I. ELECTRON DECAY OF THE π-MESON

II. A NON-LINEAR FIELD THEORY

Thesis by

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PART I.

ABSTRACT

A π-meson decays into μ-meson and neutrino at least 1000 times faster than into an electron and a neutrino. After summarizing the difficulties in assuming that electrons or μ-mesons interact with nucleons through the intermediary of the π-meson, the decay of the π is discussed for the symmetric coupling scheme in which electrons and μ-mesons interact directly with nucleons. Selection rules rigorously forbid this decay for most choices of the π-meson field and the form of nuclear β-decay. For the very special case of pseudoscalar meson and pseudovector β-decay (with arbitrary mixtures of scalar, vector and tensor) the decay rate for $\pi \longrightarrow (\mu, \nu)$ proceeds 10^4 times as fast as $\pi \longrightarrow (e, \nu)$ and 10^{+3} as fast as $\pi \longrightarrow (\text{photon}, e, \nu)$. This result is independent of perturbation theory. Agreement with the observed lifetime can be obtained if the divergent integral is cut off at the nucleon Compton wavelength.

PART II.

ABSTRACT

A unitary theory of particles is investigated, mostly on the classical level. The Dirac and the Klein-Gordon equations are augmented by simple non-linear terms. Interpreted as wave equations for classical fields they contain a much richer variety of solutions than the customary linear theories. Particles, instead of having independent existences as singularities, appear only as intense localized regions of strong field. Solutions of the field equations are subject to the boundary condition that the fields be regular everywhere and that all observable integrals be finite. For simple angular and temporal dependence the wave equation reduces to a set of ordinary differential equations. The boundary condition leads to a non-linear eigenvalue problem whose solutions are systematically described in the phase plane. Numerical solutions are found for some typical cases. The masses of the particles are positive; the number carrying unit charge is small. The scalar field variables can be interpreted in terms of operators according to the usual commutation rules, but the particles are unstable when perturbed by quantum fluctuations. The application of anti-commutation rules to the spinor fields has no classical limit. The lack of a satisfactory recipe for quantizing classical spinor fields makes the interpretation of the particle-like solutions obscure.

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I. INTRODUCTION

In proposing a description of nuclear forces through the interaction of a mesic field, Yukawa(1) suggested that a meson would decay into an electron and a neutrino, and that this was responsible for the β-decay of nuclei $\quad {}_ZN^A \longrightarrow {}_{Z+1}N^A + \pi^- \longrightarrow {}_{Z+1}N^A + e^- + \nu \qquad (1)$

That sea level mesons were observed to decay into electrons with a lifetime of 2×10^{-6} sec was considered a triumph of meson theory, but quantitative agreement with the predicted lifetime has been lacking. The discoveries of the very small interactions of sea level mesons with nuclei(2) and of the existence of at least two kinds of mesons in cosmic rays(3) has necessitated a reinterpretation of the decay schemes of mesons and nucleons. It now appears certain that the sea level or μ-mesons which were observed to be β-active are only very weakly coupled to nucleons and are not responsible for (1).

We shall present first some of the evidence which makes it appear probable that the meson field which is strongly coupled to nucleons (π-meson) transforms like a pseudoscalar (spin 0, odd parity). The possible bearing of this on the decay of the π-meson and on nuclear β-decay will then be discussed.

II. PARITY AND SPIN OF π- AND μ-MESONS*

π-mesons, which are copiously produced in nucleon-nucleon collisions or through the interaction of nucleons with gamma rays, may be charged positively, negatively, or may be neutral. The charged π-mesons decay into μ-mesons with a lifetime of 2.6×10^{-8} sec. Since the μ-meson is monoenergetic only one other particle (neutral) is emitted in the decay. The energies are consistent with assigning a zero mass to the neutral particle. The neutral meson decays into two photons in less than 10^{-14} sec. Since a system of angular

* We follow here a lecture of H. Bethe given at the California Institute of Technology, January 8, 1951.

momentum 1 is rigorously forbidden from decaying into two photons by conservation of momentum, the spin of the neutral π-meson cannot be 1. If we assume that the elementary particles have spins 0, 1/2, or 1, it follows that neutral π-mesons have spin 0.

Investigation of the capture of negative π's by Hydrogen indicates that the reactions

$$\text{(a)} \quad H + \pi^- \longrightarrow n + \gamma$$

and

$$\text{(b)} \quad H + \pi^- \longrightarrow n + \pi^0 \longrightarrow n + 2\gamma$$

proceed with about equal probability(4). Since the neutral meson is observed to be monoenergetic no other particle is emitted in addition to the neutron. Since the proton and the neutron have half integer spin, conservation of angular momentum implies the charged π-meson has spin 0 or spin 1. This agrees with the observed fact that when π-mesons are captured by nuclei stars result, since the rest mass of the π is converted into excitation energy of the nucleons. Quite the opposite behavior is observed when the μ-meson is captured by a nucleus. Very little of the rest energy of the μ becomes excitation energy. A neutral particle, other than a photon, carries off most of it. The apparent inability of a nucleon to absorb a μ-meson is usually attributed to the fact that the spin of the μ-meson is 1/2. The μ-meson decays into an electron and at least two neutral particles of small, probably 0, mass. A consistent designation of the particles involved in the $\pi - \mu$ decay and in the $\mu - e$ decay which agree with the above is:

$$\pi^{\pm} \longrightarrow \mu^{\pm} + \nu$$
$$\mu^{\pm} \longrightarrow e^{\pm} + 2\nu$$

The capture of π^- by deuterons can energetically result in any of the following processes:

$$\text{(c)} \quad \pi^- + D^+ \longrightarrow 2n$$

(d) $\pi^- + D^+ \longrightarrow 2n + \gamma$

(e) $\pi^- + D^+ \longrightarrow 2n + \pi^0 \longrightarrow 2n + 2\gamma$

From the capture by Hydrogen and the emission of a neutral π^0, the mass difference between the charged and the neutral mesons has been estimated as 10.6 ± 2 electron masses.[5] Process (e) can then proceed with an energy of 4.7 ± 2 e.m. It is not impossible that more accurate data will show (e) to be energetically impossible. A comparison of the yield (d) with that in Hydrogen gives a probability of .275 for (d)[5]. The probability of (e) is less than .05, possibly 0. One then infers that the probability of (c) is .70. It has been shown[6] that the meson will be captured into the K shell of the deuteron in a time which is very short compared to the lifetime of the π. The ground state of the deuteron has J = 1 and even parity.* If the π-meson has 0 spin and is captured into the K shell of the deuteron, the total system has J = 1 and the parity of the meson. Hence in (c), the neutrons must come out in a J = 1 state. The Pauli Exclusion principle allows only the $^3P_1$ which has odd parity. Since (c) is not forbidden, we conclude that if the charged meson has 0 spin it is a pseudoscalar. A consideration of the possibility of spin 1 mesons in (c) yields no information about their parity. If π^- and π^0 have 0 spin and odd parity (b) is allowed even for very small mass difference between the mesons. In view of the very small Q for the reaction (e) it will be forbidden. The π^--meson and the deuteron form a state with odd parity and J = 1. The low energy π^0 will come off in an S state. To conserve angular momentum, the neutrons must form a $^3P_1$ state. Since the intrinsic parity of π^0 is assumed odd, the

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* An ambiguity in the inversion properties of spinors makes it possible that protons and neutrons transform differently under reflections. The parity of the deuteron can then be made negative. $\Psi^+_P \Psi_N$ will be a pseudoscalar, etc. The result is only a renaming of the usual meson theories for charged mesons, but does not affect the physical content.

final state has even parity and will be forbidden as indicated by experiment. If π^+ and π^0 have opposite parity (b) is partially forbidden and (e) is allowed in contradiction to the data.

According to Brueckner the angular distribution of photomesons[8] (charged), and the ratio of positive to negative photomesons from carbon[8] favor the couplings of the pseudoscalar theory. Only this theory seems capable of explaining the approximate equality of photoproduction of charged and neutral π[9].

The role of the π-meson in nuclear forces is obscure especially in view of the existence of other mesons (V-mesons). The pseudoscalar theory gives spin dependent forces, and the correct sign of the tensor force.

III. COUPLING SCHEMES FOR NUCLEONS, MESONS, AND LEPTONS.

We assume that the π-meson is coupled directly to nucleons, and that the nucleons are Dirac particles. The interaction:

$$g \phi \Psi_P \, O \, \Psi_N \qquad + \text{c.c} \tag{3}$$

between the Dirac and the meson fields (or any of the derivative couplings) leads to the real capture process:

$$N + \pi^+ \longrightarrow P^+ \tag{4}$$

but it also permits the virtual decomposition of the π :

$$\pi^+ \longrightarrow P^+ + N^- \tag{5}$$

where N^- signifies an anti-neutron. On the other hand, to account for nuclear β-decay along the general lines of the Fermi or Gamow-Teller theories, one postulates the interaction:

$$g_A(\Psi_N^+ A \Psi_P)(\Psi_e^+ A \Psi_\nu) \tag{6}$$

A is a Dirac operator chosen so that the coupling is invariant under the

Improper Lorentz Group. Thus interaction leads to the observed β-process:

$$N \longrightarrow P^+ + e^- + \nu \tag{7}$$

but it also leads one to expect the reaction:

$$P^+ + N^- \longrightarrow e^+ + \nu \tag{9}$$

The virtual decomposition (5) followed by (9) leads to the real decay:

$$\pi^+ \longrightarrow e^+ + \nu \tag{10}$$

Any theory which couples π-mesons to nucleons (this need not be a direct coupling) also predicts the $\pi \longrightarrow (e, \nu)$ decay. This argument depends not on the existence of real anti-neutrons, but only on the role of such particles in virtual processes. Rainwater has looked for the possible electrons from stopped π-mesons. In 760 cases no electrons were found.* Therefore the decay of the π-meson into a μ-meson and a neutrino must proceed at a rate at least 1000 times as fast as the decay into an electron and a neutrino. In order to compare this ratio of rates with theory it is necessary to specify in some detail the coupling scheme for the interaction of nucleons, mesons, and electrons.

The Model I (Fig. 1) was proposed by Inoue and Ogawa(10). Nuclear β-decay takes place through an unstable π-meson as originally suggested by Yukawa. However, when the coupling constants are adjusted to fit the data on the lifetime for $\pi - \mu$ decay, the coupling of π-mesons to nucleons, and the Fermi constant for nuclear β-decay, the lifetime of the free π-meson for decay into an electron and neutrino is smaller than the lifetime for decay into a μ-meson and a neutrino.

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* We wish to thank Professor R. P. Feynman for reporting the unpublished results of Professor Rainwater.

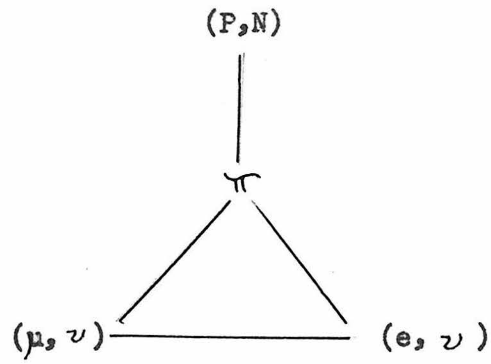
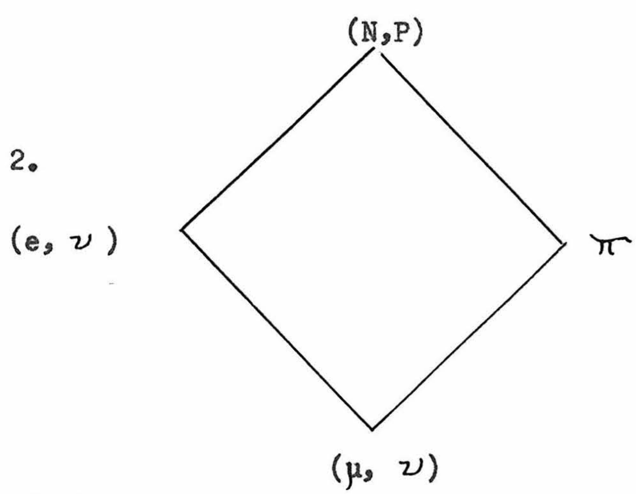


Fig. 1

Lopes[11] has shown how this can be modified if the π-meson is pseudoscalar. But in this case Sargent's Law for allowed β-decay lifetimes does not hold; the lifetimes would be proportional to the inverse seventh power of the maximum electron energy instead of the observed inverse fifth power.

Model II (Fig. 2) assumes a direct coupling between the π^- and the μ - mesons[11],[13]; the Yukawa picture of β-decay is replaced by a direct coupling between nuclei and leptons.

Fig. 2.



Latter and Christy[12] have calculated the π - μ decay rate on this model, estimating the coupling of the π-mesons to nuclei from nuclear forces and the rate of the second order reaction

$$\mu^- + P^+ \longrightarrow \begin{bmatrix} \mu^- + \pi^+ + N \\ \pi^- + \nu + P \end{bmatrix} \longrightarrow N + \nu$$

from the competition with μ - β-decay. Assuming spin 1/2 for the μ-meson the calculated lifetime for the π is: 2×10^{-8} sec. for a vector, 10^{-8} sec. for a scalar, and 2×10^{-9} sec. for a pseudoscalar. These are to be compared to the observed lifetime 2×10^{-8} sec. The value of the coupling constant between π and nucleons and the nuclear matrix elements involved in the computation of the μ capture are not known with sufficient accuracy to categorically exclude the pseudoscalar. (In the calculation of Latter and Christy both of these factors were probably overestimated. The value of $\frac{g^2}{\hbar c}$ used was 1/3 for the derivative coupling of meson to nucleons and 50 for the pseudoscalar coupling. The nuclear matrix element was taken as Z. It is certainly less than this, probably about Z/3[12]. The computed lifetimes of the π-meson would then be too large. This would make the correctly calculated lifetime for the pseudoscalar meson agree even less with experiment.*)

The π-meson can decay into an electron and a neutrino by virtue of its coupling to nucleons†:

$$\pi^+ \longrightarrow P^+ + N^- \longrightarrow e^+ + \nu \qquad (11)$$

A perturbation theory calculation of this rate diverges.

A satisfactory direct coupling model will have to show that this rate is at most 1/1000 that of the $\pi - \mu$ decay††.

Extensive calculations on Model III (Fig. 3) have been made by Wheeler and Tiomno[14] and others[15],[16].

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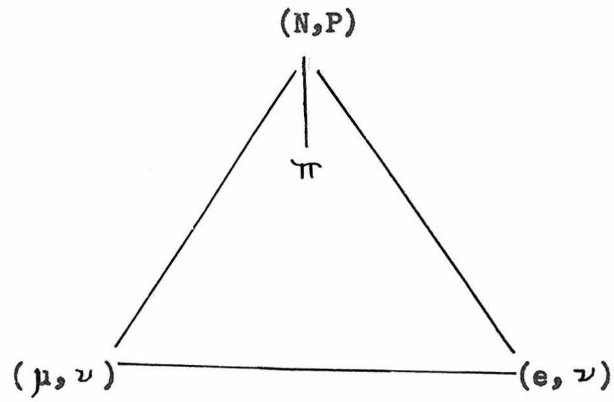
* Discussion with Dr. Latter on this point was very helpful.

† The electron decay of the π-meson through its coupling to the μ-meson will of course be negligible next to the $\pi - \mu$ decay.

†† If the divergent integrals of the perturbation theory are cut off at the nucleon Compton wavelength, the π - e decay has a lifetime of approximately 3×10^{-10} sec., except when the meson is pseudoscalar and the β-decay is pseudovector. The lifetime is then about 10^{-4} sec. in accord with experiment cf. Equation (27).

Fig. 3.



According to this symmetric coupling scheme the following three processes:

μ - capture	$\mu^- + P^+ \longrightarrow N + \nu$	(12a)
β - decay	$N \longrightarrow P + e^- + \nu$	(12b)
μ - decay	$\mu^- \longrightarrow e^- + \nu + \nu$	(12c)

result from the direct couplings:

$$g_a(\Psi_N^+ A \Psi_P)(\Psi_\nu^+ A \Psi_\mu) \tag{13a}$$

$$g_b(\Psi_N^+ B \Psi_P)(\Psi_\nu^+ B \Psi_e) \tag{13b}$$

$$g_c(\Psi_\nu^+ C \Psi_\mu)(\Psi_\nu^+ C \Psi_e) \tag{13c}$$

All of the above fields are spinor fields; A, B, and C are Dirac operators. It has been found that*:

$$g_a \cong g_b \cong g_c \cong 2 \times 10^{-49} \quad \text{erg cm}^3 \tag{14}$$

We shall adopt the attractive hypothesis:

$$g_a = g_b = g_c \tag{15}$$

$$A = B = C \tag{16}$$

The three couplings among spinor particles are thus assumed to be of the same

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* If the operator B is pseudoscalar, this is no longer true.

nature and strength.

IV. THE DECAY OF THE π -MESON.

With the symmetric coupling scheme the decay of the π-meson into a μ-meson and a neutrino is a second order process:

$$\pi^{+} \longrightarrow P^{+} + N^{-} \tag{17}$$

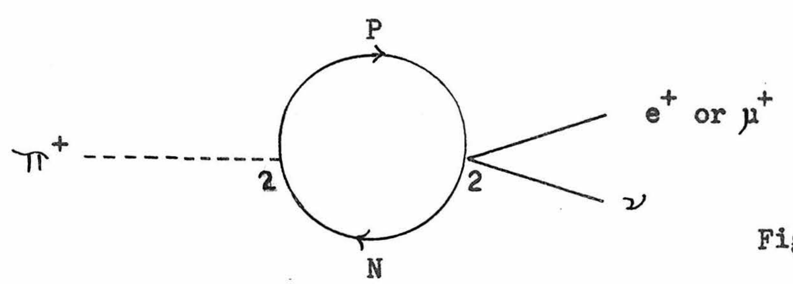
$$P^{+} + N^{-} \longrightarrow \mu^{+} + \nu \tag{18}$$

The matrix element for (18) is contributed by (13a). The matrix element for (17) can be any of the couplings (Eq 2 -Appendix II). The π-meson can decay into an electron and a neutrino through a similar second order process:

$$\pi^{+} \longrightarrow P^{+} + N^{-} \tag{19}$$

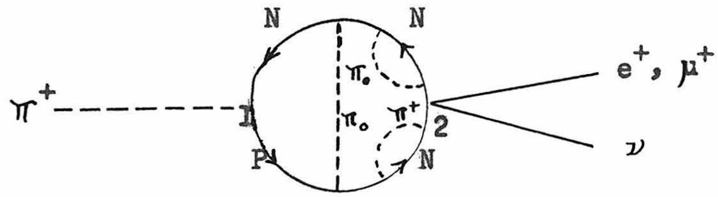
$$P^{+} + N^{-} \longrightarrow e^{+} + \nu \tag{20}$$

Here the matrix element for (20) comes from (13b). A perturbation theory calculation of the rates of the two competing decays $\pi \longrightarrow (\mu, \nu)$, $\pi \longrightarrow (e, \nu)$ gives divergent integrals. However, the ratio of these two rates will be independent of the ultimate value of the ambiguous integral. For the interaction of pseudoscalar mesons with nucleons, perturbation theory is probably inappropriate because of the large coupling constant. The spinor interactions are very weak so that for these first order perturbation theory is probably sufficient. The ratio of the decay rates does not depend on the details of the interaction; we need only make use of the transformation properties of the meson field and the choice of Dirac operator for (16). Typical Feynman diagrams for matrix elements which can lead to the decay of the π into a lepton pair are given in Fig. 4.

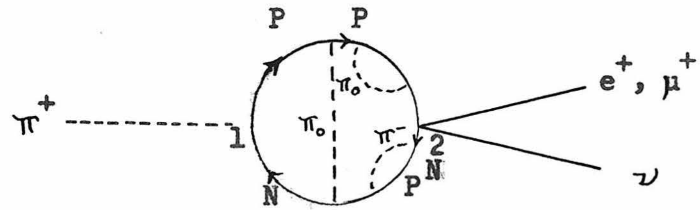


(a)

Fig. 4



(b)



(c)

The matrix element for the annihilation of the π -meson at the space-time point 1 and the creation of the lepton pair at 2 depends upon the lepton pair only through the multiplicative factor:

$$\Psi^{+}_{\substack{e\\ \mu}} (x_2,t_2) \quad A\, \Psi_{\nu}(x_2,t^2) \tag{21}$$

where A is the Dirac operator in (16). If we use a proper coordinate system in which the π-meson is at rest (21) is independent of the space coordinate 2. The time dependence part is $\exp(iE_\pi t_2)$, where E_π is the rest energy of the meson; therefore the term:

$$\Psi^{+}_{\substack{e\\ \mu}} (0,0)\; A\, \Psi_{\nu}(0,0) \tag{22}$$

can be removed from the integration over all space-time points 2. The transition probability for the transition $\pi \longrightarrow (e, \nu)$ or (μ, ν) is:

$$\frac{2\pi}{\hbar} \, |H|^2 \, \rho\,(E) \tag{23}$$

$\rho(E)$ is the density of states per unit energy for the leptons. H is the matrix element for the annihilation of the π and the creation of the pair.

The ratio of the lifetime for $\pi \longrightarrow (e, \nu)$ to that for $\pi \longrightarrow (\mu, \nu)$ is:

$$\frac{\sum_{\sigma_\mu \sigma_\nu} \left| \Psi_\mu^+ (0,0)\; 0\; \Psi_\nu (0,0) \right|^2 \rho_{\mu,\nu}(E)}{\sum_{\sigma_e \sigma_\nu} \left| \Psi_e^+ (0,0)\; 0\; \Psi_\nu (0,0) \right|^2 \rho_{e,\nu}(E)} \tag{24}$$

The summation is over the spins of the lepton pair.

Suppose the nuclear β-decay takes place through a heavy intermediate particle τ. This includes the direct interaction in the limit of an infinite mass for the intermediate particle. Instead of the creation of the lepton pair at 2 we have:

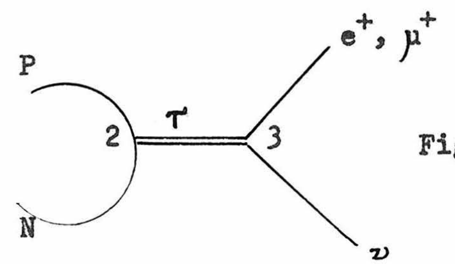


Fig. 5.

Since the coupling of the τ particle to leptons and to nucleons is Lorentz invariant, in the proper system the spin and intrinsic parity of the τ at 2 must be the same as those of the π at 1 since all of the couplings from 1 to 2 are invariant to rotation and inversion and both particles are at rest. The operator A must have the same transformation properties as the τ particle for the decay at 3 to proceed with conservation of spin and parity. Therefore only if A and the π-meson have the same behavior under space rotations and inversions will the π-lepton decay be allowed.*

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*Feynman has pointed out an alternative method for deriving this selection rule. If (cf. Fig. 4) the nucleon pair is created a 1 through the Dirac matrix 0 and the final pair is annihilated at 2 by the matrix A, the matrix element will be of the form:

$$H = \text{spur} \iint \cdot\cdot \int 0 \left[K_+(1,r_1) 0 K_+(r_1,r_2) \ldots K_+(r_n,2)\right] A \left[K_+(2,r_{n+1}) 0 \ldots K_+(r_m,1)\right]$$

$$K_+(r_1,r_2) = \int d^4p \, \frac{(-ip^\mu r_{1\mu} + ip^\mu r_{2\mu}}{p^u \gamma_\mu - m} \qquad d^4 r_1 \ldots d^4 r_m$$

The propagation kernels for the meson lines have been omitted since they contribute only a scalar numerical factor. The integration over all space coordinates insures invariance under $x \longrightarrow -x$; $y \longrightarrow -y$; $z \longrightarrow -z$. If the matrix element changes sign under any of the space inversions, it is zero. Since any spur of a product of Dirac matrices is unchanged by $\gamma_\mu \longrightarrow \gamma_\mu$ $\mu = 1,2,3$, H must be invariant to each of the three transformations:

$$\text{(a)} \quad x \longrightarrow -x, \gamma_1 \longrightarrow -\gamma_1$$

$$\text{(b)} \quad y \longrightarrow -y, \gamma_2 \longrightarrow -\gamma_2$$

$$\text{(c)} \quad z \longrightarrow -z, \gamma_3 \longrightarrow -\gamma_3$$

The kernels K_+ are by themselves invariant to these transformations. Since the factors O within the brackets occur in pairs (each meson that is created is also annihilated) the bracket is invariant, and we must have:

$$A\,O \longrightarrow A\,O \quad \text{under (a), (b), (c).}$$

This is equivalent to the selection rule.

In addition to spin and parity conservation, Furry's theorem[17] will forbid certain decay schemes to all orders. If a Feynman diagram contains a closed loop with an odd number of even matrix elements it will cancel with the matrix element from the Feynman diagram taken in the opposite sense.* Vector and anti-symmetric tensor interactions are "even"; scalar, pseudoscalar and pseudovector are "odd". For example, the matrix element for diagram (a) Fig. 4 will be zero if O is I (unit matrix) and A is γ_4 (fourth component of vector). The more involved diagrams involving the ephemeral existence of many mesons and nucleon pairs differ from (a) by an even number of operators O since every meson created by an O interaction is also annihilated. Therefore if Furry's theorem forbids a decay through virtual pairs for any order it forbids it for all orders.

The ratio of $\pi \longrightarrow (e, \nu)$ to $\pi \longrightarrow (\mu, \nu)$ decay for the direct couplings of Appendix II, (2a), (2c), (2e), and (2g) for meson-nucleon interaction any of the five β-decay interactions† is given in Table I.†† The masses of the proton and neutron have been taken equal ; the mass of the π-meson is assumed to be 286 e.m. The mass of the μ-meson is taken as 215 e.m.

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* When reversing the sense of all the Feynman diagrams it is also necessary to relabel all the protons and neutrons. For the Furry theorem to be valid the absolute value of the coupling constant between neutral mesons and neutrons must be equal to that between neutral mesons and protons. For example in Fig. (4b, 4c), for that neutral which is exchanged between a proton and a neutron only the product of the two coupling constants enters and these need not be equal. In Fig. (4b) another neutral is emitted and captured by a neutron, in Fig. (4c) by a proton. For the diagrams to be dual the square of the coupling constant must be the same. Therefore Furry's theorem will hold for the symmetric theory where the neutrals are coupled with τ_3 or if they have equal coupling constants but not for a mixture of both.

† These are:

I	scalar
$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$	pseudoscalar
γ_μ	vector
$\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$	antisymmetric tensor
$\gamma_5\gamma_\mu$	

†† See Appendix I

RATIO OF THE DECAY RATES FOR $\pi \longrightarrow (e, \nu)$ AND $\pi \longrightarrow (\mu, \nu)$

(DIRECT COUPLING)

Meson Field	Type of β-Decay				
	Scalar	P-scalar	Vector	P-vector	Tensor
Scalar	5.1	S	F	S	S and F
P-scalar	S	5.1	S	1.0×10^{-4}	S and F
Vector	S and F	S and F	4.0	S and F	2.4
P-vector	S	S	S and F	4.0	F

S indicates that the transition is forbidden to all orders by the selection rule for conservation of parity and angular momentum. F signifies that Furry's theorem forbids the decay.

The symmetric coupling scheme is in agreement with experimental facts (no $\pi \longrightarrow (e, \nu)$ decays observed) only if the meson field is pseudoscalar and β-decay coupling contains a pseudovector term. The β-decay may also contain arbitrary mixtures of scalar, vector and tensor terms since these do not contribute to the decay; a pseudoscalar term in β-decay is forbidden.* That the π-meson has to be pseudoscalar is in agreement with the conclusions of Section 2. An analysis of nuclear β-decay by Feingold and Wigner(18) have led these authors to the conclusion that pseudovector coupling is most probably responsible for allowed transitions and that vector and tensor interactions are small. The probable magnitude of scalar and pseudoscalar could not be determined.

A consideration of meson theories with derivative coupling does not give as clear cut a result as the direct coupling calculations.

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* The S-A-P coupling(19) for nuclear β-decay of Wigner and Critchfield would not be permitted.

TABLE II

RATIO OF THE DECAY RATES FOR $\pi \rightarrow (e, \nu)$ AND $\pi \rightarrow (\mu, \nu)$

(GRADIENT COUPLING)

	Type of β-decay				
	Scalar	P-scalar	Vector	P-vector	Tensor
Scalar	F and S	F and S	1.0×10^{-4}	F and S	S
P-scalar	S	5.1	F and S	1.0×10^{-4}	F
Vector	F and S	F and S	4.0	F and S	2.4
P-vector	S	S	F and S	F	2.4

On the basis of the $\pi - \mu$ decay alone one cannot exclude the possibility that the π is a scalar with derivative coupling* to nucleons. The data summarized in Section 2 indicate that if the charged π has zero spin it must be pseudo-scalar. The derivative coupling of neutral scalar mesons to nucleons is exactly equivalent to no coupling at all to all orders; the neutral is certainly not scalar with derivative coupling. The transition probability for the decay of the pseudoscalar π-meson as calculated by first order perturbation theory is:

$$\frac{1}{t} = 2\pi \; |H|^2 \rho(E) = \sum_{\sigma_\nu \sigma_\mu} 2\pi g^2 g_A^2 \left| \int \overline{dp} \sum_{\sigma_P \sigma_N} \left(\frac{1}{2\mu_\pi} \right)^{1/2} \right.$$
$$\times \left(\frac{\phi_\pi \cdot \langle u_P^+ \, O \, u_N \rangle \; \langle u_N^+ A \, u_P \rangle \cdot \langle u_\mu^+ A \, u_\nu \rangle}{2E_P - \mu_\pi} \right.$$
$$\left. \left. + \frac{\langle u_P^+ A \, u_N \rangle \cdot \langle u_\mu^+ A \, u_\nu \rangle \; \phi_\pi \cdot \langle u_N^+ \, O \, u_P \rangle}{2E_P + u_\pi} \right) \right|^2 \rho_\mu(E) \qquad (25)$$

$\hbar = c = 1$

μ_π is the mass of the π-mesons. u_P, u_N, u_μ, u_ν are the spinors

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* The diagrams (a) and (b) of Fig. 4 give no contribution to the decay of derivative coupled scalar mesons. The decay rate is proportional to $g_1^6 g_a^2$.

for free protons, anti-neutrons, μ-mesons, and anti-neutrinos respectively. ϕ_π is the amplitude of the π-meson wave function. $2E_p$ is the energy of the virtual nucleon pair and the integration is over all momenta of the nucleons. The density of states is:

$$\rho_{\mu\nu}(E) = \pi \frac{(\mu_\pi^4 - \mu_\mu^4)(\mu_\pi^2 - \mu_\mu^2)}{2\mu_\pi^4} \tag{26}$$

Performing the indicated operations we obtain the transition rate:

$$\begin{aligned} \frac{1}{t} = {} & \left(\frac{g_2^2}{\hbar c}\right)\left(g_A^2 \frac{\mu_\pi^2 \mu_P^2 c^8}{\hbar^6 c^6}\right)\left(\frac{\mu_\pi c^2}{\hbar}\right)\left(\frac{\mu_\pi^4 - \mu_\mu^4}{\mu_\pi^4}\right) \times \\ & \times \left(\frac{\mu_\pi^2 - \mu_\mu^2}{64\pi^5 \mu_\pi^2}\right)\left(1 - \frac{cP_\mu}{E_\mu}\right) \times \\ & \times \left[\ln\left(\theta + \left\{\theta^2 + 1\right\}^{1/2}\right) - \frac{\theta}{(\theta^2 + 1)^{1/2}}\right]^2 \end{aligned} \tag{27}$$

θ is the cut-off momentum in units of $\mu_P c$. A covariant calculation using an invariant cut-off prescription of Feynman[20] leads to (27) with the bracket replaced by *:

$$\ln\left(\frac{\lambda}{\mu_P}\right) + 1 - \frac{(4\mu_P^2 - \mu_\pi^2)^{1/2}}{\mu_\pi} \sin^{-1}\left(\frac{\mu_\pi}{2\mu_P}\right) \tag{27a}$$

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* Using the notation of Feynman[20] we obtain for the matrix element for Fig. 4(a) an expression of the form:

$$R(M) = g_A g \int SP\left[\gamma_1\gamma_2\gamma_3\gamma_4 \left(\frac{1}{\not p + \not q - M}\right) \gamma_1\gamma_2\gamma_3 \left(\frac{1}{\not p - M}\right)\right] d^4 p$$

$\not q$ = momentum energy vector of meson. This divergent integral is a function of M^2.

$R(M^2) - R(M^2 + \lambda^2)$ converges if the subtraction is made before the integration. This gives (27a) where terms of order $\frac{\mu_P}{\lambda}$ and higher have been dropped. λ is a cut-off with the dimensions of mass. For large cut-offs (27) and (27a) are essentially equal. Gradient coupling again gives rise to (27) and (27a) with the replacement:

$$\left(\frac{g_2}{\hbar c}\right)^2 \longrightarrow \left(\frac{2g_3\mu_P}{\hbar\,\mu_\pi c}\right)^2 \tag{28}$$

Choosing $g_A \approx 2 \times 10^{-49}$ Erg-cm^3 from β-decay and $\frac{g_3^2}{\hbar c} \approx \frac{1}{3}$ from nuclear forces we obtain the following lifetimes as a function of the cut-off Θ:

TABLE III

LIFETIME OF THE π-MESON AS A FUNCTION OF CUT-OFF

Θ in units $\mu_P c$	Lifetime of the π-Meson
1	5.7×10^{-8} sec
2	5.7×10^{-9} sec
10	3.8×10^{-10} sec

A cut-off at about the Compton wave length of the nucleon agrees with the observed lifetime of 2×10^{-8} sec.*

The π-electron decay could also proceed by any of the diagrams of Fig. (5). Since a photon is emitted in general such processes would go at least $e^2/\hbar c$ as slow as those of Fig. (4). For those cases where the electron decay is partially forbidden (pseudoscalar meson, pseudovector β-decay) the modes of Fig. (5) will compete since the electron and neutrino no longer have exactly opposite momenta so that the matrix element need no longer be small

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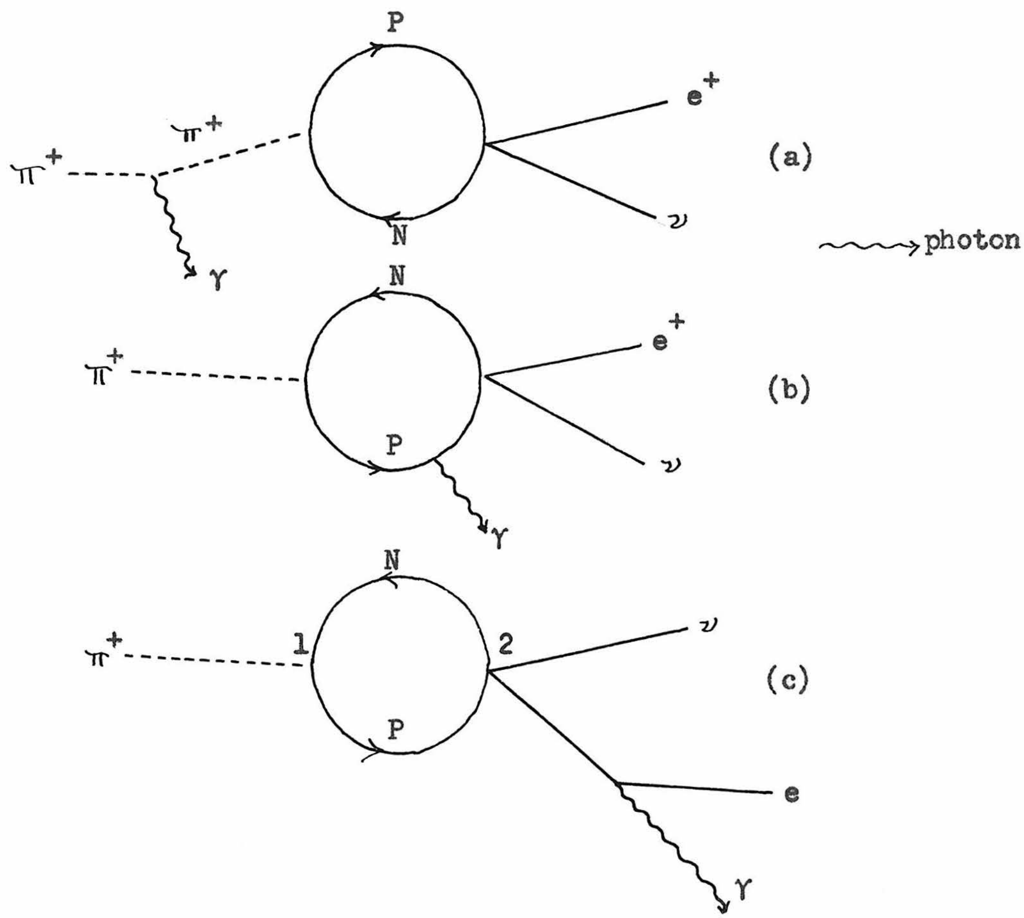
* Steinberger(21) has calculated the lifetime for this decay; after cutting off with regulators, he finds 2×10^{-2} sec. for pseudoscalar coupling. His result is slightly suspect since the lifetime for gradient coupling does not agree with that obtained from the equivalence theorem (Eq. 28). It is certainly true that regulation gives a lifetime many orders of magnitude longer than experiment.

Fig. 5.



(cf. Appendix I). Since the meson is initially at rest and the nucleons are very massive, it is reasonable that Fig. (5c) will give the dominant matrix element in the decay. Although the transition probability is given by a divergent integral, it is precisely that which was encountered in Fig. (4) so that one can estimate the ratio for the two modes of π-e decay. Designating the matrix element leading from 1 to 2 in Fig. (4a) and Fig. (5c) by S, the transition probability for the decay of (4a) is:

$$\frac{1}{\tau} \quad \frac{(2\pi)\left\langle u_e \middle| \gamma_4\gamma_5 \middle| u_\nu \right\rangle^2 S^2\, 4\pi g_A^2 E_e E_\nu^2\, 4\pi}{(2E_\pi)(2E_e)(2E_\nu) \qquad E_\pi(2\pi)^3} \tag{29}$$

The transition probability for the decay (5c) is:

$$\frac{1}{\tau} = 4\pi g_A^2\, 4\pi e^2\, 2\pi \int \frac{\left\langle u_e \middle| \not{\epsilon}\, \frac{1}{\not{A} - \not{p}_\nu - m} \gamma_4\gamma_5 \middle| u_\nu \right\rangle^2 \rho\,(E)\, S^2}{(2E_\pi)(2E_e)(2E_\nu)(2E_\gamma) \qquad (2\pi)^6} \tag{30}$$

$$\rho(E) = \frac{E_\gamma E_\nu^4 p_e^2 \, dp_e \, d\Omega_\nu \, d\Omega_e}{E_\nu^2 (E_\pi - E_e) + E_\nu \overrightarrow{p_\nu} \cdot \overrightarrow{p_\gamma}}$$

E_π, E_e, E_ν, E_γ are the energies of π-meson, electron, neutrino, and photon; $\bar{p}_\pi$, $\bar{p}_e$, $\bar{p}_\nu$, $\bar{p}_\gamma$ are the corresponding momenta. e_μ is the photon polarization vector. The integration is over all momenta of the electron, over the solid angles $d\Omega_e$ and $d\Omega_\nu$, and over the spins of the leptons and polarization of the photon. The normalization of the u_e and u_ν are such that for any operator 0, $\left[\tilde{0} = \text{Transpose conjugate}\right]$

$$\sum_{\text{spins}} \left\langle u_e^+ \, 0 \, u_\nu \right\rangle^2 = \text{spur} \left[(\not{p}_e + m_e) \, \tilde{0} \, \not{p}_\nu \, 0\right]$$

without the usual factor of $(4E_e E_\nu)^{-1}$. The ratio of (30) to (29) is approximately $(e^2/\hbar c)\left(\frac{1}{4\pi}\right)\left(\frac{1}{3}\right)\left(\frac{M_\pi}{M_e}\right)^2 \approx 15$. Therefore we should expect the π - (electron, photon, neutrino) decay to be of the order of ten times as frequent as the π - (electron, neutrino). About 0.1% of the π decays should therefore involve electrons. This is not in contradiction with experiment.

CONCLUSION AND SUMMARY

If electrons and neutrinos are coupled to nucleons through π-mesons it does not appear possible to explain both the absence of the $\pi \longrightarrow (e, \nu)$ decay and the observed facts of β-decay. The assumption that the u-meson interacts with nuclei by virtue of their coupling to π-mesons tends to rule out the possibility that the π has the pseudoscalar property indicated by experiment. The use of perturbation theory for the π-nucleon interaction, however, may not lead to reliable estimates even for the order of magnitude. In this case of direct π - μ coupling the perturbation calculation of $\pi \longrightarrow (e, \nu)$ decay through virtual nucleons diverges, so that no comparison can be made with the $\pi \longrightarrow (\mu, \nu)$ rate.

An alternative coupling scheme is to have direct interactions between fermions (symmetric coupling. The π- μ decay occurs through a nucleon anti-nucleon pair. Although the rate diverges the ratio of the $\pi \longrightarrow (e, \nu)$ to the $\pi \longrightarrow (\mu, \nu)$ lifetimes is finite and independent of perturbation theory. If the π-meson is pseudoscalar and β-decay is pseudovector the π-meson will decay into an electron-neutrino pair only 10^{-4} as often as into a μ-neutrino pair, and into an electron, neutrino, and photon about 10^{-3} as often. A perturbation theory calculation of the lifetime of π gives agreement with experiment if the divergent integrals are cut off at the nucleon Compton wavelength. These conclusions depend very crucially on the consideration of nucleons as Dirac particles especially in the prediction of the possible role of anti-particles in virtual processes.

APPENDIX I

The Matrix Element for the Production of a $\mu - \nu$ Pair

We wish $\dfrac{\int d\Omega \sum_{\sigma_\nu \sigma_\mu} \left| \phi^*_\mu A \phi_\nu \right|^2}{4}$ where $\vec{p}_\mu$ is the momentum of the μ-meson and $-\vec{p}$ the momentum of the neutrino; the integration is over all angles.

$$\text{Let } \lambda^\mu_\pm = \frac{\vec{\alpha} \cdot \vec{p}_\mu + \beta \mu \pm \sqrt{p_\mu^2 + \mu^2}}{\pm 2 \sqrt{p_\mu^2 + \mu^2}} \quad ; \quad \lambda^\nu_\pm = \frac{-\vec{\alpha} \cdot \vec{p} \pm \sqrt{p^2}}{\pm 2 \sqrt{p^2}}$$

Then (1) may be written

$$\text{spur } A \lambda^\mu_+ A \lambda^\nu_-$$

For the sixteen Dirac operators A we obtain: $p_u = \sqrt{\vec{p}_\mu \cdot \vec{p}_\mu}$

A =	I	$1 + p/E_\mu$	(a)
	α_4	$1 - p/E_\mu$	(b)
	$\alpha_1, \alpha_2, \alpha_3$	$1 + p/3E_\mu$	(c)
	$\alpha_1\alpha_2\alpha_3$	$1 - p/E_\mu$	(d)
	$\alpha_2\alpha_3, \alpha_3\alpha_1, \alpha_1\alpha_2$	$1 + p/3E_\mu$	(e)
	$\alpha_2\alpha_3\alpha_4, \alpha_3\alpha_1\alpha_4, \alpha_1\alpha_2\alpha_4$	$1 - p/3E_\mu$	(f)
	$\alpha_1\alpha_2\alpha_3\alpha_4$	$1 + p/E_\mu$	(g)
	$\alpha_1\alpha_4, \alpha_2\alpha_4, \alpha_3\alpha_4$	$1 - p/3E_\mu$	(h)

For (b) and (d) the matrix element is much smaller for an $e - \nu$ pair since

$\dfrac{p_e}{E_e} \approx 1.$

APPENDIX II.

THE INTERACTION BETWEEN π-MESONS AND NUCLEONS

Within the framework of special relativity, the only direct interaction which one knows how to construct is the contact interaction. The work on the artificial production of mesons implies that mesons can be produced singly. Therefore the Hamiltonian which describes the interaction must contain at least one odd power of the meson field. The simplest choice is to assume that the interaction is linear in the meson field in analogy to the coupling of the electromagnetic field to charges. Following the notation of Wentzel[22] the Lagrangian for the nucleon plus meson field is:

$$\mathcal{L} = \int dt\, L \quad L = -\int d^3x \left(\frac{\partial \phi^*}{\partial t}\frac{\partial \phi}{\partial t} - \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi - \frac{m^2c^2}{\hbar^2}\phi^*\phi \right) \hbar c$$

$$+ \hbar c \int \Psi^+(x) \left[\gamma_\mu \frac{\partial}{\partial x_u} - m \right] \Psi(x)\, d^3x + \quad \text{coupling}$$

If we consider those interactions which involve only the meson fields or their first derivatives, the possible invariant couplings are:

ϕ: <u>Scalar Field</u>

$$g_o \int \phi(x)\, \Psi^+(x)\, \Psi(x)\, d\tau \qquad + \text{c.c} \tag{2a}$$

$$\frac{g_1}{\mu_\pi}\int \partial_\mu \phi(x)\, \Psi^+(x)\, \gamma_\mu \Psi(x)\, d\tau \qquad + \text{c.c} \tag{2b}$$

ϕ: <u>Pseudoscalar Field</u>

$$g_2 \int \phi(x)\, \Psi^+(x)\, i\, \gamma_5 \Psi(x)\, d\tau \qquad + \text{c.c} \tag{2c}$$

$$\frac{g_3}{\mu_\pi}\int \partial_\mu \phi(x)\, \Psi^+(x)\, \gamma_5\gamma_\mu \Psi(x)\, d\tau \qquad + \text{c.c} \tag{2d}$$

ϕ_μ: <u>Vector Field</u>

$$g_4 \int \phi_u(x)\, \Psi^+(x)\, \gamma_\mu \Psi(x)\, d\tau \qquad + \text{c.c} \tag{2e}$$

$$\frac{g_5}{\mu_\pi}\int \left[\partial_\nu \phi_\mu(x) - \partial_\mu \phi_\nu(x) \right] \left[\Psi^+(x)\, \gamma_\mu\gamma_\nu\, \Psi(x) \right] d\tau + \text{c.c} \tag{2f}$$

ϕ_μ: <u>Pseudovector Field</u>

$$g_6 \int \phi_\mu(x)\, \Psi^+(x)\, \sigma_\mu\, \Psi(x) \qquad + \text{c.c} \tag{2g}$$

$$\frac{g_7}{\mu_\pi}\int \left[\partial_\nu (\phi_\mu(x) - \partial_\mu \phi_\nu(x) \right] \left[\Psi^+(x)\, \gamma_5\gamma_\mu\gamma_\nu \quad (x) \right] d\tau \quad + \text{c.c} \tag{2h}$$

$\partial_\mu = \dfrac{\partial}{\partial x_\mu}$

c.c = complex conjugate

μ_π = mass of meson

$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$

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PART II.

I. INTRODUCTION

In the Quantum Theory of Fields the elementary particles are treated in a dualistic formalism as possessing the properties of both particles and fields. In the classical limit, however, the bosons form fields while the fermions are point sources of field. The point sources give rise to infinities which carry over into the quantum theory.

On the classical level this difficulty can be removed by dispensing with the field concept. From this point of view sources and absorbers interact directly analogously to advanced and retarded Lienard-Wiechert potentials in electrodynamics. Whether the self interaction is included or not, the recoil damping accompanying transfer of energy and momentum (which is necessary if they are to be conserved) can be understood only by considering the role of all the surrounding particles as complete absorbers[1] so that the properties of both source and absorber must be analyzed at the same time. The carrying over of this action at a distance program into the quantum theory has met with great difficulties.

An alternative program is to retain the field construct but eliminate the point singularities. Particles appear only as small regions of space where energy and charge of the field are concentrated. In such a unitary theory the field is everywhere continuous, finite, and quadratically integrable; the equations of motion of the "lumps" follow from the field equations. For such lumps to be stable and capable of interacting with each other it is necessary that the field equations be non-linear.

The Maxwell, Dirac, Yukawa, and Gravitational fields, together with their usually accepted couplings form a non-linear system so that one has the possibility of a unitary theory. (Calculations with the Maxwell-Dirac*[2]

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* In this case the mass of the localized solution was negative. However calculations based on a variational integral often give misleading results concerning particle solutions of the non-linear field equations.

and Maxwell-Yukawa[3] fields have lead to particle like solutions.) A more ambitious program is to describe the elementary particles and their interactions in terms of a single underlying non-linear field. The non-linearity which accounts for the particle like solutions will also describe their interactions. (Since "lumps" of field will have some overlap even at large distances, it is not necessary to introduce an intermediary field as in the case of point particles. Efforts have been made toward such a unified theory of nuclear, electromagnetic, and gravitational fields by Einstein[4], Schroedinger[5], and others. The non-linear equations are derived from the variation of a Lagrangian invariant under the whole group of general relativity. However, the very great difficulty of calculating the interesting solutions has made their interpretation obscure.

A technically less formidable program which disregards the gravitational interaction is to investigate Lagrangians which are simple and invariant only under the Lorentz Group. The Lagrangian for the Dirac, Maxwell, and Yukawa fields plus their Lorentz invariant interactions would appear to be well suited to such an investigation. Here, however, we study the simpler problem of a single non-linear field, first to avoid the mathematical complexity of three simultaneously interacting fields, and second to explore the possibility that a simpler Lagrangian in the richer non-linear theory can accomplish as much as a more complicated Lagrangian in the linear theory.

We shall assume[6] the Lagrangian to be a function of the field quantities $\Psi^{(m)}$ $(\vec{x},t) = 1, 2, \ldots n$ and their first derivatives only so that the resulting non-linear differential equations are at most of second order. The Lagrangian is then of the general form:

$$\mathcal{L} = \int L(\Psi^{(m)}, \partial_\alpha \Psi^{(m)})\, dx_1 \ldots dx_4 \tag{1}$$

$$\partial_\alpha = \frac{\partial}{\partial x_\alpha} \qquad x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ict$$

The invariance of the Lagrangian to the inhomogeneous Lorentz Group and to gauge transformations will lead to conservation laws for a vector, a tensor of rank two, and a tensor of rank three which will be interpreted as current-charge, energy-momentum, and angular momentum respectively. The Euler equation* corresponding to (1) is

$$\sum_{\alpha=1}^{4} \partial_\alpha \frac{\partial L}{\partial(\partial_\alpha \Psi^{(m)})} - \frac{\partial L}{\partial \Psi^{(m)}} = 0 \qquad m = 1, 2, \ldots, n \tag{2}$$

We define the tensor

$$T_{\mu} = - \sum_{(m)} \frac{\partial L}{\partial(\partial_\mu \Psi^{(m)})} \left(\partial_\nu \Psi^{(m)} \right) + L\, \delta_{\mu\nu} \tag{3}$$

From (2) it follows that

$$\partial_\mu T_{\mu\nu} = 0 \tag{4}$$

Under an infinitesimal Lorentz transformation

$$\delta x_\alpha = \delta\omega_{\alpha\beta}\, x_\beta \qquad \delta\omega_{\alpha\beta} = - \delta\omega_{\beta\alpha} \tag{5}$$

the field quantities transform according to

$$\delta \Psi^{(m)} = \sum_{\alpha<\beta} S_{\alpha\beta}^{(mn)}\, \delta\omega_{\alpha\beta}\, \Psi^{(n)} \tag{6}$$

$S_{\alpha\beta}^{(mn)}$ depends on the tensor nature of $\Psi^{(m)}$.

Let the third rank tensor $M_{\lambda\,\alpha\beta}$ be defined by the equation

$$\sum_{\alpha<\beta} M_{\lambda\,\alpha\beta}\, \delta\omega_{\alpha\beta} = \sum_{(m)} \frac{\partial L}{\partial(\partial \Psi^{(m)})}\, \delta^* \Psi^{(m)} + L\, \delta x_\lambda \tag{7}$$

where $\delta^* \Psi^{(m)} = \delta \Psi^{(m)} - \partial_\mu \Psi^{(m)}\, \delta x_\mu$ (8)

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* $\delta \mathcal{L} = 0$ subject to the restriction

$\delta\Psi^{(m)} = 0$, $\delta(\partial_\alpha \Psi^{(m)}) = 0$ on the surface of the four dimensional volume

The Lagrangian leading to (2) is of course not unique

From (2)(3)(6)(7) and (8)

$$M_{\lambda\alpha\beta} = - x_\alpha T_{\lambda\beta} + x_\beta T_{\lambda\alpha} + \sum_{(m,n)} S_{\alpha\beta}^{(mn)} \frac{\partial L}{\partial(\partial_\lambda \Psi^{(m)})} \Psi^{(n)} \tag{9}$$

and

$$\partial_\lambda M_{\lambda\alpha\beta} = 0 \tag{10}$$

The first two terms of (9) will have the interpretation of orbital angular momentum, the third describes the intrinsic spin of the field.*

For the interaction of the field with an electromagnetic field A_μ we assume the usual prescription

$$\partial_\mu \phi \longrightarrow (\partial_\mu - i\, q\, A_\mu)\Psi \equiv D_\mu \Psi \tag{11a}$$

$$\partial_\mu \phi^* \longrightarrow (\partial_\mu + i\, q\, A_\mu)\Psi^* \equiv D_\mu^* \Psi \tag{11b}$$

In order for the Euler equation to be gauge invariant under the gauge transformation

$$A_\mu' \longrightarrow A_\mu + \partial_\mu \lambda \tag{12}$$

we must introduce the gauge transformations of the second kind:

$$\Psi' = \Psi\, e^{iq\lambda} \tag{13}$$

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* The $T_{\mu\nu}$ defined by (4) is in general non-symmetric. For specified values of the total energy and momentum, only the production of a gravitational field gives the $T_{\mu\nu}$ a direct physical meaning. In order to get such a correspondence we must form a symmetric $\Theta_{\mu\nu}$ such that $\partial_\mu \Theta_{\mu\nu} = 0$.

$$\Theta_{\mu\nu} = T_{\mu\nu} + \partial_p f_{p\mu\nu}$$

where

$$2 f_{\mu p \nu} = S_{p\mu\nu} + S_{\mu\nu p} - S_{\nu p \mu}$$

We shall however be interested only in $\int \overrightarrow{dx}\, \Theta_{4\nu} = \int \overrightarrow{dx}\, T_{4\nu}$

If the Lagrangian is an expression of the type:

$$L\left(D_\mu \Psi^{(m)}\ D_\mu^* \Psi^{(m)*},\quad \Psi^{(m)} \Psi^{(m)*} \right) \tag{14}$$

we have a gauge invariant theory. It then follows that the vector

$$S_\alpha = -i\varepsilon \sum_{(m)} \left(\frac{\partial L}{\partial (D_\alpha \Psi^{(m)})} \Psi^{(m)} - \frac{\partial L}{\partial (D_\alpha^* \Psi^{(m)*})} \Psi^* \right) \tag{15}$$

satisfies the conservation law

$$\partial_\alpha S_\alpha = 0 \tag{16}$$

From the continuity equations (4),(10), and (16) it follows that Q, G_α, and $M_{\alpha\beta}$ defined by

$$iQ = \int S_4 \overrightarrow{dx} \tag{17a}$$

$$iG_\alpha = \int T_{4\alpha} \overrightarrow{dx} \tag{17b}$$

$$iM_{\alpha\beta} = \int M_{4\alpha\beta} \overrightarrow{dx} \tag{17c}$$

are a scalar, vector, and anti-symmetric tensor respectively under the entire Lorentz group*. Each of the integrals (17) is independent of the time. A field confined to a small region of space will carry a definite charge Q, energy-momentum G_α, and angular momentum $M_{\alpha\beta}$. The transformation properties are the same as those for a point particle. Therefore if localized, regular solutions of the field equations exist which make the integrals (17) finite we can obtain a consistent classical description of particles as "lumps" of field.

In the canonical theory of quantization the field quantities Ψ and $\dfrac{\partial L}{\partial (ic\,\partial_4 \Psi)} = \pi$ become non-commuting operators according to the prescription: Classical Poisson Brackets = $i/\hbar$ Commutator. For Bose Fields this yields

* The transformation properties depend upon the Ψ satisfying the Euler equation at all points. The presence of a singularity can spoil the identification.

$$\left[\pi^{(m)}(x), \Psi^{(n)}(x')\right]_- = -i\hbar\, \delta_{mn}\, \delta(x - x') \tag{18a}$$

$$\left[\pi^{*(m)}(x), \Psi^{*(n)}(x')\right]_- = -i\hbar\, \delta_{mn}\, \delta(x - x')$$

All other commutators vanish.

Fields quantized according to the exclusion principle have no classical limit. The canonical commutation rule is

$$\left[\Psi^{(m)}(x), \Psi^{*(n)}(x')\right]_+ = \frac{\hbar}{i}\, \delta_{mn}\, \delta(x - x') \tag{18b}$$

The other anti-commutators vanish. With the rules (18a) or (18b) it has been shown(6) that

$$\left[G_\alpha, G_\beta\right]_- = 0 \tag{19}$$

$$\left[M_{\alpha\beta}, M_{\gamma\delta}\right]_- = \left(\delta_{\alpha\delta} M_{\beta\gamma} + \delta_{\beta\gamma} M_{\alpha\delta} - g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\delta} M_{\alpha\gamma}\right) i\hbar \tag{20}$$

$$\left[M_{\alpha\beta}, G_\gamma\right]_- = (G_\alpha \delta_{\beta\gamma} - G_\beta \delta_{\alpha\gamma})\, i\hbar \quad . \tag{21}$$

If a position operator X_i is defined by*

$$X_i \int S_4\, \overrightarrow{dx} = \int x_i S_4\, \overrightarrow{dx} \tag{22}$$

then

$$\left[G_k, X_i\right]_- \int S_4\, \overrightarrow{dx} = -i\hbar\, \delta_{ki} \int S_4\, \overrightarrow{dx} \tag{23}$$

If the rest mass is defined by the operator

$$-M^2 = G_\alpha G_\alpha = G_4^2 + \vec{G}\cdot\vec{G} \tag{24}$$

then

$$\left[G_i, M\right]_- = 0 \tag{25}$$

and

$$\left[M_{\alpha\beta}, M^2\right] = 0 \tag{26}$$

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* This definition is non-relativistic. A covariant definition is $X_i = (G_4)^{-1} \int \Theta_{4x}\, x_i\, \overrightarrow{dx}$. Then if G is the operator $\sum_{\mu=1}^{4} G_\mu G_\mu$ we obtain

$$\left[G_\alpha, X_\beta\right]_- = -i\hbar\left(\delta_{\alpha\beta} - G_\alpha G_\beta G^{-1}\right) \text{and} \left[X_\alpha, X_\beta\right]_- = i\hbar G^{-1} M_{\alpha\beta} .$$

The fundamental length appearing in these commutators is the Compton wavelength of the particle. The charge also commutes with the other observables

$$\left[Q, G_\alpha\right]_- = 0 \tag{27}$$

$$\left[Q, M_{\alpha\beta}\right]_- = 0 \tag{28}$$

$$\left[Q, M^2\right]_- = 0 \tag{29}$$

According to (19),(20),(21),(23),(25) - (29), the observables associated with a "lump" of field obey the commutation rules ordinarily assumed for the corresponding properties of particles.

The canonical quantization does not offer a very satisfactory treatment for the classical unitary theory we shall consider. The singular commutators lead to divergences and ambiguities; the spinor field when quantized according to anti-commutation rules has no classical limit; the motivation for the usual quantization is to endow a field with particle properties but this is already accomplished on the classical level in a unitary field theory. The classical unitary theory can be crudely quantized by specifying the coupling constant q [eqn. (11) and (15)] to have its usual value

$$q = \frac{\ell}{\hbar c} \tag{30}$$

where ℓ is the charge on the electron. Then since Q is an integral multiple of ℓ the solution of (2) corresponding to the lowest charge state is normalized according to

$$-i \int S_4 \, \overrightarrow{dx} = \hbar c \tag{31}$$

We shall first discuss on the classical level, the particle-like solutions of some simple non-linear Lagrangians for scalar fields. Under certain conditions an interpretation can be given from the viewpoint of the canonical quantization, Eq. (18a). Lagrangians for spinor fields which yield non-linear theories can be treated canonically, Eq. (18b). However such a theory is quite different from the classical spinor theory we shall treat.

II. NON-LINEAR SPINLESS BOSE FIELD

1. The Wave Equation

Let $\Psi(\vec{x},t)$ be a single field coupled to itself through a non-linear coupling. If there is no intrinsic spin we shall assume the Lagrangian:

$$\mathcal{L} = \int \vec{dx} \left[- \partial_\mu \Psi^* \partial_\mu \Psi + \mu^2 \Psi^* \Psi \right] - \epsilon \int \vec{dx}\, P\,(\Psi^*, \Psi\, \partial_\mu \Psi^*\, \partial_\mu \Psi) \qquad (32)$$

μ^{-1} is a fundamental length; P is some scalar function of the field strengths and their derivatives. If the coupling term were missing, we would have the usual Lagrangian for the scalar (or pseudoscalar) meson field.

When

$$P = \frac{2}{n} \Psi^n \qquad (33)$$

the Euler equation is:

$$\Box \Psi + \mu^2 \Psi - 2\epsilon \Psi^{n-1} = 0 \qquad (34)$$

where

$$\Box \equiv \nabla^2 - 1/c^2\ \partial_t^2 \qquad (35)$$

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* Non-linear Bose fields have been considered in connection with the scattering of light by light[7] and meson-meson[8] interactions. The interaction of bosons occurs through transient pairs of the coupled spinor field. The spinor field can be eliminated in various approximations giving rise to a non-linear Lagrangian involving only the Bose field variables.

Schiff[9] has considered a non-linear meson theory in connection with saturation. His equation is the same as our Equation (37) except with the sign of ϵ reversed. The sign reversal prohibits particle like solutions.

Since P is gauge invariant, if we have complex fields (charged), n must be even.

We shall look for steady state solutions with the simple time and space dependence

$$\Psi = y(r)e^{i\omega t} \tag{36}$$

y(r) is taken real so that we have no radial current. More general solutions of equation (34) with other time and space dependence have not been investigated because of computational difficulties in solving the partial differential equation. With the assumption (36), the Euler equation is the ordinary differential equation:

$$\frac{d^2y}{dr^2} + \frac{2}{r}\frac{dy}{dr} - (1 - \omega^2)\, y + \epsilon y^3 = 0 \tag{37}$$

n has been specified to 4. From (17a) the charge is:

$$\frac{4\pi e}{\hbar c}\int_0^\infty r^2\, 2\omega\, y^2(r)\, dr \tag{38}$$

From (17b) the mass is:

$$4\pi\int_0^\infty r^2\, dr \left[\frac{\epsilon}{2} y^4 + 2\,\omega^2 y^2\right] \tag{39}$$

When

$$P = \frac{1}{4}\,(\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*)(\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*) \tag{40}$$

and the solution is of the form (36), the Euler equation is:

$$\frac{d^2y}{dr^2} + \frac{2}{r}\frac{dy}{dr} - (1 - \omega^2)\, y + \epsilon\,\omega^2 y^3 = 0 \tag{41}$$

The charge is:

$$4\pi\int_0^\infty r^2\, dr\, 2\,\omega\, y^2 \left[1 + \frac{\epsilon}{2} y^2\right] \tag{42}$$

The mass is:

$$4\pi\int_0^\infty r^2\, dr \left[\frac{\epsilon}{2}\,\omega^2 y^4 + 2\,\omega^2 y^2\right] \tag{43}$$

If $y(r)$ is a solution of (37) for given ω , then $y(r)/\omega$ solves (41). Particle like solutions of (37) and (41) exist only for $\epsilon > 0$. Therefore the masses are always positive.

2. The Existence of Particle-Like Solutions

There exists a two parameter family of solutions of the differential equation:

$$y'' + \frac{2y'}{r} - y + y^3 = 0 \tag{44}$$

where ' signifies $\frac{d}{dr}$. If $y_o(r)$ is a solution of (44), then

$$\sqrt{\frac{1-\omega^2}{\epsilon}}\; y_o\left(\sqrt{1-\omega^2}\, r\right) \tag{45}$$

is a solution of

$$y'' + \frac{2y'}{r} - (1-\omega^2)\, y + \epsilon\, y^3 = 0 \tag{46}$$

Therefore if we find solutions of (44) which are everywhere differentiable and which are quadratically integrable (finite charge and mass) we also have a proper solution of the charged field equation (46). Such solutions exist* only for $\epsilon > 0$, $\omega^2 < 1$, so that the derived solutions are real (no radial current). Equation (44) is the Euler equation of the Lagrangian:

$$\mathcal{L} = \int d^3x \left[-(y')^2 + y^2 - \frac{y^4}{2} \right] \tag{47}$$

$$= \int_0^\infty L\, dr \tag{48}$$

L is defined as

$$4\pi\, r^2 \left[-(y')^2 + y^2 - \frac{y^4}{2} \right] \tag{49}$$

The conjugate momentum (with r playing the role of t) is:

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* We are greatly indebted to Professor H. F. Bohnenblust for pointing out a method of proof for the existence of quadratically integrable solutions of Equation (44). We wish to thank him for an extremely helpful discussion.

$$p = \frac{\partial\left[-r^2 (y')^2 + r^2y^2 - r^2y^4/2\right] 4\pi}{\partial(y')} = \tag{50}$$

$$- 8\pi\, r^2y' \tag{51}$$

The Hamiltonian is:

$$\int_0^\infty H\, r^2\, dr \tag{52}$$

where

$$H = y'\, p - L \tag{53}$$

$$= - r^2\left[- (y')^2 + y^2 - \frac{y^4}{2}\right] 4\pi \tag{54}$$

$$= - \left[\frac{p^2}{(8\pi r)^2} + r^2y^2 - r^2\, \frac{y^4}{2}\right]\, 4\pi \tag{55}$$

We form the pseudo-Hamiltonian

$$\overline{H} = \frac{H}{r^2} = 4\pi\left[\, (y')^2 - y^2 + \frac{y^4}{2}\,\right] \tag{56}$$

In this form $\overline{H}$ has no explicit dependence on r; it is a function of y and y' only. Differentiating,

$$\frac{dH}{dr} = r^2\, \frac{d\overline{H}}{dr} + 2r\,\overline{H} \tag{57}$$

Since H is the Hamiltonian function for the Euler Equation (44), for any solution of (44)

$$\frac{dH(y,p,r)}{dr} = \frac{\partial\, H(y,p,r)}{\partial\, r} \tag{58}$$

then from (57)

$$\frac{d\overline{H}}{dr} = \frac{1}{r^2}\frac{\partial H}{\partial r} - \frac{2H}{r^3} \tag{59}$$

Substituting from (55)

$$\frac{d\overline{H}}{dr} = - \frac{p^2}{4\pi r^5} = - \frac{16\pi\, (y')^2}{r} \tag{60}$$

Therefore

$$\frac{d\overline{H}}{dr} \leq 0 \tag{61}$$

along any trajectory. Equations (56) and (61) are sufficient for a qualitative investigation of the solution. In Figure (1) we have plotted the contours $\overline{H}$ = constant in the phase space (y,y'). If Equation (61) is neglected these are trajectories for Equation (44) with the $\frac{2y'}{r}$ term dropped. In order that y remains finite at the origin $r = 0$, the $\frac{2y'}{r}$ term in (44) restricts us to the one parameter set with $y'(0) = 0$. Therefore in the phase space we are interested only in those trajectories which originate on the axis $y' = 0$.

There are three singular points in phase space (1,0), (-1,0), and (0,0). The first two are minima of $\overline{H}$; the origin is a saddle point. All trajectories are bounded since $\overline{H}$ must decrease along a trajectory (Equation 61). Once within any of the $\overline{H}$ = constant contours, the trajectory cannot leave and must ultimately be captured on one of the three singular points.*
If the initial value of y is greater than 0 but less than the $\sqrt{2}$, $\overline{H}(r=0) < 0$, and the trajectory gets captured at A (curve a of Figure 1). If the initial value of y is sufficiently large, the capture will take place at the singular point B (curve b). For an appropriate initial value between that of a and b, the trajectory will end at the origin in phase space, i.e. as $r \longrightarrow \infty$ y and $y' \longrightarrow 0$. †

Let S be the set s_0 of initial values $y(0)$ such that $y(r,s_0)$ cross the axis $y = 0$. That this set is non-empty is easily shown by calculation. Let T be the set of initial values t_0 such that $y(r,t_0)$ do not cross the axis and get caught at (1,0). This set is non-empty since $y(r) \equiv 1$, $y' \equiv 0$

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* It is easily shown that for this case there are no limit cycles or closed trajectories

† The saddle point can only be reached in the limit $r \longrightarrow \infty$.

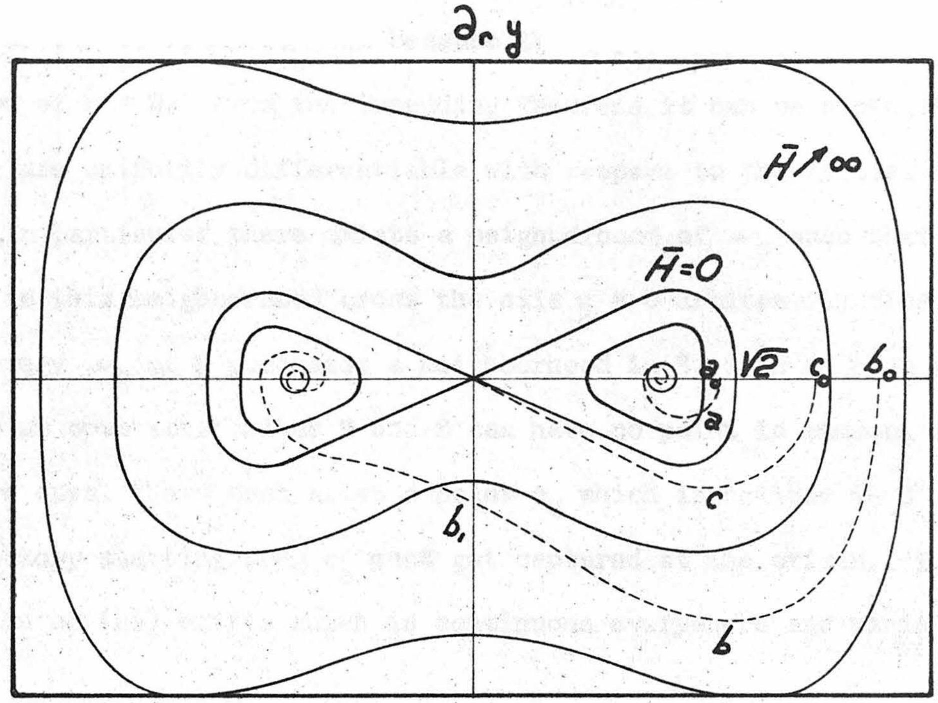


FIG. 1 ... PHASE PLOT FOR EQUATION 44.
$\epsilon = +1$

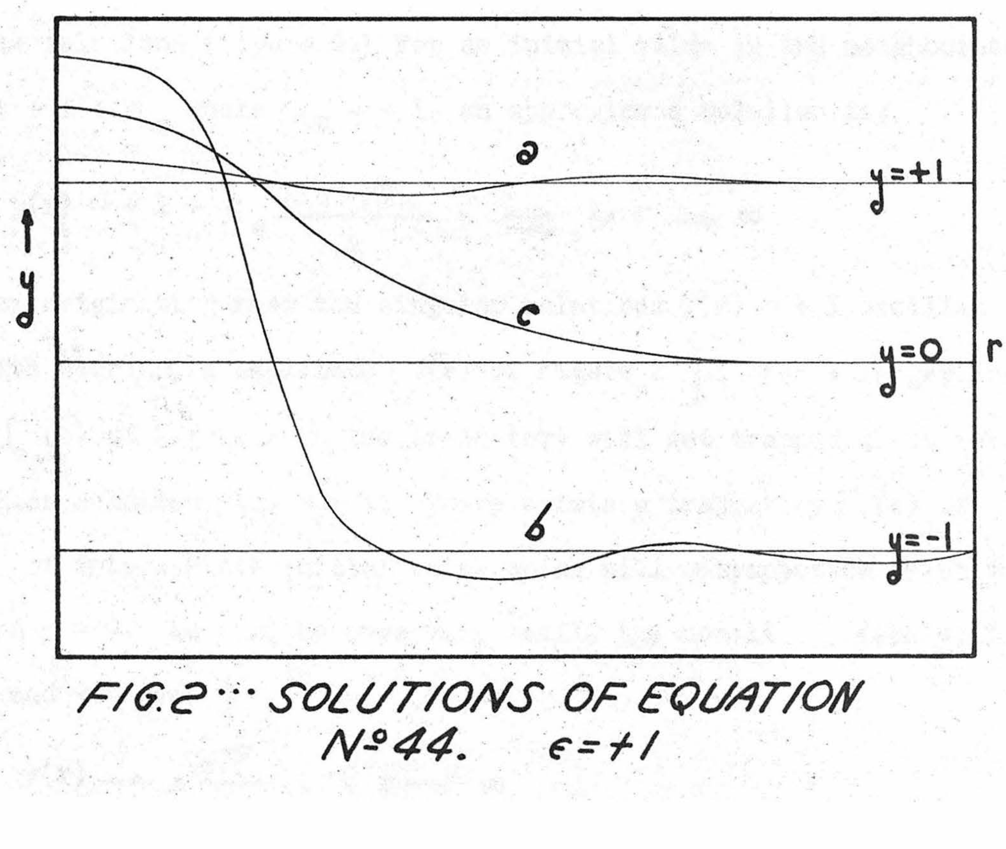


FIG. 2 ... SOLUTIONS OF EQUATION
No 44. $\epsilon = +1$

is such a solution. Now $y(r, \nu_0)$ is an analytic function of r in a neighborhood of $r = 0$. This follows because the Taylor expansion converges in a neighborhood of $r = 0$. From the Imbedding Theorems it can be shown that the solutions are uniformly differentiable with respect to the initial value ν_0 or t_0. In particular there exists a neighborhood of ν_0 such that all trajectories in this neighborhood cross the axis $y = 0$ arbitrarily close to ν. Since every ν_0 of S possesses a neighborhood in S, S is an open set. Likewise T is an open set. Since T and S can have no point in common, and since both are open, there must exist a point c_0 which is neither in S nor T. Hence a trajectory starting from c_0 must get captured at the origin. Therefore a solution of (44) exists which is continuous everywhere and vanishes as $r \longrightarrow \infty$.

The phase space description has the following mirroring in (y,r) space:

$$y(r) = +1 \qquad\qquad y(r) = -1 \qquad\qquad y(r) = 0$$
$$y'(r) = 0 \qquad\qquad y'(r) = 0 \qquad\qquad y'(r) = 0$$

are possible solutions (Figure 2.) For an initial value in the neighborhood of ± 1, say $\pm 1 + \eta_0$ where $\eta_0 << 1$, an approximate solution is:

$$y(r) = \pm 1 + \eta_0 \frac{\sin \sqrt{2}x}{x} \quad \Longrightarrow \quad 1, \ r \longrightarrow \infty \tag{62}$$

Trajectories originating near the singular solutions $y(r) = +1$ oscillate about it with decreasing amplitude [(a) of Figure 2]. For a larger initial value y(0) [(b) of Figure 2] the trajectory will get trapped about the lower singular solution $y(r) = -1$. There exists a trajectory [(c) of Figure 2] of intermediate initial value which will asymptotically approach the solution $y = 0$. As y(r) becomes very small, the non-linear term will be negligible and we have:

$$y(r) \longrightarrow A \frac{e^{-r}}{r} \ , \qquad r \longrightarrow \infty \tag{63}$$

The mass and charge integrals will exist. For solutions of (46) as $r \longrightarrow \infty$, if $\omega^2 < 1$,

$$y(r) \longrightarrow B \frac{e^{-\sqrt{1-\omega^2}\,r}}{r} \quad , \quad r \longrightarrow \infty \tag{64}$$

However, if $\omega^2 \geq 1$,

$$y(r) \longrightarrow C \frac{\sin \sqrt{\omega^2 - 1}\; r}{r} \, , \; r \longrightarrow \infty \tag{65}$$

For non-zero amplitude this does not give finite charge and mass integrals; therefore for a proper solution:

$$-1 < \omega < +1 \tag{66}$$

In a similar way the existence of solutions with a higher number of nodes can be demonstrated.

If $\epsilon = -1$ the phase space diagram is given in Figure (3). The contours of constant $\overline{H}$ are open; $\overline{H}$ runs from $+\infty$ to $-\infty$. Since solutions originate on the $y' = 0$ axis $[\overline{H} < 0]$, the restriction $\frac{d\overline{H}}{dr} \leq 0$ along any trajectory keeps it from approaching the origin. Therefore no proper solutions exist for $\epsilon \leq 0$.

We can make a qualitative investigation of more general non-linear Lagrangians using the same technique. If

$$\mathcal{L} = \int d\tau \left[-(y')^2 + y^2 - P(y) \right] \tag{67}$$

the Euler equation is:

$$y'' + \frac{2y'}{r} - (1 - \omega^2)\, y + \frac{\partial P(y)}{2\,\partial y} \tag{68}$$

For the pseudo-Hamiltonian $\overline{H}$ we obtain:

$$\overline{H} = 4\pi \left[(y')^2 - y^2 + P(y) \right] \tag{69}$$

Along any solution of the Euler equation:

$$\frac{d\overline{H}}{dr} = - \frac{16\pi}{r} (y')^2 \leq 0 \tag{70}$$

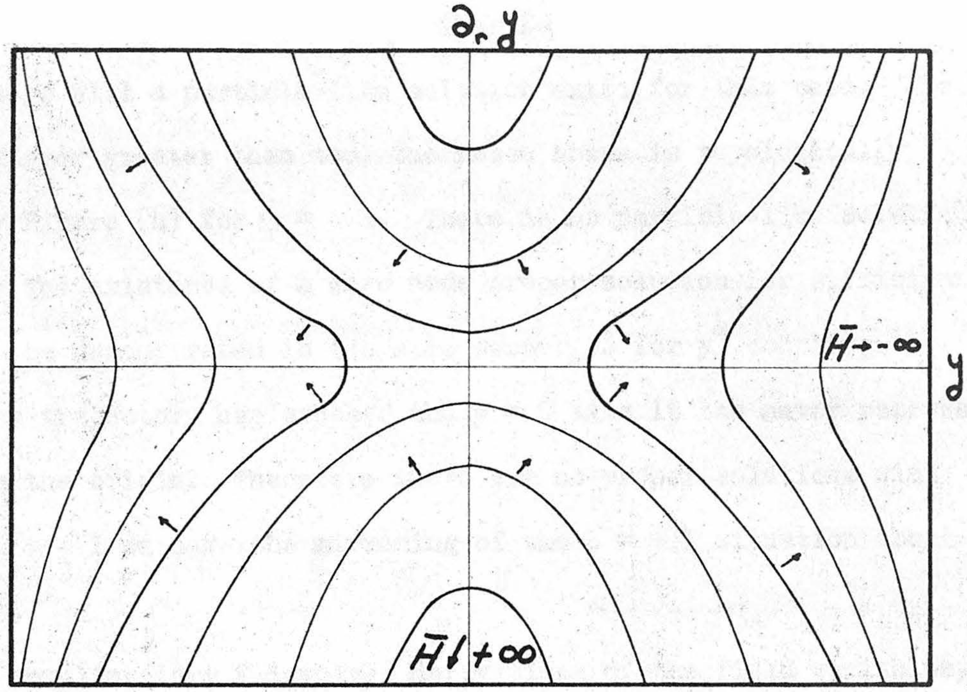


FIG. 3 ··· PHASE PLOT FOR EQUATION·44.
$\epsilon = -1$

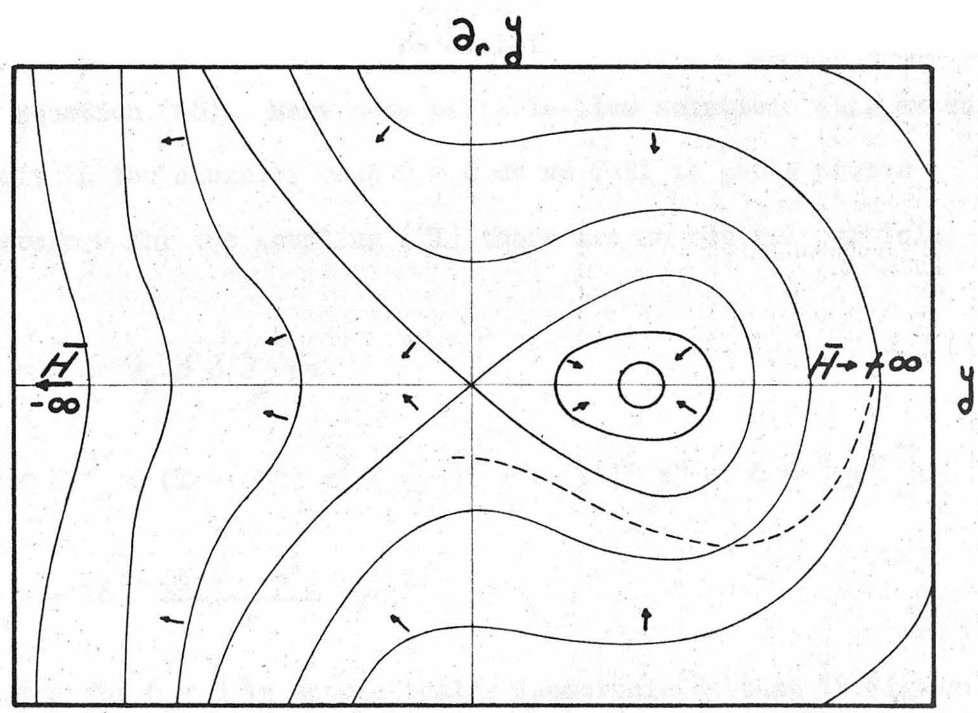


FIG. 4 ··· PHASE PLOT FOR EQUATION·68.
n odd $\qquad \epsilon = +1$

When $P(y) = \epsilon y^n$, the phase diagram is topologically equivalent to Figure (1) for n an even integer greater than two. Therefore if, and only if, ϵ is greater than zero will a particle-like solution exist for this case. When n is an odd integer greater than two, the phase space is topologically represented by Figure (4) for $\epsilon = +1$. There is no particle-like solution for $y(0) \leq 0$. The existence of a zero node proper solution for sufficiently large $y(0)$ can be demonstrated in the same manner as for y^4 coupling. However, once a trajectory has crossed the $y = 0$ axis it can never recross it or approach the origin. Therefore there are no proper solutions with nodes. When $\epsilon = -1$ we have the mirroring of the $\epsilon = +1$ situation about the $y = 0$ axis.

If the coupling term F involves derivatives of the field variables, $\frac{d}{dr}\overline{H}$ can change sign. For

$$P = \frac{\epsilon}{4}\left[\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right]\left[\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right] \tag{71}$$

we are lead to Equation (41). Solutions of this can be derived immediately from those of Equation (46). Many node particle-like solutions will exist for $\epsilon > 0$. Only in the singular case $\omega = 0$ do we fail to get a proper solution. Therefore for the coupling (71) there are no neutral particle solutions.

When
$$P = \epsilon \phi^* \partial_\mu \phi \phi \partial_\mu \phi^* \tag{72}$$
we have
$$\overline{H} = 4\pi\left[-(1-\omega^2) y^2 + (y')^2 + \epsilon (y')^2 y^2 + \epsilon \omega^2 y^4\right] \tag{73}$$
and
$$\frac{d\overline{H}}{dr} = -16\pi \frac{(1 + y^2)}{r} (y')^2 \tag{74}$$

The phase diagram for $\epsilon > 0$ is topologically isomorphic to that in Figure(2). Therefore we are lead to expect a set of multinode proper solutions. However in the special case $\omega = 0$ we have the topology of Figure (5) and no

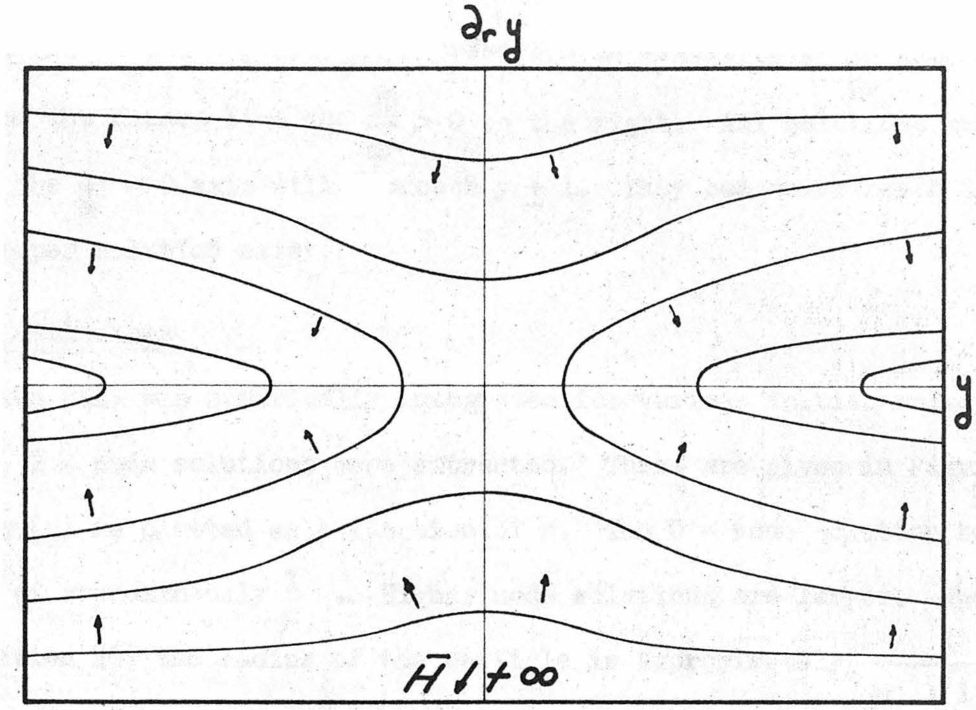


FIG. 5 ··· PHASE PLANE FOR EQUATION·73.
$\epsilon > 0$ $\omega = 0$

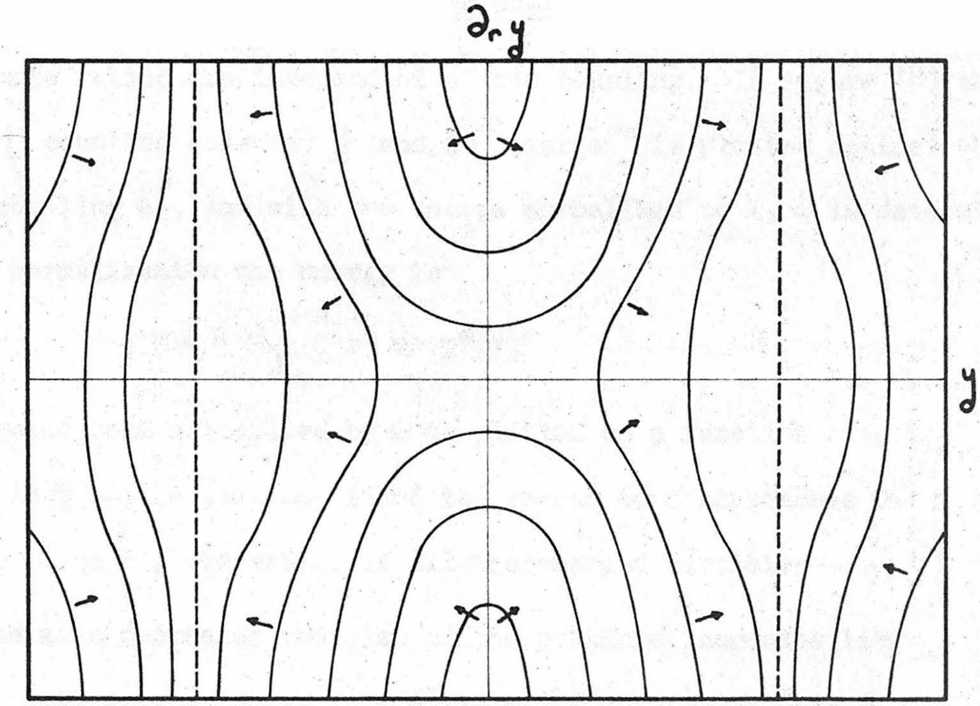


FIG. 6 ··· PHASE PLANE FOR EQUATION·73.
$\epsilon < 0$

proper solution exists. Therefore all proper solutions represent charged fields. When $\epsilon < 0$ the topology is represented in Figure (6). $\frac{d\bar{H}}{dr} < 0$ to the left of the dashed line and $\frac{d\bar{H}}{dr} > 0$ to the right. All solutions which originate on the $y' = 0$ axis will approach $y \pm 1$. They can never reach the origin; no proper solution exists.

3. Numerical Solutions

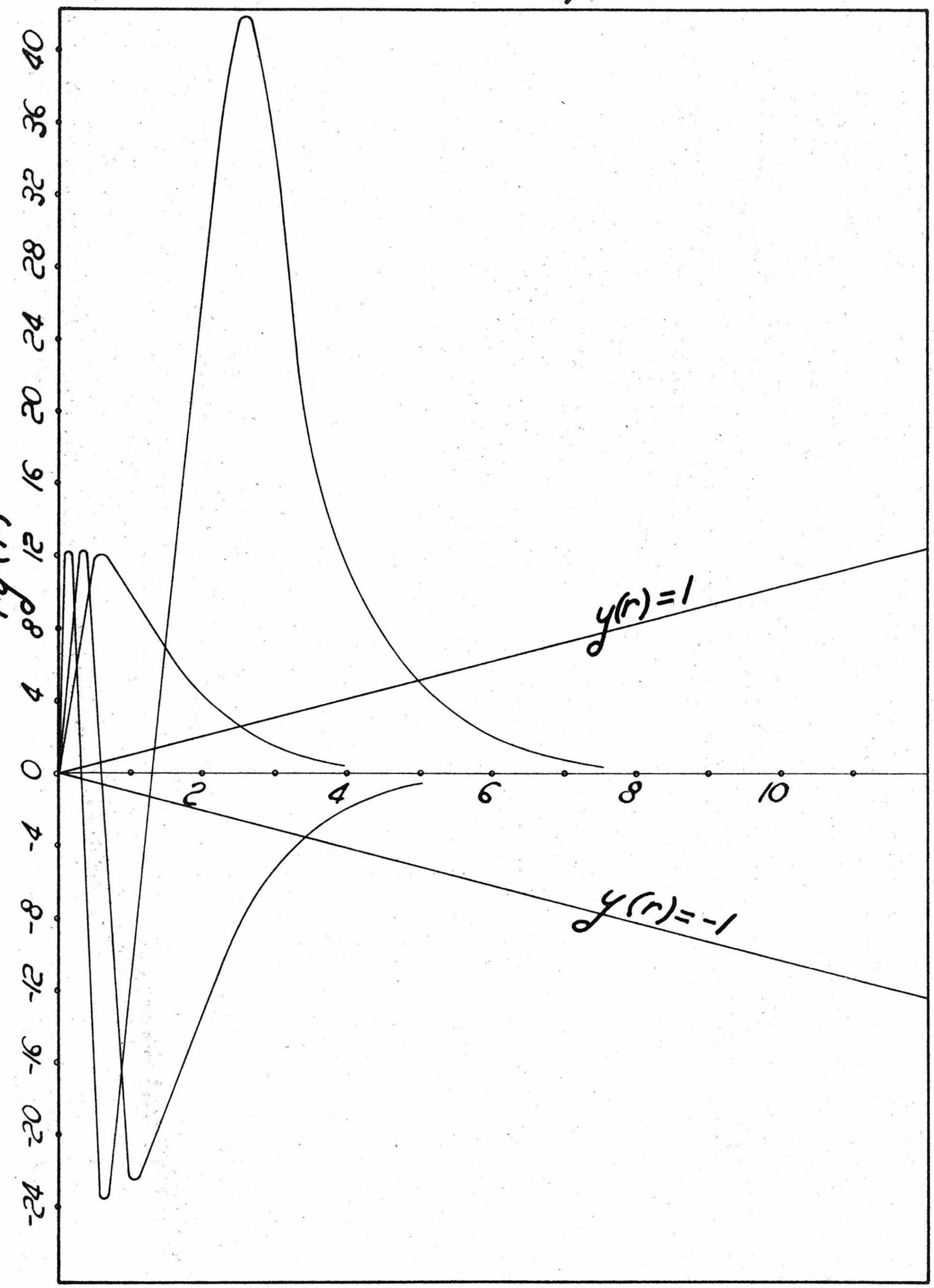
Equation (44) was numerically integrated for various initial values. The 0 -, 1 -, 2 - node solutions were extracted. These are given in Figure (7), where $ry(r)$ is plotted as a function of r. The 0 - node solution has an extension of approximately $\frac{1}{\mu}$. Higher node solutions are larger. When $\omega \neq 0$ (Equation 46) the radius of the particle is approximately $\frac{1}{\mu\sqrt{1-\omega^2}}$. In general the particle radii are greater than the fundamental length; neutral solutions are smaller than charged ones.

The masses corresponding to the neutral particles are obtained from (39) with $\omega = 0$. These are $2\pi\frac{5.63}{\epsilon}$, $2\pi\frac{38.2}{\epsilon}$, $2\pi\frac{109}{\epsilon}$, for 0,1, and 2 nodes. The mass ratios are independent of the coupling. In Figure (8) the product of ϵ [coupling constant] and Q [charge] is plotted against ω. For a fixed coupling ϵ, and with the charge normalized to 1, ω is determined. After charge normalization the energy is

$$|\omega| + 4\pi\epsilon\int_0^\infty r^2\, dr\, y^4(r) \tag{75}$$

The second term multiplied by ϵ is plotted as a function of ω in Figure (9). As $\epsilon \longrightarrow \infty$, $\omega \longrightarrow 1$ and the second term approaches 0. Therefore for large ϵ, the masses of all the charged particles $\longrightarrow |\omega| \longrightarrow 1$. As the mass decreases the size of the particle increases like $\frac{1}{\sqrt{1-\omega^2}}$. The neutral masses $\longrightarrow 0$ for large ϵ ; the size of these particles remains constant. For very small ϵ, $\omega \longrightarrow 0$, and the mass originates almost entirely in the latter term of (75) which $\longrightarrow \infty$

FIG. 7 ··· SOLUTIONS, EQUATION 44.



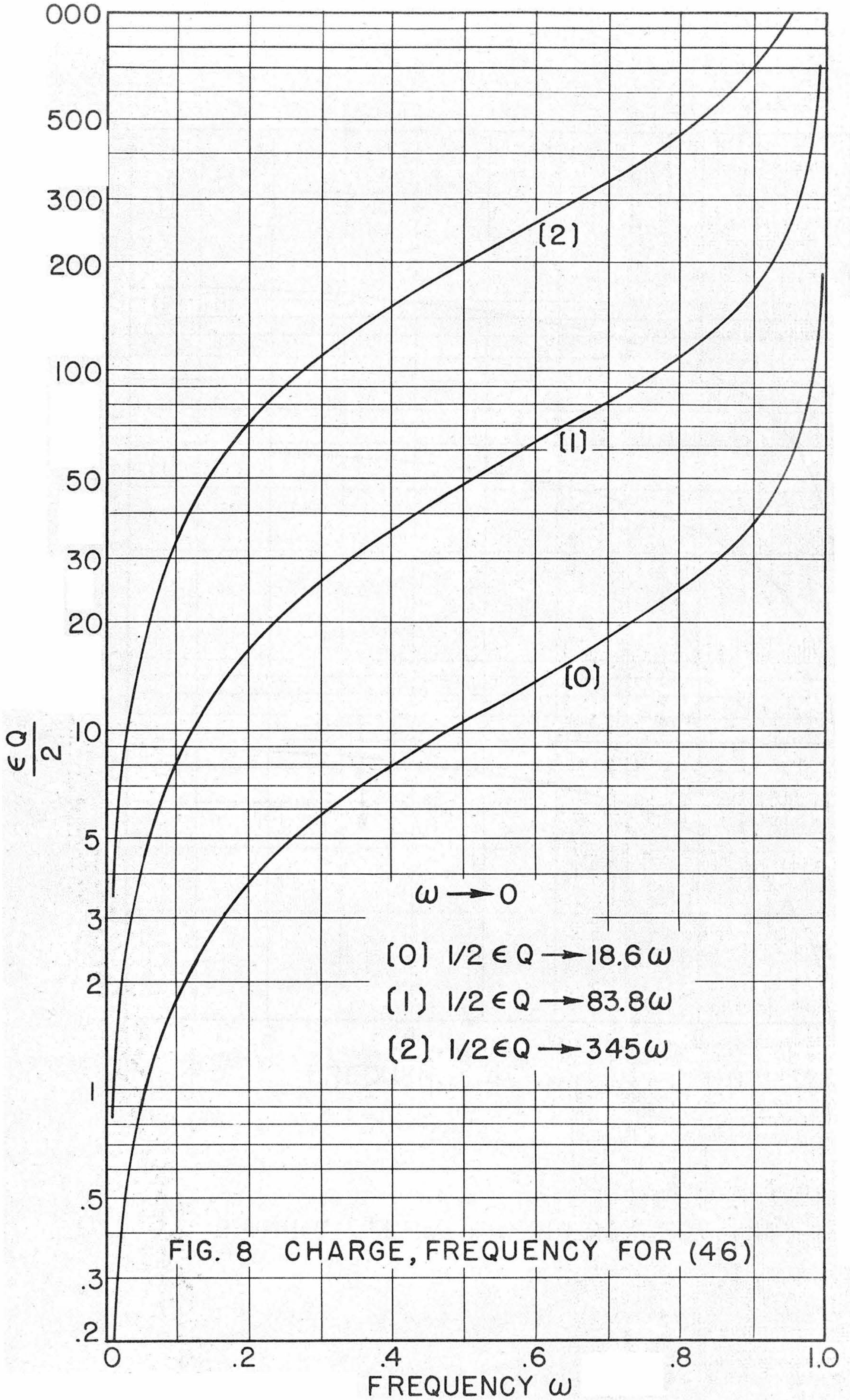


FIG. 8 CHARGE, FREQUENCY FOR (46)

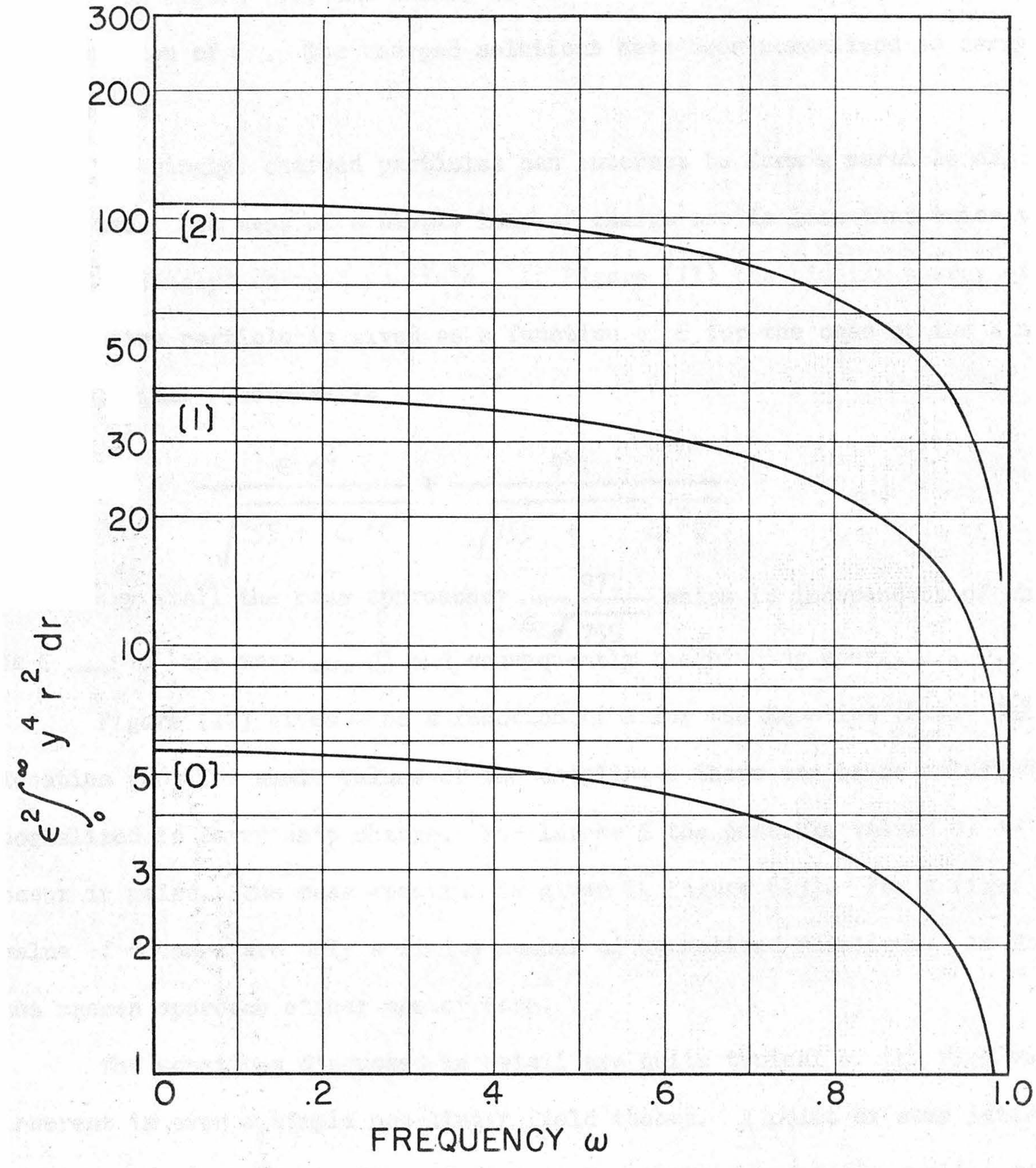


FIG. 9 BINDING TERM, FREQUENCY FOR (46)

as $\epsilon \longrightarrow 0$. Neutral and charged particles then have the same size and energy. In Figure (10) the masses for 0,1, and 2 node solutions are given as a function of ϵ . The charged solutions have been normalized to carry unit charge.

Two singly charged particles can interact to form a particle of charge two. The mass of a single lump of charge two is less than twice the mass of a singly charged particle. In Figure (11) the binding energy of a charge two particle is given as a function of ϵ for the case of the single node solution. The mass is

$$\frac{\epsilon Q^2}{\sqrt{755 + \epsilon^2 Q^2}} + \frac{971}{\epsilon\sqrt{755 + \epsilon^2 Q^2}}$$

For ϵ very small the mass approaches $\dfrac{971}{\epsilon\sqrt{755}}$ which is independent of charge. As $\epsilon \longrightarrow \infty$ the mass $\longrightarrow Q$, and consequently the binding energy $\longrightarrow 0$.

Figure (12) gives ω as a function of ϵ for the Equation (41). Unlike Equation (46) for small values of the coupling ϵ there can be no solutions normalized to carry unit charge. For larger ϵ the possible values of ω occur in pairs. The mass spectrum is given in Figure (13). For a fixed value of ϵ there are only a finite number of normalized solutions. As $\epsilon \longrightarrow \infty$, the masses approach either one or zero.

The equations discussed in detail are quite typical of the rich variety inherent in even a simple non-linear field theory. A point of some interest is the role of charge in the solutions. For certain types of coupling only charged particles or only neutral particles could exist; in those cases where both charged and neutral could be formed, they will have similar properties only for very strong coupling.

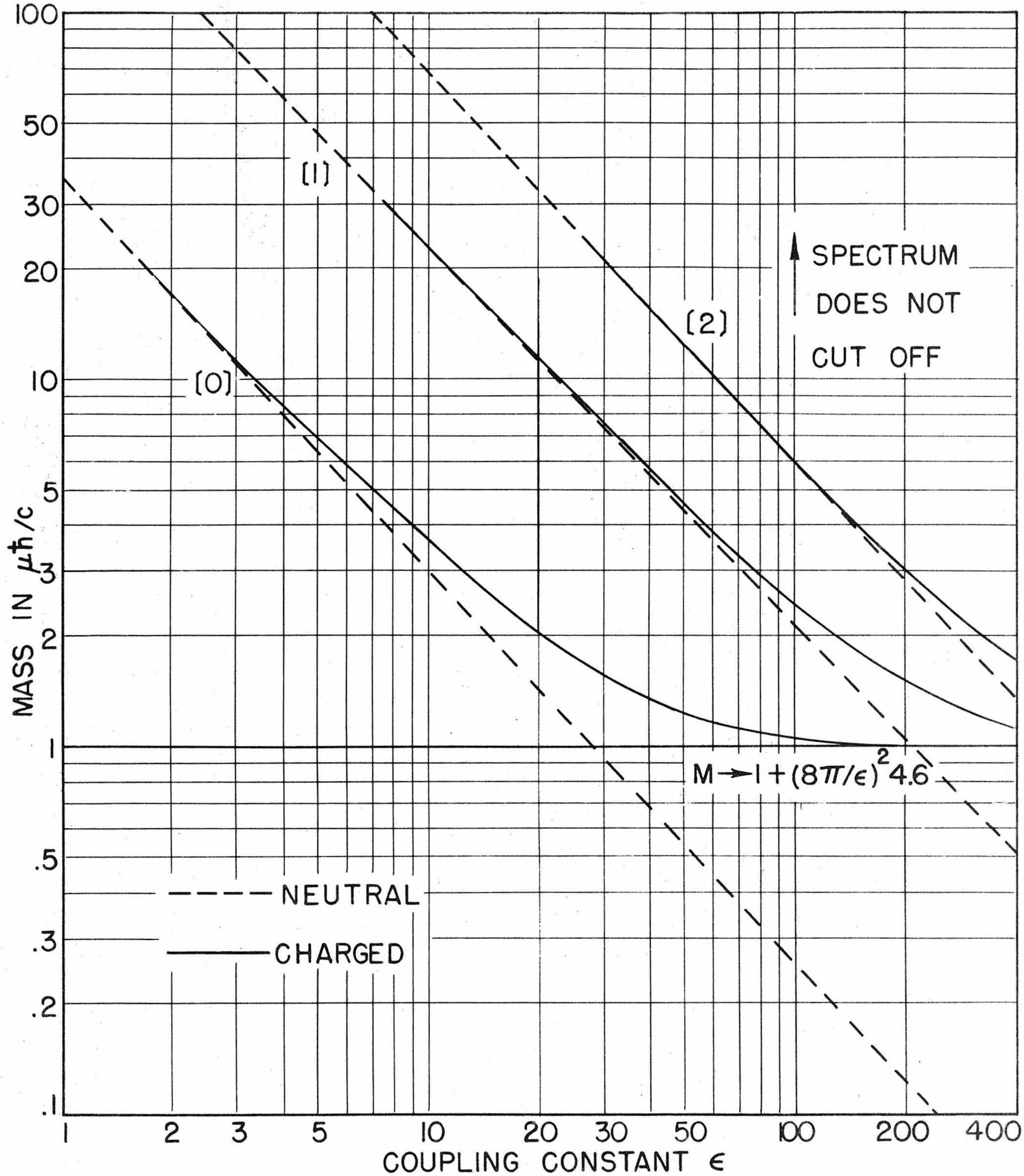


FIG. 10 MASS SPECTRUM FOR (46)

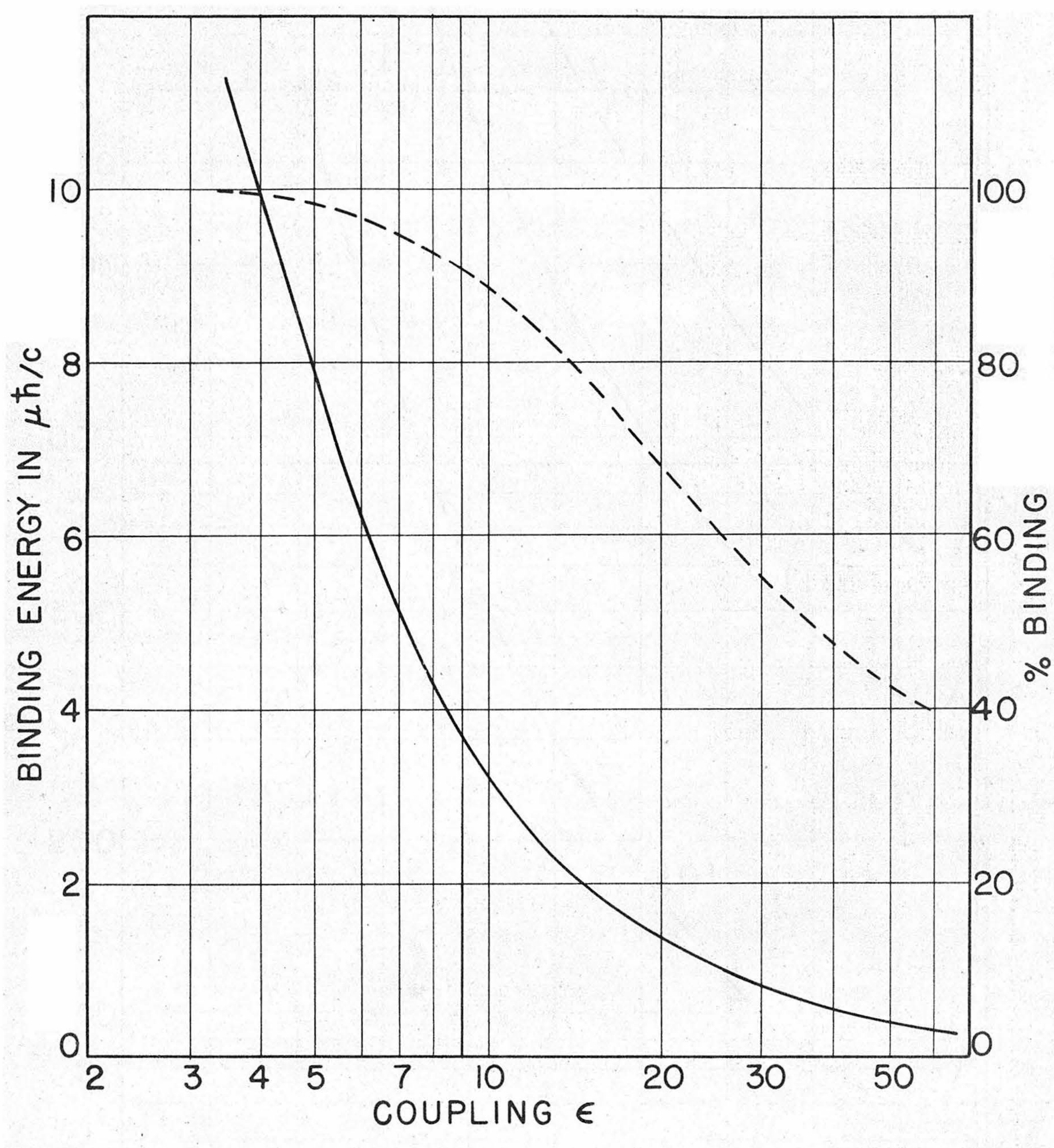


FIG. 11 BINDING ENERGY OF CHARGE-TWO PARTICLE

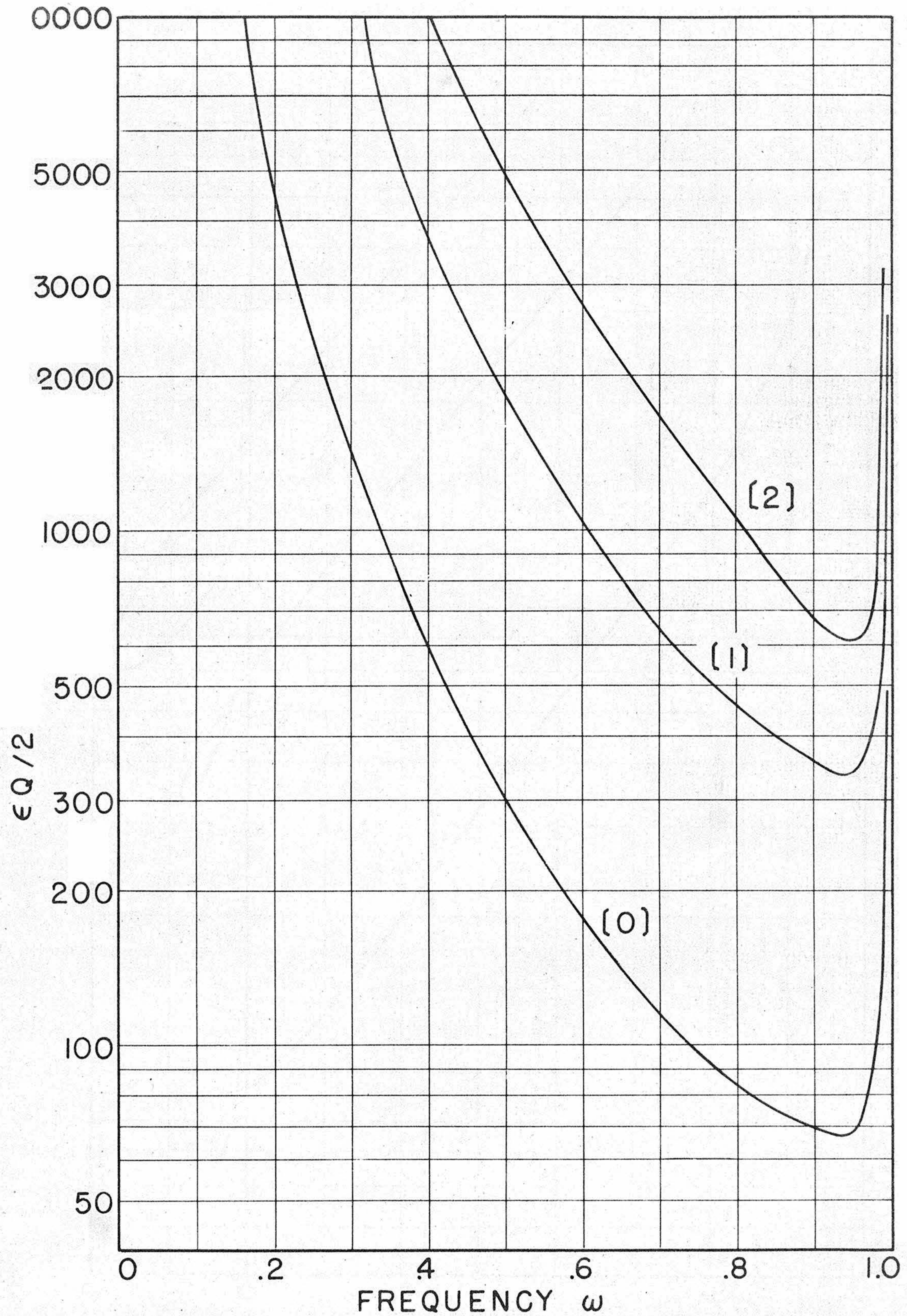


FIG. 12 CHARGE, FREQUENCY FOR (41)

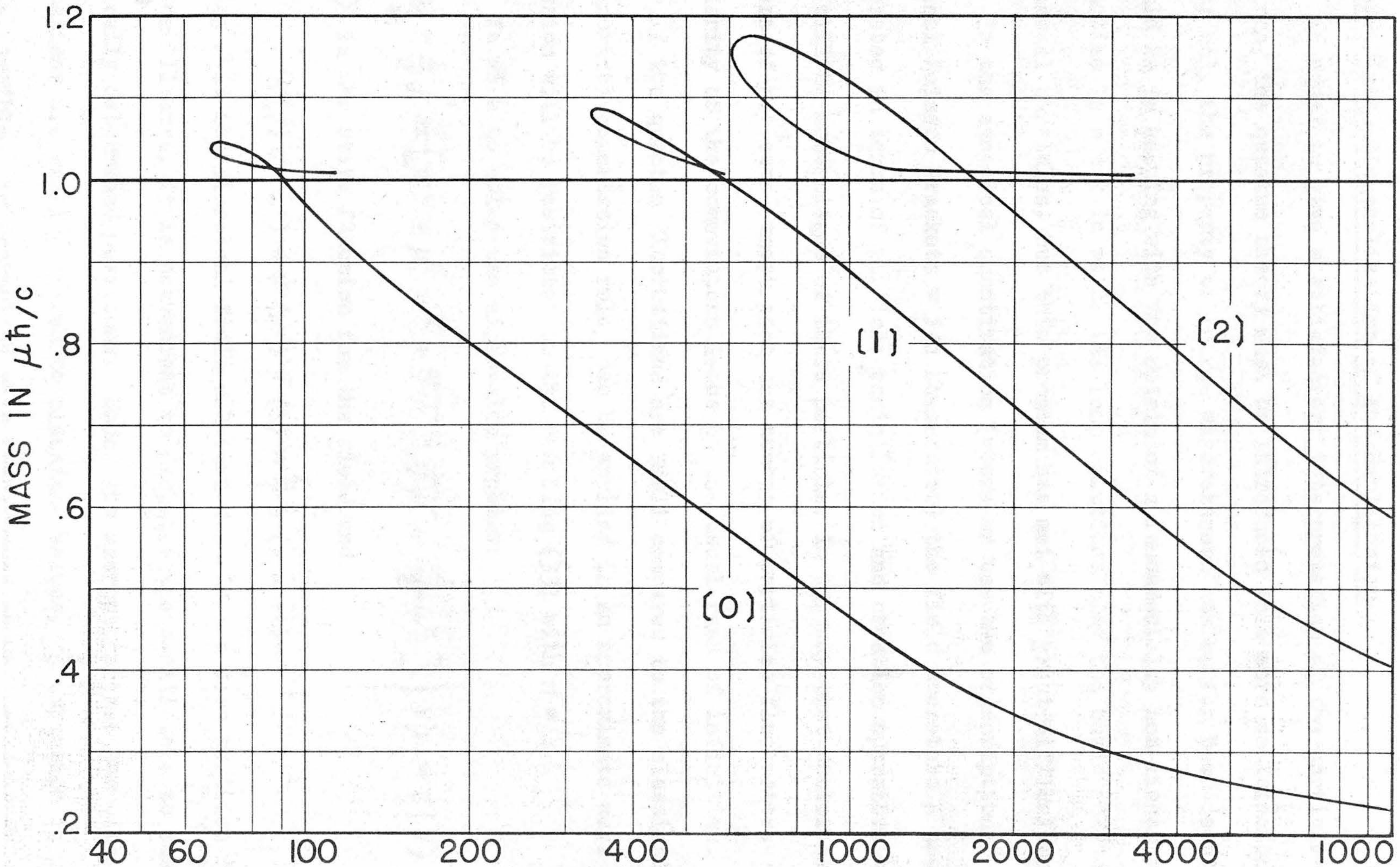


FIG. 13 MASS SPECTRUM FOR (41)

III. CANONICAL QUANTIZATION OF THE SCALAR FIELDS

1. Approximate Diagonalization of the Hamiltonian

In order to give a satisfactory interpretation of the particle-like solutions, the quantum theory must be introduced in a more satisfactory manner; only the property of charge discreteness has so far been described. It would be in keeping with the spirit of a fundamentally non-linear theory to quantize in a way in which the lump-solutions play the basic role of fundamental particles, but this program has met with great difficulties.

In the canonical quantization (where we use the prescription: Classical Poisson Brackets = $i/\hbar$ Commutator) the field operators π and ψ are interpreted in terms of particle annihilation and creation operators and the lumps will be assemblages of these particles in the way that nuclei are clusters of nucleons except that the number of particles fluctuates. The singularity of the commutators leads to the usual rash of infinities.

If the quantum fluctuations are small compared to the classical values, the canonical commutation rules can be applied in an approximate way. The discussion will be restricted to the coupling (33) with n = 4.

We wish to solve the eigenvalue problem:

$$H\,|F\rangle = \frac{1}{2}\int \vec{dx}\left[\pi^2 + \mu^2\psi^2 + \overline{\nabla\psi}\cdot\overline{\nabla\psi} - \frac{\epsilon}{2}\psi^4\right]|F\rangle = E\,|F\rangle \tag{76}$$

$|F\rangle$ is the state function for the field and

$$\pi(x)\,\psi(x') - \psi(x')\,\pi(x) = \frac{\hbar}{i}\,\delta(x - x') \tag{77}$$

Assuming that the classical field solution describes a mean position for the field oscillators, it is convenient to displace the oscillators to the classically determined positions. Under the assumption that the quantum fluctuations are small compared to classical values, an expansion in powers of h is feasible. To accomplish this transformation we introduce the unitary operator:

$$S = \exp\left[\,i\int \frac{\pi(x)\,\psi_0(x)}{\hbar}\,\vec{dx}\,\right] \tag{78}$$

where $\psi_o(x)$ is the proper solution of the classical Equation (37). If

$$|F\rangle = S|G\rangle \tag{79}$$

G satifies:

$$H'|G\rangle = S^{-1}HS|G\rangle = E|G\rangle =$$

$$\frac{1}{2}\int \overrightarrow{dx}\left[(\pi')^2 + \overrightarrow{\nabla\psi'}\cdot\overrightarrow{\nabla\psi'} + \mu^2\psi'^2 + \frac{\epsilon}{2}\psi_o^4\right. \tag{80a}$$

$$-3\epsilon\,\psi'^2\psi_o^2 \tag{80b}$$

$$-2\epsilon\,\psi'^3\psi_o \tag{80c}$$

$$\left.-\frac{\epsilon}{2}\psi'^4\right]|G\rangle = 0 \tag{80d}$$

π', ψ' satisfy the commutation rule (77). ψ' is $\psi - \psi_o$. The terms in (c) and (d) are higher order in the fluctuation. A field operator is of the order $h^{1/2}$. Therefore in (a) we have the classical energy independent of h; the other terms of (a) and (b) are linear in h; (c) $\sim h^{3/2}$; (d) $\sim h^2$.

If we had begun with the Hamiltonian for a charged field and sought a contact transformation to a form explicitly exhibiting the classical energy and containing no terms linear in the field variables, the required unitary transformation is:

$$S_c = \exp\left[\frac{i\int\pi(x)\,\psi_o(x)\overrightarrow{dx}}{h}\right]\exp\left[\frac{i\int\pi^*(x')\psi_o^*(x')\overrightarrow{dx'}}{h}\right]$$
$$\exp\left[\frac{-i\int\psi(x'')\partial_t\psi_o(x'')\,\overrightarrow{dx''}}{\hbar}\right]\exp\left[\frac{-i\quad\psi^*(x''')\partial_t\psi_o^*(x''')\overrightarrow{dx'''}}{\hbar}\right] \tag{81}$$

$|F_c\rangle$ is the state function and $\psi_o(x, t)$ is the solution of (36, 37). If $|F_c\rangle = U\ G_c\rangle$ then

$$\left[U^{-1}H_cU + U^{-1}\frac{\hbar}{i}\partial_t U\right]|G_c\rangle = E|G_c\rangle \tag{82}$$

and we obtain a transformed equation exactly similar to (80). Since the transformation involved is more complicated but involves no new feature we shall consider only neutral fields.

Keeping terms in h only (neglecting (80c) and (80d)), the Hamiltonian (80) can be diagonalized. Let $\eta_g(x)$ and λ_g be the eigenfunctions and eigenvalues of the equation

$$-\nabla^2 \eta_g(x) - 3\epsilon\, \eta_g(x)\, \psi_o^2(x) = \lambda_g\, \eta_g(x) \quad . \tag{83}$$

We define

$$E_g = + \sqrt{1 + \lambda_g} \tag{84}$$

The set η_g are complete and orthonormal. We take η_g and E_g as the normal modes and frequencies for the unperturbed field oscillators. If a_g^* and a_g are the creation and annihilation operators for the gth mode obeying the commutation rule

$$a_g^* a_{g'} - a_{g'} a_g^* = \delta_{gg'} \tag{85}$$

then we may expand

$$\psi' = \sum_g \sqrt{\frac{\hbar}{2E_g}} \left[a_r \eta_r(x) + a_r^* \eta_r^* (x) \right] \tag{86}$$

$$\pi' = \sum_g \sqrt{\frac{E_g \hbar}{2}} \left[a_g \eta_g(x) - a_g^* \eta_g^*(x) \right] \tag{87}$$

The commutation rule (77) is unaltered. Substituting into (80a,b)

$$H' = \sum_g a_g^* a_g E_g \frac{\hbar\mu}{c} + \frac{E_g}{2} \frac{\hbar\mu}{c} \tag{88}$$

E_g is dimensionless. The diagonalization can no longer be accomplished if $\lambda_g < -1$. This case will be discussed later.

The Hamiltonian possesses a complete set of solutions, that with the lowest energy being a vacuum. Considering fluctuations to order h only,

$$\exp\left[\frac{i\int \pi(x)\,\psi_o(x)\,\vec{dx}}{\hbar}\right]\Big|\,\mathrm{VAC}\Big\rangle \tag{89}$$

is a solution of (76). The quantum fluctuation energy $\sum \frac{E_g}{2}$, is infinite. This infinity is composed of two parts: the half quantum of energy which exists for each field oscillator even when no lump is present $\left(\sum_k \frac{1}{2}\,(1+k^2)^{1/2}\right)$ and an infinite contribution from each of the bosons in the lump interacting with the vacuum oscillations. Subtracting the vacuum energy,

$$E_{FLUCT} = \frac{\hbar\mu}{2c}\sum_g\left(E_g - \sqrt{k_g^2+1}\right) \tag{90}$$

This diverges like $-\frac{\hbar\mu}{2c}\epsilon\int\psi_o^2(x)\,\vec{dx}\int^{\infty} k\,dk$. It is independent of the coupling constant since $_o(x)$ is proportional to $\epsilon^{-1/2}$. The infinite term can be cut-off (the Dirac Indefinite Metric was used); the residue is an estimate of the quantum correction caused by gathering the bosons into a lump. It is $\hbar/\mu c$ times a numerical factor which depends on the low lying energy levels. The condition for the validity of the classical approximation is

$$\frac{E\ \text{quantum}}{E\ \text{classical}} \approx \frac{\hbar\mu}{CE\ \text{class}} << 1 \tag{91}$$

The lump must be large compared to its Compton wavelength because μ^{-1} is a measure of the size of the lump. Since $E_{CL} \approx \epsilon^{-1}$, the coupling must be small. This condition is equivalent to the restriction that the number of bosons in the lump be large next to the fluctuations in that number.

The state function (89) is not a satisfactory approximate solution of (76) because of the degeneracy of the classical lump. Instead of displacing the field oscillators to the classical solution $\psi_o(x)$, we could with equal justification have used $\psi_o(x + a)$. This difficulty manifests itself in the fact that (89) is not an eigenfunction of the momentum operator

$$\int (\pi \overrightarrow{\nabla \psi} + \pi^* \overrightarrow{\nabla \psi^*}) \, \overrightarrow{dx} \qquad (92)$$

which commutes with the Hamiltonian. The position of the center of mass of the lump is well defined, which gives a large spread to the momentum and kinetic energy. The state function*

$$|K\rangle = \frac{1}{\sqrt{V}} \sum_a \exp\left[i/\hbar \int \pi(x)\, \psi_o(x+a)\, \overrightarrow{dx} \right] \exp iKa \, |\, VAC\rangle \qquad (93)$$

is an eigenfunction of (92) with E-value $\hbar K$. It is a sum of solutions of (76), all of which have the same energy to order h. $\langle L | K\rangle = \delta_{LK}$ so that the eigenfunctions are orthonormal. The energy degeneracy is split by the $h^{3/2}$ term which does not couple states with different K, thus justifying this choice of eigenfunction.

2. Representation of the State Function

A convenient representation of the state function for systems with fluctuating numbers of particles has been given by Fock[10]. Instead of the language of the quantum theory of fields we make use of configuration space formalism. In configuration space the state function Φ is a function of the coordinates $r_1, r_2, \ldots, r_n$ where n is the number of particles. When the number of particles does not commute with the Hamiltonian, Φ is a superposition of states with various occupation numbers: Φ = superposition $\phi_o, \phi_1(r_1), \phi_2(r_1, r_2), \ldots\ \phi_m(r_1, r_2, \ldots, r_m)$ is the symmetrized (or anti-symmetrized) Schroedinger wave function for the state in which there are m particles. We write Φ as the column vector:

$$\Phi = \begin{bmatrix} \phi_o \\ \phi_1(r_1) \\ \phi_2(r_1, r_2) \\ \cdot \end{bmatrix}$$

* This state function no longer has a classical limit.

$$\begin{bmatrix} \vdots \\ \phi_m(r_1, r_2, \ldots, r_m) \\ \vdots \end{bmatrix}$$

It is now necessary to find a representation for the field operators ψ and π in this language. We first express these operators as functions of a(r) and $a^*(r)$ such that

$$a(r')a*(r) - \alpha\ a*(r)a(r') = \delta(r - r') \tag{94}$$

$\alpha = -1$ for Fermi statistics; $\alpha = +1$ for Bose statistics. Such an expansion has already been performed in Equations (86) and (87). Let n be an operator defined by

$$n = \int \overrightarrow{dr}\ \ a*(r)a(r) \tag{95}$$

n has the interpretation of number of bosons or fermions. For both statistics

$$na - a(n - 1) = 0 \tag{96}$$

Therefore the matrix element between two states of n and n' particles is

$$\langle n \,|\, na - a(n - 1) \,|\, n' \rangle \quad = 0 \tag{97a}$$

or

$$(n - n' + 1) \langle n \,|\, a \,|\, n' \rangle = 0 \tag{97b}$$

a(r) therefore has the form:

$$\begin{bmatrix} 0 & \langle 0 | a | 1 \rangle & 0 & 0 & \cdot\ \cdot \\ 0 & 0 & \langle 1 | a | 2 \rangle & 0 & \cdot\ \cdot \\ 0 & 0 & 0 & \langle 2 | a | 3 \rangle & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \tag{100}$$

We take as an Ansatz for $\alpha = +1$

$$\langle n - 1 \,|\, a\ (r) \,|\, n \rangle\ \phi_n(r_1, r_2, \ldots, r_n) = \sqrt{n}\ \phi_n(r_1 r_1, \ldots, r_{n-1}) \tag{101}$$

Then

$$a(r) \begin{bmatrix} \phi_o \\ \phi_1(r_1) \\ \\ \end{bmatrix} = \begin{bmatrix} \phi_1(r_1) \\ \sqrt{2}\ \phi_2(r, r_1) \\ \\ \end{bmatrix} \tag{101}$$

Alternatively we could write for either ϵ

$$\langle n - 1 \,|\, a(r) \,|\, n\rangle \; \phi_n(r_1, \ldots, r_n) =$$

$$\frac{1}{\sqrt{n}}\left[\phi(r, r_1, r_2, \ldots, r_{n-1}) + \alpha\, \phi(r_1, r, r_2, \ldots, r_{n-1}) + \ldots + \alpha^{n-1}\phi(r_1 r_2, \ldots, r_{n-1}, r)\right]$$

For $\alpha = +1$ this is obviously equivalent to (101) since each ϕ is symmetric in the coordinates. For the conjugate operator $a*(r)$ we have

$$\langle a(r)\,\Phi \,|\, a(r)\,\Phi\rangle = \langle a*(r)a(r)\,\Phi \,|\, \Phi\rangle$$

It follows that $a*(r)$ has the form:

$$a* = \begin{bmatrix} 0 & 0 & 0 & \cdot\cdot \\ \langle 1|\,a*\,|\,0\rangle & 0 & 0 & \cdot\cdot \\ 0 & \langle 2|a*|1\rangle & 0 & \cdot\cdot \\ 0 & 0 & \langle 3|a*|2\rangle & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \end{bmatrix} \tag{102}$$

In order for the commutation rule (77) to be satisfied

$$\langle n \,|\, a*(r) \,|\, n - 1\rangle \phi_{n-1}(r_1, r_2, \ldots, r_{n-1}) =$$

$$= \frac{1}{\sqrt{n}}\left[\delta(r_1 - r)\phi_n(r_2, r_3, \ldots, r_n) + \epsilon\, \delta(r_2 - r)\phi_{n-1}(r_1, r_3, \ldots, r_n) + \ldots \right.$$
$$\left. \ldots + \epsilon^{\,n-1}\, \delta(r_n - r)\phi_{n-1}(r_1, r_2, \ldots, r_{n-1})\right] \tag{103}$$

or for $\epsilon = 1$

$$a*(r)\begin{bmatrix} \phi_o \\ \phi_1(r_1) \\ \phi_2(r_1, r_2) \\ \phi_3(r_1, r_2, r_3) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_o\, \delta(r - r_1) \\ \frac{1}{\sqrt{2}}\left[\delta(r-r_1)(\phi_1(r_2) + \delta(r-r_2)\phi_1(r_1)\right] \\ \frac{1}{\sqrt{3}}\left[\delta(r-r_1)\phi_2(r_2, r_3) + \delta\,(r-r_2)\phi_2(r_1, r_3) + \delta(r-r_3)\phi_2\right. \\ \left.(r_1, r_2)\right. \\ \vdots \end{bmatrix} \tag{104}$$

Let $\sigma_o(k)$ be the Fourier transform of $\psi_o(x)$ and let $\Theta(k,g)$ be the Fourier transform of $\eta_g(x)$. Then the state function (93) in the Fock notation becomes:

$$\Phi = \frac{A}{\sqrt{V}}\begin{bmatrix} \delta(K)/V^{1/2} \\ \sqrt{2}\left(\frac{c^3}{h\epsilon}\right)^{1/2}(E_{g_1})^{1/2}\,\sigma_o(K/\mu)\,\Theta\,(K/\mu,\ g_1)\,\mu^{-3/2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{2^{n/2}}{\sqrt{n!}}\left(\frac{c^3}{h\epsilon}\right)^{n/2}\mu^{-3n/2}(E_{g_1}E_{g_2}\ldots E_{g_n})^{1/2}\int \vec{dk_1}\int \vec{dk_2}\int\ldots\int \vec{dk}_{n-1}\,\sigma_o(k_1/\mu)\Theta(k_1/\mu,g_1) \\ \times\,\sigma_o(k_2/\mu)\Theta(k_2/\mu,g_2)\ \times\ldots\sigma_o(k_{n-1}/\mu)\Theta(k_{n-1}/\mu,g_{n-1}). \\ \times\,\sigma_o\left(\frac{K-k_1-k_2-\ldots-k_{n-1}}{\mu}\right)\Theta\left(\frac{K-k_1-k_2-\ldots-k_{n-1}}{\mu},\ g_n\right) \\ \cdot \\ \cdot \end{bmatrix} \qquad (105)$$

For large E_g $\eta_g = \exp\left[i\,\vec{g}\cdot\vec{x}\,\mu\right]$. The state function takes the simpler form

$$\Phi = \frac{A}{\sqrt{V}}\begin{bmatrix} \delta(K)/\sqrt{V} \\ \sqrt{2}\left(\frac{c^3}{h\epsilon}\right)^{1/2}\left(1+k_1^2/\mu^2\right)^{1/4}\sigma_o(k_1/\mu)\mu^{-3/2}\,\delta(K-k_1) \\ \cdot \\ \cdot \\ \frac{2^{n/2}}{\sqrt{n!}}\left(\frac{c^3}{h\epsilon}\right)^{n/2}\prod_{i=1}^{n}\left(1+k_i^2/\mu^2\right)^{1/4}\sigma_o\left(\frac{k_i}{\mu}\right)\mu^{-3n/2}\delta(K-k_1-\ldots-k_n) \\ \cdot \\ \cdot \end{bmatrix} \qquad (106)$$

The probability of finding n particles in the lump is

$$P(n) = \frac{A^2}{n!}\left(\frac{2c^3}{h\epsilon}\right)^n\mu^{-3n}\int\vec{dk}_1\ldots\int\vec{dk}_n\prod_{i=1}^{n}\left(1+k_i^2/\mu^2\right)^{1/2}\sigma_o^2(k_i/\mu)\,\delta(K-k_1-\ldots-k_n) \qquad (107$$

In order for the approximate quantization to have some validity the number of particles should be large compared to the fluctuation in this number. In this case the n-fold integral can be evaluated by the central limit theorem. The integral is just that which occurs in the random walk problem for a spherical distribution of displacements. For n 1

$$P(n) \approx \frac{A^2}{n!}\left(\frac{2c^3}{h}\right)\mu^{-3n}\left[\int \vec{dy}\,\sqrt{1+y^2}\;\sigma_o^2(y)\right]\frac{\exp\left[-3K^2/2n\langle k^2\rangle\right]}{\left[2\pi\, n\langle k^2\rangle/3\right]^{3/2}} \tag{108}$$

where

$$\langle k^2\rangle = \frac{\int_o^\infty y^4\sqrt{1+y^2}\;\sigma_o^2(y)\,dy}{\int_o^\infty y^2\,\sigma_o^2\,(y)\,dy} \tag{109}$$

For large n the distribution P(n) is approximately Poisson with a mean number

$$\bar{n} = \frac{2c^3}{h}\int dy\,\sqrt{1+y^2}\;\sigma_o^2(y) =$$

$$\frac{4}{\pi}\left(\frac{\text{mass of lump}\,c}{\hbar\mu}\right)\left(\frac{\int y^2\,\sqrt{1+y^2}\;\sigma_o^2(y)\,dy}{\int y^2\,\sigma_o^4\,(y)\,dy}\right) \tag{110}$$

The second parenthesis .63 for the 0-node solution, .55 for 1-node, and .50 for 2-node. The r.m.s. fluctuation is $\sqrt{\bar{n}}$. In order that this be small next to $\bar{n}$ we have

$$\frac{\text{mass } c}{\hbar} >> \mu^{-1}$$

The lump must be much larger than its Compton wavelength, which is the condition previously stated. The fluctuation energy proportional to $h^{3/2}$ is infinite. If the infinite integral (which is the same as that in the h proportional term) is cut off, the energy correction for the K = 0 state is of the order

$$\frac{1}{\sqrt{\bar{n}}}\,\frac{\mu\hbar}{c}\,\left(\frac{\mu\hbar}{c}\right)^{3/2}\left(\frac{1}{\text{Mass}}\right)^{1/2} \tag{111}$$

For $\psi_o(x)$ given by (44), the lowest E-value λ_o of Equation (83) is less than -1. To show this we assume that λ_o corresponds to a state of zero angular momentum. Then

$$-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\eta_o(r) - 3\,\eta_o(r)\,\psi_o^{\,2}(r) = \lambda_o\,\eta_o(r) \tag{112}$$

Now $\psi_o(r)$ is very approximately $\dfrac{4}{\cosh r}$. Putting $\eta_o(r) = \dfrac{R_o(r)}{r}$,

$$\frac{d^2R_o}{dr^2} + \lambda_o R_o = -\frac{(3)(16)R_o}{\cosh^2 r} = \frac{1}{2}\frac{\tau(\tau-1)\;R_o}{\cosh^2 r}\;; \qquad \tau \approx 10 \tag{113}$$

The eigenfunctions of this equation can be expressed in terms of hypergeometric functions(11):

$$R_o = \cosh^2 r \sinh r\, F(p + iq,\, p-iq,\, 3/2;\, -\sinh^2 r) \tag{114}$$

$$p = \frac{\tau+1}{2}\;; \quad q = \frac{\sqrt{\lambda_o}}{2}$$

The E-values $\lambda_o, \lambda_1, \lambda_2, \ldots$ for the bound states ($\lambda < 0$) are $-\frac{1}{2}\left[\tau - 2n + 2\right]$ where n 0,1,2,.... , $n < \tau/2 - 1$. Therefore for $\tau = 10$,

$$\lambda_o = -32$$
$$\lambda_1 = -18$$
$$\lambda_2 = -\;2$$

The lowest E-value is so much less than -1 that there is no need to improve the approximation to $\psi_o(r)$

When $E_i^{\,2} = 1 + \lambda_i < 1$ for $0 \leq i \leq m$

and $E_i^{\,2} = 1 + \lambda_i > 1$ for $i > m$

the Hamiltonian can be put into the form:

$$\sum_{g=m+1}^{\infty} a_g^* a_g E_g + E_g/2$$

$$\sum_{g=0}^{m} \left| E_g \right| (a_g^* a_g^* + a_g a_g) \tag{115}$$

There is no eigenstate function for this Hamiltonian which represents a finite number of particles.* The potential $3\,\psi_o^{\,2}(x)$ can create real pairs

* The proof that the Hamiltonian cannot be entirely diagonalized and that the state function represents an infinite number of particles is almost identical to that of Schiff, Snyder, and Weinberg(12) for scalar mesons in sufficiently strong electric fields.

since the bosons can have binding energies greater than their rest masses. The existence of bosons in these states stimulates further emission, etc.

The breakdown of the quantum treatment mirrors a classical instability. If we look for solutions of (34) of the form $\varphi_0(r) + A(r) \cos \omega t$ for a very small perturbation amplitude A, A and ω are connected by the equation:

$$-\nabla^2 A - 3\epsilon A \varphi_0^2 = (\omega^2 - 1)A \tag{116}$$

A is continuous and vanishes as r approaches infinity. (116) is identical with the eigen-equation (112). Therefore some of the roots ω are imaginary so that the perturbation is unstable.* Solutions of (34), when perturbed, can degenerate into the usual S-wave solution of the linear Bose equation since the amplitude becomes so small that the non-linear term is negligible. Therefore the quantum fluctuations excite modes which are classically unstable.

IV. CANONICALLY QUANTIZED NON-LINEAR SPINOR FIELD

1. Canonical Formalism

A non-linear spinor field is more pregnant with possibilities for a theory of elementary particles than a bose field. Scalars and vectors can be built out of spinors, while a scalar or vector field seems incapable of describing the properties of spin 1/2 particles. Spinor theories of the π-meson[13][14] and the photon[15] have been proposed which do not discourage further investigation.

The Dirac Lagrangian may be written in terms of the two invariants I_0 and I_1.

$$\mathcal{L} = \int L_0 d^4x \tag{117}$$

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* The energy is: $H \approx \int \varphi^4 \, d\tau + \omega^2 \int A^2 \, d\tau$ for real ω so that H is a local minimum when ω equals zero. If ω is imaginary the ω^2 term is missing; the energy of the ω equals zero solution is no longer a local relative minimum.

$$L_o = \mu I_o + I_1 \tag{118}$$

where

$$I_o = i\psi^+ \psi \tag{119}$$

and

$$I_1 = i/2 \left[\psi^+ \gamma_\mu \partial_\mu \psi - (\partial_\mu \psi^+) \gamma_\mu \psi \right] \tag{120}$$

μ is the inverse of the fundamental length.* A generalization of (118) is:

$$L = \mu I_o + I_1 + \epsilon W(I_o, I_1, J) \tag{121}$$

ϵ is a coupling constant. J is any other invariant composed from the spinor field. We shall assume that J contains no derivatives of the field. W is some function of the invariants which makes the Euler equation non-linear.

For a spinor field

$$S^{(mn)}_{\alpha\beta} = \frac{1}{2} (\gamma_\alpha \gamma_\beta)_{mn} \tag{122}$$

Then (17a), (17b), and (17c) give:

$$Q = \int L_1 \psi * \psi \overrightarrow{dx} \tag{123}$$

$$Mc^2 = -\int T_{44} \overrightarrow{dx} = \int \frac{L_1}{2} \left[\psi * \partial_4 \psi - (\partial_4 \psi) * \psi \right] \overrightarrow{dx} + \mathcal{L} \tag{124}$$

$$S_z = \frac{1}{c} \int L_1 \psi * \left(\frac{1}{i} \frac{o}{o\phi} + \frac{1}{2} o_z \right) \psi \tag{125}$$

ϕ is the azimuth angle in polar coordinates and

$$L_1 = \frac{\partial L}{\partial I_1} = 1 + g \frac{\partial W}{\partial I_1} \tag{126}$$

These integrals are to be computed in the proper coordinate system $\vec{G} = 0$. The z-axis is parallel to the spin. The simplest choice of W which results in a non-linear theory is

$$W = I_o^2 \tag{127}$$

Then $L_1 = 1$ and Q, Mc^2, and S_z reduce to the usual expressions of the Dirac

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* For the classical field μ is not necessarily related to the Compton wavelength appearing in the usual formulation.

theory. In particular the conjugate momentum is given by

$$\pi = \frac{\partial L}{i\, c\, \partial(\partial_4 \psi)} = \frac{1}{ic} \frac{\partial L}{\partial I_1} \frac{\partial I_1}{\partial(\partial_4 \psi)} = \psi^* \tag{128}$$

as in the linear Dirac theory so that the usual commutator of Equation (18b) can be applied:

With the canonical anticommutation rules for the spinor field operators:

$$\psi^*_\alpha(\vec{x})\, \psi_\beta(x') + \psi_\beta(x')\, \psi^*_\alpha(x) = \frac{\hbar}{i}\, \delta(x - x')\, \delta_{\alpha\beta} \tag{129}$$

no classical limit exists. For the non-linear coupling (127) the Hamiltonian is

$$H = \vec{dx} \Big[\frac{\hbar}{i} \sum_{rs} \psi^*_r(x)\, \vec{\alpha}_{rs} \cdot \vec{\nabla} \psi_s(x) + M \psi^*_r(x)\, \beta_{rs} \psi(x)_s$$

$$- g \sum_{rstv} \psi^*_r(x)\, \beta_{rs} \psi_s(x)\, \psi^*_t(x)\, \beta_{tv}\, \psi_v(x) \Big] \tag{130}$$

The complete solution of the non-linear field problem consists in the determination of a state function $| x \rangle$ such that

$$H | x \rangle = E | x \rangle \tag{131}$$

Since the anticommutation rules (129) have the form (94), the Fock formalism can be carried over with ψ and ψ^* playing the role of annihilation and creation operator. The operator n here commutes with the Hamiltonian so that unlike the Bose case, the number of particles is a good quantum number. (The filling of the vacuum makes the number of particles infinite; we assume for the moment that the vacuum is empty).

For the two particle problem the state function is of the form:

$$| \rangle = \phi(x_1, x_2;\, \sigma_1, \sigma_2) \rangle \qquad \sigma_1, \sigma_2 = 1,2,3,4. \tag{132}$$

Substituting into (130) we obtain for the coupling term:

$$- g \sum_{jm'} (\beta)\sigma_{2j} (\beta) \sigma_1 m'\ \phi(x_1, x_2;\, j, m')\ \delta(x_1 - x_2)$$

$$+ g \sum_{jm'} (\beta)\sigma_1 j(\beta)\sigma_2 m' \; \phi(x_1,x_2;\, jm')\; \delta(x_1 - x_2)$$

$$+ g\,\Delta\,\phi(x_1,x_2\,;\,\sigma_1,\sigma_2) - g\,\Delta\,\phi(x_2,x_1;\,\sigma_2,\sigma_1) \tag{133}$$

Since $\phi(x_1,x_2;\sigma_1,\sigma_2)$ is antisymmetric in 1 and 2 we write the coupling:

$$- g \sum_{jm'} (1 - P_{12}^{\sigma})(\beta)\sigma_2 j(\beta)\sigma_1 m' \;\delta(x_1 - x_2)\phi(x_1,x_2;\,\sigma_1,\sigma_2) \tag{134a}$$

$$+ 2g\,\Delta\,\phi(x_1,x_2;\,\sigma_1,\sigma_2) \tag{134b}$$

P_{12}^{σ} exchanges the four components of the spinors 1 and 2. Δ is a divergent integral of the form $\int^{\infty} k^2\, dk$. The mass of a single particle at rest is $+ M + 2g\,\Delta$ or $- M + 2g\,\Delta$ *.

For n particles there is a contact term of the form (134a) between each pair and the self energy term $2ng\,\Delta$. No solutions exist for the many particle problem with delta function interactions.

Heisenberg(16) has suggested that if the infinities associated with point particles are modified by introducing regulated commutators in (129), it might be possible to build a series of particles with structure from a single underlying field. In particular he proposes that a non-linearity of the type (121)

$$\int \psi^{+}(x)\; 0\; \psi(x)\, \psi^{+}(x)\; 0\; \psi(x)\; \overrightarrow{dx}$$

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* It might be possible in a divergent theory for $2g\,\Delta$ to be greater than M. In this case all solutions of the single particle Hamiltonian would represent states of positive energy, and the vacuum could be empty. The heavy and light particles could transform into each other, but there is no pair creation or annihilation since the number of particles is conserved.

A similar situation exists in meson theory. The self energy of a nucleon at rest in a positive or negative energy state has as the dominant divergent term:

$$+ \frac{g^2 M}{\pi \hbar c} \int^{\infty} k\, dk$$

Only for hole theory is there symmetry between positive and negative energy states.

could describe all elementary particles (Bose and Fermi) including even photons. Computational difficulties have so far prevented any calculations.

We shall be concerned mostly with a very different point of view: the elimination of point singularities by the description of particles as localized regions of strong field. Such a classical spinor theory does not have even an approximate interpretation in terms of field variables obeying anti-commutation rules. Before turning to the classical theory we shall look at some attempts of Fermi and Yang[13] to get an approximate solution for a Hamiltonian of the type (130). Their model enables one to make a finite calculation for the lifetime of the π-meson which agrees remarkably well with experiment if the conditions specified in Part I are compiled with.

2. Composite Particles

Fermi and Yang have suggested a pair coupling between nucleons of the form (133); the π-meson appears as a composite particle formed by a nucleon and an anti-nucleon under the assumption that one is dealing essentially with a two body problem. The effect of the virtual pairs is interpreted as smearing the contact interaction to a range $\hbar/Mc$, where M is the mass of the nucleon. In order that only particle and anti-particle can be tightly bound, but not particle and particle, the coupling operator should be chosen vector or tensor.

We seek a sixteen component state function of the form $\phi(\bar{r};\ \sigma_1, \sigma_2)$ which will represent a scalar, p-scalar, vector, or p-vector composite particle. ϕ transforms like the direct product of two spinors under rotations and inversions. If U_{ij} is the transformation matrix for a single spinor under the rotation group

$$\begin{aligned} \phi'_{\sigma_1'\sigma_2'} &= U_{\sigma_1'\sigma_1} U_{\sigma_2'\sigma_2} \phi_{\sigma_1\sigma_2} \\ &= U_{\sigma_1\sigma_1'} \phi_{\sigma_1\sigma_2} U_{\sigma_2\sigma_2'} \end{aligned} \tag{135}$$

where $\tilde{U}$ is the transpose of U. We seek an operator R such that

$$(\phi' R) = U(\phi R)U^{-1} \tag{136}$$

R must satisfy
$$\tilde{U}R = RU^{-1} \tag{137}$$

Now
$$U = A\,\alpha_2\alpha_3 + B\,\alpha_3\alpha_1 + C\,\alpha_1\alpha_2 + DI \tag{138a}$$

$$\tilde{U} = A\,\alpha_2\alpha_3 - B\,\alpha_3\alpha_1 + C\,\alpha_1\alpha_2 + DI \tag{138b}$$

$$U^{-1} = -A\,\alpha_2\alpha_3 - B\,\alpha_3\alpha_1 - C\,\alpha_1\alpha_2 + DI \tag{138c}$$

where A, B, and C are functions of the rotation angles only. Therefore R can be $\alpha_1\alpha_3$ or i $\alpha_2\alpha_4$.* The most general scalar under the rotation and inversion transformation $S = USU^{-1}$ is:

$$|A(r)\alpha_4 + B(r)\,\vec{\alpha}\cdot\vec{r} + C(r) + D(r)\,\alpha_4\,\vec{\alpha}\cdot\vec{r} = S \tag{139}$$

The most general P-scalar is:

$$\alpha_1\alpha_2\alpha_3\left[\mathcal{E}(r)\,\alpha_4 + F(r)\,\vec{\alpha}\cdot\vec{r} + G(r) + H(r)\,\alpha_4\,\vec{\alpha}\cdot\vec{r}\right] = PS \tag{140}$$

The most general (P-vector) with spin one in the z direction is: [vector written above "P-vector"]

$$(x + iy)\begin{pmatrix}\text{scalar}\\ \text{P-scalar}\end{pmatrix} + (\alpha_2\alpha_3 + i\,\alpha_3\alpha_1)\begin{pmatrix}\text{P-scalar}\\ \text{scalar}\end{pmatrix} = \begin{pmatrix}\text{vector}\\ \text{P-vector}\end{pmatrix} \tag{141}$$

Under the transformation (135)

$$(S\,\alpha_2\alpha_4)' = U(S\,\alpha_2\alpha_4)U = USU^{-1}\,\alpha_2\alpha_4 = S\,\alpha_2\alpha_4 \tag{142}$$

and similarly for the other operators.

We have investigated solutions of the form PS $\alpha_2\alpha_4$ for vector coupling** with the contact interaction replaced by a square well of range $\hbar$/Mc and depth 26.4 Bev. In addition to the π-particle solution of Fermi and Yang an excited state of about 1600 electron masses was found. The effective

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* If we have a direct product of two spinors representing two particles; $\phi_i^1\,\phi_j^2$, $\phi_i^1\,\phi_k^2\,(\alpha_2\alpha_4)kj$ has the interpretation of the direct product of the wave functions of a particle and anti-particle. $\alpha_3\alpha_1$ flips the spin of one of the particles.

** A rather complete discussion of the composite particle problem has since been given by Moseley and Rosen[17].

$g^2/\hbar c$ for the coupling of this particle to nucleons is approximately 80 times that of a π-meson.*

The composite π-meson has the usual difficulty of the competition between $\pi \longrightarrow (e, \nu)$ and $\pi \longrightarrow (\mu, \nu)$ decay, since $P^+ + N^- \longrightarrow e^+ + \nu$ or $\mu^+ + \nu$. Again if the π is pseudoscalar and the β-decay interaction contains no pseudoscalar but does involve a pseudovector term, the difficulty is resolved.** The transition probability on this model is finite.

Fermi and Yang solved the problem of two particles interacting in a $^1S_0$ state. Their solution may be expressed in the form (139) as:

$$\psi_p \psi_N = \phi(r) = \left[i\left(\frac{f_1-f_4}{2}\right)\alpha_4 - \left(\frac{f_2+f_3}{2}\right)\frac{\vec{a}\cdot\vec{r}}{r} + i\left(\frac{f_1+f_4}{2}\right) + \left(\frac{f_3-f_2}{2}\right)\frac{\alpha_4\vec{\alpha}\cdot\vec{r}}{r}\right]\alpha_1\alpha_3$$

For $\quad r < \hbar/Mc \equiv r_o \qquad (143)$

$$f_1 = \frac{.0136}{r_o^{3/2}} \frac{\sin v}{v} \qquad (144)$$

$$f_4 = \frac{-.0147}{r_o^{3/2}} \frac{\sin v}{v} \qquad (145)$$

$$f_2 = f_3 = \frac{0.370}{r_o^{3/2}} \left[\frac{\cos v}{v} - \frac{\sin v}{v^2}\right] \qquad (146)$$

$$v - 2.03\,(r/r_o) \qquad (147)$$

To convert the description to that of a particle interacting with an anti-particle (the matrix then has the transformation properties of a spinor

---

* The calculation proceeds exactly as that of Fermi and Yang. However, in checking the coupling of the composite π to nucleons we obtain

$$R = i \left(\frac{2\pi \hbar^3 c^3}{r\,u\,c^2}\right)^{1/2} (.62)\ \gamma_1\gamma_2\gamma_3$$

for the first term of their Equation (18). This gives a meson coupling $g^2/\hbar c \approx 1/200$ instead of the 0.27 which was the attractive point of their calculation.

** cf. page. 14.

[annihilation operator] and a complex conjugate [creation operator]). We multiply by the charge conjugate operator $\alpha_2\alpha_4$ so that

$$\bar{\phi}_{ij} = \psi^*_{pi}\,\psi_{Nj} = \phi_{ik}(\alpha_2\alpha_4)_{kj} \tag{148}$$

$\bar{\phi}$ is now a pseudoscalar under-reflections and rotations.

The transition probability for $\pi - \mu$ decay is:

$$\frac{2\pi}{\hbar}\, g_A^2 \sum_{i,j,\mu,\nu=1}^{4} |\bar{\phi}_{ij}(0)\, A_{ij}|^2 \; |\psi^*_\mu\, A_{\mu\nu}\,\psi_\nu|^2 \, \rho(E) \tag{149}$$

Only if A is a pseudoscalar or the fourth component of a pseudovector will this transition be allowed. For P-vector β-decay, $A = \alpha_1\alpha_2\alpha_3$ (150)

$$|\,\mathrm{Trace}\ \bar{\phi}\,(0)A\,|^2 = 4\,|\,f_1 - f_4\,|^2 = \frac{5 \times 10^{-6}}{r_o^3} \tag{151}$$

The transition rate is

$$2\pi(5 \times 10^{-6})\, g_A^2\, \frac{\pi(\mathrm{Mass}_\pi^4 - \mathrm{Mass}_\mu^4)(\mathrm{Mass}_\pi^2 - \mathrm{Mass}_\mu^2)}{(2\pi)^3\ 2\ \mathrm{Mass}_\pi^4} \left(1 - \frac{cp_\mu}{E_\mu}\right) \tag{152}$$

where $\hbar = M = c = 1$

Using $g_A = 2 \times 10^{-49}$ ERG-CM^3

$$\tau = 3 \times 10^{-8}\ \text{sec.}$$

in good agreement with experiment.*

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* For P-scalar β-decay $\tau = 2 \times 10^{-11}$ sec.

V. CLASSICAL SPINOR FIELDS

1. The Wave Equation

The equations of motion of the classical field which result from the Lagrangian (121) are:

$$\frac{\delta\mathcal{L}}{\delta\psi^+} = 0 \; ; \; \gamma_\alpha \partial_\alpha \psi + \mu\psi - i\epsilon \frac{\partial W}{\partial \psi^+} + i\epsilon\, \partial_\mu \left[\frac{\partial W}{\partial I_1} \gamma_\mu \psi \right] = 0 \qquad (153a)$$

$$\frac{\delta\mathcal{L}}{\delta\psi} = 0 \; ; \; (\partial_\alpha \psi^+)\gamma_\alpha - \mu\psi + i\epsilon \frac{\partial W}{\partial \psi} + i\epsilon\, \partial_\mu \left[\frac{\partial W}{\partial I_1} \psi^+ \gamma_\mu \right] = 0 \qquad (153b)$$

We also have the useful invariant equation:

$$\psi^+ \frac{\delta\mathcal{L}}{\delta\psi^+} \quad \frac{\delta\mathcal{L}}{\delta\psi} = 0$$

$$I_1 + \mu I_0 + \frac{\epsilon}{2}\left(\frac{\partial W}{\partial\psi}\psi + \psi^+ \frac{\partial W}{\partial\psi^+} \right) = 0 \qquad (154)$$

The partial differential equations (153) are too difficult to enable a general solution to be obtained as in the case of the linear equation. However, for certain forms of W, solutions exist which make (153) separable and therefore reduce the problem to that of solving a set of ordinary differential equations.

The time separation is accomplished by the substitution

$$\psi = e^{i\zeta(t)} \chi(x,y,z) \qquad (155)$$

The linear part of (153) separates for the Ansatz:

$$\psi_{\pm} = \frac{1}{2} e^{i\zeta} \left\{ (F + iG)\,\Omega_{\pm} + (F - iG)\beta\,\Omega_{\pm} \right\} \qquad (156)$$

F and G are functions of r only.

$$\Omega_+ = \begin{pmatrix} a\, Y_{J-1/2,m-1/2}(\theta,\phi) \\ b\, Y_{J-1/2,m+1/2}(\theta,\phi) \\ c\, Y_{J+1/2,m+1/2}(\theta,\phi) \\ d\, Y_{J+1/2,m+1/2}(\theta,\phi) \end{pmatrix} \qquad \begin{aligned} a &= \left(\frac{J+m}{2J}\right)^{1/2} \\ c &= \left(\frac{J-m+1}{2J+2}\right)^{1/2} \\ a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1 \end{aligned} \qquad (157)$$

and $$\Omega_- = \gamma_5 \Omega_+ \tag{158}$$

when $J = 1/2$, $m = 1/2$ we have

$$\psi_+ = e^{i\xi(t)} \begin{bmatrix} F(r) \\ 0 \\ iG(r)\cos\theta \\ iG(r)\sin\theta\, e^{i\phi} \end{bmatrix} \quad ; \quad \psi_- = \gamma_5 \psi_+ \tag{159}$$

Simple forms for W for which (153) separates have been found only for this case. The Lagrangian for the linear part becomes

$$L_o = \mu I_o + I_1$$

$$I_o = G^2 - F^2 \tag{160}$$

$$I_1 = GF' - FG' - 2FG/r - \frac{d\xi}{dt}(F^2 + G^2) \tag{161}$$

Since F and G are time independent $d\xi/dt = \omega$ = constant. L_o is therefore a function of r only; W also depends only on r. Equation (153) reduces to a pair of ordinary differential equations. Restricting consideration to W's which are quadratic* and symmetric in the fields ψ and ψ^*, some of the simple invariants are:

$$(\psi^+ \gamma_\mu \psi)(\psi^+ \gamma_\mu \psi) = 4G^2F^2 \sin^2\theta + G^4 + F^4 + 2G^2F^2 \tag{162a}$$

$$(\psi^+ \gamma_5 \psi)(\psi^+ \gamma_5 \psi) = 4F^2G^2\cos^2\theta \tag{162b}$$

$$(\psi^+ \sigma_\mu \psi)(\psi^+ \sigma_\mu \psi) = G^4 + F^4 + 2G^2F^2(\cos^2\theta - \sin^2\theta) \tag{162c}$$

$$(\psi^+ \psi)(\psi^+ \psi) = G^4 + F^4 - 2G^2F^2 \tag{162d}$$

$$(\partial_\mu I_o)(\partial_\mu I_o) = (GG' + FF')^2 \tag{162e}$$

---

* The coupling quadratic in the spinor fields seems capable of describing all the known interactions of the spin 1/2 particles among themselves ("symmetric coupling").

$$(\psi^+ \sigma_\mu \psi)\, I_u = 0 \quad \ddagger \tag{162f}$$

$$(\psi^+ \gamma_\mu \psi) I_u = \frac{4GF^3}{r} \sin^2\theta \tag{162g}$$

$$\psi^+(\partial_\mu \psi)(\partial_\mu \psi^+)\psi = (GG' + FF')^2 - \sin^2\theta\, F^4/r^2 \tag{162h}$$

$$(\psi^+ \gamma_5 \psi) I_1 = I_1 FG \cos\theta \quad \ddagger \tag{162j}$$

$$(\psi^+ \psi) I_1 = (G^2 - F^2) I_1 \tag{162k}$$

$$I_1{}^2 = \text{See (161)}$$

$$I_\mu I_\mu = 4F^2/r^2 \sin^2\theta \tag{162m}$$

$$(\psi^+ \partial_\mu \psi)^2 + ([\partial_\mu \psi^+]\psi)^2 + 2([\partial_\mu \psi^+]\psi)(\psi^+ \partial_\mu \psi) = 2(GG' + FF')^2 \tag{162n}$$

In the above

$$I_\mu = \frac{i}{2}\left[\psi^+ \partial_\mu \psi - (\partial_\mu \psi^+)\psi\right] \tag{163}$$

All of the quadratic forms are not independent. For example from (162d), (162b), and (162c)

$$I_o{}^2 = (\psi^+ \sigma_\mu \psi)(\psi^+ \sigma_\mu \psi) - (\psi^+ \gamma_5 \psi)(\psi^+ \gamma_5 \psi) \tag{164}$$

The quadratic form

$$G^4 + F^4 + \lambda G^2 F^2 \tag{165}$$

may be written in terms of invariants* as:

$$\left[\frac{6-\lambda}{8}\right] I_o{}^2 + \left[\frac{2+\lambda}{8}\right] \sum_{\mu=1}^{5} (\psi^+ \gamma_\mu \psi)^2 \tag{166}$$

When W involves derivatives of the field variables there exists the possibility of constructing neutral particles since $L_1 \neq 1$ in (123). However for the wave function (159)

$$\psi^*\left(\frac{1}{i}\frac{\partial}{\partial\phi} + \frac{1}{2}\sigma_z\right)\psi = \frac{1}{2}\psi^*\psi \tag{167}$$

---

* For $\lambda = 6$ this is very analogous to the Møller-Rosenfeld coupling in meson theory.

‡ There seems to be no a priori reason for the coupling constant ϵ also not being pseudoscalar so that invariance is retained.

Therefore from (125)

$$S_z = \frac{\hbar}{2c} Q \tag{168}$$

The spin of a neutral particle is zero. When the normalization is performed by putting $Q = 1$, we have

$$S_z = \frac{\hbar}{2} \tag{169}$$

For this non-quantized theory

$$S_y = S_x = 0 \tag{170}$$

Since the derivative coupling introduces great complications in (153) we shall choose only the simpler derivative forms $I_o I_1$ and I_1^2.*

The boundary conditions to be imposed on F and G are that these functions be everywhere continuous, with all observable integrals finite. The linear part of (153) is:

$$\begin{aligned} G' + \frac{2G}{r} + (\mu + \omega)F + \ldots &= 0 \\ F' + (\mu - \omega)G + \ldots &= 0 \end{aligned} \tag{171}$$

In order for F and G to be finite at the origin the $2G/r$ term forces the initial condition $G(0) \quad 0$; this will also imply $F'(0) = 0$ in all cases to be studied. Since (153a) reduces to two simultaneous first order equations in F and G we have two initial values to specify. One is $G(0) = 0$; the other is the initial value of F †. A solution for arbitrary F(0) usually

---

* Perhaps the simplest generalization of (118) is $L = f(uI_o + I_1)$. The Euler equation is $f'(\mu I_o + I_1)\left[\gamma_\mu \partial_\mu \psi + \mu \psi\right] = 0$; f' is the derivative of the arbitrary function f. This equation in addition to all solutions of the linear equation is solved by any F and G for which $f' = 0$. For the linear solution $I_1 + \mu I_o = 0$ so that Q and Mc^2 have their usual form except for a multiplicative constant.

† When W involves I_1^2 the Euler equation has $\partial_\mu I_1$ which contains second order derivatives. However, the I_1 can be expressed in terms of F and G only by means of the invariant equation (154) so that we again have two first order differential equations.

does not vanish at infinity so that charge and mass integrals do not exist. Only those solutions corresponding to a discrete set of initial values for F will give proper particle-like solutions. We now turn to a discussion of the existence of solutions similar to that for the bose field.

2. The Existence of Particle-Like Solutions

Although F and G are, in general not conjugate in the sense of y and y' of Equation (46), a consideration of trajectories in the F-G plane yields a qualitative description of many properties of the solutions of the simpler Lagrangians.

The simplest Lagrangian foe which the charge density is not positive definite is (putting $\mu = 1$):

$$L = I_o + I_1 + \epsilon I_o I_1 \tag{172}$$

The Euler equation is:

$$\begin{aligned} 0 &= (1 + \epsilon I_o)(G' + 2G/r + \omega F) + F + \epsilon G(FF' - GG') + \epsilon F I_1 \\ 0 &= (1 + \epsilon I_o)(F' - \omega G) + G + \epsilon F(FF' - GG') + \epsilon G I_1 \end{aligned} \tag{173}$$

The invariant equation is given by the simple expression:

$$I_1 + I_o + 2\epsilon I_o I_1 = 0 \tag{174}$$

From (123) and (126) the charge becomes:

$$Q = 4\pi \int (1 + \epsilon I_o) \psi^* \psi \, \vec{dx} \tag{175}$$

or

$$Q = \int_o^\infty r^2 dr \, (1 + \epsilon F^2 - \epsilon G^2)(G^2 + F^2) \tag{175b}$$

A Lagrangian analogous to Equation (49) which gives (173) is:

$$\int L' dr = \int dr \, r^2 \left[F^2 - G^2 - F'G + G'F + \omega G^2 + \omega F^2 \right]$$

$$+ \epsilon (F^2 - G^2)(F'G - G'F + 2FG/r + \omega F^2 + \omega G^2) \tag{176}$$

The pseudo-Hamiltonian resulting from (176) according to the prescription (56) does not yield a useful form since $\overline{H}$ still depends explicitly on r. In order to remedy this we make the contact transformation:

$$L'' = L' - \frac{d}{dr}(r^2 FG) - \epsilon \frac{d}{dr}(F^3 G - FG^3) \tag{177}$$

Variation of L" still gives (173). The canonical Hamiltonian got from L" becomes:

$$H = -r^2 \left[I_o + \omega(F^2 + G^2) + \epsilon \omega I_o (F^2 + G^2) \right] \tag{178}$$

and

$$\overline{H} = - \left[1 + \epsilon I_o \right] \omega \left[F^2 + G^2 - I_o \right] \tag{179}$$

$$\frac{d\overline{H}}{dr} = - I_o{}' - 2\,\omega\,(FF' + GG') - 4\,\epsilon\,\omega\,(F^3F' - G^3G') \tag{180}$$

Using the differential equation (173) to eliminate the derivatives in (180) gives

$$\frac{d\overline{H}}{dr} \quad \frac{2G^2}{r(1 + 2\,\epsilon\, I_o)} \left[-1 + \omega + 2\epsilon^2 \omega I_o^2 - \epsilon I_o + 3\,\epsilon\,\omega\, I_o \right] \tag{181}$$

$d\overline{H}/dr$ changes sign when $I = -\frac{1}{\epsilon}, -\frac{1}{2\epsilon}, \frac{1-\omega}{2\omega\epsilon}$ *. The contour lines for $\overline{H}$ = constant are plotted in Figures 14 and 15 for the four combinations of ϵ and ω. It is convenient to discuss the solution in the accompanying $F^2 - G^2$ space because of the complicated behavior of (180).

Trajectories begin on the G = 0 and must terminate at the origin. The trajectory $I_o = 1$ is also a charge node. According to (180) either all trajectories are pulled toward it on both sides or can't reach it from either side. In either event trajectories can't cross, so that there can be no change in sign for the charge density along a solution which reaches the origin. For Figure 15a the arguments of Figure 1 can be applied to a trajectory for which $F(0) \leq \omega^{-1/2}$. For the cases represented by Figures 15a and 15b a trajectory originating on G = 0 could not reach the origin.

The argument preventing a node in the charge density can easily be extended to the more general Lagrangian:

$$L = I_o + I_1 + \quad I_o I_1 + \alpha(F^4 + G^4 + \quad G^2F^2) \tag{182}$$

* If the 2FG/r term were not in the Lagrangian, $\overline{H}$ = constant would describe the trajectory. $d\overline{H}/dr$ gives the effect of the damping term 2G/r in the differential equation.

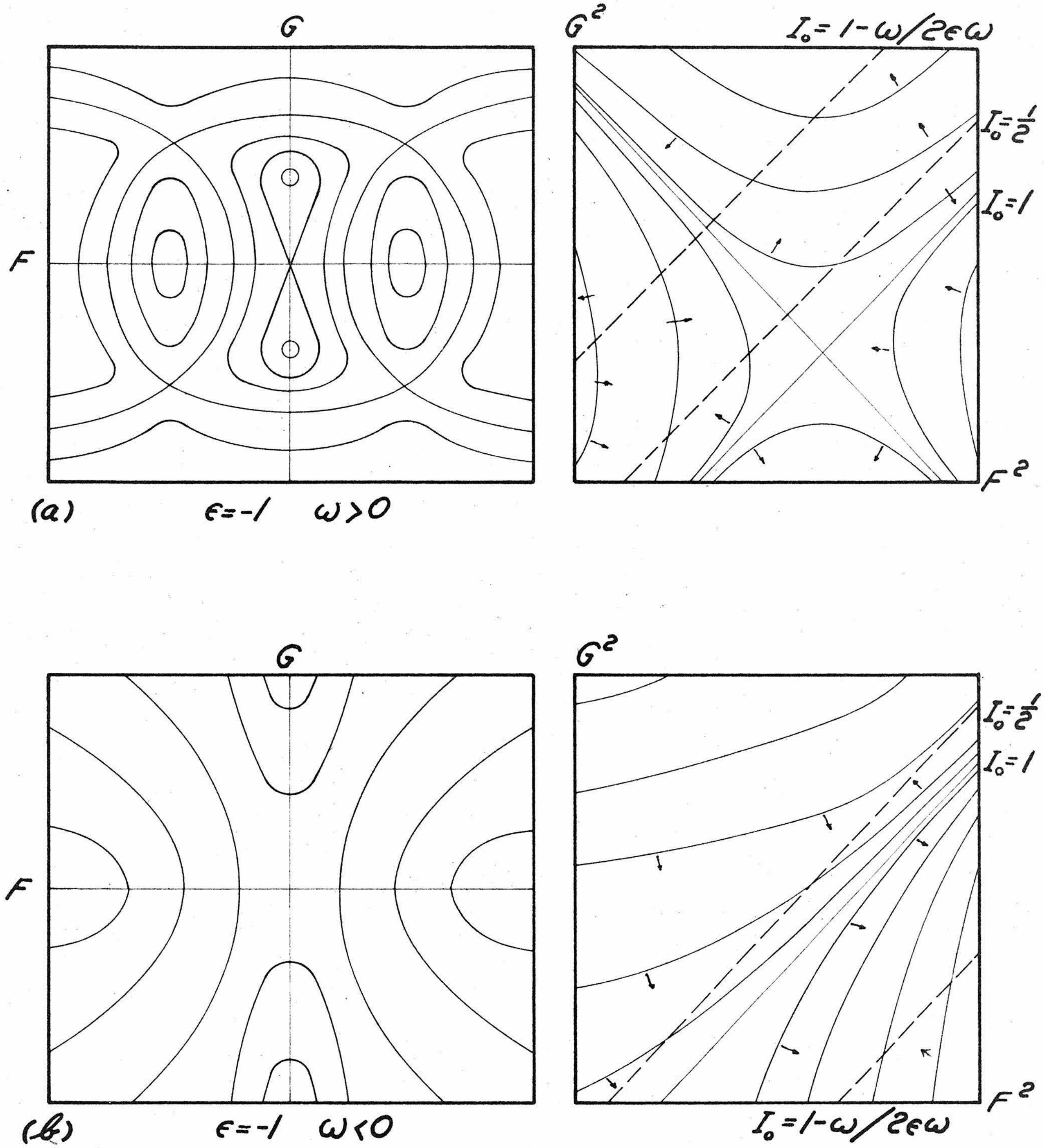


FIG. 14. PHASE PLOTS FOR $\epsilon I_0 I_1$

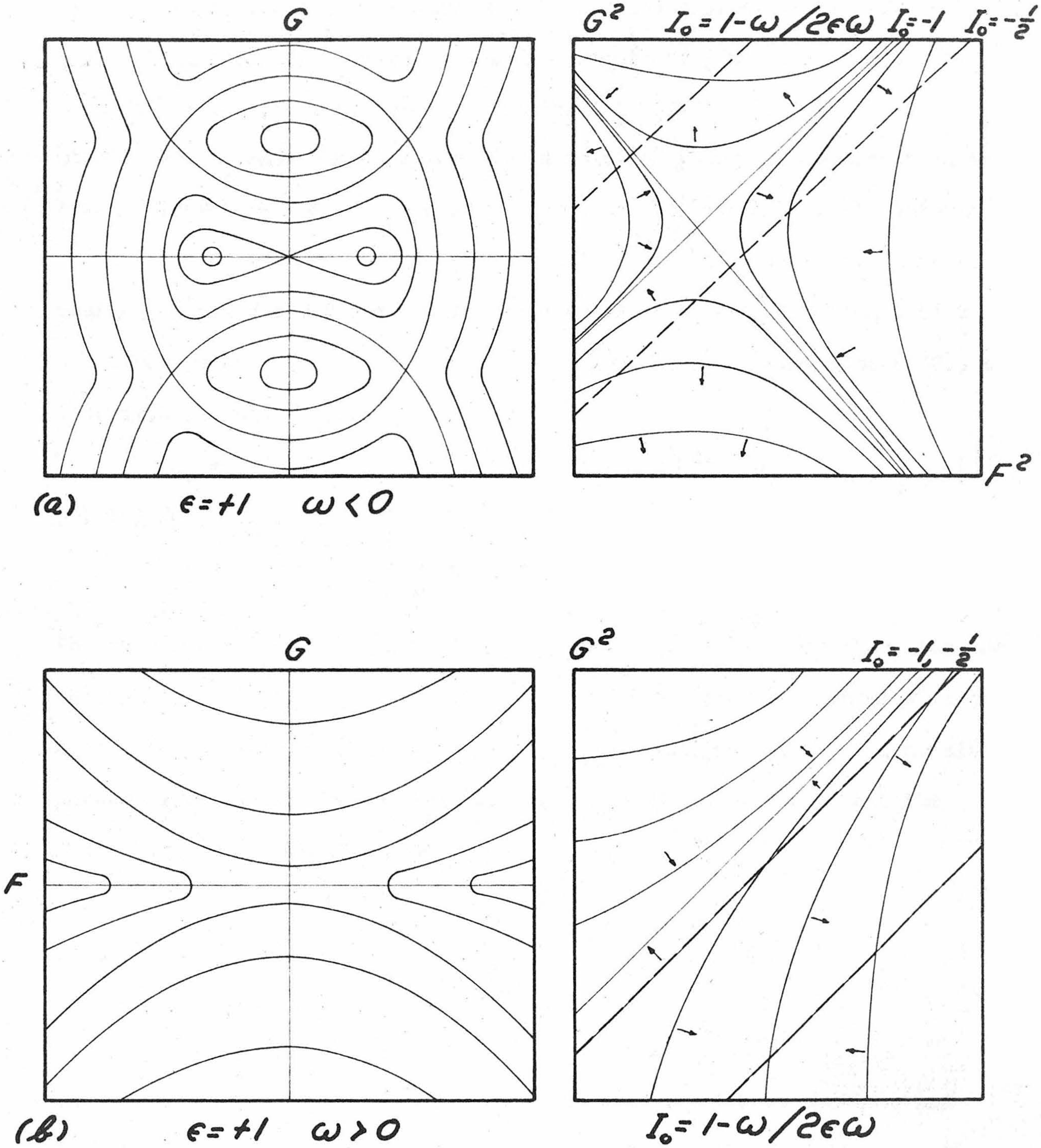


FIG. 15 PHASE PLOTS FOR $\epsilon I_0 I_1$

We look first at the invariant equation (174) for the Lagrangian (172) in $I_o - I_1$ space, plotted in Figure 16. In (a) the region to the left of $I_o = -1/\epsilon$ is one of negative charge; to the right, of positive charge. In (b) negative is to the right, positive to the left of $I_o = 1/|\epsilon|$. As $r \rightarrow \infty$, both I_o and I_1 must approach zero for a solution going to the origin in F-G space. Therefore the solution must correspond to branch 1 of the invariant equation in $I_o - I_1$ space. However, only branch 2 can have both signs of charge. Since F and G are continuous a solution cannot jump from 1 to 2 and a charge node is forbidden. With the more general Lagrangian (182), the invariant equation may be written:

$$I_1 + I_o + 2\epsilon\, I_o I_1 + 2\,\alpha\,(F^4 + G^4 + \lambda\, G^2F^2) = 0 \tag{183}$$

Solving for I_1:

$$I_1 = -\frac{I_o + 2\,\alpha\,(F^4 + G^4 + \lambda\, G^2F^2)}{I + 2\epsilon I_o} \tag{184}$$

The charge equation (175) is unaltered. I_1 still has two branches; the one through the origin runs to infinity because of the singular denominator at $I_o = -1/2\epsilon$ before it can have a charge node at $I_o = -1/\epsilon$. A possible escape is to choose the special Lagrangian $\alpha = \epsilon$, $\lambda = -2$ so that the numerator and denominator node together.

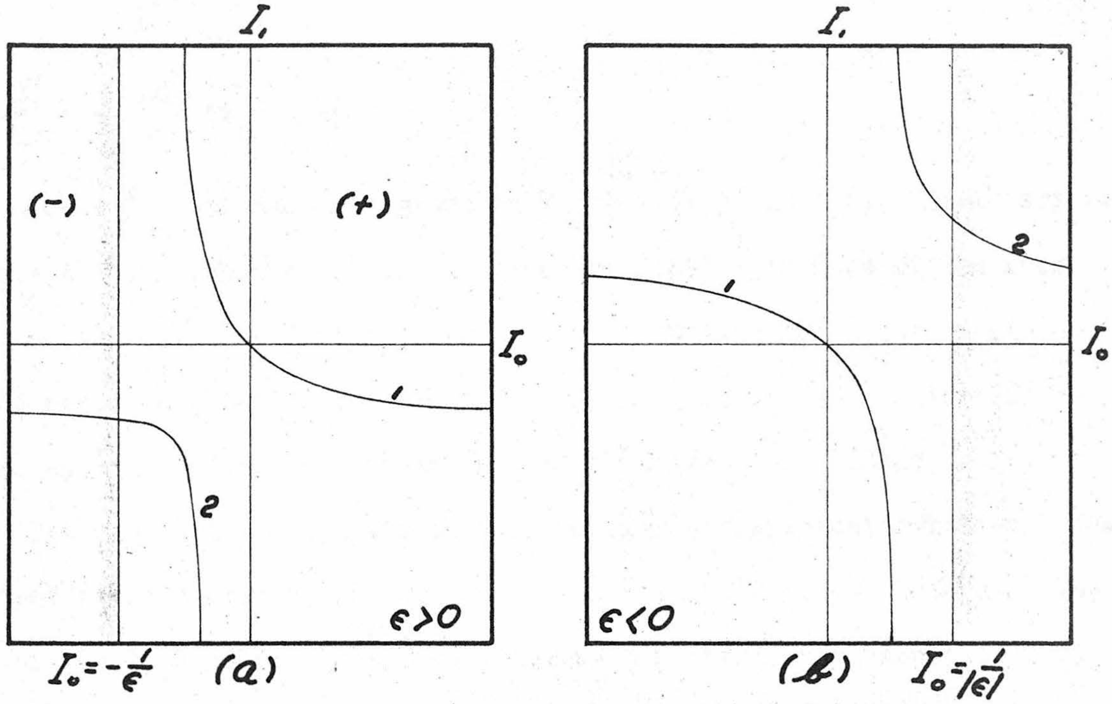
$$L = I_o + I_1 + \epsilon I_o I_1 + \epsilon I_o^{\,2} \tag{185}$$

Then (184) becomes

$$I_1 = -\frac{I_o(1 + 2\epsilon I_o)}{1 + 2\epsilon I_o} = -I_o \tag{186}$$

The curve $I_o = -I_1$ passes continuously through regions of positive and negative charge as well as the origin. The differential equations corresponding to (184) are:

$$\begin{aligned} &(G' + 2G/r + F)(1 + 2\epsilon I_o) + \omega F + 2\epsilon G^3/r + 2\epsilon\omega F^3 = 0 \\ &(F')(1 + 2\epsilon I_o) + G\,(1 - \omega) + 2\epsilon\omega\, G^3 + 2\epsilon\, G^2F/r + 2\epsilon G I_o = 0 \end{aligned} \tag{187}$$



INVARIANT EQUATION FOR $\epsilon I_0 I_1$

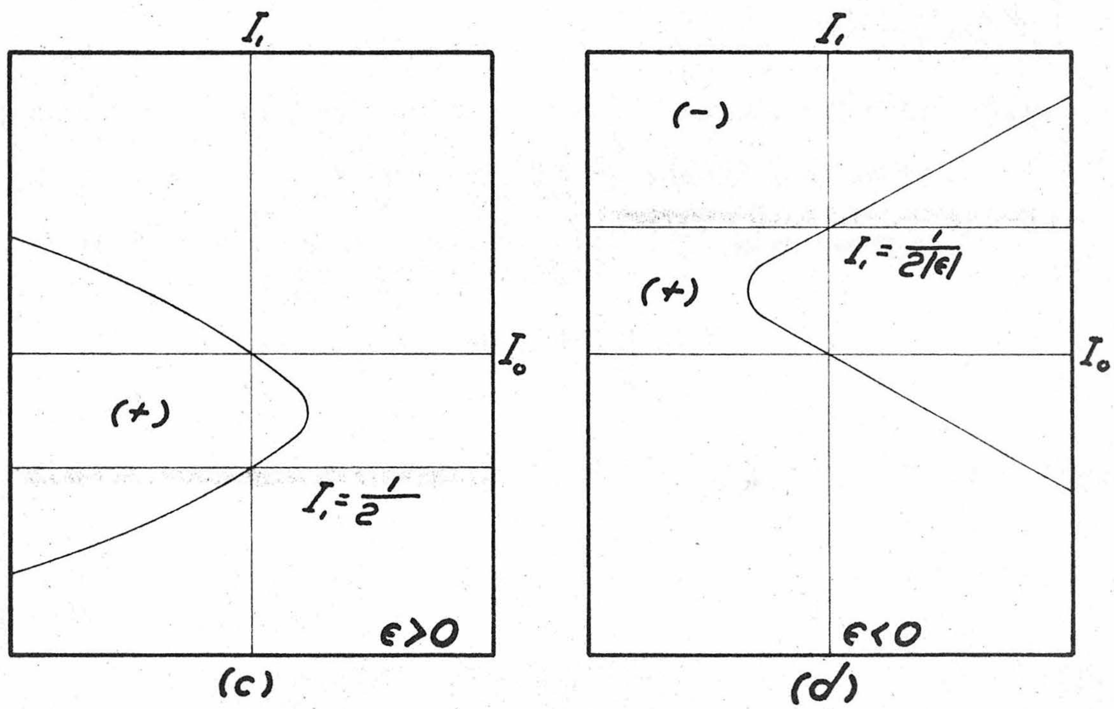


FIG. 16. INVARIANT EQUATION FOR ϵI_1^2

Proceeding as for Equation (173) we obtain:

$$\bar{H} = -\left[1 + \epsilon I_o\right]\left[(\omega - 1)\, G^2 + (\omega + 1)F^2\right] \tag{188}$$

and

$$\frac{d\bar{H}}{dr} = -\frac{4G^2}{r}(1 - \omega)(1 + \epsilon I_o) \tag{189}$$

The contours $\bar{H}$ = constant are given in Figures (17) and (18). Small arrows indicate the direction of $d\bar{H}/dr$ as given by (189). In none of the four cases can a trajectory which starts on the $G = 0$ axis reach the origin so that no particle solution can exist for the Lagrangian (185). Therefore no zero charge particle solutions exist for Lagrangians linear in I_1.

The coupling ϵI_1^2 leads to very complex differential equations. The invariant equation remains simple and is plotted in Figures (16c, d). Instead of the hyperbola of Equation (174) we now obtain the parabola:

$$I_o + I_1 + 2\epsilon I_1^2 = 0 \tag{190}$$

It might be possible in this more complicated case to find solutions which have a node in the charge density. A more detailed consideration of the differential equation has excluded case (a). Since $I_o > 0$ ar $r = 0$, any solution of the I_1^2 case with $\epsilon < 0$ will have both signs for the charge density. If the net charge were zero the spin would also vanish.

We turn now to a consideration of the Lagrangian based on the coupling (166).

$$L = I_o + I_1 + \alpha\,(G^4 + F^4 + \lambda G^2F^2) \tag{191}$$

The Euler equations are:

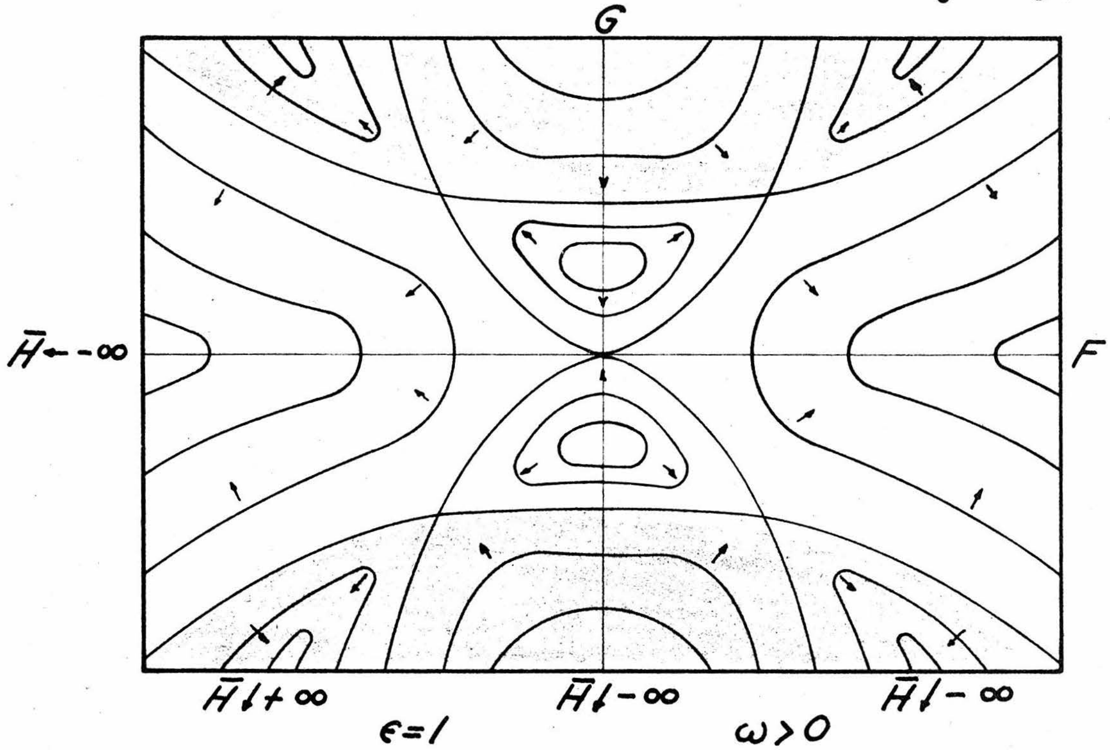
$$G' + 2G/r + (1 + \omega)F + \alpha\,(2F^3 + \lambda FG^2) = 0 \tag{192}$$

$$F' + (1 - \omega)G - \alpha\,(2G^3 + \lambda GF^2) = 0$$

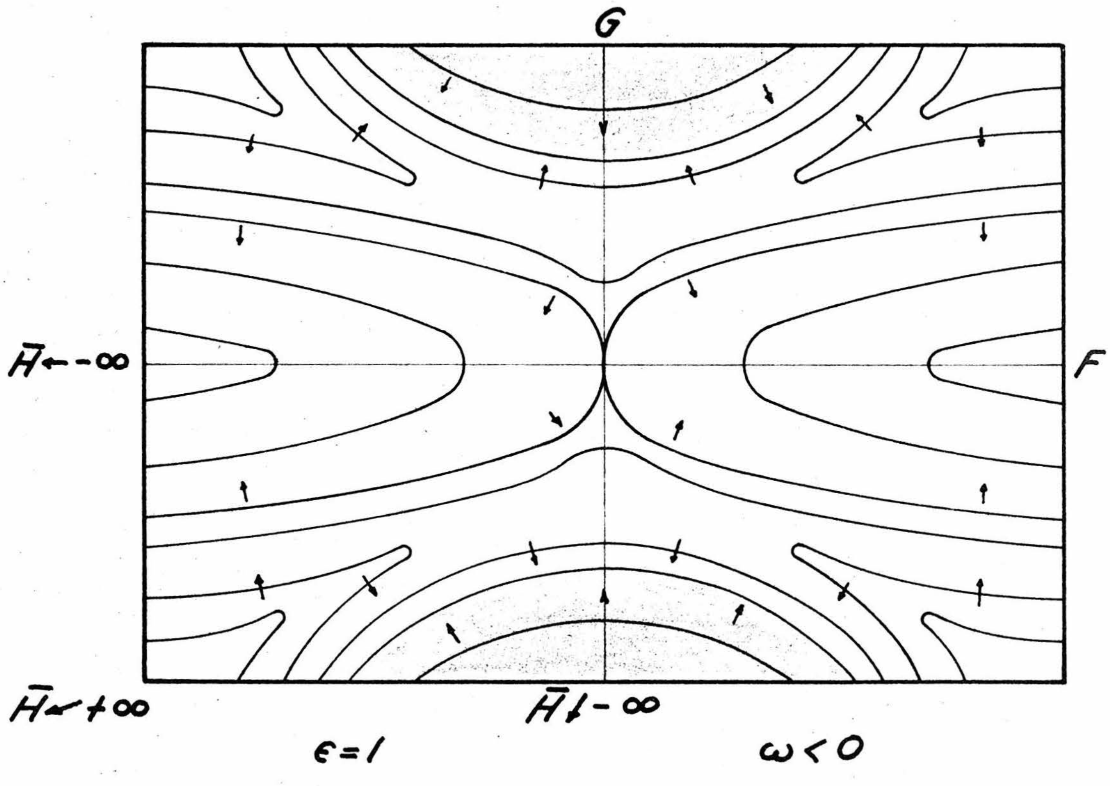
The invariant equation is:

$$I_1 + I_o + 2\alpha(F^4 + G^4 + \lambda G^2F^2) = 0 \tag{193}$$

FIG. 17·A ··· PHASE PLOT FOR $\epsilon I_0^2 + \epsilon I_0 I_1$



→ Neg. Charge Region



→ Negative Charge Region

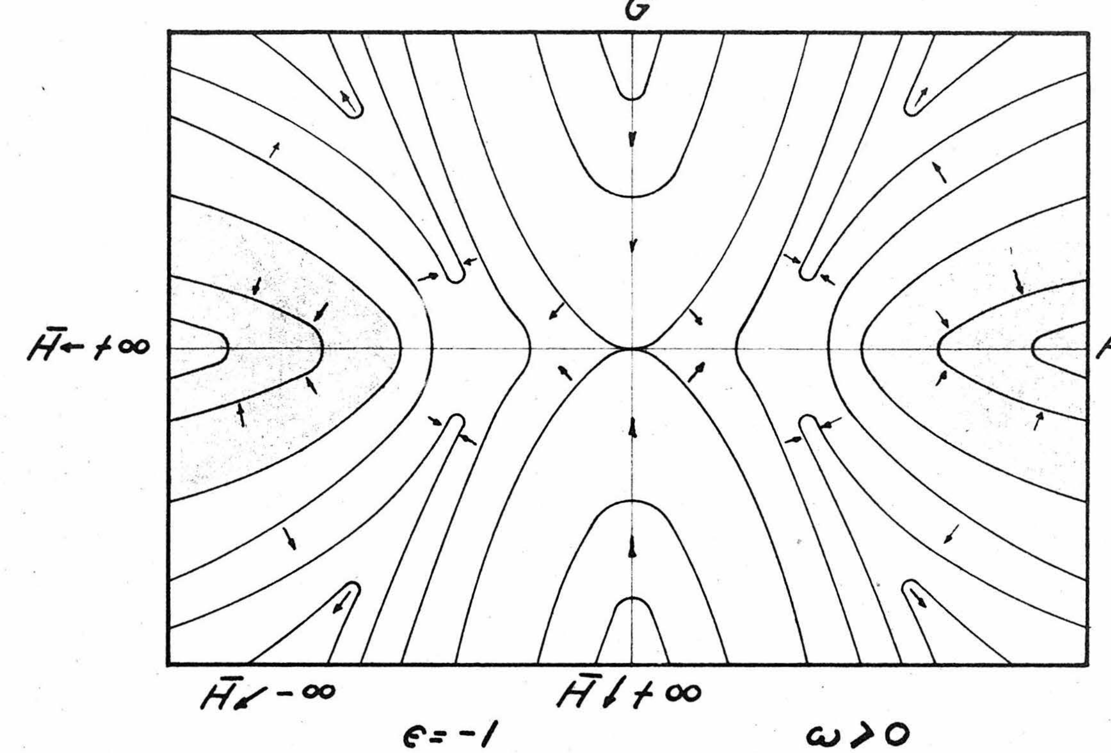
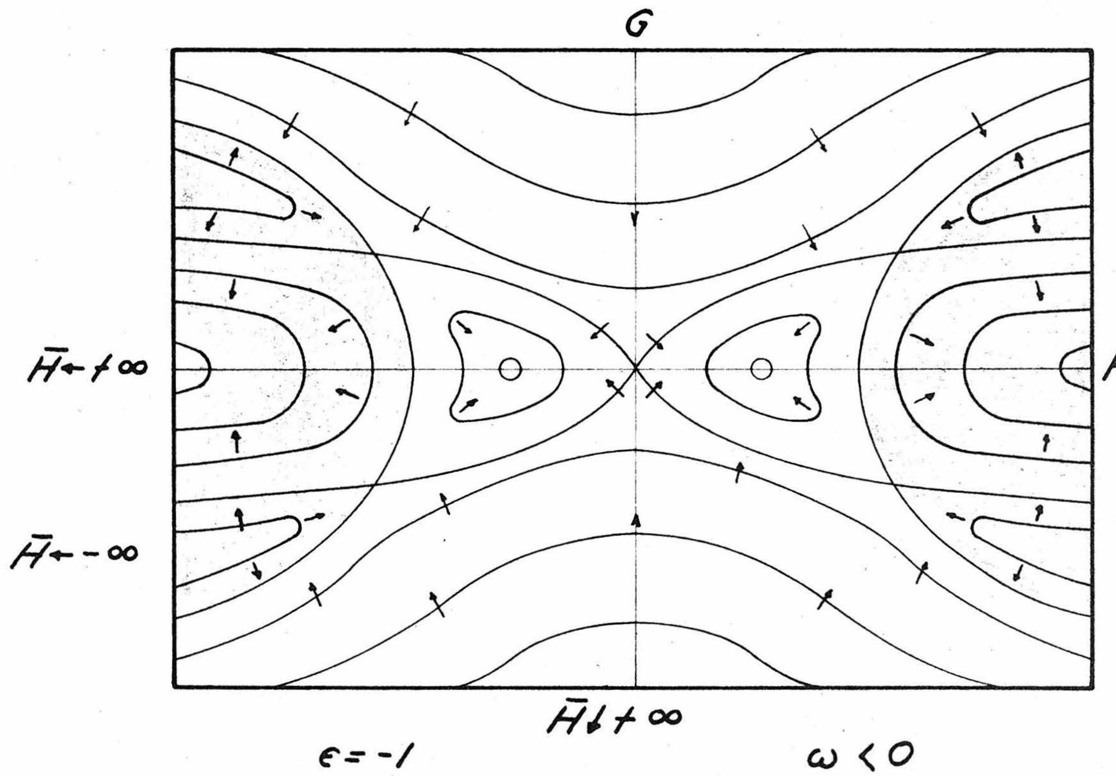


FIG. 18.··· PHASE PLOTS FOR $\epsilon I_0^2 + \epsilon I_0 I_1$



Proceeding as for Equation (172)

$$\bar{H} = -\omega(F^2 + G^2) - (F^2 - G^2) - \alpha(F^4 + G^4 + \lambda G^2F^2) \tag{194}$$

and

$$\frac{d\bar{H}}{dr} = -\frac{4G^2}{r}\left\{1 - \omega - \alpha\left[2G^2 + \lambda F^2\right]\right\} \tag{195}$$

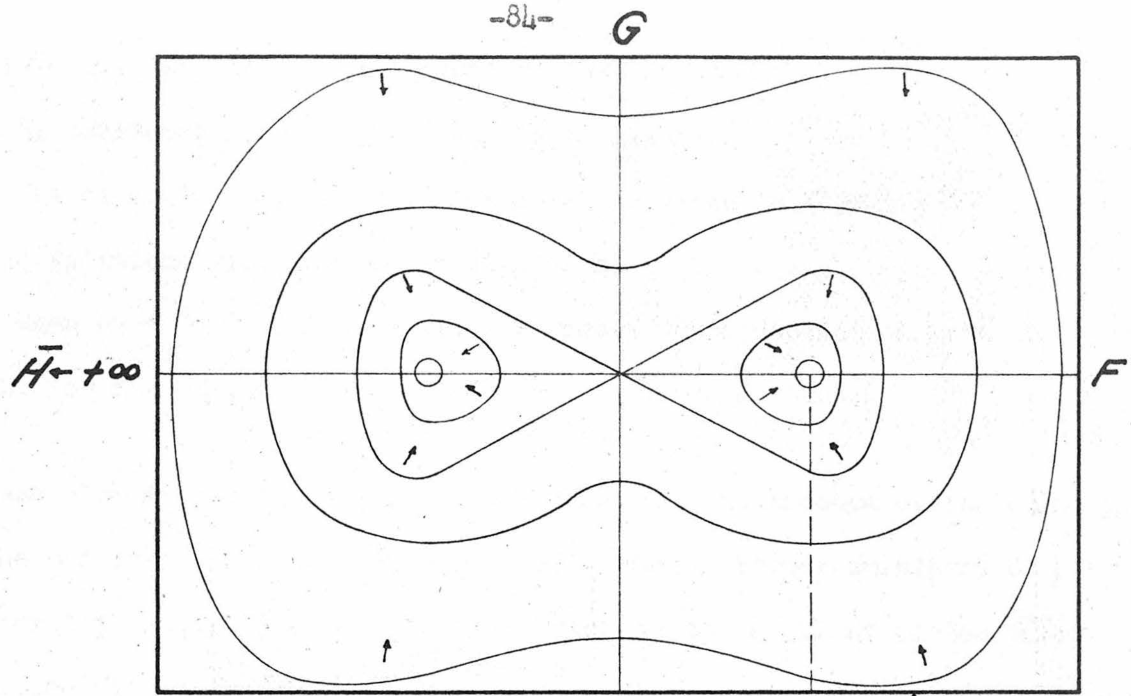
In $F^2 - G^2$ space, the family $\bar{H}$ = constant consists of conic sections confined to the first quadrant. The curves in F-G space are topologically identical but must be completed by reflection through the F and G axes. If $\lambda^2 > 4$ the conics are hyperbolae; if $\lambda^2 < 4$ they are ellipses; $\lambda^2 = 4$ gives parabolas. Another possible coupling is nG^2F^2 which gives hyperbolae in $F^2 - G^2$ space.

When $\alpha < 0$ and $\lambda \geq 0$ $d\bar{H}/dr \leq 0$. The phase space plot is given in Figure (19a); it is identical to the contour diagram of Equation (44) so that the same existence proof will hold. Therefore solutions should exist for $-1 < \omega < +1$* unless no solution crosses the $F = 0$ axis. For $\alpha > 0$ and $\lambda \geq 0$ $d\bar{H}/dr \leq 0$ outside the ellipse $2G^2 + \lambda F^2 = 1 - \omega$ and $d\bar{H}/dr \geq 0$ inside. The contours are illustrated in Figure (19b). If the $d\bar{H}/dr$ term were neglected the $\bar{H}$ = constant trajectories would be traversed in a clockwise direction. For a proper solution the origin can be reached only from the first or third quadrant; this follows from the linear part of (192). Although one cannot disprove the existence of $\alpha > 0$ solutions, the arguments which lead one to expect a solution are not valid for this case. Since all solutions remain bounded either there must exist a limit cycle or all solutions end at the origin and so are particle solutions.**

---

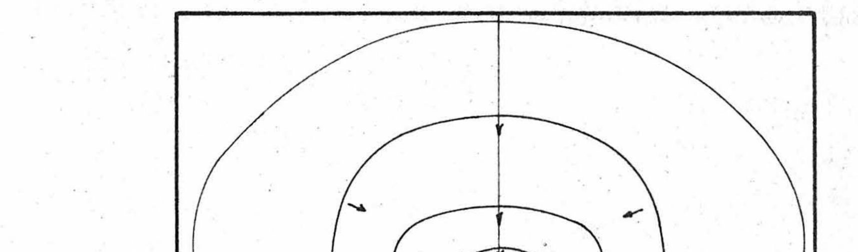
* The mass for a unit charged particle is $\omega + |\alpha| \int^{\infty} 4\pi r^2 dr\, [G^4 + F^4 + \lambda G^2F^2]$ Since ω can be negative we have the possibility of negative mass. In this case by appropriately choosing the coupling constant, the mass can be made positive but arbitrarily close to zero. The ratio of masses for the 0-node and 1-node solution could then be quite large.

** Since a trajectory gain "energy" outside of the dotted ellipse but loses energy inside, periodic solutions seem possible. We have symmetry about $G = 0$ and $F = 0$; either the cycle encloses both maxima or there exists a cycle enclosing each maximum.



(a) $\alpha = -1$ $\lambda \geq 0$ $F = \left(\frac{1+\omega}{2}\right)^{\frac{1}{2}}$

FIG. 19 PHASE PLOT, EQUATION 192.



(b)
$\alpha = +1$
$\lambda \geq 0$

If there exist two limit cycles symmetric with respect to $G = 0$, the canonical existence proof might again be applicable.

The situation for the coupling nG^2F^2 is given in Figure (20. No particle solutions exist for either sign of n.

When $-2 < \lambda < 0$ the topology in phase space depends on both α and ω. For $\alpha = -1$ and

$$2 + \lambda \geqslant (2 - \lambda)\omega \tag{196}$$

the phase plot is represented in Figure (21). In the absence of the $d\overline{H}/dr$ term the contours are traversed counter clockwise. Proper solutions do exist for this case. Even solutions originating to the right of the $d\overline{H}/dr = 0$ curve are ultimately captured at either of the two stable points; the argument concerning Figure (19) is again applicable. When $2 + \lambda < (2 - \lambda)\omega$ we have a more ambiguous situation. It seems likely that particle-solutions can be found but no calculations were made with equations giving this topology.

If $\alpha = +1$, $-2 < \lambda < 0$ and

$$-(\lambda + 2) \geq \omega(2 - \lambda)$$

we have Figure (23a); for the reverse inequality Figure (23b) gives the phase plane description. The contours are traversed in a clockwise sense. We do not know if it is possible for a particle-solution to exist in these cases; this would again seem to depend on a numerical investigation. When $\lambda < -2$ the phase plots are similar to those discussed.

For $\omega \longrightarrow -1$ the condition (196) is always satisfied; if in addition $\alpha < 0$ the phase space topology is that of Figure (1). The solution (45) of Equation (46) will then (in the limit of almost -1) satisfy Equation (192) with $F = y$. If $G \ll F$, $\omega \approx -1$, Equation (192) becomes:

$$\begin{aligned} &F' + (1 - \omega)G = 0 \\ &G' + 2G/r + (1 + \omega)F + 2\alpha F^3 = 0 \end{aligned} \tag{198}$$

Combining these equations

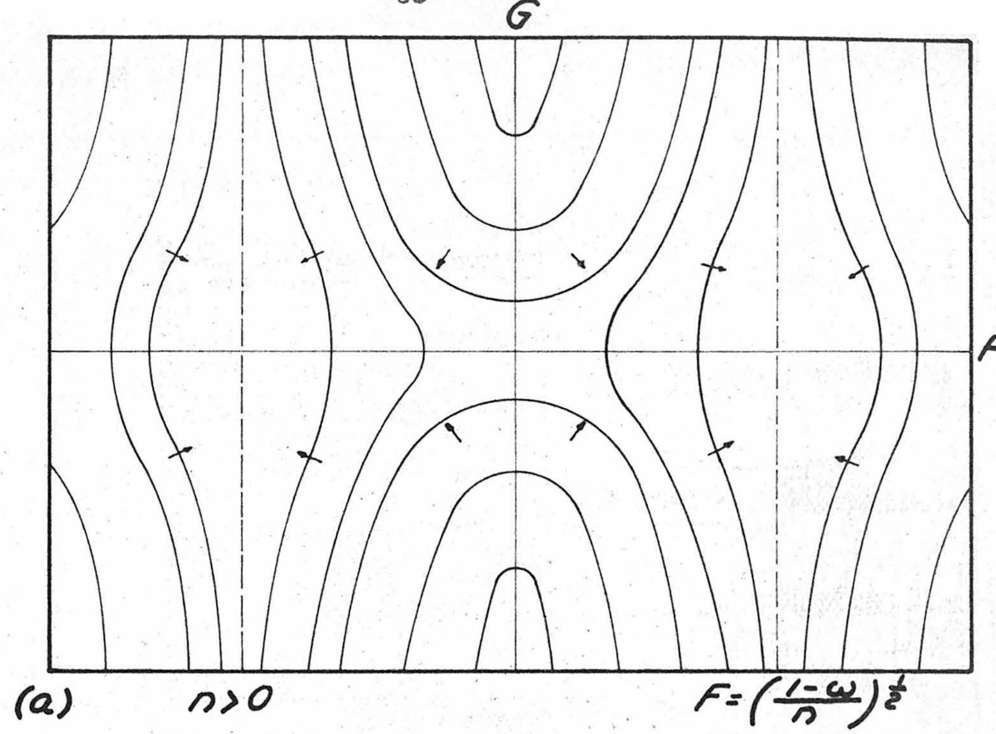


FIG. 20 ... PHASE PLOTS FOR nG^2F^2

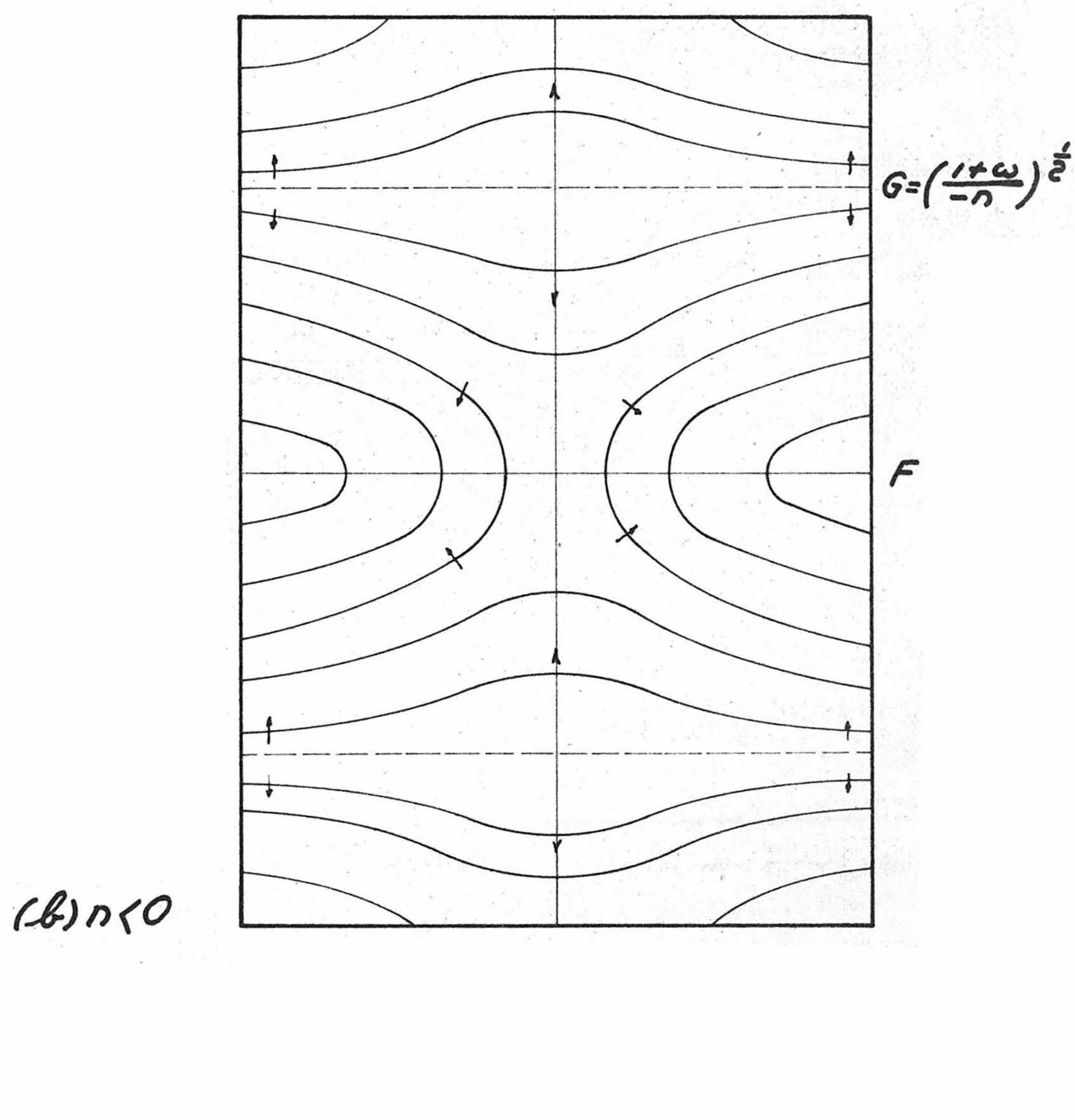
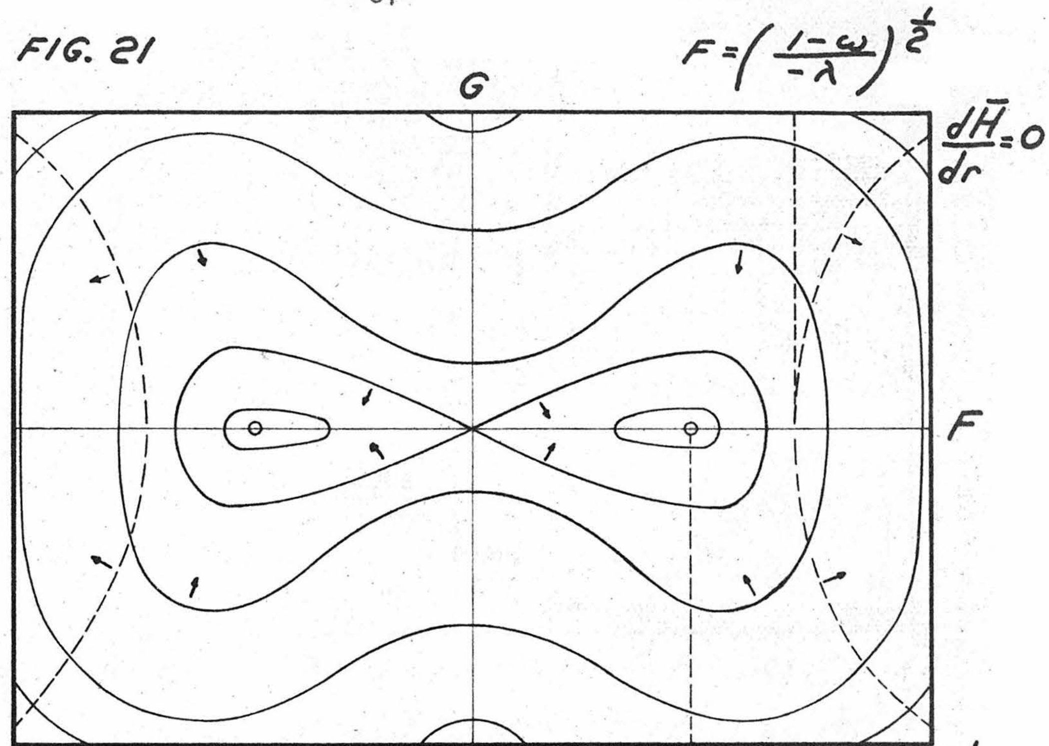


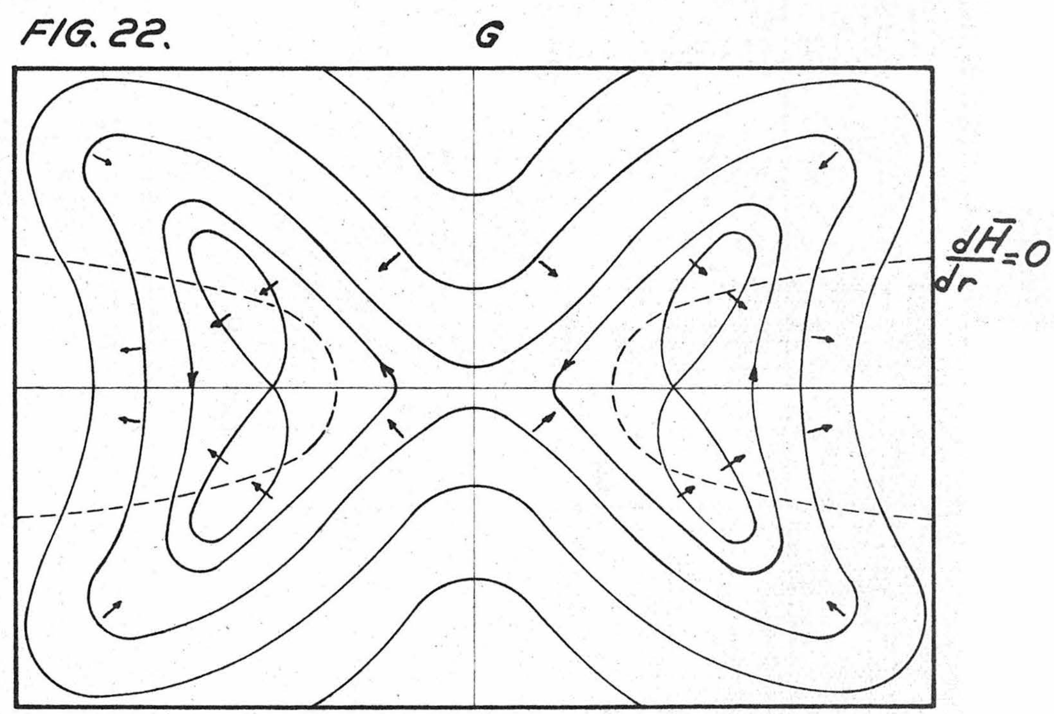
FIG. 21



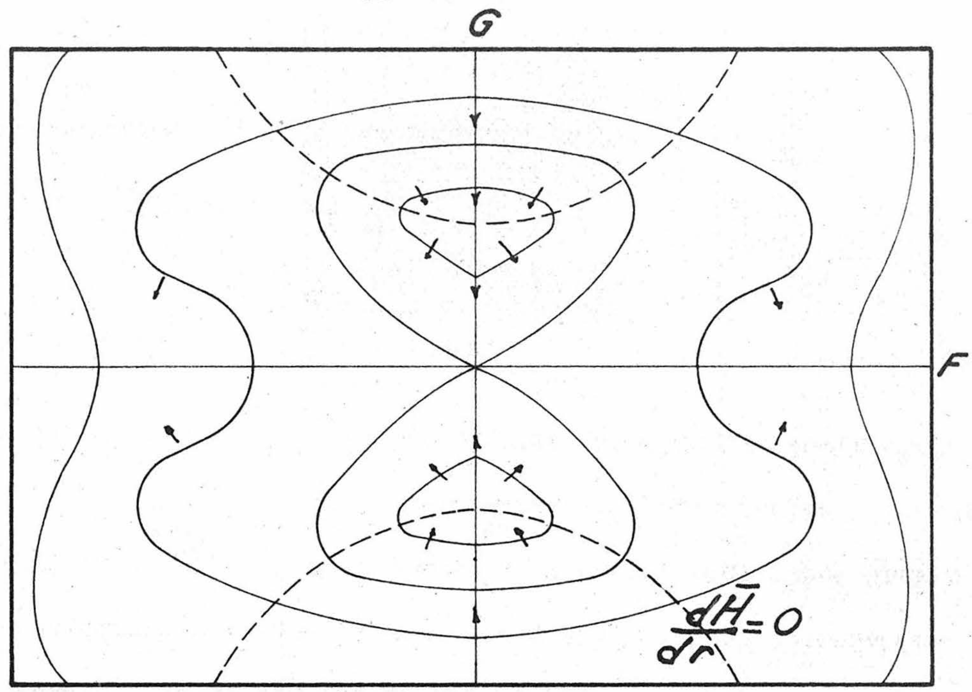
$\alpha = -1$ $\lambda \leq 0$ $2+\lambda \geq (2-\lambda)\omega$

PHASE PLOTS, EQUATION 192.

FIG. 22.

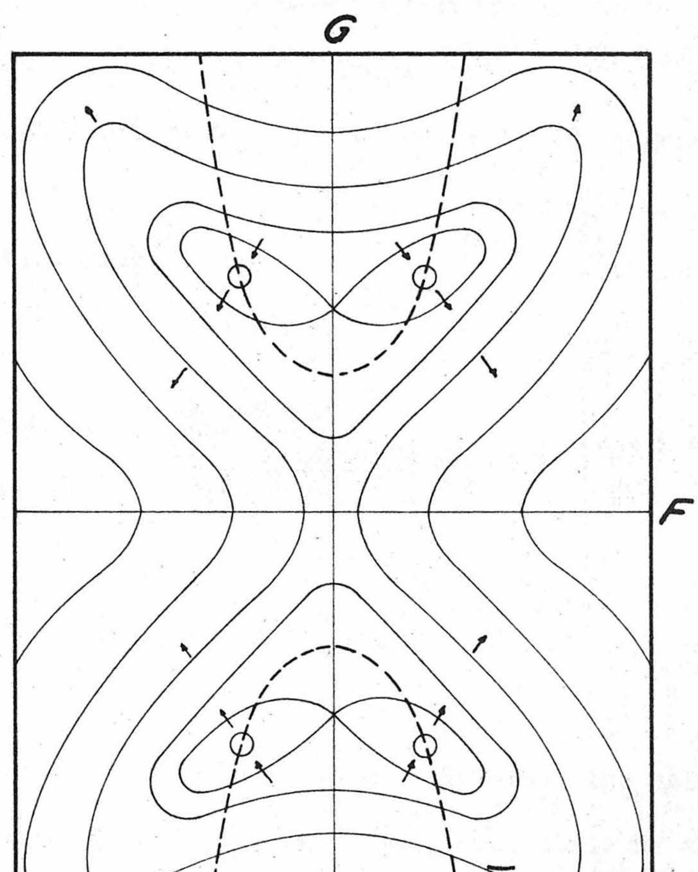


$\alpha = -1$ $\lambda \leq 0$ $2+\lambda \leq (2-\lambda)\omega$



(a) $\alpha = +1$ $\lambda < 0$ $-(\lambda+2) \geq (2-\lambda)\omega$

FIG. 23 ... PHASE PLOTS, EQUATION 192.



(b)
$\alpha = +1,\ \lambda < 0$
$-(\lambda+2) \leq (2-\lambda)\omega$

$$F'' + \frac{2F}{r} - (1 - \omega^2)\, F - 4\, \alpha\, F^3 = 0 \tag{199}$$

Since $\alpha < 0$ the solutions of this are those of (46)

$$F = \sqrt{\frac{1 - \omega^2}{4\,|\alpha|}}\; y_o\left(\sqrt{1 - \omega^2}\; r\right) \tag{200}$$

From (198)

$$G \approx -\frac{F'}{2} \tag{201}$$

Therefore G is of the order $\sqrt{1 - \omega^2}\, F$ so that for ω sufficiently close to $- 1$, $G << F$, thus making (198) a valid approximation. We conclude, therefore, that since solutions of (46) exist and have been calculated, there always exist particle-solutions of (192) for $\alpha < 0$ and any λ for an appropriate range of ω. However, as we shall see, this is not a very interesting region.

Extensive numerical calculations were performed for the special case $\lambda = - 2$, i.e. for I_o^2 coupling and $\alpha < 0$. For $\omega > 0$ no particle-solutions exist as is obvious from a consideration of the phase plot: Figure (24a). For $- 1 < \omega < 0$, the phase plot of Figure (21) is applicable so that solutions do exist for this range of ω.

3. Numerical Calculations

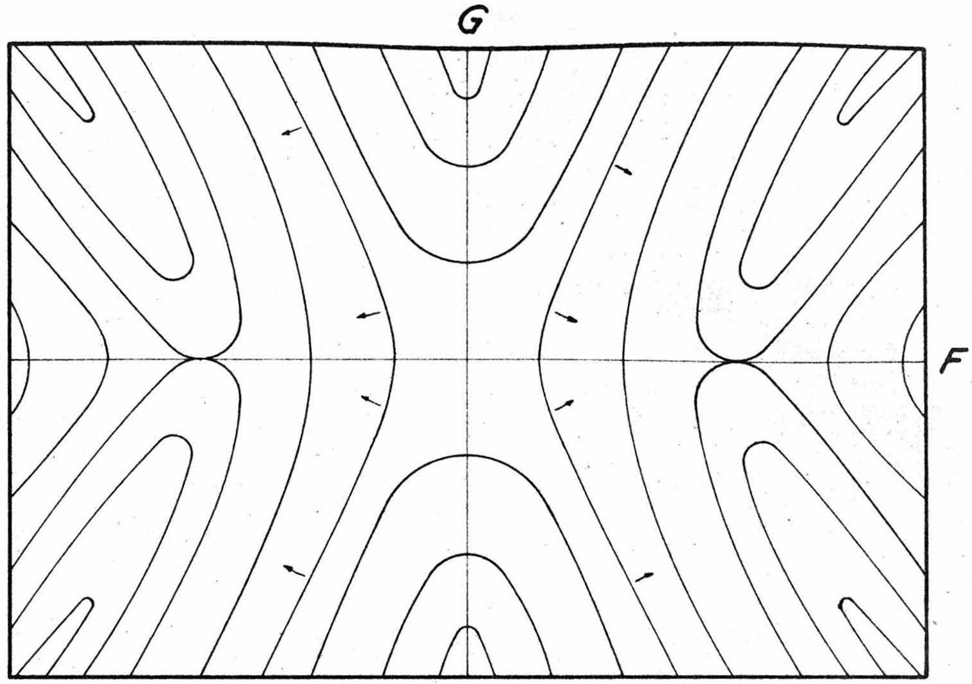
When the Lagrangian is

$$L = I_1 + \mu I_o - \frac{\epsilon}{2}\; I_o^2 \tag{202}$$

the Euler equations are

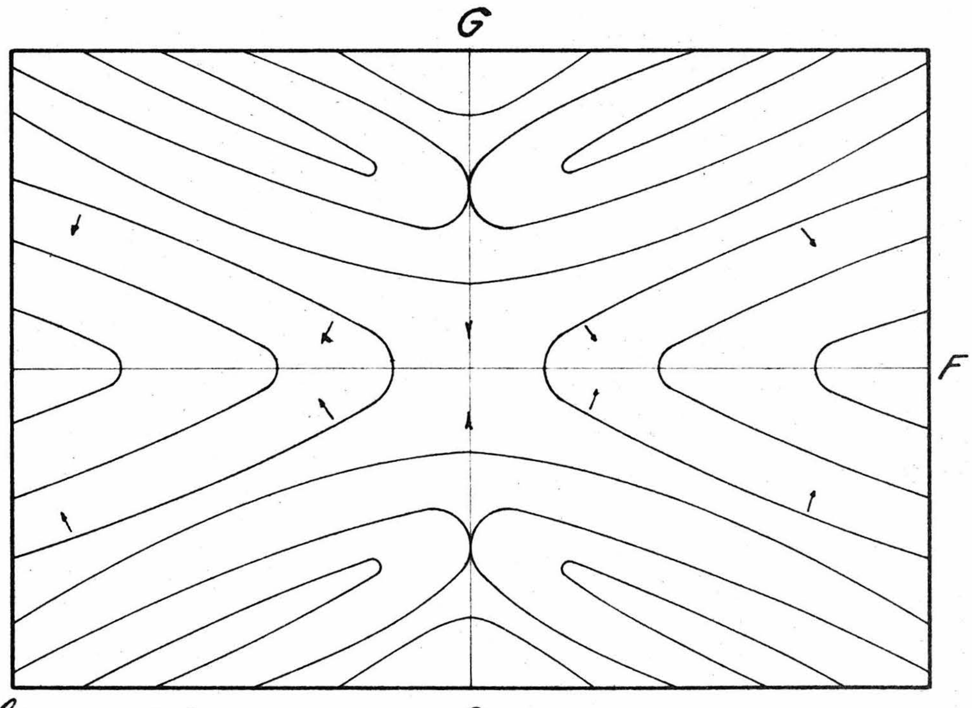
$$\begin{aligned} G' + \frac{2G}{r} + (1 + \omega)\, F - \epsilon F\, (F^2 - G^2) &= 0 \\ F' + (1 - \omega)\, G - \epsilon G\, (F^2 - G^2) &= 0 \end{aligned} \tag{203}$$

As in Equation (46) the α is just a scale factor. However, the solutions for various values of ω cannot be transformed among each other by algebraic manipulation. An investigation of the solutions of (203) involves the numerical integration for different initial conditions F(0) for different ω



(a) $\alpha < 0$ $\omega > 0$

FIG. 24 ... PHASE PLOT FOR I_o^2 COUPLING.



(b) $\alpha > 0$ $\omega < 0$

between 0 and - 1.

There are three special solutions of (203):

$$\begin{aligned} &(a) \quad G \equiv 0,\ F - 0 \\ &(b) \quad G \equiv 0,\ F = \left(\frac{1+\omega}{\epsilon}\right)^{1/2} \\ &(c) \quad G \equiv 0,\ F = -\left(\frac{1+\omega}{\epsilon}\right)^{1/2} \end{aligned} \tag{204}$$

These are the origin and the two minima of Figure (21). For an initial condition sufficiently near (b) or (c), the solution will be captured at the corresponding special solution. For a fixed ω, as the initial value F(0) is varied, the solutions were captured alternately at (b) or (c). In this manner it was possible to obtain a very close approximation to the discrete set of solutions which get captured at (a).

The equations (203) were integrated by the UCLA Differential Analyzer for $\omega = -.1, -.3, -.5, -.7, -.9$. The solutions having one, two, or three nodes in F were obtained. A typical solution in r-space is given in Figure (25).

The solutions of Equation (203) form a two parameter family depending on ϵ and ω. According to (31), in order that the solutions represent particles of unit charge, we normalize so that

$$Q = 4\pi \int_0^\infty r^2\, dr \ \left[F^2 + G^2\right] = 1 \tag{205}$$

Figure (26) gives ϵQ as a function of ω. For a fixed value of the coupling and Q = 1, ω can be read from this graph. The equation of this curve in the region very close to $\omega = -1$ was inferred from the discussion preceding (198). The dotted portion indicates some uncertainty in connecting the numerical work (up to $\omega = -0.9$) with the asymptotic part.

The mass associated with each solution is:

$$\text{Mass}\left(\text{IN } \frac{u\hbar}{c}\right) = \omega Q + \frac{|\epsilon|}{2} \int_0^\infty 4\pi r^2\, dr\, (F^2 - G^2)^2 \tag{206}$$

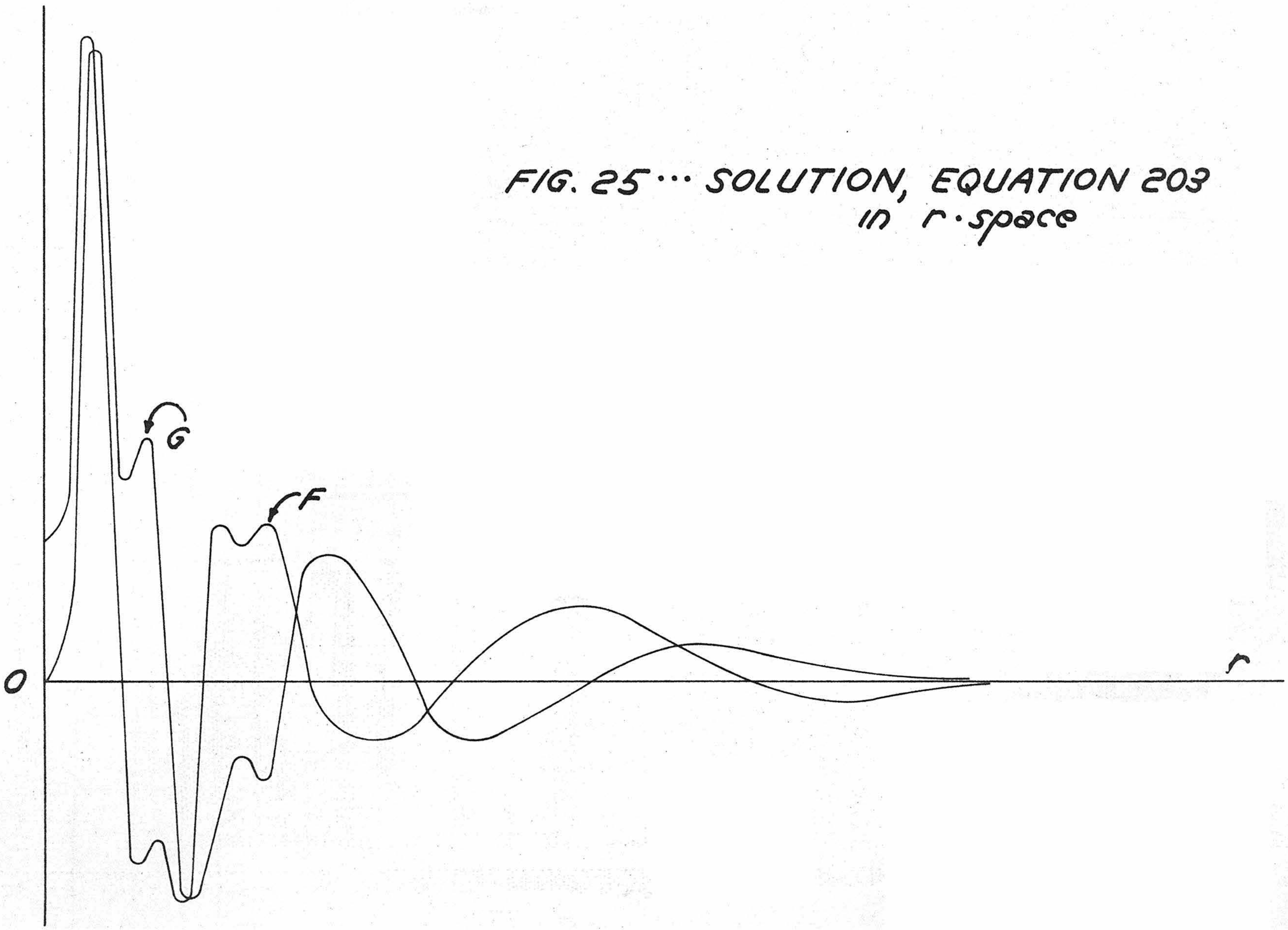


FIG. 25 ... SOLUTION, EQUATION 203
in r·space

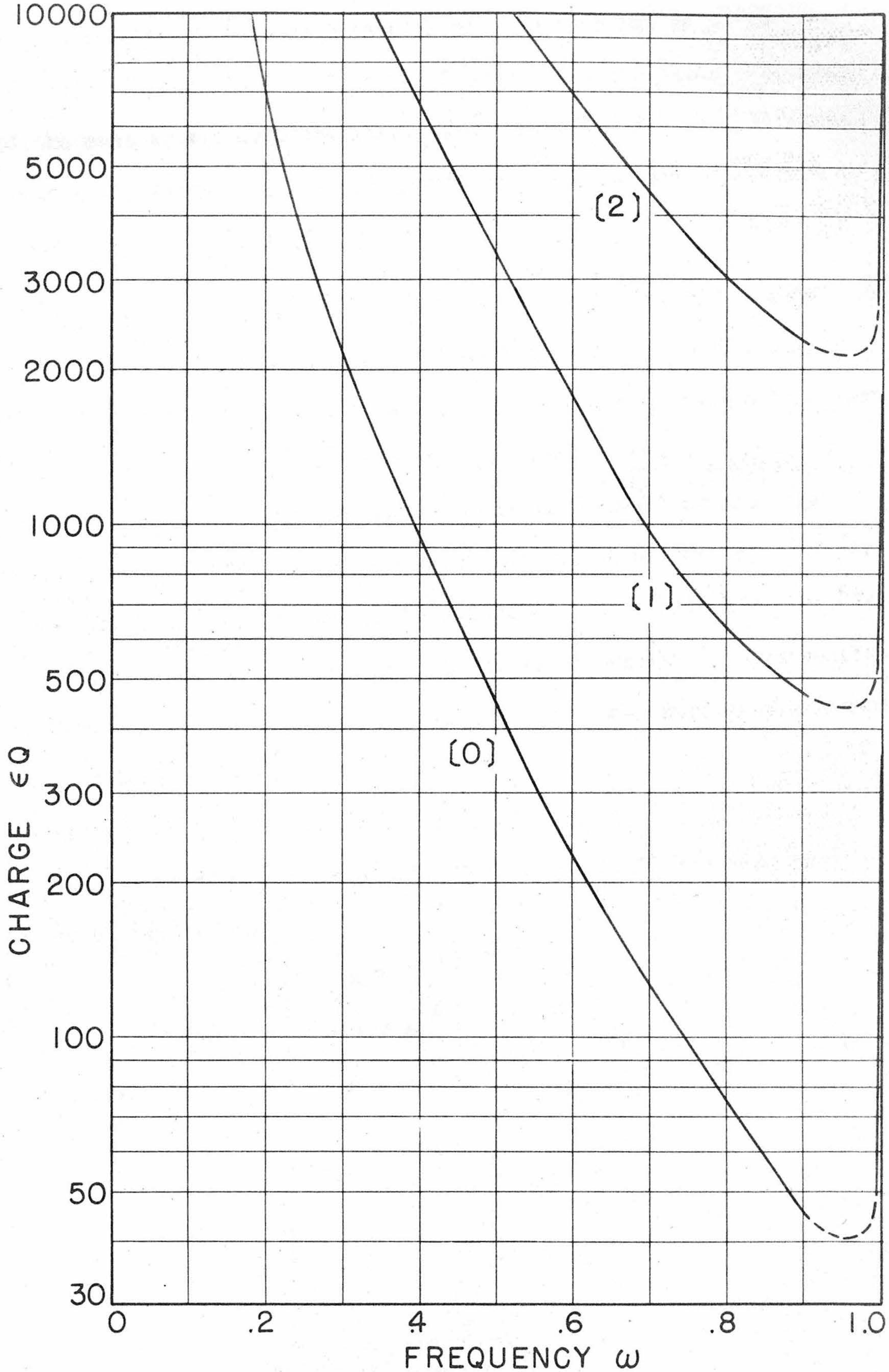


FIG. 26 CHARGE, FREQUENCY FOR I_0^2 COUPLING

The second term multiplied by ϵ is plotted as a function of ω in Figure (27). A comparison of Figures (27) and (26) indicates that over 85% of the mass arises from the first term. Finally, Figure (28) shows the spectrum of masses each carrying unit charge which are associated with each value of the coupling constant.

4. Discussion

With the crude quantization of Equation (31) there exist only a finite number of masses for each value of the coupling constant ϵ. Each particle carries a spin $\hbar/2$ and a magnetic moment which in general is anomolous. The number of masses can be made quite small by properly adjusting ϵ but the mass ratios are relatively fixed and of order two or three so that they seem to bear no relationship to the spin 1/2 particles so far observed (electron, μ - meson, proton). The great variety available in the forms of the coupling seem capable of producing much greater mass ratios than found here.*

Particle-like solutions of (203) yield positive masses since $\epsilon < 0$ and $\omega < 0$. Instead of the Ansatz (157) Equation (153) may also be separated by (158). If

$$\psi = \begin{bmatrix} G \cos \theta \\ G \sin \theta \, e^{i\phi} \\ - i F \\ 0 \end{bmatrix} \tag{207}$$

the Euler equations corresponding to (203) are:

$$\begin{aligned} G' + 2G/r + (1 - \omega)\, F + \epsilon F(G^2 - F^2) &= 0 \\ F' + (1 + \omega)\, G + \epsilon G(G^2 - F^2) &= 0 \end{aligned} \tag{208}$$

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* See for example the footnote, Page 83 .

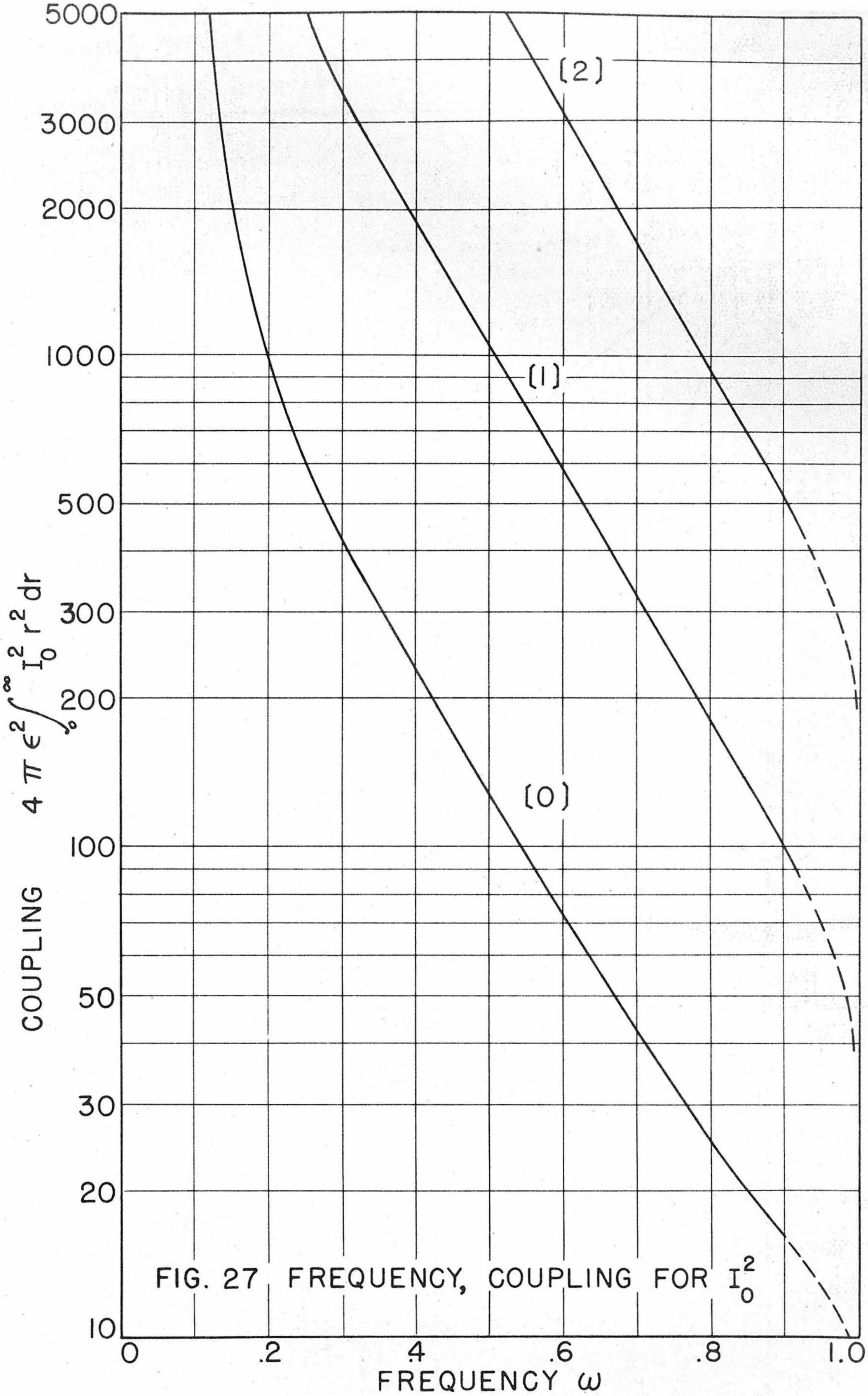

FIG. 27 FREQUENCY, COUPLING FOR I_0^2

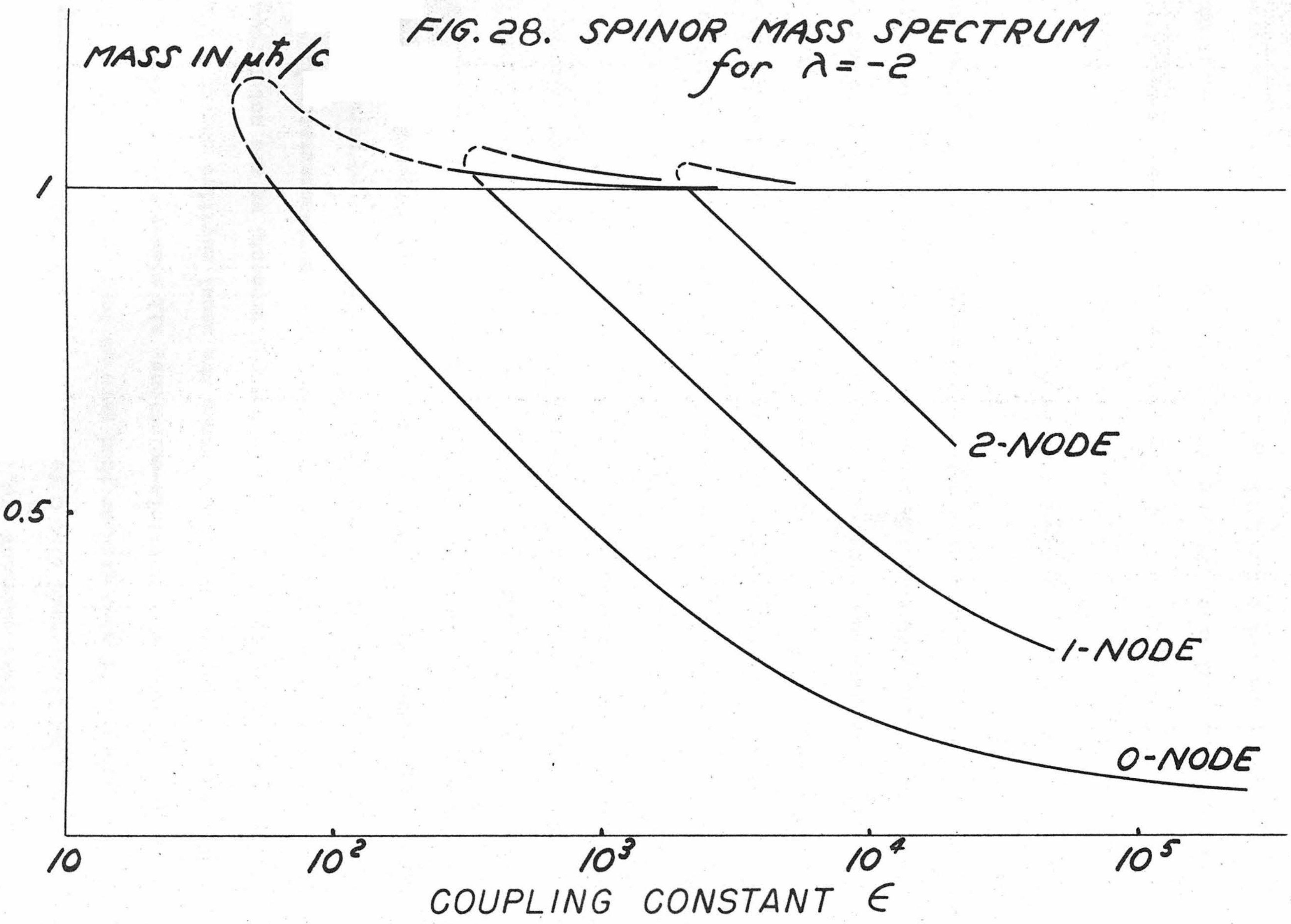

FIG. 28. SPINOR MASS SPECTRUM for $\lambda = -2$

Since ϵ is a fixed constant > 0, a comparison with Equation (203) and Figure (24) shows that $\omega < 0$ for a solution to exist. Then the mass for this solution is also greater than zero so that even if the topology of Figure (23) does give particle-solutions, all masses associated with the coupling $-\epsilon I_0^2$ are positive. Since the usual negative energy difficulties associated with spinor fields are not present, we are not forced to quantize according to the exclusion principle (anti-commutators).

The introduction of neutral or negatively charged particles does not follow from any of the very simple couplings although slightly more complicated forms, such as ϵI_1^2, may be capable of giving neutral or even negatively charged localized solutions. We would then not expect to have identical properties for the positive and negative particles as guaranteed by hole theory.

For a particular ϵ, those solutions having a mass greater than $\mu\hbar/c$ are unstable against expanding to infinity, while the amplitude approaches zero; the non-linear term then becomes negligible and the F and G functions form the usual s-wave solution of the Dirac equation. When the mass is less than $\mu\hbar/c$ the rigorous conservation of energy and charge will stabilize the solution since for very small amplitude the energy is approximately that of a free particle of charge one and therefore greater than $\mu\hbar/c$.

A reasonable interpretation of this unitary theory of particles depends upon a successful introduction of the quantum theory; to interpret in terms of the canonical rules is unsatisfactory for the reasons discussed. On the classical level the results obtained from the very simplest non-linear Lagrangians are not entirely discouraging. A small number of localized solutions exist which can interact with each other to form more complex structures.* Until some sort of quantization is available the fields that can be constructed from (121) would seem to be limited to the spinors. The lack of an attractive recipe for quantizing is the main stumbling block to further progress in this direction.

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* For example, we can normalize a lump to carry charge two. It is then energetically stable against decay into two charge one particles.

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