TRANSFORMATION OF LINEAR SPACES
AND LINEAR OPERATORS
BY INVERSE REVERSION

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1 9 5 2

Acknowledgment

The author gratefully acknowledges the inspiration, assistance and good counsel he has received from Professor A. D. Michal, under whose guidance this thesis was prepared.

Summary

This thesis develops a new method for transforming and extending the classes of operators and operands which appear in certain linear operations in such a way that restrictions on the ranges and domains of the operands and on the algebraic manipulation of the operators are reduced or removed. In particular, the method leads to a complete rationalization of the p operators and impulse 'functions' employed by Heaviside, Dirac and others in the analysis of certain linear systems.

In this method, the operators A of a primary class K are, in effect, first reversed, forming A^*, then inverted, forming A^{*-1}, the <u>inverse reverse</u> of A, and these operators are utilized to effect the remaining transformations and class extensions. The method is therefore epitomized by the phrase <u>inverse reversion</u>.

Contents

Introduction

Let F be an abelian additive group and L the class of additive operators A on subsets of F to F. A is *complete*, *unrestricted*, *univocal* according as (1) the range of A is F, (2) the domain of A is F, (3) $Af=0$ implies $f=0$ for f in the domain of A; otherwise A is respectively incomplete, restricted, equivocal. A is *perfect* on F if (1), (2), (3) hold; then A has a unique inverse $A^{-1} \in L$ (i.e. $A^{-1}Af = AA^{-1}f = f$ for $f \in F$.)

Suppose $K \subset L$ is a commutative ring containing the unit element 1 of L and such that every non-zero $A \in K$ has a (right) reverse $A^{*} \in L$ (i.e. $AA^{*}f = f$ for $f \in F$. Then A is complete, but possibly restricted or equivocal, and A^{*} is unrestricted and univocal, but possibly incomplete.) Let K^{*} be the class of all A^{*} for $0 \neq A \in K$, and $K_g = KK^{*}$, the class of all products AB^{*} for $A, B \in K$. The *commutator* of A, B is $(A,B) = (AB^{*} - B^{*}A) \in L$; it vanishes if A^{-1} exists. In particular, let $\tilde{A} = (A,A) = 1 - A^{*}A$. Then $A\tilde{A} = 0$ and $AA^{*} = 1$.

Suppose further that $(AB)^{*} = A^{*}B^{*}$ for $A, B \in K$ (then K_g is closed under addition and subtraction:

$$A_1B_1^{*} \pm A_2B_2^{*} = (A_1B_2 \pm A_2B_1)(B_1B_2)^{*} \text{ if } A_1B_1^{*}, A_2B_2^{*} \in K_g)$$

and that K_g includes a ring K_d such that (1) *for any $A \in K$, $AB^{*} \in K_d$ for some $B \in K$*, (2) *every non-zero element of K_d is unrestricted and univocal, and* (3) $(A_1B_1^{*})(A_2B_2^{*}) = A_1A_2\,(B_1B_2)^{*}$ *if* $A_1B_1^{*}, A_2B_2^{*} \in K_d$ (then K_d is commutative and contains K^{*}.)

Our main result is that under the foregoing suppositions F may be extended to an essentially unique abelian additive group $\bar{F} = F/K_d$ and K_d may be extended to a unique field $\bar{K}_d = K_d/K_d$ within the class $\bar{L}$ of additive operators on $\bar{F}$ to $\bar{F}$.

The classes $\bar{F}$ and $\bar{K}_d$ are the ranges of $A^{*-1}f$ and $A^{*-1}\chi$ respectively, with $A \varepsilon K,\ f \varepsilon F,\ \chi \varepsilon K^*$, where A^{*-1} is the inverse reverse of A. Hence the process which generates $\bar{F}$ and $\bar{K}_d$ is called inverse reversion (see Section 8).

Each non-zero element of $\bar{K}_d$ is perfect on $\bar{F}$. In particular A^{*-1} is perfect on $\bar{F}$, whereas $A \varepsilon K$ is merely complete on F (since A^* exists). We can therefore reduce the algebraic restrictions and limitations on equations involving operators $A \varepsilon K$ by replacing them with expressions involving their perfect counterparts A^{*-1}. This is the essential idea of the 'Heaviside' calculus. It is implemented by the following considerations:

(1) The correspondence between A and A^{*-1} is an isomorphism (Theorem 29):

$$(A \pm B)^{*-1} = A^{*-1} \pm B^{*-1} \quad \text{and} \quad (AB)^{*-1} = A^{*-1}B^{*-1}$$

(2) For any f in the domain of A (see Theorems 32, 41):

$$A^{*-1}f - Af = A^{*-1}\breve{A}, \text{ i.e. } Af = A^{*-1}(1-\breve{A})f$$

The form of $\breve{A}f$ will of course depend upon the interpretation of F, K, K^*, K_d. In Sections 1 - 8 below, the relationship between A and A^* is developed (Sections 2, 3, 4), the form of $(A,B) = AB^* - B^*A$ is determined (Section 5), the classes K_g and K_d are studied (Section 6), a useful 'transformation' calculus is outlined (Section 7) and finally the inverse reversion process is carried out

(Section 8) for the case where F is the class of everywhere continuous matrices $[f(t)]$ on the real numbers to a complex Banach space (see Section 1), K is the class of 'derivators' $A = a_0 D^n + a_1 D^{n-1} + \dots + a_n$, K^* is the class of 'integrators' A^* characterized by the system $AA^*f = f$, $A^*f/_0 = 0$, $DA^*f/_0 = 0, \dots, D^{n-1}A^*f/_0 = 0$ (a constructive definition is given in Section 3) and K_d is the class of 'derigrators' AB^* of non-negative rank. It is shown that the isomorphism between A and A^{*-1} provides a satisfactory rationalization of the methods employed by Heaviside for lumped linear systems, without *ad hoc* transformations of the operands. Thus Heaviside's mystical operator p becomes D^{*-1}, the inverse reverse of the elementary derivator, and Dirac's 'function' δ becomes $D^{*-1}[1]$, where $[1]$ is the matrix of the unit function.

In Section 9, the essential steps of the inverse reversion process are given for the case where F is the class of sectionally continuous matrices $[f(t)]$, with $f(t) = 0$ for all sufficiently small t, K is the class of derivators A, as above, K^* is the class of integrators A^* characterized by the equation $AA^*f = 0$ and the condition that A^*f and its first $(n-1)$ derivatives be everywhere continuous and vanish for all sufficiently small t, and K_d is the class of derigrators, as above, of non-negative rank.

'Strong' and 'weak' limits, derivatives and integrals of sectionally continuous matrix functions $[f_x(t)]$ of x are introduced in Sections 10 and 11, and the representation of elements of $\overline{F}$ and $\overline{K_d}$ as infinite superposition integrals is given. These sections barely outline the calculus of such matrix functions; the details will have to be presented elsewhere.

Various other interpretations of F, K, K^*, K_d have been studied. Thus F may be taken as the class of sectionally continuous matrices $[f(t)]$ with $f(t) = 0$ for $t < 0$ (Heaviside case), K and K_d as above, and A^* characterized by the equation $AA^*f = 0$ and the condition that A^*f and its first $(n-1)$ derivatives be continuous for $t \geqslant 0$ and vanish for $t < 0$ (and hence for $t = 0$). Or F may be taken as the class of matrices $[f(n)]$ (essentially sequences), where $f(n)$ is a function on the non-negative integers to a linear space, K as the class of operators $A = a_0 E^n + a_1 E^{n-1} + \cdots + a_n$, where $E[f(n)] = [f(n+1)]$, K^* as the class of operators A^* characterized by the system

$$AA^*f = f, \quad f/_0 = 0, \; Ef/_0 = 0, \; \ldots, \; E^{n-1}f/_0 = 0,$$

and K_d as the class of operators AB^* of rank $\geqslant 0$ (a valid interpretation also results if E is replaced by Δ, where $\Delta[f(n)] = [f(n+1) - f(n)]$. With this latter interpretation (or others which are less familiar but more convenient), the method of inverse reversion and the related concepts and techniques of reversion, commutation and 'integrator' transformation have proved very useful in the theory of linear difference equations, including fractional differencing and summation. The method may also be applied to certain 'differentiators' of the Frechet and Goteaux type in arc-wise connected spaces, the right reverses then being given by appropriate line integrals. These and other applications will be presented elsewhere.

1. General Definitions and Notations

Unless the contrary is explicitly indicated, $F(t)$ is the class of everywhere continuous functions f of t on the real numbers R to a complex Banach space S, $F_n(t)$ the subclass of $F(t)$ whose elements are n-fold continuously derivable, $F_n^*(t)$ the subclass of $F_n(t)$ whose elements, together with their first $(n-1)$ derivatives vanish initially (at $t=0$). $\Phi(t)$, $\Phi_n(t)$ and $\Phi_n^*(t)$ are the corresponding classes of functions f of t on R to C, the complex number class. The notations $V_a^t f$, $f|_{t=a}$ will be used for the value of f at $t=a$.

The mark $[f]$ is the matrix of the function f of t. F is the class of matrices corresponding to the functions $\varepsilon\, F(t)$ and similarly for F_n, F_n^*, etc. Equal matrices correspond to equal functions: $[f] = [g]$ is equivalent to $f \underset{t}{=} g$ i.e. to $f = g$ for every t. The sum $[f]+[g]$ and difference $[f]-[g]$ of $[f]$, $[g]$ and the product of $[f]$ by $\alpha \varepsilon C$ are defined as the matrices respectively equal to $[f+g]$, $[f-g]$ and $[\alpha f]$. If $f \varepsilon \Phi$ and $\sigma \varepsilon S$, $\sigma[f]$ is the matrix equal to $[\sigma f]$. When considering functions with different arguments, the more complete notation $[f]_t$ is convenient. Thus $[t(t-1)]_t = [\tau^2-\tau]_\tau$. The value $V_a[f]$, or $[f]|_a$ of $[f]$ at a is $V_a^t f$.

In practice, contrary to the usage above, single letters will be used for matrices and the same letters with the argument in parentheses for the corresponding function: $f = [f(t)]$

An operator on a subclass K of a space Σ to a subclass K' of a space Σ' is a mark A such that $A\chi$ is a function of χ on K to K'.

The domain of A in Σ and range of A in Σ' are those of $A\chi$. Operators A, B defined in a space Σ are equal on $K < \Sigma$ if $A\chi = B\chi$ for $\chi \varepsilon K$. $A = B$ if they have the same domain in Σ and are equal on that domain. If B is on $K < \Sigma$ to $K' < \Sigma'$ and A is on $K' < \Sigma'$ to $K'' < \Sigma''$, then the product AB is the operator defined by $(AB)\chi = A(B\chi)$ FOR $\chi \varepsilon K$.

We shall be mainly concerned with linear operators, i.e. additive homogeneous operators on linear manifolds of linear spaces to linear spaces. The null domain N_A of a linear operator in a linear space Λ is the linear manifold of all $\chi \varepsilon \Lambda$ such that $A\chi = 0$. A set $\lambda < \Lambda$ spans λ' with C (OR R) if every element of λ' is a finite linear combination with coefficients in C (OR R) of elements of λ; if moreover every such linear combination belongs to λ', then λ is a basis with C (OR R) of λ'. For linear operators A, B defined in a linear space Λ, the sum $A + B$ and difference $A - B$ of A and B and the product of A by $\alpha \varepsilon C$ (OR R) are the linear operators respectively defined by $(A+B)f = Af + Bf$, $(A-B)f = Af - Bf$, $\alpha(Af) = (\alpha A)f$.

The operators $\mathcal{D}, I, E_\alpha$ are defined as follows: $\mathcal{D}f = [\frac{d}{dt}f(t)]$ FOR $f \varepsilon F_1$ OR Φ_1; $If = [\int_0^t f(\tau)d\tau]$ for $f \varepsilon F$ OR Φ ($\int_0^t$ is replaced by $\int_{-\infty}^t$ after Section 9); $E_\alpha f = [e^{\alpha t}f(t)]$ FOR $f \varepsilon F$ OR Φ. These operators belong to the class L of all categorical linear operators with ranges and domains in F, i.e. operators which may be defined without specifying S.

A correspondence $P(\chi, y)$ between classes X, Y is univocal in X if for any $\chi_1, \chi_2 \varepsilon X$ and any $y \varepsilon Y$, $P(\chi_1, y)$ and $P(\chi_2, y)$ implies $\chi_1 = \chi_2$. The correspondence is biunivocal if it is univocal in χ and in y.

If A is on operator between $K \subset \Sigma$ and $K' \subset \Sigma'$, then $Ax = y$ is a correspondence between K, K'. If this correspondence is univocal in x, then A has a unique right reverse A^* on K' to K. If $Ax = y$ is equivocal in x (i.e. not universal), then a right reverse exists but is not unique (a supplemental condition may then be applied to make it unique). Similar remarks apply to univocality in y and left reverses ${}^*A$ on K' to K. If $Ax = y$ is biunivocal, then a unique inverse $A^{-1} = A^* = {}^*A$ exists on K' to K.

2. Derivators

Let $p_0, p_1, \cdots, p_m$ be the coefficients of a $\mathcal{D}$- polynomial P. The coefficient sequence of P is the sequence $\{a_k\}$ such $a_k = p_k$ for $k = 0, 1, \cdots, m$ and $a_k = 0$ for $k > m$. Two $\mathcal{D}$- polynomials P, Q are cogredient: $P \sim Q$ if they have the same coefficient sequence. Clearly cogredience is an equivalence relation (reflexive, symmetric, transitive) and cogredient $\mathcal{D}$- polynomials are equal (as elements of L).

THEOREM 1 *For any two $\mathcal{D}$- polynomials P, Q the following statements are equivalent:*

(1) $P = Q$ on F_n^* *for some* n,

(2) P *and* Q *are cogredient*

(3) $P = Q$

Proof: Suppose (1). The matrix $\left[\frac{t^{n+k}}{(n+1)(n+2)\cdots(n+k)}\right] \varepsilon F_n^*$ for $k = 0, 1, \cdots$. Let $\{a_k\}, \{b_k\}$ be the coefficient sequences of P, Q and let $R = \sum_k c_k \mathcal{D}^k$, where $c_k = a_k - b_k$. Then $R = 0$ on F_n^*, hence $\lim_{\tau \to \infty} \tau^{-n} V_\tau R[t^n] = c_0 = 0$, so that $R = c_1 \mathcal{D} + c_2 \mathcal{D}^2 + \cdots$. Hence $\lim_{\tau \to \infty} \tau^{-n} V_\tau R\left[\frac{t^{n+1}}{n+1}\right] = c_0 = 0$, so that $R = c_2 \mathcal{D}^2 + \cdots$. Continuing in this way, $c_k = 0$ for $k = 0, 1, \cdots$, i.e. (2) follows. Of course (2) implies (3), and (3) implies (1), since F_n^* is within the domains of P, Q for all sufficiently large n.

A derivator A is an element of L equal to a $\mathcal{D}$- polynomial, say P. The *coefficients*, *degree* and *coefficient sequence* of A are those of P; this is unambiguous since, by the preceding theorem, any other $\mathcal{D}$- polynomial equal to A is cogredient with P. In particular, every $\mathcal{D}$- polynomial is a derivator. Derivators are <u>cogredient</u> if they have the same coefficient sequence.

A sequence $\{a_k\}$ of complex numbers is <u>nearly null</u> if almost all of its terms vanish, i.e. $a_k = 0$ for all sufficiently large k. For each nearly null sequence $\{a_k\}$, there exists a $\mathcal{D}$- polynomial P whose coefficient sequence is $\{a_k\}$. Let $\sum_k a_k \mathcal{D}^k$ be an element of L equal to P. Then $\sum_k a_k \mathcal{D}^k$ is a derivator and $\{a_k\}$ is its coefficient sequence. Conversely, the coefficient sequence of every derivator is nearly null. Thus $A = \sum_k a_k \mathcal{D}^k$ is a biunivocal correspondence between derivators A and nearly null sequences $\{a_k\}$.

Let A, B be any derivators, and let $\{a_k\}, \{b_k\}$ be the coefficient sequences of A, B. From the algebra of L, it follows that $A \pm B = \sum_k (a_k \pm b_k)\mathcal{D}^k$; $AB = \sum_k (\sum_{0}^{k}{}_{\ell}\, a_\ell b_{k-\ell})\mathcal{D}^k$ and $\alpha A = \sum_k (\alpha a_k)\mathcal{D}^k$, and hence that the coefficient sequences of $A+B, A-B, AB, \alpha A$ are respectively $\{a_k + b_k\}, \{a_k - b_k\}$ $\{\sum_{0}^{k}{}_{\ell}\, a_\ell b_{k-\ell}\}, \{c\, a_k\}$. The class $K(D)$ of all derivators is therefore a commutative ring $\subset L$ which is isomorphic, through the cogredience correspondence $\sum_k a_k \mathcal{D}^k \sim \sum_k a_k \theta^k$ with the abstract ring $K(\theta)$ of polynomials in the 'indeterminate' θ with coefficients $\in C$. Through this isomorphism, all the factorization and distribution theorems for $K(\theta)$ are applicable to $K(D)$. In particular, if a_0 is the leading coefficient and $\alpha_1, \alpha_2, \dots, \alpha_m$ the zeros of a polynomial $P(\theta) \in K(\theta)$, then $P(D) = a_0 (D-\alpha_1)(D-\alpha_2)\dots(D-\alpha_m)$ is the factorization, unique except for the order of the α's, of the $\mathcal{D}$- polynomial $P(D)$ cogredient with $P(\theta)$ into

D-polynomials of degree ≤ 1. Moreover, $D-\alpha = \epsilon_\alpha D \epsilon_{-\alpha}$,

i.e. the <u>simple</u> derivator $D_\alpha = (D-\alpha)$ is the transform (in the group-theoretic sense) of the <u>elementary</u> derivator D by ϵ_α, whence $D_\alpha^k = \epsilon_\alpha D^k \epsilon_{-\alpha}$. We summarize the foregoing in

<u>THEOREM 2</u> <u>For any complex numbers</u> $a_0, a_1, \dots, a_m$, with

$$a_0 \neq 0, \quad P(D) \equiv a_0 D^m + a_1 D^{m-1} + \dots + a_m = a_0(D-\alpha_1)\cdots(D-\alpha_m)$$
$$= a_0 D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_m} = a_0 D_{\gamma_1}^{m_1} D_{\gamma_2}^{m_2} \cdots D_{\gamma_r}^{m_r}$$

<u>where</u> $\alpha_1, \alpha_2, \dots, \alpha_m$ <u>are the zeros in</u> C <u>of</u> $P(\theta)$ <u>and</u> $\gamma_1, \gamma_2, \dots, \gamma_r$ <u>are the distinct zeros with multiplicities</u> $m_1, m_2, \dots, m_r$. <u>Moreover, if</u> $P(D) = b_0(D-\beta_1)\cdots(D-\beta_n)$ <u>then</u> $a_0 = b_0$ <u>and</u> $\beta_1, \beta_2, \dots, \beta_n$ <u>is a permutation of</u> $\alpha_1, \alpha_2, \dots, \alpha_m$.

Thus a derivator A of degree m is characterized either by its $m+1$ coefficients $a_0, a_1, \dots, a_m$ or by its <u>module</u> a_0 and <u>indices</u> $\alpha_1, \alpha_2, \dots, \alpha_m$ where the first, second, ..., m^{th} elementary symmetric functions of the indices are respectively equal to $-\frac{a_1}{a_0}, \frac{a_2}{a_0}, \dots, (-)^m \frac{a_m}{a_0}$. The derivator is <u>normal</u> if its module $= 1$.

<u>THEOREM 3</u> <u>For any complex numbers</u> $\alpha_1, \alpha_2, \dots, \alpha_m$ <u>and</u> $\beta_1, \beta_2, \dots, \beta_m$ <u>and any</u> $f \in F_m$,

(1) $\qquad f, D_{\alpha_1} f, D_{\alpha_1} D_{\alpha_2} f, \dots, D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_m} f$

<u>are linear combinations with coefficients in</u> C <u>of</u>

(2) $\qquad f, D_{\beta_1} f, D_{\beta_1} D_{\beta_2} f, \dots, D_{\beta_1} D_{\beta_2} \cdots D_{\beta_m} f$.

<u>Proof</u>: For $k = 1, 2, \dots, m$,

(3) $\qquad D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} f = D^k f - (\alpha_1 + \alpha_2 + \dots + \alpha_k) D^{k-1} f + \dots + (-)^k \alpha_1 \alpha_2 \cdots \alpha_k f$.

These equations determine $D^k f$ recursively as a linear combination of $f, D_{\alpha_1} f, \dots, D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} f$

with coefficients depending only on $\alpha_1, \alpha_2, ---, \alpha_k$.
If $\alpha_1, \alpha_2, ---, \alpha_m$ are replaced by $\beta_1, \beta_2, ---, \beta_m$, equations (3) give $\mathcal{D}_{\beta_1} \mathcal{D}_{\beta_2} --- \mathcal{D}_{\beta_k} f$ as linear combinations of $f, \mathcal{D} f, --- \mathcal{D}^k f$, hence $\mathcal{D}_{\beta_1} \mathcal{D}_{\beta_2} --- \mathcal{D}_{\beta_k}$ is a linear combination of $f, \mathcal{D}_{\alpha_1} f, ---, \mathcal{D}_{\alpha_1} \mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_k} f$.

If $P(t)$ is a polynomial in t of degree m with coefficients in S, then $P = [P(t)]$ is a <u>polynomial</u> <u>matrix</u> and $\epsilon_\alpha P$ is an exponential-polynomial matrix or simply an <u>exponomial</u>. The <u>degree</u> and the <u>coefficients</u> of P and of $\epsilon_\alpha P$ are those of $P(t)$; the <u>exponent</u> and <u>type</u> of $\epsilon_\alpha P$ are α and (α, m) . The <u>elementary</u> exponomial of type (α, m) is $\epsilon_\alpha [t^m]$.

<u>THEOREM 4</u> <u>If</u> $P_i = \sum_{0}^{m_i}{}_j \sigma_{ij} [t^j]$ <u>for</u> $i = 1, 2, ---, n$ <u>where the</u> $\sigma_{ij} \in S$ <u>and</u> $\alpha_1, \alpha_2, ---, \alpha_n$ <u>are distinct complex numbers, then</u>

$$\epsilon_{\alpha_1} P_1 + \epsilon_{\alpha_2} P_2 + --- + \epsilon_{\alpha_n} P_n = 0$$

<u>implies</u> $P_i = 0$ i.e. $\sigma_{ij} = 0$ <u>for</u> $j = 1, 2, ---, m_i$.

<u>Proof</u>: The theorem is certainly true for $n = 1$: $\epsilon_{\alpha_1} P = 0$ implies $P_1 = 0$ which implies $\frac{1}{k!} \mathcal{D}^k P_1 \big|_0 = \sigma_{1k} = 0$ for $k = 0, 1, ---, m$. Assume the theorem for $n = k$. Suppose $\epsilon_{\alpha_1} P_1 + \epsilon_{\alpha_2} P_2 + --- + \epsilon_{\alpha_{k+1}} P_{k+1} = 0$. Then

(4) $\epsilon_{\beta_1} P_1 + \epsilon_{\beta_2} P_2 + --- + \epsilon_{\beta_k} P_k = - P_{k+1}$

where $\beta_i = \alpha_i - \alpha_{k+1}$ for $i = 1, 2, ---, k$; the β_i are distinct and $\neq 0$. Clearly the derivative of an exponomial with non-zero exponent is an exponomial of the same type. Hence if r is greater than the degree of P_{k+1}, $\mathcal{D}^r \sum_{1}^{k}{}_j \epsilon_{\beta_j} P_j = \sum_{1}^{k}{}_j \epsilon_{\beta_j} Q_j = 0$, where Q_j is a polynomial matrix of the same degree as P_j .

From the inductive assumption, the Q_j vanish, hence each is of degree 0 and so are the P_j for $j = 1, 2, ---, k$. From (4), $P_{k+1} = 0$.

Let $X_i = \epsilon_{\alpha_i} P_i$. Suppose $\sum_1^n{}_i \lambda_i X_i = 0$ where the $\lambda_i \varepsilon C$. Then by the preceding theorem $\lambda_i P_i = 0$, $i = 1, 2, ---, n$. Hence the λ_i vanish if each $P_i \neq 0$. Thus non-zero exponomials with distinct exponents and with coefficients in S are linearly independent with respect to C.

Now suppose the coefficients of the P_i are in C. If $\sum_1^n{}_i \sigma_i X_i = 0$, where the σ_i are in S then by the preceding theorem, $\sigma_i P_i$ (which equals a polynomial matrix with coefficients in S) equals zero for $i = 1, 2, ---, n$, and hence the σ_i vanish if each $P_i \neq 0$. Thus non-zero exponomials with distinct exponents and with coefficients in C are linearly independent with respect to S.

Since $\mathcal{D}_\alpha^l [e^{\alpha t} t^k] = \epsilon_\alpha \mathcal{D}^l [t^k] = 0$ if $l > k$, the linear manifold M of the elementary exponomials $[e^{\alpha_j t} t^k]$ for $j = 1, 2, ---, r$; $k = 0, 1, ---, m_j - 1$ is certainly within the null domain of the derivator $A = \mathcal{D}_{\alpha_1}^{m_1} \mathcal{D}_{\alpha_2}^{m_2} --- \mathcal{D}_{\alpha_r}^{m_r}$.
It will be shown below (see page 27) that they span this domain.
Since they are linearly independent with respect to S, they form a basis with S for the null domain of A. Other bases with S for the null domains of derivators (i.e. linear combinations of the elementary exponomials, with coefficients in C, which are linearly independent with respect to C) are given in Theorems 11 and 24.

3. Integrators

Let $I_\alpha = \epsilon_\alpha I \epsilon_{-\alpha} \varepsilon L$ clearly the domain in F of I_α, and hence of $I_\alpha{}^m$ is F. An <u>integrator</u> is an element $A \varepsilon L$ equal to $\mu I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_m}$ for some $\mu, \alpha_1, \cdots, \alpha_m \varepsilon C$. The <u>module</u> and <u>order</u> of A are μ and m; the <u>indices</u> of A are the α's. A is normal if its module $= 1$. Any complex number $\mu \neq 0$ will be regarded as an integrator of order zero, with module μ, but without indices.

It will be shown below that for $\mu, \nu \neq 0$, $\mu I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_m} = \nu I_{\beta_1} I_{\beta_2} \cdots I_{\beta_n}$ if and only if $\mu = \nu$ and the β's are a permutation of the α's so that the module and order of an integrator and the set of its indices are unambiguous (the <u>set</u> of indices $\alpha_0, \alpha_1, \cdots, \alpha_m$ always contains exactly m elements, whereas the number of elements in the <u>class</u> of indices is the number of <u>distinct</u> indices). Integrators of the first order are <u>simple</u>. The integrator I of order 1 with index 0 and module 1 is <u>elementary</u>. The product of two integrators is an integrator; the order of the product is the sum of the orders of the factors; the indices of the product are those of the factors together. The module of the product is the product of the modules.

<u>THEOREM 5</u> The product AB of <u>two integrators</u> A, B is commutative:

$$AB = BA$$

<u>Proof:</u> Suppose, without loss of generality, that A, B are normal. Let A and B be replaced by equal products of simple integrators. Since multiplication in L is associative, the theorem follows if AB is unaltered when the elementary integrators are permuted. This will be the case if $I_\alpha I_\beta = I_\beta I_\alpha$ for $\alpha \neq \beta$. But the latter is true: for any $f \varepsilon F$, $I_\alpha I_\beta = \frac{I_\alpha - I_\beta}{\alpha - \beta}$,

as may be seen from $I_\alpha I_\beta = \left[e^{\alpha t} \int_0^t e^{(\beta-\alpha)\tau} \int_0^\tau e^{-\beta\sigma} f(\sigma)\, d\sigma\, d\tau\right]$

either by integrating by parts or by reversing the order of integration.

THEOREM 6 For any distinct complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$

$$I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_m} = \frac{I_{\alpha_1}}{(\alpha_1\alpha_2 \cdots \alpha_m)} + \frac{I_{\alpha_2}}{(\alpha_2\alpha_3 \cdots \alpha_m\alpha_1)} + \cdots + \frac{I_{\alpha_m}}{(\alpha_m\alpha_1\alpha_2 \cdots \alpha_{m-1})},$$

where $(\alpha_1\alpha_2 \cdots \alpha_m) = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_m)$, and the denominators of the terms following the first are obtained from that of the first by cyclical permutation of the α's.

Proof: Since $I_{\alpha_1} I_{\alpha_2} = \dfrac{I_{\alpha_1} - I_{\alpha_2}}{\alpha_1 - \alpha_2}$, the theorem is true for $m = 2$. Assume it is true for $m = k$. Then

$$I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_{k+1}} = \left\{\frac{I_{\alpha_1}}{(\alpha_1\alpha_2 \cdots \alpha_k)} + \frac{I_{\alpha_2}}{(\alpha_2\alpha_3 \cdots \alpha_k\alpha_1)} + \cdots + \frac{I_{\alpha_k}}{(\alpha_k\alpha_1\alpha_2 \cdots \alpha_{k-1})}\right\} I_{\alpha_{k+1}}$$

$$= \left\{\frac{I_{\alpha_1} - I_{\alpha_{k+1}}}{(\alpha_1\alpha_2 \cdots \alpha_k)(\alpha_1-\alpha_{k+1})} + \frac{I_{\alpha_2} - I_{\alpha_{k+1}}}{(\alpha_2\alpha_3 \cdots \alpha_k\alpha_1)(\alpha_2-\alpha_{k+1})} + \cdots + \frac{I_{\alpha_k} - I_{\alpha_{k+1}}}{(\alpha_k\alpha_1 \cdots \alpha_{k-1})(\alpha_k-\alpha_{k+1})}\right\},$$

$$= \frac{I_{\alpha_1}}{(\alpha_1\alpha_2 \cdots \alpha_{k+1})} + \cdots + \frac{I_{\alpha_k}}{(\alpha_k\alpha_{k+1}\alpha_1 \cdots \alpha_{k-1})} - \left\{\frac{1}{(\alpha_1\alpha_2 \cdots \alpha_{k+1})} + \cdots + \frac{1}{(\alpha_k\alpha_{k+1}\alpha_1 \cdots \alpha_{k-1})}\right\}$$

The theorem follows for $m = k+1$ and hence for any $m \geq 2$ if the coefficient of $I_{\alpha_{k+1}}$ equals $\dfrac{1}{(\alpha_{k+1}\alpha_1\alpha_2 \cdots \alpha_k)}$. It does, as may be seen from the following

Lemma 1 For any distinct complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$,

$$\frac{\alpha_1^k}{(\alpha_1\alpha_2 \cdots \alpha_m)} + \frac{\alpha_2^k}{(\alpha_2\alpha_3 \cdots \alpha_m\alpha_1)} + \cdots + \frac{\alpha_m^k}{(\alpha_m\alpha_1\alpha_2 \cdots \alpha_{m-1})} = \delta_{k,m-1}$$

for $k = 0, 1, \cdots, m-1$

Proof: For any polynomial $f(z)$ on C to C of degree $< m$

$$f(z) = f(\alpha_1)\frac{(z\alpha_2 \cdots \alpha_m)}{(\alpha_1\alpha_2 \cdots \alpha_m)} + f(\alpha_2)\frac{(z\alpha_3 \cdots \alpha_m\alpha_1)}{(\alpha_2\alpha_3 \cdots \alpha_m\alpha_1)} + \cdots + f(\alpha_m)\frac{(z\alpha_1 \cdots \alpha_{m-1})}{(\alpha_m\alpha_1 \cdots \alpha_{m-1})};$$

this is the Lagrange interpolation formula. In particular, the equation holds for $k = 0, 1, \cdots, m-1$ with $f(z) = z^k$, and the result follows upon equating the coefficients of z^{m-1} in the left and right members. The case $k=0$ completes the proof of the preceding theorem.

THEOREM 7 For any $f \varepsilon F$, $I_\alpha^m f = \left[\int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} e^{\alpha(t-\tau)} f(\tau)\, d\tau\right]$

Proof: Since $I_\alpha f = \left[\int_0^t e^{\alpha(t-\tau)} f(\tau)\, d\tau\right]$, the theorem is true for $m=1$. Assume it is true for $m=k$. Then $I_\alpha^{k+1} f = I_\alpha^k I_\alpha f$

$$= \left[\int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} e^{\alpha(t-\tau)} \int_0^\tau e^{\alpha(\tau-\sigma)} f(\sigma)\, d\sigma\, d\tau\right]$$

$$= \left[\int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} \int_0^\tau e^{\alpha(t-\sigma)} f(\sigma)\, d\sigma\, d\tau\right] = \left[\int_0^t \frac{(t-\tau)^k}{k!} e^{\alpha(t-\tau)} f(\tau)\, d\tau\right.$$

after the inevitable integration by parts. Hence the theorem is true for $m=k+1$ and by induction for any $m \geqslant 1$.

Other important properties of integrators can be inferred directly from the definition by methods like those used above, in which the intimate connection between integrators and derivators does not appear conspicuously. It will be more convenient, however, to obtain these results after the derivator-integrator relationship has been fairly well developed.

4. Reversion of Derivators

The derivator $A = a\mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2}\cdots\mathcal{D}_{\alpha_n}$ and the integrator $X = \xi I_{\beta_1} I_{\beta_2}\cdots I_{\beta_n}$ are coindicial if $\beta_1, \beta_2, \cdots \beta_n$ is a permutation of $\alpha_1, \alpha_2, \cdots \alpha_n$; if moreover $a\xi = 1$, then A and X are reciprocal.

For any $f \varepsilon F$, $\mathcal{D}_\alpha I_\alpha f = \epsilon_\alpha \mathcal{D} I \epsilon_{-\alpha} f = f$, since $\mathcal{D}If = [\frac{d}{dt}\int_0^t f(t)\,dt] = f$. Hence I_α is a right reverse of $\mathcal{D}_\alpha$. More generally, suppose A and X are reciprocal. Then

(5) $$AXf = \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2}\cdots\mathcal{D}_{\alpha_m} I_{\alpha_m} I_{\alpha_{m-1}}\cdots I_{\alpha_1} f = f,$$

in applying the preceding result m times. Hence X is a right reverse of A and the notation $A^* = X$ for the integrator reciprocal to A is justified (see Sec. 1) and will be used hereafter.

From (1.5), for any $f, g \varepsilon F$, $f = A^* g$ implies $Af = g$. The converse is false for some f, g, since the correspondence is equivocal in f. But if the linear condition $f \varepsilon F_m^*$ is subjoined to $Af = g$, the result is univocal in f and equivalent to $f = A^* g$ for any $f, g \varepsilon F$. Many properties of A^* may be inferred more readily from this equivalence than from the definition in the preceding section.

<u>THEOREM 8</u> <u>For any derivator A and any $f, g \varepsilon F$, $Af = g$ and $f \varepsilon F_m^*$ if and only if $f = A^* g$, where A^* is the integrator reciprocal to A.</u>

<u>Proof</u>: Without loss of generality suppose A is normal. We remark that, since $g \varepsilon F$ is continuous, $\mathcal{D}_\alpha f = g$ is equivalent to

$$f = I_\alpha g + \epsilon_\alpha[f_0] = I_\alpha g + f_0[e^{\alpha t}].$$

Hence $\mathcal{D}_\alpha f = g$ and $f_0 = 0$ is equivalent to $f = I_\alpha g = \mathcal{D}_\alpha^* g$.

Let $\alpha_1, \alpha_2, ---, \alpha_m$ be the indices of A, so that $A = \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_m}$. Now suppose $Af = g$ and $f \varepsilon F_m^*$. Then $\mathcal{D}_{\alpha_1}(\mathcal{D}_{\alpha_2} \cdots \mathcal{D}_{\alpha_m} f) = g$ and by Theorem 3, $\mathcal{D}_{\alpha_2}\mathcal{D}_{\alpha_3}\mathcal{D}_{\alpha_4} \cdots \mathcal{D}_{\alpha_m} f/_0 = 0$. Hence by the preceding remark, $\mathcal{D}_{\alpha_2}(\mathcal{D}_{\alpha_3} --- \mathcal{D}_{\alpha_m} f) = I_{\alpha_1} g$. Again by Theorem 3, $\mathcal{D}_{\alpha_3}\mathcal{D}_{\alpha_4} --- \mathcal{D}_{\alpha_m} f/_0 = 0$, and as before $\mathcal{D}_{\alpha_3}\mathcal{D}_{\alpha_4} --- \mathcal{D}_{\alpha_m} f = I_{\alpha_2} I_{\alpha_1} g$. Repeating this argument, we finally obtain $f = I_{\alpha_m} I_{\alpha_{m-1}} --- I_{\alpha_1} g = A^* g$. Conversely, if the latter is true then

$$Af \doteq \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_m} I_{\alpha_m} I_{\alpha_{m-1}} --- I_{\alpha_1} f = f.$$

We can now show that the indices of an integrator are unique except for order:

<u>THEOREM 9</u> <u>Equal integrators have equal modules and the same set of indices.</u>

<u>Proof</u>: Suppose that the integrators $X = \lambda I_{\alpha_1} I_{\alpha_2} --- I_{\alpha_m}$ and $Y = \mu I_{\beta_1} I_{\beta_2} --- I_{\beta_n}$ are equal. Let A, B be the derivators reciprocal to X, Y. Then $AX = BY = 1$ on F, and since $X = Y$, $(A - B)Xf = 0$ for any $f \varepsilon F$. Let $r = \max(m, n)$. By the preceding theorem, $A^* = B^* = X$ hence the domain of X includes F_r^*. By Theorem 1, A and B are cogredient, and by Theorem 2 the modules of A and B are equal and their indices are the same except for order. The conclusion now follows, since the modules of A and B are λ^{-1}, μ^{-1} and their indices are $\alpha_1, \alpha_2, ---, \alpha_m$ and $\beta_1, \beta_2, ---, \beta_n$.

Let $A = \mathcal{D}^m + a_1\mathcal{D}^{m-1} + --- + a_m = \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_m}$ be a normal derivator of degree m. For $g \varepsilon F$, let

$$f = A^* g = I_{\alpha_1} I_{\alpha_2} --- I_{\alpha_m} g.$$

. By Theorem 6,

(6) $f = \{ A_1 I_{\alpha_1} + A_2 I_{\alpha_2} + \cdots + A_m I_{\alpha_m} \} g,$

where $A_1 = (\alpha_1 \alpha_2 \cdots \alpha_{m_1})^{-1}, A_2 = (\alpha_2 \alpha_3 \cdots \alpha_m \alpha_1)^{-1} \cdots, A_m = (\alpha_m \alpha_1 \alpha_2 \cdots \alpha_{m-1}$

. The individual terms on the right in this equation belong to F_1 since $g \varepsilon F$ is continuous, but in general they do not belong to F_k for $k > 1$. Yet the entire right member belongs to $F_m^* < F_m$, since by Theorem 8, $Af = g$ and $f \varepsilon F_m^*$.

It will be instructive to verify this by means of Lemma 1.

Since $\mathcal{D} I_\alpha = (\mathcal{D} - \alpha + \alpha) I_\alpha = (\mathcal{D}_\alpha + \alpha) I_\alpha = 1 + \alpha I_\alpha$ on F_1

$$\mathcal{D} f = \{ \alpha_1 A_1 I_{\alpha_1} + \alpha_2 A_2 I_{\alpha_2} + \cdots + \alpha_m A_m I_{\alpha_m} \} g$$
$$+ \{ A_1 + A_2 + \cdots + A_m \} g.$$

Now by Lemma 1.1 the second expression in braces vanishes, hence $\mathcal{D} f \varepsilon F_1$ and

$$\mathcal{D}^2 f = \{ \alpha_1^2 A_1 I_{\alpha_1} + \alpha_2^2 A_2 I_{\alpha_2} + \cdots + \alpha_m^2 A_m I_{\alpha_m} \} g$$
$$+ \{ \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_m A_m \} g.$$

Again by the lemma, the second expression in braces vanishes, so that $\mathcal{D}^2 f \varepsilon F_1$.

Continuing in this way, we obtain

(7) $\mathcal{D}^k f = \{ \alpha_1^k A_1 I_{\alpha_1} + \alpha_2^k A_2 I_{\alpha_2} + \cdots + \alpha_m^k A_m I_{\alpha_m} \} g,$ for $k = 0, 1, \cdots, m-1,$

and

(8) $\mathcal{D}^m f = \{ \alpha_1^m A_1 I_{\alpha_1} + \alpha_2^m A_2 I_{\alpha_2} + \cdots + \alpha_m^m A_m I_{\alpha_m} \} g$
$+ \{ \alpha_1^{m-1} A_1 + \alpha_2^{m-1} A_2 + \cdots + \alpha_m^{m-1} A_m \} g,$

where this time the second expression in braces $= 1$.

From (7) we verify that $\mathcal{D}^k f /_0 = 0$ for $k = 0, 1, \cdots, m-1$

i.e. $f \in F_m^*$. From (7) and (8)

$$Af = \left(\sum_0^m{}_k\, a_{m-k}\, \mathcal{D}^k\right) f = \left\{ \sum_1^m{}_\ell\, A_\ell \left(\sum_0^m{}_k\, a_{m-k}\, \alpha_\ell^k\right) I_{\alpha_\ell} \right\} g + g$$

$$= g,$$

since $\alpha_1, \alpha_2, ---, \alpha_m$ are the zeros of $\sum_0^m{}_k\, a_{m-k}\, z^k$ in C.

If the derivator $A = \prod_1^m{}_\nu\, \mathcal{D}_{\alpha_\nu}$ is applied to the matrix $h_k = I_{\alpha_1} I_{\alpha_2} --- I_{\alpha_k} [e^{\alpha_{k+1} t}]$, k of the derivator factors annihilate the integrator, i.e. reduces it to 1 and one of the remaining factors nullifies $[e^{\alpha_{k+1} t}]$ i.e. reduces it to zero. Thus $h_0 = [e^{\alpha_1 t}]$, and $h_1, h_2, ---, h_{m-1}$ are in the null domain of A and it is easy to see that they are linearly independent with respect to S: IF $h = \sum_0^{m-1}{}_k\, \sigma_k h_k$ $= \sigma_0 [e^{\alpha_1 t}] + \sigma_1 I_{\alpha_1} [e^{\alpha_2 t}] + --- + \sigma_{m-1} I_{\alpha_1} I_{\alpha_2} --- I_{\alpha_{m-1}} [e^{\alpha_m t}]$ with $\sigma_0, \sigma_1, ---, \sigma_{m-1} \in S$, then $\mathcal{D}_{\alpha_1} h|_0 = \sigma_1$, $---, \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_{m-1}} h|_0 = \sigma_{m-1}$; hence $h = 0$ implies $\sigma_0 = \sigma_1 = --- = \sigma_{m-1} = 0$. Moreover, it is shown below (Theorem 11) that the h_i span the null domain of A and hence form a basis with S for that domain.

<u>THEOREM 10</u> <u>For any</u> $f, g \in F$, $Af = \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_m} f = g$ <u>and</u> $f|_0 = \sigma_0$, $\mathcal{D}_{\alpha_1} f|_0 = \sigma_1, ---, \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_{m-1}} f|_0 = \sigma_{m-1}$ <u>is equivalent to</u> $f = A^* g + h$, <u>where</u> $h = \sigma_0 h_0 + \sigma_1 h_1 + --- + \sigma_{m-1} h_{m-1}$.

<u>Proof</u>: From the remarks in the preceding paragraph, $Ah = 0$ and the initial values of $h, \mathcal{D}_{\alpha_1} h, ---, \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} --- \mathcal{D}_{\alpha_{m-1}} h$

are respectively $\sigma_0, \sigma_1, \cdots, \sigma_{m-1}$. This and Theorem 3 imply that the simultaneous equations $Af = g$, $f/_0 = \sigma_0$, $\mathcal{D}_{\alpha_1} f/_0 = \sigma_1, \cdots, \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} \cdots \mathcal{D}_{\alpha_{m-1}} f/_0 = \sigma_{m-1}$ are equivalent to $A(f-h) = 0$ and $f - h \in F_m^*$; by Theorem 8 this conjunction is equivalent to $f - h = A^* g$, whence the theorem. The important case $g = 0$, corresponding to the null domain of A, is repeated as

<u>THEOREM 11</u> <u>For any</u> $f \in F$, $Af = 0$ <u>and</u> $f/_0 = \sigma_0$, $\mathcal{D}_{\alpha_1} f/_0 = \sigma_1, \cdots, \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} \cdots \mathcal{D}_{\alpha_{m-1}} f/_0 = \sigma_{m-1}$ <u>is equivalent to</u> $f = \sigma_0 h_0 + \sigma_1 h_1 + \cdots + \sigma_{m-1} h_{m-1}$.

Replacing g by Af in Theorem 10 yields

<u>THEOREM 12</u> <u>For any</u> $f \in F_m$

(9) $$f = A^*Af + h$$

<u>where</u> $\sigma_0, \sigma_1, \cdots \sigma_{m-1}$ <u>are the initial values of</u> $f, \mathcal{D}_{\alpha} f, \cdots, \mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} \cdots \mathcal{D}_{\alpha_{m-1}} f$ <u>and</u> $h = \sigma_0 h_0 + \sigma_1 h_1 + \cdots + \sigma_{m-1} h_{m-1}$.

Thus associated with each derivator A of degree m is the resolution (9) of $f \in F_m$ into its projection A^*Af on F_m^*, and its projection h on N_A. By Theorems 3 and 11, the intersection of F_m^* and N_A contains only $0 \in F$. Hence the resolution (9) is unique, in the sense that if $f = g_1 + h_1 = g_2 + h_2$ where the g's are in F_m^* and the h's in N_A, then $g_1 - g_2 = h_2 - h_1 = 0$, since $g_1 - g_2 \in F_m^*$ and $h_2 - h_1 \in N_A$.

There is another very important way of regarding (9): It determines the matrix $f \in F_m$ in terms of the following data: (1) the matrix $\mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} \cdots \mathcal{D}_{\alpha_m} f \in F$ and (2) the elements $\mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} \cdots \mathcal{D}_{\alpha_{m-1}} f/_0$, $\mathcal{D}_{\alpha_1}\mathcal{D}_{\alpha_2} \cdots \mathcal{D}_{\alpha_{m-2}} f/_0, \cdots, \mathcal{D}_{\alpha_1} f/_0$ of S. In many practical problems (differential equations of physics and engineering in which f is the 'unknown') these data are given and (9) solves the problem.

5. Commutation

For any two elements P, Q of L, the operator $\Gamma = PQ - QP$ is the commutator of P, Q. Let Δ_1, Δ_2 be the domains in F of PQ, QP. The domain in F of Γ is $\Delta_1 \cdot \Delta_2$. The null domain of Γ is in general a proper subset Δ of $\Delta_1 \cdot \Delta_2$. P and Q commute (see Sec. 1) if $\Delta = \Delta_1 = \Delta_2$. This iscertainly the case if P, Q are derivators of degree m, n $(\Delta = \Delta_1 = \Delta_2 = F_{m+n})$ or integrators of order m, n $(\Delta = \Delta_1 = \Delta_2 = F)$. If P is a derivator of degree $m > 0$ and Q an integrator of order $n \geq m$, then $\Delta_1 = F$ and $\Delta_2 = F_m \neq \Delta_1$, so that P and Q do not commute. The domain of Γ is then $F \cap F_m = F_m$, but the null domain of Γ is not so simple: it is F_m^* if $m \leq n$ (Theorem 13) but isharder to describe if $m > n$ (Theorem 14).

A commutation condition is a condition $C(f)$ on $f \in F$ such that $C(f)$ implies $\Gamma f = 0$. A commutation rule is an equation $\Gamma f = \phi$, where ϕ is an expression in f, valid for all f in the domain of Γ. In this section we shall give several commutation conditions and commutation rules for

$\Gamma = AB^* - B^*A$ where A, B are derivators of degree m, n respectively.

THEOREM 13 If $m \leq n$, then the null domain of Γ is F_m^*, i.e.

$(AB^* - B^*A)f = 0$ if and only if $f \in F_m^*$.

Proof: Since the domain of Γ is F_m, suppose $f \in F_m$.

$B\Gamma f = B(AB^* - B^*A)f = 0$, since operator multiplication is distributive, and associative, derivators commute and $BB^* = 1$ (we shall usually omit such formal details). Hence by

Theorems 11 and 3, $\Gamma f = 0$ is equivalent to $\Gamma f \in F_n^*$, and this is equivalent to $AB^*f \in F_n^*$ since $B^*Af \in F_n^*$ (by Theorem 8 the range of B^* is F_n^*). Clearly $\mathcal{D}^k AB^* f|_0 = 0$ if $k < n - m$, since the degree of the derivator $\mathcal{D}^k A$ is then $m + k < n$. Hence $AB^*f \in F_n^*$ if and only if $\mathcal{D}^{n-m+k} AB^* f|_0 = 0$ for $k = 0, 1, \cdots, m-1$.

There exist unique derivators Q_k of degree k and R_k of degree $< n$ such that $\mathcal{D}^{n-m+k} A = Q_k B + R_k$, as may be seen by applying the division algorithm to the abstract polynomials $A(\theta)$, $B(\theta)$ cogredient with A, B. Then $\mathcal{D}^{n-m+k} AB^* f|_0 = Q_k f|_0 + R_k B^* f|_0$, and the last term vanishes since the degree of R_k is less than the order of B^* (the degree of B). Hence $\Gamma f = 0$ if and only $Q_k f|_0 = 0$ for $k = 0, 1, \cdots, m-1$, and this is equivalent to $f|_0 = \mathcal{D} f|_0 = \cdots = \mathcal{D}^{m-1} f|_0 = 0$ since $Q_k f|_0$ is a linear combination of the initial values of $f, \mathcal{D}f, \cdots, \mathcal{D}^k f$ in which the coefficient of $\mathcal{D}^k f$ is $\neq 0$.

<u>THEOREM 14</u> <u>If</u> $m > n$, <u>then</u> $(AB^* - B^*A)f = 0$ <u>if and only</u> $f_{m-n}, f_{m-n+1}, \cdots f_{m-1}$ <u>are certain linear combinations of</u> $f_0, f_1, \cdots, f_{m-n-1}$, <u>where</u> $f_k = \mathcal{D}^k f|_0$ for $k = 0, 1, \cdots, m-1$.

<u>Proof:</u> As in the preceding proof, $\Gamma f = 0$ if and only if $AB^* f \in F_n^*$. There exist unique derivators Q of degree $m-n$ and R of degree $< n$ such that $A = QB + R$. Hence for $k = 0, 1, \cdots, n-1$ $\mathcal{D}^k AB^* f|_0 = \mathcal{D}^k Q f|_0 + \mathcal{D}^k R B^* f|_0 = 0$ is a linear equation in $f_0, f_1, \cdots, f_{m-n+k}$ in which the coefficient of f_{m-n+k} is $\neq 0$. ~~Expressions for these coefficients will be given in Section 9.~~

Let $g = R_n R_{n-1} \cdots R_k \cdots R_1 f$ where (1) $f \varepsilon F_\lambda$ and $f \varepsilon F_\rho^*$ (2) either (i) for every k ($i.e.$ $k = 1, 2, \cdots, n$), $R_k = P_k Q_k$ or (ii) for every k, $R_k = Q_k P_k$, and (3) for any k, P_k is an integrator of order p_k and Q_k is a derivator of order g_k. Let $\lambda_k = \lambda + (p_1 - g_1) + (p_2 - g_2) + \cdots + (p_{k-1} - g_{k-1})$. (To avoid exceptions, let $p_0 = g_0 = 0$).

THEOREM 15 In case (i): g exists εF if for every k, $g_k \leq \lambda_k$. If this condition is satisfied, then g is unaltered when P_k, Q_k are transposed if $g_k \leq \mu_k$ where the μ_j are defined recursively by $\mu_0 = \rho$ and $\mu_j = p_{j-1} + \langle \mu_{j-1} - g_{j-1} \rangle$. (For any integer i, $\langle i \rangle = i$ or 0 according as $i > 0$ or $i \leq 0$).

In case (ii): g exists εF if for every k, $g_k \leq \lambda_k + p_k$. If this condition is satisfied, then g is unaltered when Q_k, P_k are transposed if $g_k \leq \nu_k$, where the ν_j are defined recursively by $\nu_0 = \rho$ AND $\nu_j = \langle p_{j-1} + \nu_{j-1} - g_{j-1} \rangle$.

Proof: Since the domain of the integrators is F we need only ascertain that the operands of all the derivators are within the domains thereof. Suppose case (i). The operand of Q_k is $f_k = R_{k-1} R_{k-2} \cdots R_1 f$ if $k > 1$ and f if $k = 1$.

If $g_1 < \lambda_1 = \lambda$, clearly $Q_1 f$ exists $\varepsilon F_{\lambda_1 - g_1}$ and since the range of P_1 is $F_{p_1}^* \subset F_{p_1}$, certainly $f_1 = P_1 Q_1 f \varepsilon F_{p_1 + \lambda_1 - g_1} = F_{\lambda_2}$. Repeating the argument, $Q_2 P_1 Q_1 f$ exists $\varepsilon F_{\lambda_2 - g_2}$ and $f_2 = P_2 Q_2 P_1 Q_1 f$ $\varepsilon F_{p_2 + \lambda_2 - g_2} = F_{\lambda_3}$ etc., and finally g exists $\varepsilon F_{\lambda_{n+1}}$ Assume that the conditions $g_k \leq \lambda_k$ are satisfied.

If $q_1 \leq \mu_1 = p$, then $Q_1 f \varepsilon F^*_{\mu_1 - q_1}$ and $P_1 Q_1 f \varepsilon F^*_{p_1 + \mu_1 - q_1}$, but if $q_1 > \mu_1$, we can only assert $Q_1 f \varepsilon F$, and then $P_1 Q_1 f \varepsilon F^*_{p_1}$; in either eventuality, $f_1 = P_1 Q_1 f \varepsilon F^*_{p_1 + \langle \mu_1 - q_1 \rangle} = F^*_{\mu_2}$. Continuing the argument, we see that $f_k \varepsilon F^*_{\mu_k}$ for $k = 1, 2, \cdots, n-1$. Hence for any k, if $q_k \leq \mu_k$, then $f_k \varepsilon F^*_{q_k}$ and by Theorem 13 commutation of P_k, Q_k will not alter g. The proof for case (ii) is similar.

<u>THEOREM 16</u> <u>Let</u> $A = \sum_0^m {}_k a_{m-k} \mathcal{D}^k$ <u>be any derivator of degree</u> m, <u>and</u> B <u>any derivator of degree</u> $\geq m$. <u>Then for any</u> $f \varepsilon F_m$

$$(10) \quad (AB^* - B^*A)f = (A_0 \mathcal{D}^m + A_1 \mathcal{D}^{m-1} + \cdots + A_{m-1} \mathcal{D}) B^* [1]$$

<u>where</u> $A_k = \sum_0^k {}_l a_{k-l} f_l$

<u>and</u> $f_l = \mathcal{D}^l f |_0$.

<u>Proof</u>: Let $g = (AI^n - I^n A)f$, where $n \geq m$. Then $\mathcal{D}^n g = 0$ and hence by Theorem 11 (with the α's there equal to one)

$$g = (\sigma_0 + \sigma_1 I + \sigma_2 I^2 + \cdots + \sigma_{n-1} I^{n-1}) [1]$$

where $\sigma_k = \mathcal{D}^k g|_0 = \mathcal{D}^k A I^n f|_0$. For $k < n-m$ $\sigma_k = 0$, since the degree of $\mathcal{D}^k A$ is $< n$. For $k = 0, 1, \cdots, m-1$,

$$\sigma_{n-m+k} = \mathcal{D}^{n-m+k} A I^n f|_0 = (a_0 \mathcal{D}^{n+k} + a_1 \mathcal{D}^{n+k-1} + \cdots$$
$$+ a_m \mathcal{D}^{n-m+k}) I^n f|_0 = (a_0 f_k + a_1 f_{k-1} + \cdots + a_k f_0) = A_k$$

Moreover, $I^{n-m+k} = (\mathcal{D}^{m-k} I^{m-k}) I^{n-m+k} = \mathcal{D}^{m-k} I^n$.

Hence

$$(11) \quad (AI^n - I^n A) f = (A_0 \mathcal{D}^m + A_1 \mathcal{D}^{m-1} + \cdots + A_{m-1} \mathcal{D}) I^n [1].$$

Now $I^n = \mathcal{D}^{*n} = \mathcal{D}^{n*}$. By Theorem 8, $B^* f$ and $\mathcal{D}^{n*} f$ belong to F_m^*. By Theorems 13 and 5, $B^* I^n f$ $= A B^* \mathcal{D}^{n*} f = \mathcal{D}^{n*} A B^* f$.

Hence applying B^* to (11), with $Q \equiv (A_0 \mathcal{D}^m + A_1 \mathcal{D}^{m-1} + \cdots + A_{m-1} \mathcal{D})$,

$$(12) \quad I^n (AB^* - B^* A) f = B^* Q I^n [1]$$
$$= Q B^* I^n [1] = I^n Q B^* [1]$$

where the commutation of B^*, Q and of Q, I^n is justified by Theorem 13, since $I^n [1]$ and $B^* [1]$ belong to F_m^*. The Theorem follows on applying $\mathcal{D}^n$ to (12).

<u>THEOREM 17</u> <u>With A as in the preceding theorem, for any</u> $f \varepsilon F_m$

$$(13) \quad f = A^* A f + (A_0 \mathcal{D}^m + A_1 \mathcal{D}^{m-1} + \cdots + A_{m-1} \mathcal{D}) A^* [1].$$

<u>Proof:</u> Application of BA^* to (10) or (11) yields (13) after commutations and reductions justified by Theorem 13 and $AA^* = 1$. Or simply take $B = A$ in Theorem 16.

6. Derigrators.

For any derivator A of degree m and integrator X or order n, the operator AX is a derigrator of degree m, order n and rank $n-m$. If Y and B are the reciprocals of A and X, i.e. $Y = A^*$ and $B = {}^*X$, then $AX = {}^*YX = AB^*$.

If $AX = A_1X_1$, where A_1 is of degree m_1, and X_1 of order n_1, then ${}^*X_1A = {}^*XA_1$, after left multiplication by ${}^*X\,{}^*X_1$, or $XA_1^* = X_1A^*$, after right multiplication by $A^*A_1^*$, and by Theorems 1 or 9, $m+n_1 = m_1+n$, i.e. $n-m = n_1-m_1$. We have, in effect proved the following:

Theorem 18 Derigrators AB^* and $A_1B_1^*$ are equal if and only if $AB_1 = A_1B$. Equal derigrators have the same rank.

Theorem 19 For any derigrators AB^*, $A_1B_1^*$,

$AB^* \pm A_1B_1^* = (AB_1 \pm A_1B)(BB_1)^*$. If the rank of $A_1B_1^*$ is ≥ 0, then $(AB^*)(A_1B_1^*) = (AA_1)(BB_1)^*$.

Proof: The first part follows from $AB^* = A(B_1B_1^*)B^* = (AB_1)(BB_1)^*$. The second part follows from Theorem 13.

A derigrator AB^* is dextral or sinistral according as the rank of AB^* is ≥ 0 or ≤ 0 (thus derivators of rank zero are dextral and sinistral).

If AB^* is dextral, then by Theorem 13 $(BA^*)(AB^*) = 1$ i.e. AB^* has an unrestricted left reverse ${}^*(AB^*) = BA^*$ and is therefore univocal (see Sec. 1). In particular, the integrator B^* is a dextral derigrator of degree zero with unrestricted left reverse ${}^*(B^*) = B$.

If AB^* is sinistral, then $(AB^*)(BA^*) = 1$, i.e. AB^* has an unrestricted right reverse $(AB^*)^* = BA^*$, but AB^* is equivocal: by Theorem 11, the null domain of AB^* contains (in fact is spanned by) the non-zero matrices $h_n, h_{n+1}, ---, h_{m-1}$. In particular, the derivator A is a sinistral derigrator of order zero with unrestricted right reverse A^*.

The derigrator concept may be extended as follows. Any element of L will be called a derigrator if it equals a derivator-integrator product AB^*, hereafter called a <u>primitive</u> derigrator. The <u>rank</u> of a derigrator is that of any equal primitive derigrator (by Theorem 18, this is unambiguous). By Theorem 19, the class K of derigrators is closed under addition and subtraction, and the product of two elements of K belongs to K if the rank of the second factor is ≥ 0. Derigrators are dextral or sinistral according as equal primitive derigrators are dextral or sinistral.

<u>Theorem 20</u> <u>The class $K_d < K$ of all dextral derigrators is a commutative ring containing the unit operator 1. Every element of K_d is univocal.</u>

<u>Proof:</u> Let AB^*, $A_1B_1^*$ be primitive derigrators equal to $P, Q \varepsilon K_d$. By Theorem 19, $P \pm Q = (AB_1 \pm A_1B)(BB_1)^*$ and $PQ = (AA_1)(BB_1)^*$. The order of $(BB_1)^*$ (i.e. degree of BB_1) is the sum of the orders of B^*, B_1^*; for each of AB_1, A_1B, AA_1, the degree of the product is the sum of the degrees of the factors; the degree of $AB_1 \pm A_1B$ is not greater than the degrees of the terms. Hence the ranks of $P \pm Q$ and PQ are ≥ 0. Clearly $P+Q$ and PQ are commutative, and P

is univocal since AB^* is.

Let P be an element of K_d of positive rank. Let A, B be derivators with no common indices such that $P = AB^*$. These conditions uniquely determine the indices of A, B and the quotient μ of the module of A by that of B (if A_1, B_1 also satisfy the conditions then by Theorem 18, $AB_1 = A_1B$. By Theorem 2, the indices of A and B together must be those of A_1 and B together, hence A, A_1 are coindicial and so are B, B_1. Moreover the product of the modules of A, B_1 must equal the product of the modules of A_1, B). Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be the distinct indices of B, with multiplicities $m_1, m_2, \cdots, m_r$.

<u>Theorem 21</u> <u>There exist unique complex numbers</u> $p_{jk}\ (j = 1, 2, \cdots, r;\ k = 0, 1, \cdots m_{r-1})$ <u>such that</u>

$$(1.4) \qquad P = \sum_{1}^{r}{}_{j} \sum_{0}^{m_j-1}{}_{k}\, p_{jk}\, I_{\alpha_j}^{m_j-k}.$$

<u>Proof</u>: Let $A(\theta)$, $B(\theta)$ be abstract polynomials cogredient with A, B. Unique complex numbers p_{jk} exist ('partial fractions' algorithm) such that

$$A(\theta) = \mu \sum_{1}^{r}{}_{j} \Big(\sum_{0}^{m_j-1}{}_{k}\, p_{jk} (\theta - \alpha_j)^k \Big) \prod_{1}^{r}{}_{\ell \neq j} (\theta - \alpha_\ell)^{m_\ell}.$$

Hence

$$A = \mu \sum_{1}^{r}{}_{j} \Big(\sum_{0}^{m_j-1}{}_{k}\, p_{jk}\, D_{\alpha_j}^{k} \Big) \prod_{1}^{r}{}_{\ell \neq j} D_{\alpha_\ell}^{m_\ell}.$$

Equation (~~20~~ 14) follows on applying this expression for A to $B^* = \frac{1}{\mu} \prod_{1}^{r}{}_{\ell} I_{\alpha_\ell}^{m_\ell}$.

Any sinistral derigrator (or dextral derigrator of rank zero) may be resolved uniquely into a derivator and an element of K_d of positive rank, to which (~~20~~ 14) may be applied.

For any $P \varepsilon K_d$, the matrix $P[1]$ is the <u>indicial matrix</u> of P and of its reciprocal $^*P$. If the rank of P is positive, the matrix $DP[1]$ is the <u>weighting matrix</u> or <u>density</u> of P and of its reciprocal $^*P$. In particular $A^*[1]$ and $DA^*[1]$ are the indicial matrix and the density (if the degree of A is positive) of both the derivator A and the integrator A^*.

<u>Theorem 22</u> <u>With the notation of the preceding theorem, for any derigrator P of positive rank and any</u> $f \varepsilon F$

$$(15) \quad Pf = \left[\int_0^t \left\{ \sum_{1j}^{r} \sum_{0k}^{m_j - 1} p_{j,(m_j-1-k)} \frac{(t-\tau)^k}{k!} e^{\alpha_j (t-\tau)} \right\} f(\tau)\, d\tau\right].$$

This is an immediate consequence of Theorems 7 and 21.

<u>Theorem 23</u> <u>If the rank of P is positive, the density</u> $G = DP[1]$ is given by

$$(16) \quad G = \left[\sum_{1j}^{r} \sum_{0k}^{m_j - 1} p_{j,(m_j-1-k)} \frac{t^k}{k!} e^{\alpha_j t}\right].$$

This follows immediately from the preceding theorem. Thus equation (15) may be written as

$$(17) \quad Pf = \left[\int_0^t G(t-\tau) f(\tau)\, d\tau\right].$$

<u>Theorem 24</u> <u>With the notation of Theorem 16, if A is a derivator of positive degree m, then for any</u> $f \varepsilon F_m$ <u>and</u> $g \varepsilon F$, $Af = g$ <u>is equivalent to</u>

$$(18) \quad f = \left[\int_0^t G(t-\tau) g(\tau)\, d\tau\right] + (A_0 \mathcal{D}^{m-1} + A_1 \mathcal{D}^{m-2} + \cdots + A_{m-1}) G,$$

<u>where</u>

$$G = DA^*[1]$$

<u>Proof:</u> The direct part of the theorem follows immediately from Theorem 17 and equation (17) with $P = A^*$. For the converse, suppose (18). From (16), $G \varepsilon F_\infty$, hence applying A to

(18) , $Af = g + AQG = g + QAG$ where $Q = A_0 D^{m-1} + A_1 D^{m-2} + \cdots + A_{m-1}$. But the elementary exponomials $[t^k e^{\alpha_j t}]$ in (16) belong to the null domain of A . Hence $AG = 0$.

With $g = 0$, it is clear from (18) and (16) that every element of the null domain of A is a linear combination with coefficients in S of the elementary exponomials. Since these are linearly independent with respect to S , they form a basis with S for N_A .

<u>Theorem 25</u> <u>If</u> $A = \sum_{0}^{m}{}_k a_{m-k} D^k$ <u>is a derivator of positive degree</u> m , <u>then for any</u> $f \in F_m$, $g \in F$ <u>and elements</u> $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ <u>of</u> S , <u>the system of equations</u>

$$(19) \quad Af = g, \; f/_0 = \sigma_0, \; Df/_0 = \sigma_1, \ldots, D^{m-1}f/_0 = \sigma_{m-1},$$

<u>is equivalent to</u>

$$(20) \quad f = A^* g + QA^*[1] = \left[\int_0^t G(t-\tau) g(\tau)\, d\tau\right] + (q_0 D^{m-1} + \cdots + q_{m-1}) G,$$

<u>where</u>

$$q_k = \sum_{0}^{k}{}_\ell a_{k-\ell} \sigma_\ell \quad \underline{\text{for}} \;\; k = 0, 1, \ldots, m-1,$$
$$Q = \sum_{0}^{m-1}{}_k q_k D^{m-k} \quad \underline{\text{and}} \;\; G = DA^*[1]$$

<u>Proof:</u> Assume (19) . Then $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ are respectively equal to $f_0, f_1, \ldots, f_{m-1}$ (which are defined as $f/_0, Df/_0 \ldots D^{m-1}f/_0$ in Theorem 16), hence (20) follows by Theorem 24. Conversely, assume (20) . Then $Af = g$ follows, as in the proof of Theorem 24. We must verify $D^k f/_0 = \sigma_k$ Applying AI^m to (20) yields

$$AI^m f = AI^m A^* g + AI^k QA^*[1] = AA^* I^m g + AA^* QI^m[1]$$
$$= I^m g + QI^m[1]$$

where the commutations of Q, A^* and Q, I^m are justified by Theorem 13. Now applying D^k we obtain

$$(a_0 D^k + a_1 D^{k-1} + \cdots + a_k + a_{k+1} I + \cdots + a_m I^{m-k}) f$$
$$= I^{m-k} g + (g_k + g_{k+1} I + \cdots + g_{m-1} I^{m-k-1})[1].$$

Evaluating at 0,

as
$$a_0 f_k + a_1 f_{k-1} + \cdots + a_k f_0 = g_k = a_0 \sigma_k + a_1 \sigma_{k-1} + \cdots + a_k \sigma_0$$

for $k = 0, 1, \cdots, m-1,$ which imply $f_k = \sigma_k$.

7. The Method of Integrator Transforms

For $j, k = 1, 2, \cdots, m,$ let A_{jk} be a derivator of degree d_{jk}, $d_k = \max_j d_{jk}$, $f_k \in F_{d_k}$ and $g_j \in F$. Consider the equations

$$(21) \quad \begin{aligned} A_{11} f_1 + A_{12} f_2 + \cdots + A_{1n} f_m &= g_1, \\ A_{21} f_1 + A_{22} f_2 + \cdots + A_{2n} f_m &= g_2, \\ &\vdots \\ A_{m1} f_1 + A_{m2} f_2 + \cdots + A_{mn} f_m &= g_m, \end{aligned}$$

subject to the (initial) conditions

$$(22) \quad f_k|_0 = f_{k0},\ D f_k|_0 = f_{k1}, \cdots, D^{d_k - 1} f|_0 = f_{k, d_k - 1},$$
$$\text{for } k = 1, 2, \cdots, m.$$

Suppose $A_{jk} = \sum_{\ell=0}^{d_{jk}} a_{jk\ell} D^{d_{jk} - \ell}$. For $\ell = 1, 2, \cdots, d_{jk-1}$, let $g_{jk\ell} = \sum_{r=0}^{\ell} a_{jkr} f_{k, \ell - r}$ and

$$Q_{jk} = \sum_{\ell=0}^{d_{jk}-1} g_{jk\ell} D^{d_{jk} - \ell}$$

. Let Q be an integrator of order $N \geq \max_k d_k$

Suppose (21) and (22). Then by Theorem 16,

$$Q \sum_{k=1}^{n} A_{jk} f = \sum_{k=1}^{n} Q A_{jk} f = \sum_{k=1}^{n} \{A_{jk} Q f - Q_{jk} Q[1]\} = Q g_j,$$

and hence

$$(23) \quad \sum_{1}^{m}{}_{k} A_{jk} Qf_k = Qg_j + \sum_{1}^{m}{}_{k} Q_{jk} Q[1]$$

Conversely, if (23) holds then by an evident extension of the proof of Theorem 25, (21) and (22) follow. Thus the single system (23) is equivalent to (21) and (22).

Let M_{kj} be the cofactor of A_{jk} in the determinant $M = |A_{jk}|$ (defined in the usual way as a sum of products). Then (23) implies

$$\sum_{1}^{m}{}_{j} M_{ij} \sum_{1}^{m}{}_{k} A_{jk} Q f_k = \sum_{1}^{m}{}_{k} \left\{ \sum_{1}^{m}{}_{j} M_{ij} A_{jk} \right\} Q f_k$$
$$= MQf_i = \sum_{1}^{m}{}_{j} M_{ij} Q g_j + \sum_{1}^{m}{}_{j} \sum_{1}^{m}{}_{k} M_{ij} Q_{jk} Q[1]$$

and hence

$$(24) \quad MQf_i = \sum_{1}^{m}{}_{j} M_{ij} Q g_j + \sum_{1}^{m}{}_{j,k} M_{ij} Q_{jk} Q[1],$$

provided that the order N of Q islarge enough to ensure that Qf_i and Qg_j are within the domains of the derivators applied to them (see Theorem 15). Moreover, by Theorem 13, if N islarge enough, $(M^*M - MM^*)\phi = 0$, where ϕ is any of the Q transforms appearing in (24), and hence

$$(25) \quad Qf_i = \sum_{1}^{m}{}_{j} M_{ij} M^* Q g_j + \sum_{1}^{m}{}_{j,k} M_{ij} Q_{jk} M^* Q[1],$$

the commutations in the right member being justified, by Theorem 13, if N is sufficiently large.

Conversely, (23) follows from (25) on applying $A_{\ell i}$, summing on i and reducing the right member by $\sum_{1}^{m}{}_{i} A_{\ell i} M_{ij} = \delta_{\ell j} M$

and $MM^* = 1$, - provided that N is sufficiently large.

Let $\bar{f} = Q^* f$ for any $f \in F$, and introduce the equivalence relation $\overline{\overline{Q}}$, meaning equal for all Q with sufficiently large N . Then (23) becomes

$$(26) \qquad \sum_{1}^{m}{}_{k} A_{jk} \bar{f}_k \;\overline{\overline{Q}}\; \bar{g}_j + \sum_{1}^{m}{}_{k} Q_{jk} \;[T]$$

the <u>subsidiary systems</u> for (21) and (22); (25) becomes

$$(27) \qquad \bar{f}_i \;\overline{\overline{Q}}\; \sum_{1}^{m}{}_{j} M_{ij} M^* \bar{g}_i + \sum_{1}^{m}{}_{j,k} M_{ij} Q_{jk} M^* [T]$$

and the systems (27) and (26) are each equivalent to the double system (21), (22).

Since $\bar{f}_i \in \bar{F}_N$, a necessary condition for the existence of matrices f_i satisfying (27) (and hence (21), (22) is that the right member belongs to F_N^* , and a sufficient condition for this is, by Theorem 13, that the ranks of $M_i = (\sum_{1}^{m}{}_{j} M_{ij}) M^*$ and $Q_i = (\sum_{1}^{m}{}_{jk} M_{ij} Q_{jk}) M^*$ be positive, for then the right member of (27) may be written as

$$Q^* \left\{ \sum_{1}^{m}{}_{j} M_{ij} M^* g_j + \sum_{1}^{m}{}_{j,k} M_{ij} Q_{jk} M^* [1] \right\} \in F_n^*$$

Indeed, the condition that the M_i , Q_i be dextral derigrators is sufficient for the existence of unique roots f_i of (27) given by

$$(28) \qquad f_i = \sum_{1}^{m}{}_{j} M_{ij} M^* g + \sum_{1}^{m}{}_{jk} M_{ij} Q_{jk} M^* [1]$$

which follows from (27) upon application of Q .

In general, if the M_i , Q_i are not all dextral, then the condition that the right member of (27) belong to F_N^* becomes a necessary condition on the g_j in order that roots of (27) exist.

We have tacitly assumed that $M \neq 0$. If this is not the case, then (24) becomes a necessary condition on the g_i for the existence of solutions of (21), (22); if it is satisfied, some of the equations in (21), (22) become dependent on the others and the system (21) becomes, in effect, a rectangular rather than a square array. This more general case can be treated by the methods used above.

8. <u>The Method of Inverse Reversion</u>

In the preceding section it was shown that the system

$$A(D)f = g, \quad f/_0 = f_0, \quad Df/_0 = f_1, \ldots, D^{m-1}f/_0 = f_{m-1}$$

is equivalent to the 'equation'

(29) $$A(D)\bar{f} \mathrel{\overline{\overline{Q}}} \bar{g} + (A_0 D^m + A_1 D^{m-1} + \cdots + A_{m-1} D)[\bar{1}],$$

where $\bar{h} = Q^* h$ for any $h \in F$ and $\overline{\overline{Q}}$ means 'equal for any Q of sufficiently high degree'.

The binding variable Q can be eliminated from $\overline{\overline{Q}}$ in (29) in the same way that t was eliminated from $\overline{\overline{t}}$ in $f(t) \mathrel{\overline{\overline{t}}} g(t)$. We introduce 'matrices' $[\phi(Q)]_Q$, where $\phi(Q)$ is a function of Q on K_d to F, just as we introduced the matrices $[f(t)]_t$, where $f(t)$ is on R to S. After appropriate definitions of equality, addition, etc., the class $[F]$ of all matrices $[\phi(Q)]_Q$ is an abelian group with operators $\in K_d$ and equation (29) may be written as

$$A(D)[Q^*f] = [Q^*g] + (A_0 D^m + A_1 D^{m-1} + \cdots + A_{m-1} D)[Q^*[1]]$$

(the subscript Q having finally been dropped). The class K_d is a <u>field</u> of operators on $[F]$.

This is a very satisfactory procedure in many respects and we shall present it in detail elsewhere. We wish here to examine another way of eliminating the binding variable $\mathcal{Q}$.

Suppose that the ring K_α has been extended to a field $\bar{K}_\alpha$ and the class F correspondingly extended to a complex linear class $\bar{F}$. Then $\mathcal{Q}^{*-1}$, the inverse of the reverse of $\mathcal{Q}$, will exist in $\bar{K}_\alpha$, and applying it to (26) yields

(30) $\mathcal{Q}^{*-1}A(\mathcal{D})\mathcal{Q}^*f = g + \mathcal{Q}^{*-1}(A_0\mathcal{D}^m + --- + A_{m-1}\mathcal{D})\mathcal{Q}^*[1]$.

Assuming that $\mathcal{Q}^{*-1}\mathcal{D}\mathcal{Q}^* \varepsilon \bar{K}_\alpha$ if the order of $\mathcal{Q}$ is sufficiently large, $\mathcal{Q}^{*-1}\mathcal{D}^k\mathcal{Q}^* = (\mathcal{Q}^{*-1}\mathcal{D}\mathcal{Q}^*)^k$, and distributing $\mathcal{Q}^{*-1}$ forward and $\mathcal{Q}^*$ backward in (30) yields

(31) $A(p)f = g + (A_0 p^m + A_1 p^{m-1} + --- + A_{m-1}p)[1]$,

where $p = \mathcal{Q}^{*-1}\mathcal{D}\mathcal{Q}^*$. Thus the dependence of the equation on $\mathcal{Q}$ is lodged entirely in the operator p.

We shall show that K_α and F can be extended in such a way that the foregoing heuristic argument is justified.

The range IN F of any non-zero element $X \varepsilon K_\alpha$ of positive rank is a proper subset of F: for some $f \varepsilon F$, the equation $Xg = f$ has no root $g \varepsilon F$. We now construct an extension $\bar{F}$ of F such that this equation has a unique root $\frac{f}{X} \varepsilon \bar{F}$ for every $f \varepsilon F$.

We wish this extension $\bar{F}$ to be minimal, i.e. to contain only such elements g extraneous to F as are necessary to satisfy the equation $Xg = f$. Thus $\bar{F}$ should be the range of the quotient $\frac{f}{X}$ for $f \varepsilon F$ and $0 \neq X \varepsilon K_\alpha$, with $X\frac{f}{X} = f$. Moreover, $\bar{F}$ should be an abelian group with operators in K_α. We begin therefore with the following postulates for $\bar{F}$.

With $\phi, \psi \in \bar{F}$; $f, g \in F$; X, Y in K_d:

1) $\phi = \psi$ is a reflexive, symmetric, transitive relation.

2) Every quotient $\frac{f}{X}\ (X \neq 0)$ belongs to $\bar{F}$, and every element of $\bar{F}$ equals such a quotient.

3) $\phi \pm \psi,\ X\phi$ belong to $\bar{F}$.

4) $X = Y$ implies $X\phi = Y\phi$, and $\phi = \psi$ implies $X\phi = X\psi$

5) $X\phi = X\psi$ and $X \neq 0$ implies $\phi = \psi$.

6) $(XY)\phi = X(Y\phi)$

7) $X(\phi \pm \psi) = X\phi \pm X\psi$

8) $X\frac{f}{X} = f \quad (X \neq 0)$

From these postulates, we infer that, with $X, Y \neq 0$ (and hence $XY \neq 0$ since X, Y univocal):

9) $\frac{f}{X} = \frac{g}{Y}$ equiv $Y(X\frac{f}{X}) = X(Y\frac{g}{Y})$ equiv $Yf = Xg$

10) $XY(\frac{f}{X} \pm \frac{g}{Y}) = Y(X\frac{f}{X}) \pm X(Y\frac{g}{Y}) = XY\frac{Yf \pm Xg}{XY}$

whence $\frac{f}{X} \pm \frac{g}{Y} = \frac{Yf \pm Xg}{XY}$

11) $Y(X\frac{g}{Y}) = X(Y\frac{g}{Y}) = Y\frac{Xg}{Y}$ whence $X\frac{g}{Y} = \frac{Xg}{Y}$

These inferences suggest the following definitions:

Let $\bar{F}$ be the class of quotients $\frac{f}{X}$ for $f \in F$ and $0 \neq X \in K_d$, and for any element $\frac{f}{X}$, $\frac{g}{Y}$ of $\bar{F}$ let (i) $\frac{f}{X} = \frac{g}{Y}$ be equivalent to $Yf = Xg$, (ii) $\frac{f}{X} \pm \frac{g}{Y} \equiv \frac{Yf \pm Xg}{XY}$, (iii) $\frac{\frac{f}{X}}{Y} \equiv \frac{f}{XY}$ and (iv) $X\frac{g}{Y} \equiv \frac{Xg}{Y}$. With these definitions 1) -7) are satisfied, but instead of 8) we have

8)' $X\frac{f}{X} = \frac{f}{1} \quad (X \neq 0)$.

Let $\bar{F}'$ be the range in $\bar{F}$ of $\frac{f}{1}$ for $f \in F$. For any

$f' \varepsilon \bar{F}'$, $f \varepsilon F$, write $f' \sim f$ if and only if $f' = \frac{f}{1}$. Then $f' \sim f$ is an isomorphism (this follows from $(i), (ii), (iv)$ and 12) below) between $\bar{F}'$, F, so we may without contradiction identify $\frac{f}{1}$ and f, thereby making $\bar{F}' = F$ and $\bar{F}$ an extension of F such that 8) holds for $f \varepsilon F$ (and hence from (iii) for $f \varepsilon \bar{F}$: if $g \varepsilon F$, $0 \neq X \varepsilon K_d$, then $X \frac{\frac{g}{Y}}{X} = X \frac{g}{XY} = \frac{Xg}{XY} = \frac{\frac{Xg}{X}}{Y} = \frac{g}{Y}$).

Thus the quotient class $\bar{F}$ satisfies postulates 1) - 8), and it is easy to see that any other class $\bar{F}^*$ satisfying these postulates must be isomorphic with $\bar{F}$ (for $\phi \varepsilon \bar{F}$, $\phi^* \varepsilon \bar{F}^*$, let $\phi \sim \phi^*$ be equivalent to $\phi = \frac{f}{X} = \phi^*$ for some $f \varepsilon F$, $0 \neq X \varepsilon K_d$. The relation $\phi \sim \phi^*$ is an isomorphism between $\bar{F}$, $\bar{F}^*$). Hence the postulates are consistent and categorical (but not independent). In addition to 9), 10), 11) we shall need

12) $$\frac{\phi \pm \psi}{X} = \frac{\phi}{X} \pm \frac{\psi}{X}, \quad \text{if } X = 0,$$

and

13) $$(X \pm Y)\phi = X\phi \pm Y\phi$$

for $\phi, \psi \varepsilon \bar{F}$ and $X, Y \varepsilon K_d$. These follow readily from $(i) - (iii)$ and 1) - 8). The foregoing is summarized in

<u>THEOREM 26</u> <u>The quotient class $\bar{F}$ is an abelian group with operators X in the commutative ring K_d. F is a sub group of $\bar{F}$. For any $\phi, \psi \varepsilon \bar{F}$ and $0 \neq X \varepsilon K_d$, $X\phi = \psi$ if and only if $\phi = \frac{\psi}{X}$ (in particular $X\phi = f$ is equivalent to $\phi = \frac{f}{X}$ for $f \varepsilon F$).</u>

With g constant in F, the range of $X \frac{g}{Y}$ for $X, Y \varepsilon K_d$ and $Y \neq 0$ is of course more extensive than if only one of X, Y varied. We wish now to define <u>operators</u> $\frac{X}{Y}$ such that (v) the transform $\frac{X}{Y} g = X \frac{g}{Y}$.

Let $\bar{L}$ be the class of all linear operators on $\bar{F}$ to $\bar{F}$. For any $X, Y \varepsilon K_d$ with $Y \neq 0$, let $\frac{X}{Y}$ be an element of $\bar{L}$ such that for any $\phi \varepsilon \bar{F}$, $\frac{X}{Y}\phi = X\frac{\phi}{Y}$ ($= \frac{X\phi}{Y}$; this follows readily from 8) and (iii)). Then with $\phi \varepsilon \bar{F}$; $X, Y, Z \varepsilon K_d$; $U, V, W \varepsilon K_d$ and $\neq 0$ (and hence $UV \neq 0$ since U, V univocal):

14) $(\frac{X}{U} \pm \frac{Y}{V})\phi = \frac{X}{U}\phi \pm \frac{Y}{V}\phi$ (by definition of operator sum and difference) $= \frac{X\phi}{U} \pm \frac{Y\phi}{V} = \frac{V(X\phi) \pm U(Y\phi)}{UV}$ $= \frac{(VX \pm UY)\phi}{UV} = \frac{VX \pm UY}{UV}\phi$, by (V), 12) and the postulates.

15) $(\frac{X}{U}\frac{Y}{V})\phi = \frac{X}{U}(\frac{Y}{V}\phi)$ (by definition of operator product) $= \frac{X}{U}(\frac{Y\phi}{V}) = \frac{(XY)f}{UV} = \frac{XY}{UV}f$, by (V) and the postulates.

16) $\frac{X}{U}\frac{Y}{V} = \frac{Z}{W}$ equiv $\frac{U}{X}(\frac{X}{U}\frac{Y}{V}) = \frac{U}{X}\frac{Z}{W}$ equiv $\frac{Y}{V} = \frac{UZ}{XW}$, if $X \neq 0$.

17) $\frac{X}{1}\phi = X\phi$.

Let $\bar{K}_d$ be the class of quotients $\frac{X}{Y} \varepsilon \bar{L}$ for $X, Y \varepsilon K_d$ and $Y \neq 0$. From 14), 15), $\bar{K}_d$ is closed under addition, subtraction, multiplication, and the multiplication is commutative. From 16) $\bar{K}_d$ is closed under division by non-zero elements. Hence $\bar{K}_d$ is a field of operators on $\bar{F}$.

Let $\bar{K}_d'$ be the range in $\bar{K}_d$ of $\frac{X}{1}$ for $X \varepsilon K_d$. For $X' \varepsilon \bar{K}_d'$, $X \varepsilon K_d$ write $X' \sim X$ if and only if $X' = \frac{X}{1}$. Then $X' \sim X$ is an isomophism between $\bar{K}_d'$, K_d. By 17), we may identify $\frac{X}{1}$ and X, thereby making $\bar{K}_d' = K_d$ and the field $\bar{K}_d$ an extension of the ring K_d.

THEOREM 27 The quotient class, class $\bar{K}_d$, is a field of linear operators on $\bar{F}$ to $\bar{F}$ which contains the ring K_d. For any $\phi, \psi \varepsilon \bar{F}$ and $0 \neq X \varepsilon \bar{K}_d$, $X\phi = \psi$ if and only if $\phi = X^{-1}\psi$

Let K be the class of derivators and K^* the class of integrators (including 0, which is not the reverse of a derivator). Let K^{*-1} be the class of inverse reverses A^{*-1} of elements $A \varepsilon K$ (with $0^{*-1} \equiv 0$).

Suppose $AP^*, BQ^* \varepsilon K_d$ (i.e. are dextral derigrators) with $B \neq 0$. Then $X = \frac{AP^*}{BQ^*} \varepsilon \bar{K}_d$, $\frac{B^*}{B^*} = 1$, $\frac{AB^*}{BQ^*}\frac{B^*}{B^*} = \frac{A(BP)^*}{Q^*}$. If $A \neq 0$, multiplication by $\frac{A^*}{A^*}$ yields $X = \frac{(BP)^*}{(AQ^*)} = (AQ)^{*-1}(BP)^*$, and the last expression is still valid if $A = 0$, since $0^{*-1} = 0$. If $\phi \varepsilon \bar{F}$, then for some $BQ^* \varepsilon K_d$ with $B \neq 0$, $\phi = \frac{f}{BQ^*} = \frac{B^*f}{Q^*}$ (on multiplying on the left with $\frac{B^*}{B^*}$ and using Theorem 13) $= Q^{*-1}(B^*f)$.

Hence

THEOREM 28 The classes $\bar{F}$, $\bar{K}_d$ are the ranges of the transforms $A^{*-1}f$, $A^{*-1}X$ for $A \varepsilon K$, $f \varepsilon F$, $X \varepsilon K^*$.

This result is the main reason for referring to the process of generating the classes $\bar{F}$, $\bar{K}_d$ as inverse reversion.

The image of A by X is the group-theoretic transform $X^{-1}(AX)$. If $AP^* \varepsilon K_d$ then $P^{*-1}(AP^*) = \frac{P^*}{A^*P^*} = \frac{1}{A^*} = A^{*-1}$ independent of P (the last expression is valid for $A = 0$), i.e. the image of A by P^* equals the inverse reverse of A. Hence for all $Q \varepsilon K$ of sufficiently high degree

$$(A \pm B)^{*-1} = \frac{(A \pm B)Q^*}{Q^*} = \frac{AQ^*}{Q^*} \pm \frac{BQ^*}{Q^*} = A^{*-1} \pm B^{*-1}$$

and

$$(AB)^{*-1} = \frac{(AB)Q^*}{Q^*} = \frac{(AB)Q^*Q^*}{Q^*Q^*} = \frac{AQ^*}{Q^*}\,\frac{BQ^*}{Q^*}$$

$$= A^{*-1}B^{*-1} \qquad \text{(using Theorem 13).}$$

Thus we have proved

THEOREM 29 For any derivators A, B, $(A \pm B)^{*-1} = A^{*-1} \pm B^{*-1}$ AND $(AB)^{*-1} = A^{*-1}B^{*-1}$, i.e. the correspondence between $A \varepsilon K$ and $A^{*-1} \varepsilon K^{*-1}$ is an isomorphism.

Let $p = \mathcal{D}^{*-1}$. Then

THEOREM 30 If $A = \sum_{0}^{m}{}_{k}\, a_{m-k}\, \mathcal{D}^{k}$ is a $\mathcal{D}$- polynomial of degree m, then

$$A(\mathcal{D})^{*-1} = \left\{ \sum_{0}^{m}{}_{k}\, a_{m-k}\, \mathcal{D}^{k} \right\}^{*-1} = \sum_{0}^{m}{}_{k}\, a_{m-k} (\mathcal{D}^{k})^{*-1} = A(p),$$

and hence

$$A(\mathcal{D})^{*} = \frac{1}{A(\mathcal{D}^{*-1})} = A(p)^{-1}.$$

This follows immediately from the preceding theorem.

THEOREM 31 If $A(\mathcal{D}) = \sum_{0}^{m}{}_{k}\, a_{m-k}\, \mathcal{D}^{k}$ and $B(\mathcal{D}) = \sum_{0}^{n}{}_{k}\, b_{n-k}\, \mathcal{D}^{k}$ are $\mathcal{D}$ - polynomials of degree m, n with $m \leq n$, then

$$A(\mathcal{D})\, B(\mathcal{D})^{*} = \frac{A(p)}{B(p)}.$$

Proof: The image $B(\mathcal{D})^{*-1}\{A(\mathcal{D})\, B(\mathcal{D})^{*}\}$ OF $A(\mathcal{D})$ equals $A(\mathcal{D})^{*-1}$. Hence by the preceding theorem

$$B(p)\,\{A(\mathcal{D})\, B(\mathcal{D})^{*}\} = A(p).$$

THEOREM 32 With $A(\mathcal{D})$ as in the preceding theorem, for any $f \varepsilon F_m$

(32) $\quad A(p) f - A(\mathcal{D}) f = (A_0 p^{m} + A_1 p^{m-1} + \cdots + A_{m-1} p)\,[1],$

where $A_j = \sum_{0}^{j}{}_{k}\, a_{j-k}\, \mathcal{D}^{k} f|_0$ FOR $j = 0, 1, \cdots, m-1$.

Proof: By Theorem 31, equation (32) follows from (13) on applying $A(p)$.

Let $\delta = p\,[1]$. Then

THEOREM 33 With $A(\mathcal{D})$ as in Theorem 31, for any $f \varepsilon F_m$, $g \varepsilon F$

and any $\sigma_1, \sigma_2, ---, \sigma_{m-1} \in S$, the system

$$A(\mathcal{D})f = g, \quad f|_0 = \sigma_0, \mathcal{D}f|_0 = \sigma_1, ---, \mathcal{D}^{m-1}f|_0 = \sigma_{m-1}$$

is equivalent to

(33) $$A(p)f = g + (A_0 p^{m-1} + A_1 p^{m-2} + --- + A_{m-1})\delta,$$

where $A_j = \sum_0^j {}_k\, a_{j-k}\sigma_k$ FOR $j = 0, 1, ---, m-1$.

Proof: By Theorem 31, equation (33) follows from (20) on applying $A(p)$.

Theorems 29 - 33 provide a rational basis for the methods employed by Heaviside for lumped linear systems, with his mystical operator p defined as $\mathcal{D}^{*-1}$ and δ_k , the 'impulse matrix of the k^{th} order', defined as $p^k[1]$. Actually, Heaviside used these methods in the case to be considered in the next section, where the elements of F are sectionally continuous matrices, but the theorems of this section remained valid, mutatis mutandis.

9. Sectionally Continuous Matrices. Shift and Jump Operators*

Let $f(t)$ be a function of t on $\Delta < R$ to S, where the complement $\Delta' \subset R$ of the domain Δ of $f(t)$ is scattered, i.e. the intersection of Δ' and any finite interval of R is a finite set. Then $f(t)$ is defined nearly everywhere, or for nearly all t. Let $g(t)$ be another function of t defined nearly everywhere. $f(t)$ and $g(t)$ are equal nearly everywhere, or for nearly all t : $f(t) \underset{t}{\doteq} g(t)$ if the values of t for which $f(t) \neq g(t)$ form a scattered set.

*In this and the following sections the theorems are given without proofs. The demonstrations will be given in detail elsewhere.

THEOREM 34 $f(t) \underset{t}{\doteq} g(t)$ if and only if for any $\tau \varepsilon R$, $f(t) = g(t)$ for all t sufficiently near τ but $\neq \tau$.

$f(t)$ is continuous nearly everywhere if its points of discontinuity form a scattered set. Similarly, $f(t)$ is derivable nearly everywhere if $\mathcal{D}_\tau^t f(t)$ exists for nearly all τ.

The jump operator $J_\tau^t f(t) = (\lim_{t \to \tau^+} - \lim_{t \to \tau^-}) f(t)$.

$f(t)$ is sectionally continuous if it is continuous nearly everywhere and $J_\tau^t f(t)$ exists for all τ. Then values of τ for which $J_\tau^t f(t) \neq 0$ form a subset of the scattered set of discontinuities of $f(t)$.

$f(t)$ is scattered on a scattered set σ if $f(t)$ is defined everywhere and vanishes on the complement of σ.

THEOREM 35 For any scattered set σ, $f(t) \underset{t}{=} \sum_\tau^\sigma \delta_{t\tau} f(\tau)$ if and only if $f(t)$ is scattered on σ, where $\delta_{t\tau} = 1$ or 0 according as $t = \tau$ or $t \neq \tau$.

Let $F(t)$ be the class of sectionally continuous functions $f(t)$ such that $f(t) = 0$ for all sufficiently small t, $F_m(t) < F(t)$ the class of m-fold sectionally continuously derivable $f(t)$ and $F_m^*(t) < F_m(t)$ the class of m-fold continuously derivable $f(t)$ (i.e. $f(t)$ and its first $(m-1)$ derivatives are continuous everywhere and vanish for all sufficiently small t). Let F be the class of matrices $[f(t)]$ for $f(t) \varepsilon F(t)$. Let $[f(t)] = [g(t)]$ be equivalent to $f(t) \underset{t}{\doteq} g(t)$, and define the sum, difference and numerical multiple of elements of F as in Section 1. Let F_m, F_m^* be the classes of matrices of the elements of $F_m(t), F_m(t)^*$ respectively.

Let L be the class of all linear operators on subsets of F to F and define the elements $\mathcal{D}, \epsilon_\alpha$ as in Section 1. Define $I \varepsilon L$ by $I[f(t)] = [\int_{-\infty}^{t} f(\tau)\,d\tau]$. Define $J_\lambda \varepsilon L$ by $J_\lambda[f(t)] = [J_\lambda^t f(t)]$.

Define derivator, integrator, derigrator, the classes $K, K^*, K_g, K_s, K_d, \bar{K}_d, \bar{F}$ as in the preceding sections (except that the integrals in integrators are all from $-\infty$ to t).

For any real number λ, define the shift operator $E_\lambda \varepsilon L$ by $E_\lambda[f(t)] = [f(t-\lambda)]$. Let $U = [U(t)]$, where $U(t) = 0$ or 1 according as $t <$, $t > 0$ (undefined for $t = 0$).

THEOREM 36 For any $f \varepsilon F$, any $\lambda, \mu \varepsilon R$, $J_\lambda E_\mu f = J_{\lambda-\mu} f$. In particular $J_\lambda E_\mu U = \delta_{\lambda\mu}$.

THEOREM 37 For any derigrator $A \varepsilon K_g$ and any $x \varepsilon R$, E_λ commutes with A:

$$E_\lambda A = A E_\lambda.$$

THEOREM 38 Let $A = \sum_{0}^{m}{}_k a_{m-k} \mathcal{D}^k$ be a derivator of degree m. For $k = 0, 1, \cdots, m-1$, let σ_k be a scattered set and ϕ_λ^k a function of λ on R to S scattered on σ_k. Then for any $f \varepsilon F_m$ and $g \varepsilon F$, the system

$$Af = g,\ J_\lambda f_\lambda \underset{\lambda}{=} \phi_\lambda^0,\ J_\lambda \mathcal{D} f \underset{\lambda}{=} \phi_\lambda^1,\ \cdots,\ J_\lambda \mathcal{D}^{m-1} f \underset{\lambda}{=} \phi_\lambda^{m-1}$$

is equivalent to

$$(34)\qquad f = A^* g + (A_0 \mathcal{D}^m + A_1 \mathcal{D}^{m-1} + \cdots + A_{m-1} \mathcal{D}) A^* U,$$

where $A_j = \sum_{0}^{j}{}_k a_{j-k} \sum_{\lambda}^{\sigma_k} \phi_\lambda^k E_\lambda$ for $j = 0, 1, \cdots, m-1$.

THEOREM 39 With A as in the preceding theorem, for any $f \varepsilon F_m$

$$(35)\qquad f = A^* A f + (A_0 \mathcal{D}^m + A_1 \mathcal{D}^{m-1} + \cdots + A_{m-1} \mathcal{D}) A^* U,$$

where for $j = 1, 2, ---, m-1$, $A_j = \sum_{0}^{j}{}_k a_{j-k} \sum_{\lambda}^{\sigma_k} (J_\lambda D^k f) E_\lambda$ and σ_j is the set of discontinuities of $D^j f$.

What shall we mean by $E_\lambda \phi$ for $\phi \varepsilon \bar{F}$? Let $X, Y \varepsilon K_d$ be such that $X\phi, Y\phi \varepsilon F$ (clearly such derigrators exist, since the elements of $\bar{F}$ are quotients of elements of F by those of K_d). Then $Y E_\lambda X\phi = E_\lambda YX\phi = E_\lambda XY\phi$ $= X E_\lambda Y\phi$, i.e. $X^{-1} E_\lambda X\phi = Y^{-1} E_\lambda Y\phi$. This motivates and justifies the following definition:

For any $\phi \varepsilon \bar{F}$, let $E_\lambda \phi = X^{-1} E_\lambda X \phi$ where X is any element of K_d such that $X\phi \varepsilon F$. This reduces to the earlier meaning of $E_\lambda \phi$ if $\phi \varepsilon F$.

THEOREM 40 For any $X \varepsilon \bar{K}_d$ and $\lambda \varepsilon R$, E_λ commutes with X : $E_\lambda X = X E_\lambda$.

Let $p = D^{*-1}$. Then

THEOREM 41 If $A(D) = \sum_{0}^{m}{}_k a_{m-k} D^k$ is a D-polynomial of degree m, then for any $f \varepsilon F_m$

(36) $$A(p)f - A(D)f = (A_0 p^m + A_1 p^{m-1} + \cdots + A_{m-1} p) U,$$

where the A_j are as in Theorem 39.

Let $\delta = pU$. Then

THEOREM 42 If $A(D) = \sum_{0}^{m}{}_k a_{m-k} D^k$ is a D-polynomial of degree m and σ_k, ϕ_λ^k are as in Theorem 38, then for any $f \varepsilon F_m$ and $g \varepsilon F$, the system

$$Af = g, \quad J_\lambda f \underset{\lambda}{=} \phi_\lambda^0, \quad J_\lambda Df \underset{\lambda}{=} \phi_\lambda^1, ---, J_\lambda D^{m-1} f \underset{\lambda}{=} \phi_\lambda^{m-1}$$

is equivalent to

(37) $$A(p)f = g + (A_0 p^{m-1} + A_1 p^{m-2} + \cdots + A_{m-1})\delta$$

where the A_j are as in Theorem 38.

These theorems form the basis of a rational 'operational calculus' completely adequate for the analysis of lumped linear systems driven by linear combinations of sectionally continuous matrices ϵF and 'impulse matrices' $\delta_k = p^k U$ for $k = 1, 2, ---$.

10. Strong Limits of Matrix Functions

A function f_x on a class Σ to F is a matrix function of x on Σ. Let $\{f_n\}$ be a sequence in F, i.e. $f_n = [f_n(t)]$ is a matrix function of n on the positive integers. Then $\lim_{n\to\infty} f_n$, the strong limit of f_n as $n \to \infty$, is an element $f = [f(t)] \in F$ such that $\lim_{n\to\infty} f_n(t)$ exists uniformly in t on every finite continuity interval of $f(t)$ (i.e. every finite interval not containing a discontinuity of $f(t)$). $\lim_{n\to\infty} f_n$ is clearly unique if it exists. $\lim_{n\to\infty}$ is a linear operator on the class of matrix functions of n to F.

Similar remarks apply to the meaning and properties of $\lim_{x\to a} f_x$, the strong limit of f_x as $x \to a$, where f_x is a matrix function of x on an interval of R including a. f_x is strongly continuous in x at a if $\lim_{x\to a} f_x$ exists and equals $f(a)$.

THEOREM 43 If $f_x(t) \in F(t)$ for all x in an interval I of R including a and $f_x(t)$ is continuous in t, x for nearly all t and for all $x \in I$, then $f_x = [f_x(t)]$ is strongly continuous in x at a.

$\frac{d}{dx} f_x$, the strong derivative of f_x in x at a, equals $\lim_{x\to a} \frac{f_x - f_a}{x - a}$. $\frac{d}{dx}$ is of course a linear operator on the class of matrix functions of the real variable x.

THEOREM 44 If $\frac{\partial}{\partial x} f_x(t) \in F(t)$ for all x in an interval I of R including a and $\frac{\partial}{\partial x} f_x(t)$ is continuous in t, x for nearly all t and for all $x \in I$, then $\frac{d}{dx}[f_x(t)]$ exists.

THEOREM 45 If $\alpha(\chi)$ is a complex-valued function of χ and $f_\chi = [f_\chi(t)]$ a matrix function of χ on an interval I of R including a, and the ordinary derivative $\alpha'(\chi)$ and the strong derivative $\frac{d}{d\chi} f_\chi$ both exist at a, then $\frac{d}{d\chi}\alpha(\chi) f_\chi$ exists equal to $\alpha(\chi)\frac{d}{d\chi} f_\chi + \alpha'(\chi) f_\chi$.

$\int_a^b f_\chi \, d\chi$, the strong integral of f_χ in χ from a to b, is an element $f \varepsilon F$ such that $\int_a^b f_\chi(t)\, d\chi$ exists uniformly in t on every finite continuity interval of $f(t)$.

THEOREM 46 If $f_\chi(t) \varepsilon F(t)$ for all χ in the closed interval (a, b) of R and is continuous in t, χ for nearly all t and for all $\chi \varepsilon (a, b)$, then $\int_a^b [f_\chi(t)]\, d\chi$ exists.

It can be shown that equation (17) is valid for sectionally continuous matrices $f = [f(t)]$ (and with all integrations from $-\infty$ to t), i.e.

(38) $$P(\mathcal{D}) f = \left[\int_{-\infty}^{t} G(t-\tau) f(\tau)\, d\tau \right],$$

Where $f \varepsilon F$ and $P(\mathcal{D})$ is a primitive derigrator $A(\mathcal{D})\, B(\mathcal{D})^*$ of positive rank and $G = [G(t)] = \mathcal{D}\, P(\mathcal{D})\, U$. With the notion of strong integral introduced above, (38) may be transformed as follows:

$$\left[\int_{-\infty}^{t} G(t-\tau) f(\tau)\, d\tau \right] = \left[\int_{-\infty}^{\infty} f(\tau)\, G(t-\tau)\, d\tau \right],$$

since now $G(\xi) = 0$ if $\xi < 0$ (this was not true in the preceding sections, where $[1]$ was used instead of U),

$$= \int_{-\infty}^{\infty} f(\tau) [G(t-\tau)]\, d\tau = \int_{-\infty}^{\infty} f(\tau)\, E_\tau [G(t)]\, d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\, E_\tau \{\mathcal{D}\, P(\mathcal{D})\, U\}\, d\tau = \int_{-\infty}^{\infty} f(\tau)\, E_\tau \{p\, P(p)\, U\}\, d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\, E_\tau\, P(p)\, \delta\, d\tau, \quad \text{WHERE } \delta = pU.$$

Thus we have shown (the details will be given elsewhere),

(39) $$P(\mathcal{D})f = \int_{-\infty}^{\infty} f(\tau)\, E_\tau\, P(p)\,\delta\, d\tau,$$

which is a resolution of $P(\mathcal{D})$ into a spectrum of 'retarded impulses'. It should be carefully noted that the derigrator $P(\mathcal{D})$ on the left is proper, i.e. an element of K_d, whereas the 'derigrator' in the integrand is the inverse reverse of $P(\mathcal{D})$, i.e. the image of $P(\mathcal{D})$ in $\bar{K}_d$. Nevertheless, since the rank of $P(\mathcal{D})$ is positive $P(p)\,\delta \in F$, i.e. the integrand, and the integral, belongs to F. If the rank of $P(\mathcal{D})$ were 0, the integral ~~must~~ would be interpreted in the 'weak' sense of the next section.

11. Weak Limits of Matrix Functions

Let $\{\phi_n\}$ be a sequence in the complement of F in $\bar{F}$, i.e. the terms ϕ_n are 'improper' matrices. Since ϕ_n is not the matrix of a function $\phi_n(t)$, what meaning can be given to $\lim_{n\to\infty}\phi_n$? Consider the 'images' $\chi^{-1}\lim_{n\to\infty}\chi$ and $\Upsilon^{-1}\lim_{n\to\infty}\Upsilon$ of $\lim_{n\to\infty}$ where $\chi, \Upsilon \in K_d$. Suppose $\chi\phi_n, \Upsilon\phi_n$ are in F and $\lim_{n\to\infty}\chi\phi_n, \lim_{n\to\infty}\Upsilon\phi_n$ both exist. Since the operator $\lim_{n\to\infty}$ is homogeneous with respect to multipliers $\in K_d$,

$$\Upsilon\lim_{n\to\infty}\chi\phi_n = \lim_{n\to\infty}\Upsilon\chi\phi_n = \lim_{n\to\infty}\chi\Upsilon\phi_n = \chi\lim_{n\to\infty}\Upsilon\phi_n,$$

hence $\chi^{-1}\lim_{n\to\infty}\chi\phi_n = \Upsilon^{-1}\lim_{n\to\infty}\Upsilon\phi_n$, i.e. the image operators cannot produce unequal results. Moreover $\lim_{n\to\infty}$ itself is included, with $\chi = 1$. These considerations motivate and justify the following definition:

$\operatorname{lim\,w}_{n\to\infty}\phi_n$, <u>the weak limit of ϕ_n as $n\to\infty$</u> equals $\chi^{-1}\lim_{n\to\infty}\chi\phi_n$ for any $\chi \in K_d$ such that $\chi\phi_n \in F$ and $\lim_{n\to\infty}\chi\phi_n$ exists. Since the existential distinction between weak and strong limits is often important, we have used the distinct notation $\operatorname{lim\,w}_{n\to\infty}$.

In terms of $\operatorname{lim\,w}_{n\to\infty}$ it is possible to regard the improper elements of $\bar{F}$ as limits of sequences of proper elements $\in F$, thereby giving the improper elements an intuitive significance which they have hitherto lacked. Thus

$$\delta = pU = D^{*-1}\lim_{n\to\infty}D^{*}[n e^{-nt}U(t)], \text{ i.e. } \delta = \operatorname{lim\,w}_{n\to\infty}[n e^{-nt}U(t)]$$

This interpretation of the impulse matrices and the other improper elements of $\bar{F}$ is very helpful in building a 'picture' of $\bar{F}$ adequate for the physical applications.

The notions of weak continuity and weak derivability are evident modifications of the corresponding 'strong' notions and we shall not stop here to give the explicit definitions and state the principal properties.

We are especially interested in the notion of 'weak' integral':

$\oint_a^b f_\chi \, d\chi$, <u>the weak integral of</u> f_χ <u>in</u> χ <u>from</u> a <u>to</u> b, equals $\chi^{-1}\int_a^b \chi f_\chi d\chi$ for any $\chi \varepsilon K_d$ such that $\chi f_\chi \varepsilon F$ for any $\chi \varepsilon (a,b)$ and the strong integral $\int_a^b \chi f_\chi \, d\chi$ exists. The weak integral, if it exists, is independent of χ. The formal properties of $\oint_a^b f_\chi \, d\chi$ follow from, and are the same as those of $\int_a^b \chi f_\chi \, d\chi$,

<u>THEOREM 47</u> <u>For any</u> $f = [f(t)] \varepsilon F$

$$\oint_a^b f(\tau) E_\tau \delta \, d\tau = [f(t; a, b)],$$

<u>where</u>

$$\begin{aligned} f(t;a,b) &= f(t) \quad \underline{if} \quad a < t < b, \\ &= 0 \quad \underline{if} \quad t < a \text{ or } t > b \end{aligned}$$

<u>is undefined for</u> $t = a$ or $t = b$

The proof of this theorem depends upon the following

LEMMA <u>For any</u> $f(t) \varepsilon F(t)$

$$\int_a^b f(\tau) U(t-\tau)\, d\tau \;\overline{\overline{t}}\; \int_{-\infty}^t f(\tau; a, b)\, d\tau$$

<u>Proof</u>: It is easy to verify that

$$\int_{-\infty}^{\min(a,b)} f(\tau)\, d\tau = \int_{-\infty}^{a} f(\tau) U(b-\tau)\, d\tau = \int_{-\infty}^{b} f(\tau) U(a-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) U(b-\tau) U(a-\tau)\, d\tau$$

Hence

$$\int_a^b f(\tau)\, U(t-\tau)\, d\tau = \int_{-\infty}^{b} f(\tau) U(t-\tau) d\tau - \int_{-\infty}^{a} f(\tau)\, U(t-\tau) d\tau$$

$$= \int_{-\infty}^{t} f(\tau)\, U(b-\tau) d\tau - \int_{-\infty}^{t} f(\tau)\, U(a-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} f(\tau) \{U(b-\tau) - U(a-\tau)\} d\tau = \int_{-\infty}^{t} f(\tau; a, b)\, d\tau,$$

The proof of Theorem 47 may now be outlined as follows:

$$\oint_a^b f(\tau) E_\tau \delta\, d\tau = \mathcal{D}^{*-1} \int_a^b \mathcal{D}^* f(\tau) E_\tau \delta\, d\tau = \mathcal{D}^{*-1} \int_a^b f(\tau) E_\tau \mathcal{D}^* \delta\, d\tau$$

$$= \mathcal{D}^{*-1} \int_a^b f(\tau) E_\tau U\, d\tau = \mathcal{D}^{*-1} \int_a^b f(\tau) [U(t-\tau)]\, d\tau = \mathcal{D}^{*-1} \left[\int_a^b f(\tau) U(t-\tau)\, d\tau \right]$$

$$= \mathcal{D}^{*-1} \left[\int_{-\infty}^t f(\tau; a,b)\, d\tau \right] = \mathcal{D}^{*-1} \mathcal{D}^* [f(t; a,b)] = [f(t; a,b)]$$

<u>THEOREM 48</u> <u>For any</u> $f = [f(t)] \varepsilon F$

$$f = \oint_{-\infty}^{\infty} f(\tau) E_\tau \delta\, d\tau$$

This is comparable with the resolution (39) of the preceding section, but here the integral is weak, whereas that in (39) is strong.

12. Appendix

Satisfactory rationalizations of the 'Heaviside' operational calculus have been developed recently by L. Schwartz[1] and J. G. Mikusinski[2] (the superscripts refer to the Bibliography), using very different methods. These methods will now be compared with the method of inverse reversion by reinterpreting the fundamental equation (32) in terms of the ideas of, first, Schwartz and then Mikusinski.

Let F be as in Section 1, i.e. the linear class of all matrices $f = [f(t)]$ where $f(t)$ is everywhere continuous on R to S. Let Φ be the class of all matrices $\phi = [\phi(t)]$ where $\phi(t)$ is on R to C, is ∞-fold derivable everywhere, and vanishes for all sufficiently large $|t|$. For any sequence $\{\phi_n\}$ in Φ, let $\lim_{n\to\infty} \phi_n$ be the element $\phi \varepsilon \Phi$ such that for $m = 0, 1, 2, \cdots,\ \mathcal{D}_t^m \phi_n(t) \longrightarrow \mathcal{D}_t^m \phi(t)$ as $n \to \infty$ uniformly in t on every bounded subset of R. Let $\bar{F}$ be the class of linear operators λ on Φ to S, i.e. $\lambda\phi$ is an additive homogeneous function of ϕ on Φ to S and $\lim_{n\to\infty} \lambda\phi_n = 0$ for any null sequence $\{\phi_n\}$ in Φ. Let $\bar{L}$ be the class of linear operators on $\bar{F}$ to $\bar{F}$.

For $f \varepsilon F$, $\phi \varepsilon \Phi$, let $f\phi = \int_0^\infty f(t)\,\phi(t)\,dt.$

Then $f\phi$ is a linear function of ϕ on Φ to S, and it is easy to see that F is isomorphic with a linear subclass F' of F. Hence $F \subset \bar{F}$, upon identifying corresponding elements of F and F'.

If $\mathcal{D}f = [\mathcal{D}_t f(t)] \varepsilon F$, then

$$(\mathcal{D}f)\phi = \int_0^\infty f'(t)\,\phi(t)\,dt = f(t)\,\phi(t)\Big|_0^\infty - \int_0^\infty f(t)\,\phi'(t)\,dt,$$

i.e.

$$-\int_0^\infty f(t)\,\phi'(t)\,dt = (\mathcal{D}f)\phi + f(0)\,\phi(0).$$

More generally, if $\mathcal{D}^n f = F$, then

(40) $$(-)^n \int_0^\infty f(t)\,\phi^{(n)}(t)\,dt = (\mathcal{D}^n f)\phi + f^{(n-1)}(0)\phi(0) - f^{(n-2)}(0)\,\phi'(0) + \cdots + (-)^{n-1} f(0)\,\phi^{n-1}(0).$$

Define $p \in \bar{L}$ by $(pf)\phi = -f\mathcal{D}\phi$ for $f \in \bar{F}$, $\phi \in \Phi$. Then $(p^2 f)\phi = p(pf)\phi = -(pf)\mathcal{D}\phi = f\mathcal{D}^2\phi$, and in general, $(p^n f)\phi = (-)^n f\mathcal{D}^n\phi$. Hence if $f \in F$,

$$(p^n f)\phi = (-)^n f\mathcal{D}^n\phi = (-n)\int_0^\infty f(t)\,\phi^{(n)}(t)\,dt.$$

In particular, $p^n[1]\phi = (-)^n \int_0^\infty \phi^n(t)\,dt = (-)^{n-1}\,\phi^{(n-1)}(0)$, or $(p^n\delta)\phi = (-)^n\,\phi^{(n)}(0)$ WITH $\delta \equiv p[1]$.

Equation (40) may now be written

$$(p^n f)\phi = (\mathcal{D}^n f)\phi + \{f^{(n-1)}(0)\,p + f^{(n-2)}(0)\,p^2 + \cdots + f(0)\,p^n\}[1]\phi,$$

which is equivalent to

(41) $$p^n f = \mathcal{D}^n f + (f_{n-1}\,p + f_{n-2}\,p^2 + \cdots + f_0\,p^n)[1]$$

with $f_k = f^k(0)$. Equation (32) now follows, with the notation of Theorem 32.

The elements of $\bar{F}$ are essentially <u>distributions</u> in the sense of Schwartz[1] (except that Schwartz takes $f\phi \equiv \int_{-\infty}^\infty f(t)\,\phi(t)\,dt$ for $f \in F$, where we have $\int_0^\infty$. With Schwartz's definition and F as in Section 9, the preceding discussion, mutatis mutandis, leads to equation (36).) We have had to take some liberties with Schwartz's notation in order to preserve the form of (32), but on the whole the changes seem to be improvements.

For a detailed comparison of inverse reversion with Schwartzian distributions, the exact relationship between $\bar{F}$ as defined above, i.e. the class of linear operators on Φ to S, and $\bar{F}$ as defined in Section 8, i.e. the range of $\frac{f}{x}$ for $f \varepsilon F$, $0 \neq x \varepsilon K_d$, must be determined. This has not yet been done.

For the comparison with Mikusinski, let F be the class of matrices $f = [f(t)]$ where $f(t)$ is on R_+, the non-negative real numbers, to C, is continuous for $t > 0$ and right-continuous at $t = 0$. Define sum and difference of elements of F as in Section 1 (but not numerical multiples, for reasons given below).

For $f, g \varepsilon F$, LET $fg = [\int_0^t f(\tau) g(t-\tau) d\tau]$.

Then (c.f. Mikusinski[2]) F is a commutative ring without divisors of zero, and hence may be extended to the essentially unique quotient field $\bar{F} = F/F$. F has no unit element (there is no function $\delta(t)$ such that $\int_0^t \delta(\tau) f(t-\tau) d\tau = f(t)$, i.e. Dirac's 'function' is not a function). However, $\delta = \frac{f}{f}$, for any $0 \neq f \varepsilon F$, is the unit element of the field $\bar{F}$.

Let $p = [1]^{-1}$, so that $\delta = p[1]$. Let $F' \subset \bar{F}$ be the class of elements $p[\alpha]$ for $\alpha \varepsilon C$. The correspondence between $\alpha \varepsilon C$ and $p[\alpha] \varepsilon F'$ is an isomorphism. Hence $C \subset \bar{F}$, upon identifying α and $p[\alpha]$. Moreover, for any $f \varepsilon F$,

$$\alpha f = p[\alpha] f = p [\int_0^t \alpha f(t) dt] = [\alpha f(t)],$$

from which it follows readily that F is now a commutative algebra (linear ring) and $\bar{F}$ is a linear field (this result makes unnecessary the definition of numerical multiple of elements of F as in Section 1).

If $\mathcal{D}f \in F$, then

$$f = [\int_0^t f'(t)dt + f_0] = [\int_0^t f'(t)dt] + [f_0]$$

and $pf = \mathcal{D}f + f_0 = \mathcal{D}f + f_0 p[1]$. In general if $\mathcal{D}^n f \in F$, then

$$p^n f = \mathcal{D}^n f + (f_{n-1} p + f_{n-2} p^2 + \cdot \cdot \cdot + f_0 p^n)[1],$$

i.e. (41) holds, and (32) follows immediately.

Thus in this very special case, with $S = C$ and the domain of $f(t)$ restricted to R_+, the classes F, $\bar{F}$ play the roles of the operator classes L, $\bar{L}$ in the more general theory.

BIBLIOGRAPHY

The literature in the fields of abstract algebra, abstract analysis and operational calculus is enormous. Extensive references to the first two fields are given in Hille[13] and Michal[14] below. Gardner and Barnes[9] and Doetsch[10] contain comprehensive bibliographies of the literature on operational calculus. The history of this field is reviewed in Cooper[8]. Of the remaining references below, Banach[12] and Hasse[11] are standard classics, and to the author's knowledge, Schwartz[1] and Mikusinski[2,3,4] contain the only satisfactory rationalizations of the 'Heaviside' calculus thus far published which do not impose unnatural (i.e. physically irrelevant) conditions on $f = [f(t)] \varepsilon F$; brief appraisals follow Heaviside[5], Jeffreys[6] and Brown[7].*

Operational Calculus

1. L. Schwartz, Théorie des Distributions I, Hermann, Paris, 1950 (Actualites Scientifiques et Industrielles, No. 1091).

2. J. G. Mikusinski, "Sur les fondaments du calcul operatoire", Studia Mathematica 11 (1949) 41-70.

3. ________________, "L'Anneau Algibrique et ses applications dans l'analyse fonctionelle", Annales Univ. Mariae Curie-Sklodowska III (1949) 1-82

*Professor A. Erdelyi kindly called my attention to this paper.

4. ________________, "Une nouvelle justification du calcul operatoire", Atti dell'Acadamia Nazionale dei Lincei I (1950) 113-121.

5. O. Heaviside, Electromagnetic Theory, Longon, 1899.

Contains many applications of p-operators and impulse 'functions' to the ordinary and partial differential equations of electromagnetic circuit theory and field theory, guided by remarkable physical and mathematical insight and intuition but without adequate definitions and proofs.

6. H. Jeffreys, Operational Methods in Mathematical Physics, Camb. Univ. Press, 1931 (Cambridge Tracts in Mathematics and Mathematical Physics, No. 23).

Clearly outlines the relation between cogredient polynomials in $\mathcal{D}_t$ and $\int_0^t$ (cf. Section 7 and the proof of Theorem 25 above) but introduces and uses the p-operator with no more rigor and less vigor than Heaviside.

7. B. M. Brown, "Solution of Differential Equations by Operational Methods", Mathematical Gazette XXXI (1947) 145-153.

First 'defines' the reciprocal $\frac{1}{R(p)}$ of a p-polynomial as essentially $R(\mathcal{D})^*$ (see Section 3 above), then gives a 'definition' of $R(p)$ that can only be described as incomprehensible. Contains vague adumbrations of some of the theorems in Sections 3 and 6. Does not mention 'impulse' functions.

8. J. L. B. Cooper, "Heaviside and the Operational Calculus", Mathematical Gazette XXXVI (1952) 5-19.

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12. S. Banach, Théorie des Operations Lineares, Warszawa, 1932.

13. E. Hille, Functional Analysis and Semi-Groups, Amer. Math. Soc., New York, 1948 (Colloquium Publications Vol. XXXI)

14. A. D. Michal, "General Differential Geometries and Related Topics", Bull. Amer. Math. Soc. (1939) 529 - 563.