\dot{v}$ = Vertical acceleration of fluid

ρ = Mass density of the fluid

p = Fluid pressure between two membranes

Q = Total horizontal force between two membranes

h = The depth of fluid

$2L$ = Length of container

p_w = Pressure of fluid on one wall

p_b = Pressure of fluid on bottom of container

P = Total force acting on one wall

h_o = Distance above bottom that P acts

M_o = The equivalent mass rigidly connected to the walls of the container at height, h_o, necessary to produce a force P

M = Total mass of fluid

The assumption is made that the horizontal component of fluid velocity, u, is a function of x only and is independent of the y coordinate. Physically this is the same as if the fluid were constrained by thin, massless, vertical membranes, initially spaced at a distance dx apart, and free to move in a horizontal direction. This constraint on the fluid flow is expressed by the equation

$$V = (h-y) \frac{du}{dx} \tag{1}$$

which states the condition of conservation of mass for the element (fig. 2). Since incompressible fluid flow is assumed, the fluid accelerations are proportional to the velocities. The hydrodynamic equation expressing the pressure of the fluid between two membranes is

$$\frac{\partial p}{\partial y} = - \rho \dot{v} \tag{2}$$

and the total horizontal force between two membranes is given by

$$Q = \int_0^h p\,dy \tag{3}$$

The above equations may be written as

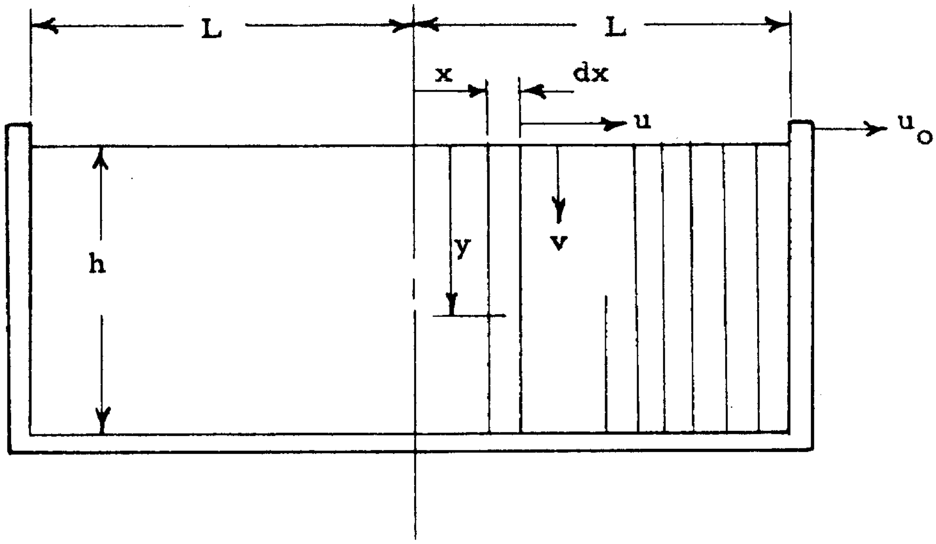


Figure 1. Rectangular Tank of Unit Width Subjected to an Impulsive Acceleration

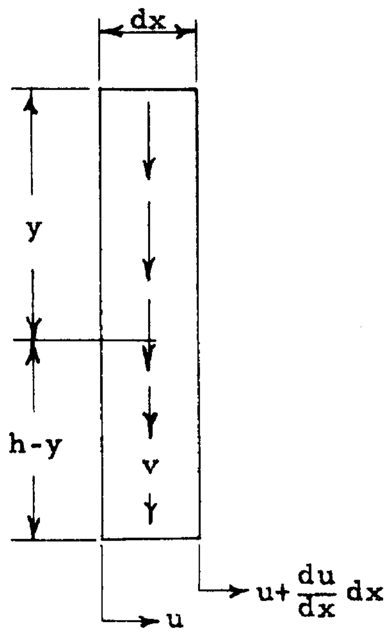


Figure 2. A Fluid Element in a Rectangular Tank (Impulsive Pressure)

$$V = (h-y)\frac{du}{dx}$$

$$p = -\rho\int_0^y (h-y)\frac{d\dot{u}}{dx}dy = -\rho h^2\left[\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right]\frac{d\dot{u}}{dx} \tag{4}$$

$$Q = -\rho h^2\int_0^h \left[\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right]\frac{d\dot{u}}{dx}dy = -\rho\frac{h^3}{3}\frac{d\dot{u}}{dx}$$

If the fluid element shown in fig. 2 is to be accelerated in the x-direction there must be a difference in pressure between the two membranes. The equation of motion of the element is

$$\frac{dQ}{dx}dx = -(\rho h dx)\dot{u} \tag{5}$$

Substituting the value of Q from equation 4 gives the following differential equation of motion

$$\frac{d^2\dot{u}}{dx^2} - \frac{3}{h^2}\dot{u} = 0 \tag{6}$$

The general solution of this equation is

$$\dot{u} = A\sinh\sqrt{3}\,\frac{x}{h} + B\cosh\sqrt{3}\,\frac{x}{h} \tag{7}$$

and the constants are determined from the boundary conditions

$$(\dot{u} = \dot{u}_o)_{x=\pm L}$$

which give

$$A = 0 \qquad B = \frac{\dot{u}_o}{\cosh\sqrt{3}\frac{L}{h}}$$

Substituting these values for the constants in equation 7 yields

$$\dot{u} = \dot{u}_o\,\frac{\cosh\sqrt{3}\,\frac{x}{h}}{\cosh\sqrt{3}\,\frac{L}{h}} \tag{8}$$

By using this value of $\ddot{u}$ in equations 4 the following expressions for the pressure and force on the membrane are obtained.

$$p = -\rho\ddot{u}_o\, h\sqrt{3}\left[\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right]\frac{\sinh\sqrt{3}\,\frac{x}{h}}{\cosh\sqrt{3}\,\frac{L}{h}} \qquad (9)$$

$$Q = -\frac{\rho\ddot{u}_o h^2}{\sqrt{3}}\;\frac{\sinh\sqrt{3}\,\frac{x}{h}}{\cosh\sqrt{3}\,\frac{L}{h}} \qquad (10)$$

The horizontal acceleration, $\ddot{u}_o$, produces an increase of pressure on one wall and a decrease of pressure on the other wall of

$$p_w = \rho\ddot{u}_o h\left[\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right]\sqrt{3}\,\tanh\sqrt{3}\,\frac{L}{h} \qquad (11)$$

and produces a pressure on the bottom of the tank

$$p_b = -\rho\ddot{u}_o\, h\,\frac{\sqrt{3}}{2}\;\frac{\sinh\sqrt{3}\,\frac{x}{h}}{\cosh\sqrt{3}\,\frac{L}{h}} \qquad (12)$$

The integral of equation 11 over the depth of the fluid gives for the resultant force acting on one wall

$$P = \frac{\rho\ddot{u}_o\, h^2}{\sqrt{3}}\;\tanh\sqrt{3}\,\frac{L}{h} \qquad (13)$$

The location of the resultant force is given by

$$\bar{y} = \frac{\int_o^h p_w y\,dy}{\int_o^h p_w\,dy} = \frac{5}{8}\,h \qquad (14)$$

or

$$h_o = h - \bar{y} = \frac{3}{8}\,h$$

where h_o is the distance measured from the bottom of the tank.

The total horizontal force on both walls is the same as would be produced by an equivalent mass, M_o, fastened to the walls of the container at a height 3/8 h above the bottom, where M_o is determined by

$$2P = M_o \dot{u}_o = \frac{2\rho \dot{u}_o h^2}{\sqrt{3}} \tanh \sqrt{3}\,\frac{L}{h}$$

$$M_o = \frac{M \tanh \sqrt{3}\,\frac{L}{h}}{\sqrt{3}\,\frac{L}{h}} \tag{15}$$

The moment exerted by the pressure on the bottom of the tank is given by

$$\int_{-L}^{+L} p_b\, x dx = -\rho \dot{u}_o h^2 L \left(1 - \frac{\tanh \sqrt{3}\frac{L}{h}}{\sqrt{3}\,\frac{L}{h}}\right) \tag{16}$$

To give the correct total moment on the tank the mass, M_o, must, therefore, be at an elevation

$$h_o = \frac{3}{8} h \left[1 + \frac{4}{3} \left(\frac{\sqrt{3}\,\frac{L}{h}}{\tanh \sqrt{3}\frac{L}{h}} - 1 \right) \right] \tag{17}$$

For tall, narrow tanks let it be assumed that there is a rigid horizontal membrane at a depth, h, below the surface. Since the fluid below this membrane is restrained so that $v = \dot{v} = o$, it exerts a moment on the membrane equal to

$$\frac{2}{3} \rho \dot{u}_o L^3 \tag{18}$$

Equation 16 gives a moment equal to this when

$$\frac{h}{L} = 1.6 \tag{19}$$

This means that the preceding equations should be applied only to tanks whose proportions are such that the ratio of $(\frac{h}{L})$ is equal to or less than 1.6. If the tank is taller, the fluid below this critical depth should be treated as a rigid body moving with the tank and exerting a pressure on the walls of

$$p_w = \rho \ddot{u}_0 L$$

Convective Pressures

The effect of the impulsive pressures induced by the horizontal accelerations of the side walls of the container is to excite the fluid into oscillations. The additional convective pressures caused by these oscillations must be considered together with the impulsive pressures to determine the total force that is exerted on the tank by the contained fluid.

To examine the first mode of oscillation of the fluid consider the fluid to be constrained between rigid, massless, horizontal membranes that are free to rotate about a horizontal axis (fig. 3). The notation used in the last section applies to the same quantities and additional notation is introduced as follows:

θ = The angular rotation of a fluid element bounded by two membranes

$\dot{\theta}$ = The angular velocity

$\ddot{\theta}$ = The angular acceleration

M_1 = The equivalent mass in a mechanical system that will produce the same total force on the container walls as will the oscillating fluid of the first mode

x_1 = The displacement of the equivalent mass M_1

F_1 = The force produced by the mass M_1

A_1 = The maximum displacement of mass M_1

T_1 = The kinetic energy of M_1

K_1 = The equivalent spring constant of mass M_1

T = The kinetic energy of the fluid

V = The potential energy of the fluid

$\dot{\theta}_o$ = Maximum angular rotation, measured at the center of the tank

The conservation of mass requires that the fluid flow into an element be equal to the increase in volume of the element (fig. 4).

$$u\,dy = \frac{1}{2}\left(x\frac{d\dot{\theta}}{dy}\,dy + L\,\frac{d\dot{\theta}}{dy}\,dy\right)(L-x)$$

This gives

$$u = \frac{(L^2 - x^2)}{2}\;\frac{d\dot{\theta}}{dy} \tag{20}$$

$$v = \dot{\theta}x \tag{21}$$

The hydrodynamic equation for the pressure of the fluid between the membranes is

$$\frac{\partial p}{\partial x} = -\rho\dot{u}$$

Differentiating equation 20 and substituting into this equation for pressure yields

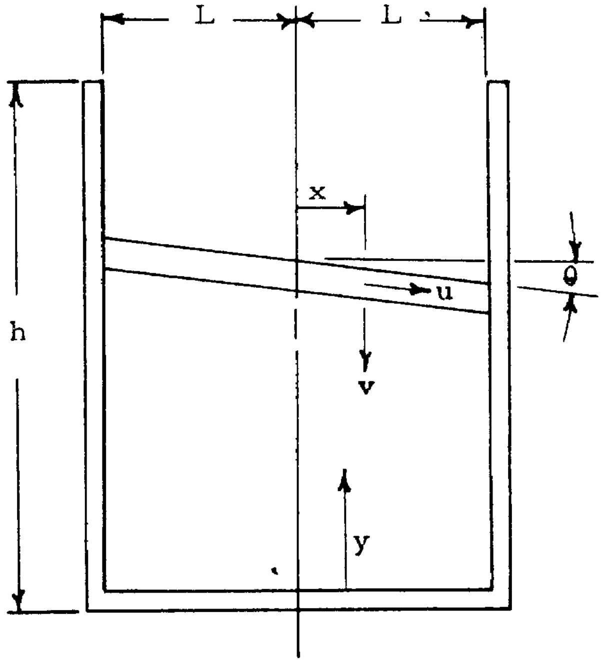


Figure 3. Rectangular Tank of Unit Width (Convective Pressure).

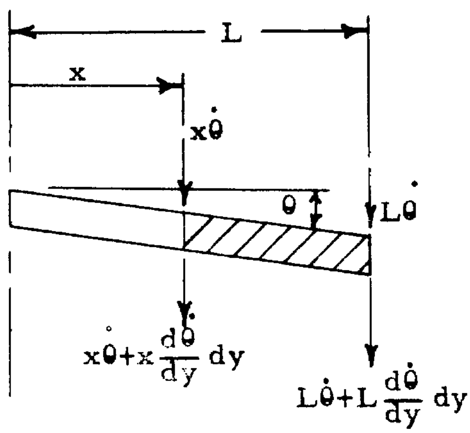


Figure 4. Fluid Element in a Rectangular Tank (Convective Pressure).

$$p = -\rho \frac{L^3}{2}\left[\frac{x}{L} - \frac{1}{3}\left(\frac{x}{L}\right)^3\right]\frac{d\ddot{\theta}}{dy} \tag{22}$$

The equation governing the motion of a slice of fluid may be derived by equating the rate of change of angular momentum of a fluid element about its axis of rotation to the moment acting on the element:

$$I\ddot{\theta} = \text{Moment}$$

For any element dxdy

$$\text{Moment} = -\frac{\partial p}{\partial y}(dydx)\, x \sin\theta$$

where for small θ

$$x \sin\theta \simeq x$$

$$\text{Moment} = -\frac{\partial p}{\partial y}(dydx)x$$

The rate of change of angular momentum of the fluid between two membranes is

$$\rho\frac{(2L)^3}{12}\ddot{\theta}\, dy$$

and equating this to the total moment of the fluid element between two membranes gives

$$\int_{-L}^{+L}\frac{\partial p}{\partial y}\, xdydx = -\rho\frac{(2L)^3}{12}\ddot{\theta}\, dy$$

Upon integration one obtains

$$\frac{d^2\ddot{\theta}}{dy^2} = \frac{5}{2}\frac{\ddot{\theta}}{L^2} \tag{23}$$

Applying the appropriate boundary conditions and considering sinusoidal oscillations yields the following expression for the fluid oscillation.

$$\theta = \theta_o \frac{\sinh\sqrt{\frac{5}{2}}\,\frac{y}{L}}{\sinh\sqrt{\frac{5}{2}}\,\frac{h}{L}} \sin\omega t \tag{24}$$

Equating the maximum kinetic energy of the oscillating fluid to its maximum potential energy will give an expression for the natural frequency of vibration of the system.

$$T = \int_0^h \int_{-L}^{+L} \tfrac{1}{2}\rho(u^2+v^2)\,dx\,dy = \frac{L^4\rho\theta_o^2\omega^2\cos^2\omega t}{3\sqrt{\frac{5}{2}}\tanh\sqrt{\frac{5}{2}}\,\frac{h}{L}}$$

$$V = \int_{-L}^{+L} \tfrac{1}{2}\rho g x^2\theta_o^2\sin^2\omega t\,dx = \frac{L^3}{3}\rho g\theta_o^2\sin^2\omega t \tag{25}$$

Substituting equations 20 and 21 into the above expressions and integrating yields

$$\omega^2 = \frac{g}{L}\sqrt{\frac{5}{2}}\tanh\sqrt{\frac{5}{2}}\,\frac{h}{L} \tag{26}$$

and in a similar fashion one may find the natural frequency of the nth mode to be

$$\omega_n^2 = \frac{g}{L}\,n\sqrt{\frac{5}{2}}\tanh\sqrt{\frac{5}{2}}\,\frac{hn}{L} \tag{27}$$

Substituting equation 24 into equation 22 gives for the pressure on the wall of the container

$$p_w = \rho\frac{L^2}{3}\sqrt{\frac{5}{2}}\,\frac{\cosh\sqrt{\frac{5}{2}}\,\frac{y}{L}}{\sinh\sqrt{\frac{5}{2}}\,\frac{h}{L}}\,\omega^2\theta_o\sin\omega t \tag{28}$$

The total horizontal force on one wall is

$$P = \int_0^h p_w \, dy = \rho \frac{L^3}{3} \omega^2 \theta_o \sin \omega t \tag{29}$$

A force equal in magnitude and in the same direction acts on the opposite wall giving a total of 2P. One may consider this total force to be produced by an equivalent mass, M_1, which is spring mounted (fig. 5). The mass will oscillate according to

$$x_1 = A_1 \sin \omega t$$

and produce a force

$$F_1 = -M_1 A_1 \omega^2 \sin \omega t$$

which is obtained by differentiating twice and multiplying by the mass. The kinetic energy is, therefore,

$$T_1 = \tfrac{1}{2} M_1 A_1 \omega^2 \sin^2 \omega t$$

Comparing these equations with equations 25 and 28 of the fluid yields

$$A_1 = \frac{\theta_o L}{\sqrt{\frac{5}{2}} \tanh \sqrt{\frac{5}{2}} \frac{h}{L}} \tag{30}$$

$$M_1 = \frac{M}{3} \sqrt{\frac{5}{2}} \frac{L}{h} \tanh \sqrt{\frac{5}{2}} \frac{h}{L} \tag{31}$$

This equation for M_1 exceeds the more exact value as given by Graham and Rodriguez[4] by less than 2%.

To determine the elevation of M_1 above the bottom of the tank one has to equate the moment exerted by the fluid pressures on the side walls and on the bottom of the tank to the moment exerted by M_1. This gives

$$h_1 = h\left(1 - \frac{\cosh\sqrt{\frac{5}{2}}\,\frac{h}{L} - 2}{\sqrt{\frac{5}{2}}\,\frac{h}{L}\,\sinh\sqrt{\frac{5}{2}}\,\frac{h}{L}}\right) \tag{32}$$

In order to obtain the corresponding quantities for the higher modes, $(\frac{L}{n})$ should be substituted for L in the equations. Only the odd modes will yield moments since even numbered modes represent symmetrical conditions.

The fluid system has now been reduced to a relatively simple mechanical system (fig. 5). When the tank is subjected to dynamic loads such as those produced by earthquakes, the various modes of vibration will be excited. The equivalent system, therefore, reduces the problem to that of solving for the response of the simple oscillators.

In the event that the container is an elevated water tower, the tower structure adds an additional degree of freedom to the system.

The Equivalent Mechanical System

Having reduced the fluid system to one of simple oscillators it becomes possible to investigate the regions of applicability and accuracy of Housner's theory(5) as the amplitudes of oscillation increase, by restricting the motion of the tank so that only the first mode of the system is excited. If this is done the forces acting in the system are due only to the oscillations of the fluid in the first mode and to the rigid mass.

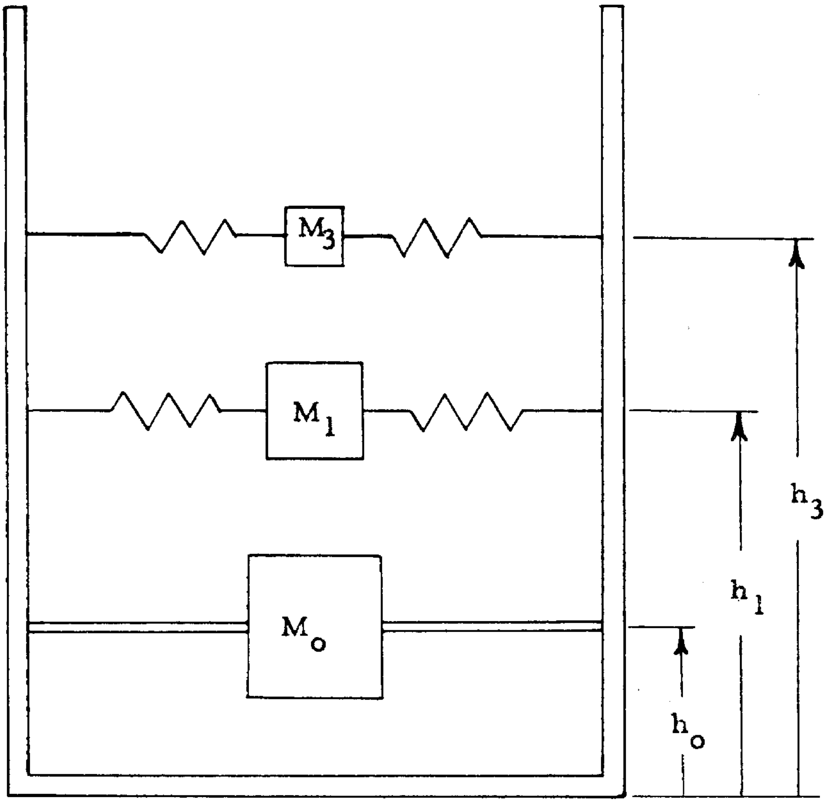


Figure 5. Equivalent Mechanical System of Simple Oscillators

The natural frequencies of the various modes are sufficiently separated so that if one investigates a narrow band of frequencies, near and including the resonant frequency of the first mode, the oscillating mass which produces forces on the tank is primarily that corresponding to the first mode.

To facilitate the measurement of the forces acting on the walls of the container a mechanical system as shown in fig. 6 was studied and experiments were designed to approach this idealized system.

Knowing the spring constant, K_2, and measuring the spring deflection will enable one to calculate the total force exerted by the tank. The force due to the oscillating fluid mass can be determined by subtracting the force produced by the rigid mass from the measured total force. The following notation will be used in analyzing the mechanical system.

M_1 = Equivalent oscillating mass of first mode

M_R = The total rigid mass of the system

M_o = The equivalent rigid mass of impulsive forces

M_o' = The rigid mass of the fluid below depth h, the tank, one third the mass of the springs, and the mass of the measuring stand

K_1 = The equivalent spring constant of mass M_1

K_2 = The spring constant of the mounting springs

x_1 = Displacement of M_1 relative to the tank walls

x_o = Displacement of M_o relative to the base

V = Potential energy

T = Kinetic energy

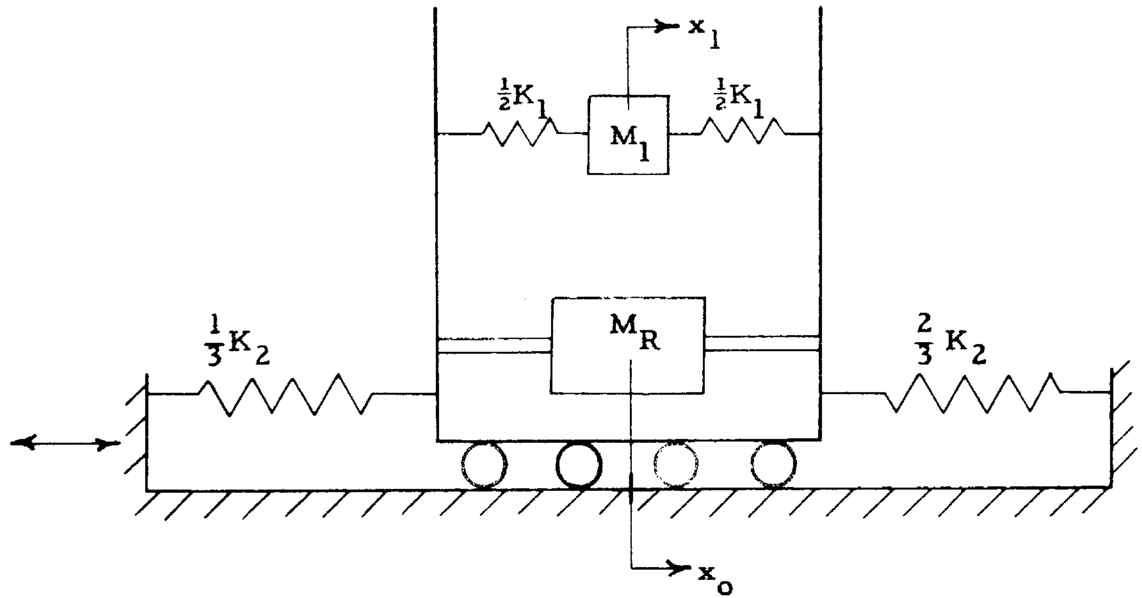


Figure 6. Equivalent Mechanical System Considering Only the First Mode of Fluid Oscillation

A_o = Amplitude of tank relative to the base

A_1 = Amplitude of mass M_1, relative to the container walls

$A_i^{(r)}$ = Amplitude of mass M_i, in the rth principal mode

The expressions for the potential energy, V, and for the kinetic energy, T, are

$$V = \tfrac{1}{2} K_1 x_1^2 + \tfrac{1}{2} K_2 x_o^2$$

$$T = \tfrac{1}{2} M_R \dot{x}_o^2 + \tfrac{1}{2} M_1 (\dot{x}_1 + \dot{x}_o)^2 \qquad (33)$$

Employing Lagrange's equation,

$$\frac{\partial}{\partial t} \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} = 0,$$

the equations of motion of the system are

$$(M_R + M_1)\ddot{x}_o + M_1 \ddot{x}_1 + K_2 x_o = 0$$

$$M_1 \ddot{x}_o + M_1 \ddot{x}_1 + K_1 x_1 = 0 \qquad (34a)$$

For simplicity the following changes in notation are made:

$K_{00} = K_2$ $\qquad$ $M_{00} = (M_R + M_1)$

$K_{11} = K_1$ $\qquad$ $M_{11} = M_1$

The equations then become

$$M_{00}\ddot{x}_0 + M_{11}\ddot{x}_1 + K_{00}x_0 = 0$$

$$M_{11}\ddot{x}_0 + M_{11}x_1 + K_{11}x_1 = 0 \qquad (34b)$$

To determine the natural frequency of the system one assumes a harmonic solution of the form

$$x_i = A_i \sin(\omega t + \beta) \qquad (35)$$

and substitutes into equations 34b. This yields the following set of equations:

$$(K_{00}-M_{00}\omega^2)\,A_0 + (-M_{11}\,\omega^2)\,A_1 = 0$$
$$(-M_{11}\,\omega^2)\,A_0 + (K_{11}-M_{11}\,\omega^2)\,A_1 = 0 \qquad (36)$$

For a nontrivial solution the determinant of the coefficients must be set equal to zero which yields the frequency equation

$$\omega^4 - \omega^2\left[\frac{(M_{00}K_{11}+K_{00}M_{11})}{(M_{00}M_{11} - M_{11}^2)}\right] + \frac{K_{00}K_{11}}{(M_{00}M_{11}-M_{11}^2)} = 0 \qquad (37)$$

from which

$$\omega^2 = +\tfrac{1}{2}\left[\frac{(M_{00}K_{11}+K_{00}M_{11})}{(M_{00}M_{11}-M_{11}^2)}\right] \mp \tfrac{1}{2}\sqrt{\left[\frac{M_{00}K_{11}+K_{00}M_{11}}{(M_{00}M_{11}-M_{11}^2)}\right]^2 - \frac{4K_{00}K_{11}}{(M_{00}M_{11}-M_{11}^2)}} \qquad (38)$$

Equation 38 gives two values of ω^2, (ω_1^2 and ω_2^2). These are the natural frequencies of the system. Substituting either of these into equation 36 one obtains

$$\frac{A_0}{A_1} = \frac{M_{11}\,\omega^2}{K_{00}-M_{00}\,\omega^2} = \frac{K_{11}-M_{11}\,\omega^2}{M_{11}\,\omega^2} \qquad (39)$$

as the ratio of the amplitudes of the two masses for the mode being considered.

The general solution is

$$x_i = \sum_{r=1}^{n} A_i^{(r)} \sin(\omega_r t + \beta_r) \qquad (40)$$

where r corresponds to the mode being considered and i to the coordinate[6].

In the experiments measurements were made when the two masses oscillated in phase with each other. This corresponds to the first mode or lower frequency. The first principal mode of oscillation can be expressed as

$$x_0 = A_0^{(1)} \sin (\omega_1 t + \beta_1)$$
$$x_1 = A_1^{(1)} \sin (\omega_1 t + \beta_1) \qquad (41)$$

and the masses are in phase when equation 39 is positive.

III. EXPERIMENTAL MEASUREMENTS

Instrumentation

The experimental model was designed to resemble the idealized system (fig. 6), as closely as possible. This was accomplished by the apparatus shown in fig. 7. The tank used to contain the fluid was made from plexiglass sheet, 3/16 of an inch in thickness, and was 8 x 8 inches in cross section and 11 inches high. Plexiglass, while supplying sufficient rigidity, had the advantage of making possible visual observation of the fluid motion. The tank rested on ball bearings which were free to move on steel tracks. A grid was scribed on one side of the tank to facilitate the determination of the shape and amplitude of the waves.

A screw type surface gage was used to determine the amplitude of the waves. This was done by measuring the difference in height of the screw thread between the level water surface and the crest of the wave with a vernier caliper which gave accuracy to 0.001 inch. A cam-shaft, whose eccentricity could be varied, supplied a harmonic, horizontal motion to the table supporting the tank. This motion excited oscillations in the fluid. Power was supplied from a Bodine variable speed D.C. motor driven by a Sola Constant Voltage Transformer. Frequency measurements were made with a Strobotac strobe light and a Federal Dial Indicator read the deflection of the tank relative to its supporting table.

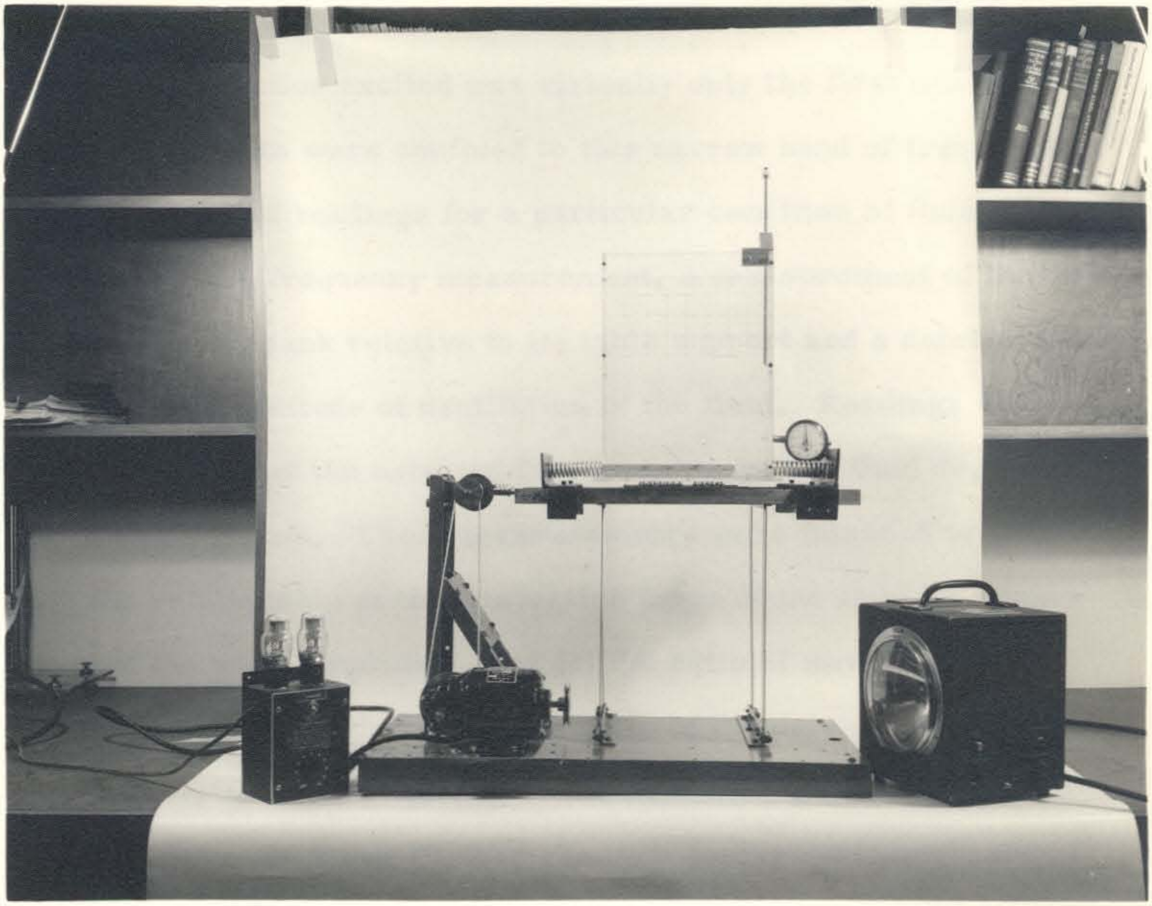


Figure 7. Experimental Apparatus

Procedure

At a given fluid depth a series of experiments was run at frequencies within $\pm$ 8 percent of the natural frequency of the first mode so that the amplitudes of oscillation induced were large enough to obtain sufficient accuracy of measurement. The mode of fluid oscillation excited was virtually only the first mode since the experiments were confined to this narrow band of frequencies. A typical set of readings for a particular condition of fluid motion consisted of a frequency measurement, a measurement of the deflection of the tank relative to its table support and a determination of the amplitude of oscillation of the fluid. Readings were taken throughout the selected frequency range for fluid depths of 5, 6 and 7 inches. These measurements were intended to give (1) the relationship of the convective force of the fluid as a function of the wave amplitude, and (2) the ratio of wave amplitude to fluid depth as a function of the ratio of the driving frequency to the linear, natural frequency of the first mode.

Figs. 8, 9 and 10 show the tank during various phases of an experiment for a fluid depth of 7 inches.

Experimental Errors

In these experiments three different measurements were made. These were a measurement of the deflection of the tank relative to the supporting table, a frequency reading, and a determination of the amplitude of fluid oscillation.

The dial gage used to read the deflection of the springs was accurate to 0.0001 of an inch and since it was used at

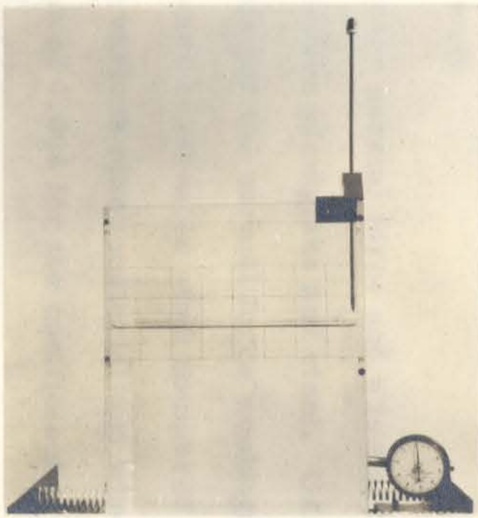


FIGURE 8.
Fluid Surface Level

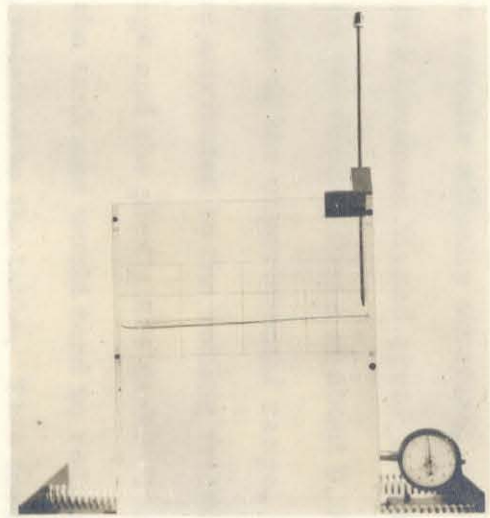


FIGURE 9.
Wave of Small Amplitude

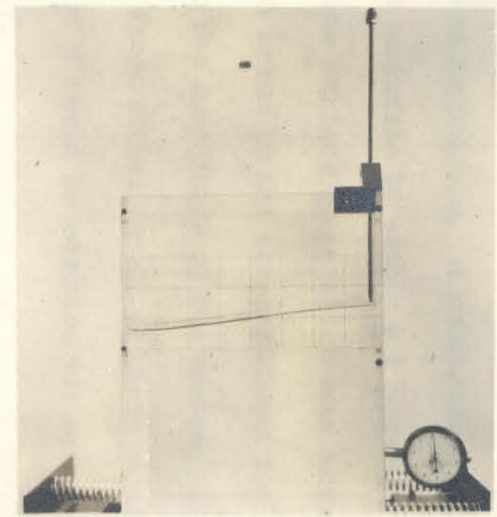


FIGURE 10.
Wave of Intermediate Amplitude

frequencies well below its own natural frequency the error it contributed was negligible. A calibration of the Strobotac indicated that it was accurate to within $\pm$ 1 percent in frequency determinations and the surface gage which was accurate to 0.001 inch contributed no error to the computations and negligible error in measuring the amplitude of the waves. It should be noted that any error from either or all of these sources would be an unbiased error since it could be positive for one computation and negative for another.

The results actually obtained, shown in figs. 12, 13 and 14, indicate a constant biased error in the experimental determination of the convective force of about 0.05 pounds throughout a major portion of the experimental range. This error can most probably be attributed to the rolling friction force between the ball bearings and the plexiglass tank. This force opposes the motion of the tank and would tend to reduce its maximum displacement relative to the table. This force may be defined as

$$F = \eta N$$

where N is the normal force and η is the coefficient of rolling friction between plexiglass and steel. Using an average value of 15 pounds for the normal force would give a value of $\eta = 0.0033$ which appears reasonable when compared to values of this coefficient for various other material and when compared to a measured value of 0.006 for the coefficient of static friction (impending motion).

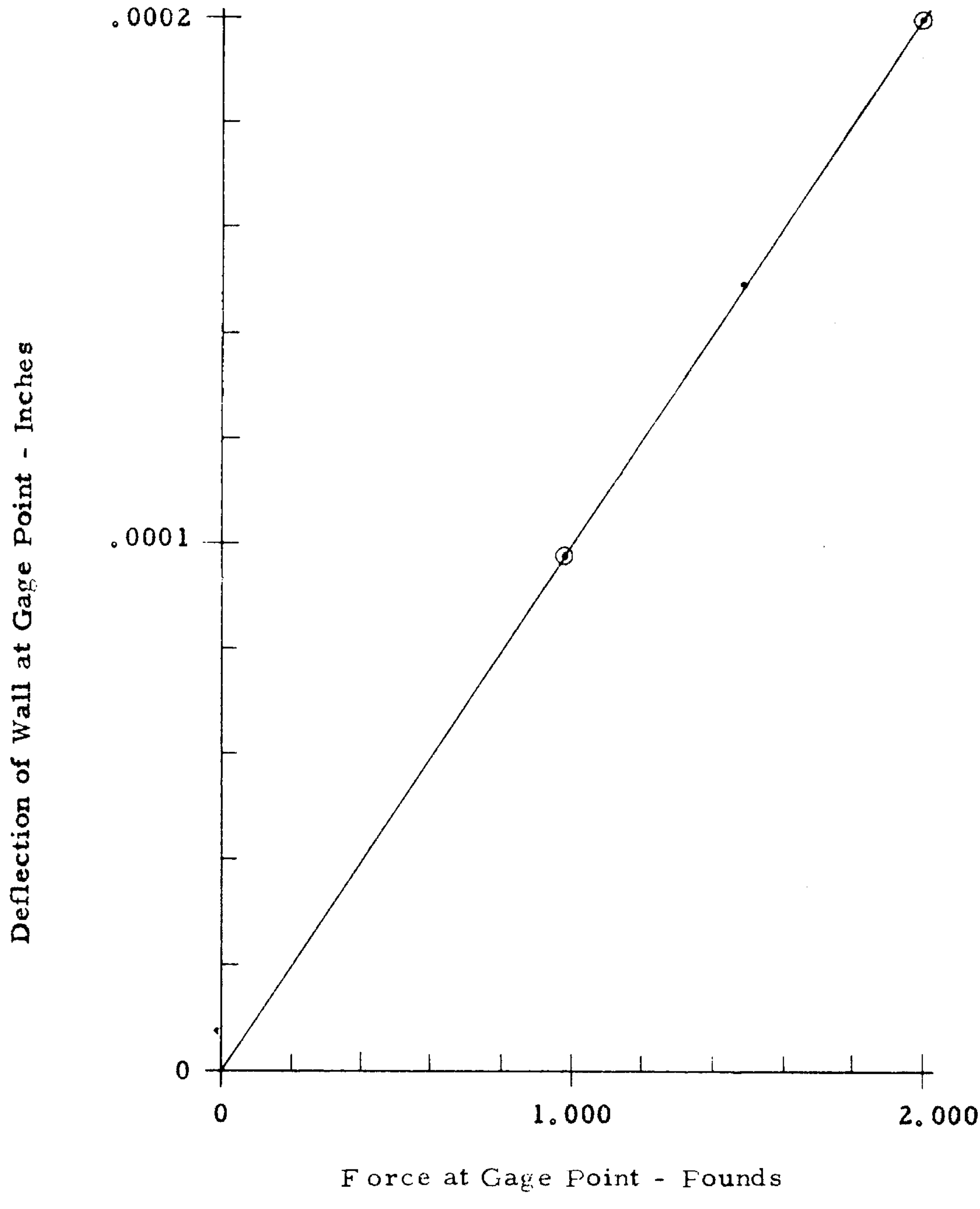


Figure 11. Deflection of Tank Wall at the Gage Point

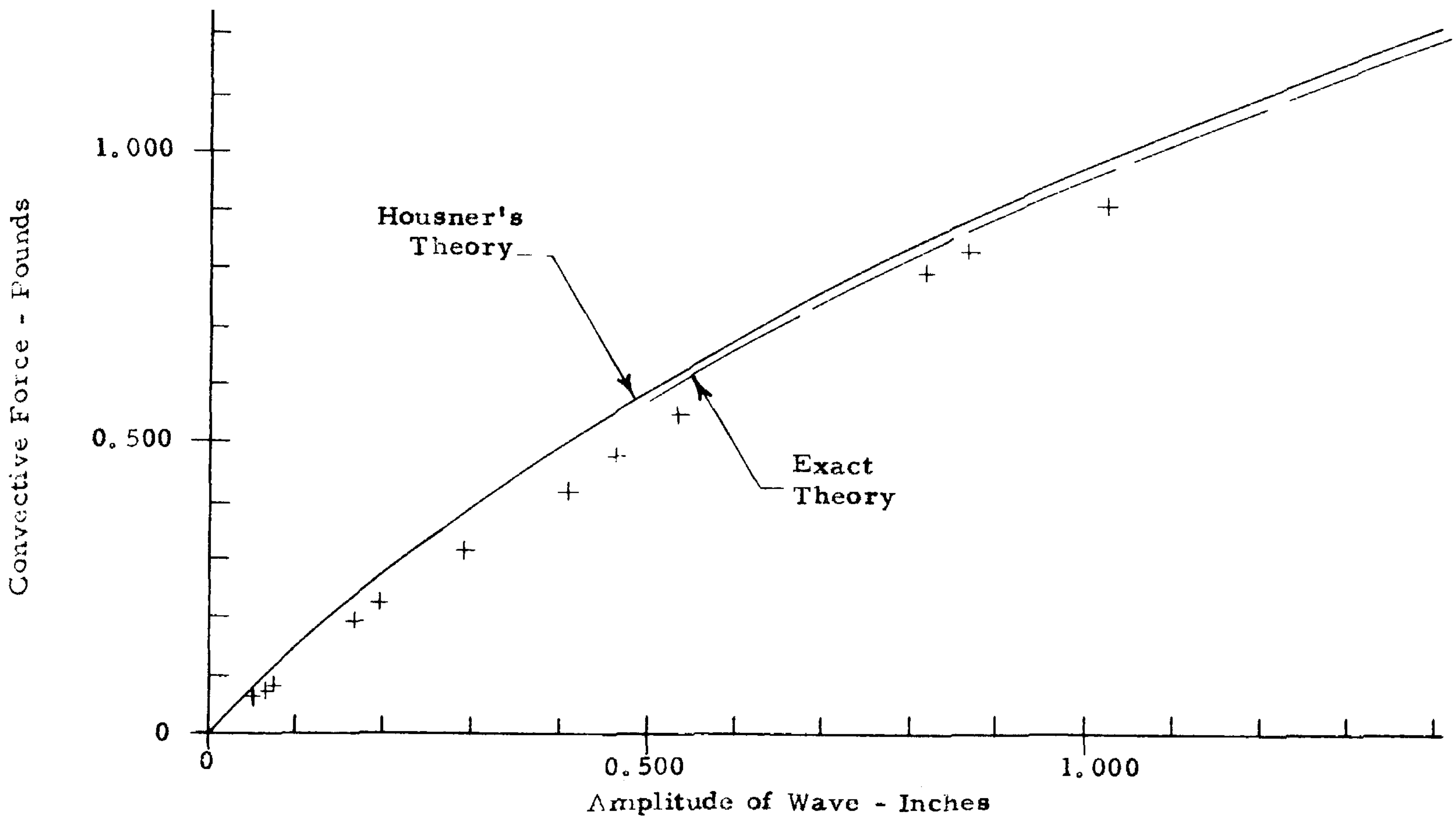


Figure 12. Convective Forces for a Fluid Depth of 5 Inches

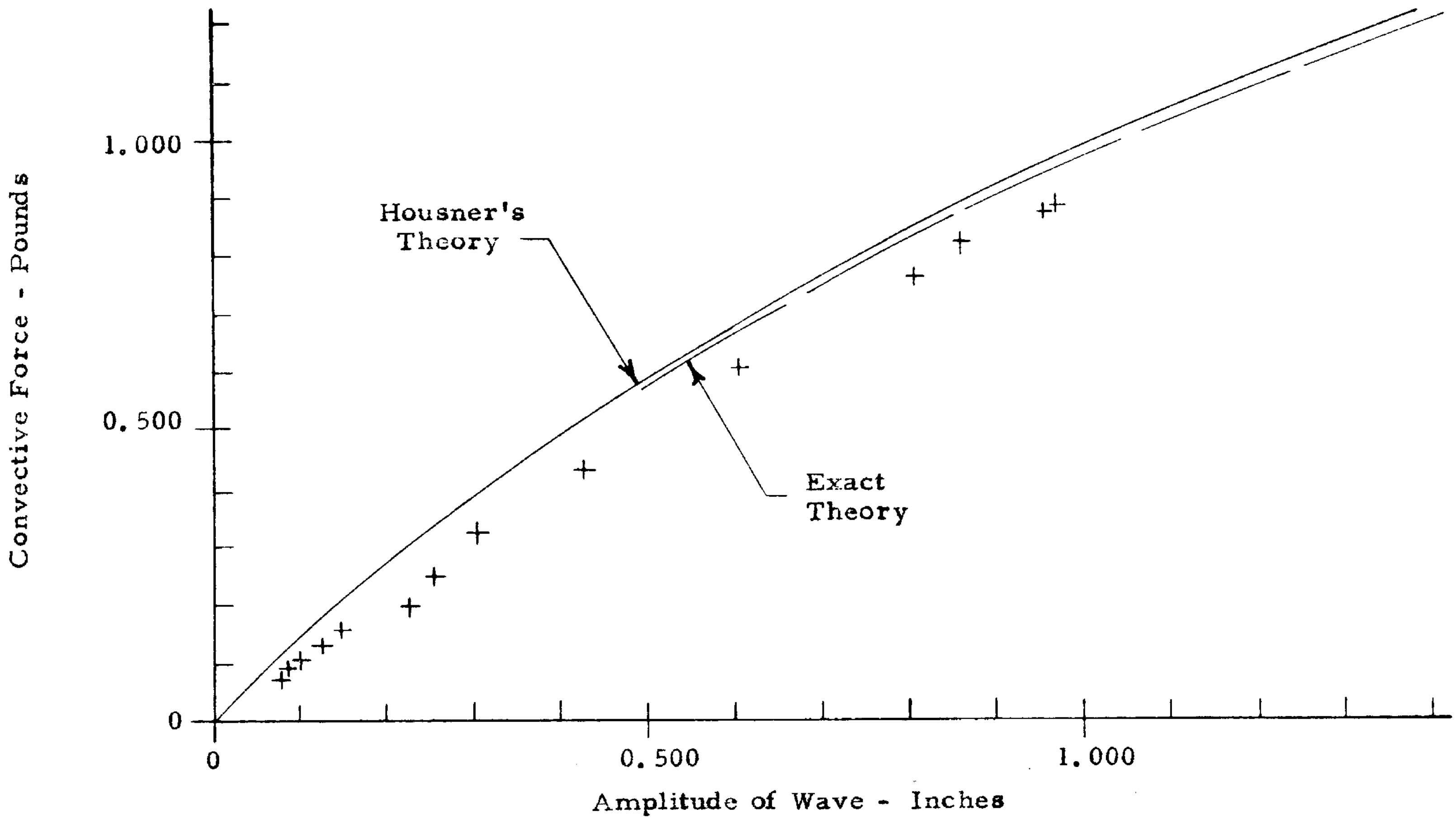


Figure 13. Convective Forces for a Fluid Depth of 6 Inches

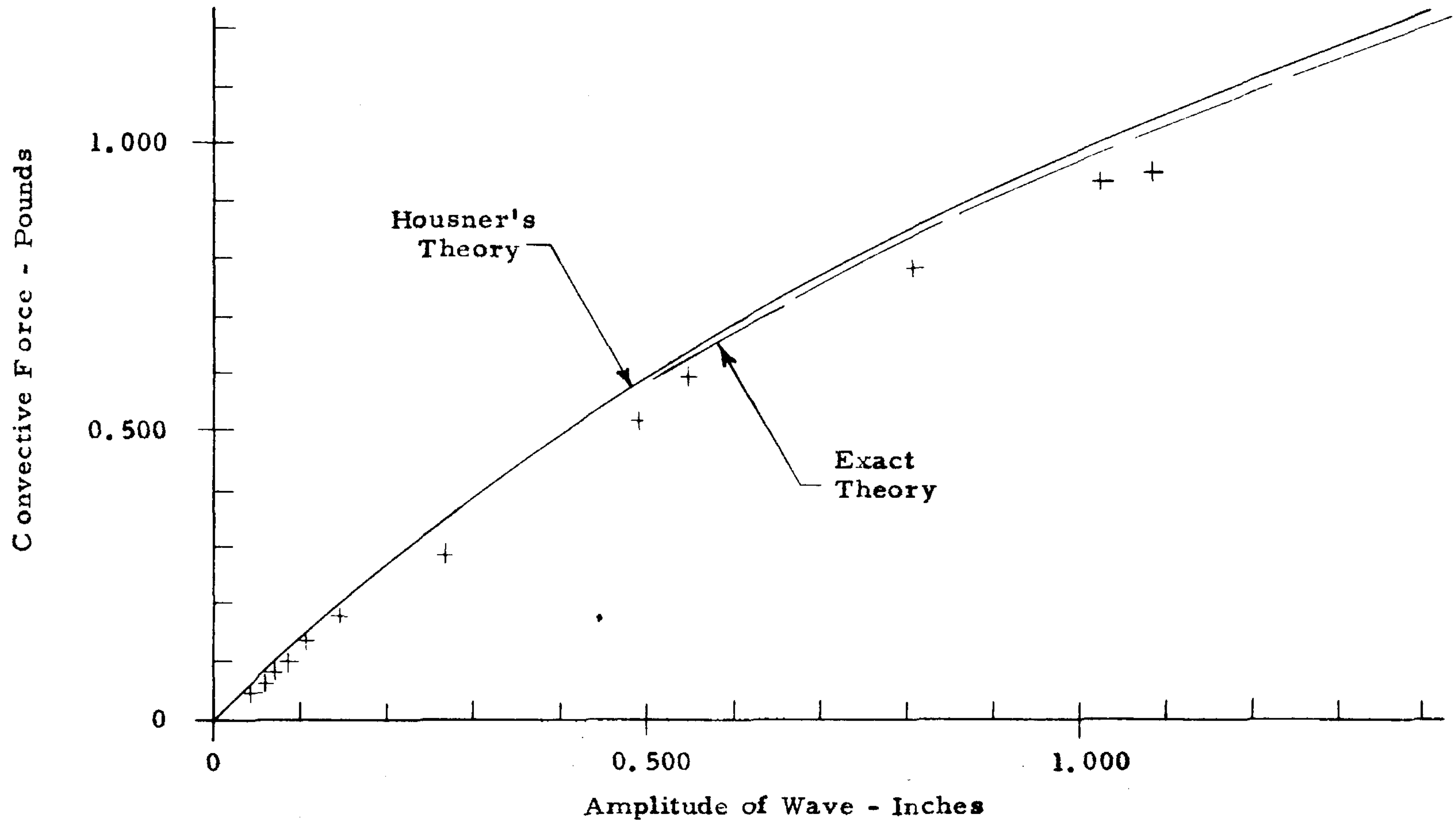


Figure 14. Convective Forces for a Fluid Depth of 7 Inches.

Method of Obtaining the Force Due to Convective Pressures

In order to compare the experimentally measured forces to those predicted by Housner's method[5] it was necessary to establish a relationship between the measured amplitude of the fluid oscillation and the displacement of the equivalent mass, M_1, relative to the container walls. To accomplish this a relationship between the measured amplitude of the fluid and the angle of rotation, θ_o, of a fluid element was derived. This was done by finding an equivalent expression for the pressure on the wall of the container by considering a fluid element dx in width and of amplitude, a, above the level surface.

The notation used previously is the same except for the addition of the following:

a = Amplitude of fluid oscillation measured at x = L

Equating equation 28 to the net inertia force of the element, dx in width, at x = L gives

$$p_w = \frac{\rho L^2}{3}\sqrt{\frac{5}{2}}\left(\frac{1}{\tanh\sqrt{\frac{5}{2}}\frac{h}{L}}\right)\omega^2\theta_o = \rho a g - a\rho\dot{v} \qquad (42)$$

where

$$v = \dot{\theta} L \qquad (43)$$

and

$$\ddot{v} = \ddot{\theta} L$$

Equation 24 becomes

$$\theta = \theta_o \sin\omega t$$

from which one obtains

$$\ddot{\theta} = -\theta_o \omega^2 \sin\omega t \tag{44}$$

Substituting equation 44 into equation 43 leads to

$$\ddot{v} = -\theta_o \omega^2 L \tag{45}$$

Using this value of $\ddot{v}$ in equation 42 and solving for θ_o yields

$$\theta_o = \frac{a g}{\dfrac{L^2\sqrt{\frac{5}{2}}\,\omega^2}{3\tanh\sqrt{\frac{5}{2}}\,\frac{h}{L}} + a\omega^2 L} \tag{46}$$

Since the experiments were confined to frequencies within $\pm$ 8 percent of the natural frequency of the first mode of vibration the frequency used for computations of the approximate curve from equation 46 was the linear natural frequency as given by equation 26. Therefore, for any given depth the above equation depends on the amplitude of oscillation. The values for the angle, θ_o, were then computed and by using equation 30 the amplitude of oscillation of the equivalent mass, M_1, could be calculated for any amplitude of wave. Multiplying this amplitude, A_1, by the equivalent spring constant, K_1, gave the force exerted by the oscillating mass for any amplitude of wave.

The resulting relationship for the force exerted by the mass, M_1, as a function of the amplitude of wave has been plotted for various depths of fluid and appears as the solid curve in figs. 12, 13, and 14 which also show the results of the experiments.

In the experiments the only force measurement is made by use of the dial gage indicator. It measures the displacement of the tank relative to the base upon which it rolls. This displacement when multiplied by the spring constant of the spring mounts gives the total force exerted by the tank. This total force includes the force exerted by the rigid masses (impulsive forces) as well as the force due to the convective pressures in the fluid. The problem then reduces to the evaluation of the portion of the total force contributed by the oscillating fluid.

The rigid mass of the system is composed of the equivalent rigid mass of water in depth h, when the depth does not exceed 1.6L, the mass of fluid below 1.6L, the mass of the tank and partition, the portion of the spring mass that contributes to the system and the mass of the measuring screw and stand.

The force exerted by the total rigid mass, M_R, is

$$F_R = M_R \ddot{u}_o \tag{47}$$

where for a sinusoidal motion of the tank relative to the base

$$x_o = A_o \sin\omega t \tag{48}$$

and the acceleration is

$$\ddot{u}_o = -A_o \omega^2 \sin\omega t \tag{49}$$

where

$$\ddot{u}_o = \ddot{x}_o$$

Since the measurements are made when the acceleration is maximum, substitution of equation 49 into equation 47, and maximizing, yields

$$F_R = - M_R A_o \omega^2 \tag{50}$$

The equation of motion for the system may then be written as

$$M_1 \ddot{x}_1 = F_T - M_R A_o \omega^2$$

or

$$F_1 = F_T - M_R A_o \omega^2 \tag{51}$$

where the force exerted by the rigid mass is the total impulsive force.

The total force may be expressed as

$$F_T = K_{00} A_0$$

where the spring constant, K_{oo}, was equal to 167 pounds per inch for all the experiments.

Previously the container had been assumed to be absolutely rigid. However, the tank does deflect due to the forces exerted by the springs at the base of the tank. The deflection of the tank at the point where the dial gage makes contact must be added to the dial gage reading to obtain the actual displacement of the tank. Fig. 11 is a plot of the deflection at the gage point as a function of the force exerted by the springs on the gage side of the tank. The amplitude used in the calculations was corrected for the deflection of the tank.

Results

The force-amplitude measurements are shown in figs. 12, 13 and 14 which correspond to fluid depths of 5, 6 and 7 inches respectively and are represented by the plotted points. The solid curves represent the force-amplitude relationship predicted by Housner[5] and the curves composed of long dashes represent the force-amplitude relationship predicted by Graham and Rodriguez[4]. The measured values agree very favorably with both predictions when one takes into account the effect of rolling friction.

Fig. 15 shows the amplitude-frequency measurements for the various depths of fluid considered. The frequency ratio is that of the driving frequency to the linear natural frequency of the first mode of oscillation. An inspection of these curves reveals that the linear natural frequency indicated by them agrees very closely with the values obtained from equation 38.

The shape of the fluid surface during oscillation changed considerably over the range of $(\frac{a}{d})$ ratios. This is shown in fig. 16 together with the approximate region of occurrence of each wave shape.

Discussion of Results

The amplitude-frequency curve for a fluid depth of 5 inches reveals a very pronounced jump phenomenon. This jump condition is also exhibited by the curves for fluid depths of 6 and 7 inches, but becomes less pronounced with increasing fluid depths and with the corresponding smaller $(\frac{a}{d})$ ratios. Due to the jump

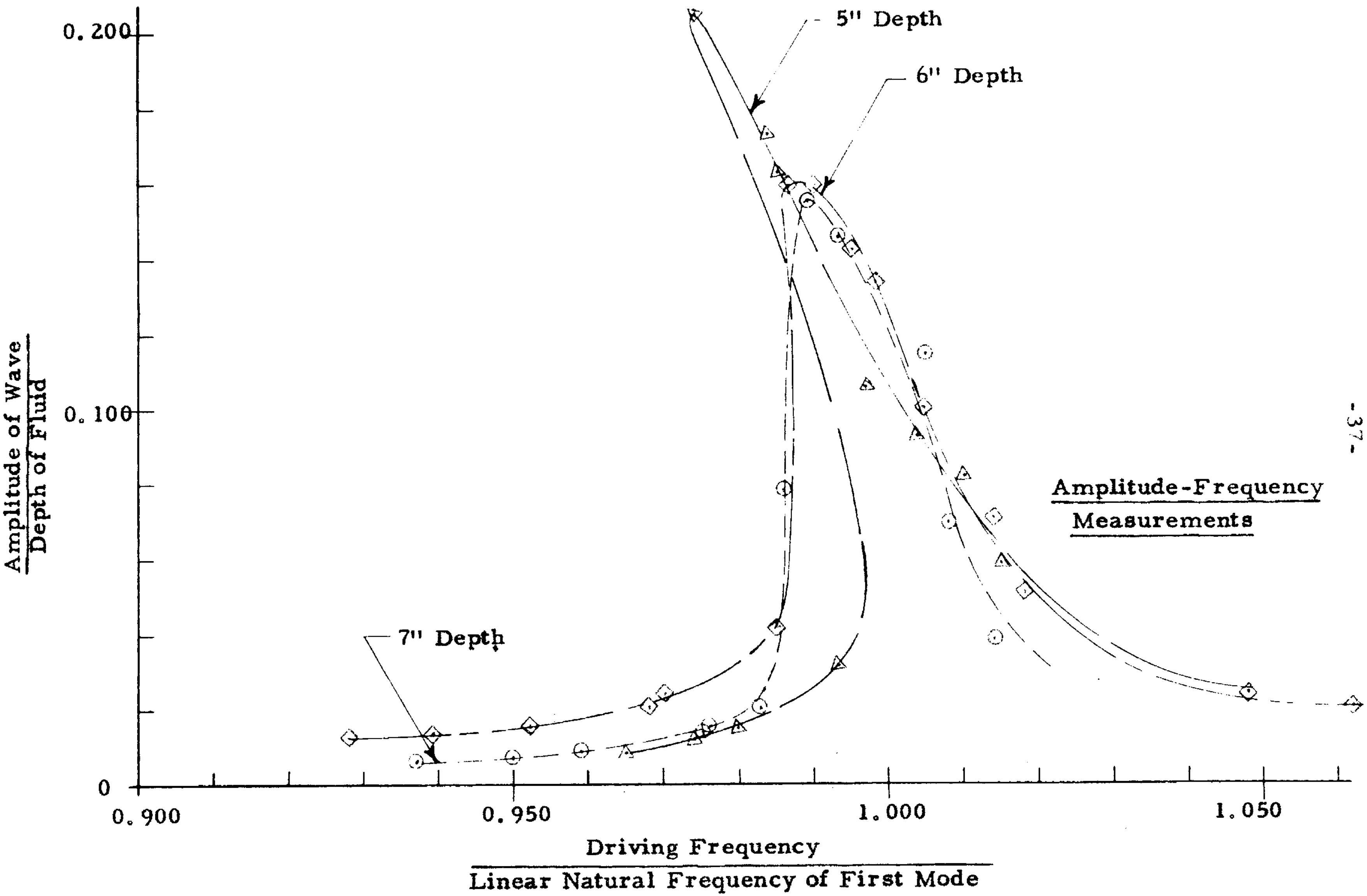
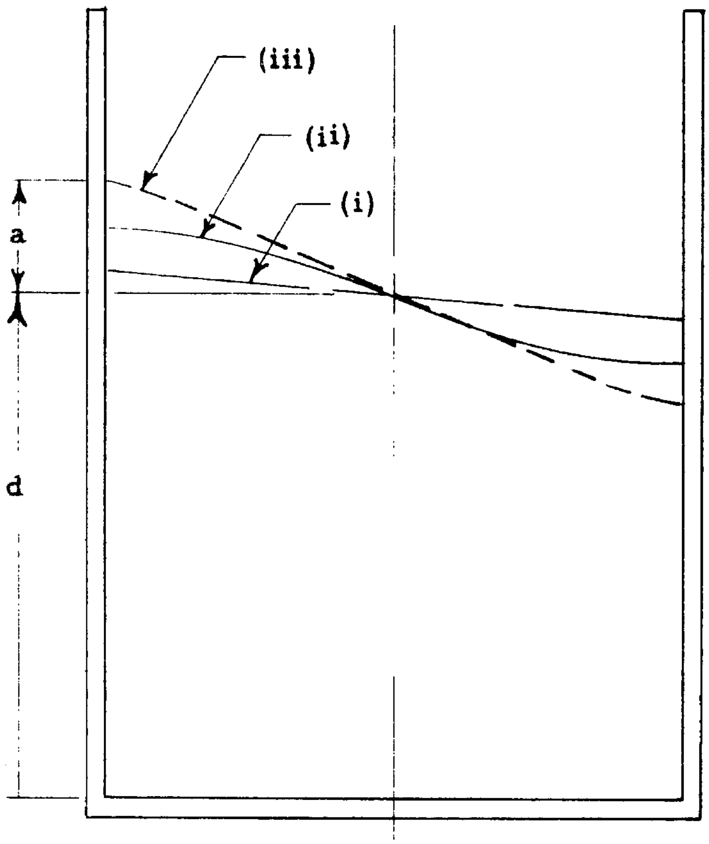

5" Depth
6" Depth
7" Depth
Amplitude of Wave / Depth of Fluid
0.200
0.100
0
0.900
0.950
1.000
1.050
Driving Frequency / Linear Natural Frequency of First Mode
Amplitude-Frequency Measurements

Figure 15



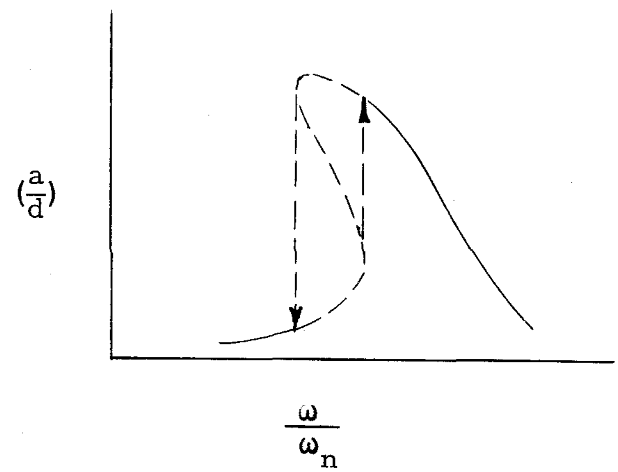
(i) $0 < \frac{a}{d} < 0.05$

(ii) $0.05 < \frac{a}{d} < 0.15$

(iii) $0.15 < \frac{a}{d}$

Figure 16. Wave Shape for Various Amplitude to Depth Ratios

condition the resonant frequency could only be attained by decreasing the driving frequency from values higher than the resonant condition until resonance had been reached. The jump is illustrated in the figure below by the dashed lines. The high degree of instability of the fluid motion at driving frequencies close to the



resonant frequency made it extremely difficult to maintain the resonant condition for any period of time and as a result of this non-linear characteristic the measurement of the wave amplitude had to be made very quickly.

An experiment was attempted for a fluid depth of 4 inches but the motion was so unstable that accurate measurements were impossible.

In exciting greater amplitudes, one would expect that at a sufficiently large $(\frac{a}{d})$ ratio a turbulent type of fluid motion could be reached where the fluid would fall back upon itself. This would add to the energy dissipation in the system and in all probability would be an upper limit to the motion of the oscillating fluid. Although these experiments did obtain an $(\frac{a}{d})$

ratio slightly higher than 0.2, this turbulent type of motion did not occur.

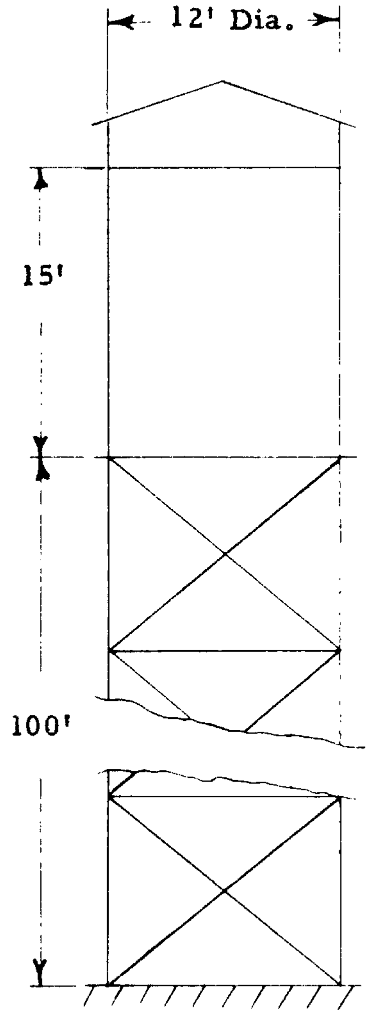
The non-linearity exhibited by the frequency curves for high $(\frac{a}{d})$ ratios shows that the region of applicability of any linear theory in describing the motion of the fluid for these regions has been exceeded. However, despite the non-linearity of the oscillation, the force-amplitude relation predicted by Housner[5] is consistent with that actually measured, at least up to an $(\frac{a}{d})$ ratio of 0.2. The theory takes into account the force exerted by the oscillating fluid from a depth h to the initial water surface, as shown by equation 29. However, in this experiment one measures the force exerted by the fluid to a depth $(h + a)$, on one side of the tank and to a depth $(h - a)$, on the other side. These experiments indicate compensating effects since the forces measured are consistent from small amplitudes to the largest ones encountered in these experiments.

IV. APPLICATIONS TO SPECIFIC TANKS

Elevated Cylindrical Water Tower

To illustrate the application of Housner's method[5] in the analysis of an elevated water tower, the typical small, cylindrical water tower shown in fig. 17 was selected. If only the first mode of fluid oscillation is considered, the system may be reduced to an equivalent mechanical system with two degrees of freedom (fig. 18). The quantities shown in this figure were obtained by using the equations in Appendix A. The tower stiffness, which corresponds to the spring constant, K_{oo}, in fig. 18, was varied from 2000 pounds per inch to 10,000 pounds per inch to provide a range of approximately one second in the undamped fundamental period of vibration. Table I gives the undamped natural period of the fundamental mode for each tower stiffness considered and equation 38 was used to calculate these periods.

Viscous damping was introduced into the mechanical system to simulate the structural damping in the tower and also the viscous damping of the fluid. The damping coefficients used were 3 percent of critical damping for the tower and 1 percent of critical damping for the fluid. The assumption of viscous damping simplifies the mathematical treatment of the problem to be discussed and has been shown by Jacobsen[7] to satisfactorily describe the behavior of vibrating systems with various types of damping.

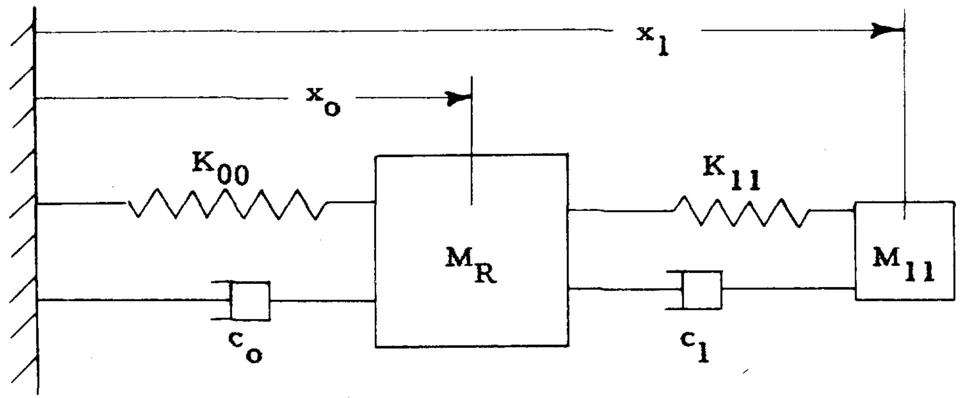


Mass of Supporting Structure = 126 $\frac{\text{lb-sec}^2}{\text{in.}}$

Mass of Tank = 42 $\frac{\text{lb-sec}^2}{\text{in.}}$

Mass of Contained Fluid = 275 $\frac{\text{lb-sec}^2}{\text{in.}}$

Figure 17. A Typical Small Cylindrical Water Tower



$M_R = 312 \frac{\text{lb-sec}^2}{\text{in}}$ $K_{00} = 2000\text{-}10000$ lb/in

$M_{11} = 42 \frac{\text{lb-sec}^2}{\text{in}}$ $K_{11} = 411.7$ lb/in

Figure 18. Equivalent Mechanical System of the Water Tower

Table 1

Tower Stiffness, K_{oo} (lb./in.)	Fundamental Period, τ_1 (seconds)
2000	2.802
2500	2.572
3000	2.421
3500	2.321
4000	2.248
4500	2.206
5000	2.172
5500	2.147
6000	2.128
6500	2.113
7000	2.102
7500	2.092
8000	2.084
8500	2.077
9000	2.072
10000	2.063

The problem considered was the determination of the maximum shear force induced in the tower structure when the water-tower system was subjected to an earthquake representative of those occurring in the West Coast areas of the United States. This requires a solution for the response of the structure when its base is subjected to an irregular, transient acceleration. The earthquake selected for this purpose was the Olympia, Washington earthquake of April 13, 1949, its recording consisting of a component in a S10°E direction and a component in a S80°W direction. The accelerograms of these components are shown in figs. 19 and 20.

Two methods of obtaining a solution to this problem were carried through. The first method[8] involved the representation of the lateral displacement, y, of the structure at any point by the sum of the normal modes of vibration. Having accomplished this each mode was then independently analyzed when acted upon by the earthquake accelerations and the sum of the maximum shear forces produced by each mode were added without regard to time of occurrence. This method gives shear forces greater than that actually occurring in the structure since the sum of the maximum values is greater than the maximum of the sums. The advantage of this method is that once the modal systems and their corresponding natural periods have been determined it is possible to use response spectra already obtained for various earthquakes[9] to calculate the maximum shear force.

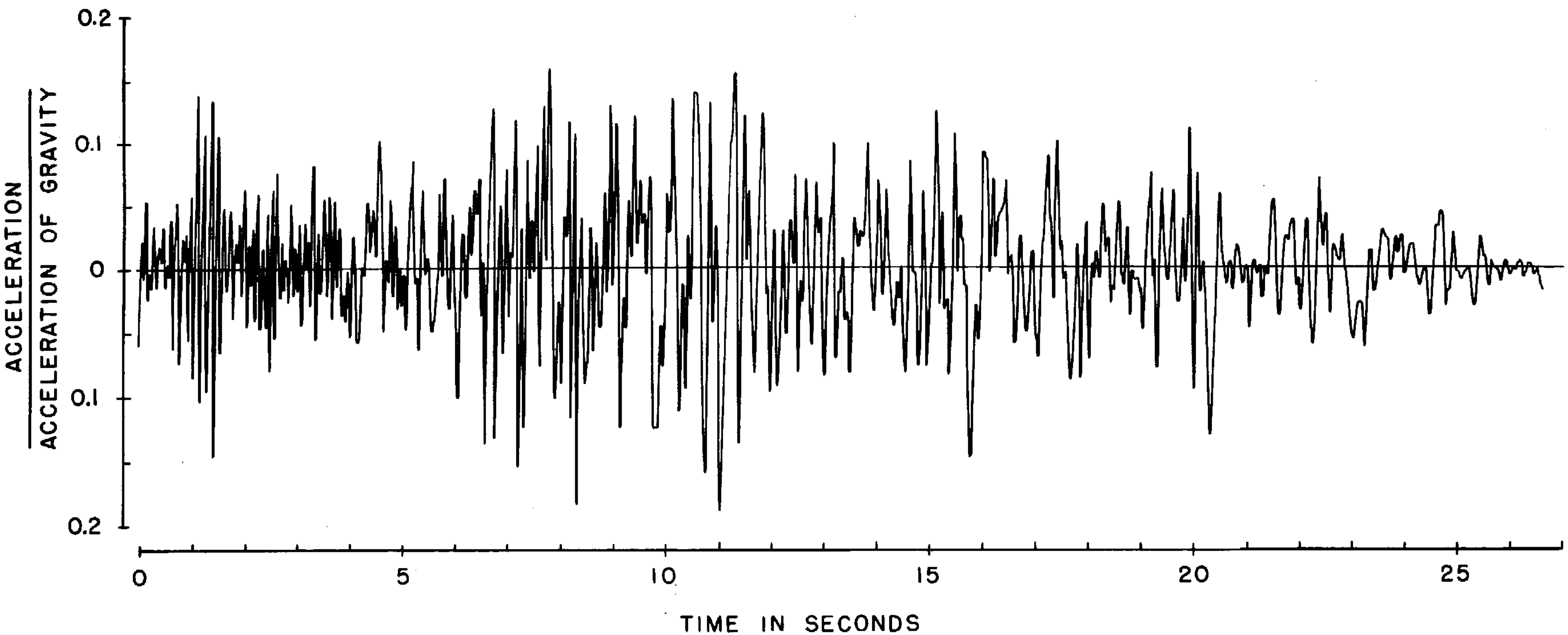


Figure 19. Accelerogram for Olympia, Washington; earthquake of April 13, 1949. Component S 10 E.

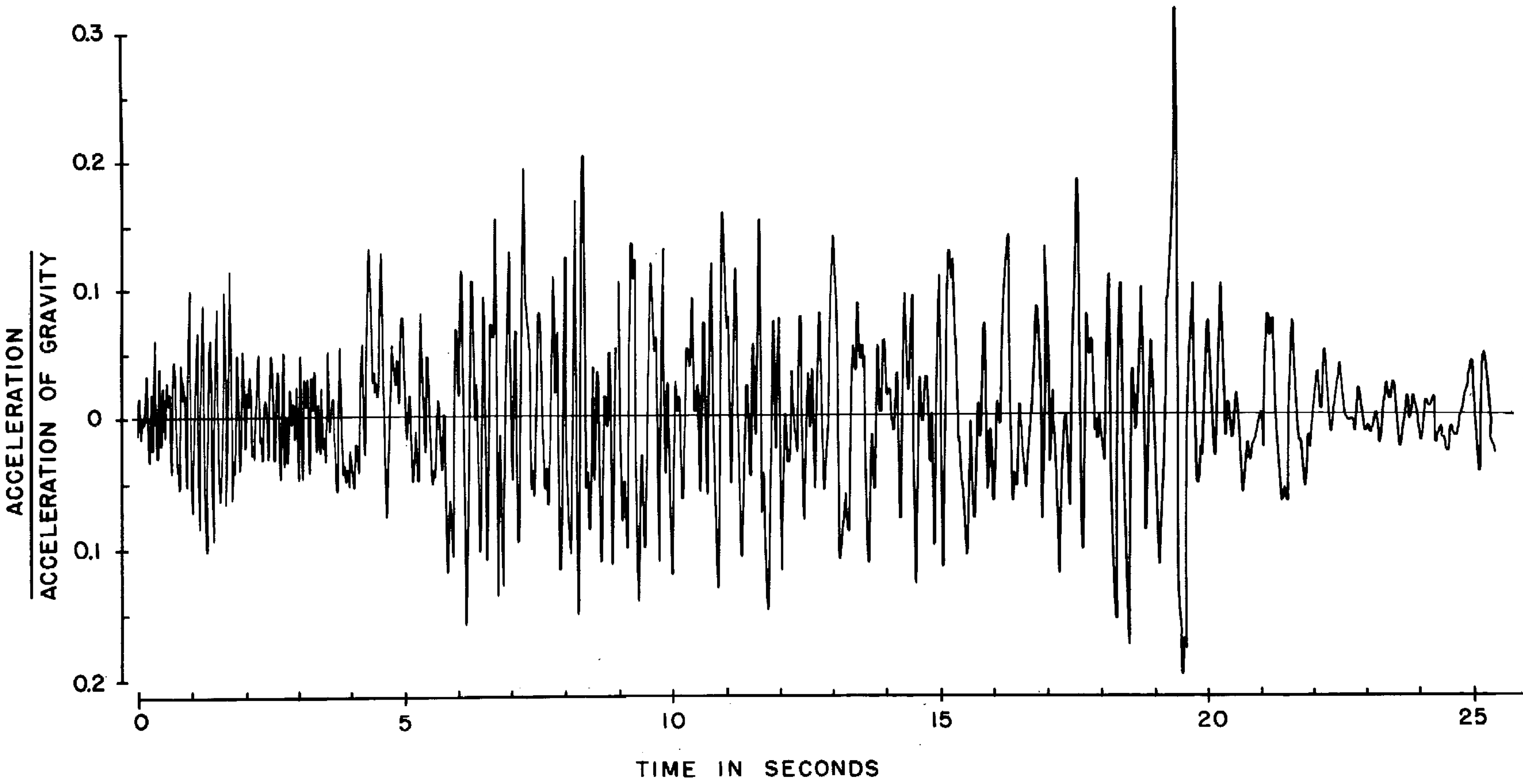


Figure 20. Accelerogram for Olympia, Washington; earthquake of April 13, 1949. Component S 80 W.

The second method was to subject a model of the water-tower system to the recorded ground accelerations of the actual earthquake components. The maximum shear force induced in the tower structure by each earthquake component was then measured for tower stiffnesses throughout the selected range. This was done electrically on the Analog Computer[10].

Modal Analysis

The calculation of the forces induced in a structure requires that the response of an elastic, damped system of many degrees of freedom to an irregular, transient base motion be determined. Considering the structure to be linearly elastic and having freedom to move in one direction only, the displacement, y, at any point in the structure can be represented by the sum of the normal modes of vibration

$$y = \sum_i c_i \phi_i e^{-n_i p_i t} \sin p_i t \qquad (52)$$

where

c_i = Undetermined coefficient

ϕ_i = ith normal mode

p_i = 2π times the frequency of vibration of the i^{th} mode

n_i = Ratio of damping in the i^{th} mode to critical damping (small damping)

t = Time

Initiating the free vibrations so that at time t = o the displacement is zero and the velocity is v_o at every point in the structure, the coefficients c_i are then evaluated in the Fourier manner

$$c_i = \frac{v_o}{p_i} \frac{\int \phi_i \rho}{\int \phi_i^2 \rho} = \frac{v_o}{p_i} W_i \tag{53}$$

where ρ is the density and the integrals are evaluated over the total mass of the structure. The free vibrations are then

$$y = \sum_i \frac{v_o}{p_i} W_i e^{-n_i p_i t} \sin p_i t \tag{54}$$

Subjecting the base of the structure to a variable acceleration, a, the displacement at time, t, is

$$y = \sum_i \frac{W_i}{p_i} \phi_i \int_o^t a\, e^{-n_i p_i (t-\tau)} \sin p_i (t-\tau) d\tau \tag{55}$$

or

$$y = \sum_i \frac{W_i}{p_i} \phi_i X_i \tag{56}$$

where

$$X_i = \int_o^t a\, e^{-n_i p_i (t-\tau)} \sin p_i (t-\tau)\, d\tau \tag{57}$$

The integral, X_i, was maximized to give the peak response of each of the two normal modes of vibration used in the solution of this problem. The factor W_i/p_i is a function of the mass, rigidity and dimensions of the structure, ϕ_i is a function of the space coordinates and X_i is a function of the ground acceleration, damping ratio, undamped period of vibration and the duration of the excitation. The square of equation 6, X_i^2, represents the kinetic energy of an oscillator with frequency $p_i/2\pi$ which is subjected to the ground acceleration.

Alford, Housner, and Martel[9] have evaluated X_i for periods of vibration between 0.1 second and 3.0 seconds for several values of damping ratio. Using the maximum response for particular values of p_i and n_i they have plotted spectra consisting of maximum response versus period of vibration, with damping ratio n, as a parameter. The maximum values of X, when plotted against periods give the "velocity spectrum" and the maximum values of $(1/p)X$ correspond to the value of acceleration response. Figs. 21 and 22 are the velocity spectrum of the earthquake components considered in this problem.

Applying modal analysis to the water tower system yields a maximum shear force of 42,000 pounds induced in the tower structure by the S10°E component of the earthquake and a maximum shear force of 40,000 pounds by the S80°W component. Both of these forces occurred when the tower stiffness was very nearly 7,000 pounds per inch.

Analog Computer Technique

The mathematical calculation of base shear forces in structures would be a formidable task even when applied to an equivalent mechanical system of two degrees of freedom (fig. 18).

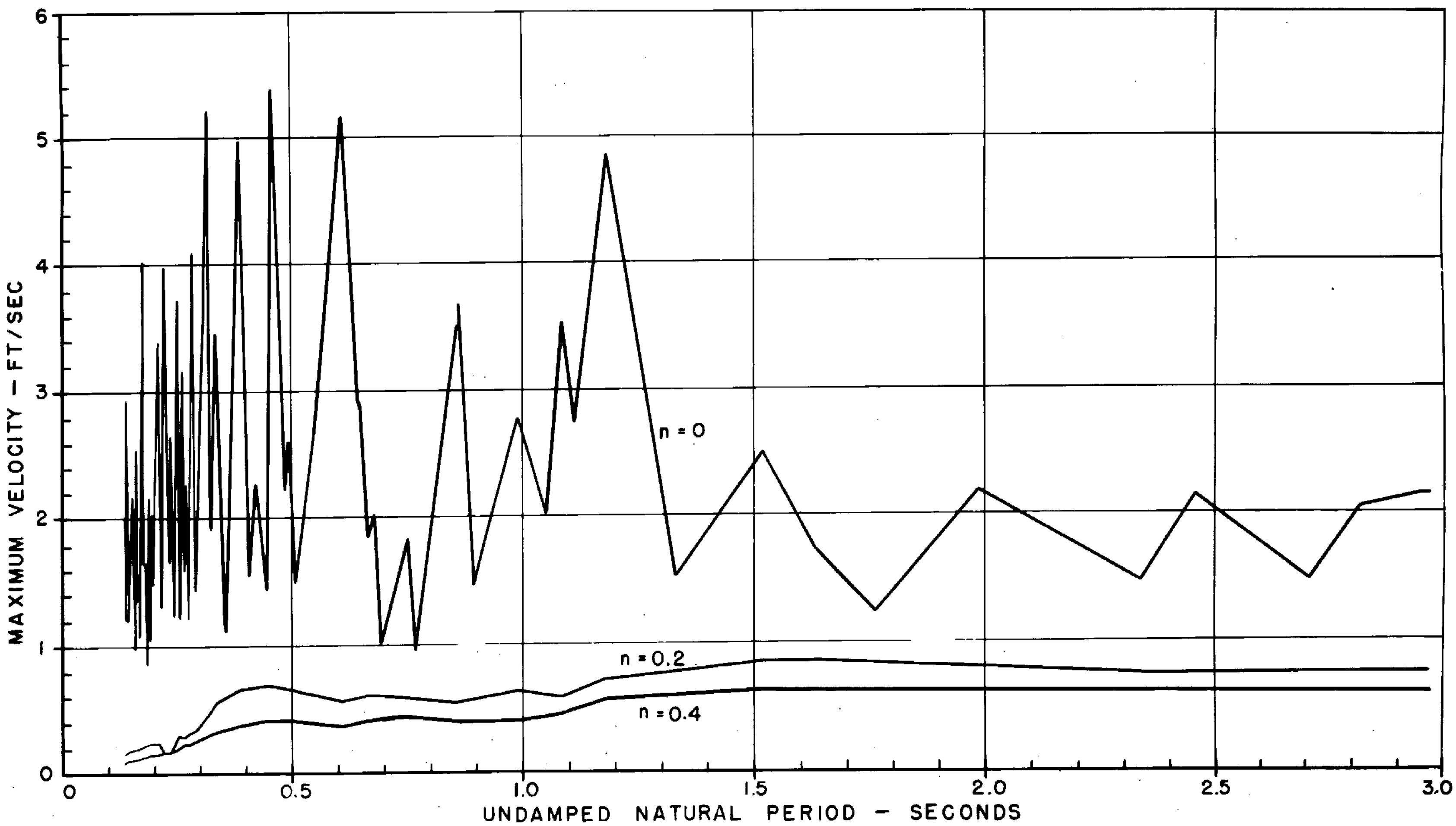


Figure 21. Velocity spectrum for Olympia, Washington; earthquake of April 13, 1949. Component S 10 E.

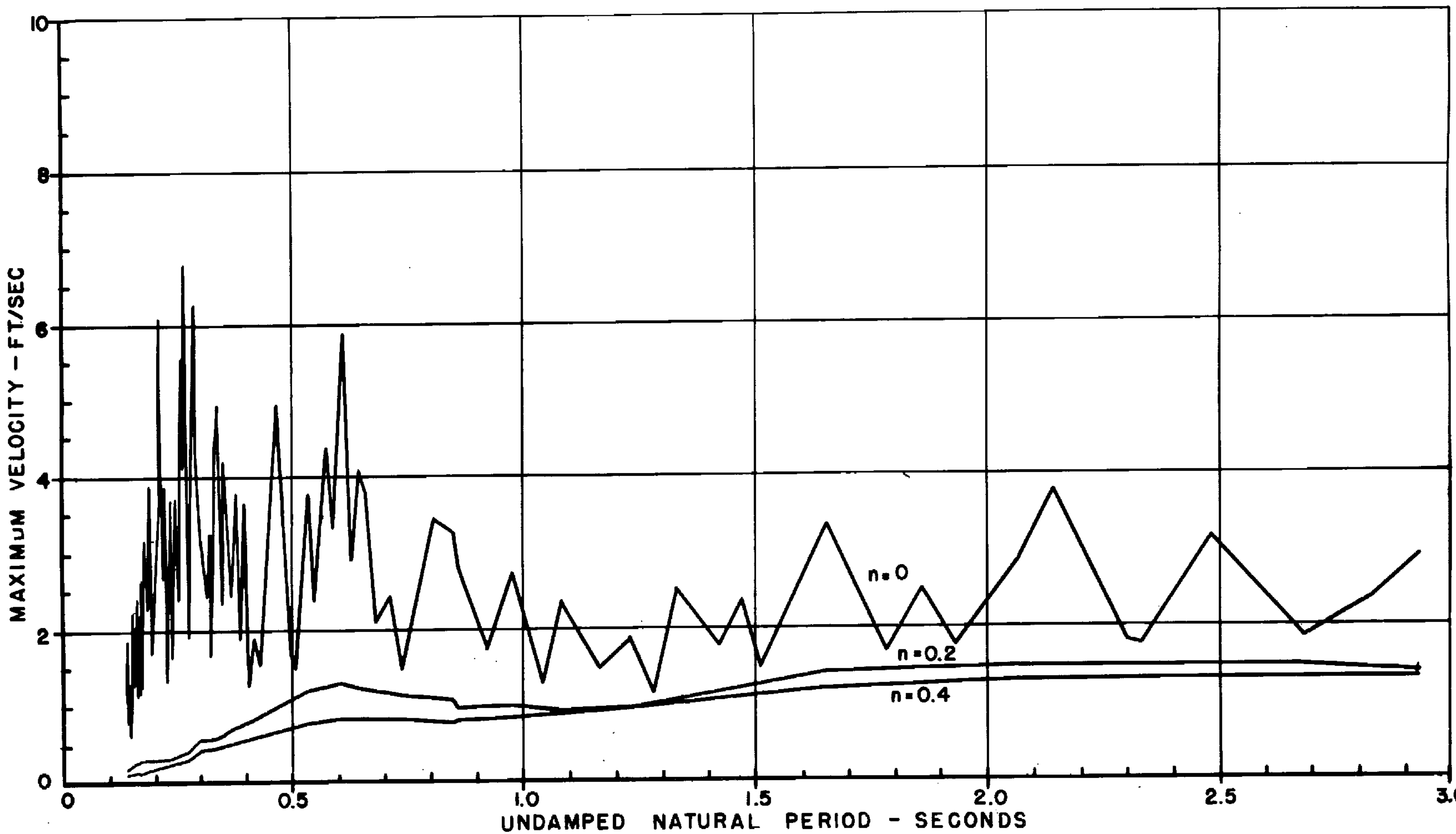


Figure 22. Velocity spectrum for Olympia, Washington; earthquake of April 13, 1949. Component S 80 W.

This would involve a solution for the response of the system when subjected to an irregular, transient ground acceleration. However, the use of an electric analog computer provides a fast and reliable method of analyzing such systems when subjected to the recorded seismic accelerograms (figs. 19 and 20). The shear forces may be read directly and the effect of tower stiffness can be observed as a single set of dials is turned.

The components of the electric analog are analogous to the components of the mechanical system and the equations governing the electric circuit are of the same form as the equations of motion of the mechanical system. The properties of the circuit can be changed rapidly and the voltage and current at any point can easily be read.

The equations of motion of the mechanical system shown in fig. 18 are

$$M_R \ddot{x}_o + c_o \dot{x}_o + c_1(\dot{x}_o - \dot{x}_1) + K_{oo} x_o + K_{11}(x_o - x_1) = \ddot{Z} M_R$$

$$M_{11} \ddot{x}_1 + c_1 (\dot{x}_1 - \dot{x}_o) + K_{11}(x_1 - x_o) = \ddot{Z} M_{11} \tag{58}$$

where

M_R = Equivalent rigid mass of water tower system

M_{11} = Equivalent mass of first mode of fluid oscillation

c_c = Critical damping

c_o = Damping in tower structure = $0.03 c_c$

c_1 = Damping of fluid = $0.01 c_c$

x_o = Displacement of M_o relative to base

x_1 = Displacement of M_1 relative to base

Z = Absolute displacement of ground

The shear force in the tower is

$$F = K_{00}x_0 \tag{59}$$

and it is the maximum value of this shear force occurring during an earthquake that is of importance in the design of the tower structure.

A list of the mechanical electrical analogies used appears in Table 2 and fig. 23 shows the electrical analog of the mechanical system (fig. 18). For this particular problem an inductance of 1 henry corresponded to a mass of 104 $\frac{\text{lb-sec}^2}{\text{in}}$, a capacitance of 1.670 microfarads corresponded to a spring constant of 411.7 pounds per inch, and the value of N for the earthquake records used was 389. The excitation was supplied from a function generator(10) which applied voltages generated from a film record of the actual earthquake components. The excitation was supplied to each circuit in proportion to its inductance and the effect of varying the tower stiffness was obtained by changing the value of C_o from 0.334 microfarads down to 0.069 microfarads. This was done for each of the earthquake components.

The shear force corresponds to the voltage at the capacitor, C_o. Since only the maximum shear force was desired the voltage was read on an oscilloscope adjusted so that a stationary

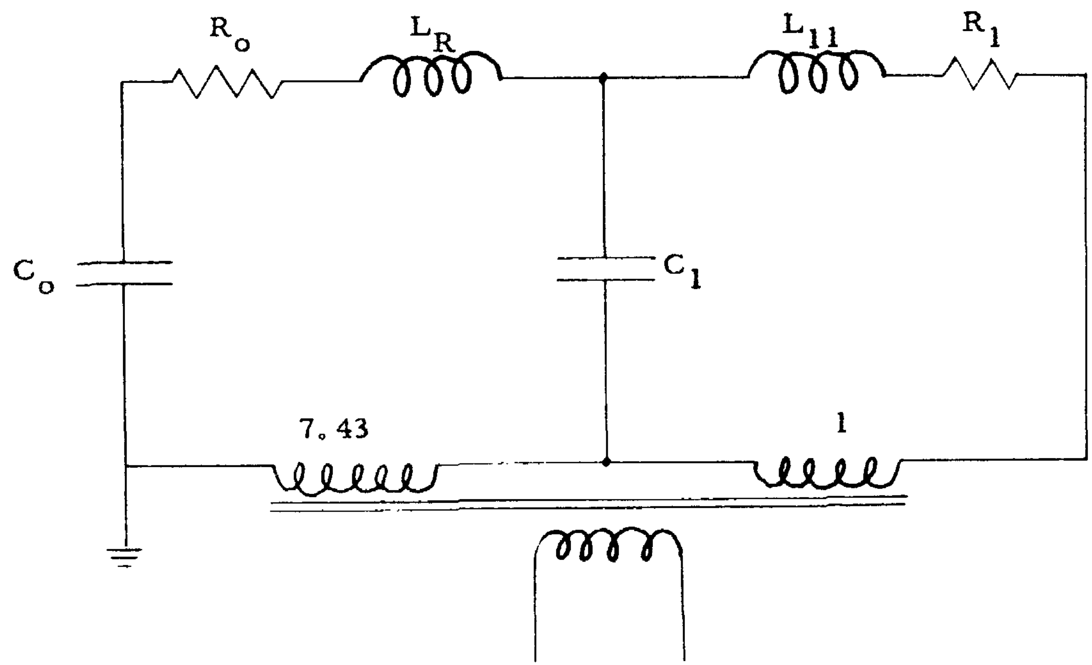


Figure 23. Electrical Analogy of the Mechanical System

Table 2

MECHANICAL SYSTEM	ELECTRICAL SYSTEM
M = mass of system	L = inductance = $\frac{a}{N^2}$ M
K = spring constant	C = capacitance $\frac{1}{aK}$
c = damping constant	R = resistance $\frac{a}{N}$ c
τ = actual period of vibration	$\tau' = \frac{\tau}{N}$ = simulated period
F = exciting force	E = applied voltage
x = displacement	q = electrical charge

$$x = \frac{aF}{E} q$$

N = ratio by which time is reduced in analog

a = arbitrary impedance base factor

transient response pattern was obtained from which the maximum response could be read directly. This maximum value was obtained as an amplification of the peak input acceleration. The maximum shear force could then be expressed as

$$F = \left(\frac{E_{out}}{E_{in}}\right) \ddot{Z} M_R \tag{60}$$

in which the maximum value of the ratio $\frac{(Eout)}{(Ein)}$ is used for each simulated tower stiffness and $\ddot{Z}$ is the peak earthquake acceleration. This was done for each of the earthquake components and the results are shown in figs. 24 and 25. The maximum shear force induced in the tower structure for the range of stiffness considered was 37,500 pounds by the S10°E component and 34,000 pounds by the S80°W component. These correspond respectively to 0.312g's and 0.284g's equivalent horizontal acceleration. The corresponding forces given by the modal analysis were 42,000 pounds and 40,000 pounds respectively.

Cylindrical Tank with Hemispherical Bottom

Most elevated water tanks of cylindrical cross-section have a rounded bottom. No exact solution for the period of the fundamental mode of fluid oscillation exists for the case of such a rounded container. However, an approximation for the period of a hemispherical container may be found by means of Rayleigh's principle. This has been done by Binnie(11) and yields

$$\tau = 5.115 \left(\frac{R}{g}\right)^{\frac{1}{2}}$$

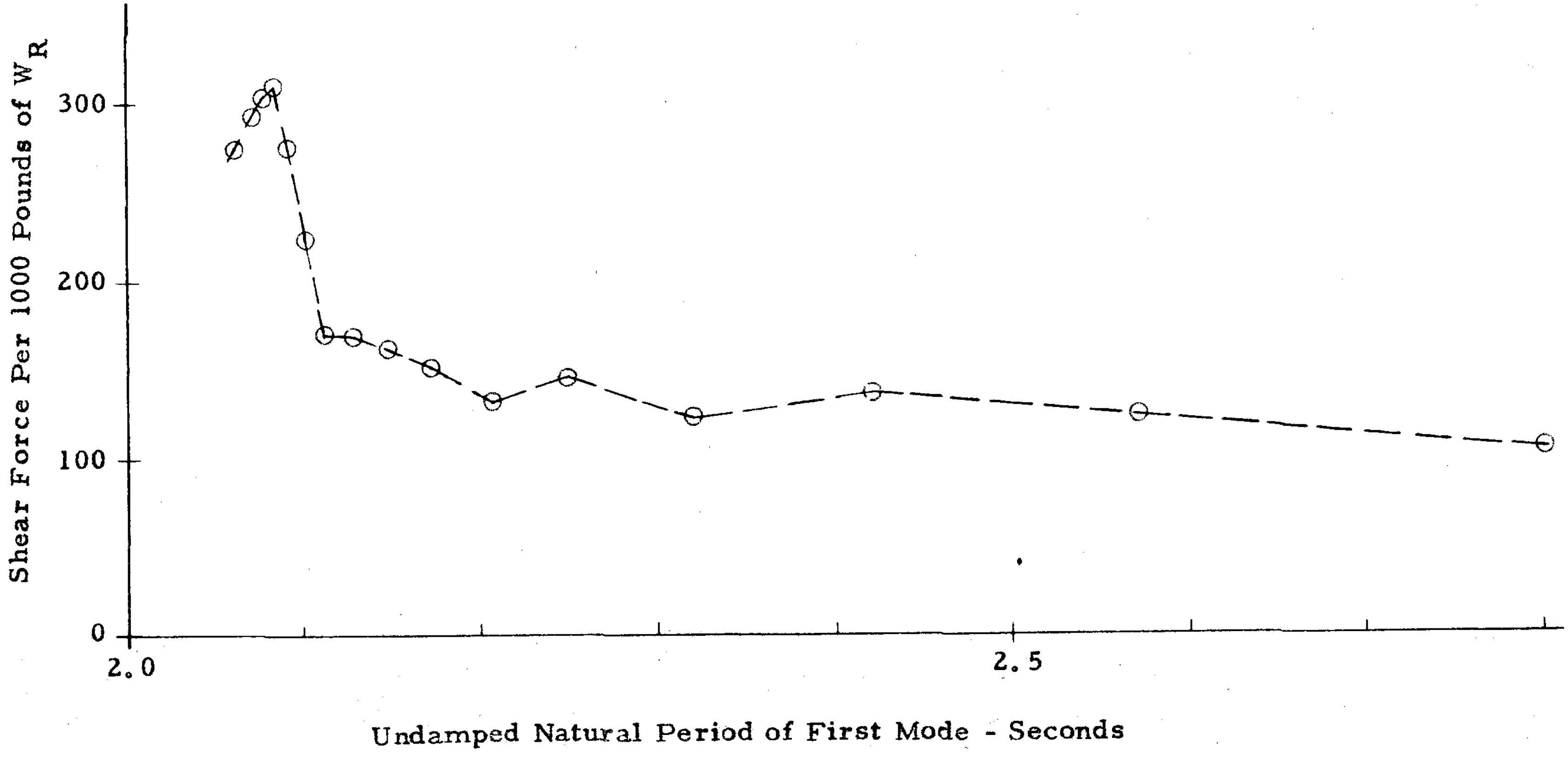


Figure 24. Shear Force Spectrum for Olympia, Washington; Earthquake of April 13, 1949. Component S 10 E

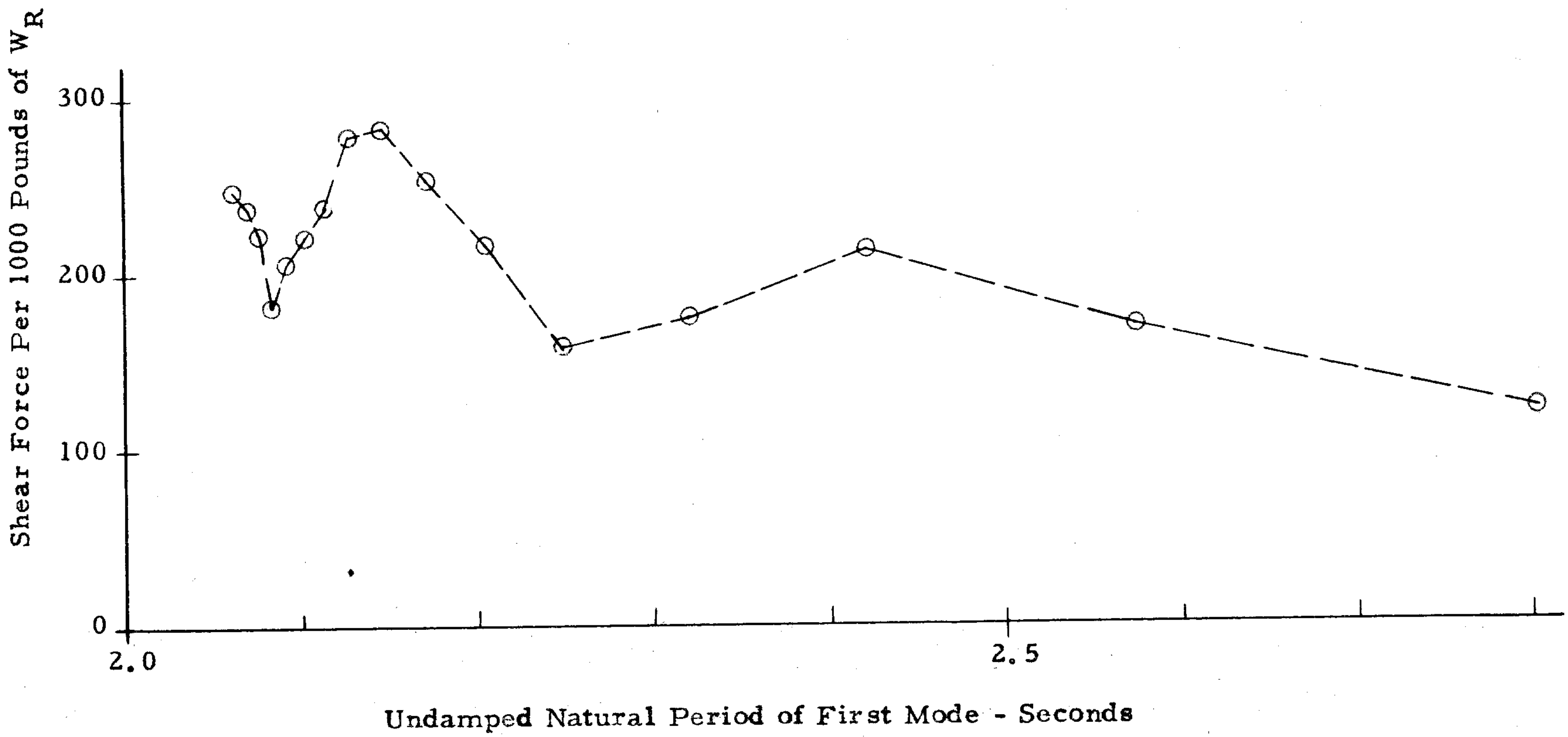


Figure 25. Shear Force Spectrum for Olympia, Washington; Earthquake of April 13, 1949. Component S 80 W

for the fundamental period where R is the radius. This gives values of τ, approximately 3 percent higher than those measured by Binnie's experiments.

Replacing a hemispherical bottom with an equivalent cylinder of radius R, and equating volumes gives a cylinder of depth, $d = \frac{2}{3} R$. Using equation A-5 in Appendix A for the fundamental period of the equivalent cylinder gives

$$\tau = 5.052 \left(\frac{R}{g}\right)^{\frac{1}{2}}$$

which is 1 percent lower than that found by Binnie.

Since the periods agree quite closely it appears that a logical approach to the analysis of this type of water-tower system would be to replace the rounded bottom with an equivalent cylinder. The problem then becomes the same as that previously studied.

V. SUMMARY

This thesis presents the results of experimental measurements of the hydrodynamic forces induced in fluid containers when the containers are subjected to varying accelerations. Experiments were performed on a model of a rectangular tank and the measured forces due to the induced convective pressures were compared with the convective forces predicted by a simplified linear theory in which the fluid system is reduced to a system of simple oscillators. An investigation was made of the limit of applicability of this simplified theory, which assumes small fluid displacements, and to observe the effect of the non-linearity of the fluid motion on the predicted forces.

The electric analog computer technique was employed to determine the shear forces induced in a water tower structure by an actual earthquake. The circuitry used was completely analogous to the physical model of the water-tower system which was reduced to a mechanical system of two degrees of freedom, and the forces obtained by this method were compared with those obtained by a modal analysis.

The conclusions drawn from the body of the material in this report are as follows:

1. The non-linear character of the fluid oscillation becomes observable at ratios of amplitude of wave to depth of fluid, $(\frac{a}{d})$, equal to 0.05.

2. The amplitude-frequency curves exhibit a very pronounced jump phenomenon during which the $(\frac{a}{d})$ ratios jump suddenly from

small values to comparatively high values at frequencies greater than the resonant condition. The non-linearity becomes exceedingly strong for shallow depths of fluid and the motion of the fluid is highly unstable at frequencies close to the resonant frequency.

3. Although experimental measurements were made for $(\frac{a}{d})$ ratios as great as 0.20, there was no indication that the velocity field of the fluid was sufficiently altered so as to give a force-amplitude relation different from that measured at small amplitudes.

4. An analysis of a typical water-tower system can be done easily with an electric analog computer. In the absence of this type of equipment, a modal analysis can be performed employing previously computed response spectra. In this paper the modal solution yielded forces up to 18 percent greater than those obtained on the analog computer. This magnitude of error is reasonably representative of the error encountered in the application of modal analysis to an earthquake type acceleration function(12).

5. The fundamental frequency of fluid motion for a hemisphere is almost the same as that for a cylinder of equal radius and volume. It is concluded, therefore, that when applying Housner's method(5) to the analysis of a tank with a rounded bottom, the bottom can be replaced by an equivalent cylinder of equal volume.

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APPENDIX A

EQUATIONS FOR A CYLINDRICAL TANK

The equations for a cylindrical tank are derived in the same manner as those for a rectangular tank. The only change in notation is as follows:

R = Radius of the cylinder

Impulsive Pressures

$$M_o = \frac{M \tanh \sqrt{3}\,\frac{R}{h}}{\sqrt{3}\,\frac{R}{h}} \tag{A-1}$$

$$\frac{h}{R} \leq 1.6 \tag{A-2}$$

$$h_o = \frac{3}{8} h \left[1 + \frac{4}{3} \left(\frac{\sqrt{3}\,\frac{R}{h}}{\tanh \sqrt{3}\,\frac{R}{h}} - 1 \right) \right] \tag{A-3}$$

Convective Pressures

$$\theta = \theta_o \frac{\sinh \sqrt{\frac{27}{8}}\,\frac{y}{R}}{\sinh \sqrt{\frac{27}{8}}\,\frac{h}{R}} \tag{A-4}$$

$$\omega^2 = \frac{g}{R} \sqrt{\frac{27}{8}} \tanh \left(\sqrt{\frac{27}{8}}\,\frac{h}{R}\right) \tag{A-5}$$

$$p_w \cong -\rho \frac{\partial \theta^*}{\partial y} \frac{R^3}{3} \left(1 - \frac{\cos^2\theta}{3} - \frac{\sin^2\theta}{4}\right) \cos\theta \qquad \text{(A-6)}$$

$$M_1 \cong \frac{M}{4} \left(\frac{11}{12}\right)^2 \sqrt{\frac{27}{8}} \frac{R}{h} \tanh \sqrt{\frac{27}{8}} \frac{h}{R} \qquad \text{(A-7)}$$

$$A_1 \cong \frac{12}{11} \theta_o h \frac{1}{\sqrt{\frac{27}{8}} \frac{h}{R} \tanh \sqrt{\frac{27}{8}} \frac{h}{R}} \qquad \text{(A-8)}$$

$$h_1 \cong h \left(1 - \frac{\cosh\sqrt{\frac{27}{8}} \frac{h}{R} - \frac{135}{88}}{\sqrt{\frac{27}{8}} \frac{h}{R} \sinh \sqrt{\frac{27}{8}} \frac{h}{R}}\right) \qquad \text{(A-9)}$$