AN ABSOLUTE METHOD FOR THE DETERMINATION OF THE NOISE FIGURE OF RADIO RECEIVERS

Thesis by

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A B S T R A C T

In the first part of this thesis the nature of random noise, which arise and theoretically cannot be eliminated from radio receivers, is described.

Nyquist equation, pertaining to the amount of noise generated in a resistor, is rederived by a simple method. The original equation as derived by Nyquist, is found to be valid, only, for physically realizable impedances of the minimum reactance type.

The general problem of a passive electric network excited by random sources is treated; a useful principle is developed for its solution.

Part II treats of the definition of the noise figure of a four terminal network and the most important methods available for its measurement. An improved noise generator for the measurement of noise figure is described.

CONTENTS

P A R T I

RANDOM NOISE AND ITS CHARACTERISTICS

1.1. Thermal Noise

In 1928 J. B. Johnson showed that:[1] "Statistical fluctuation of electric charge exists in all conductors, producing random variation of potential between the ends of the conductor." Working in conjunction with him, Nyquist[2] was able to show on the basis of the statistical theory of thermodynamics that the thermal noise voltage generated in an impedance Z is given by the equation:

$$\bar{E}^2 = 4RKT\Delta F$$

where;

$\bar{E}$	is the r.m.s. value of the thermal noise voltage
R	the real part of Z in ohms
T	the absolute temperature in degrees absolute
K	Boltzmann's constant 1.37×10^{-23} watt-sec/deg.Ab.
ΔF	bandwidth in cycles per second

Combining electric circuit theory with the equipartition theorem from thermodynamics, a simple derivation of Nyquist equation is possible. Consider the simple r-c circuit shown in Fig.1 ; if r is the site of an e.m.f. whose r.m.s. value is $\bar{v}$, there must be energy stored in the electric field across the condenser of magnitude $\frac{1}{2}C\bar{V}^2$, where $\bar{V}$ is the r.m.s. value of the potential drop across C. By the equipartition theorem, the average free energy fluctuation is equal to KT/2. Since the system has only one degree of freedom, and there can be no free energy in the resistor, we may write:

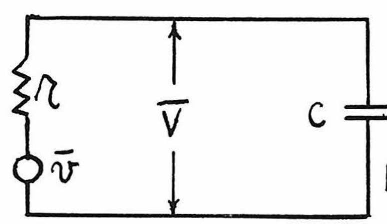


Fig. 1

$$\frac{KT}{2} = \frac{1}{2}C\bar{V}^2 \qquad \text{or} \qquad KT = C\bar{V}^2 \qquad (1)$$

From simple circuit theory it is well known that for a simple r-c circuit the following identity is true:

$$\int_0^\infty R \, d\omega = \frac{\pi}{2C} \qquad (2)$$

where R is the real part of the parallel combination of r and c. Substituting for C in the equation (2) we get:

$$2\pi \int_0^\infty R \, df = \frac{\pi \overline{V}^2}{2KT}$$

or

$$\overline{V}^2 = 4KT \int_0^\infty R \, df \qquad (3)$$

This is Nyquist equation in its integral form.

In equation (3) if we let $C \to 0$, $R \to r$, a pure resistance independent of frequency, then we may write:

$$\overline{V}^2 = \overline{v}^2 = 4KTr \int_0^\infty df \qquad (4)$$

This is another way of saying that the square of the mean noise voltage across a pure resistor is a quantity independent of frequency per unit frequency, i.e., proportional to the bandwidth of interest.

Generalized proof of Nyquist equation: The fact that the mean square fluctuations per unit frequency are independent of frequency may be proved, in general, by the following reasoning:

The mean square fluctuation $\overline{S}^2$ of random phenomena is defined as:

$$\overline{S}^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} S(t)^2 \, dt \qquad (5)$$

Also the Fourier integral energy theorem enables us to write the identity,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} S^2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega \qquad (6)$$

therefore:

$$\bar{S}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |S(\omega)|^2 d\omega \qquad (7)$$

To explain the nature of the electric disturbance, that may arise in a closed conductor in equilibrium, we follow Lorentz[3] in claiming that: "The random motion of electrons, similar to the thermal agitation of gas molecules, will give rise to spontaneous electric currents, whose direction and intensity vary continuously." Therefore, we shall assume the disturbance to be of a random impulsive nature. We already know that the Fourier transform of a single pulse of length τ is a constant independent of frequency, for frequencies $f \ll \frac{1}{\tau}$; also the mean square of the sum of random disturbances, is equal to the sum of the squares of each individual disturbance, since all cross products vanish. Hence we have the general proof, that the mean square of a random disturbance of an impulsive nature and of length τ , per unit frequency, is independent of frequency as long as we are interested in frequencies $f \ll \frac{1}{\tau}$.

It is of interest to note in this place, that the above proof, at least, is not in conflict with quantum theory. It speculates that at frequencies of the order of $\frac{1}{\tau}$ the result will not hold. According to quantum theory the limiting frequency is given by:

$$f = KT/h = 6.1 \times 10^{12} \text{ cycles/sec}$$

where h is Planck's constant and T, for room temperature, equals 293° Ab.

In what follows we shall concern ourselves only with frequencies much smaller than the limiting frequency of 6.1×10^{12} cycles/second. We are ready now to develop Nyquist equation for a generalised, physically realizable, minimum reactance impedance function $\theta = A(\omega) + jB(\omega)$ as shown in Fig. 2, where C represents all the shunt capacity that may

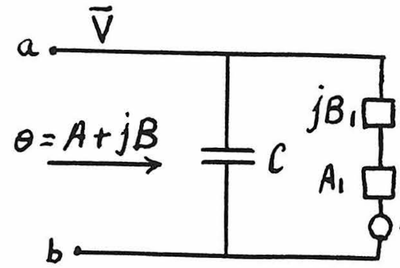


Fig. 2

be pulled out between the terminals a-b; $A_1 + jB_1$ any physically realizable impedance of the minimum reactance type, $\bar{v}$ the r.m.s. noise voltage generated within $A_1 + jB_1$, and $\bar{V}$ the r.m.s. value of the potential drop across a-b. If θ is assumed analytic at zero and infinite frequency, then it may be expanded in power series around zero and infinite frequency as follows:

$$\theta_0 = A_0 + jB_0\omega + A_1'\omega^2 + jB_1'\omega^3 + \text{-----} \qquad (8)$$

and

$$\theta_\infty = A_\infty + j\frac{B_\infty}{\omega} + \frac{A_1''}{\omega^2} + j\frac{B_1''}{\omega^3} + \text{------} \qquad (9)$$

We immediately recognize that in Fig. 2, $A_\infty = 0$, $B_\infty = -\frac{1}{C}$

From an energy point of view the terminals a-b in Fig. 2, represent one degree of freedom in the system and the mean free energy that may be associated with it, is to be equated to $KT/2$. But the average noise energy E, between a-b, is by the Fourier energy theorem proportional to:

$$E \propto \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{V(\omega)}|^2\, d\omega$$

Hence the equality:

$$E = \frac{\alpha}{2\pi}\int_{-\infty}^{\infty} |\overline{V(\omega)}|^2\, d\omega \qquad (10)$$

where

α is the proportionality constant depending on the geometry of the system, and

$\overline{V(\omega)}$ is the Fourier transform of the r.m.s. voltage drop across a-b.

By simple circuit theory, equation (10) reduces to:

$$E = \frac{\alpha}{2\pi}\int_{-\infty}^{\infty} \overline{v}^2 \frac{A}{A_1}\, d\omega \tag{11}$$

But $\overline{v}^2$ is independent of ω and A/A_1 is an even function of frequency,

Therefore:

$$E' = \frac{E}{2} = \frac{\alpha \overline{v}^2}{2\pi}\int_0^{\infty} \frac{A}{A_1}\, d\omega \tag{12}$$

This is the total average noise energy in the degree of freedom defined by a-b; hence by the equipartition theorem it represents double the free noise energy in the same degree of freedom. Now the $\int_0^{\infty}\frac{A}{A_1}\,d\omega$ may be evaluated by taking the line integral of $\frac{\theta - A_\infty}{A_1}$ in the right half of the complex frequency plane, along a contour bounded by a large semi-circular arc near infinity and the real frequency axis between ∞ and $-\infty$. If both θ and A_1 are impedances of the minimum reactance type, neither of them will contain any singularities within the path of integration and we may write:

$$\oint \frac{\theta - A_\infty}{A_1}\, d\omega = 0 = \int_{-\infty}^{\infty} \frac{A - A_\infty}{A_1}\, d\omega + j\int_{-\infty}^{\infty} \frac{B}{A_1}\, d\omega + \oint \frac{\theta - A_\infty}{A_1}\, d\omega \tag{13}$$

The second integral in equation (13) vanishes because B/A_1 is an odd function of frequency; the third integral can be easily evaluated as $\frac{\pi B_\infty}{A_{1\infty}}$. Hence we have the identity:

$$\int_0^{\infty} \frac{A - A_\infty}{A_1}\, d\omega = -\frac{\pi B_\infty}{2 A_{1\infty}} \tag{14}$$

For our circuit $A_\infty = 0$, $B_\infty = -\frac{1}{C}$, $A_{1\infty} = r$, the value of the pure resistance that may be pulled out from $A_1 + jB_1$. Therefore equation (12) reduces to :

$$E' = \frac{\pi \alpha \bar{v}^2}{4\pi r C} = \frac{\alpha \bar{v}^2}{4 r C} = KT$$

or

$$\frac{\alpha}{C} \bar{v}^2 = 4KTr \tag{15}$$

To calculate α we let $A_1 \to r$ and our circuit reduces to the simple r-c circuit shown in Fig. 1 for which we have already proved the identity:

$$\bar{v}^2 = 4KTr \qquad \text{per unit frequency.}$$

Hence we deduce that $\alpha / C = 1$ and obtain the general relation

$$\bar{v}^2 = 4KTr \qquad \text{per unit frequency} \tag{16}$$

Equation (16) definitely associates the noise voltage fluctuations, per unit frequency, with a physical resistance that may be pulled out of any two terminal, physically realizable impedance of the minimum reactance type.

It may be easily verified that, as long as θ is a minimum reactance impedance, the theorem holds immaterial of whether $A_1 + jB_1$ is a minimum reactance impedance or not.

If both θ and $A_1 + jB_1$ are non minimum reactances, then we may prove that:

$$\int_0^\infty \frac{A}{A_1} d\omega < -\frac{\pi}{2} \frac{B_\infty}{A_{1\infty}} \tag{17}$$

but we already know that

$$\bar{v}^2 \int_0^\infty \frac{A}{A_1} d\omega$$

is proportional to the noise energy; thus we deduce the interesting result:

$$\overline{v}^2 > 4KT\Omega \qquad \text{per unit frequency} \qquad (18)$$

Therefore, we may generalize Nyquist equation and write

$$\overline{v}^2 \geqslant 4KT\Omega \qquad \text{per unit frequency} \qquad (19)$$

where the equality sign holds for all physically realizable impedances of the minimum reactance type. In terms of the mean square voltage between terminals a-b, the generalized Nyquist equation may now be written as:

$$\overline{V}^2 \geqslant 4KT\int_0^\infty A\,df \qquad (20)$$

where the equality sign holds for all impedances of the minimum reactance type.

1.2. Shot Noise

In an ordinary vacuum tube, the electric current emitted from a hot cathode consists of the combined effect of a large number of independently emitted electrons. In 1918, W. Schottky[4] described the nature of noise which should theoretically be associated with the random emission of the electron convection current. The magnitude of this noise at low frequency has been calculated by various methods for both the temperature and space charge limited diode and is found, when initial velocities and secondary electron emission are neglected, to be given by:

$$\overline{i^2} = 2 e I_0 \Delta f \qquad (21)$$

for the temperature limited diode, and by

$$\overline{i^2} = 0.644 \times 4 K T g \Delta f \qquad (22)$$

for the space charge limited diode.

where:

e is the charge of an electron

I_0 is the D.C. plate current

K is Boltzmann's constant

T is the cathode absolute temperature

g is the anode conductance of the diode

A comprehensive derivation of the above two formulas is due to J. R. Pierce.[5]

1.3. Transit Time Effect On Shot Noise:[6] As shot noise is assumed to be formed of identical independent pulses, the power spectrum resulting from adding these pulses will be proportional to the absolute square of the Fourier transform of a single one of the pulses. For the temperature limited diode the current pulse may be shown to be of the form:

$$i(t) = \frac{2e}{T^2} t \qquad 0 < t < T$$
$$i(t) = 0 \qquad \text{for all other } t \qquad (23)$$

where,

e is the charge of the electron

T is the cathode to anode transit time.

Therefore the Fourier transform of $i(t)$ is:

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \times \frac{2e}{T^2} \int_0^T t\, e^{-j\omega t}\, dt \qquad (24)$$

and

$$|G(\omega)|^2 = \frac{4e^2}{2\pi\, \theta^4} \left[\theta^2 + 2\left(1 - \cos\theta - \theta \sin\theta\right)\right] \qquad (25)$$

which is proportional to the power spectrum of the noise. The constant of proportionality is computed by letting θ go to zero and the power spectrum approach its low frequency value

$$\frac{2eI_0}{2\pi}$$

Hence we get:

$$i^2 = 2eI_0\Delta f \times \frac{4}{\theta^4}\left[\theta^2 + 2\left(1 - \cos\theta - \theta\sin\theta\right)\right] \qquad (26)$$

$$= 2eI_0\Delta f\, \Gamma^2 \qquad (27)$$

Similarly for the space charge limited diode it may be shown that:

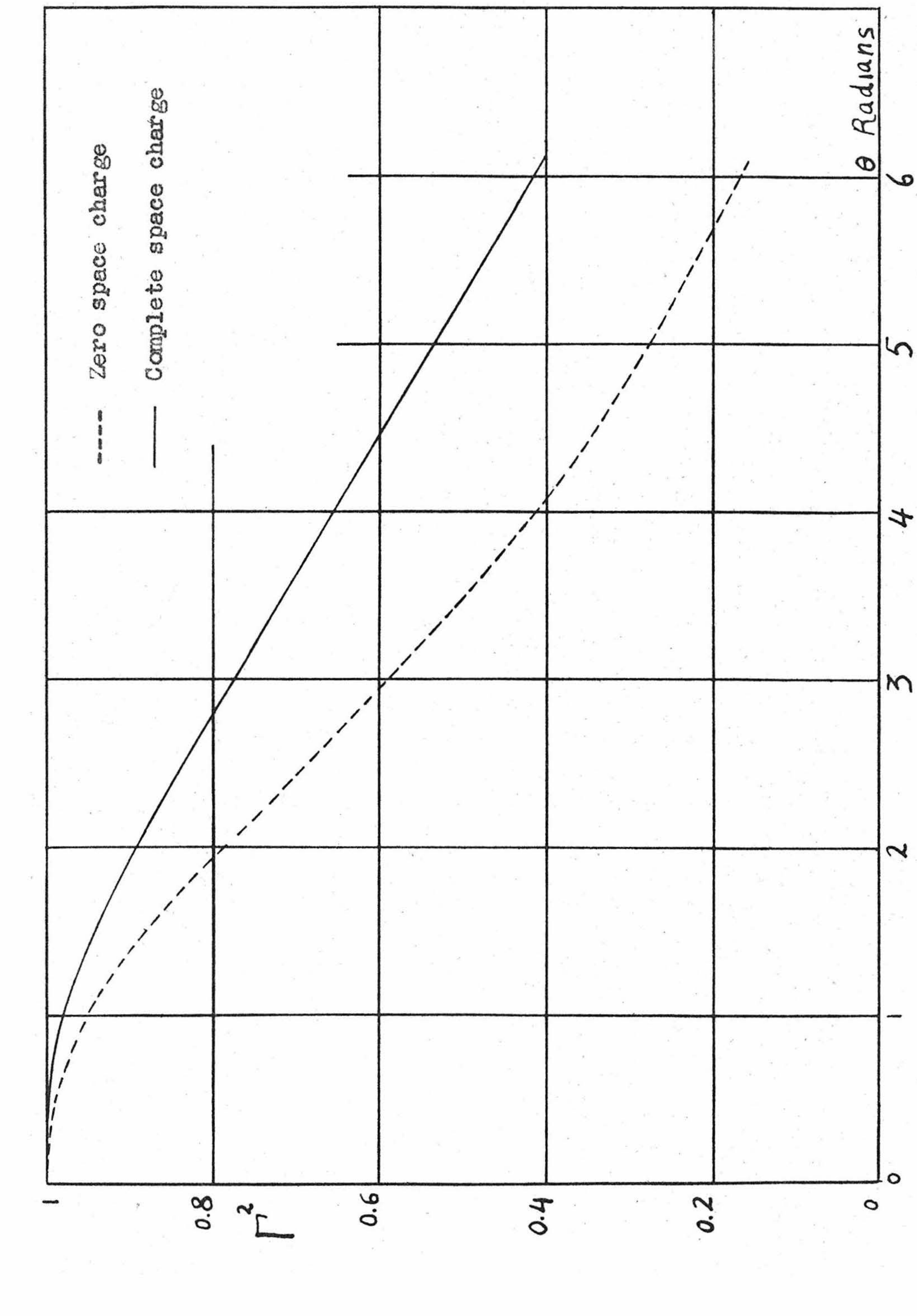


Fig. 3

Comparison of Γ^2 for zero and complete space charge

$$i(t) = \frac{4e\,t^3}{T^4} \qquad\qquad 0 < t < T$$

$$i(t) = 0 \qquad\qquad \text{for all other } t \tag{28}$$

By following the same steps as in the temperature limited case we get for the space charge limited diode:

$$\overline{i^2} = 0.644 \times 4KTg\Delta f \times \frac{16}{\theta^2}\left[1 - \frac{3}{\theta^2} + \frac{12\sin\theta}{\theta^3} + \frac{36\cos\theta}{\theta^4} - \frac{72\sin\theta}{\theta^5} + \frac{72}{\theta^6}(1-\cos\theta)\right] \tag{29}$$

$$\overline{i^2} = 0.644 \times 4KTg\Delta f\,\Gamma^2 \tag{30}$$

The factor Γ^2 for both zero and complete space charge is plotted in Fig.3., as a function of the transient angle θ .

The above results represent a summary of many articles which have been published on the subject. They have been included here for the sake of completeness.

1.4. <u>Effect Of Lead Inductance And Shunt Capacity On The Available Noise Power From A Single Diode:</u> When the frequency is high enough so that the plate resistance is effectively shunted by the cathode-plate capacity, the equivalent high frequency circuit of the matched diode used for noise measurement is as shown. From simple circuit theory it is easily found that the available noise power at the load admittance Y_ℓ is:

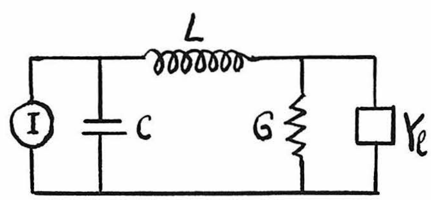


Fig 4

$$P = \frac{I^2}{4}\sqrt{\frac{L}{C}} \times \frac{1}{1 - 1.75\,\omega^2 LC + \omega^4 L^2 C^2}$$

when,

$$Y_L = G = \sqrt{\frac{C}{L}}$$

We see that the available noise power is a function of frequency. For example at 100 m.c. if,

$$C = 5\,\mu\mu f \quad ; \quad L = 0.02\,\mu h. \quad ; \quad \omega^2 LC = 0.04$$

and the power has increased by about 7% of its low frequency value; then to compare this result with the transit time reduction factor Γ^2, let us consider a plane diode with 2 mm separation between cathode and anode; let the anode voltage be 100 volts; for temperature limited operation the transit angle is about 0.42 radians at 100 m.c.; From Fig.3. the available noise power is found to be reduced by less than 1%. Thus the shunt capacity and lead inductance have a much more serious effect at 100 m.c. in reducing the available noise power from the single diode, than the transit time. It is to be pointed out also that the use of networks to tune out the shunt capacity or lead inductance is not to be recommended in noise measurements because of the selective characteristics of these networks with respect to the frequency, and it is indeed no exageration to claim that in many cases, tuning networks, if not carefully designed, affect the accuracy of noise measurements in a most arbitrary manner.

1.5. Sources Of Random Noise In Conventional Passive Circuits

Let us consider a generalized passive network excited by random noise sources. Let the available noise power at a certain point of interest in the network, called the origin, due to discrete noise sources of r.m.s. magnitudes

$$I_1 , \quad I_2 , \quad \text{--------} \quad I_n$$

and placed at points,

$$X(\xi_1) , \quad X(\xi_2) , \quad \text{-----------} \quad X(\xi_n)$$

be $P(\xi_1)$, $P(\xi_2)$, ----------- $P(\xi_n)$

when the sources are introduced individually in the network.

ξ may be considered as a space variable relating input and output.

In case the sources

$$I_1 , \quad I_2 , \quad \text{------------} \quad I_n$$

are introduced simultaneously in the system, the noise power output at the origin will be

$$\sum_{n=1}^{n=n} P(\xi) = P(\xi_1) + P(\xi_2) + \text{------} + P(\xi_n) \qquad (31)$$

because the sources are assumed to be of a random nature. In general $P(\xi)$ may be written as a fraction, the numerator and denominator of which are both functions of ξ . Hence the identity:

$$\sum_{n=1}^{n=n} P(\xi) = \sum_{n=1}^{n=n} \frac{N(\xi)}{D(\xi)} \qquad (32)$$

However if the system is linear, the different noise sources should not interact with each other, so that $D(\xi)$ must be independent of (ξ) .

Therefore we may write for a linear system,

$$\sum_{m=1}^{m=n} P(q) = \frac{1}{D} \sum_{m=1}^{m=n} N(q) \tag{33}$$

where D is a constant which depends on the geometry of the system. In case the discrete noise sources become continuous in space, the output noise power may be written in the form of the definite integral

$$P = \frac{1}{D} \int_{q_1}^{q_2} N(q)\, dq \tag{34}$$

Usually D is determined by placing a noise source at a convenient point in the network and then solving for the output power. The above result is quite general and is only restricted by the speculation that the system must be linear and the input of a random nature. It applies to mechanical as well as to electrical systems, since the two are analogous.

It should be noticed at this point, that $N(q)$ must be a scalar quantity; the network may affect, only, the magnitude of the noise power output, because this is the only quantity by which random sources are characterized.

PART II

THE DETERMINATION OF NOISE FIGURE BY THE DISTRIBUTED AND TRANSMISSION LINE NOISE GENERATOR DIODES

2.1 Noise Figure

During recent years much has been said about the definition of noise figure of four-terminal networks in general, and many methods have been described and used for its measurement. The most suitable definition of the noise figure, F, for our purpose, is that given by H. T. Friis:[7]

$$F = \frac{\text{Available input signal power/Ideal available input noise power}}{\text{Available output signal power/Available output noise power}}$$

$$F = \frac{S_g/K\,T\,B}{S/N} = \frac{N}{G\,K\,T\,B}$$

$N = F\,G\,K\,T\,B$

where $G = S/S_g$ gain of the network

For linear networks we may refer all quantities to the output and write: $N = F\,K\,T\,B$

A general procedure to measure F is to connect a square law meter to the output terminal and note the output power noise, N_1, when the input is matched and no signal present. Then a signal is added to the input whose average r.m.s. power is S_1, and the output signal plus noise power N_2 is noted. Hence we may write:

$$N_1 = F\,K\,T\,B$$

$$N_2 = F\,K\,T\,B + S_1$$

$$N_2 - N_1 = S_1$$

$$\frac{N_2 - N_1}{N_1} = \frac{S_1}{F\,K\,T\,B}$$

$$F = \frac{S_1}{(N_2/N_1 - 1)\,K\,T\,B}$$

For absolute determination of F, the bandwidth may be eliminated by choosing S_1 such that it is proportional to the bandwidth of the network whose noise figure is to be measured. Such signals are available from random noise generating sources such as hot wires or temperature limited diodes. In case a hot wire is used whose temperature is T_1

$$F = \frac{K\,T_1\,B}{(N_2/N_1 - 1)\,K\,T\,B} = \frac{T_1/T}{(N_2/N_1 - 1)}$$

In practice, T_1/T is of the order of 10;and if the measurement of noise figure is to be accurate, N_2/N_1 should be of the order of 2. Thus hot wire sources, though excellent noise generators, may only be used for the measurement of noise figures of the order of 10 or less.

The use of conventional temperature limited diodes permits the measurement of noise figures substantially larger than 10 at low frequency. Since $S = \frac{1}{2}\,e\,I_0\,B\,R$

$$F = \frac{e\,I_0\,B\,R}{2(N_2/N_1 - 1)\,K\,T\,B} = \frac{e\,I_0\,R}{2(N_2/N_1 - 1)\,K\,T}$$

If $T = 290^\circ$ Ab., then $e / 2\,K\,T = 20$ and

$$F = \frac{20\,I_0\,R}{(N_2/N_1 - 1)}$$

where R is the load resistance assumed to match the diode noise generator output resistance.

Again, for $N_2/N_1 = 2$ and $R = 50$ ohms, noise figures of the order of 30 may be easily measured at low frequencies when both transit angle effects and lead inductance and shunt capacity are negligible. However, at high frequencies the limitations on conventional diodes are such that they virtually become useless as standard noise generators at frequencies higher than 100 mc.

To remedy the effects of shunt capacity and lead inductance, R. Kompfner[8] and others have described the transmission line diode to be used as a noise source at centimeter wavelengths. They faced the problem of constructing a specially designed diode to perform their experiments. In the following pages, a noise generator consisting of a finite number of conventional high frequency diodes cascaded together, is described. It has all the advantages of the transmission line diode to which it is practically equivalent and the added advantage of being built of components easily available on the market. Also, the available noise power from the transmission line diode, operating under certain prescribed conditions is computed by simple means. A comparison is made between the loss-less transmission line diode, the one in which the losses are small and the one in which the losses are predominant.

2.2 The Distributed Diode as a Standard Noise Source

We shall solve the problem of $N+1$ similar noise generator diodes cascaded together; their plate resistance will be neglected compared to their shunt capacity; their equivalent high frequency circuit, when all losses are neglected, will be assumed as shown in Fig. 5 .

The available noise power at the load end may be computed by straight forward methods using the principle of superposition for random sources, i.e. applying one current at a time and solving for the output power at the load. This method involves the solution of $N+1$ simultaneous equations $N+1$ times. In case the number of tubes is larger than the modest number of four, the computational labor involved becomes unjustified. However a solution in a closed form is possible. It involves the solution of a simple second order difference equation and the application of the principle developed in section 1.5. concerning the superposition of random noise sources in a passive network. The complete solution, for one end terminated by its characteristic admittance $\sqrt{C/L}$ when the far end is short circuited, is developed in appendix I. All other results are developed by following similar steps.

The available noise power $P_{s.c.}$ at the load admittance Y_ℓ when the far end is short circuited is found to be:

$$P_{s.c.} = I^2 \sqrt{L/C} \, \frac{\sum_{S=0}^{S=N} \sinh^2 (N-S)\theta}{1 + \sinh\frac{\theta}{2} \, \sinh\left(2N+\frac{1}{2}\right)\theta} \qquad (35)$$

where, $\cosh\theta = 1 + \frac{1}{2} \times \frac{Y_2}{Y_1} = 1 - \frac{1}{2}\omega^2 LC$

$Y_\ell = \sqrt{C/L}$

$N+1$ is the total number of tubes

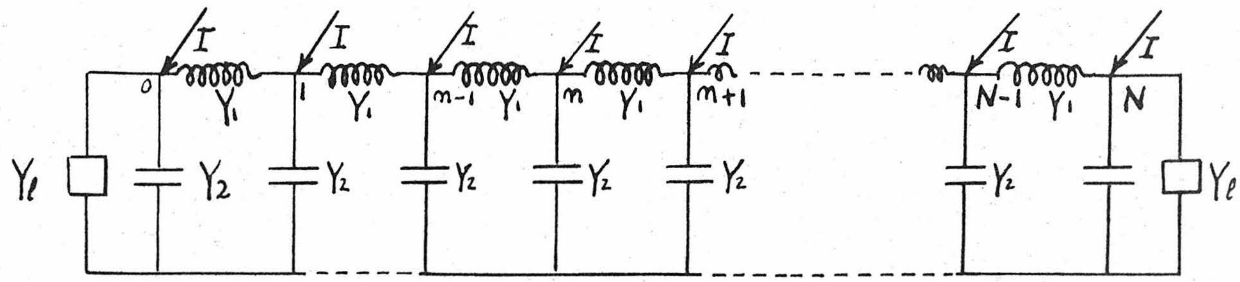


Fig. 5

Equivalent High Frequency Circuit of the Distributed

Noise Generator Diode.

Y_ℓ and Y_ℓ' are terminal admittances

Y_1 is the series admittance of a wire connecting two tubes

Y_2 is the shunt admittance to ground of each tube

I is the R. M. S. value of the noise current generated in each diode

S is an integer over which the summation is to be taken

I^2 the square of the r.m.s. of the noise current of each tube.

In the region where $0 \leq \omega^2 LC \leq 4$, to make the solution real, we put $\theta = j\phi$ and obtain:

$$\cos\phi = 1 - \frac{1}{2}\omega^2 LC$$

$$P_{s.c.} = I^2\sqrt{L/C}\;\frac{\sum_{S=0}^{S=N}\sin^2(N-S)\phi}{1 - \sin\frac{\phi}{2}\,\sin\left(2N+\frac{1}{2}\right)\phi} \qquad (36)$$

and in the region where $4 < \omega^2 LC < \infty$ we put $\theta = \phi + j\pi$ and get:

$$\cosh\phi = \frac{1}{2}\omega^2 LC - 1$$

$$P_{s.c.} = I^2\sqrt{L/C}\;\frac{\sum_{S=0}^{S=N}\sinh^2(N-S)\phi}{1 + \sinh\frac{\phi}{2}\,\sinh\left(2N+\frac{1}{2}\right)\phi} \qquad (37)$$

It can easily be seen that the available noise power tends to zero at both zero and infinite frequencies as it actually should.

The available noise power divided by $I^2\sqrt{L/C}$ is plotted as ordinates against the dimensionless quantity $\omega^2 LC$ as abscissa on semi-log paper in Fig.6., when two, six and ten identical tubes are used. This plot is interesting in that it shows that the available noise power in the region $0 \leq \omega^2 LC \leq 4$ has N maxima in each case, and that the envelopes of these curves seem to be parallel straight lines; so that actually they must be exponentials of the same index.

The case when the noise generator is matched at both of its ends by its characteristic admittance $\sqrt{C/L}$ has also been solved and the available noise power P_m at each end is found to be:

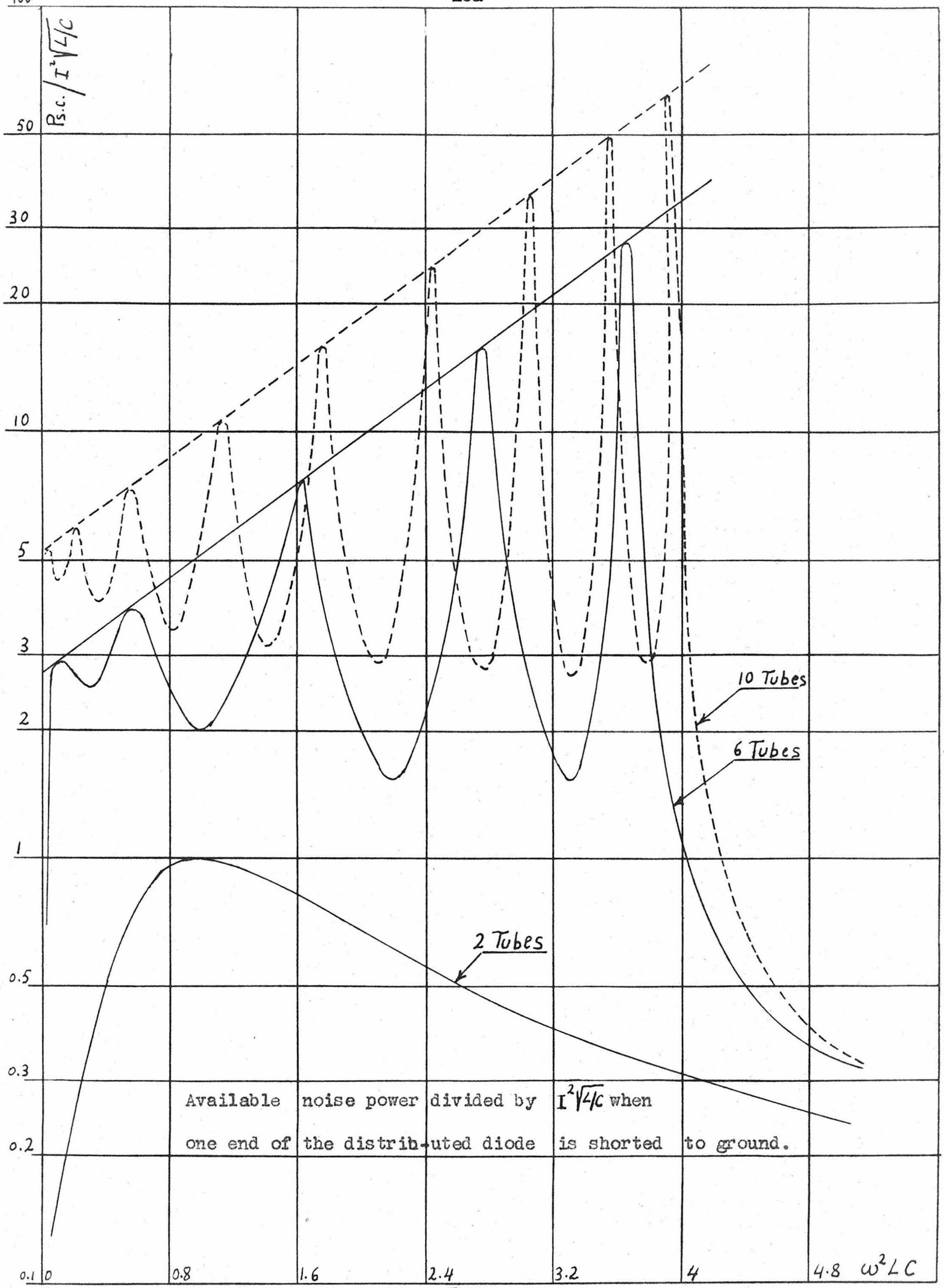

$P_{s.c.}/I^2\sqrt{L/C}$
100
50
30
20
10
5
3
2
1
0.5
0.3
0.2
0.1
0
0.8
1.6
2.4
3.2
4
4.8
$\omega^2 LC$
10 Tubes
6 Tubes
2 Tubes
Available noise power divided by $I^2\sqrt{L/C}$ when one end of the distrib-uted diode is shorted to ground.

Fig. 6

$$P_m = \frac{I^2}{4}\sqrt{L/C}\ \frac{(N+1) + \sinh\frac{\theta}{2}\sum_{S=0}^{S=N}\sinh\left(2N-2S+\frac{1}{2}\right)\theta}{1-\sinh^2\frac{\theta}{2}\ \sinh^2\left(N+\frac{1}{2}\right)\theta} \tag{38}$$

where

$$\cosh\theta = 1-\frac{1}{2}\omega^2 LC$$

and all other symbols have the same meaning as in the short circuit case. Again in the region where $0 \leq \omega^2 LC \leq 4$ to make the solution real, we put $\theta = j\phi$ and obtain:

$$\cos\phi = 1-\frac{1}{2}\omega^2 LC$$

$$P_m = \frac{I^2}{4}\sqrt{L/C}\ \frac{(N+1) - \sin\frac{\phi}{2}\sum_{S=0}^{S=N}\sin\left(2N-2S+\frac{1}{2}\right)\phi}{1-\sin^2\frac{\phi}{2}\ \sin^2\left(N+\frac{1}{2}\right)\phi} \tag{39}$$

and in the region where $4 < \omega^2 LC < \infty$ we put $\theta = \phi + j\pi$ and obtain:

$$\cosh\phi = \frac{1}{2}\omega^2 LC - 1$$

$$P_m = \frac{I^2}{4}\sqrt{L/C}\ \frac{(N+1) + \sinh\frac{\phi}{2}\sum_{S=0}^{S=N}\sinh\left(2N-2S+\frac{1}{2}\right)\phi}{1-\sinh^2\frac{\phi}{2}\ \sinh^2\left(N+\frac{1}{2}\right)\phi} \tag{40}$$

We easily find that at zero frequency,

$$P_m = I^2\sqrt{L/C}\ \frac{N+1}{4} \tag{41}$$

which is indeed the expected result. At infinite frequencies the output again falls to zero as in the short circuit case. The available noise power divided by $I^2\sqrt{L/C}$ is plotted against $\omega^2 LC$ in Fig.7 on semi-log paper. The response is found to be flatter than in the short circuit case in the region $0 \leq \omega^2 LC \leq 4$. The number of maxima is again N if the zero frequency value is excluded.

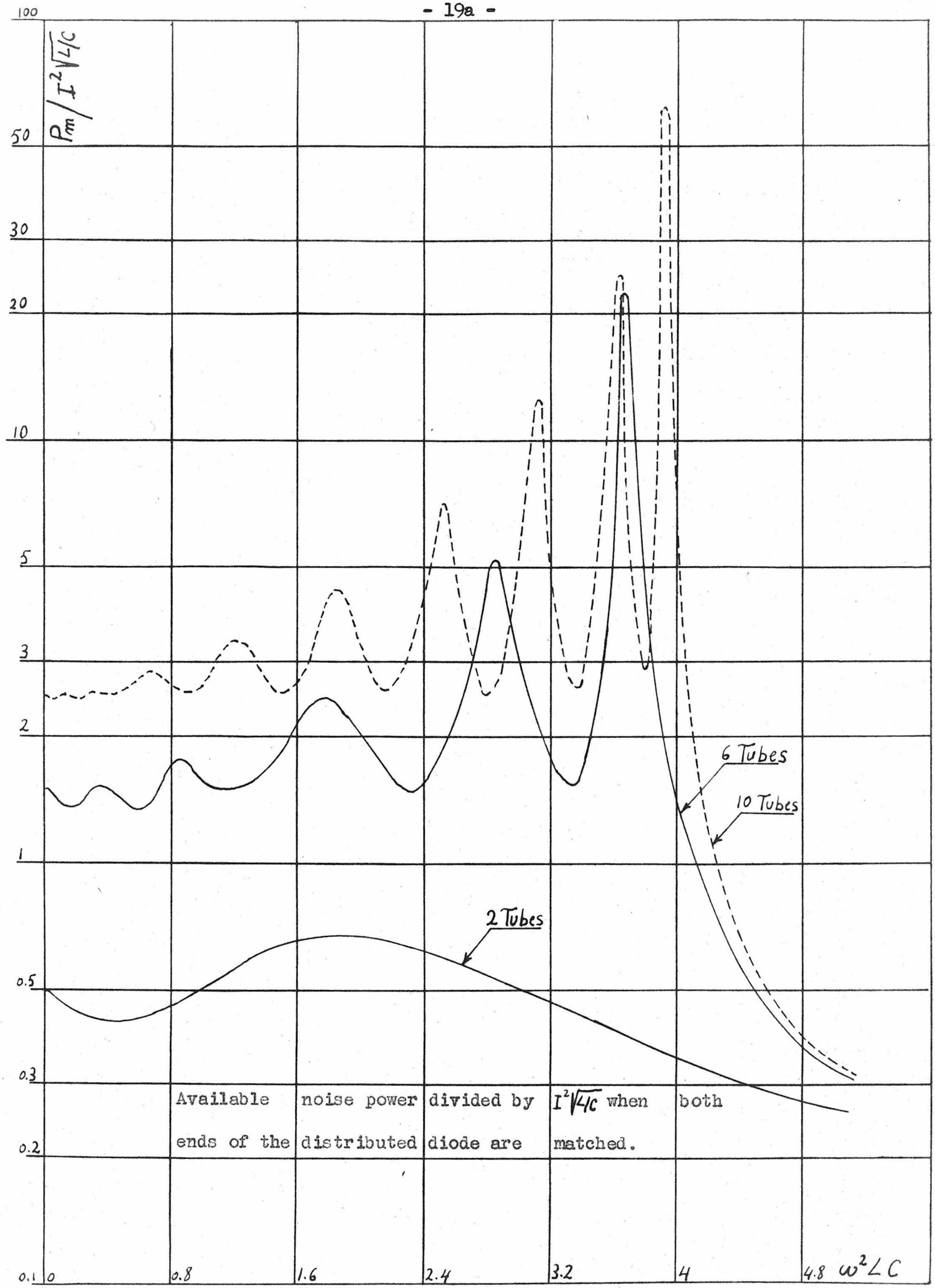

100
$P_m / I^2\sqrt{L/C}$
50
30
20
10
5
3
2
1
0.5
0.3
0.2
0.1
0
0.8
1.6
2.4
3.2
4
4.8
$\omega^2 LC$
6 Tubes
10 Tubes
2 Tubes
Available noise power divided by $I^2\sqrt{L/C}$ when both ends of the distributed diode are matched.

Fig. 7

Effect Of Losses: The general problem when both series and shunt admittances become complex has not been attempted; the effect of equal losses in each tube is expected to decrease the value of the peaks thus giving a much flatter response, in general, than in the lossless case.

Case Of A Load Limited By Its Shunt Capacity And Lead Inductance: We may easily calculate the available noise power at a load limited by a shunt capacity and lead inductance equal to that of the noise generator network, by considering the first section as dead.

2.3 The Lossless Transmission Line Diode

Having solved the problem of the lossless distributed diode with lumped elements and discrete noise sources, we now proceed to the solution of the lossless transmission line diode with distributed elements and uniformly distributed noise current along its length. We shall consider the problem when the line terminal admittances Y_ℓ and Y_ℓ' are unequal but one of them equal to the line characteristic admittance, i.e. $Y_\ell = \sqrt{C/L}$, $Y_\ell \neq Y_\ell'$

The complete solution for the transmission line diode with losses is developed in Appendix II. The lossless case may be deduced by letting $\alpha \rightarrow 0$. The available noise power P at the load admittance $Y_\ell = \sqrt{C/L}$ is found to be given by:

$$P = \frac{I^2 \ell \sqrt{L/C}}{2\left[Y_\ell' + \sqrt{C/L}\right]^2} \left\{ \frac{C}{L}\left[1 + \frac{\sin 2\ell\phi}{2\ell\phi}\right] + Y_\ell'^2 \left[1 - \frac{\sin 2\ell\phi}{2\ell\phi}\right] \right\} \tag{42}$$

In case the far end is short circuited, i.e. $Y_\ell' \rightarrow \infty$, the available noise power reduces to:

$$P_{s.c.} = \frac{1}{2} I^2 \ell \sqrt{L/C} \left[1 - \frac{\sin 2\ell\phi}{2\ell\phi}\right] \tag{43}$$

In case it is open circuited, i.e. $Y_\ell' \rightarrow 0$, the available noise power reduces to:

$$P_{o.c.} = \frac{1}{2} I^2 \ell \sqrt{L/C} \left(1 + \frac{\sin 2\ell\phi}{2\ell\phi}\right) \tag{44}$$

and in case it is matched, i.e. $Y_\ell = Y_\ell' = \sqrt{C/L}$, the available noise power becomes:

$$P_m = \frac{1}{4} I^2 \ell \sqrt{L/C} \tag{45}$$

where:

l is the length of the line

I^2 the square of the average r.m.s. noise current emitted per unit length and has the units of amperes square per unit length.

$\phi = \omega\sqrt{L/C}$

$\omega = 2\pi \times$ frequency

L the inductance of the line per unit length

C the capacity of the line per unit length.

The above results are indeed simple and interesting. They show that, as the frequency becomes high or the length of the line large, the open circuit case reduces to the short circuit and the available noise power reduces to $\frac{1}{2} I^2 l \sqrt{L/C}$, which is double that obtained from the line matched at both ends.

<u>The Transmission Line Diode With Losses</u>.- The problem of the transmission line diode with losses is no more difficult to solve analytically than the problem of the lossless transmission line diode. The only difficulty which arises is the complex problem of estimating the high frequency losses of the line. We have solved the problem for the following three cases of interest, namely: when the load end is terminated by its characteristic impedance and the far end, terminated by its characteristic impedance, short circuited and open circuited respectively. The available noise power when both ends are matched is found to be:

$$P_m = \frac{I^2 l e^{-\alpha l}}{4 Y_0^*} \times \frac{\sinh \alpha l}{\alpha l} \qquad (46)$$

where:

$\overset{*}{Y}_0$ is the conjugate of the characteristic admittance of the line

α the attenuation constant of the line

and all other symbols have the same meaning as in the lossless line. We note with interest that for a line whose $\alpha l \ll 1$ equation (46) reduces to:

$$P_m \simeq \frac{I^2 l}{4 \overset{*}{Y}_0} \left[1 - 2\alpha l\right] \tag{47}$$

Thus the transmission line diode with small losses is equivalent to a lossless line of shorter length. When the losses are large, i.e. $\alpha l \gg 1$ equation (46) reduces to:

$$P_m \simeq \frac{I^2}{4 \overset{*}{Y}_0} \times \frac{1}{2\alpha} \tag{48}$$

Thus the output power becomes independent of the length and inversely proportional to the attenuation constant of the line.

When the far end of the line is short circuited the available noise power at the load end is found to be:

$$P_{s.c.} = \frac{I^2 l}{2 \overset{*}{Y}_0} e^{-2\alpha l} \left[\frac{\sinh 2\alpha l}{2\alpha l} - \frac{\sin 2\beta l}{2\beta l}\right] \tag{49}$$

where:

β is the phase shift constant of the line.

When the far end is open circuited we obtain for the available noise power at the load end of the line:

$$P_{o.c.} = \frac{I^2 l}{2 \overset{*}{Y}_0} e^{-2\alpha l} \left[\frac{\sinh 2\alpha l}{2\alpha l} + \frac{\sin 2\beta l}{2\beta l}\right] \tag{50}$$

We again note the interesting fact that a line with $\alpha l \ll 1$ is equivalent to a lossless line of shorter length as shown by the following two equations deduced from equations (49) and (50):

$$P_{s.c.} \simeq \frac{I^2 \ell}{2 \overset{*}{Y}_0} \left[1 - 2\alpha\ell\right]\left[1 - \frac{\sin 2\beta\ell}{2\beta\ell}\right] \qquad (51)$$

$$P_{o.c.} \simeq \frac{I^2 \ell}{2 \overset{*}{Y}_0} \left[1 - 2\alpha\ell\right]\left[1 + \frac{\sin 2\beta\ell}{2\beta\ell}\right] \qquad (52)$$

For a lossy line we have $\alpha\ell \gg 1$ and we can easily deduce from equations (49) and (50) the following relation.

$$P_{s.c} = P_{o.c.} \simeq \frac{I^2}{4 \overset{*}{Y}_0} \times \frac{1}{2\alpha} \qquad (53)$$

This is the same result obtained for the line matched at both ends.

We thus are justified to conclude that for a line with $\alpha\ell \gg 1$ and terminated at one of its ends by its characteristic impedance, the available noise power at the matched end is independent of the termination at the other end and of the length of the line, and is only a function of the emission characteristics of the line and its attenuation constant. This fact enables us to build a high frequency noise generator whose power output is independent of frequency and of the terminal impedance at one of its ends. It is indeed a standard noise source if we know how to estimate accurately its attenuation constant and how to match it properly to its load.

APPENDIX I

Distributed Noise Generator Diode With One End Shorted To Ground:

We shall develop here a general solution for the amount of available noise power at the load admittance Y_ℓ of the network shown in Fig.8. We shall solve the problem when a single noise source whose r.m.s. noise current is I, is placed at node zero; then using the principle developed in section 1.5. we shall write by inspection the solution when $N+1$ sources are placed simultaneously at the $N+1$ nodes.

Applying Kirchhoff's current theorem, we may write at node n,

$$V_{n-1} + V_{n+1} = V_n\left(2 + Y_2/Y_1\right) = V_n\left(2-\omega^2 LC\right) \qquad (1)$$

$$\text{Let } \cosh\theta = 1+\tfrac{1}{2}Y_2/Y_1 = \tfrac{1}{2}\left(2-\omega^2 LC\right) \qquad (2)$$

The solution of equation (1) is of the form:

$$V_n = A\cosh n\theta + B\sinh n\theta \qquad (3)$$

where A and B are arbitrary constants to be determined from boundary conditions.

At node zero, we have by Kirchhoff's law:

$$V_0\left(Y_1+Y_2+Y_\ell\right) - V_1 Y_1 = I \qquad (4)$$

$$A\left(Y_1+Y_2+Y_\ell\right) - Y_1\left(A\cosh\theta + B\sinh\theta\right) = I \qquad (5)$$

$$A\left[\left(Y_1+Y_2+Y_\ell\right) - Y_1\cosh\theta\right] - BY_1\sinh\theta = I \qquad (6)$$

If there are $(N+1)$ nodes and the Nth node is short circuited, then,

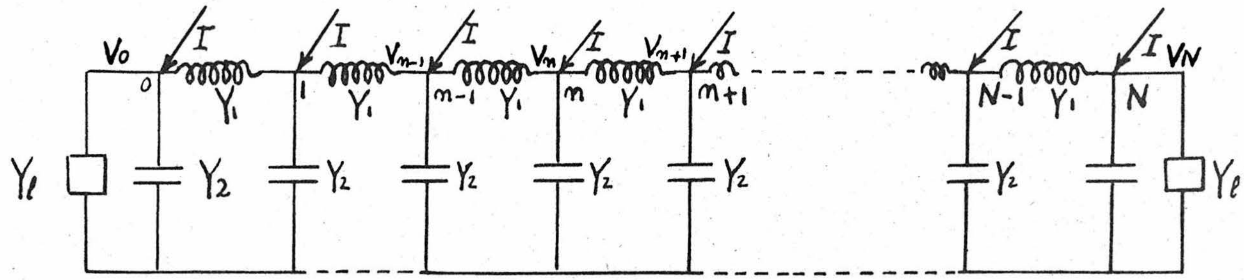


Fig. 8

Equivalent High Frequency Circuit of the Distributed Noise Generator Diode.

Y_ℓ and Y_ℓ' are terminal admittances

Y_1 is the series admittance of a wire connecting two tubes

Y_2 is the shunt admittance to ground of each tube

I is the R. M. S. value of the noise current generated in each diode

$$V_n = 0 = A \cosh N\theta + B \sinh N\theta \qquad (7)$$

$$A = -B \tanh N\theta \qquad (8)$$

$$I = -B\left\{\left[(Y_1+Y_2+Y_\ell) - Y_1 \cosh\theta\right]\tanh N\theta + Y_1 \sinh\theta\right\} \qquad (9)$$

$$B = \frac{-I}{\left[(Y_1+Y_2+Y_\ell) - Y_1 \cosh\theta\right]\tanh N\theta + Y_1 \sinh\theta} \qquad (10)$$

$$A = \frac{I \tanh N\theta}{\left[(Y_1+Y_2+Y_\ell) - Y_1 \cosh\theta\right]\tanh N\theta + Y_1 \sinh\theta} \qquad (11)$$

$$V_n = I\,\frac{\sinh N\theta \cosh n\theta - \cosh N\theta \sinh n\theta}{\left[(Y_1+Y_2+Y_\ell) - Y_1 \cosh\theta\right]\sinh N\theta + Y_1 \sinh\theta \cosh N\theta} \qquad (12)$$

$$V_n = \frac{I \sinh (N-n)\theta}{(Y_1+Y_2+Y_\ell)\sinh N\theta - Y_1 \sinh (N-1)\theta} \qquad (13)$$

If we make Y_ℓ real and $Y_1 = \frac{1}{j\omega L}$, $Y_2 = j\omega C$, then,

$$V_n = \frac{I \sinh (N-n)\theta}{Y_1\left[(1-\omega^2 LC)\sinh N\theta - \sinh(N-1)\theta\right] + Y_\ell \sinh N\theta} \qquad (14)$$

but $(1-\omega^2 LC) = 2\cosh\theta - 1$ $\qquad (15)$

therefore,

$$V_m = \frac{I \sinh (N-m)\theta}{Y_1\left[(2\cosh\theta - 1)\sinh N\theta - \sinh(N-1)\theta\right] + Y_\ell \sinh N\theta} \qquad (16)$$

$$V_m = \frac{I \sinh (N-m)\theta}{Y_1\left[\sinh (N+1)\theta - \sinh N\theta\right] + Y_\ell \sinh N\theta} \qquad (17)$$

$$V_m = \frac{I \sinh (N-m)\theta}{Y_1\left[2 \sinh\frac{\theta}{2} \cosh(N+\frac{1}{2})\theta\right] + Y_\ell \sinh N\theta} \qquad (18)$$

In particular, the voltage at node zero is:

$$V_0 = \frac{I \sinh N\theta}{Y_1\left[2 \sinh\frac{\theta}{2} \cosh(N+\frac{1}{2})\theta\right] + Y_\ell \sinh N\theta} \qquad (19)$$

The available noise power at the load is:

$$P_0 = Y_\ell V_0 \overset{*}{V_0} \qquad (20)$$

$$P_0 = \frac{I^2 Y_\ell \sinh^2 N\theta}{\frac{1}{\omega^2 L^2}\left[2 \sinh\frac{\theta}{2} \cosh(N+\frac{1}{2})\theta\right]^2 + Y_\ell^2 \sinh^2 N\theta} \qquad (21)$$

If we make $Y_\ell = \sqrt{C/L}$, then:

$$P_0 = \frac{I^2 \sqrt{L/C} \sinh^2 N\theta}{\frac{1}{\omega^2 LC}\left[4 \sinh^2\frac{\theta}{2} \cosh^2(N+\frac{1}{2})\theta\right] + \sinh^2 N\theta} \qquad (22)$$

but $4 \sinh^2\frac{\theta}{2} = -\omega^2 LC$ (23)

therefore,

$$P_o = \frac{I^2\sqrt{L/C}\ \sinh^2 N\theta}{\sinh^2 N\theta - \cosh^2\left(N+\frac{1}{2}\right)\theta} \tag{24}$$

$$P_o = \frac{I^2\sqrt{L/C}\ \sinh^2 N\theta}{1+\sinh\frac{\theta}{2}\sinh\left(2N+\frac{1}{2}\right)\theta} \tag{25}$$

The available noise power $P_{s.c.}$, in general, when $(N+1)$ sources are placed at the $(N+1)$ nodes simultaneously, may be easily written by inspection of equation (33) section 1.5 as:

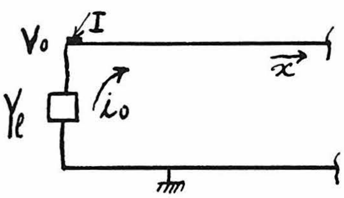
$$P_{s.c.} = \frac{I^2\sqrt{\frac{L}{C}}\sum_{S=0}^{S=N}\sinh^2(N-S)\theta}{1+\sinh\frac{\theta}{2}\sinh\left(2N+\frac{1}{2}\right)\theta} \tag{26}$$

This result has been compared with the solution obtained by placing the noise source at a generalized node of Fig. 8 and found to be correct. It also agrees with the result obtained by the straight forward solution of the simultaneous equations, in the special case when the distributed noise generator is made out of four diodes.

APPENDIX II

Transmission Line Diode Matched At One End And Terminated By An Arbitrary Impedance At The Other End:

Let us consider a transmission line whose inner conductor emits a noise current of r.m.s. intensity I. Considering an elementary length of the line at the origin, we may write by applying Kirchhoff's law at the origin:



$$V_0 Y_l + i_0 = I \qquad (1)$$

It is well known that the general steady state solution for the current and voltage along the length of a transmission line is of the form:

$$V = A \cosh px + B \sinh px \qquad (2)$$

$$-i\left(R + j\omega L\right) = \frac{\partial V}{\partial x} \qquad (3)$$

$$i = -Y_0\left[A \sinh px + B \cosh px\right] \qquad (4)$$

where $p = \sqrt{(R+j\omega L)(G+j\omega C)} = \alpha + j\beta$ $\qquad$ (5)

$$Y_0 = \sqrt{\frac{G+j\omega C}{R+j\omega L}} \qquad (6)$$

Therefore at the origin we may write:

$$i_0 = -Y_0 B \qquad (7)$$

$$V_0 = A \quad (8)$$

Substituting in equation (1), we have:

$$A\left[Y_\ell - Y_0 \frac{B}{A}\right] = I \quad (9)$$

If the far end is terminated by the arbitrary admittance Y_ℓ' then,

$$V_\ell Y_\ell' = i_\ell \quad (10)$$

$$Y_\ell'\left[A \cosh p\ell + B \sinh p\ell\right] + Y_0\left[A \sinh p\ell + B \cosh p\ell\right] = 0 \quad (11)$$

$$A\left[Y_\ell' \cosh p\ell + Y_0 \sinh p\ell\right] + B\left[Y_\ell' \sinh p\ell + Y_0 \cosh p\ell\right] = 0 \quad (12)$$

$$-\frac{B}{A} = \frac{Y_\ell' \cosh p\ell + Y_0 \sinh p\ell}{Y_\ell' \sinh p\ell + Y_0 \cosh p\ell} \quad (13)$$

Substituting from equation (13) into (9), we have:

$$I = A\left[Y_\ell + Y_0 \frac{Y_\ell' \cosh p\ell + Y_0 \sinh p\ell}{Y_\ell' \sinh p\ell + Y_0 \cosh p\ell}\right] \quad (14)$$

$$I = A\left[\frac{Y_\ell Y_\ell' \sinh p\ell + Y_\ell Y_0 \cosh p\ell + Y_0 Y_\ell' \cosh p\ell + Y_0^2 \sinh p\ell}{Y_\ell' \sinh p\ell + Y_0 \cosh p\ell}\right] \quad (15)$$

If $Y_\ell = Y_0$, then equation (15) reduces to:

$$I = A \quad \frac{Y_o Y_\ell' \left[\sinh p\ell + \cosh p\ell\right] + Y_o^2 \left[\cosh p\ell + \sinh p\ell\right]}{Y_\ell' \sinh p\ell + Y_o \cosh p\ell} \qquad (16)$$

$$A = I \quad \frac{e^{-p\ell} \left[Y_\ell' \sinh p\ell + Y_o \cosh p\ell\right]}{Y_o \left[Y_\ell' + Y_o\right]} \qquad (17)$$

$$A = \frac{I e^{-p\ell}}{Y_o\left[Y_\ell' + Y_o\right]} \left\{ \cos\beta\ell \left[Y_\ell' \sinh \alpha\ell + Y_o \cosh\alpha\ell\right] + j \sin\beta\ell \left[Y_\ell' \cosh\alpha\ell + Y_o \sinh\alpha\ell\right] \right\} \qquad (18)$$

For an arbitrarily placed noise generating element whose coordinate along the line is defined by the variable q we have:

$$A_q = \frac{I e^{-p(\ell-q)}}{Y_o\left[Y_\ell' + Y_o\right]} \left\{ \cos\beta(\ell-q) \left[Y_\ell' \sinh \alpha(\ell-q) + Y_o \cosh\alpha(\ell-q)\right] + j \sin\beta(\ell-q) \left[Y_\ell' \cosh\alpha(\ell-q) + Y_o \sinh\alpha(\ell-q)\right] \right\} \qquad (19)$$

This voltage will reach the load end multiplied by the factor e^{-pq} so that

$$A_{q_o} = \frac{I e^{-p\ell}}{Y_o\left[Y_\ell' + Y_o\right]} \left\{ \cos\beta(\ell-q) \left[Y_\ell' \sinh \alpha(\ell-q) + Y_o \cosh\alpha(\ell-q)\right] + j \sin\beta(\ell-q) \left[Y_\ell' \cosh\alpha(\ell-q) + Y_o \sinh\alpha(\ell-q)\right] \right\} \qquad (20)$$

If $Y_\ell' = Y_o$, then equation (20) reduces to:

$$A_{q_o} = \frac{I}{2 Y_o} e^{-pq} \qquad (21)$$

Therefore the element of power at the load is

$$\Delta P = A_{q_0} \cdot \overset{*}{A}_{q_0} Y_0 = \frac{I^2}{4 \overset{*}{Y}_0} e^{-2\alpha q} \tag{22}$$

Total available power at the load is

$$P_m = \frac{I^2}{4 \overset{*}{Y}_0} \int_0^l e^{-2\alpha q} dq \tag{23}$$

$$P_m = \frac{I^2}{4 \overset{*}{Y}_0} \times \frac{1}{2\alpha} \left(1 - e^{-2\alpha l} \right) = \frac{I^2 e^{-\alpha l}}{4 \overset{*}{Y}_0} \times \frac{\sinh \alpha l}{\alpha l} \tag{24}$$

If the far end is short circuited, $Y_l' \rightarrow \infty$, and equation (20) becomes

$$A_{q_0} = \frac{I e^{-pl}}{Y_0} \left[\cos\beta(l-q) \sinh\alpha(l-q) + j \sin\beta(l-q) \cosh\alpha(l-q) \right] \tag{25}$$

Element of power available at the load:

$$\Delta P = \frac{I^2 e^{-2\alpha l}}{\overset{*}{Y}_0} \left[\sinh^2 \alpha(l-q) + \sin^2 \beta(l-q) \right] \tag{26}$$

$$P_{s.c} = \frac{I^2 e^{-2\alpha l}}{\overset{*}{Y}_0} \int_0^l \left[\sinh^2 \alpha(l-q) + \sin^2 \beta(l-q) \right] dq \tag{27}$$

$$P_{s.c.} = \frac{I^2 l e^{-2\alpha l}}{2 \overset{*}{Y}_0} \left[\frac{\sinh 2\alpha l}{2\alpha l} - \frac{\sin 2\beta l}{2\beta l} \right] \qquad (28)$$

When the far end is open circuited $Y_l' \rightarrow 0$, and by a similar method we obtain for the noise power available at the load end:

$$P_{o.c.} = \frac{I^2 l e^{-2\alpha l}}{2 \overset{*}{Y}_0} \left[\frac{\sin 2\alpha l}{2\alpha l} + \frac{\sin 2\beta l}{2\beta l} \right] \qquad (29)$$

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