HOMOMORPHISMS OF FUNCTION LATTICES

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ABSTRACT

This thesis is an algebraic study of systems of real-valued functions which are closed under the operations of pointwise meets and the addition of constants.

In the first chapter, a new kind of lattice congruence is defined in terms of lattice ideals. The properties of this congruence are studied. This congruence is then applied to translation lattices, i. e., algebraic systems in which the two operations of meet and the addition of constants is defined. Results which are analogous to the isomorphism theorems of group theory are proved.

The second chapter contains the development of a representation theory for translation lattices. For this purpose, the concept of a normal lattice function is introduced. These functions are closely related to the normal functions on a topological space. It is shown that a translation lattice can always be mapped homomorphically onto a system of normal lattice functions. Uniqueness theorems are established for this representation.

Chapter three develops, briefly, a new method of constructing topological spaces from a complete Boolean algebra. In the final chapter, this construction is applied to show that a translation lattice can be represented as a translation lattice of continuous functions on a compact Hausdorff space. When suitable restrictions are imposed on the representation, this space -- called the characteristic space -- is uniquely determined. Finally, the relations between different representations by continuous functions are discussed. It is proved that the characteristic space, in an appropriate sense, is the minimal representation space.

PREFACE

A convenient way of studying an abstract algebraic system is to represent the elements of the system as continuous real-valued functions on a suitably defined topological space. This is usually accomplished by determining a complete set of homomorphisms of the abstract system onto the real numbers; from this set of homomorphisms, the original algebraic system is obtained as a sub-direct product of its homomorphic replicas, that is, as a collection of real-valued functions. Finally, the set on which the functions are defined is topologized in such a way as to make these functions continuous.

Recently (see [1] ([1])), Dilworth proved a representation theorem for a specific function lattice in a different way than this. He showed that the mapping σ , defined by

$$\sigma f(X) = \sup_{P \in X} \inf_{y \in P} f(y),$$

is an isomorphism of the set of normal lower semi-continuous functions f on a completely regular topological space S onto the continuous functions on the Boolean space of all minimal dual ideals X in the Boolean algebra of regular open sets P of the space S. Even without a clear understanding of the concepts involved, it is possible to see that this theorem gives a more precise characterization of the representative space than the traditional representation theorems described in the first paragraph above. Moreover, the above represent-

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([1]) References to the literature are indicated by numbers in square brackets.

ation theorem is not an immediate consequence of any of the known theorems on algebraic representations since the set of normal lower semi-continuous functions is not closed under the usual operations of addition, multiplication, scalar multiplication or lattice joins. However, this set of functions is closed with respect to the operations of pointwise meets and the addition of real constants. It is the purpose of this thesis to initiate a general investigation of systems of functions which are closed with respect to these two operations. We will see that the normal functions play a central role in this investigation. The result of Dilworth is, of course, included as a special case of the general theorems which will be obtained.

SUMMARY OF CHAPTER I

The first section of chapter I is devoted to an enumeration of the better known lattice theoretic results which are used in the remainder of the thesis. In the second section, a new method is given for obtaining lattice homomorphisms from lattice ideals. This method is studied in some detail and it is shown that theorems similar to the isomorphism theorems of group theory can be established.

In article one of section two, the concept of a disjunctive semi-lattice is introduced. In the second article, it is shown that the above mentioned homomorphisms are precisely those homomorphisms for which the image lattice is disjunctive. Article three contains a collection of specific examples, while article four is devoted to a compilation of general results. In the fifth article, it is shown that any disjunctive lattice can be imbedded in a complete Boolean algebra. Moreover, the process is essentially unique. Finally, in section six, it is shown that a theorem analogous to the third isomorphism theorem of group theory can be proved for the homomorphisms which we are considering.

CHAPTER I — LATTICE THEORETIC FOUNDATIONS

1.1 Fundamental definitions.

The first section of this chapter is devoted to an enumeration of the lattice theoretical concepts which will be used in the chapters to follow. A complete exposition of the theory can be found in the standard reference, Garrett Birkhoff's Lattice Theory, [2]. The terminology and notation of this book will be used wherever it is practical to do so.

1.1.1. Lattices.

A set P is said to be partially ordered by the relation $a \leq b$(1) if the postulates

P1: $a \leq a$ for all $a \in P$

P2: $a \leq b$ and $b \leq c$ implies always that $a \leq c$.

If a third postulate

P3: $a \leq b$ and $b \leq a$ implies $a = b$

is added, then P will be said to be properly partially ordered. This notation diverges from that of Birkhoff who calls any system which satisfies only P1 and P2 quasi-ordered and always assumes P3 for a partially ordered set. If a relation satisfies P1 and P2, then by identifying elements a and b which satisfy $a \leq b$ and $b \leq a$, a proper partial ordering is obtained. Though most of the partial orderings which we consider will be proper, it will not usually be

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(1) This notation is used interchangeably with $b \geq a$.

necessary to emphasize this feature.

An element a of a partially ordered set P is said to be an upper bound of a subset A of P if $b \leq a$ is true for all $b \in A$. Similarly, it is a lower bound of A if $a \leq b$ for all $b \in A$. A least upper bound of a subset A is an upper bound of A which satisfies $a \leq b$ for all other upper bounds b of A. Greatest lower bounds are similarly defined.

A <u>lattice</u> L is a properly partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound. If a and b are elements of L then the greatest lower bound, or meet, of a and b is written $a \wedge b$ while the least upper bound, or join, is denoted $a \vee b$. If this is done, then $\wedge$ and $\vee$ can be thought of as operations on L which satisfy the identities:

L1: $a \wedge a = a$ and $a \vee a = a$,

L2: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$,

L3: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$,

L4: $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

Conversely, any set L over which operations $\wedge$ and $\vee$ are defined and which satisfy L1 to L4 is a properly partially ordered set (defining $a \leq b$ if $a = a \wedge b$) in which $a \vee b$ and $a \wedge b$ are respectively the least upper and greatest lower bounds of a and b in L.

An element z is called a zero of a partially ordered set P if z is a lower bound of P itself. Similarly i is called a unit if it is an upper bound of P. In general, partially ordered sets and lattices need not have units or zeros. It should be noted that for a proper partial ordering, the unit (or zero), if it exists, is unique.

A lattice is called complete if all its (non-void) subsets have a greatest lower and a least upper bound. In a partially ordered set with a zero, the existence of a least upper bound for every subset guarantees the existence of a greatest lower bound. For lattices without a zero or a unit, the concept of completeness can be replaced with that of conditional completeness. A lattice is said to be conditionally complete in case all of its bounded, non-void subsets have both a least upper and a greatest lower bound. By the well-known process of taking Dedekind cuts, any properly partially ordered set P can be imbedded in a complete lattice. The elements of this complete lattice are just the normal subsets of P, that is, those subsets which contain all lower bounds to the set of their upper bounds. By restricting this construction to subsets which have an upper bound, one can imbed any (properly) partially ordered set in a conditionally complete lattice.

One of the most important properties of lattices is their dual nature. A glance at the postulates L1 - L4 shows that when the operations $\wedge$ and $\vee$ are interchanged, the resulting identities are still the same. To every concept or theorem of lattice theory, there corresponds a dual concept or theorem (which may be identical with the original one) obtained simply by interchanging the role of the two operations. This is not to say that any proposition which is true for a given lattice has its dual proposition also true in that lattice. For example, the lattice of positive integers (ordered by divisibility) has a zero element, but no unit. Much of the work presented in the following pages is definitely non-dual in nature. Nevertheless, the work falls within the confines of lattice theory, and therefore with every theorem and concept we can formulate a dual theorem and a dual proposition.

One of the most important concepts of lattice theory is that of an ideal. An ideal I of a lattice L is a subset of L which enjoys the properties:

I1: $a \in I$ and $b \leq a$ implies $b \in I$,

I2: $a \in I$ and $b \in I$ implies $a \vee b \in I$.

Those subsets which satisfy the relations which are dual to I1 and I2 are called quite naturally, dual ideals.

It is not hard to find examples of ideals in a lattice. For instance, we define the principal ideal associated with an element a of the lattice L to be the set of all $b \in L$ which satisfy $b \leq a$. This ideal will be denoted (a).

Closely connected with the concept of an ideal of a lattice is the notion of homomorphism. A homomorphism of a lattice L onto a lattice L' is a single-valued mapping of L onto L' such that the image of the join (meet) of two elements is the join (meet) of their images. An isomorphism is a homomorphism which is also one-to-one. Evidently any homomorphism preserves the natural partial ordering of a lattice. Conversely, any order preserving isomorphism of a lattice also preserves the join and meet operations. Another way of looking at homomorphisms is in terms of lattice congruences. An equivalence relation $\sim$ on a lattice L is a congruence if it satisfies $a \wedge c \sim b \wedge c$ and $a \vee c \sim b \vee c$ whenever $a \sim b$ and c is any element of L. Given a homomorphism h of a lattice, a congruence is obtained by writing $a \sim b$ when $h(a) = h(b)$. Conversely, any congruence on L determines a natural homomorphism of L onto the lattice of congruence classes in the well known way. These connections determine a one-to-one correspondence between the homomorphisms and congruences of a lattice.

1.1.2 Distributive lattices.

A lattice L is called distributive if it satisfies the relations

L5: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Many of the most important lattices satisfy this postulate. In particular, the so-called function lattices are distributive. These are sets of real valued functions which are closed under pointwise joins and meets: $(f \vee g)(x) = \max \{f(x), g(x)\}$, $(f \wedge g)(x) = \min \{f(x), g(x)\}$.

For complete distributive lattices, there is a natural generalization of the distributive law L5, namely

L5': $a \wedge \bigvee \{b \mid b \in B\} = \bigvee \{a \wedge b \mid b \in B\}$

and its dual. The notation $\bigvee \{b \mid b \in B\}$ is used to denote the least upper bound (or join) of the subset B of L. Not every complete completely distributive lattice satisfies L5'. However, many lattices of interest do, and in the sequal,we will be interested in several of these. Usually, however, it will be sufficient to assume that only L5' (and not its dual) is valid. Thus when the term "completely distributive" is used, only the single relation L5' will be implied.

It is in the theory distributive lattices that the <u>prime</u> ideals of lattice theory attain a position of fundamental importance. An ideal is said to be prime in a lattice L if its set complement in L is a dual ideal. Thus I is a prime ideal if it is an ideal satisfying the condition that if $a \wedge b \in I$, then at least one of $a \in I$ or $b \in I$ is valid.

Another type of ideal of importance is the maximal ideal. An ideal is called maximal if it is contained as a proper subset of no other ideal except the whole lattice (which, of course, is itself an ideal). It is not hard to show that in a distributive lattice, every maximal ideal is prime. In general the converse is not true.

Concerning the question of existence of prime and maximal ideals, it is necessary to use the full power of transfinite methods to prove any general theorems. In any lattice with a unit element, the maximal principle assures the existence of maximal ideals. In any distributive lattice, transfinite arguments can also be used to establish the existence of prime ideals, although in specific examples of lattices, it may be possible to explicitly exhibit maximal and prime ideals.

A distributive lattice L with a zero element z is called pseudo-complemented in case, for every element a, there exists in L a maximal disjoint element a^*. The pseudo-complement a^* is characterized by the properties that $a \wedge a^* = z$, and that $b \wedge a = z$ implies $b \leq a^*$. It is an easy matter to prove that a complete, distributive lattice (with a zero z) is pseudo-complemented if it satisfies the infinite distributive law L5'. The most important property of pseudo-complemented (distributive) lattices is expressed in the theorem of Glivenko: In any complete pseudo-complemented distributive lattice L, the correspondence $a \to a^{**}$ is a lattice homomorphism of L onto the complete Boolean algebra of closed elements (that is, the elements satisfying $a = a^{**}$). A proof of this theorem will be given later.

1.1.3 Boolean Algebras.

The ultimate product in the chain of specialization from the partially ordered set to the lattice to the distributive lattice is the Boolean algebra. A Boolean algebra is a distributive lattice with a zero and a unit element in which every element a has a complement a' satisfying $a \wedge a' = z$ and $a \vee a' = i$. Evidently a complement must be unique. There are many alternative definitions of a Boolean algebra. For example, Stone [3] has shown that Boolean algebras are idempotent rings with unit elements and that, conversely, every idempotent ring with a unit element is a Boolean algebra when ordered by defining $a \le b$ if $a \cdot b = a$. Boolean algebras enjoy many simple properties which are not shared by general lattices. We will enumerate some of them.

The (lattice) homomorphic image of any Boolean algebra is again a Boolean algebra. Every homomorphism of a Boolean algebra is uniquely determined by the kernel of the mapping, that is, the ideal consisting of those elements which are mapped into the zero of the homomorphic image. Conversely, any ideal of a Boolean algebra determines a unique congruence on the Boolean algebra as follows: $a \equiv b\ (I)$ if $(a \wedge b') \vee (a' \wedge b) \in I$. This congruence defines a natural homomorphism of the Boolean algebra onto the Boolean algebra of congruence classes modulo I. The kernel of the homomorphism is I. Up to isomorphism of the homomorphic image, every homomorphism of a Boolean algebra is of precisely this form. The proofs of these assertions are quite easy and will not be reproduced here.

The set of all ideals of a Boolean algebra is itself a complete (distributive) lattice. For later work, it is important to observe that every ideal is the meet of the maximal ideals which contain it. Dually, every dual ideal is the join of all minimal dual ideals contained in it. To see this, observe that if $a \notin I$, where I is an ideal of the Boolean algebra P, then $(a') \vee I \neq P$ (here (a') denotes the principal ideal generated by a': $(a') = \{b \in P \mid b \le a'\}$.) By the maximal principle, it is possible to find a maximal J such that $(a') \vee I \le J$. Then $a' \in J$, so $a \notin J$. It follows immediately that

$I = \bigvee\{J \text{ maximal } | I \leq J\}$.

In the following pages complete Boolean algebras will often appear. These arise quite naturally from a non-complete Boolean algebra when one takes the normal completion (i.e., the completion by cuts). This fact, due to Stone and Glivenko (see respectively [3] and [4]), can be easily deduced from results which will be proved later (see proposition 1.3.9). A property of complete Boolean algebras which will be used repeatedly is their infinite distributivity, i.e., every complete Boolean algebra satisfies L5' and its dual. For let A be a subset of the complete Boolean algebra P. Let $a \in L$. Since $a \wedge \bigvee\{b|b \in A\} \geq a \wedge b$ for all $b \in A$, $a \wedge \bigvee\{b|b \in A\} \geq \bigvee\{a \wedge b|b \in A\}$. On the other hand, if c is any upper bound of the set of $a \wedge b$ where $b \in A$, then $b = (b \wedge a) \vee (b \wedge a')$ $\leq c \vee a'$ for all $b \in A$. Hence, $a \wedge \bigvee\{b|b \in A\} \leq a \wedge (c \vee a') =$ $(a \wedge c) \vee (a \wedge a') = a \wedge c \leq c$. This shows that $a \wedge \bigvee\{b|b \in A\} =$ $\bigvee\{a \wedge b|b \in A\}$. A dual argument can be used to obtain the dual identity. This proof, copies from Birkhoff [2] , is included because of the importance of the result in later work.

1.1.4 Semi-lattices.

If the postulates for a lattice are weakened by not requiring the existence of the join operation, the resulting system is called a semi-lattice. Thus a semi-lattice is a properly partially ordered set in which every pair of elements has a greatest lower bound (dually, a least upper bound). An alternative characterization of a semi-lattice is the following: a semi-lattice is an idempotent, commutative semi-group.

While semi-lattices are more general than lattices, they are rarely of interest in themselves. Most of the common examples of semi-lattices turn out to be lattices as well. However the techniques developed in the following pages seem to apply more naturally to semi-lattices than to general lattices. Hence we will be concerned chiefly with systems which are either semi-lattices or distributive lattices.

Many of the remarks made above concerning general lattices also apply (with slight modifications) to semi-lattices. However one important distinction should be made between the ideals of a semi-lattice and those of a lattice. By a semi-lattice ideal we will mean a subset I with the single property I1: $a \in I$ and $b \leq a$ implies $b \in I$. In order to distinguish the ordinary ideals which satisfy both I1 and I2, the terminology "lattice ideal" will be used.

1.2 Isomorphism Theorems.

Three of the most important elementary results of group theory are the so-called isomorphism theorems. The first of these establishes a one-to-one correspondence between the homomorphisms of a group and its normal sub-groups. According to this theorem, every normal sub-group N of a group G determines a natural homomorphism of G onto the factor group G/N. Conversely, any homomorphism of a group G onto a group $\overline{G}$ determines the normal sub-group N of all elements which are mapped into the zero of $\overline{G}$. The correspondence between homomorphism and normal sub-group is then completely described by the observation that $\overline{G}$ is isomorphic to G/N. The second isomorphism theorem (sometimes called the third) states that if $\overline{G}$ is a homomorphic image of G, if $\overline{N}$ is any normal sub-group of $\overline{G}$, and if N is the normal sub-group in G consisting of all elements which map into $\overline{N}$, then G/N is isomorphic to $\overline{G}/\overline{N}$. The third isomorphism theorem asserts that, for any normal sub-group N and any sub-group H of G, $(N \cdot H)/N$ is isomorphic to $H/(N \cap H)$.

Garrett Birkhoff has given analogues for the first two of these theorems which establish a correspondence between the congruence relations and the homomorphisms of a general algebraic system. These results of Birkhoff say nothing about the relation between the homomorphisms of the algebraic system and its sub-algebras (or ideals), although this relationship is the most important part of the group theoretic theorems. It is the object of this section of the present chapter to explore some of the possible relationships between the ideals and the homomorphisms of a lattice.

It is well known that every ideal I of a lattice L determines a congruence relation $\equiv$ on the lattice. This congruence is defined by $a \equiv b$ (I) if $a \vee I = b \vee I$ (using the notation of the lattice of ideals). By the theorem of Birkhoff, every congruence corresponds to a homomorphism, so that every ideal of a lattice L determines a homomorphism of L onto a lattice $\bar{L}$. It is easy to see that $\bar{L}$ has a zero element and that I is precisely the set of elements of L which are mapped into this zero element by the homomorphism of L onto $\bar{L}$. At this point however, the analogy between the situations in groups and lattices ends. For, while in a group, a homomorphism is uniquely determined by its kernel, there may be many homomorphisms of a lattice which have the same kernel.

This section will be devoted to the description of a class of (semi-) lattice homomorphisms. In general, this class is different from the set of homomorphisms defined above. Corresponding to each ideal of a given semi-lattice a homomorphism of the class will be defined. This class of homomorphisms is characterized as the set of all meet homomorphisms onto a special type of semi-lattice (called a disjunctive semi-lattice). This property gives these homomorphisms many advantages over the ones defined above. In particular, it becomes possible to prove results analogous to the three isomorphism theorems of group theory. In the course of establishing these three theorems, enough of the properties of the homomorphisms are considered to lay the foundations for the work of the remainder of the thesis. Hence, the program for this section can be described rather simply: a new class of homomorphisms of semi-lattices will be defined and theorems analogous to the isomorphism theorems of group theory will be proved.

1.2.1 Disjunctive lattices.

It will be convenient to make the following definition. Its importance will soon become clear.

Definition 1.2.1. A semi-lattice L will be called disjunctive if:

(1) L contains a zero element z, and

(2) if $a \nleq b$ in L, then $c \in L$ exists satisfying

$$z \neq c \leq a \text{ and } b \wedge c = z.$$

These conditions are just the well-known disjunction properties defined by Wallman [5]. A disjunctive lattice is a special case of an algebraic system called a "gefuge" (see Buchi [6]). A gefuge stands in the same relation to a disjunctive semi-lattice as a partially ordered set to a semi-lattice. Several of the results proved below are special cases of Buchi's theorems for gefuges.

The hypothesis (2) of the definition is evidently equivalent to the requirement that if $a \nleq b$, then c exists such that $a \wedge c \neq z$, while $b \wedge c = z$. For if this c does not already satisfy $c \leq a$, then the element $a \wedge c$ will, and moreover, $b \wedge (a \wedge c) = z$.

1.2.2 The first and second isomorphism theorems.

In a Boolean algebra, every homomorphism is associated with an ideal I. Two elements a and b are congruent modulo this ideal if their symmetric difference $a - b = (a \wedge b') \vee (a' \wedge b)$ is contained in the ideal. It is possible to re-phrase this definition in such a way that it can be applied to an arbitrary distributive lattice (more generally, to a semi-lattice). One way is by means of the well-known congruence $a \equiv b$ (I) whenever $a \vee I = b \vee I$. Still another formulation is the following: define $a \sim b$ (I) whenever the relation $a \wedge c \in I$ is equivalent to $b \wedge c \in I$. It is easy to verify that in a Boolean algebra this is just the same as the usual definition. Notice first that I is a lattice ideal of a Boolean algebra if and only if it is a ring ideal of the algebra, considered as a Boolean ring. The distributive law $(a \wedge c) - (b \wedge c) = (a - b) \wedge c$ then shows that if $a - b$ is in I, $a \wedge c \in I$ is equivalent to $b \wedge c \in I$. Conversely, if $a \sim b$ (I), $a \wedge a' \in I$ implies that $b \wedge a' \in I$. Similarly, from $b \wedge b' \in I$, it follows that $a \wedge b' \in I$. Thus $a - b = (a \wedge b') \vee (a' \wedge b) \in I$.

In this last form, the definition of congruence can be taken over to arbitrary semi-lattices. In general, this does not give the same congruence as the conventional definition $a \vee I = b \vee I$. In fact, it is possible to show that the only distributive lattices with a unit for which these two definitions are equivalent, are precisely the Boolean algebras.

In this article, we give the formal definitions of the congruence $(\sim)$, and then proceed to prove the isomorphism theorems.

Lemma 1.2.1 Let L be an arbitrary semi-lattice and suppose that I is an ideal of L. Define:

(1) $a \supseteq b$ (I) $.\equiv.$ $a \wedge c \in I$ implies $b \wedge c \in I$ for all $c \in L$

(2) $a \sim b$ (I) $.\equiv.$ $a \supseteq b$ (I) and $b \supseteq a$ (I).

Then $\geqq$ is a partial ordering and $\sim$ is a meet congruence relation. Moreover $a \wedge b \sim b$ (I) if and only if $a \geqq b$ (I).

<u>Proof.</u> It is easily verified that $\geqq$ is a partial ordering. Also, the ordering $\geqq$ preserves the original partial ordering of L. For if $a \geq b$, then $a \wedge c \in I$ immediately implies $b \wedge c \leq a \wedge c \in I$.

Suppose that $a \geqq b$ (I) and d is any element of L. The second statement of the lemma will be proved if it can be shown that $a \wedge d \geqq b \wedge d$ (I). If $(a \wedge d) \wedge c \in I$, then $a \wedge (d \wedge c) \in I$. Hence, because $a \geqq b$ (I), $b \wedge (d \wedge c) \in I$. Consequently $(b \wedge d) \wedge c \in I$. Since c was an arbitrary element of L, $a \wedge d \geqq b \wedge d$ (I).

Finally, suppose $a \wedge b \sim b$ (I). Since $a \geq a \wedge b$, it follows that $a \geqq a \wedge b \geqq b$ (I). On the other hand, if $a \geqq b$ (I), we have $a \wedge b \geqq b \wedge b = b$ (I), while $b \geqq a \wedge b$ (I) since $b \geq a \wedge b$. This completes the proof.

<u>Lemma 1.2.2.</u> Let L be a distributive lattice. Let I be a lattice ideal of L. Then the semi-lattice congruence relation defined by (1) and (2) in lemma 1.2.1 is a lattice congruence.

<u>Proof.</u> Let $a \geqq b$ (I) and suppose d is any element of L. We will show that $a \vee d \geqq b \vee d$ (I). This will then immediately imply the assertion of the lemma. Suppose $c \in L$ is such that $(a \vee d) \wedge c \in I$. Then $(a \wedge c) \vee (d \wedge c) \in I$. Since I is a lattice ideal, this implies $a \wedge c \in I$ and $d \wedge c \in I$. Hence, also, $b \wedge c \in I$. But then $(b \vee d) \wedge c = (b \wedge c) \vee (d \wedge c) \in I$. Since c was picked at random from L, it follows

that $a \vee d \geq b \vee d$ (I).

It is well known (for details see Birkhoff [2]) that a congruence relation on an algebraic system determines a homomorphism of the system. The homomorphic image is the set of congruence classes with operations suitably defined.

Denote by L/I the homomorphic image of L which is defined by the congruence relation of lemma 1.2.1. By the natural mapping of L onto L/I will be meant that mapping which sends each element of L into its congruence class in L/I.

So far, we have only obtained a new way of defining a homomorphism on a lattice. The important property of this homomorphism is that it can be characterized in terms of its image.

Lemma 1.2.3. The semi-lattice L/I is disjunctive. Moreover, the kernel of the natural mapping of L onto L/I (i.e., the set of elements mapped into the zero of L/I) is precisely I.

Proof. Denote by h the natural mapping of L onto L/I. Suppose $a \in I$. Then $a \geq b$ (I) if, and only if, $b \in I$. For suppose $a \geq b$ (I). Then, since $a \wedge b \in I$, $b = b \wedge b \in I$. Conversely, if $b \in I$, $b \wedge c \in I$ for all $c \in L$ so trivially $a \wedge c \in I$ implies $b \wedge c \in I$. Thus if $a \in I$, $h(a) = h(b)$ if and only if $b \in I$. In other words, h(I) is the zero element of L/I and I is the kernel of h.

Suppose now that a and b are any elements of L which satisfy

$h(a) \ngeq h(b)$. Then $b \ngeq a \quad (I)$, so by definition, there is a $c \in L$ such that $b \wedge c \in I$ while $a \wedge c \notin I$. Hence $h(b) \wedge h(c) = h(b \wedge c) = h(I)$ and $h(a) \wedge h(c) = h(a \wedge c) \neq h(I)$. Thus, according to definition 1.3.1, L/I is disjunctive.

An important and rather surprising fact is that the converse of the above lemma is true.

<u>Lemma 1.2.4.</u> Let h be a (meet) homomorphism of the semi-lattice L onto the disjunctive semi-lattice $\overline{L}$. Then if I is the kernel of h, I is an ideal of L and $\overline{L}$ is isomorphic to L/I. Under this isomorphism, h(a) corresponds to the congruence class of a modulo I.

<u>Proof.</u> It is clear that I is an ideal in L. The proof will be completed by showing that $a \geq b \quad (I)$ if and only if $h(a) \geq h(b)$. If $a \geq b \quad (I)$, then for all $c \in L$, $a \wedge c \in I$ implies that $b \wedge c \in I$. Thus for all h(c), $h(a) \wedge h(c) = z$ implies $h(b) \wedge h(c) = z$. Because of the disjunctiveness of $\overline{L}$ this means $h(a) \geq h(b)$. On the other hand, if $a \ngeq b \quad (I)$, there is a $c \in L$ such that $a \wedge c \in I$ while $b \wedge c \notin I$. This means $h(a) \wedge h(c) = z$ and $h(b) \wedge h(c) \neq z$. Therefore $h(a) \ngeq h(b)$. The proof is complete.

Hereafter, any homomorphism of a lattice will be called disjunctive if it is onto a disjunctive semi-lattice. It should be noted that if L is a lattice and h is a <u>lattice</u> homomorphism, then I (the kernel of h) is a <u>lattice</u> ideal. For our theory, this is important in the case where L is a distributive lattice.

We now summarize all of these lemmas in the "first isomorphism theorem":

Theorem 1.2.1. Let L be a semi-lattice (distributive lattice). Then there is a one-to-one correspondence between the ideals (lattice ideals) of L and the homomorphisms (lattice homomorphisms) of L onto disjunctive semi-lattices (distributive lattices). Any ideal (lattice ideal) I determines a natural homomorphism (lattice homomorphism) of L onto the disjunctive semi-lattice (distributive lattice) L/I such that I is the kernel of the homomorphism. Conversely, any homomorphism (lattice homomorphism) of L onto a disjunctive semi-lattice (distributive lattice) is algebraically equivalent to the natural homomorphism of L onto L/I where I is the kernel of the given homomorphism.

The second isomorphism theorem is an elementary consequence of the first.

Theorem 1.2.2. Let g be a homomorphism (lattice homomorphism) of the semi-lattice (distributive lattice) L onto $\bar{L}$. Let $\bar{I}$ be any ideal (lattice ideal) of $\bar{L}$. Denote $I = g^{-1}(\bar{I})$. Then I is an ideal (lattice ideal) of L and L/I is isomorphic to $\bar{L}/\bar{I}$.

Proof. If $a \nleq b$ and $b \in I$, then $g(a) \leq g(b) \in \bar{I}$ so $g(a) \in \bar{I}$. Thus $a \in I$. For the distributive case, $a, b \in I$ implies that $g(a)$, $g(b) \in \bar{I}$. Then $g(a \vee b) = g(a) \vee g(b) \in \bar{I}$. Therefore $a \vee b \in I$. We have shown that I is an ideal.

Consider $\overline{L}/\overline{I}$. This semi-lattice is regular. Denote by $\overline{h}$ the natural homomorphism of $\overline{L}$ onto $\overline{L}/\overline{I}$. Map L onto $\overline{L}/\overline{I}$ by $h(a) = \overline{h}(g(a))$. The kernel of this mapping is clearly I, so by theorem 1.2.1, L/I is isomorphic to $\overline{L}/\overline{I}$. This completes the proof.

1.2.3 Some specific examples.

In order to be able to better appreciate the meaning of the first two homomorphism theorems, we will consider the results of applying them to a few special lattices.

Example 1. One of the simplest possible examples of a distributive lattice is a chain, that is, a totally ordered set. It is easy to prove that a chain is a disjunctive semi-lattice only if it is the system $\overline{2}$ consisting of the zero and the unit element. In fact, suppose that a and b are distinct elements of the chain L with $z \neq a \leq b$. If L were disjunctive, c would exist satisfying $z \neq c \leq b$ and $c \wedge a = z$. But this last relation is possible in a chain only if $c = z$ or $a = z$. This contradiction shows that two elements distinct from zero cannot exist. On the other hand, the chain $\overline{2}$ is clearly a disjunctive semi-lattice. An immediate consequence is the following fact: if I is any ideal of a chain, then L/I is isomorphic to $\overline{2}$. This demonstrates how, even for the simplest lattices, the homomorphism defined by an ideal I, as in lemma 1.2.1, differs radically from the usual definition of a homomorphism generated by I (that is, $a \equiv b \quad (I)$ if $a \vee I = b \vee I$). For example, consider the principal ideal $(c) = \{d \in L | d \leq c\}$. We have a $\vee$ (c) = b $\vee$ (c) if and only if either a and b are both less than or equal to c, or also a = b. Consequently, the homomorphism defined by this congruence is a mapping of L onto the interval $\{d \in L \mid d \geq c\}$. In general, this is different from $\bar{2}$.

Example 2. Let L be the distributive lattice of open sets of a topological space S. It is easy to see that in general L will not be disjunctive. In fact, if S is a T_1 space, L will be disjunctive only if S is discrete. The mapping $a \to a^{-o}$ is a lattice homomorphism (1) of L onto the complete Boolean algebra of regular open sets of S. (For details and definitions, see [2].) The kernel of this mapping is clearly the null set. Hence, the Boolean algebra of regular open sets of S is isomorphic to L/(z). (It is evident that any Boolean algebra is disjunctive in the sense of definition 1.3.1.)

Example 3. Let L be the lattice of continuous functions on a metric topological space S. Since L has no zero, it cannot be disjunctive. The mapping $f \to \{x \in S \mid f(x) > 0\}^{-o}$ is a lattice homomorphism of L onto the Boolean algebra of regular open sets of S, as may be easily verified. The kernel of the mapping is $(0) = \{f \in L \mid f \leq 0\}$. Hence, L/(0) is isomorphic to the Boolean algebra of regular open sets of S. When the restriction that S be a metric space is dropped, we no longer necessarily get all of this Boolean algebra. However, it

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(1) For any set $a \subseteq S$, a^- denotes the closure of a, a^o the interior of a.

is easily shown that this same mapping carries L onto a disjunctive lattice.

Example 4. Let L be the Boolean algebra of regular open sets of a topological space T. Let S be an arbitrary sub-space of T with the relative topology. For any $a \in L$, map $a \to (a \cap S)^{-o}$ (where closure and interior operations are in the topology of S). This is a meet homomorphism of L onto the Boolean algebra of regular open sets of S. Notice that in general the mapping is not a join homomorphism. The kernel of the homomorphism is the semi-lattice ideal $I = \{a \in L \mid a \cap S = \emptyset\}$. By theorem 1.2.1, the Boolean algebra of regular open sets of S is isomorphic to L/I. This example shows that, even in the case where L is a distributive lattice, it may be useful to consider semi-lattice homomorphisms of L.

Example 5. It is a consequence of the result in example 3 that the Boolean algebra of regular open sets of a metric topological space S is characterized by the lattice structure of C(S). Another result of this nature is the following. Let L be the lattice of all integrable functions on a set S for which an integration theory (in the sense of Bohnenblust [7]) has been defined. A subset A of S is called a strongly (or ring) measurable if it is of the form $A = \{x \mid f(x) > 0\}$ for some $f \in L$. Now the mapping $f \to \{x \mid f(x) > 0\}$ of L onto the collection of strongly measurable sets is clearly a lattice homomorphism. Moreover, it can be shown that the set of strongly measurable sets forms a ring of sets: if A and B are strongly measurable, then so are $A \cup B$ and $A \cap B^c$. Thus if $A \not\subseteq B$, then $\emptyset \neq A \cap B^c \subseteq A$ and $(A \cap B^c) \cap B = \emptyset$. It follows that the collection of strongly measurable sets is a disjunctive (distributive) lattice. The kernel of the homomorphism $f \rightarrow \{x | f(x) > 0\}$ is just the ideal $I = \{f \in L | f \leq 0\}$. Hence the ring of strongly measurable sets is just L/I. We have proved, incidentally, that two integration theories, which have isomorphic lattices of integrable functions, have isomorphic rings of strongly measurable sets.

1.2.3 Miscellaneous general results.

In this section are collected together some of the important properties of the homomorphism defined in lemma 1.2.1. Only those which will be needed later are included.

We first look at the homomorphism for two special kinds of ideals, namely, the zero ideal (z) and the prime ideals.

Proposition 1.2.1. A semi-lattice L with a zero z is disjunctive if and only if L is isomorphic to L/(z).

Proof. The sufficiency is clear since L/(z) is disjunctive. Conversely, if L is disjunctive, the identity mapping is a homomorphism of L onto a disjunctive semi-lattice with the kernel z. By theorem 1.2.1, L is isomorphic to L/(z).

From this last proposition, the following useful result is derived.

<u>Proposition 1.2.2.</u> Let L be a disjunctive semi-lattice. Then a semi-lattice homomorphism of L which has the kernel (z) is necessarily an isomorphism.

<u>Proof.</u> Under the stated conditions, the image is a disjunctive semi-lattice. For $h(a) \nleq h(b)$ implies $a \nleq b$, so c exists with $z \neq c \leq a$ and $c \wedge b = z$. Then $z \neq h(c) \leq h(a)$, $h(c) \wedge h(b) = z$. Consequently, $L = L/(z)$ is isomorphic to $h(L)$ by theorem 1.2.1.

<u>Proposition 1.2.3.</u> Let L be a distributive lattice and let I be an ideal of L. Then I is a prime ideal if and only if L/I is isomorphic to $\overline{2}$ (the Boolean algebra of two elements).

<u>Proof.</u> Sufficiency: suppose L/I is isomorphic to $\overline{2}$. Let h be the natural homomorphism of L onto $\overline{2}$, which has the kernel I. If $a \wedge b \in I$, then $h(a) \wedge h(b) = h(a \wedge b) = z$. Hence, either $h(a) = z$ or $h(b) = z$; that is, either $a \in I$ or $b \in I$. Since a and b could be any elements, it follows that I is a prime ideal.

Necessity: Suppose I is a prime ideal. Let $a \notin I$. Then $a \wedge c \in I$ implies $c \in I$. Hence, for any $b \in L$, $a \wedge c \in I$ implies $b \wedge c \in I$; that is, $a \geq b$ (I). This means that under the natural homomorphism of L onto L/I, a maps onto the unit of L/I. On the other hand, if $a \in I$, then a maps onto the zero of L/I. Since this exhausts the possibilities, L/I must be $\overline{2}$.

<u>Remark.</u> The above lemma is equally true for semi-lattices provided prime ideals are defined in a suitable manner. The appropriate

definition is: a semi-lattice ideal is prime if $a \notin I$ and $b \notin I$ implies $a \wedge b \notin I$.

The next topic is the problem of finding a criterion for L/I to be a Boolean algebra. The first step is a lemma.

<u>Lemma 1.2.5.</u> Let L be a semi-lattice. Let I be an ideal of L and denote by h the natural homomorphism of L onto L/I. Then h(a) is the unit of L/I if and only if $a \wedge c \in I$ holds only when $c \in I$.

<u>Proof.</u> If h(a) is the unit of L/I, then $a \geq b$ (I) is true for all $b \in L$. Thus $a \wedge c \in I$ implies $b \wedge c \in I$. In particular, $a \wedge c \in I$ implies $c = c \wedge c \in I$. Conversely, if $a \wedge c \in I$ implies $c \in I$, then also $a \wedge c \in I$ implies $b \wedge c \in I$ for any b. Thus $a \geq b$ (I) for all $b \in L$ or, in other words, h(a) is the unit of L/I.

<u>Proposition 1.2.4.</u> Let L be a distributive lattice. Let I be a lattice ideal of L. Then L/I is a Boolean algebra if and only if, for any $a \in L$, there exists a^* in L satisfying:

(i) $a \wedge a^* \in I$, and

(ii) $a \wedge b \in I$ <u>and</u> $a^* \wedge b \in I$ imply $b \in I$.

<u>Proof.</u> Condition (ii) is clearly equivalent to: $(a \vee a^*) \wedge b \in I$ implies $b \in I$. Thus, by the above lemma, $h(a) \vee h(a^*) = h(a \vee a^*) = i$ (the unit of L/I). From condition (i), it follows that $h(a) \wedge h(a^*) = h(a \wedge a^*) = z$. Thus h(a) has the complement $h(a^*)$ in L/I. Therefore L/I is a Boolean algebra.

Conversely, suppose L/I is a Boolean algebra. If $a \in L$, let a^*

be any element of L such that $h(a^*) = h(a)'$. Clearly $h(a \vee a^*) = i$ and $h(a \wedge a^*) = z$ so the conditions (i) and (ii) are satisfied.

Corollary 1.2.1. If L is a pseudo-complemented distributive lattice, then L/(z) is a Boolean algebra. In particular, a disjunctive distributive lattice is pseudo-complemented if and only if it is a Boolean algebra.

In later work, certain questions of completeness will arise. It is important to have a criterion for determining when the homomorphic image of a complete lattice is complete. For many purposes, the following result is adequate.

Proposition 1.2.5. If L is a complete, completely distributive lattice[1] and I is a principal ideal, then L/I is complete and the natural mapping preserves unlimited joins.

Proof. Denote by h the natural homomorphism of L on L/I. Suppose A is any non-empty subset of L. It will be shown that $h(\bigvee A)$ is the least upper bound of the set $h(A) = \{h(a) \mid a \in A\}$ in L/I. First of all, it is clear that $h(\bigvee A)$ is an upper bound of h(A). On the other hand, if $h(b) \geq h(a)$, that is, $b \geq a$ (I) prevails for all $a \in A$, then $b \geq \bigvee A$ (I). For $b \wedge c \in I$ implies $a \wedge c \in I$ for all $a \in A$. But since I is principal, it is closed under unlimited joins.

(1) In other words, L satisfies the infinite distributive law $b \wedge \bigvee \{a \mid a \in A\} = \bigvee \{b \wedge a \mid a \in A\}$. In particular, this implies that L is pseudo-complemented. It is not necessary to assume that L is closed under unlimited meets.

Thus $(\bigvee A) \wedge c = \bigvee (A \wedge c) \in I$. Since c was chosen arbitrarily, $b \geq \bigvee A(I)$. It follows that $h(\bigvee A)$ is the <u>least</u> upper bound of $h(A)$, and the proof is complete.

It will be noticed that the above proof uses only the fact that I is a closed ideal of L. But in a complete lattice, this immediately implies that I is principal (and conversely). The restriction to principal ideals is a necessary one. For if the natural mapping preserves unlimited joins, $h(\bigvee I) = \bigvee h(I) = z$, i.e. $\bigvee I \in I$ and therefore I is principal.

1.2.4. <u>Imbedding theorems</u>.

In this article, the general theory of disjunctive semi-lattices will be studied more systematically. It will be shown that a disjunctive semi-lattice can be imbedded uniquely in a complete Boolean algebra. Most of the theorems proved are not new. The imbedding and uniqueness theorems can be found explicitly in the paper of Buchi and implicitly to the works of Glivenko and Stone.

In order to construct the complete Boolean algebra in which the semi-lattice is to be imbedded, it is necessary to employ a theorem of Glivenko. Because of the importance of the imbedding theorem, the proof of Glivenko's theorem will be given.

Let L be an arbitrary semi-lattice with zero z. Denote by $\mathcal{J}$ the set of all ideals of L (in the sense $a \leq b$ and $b \in I$ implies $a \in I$).

Lemma 1.2.6. The set $\mathcal{Y}$ is a complete, distributive, pseudo-complemented lattice under set unions and intersections. The pseudo-complement of $A \in \mathcal{Y}$ is given by $A^* = \{b \in L | a \wedge b = z$ all $a \in A\}$. The principal ideals of L form a sub-semi-lattice of $\mathcal{Y}$, which is isomorphic to L.

Proof. If $\{A_\alpha\}$ is any collection of elements of $\mathcal{Y}$, then clearly $\bigcup_\alpha A_\alpha$ and $\bigcap_\alpha A_\alpha$ are ideals and hence in $\mathcal{Y}$. If $A \in \mathcal{Y}$, let $A^* = \{b \in L | a \wedge b = z$ all $a \in A\}$. Evidently A^* is an ideal. $A \cap A^* = \{z\}$ since if $a \in A \cap A^*$, then $a = a \wedge a = z$. Also if $A \cap B = \{z\}$, then $B \subseteq A^*$. For if $b \in B$, $a \wedge b \in A \cap B = \{z\}$ all $a \in A$. Thus A^* is the pseudo-complement of A. The last assertion of the lemma is clear.

Lemma 1.2.7. (Glivenko; proof after Birkhoff [2]).
The mapping $A \to A^{**}$ is a homomorphism of $\mathcal{Y}$ onto the complete Boolean algebra $\mathcal{B}$ of "closed" ideals (i.e., ideals A for which $A = A^{**}$).

Proof. The proof rests on a series of formulas.

(1) $A^{**} \supseteq A$

This follows from the definition of A^{**} since $A \cap A^* = \{z\}$.

(2) $A \subseteq B \Rightarrow A^* \supseteq B^*$.

For if $A \subseteq B$, $A \cap B^* \subseteq B \cap B^* = \{z\}$ and hence $B^* \subseteq A^*$.

(3) $(A \cup B)^* = A^* \cap B^*$.

$A \cup B \supseteq A, B \Rightarrow (A \cup B)^* \subseteq A^*, B^* \Rightarrow (A \cup B)^* \subseteq A^* \cap B^*$. $(A \cup B) \cap A^* \cap B^* \subseteq (A \cap A^*) \cup (B \cap B^*) = \{z\} \Rightarrow A^* \cap B^* \subseteq (A \cup B)^*$.

(4) $A^{***} = A^{*}$.

$A \subseteq A^{**} \Rightarrow A^{*} \supseteq A^{***}$ by (2). The opposite inclusion follows from (1).

(5) $(A \cap B)^{*} \supseteq A^{*} \cup B^{*}$.

This is clear from (2).

(6) $(A^{**} \cap B^{**})^{*} \supseteq (A \cap B)^{*}$.

By (5) and (3), $(A \cap B)^{**} \subseteq (A^{*} \cup B^{*})^{*} = A^{**} \cap B^{**}$. Hence the result follows from (2) and (4).

(7) $(A^{**} \cap B^{**})^{*} \supseteq (A \cap B)^{*}$.

Repeated applications of the definition of pseudo-complement give the implications: $(A \cap B)^{*} \cap A^{**} \cap B^{**} \cap A \cap B = \{z\} \Rightarrow (A \cap B)^{*} \cap$ $\cap A^{**} \cap B^{**} \cap A \subseteq B^{*} \Rightarrow (A \cap B)^{*} \cap A^{**} \cap B^{**} \cap A = \{z\} \Rightarrow (A \cap B)^{*} \cap$ $\cap A^{**} \cap B^{**} = \{z\}$, which is equivalent to (7).

(8) $(A \cap B)^{**} = A^{**} \cap B^{**}$.

For, from (6), (7) and (3), $(A \cap B)^{*} = (A^{**} \cap B^{**})^{*} = (A^{*} \cup B^{*})^{**}$. Hence, $(A \cap B)^{**} = (A^{*} \cup B^{*})^{***} = (A^{*} \cup B^{*})^{*} = A^{**} \cap B^{**}$.

(9) $(\bigcup\{A^{**} | A \in \mathfrak{A}\})^{**} = (\bigcup\{A | A \in \mathfrak{A}\})^{**}$ where $\mathfrak{A} \subseteq \mathcal{S}$.

$A^{**} \supseteq A \Rightarrow \bigcup\{A^{**} | A \in \mathfrak{A}\} \supseteq \bigcup\{A | A \in \mathfrak{A}\} \Rightarrow (\bigcup\{A^{**} | A \in \mathfrak{A}\})^{**} \supseteq$ $\supseteq (\bigcup\{A | A \in \mathfrak{A}\})^{**}$. On the other hand, $A \in \mathfrak{A} \Rightarrow A^{**} \subseteq (\bigcup\{A | A \in \mathfrak{A}\})^{**} \Rightarrow$ $\Rightarrow \bigcup\{A^{**} | A \in \mathfrak{A}\} \subseteq (\bigcup\{A | A \in \mathfrak{A}\})^{**} \subseteq (\bigcup\{A^{**} | A \in \mathfrak{A}\})^{**} \subseteq$ $\subseteq (\bigcup\{A | A \in \mathfrak{A}\})^{****} = (\bigcup\{A | A \in \mathfrak{A}\})^{**}$.

Using the formulas (1) - (9), it is possible to prove the theorem: $A \to A^{**}$ is a homomorphic mapping of $\mathcal{S}$ onto $\mathfrak{B}$; $\mathfrak{B}$ is a complete Boolean algebra with $A \wedge B = A \cap B$, $\bigvee\{A | A \in \mathfrak{A} \subseteq \mathfrak{B}\} = (\bigcup\{A | A \in \mathfrak{A} \subseteq \mathfrak{B}\})^{**}$, $A' = A^{*}$ ($\wedge$, $\vee$, $'$ denoting Boolean operations).

First $\mathcal{B}$ is a complemented lattice with these operations. For by (8) $A \cap B$ is the greatest lower bound of A and B in $\mathcal{B}$. If $\mathcal{A} \subseteq \mathcal{B}$ and $B \in \mathcal{B}$ satisfies $B \supseteq A$ for all $A \in \mathcal{A}$, then $B = B^{**} \supseteq$ $\supseteq (\cup \{A | A \in \mathcal{A}\})^{**}$. Clearly $(\cup \{A | A \in \mathcal{A}\})^{**} \supseteq A$ for all $A \in \mathcal{A}$ so $(\cup \{A | A \in \mathcal{A}\})^{**}$ is the least upper bound of $\mathcal{A}$ in $\mathcal{B}$. Finally A^* is the complement of A since $A \vee A^* = (A \cup A^*)^{**} =$ $= (A^* \cap A^{**})^* = \{z\}^* = L$, $A \wedge A^* = A \cap A^* = \{z\}$ and $A^* \in \mathcal{B}$ by (4).

Now the mapping $A \to A^{**}$ is a homomorphism by (8) and (9). The mapping is onto since if $A \in \mathcal{B}$, $A \to A^{**} = A$. Finally, $\mathcal{B}$, being the homomorphic image of a distributive lattice, must be distributive, and hence a Boolean algebra. By (9), $\mathcal{B}$ is complete. This finishes the proof of Glivenko's theorem.

Now we are ready to state and prove the imbedding theorem. It is convenient, however, to first introduce some terminology.

Definition 1.2.2. Let L be a semi-lattice with a zero z. Let M be a sub-semi-lattice of L. M will be called dense in L if $M - \{z\}$ is coinitial in L, that is, $z \neq a \in L$ implies that $b \in M$ exists with $z \neq b \leq a$.

Theorem 1.2.3. Let L be a semi-lattice with zero z. Let (z) denote the principal ideal generated by z. Then L/(z) is isomorphic to a dense sub-semi-lattice of a complete Boolean algebra. If L/(z) has a unit, it is mapped into the unit of the Boolean algebra by the isomophism. If L is distributive, the mapping is a lattice isomorphism.

Proof. The mapping $a \to (a)^{**}$ is a join homomorphism of L into the complete Boolean algebra $\mathcal{B}$ (by lemmas 1.2.6 and 1.2.7). If $z \neq a \in A \in \mathcal{B}$, then $(a)^{**} \subseteq A$. Hence the ideals $(a)^{**}$ are dense in $\mathcal{B}$. The kernel of this mapping is z, so the first assertion follows from theorem 1.2.1.

Notice that if L has a unit i, then the principal ideal (i) contains every element of L. Hence $(i)^{**} = (i) = L \supseteq A$ for all $A \in \mathcal{B}$. Hence the unit of L is mapped into the unit of $\mathcal{B}$.

To prove the last statement of the theorem, notice that $a \vee b \to$ $\to (a \vee b)^{**}$. We will show that $(a \vee b)^{**} = [(a) \cup (b)]^{**} =$ $[(a)^{**} \vee (b)^{**}]^{**} = (a)^{**} \vee (b)^{**}$. The first of these equalities is all that is needed, the others being consequences of the previous lemma.

If $c \in [(a) \cup (b)]^{*} = (a)^{*} \cap (b)^{*}$, then $a \wedge c = b \wedge c = z$. Thus $(a \vee b) \wedge c = z$, so $(a \vee b) \in [(a) \cup (b)]^{**}$. By formulas (2) and (4) of the lemma 1.2.7, $(a \vee b)^{**} \subseteq [(a) \cup (b)]^{**}$. On the other hand $(a) \cup (b) \subseteq (a \vee b)$, so $[(a) \cup (b)]^{**} \subseteq (a \vee b)^{**}$. The imbedding theorem is proved.

Corollary 1.2.2. If L is a disjunctive semi-lattice, then L is isomorphic to a dense sub-semi-lattice of a complete Boolean algebra.

Proof. The corollary is an evident consequence of proposition 1.2.1. However, it is instructive to notice that for disjunctive L, the principal ideals are closed: $(a)^{**} = (a)$. For, if $b \nleq a$, then by the disjunction property, $c \in L$ exists with $z \neq c \leq b$, $c \wedge a = z$.

Thus $c \in (a)^*$ and $b \wedge c = c \neq z$, and therefore $b \notin (a)^{**}$. Consequently $(a)^{**} \leq (a)$, while the opposite inclusion is a consequence of formula (1) of lemma 1.2.7.

For any disjunctive lattice L denote by $[L]$ the complete Boolean algebra which is obtained from L by the construction of lemmas 1.2.6, 1.2.7 and theorem 1.2.3. L is a dense sub-semi-lattice of $[L]$. It will be shown that this property characterizes $[L]$, that is, any complete Boolean algebra which contains L as a dense sub-semi-lattice is isomorphic to $[L]$. However, before proving this uniqueness theorem, let us look a little closer at the concept of denseness. The properties which will be needed for later work can be collected together here.

<u>Proposition 1.2.6.</u> (i) Any semi-lattice is dense in itself; (ii) if M is dense in N and N is dense in L, then M is dense in L; (iii) if L is disjunctive and M is dense in L, then M is disjunctive.

<u>Proof.</u> (i) and (ii) are so obvious that they need no proof. To prove (iii), suppose $a \nleq b$ in M. Then $c \in L$ exists satisfying $z \neq c \leq a$ and $b \wedge c = z$. Finally $d \in M$ exists with $z \neq d \leq c \leq a$ and $d \wedge b \leq c \wedge b = z$.

A consequence of this proposition and theorem 1.2.3 is the following characterization of a disjunctive semi-lattice: a semi-lattice is disjunctive if and only if it is isomorphic to a dense sub-semi-lattice of a complete Boolean algebra.

Now the fundamental uniqueness theorem will be formulated and proved.

Theorem 1.2.4. Let L be a disjunctive sub-semi-lattice of the complete Boolean algebra P. Then [L] is isomorphic to a sub-semi-lattice of P; if L is dense in P, [L] is isomorphic to all of P.

Proof. Recall that L consists of all the "closed" ideals of L, i.e., those ideals A satisfying $A^{**} = A$.

Map $A \to \bigvee A$ ($\bigvee$ in P). This is a semi-lattice homomorphism: $A \wedge B = A \cap B \to \bigvee(A \cap B) = \bigvee\{a \wedge b | a \in A, b \in B\} = (\bigvee\{a | a \in A\}) \wedge$ $\wedge (\bigvee\{b | b \in B\}) = (\bigvee A) \wedge (\bigvee B)$, by the infinite distributive law in a complete Boolean algebra. The kernel of this mapping is z. Thus, by proposition 1.2.2, the mapping is an isomorphism. This proves the first assertion.

Now suppose L is dense in P. Then if $a \in P$, $A = \{b \in L | b \leq a\}$ is an ideal of L. We will show that it is closed. If $b \in L$ with $b \notin A$, (i.e., $b \nleq a$) then $b \wedge a' \neq z$. Hence, $c \in L$ exists with $z \neq c \leq b \wedge a'$. Then $c \in A^*$ and $b \wedge c \neq z$. This says $b \notin A^{**}$. Hence $A^{**} \subseteq A$, which shows that A is closed and $A \in [L]$.

$\bigvee A = a$ since otherwise, $(\bigvee A)' \wedge a \neq z$ and $b \in L$ exists with $z \neq b \leq a \wedge (\bigvee A)'$. In other words, $b \in A$, while $z = b \wedge (\bigvee A) \geq$ $\geq b \wedge b = b$ — a contradiction. Since a was chosen arbitrarily from P, the mapping of [L] is onto P. The proof is complete.

The construction of $[L]$ from L is a completion process: it imbeds a partially ordered set in a complete lattice. It is of interest to compare it with another well known process for accomplishing this, namely normal completion. In particular it is desirable to know when the pseudo-complement completion process can be replaced by the process of taking the normal completion. The following result (which is a slight generalization of the theorem that the normal completion of a Boolean algebra is again a Boolean algebra) gives the desired information for a distributive lattice.

Proposition 1.2.7. Let L be a distributive lattice. A necessary and sufficient condition that the normal completion of L be a Boolean algebra is that L be disjunctive with respect to both its join and its meet operation. In other words, L must have a zero z and a unit i, and be such that if $a \nleq b$, then c and d exist satisfying $z \neq c \leq a$, $c \wedge b = z$, $i \neq d \geq b$ and $d \vee a = i$.

Proof. Necessity: Let $\mathcal{B}$ denote the lattice of (lower) normal subsets of L, that is, subsets which contain every lower bound of the set of their upper bounds. By hypothesis, $\mathcal{B}$ is a Boolean algebra. We can exclude the case where $\mathcal{B} = \bar{2}$ since this lattice can only be the completion of itself. The lattice $\mathcal{L}$ of principle ideals constitutes a sub-lattice which is isomorphic to L: $a \to \{b \in L | b \leq a\}$. Moreover $\mathcal{L}$ has the property that if $N \in \mathcal{B}$, then $N = \bigvee \{A \in \mathcal{L} \mid A \subseteq N\}$, and $N = \bigwedge \{A \in \mathcal{L} \mid A \supseteq N\}$. With these preparations out of the way, we can proceed with the proof.

If L had no zero element, the zero of $\mathfrak{B}$ would be the empty set. This leads to a contradiction. For if N and N' are non-trivial complementary sets of $\mathfrak{B}$, and if $a \in N$, $b \in N'$, then $a \wedge b \in N \cap N' = \emptyset$. Hence L must have a zero. A dual argument shows that L must also have a unit element.

Now suppose $a \nleq b$. Then if $A = (a)$ and $B = (b)$, $A \nleq B$. Thus $A \wedge B' \neq Z$ and $A' \vee B \neq I$. Hence $C, D \in \mathcal{L}$ exist so that $Z \neq C \subseteq A \wedge B'$ and $A' \vee B \subseteq D \neq I$. If $C = (c)$ and $D = (d)$, $z \neq c \leq a$ and $b \leq d \neq i$. Since $C \wedge B = Z$, $D \vee A = I$, $c \wedge b = z$ and $d \vee a = i$. This completes the proof of the necessity of the conditions.

Sufficiency: It is sufficient to show that an ideal N is a normal subset if and only if it is closed: $N^{**} = N$.

Suppose N is normal and $a \notin N$. Then b, an upper bound of N, exists satisfying $a \nleq b$. By the (lower) disjunction property, $c \in L$ exists such that $z \neq c \leq a$ and $c \wedge b = z$. Consequently $c \wedge d = z$ for all $d \in N$ and therefore $c \in N^*$. Since $a \wedge c \neq z$, $a \notin N^{**}$. Thus, since a was arbitrarily chosen, $N^{**} \subseteq N$ and N is closed (for $N \subseteq N^{**}$ always).

On the other hand, let N be closed. Let $a \notin N = N^{**}$. By definition, there exists $b \in N^*$ such that $a \wedge b \nleq z$. By the (upper) disjunctive property, c exists satisfying $i \neq c \geq z$, $(a \wedge b) \vee c = i$. If $d \in N$, $d = d \wedge i = d \wedge [(a \wedge b) \vee c] \leq (b \wedge d) \vee (c \wedge d) = z \vee (c \wedge d) =$ $= c \wedge d \leq c$. Consequently c is an upper bound of N. But $a \nleq c$ (for otherwise $i = c \vee (a \wedge b) = c \neq i$) and because a was any element not in N, N must contain all lower bounds of the set of its upper bounds. That is, N is normal. This completes the proof of the proposition.

Example 6. The following application is typical of the way in which the preceding results will be used later. Consider the lattice L/(0) where L is the lattice of continuous functions on a completely regular topological space S (vide example 2). This lattice is isomorphic to the sub-lattice of the Boolean algebra of regular open sets which consists of all sets of the form $\{x \in S | f(x) > 0\}^{-o}$ for some continuous f. If S is normal, it can be shown that these are precisely the regular open hulls of open F_σ sets. In the case of general spaces, these do not form a complete Boolean algebra. However these sets are dense in the regular open sets (in the sense of definition 1.2.2) and therefore [L/(0)] is isomorphic to the Boolean algebra of regular open sets. (Density follows from the complete regularity of S.) L/(0) is, of course, disjunctive. It is also disjunctive with respect to the join operation. For if R is any regular open set of S which does not coincide with the whole space, there exists a function $h \in C(S)$ satisfying $S \neq \{x | h(x) > 0\}^{-o} \supseteq R$. Then for continuous functions f and g which satisfy $f \notin g\ (0)$, picking $R = [\{x|f(x) > 0\}^{-c} \cup \{x|g(x) > 0\}^{-o}]^{-o}$, gives the dual disjunctive property. Hence by proposition 1.2.7, the Boolean algebra of regular open sets of S is isomorphic to the normal completion of L/(0).

1.2.5 The third isomorphism theorem.

The first isomorphism theorem established the characteristic connection between disjunctive homomorphisms (i.e., homomorphisms onto disjunctive lattices) and ideals; the second isomorphism theorem was concerned with iterated homorphisms onto disjunctive lattices; the third isomorphism theorem studies the behavior of sub-semi-lattices under disjunctive homomorphisms. This theorem requires much more machinery and gives far less satisfactory information than either of the first two homomorphism theorems. Nevertheless, it is an important result for the development of the remainder of the thesis.

Two preparatory lemmas are needed.

Lemma 1.2.8. Let M be a complete lattice satisfying the infinite distributive law. Let L be a sub-semi-lattice. Let J be a principal ideal of M and put $I = J \wedge L$. Then L/I is isomorphic to a sub-semi-lattice of M under the mapping $a \rightarrow \bigvee\{b \in L \mid a \geq b\ (I)\}$.

Remark. It should be emphasized that the congruence is being defined entirely within the semi-lattice L and has nothing to do with lattice M in which L is imbedded. Again, the definition is: $a \geq b\ (I) \cdot \equiv \cdot\ a \wedge c \in I$ implies $b \wedge c \in I$, for all $c \in L$. It may very well happen that $d \in M$ exists with $a \wedge d \in I$, but $b \wedge d \notin I$.

Proof of the lemma. First of all, the mapping is a join homomorphism. $a \wedge b \rightarrow \bigvee\{c \in L \mid a \wedge b \geq c\ (I)\} = \bigvee\{c_1 \wedge c_2 \mid a \geq c_1,\ b \geq c_2\ (I)\}$

$= [\bigvee\{c_1 | a \geq c_1 \ (I)\}] \wedge [\bigvee\{c_2 | b \geq c_2 \ (I)\}]$.

Next $\bigvee\{c \in L | a \geq c \ (I)\} \leq \bigvee I$ if and only if $a \in I$. For when $a \in I$, $a \geq c \ (I)$ implies $c \in I$ and hence $\bigvee\{c \in L | a \geq c \ (I)\} \leq$ $\leq \bigvee I$. On the other hand, if this last holds, $a \leq \bigvee I \leq \bigvee J$ (since $a \geq a \ (I)$) and hence $a \in I$.

In order to apply theorem 1.2.1 to the mapping defined in the lemma, it must be shown that the image of the mapping is disjunctive. To this end, suppose $\bigvee\{c \in L | a \geq c \ (I)\} \nleq \bigvee\{c \in L | b \geq c \ (I)\}$. Then there is an element c of L satisfying $a \geq c$ and $b \ngeq c \ (I)$. That is, $d \in L$ exists with $b \wedge d \in I$ and $c \wedge d \notin I$. Then $(c \wedge d) \wedge b \in I$ so $\bigvee\{e | c \wedge d \geq e \ (I)\} \nleq \bigvee I$, $[\bigvee\{e | d \wedge c \geq e \ (I)\}] \wedge$ $[\bigvee\{e | b \geq e \ (I)\}] = \bigvee\{e | c \wedge d \wedge b \geq e \ (I)\} \leq \bigvee I$; and obviously $\bigvee\{e | c \wedge d \geq e \ (I)\} \leq \bigvee\{e | a \geq e \ (I)\}$. Hence the image is disjunctive.

Apply theorem 1.2.1. The zero of the image is $\bigvee I$ and the kernel of the mapping is I. Consequently L/I is isomorphic to this image.

<u>Corollary 1.2.3.</u> With the hypotheses of the lemma, and if in addition $I = J = (z)$, z being the common zero of M and L, then the mapping of the lemma carries the zero of L/I into the zero of M.

Suppose we pose the question: when will the image of L/I under the mapping defined in the previous lemma be dense in M? Since the zero element of this image is $\bigvee I$, this can certainly never happen if $I \neq (z)$. But the case where $I = (z)$ is of sufficient importance for a later application to warrant special consideration. To answer the question necessitates the introduction of a new concept. This condition is a weakening of the disjunction property on a semi-lattice. Hence we call it the "quasi-disjunctive" property.

Definition 1.2.3. A semi-lattice L will be called quasi-disjunctive if it satisfies the condition:

for any $a \neq z$ in L, $b \leq a$ exists with $b \neq z$, and such that if $c \nleq a$, then $d \neq z$ exists satisfying $d \leq c$, $d \wedge b = z$. In other words, $b \supseteq c\ (z)$ implies $c \leq a$.

Lemma 1.2.9. Let M be a complete lattice satisfying the infinite distributive law. Let L be a sub-semi-lattice. Suppose M and L have a zero element z in common. Then, in the imbedding of L/(z), as defined in lemma 1.2.8, the image of L/(z) is dense in M if and only if

(i) L is dense in M, and

(ii) L is quasi-disjunctive.

Proof. Sufficiency. Suppose $e \neq z$ in M. Then $a \neq z$ exists in L such that $a \leq e$ (by (i)). Next $b \leq a$ exists with $b \neq z$ and such that $b \supseteq c\ (z)$ implies $c \leq a$. It follows that $z \neq \bigvee\{c \in L \mid b \supseteq c\ (z)\} \leq \leq a \leq e$. Thus the image of L/(z) is dense in M.

Necessity. If $a \neq z$ in M, then by hypothesis $b \in L$ exists satisfying $z \neq \bigvee\{c \mid b \supseteq c\ (z)\} \leq a$. Hence $z \neq b \leq a$, so L is dense in M. With the added hypothesis that a is in L, we get $c \leq a$ for all c satisfying $b \geq c$ (z). This establishes (ii).

If L is disjunctive, condition (ii) is automatically satisfied. In particular, if M is disjunctive (i.e., is a Boolean algebra) then (i) implies the disjunctivity of L which in turn implies (ii). Hence,

Corollary 1.2.4. Let M be a complete Boolean algebra and let L be a sub-semi-lattice of M. The necessary and sufficient condition that L/(z) be dense in M is that L be dense in M (and hence L is disjunctive: L/(z) = L).

Specializing the lemma to the case where L = M gives a characterization of the quasi-disjunctive condition.

Corollary 1.2.5. Let L be a complete, completely distributive lattice. Then L/(z) is dense in L if and only if L is quasi-disjunctive.

Now it is possible to prove the third isomorphism theorem.

Theorem 1.2.5. Let M be a semi-lattice and let L be a sub-semi-lattice of M. Let J be an ideal of M and put $I = J \cap L$. Then $[L/I]$ is isomorphic to a sub-semi-lattice of $[M/J]$.

Remark. The analogy between this theorem and the corresponding third isomorphism theorem of group theory is not too clear. Even putting aside the fact that, in theorem 1.2.5, we are dealing with closures of quotient lattices, the correspondence is not obvious. This point will be clarified somewhat when we later consider conditions which will insure that [L/I] is isomorphic to [M/J].

Proof. It will first be shown that L/I is isomorphic to a sub-semi-lattice of [M/J]. The proof is then completed by an application of the uniqueness theorem, 1.2.4.

Let h be the natural mapping of M onto M/J. Restriction of h to L gives a (semi-lattice) homomorphism of L into M/J. Denote the image of this mapping by $\bar{L}$. Notice that $\bar{L}$ contains the zero of M/J and in fact h(a) = z if and only if a ϵ I. By the second isomorphism theorem, $\bar{L}$/(z) is isomorphic to L/I. Now $\bar{L}$ is a sub-semi-lattice of [M/J]. Thus by lemma 1.2.8, $\bar{L}$/(z) is isomorphic to a sub-semi-lattice of [M/J]. Hence L/I is isomorphic to a sub-semi-lattice of [M/J]. Applying the uniqueness theorem, the proof is complete.

Remark: The proof shows that the zeros in [M/J] and [L/I] correspond in the mapping.

A very simple example may help clarify this theorem.

Example 7. Let M be the Borel field generated from the open sets of a metric space. Let L be the lattice of open sets of the space. Let J = I = (∅) be the zero ideal. M is disjunctive and [M] is isomorphic to the Boolean algebra of all subsets of the space. (This is an obvious application of theorem 1.2.4.) L/(∅) is the Boolean algebra of regular open sets of the space. Theorem 1.2.5 merely expresses the fact that the class of regular open sets is closed under intersections. It should be noticed that although we started in this example with distributive lattices L and M, and L was actually a sub-lattice of M, the resulting $[L/(\emptyset)]$ turned out to be only a sub-semi-lattice of $[M]$. In other words, there is apparently no improvement of the theorem for the distributive lattice case.

It is desirable to have a criterion for determining when $[L/I]$ will be isomorphic to $[M/J]$. This is furnished by the definition

Definition 1.2.4. Let L be a sub-semi-lattice of M. L is called dense in M relative to the ideal J (of M) if the image of L under the natural mapping $M \to M/J$ is dense in M/J. In other words, if $a \in M$ and $a \notin J$, then $b \in L$ exists satisfying $b \notin J$ and $a \supseteq b \quad (J)$.

Lemma 1.2.10. Let M be a semi-lattice and let L be a sub-semi-lattice of M. Let J be an ideal of M and let $I = J \frown L$. Then the mapping in theorem 1.2.6 of $[L/I]$ into $[M/J]$ is an isomorphism onto if and only if L is dense in M relative to J.

Proof. Use the notation of theorem 1.2.5. By theorem 1.2.4 and corollary 1.2.4, $[M/J]$ is isomorphic to $[L/I]$ if and only if $\bar{L}$ is dense in $[M/J]$. But since $\bar{L}$ is actually a sub-semi-lattice of M/J, this means dense in M/J. This is precisely the definition that L is dense in M relative to J.

Now it is possible to clarify the analogy between theorem 1.2.5 and the third isomorphism theorem of the theory of groups. The group theoretic theorem asserts that if N is a normal subgroup of a group

G, and if H is any subgroup of G, then $(N \cdot H)/N$ is isomorphic to $H/(N \cap H)$. For the purpose of the analogy, let G correspond to a semi-lattice M, N to an ideal I of M, H to a sub-semi-lattice L of M. Now $N \cdot H$ is the smallest sub-group of G which contains both N and H. Let $L \cup I$ denote the smallest sub-semi-lattice of M which contains both L and I. It is easy to see that $L \cup I$ is just the set union of L and I. (In the distributive case, $L \cup I$ must be the smallest sub-lattice of M which contains both L and I. For this case, $L \cup I = \{ a \vee b | a \in L, b \in I \}$.) If the analogy were perfect, we should have $(L \cup I)/I$ isomorphic to $L/(L \cap I)$. Unfortunately, the situation is not quite that nice. Instead, $[(L \cup I)/I]$ is isomorphic to $[L/(L \cap I)]$.

To prove this, it is only necessary to show that L is dense in $L \cup I$ relative to I. But this is a very special case of the situation described in definition 1.2.4. If $a \in L \cup I$ and $a \notin I$, then $a \in L$. Thus L is dense in $L \cup I$. (The distributive case is not much more difficult. If $c \in L \cup I$, then $c = a \vee b$, where $a \in L$ and $b \in I$. If $c \notin I$, $a \notin I$, while $c = a \vee b \supseteq a \not\vdash z$ (I).)

SUMMARY OF CHAPTER II

Chapter two is devoted to the study of translation lattices. A translation lattice can be thought of as a semi-lattice of functions which is closed under the operations of adding any constant to the functions.

In the first section, the elementary properties of a translation lattice are developed. Also a number of examples are introduced to show that the concept of a translation lattice is not devoid of intrinsic interest.

Section two is concerned with the extension of the results of chapter one to translation lattices. An anologue for the definition of a disjunctive lattice is found. We also define a congruence relation, using the ideas and theorems of section two of chapter one, which resembles, in many of its properties, the congruence of chapter one. Finally, it is shown that theorems analogous to the first and second isomorphism theorems can be proved for this congruence relation.

In section three, the problem of representing a translation lattice is studied. With the eventual objective of representing translation lattices by means of lattices of continuous functions, an object called a normal lattice function is defined. A normal lattice function is nothing other than a bounded, real-valued function on a complete Boolean algebra with the property that it is a (dual) join homomorphism of the Boolean algebra onto the real numbers. It is shown that certain translation lattices can be represented in a natural way as translation lattices of normal lattice functions. The section ends with a detailed discussion of the unicity of this representation.

CHAPTER II — TRANSLATION LATTICES

2.1 Preliminaries.

The aim of the first section of this chapter is to acquaint the reader with the concept of a translation lattice. Consequently, it will be devoted largely to examples and discussion of the postulates. In the study of function lattices, we are faced with the difficulty of not having enough algebraic structural features. It is natural to look around for some other algebraic feature possessed by large numbers of function classes. Thus one is led to the study of lattice ordered rings, lattice ordered vector spaces, lattice groups and so forth. Kaplansky has introduced another structure of this nature (see [8]). He calls his algebraic system a translation lattice. Intuitively, we may think of a translation lattice as a function lattice which is closed under the operation of adding any real number. Thus, with any function f in the lattice and any real number α is associated the function $f + \alpha$ defined by $(f + \alpha)(x) = f(x) + \alpha$. To justify the consideration of this algebraic system, it will be necessary to exhibit a large number of significant examples of translation lattices of functions. Moreover, it is desirable that some of these examples be function classes which are not amenable to techniques developed in the study of lattice rings, lattice groups and the like. In other words, we want to find translation lattices which are not also groups. It is remarkable that such systems exist in in considerable abundance, and their existence forms a partial justification for the present study.

2.1.1 Definition of a translation lattice.

Definition 2.1.1. An algebraic system L is called a translation lattice if it satisfies the following conditions:

(1) L is a semi-lattice (with the operation written as meet),

(2) Corresponding to each real number α, there is an automorphism T_α of L (called a translation) such that

(a) $T_0 f = f$;

(b) $T_\alpha (T_\beta f) = T_{\alpha+\beta} f$;

(c) if $\alpha > 0$, $T_\alpha f > f$;

(d) if $f \geq g$, $T_\alpha f \geq T_\alpha g$ for all α;

(e) for any f and g in L, an α exists such that $T_\alpha f \geq g$;

(f) $f \leq T_\alpha g$ for all $\alpha > 0$ implies $f \leq g$.

This definition differs from the one of Kaplansky only in the assumption (1). Kaplansky assumes that L is a distributive lattice. However, for the use which we will make of translation lattices, it is more natural to assume only that the system forms a semi-lattice. When it is desirable to add the assumption that the system also forms a distributive lattice, the term distributive translation lattice will be used.

The strict inequality sign in postulate (c) is to be interpreted as follows: $f > g$ means that $f \geq g$ and $f \neq g$. This excludes the possibility of trivial systems containing only a single element.

These postulates need little discussion. The conditions (a) - (c) express the fact that the set $\{T_\alpha\}$ forms an ordered group of automorphisms which is isomorphic to the real numbers. Postulate (d) merely expresses the fact that the T are homomorphisms of L. (f) asserts that for any g, the mapping $\alpha \to T_\alpha g$ is continuous on the reals into L. Finally, (e) just says that for any f, the set of $T_\alpha f$ is cofinal in L.

All of these postulates seem quite natural except possibly (e). In terms of function lattices, this restriction means that only lattices of bounded functions are being considered. While it would be desirable to avoid this postulate, it is very doubtful whether one could expect to get as many interesting results as can be obtained for bounded function lattices.

Another possible direction of generalization is the substitution of another simply ordered group in place of the real numbers as the index group of the automorphisms T_α. However, in order to obtain interesting theorems, it is necessary to assume some kind of completeness properties for this group. But with the added assumption of (conditional) completeness, the only possible simply ordered group other than the real number system is the additive group of integers. It would be interesting to try to carry out a study of the system obtained when definition 2.1 is altered with the replacement of the real number system by the integers, but this will not be done.

Before proceeding with our study, it is convenient to introduce a change of notation. In the definition 2.1.1, the notation $T_\alpha f$ was used to denote the result of the application of the automorphism T_α to the element f. This notation is different from that of Kaplansky. In his paper [8], Kaplansky writes $f + \alpha$ instead of $T_\alpha f$. This notation is evidently justified and preferable once it is realized that when we speak of a translation lattice, a function (semi-) lattice is always what we have in mind. Indeed, it will be shown later that every abstract algebraic system satisfying postulates (1) and (2) of definition 2.1.1 is isomorphic to a translation lattice of functions by an isomorphism which makes the operation T_α correspond to the addition of the constant real number α. In the remainder of the thesis, the notation of Kaplansky will be used. Instead of $T_\alpha f$, we will always write $f + \alpha$.

It is an important consequence of this notational convention that the following formula is valid for any translation lattice:

$$(f \wedge g) + \alpha = (f + \alpha) \wedge (g + \alpha).$$

The equality is seen immediately when it is recalled that the translation T_α is an automorphism of the translation lattice. Because of the elementary nature of this formula, its use in the sequal will be unattended by any special reference.

2.1.2 Examples.

Example 1. As a first example, consider the set C(S) of all bounded, real-valued, continuous functions on a completely regular topological space S. This set is, of course, a translation lattice with the usual operations taken pointwise. Of course at the same time, it is also a ring, a vector space, and so forth.

Example 2. For the second example, take the set of all bounded, lower semi-continuous functions on the completely regular space S. With respect to the usual meet and addition of a constant, this is a translation lattice. In fact it is a distributive translation lattice. We denote it by L(S).

Example 3. It is possible to treat the set L(S) defined above as a translation lattice with respect to its join operation. However, it is more convenient to study the equivalent system of all bounded (real valued) upper semi-continuous functions as a translation lattice with respect to its meet operation. This system will be denoted U(S). It is a remarkable fact that, as will be shown later, the theory of U(S) differs essentially from that of L(S).

Example 4. All of the above examples are distributive function lattices. That is, they are closed under pointwise join of elements. For an example where this is not the case, we may take the lattice of (bounded) normal lower semi-continuous functions on a completely regular topological space S. Denote this function lattice by N(S). N(S) is actually a (conditionally) complete, completely distributive lattice, but only the meet operation is taken pointwise. Dilworth [1] has proved that N(S) is isomorphic to $C(\gamma)$ where γ is the Boolean space associated with the Boolean algebra of regular open sets of S.

Example 5. A very natural example of a translation lattice is furnished by the bounded Lebesgue measurable functions on a space of finite total measure. This system, of course, also forms a vector space, but for many purposes it is more convenient to study it from the viewpoint of its lattice properties.

Example 6. It is not hard to find examples of translation lattices which are not groups. An interesting one is the set of all non-decreasing functions on the real line interval $[0,1]$.

Example 7. An example of a non-distributive translation lattice is furnished by the set of all concave functions on the real interval. This set forms a complete (non-distributive) lattice with meets taken pointwise. It is of course also a translation lattice. As a generalization of this example, one can consider the set of super-harmonic functions in Euclidean space.

Example 8. For an example of a translation lattice which is of a somewhat different nature, consider, in a Banach space, the set of all convex open sets which contain the origin. Translation for this system is defined as magnification by 2^{α} while the meet operation is just the ordinary set intersection.

2.1.3 Metric properties.

Every translation lattice has a natural metric topology defined by

$$\rho(f,g) = \inf\{\lambda \mid f - \lambda \leq g \leq f + \lambda\} .$$

Proposition 2.1.1. If L is a translation lattice, then the function ρ defined above is a distance function on L.

Proof. Clearly $\rho(f,g) = \rho(g,f)$ and $\rho(f,f) = 0$. If $\rho(f,g) = 0$, then $f = g$ is a consequence of (f) in definition 2.1.1. To complete the proof, the triangular inequality must be established. Let $f, g, h \in L$. Pick any $\delta > 0$. Real numbers λ and μ exist so that $f - \lambda \leq g \leq f + \lambda$, $g - \mu \leq h \leq g + \mu$ and $\rho(f,g) > \lambda - \delta$, $\rho(g,h) > \mu - \delta$. Then $f - (\lambda + \mu) \leq g - \mu \leq$ $\leq h \leq g + \mu \leq f + (\lambda + \mu)$, so $\rho(f,h) \leq \lambda + \mu < \rho(f,g) + \rho(g,h) +$ 2δ . Because δ was arbitrarily chosen, $\rho(f,h) \leq \rho(f,g) + \rho(g,h)$.

While the metric topology defined by ρ is not the only possible topology which can be imposed on a translation lattice (another is the interval topology defined in terms of the partial ordering of L), examples show that the topology of ρ is one of the most important. For a translation lattice of functions, convergence in the metric is just uniform convergence. Without explicit mention, it will always be assumed in the future that a translation lattice is topologized by its metric topology as defined in lemma 2.1.1. Thus a subset of a translation lattice will be called closed only if it is closed with respect to the topology of ρ . In particular, if an ideal I is closed, then whenever $f \notin I$, there is a $\delta > 0$ such that $f - \delta \notin I$.

<u>Proposition 2.1.2.</u> If L is a translation lattice, then with the topology defined by ρ, L is a topological algebra. That is, the operations of translation and join are continuous.

<u>Proof.</u> Suppose $f_n \to f$ and $g_n \to g$ as $n \to \infty$. Let $\delta > 0$ be arbitrarily small. Choose N_δ so large that $f - \delta \leq f_m \leq f + \delta$ and $g - \delta \leq g_n \leq g + \delta$ whenever $m > N_\delta$ and $n > N_\delta$. Then $(f \wedge g) - \delta \leq f_m \wedge g_n \leq (f \wedge g) + \delta$. Therefore $f_m \wedge g_n \to f \wedge g$ as $m, n \to \infty$. In case L is a distributive lattice, the same argument shows that $f_m \vee g_n \to f \vee g$. The fact that translation is a continuous operation is easily seen: if $f_n \to f$ and $\alpha_m \to \alpha$, then $f_n + \alpha_m \to f + \alpha$ as n and m go to ∞.

In a translation lattice L, since L is a metric space, it is meaningful to speak of completeness in the sense that every Cauchy sequence in L has a limit in L. A fundamental result is

<u>Proposition 2.1.3.</u> Let L be a translation lattice. Then there exists a <u>unique</u> complete translation lattice $\bar{L}$ of which L is isomorphic and isometric to a dense sub-translation lattice.

<u>Proof.</u> Let the points of $\bar{L}$ be just the points of the unique complete metric space of which L is a dense sub-space, (see Hausdorff [9] p. 106). The operations in $\bar{L}$ can be defined in the usual way: if $f_n \to \bar{f}$ and $g_n \to \bar{g}$ where $f_n, g_n \in L$ and $\bar{f}, \bar{g} \in \bar{L}$, define $\bar{f} \wedge \bar{g} = \lim f_n \wedge g_n$; also define $\bar{f} + \alpha = \lim f_n + \alpha$. Using the previous lemma, it is easily verified that these limits exist and are

independent of the choice of the defining sequence. At the same time (using the usual techniques) all of the identities which are the postulates for a translation lattice can be established. Finally, it must be shown that L is a sub-translation lattice of $\overline{L}$. All of the necessary manipulations are so familiar that they can be safely omitted.

2.2 Homomorphisms of Translation Lattices.

Since every translation lattice is a semi-lattice, one can naturally expect the theory developed in 1.2 to have an extension to the theory of translation lattices. The direct application of the results of 1.2 does not make full use of the potentialities of the techniques which have been developed. In the first place, a translation lattice never has a zero element and therefore can never be disjunctive. In the second place (and this is not unrelated to the first difficulty) the natural lattice homomorphism defined in 1.2 preserves the translation operations of a lattice only in special cases. Thus it will often happen that $f \sim g$ (I) while, for some $\lambda \neq 0, f - \lambda \not\sim g - \lambda$ (I). Fortunately, there is a natural way to avoid these difficulties. The first part of this section will be devoted to the definition of a semi-lattice homomorphism which also preserves the translations.

The idea behind this homomorphism can be explained rather simply. As before, homomorphisms are constructed out of equivalence relations. The equivalences are defined by ideals of the lattice.

Let L be a translation lattice and let I be an arbitrary ideal of L. With respect to this ideal, a congruence relation is defined in the following manner: f and g in L are called equivalent if $f - \lambda \sim g - \lambda (I)$ for all real λ. The equivalence $\sim$ is just the one defined in lemma 1.3.1. It is immediately clear that this device gives a translation congruence: if f is equivalent to g, then for any real $f - \lambda$ is equivalent to $g - \lambda$. Unfortunately, there are complications which necessitate restrictions on the ideal I. These problems will be dealt with when they arise. Of course, not every homomorphism of a translation lattice will be of this form. However, the development below shows that a remarkably large number of translation lattice homomorphisms are of this nature.

2.2.1 Divisible translation lattices.

In this article, we will define the analogue of the disjunctive property for translation lattices. The above discussion makes the following plausible.

Definition 2.2.1. Let L be a translation lattice. Let I be an ideal of L. Then L will be called divisible with respect to I if, whenever $f \not\geq g$ in L, there exists a real λ such that $f - \lambda \not\geq g - \lambda (I)$. In other words, $h \in L$ exists satisfying $(f - \lambda) \wedge h \in I$ and $(g - \lambda) \wedge h \notin I$.

The property of being divisible with respect to some ideal is an important one — one which merits rather close consideration. At the same time, it is a rather general property for translation lattices.

Lemma 2.2.1. Let L be a lattice ordered group of functions which contains all constant functions (operations taken pointwise, of course). Then L is divisible with respect to any principal ideal.

Proof. First notice that it is sufficient to prove the lemma for the case of the principal ideal (0), since for any fixed function k, the mapping $f \to f - k$ is a translation lattice automorphism of L which carries k into the zero function 0. Then from the validity of the lemma for the ideal (0) follows its validity for the ideal (k).

Now $f \not\geq g$ implies that there is an x such that $f(x) < g(x)$. Choose a $\delta > 0$ so that $f(x) + \delta < g(x)$. Take $h = -f + f(x) + \delta$, $\lambda = f(x) + \delta$. Evidently $(f - \lambda) \wedge h = (f - f(x) - \delta) \wedge$ $\wedge (-f + f(x) + \delta) \leq 0$. On the other hand, $[(g - \lambda) \wedge h](x) =$ $= [g(x) - f(x) - \delta] \wedge \delta > 0$, so $(g - \lambda) \wedge h \not\leq 0$. Thus the hypothesis of definition 2.2.1 is fulfilled.

As a corollary of this lemma, it follows that the translation lattices described in examples 1, 4 and 5 of the previous section are all divisible with respect to their principal ideals. The other examples require specific consideration.

Example 9. The set U(S) of upper semi-continuous functions on a topological space S is divisible with respect to any principal ideal (k).

In fact, suppose $f \not\geq g$. This means that $x \in S$ exists so that

$f(x) < g(x)$. Choose λ to satisfy $f(x) - \lambda < k(x) < g(x) - \lambda$. Let $h \in U(S)$ be defined by $h(y) = k(y)$ for $y \neq x$, $h(x) = g(x) - \lambda$. Then $(f - \lambda) \wedge h \leq k$ is clear. Since $[(g - \lambda) \wedge h](x) = g(x) - \lambda > k(x)$, it follows that $(g - \lambda) \wedge h \nleq k$.

Example 10. In contrast to the behavior of U(S), the translation lattice L(S) defined in example 2 is not divisible with respect to any of its closed ideals, provided certain restrictions are put on the space S. It is enough, for example, to assume that S is a compact metric space which is dense in itself. We will not prove this fact, but instead will be content to show that if S is not discrete, L(S) is not divisible with respect to the principal ideal (0).

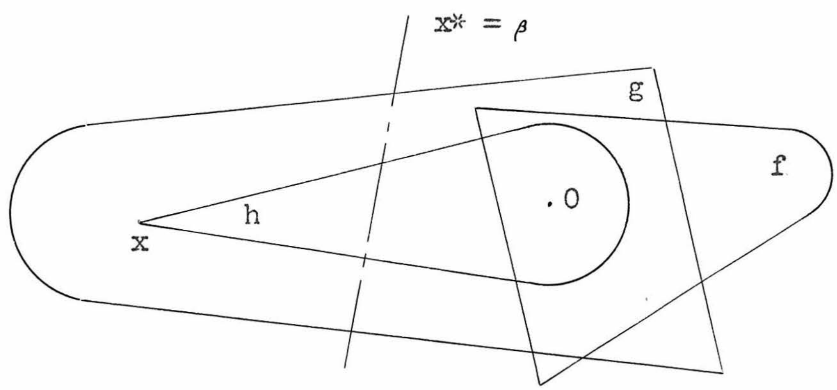
To make this proof, it is sufficient to exhibit functions f and g in L(S) which are such that $f \nleq g$, while $(f - \lambda) \wedge h \leq 0$ and $(g - \lambda) \wedge h \nleq 0$ can never be simultaneously true when λ is a real number and $h \in L(S)$. For g, choose the function which has the constant value zero on S. Let f be the function which is zero except at a single non-isolated point x. Let $f(x) = -1$. Clearly $f \nleq g$. If $h \in L(S)$ and λ are such that $(g - \lambda) \wedge h \nleq 0$, then necessarily $\lambda > 0$. If also $(f - \lambda) \wedge h \leq 0$, $\lambda > 0$ would require that $h(y) \leq 0$ whenever $y \neq x$. Then, because h has to be lower semi-continuous, and because x is not isolated, this would imply $h(x) \leq 0$. In other words, $h \leq 0$. Then $(g - \lambda) \wedge h \leq 0$, contrary to the hypothesis.

While L(S) is not divisible with respect to any closed ideal, it still contains rather simple ideals with respect to which it is divisible. For example if S is a metric space, then L(S) is divisible with respect to the ideal $I = \{ f \mid f(x) < 0 \text{ all } x \}$. For suppose $f(x) < g(x)$. Choose $\lambda = g(x)$. Define h by the formula $h(y) = \max \{ -1, -\rho(x,y) \}$, where ρ is the distance function on S. Then $h(y) < 0$ if $y \neq x$ and $h(x) = 0$. With these choices $(f - \lambda) \wedge h \in I$ and $(g - \lambda) \wedge h \notin I$.

The consideration of this example raises the following question: is it possible to find examples of distributive translation lattices which are not divisible with respect to any of their lattice ideals? This has been answered affirmatively. The example, however, is extremely special, and the proof that it is not divisible with respect to any ideal is rather tedious. For this reason, no attempt will be made to reproduce it here.

Example 11. The translation lattice of example 6 has the interesting property that it is not divisible with respect to the principal ideal (0), while it is divisible with respect to the principal ideal (k), k being the function defined by $k(x) = x$. The proof of these facts are left for the reader to supply.

Example 12. Consider the translation lattice L (defined in example 8) whose elements are the convex open sets containing the origin in a Banach space B. (For a reference on Banach spaces, see [10].) We will show that this lattice is divisible with respect to the principal ideal generated by the unit sphere. Suppose f and g are elements of L with $f \not\geq g$. Since f and g are convex open sets, it follows that they are regular open sets and thus $f^{-} \not\geq g$. In other words $x \in B$ exists with $x \in g$, $x \notin f^{-}$. Define h in L to be the set of all z of the form $z = \lambda \cdot x + (1 - \lambda)y$ where $0 < \lambda < 1$ and $\|y\| < \delta$, δ being a small positive constant which will be determined later. Clearly $h \leq g$ (provided δ is sufficiently small). It will be shown that there exists a sphere with center at the origin which contains $h \wedge f$ but not h. From this, the desired result follows (after appropriate magnification).



By the separation theorem for normed linear spaces, a continuous linear functional x^* exists satisfying $x^*(z) \leq \beta < \alpha = x^*(x)$ for all $z \in f^{-}$ (see Tuckey [11]). Thus if $z \in f \wedge h$, we can write $z = \lambda \cdot x + (1 - \lambda)y$ where $0 < \lambda < 1$ and $\|y\| < \delta$. Then $\beta \geq x^*(z) = \lambda \cdot x^*(x) + (1 - \lambda)x^*(y) \geq \lambda\alpha - (1 - \lambda)\|x^*\| \cdot \|y\| >$ $> \lambda\alpha - (1 - \lambda)\|x^*\|\delta$. Consequently $\lambda < \dfrac{\beta + \|x^*\| \cdot \delta}{\alpha + \|x^*\| \cdot \delta}$ and thus $\|z\| \leq \lambda\|x\| + (1 - \lambda)\|y\| \leq \left[\dfrac{\beta + \|x^*\| \cdot \delta}{\alpha + \|x^*\| \cdot \delta}\right]\|x\| + \delta$. If δ is sufficiently small, then $\|z\| \leq \sigma < \|x\|$ for all $z \in h \wedge f$

(where σ is a fixed constant). Thus the (closed) sphere of radius σ about the origin contains $h \wedge f$ but not h. The proof is then completed by a magnification of magnitude $1/\sigma$.

Example 13. The simplest possible translation lattice is the set of real numbers in their natural ordering. For this system, the ideals are of the form $I_\beta = \{\alpha \mid \alpha \leq \beta\}$ and $I'_\beta = \{\alpha \mid \alpha < \beta\}$. One easily verifies that the real numbers are divisible with respect to any of these ideals. Conversely, if a distributive translation lattice is divisible with respect to all of its ideals, then it is isomorphic to the real numbers. In fact, we will prove later that a distributive translation lattice is divisible with respect to a prime ideal only if it is isomorphic to the lattice of real numbers.

2.2.2. The isomorphism theorems.

The main result of the previous chapter was the first isomorphism theorem -- theorem 1.2.1. We will now proceed to develop the analogue of this theorem for translation lattices. The central idea (which was presented in the introduction to this section) will be developed in detail, following article 1.2.2 as a pattern.

Lemma 2.2.2. Let L be a translation lattice and let I be an ideal of L. Define: $f \approx g$ (I) if $f - \lambda \sim g - \lambda$ (I) for all real λ. Then $\approx$ is a congruence relation (preserving meets and translations) on L. If L is distributive and I is a lattice ideal, $\approx$ also preserves joins.

Proof. The fact that $\approx$ is an equivalence relation is an immediate consequence of the fact that $\sim$ is an equivalence. If $f \approx g$ (I) and h is any element of L, then $f \wedge h - \lambda = (f - \lambda) \wedge (h - \lambda) \sim (g - \lambda) \wedge (h - \lambda) = g \wedge h - \lambda (I)$ for any λ. Consequently $f \wedge h \approx g \wedge h$ (I). Similarly if L is distributive and I is a lattice ideal, $f \vee h \approx g \vee h$ (I). To complete the proof, it must be shown that $f \approx g$ implies $f + \alpha \approx g + \alpha$ for any real α. But this fact is an immediate consequence of the definition, since $(f + \alpha) - \lambda \sim (g + \alpha) - \lambda$ (I) is equivalent to $f - (\lambda - \alpha) \sim g - (\lambda - \alpha)(I)$ for any λ.

As usual, the elements of a translation lattice L which are congruent by the equivalence $\approx$ can be identified to obtain a new semi-lattice which will be denoted L//I.

The semi-lattice L//I need not be a translation lattice. Indeed, if I = L, then L//I contains only a single element. Nevertheless, it is always possible to define an operation in L//I which has most of the properties of translation. The procedure is as follows: denote the congruence class of the element f by $\bar{f}$; then define, for $\bar{f} \in L//I$ and α real, $\bar{f} + \alpha = \overline{f + \alpha}$. By the preceding lemma, this is a valid definition -- it is independent of the choice of the representative f in the congruence class $\bar{f}$. It is also easy to verify that the postulates (a), (b), (d) and (e) in definition 2.1.1 are satisfied. The other two postulates (c) and (f) may not always be satisfied.

Lemma 2.2.3. Let L be a translation lattice. Let I be a non-trivial ideal of L (neither empty nor all of L) which is a closed subset of L in the metric topology. Then L//I is a translation lattice and the natural mapping $f \to \bar{f}$ of the elements of L onto their congruence classes is a translation lattice homomorphism.

Proof. Define $\bar{f} + \alpha = \overline{f + \alpha}$ as above. Clearly the mapping $f \to \bar{f}$ is a homomorphism. It is only necessary to show that the postulates (c) and (f) of definition 2.1.1 are satisfied.

Postulate (c) states that if $\alpha > 0$, then $\bar{f} + \alpha > \bar{f}$. It is clear at any rate that $\bar{f} + \alpha \geq \bar{f}$. Suppose that for some $\alpha > 0$, $\bar{f} + \alpha = \bar{f}$. By induction $\bar{f} + n\alpha = \bar{f}$ for any integer n. Then also $\bar{f} = (\bar{f} + n\alpha) - n\alpha = \bar{f} - n\alpha$. Since I is non-trivial, postulate (e) of definition 2.1.1 shows that it is possible to pick n large enough so that $f - n\alpha \in I$ and $f + n\alpha \notin I$. Then $f + n\alpha \not\leq f - n\alpha$ (I) and consequently $f + n\alpha \not\equiv f - n\alpha$ (I). But this contradicts $\bar{f} + n\alpha = \bar{f} - n\alpha$. Thus $\bar{f} + \alpha = \bar{f}$ is impossible if $\alpha > 0$.

Postulate (f) states that if $\bar{f} \leq \bar{g} + \alpha$ for all $\alpha > 0$, then $\bar{f} \leq \bar{g}$. Let us prove this. Suppose $\bar{f} \leq \bar{g} + \alpha$ for all $\alpha > 0$. This means (referring back to the definition) that for $\alpha > 0$ $[(g + \alpha) - \lambda] \wedge h \in I$ implies $(f - \lambda) \wedge h \in I$. Suppose that $(g - \lambda) \wedge h \in I$. Then for any $\alpha > 0$, $(g - \lambda) \wedge (h - \alpha) \in I$. Thus $[(g + \alpha) - (\lambda + \alpha)] \wedge (h - \alpha) \in I$, so by the hypothesis

$[f - (\lambda + \alpha)] \wedge (h - \alpha) \in I$. Rearranging this gives $[(f - \lambda) \wedge h] - \alpha \in I$ and since α was any positive number $(f - \lambda) \wedge h \in I^{-} = I$. Summarizing, we have shown that for any λ and $h \in L$: $(g - \lambda) \wedge h \in I$ implies $(f - \lambda) \wedge h \in I$. In short $\bar{f} \leq \bar{g}$. This completes the proof.

The question now arises what characterizes the natural mappings of a translation lattice L onto its quotient L//I? The following lemma gives some conditions which are certainly satisfied by these mappings.

Lemma 2.2.4. Let L be a translation lattice and let I be a closed non-trivial ideal of L. Denote by H the natural homomorphism of L onto L//I. Then

(a) $H^{-1}(H(I)) = I$.

(b) H(I) is a closed ideal of L//I (in its natural metric topology).

(c) L//I is divisible with respect to H(I).

Proof. (a): It is clear that $I \subseteq H^{-1}(H(I))$. Suppose $g \in H^{-1}(H(I))$. Then $H(g) = H(f)$ for some $f \in I$. Going back to the definition in lemma 2.2.2 this implies in particular that $f \sim g$ (I). Since $f \in I$, this means $g \in I$ (by lemma 1.2.3). Consequently, $H^{-1}(H(I)) \subseteq I$ and (a) is proved.

(b): It is very easy to check that H(I) is an ideal of L//I and the computation will be omitted. We must show that H(I) is a closed subset of L//I. Suppose that f_n is a sequence of elements of I such that $H(f_n)$ converges to H(f). In other words, if $\delta > 0$, $H(f) - \delta \leq H(f_n) \leq H(f) + \delta$ provided n is sufficiently large. Then using assertion (a) of this lemma (proved above) it follows from $H(f - \delta) = H(f) - \delta \leq H(f_n)$ that $f - \delta \in I$. Since δ can be arbitrarily small, $f \in I^- = I$. Thus $H(f) \in H(I)$.

(c): Suppose $H(f) \not\geq H(g)$. This means that $f \wedge g \not\geq g$ (I). Hence λ exists such that $(f \wedge g) - \lambda \not\geq g - \lambda$ (I). By lemma 1.2.1 this implies $(f - \lambda) \not\geq (g - \lambda)$ (I). Thus $h \in L$ exists satisfying $(f - \lambda) \wedge h \in I$ and $(g - \lambda) \wedge h \notin I$. Using (a) above, this tells us that $H[(f - \lambda) \wedge h] \in H(I)$ and $H[(g - \lambda) \wedge h] \notin H(I)$. But H is a homomorphism so $(H(f) - \lambda) \wedge H(h) \in H(I)$ while $(H(g) - \lambda) \wedge$ $\wedge H(h) \notin H(I)$. This is the condition in definition 2.2.1 for divisibility.

<u>Remark 1.</u> It should be noticed that the properties (a) and (c) and the fact that H(I) is an ideal do not depend on the initial assumption that I is a closed ideal of L. In the case that it is known that L//I is a translation lattice, these properties will still obtain.

<u>Remark 2.</u> Another property which is a characteristic of H is that it is a continuous mapping when L and L//I are considered as topological spaces with their natural metric topologies. However, this fact is an immediate consequence of the fact that H is a homo-

morphism, as the reader may verify.

The properties proved in the previous lemma characterize the natural homomorphism H in the following sense:

Lemma 2.2.5. Let L be a translation lattice. Let $\overline{H}$ be a (translation lattice) homomorphism of L onto the translation lattice $\overline{L}$. Suppose moreover that $\overline{L}$ is divisible with respect to one of its ideals $\overline{I}$. Let $I = \overline{H}^{-1}(\overline{I})$. Then there is an isomorphism M of L//I onto $\overline{L}$ such that if H is the natural homomorphism of L onto L//I, then for any $f \in L$, $\overline{H}(f) = M(H(f))$.

Proof. We will first show that for two elements f and g of L $H(f) = H(g)$ if and only if $\overline{H}(f) = \overline{H}(g)$. Now $H(f) = H(g)$ means $f \approx g \quad (I)$, or $f - \lambda \sim g - \lambda \quad (I)$ for all λ. In other words, this says $(f - \lambda) \wedge h \in I$ if and only if $(g - \lambda) \wedge h \in I$. By the definition of I this is equivalent to: $(\overline{H}(f) - \lambda) \wedge \overline{H}(h) \in \overline{I}$ if and only if $(\overline{H}(g) - \lambda) \wedge \overline{H}(h) \in \overline{I}$. But $\overline{L} = \overline{H}(L)$ is divisible with respect to $\overline{I}$ so this is possible if and only if $\overline{H}(f) = \overline{H}(g)$.

Now M is defined as that mapping which carries H(f) of L//I into $\overline{H}(f)$ in $\overline{L}$. Then M is a uniquely defined, one-to-one correspondence between L//I and $\overline{L}$. Moreover $M(H(f) - \lambda) = M(H(f - \lambda)) = \overline{H}(f - \lambda) = \overline{H}(f) - \lambda = M(H(f)) - \lambda$. Similarly $M(H(f) \wedge H(g)) = M(H(f)) \wedge M(H(g))$ (and $M(H(f) \vee H(g)) = M(H(f)) \vee M(H(g))$ provided L is distributive and $\overline{H}$ is a distributive translation lattice homomorphism). Thus M is an isomorphism which has the properties described in the lemma. The proof is complete.

Remark. For the proof of this lemma it was not necessary to assume that $\bar{I}$ was closed. Of course if $\bar{I}$ happened to be closed in $\bar{L}$, then so would I be closed in L.

An immediate and useful corollary of this lemma is the following:

Corollary 2.2.1. A necessary and sufficient condition that a translation lattice L be divisible with respect to an ideal I is that L//I be isomorphic to L by the natural homomorphism.

Proof. The necessity is a consequence of the lemma where $\bar{H}$ is taken to be the identity mapping. The sufficiency follows from lemma 2.2.4 when account is taken of the remark 1 following that lemma.

Before proceeding, let us summarize the results of the past few lemmas in the first isomorphism theorem.

Theorem 2.2.1. Let L be a translation lattice. Then there is a many-to-one correspondence between the closed, non-trivial ideals of L and the homomorphisms of L onto translation lattices which are divisible with respect to some closed ideal. Any closed, non-trivial ideal I determines a natural homomorphism H of L onto the translation lattice L//I which is divisible with respect to the closed ideal H(I). A second ideal J of L determines the same homomorphism as I if and only if L//I is also divisible with respect to H(J). Conversely, any homomorphism of L onto a translation lattice which is divisible with respect to one of its closed ideals is algebraically equivalent to the natural homomorphism of L onto L//I where I is a certain closed ideal of L.

Remark. It should be emphasized again the the terminology translation lattice above envisages only a semi-lattice. The ideals are only semi-lattice ideals. Moreover, the homomorphisms are assumed to preserve translations and meets, but not necessarily joins (where they exist). On the other hand, the above theorem is true if a translation lattice is interpreted as a distributive translation lattice, provided that ideals are meant lattice ideals; in this case the homomorphisms are true lattice homomorphisms.

Example 14. Let S be a completely regular topological space. Define a mapping of L(S) -- the lower semi-continuous functions on S -- onto N(S) -- the normal lower semi-continuous functions on S -- by letting $f \to (f^*)_*$ (where $(f^*)_*(x) = \liminf_{y \to x} (\limsup_{z \to y} f(z))$ -- see Dilworth [1]). It is easily shown that this mapping preserves meets and translations. By lemma 2.2.1, N(S) is divisible with respect to the closed ideal $\{f \in N(S) | f \leq 0\}$, and if f is lower semi-continuous, it is easy to see that $(f^*)_* \leq 0$ if and only if $f \leq 0$. Thus, denoting the ideal $\{f \in L(S) | f \leq 0\}$ by (0), theorem 2.2.1 shows that N(S) is isomorphic to L(S)//(0).

It is convenient to formulate and prove the analogue of the second isomorphism theorem now.

Theorem 2.2.2. Let $\bar{H}$ be a homomorphism of the translation lattice L onto $\bar{L}$ (also a translation lattice). Let $\bar{I}$ be a closed, non-trivial ideal of $\bar{L}$. Define $I = \bar{H}^{-1}(\bar{I})$. Then I is a closed, non-trivial ideal of L and L//I is isomorphic to $\bar{L}//\bar{I}$.

Proof. The proof that I is a closed non-trivial ideal is just a straightforward computation and will be omitted.

We must prove L//I isomorphic to $\bar{L}//\bar{I}$. Denote by H the natural homomorphism of $\bar{L}$ onto $\bar{L}//\bar{I}$. Map L onto $\bar{L}//\bar{I}$ by the homomorphism $\bar{H}(H(f))$. The inverse image of $H(\bar{I})$ under this mapping is precisely I (by lemma 2.2.4). Since $\bar{L}//\bar{I}$ is divisible with respect to $H(\bar{I})$, it follows from lemma 2.2.5 that L//I is isomorphic to $\bar{L}//\bar{I}$. The proof is complete.

2.2.3 Prime ideals.

In this section, the homomorphisms defined with respect to a special class of ideals will be studied in some detail. Out of this study will come representation theorems for abstract translation lattices.

Definition 2.2.2. Let L be a semi-lattice. An ideal I of L will be called prime if its set complement in L is a lattice dual ideal. In other words, if $f \notin I$ and $g \notin I$, then $f \wedge g \notin I$.

Notice that if L is a distributive lattice and I is a lattice ideal, this is just the usual definition of a prime ideal.

Definition 2.2.2 makes it clear that prime ideals can always be found in a semi-lattice. For example, if $f \in L$, then $\{g \in L \mid g \not\geq f\}$ is a prime ideal. On the other hand, the existence of prime lattice ideals is not so easy to prove and in general the proof requires the use of transfinite methods.

Now the work of the previous article will be specialized to the case where I is a prime ideal. The key lemma is the following:

Lemma 2.2.6. Let L be a translation lattice. Let I be a closed ideal of L. A necessary and sufficient condition that I be prime is that L//I be isomorphic to the translation lattice of real numbers.

Proof. First suppose that I is prime. Define the mapping H: $f \to \inf \{ \lambda / f - \lambda \in I \}$. This clearly preserves translations. Also $H(f \wedge g) = \inf \{ \lambda \mid f \wedge g - \lambda \in I\} = \inf \{ \lambda \mid (f - \lambda) \wedge \wedge (g - \lambda) \in I \}$. Since I is prime, this equals $\inf \{ \lambda \mid (f - \lambda) \in \in I$ or $(g - \lambda) \in I \} = \min \{ H(f), H(g) \}$. Similarly, if I is a lattice ideal, one easily proves that $H(f \vee g) = \max \{ H(f), H(g) \}$. Now the inverse image under H of the ideal (0) consisting of all non-positive numbers is precisely $I^{-} = I$. Thus by theorem 2.2.1, L//I is isomorphic to the real numbers.

Conversely, if L//I is isomorphic to the real numbers, the natural homomorphism H carries I onto a closed, non-trivial ideal of the real numbers and $H^{-1}(H(I)) = I$. But every ideal of the real number system is prime (see example 13) and the inverse image of a prime ideal is clearly prime. Hence, I is prime.

Remark. The homomorphism which was set up in the first part of the above proof depended in no way on the closed property of the ideal I.

Moreover, the construction used there to define H (i.e., $H(f) = \inf \{ \lambda \mid f - \lambda \in I \}$) is more convenient than the construction of the natural homomorphism of L onto L//I. Hence, for this article, we adopt this formula for the definition of the natural homomorphism with respect to a prime ideal.

The homomorphisms of a translation lattice onto the real numbers are of sufficient importance to deserve a name. Following the terminology of group theory, we will call them characters of the translation lattice. It follows from the above lemma that the characters of a translation lattice are intimately connected with the ideals of the lattice.

It is instructive to look at translation lattices of functions in order to get a better idea of the meaning of the concepts of prime ideal and character. Let L be any translation lattice of functions (i.e. closed under pointwise meets and addition of constants) on a set S. Let x be any point of S. Then the mapping $f \to f(x)$ is, by the very meaning of the concept "translation lattice of functions", a character of the lattice. Associated with this character is the closed prime ideal $I = \{f \mid f(x) \leq 0\}$. It is an easy matter to show that the mapping $f \to f(x)$ is precisely the natural mapping of L onto L//I. Conversely, if S is a compact Hausdorff topological space and L is all of C(S), then, as Kaplansky has shown (see [12] and [8]), every closed ideal is a translation of an ideal of this form. Thus, for function lattices, the points of the base set are closely related to the characters and the prime ideals of the lattice. This is the idea which is the basis of most representation theorems for algebraic systems. Such a representation theorem will not be proved for translation lattices. That is, it will be shown that every translation lattice is isomorphic to a translation lattice of functions on some set S. If the translation lattice is distributive, this representation can be made in such a way that joins as well as meets are preserved. This latter result is due to Kaplansky [8].

Lemma 2.2.7. Let L be a translation lattice and let $f \ngeq g$ in L. Then there is a character F of L such that $F(f) < F(g)$. If L is also distributive, the character can be so chosen that it preserves joins in L.

Proof. Let $I = \{h \in L | h \ngeq g\}$. Then I is a closed prime ideal, $f \in I$ and $g \notin I$. Also by postulate (f) of definition 2.1.1 there is a $\delta > 0$ such that $f + \delta \ngeq g$, that is, $f + \delta \in I$. Define the character F by $F(h) = \inf \{\lambda | h - \lambda \in I\}$. As is the proof of lemma 2.2.6, this is actually a character. $F(f) = -\delta < 0 = F(g)$. Thus for a translation (semi-) lattice the proof is complete.

In order to prove the distributive translation lattice case, the above argument can be applied except that it is necessary to establish the existence of a prime lattice ideal J with the properties $f + \delta \in I$ and $g \notin I$. Choose for J the ideal which is maximal with the properties $I \subseteq J$ and $g \notin J$ (I being the same ideal as defined above). The existence of such a J follows from the maximal principle. It must be shown that J is prime. Suppose h_1 and h_2 are not in J. Then by the maximality $(h_1) \cup J$ and $(h_2) \cup J$ must both contain g. So must the intersection of these two ideals. Since the lattice of ideals of a distributive lattice is itself distributive (see [2]), this means that $g \in [(h_1) \cap (h_2)] \cup J = (h_1 \wedge h_2) \cup J$. Therefore $h_1 \wedge h_2 \notin J$. Since h_1 and h_2 were any elements not in J, it follows that J is prime. The lemma is proved.

Using this lemma, we can easily deduce a representation theorem for abstract translation lattices.

<u>Theorem 2.2.3.</u> Let L be an abstract translation lattice. Let f_0 be any element of L. Then there is a set S and a translation lattice L' of bounded real valued functions on S such that L and L' are isomorphic by a mapping which sends f_0 into the zero function on S.

<u>Proof.</u> Let S be the set $\{F\}$ of all characters of L which satisfy $F(f_0) = 0$. For $f \in L$, let $f \to f'$, where f' is that function on S defined by $f'(F) = F(f)$ for all $F \in S$. Let L' be the set of all F' so obtained. The mapping $f \to f'$ is a homomorphism since $(f \wedge g)'(F) =$ $= F(f \wedge g) = \min \{F(f), F(g)\} = \min \{f'(F), g'(F)\}$ and $(f + \alpha)'(F) = F(f + \alpha) = F(f) + \alpha$ for all $F \in S$. The mapping is one-to-one by lemma 2.2.7 since if $f \neq g$, a character G exists satisfying $G(f) < G(g)$. Put $F = G - G(f_0)$. Then $F \in S$ and $f'(F) = F(f) < F(g) = g'(F)$. It is clear that f_0 goes into the zero

function. Finally, it follows from postulate (e) of definition 2.1.1 that every f' is a bounded function on S. The proof is complete.

Remark. In case L is a distributive translation lattice, the argument used also shows that the representation is one which preserves joinsof elements. The extra details are omitted.

Theorem 2.2.3 shows that no generality will be lost if, in studying translation lattices we restrict ourselves to the consideration of translation lattices of functions. However, there is little to be gained by such a specialization, so we will continue to work with general translation lattices.

Example 15. Theorem 2.2.3 implies that the translation lattice of example 8 must be isomorphic to a translation lattice of bounded, real-valued functions on a set. It is very easy to obtain one such representation explicitly in this case. Let S be the set of all points on the unit sphere of the Banach space from which L is constructed. For any $f \in L$ and $x \in S$, define $f(x) = \sup \{ \log \lambda | \lambda x \in f \}$. There is no difficulty in verifying that this convention makes a bounded, real-valued function on S correspond to each $f \in L$.

2.3 Representation Theory — Lattice Functions.

One of the shortcomings of the theory developed in section 2.2 is its failure to give a very concrete picture of the image translation lattices L//I formed with respect to a closed (non-trivial) ideal I. The present section is devoted to an attempt to fill this gap. Full use will be made of the theory developed in section 1.2. Instead of representing the translation lattice as a translation lattice of point functions, the representation will be in terms of collections (still translation lattices of course) of objects which are generalizations of set functions. These will be called lattice functions. This procedure has an advantage over the usual representation in terms of point functions. Namely, each lattice function carries with it part of the structure of the original translation lattice -- a virtue which is not shared by point functions. In fact, enough structure is possessed by the individual lattice functions that it will be possible to prove significant theorems, even when the assumption that the whole system forms a translation lattice is weakened.

2.3.1 Definitions -- The representation theorem.

Definition 2.3.1. Let P be a complete Boolean algebra. Then a bounded real valued function F defined on $P - \{z\}$ will be called a lattice function if it satisfies:

$$F(a) \leq F(b) \text{ whenever } z \neq b \leq a.$$

This definition calls for several remarks. First, it should be emphasized that the characteristic property $F(a) \leq F(b)$ when $z \neq b \leq a$ makes F a dual order homomorphism of P into the reals. The reason for this dualization will be made clear later. Second, it may look strange that F is not defined on z. This convention undeniably causes notational inconvenience in many places in the following pages. But, at the same time, to define $F(z)$ would lead to notational difficulties in many other parts of the thesis. We have only to choose between the lesser of evils and the present choice seems best. As a final remark, we note that the concept of a lattice function could be generalized by assuming only that P is a partially ordered set. However, for the uses which will be made of lattice functions, the present definition is more appropriate.

Now we will show how a translation lattice L can be represented as a translation lattice of lattice functions on a certain Boolean algebra P, provided L is divisible with respect to one of its closed ideals (see definition 2.2.1). A collection of lattice functions F on a Boolean algebra P forms a translation lattice in a natural way if the collection is closed under the elementwise operations of meets ($(F \wedge G)(a) = \min\{F(a), G(a)\}$) and translations ($(F + \alpha)(a) = F(a) + \alpha$). It is easily seen that $F \wedge G$ and $F + \alpha$ defined in this way are again lattice functions.

Before beginning the theory, let us recall some of the notation from chapter 1. Corresponding to any semi-lattice L and ideal I of L, a disjunctive lattice L/I was defined. There was a natural homomorphic mapping h_I of L onto L/I. Recall also that L/I could be imbedded in a complete Boolean algebra which was denoted $[L/I]$. The mapping h_I can therefore be thought of as a homomorphism of L

into $[L/I]$. It will be of importance to know when L/I has a unit. This question was answered in lemma 1.2.5: L/I has a unit if and only if there is an element $f \in L$ such that $f \wedge g \in I$ implies $g \in I$ for all $g \in L$. This condition is certainly satisfied if I is a non-trivial prime ideal. Also if L is a translation lattice, the condition is satisfied for all principal ideals. In fact

<u>Lemma 2.3.1.</u> Let L be a translation lattice. Let I be a closed ideal of L. Suppose that I is bounded, that is, there exists an element $f \in L$ such that $f \geq g$ for all $g \in I$. Then L/I has a unit.

<u>Proof.</u> The criterion of lemma 1.2.5 will be used. Suppose $f \geq g$ for all $g \in I$. we will show that for any $\delta > 0$, if $(f + \delta) \wedge g \in I$, then $g \in I$. By hypothesis, $(f + \delta) \wedge g \in I$ implies that $f \geq (f + \delta) \wedge g$. Applying this inequality to itself gives $f \geq ([(f + \delta) \wedge g] + \delta) \wedge g = (f + 2\delta) \wedge (g + \delta) \wedge g =$ $= (f + 2\delta) \wedge g$. Repeating this process, an induction shows that $f \geq (f + n\delta) \wedge g$ for all integers n. By postulate (e) of definition 2.1.1, n can be chosen large enough so that $(f + n\delta) \wedge g = g$. Thus $f \geq g$ and therefore $(f + \delta) \wedge g = g$. But the original assumption was that $(f + \delta) \wedge g \in I$. Therefore $g \in I$. The proof is complete, since this means that the image of $f + \delta$ in L/I is a unit of L/I.

Now the main part of the representation theorem can be formulated and proved.

Lemma 2.3.2. Let L be a translation lattice. Let I be a closed ideal of L which is such that L/I has a unit element. Suppose, moreover, that L is divisible with respect to I. Define for $a \in [L/I] - \{z\}$ and $f \in L$:

$$F_f(a) = \sup \{ \lambda \mid h_I(f - \lambda) \geq a \}.$$

Then F_f is a lattice function on $[L/I]$ and the mapping $f \to F_f$ is a translation lattice isomorphism.

Proof. The first thing to show is that F_f is well defined on $[L/I]$ by the above formula; it must be proved that $\{ \lambda \mid h_I(f - \lambda) \geq a\}$ is not empty. This follows from the assumption that L/I has a unit. Indeed if $g \in L$ is such that $h_I(g) = i$, (the common unit of L/I and $[L/I]$), then there exists a λ_o so small (negatively) that $f - \lambda_o \geq g$, $h_I(f - \lambda_o) \geq h_I(g) = i \geq a$. Thus the set of λ's for which $h_I(f - \lambda) \geq a$ always contains λ_o. Hence $F_f(a) \geq \lambda_o$ for all a. On the other hand, it is always possible to choose λ_1 so large that $f - \lambda_1 \in I$. Therefore $h_I(f - \lambda_1) = z$, so that $F_f(a) \leq \lambda_1$ (assuming of course that $a \neq z$). Thus F_f is bounded on $[L/I] - \{z\}$.

Now suppose $z \neq b \leq a$. Then $\{ \lambda \mid h_I(f - \lambda) \geq a\} \subseteq \{ \lambda \mid h_I(f - \lambda) \geq b\}$, so that $F_f(a) \leq F_f(b)$. This means that all the conditions of definition 2.3.1 are satisfied and F_f is a lattice function on $[L/I]$.

Next, it will be shown that $f \to F_f$ is a translation lattice homomorphism. $F_{f+\alpha}(a) = \sup \{ \lambda \mid h_I(f + \alpha - \lambda) \geq a\} = \sup\{ \lambda - \alpha \mid$ $\mid h_I(f - (\lambda - \alpha)) \geq a \} + \alpha = F_f(a) + \alpha$. Thus translations are preserved. $F_{f \wedge g}(a) = \sup \{ \lambda \mid h_I(f \wedge g - \lambda) \geq a\} = \sup$ $\{ \lambda \mid h_I(f - \lambda) \wedge h_I(g - \lambda) \geq a \}$. Now suppose for definiteness that $\alpha = F_f(a) \leq F_g(a) = \beta$. Then if $\delta > 0$, by definition $h_I(f - \alpha + \delta) \geq a$, $h_I(f - \alpha - \delta) \not\geq a$ and $h_I(g - \alpha + \delta) \geq$ $\geq h_I(g - \beta + \delta) \geq a$. Hence $h_I(f - \alpha + \delta) \wedge h_I(g - \alpha + \delta) \geq a$ and $h_I(f - \alpha + \delta) \wedge h_I(g - \alpha + \delta) \not\geq a$. Since δ can be arbitrarily small, it follows that $F_{f \wedge g}(a) = \sup \{ \lambda \mid h_I(f - \lambda) \wedge$ $\wedge h_I(g - \lambda) \geq a \} = \min \{ F_f(a), F_g(a) \}$.

To complete the proof it must be shown that $f \rightarrow F_f$ is a one-to-one mapping. Suppose $f_1 \not\equiv f_2$. Then since L is divisible with respect to I, there exists a real λ and an element $g \in L$ such that $(f_1 - \lambda) \wedge g \in I$, $(f_2 - \lambda) \wedge g \notin I$. Then since I is closed, $\delta > 0$ exists so that $(f_2 - \lambda - \delta) \wedge (g - \delta) \geq [(f_2 - \lambda) \wedge g] - \delta \in I$. Hence, $(f_2 - \lambda - \delta) \wedge g \notin I$. Consequently, $f_1 - \lambda \not\equiv f_2 - \lambda - \delta$ (I), that is, $h_I(f_1 - \lambda) \not\geq h_I(f_2 - \lambda - \delta)$. Now call $a = h_I(f_2 - \lambda - \delta)$. Then $F_{f_1}(a) \leq \lambda$ and $F_{f_2}(a) \geq \lambda + \delta$. This says finally that $F_{f_1} \not\equiv F_{f_2}$, proving that the mapping $f \rightarrow F_f$ is one-to-one. All of the assertions of lemma 2.3.2 have now been established.

<u>Remark.</u> The formula $F_f(a) = \sup \{ \lambda \mid h_I(f - \lambda) \geq a \}$ is meaningful only as long as $a \neq z$; for every real number λ satisfies $h_I(f - \lambda) \geq z$. This is one of the reasons that the convention of not

defining a lattice function on the zero element was adopted.

The lattice functions defined in the preceeding lemma satisfy a rather special condition.

Lemma 2.3.3. The lattice functions F_f defined in the previous lemma by $F_f(a) = \sup\{\lambda \mid h_I(f - \lambda) \geq a\}$ invert unlimited joins. That is, if A is any non-empty subset of $[L/I] - \{z\}$, then $F_f(\vee A) = \inf\{F_f(a) \mid a \in A\}$.

Proof. It is an immediate consequence of the characteristic property of lattice functions that $F_f(\vee A) \leq \inf\{F_f(a) \mid a \in A\}$. To prove the reverse inequality, denote $\alpha = F_f(\vee A)$ and pick any $\delta > 0$. Then $h_I(f - \alpha - \delta) \ngeq \vee A$, so there exists $a \in A$ such that $h_I(f - \alpha - \delta) \ngeq a$. This means $F_f(a) \leq \alpha + \delta$. Since δ was arbitrarily small, $\inf\{F_f(a) \mid a \in A\} \leq \alpha = F_f(\vee A)$.

Definition 2.3.2. A lattice function F which inverts unlimited joins (that is, $F(\vee A) = \inf\{F(a) \mid a \in A\}$, for any non-empty subset A of $P - \{z\}$) is called normal.

The reason for this terminology will become clear when the relation between lattice functions and point functions on a topological space is studied. The lemma 2.3.3 can now be expressed by saying that the lattice functions F_f are normal.

An alternative characterization of normality for a lattice function can be given. The criterion is due essentially to Dilworth (see [1]).

Proposition 2.3.1. Let F be a lattice function on P. Then F is normal if and only if the following condition is satisfied:

N: if $a \neq z$ and $F(a) < \lambda$, then b exists with $z \neq b \leq a$ such that

$$F(c) < \lambda \text{ for all } c \leq b,\ c \neq z.$$

Proof. Suppose F is a normal lattice function. If, for all $b \leq a$, there is a $c \leq b$ such that $F(c) \geq \lambda$, then clearly $a = \bigvee\{c \mid F(c) \geq \lambda,\ c \leq a\}$. Hence $\lambda > F(a) = \inf\{F(c) \mid F(c) \geq \lambda,\ c \leq a\} \geq \lambda$. This contradiction proves that N must be satisfied.

Conversely, suppose F satisfies the condition N. Let A be a non-empty subset of $P - \{z\}$. Call $\alpha = \inf\{F(a) \mid a \in A\}$. Suppose $F(\bigvee A) < \alpha$. By N, b exists in P with $z \neq b \leq \bigvee A$ and $F(c) < \alpha$ for all $c \leq b$. Since $b = b \wedge \bigvee A = \bigvee\{b \wedge a \mid a \in A\}$, $b \wedge a \neq z$ for some $a \in A$. For this a, $\alpha > F(b \wedge a) \geq F(a) \geq \inf\{F(a) \mid a \in A\} = \alpha$. This contradiction proves $F(\bigvee A) \geq \alpha$. Clearly $F(\bigvee A) \leq \alpha$. Since A was an arbitrarily chosen subset of $P - \{z\}$, the condition for F to be normal has been satisfied.

Corresponding to a given complete Boolean algebra P, there are usually many translation lattices of normal lattice functions which can be defined on P. Suppose L is one such translation lattice. The question naturally arises: is there an ideal I in L such that P is isomorphic to $[L/I]$ and the construction of lemma 2.3.2 (applied to L, where L is considered as an abstract translation lattice) yields just the lattice functions of L? In short, what kind of uniqueness theorems can be proved for the representation of a translation lattice as a collection of lattice functions? A later article will be devoted to answering this question. Here we lay the foundations by proving some necessary conditions.

Lemma 2.3.4. In the mapping $f \to F_f$ of lemma 2.3.2, the image of I is precisely $\{F_f | F_f(a) \leq 0$ for all $a \in [L/I] - \{z\}\}$.

Proof. Suppose $f \in I$. Then $h_I(f - \lambda) = z$ for all $\lambda \geq 0$. Therefore $F_f(a) \leq 0$ holds for all $a \neq z$. Conversely, if $F_f(a) \leq 0$ for all $a \neq z$, then $h_I(f - \lambda) = z$ is true for all $\lambda > 0$. In other words, $f \in I^- = I$.

A consequence of this lemma is:

Corollary 2.3.1. The image set in the mapping $f \to F_f$ contains the zero lattice function if and only if I is a principal ideal. If I is principal, then its generator maps into the zero function.

Another condition which must be satisfied by the image set of lattice functions in the mapping $f \to F_f$ is the following:

Lemma 2.3.5. Suppose that the conditions of lemma 2.3.2 prevail; construct the lattice functions F_f as described. Then if $a \neq z$ in $[L/I]$, there is an F_f such that $F_f(b) > 0$ for some b with $z \neq b \leq a$ and $F_f(c) \leq 0$ for all c satisfying $c \nleq a$.

Proof. Since L/I is dense in $[L/I]$, if $a \neq z$ in $[L/I]$, there is an $f \in L$ such that $z \neq h_I(f) \leq a$. Now $h_I(f) \neq z$ implies that $f \notin I$.

Then since I is closed by assumption, $f - \delta \notin I$ holds for some $\delta > 0$. Put $b = h_I(f - \delta) \neq z$. Then $F_f(b) \geq \delta > 0$ by definition. On the other hand, if $c \nleq a$, $h_I(f) \ngeq c$ so $F_f(c) \leq 0$. The proof is complete.

The condition expressed in the lemma is sufficiently important to be given a name.

Definition 2.3.3. Let M be a collection of normal lattice functions on a complete Boolean algebra P. Then M will be said to generate P if for any $a \neq z$ in P, there is an $F \in M$ and an element $b \in P$ such that (1): $F(b) > 0$, and (2): $F(c) \leq 0$ for all c satisfying $c \nleq a$.

Lemma 2.3.5 implies that if the translation lattice L is divisible with respect to the closed bounded ideal I, then the natural representation of L as a set of lattice functions on $[L/I]$ generates the Boolean algebra $[L/I]$. It is possible to prove the following converse result.

Proposition 2.3.2. Let L be a translation lattice of normal lattice functions on a complete Boolean algebra P. Suppose that L generates P. Then L is divisible with respect to the ideal $I = \{F | F(a) \leq 0 \text{ all } a \in P - \{z\}\}$.

Proof. Suppose $F \ngeq G$. This means that $F(a) < G(a)$ for some a. Choose λ so that $F(a) < \lambda < G(a)$. According to proposition 2.3.1, this means that $b \in P$ exists satisfying $z \neq b \leq a$ and such that $F(c) < \lambda$ for all $c \leq b$. In particular $F(b) < \lambda < G(a) \leq G(b)$. Since L generates P, there is an element $H \in L$ satisfying $H(c) > 0$ where $c \leq b$ and $H(d) \leq 0$ for all d such that $d \nleq b$. It then follows that $[(F - \lambda) \wedge H](d) \leq H(d) \leq 0$ if $d \nleq b$, $\leq F(d) - \lambda \leq 0$ if $d \leq b$. Thus $(F - \lambda) \wedge H \leq 0$ (i.e., $(F - \lambda) \wedge H \in I$). Also $[(G - \lambda) \wedge H](c) > 0$, so that $(G - \lambda) \wedge H \nleq 0$ (i.e., $(G - \lambda) \wedge H \notin I$). Since F and G were any two elements satisfying $F \ngeq G$, the conditions of divisibility with respect to I are satisfied.

For convenience, the results of the past few lemmas will be collected together as the "representation theorem".

<u>Theorem 2.3.1.</u> Let L be a translation lattice. Let I be a closed, bounded ideal of L such that L is divisible with respect to I. Define for $a \in [L/I] - \{z\}$ and $f \in L$

$$F_f(a) = \sup\{\lambda \mid h_I(f - \lambda) \geq a\}.$$

Then F_f is a normal lattice function on $[L/I]$ and the mapping $f \to F_f$ is a translation lattice isomorphism of L onto a set L' of normal lattice functions which generates $[L/I]$. The image of I in this mapping is the set $\{F \in L' \mid F \leq 0\}$.

<u>Corollary 2.3.2.</u> Let L be a translation lattice. Let I be a closed, bounded ideal of L. Define, for $a \in [L/I] - \{z\}$ and $f \in L$,

$$F'_f(a) = \sup\{\lambda \mid h_I(f - \lambda) \geq a\}.$$

Then the mapping $f \to F'_f$ is a translation lattice homomorphism of L onto a set L' of normal lattice functions on $[L/I]$. The collection L' generates $[L/I]$. Moreover, there is a natural isomorphism $\emptyset$ of

$[L/I]$ onto $[(L//I)/H(I)]$ such that

$$F'_f(a) = F_{H(f)}(\emptyset(a))$$

where H is the natural homomorphism of L onto L//I and F denotes the representation of L//I constructed in theorem 2.3.1 (with respect to the ideal H(I)).

Proof. Using theorem 2.3.1, the first part of the corollary will be an immediate consequence of the last assertion (since L//I is divisible with respect to H(I)).

Notice that by theorem 1.2.2, $h_I(f) = h_I(g)$ if and only if $h_{H(I)}(H(f)) = h_{H(I)}(H(g))$. Thus $\emptyset$ is defined as an isomorphism of L/I onto (L//I)/H(I) by putting $\emptyset(h_I(f)) = h_{H(I)}(H(f))$. By theorem 1.2.4, $\emptyset$ extends immediately to an isomorphism between the complete Boolean algebras, $[L/I]$ and $[(L//I)/H(I)]$. With this definition, $h_I(f - \lambda) \geq a$ if and only if $h_{H(I)}(H(f) - \lambda) = \emptyset(h_I(f - \lambda)) \geq \emptyset(a)$. From this, the last assertion of the corollary follows readily.

Definition 2.3.4. Let L be a translation lattice and let I be a bounded closed ideal of L. The mapping $f \to F_f$ of L into the set of normal lattice functions on L/I (where F_f is defined by $F_f(a) = \sup \{ \lambda \mid h_I(f - \lambda) \geq a \}$), will be called the natural representation of L (relative to the ideal I) as a translation lattice of normal lattice functions.

An example may help to clarify some of the concepts which have just been introduced.

Example 16. Let L be C(S), the translation lattice of bounded, real valued functions on a completely regular topological space S. Let (0) be the principal ideal generated by the zero function. Then $[L/(0)]$ is isomorphic to the Boolean algebra of regular open sets of S (see example 6 of chapter 1). The mapping $f \to \{x \mid f(x) > 0\}^{-o}$ is the natural mapping, h_I, of L into this Boolean algebra. We will show that $F_f(a) = \inf \{f(x) \mid x \in a\}$ holds for all regular open sets a.

By definition, $F_f(a) = \sup \{\lambda \mid h_I(f - \lambda) \geq a\} =$ $= \sup \{\lambda \mid \{x \mid f(x) > \lambda\}^{-o} \geq a\}$. To save writing, denote $\alpha = \inf \{f(x) \mid x \in a\}$. If $\delta > 0$, $\{x \mid f(x) > \alpha - \delta\} \geq a$ by the definition of α. Hence, $\{x \mid f(x) > \alpha - \delta\}^{-o} \geq a$. On the other hand, $x \in a$ exists satisfying $f(x) < \alpha + \delta$. By the continuity of f, this inequality holds in a neighborhood of x and thus $\{x \mid f(x) > \alpha + \delta\}^{-o} \not\geq a$. Since δ could be arbitrarily small, $\sup \{\lambda \mid \{x \mid f(x) > \lambda\}^{-o} \geq a\} = \alpha$, q.e.d.

It is important to know that normal lattice functions can be constructed on any complete Boolean algebra. The following proposition gives a method for constructing a special kind of normal lattice function.

Proposition 2.3.3. Let P be a complete Boolean algebra. Let b be any non-zero element of P. Then there is a normal lattice function F on P such that $F(c) = 1$ if $z \neq c \leq b$ and $F(c) = 0$ if $c \not\leq b$.

Proof. Let F be precisely as defined in the proposition. We must show that it is a normal lattice function. If $z \neq c \leq d$ and $F(d) = 1$, then $d \leq b$ so also $c \leq b$. Hence $F(c) = 1$. It follows that $c \leq d$ implies $F(c) \geq F(d)$. Suppose A is any non-empty subset of $P - \{z\}$. Clearly, $\vee A \nleq b$ if and only if some $a \in A$ satisfies $a \nleq b$. Hence $F(\vee A) = \inf\{F(a) \mid a \in A\}$. Thus F is normal.

Using this, it is possible to deduce an important fact.

Proposition 2.3.4. Let P be a complete Boolean algebra. Then the set of all normal lattice functions on P forms a translation lattice which generates P.

Proof. First, the set of all normal lattice functions forms a translation lattice. The fact that the system is closed under translation can be readily checked. The details are omitted. Suppose F and G are two normal lattice functions. Then $F \wedge G$ is clearly a lattice function. To prove normality, suppose A is any non-empty subset of $P - \{z\}$. Then $(F \wedge G)(\vee A) = F(\vee A) \wedge G(\vee A) = [\inf\{F(a) \mid a \in A\}] \wedge [\inf\{G(a) \mid a \in A\}] \geq \inf\{F(a) \wedge G(a) \mid a \in A\} = \inf\{(F \wedge G)(a) \mid a \in A\}$. The opposite inequality follows from the fact that $(F \wedge G)(\vee A) \leq (F \wedge G)(a)$ for all $a \in A$.

The assertion, that P is generated by the set of all normal lattice functions, is a direct consequence of the preceding proposition. The proof is, therefore, complete.

Hereafter the translation lattice of all normal lattice functions on the complete Boolean algebra P will be denoted N(P). This should not cause any confusion with the notation N(S) introduced in example 4. P always denotes a complete Boolean algebra.

2.3.3. The uniqueness theorem -- special case.

In this article, a uniqueness theorem for the representation of translation lattices as lattice functions will be deduced. Suppose a translation lattice L is given. Assume that L can be represented as a sub-translation lattice of N(P) so that the representative of L generates P. Then by proposition 2.3.2, L is divisible with respect to the ideal $\{F \in L | F \leq 0\}$. We shall answer the following question: what is the relation between the given representation and the natural representation of definition 2.3.4?

Theorem 2.3.2. Let L be a translation lattice of normal lattice functions on a complete Boolean algebra P. Suppose L is dense in P. Let $I = \{F \in L | F \leq 0\}$. Then

(1) $[L/I]$ is isomorphic to P by a mapping $\emptyset$ on $[L/I]$ to P;

(2) The natural representation, F_G of $G \in L$ as a lattice function on $[L/I]$, which is defined by $F_G(a) = \sup\{\lambda | h_I(G - \lambda) \geq a\}$ (where h_I is the natural homomorphism of L onto L/I, satisfies $F_G(a) = G(\emptyset(a))$.

Proof. For the notation in the proof of this theorem, write a for the generic element of $[L/I]$, b for the generic element of P.

Map L into P by $G \to \bigvee \{b \in P | G(b) > 0\}$. One easily verifies that this is a meet homomorphism. The image of this mapping is dense in P (see definition 1.2.3), since L generates P. The kernel of the homomorphism is $\{G \in L | G \leq 0\} = I$. Hence the mapping determines an isomorphism of L/I into P. Denote it by $\emptyset$. To be precise, $\emptyset$ is defined by:

$$\emptyset(h_I(G)) = \bigvee \{b \in P | G(b) > 0\} .$$

$\emptyset$ is extended to an isomorphism of $[L/I]$ onto P by writing $\emptyset(a) = \bigvee \{\emptyset(h_I(G)) | h_I(G) \leq a\}$. (This clearly makes $\emptyset$ a meet homomorphism with kernel z of $[L/I]$ onto a dense subset of P. By proposition 1.2.2, the mapping is an isomorphism. By theorem 1.2.5, it is onto P.)

To complete the proof, it is necessary to show that $F_G(a) = G(\emptyset(a))$ holds for all $a \in [L/I] - \{z\}$. (We notice that I is a bounded, closed ideal of L and L is divisible with respect to I.)

$F_G(a) = \sup\{\lambda | h_I(G - \lambda) \geq a\} = \sup\{\lambda | \emptyset(h_I(G - \lambda)) \geq \emptyset(a)\} = \sup\{\lambda | \bigvee\{b \in P | G(b) > \lambda\} \geq \emptyset(a)\}$.

Now $\bigvee\{b \in P | G(b) > \lambda\} \geq \emptyset(a)$ implies (by the normality of G) that $\lambda \leq \inf\{G(b) | G(b) > \lambda\} = G(\bigvee\{b \in P | G(b) > \lambda\}) \leq G(\emptyset(a))$. Hence, $F_G(a) \leq G(\emptyset(a))$.

To reverse this inequality, notice that if $\lambda < G(\emptyset(a))$, $\bigvee\{b \in P | G(b) > \lambda\} \geq \emptyset(a)$. Consequently $F_G(a) \geq \sup\{\lambda | \lambda < G(\emptyset(a))\} = G(\emptyset(a))$. This completes the proof of the theorem.

2.3.4. The general uniqueness theorem -- preliminary results.

The ultimate object of this and the remaining articles of section 2.3 is to determine the modifications which must be made in theorem 2.3.2 when the assumption that L generates P is dropped. The main result of this article, theorem 2.3.3 below, indicates the direction to be followed in the remainder of the section. The proof given could be much shortened by the application of the third isomorphism theorem of chapter 1. However, we will give a more detailed proof since some of the intermediate notations and results will be useful for later work.

Let L be a translation lattice of normal lattice functions on a complete Boolean algebra P. As usual, let $I = \{F \in L | F \leq 0\}$.

Definition 2.3.5. Denote by R the subset of P consisting of all elements of the form $a_F = \vee \{b \in P | F(b) > 0\}$ where $F \in L$.

Lemma 2.3.7. R is a sub-semi-lattice of P with the same zero and unit as P. The mapping $F \to a_F$ is a semi-lattice homomorphism whose kernel is I. R is dense in P if and only if L generates P.

Proof. We have $a_F \wedge a_G = [\vee \{b \in P | F(b) > 0\}] \wedge [\vee \{c \in P | G(c) > 0\}] = \vee \{b \wedge c \neq z | F(b) > 0, G(c) > 0\} \leq$ $\leq \vee \{b | (F \wedge G)(b) > 0\} = a_{F \wedge G}$. On the other hand, $a_F = \vee \{b \in P | F(b) > 0\} \geq \vee \{b \in P | (F \wedge G)(b) > 0\} = a_{F \wedge G}$ and similarly $a_G \geq a_{F \wedge G}$. Hence $a_F \wedge a_G \geq a_{F \wedge G}$. These two inequalities prove that the mapping $F \to a_F$ is a homomorphism and the image is a semi-lattice. Clearly z and i are elements of R. It is also evident that the kernel of the mapping is I.

The last statement of the lemma is just an expression of the equivalence of the two definitions: R is dense in P if for any $a \neq z$ in P, there is an $F \in L$ such that $z \neq \vee\{b \mid F(b) > 0\} \leq a$; L generates P if for any $a \neq z$ in P, there is an $F \in L$ such that $F(b) > 0$ for some $b \leq a$ and $F(c) \leq 0$ whenever $c \nleq a$.

Definition 2.3.6. Denote for $a_F \in R$, $\bar{a}_F = \vee\{b \in R \mid a_F \geq b\ (z) \text{ in } R\}$. The join is taken in P of course. (The notation $a_F \geq b\ (z)$ refers to the definition of lemma 1.2.1.)

Lemma 2.3.8. $a_F \to \bar{a}_F$ is a semi-lattice homomorphism of R into a disjunctive sub-semi-lattice of P. The mapping satisfies $a_F \leq \bar{a}_F$, $\bar{a}_{F \wedge G} = \bar{a}_F \wedge \bar{a}_G$, $\bar{z} = z$ and $\bar{i} = i$.

Proof. This is a special case of lemma 1.3.9.

Theorem 2.3.3. [L/I] is isomorphic to a sub-semi-lattice of P. By the isomorphism, $z \to z$, $i \to i$ and $h_I(F) \to \bar{a}_F$. The isomorphism is onto P if and only if L generates P.

Proof. The first two statements are consequences of lemma 2.3.8 and the first and second isomorphism theorems (theorems 1.2.1 and 1.2.2, respectively), together with theorem 1.2.5.

To prove the last assertion, notice $a_F \leq \bar{a}_F$. Now [L/I] will be isomorphic to all of P if and only if R/(z) is dense in P, and, because of the inequality, this means that R must be dense in P. Conversely, if R is dense in P, it is disjunctive and hence $R/(z) = R$ is dense in P. By lemma 2.3.7, the final statement follows.

Let P_1 denote the image of $[L/I]$ in P under the mapping of theorem 2.3.3. For every $F \in L$, we may ask the question: what is the relation between the lattice function which is defined as the restriction of F to P_1 and the natural representation of F as a normal lattice function on $[L/I]$. There will be no simple correspondence such as the one in theorem 2.3.2, since in general, the restriction of F to P_1 will not be normal. Hence it is necessary to devote some attention to the relation between normal lattice functions defined on two complete Boolean algebras -- one of them being a sub-semi-lattice of the other. The next article is devoted to this subject.

2.3.5. The extension and restriction of lattice functions.

In this article, P_1 and P_2 will denote complete Boolean algebras with P_1 a sub-semi-lattice of P_2. It will also be assumed that P_1 and P_2 have the same unit elements. For convenience, the fact that P_1 is related to P_2 in this way will be abbreviated as $P_1 \subseteq P_2$. Among the elements of P_2, those which also belong to P_1 will be distinguished by a bar. Thus for instance, $\bar{a}$ and $\bar{b}$ will designate elements of P_1 while a and b denote elements of P_2, (which may also be in P_1, of course).

Two questions will be treated. First, suppose that a normal lattice function on P_1 is given. Can this function be extended in a natural way to a normal lattice function of P_2? The second question is concerned with the reverse situation. Suppose that a normal lattice function in P_2 is given. Is there a natural way of associating a normal lattice function on P_1 with the given function over P_2? It seems likely that there may be many "natural" ways of making these associations. However, for our purposes, the methods outlined below are quite adequate.

In the work which follows, repeated use will be made of two different join operations in the Boolean algebra P_1. While it is assumed that P_1 is a sub-semi-lattice of P_2, P_1 is also a complete Boolean algebra in its own right (although not a sub-algebra of P_2). Thus if A is any subset of P_1, A has two (generally different) least upper bounds: the bound in P_1 and the one in P_2. These will be written respectively $\overset{1}{\vee} A$ and $\overset{2}{\vee} A$. It is important to notice that for any subset A of P_1, $\overset{1}{\vee} A \geq \overset{2}{\vee} A$. For certainly $\overset{1}{\vee} A$ is an upper bound of A in P_2 as well as P_1, and $\overset{2}{\vee} A$ is the least upper bound.

We can now formulate and prove the extension theorem.

<u>Proposition 2.3.5.</u> Let P_1 and P_2 be complete Boolean algebras with $P_1 \subseteq P_2$. Let F be a normal lattice function defined on P_1. Define F' on P_2 by

$$F'(a) = \sup \{ F(\overline{a}) | \overline{a} \geq a, \overline{a} \in P_1 \} \quad \text{for } a \in P_2,\ a \neq z.$$

Then F' is a normal lattice function on P_2 and satisfies $F'(\overline{a}) = F(\overline{a})$ for all $\overline{a} \in P_1$.

Proof. The facts that F' is a lattice function and that F' takes the same value as F on all elements of P_1 are easy consequences of the definition of F'. It is necessary to show that F' is a normal lattice function.

Let A be an arbitrarily chosen subset of P_2. Let $\delta > 0$. By the definition of F', for any $a \in A$, there is an $\bar{a} \in P_1$ such that $\bar{a} \geq a$ and $F(\bar{a}) \geq F'(a) - \delta$. Corresponding to each $a \in A$, choose an $\bar{a}$ in P_1 satisfying these conditions and denote by $\bar{A}$ the set of these $\bar{a}$'s. Then $\overset{1}{\vee} \bar{A} \geq \overset{2}{\vee} \bar{A} \geq \overset{2}{\vee} A$. Thus $F'(\overset{2}{\vee} A) \geq F(\overset{1}{\vee} \bar{A}) =$ $= \inf \{F(\bar{a}) | \bar{a} \in \bar{A}\} \geq \inf \{F'(a) - \delta \mid a \in A\} = \inf \{F'(a) | a \in A\} - \delta$. Since δ can be chosen as small as we please, $F'(\overset{2}{\vee} A) \geq$ $\geq \inf \{F'(a) | a \in A\}$. But the opposite inequality is valid for any lattice function, so it follows that F' is normal.

Now we consider the restriction problem. In order to motivate the next result, it is necessary to refer to the paper of Dilworth [1], and to understand the ideas which will be presented in the fourth chapter of this thesis. For this reason, the result is presented without attempting to show that it arises in a natural way.

Proposition 2.3.6. Let P_1 and P_2 be complete Boolean algebras with $P_1 \subseteq P_2$. Let F be a lattice function on P_2 (not assumed normal). Define F_o on P_1 by:

$$F_o(\bar{a}) = \inf_{\bar{c} \leq \bar{a}} \sup_{a \leq \bar{c}} F(a).$$

Then F_o is a normal lattice function on P_1.

Proof. Suppose $a \le b$. Then $F_0(a) = \inf_{\bar{c} \le \bar{a}} \sup_{a \le \bar{c}} F(a) \ge$ $\ge \inf_{\bar{c} \le \bar{b}} \sup_{a \le \bar{c}} F(a) = F_0(\bar{b})$. Hence F_0 is a lattice function.

To prove that F_0 is normal, the criterion of proposition 2.3.1 will be used. Suppose $F_0(\bar{a}) < \lambda$. Then $\bar{c} \le \bar{a}$ exists (by the definition of F_0) such that $\sup_{a \le \bar{c}} F(a) < \lambda$. Thus if $z \ne \bar{b} \le \bar{c}$, it follows that $F_0(\bar{b}) = \inf_{\bar{d} \le \bar{b}} \sup_{a \le \bar{d}} F(a) \le \sup_{a \le \bar{c}} F(a) < \lambda$. By the above mentioned proposition, F_0 must be normal. This completes the proof.

2.3.6. The uniqueness theorem -- general case.

As hypotheses for all the lemmas which follow, the standard assumptions are made: P is a complete Boolean algebra; L is a given translation lattice of normal lattice functions on P; I is the ideal $\{ F \in L \mid F \le 0 \}$. Again R is the sub-semi-lattice of P which consists of all elements of the form $a_F = \bigvee \{ b \mid F(b) > 0 \}$, defined for the $F \in L$. Also, P_1 will denote a sub-semi-lattice of P which is isomorphic to $[L/I]$ and such that there is a meet homomorphism $a_F \to \bar{a}_F$ of R onto a dense subset of P_1. It will also be assumed that this homomorphism satisfies $a_F \le \bar{a}_F$, and $\bar{z} = z$. The existence of at least one such P_1 is assured by lemma 2.3.8. Later we will have occasion to use a P_1 which is different from the one constructed in the definition 2.3.6, but which still satisfies these conditions. The normal lattice function F_0 (on $P_1 = [L/I]$) is defined by:

$$F_0(\bar{a}) = \inf_{\bar{c} \le \bar{a}} \sup_{a \le \bar{c}} F(a)$$

for $\overline{a} \in P_1$.

Lemma 2.3.9. $F_0(\overline{a}) > \lambda$ if and only if there exists $\lambda' > \lambda$ such that $\overline{a}_{F-\lambda'} \geq \overline{a}$.

Proof. Suppose first that $F_0(\overline{a}) > \lambda$ and choose λ' so that $F_0(\overline{a}) > \lambda' > \lambda$. Then by the definition of F_0, $\sup\{F(a) \mid a \leq \overline{c}\} > \lambda'$ holds for all $\overline{c} \leq \overline{a}$, $\overline{c} \neq z$. Hence, $a_{F-\lambda'} \wedge \overline{c} \neq z$ and consequently $\overline{a}_{F-\lambda'} \wedge \overline{c} \neq z$ are valid for all $\overline{c} \leq \overline{a}$, $\overline{c} \neq z$. Since [L/I] is disjunctive, this is possible only if $\overline{a}_{F-\lambda'} \geq \overline{a}$.

To prove the converse, suppose that for some $\lambda' > \lambda$, $F_0(\overline{a}) < \lambda'$. By definition of F_0, there exists $\overline{c} \leq \overline{a}$ such that $\sup\{F(a) \mid a \leq \overline{c}\} < \lambda'$. This means that $a_{F-\lambda'} \wedge \overline{c} = z$. From this relation, it follows that $\overline{a}_{F-\lambda'} \wedge \overline{c} = z$. Indeed, $a_{F-\lambda'} \wedge \overline{c} = z$ implies that $a_{F-\lambda'} \wedge a_G = z$ for all a_G with $\overline{a}_G \leq \overline{c}$. Then $\overline{a}_{F-\lambda'} \wedge \overline{c} = \overline{a}_{F-\lambda'} \wedge \bigvee\{\overline{a}_G \mid \overline{a}_G \leq \overline{c}\} = \bigvee\{\overline{a}_{F-\lambda'} \wedge \overline{a}_G \mid \overline{a}_G \leq \overline{c}\} = \bigvee\{\overline{z} \mid \overline{a}_G \leq \overline{c}\} = z$. But $\overline{a}_{F-\lambda'} \wedge \overline{c} = z$ for $\overline{c}$ satisfying $z \neq \overline{c} \leq \overline{a}$ implies $\overline{a}_{F-\lambda'} \not\geq \overline{a}$. This completes the proof.

Now one can conclude rather easily all of the necessary preliminaries for the general uniqueness theorem.

Lemma 2.3.10. The mapping $F \to F_0$ is a translation lattice homomorphism of L into N([L/I])(the normal lattice functions on [L/I]).

Proof. It is clear from the definition of F_0 that $(F + \alpha)_0 = F_0 + \alpha$, and that $F \leq G$ implies $F_0 \leq G_0$. A direct consequence of this last relation is the fact that if F and G are any two elements of L,

$(F \wedge G)_0 \leq F_0 \wedge G_0$. It remains only to show that $(F \wedge G)_0 \geq F_0 \wedge G_0$.

If $\lambda < F_0(\bar{a})$ and $\lambda < G_0(\bar{a})$, then by the previous lemma, there is a $\lambda' > \lambda$ such that $\bar{a}_{F - \lambda'} \geq \bar{a}$ and $\bar{a}_{G - \lambda'} \geq \bar{a}$. Consequently, $\bar{a}_{(F \wedge G) - \lambda'} = \bar{a}_{F - \lambda'} \wedge \bar{a}_{G - \lambda'} \geq \bar{a}$. Again by lemma 2.3.9, this last relation is possible only if $(F \wedge G)_0(\bar{a}) > \lambda$. Since λ can be taken arbitrarily close to $F_0(\bar{a}) \wedge G_0(\bar{a})$, it follows that $(F \wedge G)_0(\bar{a}) \geq F_0(\bar{a}) \wedge G_0(\bar{a})$. $\bar{a}$ being a generic element, the proof is complete.

Lemma 2.3.11. $F_0 \leq 0$ if and only if $F \leq 0$.

Proof. It is clear from the definition of F_0 that if $F \leq 0$, then $F_0 \leq 0$. Conversely, $F \nleq 0$ implies that there is a number $\delta > 0$ such that $F - \delta \nleq 0$. Then $a_{F-\delta} \neq z$ so that $\bar{a}_{F-\delta} \neq z$. Thus by lemma 2.3.9, $F_0(\bar{a}_{F-\delta}) > 0$. This shows that $F_0 \leq 0$ implies $F \leq 0$.

Lemma 2.3.12. The image of L under the mapping $F \to F_0$ generates [L/I].

Proof. Suppose $\bar{a} \neq z$. Then $F \in L$ exists so that $z \neq \bar{a}_F \leq \bar{a}$. Now if $F_0(\bar{c}) > 0$, then by lemma 2.3.9, $\bar{c} \leq \bar{a}_F \leq \bar{a}$. At the same time $\bar{a}_F \neq z$ means that $F \nleq 0$, so by lemma 2.3.11, $F_0 \nleq 0$. In other words, there is a $\bar{b} \neq z$ such that $F_0(\bar{b}) > 0$. Necessarily $\bar{b} \leq \bar{a}$. We see that the conditions are satisfied for the image of L to generate [L/I].

Lemma 2.3.13. The relation $a_F \leq \bar{c}$ holds if and only if $\bar{a}_F \leq \bar{c}$ is true. In particular, $a_F \leq \bar{a}_G$ implies $G \geq F$ (I).

Proof. If $\bar{a}_F \leq \bar{c}$, then $G \in L$ exists satisfying $z \neq \bar{a}_G \leq \bar{a}_F$ and $\bar{c} \wedge a_G = z$. From the first of these relations $z \neq \bar{a}_F \wedge \bar{a}_G = \overline{a_F \wedge a_G}$; therefore, $a_F \wedge a_G \neq z$. Also $z = \bar{c} \wedge \bar{a}_G \geq \bar{c} \wedge a_G \wedge a_F = (\bar{c} \wedge a_F) \wedge a_G = a_F \wedge a_G \neq z$. This contradiction shows $\bar{a}_F \leq \bar{c}$. The converse implication follows from the assumption that $a_F \leq \bar{a}_F$. Finally, the last assertion is a consequence of the fact that the mapping $F \rightarrow \bar{a}_F$ generates an isomorphism of L/I into P_1.

This completes the preparations for the main theorem:

Theorem 2.3.4. Let P be a complete Boolean algebra. Let L be a translation lattice of normal lattice functions on P. Let $I = \{F \in L | F \leq 0\}$. Denote by R the sub-semi-lattice of P which consists of all elements of the form $a_F = \bigvee\{b | F(b) > 0\}$, where $F \in L$. Then,

(1) there exists a sub-semi-lattice P_1 of P which is isomorphic to $[L/I]$ and such that there is a meet homomorphism $a_F \rightarrow \bar{a}_F$ of R onto a dense subset of P_1 satisfying $a_F \leq \bar{a}_F$ and $\bar{z} = z$;

(2) if L generates P, the only P_1 satisfying (1) above is P itself and $\bar{a}_F = a_F$;

(3) conversely, if $P_1 = P$ satisfies the conditions of (1), then L generates P;

(4) if P_1 satisfies (1), denote by $\emptyset$ the isomorphism of P_1 onto $[L/I]$ satisfying $\emptyset(\bar{a}_F) = h_I(F)$ (where h_I denotes the natural homomorphism of L on L/I); let F_0 be the restriction of F to P_1 defined by

$$F_o(\overline{a}) = \inf_{\overline{c} \leq \overline{a}} \sup_{a \leq \overline{c}} F(a) \qquad (\overline{a}, \overline{c} \in P_1, a \in P).$$

Then, $F_o(\overline{a}) = F_F(\emptyset(\overline{a}))$ where F_F is the image of F in the natural representation $F \to F_F$ of L relative to I (see definition 2.3.4).

Proof. The conclusion (1) is an expression of lemma 2.3.8 and theorem 2.3.3. To prove (2), notice that by lemma 2.3.7, if L generates P, R is dense in P. If $\overline{a}_F \nleq a_F$, it follows that since R is dense in P, a_G exists satisfying $z \neq a_G \leq \overline{a}_F$ and $a_G \wedge a_F = z$. Then $\overline{a}_F \wedge \overline{a}_G = z$ and $z \neq a_G = a_G \wedge \overline{a}_F \leq \overline{a}_G \wedge \overline{a}_F = z$. This contradiction proves (2). The result (3) is an immediate consequence of lemma 2.3.7.

The difficult parts of the proof of the conclusion (4) have already been carried out. It is now largely a matter of assembling the pieces of the proof.

First notice that the mapping $F \to \overline{a}_F$ is a meet homomorphism of L onto a dense subset of P_1 with the kernel I. By theorem 1.2.1, it follows that the mapping $\emptyset$ defined by $\emptyset(h_I(F)) = \overline{a}_F$ is a uniquely defined isomorphism of L/I onto a dense subset of P_1. Hence it can be extended to a unique isomorphism of [L/I] onto P_1 (theorem 1.2.4).

The lemmas 2.3.10, 2.3.11, and 2.3.12, in conjunction with theorem 2.2.1, show that $F \to F_o$ is algebraically equivalent to the natural homomorphism of L onto L//I. We can, therefore, define unambiguously $H_I(F) = F_o$.

Denote $I' = H_I(I)$. By corollary 2.3.2, if $a \in$ [L/I] ,

(i) $$F_F(a) = F_{F_o}(\emptyset'(a)),$$

where $\emptyset'$ is determined by the condition

(ii) $$\emptyset'(h_I(F)) = h_{I'}(H_I(F)) = h_{I'}(F_o).$$

Theorem 2.3.2 (with lemma 2.3.12) shows that

(iii) $$F_{F_o}(\emptyset'(a)) = F_o(\emptyset''(\emptyset'(a))),$$

where $\emptyset''$ is uniquely determined by the requirement that

(iv) $$\emptyset''(h_{I'}(F_o)) = \overset{1}{\vee} \{\bar{b} \in P_1 \mid F_o(\bar{b}) > 0\}.$$

Here $\overset{1}{\vee}$ again symbolizes that the join is taken in P_1.

Relations (i) to (iv) show that for any $a \in [L/I]$,

$$F_F(a) = F_o(\emptyset''(\emptyset'(a)))$$

where

$$\emptyset''(\emptyset'(h_I(F))) = \overset{1}{\vee}\{\bar{b} \in P_1 \mid F_o(\bar{b}) > 0\}.$$

In order to complete the proof, we need only show that $\bar{a}_F = \overset{1}{\vee} \{\bar{b} \in P_1 \mid F_o(\bar{b}) > 0\}$. For then, defining $\emptyset = (\emptyset''\cdot\emptyset')^{-1}$ will give $F_F(\emptyset(\bar{a})) = F_o(\bar{a})$ and $\emptyset(\bar{a}_F) = h_I(F)$.

By lemma 2.3.9, $F_o(\bar{b}) > 0$ if and only if $\bar{a}_{F-\delta} \geq \bar{b}$ for some $\delta > 0$. Hence $\overset{1}{\vee} \{\bar{b} \in P_1 \mid F_o(\bar{b}) > 0\} = \overset{1}{\vee} \{\bar{a}_{F-\delta} \mid \delta > 0\}$. However, $\bar{a}_F \geq \overset{1}{\vee} \{\bar{a}_{F-\delta} \mid \delta > 0\} \geq \overset{2}{\vee} \{\bar{a}_{F-\delta} \mid \delta > 0\} \geq \overset{2}{\vee}\{a_{F-\delta} \mid \delta > 0\} = a_F$, where $\overset{2}{\vee}$ denotes the join operation in P. By lemma 2.3.13, it follows that $\bar{a}_F \leq \overset{1}{\vee}\{\bar{a}_{F-\delta} \mid \delta > 0\} \leq \bar{a}_F$. Consequently, $\bar{a}_F = \overset{1}{\vee} \{\bar{b} \in P_1 \mid F_o(\bar{b}) > 0\}$. The proof is complete.

2.3.7. Final remarks.

Before closing this section on the representation of translation lattices, it seems advisable to briefly summarize the results which have been obtained. Our interest has been centered on the problem of representing translation lattices as sets of normal lattice functions. It was shown that for large classes of function lattices such a representation can be obtained. Indeed, corresponding to each (closed and bounded) ideal of the translation lattice, there exists an intrinsic homomorphism of the translation lattice onto a translation lattice of normal lattice functions. In the case where the original translation lattice is divisible with respect to the given ideal, the representation is a true one, that is, an isomorphism. The uniqueness of this representation was studied in some detail, with the relation between the intrinsically defined representation and arbitrary representations being given special attention. While all this work was of interest in itself, the main purpose of the study was to lay the foundation for the next chapters.

In chapters three and four, the problem of representing translation lattices as sets of continuous functions on a compact Hausdorff topological space will be our chief concern. Two main problems will be studied. The first question is one of existence. We might ask whether or not it is always possible to map an abstract translation lattice of functions into the continuous functions on a compact Hausdorff space. The answer is very easily found to be affirmative. Indeed, it is known from section 2.2.3 that any translation lattice L is isomorphic to a translation lattice L' of bounded, real-valued functions on a set S. If S is made into a topological space by taking all the subsets $\{x|f(x) > 0\}$ and $\{x|f(x) < 0\}$ where $f \in L'$, as a sub-basis for the open sets, the functions in L' become continuous. By well known methods, it is then possible to imbed S in a compact Hausdorff space in a way which preserves continuous functions.

It is not enough, usually, to know merely that a representation exists. In general there will be many different representations. It is desirable then to find a representation which is "minimal" in some appropriate sense. This is the subject of the second question to be treated in the next two chapters. What requirements can be imposed on a representation in order that it may be said to be minimal? It is possible to formulate some general requirements which should be satisfied by the topological space S over which the representation is being made. It is natural to require that S be a uniquely determined compact Hausdorff space. Also it is to be hoped that any other space over which the lattice can be represented will bear some distinguished relationship to the minimal space S.

For the case where L is a distributive translation lattice and where only representations which preserve the join operation are considered, the problem of determining a minimal representative space can be solved successfully by methods entirely different from the one which will be presented in the following pages. It is possible to prove the following result: If L is a distributive translation lattice, then L can be imbedded in a uniquely determined (to isomorphism)

Archimedean ordered, vector lattice L' with a strong unit. Moreover, every lattice isomorphism of L into an Archimedean vector lattice V can be extended to an isomorphism of L' into V. Thus the study of lattice isomorphisms of a distributive translation lattice is equivalent to the study of the isomorphisms of an Archimedean ordered vector lattice with a strong unit. For such systems, the representation theory is well known (see for instance Kakutani [13] or Kadison [14]). The procedure, described above, for obtaining a representation by continuous functions will actually give a space S in which the points are separated by the functions of L'. Here the description "minimal" can be made precise as follows: if L is (lattice) isomorphic to a sub-(distributive) translation lattice of C(S'), where S' is compact Hausdorff, then S is homeomorphic to a factor space of S'.

It is unfortunate that the methods used to prove this general result cannot be applied to the problem of the representation of arbitrary translation lattices. Unless it is assumed that both the meet and the join operation are preserved, and that these operations distribute between each other, then the techniques used to prove the imbedding theorem will not work. For this reason, no attempt will be made to present here the proof that a distributive translation lattice can be uniquely imbedded in an Archimedean vector lattice. Instead, we will exploit the results of chapter II to obtain a reasonable definition of a minimal representation. It is shown that the space over which this representation is made is uniquely determined up to homeomorphisms, and that it fits the description "minimal" in a sense which will be explained later.

SUMMARY OF CHAPTER III

Chapter three is devoted to the description of the topological prerequisites for the representation theory to be developed in the final chapter. The first section is a discussion of well known theorems on point set topology. No more is included than will be used later in the thesis. In the second section, a method of constructing topological spaces from complete Boolean algebras is presented. Again the policy of presenting only the absolute essentials is followed.

CHAPTER III — TOPOLOGICAL FOUNDATIONS

3.1 Fundamental definitions.

In this section, the definitions and the notation which will be used in chapter IV will be outlined. It will be assumed that the reader already possesses a working knowledge of the fundamental ideas of point set topology. References on topology from which the notation and definitions used in this thesis are taken include Alexandroff and Hopf [15], Bohnenblust [7], Bourbaki [16] and Lefschetz [17].

3.1.1 Definitions of a topological space.

Definition 3.1.1. A set S is called a topological space if a distinguished family $\mathfrak{F}$ of subsets of S is defined satisfying:

(1) the union of any sub-collection of $\mathfrak{F}$ is in $\mathfrak{F}$;

(2) the intersection of any finite sub-collection of $\mathfrak{F}$ is in $\mathfrak{F}$;

(3) the empty set and the whole set S are in $\mathfrak{F}$;

The subsets of the distinguished family are called the open sets of the space S.

A topology on a set can also be defined in terms of a neighborhood system.

Definition 3.1.2. A family $\mathfrak{Z}$ of subsets of S is called a neighborhood system (or a basis for the open sets) whenever:

(1) $x \in A \cap B$, where A and B are in $\mathfrak{F}$, implies that C in $\mathfrak{F}$ exists such that $x \in C \subseteq A \cap B$;

(2) $\bigcup\{A \mid A \in \mathfrak{F}\} = S$.

If $\mathfrak{F}$ is a neighborhood system for a set S, then S is a topological space when the open sets are defined to be the unions of sets in $\mathfrak{F}$.

For any topological space S, it is possible to define a closure operation on all the subsets of S in the following way: For any subset T of S, a point x is said to belong to the closure of T -- denoted T^- -- in case every open set containing x also contains a point of T. With this definition, it is easily verified that the closure postulates are satisfied: (1) $T^- \supseteq T$; (2) $T^{--} = T^-$; (3) $T_1^- \cup T_2^- = (T_1 \cup T_2)^-$; (4) $\emptyset^- = \emptyset$. The symbol $\emptyset$, here as in all that follows, denotes the empty set. A closed set is defined to be one which is identical with its closure. It can then be proved that a set is closed if and only if its complement is open. The topology of a space may also be defined either in terms of its closure operation or in terms of its collection of closed sets. A basis for the closed sets can be defined in a way analogous to the neighborhood system in definition 3.1.2. It is well known that all of these definitions of a topological space are equivalent.

The dual of the concept of closure is important for our later work. If T is an arbitrary subset of S, the interior of T is defined to be the set T^{c-c}. (Here, as always, the superscript c denotes the operation of taking the complement of the set relative to all of S. Thus T^c is the set consisting of all points of S which are not contained in T.) For the purpose of abbreviation, T^o will be written for T^{c-c}. Closely connected with the interior operation is $T^{-o} = T^{-c-c}$. A set T is open if and only if $T^o = T$. T is called a regular open set if $T^{-o} = T$. An important property of the operation $T \to T^{-o}$ is the identity $T_1^{-o} \cap T_2^{-o} = (T_1 \cap T_2)^{-o}$ which is valid for every pair of open sets T_1 and T_2. Another fact of importance is that the collection of all regular open sets of a topological space forms a complete Boolean algebra in which the (finite) meet operation is just set intersection.

Two of the most important concepts of topology are continuity and homeomorphism.

Definition 3.1.3. A mapping from one topological space into another is called continuous if the inverse image of every open set is an open set. Two topological spaces are said to be homeomorphic if there is a one-to-one mapping of one of them onto the other which is continuous and such that its inverse is continuous.

An important special case of this definition is the continuous mapping of a topological space into the real number system. Such a mapping is called a real-valued continuous function. A more convenient criterion for continuity of a real-valued function is the requirement that all sets of the form $\{x | f(x) > \lambda\}$ and $\{x | f(x) < \lambda\}$ be open.

3.1.2. Additional properties of topological spaces.

It will be assumed that the reader is familiar with the standard separation axioms for a topological space, namely, T_0, T_1, T_2 (= Hausdorff), regular and normal. A form of separation which is not so well known is that of semi-regularity (see Stone [18]):

Definition 3.1.4. A topological space is called semi-regular if its regular open sets form a basis for the topology of the space.

Most of the interest of chapter IV will be centered on that very important class of spaces — the compact Hausdorff topological spaces.

Definition 3.1.5. A topological space S will be called compact (bi-compact in the terminology of Alexandroff and Hopf), if it satisfies the condition that from every covering of the space by open sets (a covering by open sets is a collection of open sets such that every point of the space is contained in at least one open set of the collection), a finite covering can be selected.

An alternative definition of a compact space is the following: if a collection of closed sets of the space has the property that no finite intersection of them is empty, then there is at least one point of the space which is common to all the sets of the collection.

The properties of a compact space are in many ways quite simple. Thus for example, every compact Hausdorff space satisfies all of the separation axioms named above. At the same time, compact Hausdorff spaces are sufficiently general that much of the study of (bounded) real-valued functions on an arbitrary topological space can be reduced to the study of functions on a compact Hausdorff space (as Stone [18] and Cech [19] have shown).

Another important concept which may not be too familiar is the idea of a factor space of a topological space.

Definition 3.1.6. Let S be a topological space. Let S' be a set of disjoint closed subsets of S whose union contains S. Topologize S' by calling a collection of sets in S' open if its union in S is an open set. Then S' is called a factor space of S.

An alternative characterization of a factor space of a compact space can be given as follows: A Hausdorff space S' is a factor space of the compact space S if and only if there is a continuous mapping of S onto S'.

Let S be a topological space. Let T be a subset of S. Then T can be topologized by taking all the sets of the form $A \cap T$ with A open in S as the collection of open sets of T. The topology so obtained is called the relative topology of T. An important property of compact spaces is that every closed subset of a compact space is compact in its relative topology.

3.2 The construction of topological spaces.

The relationship between a topological space and its lattice of open (or dually, its closed) sets has been studied by several authors. The pioneer work in this field is that of Stone [18]. Stone considered the topological space obtained in a certain way from a Boolean algebra. The points of this space are the minimal dual ideals of the given Boolean algebra. The space is topologized by taking as a basis for the open (and closed) sets, those collections of ideals which contain a given element of the Boolean algebra. The spaces obtained in this way are precisely the zero-dimensional compact Hausdorff spaces -- the so called Boolean spaces. Since Stone's original work, many generalizations of the method have been studied. Of particular importance is the work of Wallman [5]. Wallman generalized Stone's ideas by constructing the space from a distributive lattice rather than a Boolean algebra. For Wallman's space, the collection of sets $\{X | a \in X\}$ (where a is an element of the lattice and X is a minimal dual ideal) are taken as a basis for the closed sets. The class of spaces obtained in this way is just the set of all compact T_1 spaces. However, this is not the only possible way of generalizing the idea of Stone. In the few pages that follow, a different means of constructing topological spaces from a given (complete) Boolean algebra will be described. The technique has some advantages over the Wallman construction and, of course, some disadvantages.

3.2.1 The construction of topological spaces.

The process which is to be used can be motivated as follows. Consider a given T_0 topological space. This system can be conceived as a set of points together with a collection of distinguished subsets called the open sets of the space. This collection of subsets enjoys certain lattice properties: it is closed under finite intersections and unlimited unions; it contains the whole space of points and the empty set. From this point of view, the points of the space are assumed to be things which are given in advance. The open sets are certain collections of the points. The observation that the points of a T_0 space are distinguished by the open sets which contain them, leads to another characterization of topological spaces. In this characterization, the lattice of open sets (considered as an abstract lattice) is the primitive notion. Points are then distinguished subsets of this lattice -- in fact they are dual ideals of the lattice. This is the basic idea behind the remainder of the work of this chapter. The fundamental idea of Wallman's paper differs from this only by replacing the open sets containing a point by the closed sets containing it. The difference between the resulting theories, however, is remarkably great.

Proposition 3.2.1. Let P be a semi-lattice with zero z. Let S be any non-empty collection of non-trivial (not empty and not all of P) dual ideals X. Call the distinct ideals of S its points and take the sets of the form

$$S(a) = \{ X \in S | a \in X \}$$

as open sets in S. Then these sets constitute a basis for the open sets of a T_0 topologization of S. Moreover, $S(a) \frown S(b) = S(a \wedge b)$ and S(z) is the empty set.

The relation of this conception of topological spaces to the more conventional one can be seen from the following:

Proposition 3.2.2. Let S be a T_o topological space and let P be a basis for the open sets of S. Choose P so that it is closed under set intersection and contains the empty set. Let S' be the collection of all dual ideals of the form

$$X_x = \{ a \in P | x \in a \}.$$

If S' is topologized by taking the sets of the form

$$S'(a) = \{ X_x \in S' | a \in X_x \}$$

as a basis P' for the open sets, then S' is a T_o topological space which is homeomorphic to S. Moreover, $a \to S(a)$ is a meet isomorphism of P onto P'.

Proof of proposition 3.2.1. First it will be shown that $S(a) \cap S(b) = S(a \wedge b)$. If $X \in S(a) \cap S(b)$, then $a \in X$ and $b \in X$. Since X is a dual ideal, this means that $a \wedge b \in X$ and, therefore, $S(a) \cap S(b) \subseteq S(a \wedge b)$. If $a \wedge b \in X$, then $a \in X$ and $b \in X$ so $X \in S(a)$ and $X \in S(b)$. Thus $S(a \wedge b) = S(a) \cap S(b)$. Also, S(z) is empty, since no $X \in S$ is all of P. That is, no X contains z.

It follows immediately that the first postulate for a neighborhood system (see definition 3.1.2) is satisfied. The second postulate is also satisfied since $\bigcup \{S(a) | a \in P\} = S$, every $X \in S$ being non-empty.

Finally S is a T_o space. For if $X \neq Y$ are in S, then either a exists in X and not Y, or there is an element b in Y and not X. In the first case $X \in S(a)$, $Y \notin S(a)$, and in the second $Y \in S(b)$ and $X \notin S(b)$.

Proof of proposition 3.2.2. It is a consequence of proposition 3.2.1 that S' is a T_0 topological space. Moreover, $a \rightarrow S'(a)$ is a meet homomorphism of P onto P'. It is one-to-one since if $a \nleq b$, there is an x of S such that $x \in a$ and $x \notin b$. Then $X_x \in S'(a)$ and $X_x \notin S'(b)$. Finally S' is homeomorphic to S since $x \rightarrow X_x$ is a one-to-one mapping of S onto S' which carries the basis P onto the basis P'.

3.2.2. Spaces constructed from Boolean algebras.

In the development of the theory of the spaces which are defined by proposition 3.2.1, it is convenient to impose a restriction on the set P. Instead of using an arbitrary semi-lattice P, it will always be assumed that P is a complete Boolean algebra. This has the advantage of simplifying the study somewhat. Thus the statements of the results are much simpler, and at the same time, there is little loss of generality. Moreover, the previous work led quite naturally to lattice functions constructed on complete Boolean algebras. This suggests that Boolean algebras are the appropriate systems from which to construct our topological spaces.

Lemma 3.2.1. Let P be a complete Boolean algebra. Let S be a non-empty collection of non-trivial dual ideals of P. Topologize S as in proposition 3.2.1. Then the mapping $a \rightarrow S(a)$ of P onto the basis for the open sets of S is an isomorphism if and only if the following condition is satisfied:

R: if $a \in P$ and $a \neq z$, there is an element X of S with $a \in X$.

Proof. By proposition 3.2.1, the mapping $a \rightarrow S(a)$ is a meet homomorphism of P onto the set of S(a)'s. The condition R is just an expression of the requirement that the kernel of this homomorphism be z. Hence, condition R is necessary for the mapping to be one-to-one. Its sufficiency is a consequence of proposition 1.2.2.

Definition 3.2.1. Let P be a complete Boolean algebra. Let S be a T_0 topological space which is constructed with non-trivial dual ideals X as its points; let the sets of the form $S(a) = \{X | a \in X\}$ be a basis for the open sets of S (i.e., according to proposition 3.2.1); assume that the condition R, i.e., that S(a) is non-empty whenever $a \neq z$ is satisfied. Then S will be called a representative space for P. The symbol S(P) will always denote such a space.

The justification for this terminology will be furnished by theorem 3.2.1 below.

Shortly, a topological criterion that a space be homeomorphic to an S(P) will be obtained. First, however, it is convenient to consider the closure topology of a space S(P).

Lemma 3.2.2. Let P be a complete Boolean algebra and let S(P) be a representative space for P (according to definition 3.2.1). Let T be an arbitrary subset of S(P). Then the closure of T is given by $T^- = \{X \in S(P) | X \subseteq \cup T\}$ where $\cup T = \cup \{X | X \in T\}$ (set operations).

Proof. If $X \subseteq \bigcup T$, and if $a \in X$, then $a \in Y$ for some $Y \in T$. Thus every neighborhood of X contains a point of T so that $X \in T^-$. Conversely, if $X \in T^-$, every neighborhood of X contains a point of T. Hence if $a \in X$, then Y in T exists so that $a \in Y$, that is, $X \subseteq \bigcup T$.

Corollary 3.2.1. $X \in S(a)^-$ if and only if $a \wedge b \neq z$ holds for all $b \in X$.

Proof. If $a \wedge b \neq z$ whenever $b \in X$, then it follows from the condition R that Y exists in S(P) satisfying $a \wedge b \in Y$. Since $a \in Y$, $Y \in S(a)$. Because b was chosen arbitrarily from X, it follows that $X \subseteq \bigcup S(a)$. Conversely, if $X \subseteq \bigcup S(a)$ and $b \in X$, there is a $Y \in S(a)$ so that $b \in Y$. Since a and b are both in Y, and since Y is a non-trivial dual ideal, it follows that $a \wedge b \in Y$ and consequently $a \wedge b \neq z$. The proof is complete.

Now it is possible to give a topological characterization of the S(P) spaces. One preliminary lemma is needed.

Lemma 3.2.3. Let P be a complete Boolean algebra. Let S(P) be any representative space for P. Then the regular open sets of S(P) are precisely those of the form S(a).

Proof. By corollary 3.2.1, $S(a)^- = \{ X \in S(P) | a \wedge b \neq z \text{ all } b \in X \}$. Thus $S(a)^{-c} = \{ X \in S(P) \mid a \wedge b = z \text{ some } b \in X \} = \{ X \in S(P) | a' \in X \}$. It follows that $S(a)^{-c-c} = \{ X \in S(P) | a' \in X \}^{-c} = \{X \in S(P) | (a')' \in X \} = \{ X \in S(P) | a \in X \} = S(a)$. Thus S(a) is a regular open set.

Conversely, suppose R is any regular open set of S(P). Then by the definition of the topology of S(P), we can write

$$R = \bigcup \{S(a) \mid a \in A\},$$

A being a certain subset of P. Since P is a complete Boolean algebra, it is possible to define $b = \bigvee \{a \mid a \in A\}$. The proof will be completed by showing that $R = S(b)$.

By lemma 3.2.2, $X \in R^-$ if and only if $X \subseteq \bigcup R$. Thus $Y \in R^{-c}$ if and only if $a_o \in Y$ exists so that $a_o \notin X$ for all $X \in R$. Now it will be shown that $a_o \notin X$ holds for all $X \in R$ if and only if $a_o \wedge b = z$ (where, of course, $b = \bigvee \{a \mid a \in A\}$). Clearly, since $b \in X$ holds for all $X \in R$, if $a_o \wedge b = z$, then $a_o \notin X$ for all $X \in R$. Suppose $a_o \wedge b \neq z$. Then $a_o \wedge \bigvee\{a \mid a \in A\} = \bigvee\{a_o \wedge a \mid a \in A\} \neq z$, so $a \in A$ exists satisfying $a_o \wedge a \neq z$. Because of the condition R, it is possible to find $X_o \in S(P)$ containing $a_o \wedge a$. Then $a \in X_o$ and $a_o \in X_o$. From these, $X_o \in S(a) \subseteq R$ and $a_o \in X_o \in R$. This proves the assertion that $a_o \notin X$ for all $X \in R$ if and only if $a_o \wedge b = z$.

The consequence of this is $R^{-c} = \{Y \in S(P) \mid a_o \wedge b = z$ some $a_o \in Y\} = \{Y \in S(P) \mid b' \in Y\}$. Applying the result of the first paragraph and using the fact that R is a regular open set gives $R = R^{-c-c} = \{Y \in S(P) \mid b' \in Y\}^{-c} = \{X \in S(P) \mid b \in X\} = S(b)$. This completes the proof.

<u>Theorem 3.2.1</u>. Let P be a complete Boolean algebra. Then the class of all representative spaces for P is just the class of semi-regular T_o topological spaces whose Boolean algebra of regular open

sets is isomorphic to P.

Proof. By lemma 3.2.3, if S(P) is a representative space for P, its Boolean algebra of regular open sets is just the collection of S(a). These open sets form a basis for the open sets of S(P) by definition. Therefore S(P) is semi-regular. Also, by lemma 3.2.1, this collection is isomorphic to P.

Conversely, if S is a semi-regular topological space whose Boolean algebra of regular open sets is isomorphic to P, then by proposition 3.2.2, S is homeomorphic to a space S(P). This completes the proof.

3.2.3. Final remarks.

It is not our intention to develop here the theory of representative spaces of a Boolean algebra. In this chapter, hardly more than the essential definitions have been presented. The following chapter will add slightly to the theory, but no more will be included than is needed for the development of the central subject of the thesis. Before beginning the next chapter, there is one more result which belongs to the general theory of representative spaces and which is necessary for the work to follow. This would seem to be the correct place to present it.

Lemma 3.2.4. Let P be a complete Boolean algebra and let S(P) be a representative space for P. Then S(P) is a Hausdorff topological space if and only if, any two distinct points X and Y in S(P), there exist a and b in P such that $a \in X$, $b \in Y$ and $a \wedge b = z$.

Proof. If the conditions are satisfied, $X \in S(a)$, $Y \in S(b)$ and $S(a) \frown S(b) = S(z)$ is empty. Thus $S(P)$ is a Hausdorff space.

Conversely, suppose $S(P)$ is a Hausdorff space and $X \neq Y$. Then, since the sets $S(a)$ form a basis for $S(P)$, a and b exist in P satisfying $X \in S(a)$, $Y \in S(b)$ and $\emptyset = S(a) \frown S(b) = S(a \wedge b)$. Consequently $a \in X$, $b \in Y$ and $a \wedge b = z$. This completes the proof.

Example 1. Let P be any complete Boolean algebra. We may ask the question: is it always possible to find a representative space $S(P)$ for P? The answer is, of course, yes. We need merely take $S(P)$ to be the set of all principal ideals. However, it may still be asked whether it is possible to obtain spaces with specific topological characteristics. Is it always possible, for example, to find a compact Hausdorff representative space for P? The answer is again yes. The space $\gamma(P)$ constructed from all minimal dual ideals of P is a compact Hausdorff representative space for P. This fact is a corollary of the next example.

Example 2. Let P be a complete Boolean algebra. Let $\overline{P}$ be a sub-algebra of P which is dense in P. The algebra $\overline{P}$ will not be complete unless it coincides with P. Let $S(P)$ be the set of all dual ideals X which are such that $X \frown \overline{P}$ is a minimal dual ideal in $\overline{P}$. In other words, $S(P)$ is the set of all dual ideals of P generated by a minimal dual ideal of $\overline{P}$. Make $S(P)$ into a topological space by the method of definition 3.2.1. Then $S(P)$ is a representative space for P. For if $b \neq z$ in P, there is an element $\overline{a}$ of $\overline{P}$ with $z \neq a \leq b$.

By the maximal principle, a minimal dual ideal $\overline{X}$ of $\overline{P}$ exists with $\overline{a} \in \overline{X}$. If $X = \{a \in P | a \geq \overline{b} \text{ some } \overline{b} \in \overline{X}\}$, $X \in S(P)$, and $a \in X$.

It is clear that the sets of the form $S(\overline{a})$, where $\overline{a} \in \overline{P}$, constitute a basis for the open sets of $S(P)$. We will show that these are precisely open and closed sets of $S(P)$. Indeed, $X \in S(\overline{a})^-$ implies $\overline{a} \wedge \overline{b} \neq z$ for all $\overline{b} \in X$. But since $X \cap \overline{P}$ is minimal, it follows that $\overline{a} \in X$. In other words, $X \in S(\overline{a})$. This proves that $S(\overline{a})$ is closed. An immediate consequence is the fact that $S(P)$ is a totally disconnected, Hausdorff space.

We have yet to prove that the only open and closed sets are those of the form $S(\overline{a})$, where $\overline{a} \in \overline{P}$. Every open and closed set is regular open and hence of the form $S(a)$ for some $a \in P$. If $a \notin \overline{P}$, it is possible to construct a minimal dual ideal $\overline{X}$ of $\overline{P}$ with the property that $a \wedge \overline{b} \neq z$ for all $\overline{b} \in \overline{X}$, while $\overline{b} \leq a$ holds for no $\overline{b} \in \overline{X}$. To do this, let $\overline{Y} = \{\overline{c} \wedge \overline{d} | a \leq \overline{c} \text{ and } a' \leq \overline{d}\}$. Clearly $\overline{Y}$ is closed under meets. Since $a \in \overline{P}$, $\overline{Y}$ does not contain z. Hence it is possible to extend $\overline{Y}$ to a minimal dual ideal $\overline{X}$. It is easy to see that $\overline{X}$ has the desired properties: $a \wedge \overline{b} \neq z$ and $\overline{b} \nleq a$ for all $\overline{b} \in \overline{X}$. Now if X is the point of $S(P)$ generated by $\overline{X}$, $X \in S(a)^-$, while $X \notin S(a)$. Thus $S(a)$ is not open and closed. This completes the proof that the sets $S(\overline{a})$ are precisely the open and closed subsets of $S(P)$. In conclusion, it will be shown that $S(P)$ is compact.

As we proved above, the sets of the form $S(\overline{a})$ constitute a basis for the <u>closed</u> sets of $S(P)$ as well as a basis for the open sets.

Suppose $\mathfrak{A} = \{ S(\bar{a}) | \bar{a} \in \bar{A} \}$ is a collection of sets of this form, and that the sets of $\mathfrak{A}$ have the finite intersection property. Then $\bar{a} \in \bar{A}$ and $\bar{b} \in \bar{A}$ implies that $\bar{a} \wedge \bar{b} \neq z$. By the maximal principle, it is possible to find a minimal dual ideal $\bar{X}$ of $\bar{P}$ with $\bar{X} \leq \bar{A}$. Letting X be the dual ideal of P which is generated by $\bar{X}$, X has the property that if $\bar{a} \in \bar{A}$, then $\bar{a} \in X$. Hence $X \in \bigcap \{ S(\bar{a}) | \bar{a} \in \bar{A} \}$. Because the sets of the form $S(\bar{a})$ constitute a basis for the closed sets of $S(P)$, it follows that $S(P)$ is compact.

Summarizing these results: we have proved that $S(P)$ is precisely the Boolean space associated with the Boolean algebra $\bar{P}$ (see Stone [18]).

SUMMARY OF CHAPTER IV

This chapter deals with the problem of representing translation lattices by means of continuous functions. Both the existence and the uniqueness of such representations are discussed.

In section one, representation by means of translation lattices of normal lower semi-continuous functions on a topological space is considered. We show that this problem is completely equivalent to the problem of representation by means of a translation lattice of normal lattice functions. Thus, all of the theory which was developed in the last section of the previous chapter can be transferred bodily to the problem under consideration.

Section two of this chapter is devoted to the proof of the existence of a representation by continuous functions for translation lattices which are divisible with respect to a bounded closed ideal. It is shown that any translation lattice L of normal lattice functions, which is divisible with respect to the bounded closed ideal $\{F \in L | F \leq 0\}$, can be represented as a translation lattice of continuous functions on a compact Hausdorff topological space in such a way that: (1) the functions of the representation separate the points of the space; (2) these functions generate the topology of the space in the sense that the sets of the form $\{x | f(x) > 0\}$, $f \in L$, are coinitial in the opensets of the space. A uniqueness theorem is established for representations of this nature.

The final section of this chapter is a study of the relation between the spaces S which are such that the lattices C(S) contain a sub-semi-lattice isomorphic to a given translation lattice L.

CHAPTER IV — REPRESENTATION BY CONTINUOUS FUNCTIONS

4.1 Function lattices.

In this section the relation between normal lattice functions and normal lower semi-continuous functions (see definition 4.1.1) on a topological space will be studied. In particular, the results of chapter II will be interpreted in terms of translation lattices of point functions. It will be shown that the problem of representing a translation lattice as a translation lattice of normal lower semi-continuous functions is completely equivalent to the problem of representation in terms of normal lattice functions. The following two sections will then be devoted to the problem of representation by means of continuous functions.

4.1.1. Normal lower semi-continuous functions.

Let P be a complete Boolean algebra. In the last chapter, it was shown that it is possible to construct from P a topological space which is semi-regular and has its Boolean algebra of regular open sets isomorphic to P. Suppose S(P) is such a representative space for P. Let f be a bounded, real-valued, point function on S(P). Then if we define, for $a \in P$, $F(a) = \inf\{f(X) | X \in S(a)\} = \inf\{f(X) | a \in X\}$, F(a) is evidently a lattice function. (It is a bounded, real-valued function on P which satisfies $F(a) \leq F(b)$ whenever $a \geq b$.) The question then comes naturally to mind: which lattice functions on P are of this form? Without trying to answer this question, we will prove a related result.

Proposition 4.1.1. Let F be a normal lattice function on the complete Boolean algebra P. If S(P) is a representative space for P, then there is a point function f on S(P) such that for all $a \in P$, $F(a) = \inf \{ f(X) | a \in X \}$. Conversely, suppose that F is a lattice function with the property that if S(P) is any representative space for P, there is a point function on S(P) such that $F(a) = \inf\{f(X) | a \in X\}$ for all $a \in P$. Then F is normal.

Proof. First suppose F is a normal lattice function. Let S(P) be a representative space for P. Define the point function f on S(P) by $f(X) = \sup \{ F(a) | a \in X \}$. We will show that $F(a) = \inf \{ f(X) | a \in X \}$ is true for all $a \in P$.

If $a \in X$, then $f(X) \geq F(a)$. Hence $F(a) \leq \inf \{ f(X) | a \in X \}$. Suppose $\inf \{ f(X) | a \in X \} > F(a) + \delta$ where $\delta > 0$. Then for every X with $a \in X$, b_X exists satisfying $b_X \in X$ and $F(b_X) > F(a) + \delta$ (by the definition of f). Let $b = \bigvee \{ b_X | a \in X \}$. Then $b \in X$ for all $X \in S(a) = \{ X \in S(P) | a \in X \}$. Thus $S(a) \subseteq S(b)$ and, because $a \to S(a)$ is an isomorphism, $a \leq b$. Hence $F(a) \geq F(b) = \inf \{ F(b_X) | a \in X \} \geq F(a) + \delta$ (by the normality of F). This impossibility shows that $F(a) \geq \inf \{ f(X) | a \in X \}$. Thus the first assertion is proved.

To prove the converse, let A be an arbitrarily chosen non-empty subset of $P - \{ z \}$. Put $a = \bigvee \{ b | b \in A \}$. It must be shown that $F(a) = \inf \{ F(b) | b \in A \}$.

Define a representative space S(P) for P in the following way: S(P) is the collection of all minimal dual ideals which contain either a' or an element b from A. Then S(P) is a representative space according to the definition 3.2.1. To prove this it is necessary to show that the condition R (every $c \in P$ is contained in at least one X of S(P)) is satisfied. If $c \in P$, then either $c \wedge a' \neq z$ or $c \wedge b \neq z$ for some $b \in A$. This means that there is a minimal dual ideal (by the maximum principle) X which contains $c \wedge a'$ or $c \wedge b$ (some $b \in A$). Then $c \in X$ and $X \in S(P)$, so the condition R is fulfilled.

By the assumption of the theorem, there is a function f on S(P) such that $F(c) = \inf \{ f(X) | c \in X \}$ holds for all $c \in P$. But $F(a) = \inf \{ f(X) | a \in X \} = \inf \{ f(X) | b \in X, \text{ some } b \in A \}$, (since $a \in X$ if and only if there is a $b \in A$ with $b \in X$). This last term is equal to $\inf \{ \inf \{ f(X) | b \in X \} | b \in A \} = \inf \{ F(b) | b \in A \}$. Since this is what had to be shown, it follows that F is normal. The proof is complete.

In the first part of this proof, more was demonstrated than was stated in the proposition. The excess can be formulated as a corollary.

Corollary 4.1.1. Let F be a normal lattice function on the complete Boolean algebra P. Let S(P) be an arbitrary representative space for P. Then the point function f, defined by

$$f(X) = \sup \{ F(a) | a \in X \},$$

has the property that, for all $a \in P$,

$$F(a) = \inf \{ f(X) | a \in X \} .$$

The functions obtained from normal functions in the manner of the corollary 4.1.1 by defining $f(X) = \sup \{ F(a) | a \in X \}$ are of a rather special kind. We will now show that they are precisely the normal lower semi-continuous functions of the representative space S(P).

<u>Definition 4.1.1.</u> Let S be a topological space. Let f be a bounded real valued function on S. Define the functions f^* and f_* by

$$f^*(x) = \overline{\lim_{y \to x}}\, f(y) = \inf_{x \in N} \sup_{y \in N} f(y) \quad (N \text{ open})$$

$$f_*(x) = \underline{\lim_{y \to x}}\, f(y) = \sup_{x \in N} \inf_{y \in N} f(y) \quad (N \text{ open})$$

The function f is called lower (upper) semi-continuous if $f_* = f$ (if $f^* = f$). The function will be called (see Dilworth [1]) normal lower (upper) semi-continuous if $(f^*)_* = f$ (if $(f_*)^* = f$).

The properties of upper and lower semi-continuous functions are sufficiently well known that they need not be enumerated here. We note only that the following condition is equivalent to the definition above for lower semi-continuity:

$$\{ x \in S | f(x) > \lambda \} \quad \text{is open for all } \lambda .$$

A dual characterization can be given for lower semi-continuous functions. On the other hand, familiarity with normal lower semi-continuous functions cannot be assumed. We will, therefore, reproduce from [1] a convenient characterization of normal functions in the set of lower semi-continuous functions.

Before stating and proving this result, it will be helpful to collect some of the well known properties of the $\overline{\lim}$ and $\underline{\lim}$ operations. These are most easily stated in terms of the (*) operations.

(a) $f^* \geq f \geq f_*$;

(b) $(f^*)^* = f^*$, $(f_*)_* = f_*$;

(c) $f \geq g$ implies $f^* \geq g^*$ and $f_* \geq g_*$;

(d) $(((f^*)_*)^*)_* = (f^*)_*$, $(((f_*)^*)_*)^* = (f_*)^*$;

Of these identities, only the last needs proof. Notice $(f^*)_* \leq f^*$, so that $((f^*)_*)^* \leq (f^*)^* = f^*$. But again $((f^*)_*)^* \geq (f^*)_*$, so $(((f^*)_*)^*)_* = ((f^*)_*)_* = (f^*)_*$. The dual relation is proved similarly.

Lemma 4.1.1. (Dilworth) Let f be a lower semi-continuous function on a topological space S. Then f is normal if and only if the following condition is satisfied:

If $x \in S$ and $f(x) < \lambda$, and if N is an arbitrary neighborhood of x, then a non-empty open set $A \subseteq N$ exists such that $f(y) < \lambda$ for all $y \in A$.

Proof. Suppose f is normal, $x \in S$, $f(x) < \lambda$ and N is a neighborhood of x. Then $\inf_{z \in N} f^*(z) \leq \sup_{x \in N} \inf_{z \in N} f^*(z) = (f^*)_*(x) = f(x)$. Thus $z \in N$ exists so that $f^*(z) < \lambda$. Consequently, there is an open B containing z such that $\sup_{y \in B} f(y) < \lambda$. Taking $A = N \cap B \neq \emptyset$ gives the asserted condition.

Conversely, let the condition be satisfied for f. Let $x \in S$; suppose N is a neighborhood of x; choose $\lambda = f(x) + \delta$ (where δ is an arbitrarily small positive number). Then by hypothesis a non-empty open $A \subseteq N$ exists such that $\sup_{y \in A} f(y) < f(x) + \delta$. If $z \in A$, $f^*(z) < f(x) + \delta$. Thus $\inf_{z \in N} f^*(z) < f(x) + \delta$, and since N was arbitrarily chosen, $(f^*)_*(x) \leq f(x) + \delta$. Finally since δ and x were arbitrary, $(f^*)_* \leq f$. It is clear on the other hand that $f^*(x) \geq f(x)$. Hence $f^* \geq f$. But the (*) operations obviously preserve order, so it follows that $(f^*)_* \geq f_* = f$ (since f is lower semi-continuous). This completes the proof that $(f^*)_* = f$.

The simplest examples of normal functions are the continuous functions on a topological space. The fact that every continuous function is normal, follows from the characterization of continuous functions as those which satisfy $f = f^* = f_*$.

Using the above characterization of normal lower semi-continuous functions, it is possible to establish the relationship between normal lattice functions and normal lower semi-continuous functions.

<u>Theorem 4.1.1.</u> Let P be a complete Boolean algebra. Let S(P) be an arbitrary representative space for P (i.e., a semi-regular topological space which has its Boolean algebra of regular open sets isomorphic to P). Then the mapping $F \to f$, where

(a) $\quad f(X) = \sup \{ F(a) | a \in X \}$, is an isomorphism of the translation lattice of normal lattice functions on P onto the set of all normal lower semi-continuous functions on S(P). Moreover, if $F \to f$ by this mapping, then

(b) $$F(a) = \inf \{ f(X) | a \in X \} .$$

Proof. The proof will be carried out in three steps. First, it will be shown that the function f, defined by (a), is normal lower semi-continuous. Next, it will be proved that if f is any normal lower semi-continuous function on S(P), and if F is defined from F by the equation (b), then F is a normal lattice function and (a) is satisfied. These two results, together with corollary 4.1.1, show that $F \to f$ defined by (a) is a one-to-one mapping of N(P) onto N(S(P)). The proof is completed by showing that this mapping is also an isomorphism.

First, suppose that F is a normal lattice function. Define f by (a). If $f(X) > \lambda$, there is an $a \in P$ $(z \neq a)$ such that $F(a) > \lambda$. Then if $Y \in S(a)$, $f(Y) \geq F(a) > \lambda$. Thus f is lower semi-continuous. To prove that it is also normal, suppose now that $f(X) < \lambda$. Let λ' be such that $f(X) < \lambda' < \lambda$. By the definition of f, $F(a) < \lambda'$, for all $a \in X$. Consider an $a \in X$. By proposition 2.3.1, $b \in P$ exists with $z \neq b \leq a$ and $F(c) < \lambda'$ for all $c \leq b$. Suppose Y is a point of S(b). If $c \in Y$, $F(c) \leq F(c \wedge b) < \lambda'$. Therefore, $f(Y) = \sup \{ F(c) | c \in Y \} \leq \lambda' < \lambda$. By lemma 4.1.1, this means that f is normal.

Next suppose that f is a given normal lower semi-continuous function on $S(P)$. Let F be defined on P by the equation (b). Clearly F is a lattice function. To prove that it is normal, the criterion of proposition 2.3.1 will be used. Suppose then that $F(a) < \lambda$. By the definition of F, there is a point X of $S(a)$ such that $f(X) < \lambda$. By the criterion of lemma 4.1.1, there exists a non-empty open set N with $N \subseteq S(a)$ such that $f(Y) < \lambda$ for all $Y \in N$. Since $S(P)$ is a semi-regular space, there is not loss of generality in assuming that N is a regular open set. In other words, it may be supposed that $N = S(b)$ where $z \neq b \leq a$. If $z \neq c \leq b$, $F(c) = \inf\{f(Y) \mid c \in Y\} < \lambda$. Thus the hypothesis of proposition 2.3.1 is fulfilled and F is normal. Moreover, for this F, $f(X) = f_*(X) = \sup_{a \in X} \inf_{a \in Y} f(Y) = \sup_{a \in X} F(a)$.

Finally, we will show that the mapping $F \to f$ defined in (a) is a homomorphism: $\sup\{(F - \lambda)(a) \mid a \in X\} = \sup\{F(a) \mid a \in X\} - \lambda = f(X) - \lambda$. Thus translation is preserved. It is also clear that $\sup\{(F \wedge G)(a) \mid a \in X\} \leq \min[\sup\{F(a) \mid a \in X\}, \sup\{G(a) \mid a \in X\}]$. The computation which completes the proof is the following: $\sup\{(F \wedge G)(a) \mid a \in X\} = \sup\{F(a) \wedge G(a) \mid a \in X\} = \sup\{F(a \wedge b) \wedge G(a \wedge b) \mid a \in X, b \in X\} \geq \sup\{F(a) \wedge G(b) \mid a \in X, b \in X\} = \min[\sup\{F(a) \mid a \in X\}, \sup\{G(b) \mid b \in X\}]$. The proof of the theorem is complete.

This theorem shows the equivalence between the problems of representing a translation lattice by means of normal lattice functions and by means of normal lower semi-continuous functions. Thus all of the theorems obtained in the previous section can be immediately transferred to theorems on representations by normal lower semi-continuous functions. The results are collected in the next article.

4.1.2. Translation lattices of normal functions.

In this article, we will study the representations of a translation lattice by means of isomorphic translation lattices of normal lower semi-continuous functions. As usual, I will denote a closed, bounded ideal of L. In order to simplify the statements of the theorems, it will be assumed throughout that L is divisible with respect to I.

Definition 4.1.2. A set M of normal lower semi-continuous functions on a topological space S will be said to generate the topology if the sets of the form $\{ x | f(x) > \lambda \}$, for $f \in L$ and λ real, are dense (in the sense of definition 1.2.2) in the open sets of S.

Notice that if M is a translation lattice, it is only necessary to consider sets of the form $\{ x | f(x) > 0 \}$ in the above definition.

The concept introduced in definition 4.1.2 can be correlated with the idea expressed in definition 2.3.3 of a collection of normal lattice functions on the Boolean algebra P having the property that they generate P.

Lemma 4.1.2. Let L be a translation lattice of normal lattice functions on a complete Boolean algebra P. Let S(P) be an arbitrary representative space for P. Map the functions F of L into normal lower semi-continuous functions f on S(P) by the definition $f(X) = \sup \{ F(a) | a \in X \}$. Denote the image of this mapping by L'. Then L' generates the topology of S(P) if and only if L generates P.

Proof. Assume L generates the Boolean algebra P. Then if $\emptyset \neq S(a) \in S(P)$, $F \in L$ exists together with $b \in P$ such that $z \neq b \leq a$, $F(b) > 0$ and $F(c) \leq 0$ if $c \not\geq a$. Define $f \in L'$ by $f(X) = \sup\{F(b) \mid b \in X\}$. Then if $X \in S(b)$, $f(X) \geq F(b) > 0$. If $X \notin S(a)$, $a \notin X$. Hence $c \not\geq a$ holds for all $c \in X$. Therefore $f(X) = \sup\{F(c) \mid c \in X\} \leq \sup\{F(c) \mid c \not\geq a\} \leq 0$.

Conversely, suppose L' generates the topology of S(P). If $a \in P$ and $a \neq z$, then there exists $f \in L'$ satisfying $\emptyset \neq \{X \mid f(X) > 0\} \subseteq S(a)$. Let $F(b) = \inf\{f(X) \mid b \in X\}$, so that $f(X) = \sup\{F(b) \mid b \in X\}$. If X is chosen so that $f(X) > 0$, then $a \in X$ and there exists $b \in X$ so that $F(a \wedge b) \geq F(b) > 0$. On the other hand, $c \not\geq a$ implies that $X \in S(c)$ exists with $X \notin S(a)$. By the hypothesis on f, $f(X) \leq 0$. Thus $F(c) = \inf\{f(X) \mid c \in X\} \leq 0$. This completes the proof.

The theorem 2.3.1 on the existence of a representation by means of normal lattice functions can be immediately translated into a theorem on the existence of a representation by means of normal lower semi-continuous functions.

Theorem 4.1.2. Let L be a translation lattice. Choose I to be any closed bounded ideal of L such that L is divisible with respect to I (if such an ideal exists). Let $S([L/I])$ be any representative space for the complete Boolean algebra $[L/I]$. Then L is isomorphic to a translation lattice L' of normal lower semi-continuous functions on $S([L/I])$ such that L' generates the topology of $S([L/I])$ and, under the mapping, the image of I is the set $\{f \mid f \leq 0\}$.

In a similar way, it is possible to translate theorem 2.3.2 on the uniqueness of representations of translation lattices into the language of normal functions.

Theorem 4.1.3. Let L_1 and L_2 be translation lattices of normal lower semi-continuous functions on the respective semi-regular T_0 spaces $S_1(P_1)$ and $S_2(P_2)$, P_1 and P_2 being their Boolean algebras of regular open sets. Suppose that L_1 generates the topology of $S_1(P_1)$ and L_2 generates the topology of $S_2(P_2)$. Finally, suppose there is an isomorphic mapping σ of L_2 onto L_1. Then there is an isomorphism $\emptyset$ of P_1 onto P_2 such that

$$(\sigma f)(X) = \sup_{X \in a} \inf_{x \in \emptyset(a)} f(x)$$

where $X \in S_1(P_1)$, $x \in S_2(P_2)$ and $a \in P_1$.

Proof. This theorem is a direct consequence of theorem 2.3.2, theorem 4.1.1 and lemma 4.1.2.

The converse of this theorem can be stated as follows:

Proposition 4.1.2. Let P be a complete Boolean algebra. Suppose that $\emptyset$ is an automorphism of P onto itself. If $S_1(P)$ and $S_2(P)$ are two representative spaces for P, then σ defined by

$$(\sigma f)(X) = \sup_{X \in S_1(a)} \inf_{x \in S_2(\emptyset(a))} f(x)$$

is an isomorphism of $N(S_2(P))$ onto $N(S_1(P))$.

Proof. This is a corollary of theorem 4.1.1.

Now consider a translation lattice of normal lower semi-continuous functions which does not necessarily generate the topology of the space. For this situation, the simple result of theorem 2.3.2 can no longer be applied. It is necessary to look to theorem 2.3.4 for information. However, before this theorem can be used to the best advantage, it is necessary to obtain some more information on the problem of extension and restriction of lattice functions, and on the relation between representative spaces of different Boolean algebras.

Proposition 4.1.3. Let P_1 and P_2 be complete Boolean algebras. Suppose that P_1 is a sub-semi lattice of P_2 with the same zero and unit. Let $S(P_2)$ be a representative space for P_2 (see definition 3.2.1). Then if $S(P_1)$ is the collection of all ideals of the form $X \cap P_1$, where $X \in S(P_2)$, and with a topology defined on this set according to definition 3.2.1, $S(P_1)$ is a representative space for P_1 and there is a continuous mapping of $S(P_2)$ on $S(P_1)$.

Proof. It is clear that all the sets of the form $X \cap P_1$ are non-trivial dual ideals of P_1 (non-trivial since they always contain i). Also, if $\bar{a} \in P_1$, there is an $X \in S(P_2)$ such that $\bar{a} \in X$. Hence $\bar{a} \in X \cap P_1$. Thus $S(P_1)$ is a representative space for P_1.

To prove that $S(P_1)$ is a continuous image of $S(P_2)$, observe that in the mapping $X \rightarrow X \cap P_1$, the inverse image of the set $\{X \cap P_1 | \bar{a} \in X \cap P_1\}$ is $\{X | \bar{a} \in X\}$ (whenever $\bar{a} \in P_1$). Since the former sets constitute an open basis for $S(P_1)$, while the latter are open in $S(P_2)$, the mapping is continuous. The proof is complete.

The problem of continuous mappings of topological spaces will be considered in more detail when we get to section three of the present chapter.

Now consider the interpretation of propositions 2.3.5 and 2.3.6 on extension and restriction for normal lower semi-continuous functions.

Proposition 4.1.4. Let P_1 and P_2 be complete Boolean algebras. Let F be a normal lattice function on P_1 and let F' denote its extension to P_2 (defined by $F'(a) = \sup_{a \leq \overline{a}} F(\overline{a})$). Suppose $S(P_2)$ is any representative space for P_2 and $S(P_1)$ is the continuous image of $S(P_2)$ consisting of all the ideals $X \cap P_1$, where $X \in S(P_2)$. If f and f' are the normal lower semi-continuous functions associated with F and F' respectively, then $f'(X) = f(X \cap P_1)$ for all $X \in S(P_2)$.

Proof. $f'(X) = \sup_{a \in X} F'(a) = \sup_{a \in X} \sup_{a \leq \overline{a}} F(\overline{a}) = \sup_{\overline{a} \in X} F(\overline{a}) = f(X \cap P_1)$. This completes the proof.

In order to treat the restriction problem, it is necessary to introduce a new notion. Assume as before that P_1 and P_2 are complete Boolean algebras and that P_1 is a sub-semi-lattice of P_2 with the same zero and unit. Let $S(P_2)$ be a representative space for P_2. Suppose that f is a bounded, real valued function on $S(P_2)$. Define two new functions f^+ and f_+ by

$$f^+(X) = \inf_{\overline{b} \in X} \sup_{\overline{b} \in Y} f(Y),$$

$$f_+(X) = \sup_{\overline{b} \in X} \inf_{\overline{b} \in Y} f(Y).$$

It is an easy matter to verify that the operations $(^{+})$ and $(_{+})$ satisfy relations similar to those satisfied by the operations $(^{*})$ and $(_{*})$. In particular;

$$f^{+} \geq f \geq f_{+}$$

$$f \geq g \text{ implies } f^{+} \geq g^{+} \text{ and } f_{+} \geq g_{+}$$

$$f^{++} = f^{+} \quad \text{and} \quad f_{++} = f_{+}$$

$$(((f^{+})_{+})^{+})_{+} = (f^{+})_{+} \text{ and dually.}$$

What we are going to prove is that if $S(P_1)$ is the continuous image of $S(P_2)$ defined by proposition 4.1.3, and if f_o is the point representative on $S(P_1)$ of the restriction F_o of F to P_1, then $f_o(X \wedge P_1) = ((f^{+})_{+})(X)$. For this proof, a simple lemma is needed.

<u>Lemma 4.1.3</u>. Let P be a complete Boolean algebra; let S(P) be a representative space for P; suppose F is a lattice function on P. Then

$$\sup_{X \in S(b)} \ \sup_{a \in X} F(a) = \sup_{a \leq b} F(a).$$

If F is a dual lattice function (i.e., $F(a) \geq F(b)$ whenever $a \geq b$), then

$$\inf_{X \in S(b)} \ \inf_{a \in X} F(a) = \inf_{a \leq b} F(a).$$

<u>Proof</u>. Suppose $\sup_{X \in S(b)} \ \sup_{a \in X} F(a) = \lambda$. Then if $\delta > 0$, X exists such that $b \in X$, $a \in X$ and $F(a) > \lambda - \delta$. Then $F(a \wedge b) \geq$ $\geq F(a) > \lambda - \delta$. Thus $\sup_{a \leq b} F(a) > \lambda - \delta$. Since δ was arbitrary, $\sup_{a \leq b} F(a) \geq \lambda$.

If $\sup_{a \leq b} F(a) > \lambda$, $c \leq b$ exists satisfying $F(c) > \lambda$. Also $X \in S(P)$ exists with $c \in X$. Then $b \in X$, so $\sup_{X \in S(b)} \sup_{a \in X} F(a) \geq$ $\geq F(c) > \lambda$. This contradiction proves the first assertion. The dual result is obtained when F is replaced by $-F$.

Proposition 4.1.5. Let P_1 and P_2 be complete Boolean algebras. Let P_1 be a sub-semi-lattice of P_2 with the same zero and unit. Suppose that $S(P_2)$ is a representative space for P_2. Let $S(P_1)$ be the continuous image of $S(P_2)$ defined by $X \to X \cap P_1$. Take F to be any normal lattice function on P_2; denote by F_o the restriction of F to P_1 (defined in proposition 2.3.6). Finally, let f and f_o be the point functions corresponding to F and F_o respectively. Then

$$f_o(X \cap P_1) = ((f^+)_+)(X).$$

Proof. We compute: $f^+(X) = \inf_{\overline{b} \in X} \sup_{Y \in S(\overline{b})} f(Y) =$ $\inf_{\overline{b} \in X} \sup_{Y \in S(\overline{b})} \sup_{a \in Y} F(a) = \inf_{\overline{b} \in X} \sup_{a \leq \overline{b}} F(a)$. This last step is a consequence of lemma 4.1.3. Hence $\inf_{X \in S(\overline{a})} f^+(X) = \inf_{X \in S(\overline{a})} \inf_{\overline{b} \in X} \sup_{a \leq \overline{b}} F(a) =$ $\inf_{\overline{b} \leq \overline{a}} \sup_{a \leq \overline{b}} F(a) = F_o(\overline{a})$. Here again lemma 4.1.3 has been used. In this case, it was applied to the dual lattice function $\sup_{a \leq \overline{b}} F(a)$, considered as a lattice function on P_1.

Finally, for any $Y \in S(P_2)$ $((f^+)_+)(Y) = \sup_{\overline{a} \in Y} \inf_{\overline{a} \in X} f^+(X) =$ $\sup_{\overline{a} \in Y} F_o(\overline{a}) = f_o(Y \cap P_1)$. Thus the proof is complete.

Corollary 4.1.2. If $X \cap P_1 = Y \cap P_1$, then $((f^+)_+)(X) = ((f^+)_+)(Y)$. The extension (defined in proposition 4.1.4) of f_o to $S(P_2)$ is $(f^+)_+$. Hence $(f^+)_+$ is a normal lower semi-continuous function on $S(P_2)$.

Corollary 4.1.3. If $F'(a) = \inf\{((f^+)_+)(X) \mid a \in X\}$, then $F'(\bar{a}) = F_0(\bar{a})$ for all $\bar{a} \in P_1$. Hence F' defined only for elements of P_1 is a normal lattice function on P_1.

Collecting all of these results together with theorem 2.3.4, theorem 4.1.1 and proposition 4.1.2, we can state

Theorem 4.1.4. Let P be a complete Boolean algebra. Let L be a translation lattice of normal lower semi-continuous functions on a representative S(P) of P. Suppose that L is divisible with respect to the ideal $\{f \in L \mid f \leq 0\}$.

Denote by R the class of all regular open sets of the form $a_f = \{x \mid f(x) > 0\}^{-0}$ where $f \in L$. Suppose P_1 is any sub-semi-lattice of P which is isomorphic to $[L/I]$, and that there is a mapping $a_f \to \bar{a}_f$ of R onto a dense sub-semi-lattice of P_1 with the properties $a_f \leq \bar{a}_f$ and $\bar{z} = z$.

Define for any bounded real valued function f:

$$(f^+)(X) = \inf_{\bar{a} \in X} \sup_{Y \in S(\bar{a})} f(Y),$$

$$(f_+)(X) = \sup_{\bar{a} \in X} \inf_{Y \in S(\bar{a})} f(Y).$$

Then the mapping $f \to (f^+)_+$ is an isomorphism of L onto a sub-translation lattice of N(S(P)). If $S(P_1)$ is the continuous image of S(P) consisting of the points $X \cap P_1$ $(X \in S(P))$, then every $(f^+)_+$ is uniquely defined on $S(P_1)$ by $((f^+)_+)(X \cap P_1) = ((f^+)_+)(X)$. In this way L maps isomorphically onto a subset of $N(S(P_1))$ which generates the topology.

Remark: This theorem does not have the same content as theorem 2.3.4. The two theorems do have one thing in common however. They show how it is possible to obtain from a translation lattice of point functions (respectively, lattice functions) isomorphic translation lattices of point functions (lattice functions) which generate the topology (generate the Boolean algebra) over which they are defined. The property of a translation lattice of normal upper semi-continuous functions generating the topology is very important. For when this condition is satisfied, there is a close relationship between the lattice structure of the functions and the topological structure of the space (as will be shown in the next section). When the functions no longer generate the topology, this strong bond between lattice structure and topological structure is broken, a fact which is amply demonstrated by the difficulties which will be met in section three below.

4.2 Representation by continuous functions.

The final two sections of this thesis are devoted to the study of translation lattices of continuous functions. In particular, the relation between the topology of a space and the lattice structure of translation lattices of continuous functions defined on the space will be investigated. It has already been shown that every translation lattice can be represented as a translation lattice of functions on a set. It is an elementary matter to introduce a topology into the set so that the functions become continuous. Finally, the well known Stone-Čech compactification method shows that there is no loss in assuming the resulting space to be a compact Hausdorff space. Thus the problem of proving the existence of at least one continuous representation is a very simple one. The trouble is that there may be very many representations and the method of obtaining them, which we have just outlined, does not give much insight into the relations between these representations. (As indicated before, this statement does not apply in the case of distributive translation lattices. Indeed, for these, such elementary considerations do give very good information. It might even be said that there is no real problem until the assumption of distributivity is dropped.) By applying the techniques developed above, we will be able to obtain a representation and a uniqueness theorem which will be more suitable for studying the relation between the representing spaces of a given translation lattice.

In order to simplify the statement of theorems, it will be assumed in this section that the translation lattice L under consideration is divisible with respect to one of its closed, bounded ideals, I. By theorem 2.3.1, we can then assume that L is a translation lattice of normal lattice functions on a complete Boolean algebra P (isomorphic to [L/I]), and that L generates P. It will be shown that there is a representative space S(P) for P which is compact Hausdorff, for which the point representatives of L on S(P) are continuous, and which is such that these point representatives separate the points of S(P).

4.2.1. Continuity spaces.

Definition 4.2.1. Let P be a complete Boolean algebra and let L be a set of normal lattice functions on P. A representative space for P, S(P), will be called a continuity space for L if the point representatives of the functions of L, that is, the functions f defined by

$$f(X) = \sup \{ F(a) | a \in X \}$$

are all continuous on S(P).

It is desirable to get a more usable characterization of the continuity spaces of a given set of normal lattice functions. For this purpose, the following definition is introduced.

Definition 4.2.2. Let P be a complete Boolean algebra and let F be a normal lattice function on P. Define the oscillation of F on an element a of P by the formula:

$$O_F(a) = \sup_{b \leq a} F(b) - F(a).$$

Proposition 4.2.1. Let P be a complete Boolean algebra. Suppose F is a normal lattice function on P. Let S(P) be a representative space for P. Denote by f the point function on S(P) associated with the lattice function F. A necessary and sufficient condition that f be continuous at a point X of S(P) is that for every $\delta > 0$, there is an element $a \in X$ such that $O_F(a) < \delta$.

Proof. The function f is continuous at X if and only if there is a neighborhood of X in which the oscillation of f is less than any pre-assigned positive δ . Since the $a \in P$ constitute a basis for the open sets, this means that there is an a of P with $a \in X$ satisfying

$$\sup_{a \in Y} f(Y) - \inf_{a \in Y} f(Y) < \delta \;.$$

This means (using lemma 4.1.3) that

$$O_F(a) = \sup_{b \leq a} F(b) - F(a) < \delta \;.$$

The proof is complete.

4.2.2. The characteristic space.

In this article, P will, as usual, denote a complete Boolean algebra. L will designate a set (not necessarily a translation lattice) of normal lattice functions. Later we will assume that L generates P (see definition 2.3.3). The existence of a continuity space with the properties described at the end of the introduction to this section will now be proved.

The first step is a proof of the existence of a certain kind of dual ideals in P.

Definition 4.2.3: A dual ideal X will be said to satisfy the condition C if it has the property that, for any $\delta > 0$, and for any $F \in L$, there is an $a \in X$ such that $O_F(a) \leq \delta$.

Proposition 4.2.2. Let X be a dual ideal of P which satisfies the condition C. Then there is a dual ideal Z with $X \leq Z$ (in the ordering of dual ideals; this means that Z is a subset of X) which is maximal (in the dual ideal ordering) with respect to the property of satisfying the condition C.

Proof. Let $\lambda_F = \sup\{F(a) | a \in X\}$. Denote by $\mathfrak{a}$ the set of all minimal dual ideals U which are such that $\sup\{F(a) | a \in U\} = \lambda_F$ for all $F \in L$. It will be shown below that $\mathfrak{a}$ is non-empty. Define $Z = V\mathfrak{a}$. It must be shown that $X \leq Z$ and that Z is maximal satisfying the condition C. The proof will be carried out in several lemmas.

Lemma 4.2.1. If Y_1 and Y_2 are any two dual ideals with $Y_1 \leq Y_2$, and if Y_2 satisfies the condition C, then

$$\sup\{F(a) | a \in Y_1\} = \sup\{F(a) | a \in Y_2\},$$

for all $F \in L$.

Proof. Since Y_2 is a subset of Y_1, $\sup\{F(a) | a \in Y_2\} \leq \sup\{F(a) | a \in Y_1\}$. To reverse this inequality, notice that if $\delta > 0$, there is an $a \in Y_2$ such that $0_F(a) < \delta$; that is, $\delta > F(b) - F(a)$ for all $b \leq a$, $b \neq z$. Thus if $c \in Y_1$, $F(c) \leq F(c \wedge a) < F(a) + \delta \leq \sup\{F(a) | a \in Y_2\} + \delta$. Hence, $\sup\{F(c) | c \in Y_1\} \leq \sup\{F(a) | a \in Y_2\} + \delta$. Since δ was arbitrary, $\sup\{F(a) | a \in Y_1\} \leq \sup\{F(a) | a \in Y_2\}$. This completes the proof of the lemma.

This result has two immediate consequences of importance. First, it shows that the set $\mathfrak{a}$ is not empty. For by the maximal principle, it is always possible to find a minimal dual ideal U satisfying $U \leq X$. By the lemma $\sup\{F(a) | a \in U\} = \sup\{F(a) | a \in X\} = \lambda_F$, for all $F \in L$.

A second important consequence of lemma 4.2.1 is the fact that $X \leq Z$. For it implies that $\mathfrak{A} \supseteq \{U | U \leq X\}$ and in the lattice of dual ideals of any Boolean algebra, $\vee\{U | U \leq X, U \text{ minimal}\} = X$ is valid for any dual ideal X. (The proof of this fact is an elementary application of the maximal principle.) Hence $Z = \vee\{U | U \in \mathfrak{A}\} \geq \vee\{U | U \leq X\} = X$.

Lemma 4.2.2. Let $U \in \mathfrak{A}$. Then if $\delta > 0$ and $F \in L$, there is an $a_U \in U$ such that $F(a_U) \geq \lambda_F - \delta/2$ and $F(b) \leq \lambda_F + \delta/2$ for all $b \leq a_U$.

Proof. Suppose, first, that every $a \in U$ contains a non-zero b_a such that $F(b_a) > \lambda_F + \delta/2$. Then $\vee\{b_a | a \in U\}$ has non-empty intersection with every $a \in U$ and hence (since U is minimal), is itself contained in U. Moreover $F(\vee\{b_a | a \in U\}) = \inf F(b_a) \geq \lambda_F + \delta/2$. This is contrary to the fact that (by definition of $\mathfrak{A}$) $\sup\{F(a) | a \in U\} = \lambda_F$. Consequently some $a_0 \in U$ is such that $F(b) \leq \lambda_F + \delta/2$ for all $b \leq a_0$.

Since $\sup\{F(a) | a \in U\} = \lambda_F$, there exists $a_1 \in U$ such that $F(a_1) \geq \lambda_F - \delta/2$. Putting $a_U = a_0 \wedge a_1$ gives an element of U with all the properties claimed.

Notice that the above proof depended in no way on the ideal X or the nature of the number λ_F beyond the fact that $\lambda_F = \sup\{F(a) | a \in U\}$. Hence,

<u>Corollary 4.2.1.</u> Every minimal dual ideal satisfies the condition C.

<u>Lemma 4.2.3.</u> The ideal Z satisfies the condition C.

<u>Proof.</u> Corresponding to $\delta > 0$ and $F \in L$, choose a_U as in lemma 4.2.2. Put $a = \vee \{a_U | U \in \mathfrak{A}\}$. Since $a \geq a_U$, it follows that $a \in U$ for all $U \in \mathfrak{A}$. Hence $a \in \vee \mathfrak{A} = Z$. To complete the proof, it will suffice to show that $O_F(a) \leq \delta$.

Since F is normal, $F(a) = \inf \{F(a_U) | U \in \mathfrak{A}\} \geq \lambda_F - \delta/2$. If $z \neq b \leq a$, then $z \neq b = b \wedge a = b \wedge \vee \{a_U | U \in \mathfrak{A}\} = \vee \{b \wedge a_U | U \in \mathfrak{A}\}$. Thus $b \wedge a_U \neq z$ for some $U \in \mathfrak{A}$. Consequently $F(b) \leq F(b \wedge a_U) \leq$ $\leq \lambda_F + \delta/2$. Combining these inequalities, $O_F(a) \leq (\lambda_F + \delta/2) - (\lambda_F - \delta/2) = \delta$. The proof is complete.

The preceding lemmas show that Z is a dual ideal containing X and satisfying the condition C. The ideal Z is also maximal with these properties. For if $Y \geq Z$ satisfies the condition C, by lemma 4.2.1, $\sup \{F(a) | a \in Y\} = \sup \{F(a) | a \in Z\} = \lambda_F$. Then by another application of lemma 4.2.1, $Y = \vee \{U | U \leq Y\} \leq \vee \{U | U \in \mathfrak{A}\} = Z$. Consequently $Y = Z$. Proposition 4.2.2 is finally proved.

Proposition 4.2.2 leads immediately to the main existence theorem for continuity spaces.

<u>Theorem 4.2.1.</u> Let L be a set of normal lattice functions on the complete Boolean algebra P. Suppose that L generates P. Then the

space $S(P)$ of all dual ideals which are maximal satisfying condition C constitutes a compact Hausdorff topological space which is a representative space for P. The point functions on $S(P)$, corresponding to the lattice functions $F \in L$, have the properties of (1) being continuous on $S(P)$, (2) generating the topology of $S(P)$, and (3) separating the points of $S(P)$.

Proof. There are several things to prove. First it must be shown that $S(P)$ is a representative space for P. In other words, it is necessary to verify that every element $a \in P$ is contained in at least one dual ideal X of $S(P)$. Next it should be shown that every $F \in L$ corresponds to a continuous point function. But this fact is an immediate consequence of proposition 4.2.1 and the fact that every $X \in S(P)$ satisfies the condition C. The fact that the point functions corresponding to the lattice functions of L generate the topology of $S(P)$ is an immediate consequence of lemma 4.1.2. The next step of the proof is to show that the point functions corresponding to L separate the points of $S(P)$. An immediate consequence of this, and the fact that these functions are continuous, is that $S(P)$ is a Hausdorff space. The final and most difficult part of the theorem is the proof that $S(P)$ is a compact space.

Following this outline, we proceed to prove the theorem by means of three lemmas.

Lemma 4.2.4. For any $a \in P$, there is an $X \in S(P)$ such that $a \in X$.

Proof. Since L generates P, F in L exists, together with $b \in P$ with $z \neq b \leq a$ such that $F(b) > 0$ and $F(c) > 0$ only if $c \leq a$. Let U be a minimal dual ideal containing b. By corollary 4.2.1, U satisfies condition C. Hence $X \geq U$ exists with X maximal satisfying condition C, that is, $X \in S(P)$. By leamm 4.2.1 $\sup\{F(c) \mid c \in X\} = \sup\{F(c) \mid c \in U\} \geq F(b) > 0$. Hence $c \in X$ exists with $F(c) > 0$. This implies $c \leq a$, so therefore $a \in X$. This is what was to be proved.

Lemma 4.2.5. The point functions f, defined by

$$f(X) = \sup\{F(a) \mid a \in X\},$$

where $F \in L$, separate the points of $S(P)$. That is, if $X \neq Y$ in $S(P)$, there is an f of this form such that $f(X) \neq f(Y)$.

Proof. Suppose that $X \neq Y$ in $S(P)$. By the maximality of X and Y, it follows that $X \vee Y$ fails to satisfy the condition C. Hence $F \in L$ and $\delta > 0$ exist so that $O_F(c) > \delta$ for all $c \in X \vee Y$.

Now since both X and Y satisfy the condition C, it is possible to find $a \in X$ and $b \in Y$ so that $O_F(a) < \delta/3$ and $O_F(b) < \delta/3$. From these, it follows that $|F(a) - f(X)| \leq \delta/3$ and $|F(b) - f(Y)| \leq \delta/3$. For, $f(x) = \sup\{F(c) \mid c \in X\} \geq F(a)$ and $f(X) = \sup\{F(c) \mid c \in X\} \leq$ $\leq \sup\{F(a \wedge c) \mid c \in X\} \leq F(a) + \delta/3$. A similar argument proves the other inequality.

Now $a \vee b \in X \vee Y$, so by hypothesis $O_F(a \vee b) > \delta$. In other words, a non-zero c exists with $c \leq a \vee b$ and $F(c) - F(a \vee b) > \delta$. The relation $c \leq a \vee b$ implies that $c = (a \wedge c) \vee (b \wedge c)$. Thus, at least one of $a \wedge c$, $b \wedge c$ is not z and (by normality) one of $F(a \wedge c) = F(c)$, $F(b \wedge c) = F(c)$ holds -- say $F(c) = F(a \wedge c)$. Hence $F(a \wedge c) - \min\{F(a), F(b)\} = F(c) - F(a \vee b) > \delta$. But since $O_F(a) < \delta/3$, $F(a \wedge c) - F(a) < \delta/3$, and this relation implies $F(a \wedge c) - F(b) > \delta$. Also $\delta < F(a \wedge c) - F(b) \leq F(a) + \delta/3 - F(b)$. Transposing, $F(a) - F(b) > 2(\delta/3)$.

Combining the results of the last two paragraphs gives $|f(X) - f(Y)| \geq |F(a) - F(b)| - |F(a) - f(X)| - |F(b) - f(Y)| > 2(\delta/3) - \delta/3 - \delta/3 = 0$. This inequality shows that the points X and Y are separated and the proof is complete.

<u>Corollary 4.2.2.</u> $S(P)$ is a Hausdorff topological space.

The proof of theorem 4.2.1 is completed by

<u>Lemma 4.2.6.</u> The space $S(P)$ is compact.

<u>Proof.</u> Let $\mathcal{J} = \{T\}$ be an arbitrarily chosen collection of closed subsets of $S(P)$ with the finite intersection property: every finite collection of sets in $\mathcal{J}$ has non-empty intersection. The compactness of $S(P)$ will follow if it is shown that $\cap \mathcal{J}$ is not empty.

The first thing to notice is that no loss in generality is incurred by assuming that $\mathcal{J}$ is a minimal dual ideal of closed subsets of $S(P)$. For, by the maximal principle, it is always possible to find a collection $\mathcal{J}'$ of closed subsets of $S(P)$ which contains $\mathcal{J}$ and is maximal with respect to the finite intersection property.

Then if $\cap \mathfrak{J}'$ is non-empty, the same is surely true for $\cap \mathfrak{J}$. Hence, hereafter it will be assumed that $\mathfrak{J}$ is a minimal dual ideal.

(1). Let $F \in L$ and denote by f the point function associated with F. Put $\lambda = \sup_{T \in \mathfrak{J}} \inf_{X \in T} f(X)$. Choose $\delta > 0$. We will show that for all $T \in \mathfrak{J}$, an element a_T of P can be found satisfying $a_T \in \bigcup \{X | X \in T\}$, $\lambda - \delta/4 < F(a_T) < \lambda + \delta/4$ and $O_F(a_T) < \delta/4$.

To prove this, notice that T_o exists satisfying $\lambda \geq \inf\{f(X) | X \in T_o\} > \lambda - \delta/4$. Then, for any $T \in \mathfrak{J}$, $\lambda \geq \inf\{f(X) | X \in T \cap T_o\} > \lambda - \delta/4$. Hence $\lambda + \delta/4 > f(X) > \lambda - \delta/4$ for some $X \in T$. As a consequence, a_T in P exists with $a_T \in X \subseteq \bigcup\{Y | Y \in T\}$ and such that $F(a_T) > \lambda - \delta/4$. Since f is continuous, it is possible to pick a_T so that $O_F(a_T) < \delta/4$. Finally $\lambda + \delta/4 > f(X)$ implies that $F(a_T) <$ $< \lambda + \delta/4$. This proves the assertion (1).

(2). Corresponding to any $F \in L$ and $\delta > 0$, there is an element a of P such that $a \in \bigcap\{(\bigcup T) | T \in \mathfrak{J}\}$ and $O_F(a) < \delta$.

To see this, choose a_T (corresponding to each $T \in \mathfrak{J}$) so that the conditions of (1) are satisfied. Put $a = \bigvee\{a_T | T \in \mathfrak{J}\}$. Then, since F is normal, $\lambda + \delta/4 \geq F(a) \geq \lambda - \delta/4 > \lambda - \delta/2$. If $z \neq b \leq a$, then $b \wedge a_T \neq z$ holds for some $T \in \mathfrak{J}$. This means that $F(b) \leq F(b \wedge a_T) \leq F(a_T) + \delta/4 < \lambda + \delta/2$. Thus $O_F(a) < \delta$. Finally $a \in \bigcup T \equiv \{X | X \in T\}$ for all $T \in \mathfrak{J}$. Consequently $a \in \bigcap\{(\bigcup T) | T \in \mathfrak{J}\}$. Thus (2) is proved.

(3). For each $F \in L$ and $\delta > 0$, choose $a_{F,\delta}$ (axiom of choice) so that $a_{F,\delta} \in \bigcap\{(\bigcup T) | T \in \mathfrak{J}\}$ and $O_F(a_{F,\delta}) < \delta$. Let Y be the dual ideal generated by the set of all $a_{F,\delta}$, that is, $Y = \{ b \in P | b \geq a_1 \wedge \dots \wedge a_n \}$, where the a_j are of the form $a_{F,\delta}$. Then Y satisfies the condition C and $Y \subseteq \bigcap\{(\bigcup T) | T \in \mathfrak{J}\}$. In particular, Y is non-trivial.

The fact that Y satisfies the condition C is evident since all of the $a_{F,\delta}$ are in Y. The relation $Y \subseteq \bigcap\{(\bigcup T) | T \in \mathfrak{J}\}$ is an immediate consequence of the following fact: If a_j, for $j = 1,\dots,n$, are elements of the form $a_{F,\delta}$, then $a_1 \wedge \dots \wedge a_n \in \bigcap\{(\bigcup T) | T \in \mathfrak{J}\}$.

Indeed, suppose $a_1 \wedge \dots \wedge a_n \notin \bigcup T$ for some $T \in \mathfrak{J}$. Denoting $S(a_j) = \{ X \in S(P) | a_j \in X \}$, this implies that $S(a_1) \cap \dots \cap S(a_n) \cap T$ is empty. Hence $T \subseteq S(a_1)^c \cup \dots \cup S(a_n)^c$. Since $\mathfrak{J}$ is a dual ideal, this is possible only if $S(a_1)^c \cup \dots \cup S(a_n)^c \in \mathfrak{J}$. Then $\mathfrak{J}$, being minimal, must be prime, so, for some index j, $S(a_j)^c \in \mathfrak{J}$. This leads to the following contradiction: $a_j \in \bigcap\{(\bigcup T) | T \in \mathfrak{J}\} \subseteq \bigcup(S(a_j)^c)$, contrary to the obvious fact that if $X \in S(a_j)^c$, then $a_j \notin X$. These considerations constitute the proof of (3).

Let Y be the ideal constructed in (3). By proposition 4.2.2, it is possible to find $Z \in S(P)$ satisfying $Z \geq Y$. Then, in terms of set inclusions, $Z \subseteq Y \subseteq \bigcap\{(\bigcup T) | T \in \mathfrak{J}\}$. By the criterion of lemma 3.2.2, this means that $Z \in T^- = T$ for all $T \in \mathfrak{J}$. Thus, we have reached our ultimate goal: $\bigcap \mathfrak{J}$ is not empty. This completes the proof that S(P) is compact, and also the proof of theorem 4.2.1.

Corollary 4.2.3. Let L be an abstract translation lattice. Suppose I is a bounded, closed ideal of L, and that L is divisible with respect to I. Then there is an isomorphism of L onto a set L' of continuous, real-valued functions on a compact Hausdorff topological space S such that : (1) the image of I under this isomorphism is just the set of those functions of L' which are less than or equal to the zero function on S; (2) L' generates the topology of S; (3) the functions of L' separate the points of S.

Definition 4.2.4. Let L be a set of normal lattice functions on the complete Boolean algebra P. Suppose that L generates P. Then the space of all dual ideals which are maximal satisfying condition C will be called the characteristic space of L and will be denoted $S_L(P)$. The same notation and terminology will be applied to describe the space constructed from an abstract translation lattice L by means of theorems 2.3.1 and 4.2.1. (Of course this construction depends on the choice of an ideal I.)

4.2.2. Uniqueness of the characteristic space.

In this article, a uniqueness theorem for the characteristic space will be proved. Also, we will consider the relationship between the characteristic space of a collection L of normal lattice functions and other continuity spaces of L which are constructed from the same Boolean algebra.

Theorem 4.2.2. Let P be a complete Boolean algebra. Suppose L is a set of normal lattice functions on P. Assume that L generates P. Then if S(P) is any compact Hausdorff continuity space for L, and if $S_L(P)$ is the characteristic space for L, $S_L(P)$ is homeomorphic to a factor space of S(P). If the points of S(P) are separated by the point functions corresponding to the lattice functions of L, then S(P) is homeomorphic to $S_L(P)$.

Proof. Let $X' \in S(P)$. Then X' satisfies condition C (proposition 4.2.1), so by proposition 4.2.2, there is an ideal X of $S_L(P)$ such that $X \geq X'$. This X is unique. For suppose Y is a point of $S_L(P)$ distinct from X. Since $S_L(P)$ is a Hausdorff space, $a \in X$ and $b \in Y$ exist such that $a \wedge b = z$. Hence $Y \ngeq X'$, because otherwise $z = a \wedge b \in X'$. Denote by $\emptyset(X')$ the unique $X \in S_L(P)$ satisfying $X \geq X'$. Also, for what follows, denote $S_L(a) = \{X \in S_L(P) | a \in X\}$ and $S(a) = \{X' \in S(P) | a \in X'\}$.

(1). If $S_L(b)^- \subseteq S_L(a)$, then $\emptyset(S(b)) \subseteq S_L(a)$.

This preliminary result is proved by reasoning to a contradiction. Suppose $X' \in S(b)$ and $a \notin \emptyset(X')$. Then $\emptyset(X') \notin S_L(b)^-$, so by corollary 3.2.1, $c \in \emptyset(X')$ exists satisfying $b \wedge c = z$. But $c \in \emptyset(X') \subseteq X'$ and $b \in X'$ together imply an impossibility: $z = b \wedge c \in X'$.

(2). Using (1), we can show that the mapping $\emptyset$ is continuous. It suffices to prove that $\emptyset^{-1}(S_L(a))$ is open for all $a \in P$. Let $X' \in \emptyset^{-1}(S_L(a))$. Then $a \in \emptyset(X')$. Since $S_L(P)$ is a regular topological space, it is possible to pick $b \in P$ so that $\emptyset(X') \in S_L(b) \subseteq S_L(b)^- \subseteq$ $\subseteq S_L(a)$. Then $\emptyset(S(b)) \subseteq S_L(a)$, so $X' \in S(b) \subseteq \emptyset^{-1}(S_L(a))$. Thus the set $\emptyset^{-1}(S_L(a))$ is open.

(3). A second consequence of (1) is the fact that $\emptyset(S(P))$ is dense in $S_L(P)$. Indeed, if $a \in P$, pick $b \neq z$ so that $S_L(b)^- \subseteq S_L(a)$. Then $\emptyset(S(b)) \subseteq S_L(a)$ and $\emptyset(S(b))$ is not empty. Hence $S_L(a) \wedge \emptyset(S(P))$ is not empty and the assertion follows.

From this it is immediate that $\emptyset(S(P)) = S_L(P)$. For $\emptyset(S(P))$, being the continuous image of a compact space, is closed, and therefore, $\emptyset(S(P)) = S_L(P)$. This proves the first assertion of the theorem. To complete the proof, we will show that when the point functions obtained from L generate the topology of S(P), $\emptyset(X') = X'$.

(4). If $\emptyset(X') \neq X'$, there is a minimal dual ideal U satisfying $U \nleq X'$ and $U \vee X' \leq \emptyset(X')$ (since in a Boolean algebra, every dual ideal is the join of minimal dual ideals). By the compactness of S(P), there is a point $Y' \in \bigcap \{S(a)^- | a \in U\}$. If $b \in Y'$, $b \wedge a \neq z$ for all $a \in U$ (by corollary 3.2.1). Hence, since U is minimal, $b \in U$. Because b was an arbitrary element of Y', it follows that $Y' \geq U$. By lemma 4.2.1, $\sup \{F(a) | a \in X'\} = \sup \{F(a) | a \in \emptyset(X')\} = \sup \{F(a) | a \in U\} = \sup \{F(a) | a \in Y'\}$ for any $F \in L$. Then the hypothesis that the points of S(P) are separated by the point functions generated from L implies that $X' = Y' \geq U$. But this contradicts the original choice of U and proves that $\emptyset(X') = X'$. The proof of the theorem is complete.

An immediate consequence of theorems 4.2.2 and 4.1.3 is

Corollary 4.2.4. Let L be an abstract translation lattice.

Suppose L is isomorphic to the set L' of continuous functions on a compact Hausdorff space S. Suppose L' generates the topology and separates the points of S. Then S is homeomorphic to the characteristic space of L.

This corollary gives a method for determining the characteristic spaces of many specific examples of translation lattices.

Example 1. Let L be the translation lattice C(S), the continuous functions on a completely regular topological space. The characteristic space is easily determined. It is precisely the Stone-Čech compactification of S. For (see Čech [19]) there is an isomorphic (translation and lattice preserving) mapping of C(S) onto the set of all continuous functions on the compactification.

Example 2. We can now prove Dilworth's theorem on the representation of N(S) -- the normal lower semi-continuous functions on a semi-regular T_0-space. By theorem 4.1.1, N(S) is (translation lattice) isomorphic to the set of all normal lattice functions on the complete Boolean algebra P of regular open sets of S. Let $\gamma(P)$ be the space defined by the set of all minimal dual ideals of P. By example 1 of chapter three, $\gamma(P)$ is a compact Hausdorff space. By corollary 4.2.1, every normal lattice function on P has a continuous point function representative on $\gamma(P)$. Conversely, every continuous function on $\gamma(P)$ is normal, lower semi-continuous and hence corresponds to some normal lattice function. We have therefore proved the theorem of

Dilworth [1] : the translation lattice N(S) is isomorphic to C(γ(P)), the set of all continuous functions on the Boolean space associated with the Boolean algebra of regular open sets of S.

4.3 The characteristic space.

In this section, the relationship between an arbitrary continuity space of a translation lattice L and the characteristic space of L will be studied. Corollary 4.2.4 shows that if a translation lattice L of continuous functions on a compact Hausdorff space S generates the topology, then the characteristic space of L is homeomorphic to a factor space of S. Now the requirement that L generate the topology will be weakened to the assumption that L be divisible with respect to the ideal $I = \{ f \in L | f \leq 0 \}$. What we will prove is that, under these conditions, the characteristic space is homeomorphic to a factor space of a closed sub-space of S.

We know already from theorems 2.3.4 and 4.1.1 that [L/I] is isomorphic to a sub-semi-lattice of the Boolean algebra of regular open sets of S. This fact leads to a consideration of the relationship between representative spaces for Boolean algebras P_1 and P_2 with P_1 a sub-semi-lattice of P_2. This is the situation which will be studied in the first article below. The final article will be devoted to the proof of the result mentioned above.

4.3.1. Projections of topological spaces.

The following definition is patterned after one of Stone's [18] :

Definition 4.3.1. Let S_1 and S_2 be Hausdorff topological spaces. A mapping $x \rightarrow T_x$ which associates with every $x \in S_1$ a closed non-empty subset T_x of S_2 is called a (closed) projection of S_1 into S_2 if it has the properties:

(a) if T_2 is a closed subset of S_2, then $\{x \in S_1 \mid T_2 \cap T_x \neq \emptyset\}$ is closed in S_1;

(b) if T_1 is a closed subset of S_1, then $\cup\{T_x \mid x \in T_1\}$ is closed in S_2.

A projection is called simple if $T_x \cap T_y = \emptyset$ whenever $x \neq y$.

An immediate consequence of this definition is the following:

Proposition 4.3.1. If $x \rightarrow T_x$ is a closed simple projection of S_1 into S_2, then S_1 is homeomorphic to a factor space of a closed sub-space of S_2. Conversely, if S_1 is a factor space of a closed sub-space of S_2, and if $\emptyset$ denotes the natural mapping of this sub-space onto S_1, then $x \rightarrow \emptyset^{-1}(x)$ defines a simple projection of S_1 into S_2.

This proposition explains our interest in projections. The reason for considering projections, rather than directly studying factor spaces of a closed sub-space, is the following: it will be shown in this article that if P_1 (a complete Boolean algebra) is a sub-semi-lattice of the complete Boolean algebra P_2, and if $S(P_1)$, $S(P_2)$ are respectively compact Hausdorff representative spaces for P_1 and P_2, then there is a projection of $S(P_1)$ into $S(P_2)$. In this way, it is possible to reduce the proof of the main theorem to the verification that a certain projection is simple. This will be done in the final article.

Let P_1 and P_2 be complete Boolean algebras with $P_1 \subseteq P_2$ (i.e., P_1 is a sub-semi-lattice of P_2, both having the same zero and unit element). As before, joins in P_1 and P_2 will be distinguished by the different symbols $\overset{1}{\vee}$ and $\overset{2}{\vee}$ respectively. Also, a bar will be placed over elements of P_1 in order to distinguish them from those of P_2.

Denote by Q the set of all elements of the form $\overset{2}{\vee} A$, where A is an arbitrary subset of P_1.

<u>Lemma 4.3.1.</u> The set Q is a complete, completely distributive sub-lattice of P_2.

<u>Proof.</u> This is obvious.

Suppose now that $S_1(P_1)$ and $S_2(P_2)$ are respectively representative spaces for P_1 and P_2. Assume also that they are both compact Hausdorff spaces.

Define, for $\bar{a} \in P_1$, $S_1(\bar{a}) = \{ X_1 \in S_1(P_1) | \bar{a} \in X_1 \}$. Now extend this definition to elements of Q by setting $S_1(b) = \bigcup \{ S_1(\bar{a}) | \bar{a} \leq b \}$, whenever $b \in Q$. Then $S_1(b)$ is an open set of $S_1(P_1)$. It is clear that the following holds:

$$S_1(b) \cap S_1(c) = S_1(b \wedge c),$$

for all b, c $\in$ Q.

<u>Lemma 4.3.2.</u> Let $b \in Q$ and $X_1 \in S_1(P_1)$. Then $X_1 \in S_1(b)^-$ if and

only if $b \wedge \bar{a} \neq z$ for all $\bar{a} \in X_1$.

Proof. Suppose $\bar{a} \wedge b \neq z$ for all $\bar{a} \in X_1$. Then, for any $\bar{a} \in X_1$, there exists $\bar{b} \leq b$ such that $\bar{a} \wedge \bar{b} \neq z$. Let Y_1 be an element of $S_1(P_1)$ with $\bar{a} \wedge \bar{b} \in Y_1$. Then $\bar{b} \in Y_1$, so $Y_1 \in S_1(\bar{b}) \subseteq S_1(b)$; also $\bar{a} \in Y_1$, so $\bar{a} \in \bigcup(S_1(b))$. Since $\bar{a}$ was arbitrary, $X_1 \subseteq \bigcup(S_1(b))$. Thus by lemma 3.2.2, $X_1 \in S_1(b)^-$.

Conversely, if $b \wedge \bar{a} = z$ for some $\bar{a} \in X_1$, then $\bar{a} \notin \bigcup(S_1(b))$. For if $\bar{a} \in Y_1 \in S_1(b)$, $\bar{b} \leq b$ exists with $\bar{b} \in Y_1$, and this means $z \neq \bar{a} \wedge \bar{b} \leq \bar{a} \wedge b$. Hence $X_1 \notin S_1(b)^-$.

Lemma 4.3.3. Let a and b be elements of Q. Then (in the topology of $S_1(P_1)$):

$$S_1(a)^{-o} \cap S_1(b)^{-o} = S_1(a \wedge b)^{-o}.$$

Proof. The sets $S_1(a)$ and $S_1(b)$ are open in $S_1(P_1)$. Consequently $S_1(a)^{-o} \cap S_1(b)^{-o} = (S_1(a) \cap S_1(b))^{-o} = S_1(a \wedge b)^{-o}$. This is what was to be proved.

Denote $S_2(a) = \{X_2 \in S_2(P_2) \mid a \in X_2\}$ whenever $a \in P_2$. For the applications below, a will always be an element of Q.

Now suppose that W is a mapping from Q into the closed subsets of $S_2(P_2)$ with the properties:

(a) if $z \neq a$, then $\emptyset \neq W(a) \subseteq S_2(a)^-$;

(b) $W(a) \subseteq W(b)$ whenever $a \leq b$.

An example of a mapping satisfying these conditions is $a \to S_2(a)^-$. However, it will turn out that the choice of W is rather delicate, and that this choice is not the one which gives us the desired results. For the present, the mapping W can be left unspecified.

Lemma 4.3.4. The mapping $X_1 \to \cap \{W(a) | a \in Q, X_1 \in S_1(a)^{-o}\}$ associates with every $X_1 \in S_1(P_1)$ a non-empty, closed subset of $S_2(P_2)$.

Proof. If $a_1, \ldots, a_n$ are elements of Q with $X_1 \in S_1(a_j)^{-o}$ for $j = 1, \ldots, n$, then by lemma 4.3.3, $X_1 \in S_1(a_1 \wedge \ldots \wedge a_n)^{-o}$. Hence, $a_1 \wedge \ldots \wedge a_n \neq z$. Thus $\emptyset \neq W(a_1 \wedge \ldots \wedge a_n) \subseteq W(a_1) \cap \ldots \cap W(a_n)$. Lemma 4.3.4 then follows from the assumed compactness of $S_2(P_2)$.

Denote the set $\bigcap \{W(a) | a \in Q, X_1 \in S_1(a)^{-o}\} = T_{X_1}$. It will now be shown that $X_1 \to T_{X_1}$ is a projection of $S_1(P_1)$ into $S_2(P_2)$.

Lemma 4.3.5. Suppose T_2 is a closed subset of $S_2(P_2)$. Then the set $\{X_1 \in S_1(P_1) | T_{X_1} \cap T_2 \neq \emptyset\}$ is closed in $S_1(P_1)$.

Proof. Suppose $T_{X_1} \cap T_2 = \emptyset$. This means $\bigcap \{W(a) \cap T_2 | a \in Q, X_1 \in S_1(a)^{-o}\} = \emptyset$, so, by compactness, $a_1, \ldots, a_n$ exist in Q such that $W(a_1) \cap \ldots \cap W(a_n) \cap T_2 = \emptyset$ and $X_1 \in \bigcap_{j=1}^{n} S_1(a_j)^{-o}$. But if $Y_1 \in \bigcap_{j=1}^{n} S_1(a_j)^{-o}$, then $T_{Y_1} \cap T_2 \subseteq W(a_1) \cap \ldots \cap W(a_n) \cap T_2 = \emptyset$. Hence the set $\{X_1 \in S_1(P_1) | T_{X_1} \cap T_2 = \emptyset\}^c$ is open, and the lemma is proved.

Lemma 4.3.6. Suppose T_1 is a closed subset of $S_1(P_1)$. Then the set $\bigcup \{T_{X_1} | X_1 \in T_1\}$ is closed in $S_2(P_2)$.

Proof. Suppose the point Y_2 of $S_2(P_2)$ is not in $\bigcup \{T_{X_1} | X_1 \in T_1\}$. Then if $X_1 \in T_1$, there exists $a \in Q$ such that $X_1 \in S_1(a)^{-o}$ and

$Y_2 \notin W(a)$. By compactness, pick $a_1, \ldots, a_n$ such that every $X_1 \in T_1$ satisfies $X_1 \in S_1(a_j)^{-o}$ for some j, and such that $Y_2 \notin W(a_j)$ for all j. Then $\bigcup \{T_{X_1} | X_1 \in T_1\} \subseteq W(a_1) \cup \ldots \cup W(a_n)$ and $Y_2 \notin W(a_1) \cup \cup \ldots \cup W(a_2)$. Thus (since Y_2 was arbitrary) $(\bigcup \{T_{X_1} | X_1 \in T_1\})^c$ is open, and the lemma follows.

Summarizing the result of these lemmas, we can write:

Proposition 4.3.2. The mapping $X_1 \rightarrow T_{X_1}$ is a projection of $S_1(P_1)$ into $S_2(P_2)$. The projection is simple if and only if, for any two distinct points X_1 and Y_1 of $S_1(P_1)$, b and c in Q exist satisfying $X_1 \in S_1(b)^{-o}$, $Y_1 \in S_1(c)^{-o}$, while $W(b) \cap W(c) = \emptyset$.

Proof. The only assertion of the proposition which needs proving is the necessity of the simplicity criterion. The proof is a routine compactness argument. We omit it, since no use will be made of the result in the following pages.

4.3.2. The main theorem.

In this article it will be shown that by suitably choosing the mapping W, the projection defined above will be simple. All the notation of the above article will be continued.

In the work leading up to, and including theorem 4.3.1, a uniform set of hypotheses will be used. For convenience, these will be assembled before starting the proofs.

Hypotheses: assume that

(a) there is given a translation lattice L of normal lattice functions on a complete Boolean algebra P_2; denote by I the ideal $\{F \in L | F \leq 0\}$;

(b) the collection of all elements in P_2 of the form $a_F = \vee\{a | F(a) > 0\}$ is denoted by R; there is a complete Boolean algebra P_1 which is a sub-semi-lattice of P_2 (with the same zero and unit), which is isomorphic to $[L/I]$, and which is such that there is a homomorphism $a_F \to \bar{a}_F$ of R onto a dense subset of P_2 with the properties $\bar{z} = z$ and $a_F \leq \bar{a}_F$;

(c) the set of all restrictions of the functions of L to P_1, defined by

$$F_o(\bar{a}) = \inf_{\bar{c} \leq \bar{a}} \sup_{a \leq \bar{c}} F(a),$$

is denoted by L_o; by theorem 2.3.4, L_o is isomorphic to $L//I$;

(d) assume $S_2(P_2)$ is any compact Hausdorff continuity space for L, and that $S_1(P_1)$ is the characteristic space $S_{L_o}(P_1)$ for L_o; in particular, the point functions corresponding to L_o are continuous, generate the topology and separate the points of $S_1(P_1)$.

Remark: It should be noticed that we have not assumed L to be divisible with respect to the ideal I.

Before defining W, it is convenient to prove two preliminary results. For these lemmas, let $F \in L$ and F_o be the restriction of F to P_1. Denote by f and f_o the (continuous) point functions corresponding to F and F_o respectively.

Lemma 4.3.7. If $f_o(X_1) < \lambda$, then there exists $b \in Q$ such that

$X_1 \in S_1(b)^-$ and $S_2(b) \subseteq \{Y_2 \in S_2(P_2) | f(Y_2) \leq \lambda\}$.

Proof. By the definition of f_o, $f_o(X_1) < \lambda$ implies that $F_o(\overline{a}) < \lambda$ for some $\overline{a} \in X_1$. Then $\overline{b}_{\overline{a}} \leq \overline{a}$ exists so that $\sup\{F(a) | a \leq \overline{b}_{\overline{a}}\} < \lambda$. Let $b = \overset{2}{\vee} \{\overline{b}_{\overline{a}} | \overline{a} \in X_1\}$. Then $b \in Q$ and $b \wedge \overline{a} = z$ for all $\overline{a} \in X_1$. Hence $X_1 \in S_1(b)^-$.

If $b \in Y_2$, then $Y_2 \in [\bigcup\{S_2(\overline{b}_{\overline{a}}) | \overline{a} \in X_1\}]^-$. For otherwise, $c \in Y_2$ exists satisfying $c \notin \bigcup[\bigcup\{S_2(\overline{b}_{\overline{a}}) | \overline{a} \in X_1\}]$. This means $c \wedge \overline{b}_{\overline{a}} = z$ for all $\overline{a} \in X_1$. But then $c \wedge b = z$, contrary to $b \in Y_2$. By the assumed continuity of f, $f(Y_2) \leq \sup\{f(X_2) | \overline{b}_{\overline{a}} \in X_2$, some $\overline{a} \in X_1\}$ $\leq \sup\{F(a) | a \leq \overline{b}_{\overline{a}}$, some $\overline{a} \in X_1\} \leq \lambda$. Thus $S_2(b) \subseteq$ $\{Y_2 \in S_2(P_2) | f(Y_2) \leq \lambda\}$. This completes the proof.

Lemma 4.3.8. If $f_o(X_1) < \lambda$, $a \in Q$ exists so that $X_1 \in S_1(a)^{-o}$ and $W(a) \subseteq \{Y_2 \in S_2(P_2) | f(Y_2) \leq \lambda\}$.

Proof. By the continuity of f_o, an element $\overline{a} \in X_1$ exists so that $f_o(Y_1) < \lambda$ for all Y_1 containing $\overline{a}$. By the previous lemma, for each Y_1, there exists $b_{Y_1} \in Q$ such that $Y_1 \in S_1(b_{Y_1})^-$ and $S_2(b_{Y_1}) \subseteq$ $\subseteq \{Y_2 \in S_2(P_2) | f(Y_2) \leq \lambda\}$. Let $a = \overset{2}{\vee} \{b_{Y_1} | \overline{a} \in Y_1\}$. Then $a \in Q$ and $Y_1 \in S_1(b_{Y_1})^- \subseteq S_1(a)^-$ for all Y_1 containing $\overline{a}$. Thus $S_1(\overline{a}) \subseteq$ $\subseteq S_1(a)^-$, and so $X_1 \in S_1(a)^{-o}$.

If $a \in Y_2$, then $c \wedge a \neq z$ whenever $c \in Y_2$. This means that $c \wedge b_{Y_1} \neq z$ for some Y_1 with $\overline{a} \in Y_1$. If $X_2 \in c \wedge b_{Y_1}$, $c \in X_2 \in S_2(b_{Y_1}) \subseteq$ $\subseteq \{Z_2 \in S_2(P_2) | f(Z_2) \leq \lambda\}$. Since c was an arbitrary element of Y_2, $Y_2 \subseteq \bigcup\{Z_2 \in S_2(P_2) | f(Z_2) \leq \lambda\}$. Thus $Y_2 \in \{Z_2 \in S_2(P_2) | f(Z_2) \leq \}^- =$

$\{Z_2 \in S_2(P_2) | f(Z_2) \leq \lambda\}$. This proves $W(a) \subseteq S_2(a)^- \subseteq \{Y_2 \in S_2(P_2) | f(Y_2) \leq \lambda\}^- = \{Y_2 \in S_2(P_2) | f(Y_2) \leq \lambda\}$. The proof is complete.

Now, before any more progress toward the ultimate goal can be made, it is necessary to define the mapping W.

Definition 4.3.2. For $b \in Q$, put

$$W(b) = \bigcap \{S_2(a_F)^- | \bar{a}_F \geq b\} \cap S_2(b)^- ,$$

where the closure is in the topology of $S_2(P_2)$.

Lemma 4.3.9. If $b \neq z$ is any element of Q, W(b) is a non-empty, closed subset of $S_2(b_2)^-$.

Proof. It is only necessary to show that W(b) is non-empty. This means that we have to establish a finite intersection property.

Suppose $\bar{a}_{F_1}, \ldots, \bar{a}_{F_n} \geq b$. Because $b \in Q$, $\bar{a} \in P_1$ exists with $z \neq \bar{a} \leq b$. The sets of the form $\bar{a}_G$ are assumed to be dense in P_1 so that $\bar{a}_G \neq z$ exists satisfying $\bar{a}_G \leq \bar{a} \leq b \leq \bar{a}_{F_1} \wedge \ldots \wedge \bar{a}_{F_n}$. Thus $z \neq a_G \wedge a_{F_1} \wedge \ldots \wedge a_{F_n} \leq b \wedge a_{F_1} \wedge \ldots \wedge a_{F_n}$. If $X_2 \in S_2(P_2)$ satisfies $X_2 \in b \wedge a_{F_1} \wedge \ldots \wedge a_{F_n}$, then $X_2 \in S_2(b)^- \cap S_2(a_{F_1})^- \cap \ldots \cap S_2(a_{F_n})^-$. By the compactness of $S_2(P_2)$, it follows that $W(b) = \bigcap \{S_2(a_F)^- | \bar{a}_F \geq b\} \cap S_2(b)^-$ is not empty.

Lemma 4.3.10. If $f_o(X_1) > \lambda$, then $\bar{a} \in P_1$ exists with $\bar{a} \in X_1$ and $W(\bar{a}) \subseteq \{Y_2 \in S_2(P_2) | f(Y_2) \geq \lambda\}$.

Proof. If $f_o(X_1) > \lambda$, then $\bar{a} \in X_1$ exists such that $F_o(\bar{a}) > \lambda$. By lemma 2.3.9, this means that $\bar{a}_{F - \lambda'} \geq \bar{a}$ for some $\lambda' > \lambda$.

Consequently $W(\overline{a}) \subseteq S_2(a_{F-\lambda'})^-$.

If $a_{F-\lambda'} \in X_2$, then $f(X_2) = \sup\{F(b) | b \in X_2\} \geq F(a_{F-\lambda'}) = F(\overset{2}{\bigvee}\{c | F(c) > \lambda'\}) = \inf\{F(c) | F(c) > \lambda'\} \geq \lambda'$. Hence $W(\overline{a}) \subseteq$ $\subseteq S_2(a_{F-\lambda'})^- \subseteq \{X_2 \in S_2(P_2) | f(X_2) \geq \lambda'\}^- \subseteq \{X_2 \in S_2(P_2) | f(X_2) \geq \lambda\}$. This completes the proof.

Collecting these results together, it is now possible to prove the fundamental theorem.

Theorem 4.3.1. Assume that all the hypotheses listed at the beginning of this article are satisfied. Then the space $S_{L_o}(P_1)$ is homeomorphic to a factor space of a closed sub-space of $S_2(P_2)$.

Proof. It is only necessary to show that the projection $X_1 \to T_{X_1}$ defined with respect to the mapping W of definition 4.3.2 is simple. For this, the criterion of proposition 4.3.2 is used.

By theorems 2.3.4 and 4.2.1, the continuous point functions corresponding to the lattice functions of the translation lattice L_o separate the points of $S_{L_o}(P_1)$. Hence if $X_1 \neq Y_1$, and f_o exists such that $f_o(X_1) \neq f_o(Y_1)$. Suppose for definiteness that $f_o(X_1) <$ $< \lambda_1 < \lambda_2 < f_o(Y_1)$. Then by lemmas 4.3.8 and 4.3.10 $a \in Q$ and $\overline{b} \in P_1 \subseteq Q$ exist so that $X_1 \in S_1(a)^{-o}$, $Y_1 \in S_1(\overline{b}) \subseteq S_1(\overline{b})^{-o}$ and $W(a) \cap W(\overline{b}) \subseteq \{Y_2 | f(Y_2) \leq \lambda_1\} \cap \{Y_2 | f(Y_2) \geq \lambda_2\} = \emptyset$. According to proposition 4.3.2, $X_1 \to T_{X_1}$ is a simple projection. The proof is complete.

When combined with the results of the previous chapters, theorem 4.3.1 gives very general results on the relationship between representations of translation lattices as sets of continuous functions.

Corollary 4.3.1. Let L be an abstract translation lattice. Let I be a bounded closed ideal of L such that L is divisible with respect to I. Suppose there is an isomorphic mapping of L onto a subset L' of C(S) (the continuous functions on the space S) where S is a compact Hausdorff topological space. Assume, moreover, that this isomorphism carries I into $\{f \in L' | f \leq 0\}$. Then the characteristic space of L (formed with respect to I) is homeomorphic to a factor space of a closed sub-space of S.

Proof. This is a direct consequence of theorems 4.3.1 and 2.3.4.

Relaxation of the restriction that L be divisible with respect to the ideal I gives the following:

Corollary 4.3.2. Let L be an abstract translation lattice. Let I be a bounded closed ideal of L. Suppose there is a homomorphic mapping h of L onto a subset L' of C(S), where S is a compact Hausdorff topological space. Assume that $I = h^{-1}\{f \in L' | f \leq 0\}$. Then the characteristic space of L//I (formed with respect to I) is homeomorphic to a factor space of a closed sub-space of S.

Proof. This follows from theorems 2.2.1, 2.3.4 and 4.3.1.

Remark: Corollary 4.3.1 was proved for the special case of the translation lattice N(S), and in a slightly different form, by Professor R. P. Dilworth in his seminar at Caltech in 1951. The theorem presented here owes a large debt of gratitude for the inspiration of Professor Dilworth's work.

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