COMPLEX FUNCTION THEORY
FOR FUNCTIONS WITH ARGUMENTS AND VALUES IN
LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

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ABSTRACT

In Chapter I a brief introduction to the basic notions of locally convex linear topological spaces is given. In Chapter II, a theory of analytic functions is developed for functions of a complex variable with values in a sequentially complete locally convex complex linear topological space (l.t.s). The theory is sketched for the case when the function values lie in the linear space of continuous linear functions on one sequentially complete locally convex complex l.t.s. to a second such space. In the same spirit some theorems relating to functions of several complex variables taking their values in a sequentially complete locally convex complex l.t.s. are developed. In Chapter III, functions on one sequentially complete locally convex complex l.t.s. to a second such space are studied and in particular notions of differentiability and analyticity. An analogue of the Cauchy-Riemann theory of functions of a complex variable is discussed.

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Chapter I

Locally Convex Linear Topological Spaces

1.1. Introduction. The main purpose of this chapter is to give a brief introduction to the basic notions in locally convex linear topological spaces. As much of the material is taken from the current literature, proofs will, in general, be omitted or merely sketched. Where proofs are included they are intended either to make up a deficit in this respect in the literature or to correspond to results which are more or less new.

§ 1. Definitions and Fundamental Properties

1.2. Postulates for a Locally Convex Linear Topological Space.

Definition 1.2.1. Let L be a non-empty set of elements x, y, z,..., and let A denote either the real or complex numbers. L will be called a linear system over A, if its elements admit of two operations, addition and scalar multiplication, subject to the following conditions.

Addition of elements satisfies:

A_1. Every ordered pair of elements x, y of L has a unique sum $x + y$ belonging to L.

A_2. Addition is commutative, i.e., $x + y = y + x$.

A_3. Addition is associative, i.e., $(x + y) + z = x + (y + z)$.

A_4. There is an element θ in L, called the zero element, such that $x + \theta = \theta + x = x$ for all x in L.

A_5. To every x there corresponds an element $-x$, its negative, such that $x + (-x) = \theta$.

Scalar multiplication satisfies:

S_1. To every number a in A, and every x in L, there corresponds a uniquely defined scalar product ax in L.

S_2. $(a + b)x = ax + bx$, a and b in A, x in L.

S_3. $a(x + y) = ax + ay$, a in A, x and y in L.

S_4. $a(bx) = (ab)x$, a and b in A, x in L.

S_5. $1x = x$.

For the elementary properties of linear systems see (1) and (2).

If M and N are subsets of L, the symbols $-M$, $M + N$, $M - N$, denote the sets $\{-x\}$, $\{x + y\}$, $\{x - y\}$ respectively where x is in M and y is in N. If B is a subset of A, the symbol BM stands for the set $\{bx\}$ where b is in B and x in M.

According as A is the real or complex numbers, a linear system L will be called a real or a complex linear system. R will always denote the real numbers and K the complex numbers.

<u>Definition 1.2.2.</u> Let L be any complex linear system and let M be a subset of L. M is said to be

(i) convex, if x,y in M implies $ax + (1-a)y$ is in M, $0 < a < 1$,

(ii) circular, if for x in M, $e^{i\phi}x$ is in M, for $0 \leq \phi \leq 2\pi$, and

(iii) balanced, if for each $x \neq \theta$, there exists a real number $a \neq 0$, such that ax is in M.

In the case of a real linear system, (ii) is replaced by

(iv) M is said to be symmetric, if whenever x is in M, then so is -x.

<u>Definition 1.2.3.</u> A complex linear system L is called a locally convex complex linear topological space (3) if the following hold.

(i) L is a Hausdorff topological space (2).

(ii) There exists a fundamental system of neighbourhoods of θ, denoted by $\mathcal{C} = \{C\} = \{C(\theta)\}$ formed of sets which are convex, circular and balanced. The neighbourhoods $\mathcal{N} = \{N(x)\}$ of an arbitrary point x of L are obtained by translating the neighbourhoods of the origin, $N(x) = x + C(\theta)$.

(iii) In L the operations of addition and scalar multiplication are continuous under the given topology, i.e., $x + y$ and ξx (x, y in L, ξ in K) are each continuous jointly in both arguments. The phrase linear topological space(s) will always be abbreviated to l.t.s.

To define a locally convex real l.t.s. circular is replaced by symmetric and K by R in Definition 1.2.3. Various authors (4,5) have given different sets of postulates for such spaces and a discussion of the relationship among them is to be found in (6). In Definition 1.2.3 it would have been sufficient to require only that the neighbourhoods defining the topology be convex sets but then such a topology can always be replaced by an equivalent topology in which the defining neighbourhoods are convex, circular and balanced.

Theorem 1.2.1. The Hausdorff separation axiom is equivalent to the statement that the intersection of all the neighbourhoods of the element θ consists only of θ.

Proof. Suppose the separation axiom holds but $\bigcap_{\mathcal{C}} C$ contains, in addition to θ, some element $x \neq \theta$. Then there exist neighbourhoods C of θ and $N(x)$ of x such that $C \cap N(x) = \emptyset$, the empty set, giving a contradiction.

If $\bigcap_{\mathcal{C}} C = \theta$, take $x \neq \theta$. Then there exists some C in $\mathcal{C}$ such that x is not in C. On the other hand since $\bigcap_{\mathcal{N}} N(x) = \bigcap_{\mathcal{C}} x + C(\theta) = x + \bigcap_{\mathcal{C}} C(\theta) = x$ there is some neighbourhood $N(x)$ of x not containing θ. These last two statements together say that L is a T_1-space, i.e., if $x \neq y$ there is a neighbourhood $N(x)$ of x to which y does not belong. This last implies (7) that L not only satisfies the Hausdorff separation axiom but is indeed regular.

Definition 1.2.4. A sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of a locally convex l.t.s. is a convergent sequence if subsequently to the choice of an arbitrary neighbourhood C of θ, a number $n_o = n_o(C)$ may always be assigned so that $m > n \geqslant n_o(C)$ implies $x_m - x_n$ is in C.

Definition 1.2.5. A sequence $x_1, x_2, \ldots, x_n, \ldots$ in a locally convex l.t.s. L is convergent to an element x of L, called its limit, if subsequently to the choice of an arbitrary neighbourhood C of θ a number $n_o = n_o(C)$ may always be assigned such that $n \geqslant n_o(C)$ implies $x - x_n$ is in C. As usual, this is expressed by writing

$$\lim_{n \to \infty} x_n = x.$$

Definition 1.2.6. A locally convex l.t.s. is said to be sequentially complete if every convergent sequence in L is convergent to an element in L.

1.3. Pseudo-norms and F-metrics. An alternative method of defining a locally convex l.t.s. will be given here.

Definition 1.3.1. A real valued function p defined on a complex linear system L is called a complex (triangular) pseudo-norm on L if

(i) $p(x) \geqslant 0$,

(ii) $p(\xi x) = |\xi| p(x)$, where ξ is in K, and

(iii) $p(x + y) \leq p(x) + p(y)$.

If L is a real linear system (ii) is replaced by

(iv) $p(ax) = | a | p(x)$, where a is in R,

and p is called a real (triangular) pseudo-norm on L.

The notion of a triangular pseudo-norm was introduced by J. von Neumann (4) and generalized by Hyers (8). It differs from a norm only in that the vanishing of the pseudo-norm of an element x does not imply $x = \theta$.

Definition 1.3.2. An arbitrary set D with elements d,e,f,... together with a binary relation, $>$, is called a directed system if:

(i) Either $d > e$ or $d \not> e$, for every pair, d,e in D.

(ii) If $d > e$, and $e > f$, then $d > f$.

(iii) Given d,e in D, there exists f in D, such that $f > d$, $f > e$.

<u>Definition 1.3.3.</u> A complex (real) linear system L will be said to be pseudo-normed with respect to a directed system D if there exists a real valued function $\| x \|_d$, defined for all x in L and d in D such that

(i) $\| x \|_d$ is a complex (real) triangular pseudo-norm,

(ii) $\| x \|_d = 0$ for all d in D implies $x = \theta$, and

(iii) $\| x \|_d \geqslant \| x \|_e$ whenever $d > e$, for all x in L.

Given a complex (real) linear system L pseudo-normed with respect to a directed system D, the sets

$$C(\theta;\ d, \epsilon\ ) \equiv \left\{ x \text{ in } L \ \middle|\ \| x \|_d < \epsilon \right\},$$

defined for every d in D and $\epsilon > 0$, form a fundamental system of convex, circular (symmetric) and balanced neighbourhoods of θ. The neighbourhoods of an arbitrary point x of L are the sets $N(x_o;d, \epsilon\ ) \equiv \left\{ x \text{ in } L \ \middle|\ \| x - x_o \|_d < \epsilon \right\}$ as d ranges over D and $\epsilon > 0$ takes on all possible values. That these neighbourhoods satisfy the Hausdorff separation axiom is clear. For, if $x_o \neq y_o$, by Definition 1.3.3 there is some d in D for which $\| x_o - y_o \|_d \neq 0$. Suppose $\| x_o - y_o \|_d = 1$. Then $N(x_o;d, \frac{1}{3})$ and $N(y_o;d, \frac{1}{3})$ are disjoint neighbourhoods of x_o and y_o respectively. Since otherwise, x being common to both, it would follow that $1 = \| x_o - y_o \|_d = \| x_o - x + x - y_o \|_d \leq \| x_o - x \|_d + \| x_o - y_o \|_d < \frac{2}{3}$, leading to a contradiction. The continuity of addition is a consequence of (iii) of Definition 1.3.1 and that of scalar multiplication of (ii) and (iii) of the same definition. Hence, using the pseudo-norms, a locally convex topology can be defined in L called the locally convex topology generated by the pseudo-norms.

Theorem 1.3.1. Every pseudo-normed linear system is a locally convex l.t.s. Conversely, given any locally convex l.t.s. L there exists a directed system D, called the associated directed system, with respect to which L may be pseudo-normed in such a way that the pseudo-norms generate a topology equivalent to the given topology in L (8).

Proof. The proof of the first statement has already been indicated. As for the second, the directed system mentioned there may be identified with the given system of neighbourhoods $\mathscr{C} = \{ C(\theta) \} = \{ C \}$. $C_1 > C_2$ then means $C_1 \subset C_2$, C_1 and C_2 in $\mathscr{C}$. If C is in $\mathscr{C}$ the corresponding pseudo-norm is defined as follows:

$$\| x \|_C = \inf \{ a > 0 \mid x \text{ is in } aC \} .$$

It is clear that $\| x \|_C \leq 1$ for x in C. That $\| x \|_C$ is actually a triangular pseudo-norm follows from the fact that C is convex, circular (symmetric) and balanced; that $\| x \|_C = 0$ for all C implies $x = \theta$ follows from the Hausdorff separation axiom or $\bigcap_{\mathscr{C}} C = \theta$. Finally since $C_1 > C_2$ means $C_1 \subset C_2$ it is true that $\| x \|_{C_1} \geq \| x \|_{C_2}$ if $C_1 > C_2$.

It is now evident that locally convex l.t.s. and pseudo-normed linear systems are identical objects. Hence, in speaking of locally convex l.t.s. instead of discussing questions of topology and analysis in terms of the given neighbourhoods, they will be discussed in terms of the pseudo-norms. The question as to when two systems of pseudo-norms define equivalent topologies (in which case the systems are said to be equivalent) then arises and is answered by

Theorem 1.3.2. Let L be a locally convex l.t.s. and D, D' two associated directed systems. In order that the systems of pseudo-norms $\| x \|_d$, d in D, and $\| x \|_{d'}$, d' in D', be equivalent it is necessary and sufficient that for each d in D there exist numbers $M_1(d) > 0$, $M_2(d) > 0$ and d' in D' such that for all x in L

$$M_1(d) \| x \|_d \leq \| x \|_{d'} \leq M_2(d) \| x \|_d \; .$$

Proof. Let $N_1' = \{ x \mid \|x\|_{d'} < M_1(d)\,\epsilon \}$, $N = \{ x \mid \|x\|_d < \epsilon \}$ and $N_2' = \{ x \mid \|x\|_{d'} < M_2(d)\epsilon \}$. It follows from the above inequalities that $N_1' \subset N \subset N_2'$ and the topologies are equivalent.

Suppose D and D' are equivalent. Consider $N = \{ x \mid \|x\|_d < 1 \}$. Then there exists a neighbourhood of θ, $N' = \{ x \mid \|x\|_{d'} < \epsilon, \; \epsilon = \epsilon(d) \}$ such that $N \subset N'$, i.e., $\| x\|_d < 1$ implies $\| x \|_{d'} < \epsilon$. With $0 < |t_o| < 1$ and t_o real, there is one and only one integer $k = k(d)$ for which $|t_o| < \| t_o^k x \|_d < 1$ as long as $x \neq \theta$, or $\| x \|_d \neq 0$. Thus $\| t_o^k x \|_{d'} < \epsilon$ and $\| x \|_{d'} < \frac{\epsilon}{|t_o|^k} < \frac{\epsilon}{t_o} \| x \|_d$. Putting $M_2(d) = \frac{\epsilon}{t_o} = \frac{\epsilon(d)}{t_o}$ this gives $\|x\|_{d'} \leq M_2(d) \| x \|_d$. This relation is true for $x = \theta$, and for those x for which $\| x \|_d = 0$ since this last implies $\| x \|_{d'} = 0$. Otherwise there would exist a real number a such that $\| ax \|_{d'} = | a | \| x \|_{d'} \geqslant \epsilon$ while $\| ax \|_{d} = | a | \| x \|_{d} = 0$, a contradiction.

In terms of pseudo-norms Definitions 1.2.4 and 1.2.5 read as follows.

Definition 1.3.4. Let L be a locally convex l.t.s. with associated directed system D. A sequence $\{ x_n \}$ of elements of L is a convergent sequence if given $\epsilon > 0$, d in D, there is a number $n_o = n_o(d, \epsilon)$ such that $m > n \geqslant n_o(d, \epsilon)$ implies $\| x_m - x_n \|_d < \epsilon$.

Definition 1.3.5. With L and D as in Definition 1.3.4, a sequence $\{ x_n \}$ in L is convergent to an element x in L, called its limit if given $\epsilon > 0$, d in D, a number $n_o = n_o(d, \epsilon)$ may always be assigned such that $n \geqslant n_o(d, \epsilon)$ implies $\| x - x_n \|_d < \epsilon$.

Definition 1.3.6. A real valued function defined on a complex linear system is called a complex F-metric (1, 9) and denoted by $| x |$, x in L, if

(i) $| x | \geqslant 0$ and $| x | = 0$ if and only if $x = \theta$,

(ii) $| x_1 + x_2 | \leqslant | x_1 | + | x_2 |$,

(iii) $| e^{i\phi} x | = | x |$, $0 \leqslant \phi \leqslant 2\pi$, and

(iv) $\zeta_n \rightarrow \zeta$ (ζ_n, ζ in K) and $| x_n - x | \rightarrow 0$ implies $| \zeta_n x_n - \zeta x | \rightarrow 0$.

For a real linear system L (iii) is replaced by

(iii') $| -x | = | x |$

and the complex sequence $\{\zeta_n\}$ in (iv) is replaced by a real sequence $\{a_n\}$ with $a_n \to a$.

If $|x|$ is an F-metric on a linear system L a topology can be introduced in L using neighbourhoods $N(\theta, \epsilon)$ of the origin where $N(\theta, \epsilon) \equiv \{x \text{ in } L \mid |x| < \epsilon\}$.

Definition 1.3.7 (10). A complex (real) $\mathcal{F}$-space is a sequentially complete locally convex complex (real) l.t.s. in which a complex (real) F-metric can be introduced defining an equivalent topology.

A result due to G. Birkhoff on F-metrics (11) implies that the $\mathcal{F}$-spaces satisfy the first axiom of countability. Hence the topology of $\mathcal{F}$-spaces can be defined by a sequence $\|x\|_m$ of pseudo-norms such that $\|x\|_n \leqslant \|x\|_{n+1}$.

The question of how locally convex l.t.s. differ from normed spaces can be answered by introducing bounded sets.

Definition 1.3.8 (4). A set S contained in a locally convex l.t.s. with directed system D is said to be bounded if $\sup_{x \text{ in } S} \|x\|_d < \infty$, for every d in D.

Equivalent definitions have been given by Kolmogoroff (7) and Michal and Paxson (12, 13).

Theorem 1.3.3 (7). In order that a norm defining an equivalent topology may be introduced into a locally convex l.t.s. it is necessary and sufficient that L contain a bounded open set having the origin as an interior point.

1.4. Examples of Locally Convex Linear Topological Spaces.

Example 1.4.1. Let D be a directed set and B_d, d in D, a family of Banach spaces, the norm of an element B_d being denoted by $\|\cdot\|_d$. Suppose

(i) $B_e \subset B_d$ whenever $e > d$, e and d in D, and

(ii) if x is in B_e and $e > d$ then $\|x\|_d < \|x\|_e$.

Set $B = \bigcap_D B_d$. B is clearly a sequentially complete locally convex l.t.s. pseudo-normed with respect to the directed system D with pseudo-norms $\|x\|_d$ defined for all x in B and d in D. As particular instances of this construction D may be taken as the set of real numbers $1 \leq p \leq \infty$ and the corresponding family of Banach spaces as the $L^p(0,1)$ spaces or the l^p spaces. Properties (i) and (ii) follow from the known properties of L^p and l^p.

Examples of $\mathcal{F}$-spaces may be constructed by taking for D the set of positive integers with B_k, k = 1, 2, ... the corresponding family of Banach spaces (9). $B = \bigcap_{k=1}^{\infty} B_k$ is an $\mathcal{F}$-space and the F-metric $|x|$ is given by

$$|x| = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|x\|_k}{1 + \|x\|_k}, \quad x \text{ in } B.$$

It is easily verified that this F-metric defines a topology equivalent to that generated by the pseudo-norms $\|x\|_k$. A non-normable sequentially complete locally convex l.t.s. L^{ω} (14) is obtained by taking $L = \bigcap_{k=1}^{\infty} L^k(0,1)$.

Let $\mathcal{D}_k$, $k = 1,2,\ldots$ be the Banach spaces of functions defined on $0 \leq t \leq 1$ having continuous derivatives of order k and with norm given by

$$\| x \|_k = \sup_{\substack{0 \leq i \leq k \\ 0 \leq t \leq 1}} | x^{(i)}(t) | , \qquad x = x(t) \text{ in } \mathcal{D}_k.$$

Set $\mathcal{D} = \bigcap_{k=1}^{\infty} \mathcal{D}_k$. $\mathcal{D}$ is the non-normable sequentially complete $\mathcal{F}$-space of infinitely differentiable functions defined on the interval $0 \leq t \leq 1$ of the real line (1,10).

Example 1.4.2. Let D be a directed set and B_d, d in D, a family of Banach spaces. If x_d is in B_d its norm will be denoted by $\| x_d \|$. Suppose conditions (i) and (ii) of example 1.4.1 are fulfilled. If x_e is in B_e with $e > d$, then x_e also belongs to B_d and as a member of B_d may be denoted by x_d. Condition (ii) then reads $\|x_d\| < \| x_e\|$. Set $B = \prod_D B_d$, that is, B is the Cartesian product of the spaces B_d. B is a linear system if addition of elements in B, say $x = \{x_d\}$, $y = \{y_d\}$ is defined by $x + y = \{x_d\} + \{y_d\} = \{x_d + y_d\}$, and scalar multiplication by $\xi\{x_d\} = \{\xi x_d\}$. The scalar multiplier ξ is taken to be real or complex according as the Banach spaces B_d are real or complex. The pseudo-norm $\| x \|_d$ of an element x in B is the norm $\| x_d \|$ of x_d in the space B_d. B is then a sequentially complete locally convex l.t.s. pseudo-normed with respect to the directed system D. As in Example 1.4.1, specific instances of this construction are obtained by taking for D the set of real numbers

$1 \leq p \leq \infty$ and for the corresponding family of Banach spaces the $L^p(0,1)$ or the l^p spaces.

Further examples of $\mathcal{F}$-spaces are obtained by taking for D the set of positive integers with B_k, $k = 1,2,\ldots$ the corresponding family of Banach spaces (9). $B = \prod_k B_k$ is now the space of all sequences $\{x_k\}$, where x_k is in B_k, and is an $\mathcal{F}$-space, the F-metric being given by

$$|x| = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|x_k\|}{1 + \|x_k\|} .$$

If B_k, $k = 1,2,\ldots.$ is the normed linear space of real numbers then the space B is the space (s) of all numerical sequences (1).

Example 1.4.3 (10). Let G be a locally compact Hausdorff space which is the union of a countable number of compact sets G_n where the closure of G_n is contained in the interior of G_{n+1}. Each compact subset of G is then contained in some G_n. Let L be the complex linear system of continuous complex valued functions defined on G. A non-normable sequentially complete locally convex l.t.s. on L is defined by the countable family of pseudo-norms

$$\|x\|_n = \sup_{t \text{ in } G_n} |x(t)| , \quad x = x(t) \text{ in } L,\ n = 1,2,\ldots .$$

Hence L is an $\mathcal{F}$-space. Particular instances of this construction are obtained by taking G to be the complex plane and G_n to be $\{\zeta \text{ in } K \mid |\zeta| \leq n\}$; or by taking G to be the real line and G_n to be $\{t \text{ in } R \mid |t| \leq n\}$.

1.5. Couple and Composite Locally Convex Linear Topological Spaces.

Given a locally convex real l.t.s., say L(R), it is possible to construct from it other locally convex l.t.s. Of particular interest are two spaces which will be called the associated couple and composite spaces and will be denoted by L(K) and L(C) respectively. L(K) is a locally convex complex l.t.s. while L(C) is a locally convex real l.t.s.

Theorem 1.5.1. Let L(R) be a locally convex real l.t.s. with directed system D. L(K) will denote the class of all pairs $z = (x,y)$ of elements of L(R) with equality, addition, multiplication by complex numbers and pseudo-norms defined as follows for $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $\zeta = (a,b)$ complex, and the directed set D:

(i) $(x_1,y_1) = (x_2,y_2)$ if and only if $x_1 = x_2$, $y_1 = y_2$,

(ii) $(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$,

(iii) $(a,b)(x_1,y_1) = (ax_1 - by_1, ay_1 + bx_1)$ where a,b are real, and

(iv) $\|(x,y)\|_d = \sup_{0 \leq \phi \leq 2\pi} \|x \cos \phi + y \sin \phi\|_d$, each d in D.

Then L(K), the associated couple space, is a locally convex complex l.t.s. pseudo-normed with respect to the directed system D. If L(R) is sequentially complete then so is L(K).

Proof. Addition clearly satisfies A_1, A_2, A_3 of Definition 1.2.1. The zero element of L(K) is (θ,θ) while the negative of (x,y) is (-x, -y). S_2, S_3, S_4 are easily checked. For example let (a_1,b_1) and (a_2,b_2) be two complex numbers. Then

$$((a_1,b_1) + (a_2,b_2))(x,y) = (a_1 + a_2, b_1 + b_2)(x,y)$$
$$= ((a_1 + a_2)x - (b_1 + b_2)y, (a_1 + a_2)y + (b_1 + b_2)x)$$
$$= (a_1x - b_1y, a_1y + b_1x) + (a_2x - b_2y, a_2y + b_2x)$$
$$= (a_1,b_1)(x,y) + (a_2,b_2)(x,y).$$

As for S_5, $1(x,y) = (1,0)(x,y) = (x,y)$.

Similarly it is easily verified by a straightforward computation that for each d in D $\|(x,y)\|_d$ is actually a triangular pseudo-norm. However, instead of doing this it will be shown by using the Minkowski functional that (iv) gives the natural definition of pseudo-norms for L(K).

Consider the family $\mathcal{C} = \{C\}$ of neighbourhoods of the origin in L(R). A topology for L(K) is most naturally constructed by taking as neighbourhoods of (θ,θ) the set of all pairs (C,C') of elements of $\mathcal{C}$. These are in a manner of speaking "rectangular" neighbourhoods. They can be replaced by neighbourhoods of the form $S = (C,C)$, C in $\mathcal{C}$, the "square" neighbourhoods, defining an equivalent topology on L(K). Each S is a convex, symmetric and balanced neighbourhood of (θ,θ). But L(K) is to be a locally convex complex l.t.s. and so the defining neighbourhoods for it must be circular. Let $W = \bigcap_{0 \leq \phi \leq 2\pi} e^{i\phi} S$. Then as S ranges over the square neighbourhoods the sets W form a fundamental system of convex, circular and balanced neighbourhoods of (θ,θ) defining an equivalent Hausdorff topology on L(K).

The neighbourhoods $\{W\}$ form a directed set when $W_1 > W_2$ is taken to mean $W_1 \subset W_2$. Let z be in L(K). The pseudo-norm of z with respect to W is defined by

$$\|z\|_W = \inf \{ h > 0 \mid z \text{ is in } hW,\ z = (x,y) \} .$$

Then,

$$\begin{aligned}
\|z\|_W &= \inf \{ h > 0 \mid z \in he^{i\emptyset} S, \quad 0 \leq \emptyset \leq 2\pi \} \\
&= \inf \{ h > 0 \mid ze^{-i\emptyset} \in hS, \quad 0 \leq \emptyset \leq 2\pi \} \\
&= \inf \{ h > 0 \mid (x \cos \emptyset + y \sin \emptyset,\ y \cos \emptyset - x \sin \emptyset \in (hC,hC), 0 \leq \emptyset \leq 2\pi \} \\
&= \inf \left\{ h > 0 \;\middle|\; \begin{array}{l} x \cos \emptyset + y \sin \emptyset \in hC, \quad 0 \leq \emptyset \leq 2\pi \\ y \cos \emptyset - x \sin \emptyset \in hC \end{array} \right\} \\
&= \sup_{0 \leq \emptyset \leq 2\pi} \{ \|x \cos \emptyset + y \sin \emptyset\|_C,\ \|x \cos(\pi + \emptyset) + y \sin(\pi + \emptyset)\|_C \} \\
&= \sup_{0 \leq \emptyset \leq 2\pi} \|x \cos \emptyset + y \sin \emptyset\|_C .
\end{aligned}$$

The elements C of $\mathscr{C}$ form a directed set if $C_1 > C_2$ means $C_1 \subset C_2$. Since $W = \bigcap_{0 \leq \emptyset \leq 2\pi} e^{i\emptyset} S$ and $S = (C,C)$ it is clear that the directed set $\{W\}$ may be replaced by the directed set $\mathscr{C}$ to obtain

$$\|z\|_C = \|(x,y)\|_C = \sup_{0 \leq \emptyset \leq 2\pi} \|x \cos \emptyset + y \sin \emptyset\|_C ,$$

that is, the expression (iv) has been derived naturally from the neighbourhood definition of the topology in L(K).

The mapping $x \rightarrow (x,\theta)$ is an isomorphism between the space L(R) and the linear sub-space of L(K) of elements of the form (x,θ). These may then be identified and $z = (x,y) = (x,\theta) + (0,1)(y,\theta)$ written as $z = x + iy$. The following inequalities are useful in this connection.

$$\|x\|_d \leqslant \|x + iy\|_d$$
$$\|y\|_d \leqslant \|x + iy\|_d$$
$$\|x + iy\|_d \leqslant \|x\|_d + \|y\|_d$$

Let L(R) be sequentially complete and suppose $\{x_n, y_n\}$ is a convergent sequence in L(K) so that given d in D, $\epsilon > 0$, there exists $n_o = n_o(d, \epsilon)$ such that for $m > n \geqslant n_o(d, \epsilon)$ it is true that

$$\|(x_m, y_m) - (x_n, y_n)\|_d = \|(x_m - x_n, y_m - y_n)\|_d < \epsilon \quad .$$

The first two inequalities above then imply

$$\|x_m - x_n\|_d < \epsilon \qquad \text{and} \qquad \|y_m - y_n\|_d < \epsilon \quad ,$$

that is, $\{x_n\}$ and $\{y_n\}$ are convergent sequence in L(R). Let $x = \lim x_n$ and $y = \lim y_n$. Then using the third inequality,

$$\|(x_n, y_n) - (x, y)\|_d = \|(x_n - x, y_n - y)\|_d \leqslant \|x_n - x\|_d + \|y_n - y\|_d$$

and this last can be made arbitrarily small for given d in D by the choice of n. L(K) is thus sequentially complete.

Theorem 1.5.2. Let L(R) be a locally convex real l.t.s. and L(K) the associated couple space. Let $z = x + iy$. Then the function $\bar{z} = x - iy$, called the conjugate of z is a continuous function of z and $\bar{\bar{z}} = z$.

Proof. For any d in D, $\|\bar{z} - \bar{z}_o\|_d = \|z - z_o\|_d$ and the result follows.

Theorem 1.5.3. Let L(R) be a locally convex real l.t.s. with directed system D. Let L(C) denote the class of all pairs (x,y) of elements of L(C) with equality, addition, multiplication by real numbers, and pseudo-norms defined as follows for the pairs (x_1, y_1), (x_2, y_2), the real number a, and the directed set D:

(i) $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$, $y_1 = y_2$,

(ii) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$,

(iii) $a(x,y) = (ax, ay)$,

(iv) $\| (x,y) \|_d = \| x \|_d + \| y \|_d$, each d in D.

Then L(C), the associated composite space, is a locally convex real l.t.s. pseudo-normed with respect to the directed system D. If L(R) is sequentially complete then so is L(C).

1.6 Continuous Linear Functionals and Weak Topologies.

Definition 1.6.1. Let L and L' be two locally convex complex l.t.s. and D and D' their associated directed systems respectively. Let T(x) be a transformation on L into L'.

(i) T is said to be linear if for all x_1 and x_2 in L

$$T(x_1 + x_2) = T(x_1) + T(x_2),$$

and for all ζ in K, x in L

$$T(\zeta x) = \zeta T(x).$$

(ii) T is said to be bounded with respect to the pseudo-norms if given d' in D' there exists d in D and $M(d,d') > 0$ such that for all x in L

$$\| T(x) \|_{d'} \leq M(d,d') \| x \|_d .$$

(iii) If L' is the space of complex numbers K, and T satisfies the conditions of (i), then T is said to be a complex linear functional.

If L and L' are locally convex real l.t.s. Definition 1.6.1 (ii) remains as it is while the second statement of Definition 1.6.1 (i) must be changed to read - for all a in R and x in L, $T(ax) = aT(x)$. If in particular L' is the space of real numbers and T satisfies the amended form of Definition 1.6.1 (i) T is said to be a real linear functional.

Theorem 1.6.1 (6). Let L and L' be two locally convex l.t.s. and let D and D' be the associated directed systems for L and L' respectively. A linear transformation T(x) on L into L' is continuous everywhere if and only if it is bounded with respect to the pseudo-norms.

Corollary. Let f(x) be a linear functional defined on a locally convex l.t.s. L with associated directed system D. f(x) is continuous if and only if there exists d in D and a real number $M(d) > 0$ such that $|f(x)| \leq M(d) \, \|x\|_d$.

It is of fundamental importance for later applications to know if there are any continuous linear functionals on a locally convex l.t.s. and if so, if there are sufficiently many to distinguish between elements of L. The first question is solved in

Theorem 1.6.2 (15). Non-zero continuous linear functionals exist on a l.t.s. if and only if the space contains an open convex set containing the origin but not the whole space.

Hence non-zero continuous linear functionals certainly exist on locally convex l.t.s. As in Banach spaces the second question may be answered with the aid of

<u>Theorem 1.6.3 (1)</u>. Let L be a real linear system and let p(x) be a positive real valued function defined on L such that

$$p(x+y) \leq p(x) + p(y), \quad p(ax) = ap(x) \text{ for } a \geq 0.$$

Let f(x) be a real linear functional f(x) defined on a linear sub-space M of L such that $f(x) \leq p(x)$ on M. Then there exists a real linear functional F(x) on L such that $F(x) \leq p(x)$ for all x in L and $F(x) = f(x)$ in M.

The inequality $F(x) \leq p(x)$ implies $F(-x) \leq p(-x)$ or $-F(x) \leq p(-x)$. Hence $-p(-x) \leq F(x) \leq p(x)$.

<u>Theorem 1.6.4</u>. Let L be a locally convex real l.t.s., D its directed system and M a linear sub-space of L. Let f be a real linear functional defined on M and bounded with respect to the pseudo-norms there. Then there exists a real linear functional $x^*(x)$ on L such that $x^*(x) = f(x)$ on M and furthermore $x^*(x)$ is bounded with respect to the pseudo-norms (with the same bound).

<u>Proof</u>. By the corollary to Theorem 1.6.1 there exists d in D and $M(d) > 0$ such that $|f(x)| \leq M(d) \|x\|_d$ for all x in M. Taking $p(x) = M(d)\|x\|_d$, Theorem 1.6.3 shows the existence of $x^*(x)$ and further that

$$-M(d) \|-x\|_d \leq x^*(x) \leq M(d) \|x\|_d.$$

Since $\|-x\|_d = \|x\|_d$, $|x^*(x)| \leq M(d) \|x\|_d$ for all x in L.

<u>Theorem 1.6.5 (6)</u>. If L is a locally convex real l.t.s. and D the associated directed system, then for any x_0 in L, and any d in D,

there exists a real linear continuous functional defined on L with the property $x^*(x_o) = \|x_o\|_d \neq 0$ and $|x^*(x)| \leq \|x\|_d$ all x in L.

Proof. Let d be in D. It can clearly be assumed that $\|x_o\|_d \neq 0$. Then the elements of the form ax_o, a real, form a real linear subspace M of L on which f(x) is defined by $f(ax_o) = a\|x_o\|_d$ and $|f(ax_o)| = |a| \|x_o\|_d$. Hence by Theorem 1.6.4, there exists an extension $x^*(x)$ of f(x) to L such that $x^*(ax_o) = a \|x_o\|_d$ and so $x^*(x_o) = \|x_o\|_d$. Furthermore $|x^*(x)| \leq \|x\|_d$ for all x in L. This shows that there exists on L infinitely many linear functionals bounded with respect to the pseudo-norms.

Corollary. Let L^* be the set of continuous linear functionals on L. Then $x^*(x) = 0$ all x^* in L implies $x = \theta$.

It is clear that Theorems 1.6.3 and 1.6.4 may still be used to prove the existence of real continuous linear functionals on L even when L is a locally convex complex l.t.s. but they no longer apply to complex continuous linear functionals on L. The extension of Theorem 1.6.4 to the case of locally convex complex l.t.s. and complex linear functionals is obtained as in the Banach space case (16). This theorem which will be called Theorem 1.6.6 is obtained from Theorem 1.6.4 by replacing real by complex in the wording of that theorem.

Proof of Theorem 1.6.6. Let $f(x) = f_1(x) + if_2(x)$ be a complex linear functional given on M. $f_1(x)$ and $f_2(x)$ are real linear functionals and a simple calculation shows that $|f_1(x)| \leq M(d) \|x\|_d$

and $|f_2(x)| \leq M(d) \, \|x\|_d$ on M. Further, $f_2(x) = -f_1(ix)$. Theorem 1.6.4 may be used to obtain a real continuous linear functional $F_1(x)$ on L such that $F_1(x) = f_1(x)$ on M and $|F_1(x)| \leq M(d) \, \|x\|_d$ on L. Set $x^*(x) = F_1(x) - iF_1(ix)$. Since $x^*(ix) = ix^*(x)$ this is a complex continuous linear functional coinciding with f(x) on M. If $x^*(x) = re^{i\theta}$, then

$$|x^*(x)| = r = e^{-i\theta}x^*(x) = x^*(e^{-i\theta}x) = F_1(e^{-i\theta}x) \leq M(d) \, \|x\|_d.$$

Theorem 1.6.5 still holds for locally convex complex l.t.s. In constructing the auxiliary functional f(x) the real number a is replaced by a complex variable ζ ranging over K and Theorem 1.6.6 is used instead of Theorem 1.6.4. This analogue of Theorem 1.6.5 will be referred to as Theorem 1.6.7. Further, with these changes the corollary carries over to Theorem 1.6.7.

Let L^* be the set of continuous linear functionals defined on a locally convex l.t.s. L^* is a linear system. A bilinear functional $B(x,x^*)$ is defined on $L \times L^*$ by $B(x,x^*) = x^*(x)$ having the properties (3)

(i) $B(x,x^*) = 0$ for all x in L implies $x^* = \theta^*$, and

(ii) $B(x,x^*) = 0$ for all x^* in L^* implies $x = \theta$.

The former is merely the definition of the zero element in L^* while the latter is a consequence of the corollary to Theorem 1.6.7 in the complex case and of the corollary to Theorem 1.6.5 in the real case. In either event for each x^* in L^*, $|x^*(x)|$ is a triangular pseudo-norm on L and by property (ii) defines a locally convex topology on L.

Similarly, for each x in L, $|x^*(x)|$ is a triangular pseudo-norm on L^* defining a locally convex topology on L.

<u>Definition 1.6.2.</u> The locally convex topology defined on L by the triangular pseudo-norms $|x^*(x)|$, x^* in L^*, is called the weak topology on L. The given topology on L is called the strong topology.

<u>Definition 1.6.3.</u> The locally convex topology defined on L^* by the triangular pseudo-norms $|x^*(x)|$, x in L, is called the weak-* topology on L^*.

<u>Theorem 1.6.8 (17).</u> A set $S \subset L$, a locally convex l.t.s., is bounded in the strong topology if and only if it is bounded in the weak topology.

<u>Proof.</u> The necessity is clear. As for the sufficiency of the condition, S weakly bounded means $\sup_{x \text{ in } S} |x^*(x)| < \infty$, for each x^* in L^*. Let d be any element of D. Consider all x^* satisfying $|x^*(x)| \leq M(d)\|x\|_d$ for all x in L. These certainly satisfy $\sup_{x \text{ in } S} |x^*(x)| < \infty$. Let $N \equiv \{x \mid \|x\|_d = 0\}$. N is a linear sub-space of L. Then L can be written as L = N + M where M is a linear sub-space of L, and for y in M, $\|y\|_d = 0$ implies $y = \theta$. Hence $\|\cdot\|_d$ is a norm on M and by the theorem of uniform boundedness for Banach spaces (2), $\|x\|_d < \infty$ for all x in S.

Let L and L' be two locally convex l.t.s. with associated directed systems D and D' respectively. $\mathcal{T}(L,L')$, the set of all continuous linear functions on L into L' is clearly a linear system. $\mathcal{T}(L,L')$ may be topologized in various ways, three of which will be dealt with here.

If E and F are any two directed systems then the set G of all pairs $g = (e,f)$, e in E, f in F is a directed system if $g > g'$, where $g' = (e', f')$ is defined to mean that $e > e'$ and $f > f'$. The following choices for E and E' lead to directed systems G by means of which the desired topologies may be constructed.

(i) Let E be D' and F the family $\{B\}$ of all bounded sets B of L. F is a directed system if $B_1 > B_2$ is defined to mean $B_1 \subset B_2$.

(ii) Let E be D' and F the family $\{\Phi\}$ of finite sets of elements of L. F is again a directed system if $\Phi_1 > \Phi_2$ means $\Phi_1 \subset \Phi_2$.

(iii) Let E be the family of all finite sets $\{\Phi^*\}$ of elements of $(L')^*$, where $(L')^*$ is the linear system of continuous linear functionals defined on the locally convex l.t.s. L'. E is a directed set if $\Phi_1^* > \Phi_2^*$ means $\Phi_1^* \subset \Phi_2^*$. F is taken as in (ii).

<u>Theorem 1.6.9 (10, 15)</u>. If the following pseudo-norms corresponding respectively to the choices (i), (ii), (iii) as directed systems are defined on $\mathcal{T}(L,L')$

(i) the $\mathcal{T}_b$ pseudo-norm: $\|T\|_{(d',B)} = \sup\limits_{x \text{ in } B} \|T(x)\|_d$, T in $\mathcal{T}(L,L')$,

(ii) the $\mathcal{T}_s$ pseudo-norm: $\|T\|_{(d',\Phi)} = \sup\limits_{x \text{ in } \Phi} \|T(x)\|_{d'}$, T in $\mathcal{T}(L,L')$

and

(iii) the $\mathcal{T}_W$ pseudo-norm: $\|T\|_{(\Phi^*,\Phi)} = \sup\limits_{\substack{x \text{ in } \Phi \\ y^* \text{ in } \Phi^*}} |y^*[T(x)]|$, T in $\mathcal{T}(L,L')$, then in each case a locally convex topology is defined on $\mathcal{T}(L,L')$ referred to respectively as the $\mathcal{T}_b$, $\mathcal{T}_s$ and $\mathcal{T}_W$ topologies on $\mathcal{T}(L,L')$. If Φ and Φ^* are each taken to contain one point only, equivalent pseudo-norms are obtained.

Theorem 1.6.10 (10). With the same notation as above if L is an $\mathcal{F}$-space and L' is sequentially complete then the linear system $\mathcal{T}(L,L')$ is sequentially complete for the $\mathcal{T}_b$ topology.

Theorem 1.6.11 (10). With the same notation as above if L is an $\mathcal{F}$-space, L' any locally convex l.t.s. and H a subset of $\mathcal{T}(L,L')$, then the following statements are equivalent:

(i) H is bounded in the $\mathcal{T}_b$ topology,

(ii) H is bounded in the $\mathcal{T}_s$ topology, and

(iii) H is bounded in the $\mathcal{T}_W$ topology.

§ 2. Locally Convex Topological Linear Rings

1.7. Multilinear Functions. A function of n variables $f(x_1,x_2,\ldots,x_n)$ on $L_1,L_2,\ldots,L_n$ to L, where $L_1,\ldots,L_n$ are locally convex complex (real) l.t.s. will be said to be complex (real) multilinear if:

(i) $f(x_1,x_2,\ldots,x_n)$ is additive in each x_i,

(ii) $f(x_1,x_2,\ldots,x_n)$ is complex (real) homogeneous of degree one in each variable.

Definition 1.7.1. If D, D_1, $D_2,\ldots,D_n$ are associated directed systems for the locally convex l.t.s. L, L_1, $L_2,\ldots,L_n$ respectively, then the multilinear function $f(x_1,x_2,\ldots,x_n)$ is said to be

(i) continuous at $(y_1,y_2,\ldots,y_n)$ if given d in D and $\epsilon > 0$ there exists $d_1,d_2,\ldots,d_n$ belonging to $D_1,D_2,\ldots,D_n$ respectively and $\delta_1 > 0$, $\delta_2 > 0,\ldots,\delta_n > 0$ such that $\| x_i - y_i \|_{d_i} < \delta_i$, $i = 1,2,\ldots,n$ implies

$$\| f(x_1,x_2,\ldots,x_n) - f(y_1,y_2,\ldots,y_n) \|_d < \epsilon ,$$

(ii) bounded with respect to the pseudo-norms if to any d in D there corresponds $d_1,d_2,\ldots,d_n$ in $D_1,D_2,\ldots,D_n$ respectively and a real number $M(d,d_1,d_2,\ldots,d_n) > 0$ such that

$$\| f(x_1,x_2,\ldots,x_n) \|_d \leq M(d,d_1,d_2,\ldots,d_n) \| x_1 \|_{d_1} \| x_2 \|_{d_2} \cdots \| x_n \|_{d_n} ,$$

all x_i in L_i.

The notation of Definition 1.7.1 will be used in the following.

Theorem 1.7.1. A multilinear function is continuous everywhere if and only if it is bounded with respect to the pseudo-norms.

Proof: The condition is necessary. Indeed the continuity of $f(x_1,\ldots,x_n)$ at $(\theta,\ldots,\theta)$ implies, for given d in D, the existence of d_i in D_i and numbers $b_i > 0$, $1 \leq i \leq n$, such that for $\| x_i \|_{d_i} \leq b_i$, $\| f(x_1,\ldots,x_n) \|_d \leq 1$. Let t_o be a real number satisfying $0 < |t_o| < 1$. For every point $(x_1,x_2,\ldots,x_n)$ with $\| x_i \|_{d_i} \neq 0$, $i = 1,\ldots,n$, there exist n integers $k_1,k_2,\ldots,k_n$ such that $b_i |t_o| < \| t_o^{k_i} x_i \|_{d_i} \leq b_i$. Hence it is true that

$|t_o|^{k_1 + k_2 + \cdots + k_n} \| f(x_1,\ldots,x_n) \|_d \leq 1$, and using

$$\frac{1}{|t_o|^{k_i}} \leqslant \frac{1}{b_i |t_o|} \| x_i \|_{d_i} ,$$

$$\| f(x_1,\ldots,x_n) \|_d \leqslant \frac{1}{|t_o|^n} \frac{1}{b_1 . b_2 \ldots b_n} \| x_1 \|_{d_1} \| x_2 \|_{d_2} \ldots \| x_n \|_{d_n}$$

$$= M(d,d_1,\ldots,d_n) \| x_1 \|_{d_1} \| x_2 \|_{d_2} \ldots \| x_n \|_{d_n} ,$$

where $M(d,d_1,d_2,\ldots,d_n) = \dfrac{1}{|t_o|^n} \cdot \dfrac{1}{b_1 b_2 \ldots b_n}$.

This result is still correct for those points $(x_1,x_2,\ldots,x_n)$ having the property that $\| x_i \|_{d_i} = 0$ for some x_i since this implies $\| f(x_1,x_2,\ldots,x_n) \|_d = 0$. Otherwise there exists some real number a such that

$$\| f(x_1,\ldots,ax_i,\ldots,x_n) \|_d = |a| \| f(x_1,\ldots,x_n) \|_d > 1,$$

while $\| ax_i \|_{d_i} = |a| \| x_i \|_{d_i} = 0$ and $\| x_k \|_{d_k} \leqslant b_k$, $k \neq i$.

The condition is sufficient. For

$$f(x_1,\ldots,x_n) - f(y_1,\ldots,y_n) = \sum_{i=1}^{n} f(y_1,\ldots,y_{i-1},x_i - y_i, x_{i+1},\ldots,x_n),$$

and given d in D, $\epsilon > 0$, it is true that $\| x_i - y_i \|_{d_i} < \delta_i$, $i = 1,2,\ldots,n$, implies

$$\| f(y_1,\ldots,y_{i-1}, x_i - y_i, x_{i+1},\ldots,x_n) \|_d$$
$$\leqslant M(d,d_1,\ldots,d_n) \| y_1 \|_{d_1} \ldots \| y_{i-1} \|_{d_{i-1}} \| x_i - y_i \|_{d_i} \| x_{i+1} \|_{d_{i+1}} \ldots \| x_n \|_{d_n}$$
$$\leqslant M(d,d_1,\ldots,d_n) \, \delta_i \prod_{k \neq i} (\| y_k \|_k + \delta_k).$$

Setting $\delta_1 = \delta_2 = \dots = \delta_n = \delta$ and $C = \sup_k \| y_k \|_{d_k}$, it follows that

$$\| f(x_1,x_2,\dots,x_n) - f(y_1,y_2,\dots,y_n) \|_d \leqslant n \cdot M(d,d_1,\dots,d_n)\, \delta\, (C+\delta)^{n-1}$$

when δ is sufficiently small. Hence $f(x_1,x_2,\dots,x_n)$ is continuous at $(y_1,y_2,\dots,y_n)$ and so everywhere.

If $f(x_1,x_2,\dots,x_n)$ is a multilinear function on the locally convex l.t.s. L to itself Definition 1.7.1 (ii) implies the following statement more suitable in that situation. Since $D = D_1 = D_2 = \dots = D_n$ there exists d' in D such that $d' > d_i$ and $\| x \|_{d'} > \| x \|_{d_i}$, $i = 1,2,\dots,n$, for all x in L.

<u>Definition 1.7.2.</u> If $f(x_1,x_2,\dots,x_n)$ is a multilinear function on the locally convex l.t.s. L to itself and D is the associated directed system then f is bounded with respect to the pseudo-norms if to each d in D corresponds d' in D and a real number $M(d,d') > 0$ such that

$$\| f(x_1,x_2,\dots,x_n) \|_d \leqslant M(d,d')\, \| x_1 \|_{d'}\, \| x_2 \|_{d'} \dots \| x_n \|_{d'}.$$

1.8. <u>Extended Locally Convex Topological Linear Rings.</u>

<u>Definition 1.8.1.</u> $\mathfrak{A}$ is a complex linear ring if $\mathfrak{A}$ is a complex linear system and if, in addition, there is defined a multiplication of elements of $\mathfrak{A}$ such that:

M_1. Every pair of elements x,y in $\mathfrak{A}$ has a unique product x,y in $\mathfrak{A}$.

M_2. Multiplication is associative, i.e., $x(yz) = (xy)z$, x,y,z in $\mathfrak{A}$.

M_3. There exists a unit element e such that ex = xe = e for each x in $\mathfrak{A}$.

S. $(\xi x)(\eta y) = (\xi\eta)xy$ with ξ , η in K, x,y in $\mathfrak{A}$.

D. Multiplication and addition are related by

$$x(y + z) = xy + xz$$

$$(y + z)x = yx + zx$$

$\mathfrak{A}$ is a real linear ring if $\mathfrak{A}$ is a real linear system and the postulate S is replaced by

S'. $(ax)(by) = (ab)xy$, a,b in R, x,y in $\mathfrak{A}$.

Definition 1.8.2. $\mathfrak{A}$ is called a locally convex complex (real) topological linear ring if $\mathfrak{A}$ is a locally convex complex (real) l.t.s. and if multiplication is a continuous function of both variables under the assigned topology.

By Theorem 1.7.1 applied to the continuous multilinear function xy on $\mathfrak{A}$ to $\mathfrak{A}$, where $\mathfrak{A}$ is a locally convex complex (real) topological linear ring and D is the directed system for $\mathfrak{A}$, corresponding to each d in D there is a d' in D and a real number $M(d,d') > 0$ such that

$$\| xy \|_d \leq M(d,d') \; \| x \|_{d'} \; \| y \|_{d'} .$$

In the case of the locally convex complex (real) topological linear ring $\mathfrak{A}$ it is desirable to have the stronger property that for d in D and all x,y in $\mathfrak{A}$

$$\| xy \|_d \leq M(d) \; \| x \|_d \; \| y \|_d .$$

Replacing $\| x \|_d$ by the equivalent pseudo-norm $M(d) \| x \|_d$ a pseudo-norm satisfying the inequality

$$\| xy \|_d \leq \| x \|_d \| y \|_d ,$$

for each d in D, is obtained (Theorem 1.3.2).

Theorem 1.8.2. The inequality $\| xy \|_d \leq \| x \|_d \| y \|_d$, for each d in D, is equivalent to the existence in the locally convex complex (real) topological linear ring $\mathfrak{A}$ of an equivalent set of convex, circular (symmetric) and balanced neighbourhoods $\{ C \}$ of the origin, such that for each C

$$CC \subset C, \text{ when } CC = \{ xy \mid x,y \text{ belong to } C \} .$$

Proof. Let D be the directed system for $\mathfrak{A}$ and suppose $\| xy \|_d \leq \| x \|_d \| y \|_d$. Then suitable neighbourhoods C are given by $C \equiv \{ x \mid \| x \|_d < \delta < 1 \}$ as d ranges over D and δ takes on all values $0 < \delta < 1$.

Conversely, if such a neighbourhood system exists, then defining

$$\| x \|_C = \inf \{ h > 0 \mid x \text{ is in } hC \}$$

for each x in $\mathfrak{A}$ and C in $\{ C \}$, it follows that $\| xy \|_C \leq \| x \|_C \| y \|_C$. For given $\delta > 0$, there exist positive numbers a and b so that $\| x \|_C \leq a < \| x \|_C + \delta$, $\| y \|_C \leq b < \| y \|_C + \delta$ while x and y are in aC and bC respectively. Then xy is in aCbC = abCC $\subset$ abCC $\subset$ $(\| x \|_C + \delta)$ $(\| y \|_C + \delta)C$ and consequently

$$\| xy \|_C \leq \| x \|_C \| y \|_C + \delta (\| x \|_C + \| y \|_C) + \delta^2 .$$

But δ is arbitrary and so $\| xy \|_C \leq \| x \|_C \| y \|_C$.

The question whether or not every locally convex topological linear ring contains an equivalent system $\{ C \}$ of neighbourhoods satisfying $CC \subset C$ is answered in the negative by the space L^{ω} of Example 1.4.1.

<u>Definition 1.8.3.</u> A locally convex complex (real) topological linear ring $\mathfrak{A}$ will be called extended if there exists an equivalent system $\{ C \}$ of convex, circular (symmetric) and balanced neighbourhoods of the origin such that $CC \subset C$.

This additional condition insures, for example, that x^{-1} (the inverse of x in $\mathfrak{A}$) exists for some x in $\mathfrak{A}$ and is a continuous function of x on the set of elements having inverses. It also supplies examples to illustrate the theory developed in Chapter III.

1.9. <u>Examples of Extended Locally Convex Topological Linear Rings.</u>

<u>Example 1.9.1.</u> Let D be a directed system and let $\mathfrak{A}_d$, d in D, be a family of complete normed linear rings which as Banach spaces satisfy the conditions (i) and (ii) of Example 1.4.1. Then $\mathfrak{A} = \bigcap_D \mathfrak{A}_d$ is an extended sequentially complete locally convex topological linear ring. With the same notation as in Example 1.4.1, if x,y are in $\mathfrak{A}$, $\| xy \|_d \leq \| x \|_d \| y \|_d$ for every d in D.

As a particular instance of this construction D may be taken as the set of real numbers $1 \leq p \leq \infty$. Let $a = \{ a_i \}$ satisfy $\| a \|_p = (\sum_i | a_i |^p)^{\frac{1}{p}} < \infty$, that is $\{ a_i \}$ is in l^p. A product of elements a,b in l^p is defined by $ab = \{a_i\} \{b_i\} = \{a_i b_i\}$. Then

$$\|ab\|_p = (\sum_i |a_i b_i|^p)^{\frac{1}{p}} \leq (\sum |a_i|^p)^{\frac{1}{p}} (\sum |b_i|^p)^{\frac{1}{p}} = \|a\|_p \|b\|_p.$$

The normed rings $\mathfrak{A}_p$ are now the normed rings l^p, and $\mathcal{L} = \bigcap_{1 \leq p \leq \infty} l^p$ is an extended sequentially complete locally convex topological linear ring.

Another example of the same construction is the following. Let G be a compact topological group and f,g real-valued functions in G. The convolution of f with g is defined by $(f*g)x = \int f(g)g(y^{-1}x)dy$, where x and y are in G and dy denotes the element of Haar measure. The L^p spaces on G for $1 \leq p \leq \infty$ constitute complete normed linear rings under the usual definitions of addition and scalar multiplication and with convolution as multiplication of elements and hence qualify as a family $\mathfrak{A}_p$ as described above.

Example 1.9.2. Let D be a directed system and let $\mathfrak{A}_d$, d in D, be a family of complete normed linear rings which as linear spaces satisfy the conditions (i) and (ii) of Example 1.4.1 as modified by the notation of Example 1.4.2. Let $\mathfrak{A}$ be the cartesian product $\prod \mathfrak{A}_d$, d in D, that is the set of all elements $x = \{x_d\}$, d in D. Addition and scalar multiplication of elements in $\mathfrak{A}$ are defined as in Example 1.4.2. Multiplication of $x = \{x_d\}$, $y = \{y_d\}$ is given by $xy = \{x_d\}\{y_d\} = \{x_d y_d\}$. Hence, since $\|x\|_d = \|x_d\|$, $\|xy\|_d = \|x_d y_d\| \leq \|x_d\| \|y_d\| = \|x\|_d \|y\|_d$. Thus $\mathfrak{A}$ is an extended sequentially complete locally convex topological linear ring.

Example 1.9.3 (14). Let $\mathcal{S}$ be any locally compact Hausdorff topological space and $\{C\}$ the family of compact sets of $\mathcal{S}$. The family $\{C\}$ forms a directed set if $C_1 > C_2$ means $C_1 \subset C_2$. Let $\mathcal{F}$ denote the family of continuous numerically valued functions defined on $\mathcal{S}$. $\mathcal{F}$ is a linear ring under the usual definitions of addition, scalar multiplication and multiplication of elements of $\mathcal{F}$. For each C in $\{C\}$, a pseudo-norm on $\mathcal{F}$ is defined by

$$\| f \|_C = \sup_{x \text{ in } C} | f(x) | , \quad f \text{ in } \mathcal{F} ,$$

giving rise to a sequentially complete locally convex topological linear ring.

$$\| fg \|_C = \sup_{x \text{ in } C} | fg(x) | = \sup_{x \text{ in } C} | f(x) \cdot g(x) | \leq \sup_{x \text{ in } C} |f(x)| \sup_{x \text{ in } C} | g(x) |$$

$$= \| f \|_C \| g \|_C .$$

In particular, let $\mathcal{S}$ be the space G of Example 1.4.3.

<u>Example 1.9.4.</u> In this example the space $\mathcal{D}$ of Example 1.3.1 is used. $\mathcal{D}$ is the non-normable sequentially complete $\mathcal{F}$-space of infinitely differentiable functions defined on the interval $0 \leq t \leq 1$ of the real line. Let addition, scalar multiplication and multiplication be defined as usual. This means that for x,y in $\mathcal{D}$, xy is the function defined by $xy(t) = x(t) \cdot y(t)$. Recalling that for x in $\mathcal{D}$, k an integer

$$\| x \|_k = \sup_{\substack{0 \leq i \leq k \\ 0 \leq t \leq 1}} | x^{(i)}(t) | ,$$

it is true that $\| xy \|_k \leq \| x \|_k \| y \|_k$. For,

$$\|xy\|_k = \sup_{\substack{0 \leq i \leq k \\ 0 \leq t \leq 1}} | xy^{(i)}(t) |$$

and $xy^{(i)}(t) = \frac{d^i}{dt^i} x(t)\, y(t) = \sum_{n=0}^{i} \binom{i}{n} x^{(n)}(t) y^{(i-n)}(t)$.

Now $\sup_{0 \leq t \leq 1} | x^{(n)}(t) | \leq \sup_{\substack{0 \leq m \leq k \\ 0 \leq t \leq 1}} | x^{(m)}(t) | = \| x \|_k$ if $0 \leq n \leq k$.

Hence $\sup_{0 \leq t \leq 1} | xy^{(i)}(t) | \leq \| x \|_k \| y \|_k (1 + \frac{i}{1} + \frac{i(i-1)}{2} + \ldots + 1)$

and so $\| xy \|_k = \sup_{\substack{0 \leq i \leq k \\ 0 \leq t \leq 1}} | xy^{(i)}(t) | \leq (\sup_{0 \leq i \leq k} 2^i) \| x \|_k \| y \|_k$

$= 2^k \| x \|_k \| y \|_k$. Putting $\| x \|_k' = 2^k \| x \|_k$ equivalent pseudo-norms are obtained such that $\| xy \|_k \leq \| x \|_k \| y \|_k$.

Chapter II

Analytic Functions of a Complex Variable

2.1. Introduction. The purpose of this chapter is to develop complex function theory in three (related) situations. In § 1, the theory is developed in detail for the case when the function values lie in a sequentially complete locally convex complex l.t.s. The case when the function values lie in the linear space of continuous linear functions on one sequentially complete locally convex complex l.t.s. to a second such space is sketched in § 2. Finally, in § 3 some theorems relating to functions of several complex variables taking their values in a sequentially complete locally convex complex l.t.s. are given. The proofs in Chapter II rest heavily on Theorem 1.6.7 and its corollary and on Theorems 1.6.8 and 1.6.11.

§ 1. Vector-Valued Functions of a Complex Variable

2.2. Continuity and Differentiability. In this section the linear system structure of the sequentially complete locally convex complex l.t.s. under consideration will be denoted by L. When referring to this space in its strong (given) topology, with respect to which the space is sequentially complete, the symbol L_s will be used and the corresponding associated directed system will be denoted by D. When referring to this space in its weak topology, with respect to which the space is not necessarily complete, the symbol L_W will be used. The symbol $\mathcal{D}$ will always denote a domain, that is, an open, connected set in the complex plane.

<u>Definition 2.2.1.</u> If $x(\zeta)$ is defined on $\mathcal{D}$ with values in L then $x(\zeta)$ is called a vector function of ζ.

<u>Definition 2.2.2.</u> A vector function $x(\zeta)$ on $\mathcal{D}$ is

(i) strongly or L_s continuous at $\zeta = \zeta_o$ in $\mathcal{D}$, if given d in D, and $\epsilon > 0$, there exists $\delta = \delta(\zeta_o, d, \epsilon)$ such that $|\zeta - \zeta_o| < \delta$ implies $\| x(\zeta) - x(\zeta_o) \|_d < \epsilon$,

(ii) weakly or L_W continuous at $\zeta = \zeta_o$ in $\mathcal{D}$, if given x^* in L^*, $\epsilon > 0$, there is a $\delta = \delta(\zeta_o, x^*, \epsilon)$ such that $|\zeta - \zeta_o| < \delta$ implies $|x^*[x(\zeta) - x(\zeta_o)]| < \epsilon$, and

(iii) $L_s(L_W)$ continuous in $\mathcal{D}$ if it is $L_s(L_W)$ continuous at each point of $\mathcal{D}$.

<u>Definition 2.2.3.</u> A vector function $x(\zeta)$ on $\mathcal{D}$ is strongly or L_s (weakly or L_W) differentiable at $\zeta = \zeta_o$ if there is an element $x'(\zeta_o)$ in L such that the difference quotient $\frac{1}{\zeta}\left\{ x(\zeta_o + \zeta) - x(\zeta_o) \right\}$ tends strongly (weakly) to $x'(\zeta_o)$ when $\zeta \to 0$. $x'(\zeta)$ is called the strong or L_s (weak or L_W) derivative of $x(\zeta)$ at ζ_o. $x(\zeta)$ is $L_s(L_W)$ differentiable in $\mathcal{D}$ if it is $L_s(L_W)$ differentiable at each point of $\mathcal{D}$.

<u>Definition 2.2.4.</u> A vector function $x(\zeta)$ defined on $\mathcal{D}$ is topologically bounded if its values lie in a bounded set.

The notions of weakly and strongly topologically bounded need not be distinguished for vector functions since the notions of weakly bounded and strongly bounded sets in L coincide.

Theorem 2.2.1. If the vector function $x(\zeta)$ is L_s continuous in a bounded closed set B of the complex plane, then it is uniformly L_s continuous there, i.e., having chosen $\epsilon > 0$, d in D it is always possible to assign a number $\delta = \delta(d,\epsilon)$ in such a manner, that for any two points ζ' and ζ'' of B for which $|\zeta'' - \zeta'| < \delta$ it is true that $\| x(\zeta'') - x(\zeta') \|_d < \epsilon$.

Proof. A circle $C(\zeta)$ of radius $r(\zeta, d, \epsilon)$ can be drawn about every point ζ of B as centre such that the

$$\sup_{\zeta'', \zeta' \text{ in } C(\zeta)} \| x(\zeta'') - x(\zeta') \|_d < \frac{\epsilon}{2},$$

because of the continuity of $x(\zeta)$ at ζ. Now to every ζ of B let correspond the circle about ζ with radius $\frac{1}{2} r(\zeta, d, \epsilon)$. By compactness a finite number of these circles is sufficient to cover B. If the radius of the smallest of these circles is $\delta = \delta(d, \epsilon)$, this number satisfies the conditions of the theorem. For, if $|\zeta'' - \zeta'| < \delta$, and if ζ' lies, say, in the circle about ζ as centre with radius $\frac{1}{2} r(\zeta, d, \epsilon)$, then since $\delta \leq \frac{1}{2} r(\zeta, d, \epsilon)$ it follows that ζ' and ζ'' lie within the circle about ζ as centre and with radius $r(\zeta, d, \epsilon)$. Hence $\| x(\zeta'') - x(\zeta') \|_d < \epsilon$.

Theorem 2.2.2. If the vector function $x(\zeta)$ is L_s continuous on a bounded, closed, connected region B of the complex plane, then it is topologically bounded there.

Proof. Given $\epsilon > 0$, d in D, determine $\delta = \delta(d, \epsilon)$ by Theorem 2.2.1. B can be covered by a finite number, say n, of circles

$C_1, C_2, \ldots, C_n$ each of radius $\frac{\delta}{3}$. Let ζ_o be a fixed point in B and ζ any point in B. Join these points to the centres of the respective circles in which they lie. These centres can then be joined by means of a polygonal line defined by $r (r \leq n)$ points $\zeta_1, \zeta_2, \ldots, \zeta_r$, which are themselves centres of circles C_i, the distance between consecutive points being at most $\frac{2}{3}\delta$, and ζ_1 and ζ_r being the centres of the circles in which ζ_o and ζ lie respectively. Hence,

$$\begin{aligned} \left| \|f(\zeta)\|_d - \|f(\zeta_o)\|_d \right| &\leq \|f(\zeta) - f(\zeta_o)\|_d \\ &= \|f(\zeta_1) - f(\zeta_o) + f(\zeta_2) - f(\zeta_1) + \ldots + f(\zeta) - f(\zeta_r)\|_d \\ &\leq \|f(\zeta_1) - f(\zeta_o)\|_d + \|f(\zeta_2) - f(\zeta_1)\|_d + \ldots + \|f(\zeta) - f(\zeta_r)\|_d \\ &\leq (r+1)\epsilon \leq (n+1)\epsilon . \end{aligned}$$

Thus, $\|f(\zeta)\|_d \leq \|f(\zeta_o)\|_d + (n+1)\epsilon$, where n is a fixed number and ζ is arbitrary in B. This implies that for given $d \in D$ $\sup_{\zeta \in B} \|x(\zeta)\|_d < \infty$ or $x(\zeta)$ is topologically bounded.

The proofs of Theorems 2.2.1 and 2.2.2 use only the fact that the L_s topology is locally convex. Hence the corresponding theorems for the L_W topology (which is also locally convex) must be valid.

2.3. Complex Integration. In order to obtain a suitable definition of a contour integral the notion of a Stieltjes integral must be extended to vector functions of a real variable. Only that case in which the integrand is a vector-valued and the integrator a complex-valued function of a real variable is considered.

Let $x(a)$ be a topologically bounded vector function of the real variable a, and let $f(a)$ be a bounded complex-valued function of the real variable a, both defined for $[a_o,b_o] \equiv \{ a | a_o \leq a \leq b_o \}$. The points $a_o, a_1, a_2, \ldots, a_n$, where $a_o < a_1 < a_2 < \ldots . a_n = b_o$, define a partition of $[a_o, b_o]$. Choosing intermediate points c_i so that $a_{i-1} \leq c_i \leq a_i$, $i = 1,2,\ldots,n$, the so-called approximating sums $x_n = \sum_{i=1}^{n} x(c_i) \left[f(a_i) - f(a_{i-1}) \right]$ may be formed. By the norm η of a partition is meant the largest of the numbers $a_i - a_{i-1}$, $i = 1,2,\ldots,n$.

Definition 2.3.1. If the limit

$$\lim_{\substack{n \to \infty \\ \eta \to 0}} x_n = \lim_{\substack{n \to \infty \\ \eta \to 0}} \sum_{i=1}^{n} x(c_i) \left[f(a_i) - f(a_{i-1}) \right],$$

exists (call it $\bar{x}$) in the $L_s(L_W)$ sense independently of the manner of partitioning and of the choice of the numbers c_i, $a_{i-1} \leq c_i \leq a_i$, then the limit is called the $L_s(L_W)$ Stieltjes integral of $x(a)$ with respect to $f(a)$ from a_o to b_o and is denoted by

$$\bar{x} = \int_{a_o}^{b_o} x(a)\, df(a).$$

In order to show that this definition is satisfactory the following theorem is necessary:

Theorem 2.3.1. Let $x(a)$ be an L_s continuous vector function on $[a_o, b_o]$, and let $f(a)$ be a complex-valued function of bounded variation on the same interval, then the approximating sums x_n converge in the L_s topology and hence $\bar{x}$ exists.

Since L is not necessarily sequentially complete in the L_W topology the existence of the L_W Stieltjes integral is not ensured when x(a) is merely L_W continuous. However, it so happens that in the case at hand the two notions of Stieltjes integral coincide.

The proof of Theorem 2.3.1 is based on the following lemmas in which it is assumed that the hypotheses of that theorem are valid.

<u>Definition 2.3.2.</u> If I is any sub-interval of $[a_o, b_o]$ and d is any member of D then by the d-oscillation of x(a) on I is meant $\sigma(d) = \sup \| x(a'') - x(a') \|_d$, a'', a' in I.

<u>Lemma 2.3.1.</u> Let $V = V(a', b')$ denote the total variation of the function f(a) on the closed sub-interval $[a', b']$ of $[a_o, b_o]$ and $\sigma(d)$ the d-oscillation of x(a) on that sub-interval for given d in D. Then for two approximating sums S_1 and S_p, which are formed on the sub-interval $[a', b']$ for $n = 1$ and $n = p(\geq 1)$ it is true that $\| S_p - S_1 \|_d < V\sigma(d)$.

<u>Proof.</u> Let $S_1 = x(c_o)\left[f(b') - f(a')\right]$ and

$$S_p = x(c_1)\left[f(a_1) - f(a')\right] + x(c_2)\left[f(a_2) - f(a_1)\right] + \ldots + x(c_p)\left[f(b') - f(a_{p-1})\right]$$

be the two approximating sums. Here $a_1, a_2, \ldots, a_{p-1}$ denote the partition points and $c_o, c_1, \ldots, c_p$ the intermediate points. By hypothesis $\| x(c_i) - x(c_o) \|_d \leq \sigma(d)$, $i = 1, \ldots, p$. Since S_1 can be written in the form

$$S_1 = x(c_o)\left[f(a_1) - f(a')\right] + x(c_o)\left[f(a_2) - f(a_1)\right] + \ldots + x(c_o)\left[f(b') - f(a_{p-1})\right],$$

it follows that for d in D

$$\| S_p - S_1 \|_d \leqslant \sigma(d)(|f(a_1)-f(a')| + |f(a_2)-f(a_1)| + \dots + |f(b')-f(a_{p-1})|).$$
$$\leqslant \sigma(d)V.$$

Lemma 2.3.2. Let S_n be a fixed approximating sum on $[a_o,b_o]$ and for d in D let the d-oscillations of x(a) on the n segments of the partition all be less than $\sigma_o(d)$. Let S be a new approximating sum derived from S_n by adding new partition points to the old ones. Then if $V = V(a_o,b_o)$ denotes the total variation of the function f(a) on $[a_o,b_o]$ it follows that for d in D $\| S_n - S \|_d \leqslant V\,\sigma_o(d)$, no matter how the intermediate points defining S are chosen.

Proof. Lemma 2.3.1 holds for each of the n segments of the partition defining S, so that for d in D

$$\| S_n - S \|_d \leqslant V_1\,\sigma_o(d) + V_2\,\sigma_o(d) + \dots + V_n\,\sigma_o(d) = V\,\sigma_o(d)$$

if $V_1,V_2,\dots,V_n$ denote the variations of the function f(a) on these segments.

Lemma 2.3.3. Given $\epsilon > 0$, d in D, there exists a $\delta = \delta(d,\epsilon)$ such that if S_1 and S_2 are any two approximating sums defined by means of partitions of $[a_o,b_o]$ of norm less than δ, then

$$\| S_1 - S_2 \|_d < \frac{\epsilon}{2}.$$

Proof. Choose $\delta = \delta(\epsilon,d)$ so that $\| x(a'') - x(a') \|_d < \frac{\epsilon}{4V(a_o,b_o)}$ for any two points a' and a'' of $[a_o,b_o]$ for which $|a'' - a'| < \delta$. This can be done due to Theorem 2.2.1 on uniform continuity. With approximating sums S_1 and S_2 as in the hypothesis, a third approximating sum S_3 is obtained from them by taking as the partition points defining it those of the first two partitions combined. This third can evidently be considered as derived from them by means of adding partition points. Hence, it follows from Lemma 2.3.2 that for d in D $\| S_1 - S_3 \|_d < V \cdot \frac{\epsilon}{4V} = \frac{\epsilon}{4}$. and similarly $\| S_2 - S_3 \|_d < V \cdot \frac{\epsilon}{4V} = \frac{\epsilon}{4}$.

Consequently,

$$\| S_1 - S_2 \|_d = \| (S_1 - S_3) - (S_2 - S_3) \|_d \leq \| S_1 - S_3 \|_d + \| S_2 - S_3 \|_d < \frac{\epsilon}{2} .$$

Lemma 2.3.4. Let an approximating sum S_n be formed for each integral value of n. If the norms of the respective partitions decrease to zero with increasing n, then $\lim_{n \to \infty} S_n$ exists.

Proof. Given $\epsilon > 0$, d in D, determine $\delta = \delta(d, \epsilon)$ according to Lemma 2.3.3. Take $N = N(d, \epsilon)$ so large that the norms of the partitions corresponding to the S_n with $n \geqslant N$ are all less than δ . Lemma 2.3.3 is then applicable to all these S_n, and $\| S_m - S_n \|_d < \frac{\epsilon}{2} < \epsilon$ for $m > n \geqslant N$. Since L is strongly sequentially complete $\lim_{n \to \infty} S_n$ exists.

Proof of Theorem 2.3.1. It remains to show that if $\epsilon > 0$, d in D are given and $\delta = \delta(d, \epsilon)$ is determined according to Lemma 2.3.3 then the relation $\| x_n - \lim_{m \to \infty} S_m \|_d < \epsilon$ holds for every approximating sum x_n for which the norm of the defining partition is less than δ . For in the proof of Lemma 2.3.4, letting $m \to \infty$ it follows that $\| \lim_{m \to \infty} S_m - S_n \|_d < \frac{\epsilon}{2}$ for $n \geqslant N$. Hence

$$\| x_n - \lim_{m \to \infty} S_m \|_d = \| (S_n - \lim_{m \to \infty} S_m) - (S_n - x_n) \|_d$$

$$\leq \| S_n - \lim_{m \to \infty} S_m \|_d + \| S_n - x_n \|_d < \epsilon \quad .$$

Thus $\lim_{n\to\infty} S_n$ is actually $\bar{x}$ and the existence of the Stieltjes integral has been proved completely.

Theorem 2.3.2. If $x_1(a)$ and $x_2(a)$ are strongly continuous vector functions and if $f(a)$ is a numerical-valued function of bounded variation in $a_o \leq a \leq b_o$, then

$$\int_{a_o}^{b_o} \left[x_1(a) + x_2(a) \right] df(a) = \int_{a_o}^{b_o} x_1(a)df(a) + \int_{a_o}^{b_o} x_2(a)df(a).$$

Theorem 2.3.3. If $x(a)$ is a strongly continuous vector function and if $f(a)$ is a numerical-valued function of bounded variation in $a_o \leq a \leq b_o$, then for any d in D,

$$\left\| \int_{a_o}^{b_o} x(a)df(a) \right\|_d \leq \int_{a_o}^{b_o} \| x(a) \|_d \, dV(a) \leq \sup_{a_o \leq a \leq b_o} \| f(a) \|_d V(a_o,b_o),$$

where $V(a)$ is the total variation of $f(a)$ on the interval $[a_o,a]$ and $V(a_o,b_o)$ is the total variation of $f(a)$ on $[a_o,b_o]$.

Proof. It is clear that $V(a)$ is again a numerical-valued function of bounded variation in $a_o \leq a \leq b_o$ and that for any d in D, $\| x(a) \|_d$ is a continuous numerical-valued function in $a_o \leq a \leq b_o$. Hence $\int_{a_o}^{b_o} \| x(a) \|_d \, dV(a)$ makes sense and is an ordinary Stieltjes integral.

Let $a_o < a_1 < a_2 < \dots < a_n = b_o$ be a partition of $[a_o,b_o]$ and let c_i, $a_{i-1} \leq c_i \leq a_i$, $i = 1,2,\dots,n$ be intermediate points. Each interval of the partition is further subdivided say $[a_{i-1},a_i]$ by the points $b_1^{(i)}, b_2^{(i)},\dots,b_{k_i}^{(i)}$. $\sum_{i=1}^{n} x(c_i) \left[f(a_i)-f(a_{i-1}) \right]$ is an approximating sum to the first integral and the following inequalities are then valid for any d in D.

$$\left\| \sum_{i=1}^{n} x(c_i)\left[f(a_i)-f(a_{i-1})\right]\right\|_d \leq \sum_{i=1}^{n} \|x(c_i)\|_d \left[|f(a_i)-f(a_{i-1})|\right]$$

$$\leq \sum_{i=1}^{n} \|x(c_i)\|_d \left[|f(a_i)-f(b_{k_i}^{(i)})| + \dots + |f(b_2^{(i)})-f(b_1^{(i)})| + |f(b_1^{(i)})-f(a_{i-1})|\right]$$

$$\leq \sum_{i=1}^{n} \|x(c_i)\|_d V\left[a_i,a_{i-1}\right] = \sum_{i=1}^{n} \|x(c_i)\|_d \left[V(a_i) - V(a_{i-1})\right]$$

$$\leq \sup_{a_o \leq a \leq b_o} \|x(a)\|_d V(a_o,b_o).$$

Hence the inequalities follow. $\int_{a_o}^{b_o} \|x(a)\|_d \, dV(a)$ is sometimes written as $\int_{a_o}^{b_o} \|x(a)\|_d |df(a)|$.

<u>Theorem 2.3.4.</u> Let T be an L_s continuous linear transformation on L to L', a sequentially complete locally convex complex l.t.s. Then under the hypothesis of theorem 2.3.1

$$T\left[\int_{a_o}^{b_o} x(a)df(a)\right] = \int_{a_o}^{b_o} T\left[x(a)\right] df(a).$$

<u>Proof:</u> From the continuity and linearity of T it follows that

$$T\left[\int_{a_o}^{b_o} x(a)df(a)\right] = T\left[\lim_{\substack{n\to\infty \\ \eta\to 0}} \sum_{i=1}^{n} x(c_i)\left[f(a_i) - f(a_{i-1})\right]\right]$$

$$= \lim_{\substack{n\to\infty \\ \eta\to 0}} \sum_{i=1}^{n} T\left[x(c_i)\right]\left[f(a_i) - f(a_{i-1})\right] = \int_{a_o}^{b_o} T\left[x(a)\right] df(a).$$

Corollary. Let Γ be a rectifiable curve of length $\mathfrak{L}$ in the complex plane given by $\zeta = \zeta(a)$, $0 \leq a \leq a_0$, where $\zeta(a)$ is continuous and of bounded variation in $[0, a_0]$. If $x(\zeta)$ is any L_s continuous function on Γ to L then the (contour) integral $\int_\Gamma x(\zeta) d\zeta = \int_0^a x[\zeta(a)] d\zeta(a)$ exists. Further $\| \int_\Gamma x(\zeta) d\zeta \|_d \leq \sup_\Gamma \|x(\zeta)\|_d \mathfrak{L}$ for any d in D. If T is any L_s continuous linear transformation on L to L' (as in Theorem 2.3.4) then

$$T \left[\int_\Gamma x(\zeta) d\zeta \right] = \int_\Gamma T[x(\zeta)] d\zeta .$$

2.4 Regular Functions and Cauchy's Integral Theorem. In classical function theory a function is regular in a domain $\mathcal{D}$ if it is single-valued, continuous and differentiable in $\mathcal{D}$. The first of these notions carries over without ambiguity but the second and third notions have different meanings for vector functions depending on which topology is used for L. The only topologies considered here are the L_s and the L_W topologies. However, while the notions of continuity and differentiability do have different meanings in the different topologies a unique definition of regularity is obtained.

Definition 2.4.1. A vector function $x(\zeta)$ is regular in the domain $\mathcal{D}$ if $x^*[x(\zeta)]$ is regular in the classical sense for every choice of x^* in L^*.

This is the weakest of several possible definitions. For example, $x(\zeta)$ might have been defined to be analytic in $\mathcal{D}$ if it is strongly differentiable there. However this can be proved from the weaker condition using Theorem 1.6.8. The following Lemma is needed.

Lemma 2.4.1 (7). If $f(\zeta)$ is a complex-valued function regular in the domain $\mathcal{D}$ and $\mathcal{D}_0$ is any domain which is bounded and strictly interior to $\mathcal{D}$, then there is a finite positive number $M(f;\mathcal{D}_0)$ such that for every choice of ζ, $\zeta+\alpha$, $\zeta+\beta$ in $\mathcal{D}_0$

$$\left| \frac{1}{\alpha-\beta} \left\{ \frac{1}{\alpha}\left[f(\zeta+\alpha) - f(\zeta) \right] - \frac{1}{\beta}\left[f(\zeta+\beta) - f(\zeta) \right] \right\} \right| \leq M(f;\mathcal{D}_0).$$

Theorem 2.4.1. Let the vector function $x(\zeta)$ be regular in the domain $\mathcal{D}$, then $x(\zeta)$ is L_s continuous and L_s differentiable in $\mathcal{D}$, uniformly with respect to ζ in any domain $\mathcal{D}_0$ which is bounded and strictly interior to $\mathcal{D}$.

Proof. Apply Lemma 2.4.1 to the function $x^*[x(\zeta)]$ where x^* is an arbitrary element of L^*. The lemma then asserts

$$\left| \frac{1}{\alpha-\beta} \left\{ \frac{1}{\alpha}(x^*[x(\zeta+\alpha)] - x^*[x(\zeta)]) - \frac{1}{\beta}(x^*[x(\zeta+\beta)] - x^*[x(\zeta)]) \right\} \right|$$
$$= \left| x^* \left[\frac{1}{\alpha-\beta} \left(\frac{x(\zeta+\alpha)-x(\zeta)}{\alpha} - \frac{x(\zeta+\beta)-x(\zeta)}{\beta} \right) \right] \right| \leq M(x^*,x;\mathcal{D}_0)$$

for every choice of ζ, $\zeta+\alpha$, $\zeta+\beta$ in $\mathcal{D}_0$ and x^* in L^*. By Theorem 1.6.8, for every d in D and every ζ, $\zeta+\alpha$, $\zeta+\beta$ in $\mathcal{D}_0$ there exists a finite number $M(x,d;\mathcal{D}_0)$ such that

$$\left\| \frac{1}{\alpha-\beta} \left(\frac{x(\zeta+\alpha)-x(\zeta)}{\alpha} - \frac{x(\zeta+\beta)-x(\zeta)}{\beta} \right) \right\|_d \leq M(x,d;\mathcal{D}_0)$$

or,

$$\left\| \frac{x(\zeta+\alpha)-x(\zeta)}{\alpha} - \frac{x(\zeta+\beta)-x(\zeta)}{\beta} \right\|_d \leq |\alpha-\beta| \quad M(x,d;\mathcal{D}_o).$$

In particular, let $\{\alpha_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} \alpha_n = 0$. For given $\epsilon > 0$, d in D, there exists $N = N(d,\epsilon)$ such that for $m > n \geq N$, $\zeta+\alpha_m$, $\zeta+\alpha_n$ are in $\mathcal{D}_o$ and $|\alpha_m - \alpha_n| < \epsilon/M(x,d;\mathcal{D}_o)$. Hence,

$$\left\| \frac{x(\zeta+\alpha_m)-x(\zeta)}{\alpha_m} - \frac{x(\zeta+\alpha_n)-x(\zeta)}{\alpha_n} \right\|_d \leq |\alpha_m-\alpha_n| \, M(x,d;\mathcal{D}_o) < \epsilon .$$

and the elements $\left\{ \frac{x(\zeta+\alpha_n)-x(\zeta)}{\alpha_n} \right\}$ form a strongly convergent sequence in L. Since L is complete in the L_s topology they converge strongly to an element $x'(\zeta)$ in L for any ζ in $\mathcal{D}_o$.

Moreover, ζ being fixed, $x'(\zeta)$ is independent of the sequence $\{\alpha_n\}$. For let $\{\beta_n\}$ be any other sequence of complex numbers such that $\lim_{n\to\infty} \beta_n = 0$. As before the elements $\left\{ \frac{x(\zeta+\beta_n)-x(\zeta)}{\beta_n} \right\}$ form a convergent sequence in L and hence converge to an element $\overline{x}'(\zeta)$ in L for any ζ in $\mathcal{D}_o$. For the same ζ then and any d in D,

$$\|\overline{x}'(\zeta)-x'(\zeta)\|_d \leq \left\| \overline{x}'(\zeta) - \frac{x(\zeta+\beta_n)-x(\zeta)}{\beta_n} \right\|_d +$$

$$\left\| \frac{x(\zeta+\beta_n)-x(\zeta)}{\beta_n} - \frac{x(\zeta+\alpha_m)-x(\zeta)}{\alpha_m} \right\|_d +$$

$$\left\| \frac{x(\zeta+\alpha_m)-x(\zeta)}{\alpha_m} - x'(\zeta) \right\|_d .$$

Now given $\epsilon > 0$, d in D, there exists $N_1(d, \epsilon)$ such that $n \geq N_1(d, \epsilon)$ implies $\left\| \bar{x}'(\zeta) - \frac{x(\zeta + \beta_n) - x(\zeta)}{\beta_n} \right\|_d < \frac{\epsilon}{3}$.

Similarly, there exists $N_2(d, \epsilon)$ such that $n,m \geq N_2(d, \epsilon)$ implies

$$\left\| \frac{x(\zeta + \beta_n) - x(\zeta)}{\beta_n} - \frac{x(\zeta + \alpha_m) - x(\zeta)}{\alpha_m} \right\|_d \leq |\beta_n - \alpha_m| M(x,d; \mathcal{D}_o)$$

$$\leq (|\beta_n| + |\alpha_m|) M(x,d; \mathcal{D}_o) \leq \left(\frac{\epsilon}{6M(x,d; \mathcal{D}_o)} + \frac{\epsilon}{6M(x,d; \mathcal{D}_o)}\right) M(x,d; \mathcal{D}_o) = \frac{\epsilon}{3}.$$

Finally, there exists $N_3(d, \epsilon) > 0$ such that $m \geq N_3(d, \epsilon)$ implies

$$\left\| \frac{x(\zeta + \alpha_m) - x(\zeta)}{\alpha_m} - x'(\zeta) \right\|_d < \frac{\epsilon}{3}.$$

If $N(d, \epsilon) = \max(N_1(d, \epsilon), N_2(d, \epsilon), N_3(d, \epsilon))$, then for $m,n \geq N(d, \epsilon)$

$$\| \bar{x}'(\zeta) - x'(\zeta) \|_d < \epsilon ,$$

that is, $\bar{x}'(\zeta) = x'(\zeta)$.

It remains now to show that for fixed ζ, $\lim_{\alpha \to 0} \frac{x(\zeta + \alpha) - x(\zeta)}{\alpha}$ exists in the L_s topology and is just $x'(\zeta)$. Given $\epsilon > 0$, d in D

$$\left\| \frac{x(\zeta + \alpha) - x(\zeta)}{\alpha} - x'(\zeta) \right\|_d = \left\| \frac{x(\zeta + \alpha) - x(\zeta)}{\alpha} - \frac{x(\zeta + \alpha_n) - x(\zeta)}{\alpha_n} \right\|_d + \left\| \frac{x(\zeta + \alpha_n) - x(\zeta)}{\alpha_n} - x'(\zeta) \right\|_d$$

$$< |\alpha - \alpha_n| M(x,d; \mathcal{D}_o) + \frac{\epsilon}{3}$$

$$\leq (|\alpha| + |\alpha_n|) M(x,d; \mathcal{D}_o) + \frac{\epsilon}{3}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon ,$$

for $|\alpha| < \dfrac{\epsilon}{3M(x,d;\mathcal{D}_o)}$, $n > N$. Thus $x(\zeta)$ has a strong derivative in $\mathcal{D}$ and so is L_s continuous in $\mathcal{D}$.

In the expression,

$$\left\| \frac{x(\zeta+\alpha)-x(\zeta)}{\alpha} - \frac{x(\zeta+\beta)-x(\zeta)}{\beta} \right\|_d \leq |\alpha-\beta| \; M(x,d,\mathcal{D}_o)$$

let $\beta \to 0$. By the continuity of the pseudo-norm it follows that

$$\left\| \frac{x(\zeta+\alpha)-x(\zeta)}{\alpha} - x'(\zeta) \right\|_d \leq |\alpha| \; M(x,d;\mathcal{D}_o),$$

i.e., the difference quotient approaches the derivative uniformly with respect to ζ in $\mathcal{D}_o$. This in turn implies strong uniform continuity and weak differentiability and continuity.

Theorem 2.4.2. If $x(\zeta)$ is a regular vector function on the domain $\mathcal{D}$ to L then

$$\int_\Gamma x(\zeta)d\zeta = \theta$$

for every simple closed rectifiable contour Γ in $\mathcal{D}$ such that the interior of Γ belongs to $\mathcal{D}$.

Proof. Take L' of Theorem 2.3.5 as the space of complex numbers and $T = x^*$ in L^*. Hence,

$$0 = \int_\Gamma x^*\left[x(\zeta)\right]d\zeta = x^*\left[\int_\Gamma x(\zeta)d\zeta\right]$$ for every x^* in L^*. The corollary to Theorem 1.6.7 then implies

$$\int_\Gamma x(\zeta)\,d\zeta = \theta.$$

If $\mathcal{D}$ is simply connected it follows from this theorem that $\int_\Gamma x(\zeta)d\zeta$ is independent of the path and depends only on the initial and final points.

The procedure taken in arriving at Theorem 2.4.2 may be summarized quickly as follows. A definition of regular vector function was given which threw this concept back on the concept of regular function in classical complex function theory. The definition of contour integral on the other hand concerned itself only with L and $\int_\Gamma x(\zeta)d\zeta$ was an element of L. By means of Theorem 2.3.4 which states that continuous linear operators (in particular, continuous linear functionals) commute with integration these concepts were related in Theorem 2.4.2. At this stage then the following situation presents itself. Methods exist in L for proving theorems about the regular vector functions while on the other hand methods exist whereby the proofs of theorems on regular vector functions are thrown back on classical complex function theory. This is illustrated in Theorem 2.4.3 where two proofs are given.

Theorem 2.4.3. Let $x(\zeta)$ be a regular vector function defined on $\mathcal{D}$. If Γ is a simple closed rectifiable curve in $\mathcal{D}$, the interior of which is contained in $\mathcal{D}$, then

$$x(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{x(\tau)}{\tau-\zeta}\,d\tau$$

if ζ is interior to Γ and $\frac{x(\tau)}{\tau-\zeta}$ means $(\tau-\zeta)^{-1}x(\tau)$.

Proof: 1. Here the proof will be thrown back on classical complex function theory. Since $x^*\left[x(\zeta)\right]$ is regular for each x^* in L^* then

$$x^*\left[x(\zeta)\right] = \frac{1}{2\pi i}\int_\Gamma \frac{x^*[x(\tau)]}{\tau-\zeta}\,d\tau$$

and by the corollary to Theorem 2.4.3

$$x^*\left[x(\zeta)\right] = x^*\left[\frac{1}{2\pi i}\int_\Gamma \frac{x(\tau)}{\tau-\zeta}\,d\tau\right]$$

or, $$x^*\left[x(\zeta) - \frac{1}{2\pi i}\int_\Gamma \frac{x(\tau)}{\tau-\zeta}\,d\tau\right] = 0, \text{ for every } x^* \text{ in } L^*.$$

The corollary to Theorem 1.6.7 then implies $x(\zeta) - \frac{1}{2\pi i}\int_\Gamma \frac{x(\tau)}{\tau-\zeta}\,d\tau = \theta$

or $$x(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{x(\tau)}{\tau-\zeta}\,d\tau \quad .$$

2. $\frac{x(\tau)}{\tau-\zeta}$ is a regular vector function in $\mathcal{D}$ except at $\tau = \zeta$. If Γ' is any simple closed rectifiable curve inside Γ containing ζ in its interior then (as in the classical case)

$$\int_\Gamma \frac{x(\tau)}{\tau-\zeta}\,d\tau = \int_{\Gamma'} \frac{x(\tau)}{\tau-\zeta}\,d\tau \quad .$$

In particular let Γ' be the circle with centre ζ and radius ρ , then

$$\int_{\Gamma'}\frac{x(\tau)}{\tau-\zeta}\,d\tau = \int_{\Gamma'} \frac{x(\zeta) + x(\tau) - x(\zeta)}{\tau-\zeta}\,d\tau$$
$$= x(\zeta)\int_{\Gamma'}\frac{d\tau}{\tau-\zeta} + \int_{\Gamma'} \frac{x(\tau) - x(\zeta)}{\tau-\zeta}\,d\tau \quad .$$

The first expression on the right hand side is $2\pi i\left[x(\zeta)\right]$. Hence, for any d in D

$$\left\| \int_\Gamma \frac{x(\tau)}{\tau-\zeta}\,d\tau - 2\pi i\left[x(\zeta)\right]\right\|_d = \left\|\int_{\Gamma'} \frac{x(\tau)-x(\zeta)}{\tau-\zeta}\,d\tau\right\|_d$$

$$\leq \max_{\Gamma'} \frac{\|x(\tau) - x(\zeta)\|_d}{|\tau-\zeta|} \quad 2\pi\rho < \epsilon \quad .$$

This last result holds since given $\epsilon > 0$, d in D, there exists $\delta = \delta(d, \epsilon)$ such that for $\rho = |\tau - \zeta| < \delta$,

$$\max_{\Gamma'} \| x(\tau) - x(\zeta) \|_d < \frac{\epsilon}{2\pi} .$$

Thus for any d in D it is true that

$$\left\| \int_\Gamma \frac{x(\tau)}{\tau - \zeta} d\tau - 2\pi i \,[x(\zeta)] \right\|_d = 0$$

or,

$$x(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{x(\tau)}{\tau - \zeta} d\tau .$$

Theorem 2.4.4. Let $x(\zeta)$ be a regular vector function on the domain $\mathcal{D}$ to the space L. Let Γ be a simple closed rectifiable curve in $\mathcal{D}$, the interior of which is contained in $\mathcal{D}$, then

$$x^n(\zeta) = \frac{n!}{2\pi i} \int_\Gamma \frac{x(\tau)}{(\tau - \zeta)^{n+1}} d\tau , \quad n = 0,1,2,\ldots$$

Proof: It is clear that for every x^* in L^*,

$$\frac{d}{d\zeta} x^*[x(\zeta)] = x^*\left[\frac{d}{d\zeta} x(\zeta)\right],$$ where the derivative on the right hand side refers to the strong derivative of $x(\zeta)$ and that on the left to the ordinary derivative of the complex-valued function $x^*[x(\zeta)]$. This follows from that fact that

$$\frac{1}{\alpha}\left\{ x^*[x(\zeta + \alpha)] - x^*[x(\zeta)]\right\} = x^*\left[\frac{1}{\alpha}[x(\zeta + \alpha) - x(\zeta)]\right]$$

by letting $\alpha \to 0$ and recalling that $x(\zeta)$ has a strong derivative. Further, the strong derivative itself has a second strong derivative for $x(\zeta)$ being a regular vector function $x^*[x(\zeta)]$ has, for every

x^* in L^*, derivatives of all orders, and so $\frac{d}{d\zeta}\, x(\zeta)$ is a regular vector function. Similarly the existence of higher stronger derivatives may be shown and

$$\frac{d^2}{d\zeta^2}\, x^*\left[x(\zeta)\right] = \frac{d}{d\zeta}\, x^*\left[\frac{d}{d\zeta}\, x(\zeta)\right] = x^*\left[\frac{d^2}{d\zeta^2}\, x(\zeta)\right],$$

$$\frac{d^n}{d\zeta^n}\, x^*\left[x(\zeta)\right] = x^*\left[\frac{d^n}{d\zeta^n}\, x(\zeta)\right].$$

Also, $$\frac{d^n}{d\zeta^n}\, x^*\left[x(\zeta)\right] = \frac{n!}{2\pi i}\int_\Gamma \frac{x^*[x(\tau)]}{(\tau-\zeta)^{n+1}}\, d\tau$$

$$= x^*\left[\frac{n!}{2\pi i}\int_\Gamma \frac{x(\tau)}{(\tau-\zeta)^{n+1}}\, d\tau\right]$$

for all n and x^* in L^*,

i.e. $$\frac{d^n}{d\zeta^n}\, x(\zeta) = \frac{n!}{2\pi i}\int_\Gamma \frac{x(\tau)}{(\tau-\zeta)^{n+1}}\, d\tau .$$

If $x(\zeta)$ is regular in $|\zeta - \zeta_o| < r$, and if for any d in D, $\| x(\zeta)\|_d \leq M(d)$ in this region, then taking Γ to be the circle $|\zeta - \zeta_o| = r - \epsilon$, $\epsilon > 0$ it follows that

$$\| x^n(\zeta_o)\|_d \leq \frac{n!}{2\pi} \max_\Gamma \frac{\| x(\zeta)\|_d}{|\zeta - \zeta_o|^{n+1}}\, 2\pi(r-\epsilon) \leq \frac{n!\, M(d)}{(r-\epsilon)^n}, \text{ all}$$

$\epsilon > 0$. Hence, $\| x^n(\zeta_o)\|_d \leq n!\,\frac{M(d)}{r^n}$, for any d in D.

The identity theorem for regular vector functions is easily obtained.

<u>Theorem 2.4.5.</u> If $x(\zeta)$ and $y(\zeta)$ are regular vector functions in $\mathcal{D}$ and if $x(\zeta_n) = y(\zeta_n)$, $n = 1,2,\ldots$, the points $\{\zeta_n\}$

having a limit point in $\mathcal{D}$, then $x(\zeta)$ is identically equal to $y(\zeta)$.

Proof: Since $x^*[x(\zeta)]$ and $x^*[y(\zeta)]$ are regular in for all x^* in L^* and $x^*[x(\zeta_n)] = x^*[y(\zeta_n)]$ it follows from the identity theorem for regular functions in the classical theory that $x^*[x(\zeta)] = x^*[y(\zeta)]$ all ζ in $\mathcal{D}$ and x^* in L^*. Hence $x(\zeta) = y(\zeta)$ for all ζ in $\mathcal{D}$.

The converse of Theorem 2.4.2 holds in the following form.

Theorem 2.4.6. If $x(\zeta)$ is a strongly continuous vector function on the simply connected domain $\mathcal{D}$ to L and if

$$\int_\Gamma x(\zeta)\,d\zeta = \theta$$

for every simple closed rectifiable contour Γ in $\mathcal{D}$, then $x(\zeta)$ is regular in $\mathcal{D}$.

Proof: For every x^* in L^*,

$$0 = x^*(\theta) = x^*\left[\int_\Gamma x(\zeta)\,d\zeta\right] = \int_\Gamma x^*[x(\zeta)]\,d\zeta .$$

By Morera's Theorem $x^*[x(\zeta)]$ is a regular function of ζ for each x^* in L^* and hence $x(\zeta)$ is a regular vector function on $\mathcal{D}$.

2.5. Power Series and Taylor Expansion. In what follows a situation similar to the one commented on below will prevail. It will be possible to treat regular vector functions defined by power series by working only in L but it will also be possible to throw most results back on the corresponding results for complex-valued functions.

Definition 2.5.1. Let $\{x_n\}$ be a sequence of elements of L.

(i) If $S_n = \sum_{r=0}^{n} x_r$ converges strongly (weakly), then $\sum_{r=0}^{\infty} x_r$ is said to converge strongly (weakly).

(ii) If for every d in D, $\sum_{n=0}^{\infty} \| x_n \|_d$ converges then $\sum_{n=0}^{\infty} x_n$ is strongly absolutely convergent.

(iii) If for every x^* in L^*, $\sum_{n=0}^{\infty} | x^*(x_n) |$ converges then $\sum_{n=0}^{\infty} x_n$ is weakly absolutely convergent.

(iv) If $S_n(\zeta) = \sum_{r=0}^{n} x_r(\zeta)$, where the $x_r(\zeta)$, r = 1,2,..., are vector functions on $\mathcal{D}$, is strongly (weakly) convergent then $\sum_{r=0}^{\infty} x_r(\zeta)$ is strongly (weakly) convergent.

Since L is sequentially complete in the strong topology then the strong convergence of the partial sums $S_n = \sum_{r=0}^{n} x_r$ implies these converge strongly to a limit x which will be called the strong sum of the series $\sum_{n=0}^{\infty} x_n$. To write $x = \sum_{n=0}^{\infty} x_n$ is another way of saying this. Similarly, $\sum_{n=0}^{\infty} x_n(\zeta)$ strongly convergent will also be expressed by writing $x(\zeta) = \sum_{n=0}^{\infty} x_n(\zeta)$ where $x(\zeta)$ is the vector function defined by taking the strong sum of $\sum_{n=0}^{\infty} x_n(\zeta)$ for each ζ in $\mathcal{D}$.

(v) $\sum_{n=0}^{\infty} x_n(\zeta)$ converges strongly (weakly) uniformly for ζ in any closed subset of $\mathcal{D}$ to a function $x(\zeta)$ if its partial sums $S_n(\zeta)$ converge strongly (weakly) uniformly to $x(\zeta)$ for ζ in $\mathcal{D}$. Written in full this means:

$\sum_{n=0}^{\infty} x_n(\zeta)$ is strongly uniformly convergent if given $\epsilon > 0$, there is determined for any d in D, a number $n_o = n_o(d, \epsilon)$ such that for all $n \geq n_o(d, \epsilon)$

$$\sup_{\zeta \text{ in } \mathcal{D}} \left\| \sum_{r=0}^{\infty} x_r(\zeta) - \sum_{r=0}^{n} x_r(\zeta) \right\|_d < \epsilon .$$

$\sum_{n=0}^{\infty} x_n(\zeta)$ is weakly uniformly convergent if given $\epsilon > 0$, there is determined for any x^* in L^*, a number $n_o = n_o(x^*, \epsilon)$ such that for all $n \geq n_o(x^*, \epsilon)$

$$\sup_{\zeta \text{ in } \mathcal{D}} \left| x^*\left[\sum_{r=0}^{\infty} x_r(\zeta) - \sum_{r=0}^{n} x_r(\zeta) \right] \right| < \epsilon .$$

<u>Theorem 2.5.1.</u> If $\sum_{n=0}^{\infty} x_n$ is strongly (weakly) absolutely convergent then it is strongly (weakly) convergent.

<u>Theorem 2.5.2.</u> Let $\{x_n\}$ be a sequence of elements in L. If $\sum_{n=0}^{\infty} x_n$ is strongly (weakly) convergent then the sequence of partial sums $S_n = \sum_{n=0}^{n} x_r$ is strongly (weakly) bounded as well as the sequence $\{x_n\}$.

<u>Proof:</u> Given $\epsilon > 0$, for any d in D there exists an integer $N = N(d, \epsilon)$ such that for $m > N$

$$\left| \| S_m \|_d - \| S_N \|_d \right| \leq \| S_m - S_N \|_d < \epsilon .$$

Hence $\| S_m \|_d < \epsilon + \| S_N \|_d$ for $m > N$. Setting $M(d) = \max(\| S_1 \|_d, \ldots, \| S_{N-1} \|_d, \epsilon + \| S_N \|_d)$ it follows that $\| S_n \|_d < M(d)$, $n = 0,1,2,\ldots$, and since $x_n = S_{n+1} - S_n$ the sequence $\{x_n\}$ is also strongly bounded. The proof for weak boundedness is similar.

Definition 2.5.2. By a power series on K to L is meant an expression of the form

$$\sum_{n=0}^{\infty} (\zeta - \zeta_0)^n x_n \qquad 2.1$$

where x_n is a fixed element of L, where ζ_0 is a fixed complex number and where ζ is complex. If all but a finite number of the x_n are the zero element then 2.1 is called a polynomial, that is, $\sum_{r=0}^{n} (\zeta - \zeta_0)^r x_r$ is a polynomial (of degree n if $x_n \neq \theta$).

The first important question concerning power series is for what values of ζ does 2.1 converge strongly (weakly) and for what values of ζ does it diverge strongly (weakly)? ζ_0 will be taken equal zero in the following, but the results hold in the usual manner if $\zeta_0 \neq 0$.

Theorem 2.5.3. If $\sum_{n=0}^{\infty} \zeta^n x_n$ is any power series which does not merely converge strongly anywhere or nowhere (i.e. only for $\zeta = 0$) then a definite positive number r_s exists such that $\sum_{n=0}^{\infty} \zeta^n x_n$ converges strongly (indeed strongly absolutely) for $|\zeta| < r_s$ but not for $|\zeta| > r_s$. The number r_s is called the radius of strong convergence.

The proof is based on two lemmas.

Lemma 2.5.1. If a given power series $\sum_{n=0}^{\infty} \zeta^n x_n$ converges strongly for $\zeta = \zeta_0$ ($\zeta_0 \neq 0$), or even if the sequence

$\{\zeta_0^n x_n\}$ of its terms is only strongly bounded there, then $\sum_{n=0}^{\infty} \zeta^n x_n$ is strongly absolutely convergent for every $\zeta = \zeta_1$ such that $|\zeta_1| < |\zeta_0|$.

Proof: If $\sum_{n=0}^{\infty} \zeta_0^n x_n$ is strongly convergent then, by Theorem 2.5.2, the sequence $\{\zeta_0^n x_n\}$ is strongly bounded. For d in D there exists some positive number M(d) such that $\|\zeta_0^n x_n\|_d < M(d)$, all n. Hence

$$\|\zeta_1^n x_n\|_d = \left|\frac{\zeta_1}{\zeta_0}\right|^n \|\zeta_0^n x_n\|_d < M(d)\,\vartheta^n,$$

where $\vartheta = \left|\frac{\zeta_1}{\zeta_0}\right| < 1$. Thus for each d in D, and ζ such that $|\zeta| < |\zeta_0|$, $\sum_{n=0}^{\infty} \|\zeta^n x_n\|_d$ is a convergent series of positive real numbers and the result follows.

Lemma 2.5.2. If the given power series $\sum_{n=0}^{\infty} \zeta^n x_n$ is not strongly convergent for $\zeta = \zeta_0$ then it does not converge (strongly) for any $\zeta = \zeta_1$ with $|\zeta_1| > |\zeta_0|$.

Proof. If the series were strongly convergent for $|\zeta_1| > |\zeta_0|$ then by Lemma 2.5.1 it would have to converge for the point ζ_0 contradicting the hypothesis.

The proof of Theorem 2.5.3 then follows in the usual manner. A nest of intervals is constructed defining the number r_s mentioned in that theorem. This proof however does not supply any information as to the magnitude of this number. This is obtained in

Theorem 2.5.4. Let d belong to D and set

$$\mu_s(d) = \overline{\lim}\ \sqrt[n]{\| x_n \|_d}\ , \qquad r_s = \inf_{d \text{ in } D} \left(\frac{1}{\mu_s(d)}\right) = \frac{1}{\sup_d \mu_s(d)}\ .$$

Then the power series $\sum_{n=0}^{\infty} \zeta^n x_n$ converges strongly absolutely for every ζ such that $|\zeta| < r_s$, but does not converge strongly for any ζ such that $|\zeta| > r_s$.

Proof. If ζ_1 is any complex number for which $|\zeta_1| < r_s$, choose r_1 such that $|\zeta_1| < r_1 < r_s$. Now $\mu_s(d) \leq \frac{1}{r_s}$, all d in D. Hence given d in D, $\overline{\lim}\ \sqrt[n]{\| x_n \|_d} = \mu_s(d) \leq \frac{1}{r_s} < \frac{1}{r_1}$, and so $\sqrt[n]{\| x_n \|_d} < \frac{1}{r_1}$ for $n \geqslant n_o(d)$. Finally, $\sqrt[n]{\| \zeta_1^n\ x_n \|_d} < \frac{\zeta_1}{r_1} < 1$ for $n \geqslant n_o(d)$ and $\sum_{n=0}^{\infty} \zeta_1^n\ x_n$ is strongly absolutely convergent.

On the other hand, if $|\zeta_1| > r_s$, then $|\zeta_1| > \inf_d \left(\frac{1}{\mu_s(d)}\right)$ $= \frac{1}{\sup_d \mu_s(d)}$, that is, $\frac{1}{|\zeta_1|} < \sup_{d \text{ in } D} \mu_s(d)$. Hence there exists some e in D such that $\frac{1}{|\zeta_1|} < \mu_s(e) = \overline{\lim}\ \sqrt[n]{\| x_n \|_e}$, and consequently $\sqrt[n]{\| x_n \|_e} > \frac{1}{\zeta_1}$ for an infinity of n, or $\| \zeta_1^n\ x_n \|_e > 1$ for an infinity of n. The series $\sum_{n=0}^{\infty} \| \zeta_1^n\ x_n \|_e$ certainly cannot converge then.

The situation with regard to weak convergence is similar.

Theorem 2.5.5. If $\sum_{n=0}^{\infty} \zeta^n\ x_n$ is any power series which does not merely converge weakly anywhere or nowhere (i.e. only for $\zeta = 0$) then a definite positive number r_w exists such that $\sum_{n=0}^{\infty} \zeta^n\ x_n$ converges weakly (indeed weakly absolutely) for every $|\zeta| < r_w$ and does not converge weakly for $|\zeta| > r_w$. The number r_w is called the weak radius of convergence.

The proof follows as before. Lemmas 2.5.1 and 2.5.2 hold with strong replaced by weak. In the proofs of these theorems the pseudo-norms $\|\cdot\|_d$, d in D, are replaced by the pseudo-norms $|x^*(\cdot)|$, x^* in L^*. The radius of weak convergence is obtained in

Theorem 2.5.6. Let x^* belong to L^* and set

$$\mu_w(x^*) = \overline{\lim} \sqrt[n]{|x^*(x_n)|}, \quad r_w = \inf_{x^* \text{ in } L^*} \left(\frac{1}{\mu_w(x^*)}\right) = \frac{1}{\sup\limits_{x^* \text{ in } L^*} \mu_w(x^*)} .$$

Then the power series $\sum_{n=0}^{\infty} \zeta^n x_n$ converges weakly absolutely for every ζ such that $|\zeta| < r_w$ but does not converge (weakly) for any ζ such that $|\zeta| > r_w$.

Theorem 2.5.7. The numbers r_s and r_w are equal, that is, the radii of strong and weak convergence are the same and equal r say which will be referred to as the radius of convergence. The notions of strong and weak convergence for power series are equivalent and need not be distinguished.

Proof. It is clear that strong convergence implies weak but the converse is also true. If $\sum_{n=0}^{\infty} \zeta^n x_n$ is weakly convergent for $\zeta = \zeta_0$ then the sequence $\{\zeta_0^n x_n\}$ is weakly bounded and hence, by Theorem 1.6.8, strongly bounded. Thus for $|\zeta| < |\zeta_0|$, $\sum_{n=0}^{\infty} \zeta^n x_n$ is strongly convergent (Lemma 2.5.1).

Theorem 2.5.8. The power series $\sum_{n=0} \zeta^n x_n$ represents, for

$|\zeta| < r$, its radius of convergence, a regular vector function $x(\zeta)$, whose derivatives are obtained by differentiating the power series term by term and these derived power series have the same radius of convergence as the given series.

Proof: For $|\zeta| < r$ and each x^* in L^*, $\sum_{n=0}^{\infty} \zeta^n x^*[x_n]$ is a convergent power series with complex coefficients and hence defines a regular function of a complex variable there. Further

$$x^*[x(\zeta)] = x^*[\sum_{n=0}^{\infty} \zeta^n x_n] = x^*[\lim_{N\to\infty} \sum_{n=0}^{N} \zeta^n x_n] = \lim_{N\to\infty} x^*\left[\sum_{n=1}^{N} \zeta^n x_n\right]$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \zeta^n x^*[x_n] = \sum_{n=1}^{\infty} \zeta^n x^*[x_n] \text{ for } |\zeta| < r \text{ and each } x^* \text{ in } L^*.$$

Hence $x(\zeta)$ is a regular vector function of ζ in $|\zeta| < r$.

Examples of regular vector functions are easily obtained. For instance let $L = \mathcal{J}$ the space of integral functions (19). For each n, $\frac{z^n}{n}$ is an integral function and $\sum_{n=1}^{\infty} \zeta^n \frac{z^n}{n}$ is a power series on the complex numbers to $\mathcal{J}$. Its radius of convergence can be computed in two ways.

(i) $$\mu_s(m) = \overline{\lim_{n\to\infty}} \sqrt[n]{\left\| \frac{z^n}{n^n} \right\|_m} = \overline{\lim_{n\to\infty}} \frac{1}{n} \sqrt[n]{\| z^n \|_m}.$$

But, $\| z^n \|_m = \sup_{|z|=m} | z^n | = m^n$ and hence $\mu_s(m) = \overline{\lim_{n\to\infty}} \frac{m}{n} = 0$. Thus $r = \infty$, and $\sum_{n=1}^{\infty} \zeta^n \frac{z^n}{n}$ represents a regular vector function (with values in $\mathcal{J}$) for all ζ.

(ii) Let $x(z) = \sum_{n=0}^{\infty} a_n z^n$ be any integral function, that is, an element of $\mathcal{J}$. This is an integral function if and only if $|a_n|^{\frac{1}{n}} \to 0$

as $n \to \infty$. Every continuous linear functional x^* in $\mathcal{I}^*$ is of the form

$$x^*[x(z)] = \sum_{n=0}^{\infty} c_n a_n, \text{ where } x(z) = \sum_{n=0}^{\infty} a_n z^n. \text{ Here } |c_n|^{\frac{1}{n}}$$

must be a bounded sequence. Treating again the power series $\sum_{n=0}^{\infty} \zeta^n \frac{z^n}{n}$ on K to $\mathcal{I}$ the following results are valid.

$$\mu_w(x^*) = \overline{\lim_{n\to\infty}} \sqrt[n]{\left|x^*\left(\frac{z^n}{n}\right)\right|} = \overline{\lim} \frac{1}{n} \sqrt[n]{|x^*(z^n)|}$$

$$= \overline{\lim_{n\to\infty}} \frac{|c_n|^{\frac{1}{n}}}{n} = 0 \text{ since } \left\{|c_n|^{\frac{1}{n}}\right\} \text{ is bounded. Thus}$$

$r = \infty$ as before.

According to Theorem 2.5.8, within its radius of convergence, the power series $\sum_{n=0}^{\infty} \zeta^n x_n$ represents a regular vector function of the complex variable ζ. The converse statement is contained in the following theorem.

Theorem 2.5.9. Let $x(\zeta)$ be a regular vector function in the domain $\mathcal{D}$ and let ζ_0 be an interior point of $\mathcal{D}$. Then there is one and only one power series of the form $\sum_{n=0}^{\infty} x_n(\zeta - \zeta_0)^n$ which converges for a certain neighbourhood of ζ_0 and represents the function $x(\zeta)$ in that neighbourhood. Moreover, $x_n = \frac{x^n(\zeta_0)}{n!}$. The series converges at least in the largest circle $\mathcal{C}$ about ζ_0 which contains only points of $\mathcal{D}$.

Proof: For each x^* in L^*, $x^*[x(\zeta)]$ is a regular function in $\mathcal{D}$ and so can be expanded in a Taylor series about the point ζ_0 converging (absolutely) at least in $\mathcal{C}$.

$$x^*[x(\zeta)] = \sum_{n=0}^{\infty} \frac{d^n}{d\zeta^n} x^*[x(\zeta_0)] \frac{(\zeta - \zeta_0)^n}{n!} = \sum_{n=0}^{\infty} x^*\left[\frac{x^{(n)}(\zeta_0)}{n!}\right] (\zeta - \zeta_0)^n.$$

The absolute convergence of this series in $\mathcal{L}$ implies the convergence of $\sum_{n=0}^{\infty} \frac{x^{(n)}(\zeta_0)}{n!}(\zeta - \zeta_0)^n$ in $\mathcal{L}$ with sum $x(\zeta)$. The uniqueness follows from Theorem 2.4.5.

The proof of the following theorem on uniformly convergent sequences of regular vector functions again illustrates the technique of throwing proofs back on the classical case.

Theorem 2.5.10. Let $x_n(\zeta)$ be a sequence of regular functions on $\mathcal{D}$ to L which converge strongly uniformly with respect to ζ on a simple closed rectifiable curve Γ, the interior of which, $\mathcal{D}_0$ say, is also in $\mathcal{D}$. Then $x_n(\zeta)$ converges to an analytic function $x(\zeta)$ in $\mathcal{D}_0$, and moreover, $x_n^k(\zeta) \to x^k(\zeta)$ in $\mathcal{D}_0$ for every k. The convergence is uniform with respect to in any fixed closed domain interior to $\mathcal{D}_0$.

Proof: $x_n(\zeta) \to x(\zeta)$ in the strong topology implies $x_n(\zeta) \to x(\zeta)$ in the weak topology and so $\lim_{n \to \infty} x^*[x_n(\zeta)] = x^*[x(\zeta)]$. The sequence $\{x^*[x_n(\zeta)]\}$ being uniformly convergent $x^*[x(\zeta)]$ is regular for every x^* in L^* and so $x(\zeta)$ is a regular vector function. Similarly the rest of the theorem follows.

The principle of the maximum can also be extended to vector valued functions.

Theorem 2.5.11. Let $x(\zeta)$ be a vector function defined on a bounded domain $\mathcal{D}$ and on its boundary $\mathcal{B}$, regular in $\mathcal{D}$ and strongly continuous in $\mathcal{D} \cup \mathcal{B}$. If for any d in D, $\sup \| x(\zeta) \|_d = M(d)$

for ζ on $\mathcal{B}$ then either $\|x(\zeta)\|_d = M(d)$ for all ζ in $\mathcal{D} \cup \mathcal{B}$ or $\|x(\zeta)\|_d < M(d)$ in $\mathcal{D}$.

Proof. The proof of this theorem needs the result that if f(a) is a continuous real-valued function of a real variable, $f(a) \leq k$, and $\frac{1}{b_o - a_o} \int_{a_o}^{b_o} f(a)da \geqslant k$, then $f(a) = k$(20).

Suppose now that at an interior point ζ_o of $\mathcal{D}$, $\|x(\zeta_o)\|_d \geqslant \|x(\zeta)\|_d$, ζ being any other point in $\mathcal{D} \cup \mathcal{B}$. Let Γ be a circle which lies entirely in $\mathcal{D}$ and which has its centre at ζ_o and radius r. Then

$$x(\zeta_o) = \frac{1}{2\pi i} \int_\Gamma \frac{x(\tau)}{\tau - \zeta_o} d\tau = \frac{1}{2\pi i} \int_0^{2\pi} \frac{x(\zeta_o + re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta.$$

Hence, $\|x(\zeta_o)\|_d \leq \frac{1}{2\pi} \int_0^{2\pi} \|x(\zeta_o + re^{i\theta})\|_d \, d\theta$

or, $$1 \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\|x(\zeta_o + re^{i\theta})\|_d}{\|x(\zeta_o)\|_d} d\theta, \text{ if } \|x(\zeta_o)\|_d \neq 0.$$

But by hypothesis $f(\theta) = \dfrac{\|x(\zeta_o + re^{i\theta})\|_d}{\|x(\zeta_o)\|_d} \leq 1$ and applying the above result $\|x(\zeta_o + re^{i\theta})\|_d = \|x(\zeta_o)\|_d$, that is, $\|x(\zeta)\|_d = \|x(\zeta_o)\|_d$ on Γ and hence everywhere. If $\|x(\zeta_o)\|_d = 0$ then by hypothesis $\|x(\zeta)\|_d = 0$ for ζ in $\mathcal{D} \cup \mathcal{B}$ and hence it is trivially true that $\|x(\zeta)\|_d = \|x(\zeta_o)\|_d$ everywhere.

2.6 Laurent's Expansion and Singularities. So far functions have been examined exclusively from the point of view of their regularity.

Now the influence of points at which the functions are not regular will be examined.

Theorem 2.6.1. If $x(\zeta)$ is a regular vector function in $0 \leq R_1 < |\zeta - \zeta_0| < R_2 \leq \infty$, then, for such ζ,

$$x(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - \zeta_0)^n + \sum_{n=1}^{\infty} a_{-n}(\zeta - \zeta_0)^{-n} \qquad 2.2$$

or more briefly,

$$x(\zeta) = \sum_{-\infty}^{\infty} a_n(\zeta - \zeta_0)^n$$

where, $$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{x(\tau)}{(\tau - \zeta_0)^{n+1}} \, d\tau ,$$

and Γ is, for instance, the circle $|\zeta - \zeta_0| = r$, $R_1 < r < R_2$.

Proof: For any x^* in L^*, $x^*[x(\zeta)]$ is regular in $R_1 < |\zeta - \zeta_0| < R_2$. Consequently in this annulus

$$x^*[x(\zeta)] = \sum_{n=0}^{\infty} b_n(\zeta - \zeta_0)^n + \sum_{n=1}^{\infty} b_{-n}(\zeta - \zeta_0)^{-n} ,$$

with $$b_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{x^*[x(\tau)]}{(\tau - \zeta_0)^{n+1}} \, d\tau = x^*(a_n), \quad n = 0,1,2,\ldots$$

and $$b_{-n} = \frac{1}{2\pi i} \int_{\Gamma} \frac{x^*[x(\tau)]}{(\tau - \zeta_0)^{-n+1}} \, d\tau = x^*(a_{-n}), \quad n = 1,2,\ldots$$

$x^*[x(\zeta)]$ is then the sum of a numerical power series of ascending powers of $\zeta - \zeta_0$ and a numerical power series of descending powers of $\zeta - \zeta_0$. The convergence of both these numerical power series in the given annulus implies the convergence, in the annulus, of both vector power series in 2.2, to regular vector functions $x_1(\zeta)$ and

$x_2(\zeta)$ respectively. Hence, $x^*[x(\zeta)] = x^*[x_1(\zeta)] + x^*[x_2(\zeta)] = x^*[x_1(\zeta) + x_2(\zeta)]$ for all x^* in L^* and all ζ such that $R_1 < |\zeta - \zeta_0| < R_2$. Thus $x(\zeta) = x_1(\zeta) + x_2(\zeta)$.

Definition 2.6.1. If $R_1 = 0$ and there is actually at least one $a_n \neq \theta$ with a negative subscript in (2), then $\zeta = \zeta_0$ is a singular point of $x(\zeta)$, namely, a pole of order m if $a_{-m} \neq \theta$ but $a_n = \theta$ for $n < -m$, otherwise an (isolated) essential singularity.

The behaviour of a vector function near a pole is contained in

Theorem 2.6.2. If the vector function $x(\zeta)$ is regular in $0 < |\zeta - \zeta_0| < r$ but has a pole of order m at $\zeta = \zeta_0$ then $x(\zeta)$ is not topologically bounded as $\zeta \to \zeta_0$.

Proof:
$$x(\zeta) = \frac{a_{-m}}{(\zeta - \zeta_0)^m} + \dots + \frac{a_{-1}}{(\zeta - \zeta_0)} + a_0 + a_1(\zeta - \zeta_0) + \dots$$
$$= \frac{1}{(\zeta - \zeta_0)^m}\left\{a_{-m} + a_{-m+1}(\zeta - \zeta_0) + \dots\right\} = \frac{1}{(\zeta - \zeta_0)^m} y(\zeta),$$
where $y(\zeta)$ is a regular vector function in a neighbourhood of $\zeta = \zeta_0$ and $y(\zeta_0) = a_{-m} \neq \theta$. $y(\zeta)$ is continuous at $\zeta = \zeta_0$ so that
$$\| y(\zeta)\|_d = \| y(\zeta_0) + y(\zeta) - y(\zeta_0)\|_d$$
$$\geqslant \| y(\zeta_0)\|_d - \| y(\zeta) - y(\zeta_0)\|_d$$
$$\geqslant \| a_{-m}\|_d - \frac{\|a_{-m}\|_d}{2} = \frac{\|a_{-m}\|_d}{2}$$
for any d in D and for ζ sufficiently close to ζ_0. Since $a_{-m} \neq \theta$ there is some e in D such that $\|a_{-m}\|_e \neq 0$. Then
$$\| x(\zeta)\|_e = \frac{1}{|\zeta - \zeta_0|^m} \| y(\zeta)\|_e \geqslant \frac{\|a_{-m}\|_e}{2} \cdot \frac{1}{|\zeta - \zeta_0|^m},$$

so that as $\zeta \to \zeta_o$, $\|x(\zeta)\|_e$ becomes infinite and $x(\zeta)$ is not topologically bounded.

Theorem 2.6.3. If $x(\zeta)$ is a regular vector function in $0 < |\zeta - \zeta_o| < r$ and if $(\zeta - \zeta_o)^m x(\zeta)$ is topologically bounded as $|\zeta - \zeta_o| \to 0$ then $\zeta = \zeta_o$ is a pole of order m.

Proof: $x(\zeta)$ can be expanded as a Laurent Series in $0 < |\zeta - \zeta_o| < r$:

$$x(\zeta) = \sum_{n=-\infty}^{\infty} a_n(\zeta - \zeta_o)^n, \quad a_n = \frac{1}{2\pi i}\int_\Gamma \frac{x(\tau)}{(\tau - \zeta_o)^{-n+1}}\, d\tau .$$

For any d in D, setting $\tau - \zeta_o = re^{i\theta}$ on Γ, it follows that

$$\|a_{-n}\|_d \leq \frac{1}{2\pi} \sup_\Gamma \frac{\|x(\tau)\|_d}{r^{-n+1}} 2\pi r \leq \frac{M(d)}{r^{m-n}} = M(d)r^{n-m}$$

for r sufficiently small using the hypothesis of the theorem. If $n - m > 0$, $\|a_{-n}\|_d \to 0$ as $r \to 0$, any d in D and so for $n > m$, $a_{-n} = \theta$.

Definition 2.6.2. A function which is regular in the entire ζ - plane is said to be an integral function.

Theorem 2.6.4. An entire function $x(\zeta)$ which is topologically bounded is a constant function (i.e. only takes on one fixed value in L).

Proof. Since $x(\zeta)$ is topologically bounded then for any d in D there exists $M(d) > 0$ such that $\|x(\zeta)\|_d \leq M(d)$ for all finite ζ. $x(\zeta)$ can be expanded in a Taylor series about $\zeta = 0$ with infinite radius of convergence (Theorem 2.5.9),

$$x(\zeta) = \sum_{n=0}^{\infty} \frac{x^{(n)}(0)}{n!} \zeta^n.$$

The estimates of Theorem 2.4.4 give for any d in D

$$\| x^{(n)}(0) \|_d \leq n! \frac{M(d)}{r^n} .$$

Letting $r \to \infty$ makes the right-hand side tend to zero if $n > 0$. Hence for any d in D, $\| x^{(n)}(0) \|_d = 0$, i.e., $x^{(n)} = \theta$ for $n > 0$ and so $x(\zeta) = x(0) = x_0$ say.

The following is a more general result of the same kind.

Theorem 2.6.5. If $x(\zeta)$ is an entire function such that $\frac{x(\zeta)}{\zeta^k}$ is topologically bounded for $|\zeta| > 0$, k being fixed and non-negative, then $x(\zeta)$ is a polynomial of degree at most k. It is a constant function if $k < 1$.

The "point at infinity" in the complex plane may be introduced in the usual way. For the behaviour of $x(\zeta)$ at $\zeta = \infty$ the behaviour of $x(\frac{1}{\zeta})$ at $\zeta = 0$ will be consulted. To be precise let $x(\zeta)$ be single-valued and regular for $|\zeta| > R$. Setting $\zeta = \frac{1}{\zeta'}$, then the function $y(\zeta')$ defined for $|\zeta'| < \frac{1}{R}$ by $x(\zeta) = x(\frac{1}{\zeta'}) = y(\zeta')$ is single-valued and regular there with the possible exception (as to its regularity) of the point $\zeta' = 0$ itself.

Definition 2.6.3. That behaviour is assigned to the function $x(\zeta)$ at infinity which $y(\zeta')$ exhibits at $\zeta' = 0$.

It follows directly from the definition that an integral vector function having a pole of order m at infinity is a polynomial of degree m. Conversely a polynomial of degree $m > 0$ has as its only singularity a pole at infinity of order m.

As an example of this theory take $L = \mathcal{I}$ the space of integral functions (19) and, let $x(\zeta) = e^{\zeta z}$. This is a vector function on the ζ-plane to $\mathcal{I}$ which is regular for all finite ζ and $x'(\zeta) = ze^{\zeta z}$. The Taylor expansion of $x(\zeta)$ is $1 + \zeta\frac{z}{1!} + \zeta^2\frac{z^2}{2!} + \ldots$. Thus $x(\zeta)$ has an isolated essential singularity at $\zeta = \infty$. Now $\|x(\zeta)\|_m = \sup_{|z|=m} |e^{\zeta z}| =$

$$\sup_{x^2+y^2=m^2} e^{xR(\zeta)-yI(\zeta)}$$

where m is a positive integer, $z = x + iy$ and $R(\zeta)$, $I(\zeta)$ are the real and imaginary parts of ζ respectively. If $\zeta \neq 0$, setting $x = m\frac{R(\zeta)}{|\zeta|}$, $y = -m\frac{I(\zeta)}{|\zeta|}$ it follows that $\|x(\zeta)\|_m \geq e^{m|\zeta|} \geq 1$, an inequality which remains true when $\zeta = 0$. $x(\zeta)$ fails to assume values whose pseudo-norms are all definitely less than one and so Weierstrass' theorem on essential singularities is not true in this theory.

$x(\zeta) = e^{\frac{z}{\zeta}}$ on K to $\mathcal{I}$ is regular for all $\zeta \neq 0$ with derivative $x'(\zeta) = \frac{-z}{\zeta^2}e^{z/\zeta}$. Its Laurent expansion is $x(\zeta) = 1 + \frac{1}{\zeta}z + \frac{1}{\zeta^2}\frac{z^2}{2!} + \ldots$ and so $x(\zeta)$ has an isolated essential singularity at $\zeta = 0$ while $\|x(\zeta)\|_m \geq e^{\frac{m}{|\zeta|}} \geq 1$ showing once more that Weierstrass theorem on essential singularities is not true in this theory.

The reason for this is that in general a locally convex l.t.s. is a much larger space than the complex numbers. For instance, $\mathcal{I}$ may be thought of as a sequence space $(\zeta_1, \zeta_2, \ldots \zeta_n, \ldots)$ with

ζ_i complex and $|\zeta_n|^{\frac{1}{n}} \to 0$ as $n \to \infty$ while the complex numbers form a one-dimensional linear sub-space of $\mathcal{A}$, namely all elements of the form $(a_o,0,\ldots,0,\ldots)$.

§ 2. Operator Functions of a Complex Variable

2.7. Regular Operator Functions. Let L be a sequentially complete complex $\mathcal{F}$-space, L', a sequentially complete locally convex complex l.t.s., and $\mathcal{T}(L,L')$ the linear system of continuous linear functions on L into L'. Using the notation of paragraph 1.6, only the $\mathcal{T}_b$, $\mathcal{T}_s$ and $\mathcal{T}_w$ topologies will be considered. By Theorem 1.6.10 $\mathcal{T}(L,L')$ is sequentially complete in the $\mathcal{T}_b$ topology.

Definition 2.7.1. Let $\mathcal{D}$ be a domain in the complex plane. $U(\zeta)$ defined on $\mathcal{D}$ with values in $\mathcal{T}(L,L')$ is called an operator function of ζ.

There are three notions of continuity of operator functions available each corresponding to one of the topologies $\mathcal{T}_b$, $\mathcal{T}_s$ or $\mathcal{T}_w$. Similarly there are three notions of differentiability for operator functions. These definitions will not be stated explicitly since it is clear what they must be.

In the case of an operator function U(a) on $[a_o,b_o]$ to $\mathcal{T}(L,L')$ the integral is obtained by considering sums of the form $U_n = \sum_{i=1}^{n} U(c_i)[f(a_i) - f(a_{i-1})]$ where f(a) is a numerically valued function of bounded variation in $[a_o,b_o]$. If these sums converge in the $\mathcal{T}_b(\mathcal{T}_s, \mathcal{T}_w)$ sense to an element U in $\mathcal{T}(L,L')$, then U is called the $\mathcal{T}_b(\mathcal{T}_s, \mathcal{T}_w)$ integral of U(a) and written

$$U = \int_{a_0}^{b_0} U(a)df(a).$$

<u>Theorem 2.7.1.</u> If U(a) is $\mathcal{T}_b$ continuous then the sums U_n converge in the $\mathcal{T}_b$ sense and so U exists.

$\mathcal{T}(L,L')$ is not necessarily complete in the $\mathcal{T}_s$ and $\mathcal{T}_w$ topologies and so the existence of the $\mathcal{T}_s$ and $\mathcal{T}_w$ integrals is not ensured when U(a) is $\mathcal{T}_s$ or $\mathcal{T}_w$ continuous. However, in the case when U(a) is regular, the three notions of integration coincide.

<u>Definition 2.7.2.</u> $U(\zeta)$ is said to be regular in the domain of the complex plane if $y^*[U(\zeta)(x)]$ is regular in the classical sense for every choice of x in L, y^* in $(L')^*$.

<u>Theorem 2.7.2.</u> If $U(\zeta)$ is regular in $\mathcal{D}$, then $U(\zeta)$ is $\mathcal{T}_b$ continuous and $\mathcal{T}_b$ differentiable (i.e. with respect to the $\mathcal{T}_b$ topology on $\mathcal{T}(L,L')$) in $\mathcal{D}$ uniformly with respect to ζ in any domain $\mathcal{D}_0$ which is bounded and strictly interior to $\mathcal{D}$.

<u>Proof.</u> The proof proceeds along lines similar to that of Theorem 2.4.1. It requires Lemma 2.4.1 applied to the function $y^*[U(\zeta)(x)]$ where x and y^* are arbitrary elements of L and $(L')^*$ respectively, and also Theorem 1.6.11.

Let Γ be a rectifiable curve in the complex plane given by the equation $\zeta = \zeta(a)$, $0 \leq a \leq a_0$ where $\zeta(a)$ is continuous and of bounded variation in $[0, a_0]$. If $U(\zeta)$ is a $\mathcal{T}_b$ continuous operator function on Γ then $\int_0^{a_0} U[\zeta(a)]\, d\zeta(a) \equiv \int_\Gamma U(\zeta)\,d\zeta$ exists. Since $\int_\Gamma U(\zeta)\,d\zeta$ defines a linear transformation then $\left[\int_\Gamma U(\zeta)\,d\zeta\right](x)$ has meaning and

$$\left[\int_\Gamma U(\zeta)\,d\zeta\right](x) = \int_\Gamma \left[U(\zeta)(x)\right] d\zeta \quad .$$

In particular if $U(\zeta)$ takes its values in $\mathcal{T}(L,L)$, where L is a sequentially complete $\mathcal{F}$-space, then, if T is in $\mathcal{T}(L,L')$ and $U(\zeta)$ is $\mathcal{T}_b$ continuous in $\mathcal{T}(L,L)$,

$$T\left[\int_\Gamma U(\zeta)(x)\,d\zeta\right] = \int_\Gamma T\left[U(\zeta)(x)\right] d\zeta \quad .$$

Theorem 2.7.3. If $U(\zeta)$ is a regular operator function on $\mathcal{D}$ to $\mathcal{T}(L,L)$ then

$$\int_\Gamma U(\zeta)\,d\zeta = \theta$$

for every simple closed rectifiable curve Γ in $\mathcal{D}$ and such that the interior of Γ belongs to $\mathcal{D}$.

Proof. Take L' as the space of complex numbers and $T = x^*$, an arbitrary element of L^*. Then

$$0 = \int_\Gamma x^*[U(\zeta)(x)]\, d\zeta = x^*\left[\int_\Gamma U(\zeta)(x)\,d\zeta\right]$$

for every x^*. Hence by the corollary to Theorem 1.6.7

$$\theta = \int_\Gamma U(\zeta)(x)\,d\zeta = \left[\int_\Gamma U(\zeta)\,d\zeta\right](x)$$

for every x, that is, $\int_\Gamma U(\zeta)\,d\zeta$ must be the zero element of $\mathcal{T}(L,L)$.

From here the development of the theory for function on the complex plane to $\mathcal{T}(L,L)$ proceeds as in paragraphs 2.4, 2.5 and 2.6.

§ 3. Vector Functions of Several Complex Variables

2.8. Vector Functions of Several Complex Variables. Much of the theory developed in § 1 of this chapter may be extended to the case of functions of several complex variables taking their values in a sequentially complete locally convex complex l.t.s. Let Z^n be the linear space of elements $Z = (\zeta_1, \ldots, \zeta_n)$ where $\zeta_1, \ldots, \zeta_n$ are complex numbers and addition and scalar multiplication are defined by the usual methods.

$$Z_1 + Z_2 = (\zeta_{11}, \ldots, \zeta_{n1}) + (\zeta_{12}, \ldots, \zeta_{n2}) = (\zeta_{11} + \zeta_{12}, \ldots, \zeta_{n1} + \zeta_{n2}).$$

$$\zeta Z = \zeta(\zeta_1, \zeta_2, \ldots, \zeta_n) = (\zeta\zeta_1, \zeta\zeta_2, \ldots, \zeta\zeta_n).$$

Z^n can be made into a sequentially complete locally convex complex l.t.s. (indeed a Banach space) by defining $\|Z\| = \sqrt{\sum_{i=1}^{n} |\zeta_i|^2}$.

Let $x(Z) = x(\zeta_1, \ldots, \zeta_n)$ be a function defined on some domain, that is an open connected set, $\mathcal{D}$, of Z^n with values in a sequentially complete locally convex l.t.s. L. The (strong) partial derivative of $x(\zeta_1, \ldots, \zeta_n)$ with respect to ζ_k is defined by

$$\frac{\partial x(Z)}{\partial \zeta_k} = \lim_{\alpha_k \to 0} \frac{1}{\alpha_k} \left\{ x(\zeta_1, \ldots, \zeta_k + \alpha_k, \ldots, \zeta_n) - x(\zeta_1, \ldots, \zeta_n) \right\},$$

the limit, which is taken in the sense of strong convergence in L, being independent of the manner in which the complex numbers ζ_k approach zero.

Theorem 2.8.1. Let $x(\zeta_1, \ldots, \zeta_n)$ on Z^n to L have first order partial derivatives in each ζ_k where $Z = (\zeta_1, \ldots, \zeta_n)$ lies in some domain $\mathcal{D}$ containing the origin. Then,

(i) $x(\zeta_1, \ldots, \zeta_n)$ has partial derivatives of all orders and the mixed partials are independent of the order of differentiation,

(ii) $x(\zeta_1, \ldots, \zeta_n)$ is continuous and topologically bounded on every closed bounded subset of $\mathcal{D}$,

(iii) if for $\|Z\| \leq r$, $\|x(Z)\|_d \leq M(d)$ for each d in D, and if $s < r$, then for $\|Z\| \leq s$

$$\left\| x(Z) - x(\theta) - \sum_{k=1}^{n} \left(\frac{\partial x}{\partial \zeta_k}\right)_\theta \zeta_k \right\|_d \leq \frac{M(d)}{r(r-s)} \|Z\|^2,$$

(iv) $x(\zeta, \zeta, \ldots, \zeta)$ is differentiable with respect to ζ.

Proof. The theorem is assumed known for the special case in which L is the complex plane and the proof of the theorem consists in reducing the general case to this particular case with the aid of the continuous linear functionals x^* in L^*.

(i) The numerical function $x^*[x(\zeta_1, \ldots, \zeta_n)]$ is partially differentiable in each ζ_k since strong differentiability implies the weak kind. Hence $\frac{\partial}{\partial \zeta_j} x^*[x(\zeta_1, \ldots, \zeta_n)] = x^*\left[\frac{\partial}{\partial \zeta_j} x(\zeta_1, \ldots, \zeta_n)\right]$. This implies $\frac{\partial}{\partial \zeta_j} x^*[x(\zeta_1, \ldots, \zeta_n)]$ is partially differentiable (for each fixed j) with respect to ζ_k, $k = 1, 2, \ldots, n$, for every x^* in L^* and $\frac{\partial}{\partial \zeta_k} \frac{\partial}{\partial \zeta_j} x^*[x(\zeta_1, \ldots, \zeta_n)] = \frac{\partial}{\partial \zeta_k} x^*\left[\frac{\partial x}{\partial \zeta_j}(\zeta_1, \ldots, \zeta_n)\right]$ for every x^* in L^*. The right hand side of this last equality says that

$\frac{\partial x}{\partial \zeta_j}(\zeta_1,\ldots,\zeta_n)$ is weakly differentiable (for each fixed j) with respect to each variable and hence strongly differentiable i.e. $\frac{\partial}{\partial \zeta_k}\frac{\partial}{\partial \zeta_j} x(\zeta_1,\ldots,\zeta_n)$ exists for $k = 1,2,\ldots,n$. The existence of the higher order derivatives can now be handled by an induction argument. The relation

$$x^*\left[\frac{\partial^2 x}{\partial \zeta_j \partial \zeta_k}\right] = \frac{\partial^2}{\partial \zeta_j \partial \zeta_k} x^*[x] = \frac{\partial^2}{\partial \zeta_k \partial \zeta_j} x^*[x] = x^*\left[\frac{\partial^2 x}{\partial \zeta_k \partial \zeta_j}\right]$$

which is valid for every x^* in L^* shows that the order of differentiation is immaterial.

(ii) It will now be shown that x(Z) is topologically bounded in $\mathcal{D}_o$, an arbitrary closed bound subset of $\mathcal{D}$, using the known result when the values of the function are in the complex plane. For every fixed x^* in L^*, $x^*[x(Z)]$ is bounded, Z in $\mathcal{D}_o$, i.e., x(Z) is weakly bounded and hence strongly bounded in $\mathcal{D}_o$.

It is now sufficient to show (iii) and this will imply the second part of (ii), namely $x(\zeta)$ is continuous, and also (iv).

(iii) Consider $r > 0$ such that $\|Z\| \leq r$ implies Z is in $\mathcal{D}$. This set may therefore serve as a $\mathcal{D}_o$. Now by (ii) for each d in D there exists $M(d) > 0$ such that $\|x(Z)\|_d \leq M(d)$ for Z in $\mathcal{D}_o$. Corresponding to each x^* in L^* there is a d in D and $\mu(d) > 0$ so that $|x^*[x(Z)]| \leq \mu(d) \|x(Z)\|_d$. Hence $|x^*[x(Z)]| \leq \mu(d)M(d)$ in $\mathcal{D}_o$ and by the numerical case

$$\left| x^*\left[x(Z)\right] - x^*\left[x(\theta)\right] - \sum_{k=1}^{n}\left\{\frac{\partial}{\partial \zeta_k} x^*[x]\right\}\zeta_k\right|$$

$$= \left| x^*\left[x(Z) - x(\theta) - \sum_{k=1}^{n} \left(\frac{\partial x}{\partial \zeta_k}\right)_o \zeta_k\right]\right| \leq \mu(d)\,\frac{M(d)}{r(r-s)}\,\|Z\|^2$$

for $\|Z\| \leq s \leq r$. Applying Theorem 1.6.7 this gives

$$\left\| x(Z) - x(\theta) - \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \zeta_k}\right)_o \zeta_k \right\|_d \leq \frac{M(d)}{r(r-s)}\,\|Z\|^2 .$$

Chapter III

Functions on Vectors to Vectors

3.1. Introduction. In this chapter, functions on one sequentially complete locally convex complex l.t.s. to a second such space will be studied and in particular functions which are analytic in a sense to be specified later. The necessary concepts of differentiability are introduced in § 1 and the properties of differentiable functions are studied there. The properties of analytic functions are treated in § 2 while in § 3 the analogue of the Cauchy-Riemann theory of functions of a complex variable is discussed.

§ 1 Differentiable Functions

3.2. Gateaux Differentiability. In this section, unless otherwise stated explicitly, L and L' will always denote sequentially complete locally convex complex l.t.s. and D and D' the corresponding associated directed systems.

Definition 3.2.1. Let $f(x)$ be a function defined on some open set Ω in L and taking its values in L'. Suppose that for x_0 in Ω and each y in L,

$$\delta f(x_0;y) = \lim_{\zeta \to 0} \frac{f(x_0 + \zeta y) - f(x_0)}{\zeta}$$

exists, where ζ is complex. Then

(i) $f(x)$ is said to be G-differentiable at $x = x_0$ and $\delta^1 f(x_0;y) = \delta f(x_0;y)$ is called the G-differential of $f(x)$ at $x = x_0$

with increment y,

(ii) f(x) is said to be G-differentiable in Ω if it is G-differentiable at each point of Ω,

(iii) the G-differential $\delta^{n+1} f(x;y_1;y_2;\dots;y_{n+1})$ of $\delta^n f(x;y_1;\dots;y_n)$ is defined by

$$\delta^{n+1} f(x;y_1;y_2;\dots;y_{n+1}) = \delta\,[\delta^n f(x;y_1;y_2;\dots;y_n)]$$

$$\delta^{0} f(x;y_1;y_2;\dots;y_n) = f(x), \text{ and}$$

(iv) $\delta^n f(x;y) = \delta^n f(x;y;y;\dots;y)$.

In much of this theory the domain of the functions considered is immaterial while their range can be extended to linear topological spaces. Thus almost all the theorems of the Banach space case are valid (2,21). The notion of G-differential is closely related to the notion of derivative of a vector function given in Definition 2.2.3.

Theorem 3.2.1. f(x) defined on the open set Ω in L to L' is G-differentiable in Ω if and only if for every x in Ω and y in L' $f(x + \zeta y)$ is a regular function of ζ whenever $x + \zeta y$ is in Ω.

Proof.
$$\frac{d}{d\zeta} f(x + \zeta y) = \lim_{\tau \to 0} \frac{1}{\tau} \left[f(x + (\zeta + \tau)y - f(x + \zeta y) \right]$$
$$= \lim_{\tau \to 0} \frac{1}{\tau} \left[f((x + \zeta y) + \tau y) - f(x + \zeta y) \right]$$
$$= \delta f(x + \zeta y;y)$$

In particular,

$$\left[\frac{d}{d\zeta} f(x + \zeta y)\right]_{\zeta = 0} = \delta f(x;y).$$

Theorem 3.2.2. f(x) is G-differentiable in the open set Ω if and only if for every x in Ω and $y_1, y_2, \ldots, y_n$ in L the functions $f(x + \zeta_1 y_1 + \ldots + \zeta_n y_n)$ is partially differentiable with respect to the ζ_k, $k = 1,2,\ldots,n$, whenever $x + \zeta_1 y_1 + \ldots + \zeta_n y_n$ is in Ω.

Proof. $\frac{\partial}{\partial \zeta_k} f(x + \zeta_1 y_1 + \ldots + \zeta_k y_k + \ldots + \zeta_n y_n)$

$$= \lim_{\tau_k \to 0} \frac{1}{\tau_k} \left[f(x + \zeta_1 y_1 + \ldots + (\zeta_k + \tau_k) y_k + \ldots + \zeta_n y_n) - f(x + \zeta_1 y_1 + \ldots + \zeta_k y_k + \ldots + \zeta_n y_n) \right]$$

$$= \lim_{\tau_k \to 0} \frac{1}{\tau_k} \left[f(x + \zeta_1 y_1 + \ldots \zeta_k y_k + \ldots + \zeta_n y_n) + \tau_k y_k) - f(x + \zeta_1 y_1 + \ldots + \zeta_k y_k + \ldots + \zeta_n y_n) \right]$$

$$= \delta f(x + \zeta_1 y_1 + \ldots + \zeta_k y_k + \ldots + \zeta_n y_n; y_k).$$

In particular,

$$\left[\frac{\partial}{\partial \zeta_i} f(x_1 + \zeta_1 y_1 + \zeta_2 y_2)\right]_{0,0} = \delta f(x;y_i),\ i = 1,2.$$

Theorem 3.2.3. Let f(x) be defined and G-differentiable on the open set Ω in L with values in L'. Then, for every y in L, $\delta f(x;y)$ is a G-differentiable function of x in Ω.

Proof. $\delta f(x;y_1) = \left[\frac{d}{d\zeta_1} f(x + \zeta_1 y_1)\right]_{\zeta_1 = 0}$.

Let $\emptyset(x) = \delta f(x;y_1)$. Then,

$$\delta\,\emptyset(x;y_2) = \left[\frac{d}{d\zeta_2}\,\emptyset(x + \zeta_2 y_2)\right]_{\zeta_2 = 0},$$

and hence,

$$\delta^2 f(x;y_1;y_2) = \left[\frac{\partial^2}{\partial\zeta_2\,\partial\zeta_1}\,f(x + \zeta_1 y_1 + \zeta_2 y_2)\right]_{\zeta_1 = \zeta_2 = 0}.$$

By Theorem 3.2.2 $f(x + \zeta_1 y_1 + \zeta_2 y_2)$ is a partially differentiable function with respect to ζ_1 and ζ_2 and Theorem 2.8.1 ensures the existence of higher derivatives.

Theorem 3.2.4. Let $f(x)$ be defined and G-differentiable on the open set Ω in L to L'. Then

(i) $$\left[\frac{\partial^n}{\partial\zeta_1\ldots\partial\zeta_n}\,f(x + \sum_{k=1}^{n}\zeta_k y_k)\right]_{\zeta_k=0} = \delta^n f(x;y_1;\ldots;y_n),$$

(ii) $$\left[\frac{\partial^n}{\partial\zeta_1\ldots\partial\zeta_n}\,f(x + \sum_{k=1}^{n}\zeta_k y_k)\right]_{y_k=y,\ \zeta_k=0} = \left[\frac{d^n}{d\zeta^n}\,f(x + \zeta y)\right]_{\zeta=0}$$

$$= \delta^n f(x;y).$$

Proof. The result (i) will be proved by induction. For $n = 1$ it is true by Theorem 3.2.1 and for $n = 2$ by Theorem 3.2.3. Suppose now that (i) is true for all values of n up to $n - 1$. Set $\delta^{n-1} f(x;y_1;\ldots;y_{n-1}) = \emptyset(x)$. Then,

$$\delta\,\emptyset(x;y_n) = \left[\frac{d}{d\zeta_n}\,\emptyset(x + \zeta_n y_n)\right]_{\zeta_n=0}$$

that is,

$$\delta^n f(x;y_1;y_2;\ldots;y_n) =$$

$$=\left\{\frac{\partial}{\partial \zeta_n}\left[\frac{\partial^{n-1}}{\partial \zeta_1 \ldots \partial \zeta_{n-1}} f((x+\zeta_n y_n)+\zeta_1 y_1+\ldots+\zeta_{n-1}y_{n-1})\right]_{0,\ldots,0}\right\}_0$$

$$=\left[\frac{\partial^n}{\partial \zeta_1 \ldots \partial \zeta_n} f(x+\zeta_1 y_1+\ldots+\zeta_n y_n)\right]_{0,0,\ldots,0} .$$

(ii) is an immediate consequence of (i) and Theorem 3.2.1.

Theorem 3.2.5. Let $f(x)$ be defined on the open set Ω in L to L'. Then $\delta f(x;y)$ is a linear function on L into L'.

Proof. It is clear that $\delta f(x;\zeta y) = \zeta \delta f(x;y)$. Consider now the function $f(x+\zeta_1 y_1+\zeta_2 y_2)$ of (ζ_1, ζ_2). By Theorem 2.8.1, for any d' in D'

$$\left\| f(x+\zeta_1 y_1+\zeta_2 y_2) - f(x) - \zeta_1\left(\frac{\partial f}{\partial \zeta_1}\right)_{0,0} - \zeta_2\left(\frac{\partial f}{\partial \zeta_2}\right)_{0,0}\right\|_{d'} \leq K(d') \|(\zeta_1, \zeta_2)\|^2$$

where $K(d')$ is a real number. By Theorem 3.2.2 this gives

$$\| f(x+\zeta_1 y_1+\zeta_2 y_2)-f(x)-\zeta_1 \delta f(x;y_1)-\zeta_2 \delta f(x;y_2)\|_{d'} \leq K(d') \|(\zeta_1, \zeta_2)\|^2 .$$

Letting $\zeta_1 = \zeta_2 = \zeta$ this last inequality now reads

$$\|f(x+\zeta(y_1+y_2))-f(x)-\zeta[\delta f(x;y_1)+\delta f(x;y_2)]\|_{d'} \leq K(d')2|\zeta|^2$$

or,

$$\left\| \frac{f(x + \zeta (y_1 + y_2)) - f(x)}{\zeta} - [\delta f(x;y_1) + \delta f(x;y_2)] \right\|_{d'} \leq 2K(d') |\zeta| .$$

Hence,

$$\delta f(x;y_1 + y_2) = \lim_{\zeta \to 0} \frac{1}{\zeta} \left[f(x + \zeta (y_1 + y_2)) - f(x) \right] = \delta f(x;y_1) + \delta f(x;y_2).$$

Theorem 3.2.6. Let f(x) be defined on the open set Ω in L to L'. Then $\delta^n f(x;y_1;y_2;\ldots;y_n)$ is a completely symmetric multilinear form in $y_1, y_2, \ldots, y_n$ which is G-differentiable with respect to x when x is in Ω . $\delta^n f(x; \lambda y) = \lambda^n \delta^n f(x;y)$.

Proof. By Theorem 3.2.4 (i) $\delta^n f(x;y_1;y_2;\ldots;y_n)$ is symmetric in $y_1, y_2, \ldots, y_n$. This combined with the fact that $\delta^n f(x;y_1;y_2;\ldots;y_n)$ is linear in the last argument (Theorem 3.2.5) implies that it is linear in each y_k. The G-differentiability follows from Theorem 3.2.3.

3.3 Series Expansions of G-differentiable Functions. Using the relations between G-differentials and derivatives the expansion theorem for analytic vector functions of a complex variable can be applied to obtain a series expansion for G-differentiable functions.

Theorem 3.3.1. Let f(x) be defined and G-differentiable on an open set Ω in L to L'. Let x_o be in Ω . Then f(x) may be expanded about the point x_o in the form

$$f(x) = f(x_o) + \frac{\delta f(x_o;x-x_o)}{1!} + \ldots + \frac{\delta^n f(x_o;x-x_o)}{n!} + \ldots ,$$

the series being convergent (at least) for all x in any neighbourhood contained in Ω about x_o.

Let $N(x_o;d,r_o) \equiv \{ x \text{ in } \Omega \mid \| x-x_o \|_d < r_o, d \text{ in } D \}$.

Let $0 < \epsilon < r_o$ and let x be chosen so that $\| x-x_o \|_d \leq \epsilon < r_o$. If r is now taken satisfying the inequality $1 < r < \frac{r_o}{\epsilon}$, then by Theorem 3.2.1 the function $f(x_o + \zeta y)$, $y = x - x_o$, is analytic for $|\zeta| < r$ and by Theorem 2.5.9 can be expanded (for $|\zeta| < r$) as

$$f(x_o+\zeta y)= \sum_{n=0}^{\infty} \left[\frac{d^n}{d\alpha^n} f(x_o+\alpha y)\right]_{\alpha=0} \frac{\zeta^n}{n!} = \sum_{n=0}^{\infty} \frac{\delta^n f(x_o;y)}{n!} \zeta^n$$

using Theorem 3.2.4 (ii). In particular for $\zeta = 1$,

$$f(x) = f(x_o+ x - x_o) = \sum_{n=0}^{\infty} \frac{\delta^n f(x_o;y)}{n!} = \sum_{n=0}^{\infty} \frac{\delta^n f(x_o;x-x_o)}{n!} .$$

From Theorem 2.4.4,

$$\delta^n f(x_o;y) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(x_o+\zeta y)}{\zeta^{n+1}} d\zeta ,$$

where Γ is any circle $|\zeta| = \rho < r$, and in particular may be taken to be of unit radius.

If for all x in $N(x_o;d,r_o)$, f(x) is topologically bounded, that is, its values for such x form a bounded set in L', then for every d' in D' there is a number $M(d') > 0$ such that

$$\| f(x) \|_{d'} \leq M(d') \quad \text{and} \quad \| \delta^n f(x_o;y) \|_{d'} \leq M(d')n!$$

for x in $N(x_0;d,r_0)$. In general while it is impossible to make any statement about the topological boundedness of $f(x)$, the following is a step in that direction.

Theorem 3.3.2. Suppose $f(x)$ with values in L' is defined and G-differentiable on an open set Ω of L where θ is in Ω. If $N(\theta;d,r_0)$ is determined as in Theorem 3.3.1 and if S is a subset of $N(\theta;d,r_0)$ which is contained in a finite, say n-dimensional linear subspace L^n of L and compact there, then $f(x)$ is bounded in S and the series $\sum_{n=0}^{\infty} \frac{\delta^n f(\theta;x)}{n!}$ converges uniformly in $S_1 = S \cap N(\theta;d, \eta r_0)$ where $0 < \eta < 1$. Further, for each d' in D', the series

$$\sum_{n=0}^{\infty} \frac{\|\delta^n f(\theta;x)\| d'}{n!}$$ converges uniformly in S_1.

Proof. L^n is homeomorphic to Z^n (see paragraph 2.8). Let $\Omega^n = L^n \cap N(\theta;d,r_0)$. Ω^n is an open connected set in L^n and contains θ. Every x in Ω^n is of the form $x = \zeta_1 x_1 + \dots + \zeta_n x_n$ where $x_1, \dots, x_n$ is a fixed basis of L^n and the ζ_i are complex numbers. The set $\mathfrak{D}$ of all these $(\zeta_1, \dots, \zeta_n)$ forms a homeomorphic image of Ω^n in Z^n containing the origin in Z^n, that is, $\mathfrak{D}$ is a domain in Z^n containing the origin. $f(\zeta_1 x_1 + \dots + \zeta_n x_n)$ is a partially differentiable function with respect to $\zeta_1, \zeta_2, \dots, \zeta_n$ for $\zeta_1 x_1 + \dots + \zeta_n x_n$ in Ω_n, i.e., for $(\zeta_1, \dots, \zeta_n)$ in $\mathfrak{D}$ (Theorem 3.2.2). Now $S \subset \Omega_n$ is a compact set in L^n and so its homeomorphic image, which is contained in $\mathfrak{D}$, is compact in Z^n and hence closed and bounded there. By Theorem 2.8.1 (ii), the values of f(x) then lie in a bounded set for x in S, that is, given d' in D' there is $M(d') > 0$ such that $\| f(x) \|_{d'} < M(d')$ for all x in S.

Choose r such that $1 < r < \frac{1}{\eta}$, where $0 < \eta < 1$. Then

$$\delta^n f(\theta;x) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta x)}{\zeta^{n+1}} \, d\zeta$$

where Γ is a circle of radius r about $\zeta = 0$ and $\| x \|_d \leq \eta r_o$. Consequently for x in $S_1 = S \cap N(\theta;d, \eta r_o)$ and d' in D'

$\frac{\| \delta^n f(\theta,x) \|_{d'}}{n!} \leq \frac{M(d')}{r^n}$. Since $r > 1$, $\sum_{n=0}^{\infty} \frac{M(d')}{r^n}$ is convergent and hence

$$\sum_{n=0}^{\infty} \frac{\| \delta^n f(\theta,x) \|_{d'}}{n!} \quad \text{converges uniformly for all x in } S_1.$$

The expansion of Theorem 3.3.1 is valid in a larger region than the neighbourhoods considered there. This larger region exploits fully the character of the G-differential as a directional derivative.

Definition 3.3.1(2). The set $C^*(x_o) \subset L$ is called a c-star about x_o if $C^*(x_o) = x_o + X$, where x in X and $|\zeta| \leq 1$ implies ζx is in X.

Definition 3.3.2(2). If Ω is an open set in L, then for x in Ω, r(x,y) will denote the supremum of all numbers r such that $|\zeta| \leq r$ implies $x + \zeta y$ is in Ω.

If $\Omega \subset L$ is open and x_0 is in Ω then the set of all points $x_0 + y$ for which $|\zeta| \leq 1$ implies $x + \zeta y$ is in Ω is a c-star about x_0, <u>the</u> c-star in Ω about x_0. That this set of points is actually a c-star is easily verified. Let y in L be such that $|\zeta| \leq 1$ implies $x + \zeta y$ is in Ω. If $|\xi| \leq 1$ then $|\zeta \xi| \leq 1$. Hence $x_0 + (\zeta \xi)y$ is in Ω. In particular a neighbourhood about x_0 is certainly a c-star about x_0.

Let f(x) on L to L' be defined and G-differentiable on an open set Ω. If x_0 is in Ω, then for $|\zeta| < r(x_0, x-x_0)$ the Taylor expansion (Theorem 2.5.9)

$$f(x_0 + \zeta(x-x_0)) = \sum_{n=0}^{\infty} \left[\frac{d^n}{d\alpha^n} f(x_0 + \alpha(x-x_0))\right]_{\alpha=0} \frac{\zeta^n}{n!}$$

$$= \sum_{n=0}^{\infty} \delta^n f(x_0; x-x_0) \frac{\zeta^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{\delta^n f(x_0; \zeta(x-x_0))}{n!},$$

holds.

The point x_0 is arbitrary. If $\zeta(x-x_0)$ is replaced by $(x-x_0)$ then this may be stated as

<u>Theorem 3.3.3.</u> For x_0 in Ω and x in <u>the</u> c-star about x_0 in Ω,

$$f(x) = \sum_{n=0}^{\infty} \frac{\delta^n f(x_0; x-x_0)}{n!}.$$

Theorem 3.3.4. If $\{f_n(x)\}$ is a sequence of G-differentiable functions on L to L' defined and G-differentiable on an open set Ω and if $\{f_n(x)\}$ converges uniformly on the "discs" $x + \zeta y$, y in L and $|\zeta| \leq r' < r(x,y)$, of Ω, then the limit will also be G-differentiable in Ω. Further, under these conditions for x in Ω, y in L,

$$\delta^k f(x;y) = \lim_{n \to \infty} \delta^k f_n(x;y).$$

Proof. The proof of both these statements is a consequence of Theorem 2.5.10. If $f(x) = \lim_{n \to \infty} f_n(x)$ it must be shown that for x in Ω and y in L, $f(x + \zeta y)$ is differentiable with respect to ζ for $|\zeta| < r(x,y)$. Then by Theorem 3.2.1 it will follow that f(x) is G-differentiable in Ω. But $\{f_n(x + \zeta y)\}$ being a sequence of analytic functions for $|\zeta| < r(x,y)$ converging uniformly for $|\zeta| \leq r'$ then by Theorem 2.5.10, the limit function $f(x + \zeta y)$ is an analytic vector function of a complex variable for $|\zeta| < r(x,y)$.

3.4. F-differentiability and Partial Differentials. The first non-metrical definition of a differential (the M-differential) for functions whose arguments are in a l.t.s. and whose values are in a second (possibly distinct) l.t.s. was given by A. D. Michal in (22). When the l.t.s. are normable spaces every Frechet differentiable function is also an M-differentiable function but not conversely. In (23) however, Michal defined a "first order differential" for functions f(x) with arguments and values in topological abelian groups with the property that if the topological abelian groups are Banach spaces then f(x) is Frechet differentiable and the two differentials are equal. Even earlier in (12) and (13) Michal and Paxson had defined a differential for functions on a l.t.s. to the same l.t.s. but this differential did not have the composition property. A comprehensive account (with extensive bibliography) of these matters, that is, of topological differential calculus, appears in (24). The differential used here is due to Hyers (25) and is a differential of the type defined in (22). Further, in Banach spaces, this differential reduces to the Frechet differential.

Definition 3.4.1. The function f(x) defined on an open set $\Omega \subset L$ to L' will be said to be F-differentiable at the point x_0 of Ω with increment y in L and $df(x_0;y) = d'f(x_0;y)$ will denote its F-differential if

(i) $df(x_0;y)$ is linear and continuous in y on L to L',

(ii) for every d' in D', there corresponds d in D with the property that given $\eta' > 0$ there exists $\delta > 0$ such that $\| f(x_0+y)-f(x_0)-df(x_0;y) \|_{d'} \leq \eta' \| y \|_d$ whenever $\|y\|_d \leq \delta$. Here $d = d(d')$ and $\delta = \delta(d,d',\eta')$. f(x) will be said to be F-differentiable in Ω if it is F-differentiable at each point of Ω. $d^{n+1}f(x;y_1;\ldots;y_{n+1}) = d(d^n f(x;y_1;\ldots;y_n))$ and

$d^o f(x;y_1;\dots;y_n) = f(x)$. $d^n f(x;y;\dots;y)$ will be written as $d^n f(x;y)$.

Theorem 3.4.1. If $f_1(x)$ and $f_2(x)$ are F-differentiable at $x = x_o$, then $f_3(x) = \xi_1 f_1(x) + \xi_2 f_2(x)$ is F-differentiable at $x = x_o$ and

$$df_3(x_o;y) = \xi_1 df_1(x_o;y) + \xi_2 df_2(x_o;y).$$

Theorem 3.4.2. If $f(x)$ is defined and F-differentiable on an open set Ω in L to L' then $f(x)$ is continuous in Ω.

Theorem 3.4.3. If $f(x)$ is defined and F-differentiable on an open set Ω in L with values in L', then $f(x)$ is G-differentiable in Ω and the two differentials are equal. Thus the F-differential when it exists is unique.

Theorem 3.4.4. Let $f(x)$ be defined on an open set Ω of L with values in L'. Let D,E be two associated directed systems for L defining equivalent systems of pseudo-norms for L and D', E' be two associated directed systems for L' defining equivalent systems of pseudo-norms for L'. If $f(x)$ is F-differentiable at x_o with respect to D and D' then it is F-differentiable there with respect to E and E' and the F-differentials are the same.

The proofs of these theorems are straightforward and will not be given here but the methods used are illustrated in the following theorem.

Theorem 3.4.5. Let L_1, L_2, L_3 be locally convex l.t.s. with associated directed systems D_1, D_2, D_3 respectively. Let $f(x)$ be defined on the open set $\Omega_1 \subset L_1$ to the open set $\Omega_2 \subset L_2$

and let $g(y)$ be defined on Ω_2 to L_3. If $f(x)$ has an F-differential at $x = x_0$ in G_1 and $g(y)$ has an F-differential at $y = y_0 = f(x_0)$ in G_2, then $h(x) = g(f(x))$ has an F-differential at $x = x_0$ given by

$$dh(x_0;x) = dg(f(x_0);\ df(x_0;x)).$$

Proof. The proof of the theorem consists in showing that given d_3 in D_3 there exists $d_1 = d_1(d_3)$ in D_1 such that for any $\eta_3 > 0$, there is $\delta_1 = \delta_1(d_3, d_1, \eta_3) > 0$ so that $\|x\|_{d_1} < \delta_1$ implies

$$\|h(x_0 + x) - h(x_0) - dg(f(x_0); df(x_0;x))\|_{d_3} \leq \eta_3 \|x\|_{d_1}.$$

Corresponding to d_3 there exists $d_2 = d_2(d_3)$ in D_2 such that for $\eta_3' > 0$, $\delta_2 = \delta_2(d_3, d_2, \eta_3')$ can be found so that $\|y\|_{d_2} < \delta_2$ implies

$$\|g(y_0 + y) - g(y_0) - dg(y_0;y)\|_{d_3} \leq \eta_3' \|y\|_{d_2}.$$

Since $f(x)$ is differentiable at x_0 it is continuous there (Theorem 3.4.2). Hence $\|f(x_0 + x) - f(x_0)\|_{d_2} < \delta_2$ for $\|x\|_{e_1} < \alpha$, where $e_1 = e_1(d_2)$ is in D_1 and $\alpha = \alpha(\delta_2) > 0$. Taking $y = f(x_0 + x) - f(x_0)$ it follows that

$$\|g(f(x_0+x)) - g(f(x_0)) - dg(f(x_0); f(x_0 + x) - f(x_0))\|_{d_3} \leq \eta_3' \|f(x_0 + x) - f(x_0)\|_{d_2}$$

for $\|x\|_{e_1} < \alpha$. Then, using this last inequality, the properties of the pseudo-norm and the linearity and continuity (as a function of y) of $dg(y_0;y)$ there is an $e_2 = e_2(d_3)$ and $M(d_3, e_2) > 0$ for which

$$\| g(f(x_o+ x))-g(f(x_o))-dg(f(x_o);df(x_o;x))\|_{d_3}$$

$$\leq \eta_3' \| f(x_o+ x)-f(x_o)\|_{d_2} + \| dg(f(x_o);f(x_o+ x)-f(x_o)-df(x_o;x))\|_{d_3}$$

$$\leq \eta_3' \| f(x_o+ x)-f(x_o)\|_{d_2} + M(d_3,e_2) \| f(x_o+ x)-f(x_o)-df(x_o;x)\|_{e_2},$$

for $\| x\|_{e_1} < \alpha$. Since $f(x)$ is differentiable at $x = x_o$, corresponding to d_2 and e_2 there exists $b_1 = b_1(d_2)$ in D_1 and $c_1= c_1(e_2)$ in D_1 such that given $\eta_2 > 0$ there are $\beta = \beta(d_2,b_1, \eta_2)$, $\gamma = \gamma(e_2,c_1, \eta_2)$ respectively so that

$$\| f(x_o+ x)-f(x_o)\|_{d_2} \leq \eta_2 \| x\|_{b_1} + \| df(x_o;x)\|_{d_2} \leq \eta_2 \|x\|_{b_1} + M(d_2,k_1)\|x\|_{k_1}$$

where $\| x\|_{b_1} < \beta$, $k_1= k_1(d_2)$ is in D_1 and $M(d_2,k_1) > 0$, and

$$\| f(x_o+ x)-f(x_o)-df(x_o;x)\|_{e_2} \leq \eta_2 \|x\|_{c_1}$$

for $\| x\|_{c_1} < \gamma$. These last inequalities then give

$$\| h(x_o+ x)-h(x_o)-dg(f(x_o);df(x_o;x) \|_{d_3}$$

$$\leq \eta_3'(\eta_2\| x\|_{b_1} + M(d_2,k_1) \| x\|_{k_1}) + M(d_3,e_2) \eta_2 \| x\|_{c_1},$$

for $\|x\|_{e_1} < \alpha$, $\| x\|_{b_1} < \beta$ and $\|x\|_{c_1} < \gamma$. Taking $d_1 > e_1,b_1,c_1,k_1,d_1$ in D_1, then since these last depend on d_3 so does d_1. η_3', η_2 can be determined so that

$$\eta_3' (\eta_2 + M(d_2,k_1)) + M(d_3,e_2) \eta_2 < \eta_3.$$

Finally if $\delta_1 = \min(\alpha , \beta , \gamma)$, then for $\| x\|_{d_1} < \delta_1$

$$\| h(x_o+ x)-h(x_o)-dg(f(x_o);df(x_o;x)) \|_{d_3} \leq \eta_3 \|x\|_{d_1}.$$

It is clear that $dgf(x_0;df(x_0;x))$ is a continuous linear function x and hence it is the F-differential of $h(x)$ at $x = x_0$ with x as increment.

Let $L(R)$ and $L'(R)$ be locally convex real l.t.s. and $L(C)$, $L'(C)$ and $L(K)$ and $L'(K)$ the associated couple and composite spaces respectively. If $f(x,y)$ is defined on an open set in $L(K)$ to $L'(R)$ partial G- and partial F-differentials with respect to the variables x and y can be defined. Thus the partial G-differential of $f(x,y)$ with respect to x is the ordinary G-differential with respect to x of $f(x,y)$ when y is held fixed in $L(R)$. By the total F-differential of $f(x,y)$ is meant the F-differential of $f(x,y)$ with respect to the composite variable (x,y). It is written $df(x,y;(\Delta x, \Delta y))$ where $(\Delta x, \Delta y)$ is the (composite) increment. This is to be contrasted with the F-differential $dg(z; \Delta z)$ of a function $g(z)$ defined on an open set of $L(C)$ to $L'(C)$. In this last z is the couple element $z = (x,y)$ and Δz is the couple increment.

§ 2. Analytic Functions

3.5. Polynomials. As before L and L' will denote sequentially complete locally convex complex l.t.s. and D and D' the corresponding associated directed systems. Polynomials on the complex numbers to a sequentially complete locally convex l.t.s. have already been defined in Definition 2.5.2.

Definition 3.5.1. A function $p(x) \not\equiv \theta$ on L to L' is called a polynomial if

(i) p(x) is continuous at every x,

(ii) there exists an integer n such that for any pair (x,y), x,y in L, $p(x + \zeta y)$ is a polynomial on the complex numbers to L'. The least number satisfying (ii) is the degree of p(x).

A polynomial p(x), homogeneous of degree n, that is one for which $p(\zeta x) = \zeta^n p(x)$ is a polynomial of degree n. Hence the phrase homogeneous polynomial of degree n is used to describe such polynomials.

Definition 3.5.2. Let f(x) be a function on L to L' and let y be an element of L. Then $\Delta_y f(x) = f(x + y) - f(x)$ is called the first difference of f(x) with respect to y. If $x_1, x_2, \ldots, x_n$ are elements of L, then the nth difference of f(x) with respect to $x_1, x_2, \ldots, x_n$ is defined to be

$$\Delta^n_{x_1 x_2 \ldots x_n} f(x) = \Delta^n_{x_n} (\Delta^{n-1}_{x_1 x_2 \ldots x_{n-1}} f(x))$$

$$= \Delta^{n-1}_{x_1 x_2 \ldots x_{n-1}} f(x + x_n) - \Delta^{n-1}_{x_1 x_2 \ldots x_{n-1}} f(x).$$

Definition 3.5.3. Let h(x) be a homogeneous polynomial of degree n on L to L'. Then the function $h(x_1, x_2, \ldots, x_n)$ on LxLx...xL to L' defined by

$$h(x_1, x_2, \ldots, x_n) = \frac{\Delta^n_{x_1 x_2 \ldots x_n} h(\theta)}{n!}$$

is the polar of $h(x)$ with respect to $x_1, x_2, \ldots, x_n$.

The algebraic properties of polynomials, homogeneous polynomials and polars are treated at length in (2,26). With regard to properties involving the topology of the space L and L$'$ certain changes from the Banach space case are in order.

Theorem 3.5.1. Let $h(x)$ be a homogeneous polynomial of degree n on L to L$'$. Then given d' in D$'$, there exists d in D and a least $M(d,d') > 0$ such that

$$\| h(x) \|_{d'} \leq M(d,d') \| x \|_d^n , \text{ all } x \text{ in } L.$$

Proof. By definition $h(x)$ is continuous at $x = \theta$. It follows that given d' in D$'$, there exists d in D and $\delta > 0$ such that $\| x \|_d < \delta$ implies $\| h(x) \|_{d'} \leq 1$. Let x be any element of L such that $x \neq \theta$ or $\| x \|_d \neq 0$. Letting $\bar{x} = \frac{\delta}{2} \cdot \frac{x}{\| x \|_d}$, then

$$\| \bar{x} \|_d = \left\| \frac{\delta}{2} \cdot \frac{x}{\| x \|_d} \right\|_d = \frac{\delta}{2} < \delta ,$$

and consequently

$$1 \geq \| h(\bar{x}) \|_{d'} = \left\| h\left(\frac{\delta}{2} \cdot \frac{x}{\| x \|_{d'}} \right) \right\|_{d'} = \left(\frac{\delta}{2} \frac{1}{\| x \|_d} \right)^n \| h(x) \|_{d'} ,$$

that is,

$$\| h(x) \|_{d'} \leq \left(\frac{2}{\delta}\right)^n \| x \|_d^n .$$

The inequality is trivially true when $x = \theta$ or $\| x \|_d = 0$.

This theorem makes possible the following definition.

Definition 3.5.4. If d' and d satisfy the statement of Theorem 3.5.1 then $\sup\limits_{\substack{x \text{ in } L \\ \|x\|_{d} \neq 0}} \dfrac{\|h(x)\|_{d'}}{\|x\|^{n}_{d_1}}$ exists and is denoted by $M_h(d,d')$. $M_h(d,d')$ is called the modulus of $h(x)$ with respect to d and d'.

If $h(x_1,\ldots,x_n)$ is the polar of the homogeneous polynomial $h(x)$ then it is a linear homogeneous polynomial in each of the arguments and is symmetric in the arguments. Further $h(x) = h(x,\ldots,x)$.

Theorem 3.5.2. Let $h(x)$ be a homogeneous polynomial of degree n on L to L' and let $h(x_1,x_2,\ldots,x_n)$ be its polar. Then $h(x_1,\ldots,x_n)$ is continuous at $(\theta,\ldots,\theta)$ and hence bounded with respect to the pseudo-norms.

Proof. $n!h(x_1,x_2,\ldots,x_n) = \Delta^{n}_{x_1x_2\ldots x_n} h(\theta)$ is of degree zero in x so that $\Delta^{n}_{x_1x_2\ldots x_n} h(\theta) = \Delta^{n}_{x_1x_2\ldots x_n} h(x)$. Let $x_1,\ldots,x_n$ be and arbitrary set of increments and take $x = -\frac{1}{2}\sum\limits_{i=1}^{n} x_i$. Forming the successive differences of $h(x)$ gives rise to

$$\Delta^{n}_{x_1x_2\ldots x_n} h(x) = \sum_{2^n \text{ terms}} h\Big(\frac{1}{2}\sum_{i=1}^{n} \epsilon_i x_i\Big), \quad \epsilon_i = \pm 1.$$

By Theorem 3.5.2 given d' in D' there exists d in D and $M(d,d') > 0$ such that

$$\Big\| h\Big(\frac{1}{2}\sum_{i=1}^{n} \epsilon_i x_i\Big)\Big\|_{d'} \leq M(d,d')\Big(\Big\|\frac{1}{2}\sum_{i=1}^{n} \epsilon_i x_i\Big\|_{d}\Big)^n \leq \frac{M(d,d')}{2^n}\Big(\sum_{i=1}^{n}\|x_i\|_{d}\Big)^n$$

$$\leq \frac{M(d,d')}{2^n}\, n^n \big(\max_i \|x_i\|_{d}\big)^n.$$

Consequently

$$\| h(x_1,x_2,\ldots,x_n) \|_{d'} \leq \frac{1}{n!} \sum_{2^n \text{ terms}} \| h(\frac{1}{2} \sum_{i=1}^{n} \epsilon_i x_i) \|_{d'}$$

$$\leq M(d,d') \frac{n^n}{n!} (\max_i \| x_i \|_d)^n$$

which can be made arbitrarily small for $\max_i \| x_i \|_d$ sufficiently small. Thus $h(x_1,\ldots,x_n)$ is bounded with respect to the pseudo-norms (Theorem 1.7.1).

This theorem makes possible the following definition.

Definition 3.5.5. By theorem 3.5.2 d' and d exist such that

$$\sup_{\substack{x_i \text{ in } L \\ \| x_i \|_d \neq 0}} \frac{\| h(x_1,x_2,\ldots,x_n) \|_{d'}}{\| x_1 \|_d \cdots \| x_n \|_d} = M_n(d,d') \text{ is finite. } M_n(d,d') \text{ is}$$

called the modulus of the polar $h(x_1,x_2,\ldots,x_n)$ of the homogeneous polynomial $h(x)$ of degree n with respect to d and d'.

Theorem 3.5.3. Let $M_h(d,d')$ be the modulus of the homogeneous polynomial $h(x)$ of degree n on L to L'. The modulus $M_n(d,d')$ of its polar exists and conversely. Further

$$1 \leq \frac{M_n(d,d')}{M_h(d,d')} \leq \frac{n^n}{n!} .$$

Proof. Taking $\| x_i \|_d = 1$, $i = 1,2,\ldots,n$ it follows from the last inequality in Theorem 3.5.2 that

$$\| h(x_1,x_2,\ldots,x_n) \|_{d'} \leq M_h(d,d') \frac{n^n}{n!} .$$

Then, $M_n(d,d') = \sup_{\|x_i\|_d = 1} \| h(x_1,x_2,\ldots,x_n) \|_{d'} \leq M_h(d,d') \frac{n^n}{n!}$.

Finally, $M_n(d,d') = \sup \frac{\| h(x_1,x_2,\ldots,x_n) \|_{d'}}{\|x_1\|_d \|x_2\|_d \cdots \|x_n\|_d} \geq \sup \frac{\| h(x,x,\ldots,x)\|_{d'}}{\|x\|_d^n} =$

$= M_h(d,d')$

3.6. The Series Expansion for Analytic Functions. In the remainder of this chapter the word domain will be used, as usual, to designate an open connected set. In a locally convex space the notions of connected and arc-wise connected are equivalent.

Definition 3.6.1. A function $f(x)$ on L to L' defined in the domain $\mathcal{D}$ is said to be analytic in $\mathcal{D}$ if it is single-valued, continuous and G-differentiable in $\mathcal{D}$.

Theorem 3.6.1. If $f(x)$ on L to L' is analytic in $\mathcal{D}$, then

(i) $f(x)$ is F-differentiable in $\mathcal{D}$ and has F-differentials $d^n f(x;x_1;x_2;\ldots;x_n)$ of all orders which are analytic functions of x in $\mathcal{D}$ for fixed $(x_1,x_2,\ldots,x_n)$ and jointly continuous in the increments $(x_1,x_2,\ldots,x_n)$ for fixed x,

(ii) the series expansion

$$f(x_0 + y) = \sum_{n=0}^{\infty} \frac{d^n f(x_0;y)}{n!}$$

is valid for every x_0 in $\mathcal{D}$ and $x_0 + y$ in any neighbourhood of x_0 lying entirely in $\mathcal{D}$.

Proof. By Theorem 3.2.6, since $f(x)$ is G-differentiable $\delta^n f(x;x_1;x_2;\ldots;x_n)$ exists for all n and is a completely symmetric multilinear form in the increments $x_1,x_2,\ldots,x_n$. $f(x)$ may be expanded as a Taylor Series of G-differentials about any point x in $\mathcal{D}$ (Theorem 3.3.1). Let x_o be an element of $\mathcal{D}$. Given d' in D' and $\epsilon' > 0$, there exists d in D and $\eta > 0$ such that $\| x-x_o \|_d < \eta$ implies $\| f(x)-f(x_o) \|_{d'} < \epsilon'$. Pick any point x such that $\| x-x_o \|_d < \frac{\eta}{2}$. Then for $\|y\|_d \leq \frac{\eta}{2}$ and such an x it is true that

$$f(x + y) = \sum_{n=0}^{\infty} \frac{\delta^n f(x;y)}{n!}, \quad \text{where} \quad \frac{\delta^n f(x;y)}{n!} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(x+\zeta y)}{\zeta^{n+1}} d\zeta ,$$

Γ being the circle with unit radius. Since $\| x + \zeta y-x_o \|_d < \eta$, then $\| f(x+\zeta y)-f(x_o) \|_{d'} < \epsilon'$ and so $\| f(x+\zeta y) \|_{d'} < \epsilon' + \|f(x_o)\|_{d'} = M(x_o,d', \epsilon')$. Thus,

$$\left\| \frac{\delta^n f(x;y)}{n!} \right\|_{d'} \leq M(x_o,d', \epsilon') \text{ for } \| x-x_o \|_d < \frac{\eta}{2}, \ \|y\|_d \leq \frac{\eta}{2}.$$

Let y be any element of L such that $y \neq \theta$ and $\|y\|_d \neq 0$. Then,

$$\left\| \delta^n f(x; \frac{\eta}{2} \frac{y}{\|y\|_d}) \right\|_{d'} \leq n!\, M(x_o,d', \epsilon'), \text{ or}$$

$$\| \delta^n f(x;y) \|_{d'} \leq n!\, M(x_o,d', \epsilon')(\tfrac{2}{\eta})^n \| y \|_d^n , \ \| x-x_o \|_d < \frac{\eta}{2} .$$

This same inequality is trivially true when $y = \theta$ or $\| y \|_d = 0$. If this were not the case then $a > 0$ could be determined so that

$\|\delta^n f(x;ay)\|_{d'} = a^n \|\delta^n f(x;y)\|_{d'} > n!\, M(x_0,d',\epsilon')$ while $\|ay\|_d = 0$, $\|x-x_0\|_d < \frac{\eta}{2}$, giving a contradiction. Thus $\delta^n f(x;y)$ as a (linear) function of y is bounded with respect to the pseudo-norms and hence (Theorem 1.6.1) is continuous in y, locally uniformly in x.

This same inequality combined with the Taylor expansion implies that for given $\eta' > 0$,

$$\| f(x+y)-f(x)- \delta f(x;y) \|_{d'} \leq \frac{4M(x_0,d',\epsilon')\|y\|_d^2}{\eta(\eta-2\|y\|_d)}$$

$$\leq \eta'\|y\|_d ,$$

when $\|x-x_0\|_d < \frac{\eta}{2}$ and $\|y\|_d < \min(\frac{\eta}{2}, \frac{\eta^2\eta'}{2\eta'+4M(x_0,d',\epsilon')})$.

With this last relation the proof that f(x) is F-differentiable with $df(x;y) = \delta f(x;y)$ is complete.

It will now be shown that under the hypothesis of the theorem that $\delta^n f(x;y)$ is an analytic function of x for fixed y. Since by Theorem 3.2.6, $\delta^n f(x;y)$ is G-differentiable it remains only to prove the continuity of $\delta^n f(x;y)$ with respect to x. This requires

<u>Lemma 3.6.1.</u> Let f(x,y) be a function on $L_1 \times L_2$ to a space L_3 where L_1, L_2, L_3 are sequentially complete locally convex complex l.t.s. Suppose f(x,y) is defined and continuous in the pair (x,y)

for x in a domain $\mathcal{D}$ of L_1 and y in a compact set G of L_2, then if x_o is in $\mathcal{D}$, $f(x_o,y)$ is continuous at x_o, uniformly with respect to y in G.

<u>Proof</u>. Suppose the theorem is false. Then, given d_3 in D_3, $\delta_3 > 0$, there corresponds to any neighbourhood $N(x_o)$ of x_o at least one x in $N(x_o)$ and at least one y in G such that $\| f(x,y)-f(x_o,y) \|_{d_3} > \delta_3$. $f(x_o,y)$ being jointly continuous for $x = x_o$ and any y in G, then for each pair (x_o,y), y in G, there are neighbourhoods $N(x_o)$ in L_1, $N(y)$ in L_2 such that x' in $N(x_o)$, y' in $N(y)$ implies

$$\| f(x',y') - f(x_o,y) \|_{d_3} < \frac{\delta_3}{2} .$$

These neighbourhoods N(y) of y, for all y in G, constitute an open covering of G. By compactness there is a finite number of them $N(y_1)$, $N(y_2)$,...,$N(y_n)$, say, constituting a covering of G. Let $N_1(x_o)$, $N_2(x_o)$,...,$N_n(x_o)$ be the corresponding neighbourhoods (in L_1) of x_o. There exists $N(x_o) \subset \bigcap_{i=1}^{n} N_i(x_o)$. The continuity condition then says that for x in $N(x_o)$, y in $N(y_i)$,

$$\| f(x,y)-f(x_o,y_i) \|_{d_3} < \frac{\delta_3}{2} .$$

Corresponding to this same neighbourhood $N(x_o)$ of x_o there is some x in $N(x_o)$ and y in G such that

$$\| f(x,y)-f(x_o,y) \|_{d_3} > \delta_3. \qquad 3.1$$

On the other hand y must lie in some neighbourhood $N(y_j)$, $j = 1,2,\dots,n$. Calling it $N(y_0)$ then for x in $N(x_0)$, y in $N(y_0)$ it is true that

$$\| f(x,y)-f(x_0,y_0) \|_{d_3} < \frac{\delta_3}{2} \quad \text{and} \quad \| f(x_0,y_0)-f(x_0,y) \|_{d_3} < \frac{\delta_3}{2},$$

and these contradict the inequality 3.1.

With this it can be shown that $\delta^n f(x;y)$ is a continuous function of x, for fixed y. Suppose that x_0 is in $\mathcal{D}$ and let y be an arbitrary fixed element of L. Using Theorem 3.3.1 an element d in D and positive numbers r_0, r can be found so that $x + \zeta y$ is in $\mathcal{D}$ when $\| x-x_0 \|_d < r_0$ and $|\zeta| < r$ and with these restrictions

$$\delta^n f(x;y) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(x+\zeta y)}{\zeta^{n+1}} \, d\zeta ,$$

Γ being the circle $|\zeta| = \rho < r$. Then given d' in D', $\delta' > 0$,

$$\| \delta^n f(x;y) - \delta^n f(x_0;y) \|_{d'} \leq \frac{1}{2\pi} \int_\Gamma \left\| \frac{f(x+\zeta y)-f(x_0+\zeta y)}{\zeta^{n+1}} \right\|_{d'} |d\zeta|$$

From this it follows that $\delta^n f(x;y)$ is continuous at $x = x_0$ provided that for a given $\epsilon > 0$, e in D and $\eta > 0$ can be picked so that $\| x-x_0 \|_e < \eta$ implies $\| f(x+\zeta y)-f(x_0+\zeta y) \|_{d'} < \epsilon$ for all ζ of Γ. This can be done as a consequence of Lemma 3.6.1 since Γ is compact. Thus for $\| x-x_0 \|_d < r_0$, $\| x-x_0 \|_e < \eta$,

$$\| \delta^n f(x;y) - \delta^n f(x_0;y) \|_{d'} \leq \frac{\epsilon}{\rho^n} = \delta' ,$$

and $\delta^n f(x;y)$ considered as a function of x for fixed y is analytic in $\mathcal{D}$.

Applying the argument which was used to show that the analytic function f(x) is F-differentiable it follows that the analytic function $\delta^n f(x;y)$ is F-differentiable with respect to x in $\mathcal{D}$. But the same conclusion then extends to $\delta^n f(x;x_1;x_2;\ldots;x_n)$ which is the polar form of the homogeneous polynomial $\delta^n f(x;y)$ and hence linearly expressible in terms of the functions $\delta^n f(x;\epsilon_1 x_1+\ldots+\epsilon_n x_n)$, $\epsilon_i = 0,1$, all of which have just been shown to be continuous F-differentiable functions of x. Further, as in Theorem 3.5.2, $\delta^n f(x;x_1;\ldots;x_n)$ is jointly continuous in $(x_1,x_2,\ldots,x_n)$. The notation $d^n f(x;y)$ may be used in place of $\delta^n f(x;y)$ to obtain the expansion

$$f(x_0+y) = \sum_{n=0}^{\infty} \frac{d^n f(x_0;y)}{n!} .$$

Theorem 3.6.2. A homogeneous polynomial p(x) on L to L' is an analytic function in L and hence it is F-differentiable.

Let $\mathcal{R}$ be a sequentially complete extended locally convex complex topological linear ring with associated directed system D (Definition 1.8.3). Let $\mathcal{R}$ be commutative and have a unit element u.

Definition 3.6.2. $\exp x = u + \frac{x}{1} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$,

where $x^n = xx \ldots x$, a product of n factors and $x^0 = u$.

As stated in paragraph 1.8 part of the motivation for introducing extended locally convex complex topological linear rings is to ensure the existence of entire functions (in particular the exponential function) on these rings. If the ring is not extended it may not be possible to define an exponential function (14).

Theorem 3.6.3. For each x in $\mathcal{R}$, the sequence of partial sums of exp x converges and defines an analytic function on $\mathcal{R}$ into itself.

Proof: Given $\epsilon > 0$, d in D, let x be any element in $\mathcal{R}$. Then

$$\left\| \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \dots + \frac{x^n}{n!} \right\|_d \leq \frac{\|x\|_d^m}{m!} + \frac{\|x\|_d^{m+1}}{(m+1)!} + \dots + \frac{\|x\|_d^n}{n!} < \epsilon ,$$

for $n > m \geq m_0(d, \epsilon)$. Hence the sequence of partial sums converges and since $\mathcal{R}$ is sequentially complete, converges to an element of $\mathcal{R}$.

Further exp x is a continuous function of x. For, given d in D, $\epsilon > 0$

$$\exp(x + y) - \exp(x) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} - \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(x+y)^n - x^n}{n!} .$$

Picking $\delta = \dfrac{\epsilon}{e^{1 + \|x\|_d} - 1} < 1$ it follows that for $\|y\|_d < \delta$,

$$\| \exp(x+y) - \exp(x) \|_d = \Big\| \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\nu=1}^{n} \binom{n}{\nu} x^{n-\nu} y^{\nu} \Big\|_d .$$

$$< \sum_{n=1}^{\infty} \frac{\delta}{n!} \sum_{\nu=1}^{n} \binom{n}{\nu} \|x\|_d^{n-\nu}$$

$$= \sum_{n=1}^{\infty} \frac{\delta}{n!} \left[(1 + \|x\|_d)^n - \|x\|_d^n \right]$$

$$\leq \sum_{n=1}^{\infty} \frac{\delta}{n!} (1 + \|x\|_d)^n$$

$$= \delta\,(e^{1 + \|x\|_d} - 1) = \epsilon .$$

exp x is also G-differentiable.

$$\lim_{\zeta \to 0} \frac{\exp(x + \zeta y) - \exp x}{\zeta} = \lim_{\zeta \to 0} \frac{\sum_{n=0}^{\infty} \frac{(x+\zeta y)^n}{n!} - \sum_{n=0}^{\infty} \frac{x^n}{n!}}{\zeta}$$

$$= \lim_{\zeta \to 0} \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(x+\zeta y)^n - x^n}{\zeta} = \sum_{n=1}^{\infty} \frac{1}{n!} \lim_{\zeta \to 0} \frac{(x+\zeta y)^n - x^n}{\zeta}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} n y x^{n-1} = y \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = y \exp x .$$

Thus $\delta \exp(x;y) = y \exp x$ and the F-differential $d \exp(x;y) = y \exp x$.

3.7 Further Properties of Analytic Functions.

Theorem 3.7.1. Let $f(x)$ be an analytic function on L to L' which vanishes in some neighbourhood in its domain of analyticity $\mathcal{D}$. Then $f(x)$ vanishes identically in $\mathcal{D}$.

Proof. $f(x) = \theta$ for all x in $N(x_0, d, \epsilon) \equiv \{ x \mid \|x - x_0\|_d < \epsilon \}$, where $N(x_o, d, \epsilon) \subset \mathcal{D}$. The expression for $\delta^n f(x_o; x-x_o)$ as an integral shows that $\delta^n f(x_o; x-x_o) = \theta$ for $\|x-x_o\|_d < \epsilon$. Let x be any element of L such that $x - x_o \neq \theta$ and $\|x - x_o\|_d \neq 0$. Setting

$$y = \frac{\epsilon}{2} \frac{x - x_o}{\|x-x_o\|_d} + x_o, \text{ then } \|y-x_o\|_d = \left\| \frac{\epsilon}{2} \frac{x - x_o}{\|x-x_o\|_d} \right\|_d = \frac{\epsilon}{2} < \epsilon ,$$

and consequently

$$\theta = \delta^n f(x_o; \frac{\epsilon}{2} \frac{x - x_o}{\|x-x_o\|_d}) = (\frac{\epsilon}{2})^n \frac{1}{\|x-x_o\|_d^n} \delta^n f(x_o; x-x_o),$$

that is $\delta^n f(x_o; x-x_o) = \theta$. For $x - x_o = \theta$ or $\|x-x_o\|_d = 0$ this is trivially true. The Taylor expansion for $f(x)$ shows that $f(x) = \theta$ in any neighbourhood of x_o contained in $\mathcal{D}$, say $N(x_o)$. Let y be any point of $\mathcal{D}$ not in $N(x_o)$. Since $\mathcal{D}$ is a connected open set it is here also arc-wise connected and the points x_o and y may be joined by a finite chain of neighbourhoods $N(x_o)$, $N(x_1), \ldots, N(x_n)$, such that y is in $N(x_n)$ and $N(x_i)$ contains x_{i+1}. Since x_1 lies in $N(x_o)$, a neighbourhood $N'(x_1) \subset N(x_o)$ can be found in which $f(x)$ vanishes. The preceding shows that $f(x)$ vanishes in $N(x_1)$ and hence by induction $f(y) = \theta$, that is, $f(x) = \theta$ for all x in $\mathcal{D}$.

<u>Theorem 3.7.2.</u> If $f(x)$ on L to L' is analytic at all points of L and if its values lie in a bounded set in L' then $f(x)$ is constant in L that is $f(x) = y_o$, where y_o is a fixed element of L'.

Proof. Let x_0 and x_1 be an arbitrary pair of points in L_1, and consider the function $f(x_0 + \zeta(x_1 - x_0))$ of the complex variable ζ. This is an entire function of ζ whose values lie in a bounded set, and hence by Theorem 2.6.4 takes on a fixed value, say y_0 in L', for all ζ. In particular for $\zeta = 0$, $f(x_0) = y_0$ and for $\zeta = 1$, $f(x_1) = y_0$. Thus $f(x_0) = f(x_1)$.

Theorem 3.7.3. Let $f(x)$ be a function on L to L' analytic in the domain $\mathcal{D}$ and continuous in $\mathcal{D} \cup \mathcal{B}$, where $\mathcal{B}$ is the boundary of $\mathcal{D}$. Then if for any d' in D', $\sup_{x \text{ in } \mathcal{B}} \|f(x)\|_{d'} = M(d')$, either $\|f(x)\|_{d'} < M(d')$ in $\mathcal{D}$ or else $\|f(x)\|_{d'} \equiv M(d')$ in $\mathcal{D}$.

Proof. Suppose x_0 is in $\mathcal{D}$ and $\|f(x_0)\|_{d'} \geqslant M(d')$. Let y in L be fixed but arbitrary and consider the linear manifold $S \equiv \{x \text{ in } L \mid x = x_0 + \zeta y\}$. $S \cap \mathcal{D}$ is open in the relative topology induced in S by L and since x_0 and y are fixed it corresponds to an open set Δ in the complex plane. $S \cap \mathcal{B}$ corresponds to the boundary of Δ, i.e., to $\bar{\Delta} - \Delta$, $\bar{\Delta}$ denoting the closure of Δ. Now x_0 is in $S \cap \mathcal{D}$ and hence Δ must contain $\zeta = 0$. Let Δ_0 be the (connected) component of Δ containing $\zeta = 0$. Then $(\bar{\Delta}_0 - \Delta) \subset (\bar{\Delta} - \Delta)$. The following assertions are then valid:

(i) $f(x_0 + \zeta y)$ is an analytic function of ζ in Δ_0,

(ii) $f(x_0 + \zeta y)$ is continuous in $\bar{\Delta}_0$,

(iii) $\sup_{\zeta \text{ in } \bar{\Delta}_0 - \Delta_0} \|f(x_0 + \zeta y)\|_{d'} \leq M(d')$.

By Theorem 2.5.11, $\| f(x_0 + \zeta y) \|_{d'} \leq M(d')$ for ζ in Δ_0. Hence $\| f(x_0) \|_{d'} = M(d')$. However, the same theorem asserts that $\| f(x_0 + \zeta y) \|_{d'} = M(d')$ everywhere in Δ_0. But y was arbitrary so we may take for y any point x in $\mathcal{D}$ which may be joined with x_0 by a straight line segment lying in $\mathcal{D}$. The above argument then amounts to saying that the assumption $\| f(x_0) \|_{d'} > M(d')$ leads to a contradiction while $\| f(x_0) \|_{d'} = M(d')$ implies $\| f(x) \|_{d'} = M(d')$ for every x having the property just stated. This in turn implies $\| f(x) \|_{d'} = M(d')$ for every x in $\mathcal{D}$ which may be joined with x_0 by a polygonal line lying entirely in $\mathcal{D}$. Hence $\| f(x) \|_{d'} \equiv M(d')$ if equality holds at a single point of $\mathcal{D}$.

Theorem 3.7.3. Let f(x) on L to L' be analytic in the domain $\mathcal{D}$: $0 \leq r_1 < \| x \|_d < r_2$, d in D. Then f(x) may be expanded in the form $f(x) = \sum_{-\infty}^{\infty} p_n(x)$, where $p_n(x) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta x)}{\zeta^{n+1}} d\zeta$, $n = 0, \pm 1, \pm 2, \ldots$, where $r_1 < \| x \|_d < r_2$ and Γ is a circle of radius ρ, $\frac{r_1}{\| x \|_d} < \rho < \frac{r_2}{\| x \|_d}$ about the origin. The functions $p_n(x)$ are analytic when $r_1 < \| x \|_d < r_2$, and $p_n(\zeta x) = \zeta^n p_n(x)$ when ζx and x both satisfy this inequality.

Proof. For a fixed x in $\mathcal{D}$, $f(\zeta x)$ is analytic in ζ when $\frac{r_1}{\| x \|_d} < |\zeta| < \frac{r_2}{\| x \|_d}$. Hence by Theorem 2.6.1, $f(\zeta x) = \sum_{-\infty}^{\infty} p_n(x) \zeta^n$ with $p_n(x)$ as given above. Since the region of convergence of this series includes the point $\zeta = 1$, setting $\zeta = 1$ gives the desired result.

§ 3. The Generalized Cauchy-Riemann Equations

3.8 The Generalized Cauchy-Riemann Equations (21). Let L(C) and L'(C) be the associated couple spaces corresponding to the sequentially complete locally convex real l.t.s. L(R) and L'(R) respectively. D will denote the associated directed system for L(R) and L(C) while D' will denote the associated directed system for L'(R) and L'(C). A function f(z) on L(C) to L'(C) can be written as $f(z) = f_1(x,y) + if_2(x,y)$ where $f_1(x,y)$ and $f_2(x,y)$ are functions of two variables on L(R) to L'(R). Let f(z) be defined in a domain of L(C) to L'(C). The analyticity of f(z) can be discussed in terms of the properties of $f_1(x,y)$ and $f_2(x,y)$.

Theorem 3.8.1. In order that f(z) be analytic in $\mathcal{D}$ it is necessary and sufficient that the functions $f_1(x,y)$, $f_2(x,y)$ be continuous jointly in (x,y) and admit at all points of $\mathcal{D}$ first partial G-differentials which are jointly continuous in (x,y) and that the equations

$$\delta_x f_1(x,y;\ \Delta x) = \delta_y f_2(x,y;\ \Delta x) \qquad 3.2$$

$$\delta_y f_1(x,y;\ \Delta x) = -\delta_x f_2(x,y;\ \Delta x) \qquad 3.3$$

are satisfied in $\mathcal{D}$ for an arbitrary increment Δx in L(R). The partial G-differentials are in fact partial F-differentials.

Proof. If $f(z)$ is analytic in $\mathcal{D}$ it is continuous there, and the differential $df(z; \Delta z)$ is linear in Δz and continuous (separately) in z and Δz. The continuity of $f_1(x,y)$ and $f_2(x,y)$ is then an immediate consequence of that of $f(z)$. Now,

$$df(z; \Delta z) = \lim_{\zeta \to 0} \frac{f(z + \zeta \Delta z) - f(z)}{\zeta}.$$

Hence in particular taking $\Delta z = \Delta x + i \cdot 0$, $\zeta = t$, where t is real,

$$df(z; \Delta x) = \lim_{t \to 0} \frac{f_1(x + t\,\Delta x,y)-f_1(x,y) + if_2(x + t\,\Delta x,y)-if_2(x,y)}{t}$$

This limit will exist only if the separate parts have limits. Hence,

$$df(z; \Delta x) = \delta_x f_1(x,y; \Delta x) + i\,\delta_x f_2(x,y; \Delta x). \qquad 3.4$$

Similarly,

$$df(z; \Delta x) = \delta_y f_2(x,y; \Delta x) - i\,\delta_y f_1(x,y; \Delta x). \qquad 3.5$$

Since the left number of each of these equations is continuous in z the G-differentials in the right members are continuous jointly in (x,y), for fixed Δx, when $x + iy$ is in $\mathcal{D}$. For fixed (x,y) in $\mathcal{D}$ the G-differentials in the right members are continuous in Δx. Equations 3.2 and 3.3 follow by equating real and imaginary parts of the two expressions 3.4 and 3.5.

The partial G-differentials in 3.2 and 3.3 are in fact partial F-differentials. Thus,

$$df(z;a\Delta_1 x + b\Delta_2 x) = \delta_x f_1(x,y;a\Delta_1 x + b\Delta_2 x) + i\,\delta_x f_2(x,y;a\Delta_1 x + b\Delta_2 x),$$

while, $df(z;a\Delta_1 x + b\Delta_2 x) = adf(z;\Delta_1 x) + bdf(z;\Delta_2 x)$

$$= a[\delta_x f_1(x,y;\Delta_1 x) + i\,\delta_x f_2(x,y;\Delta_1 x)]$$

$$+ b[\delta_x f_1(x,y;\Delta_2 x) + i\,\delta_x f_2(x,y;\Delta_2 x)].$$

From these relations, by equating real and imaginary parts of the expressions on the right hand side, it follows that

$$\delta_x f_1(x,y;a\Delta_1 x + b\Delta_2 x) = a\,\delta_x f_1(x,y;\Delta_1 x) + b\,\delta_x f_1(x,y;\Delta_2 x),$$

$$\delta_x f_2(x,y;a\Delta_1 x + b\Delta_2 x) = a\,\delta_x f_2(x,y;\Delta_1 x) + b\,\delta_x f_2(x,y;\Delta_2 x).$$

If $z = x + iy$, then $\|x\|_d \leq \|z\|_d$. Hence

$$\left.\begin{array}{l} \|f_1(x+\Delta x,y) - f_1(x,y) - \delta_x f_1(x,y;\Delta x)\|_{d'} \\ \\ \|f_2(x+\Delta x,y) - f_2(x,y) - \delta_x f_2(x,y;\Delta x)\|_{d'} \end{array}\right\} \leq \|f(z+\Delta x) - f(z) - df(z;\Delta x)\|_{d'},$$

for d' in D'. d in D exists so that given $\eta' > 0$, there is a $\epsilon > 0$ such that for $\|\Delta x\|_d < \epsilon$,

$$\|f(z+\Delta x) - f(z) - df(z;\Delta x)\|_{d'} \leq \eta' \|\Delta x\|_d.$$

Consequently, for $\|\Delta x\|_d < \epsilon$,

$$\|f_1(x+\Delta x,y) - f_1(x,y) - \delta_x f_1(x,y;\Delta x)\|_{d'} \leq \eta' \|\Delta x\|_d,$$

$$\|f_2(x+\Delta x,y) - f_2(x,y) - \delta_x f_2(x,y;\Delta x)\|_{d'} \leq \eta' \|\Delta x\|_d$$

and the partial G-differentials are indeed partial F-differentials.

To prove the sufficiency of the conditions suppose that

$\Delta z = \Delta x + i \Delta y$ is an arbitrary element of L(C) and consider the expression:

$$\frac{f(z+\zeta\Delta z)-f(z)}{\zeta} = \frac{f_1(x+s\Delta x-t\Delta y,\ y+t\Delta x+s\Delta y)-f_1(x,y)}{s+it} + i\,\frac{f_2(x+s\Delta x-t\Delta y,\ y+t\Delta x+s\Delta y)-f_2(x,y)}{s+it}$$

where z is in $\mathcal{D}$ and $\zeta = s + it$ is a sufficiently small complex number.

The function $F_1(s,t,u,v) = f_1(x + s\Delta x - t\Delta y, y + u\Delta x + v\Delta y)$ of four real variables with values in $L'(R)$ is continuous and admits continuous first partial derivatives near (0,0,0,0) (Theorem 3.2.2 and the hypotheses of the theorem.) For a function with these properties, the following mean-value theorem holds, namely,

$$F_1(s,t,u,v) - F_1(0,0,0,0) - s\frac{\partial}{\partial s}F_1(0,0,0,0) - t\frac{\partial F_1}{\partial t}(0,0,0,0) - u\frac{\partial F_1}{\partial u}(0,0,0,0) - v\frac{\partial F_1}{\partial v}(0,0,0,0) = \xi(s,t,u,v) \qquad 3.6$$

where $\lim\limits_{(s,t,u,v)\to(0,0,0,0)} \dfrac{\|\xi(s,t,u,v)\|_{d'}}{|s|+|t|+|u|+|v|} = 0$ for all d' in D'.

Writing $\xi(s,t,u,v) = (F_1(s,t,u,v) - F_1(s,t,u,0) - v\frac{\partial F_1}{\partial v}(s,t,u,0))$

$$+ (F_1(s,t,u,0)-F_1(0,t,u,0)-s\frac{\partial F_1}{\partial s}(0,t,u,0)) + (F_1(0,t,u,0) -$$

$$- F_1(0,0,u,0)-t\frac{\partial F_1}{\partial t}(0,0,u,0))+(F_1(0,0,u,0)-F_1(0,0,0,0)-u\frac{\partial F_1}{\partial u}(0,0,0,0))+$$

$$+v\left(\frac{\partial F_1}{\partial v}(s,t,u,0)-\frac{\partial F_1}{\partial v}(0,0,0,0)\right)+s\left(\frac{\partial F_1}{\partial s}(0,t,u,0)-\frac{\partial F}{\partial s}(0,0,0,0)\right)+$$
$$+t\left(\frac{\partial F_1}{\partial s}(0,0,u,0)-\frac{\partial F_1}{\partial t}(0,0,0,0)\right).$$

The result now follows on making use of the triangular property of the pseudo-norms, the joint continuity of the partial derivatives near (0,0,0,0) and the properties of integration for functions of a real variable with values in a locally convex l.t.s.(12, 18). This last is used to write

$$F_1(s,t,u,v)-F_1(s,t,u,0)-v\frac{\partial F_1}{\partial v}(s,t,u,0)=\int_0^v\left(\frac{\partial F_1}{\partial v}(s,t,u,v)-\frac{\partial F_1}{\partial v}(s,t,u,0)\right)dv.$$

Consequently, when 3.6 is expressed in terms of partial G-differentials, it reads

$$F_1(x+s\,\Delta x-t\,\Delta y,\ y+t\,\Delta x+s\,\Delta y)-F_1(x,y)=s\ \delta_x f_1(x,y;\ \Delta x)$$
$$-t\ \delta_x f_1(x,y;\ \Delta y)+t\ \delta_y f_1(x,y;\ \Delta x)+s\ \delta_y f_1(x,y;\ \Delta y)+\xi\,(s,t,t,s).$$

There is a similar relation involving $f_2(x,y)$ and a quantity $\eta(s,t,t,s)$ such that for any d' in D'

$$\lim_{(s,t)\to(0,0)}\frac{\|\eta(s,t,t,s)\|_{d'}}{|s|+|t|+|t|+|s|}=0\ .$$

Using equations (1) and (2)

$$\frac{f(z+\zeta\Delta z)-f(z)}{\zeta}=\ [(\ \delta_x f_1(x,y;\ \Delta x)-\delta_x f_2(x,y;\ \Delta y))$$
$$+i(\ \delta_x f_1(x,y;\ \Delta y)+\ \delta_x f_2(x,y;\ \Delta x))]\ \frac{(s+it)}{\zeta}+\frac{\xi+i\eta}{\zeta}\ .$$

Now $|\zeta|\geqslant\frac{1}{\sqrt{2}}\ (\ |s|\ +\ |t|\ )$ and for any d' in D',

$\| \xi + i\eta \|_{d'} \leq \| \xi \|_{d'} + \| \eta \|_{d'}$. Hence,

$$\lim_{\zeta \to 0} \frac{\| \xi + i\eta \|_{d'}}{|\zeta|} \leq \sqrt{2} \left(\lim_{(s,t)\to(0,0)} \frac{\| \xi \|_{d'}}{|s| + |t|} + \lim_{(s,t)\to(0,0)} \frac{\| \eta \|_{d'}}{|s| + |t|} \right) = 0.$$

Thus f(z) has the differential 3.7

$$\delta(f;\Delta z) = \delta_x f_1(x,y;\Delta x) - \delta_x f_2(x,y;\Delta y) + i(\delta_x f_1(x,y;\Delta y) + \delta_x f_2(x,y;\Delta x)).$$

Since $f_1(x,y)$ and $f_2(x,y)$ are continuous so is f(z) and f(z) is analytic in $\mathcal{D}$.

<u>Theorem 3.8.2.</u> If $f_1(x,y)$ and $f_2(x,y)$ are two functions on L(R) to L'(R) where L'(R) is complete, and $f(z) = f_1(x,y) + if_2(x,y)$ is analytic in the domain $\mathcal{D}$ of the couple space L(C), then f_1 and f_2 admit total F-differentials in $\mathcal{D}$ (considered as a domain in the composite space L(K)) and

$$df(z;\Delta z) = df_1(x,y;(\Delta x, \Delta y) + idf_2(x,y;(\Delta x, \Delta y)) \qquad 3.8$$

where $\Delta z = \Delta x + i\Delta y$, Δx and Δy being in L(R). The partial F-differentials of f_1 and f_2 of all orders exist and are continuous in $\mathcal{D}$. They have certain symmetry properties (see below) in particular, the generalized or abstract Laplace equation satisfied by the real and imaginary parts of f(z),

$$d^2_{xx} f_k(x,y;(\Delta x;\Delta y)) + d^2_{yy} f_k(x,y;(\Delta x;\Delta y)) = 0 \qquad 3.9$$

$$d^2_{xy} f_k(x,y;(\Delta x;\Delta y)) - d^2_{yx} f_k(x,y;(\Delta x;\Delta y)) = 0 \qquad 3.10$$

k = 1,2.

<u>Proof.</u> Using 3.2 and 3.3, 3.7 may be written as 3.11

$$df(z;\Delta z) = \delta_x f_1(x,y;\Delta x) + \delta_y f_1(x,y;\Delta y) + i(\delta_x f_2(x,y;\Delta x) + \delta_y f_2(x,y;\Delta y).$$

For 3.8 to hold it must be shown that

$$\delta_x f_1(x,y;\ \Delta x) + \delta_y f_1(x,y;\ \Delta y) = df_1(x,y;(\Delta x,\ \Delta y)) \qquad 3.12$$

and, $$\delta_x f_2(x,y;\ \Delta y) + \delta_y f_2(x,y;\ \Delta y) = df_2(x,y;(\Delta x,\ \Delta y)). \qquad 3.13$$

The partial G-differentials occuring in 3.12 and 3.13 are, by Theorem 3.8.1, partial F-differentials. To prove the contention 3.12 $df_1(x,y;(\Delta x, \Delta y))$ must be shown to be linear in the increment

$$df_1(x,y;(\Delta_1 x,\ \Delta_1 y)+(\Delta_2 x,\ \Delta_2 y)) = df_1(x,y;(\Delta_1 x + \Delta_2 x,\ \Delta_1 y + \Delta_2 y))$$
$$= \delta_x f_1(x,y;\ \Delta_1 x + \Delta_2 x) + \delta_y f_1(x,y;\ \Delta_1 y + \Delta_2 y)$$
$$= \delta_x f_1(x,y;\ \Delta_1 x) + \delta_y f_1(x,y;\ \Delta_1 y) + \delta_x f_1(x,y;\ \Delta_2 x) + \delta_y f_1(x,y;\Delta_2 y)$$
$$= df_1(x,y;(\Delta_1 x,\ \Delta_1 y)) + df_2(x,y;(\Delta_2 x,\ \Delta_2 y)).$$

To verify that $df_1(x,y;(\Delta x, \Delta y))$ is a continuous function of the increment it is sufficient to verify continuity at $(\Delta x, \Delta y) = (\theta,\theta)$. Thus given d' in D', $\epsilon > 0$, there exists d in D, $\eta > 0$ such that $\|\Delta x\|_d < \frac{\eta}{2}$, $\|\Delta y\|_d < \frac{\eta}{2}$ implies $\|\delta_x f_1(x,y;\ \Delta x)\|_{d'} < \frac{\epsilon}{2}$ and $\|\delta_y f_1(x,y;\ \Delta y)\|_{d'} < \frac{\epsilon}{2}$. Consequently $\|df_1(x,y;(\Delta x, \Delta y))\|_{d'} < \epsilon$ for $\|(\Delta x, \Delta y)\|_d = \|x\|_d + \|y\|_d < \eta$. Further given d' in D' there corresponds d in D with the property that given $\eta' > 0$ there exists $\delta > 0$ such that $\|\Delta z\|_d < \delta$ implies $\|f(z + \Delta z)-f(z)-df(z;\ \Delta z)\|_{d'} \leq \eta' \|\Delta z\|_d$. Hence if $\|(\Delta x, \Delta y)\|_d < \delta$, and recalling that if $x' + iy'$ is an element of $L'(C)$ then $\|x'\|_{d'} \leq \|x' + iy'\|_{d'}$, it follows that

$$\|f_1(x+\Delta x, y+\Delta y) - f_1(x,y) - df_1(x,y;(\Delta x, \Delta y))\|_{d'}$$
$$\leq \|f(z+\Delta z) - f(z) - df(z;\Delta z)\|_{d'}$$
$$\leq \eta' \|\Delta z\|_d$$
$$\leq \eta'(\|\Delta x\|_d + \|\Delta y\|_d) = \eta' \|(\Delta x, \Delta y)\|_d.$$

Thus $df_1(x,y;(\Delta x, \Delta y))$ is in fact the total F-differential of $f_1(x,y)$. Similarly $df_2(x,y;(\Delta x, \Delta y))$ is the total F-differential of $f_2(x,y)$.

When $f(z)$ is analytic it has F-differentials of all orders which are jointly continuous in the increments and symmetric multilinear forms in these increments. From this the further properties of f_1 and f_2 stated in the theorem follow. Thus when $\Delta z = (\Delta x, \Delta y)$, $df(z;\Delta x)$ and $df(z;\Delta y)$ are differentiable functions of z and the expression 3.4 applied to them leads to

$$d^2f(z;\Delta x;\Delta y) = d^2_{xx}f_1(x,y;\Delta x;\Delta y) + id^2_{xx}f_2(x,y;\Delta x;\Delta y) \qquad 3.14$$

and

$$d^2f(z;\Delta y;\Delta x) = d^2_{xx}f_1(x,y;\Delta x;\Delta y) + id^2_{xx}f_2(x,y;\Delta x;\Delta y). \qquad 3.15$$

But $d^2f(z;\Delta x;\Delta y) = d^2f(z;\Delta y;\Delta x)$ and on comparing 3.14 and 3.15

$$d^2_{xx}f_1(x,y;\Delta x;\Delta y) = d^2_{xx}f_1(x,y;\Delta y;\Delta x) \qquad 3.16$$

and
$$d^2_{xx}f_2(x,y;\Delta x;\Delta y) = d^2_{xx}f_2(x,y;\Delta y;\Delta x). \qquad 3.17$$

Similarly using the equation 3.5, the expressions

$$d^2_{yy}f_1(x,y;\Delta x;\Delta y) = d^2_{yy}f_1(x,y;\Delta y;\Delta x) \qquad 3.18$$

and
$$d^2_{yy}f_2(x,y;\Delta x;\Delta y) = d^2_{yy}f_2(x,y;\Delta y;\Delta x). \qquad 3.19$$

are valid.

By considering 3.4 as the given analytic function and writing down for it the form of the differential given by 3.5 it is seen that

$$d^2f(z;\ \Delta x;\ \Delta y) = d^2_{xy}f_2(x,y;\ \Delta x;\ \Delta y) - i\ d^2_{xy}f_1(x,y;\ \Delta x;\ \Delta y) \quad 3.20$$

On the other hand the expression 3.5 may be considered as the given analytic function whose differential is given according to the expression 3.4 by

$$d^2f(z;\ \Delta x;\ \Delta y) = d^2_{yx}f_2(x,y;\ \Delta x;\ \Delta y) - i\ d^2_{yx}f_1(x,y;\ \Delta x;\ \Delta y) \quad 3.21$$

Then comparing 3.20 and 3.21, the expressions of 3.10 are obtained in

$$d^2_{xy}f_1(x,y;\ \Delta x;\ \Delta y) = d^2_{yx}f_1(x,y;\ \Delta x;\ \Delta y) \quad 3.22$$

and

$$d^2_{xy}f_2(x,y;\ \Delta x;\ \Delta y) = d^2_{yx}f_2(x,y;\ \Delta x;\ \Delta y) \quad 3.23$$

Other relations, of the form

$$d^2_{xy}f_1(x,y;\ \Delta x;\ \Delta y) = d^2_{xy}f_1(x,y;\ \Delta y;\ \Delta x), \quad 3.24$$

$$d^2_{xy}f_2f(x,y;\ \Delta x;\ \Delta y) = d^2_{xy}f_2(x,y;\ \Delta y;\ \Delta x), \quad 3.25$$

and

$$d^2_{xy}f_1(x,y;\ \Delta x;\ \Delta y) = d^2_{yx}f_1(x,y;\ \Delta y;\ \Delta x), \quad 3.26$$

$$d^2_{xy}f_2(x,y;\ \Delta x;\ \Delta y) = d^2_{yx}f_2(x,y;\ \Delta y;\ \Delta x), \quad 3.27$$

can be obtained similarly.

Taking the partial F-differential with respect to x, with increment Δ y, of both sides of 3.2 gives

$$d^2_{xx}f_1(x,y;\ \Delta x;\ \Delta y) = d^2_{yx}f_2(x,y;\ \Delta x;\ \Delta y), \quad 3.28$$

and from 3.3 on using 3.23

$$d^2_{yy}f_1(x,y;\ \Delta x;\ \Delta y) = -d^2_{xy}f_2(x,y;\ \Delta x;\ \Delta y)$$

$$= -d^2_{yx}f_2(x,y;\ \Delta x;\ \Delta y) \quad 3.29$$

Comparing 3.28 and 3.29 gives

$$d^2_{xx}f_1(x,y;\ \Delta x;\ \Delta y) + d^2_{yy}f_1(x,y;\ \Delta x;\ \Delta y) = 0. \qquad 3.30$$

Similarly

$$d^2_{xx}f_2(x,y;\ \Delta x;\ \Delta y) + d^2_{yy}f_2(x,y;\ \Delta x;\ \Delta y) = 0. \qquad 3.31$$

In concluding this thesis some comments of a historical nature are appropriate. The earliest work on a theory of analytic functions in normed linear spaces was done during the year 1931-1932 by Professor A. D. Michal in collaboration with R. S. Martin. In a statement in (27) Professor Michal himself supplies details on this work which was done in his seminar at the California Institute of Technology. These same men continued with this study and in 1934 introduced the concept of a normed linear ring (28).

With regard to couple spaces the pseudo-norms defined in Theorem 1.5.1 are analogous to certain norms used in couple Banach spaces. Michal and Wyman, in 1941 (29), gave another definition of a norm for couple Banach spaces.

The Cauchy-Riemann theory considered in Chapter III is a generalization of work done by Taylor (21). Further work along this line was done by Michal, Davis and Wyman (30), who gave an introduction to a theory of polygenic functions in normed complex couple spaces.

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