ON TWO-DIMENSIONAL WAVES OF FINITE AMPLITUDE IN ELASTIC MATERIALS OF HARMONIC TYPE

Thesis by

Robert Francis Jeffers

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1971

(Submitted August 6, 1970)

ACKNOWLEDGEMENTS

The author gratefully acknowledges the help and encouragement provided by his research adviser, Professor J. K. Knowles, at all stages of this work. He suggested the topic and gave generously of his time throughout.

This research was supported by a Graduate Teaching Assistantship and a number of summer grants from the California Institute of Technology. The author takes pleasure in acknowledging this support.

ABSTRACT

In this thesis, two-dimensional waves of finite amplitude in elastic materials of harmonic type are considered. After specializing the basic equations of finite elasticity to these materials, attention is restricted to plane motions and a new representation theorem (analogous to the theorem of Lamé in classical linear elasticity) for the displacements in terms of two potentials is derived.

The two-dimensional problem of the reflection of an obliquely incident periodic wave from the free surface of a half-space composed of an elastic material of harmonic type is formulated. The incident wave is a member of a special class of exact one-dimensional solutions of the nonlinear equations for elastic materials of harmonic type, and reduces upon linearization to the classical periodic "shear wave" of the linear theory.

A perturbation procedure for the construction of an approximate solution of the reflection problem, for the case where the incident wave is of small but finite amplitude, is constructed. The procedure involves series expansions in powers of the ratio of the amplitude to the wavelength of the incident wave and is of the so-called two-variable type. The perturbation scheme is carried far enough to determine the second-order corrections to the linearized theory.

A summary of results for the reflection problem is provided, in which nonlinear effects on the reflection pattern, on the particle displacements at the free surface and on the behavior at large depth in the half-space are detailed.

TABLE OF CONTENTS

PART	TITLE	PAGE
	Acknowledgements	ii
	Abstract	iii
	Table of Contents	v
I.	INTRODUCTION	1
	§1. Purpose and Scope of the Present Work	1
	§2. Previous Work on Nonlinear Elastic Waves	2
II.	FUNDAMENTAL EQUATIONS	5
	§3. Field Equations of Finite Elasticity	5
	§4. Materials of Harmonic Type	7
	§5. Plane Motions of Harmonic Materials	12
	§6. Potential Representations for Plane Motions of Harmonic Materials	15
III.	WAVES OF CONSTANT $\bar{r}$	22
	§7. Plane Periodic Waves with Constant $\bar{r}$	22
	§8. Reflection of Waves of Constant $\bar{r}$ From a Free Surface	29
IV.	PERTURBATION PROCEDURE FOR THE REFLECTION PROBLEM	31
	§9. Form of the Expansions	31
	§10. The First Order Problem - Linear Theory	36
	§11. Modification of the First Order Solution	40
	§12. Determination of Depth-Dependent Amplitudes $B_n(\tilde{\eta})$	46
	§13. Modification of the Second-Order Solution	53
	§14. Determination of Depth-Dependent Amplitudes $D_n(\tilde{\eta})$	57

TABLE OF CONTENTS (Continued)

PART	TITLE	PAGE
	§15. Final Form of the First and Second Order Solutions	66
V.	RESULTS	71
	§16. Summary of Results	71
	§17. Nonlinear Effects on the Reflection Pattern	79
	§18. Nonlinear Effects on the Surface Displacements	85
	§19. The Case of Normal Incidence	92
	§20. Nonlinear Effects on the Solution at Large Depth	95
	FIGURES	98
	REFERENCES	107

INTRODUCTION

§1. Purpose and Scope of the Present Work

In recent years there have been several studies concerned with the propagation of one-dimensional waves of finite amplitude in elastic solids. However, little or no attention has yet been given to two- or three-dimensional problems in this subject, presumably because of the complexity involved.

In the present work we consider two-dimensional (plane strain) waves for a class of elastic materials introduced by John in [1] and referred to as materials of harmonic type. These materials are characterized by a special class of stored energy functions, and, although they do not exhaust the class of elastic solids, it is hoped that some of the qualitative features of their behavior will be typical of elastic materials in general. The theory of finite deformations of elastic materials of harmonic type, developed in [1] and [2], appears to be simpler in many respects than that of more general elastic materials. The present investigation was undertaken with the expectation that some of these simpler features would make it possible to examine finite elastic waves for such materials more explicitly that is possible for more general elastic solids, and yet without sacrifice of any major qualitative characteristics.

We begin in Chapter II by stating the basic equations of finite elasticity in general. After specializing these to materials of harmonic type, we restrict attention to plane motions and derive a new representation theorem for the displacements in terms of two potentials. This representation is analogous to the theorem of Lamé in classical linear elasticity (see [3]) in the case of plane strain, and indeed when linearized, reduces to this result.

In Chapter III we consider a special class of exact one-dimensional solutions of the nonlinear equations for elastic materials of harmonic type. This special class of solutions, given originally by John 1], [2], includes a periodic wave which reduces upon linearization to the classical periodic "shear wave" of the linear theory. We then formulate the two-dimensional problem of the reflection of an obliquely incident periodic wave of this kind from the free surface of a half-space composed of an elastic material of harmonic type.

For an incident wave of small but finite amplitude we describe in Chapter IV a perturbation procedure for the construction of an approximate solution of the reflection problem formulated in Chapter III. This procedure involves series expansions in powers of the ratio of the amplitude to the wave length of the incident wave and is of the so-called two-variable type (see [4], Chapter III). The perturbation scheme is carried far enough to determine the second-order corrections to the linearized theory.

In Chapter V we summarize the results for the reflection problem, with emphasis on the effect of nonlinearity on the reflection pattern and on the particle displacements on the free surface of the half-space.

§2. Previous Work on Nonlinear Elastic Waves

Recent work in the field of nonlinear elastic waves falls into three main categories. The first of these is concerned with the propagation of singular surfaces of the second and higher orders. First studied by Hadamard [5]), this subject has been investigated by Truesdell [6], Green [7] and Varley and Dunwoody [8], among others. Secondly, plane infinitesimal disturbances superimposed on a finite uniform deformation of an elastic body have been studied by Toupin and Bernstein [9], Hayes and Rivlin [10] and others. Thirdly, there have been such investigations as those of Bland [11], [12], [13], Chu [14], Collins [15], Davison [16], [17], Varley [18], Varley, Mortell and Trowbridge [19], into the propagation of shocks and simple waves in elastic solids.

In another direction, Fine and Shield [20] have used a straightforward perturbation analysis for general three-dimensional problems. Their results hold, however, only for a finite time-interval because of the presence of secular terms in their solutions.

The present work differs from previous investigations involving nonlinear waves in elasticity in two respects. First we are concerned with two-dimensional rather than one-dimensional waves. Secondly, our emphasis is on periodic solutions of the underlying differential equations rather than on boundary-initial value problems. As indicated in the preceding section we deal with a class of elastic materials, called "harmonic," and defined by John in [1]. That reference contains, among other results, a description of the propagation of two kinds of elastic waves in harmonic materials, in which time dependent infinitesimal perturbations from a finitely strained state are involved. The formulation of the equations of motion given there enables us to derive the potential representation theorem referred to earlier.

The present work is also influenced by a second paper [2] of John, which is concerned with the study of polarized plane waves and irrotational motions of an elastic material. In that paper two special classes of materials prove to be of particular interest with respect to such motions. One such class of materials, called "Hadamard materials" by John, is shown to be the only one for which there exist three polarized plane waves for arbitrary orientation of the wave front. Hadamard [5] had shown that, for these materials, there exist infinitesimal plane waves polarized perpendicular to the wave front for every wave front orientation.

The other class of materials considered by John in [2] is the class of harmonic materials. Defining a deformation to be "pseudo-irrotational" if its curl with respect to the material coordinates is zero (in which case the deformation can be expressed as the gradient of a scalar function) he proceeds to show that harmonic materials are exactly those materials which remain pseudo-irrotational in the absence of body forces when their initial conditions are pseudo-irrotational. A discussion of these materials is given in §4 of the present work.

II. FUNDAMENTAL EQUATIONS

§3. Field Equations of Finite Elasticity

In this section we assemble the basic equations of the nonlinear theory of elasticity.* These equations include statements of the mechanical principles of conservation of mass, linear and angular momentum, and the mechanical constitution of the material.

We consider a body of homogeneous elastic material which occupies a region R of three-dimensional space in its natural (or undeformed) state. Let X be a rectangular Cartesian coordinate frame, fixed in space, and let X_i $(i = 1, 2, 3)$ be the coordinates in this frame of a typical point in the body in the natural state. We shall be concerned with motions in which a particle located at the point X_i in the natural state is located at time t at the point whose coordinates are $x_i(X_1, X_2, X_3, t)$ in the frame X. We employ the standard notation for Cartesian tensors in which subscripts have range 1, 2, 3, and the summation convention holds for repeated subscripts.

We denote the displacement gradients by

$$c_{ij} = \frac{\partial x_i}{\partial X_j} , \tag{3.1}$$

and we let c stand for the matrix (c_{ij}). Conservation of mass requires that

$$J\rho = \rho_o , \tag{3.2}$$

where ρ_o is the constant mass density in the natural configuration, $\rho(X_1, X_2, X_3, t)$ is the mass density (per unit volume of the deformed body) at time t, and J is the Jacobian of the transformation

*See [21] for a complete discussion of the theory.

$(X_1, X_2, X_3) \rightarrow (x_1, x_2, x_3)$. Thus,

$$J = \det (c_{ij}) \quad . \tag{3.3}$$

J is assumed to be positive for all times and all particles in the body.

Let t_{ij} be the components of the stress tensor in the coordinate system X. Conservation of angular momentum requires that the stress tensor be symmetric, and conservation of linear momentum requires that, in the absence of body forces, the equations of motion

$$\frac{\partial t_{ij}}{\partial x_j} = \rho \frac{\partial^2 x_i}{\partial t^2} \tag{3.4}$$

be satisfied at all times t and at all points in the region occupied by the body at time t.

Let the unit normal $\underline{n}$ at a point P on a surface in the deformed body at time t have components n_i in the frame X. If $\underline{T}$ is the traction, i. e., the force per unit area of deformed surface, exerted on the surface at P by the material into which $\underline{n}$ is directed, then its components T_i, in X, are given by

$$T_i = t_{ij} n_j \, . \tag{3.5}$$

We shall confine our attention here to perfectly elastic materials which are characterized by the existence of a strain - energy density W per unit volume of the undeformed body, from which the stresses may be derived. W depends only on the displacement gradients c_{ij}. The constitutive equations for such materials are

$$t_{ij} = \frac{1}{J} c_{ik} \frac{\partial W}{\partial c_{jk}} \quad . \tag{3.6}$$

When the material is isotropic, W can be written as a function of the three principal invariants I_1, I_2, I_3 of the matrix $g = cc^T$. The elements g_{ij} of g are given by

$$g_{ij} = c_{ik}c_{jk} \ . \tag{3.7}$$

The invariants I_1, I_2, I_3 are given by

$$\left.\begin{aligned} I_1 &= g_{ii} = \mathrm{Tr}\, g \ , \\ I_2 &= \tfrac{1}{2}(g_{ii}g_{jj} - g_{ij}g_{ji}) = \tfrac{1}{2}(\mathrm{Tr}\, g)^2 - \tfrac{1}{2}\mathrm{Tr}(g^2) \ , \\ I_3 &= \det(g_{ij}) = \det g = J^2 \ . \end{aligned}\right\} \tag{3.8}$$

In this case the stresses are given by

$$t_{ij} = 2I_3^{-\frac{1}{2}}\left\{\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)g_{ij} - 2\frac{\partial W}{\partial I_2} g_{ik}g_{jk} + I_3 \frac{\partial W}{\partial I_3} \delta_{ij}\right\} \ ,$$

where δ_{ij} is the Kronecker delta.

§4. Materials of Harmonic Type

We now define a special class of materials known as those of harmonic type, and we summarize the basic equations governing their finite motions. These special materials were first considered by John, and the description of their properties which we provide in this section and in §5 is adapted, with occasional changes of notation, from his work [1], [2].

In what follows it will be more convenient to use Lagrangian rather than Eulerian coordinates. The "Lagrange stresses" q_{ij} are defined by

$$q_{ij} = J \frac{\partial X_j}{\partial x_k} t_{ki} \ . \tag{4.1}$$

By (3.6), (3.1) we then have

$$q_{ij} = \frac{\partial W}{\partial c_{ij}} \ . \tag{4.2}$$

Unlike the stress tensor, the matrix of Lagrange stresses is not symmetric. Differentiating (4.1), we obtain

$$\frac{\partial q_{ij}}{\partial X_j} = t_{ki} \frac{\partial}{\partial X_j}\left(J \frac{\partial X_j}{\partial x_k}\right) + J \frac{\partial X_j}{\partial x_k} \frac{\partial t_{ki}}{\partial x_m} \frac{\partial x_m}{\partial X_j} \ .$$

Since

$$\frac{\partial}{\partial X_j}\left(J \frac{\partial X_j}{\partial x_i}\right) = 0 \ , \tag{4.3}$$

it follows from (3.2), (3.4) that the equations of motion can be written as

$$\frac{\partial q_{ij}}{\partial X_j} = \rho_o \frac{\partial^2 x_i}{\partial t^2} \ . \tag{4.4}$$

Let S_o be any surface in the undeformed configuration and let P_o be any point on S_o. Let the unit normal $\underline{n}^o$ to S_o at P_o have components n_i^o with respect to the fixed Cartesian coordinate system X. In the deformed body at time t, the particles which lay on S_o in the undeformed state lie on a surface S, and the particle which was at P_o now occupies the position P (say) on S. Let n_i be the components of the normal $\underline{n}$ to S at P. Let T_i be the components of the traction $\underline{T}$ exerted on S at P by the material on the side into which $\underline{n}$ is directed. We now outline the computation involved in determining the counterpart of (3.5). Consider a surface element containing P_o which has unit normal $\underline{n}^o$ and area dS_o in the undeformed state and has unit normal $\underline{n}$, area dS and contains P in the deformed state. Let the force acting on the surface element dS have components dF_i in X. Then, by (3.5), on taking the limit as $dS \to 0$, we have

$$T_i = \frac{dF_i}{dS} = t_{ij}n_j .$$

On using the vector transformation law one finds that

$$n_i dS = J \frac{\partial X_r}{\partial x_i} n_r^o dS_o , \tag{4.5}$$

from which, by (4.1), one obtains

$$dF_i = t_{ij}n_j dS = J \frac{\partial X_r}{\partial x_j} t_{ij} n_r^o dS_o = q_{ir} n_r^o dS_o . \tag{4.6}$$

If we now multiply each side of (4.5) by $\frac{1}{J}\frac{\partial x_i}{\partial X_j}$ and note that

$$n_j^o = \frac{\partial x_k}{\partial X_j} n_k \left(\frac{\partial x_r}{\partial X_i} \frac{\partial x_s}{\partial X_i} n_p n_q\right)^{-\frac{1}{2}} ,$$

we find that

$$dS_o = \frac{1}{J}\left(\frac{\partial x_i}{\partial X_j} \frac{\partial x_k}{\partial X_j} n_i n_k\right)^{\frac{1}{2}} dS .$$

From this and (3.5) it follows that, on taking the limit as $dS \to 0$,

$$T_i = q_{ij} n_j^o \frac{1}{J}\left(\frac{\partial x_r}{\partial X_k} \frac{\partial x_s}{\partial X_k} n_r n_s\right)^{\frac{1}{2}} . \tag{4.7}$$

As noted in section 3, for an isotropic material, the strain - energy W is a function of I_1, I_2, I_3 , which are the elementary symmetric functions of the eigenvalues λ_i of the matrix $g = cc^T$. In discussing harmonic materials, however, it turns out to be more convenient to work with the elementary symmetric functions r, s, J of the eigenvalues μ_i of the positive - definite symmetric matrix e that satisfies the relation

$$e^2 = c^T c . \tag{4.8}$$

Since the eigenvalues of $c^T c$ coincide with those of cc^T , we have

$$\left.\begin{aligned} \mu_i &= \sqrt{\lambda_i}\,, \\ r = \mu_1+\mu_2+\mu_3\,, \quad s = \mu_2\mu_3+\mu_3\mu_1+\mu_1\mu_2\,, \quad J &= \mu_1\mu_2\mu_3 = \sqrt{\det c^T c} \end{aligned}\right\} \qquad (4.9)$$

J is, of course, the Jacobian of the transformation $(X_1, X_2, X_3) \rightarrow (x_1, x_2, x_3)$, defined in (3.3).

It can be shown [2, p.328] that the invariants r, s, J of (4.9) are related to the invariants I_1, I_2, I_3 of (3.8) by the equations

$$I_1 = r^2 - 2s\,, \quad I_2 = s^2 - 2rJ\,, \quad I_3 = J^2\,. \qquad (4.10)$$

A harmonic elastic material is one for which the strain-energy per unit volume in the undeformed state is of the form

$$W = F(r) + as + bJ\,, \qquad (4.11)$$

where a, b are arbitrary constants and F is an arbitrary function.*

The matrix $h = h_{ij}$ defined through

$$h = ce^{-1} \qquad (4.12)$$

satisfies

$$h^T h = (e^{-1})^T c^T c\, e^{-1} = I\,, \qquad (4.13)$$

where I is the unit matrix of order 3, and so h is orthogonal. We have, in

$$c = he\,,$$

a "polar decomposition" of the deformation into a pure strain, described by e, followed by a pure rotation, described by h.

We now substitute for W from (4.11) into (4.2), (4.4) and determine the constitutive equations and the equations of motion for harmonic materials. After some computation, the Lagrange stresses

---

*Restrictions on a, b and F which are required for later purposes are discussed in §5.

q_{ij} are seen to be given by

$$q_{ij} = \big(F'(r)+ar\big)h_{ij} - ac_{ij}+bJc^{ji} , \qquad (4.14)$$

where c^{ij} is the element in the i^{th} row, j^{th} column of the matrix c^{-1}. From (3.1) it follows that

$$c^{ij} = \frac{\partial X_i}{\partial x_j} . \qquad (4.15)$$

By (4.14), (4.15), (4.3) we then have the following as equations of motion for materials of harmonic type:

$$\rho_o \frac{\partial^2 x_i}{\partial t^2} = \big(F'(r)+ar\big)_{,j} h_{ij} + \big(F'(r)+ar\big)h_{ij,j} - ax_{i,jj} \qquad (4.16)$$

where a subscript j preceded by a comma indicates partial differentiation with respect to X_j.

Consistency with the stress-strain law of linear elasticity requires that

$$F'(3)+2a+b = 0 , \quad a+b = -2\mu , \quad F''(3) = \lambda+2\mu , \qquad (4.17)$$

where λ, μ are the Lamé constants.

In [2], a vector field with components $f_i(X_1, X_2, X_3)$, defined on R , is said to be <u>pseudo-irrotational</u> if

$$f_{i,j} - f_{j,i} = 0 \qquad \text{on} \quad R .$$

Harmonic materials are precisely those elastic materials with the property that, in the absence of body-forces, pseudo-irrotationality of the displacement field at a particular instant of time implies pseudo-irrotationality of the acceleration field at that instant.

As will be seen in the next section, choice of the name "harmonic" is prompted by the fact that, when these materials are in equilibrium in a state of plane strain with no body-forces present, the local rotation angle θ is a harmonic function of the Lagrangian co-ordinates.

§5. Plane Motions of Harmonic Materials

We now confine our attention to the case of plane motions of a harmonic material in which the third coordinate remains unchanged, i. e., to motions of the form

$$x_1 = x_1(X_1, X_2, t), \quad x_2 = x_2(X_1, X_2, t), \quad x_3 = X_3 . \tag{5.1}$$

In this case, the matrix c of displacement gradients is given by

$$c = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & 0 \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2}$$

The matrices h and e defined in (4.12) and (4.8), respectively, now have the form

$$h = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.3}$$

$$e = \begin{pmatrix} c_{11}\cos\theta + c_{21}\sin\theta , & c_{12}\cos\theta + c_{22}\sin\theta , & 0 \\ -c_{11}\sin\theta + c_{21}\cos\theta , & -c_{12}\sin\theta + c_{22}\cos\theta , & 0 \\ 0 , & 0 , & 1 \end{pmatrix} \tag{5.4}$$

where

$$\cos\theta = \frac{c_{11}+c_{22}}{\bar{r}} , \qquad \sin\theta = \frac{c_{21}-c_{12}}{\bar{r}} , \tag{5.5}$$

and

$$\bar{r} = \left[(c_{11}+c_{22})^2 + (c_{21}-c_{12})^2\right]^{\frac{1}{2}} . \tag{5.6}$$

From (5.4), the eigenvalues of e are the constant 1 and the two values e_1, e_2 satisfying

$$e_1+e_2 = \bar{r}\ , \quad e_1e_2 = \bar{s} \equiv c_{11}c_{22}-c_{21}c_{12} = J\ . \tag{5.7}$$

The quantities $\bar{r}, \bar{s}$ and the principal invariants r, s, J of e are related by

$$r = e_1+e_2+1 = \bar{r}+1\ , \quad s = e_2+e_1+e_1e_2 = \bar{r}+\bar{s},\ J = e_1e_2 = \bar{s}\ , \tag{5.8}$$

and so, for the plane motions (5.1) of harmonic materials, W is a function of $\bar{r}, \bar{s}$ only. Using (5.8) in (4.11) we have

$$W = F(\bar{r}+1) + a(\bar{r}+\bar{s}) + b\bar{s}\ . \tag{5.9}$$

Thus, on setting

$$2\mu\bar{F}(\bar{r}) = F(\bar{r}+1) + a\bar{r}\ , \tag{5.10}$$

and using (4.17), we have

$$W = 2\mu(\bar{F}(\bar{r})-\bar{s})\ , \tag{5.11}$$

where

$$\bar{F}'(2) = 1\ , \quad \bar{F}''(2) = \frac{\lambda+2\mu}{2\mu}\ , \tag{5.12}$$

and $\bar{F}(\bar{r})$ is otherwise arbitrary.

We impose the further condition

$$\bar{F}(2) = 1\ , \tag{5.13}$$

which corresponds to $W = 0$ for rigid-body motions. Use of (5.8), (5.10), (5.11) in (4.14) leads to the following expressions for the Lagrange stresses q_{ij} :

$$q_{ij} = 2\mu\bar{F}'(\bar{r})h_{ij}-2\mu\bar{s}c^{ji}+a(\bar{r}-1-\bar{s})\delta_{i3}\delta_{j3}\ . \tag{5.14}$$

The elements of the Lagrange stress matrix, written individually, are

$$\left.\begin{aligned}
q_{11} &= 2\mu\left(\overline{F}'(\overline{r})\frac{c_{11}+c_{22}}{\overline{r}} - c_{22}\right) , \\
q_{12} &= 2\mu\left(-F'(\overline{r})\frac{c_{21}-c_{12}}{\overline{r}} + c_{21}\right) , \\
q_{21} &= 2\mu\left(\overline{F}'(\overline{r})\frac{c_{21}-c_{12}}{\overline{r}} + c_{12}\right) , \\
q_{22} &= 2\mu\left(\overline{F}'(\overline{r})\frac{c_{11}+c_{22}}{\overline{r}} - c_{11}\right) , \\
q_{13} &= q_{31} = q_{23} = q_{32} = 0 , \\
q_{33} &= 2\mu\left(\overline{F}'(\overline{r})-\overline{s}\right) + a(\overline{r}-\overline{s}-1) .
\end{aligned}\right\} \qquad (5.15)$$

Since $\overline{s} = J$ and $c^{ji} = \dfrac{\partial X_j}{\partial x_i}$, it follows from (4.3) that

$$\frac{\partial}{\partial X_j}(\overline{s}\, c^{ji}) = 0 . \qquad (5.16)$$

Equations (5.14), (5.16), (5.3) together imply that, for the plane motions (5.1) of harmonic materials, the equations of motion (4.4) take the form

$$\left.\begin{aligned}
\frac{\partial^2 x_1}{\partial t^2} &= \frac{\partial A}{\partial X_1} + \frac{\partial B}{\partial X_2} , \\
\frac{\partial^2 x_2}{\partial t^2} &= -\frac{\partial B}{\partial X_1} + \frac{\partial A}{\partial X_2} ,
\end{aligned}\right. \qquad (5.17)$$

where

$$A = \frac{2\mu}{\rho_o}\overline{F}'(\overline{r})\cos\theta , \qquad B = -\frac{2\mu}{\rho_o}\overline{F}'(\overline{r})\sin\theta , \qquad (5.18)$$

with θ, $\overline{r}$ given by (5.5), (5.6).

When the material is in equilibrium in plane strain with no body-forces present, we have, instead of (5.17),

$$\frac{\partial A}{\partial X_1} + \frac{\partial B}{\partial X_2} = 0 ,$$

$$\frac{\partial A}{\partial X_2} - \frac{\partial B}{\partial X_1} = 0 .$$

These imply

$$\frac{\partial^2 \theta}{\partial X_1^2} + \frac{\partial^2 \theta}{\partial X_2^2} = 0 ,$$

where θ is the local rotation angle, given by

$$\theta = - \arctan \frac{B}{A} .$$

Hence the name "harmonic."

§6. Potential Representations for Motions of Harmonic Materials

We present here an analog, for the case of plane motions of harmonic materials, of the Lamé potential representation of the solutions of the displacement equations of motion in the linear theory of elasticity (see e. g. [3]).

Before beginning our discussion, we make some notation changes. There is no further advantage in using indicial notation, so we replace the coordinates X_1, X_2 by x, y, respectively; and $x_1(X_1, X_2, t)$, $x_2(X_1, X_2, t)$ by $x+u(x, y, t)$, $y+v(x, y, t)$, respectively. The functions $u(x, y, t)$, $v(x, y, t)$ are the displacements, at time t, of the particle which occupied the point x, y in the undeformed state.

We shall assume here that u and v, together with their partial derivatives of the first and second order, are continuous for all x, y in some (two-dimensional) region D and for all times t. The quantities $\bar{r}, \bar{s}$ are now given by

$$\bar{r} = \left((2+u_x+v_y)^2+(v_x-u_y)^2\right)^{\frac{1}{2}} , \quad \bar{s} = (1+u_x)(1+v_y)-v_x u_y . \tag{6.1}$$

Here, subscripts x, y denote partial differentiation with respect to x, y. We shall also use the subscript t to denote partial differentiation with respect to t. Equations (5.5) become

$$\cos\theta = \frac{2+u_x+v_y}{\bar{r}}, \qquad \sin\theta = \frac{v_x-u_y}{\bar{r}}, \tag{6.2}$$

and the equations (5.17) for plane motions of materials of harmonic type now read

$$u_{tt} = A_x+B_y, \quad v_{tt} = -B_x+A_y \tag{6.3}$$

with A, B given by (5.18), (6.1), (6.2).

We shall now show that every solution u, v of (6.3) admits the representation

$$u = \Phi_x-\Psi_y, \qquad v = \Phi_y+\Psi_x, \tag{6.4}$$

where Φ, Ψ satisfy

$$\Phi_{tt} = \frac{2\mu}{\rho_o}\frac{\bar{F}'(\bar{r})}{\bar{r}}(2+\Delta\Phi), \quad \Psi_{tt} = \frac{2\mu}{\rho_o}\frac{\bar{F}'(\bar{r})}{\bar{r}}\Delta\Psi \tag{6.5}$$

in D, for all t, Δ being the Laplacian operator with respect to x, y. Conversely, if Φ, Ψ satisfy (6.5), then u, v, defined through (6.4), constitute a solution of (6.3).

Suppose u, v is a solution of (6.3) with the following values at the time t = 0 :

$$\left.\begin{aligned} u(x,y,0) &= \mathring{u}(x,y), \quad & v(x,y,0) &= \mathring{v}(x,y), \\ u_t(x,y,0) &= \mathring{u}_t(x,y), \quad & v_t(x,y,0) &= \mathring{v}_t(x,y). \end{aligned}\right\} \tag{6.6}$$

Then, on integrating (6.3) twice with respect to t,

$$\left.\begin{aligned} u-\mathring{u}-t\mathring{u}_t &= \frac{\partial}{\partial x}\int_0^t(t-\tau)A(x,y,\tau)d\tau + \frac{\partial}{\partial y}\int_0^t(t-\tau)B(x,y,\tau)d\tau, \\ v-\mathring{v}-t\mathring{v}_t &= -\frac{\partial}{\partial x}\int_0^t(t-\tau)B(x,y,\tau)d\tau + \frac{\partial}{\partial y}\int_0^t(t-\tau)A(x,y,\tau)d\tau. \end{aligned}\right\} \tag{6.7}$$

By the Helmholz vector decomposition theorem, there exist functions $U(x,y)$, $V(x,y)$, $\dot{U}(x,y)$, $\dot{V}(x,y)$ such that

$$\begin{aligned} \overset{o}{u}(x,y) &= U_x - V_y , \quad \overset{o}{v}(x,y) = U_y + V_x , \\ \overset{o}{u}_t(x,y) &= \dot{U}_x - \dot{V}_y , \quad \overset{o}{v}_t(x,y) = \dot{U}_y + \dot{V}_x . \end{aligned} \tag{6.8}$$

Using (6.8) in (6.6), (6.7) we have

$$u = U_x - V_y + t\dot{U}_x - t\dot{U}_y + \frac{\partial}{\partial x}\int_0^t (t-\tau)A\,d\tau + \frac{\partial}{\partial y}\int_0^t (t-\tau)B\,d\tau = \Phi_x - \Psi_y , \tag{6.9}$$

$$v = U_y + V_x + t\dot{U}_y + t\dot{V}_x - \frac{\partial}{\partial x}\int_0^t (t-\tau)B\,d\tau + \frac{\partial}{\partial y}\int_0^t (t-\tau)A\,d\tau = \Phi_y + \Psi_x , \tag{6.10}$$

where

$$\Phi = U + t\dot{U} + \int_0^t (t-\tau)A\,d\tau , \tag{6.11}$$

$$\Psi = V + t\dot{V} - \int_0^t (t-\tau)B\,d\tau . \tag{6.12}$$

Further, by (6.9) - (6.12), (6.2) and the definition (5.18) of A, B, we have

$$\Phi_{tt} = A = \frac{2\mu}{\rho_o}\frac{\overline{F}'(\overline{r})}{\overline{r}}(2+\Delta\Phi) , \quad \Psi_{tt} = -B = \frac{2\mu}{\rho_o}\frac{\overline{F}'(\overline{r})}{\overline{r}}\Delta\Psi .$$

We have thus shown that, if u, v are a solution of (6.3), then there exist Φ, Ψ satisfying (6.5), for which (6.4) is true.

Conversely, if Φ, Ψ satisfy (6.5) and if we define u, v through (6.4), then

$$\Phi_{tt} = \frac{2\mu}{\rho_o}\frac{\overline{F}'(\overline{r})}{\overline{r}}(2+\Delta\Phi) = \frac{2\mu}{\rho_o}\overline{F}'(\overline{r})\cos\theta = A ,$$

$$\Psi_{tt} = \frac{2\mu}{\rho_o}\frac{\overline{F}'(\overline{r})}{\overline{r}}\Delta\Psi = \frac{2\mu}{\rho_o}\overline{F}'(\overline{r})\sin\theta = -B ,$$

and so

$$u_{tt} = \Phi_{xtt} - \Psi_{ytt} = A_x + B_y , \quad v_{tt} = \Phi_{ytt} + \Psi_{xtt} = A_y - B_x ,$$

i.e., u, v satisfy (6.3). We have thus proved our assertions concerning the potential representation of the displacements $u(x, y, t)$, $v(x, y, t)$.

Since, in the undeformed state, $\bar{r} = 2$ and $\frac{\bar{F}'(2)}{2} = \frac{1}{2}$, it follows from (6.5) that

$$\Phi = \frac{\mu}{\rho_o} t^2 , \quad \Psi = 0$$

correspond to the undeformed state. In subsequent chapters we shall have occasion to use a procedure based on a perturbation from the undeformed state. It is therefore convenient to set

$$\Phi \equiv \frac{\mu}{\rho_o} t^2 + \phi , \quad \Psi \equiv \psi . \tag{6.13}$$

We also set

$$G(\bar{r}) \equiv \frac{2\bar{F}'(\bar{r})}{\bar{r}} , \quad c_2^2 \equiv \frac{\mu}{\rho_o} . \tag{6.14}$$

$G(\bar{r})$ is assumed positive for all values of $\bar{r}$.

In terms of the notation introduced at the beginning of this section, the matrix c is now given by

$$c = \begin{pmatrix} 1+u_x , & u_y , & 0 \\ v_x , & 1+v_y , & 0 \\ 0 , & 0 , & 1 \end{pmatrix} . \tag{6.15}$$

The Lagrange stresses of (5.15) then have the following representation in terms of u, v :

$$\left.\begin{aligned}
\frac{q_{11}}{2\mu} &= \tfrac{1}{2}G(\bar{r})(u_x+v_y)+G(\bar{r})-1-v_y \ , \\
\frac{q_{12}}{2\mu} &= -\tfrac{1}{2}G(\bar{r})(v_x-u_y)+v_x \ , \\
\frac{q_{21}}{2\mu} &= \tfrac{1}{2}G(\bar{r})(v_x-u_y)+u_y \ , \\
\frac{q_{22}}{2\mu} &= \tfrac{1}{2}G(\bar{r})(u_x+v_y)+G(\bar{r})-1-u_x \ , \\
q_{13} &= q_{23} = q_{31} = q_{32} = 0 \ , \\
\frac{q_{33}}{2\mu} &= \tfrac{1}{2}\bar{r}G(\bar{r}) - \frac{a+2\mu}{2\mu}\bar{s} + \frac{a}{2\mu}(\bar{r}-1)
\end{aligned}\right\} \qquad (6.16)$$

where $\bar{r}, \bar{s}$ are given by (6.1).

We now collect the basic equations in terms of the potentials ϕ, ψ for purposes of subsequent reference. From (6.4), (6.13) the displacements are given by

$$u = \phi_x - \psi_y \ , \quad v = \phi_y + \psi_x \ . \qquad (6.17)$$

The differential equations (6.5) take the form

$$\frac{1}{c_2^2}\phi_{tt} = G(\bar{r})\Delta\phi + 2(G(\bar{r})-1) \ , \quad \frac{1}{c_2^2}\psi_{tt} = G(\bar{r})\Delta\psi \ , \qquad (6.18)$$

where, by (6.1), (6.17),

$$\bar{r} = 2\left\{(1+\tfrac{1}{2}\Delta\phi)^2 + \tfrac{1}{4}(\Delta\psi)^2\right\}^{\frac{1}{2}} \ . \qquad (6.19)$$

By (6.14), (6.17), the Lagrange stresses $q_{11}, q_{12}, q_{21}, q_{22}$ have the following representation in terms of ϕ, ψ :

$$
\left.
\begin{aligned}
\frac{q_{11}}{2\mu} &= \tfrac{1}{2}G(\bar{r})\Delta\phi + G(\bar{r}) - 1 - \phi_{yy} - \psi_{xy}\,, \\
\frac{q_{12}}{2\mu} &= -\tfrac{1}{2}G(\bar{r})\Delta\psi + \phi_{xy} + \psi_{xx}\,, \\
\frac{q_{21}}{2\mu} &= \tfrac{1}{2}G(\bar{r})\Delta\psi + \phi_{xy} - \psi_{yy}\,, \\
\frac{q_{22}}{2\mu} &= \tfrac{1}{2}G(\bar{r})\Delta\phi + G(\bar{r}) - 1 - \phi_{xx} + \psi_{xy}\,.
\end{aligned}
\right\} \tag{6.20}
$$

Using (6.18) we also have the following representation for q_{11}, q_{12}, q_{21}, q_{22} :

$$
\left.
\begin{aligned}
\frac{q_{11}}{2\mu} &= \frac{1}{2c_2^2}\phi_{tt} - \phi_{yy} - \psi_{xy}\,, \\
\frac{q_{12}}{2\mu} &= -\frac{1}{2c_2^2}\psi_{tt} + \psi_{xx} + \phi_{xy}\,, \\
\frac{q_{21}}{2\mu} &= \frac{1}{2c_2^2}\psi_{tt} - \psi_{yy} + \phi_{xy}\,, \\
\frac{q_{22}}{2\mu} &= \frac{1}{2c_2^2}\phi_{tt} - \phi_{xx} + \psi_{xy}\,.
\end{aligned}
\right\} \tag{6.21}
$$

The expressions in (6.21) are linear in ϕ, ψ and so must be identical with the corresponding expressions in the linear theory.

We conclude this section by showing that, when linearized, the equations (6.18) for plane motions of harmonic materials reduce to those of the classical linear theory of elasticity. Let

$$
c_1^2 = \frac{\lambda+2\mu}{\rho_o}\quad . \tag{6.22}
$$

Equations (6.14), (5.12) imply

$$
G(2) = 1\;, \quad G'(2) = \tfrac{1}{2}\,(\,c_1^2/c_2^2 - 1\,)\,, \tag{6.23}
$$

and so

$$G(\bar{r}) = 1 + \tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)(\bar{r}-2) + k_2(\bar{r}-2)^2 + k_3(\bar{r}-2)^3 + O((\bar{r}-2)^4) \quad \text{as } \bar{r} \to 2 , \qquad (6.24)$$

where k_2, k_3 are material constants. By (6. 19) we have

$$\bar{r} \approx 2 + \Delta\phi \qquad (6.25)$$

on linearizing, and so, by (6.24),

$$G(\bar{r}) \approx 1 + \tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\Delta\phi . \qquad (6.26)$$

Thus, when linearized, equations (6.18) become

$$\frac{1}{c_2^2}\phi_{tt} = \Delta\phi + \left(\frac{c_1^2}{c_2^2} - 1\right)\Delta\phi , \quad \frac{1}{c_2^2}\psi_{tt} = \Delta\psi ,$$

i. e.,

$$\frac{1}{c_1^2}\phi_{tt} = \Delta\phi , \quad \frac{1}{c_2^2}\psi_{tt} = \Delta\psi \qquad (6.27)$$

as in the classical linear theory.

III. WAVES OF CONSTANT $\overline{r}$

§7. Plane Periodic Waves with Constant $\overline{r}$

In this section we show that equations (6.18) possess periodic plane wave solutions for which the corresponding $\overline{r}$, given by (6.19), is constant. Such solutions will be seen to reduce upon linearization to the periodic one-dimensional shear waves of the classical linear theory.

If $\overline{r} = r^*$ is constant, equations (6.18), (6.19) become

$$\frac{1}{c_2^2}\phi_{tt} = G(r^*)\Delta\phi + 2\left(G(r^*) - 1\right), \tag{7.1}$$

$$\frac{1}{c_2^2}\psi_{tt} = G(r^*)\Delta\psi, \tag{7.2}$$

where

$$\{(2+\Delta\phi)^2 + (\Delta\psi)^2\}^{\frac{1}{2}} = r^* . \tag{7.3}$$

Let

$$z = x\cos\alpha + y\sin\alpha, \tag{7.4}$$

where α is a given angle. We seek solutions ϕ, ψ of (7.1) - (7.3) which have the form

$$\phi = \phi(z,t), \quad \psi = \psi(z,t), \quad -\infty < z < \infty, \quad -\infty < t < \infty, \tag{7.5}$$

and for which the corresponding displacements u and v, given by (6.17), are bounded for all times t and periodic in z with a given period L. By (6.17) and (7.4),

$$u = \phi_z\cos\alpha - \psi_z\sin\alpha, \quad v = \phi_z\sin\alpha + \psi_z\cos\alpha . \tag{7.6}$$

Since, from this,

$$\phi_z = u\cos\alpha + v\sin\alpha, \quad \psi_z = -u\sin\alpha + v\cos\alpha,$$

it follows that ϕ_z and ψ_z are periodic of period L in z if and only if u and v are periodic of period L in z.

Let

$$c_3 = c_2[G(r^*)]^{\frac{1}{2}} . \qquad (7.7)$$

By (7.5), (7.4), (7.7), the equations (7.1) - (7.3) can be written

$$\frac{1}{c_3^2}\phi_{tt} = \phi_{zz} + 2\left(1 - \frac{1}{G(r^*)}\right) , \qquad (7.8)$$

$$\frac{1}{c_3^2}\psi_{tt} = \psi_{zz} , \qquad (7.9)$$

$$[(2+\phi_{zz})^2+(\psi_{zz})^2]^{\frac{1}{2}} = r^* , \text{ a constant.} \qquad (7.10)$$

We now assume that $\phi(z,t)$, $\psi(z,t)$ and r^* satisfy (7.8) - (7.10) and that ϕ_z, ψ_z are periodic in z. Our procedure involves the determination of necessary conditions on ϕ, ψ and r^* which follow from these assumptions.

Since ψ_z is to be periodic with period L in z, so also must ψ_{zz} be periodic with the same period in z. Since, by (7.10),

$$\phi_{zz} = -2 + [r^{*2} - (\psi_{zz})^2]^{\frac{1}{2}} , \qquad (7.11)$$

it follows that ϕ_{zz} must also be periodic with that period in z. From this, it further follows that, to guarantee periodicity of ϕ_z, we must have

$$\int_0^L [r^{*2} - (\psi_{zz})^2]^{\frac{1}{2}} dz = 2L , \qquad -\infty < t < \infty . \qquad (7.12)$$

Let

$$\epsilon = \max_{\substack{0 \le z \le L \\ -\infty < t < \infty}} |\psi_{zz}(z,t)| \qquad (7.13)$$

and define p(z, t) by

$$\epsilon p = \psi_{zz} . \tag{7.14}$$

Then ϵ, p and r^* satisfy

$$p(z+L, t) = p(z, t) , \tag{7.15}$$

$$\int_0^L p(z,t)dz = 0 , \quad -\infty < t < \infty , \tag{7.16}$$

$$\int_0^L [r^{*2} - \epsilon^2 p^2(z,t)]^{\frac{1}{2}} dz = 2L , \quad -\infty < t < \infty , \tag{7.17}$$

$$\max_{\substack{0 \le z \le L \\ -\infty < t < \infty}} |p(z,t)| = 1 . \tag{7.18}$$

We observe that (7.17), (7.18) imply that

$$\epsilon \le r^* , \qquad r^* \ge 2 . \tag{7.19}$$

If we add the assumption that ϕ, ψ represent waves traveling in the positive z-direction, it follows from (7.14) and (7.9) that

$$p(z,t) = p(z - c_3 t) . \tag{7.20}$$

Then (7.17) can be written

$$\int_0^L \left[\frac{r^{*2}}{\epsilon^2} - p^2(s) \right]^{\frac{1}{2}} ds = \frac{2L}{\epsilon} . \tag{7.21}$$

It follows from (7.21) and the first of (7.19) that ϵ must satisfy

$$\frac{2L}{\epsilon} \ge \int_0^L [1 - p^2(s)]^{\frac{1}{2}} ds . \tag{7.22}$$

Thus, solutions ψ of the assumed form can only exist for sufficiently small amplitudes ϵ.

From (7.11), (7.14), and (7.20) we find that ϕ must be given by

$$\phi(z,t) = \int_0^{z-c_3 t} (z - c_3 t - s)\left\{-2 + [r^{*2} - \epsilon^2 p^2(s)]^{\frac{1}{2}}\right\} ds + A(t) + B(t)z \tag{7.23}$$

for suitable functions A(t), B(t). Substitution of (7.23) in (7.8) leads immediately to the conclusion that

$$A(t) = \left[1 - \frac{1}{G(r^*)}\right] c_3^2 t^2 + a_o + a_1 t \ , \qquad B(t) = b_o + b_1 t \ , \tag{7.24}$$

where a_o, a_1, b_o, b_1 are arbitrary constants. The constants a_o and b_o have the dimension of length, while a_1 and b_1 have dimension length per unit time. The condition that ϕ_z be bounded for all t requires that $b_1 = 0$. Since a_o and a_1 do not contribute to the displacements, we may take $a_o = a_1 = 0$ without loss of generality. Thus, ϕ is given by

$$\phi(z,t) = \int_0^{z-c_3 t} (z-c_3 t-s)\left\{-2+[r^{*2} - \varepsilon^2 p^2(s)]^{\frac{1}{2}}\right\}ds + \left[1 - \frac{1}{G(r^*)}\right] c_3^2 t^2 + b_o z \ , \tag{7.25}$$

while ψ is given according to (7.14) by

$$\psi(z,t) = \varepsilon \int_0^{z-c_3 t} (z-c_3 t-s)p(s)ds + c_o z \ , \tag{7.26}$$

where c_o is an arbitrary constant with the dimension of length. The constants b_o, c_o correspond to rigid body displacements.

Conversely, let $p(s)$ be a given continuous function with the properties

$$\left.\begin{aligned} p(s+L) &= p(s) \ , \quad \text{all } s \ , \\ \int_0^L p(s)ds &= 0 \ , \\ \max_{0 \le s \le L} |p(s)| &= 1 \end{aligned}\right\} \tag{7.27}$$

and let ε be a given number satisfying (7.22). Then there exists a unique value $r^* = r^*(\varepsilon)$ satisfying (7.21). Define ϕ, ψ by (7.25), (7.26), with b_o and c_o arbitrary. It is easily verified that ϕ, ψ and r^*, constructed in this way, satisfy (7.8), (7.9), and (7.10), and hence provide solutions for which r^* is constant.†

The particular case

$$p(z-c_3t) = \sin\frac{2\pi}{L}(z-c_3t) \tag{7.28}$$

will be of interest later; p as given by (7.28) satisfies (7.27). The amplitude ϵ is now required by (7.22) to satisfy

$$\epsilon \le \pi . \tag{7.29}$$

Equation (7.21) for r^* becomes

$$E\left(\frac{\epsilon}{r^*}\right) = \frac{2\pi}{r^*} \tag{7.30}$$

where

$$E(k) = \int_0^{\pi/2}\left(1-k^2\sin^2\xi\right)^{\frac{1}{2}} d\xi \tag{7.31}$$

is the complete elliptic integral of the second kind.

We include here the displacements and Lagrange stresses corresponding to the ϕ, ψ of (7.25), (7.26). From (7.6), (7.25), (7.26) we have

$$\left.\begin{aligned} u &= \int_0^{z-c_3t} \{(-2+[r^{*2}-\epsilon^2p^2(s)]^{\frac{1}{2}})\cos\alpha-\epsilon p(s)\sin\alpha\}ds , \\ v &= \int_0^{z-c_3t} \{(-2+[r^{*2}-\epsilon^2p^2(s)]^{\frac{1}{2}})\sin\alpha+\epsilon p(s)\cos\alpha\} ds , \end{aligned}\right\} \tag{7.32}$$

to within an arbitrary rigid body displacement. By (6.21) and (7.7) we have the following expressions for the Lagrange stresses $q_{11}, q_{12}, q_{21}, q_{22}$ corresponding to the ϕ, ψ of (7.25), (7.26):

---

†It is also possible to prescribe p and $r^* > 2$, rather than p and ϵ, and to use (7.21) to determine $\epsilon = \epsilon(r^*)$.

$$
\left.\begin{aligned}
\frac{q_{11}}{2\mu} &= \left(\tfrac{1}{2}G(r^*)-\sin^2\alpha\right)\left(-2+\{r^{*2}-\epsilon^2p^2(z-c_3t)\}^{\frac{1}{2}}\right) \\
&\qquad -\epsilon\sin\alpha\cos\alpha p(z-c_3t)+G(r^*)-1 \, , \\
\frac{q_{12}}{2\mu} &= \sin\alpha\cos\alpha\left(-2+\{r^{*2}-\epsilon^2p^2(z-c_3t)\}^{\frac{1}{2}}\right) \\
&\qquad -\epsilon\left(\tfrac{1}{2}G(r^*)-\cos^2\alpha\right)p(z-c_3t) \, , \\
\frac{q_{21}}{2\mu} &= \sin\alpha\cos\alpha\left(-2+\{r^{*2}-\epsilon^2p^2(z-c_3t)\}^{\frac{1}{2}}\right) \\
&\qquad +\epsilon\left(\tfrac{1}{2}G(r^*)-\sin^2\alpha\right)p(z-c_3t) \, , \\
\frac{q_{22}}{2\mu} &= \left(\tfrac{1}{2}G(r^*)-\cos^2\alpha\right)\left(-2+\{r^{*2}-\epsilon^2p^2(z-c_3t)\}^{\frac{1}{2}}\right) \\
&\qquad +\epsilon\sin\alpha\cos\alpha p(z-c_3t)+G(r^*)-1 \, .
\end{aligned}\right\} \quad (7.33)
$$

The Lagrange stresses are thus seen to be the sum

$$q_{\alpha\beta} = \hat{q}_{\alpha\beta}+2\mu(G(r^*)-1)\delta_{\alpha\beta} \quad , \quad \alpha, \beta = 1, 2$$

of a periodic Lagrange stress system $\hat{q}_{\alpha\beta}$ and a uniform hydrostatic stress.

We show now that our solutions ϕ, ψ, given by (7.25), (7.26), reduce, when linearized, to the periodic one-dimensional waves of the linear theory. Linearization will be with respect to the small amplitude ϵ. Equation (7.17) determines r^* as a function of ϵ. The constant r^* thus possesses an expansion of the form

$$r^* = 2+\epsilon r_1^*+\epsilon^2 r_2^*+O(\epsilon^3) \qquad (7.34)$$

for sufficiently small ϵ. Here, $r_1^*, r_2^*, \ldots$ are constants independent of ϵ. Then

$$[r^{*2}-\epsilon^2p^2(s)]^{\frac{1}{2}} = 2+\epsilon r_1^*+\epsilon^2\left(r_2^*-\tfrac{1}{4}p^2(s)\right) + O(\epsilon^3) \, . \qquad (7.35)$$

Using this in (7.21), it follows that

$$r_1^* = 0 \, . \qquad (7.36)$$

Thus, $r^* - 2$ is of order ϵ^2, and so, by (6.24),

$$G(r^*) = 1 + O(\epsilon^2) \; . \tag{7.37}$$

Using (7.35), (7.37) in (7.25) we see that

$$\phi = O(\epsilon^2) \; . \tag{7.38}$$

By (7.7), (7.37),

$$c_3 t = c_2 t + O(\epsilon^2) \; . \tag{7.39}$$

From (7.31), (7.38), (7.39), and the properties (7.27) of the function p, it now follows that, correct to order ϵ, $\phi \equiv 0$ and ψ is an arbitrary periodic function of $z - c_2 t$. This is identical with the result of the classical linear theory with reference to a one-dimensional periodic shear wave progressing in the positive z-direction.

The analysis of this section, besides proving the original assertion concerning existence of plane wave solutions of equations (6.18), also shows that every periodic plane wave with $\overline{r} = r^*$, a constant, traveling in a direction $(\cos\alpha, \sin\alpha)$ with respect to a fixed Cartesian frame X, has displacements given by (7.32) for some function p satisfying (7.27). Once p is known, r^* is uniquely determined as a function of ϵ (and vice versa) by (7.21).

John [2, p. 336] presents a solution of the equations of motion of harmonic materials, for which $\overline{r} = \overline{r}_o$, a known constant. Replacement of the -1 in the first of his equation (109) by $-\dfrac{\pi_1 + \pi_2}{\overline{r}_o}$ gives a solution for which $\overline{r} = \overline{r}_o$, an <u>arbitrary</u> constant. The displacements of (7.32), for the case $\alpha = 0$, are identical with those given by this modified solution if we set $\pi_1 = \pi_2 = 1$, $r^* = \overline{r}_o$ and $\epsilon p(s) = \overline{r}_o \sin g(s)$.

§8. Reflection of Waves of Constant $\bar{r}$ from a Free Surface

We now wish to consider the problem of reflection by a free surface of waves of the type discussed in the previous section. The region occupied by the elastic body in its undeformed state is now assumed to be a half-space. Since the problem is one of plane strain, we consider the half-space $y \le 0$, $-\infty < x < \infty$.

Let the dimensionless variables $\xi, \eta, \zeta, \tau, \tilde{\phi}, \tilde{\psi}$ be defined by

$$\left.\begin{aligned} &\xi = \frac{2\pi}{L} x, \quad \eta = \frac{2\pi}{L} y, \quad \zeta = \frac{2\pi}{L} z = \xi\cos\alpha + \eta\sin\alpha, \\ &\tau = \frac{2\pi}{L} c_3 t, \quad \tilde{\phi} = \frac{4\pi^2}{L^2}\phi, \quad \tilde{\psi} = \frac{4\pi^2}{L^2}\psi . \end{aligned}\right\} \tag{8.1}$$

We suppose the half-space to be disturbed by an incident wave of constant $\bar{r}$ of the form (7.25), (7.26); in terms of the new variables, we take

$$\left.\begin{aligned} \tilde{\phi} = \phi^*(\zeta,\tau;\epsilon) &= \int_0^{\zeta-\tau} (\zeta-\tau-s)\{-2+[r^{*2}-\epsilon^2 p^2(s)]^{\frac{1}{2}}\}ds \\ &\quad + \left[1 - \frac{1}{G(r^*)}\right]\tau^2 , \\ \tilde{\psi} = \psi^*(\zeta-\tau;\epsilon) &= \epsilon \int_0^{\zeta-\tau} (\zeta-\tau-s)p(s)ds \end{aligned}\right\} \tag{8.2}$$

where

$$p(s) = \sin s , \tag{8.3}$$

ϵ is a given number less than π, and $r^* = r^*(\epsilon)$ is determined by

$$\int_0^{2\pi} [r^{*2} - \epsilon^2 p^2(s)]^{\frac{1}{2}} ds = 4\pi . \tag{8.4}$$

The direction of propagation of the incident wave is that of the vector $(\cos\alpha, \sin\alpha)$. (See Fig. 1.) The surface $y = 0$ is required to remain free of traction during the subsequent deformation, so that

$$q_{12} = q_{22} = 0 \quad \text{at} \quad y = 0 \ . \tag{8.5}$$

The reflection process in general will give rise to waves for which $\overline{r}$ is not necessarily constant.

Let the operator Δ denote the Laplacian with respect to ξ and η from now on. We consider the differential equations

$$\left.\begin{aligned} G(r^*)\widetilde{\phi}_{\tau\tau} &= G(\overline{r})\Delta\widetilde{\phi} + 2[G(\overline{r})-1] \ , \\ G(r^*)\widetilde{\psi}_{\tau\tau} &= G(\overline{r})\Delta\widetilde{\psi} \ , \end{aligned}\right\} \tag{8.6}$$

where

$$\overline{r} = [(2+\Delta\widetilde{\phi})^2 + (\Delta\widetilde{\psi})^2]^{\frac{1}{2}} \ , \tag{8.7}$$

and we seek solutions of the form

$$\begin{aligned} \widetilde{\phi} &= \phi^*(\zeta, \tau; \epsilon) + \overline{\phi}(\xi, \eta, \tau; \epsilon) \ , \\ \widetilde{\psi} &= \psi^*(\zeta - \tau; \epsilon) + \overline{\psi}(\xi, \eta, \tau; \epsilon) \ . \end{aligned} \tag{8.8}$$

From (8.5), (6.21) we have the boundary conditions

$$\begin{aligned} \eta = 0: \ \tfrac{1}{2}G(r^*)\widetilde{\phi}_{\tau\tau} - \widetilde{\phi}_{\xi\xi} + \widetilde{\psi}_{\xi\eta} &= 0 \ , \\ \eta = 0: \ \tfrac{1}{2}G(r^*)\widetilde{\psi}_{\tau\tau} - \widetilde{\psi}_{\xi\xi} - \widetilde{\phi}_{\xi\eta} &= 0 \ , \end{aligned} \tag{8.9}$$

and we require that $\overline{\phi}$ and $\overline{\psi}$ represent outgoing waves at $\eta = -\infty$.

In the following section we formulate a perturbation procedure for the reflection problem based on the assumption that ϵ is small.

IV. PERTURBATION PROCEDURE FOR THE REFLECTION PROBLEM

§9. Form of the Expansions

In this chapter we introduce a perturbation procedure for the purpose of obtaining a solution, in the form of an expansion in powers of the small amplitude of the incident wave, to the problem formulated in §8. The procedure is motivated by the fact that, upon linearization, the first term in this expansion is identical with the solution of the corresponding problem in the classical linear theory. We assume that the reflected wave is periodic in time with the same period as the incident wave.

On expanding $\overline{\Phi}, \phi^{*}, \overline{\Psi}, \psi^{*}$ in powers of the amplitude ϵ, the problem (8.6)-(8.9) reduces to a sequence of linear boundary value problems, the first of which is homogeneous and is essentially the usual reflection problem of the classical linear theory, the rest non-homogeneous. Secular terms arise in the solutions of all of the non-homogeneous problems. Elimination of these terms is achieved by allowing amplitudes to depend on a slowly varying function of depth in the half-space. A full discussion of this matter is included in sections 11 through 14.

For given ϵ and with ϕ^{*}, ψ^{*} given by (8.2), (8.3) we assume existence of non-trivial $\overline{\Phi}, \overline{\Psi}$ satisfying (8.6)-(8.9). We consider the case $0 < \epsilon << 1$ and try expansions of the form

$$\begin{aligned} \overline{\Phi} &= \epsilon\, \overline{\Phi}_0(\xi,\eta,\tau) + \epsilon^2\, \overline{\Phi}_1(\xi,\eta,\tau) + \epsilon^3\, \overline{\Phi}_2(\xi,\eta,\tau) + \ldots \\ \overline{\Psi} &= \epsilon\, \overline{\Psi}_0(\xi,\eta,\tau) + \epsilon^2\, \overline{\Psi}_1(\xi,\eta,\tau) + \epsilon^3\, \overline{\Psi}_2(\xi,\eta,\tau) + \ldots \end{aligned} \tag{9.1}$$

for the reflected wave $\overline{\Phi}, \overline{\Psi}$. For simplicity we shall agree to omit the bar over ϕ, ψ in the remainder of this chapter. It is thus understood that ϕ, ψ now denote the reflected waves arising from the incident disturbance ϕ^*, ψ^*.

We begin by calculating the ϵ-expansions of the quantities ϕ^*, ψ^*, r^* and $G(r^*)$ associated with the given incident wave. Equations (8.3) and (8.4) determine r^* as a function of ϵ which possesses an expansion of the form

$$r^* = 2 + \epsilon r_1^* + \epsilon^2 r_2^* + \epsilon^3 r_3^* + \dots \tag{9.2}$$

where

$$r_1^* = 0, \qquad r_2^* = \frac{1}{8}, \qquad r_3^* = -\frac{1}{16}. \tag{9.3}$$

Thus, by (6.24) ,

$$G(r^*) = 1 + \epsilon G_1^* + \epsilon^2 G_2^* + \epsilon^3 G_3^* + \dots. \tag{9.4}$$

where

$$G_1^* = 0, \quad G_2^* = \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right), \quad G_3^* = -\frac{1}{32}\left(\frac{c_1^2}{c_2^2} - 1\right). \tag{9.5}$$

By (8.2) and (8.3), ϕ^* and ψ^* can be shown to have the ϵ-expansions

$$\begin{aligned} \phi^* &= \epsilon \phi_o^*(\zeta, \tau) + \epsilon^2 \phi_1^*(\zeta, \tau) + \epsilon^3 \phi_2^*(\zeta, \tau) + \dots \\ \psi^* &= \epsilon \psi_o^*(\zeta, \tau) + \epsilon^2 \psi_1^*(\zeta, \tau) + \epsilon^3 \psi_2^*(\zeta, \tau) + \dots \end{aligned} \tag{9.6}$$

where

$$\phi_o^* = 0, \qquad \psi_o^* = \int_o^{\zeta-\tau} (\zeta-\tau-s)\sin s \, ds , \tag{9.7}$$

$$\phi_1^* = \frac{1}{8}\int_o^{\zeta-\tau}(\zeta-\tau-s)\cos 2s\,ds + \frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)\tau^2, \quad \psi_1^* = 0\,, \tag{9.8}$$

$$\phi_2^* = -\tfrac{1}{2}\phi_1^*\,, \quad \psi_2^* = 0 \tag{9.9}$$

We now determine the ϵ-expansions of $\bar{r}$ and $G(\bar{r})$. Using (9.1), (9.6) and (8.8) in (8.7) we obtain

$$\bar{r} = 2 + \epsilon\,\bar{r}_1 + \epsilon^2\,\bar{r}_2 + \epsilon^3\,\bar{r}_3 + \ldots. \tag{9.10}$$

where

$$\left.\begin{aligned}
\bar{r}_1 &= \Delta\phi_o + \phi^*_{o_{\zeta\zeta}}\,, \\
\bar{r}_2 &= \Delta\phi_1 + \phi^*_{1_{\zeta\zeta}} + \frac{1}{4}\left(\Delta\psi_o + \psi^*_{o_{\zeta\zeta}}\right)^2, \\
\bar{r}_3 &= \Delta\phi_2 + \phi^*_{2_{\zeta\zeta}} + \tfrac{1}{2}\left(\Delta\psi_o + \psi^*_{o_{\zeta\zeta}}\right)\left(\Delta\psi_1 + \psi^*_{1_{\zeta\zeta}}\right) - \frac{1}{8}\left(\Delta\phi_o + \phi^*_{o_{\zeta\zeta}}\right)\left(\Delta\psi_o + \psi^*_{o_{\zeta\zeta}}\right)^2.
\end{aligned}\right\} \tag{9.11}$$

The ϵ-expansion of $G(\bar{r})$ is now found by substituting from (9.10), (9.11) for $\bar{r}$ in (6.24). On doing so we find that

$$G(\bar{r}) = 1 + \epsilon\,G_1 + \epsilon^2\,G_2 + \epsilon^3\,G_3 + \ldots. \tag{9.12}$$

where

$$G_1 = \tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\left(\Delta\phi_o + \phi^*_{o_{\zeta\zeta}}\right)\,, \tag{9.13}$$

$$G_2 = \tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\left(\Delta\phi_1 + \phi^*_{1_{\zeta\zeta}}\right) + \frac{1}{8}\left(\frac{c_1^2}{c_2^2} - 1\right)\left(\Delta\psi_o + \psi^*_{o_{\zeta\zeta}}\right)^2 + k_2\left(\Delta\phi_o + \phi^*_{o_{\zeta\zeta}}\right)^2, \tag{9.14}$$

$$\begin{aligned}
G_3 = {} & \tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\left(\Delta\phi_2 + \phi^*_{2_{\zeta\zeta}}\right) + \frac{1}{4}\left(\frac{c_1^2}{c_2^2} - 1\right)\Big[\left(\Delta\psi_o + \psi^*_{o_{\zeta\zeta}}\right)\left(\Delta\psi_1 + \psi^*_{1_{\zeta\zeta}}\right) \\
& \qquad - \frac{1}{4}\left(\Delta\phi_o + \phi^*_{o_{\zeta\zeta}}\right)\left(\Delta\psi_o + \psi^*_{o_{\zeta\zeta}}\right)^2\Big] \\
& + 2k_2\left(\Delta\phi_o + \phi^*_{o_{\zeta\zeta}}\right)\Big[\left(\Delta\phi_1 + \phi^*_{1_{\zeta\zeta}}\right) + \frac{1}{4}\left(\Delta\psi_o + \psi^*_{o_{\zeta\zeta}}\right)^2\Big] + k_3\left(\Delta\phi_o + \phi^*_{o_{\zeta\zeta}}\right)^3.
\end{aligned} \tag{9.15}$$

On using (9.4)-(9.9) and (9.1), (9.12)-(9.15) in (8.6)-(8.9) we obtain the first, second and third order boundary value problems for the pairs (ϕ_o, ψ_o), (ϕ_1, ψ_1) and (ϕ_2, ψ_2) respectively. Those problems are as follows:

First Order Problem

$$\eta < 0: \qquad \phi_{o_{\tau\tau}} - \frac{c_1^2}{c_2^2}\,\Delta\phi_o = 0 \ , \tag{9.16}$$

$$\eta < 0: \qquad \psi_{o_{\tau\tau}} - \Delta\psi_o = 0 \ , \tag{9.17}$$

$$\eta = 0: \ \tfrac{1}{2}\phi_{o_{\tau\tau}} - \phi_{o_{\xi\xi}} + \psi_{o_{\xi\eta}} = -\sin a \cos a \sin(\xi\cos a - \tau) \ , \tag{9.18}$$

$$\eta = 0: \ \tfrac{1}{2}\psi_{o_{\tau\tau}} - \psi_{o_{\xi\xi}} - \phi_{o_{\xi\eta}} = (\cos^2 a - \tfrac{1}{2})\sin(\xi\cos a - \tau) \ . \tag{9.19}$$

$$\phi_{o_\xi} - \psi_{o_\eta} \text{ and } \phi_{o_\eta} + \psi_{o_\xi} \text{ periodic of period } 2\pi \text{ in } \tau \ . \tag{9.20}$$

Second Order Problem

$$\eta < 0: \phi_{1_{\tau\tau}} - \frac{c_1^2}{c_2^2}\Delta\phi_1 = \left[\tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right) + 2k_2\right](\Delta\phi_o)^2 + \tfrac{1}{4}\left(\frac{c_1^2}{c_2^2} - 1\right)\left[(\Delta\psi_o)^2 + 2\Delta\psi_o\sin(\xi-\tau)\right], \tag{9.21}$$

$$\eta < 0: \ \psi_{1_{\tau\tau}} - \Delta\psi_1 = \tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\Delta\phi_o[\Delta\psi_o + \sin(\xi-\tau)] \ , \tag{9.22}$$

$$\eta = 0: \ \tfrac{1}{2}\phi_{1_{\tau\tau}} - \phi_{1_{\xi\xi}} + \psi_{1_{\xi\eta}} = \frac{1}{8}(\cos^2 a - \tfrac{1}{2})\cos 2(\xi\cos a - \tau) - \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right) , \tag{9.23}$$

$$\eta = 0: \ \tfrac{1}{2}\psi_{1_{\tau\tau}} - \psi_{1_{\xi\xi}} - \phi_{1_{\xi\eta}} = \frac{1}{8}\sin a \cos a \cos 2(\xi\cos a - \tau) \ , \tag{9.24}$$

$$\phi_{1_\xi} - \psi_{1_\eta} \text{ and } \phi_{1_\eta} + \psi_{1_\xi} \text{ periodic of period } 2\pi \text{ in } \tau, \tag{9.25}$$

Third Order Problem

$$\eta < 0: \phi_{2_{\tau\tau}} - \frac{c_1^2}{c_2^2}\Delta\phi_2 = \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right) - \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)\cos 2(\zeta-\tau) - \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)\phi_{o_{\tau\tau}}$$

$$+\left(4k_2 + \frac{c_1^2}{c_2^2} - 1\right)\Delta\phi_o\left[\Delta\phi_1 + \frac{1}{8}\cos 2(\zeta-\tau)\right] + (k_2+2k_3)(\Delta\phi_o)^3$$

$$+\frac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\Delta\psi_1\left[\Delta\psi_o + \sin(\zeta-\tau)\right] + k_2\,\Delta\phi_o\left[\Delta\psi_o + \sin(\zeta-\tau)\right]^2,$$

(9.26)

$$\eta < 0: \psi_{2_{\tau\tau}} - \Delta\psi_2 = -\frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)\left[\psi_{o_{\tau\tau}} + \sin(\zeta-\tau)\right] + k_2(\Delta\phi_o)^2\left[\Delta\psi_o + \sin(\zeta-\tau)\right]$$

$$+\frac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\Big[\left(\Delta\psi_o + \sin(\zeta-\tau)\right)\Big\{\Delta\phi_1 + \frac{1}{8}\cos 2(\zeta-\tau) + \frac{1}{4}\left(\Delta\psi_o + \sin(\zeta-\tau)\right)^2\Big\}$$

$$+\Delta\phi_o\,\Delta\psi_1\Big], \qquad (9.27)$$

$$\eta = 0: \frac{1}{2}\phi_{2_{\tau\tau}} - \phi_{2_{\xi\xi}} + \psi_{2_{\xi\eta}} = -\frac{1}{32}\left(\frac{c_1^2}{c_2^2} - 1\right)\phi_{o_{\tau\tau}} + \frac{1}{16}\left(\cos^2\alpha - \frac{1}{2}\right)\cos 2(\xi\cos\alpha - \tau)$$

$$+\frac{1}{32}\left(\frac{c_1^2}{c_2^2} - 1\right), \qquad (9.28)$$

$$\eta = 0: \frac{1}{2}\psi_{2_{\tau\tau}} - \psi_{2_{\xi\xi}} - \phi_{2_{\xi\eta}} = -\frac{1}{32}\left(\frac{c_1^2}{c_2^2} - 1\right)\left[\psi_{o_{\tau\tau}} + \sin(\xi\cos\alpha - \tau)\right]$$

$$-\frac{1}{16}\sin\alpha\cos\alpha\cos 2(\xi\cos\alpha - \tau),$$

(9.29)

$\phi_{2_\xi} - \psi_{2_\eta}$ and $\phi_{2_\eta} + \psi_{2_\xi}$ periodic of period 2π in τ , (9.30)

and ϕ, ψ must represent outgoing waves at $\eta = -\infty$.

§10. The First Order Problem -- Linear Theory

We discuss here the structure of the solution of the problem described by equations (9.16)-(9.20).

Taking account of the boundary conditions (9.18) and (9.19), we seek a solution ϕ_o, ψ_o of the form

$$\phi_o = f(\eta)\sin(\xi \cos\alpha - \tau), \quad \psi_o = g(\eta)\sin(\xi\cos\alpha - \tau) \quad , \tag{10.1}$$

for some suitable functions f and g. On substituting for ϕ_o, ψ_o in (9.16), (9.17) we find that $f(\eta)$ and $g(\eta)$ must satisfy

$$f''(\eta) + \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)f(\eta) = 0, \quad \eta < 0 \quad , \tag{10.2}$$

and

$$g''(\eta) + \sin^2\alpha\, g(\eta) = 0 \quad , \quad \eta < 0 \quad . \tag{10.3}$$

Clearly $f(\eta)$ is sinusoidal or exponential depending on whether $\frac{c_2^2}{c_1^2} - \cos^2\alpha$ is positive or negative. If $\frac{c_2^2}{c_1^2} - \cos^2\alpha$ is negative then (10.2) has solutions

$$f(\eta) = \exp\left\{\pm\eta\left(\cos^2\alpha - c_2^2/c_1^2\right)^{\frac{1}{2}}\right\} . \tag{10.4}$$

The negative exponential is excluded by the requirement that the displacements corresponding to (10.1) be bounded for $\eta < 0$. Thus, by (10.4), (10.1), ϕ_o decays exponentially with η, when $c_2^2/c_1^2 - \cos^2\alpha$ is negative.

If $c_2^2/c_1^2 - \cos^2\alpha$ is positive, then (10.2) has solutions

$$f(\eta) = a_1 \cos\left\{\eta\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}}\right\} + a_2 \sin\left\{\eta\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}}\right\} \qquad (10.5)$$

where a_1 and a_2 are arbitrary constants. Then, by (10.5), (10.1), ϕ_o is a linear combination of the four terms

$$\sin\left(\xi \cos a \pm \eta\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} - \tau\right), \quad \cos\left(\xi \cos a \pm \eta\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} - \tau\right) .$$

The terms with argument $\xi \cos a + \eta\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} - \tau$ are excluded on the grounds that they represent incoming waves at $\eta = -\infty$.

In the case where $c_2^2/c_1^2 - \cos^2 a$ is zero, all solutions of (10.2) are constant or linear in η. The solution $f(\eta)$ must be chosen to be constant since otherwise, by (10.1), the resulting displacements are unbounded in η .

We confine our attention here to unattenuated plane wave solutions of the problem (9.16)-(9.20) and thus assume that

$$0 < a_o \leq a \leq \frac{\pi}{2} \quad , \qquad a_o = \arc\cos \frac{c_2}{c_1} \quad . \qquad (10.6)$$

We deal here with materials for which Poisson's ratio is positive. In this case, $c_2^2/c_1^2 < \frac{1}{2}$, and hence, $a_o > \pi/4$.

Equation (10.3) has solutions

$$g(\eta) = b_1 \cos(\eta \sin a) + b_2(\eta \sin a) \quad , \qquad (10.7)$$

where b_1 and b_2 are arbitrary constants. By (10.7), (10.1) it then follows that ψ_o is a linear combination of $\sin(\xi \cos a - \eta \sin a - \tau)$ and $\cos(\xi \cos a - \eta \sin a - \tau)$, the solutions with argument $\xi \cos a + \eta \sin a - \tau$ being excluded by reason of the fact that they represent incoming waves at $\eta = -\infty$. Thus, when a satisfies (10.6)and solutions of the form (10.1) are assumed, ϕ_o and ψ_o are given by

$$\phi_o = B\sin\left(\xi\cos\alpha - \eta\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} - \tau\right) + B'\cos\left(\xi\cos\alpha - \eta\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} - \tau\right), \tag{10.8}$$

$$\psi_o = C\sin\left(\xi\cos\alpha - \eta\sin\alpha - \tau\right) + C'\cos(\cos\alpha - \eta\sin\alpha - \tau), \tag{10.9}$$

where B, B', C, C' are constants to be determined by the boundary conditions (9.18), (9.19). On applying the boundary conditions we obtain the following systems of equations for B, C and B', C' respectively:

$$\left.\begin{aligned} B(\cos^2\alpha - \tfrac{1}{2}) + C\sin\alpha\cos\alpha &= -\sin\alpha\cos\alpha, \\ -B\cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} + C(\cos^2\alpha - \tfrac{1}{2}) &= \cos^2\alpha - \tfrac{1}{2}, \end{aligned}\right\} \tag{10.10}$$

$$\left.\begin{aligned} B'(\cos^2\alpha - \tfrac{1}{2}) + C'\sin\alpha\cos\alpha &= 0 \\ -B'\cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} + C'(\cos^2\alpha - \tfrac{1}{2}) &= 0 \end{aligned}\right\} \tag{10.11}$$

The determinant of the matrix of coefficients in (10.10) and (10.11) is

$$(\cos^2\alpha - \tfrac{1}{2})^2 + \sin\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} > 0 ; \tag{10.12}$$

the fact that the determinant is positive follows from (10.6). Thus

$$B' = C' = 0 \tag{10.13}$$

and

$$B = \frac{2\sin\alpha\cos\alpha(\tfrac{1}{2} - \cos^2\alpha)}{(\cos^2\alpha - \tfrac{1}{2})^2 + \sin\alpha\cos^2\alpha\left(c_2^2/c_1^2 - \cos^2\alpha\right)^{\frac{1}{2}}}, \tag{10.14}$$

$$C = \frac{(\cos^2 a - \frac{1}{2})^2 - \sin a \cos^2 a \left(c_2^2/c_1^2 - \cos^2 a\right)^{\frac{1}{2}}}{(\cos^2 a - \frac{1}{2})^2 + \sin a \cos^2 a \left(c_2^2/c_1^2 - \cos^2 a\right)^{\frac{1}{2}}} \, . \tag{10.15}$$

Hence, by (10.13), (10.8) and (10.9),

$$\phi_o = B \sin\left(\xi \cos a - \eta\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} - \tau\right) , \tag{10.16}$$

$$\psi_o = C \sin(\xi \cos a - \eta \sin a - \tau) \qquad , \tag{10.17}$$

where B and C are given by (10.14), (10.15).

On recalling the definitions of ξ, η and τ (see (8.1) and (7.7)), the function ϕ_o of (10.16) is seen to represent a plane wave travelling with the wave speed $c_1[G(r^*)]^{\frac{1}{2}}$ in the direction $(\cos\beta, -\sin\beta)$, where

$$\cos\beta = \frac{c_1}{c_2} \cos a \, , \quad 0 < \beta < a \quad , \tag{10.18}$$

while the ψ_o of (10.17) represents a plane wave travelling with the wave speed $c_2[G(r^*)]^{\frac{1}{2}}$ in the direction $(\cos a, -\sin a)$. The situation is illustrated in Figure 1. Since, by (9.4), (9.5),

$$G(r^*) = 1 + O(\varepsilon^2) \, ,$$

the wave speeds $c_1[G(r^*)]^{\frac{1}{2}}$, $c_2[G(r^*)]^{\frac{1}{2}}$ reduce, in the limit of small ε, to c_1 and c_2, the respective propagation speeds of dilatation and shear waves in the classical linear theory.

Similarly a plane wave of dilatation, when reflected from the free surface of a half-space, gives rise, in general, to both a reflected shear wave and a reflected dilatation wave. This subject is discussed in Chapter 2 of Ewing, Jardetzky and Press [22].

§11. Modification of the First Order Solution

We now consider the problem described by equations (9.21)-(9.25) with ϕ_o and ψ_o given by (10.16), (10.17), (10.14), (10.15).

First we introduce some simplifying notation. Let

$$\zeta_1 = \xi \cos\alpha - \eta \sin\alpha \quad , \tag{11.1}$$

$$\zeta_2 = \frac{c_2}{c_1} (\xi \cos\beta - \eta \sin\beta) \tag{11.2}$$

where β is given by (10.18). Then, by (10.16), (10.17),

$$\phi_o = B \sin(\zeta_2 - \tau), \quad \psi_o = C \sin(\zeta_1 - \tau) \tag{11.3}$$

where B and C are given by (10.14), (10.15) .

On substituting from (11.3) for ϕ_o and ψ_o on the right-hand sides of (9.21) and (9.22) we find that the terms involving $(\Delta\phi_o)^2$ and $(\Delta\psi_o)^2$ in (9.21) contribute, among other terms, a constant term and a term proportional to $\cos 2(\zeta_2 - \tau)$. The constant term produces contributions to ϕ_1 which are quadratic in ξ, η and τ. The term proportional to $\cos 2(\zeta_2-\tau)$ on the right-hand side of the differential equation for ϕ_1 contributes to ϕ_1 a term proportional to $\xi \sin 2(\zeta_2-\tau)$ or $\eta \sin 2(\zeta_2-\tau)$; a term proportional to $\tau \sin 2(\zeta_2-\tau)$ in ϕ_1 is ruled out by the periodicity requirement (9.25). The second order displacements are given by first partial derivatives of ϕ_1 and ψ_1. Thus it follows that the presence of a constant term and a term proportional to $\cos 2(\zeta_2-\tau)$ on the right-hand side of (9.21) gives rise to terms linear in the space variables and to a divergent oscillatory term in the expressions for the displacements. The right-hand side of (9.22) contains neither constant terms nor sinusoidal terms with argument proportional to $\zeta_1-\tau, \zeta-\tau$, so no such difficulties arise there.

The question of the presence of constant terms on the right-hand side of (9.21) and (9.23) will be discussed further in §13 and in §17. We confine our attention here to elimination of the term proportional to $\cos 2(\zeta_2-\tau)$ on the right-hand side of (9.21). The method used is a certain two-variable expansion procedure to be described below. A full description of two-variable expansion procedures is to be found in Chapter 3 of Cole [4].

We assume that ϕ and ψ depend on depth in the half-plane through the "slow" variable

$$\tilde{\eta} \equiv \epsilon \eta \tag{11.4}$$

as well as through η. The variables η and $\tilde{\eta}$ are treated formally as independent in differentiation and, on expanding ϕ and ψ in powers of ϵ, a sequence of boundary value problems is obtained. The second and higher order problems in this sequence will differ from those previously obtained in §9 in a manner to be determined below. By appropriate choice of the $\tilde{\eta}$-dependence of ϕ_o, ψ_o, that term in the second order problem which leads to a divergent oscillatory contribution to ϕ_1 is eliminated. Similarly, divergent oscillatory behavior in the solution ϕ_2, ψ_2 of the third order problem is avoided by choosing the $\tilde{\eta}$-dependence of ϕ_1, ψ_1 appropriately. The method is expected to be equally effective in eliminating divergent oscillatory behavior in all of the solutions of the higher order problems, although we do not here pursue problems of order greater than two.

With

$$\phi = \phi\ (\xi, \eta, \tilde{\eta}, \tau; \epsilon), \qquad \psi = \psi(\xi, \eta, \tilde{\eta}, \tau; \epsilon) \quad , \tag{11.5}$$

and treating η and $\tilde{\eta}$ as independent in differentiation, we obtain, as before, a sequence of boundary-value problems on substituting the ϵ-expansions

$$\left.\begin{aligned} \phi &= \epsilon\, \phi_o(\xi, \eta, \tilde{\eta}, \tau) + \epsilon^2\, \phi_1(\xi, \eta, \tilde{\eta}, \tau) + \epsilon^3 \phi_2(\xi, \eta, \tilde{\eta}, \tau) + \ldots \\ \psi &= \epsilon\, \psi_o(\epsilon, \eta, \tilde{\eta}, \tau) + \epsilon^2\, \psi_1(\xi, \eta, \tilde{\eta}, \tau) + \epsilon^3 \psi_2(\xi, \eta, \tilde{\eta}, \tau) + \ldots \end{aligned}\right\} \tag{11.6}$$

in the problem formulated in §8. The changes due to the introduction of $\tilde{\eta}$ are as follows:

$$\left.\begin{aligned} &\frac{\partial}{\partial \eta} \quad \text{becomes} \quad \frac{\partial}{\partial \eta} + \epsilon \frac{\partial}{\partial \tilde{\eta}} \quad , \\ &\frac{\partial^2}{\partial \eta^2} \quad \text{becomes} \quad \frac{\partial^2}{\partial \eta^2} + 2\epsilon \frac{\partial^2}{\partial \eta \partial \tilde{\eta}} + \epsilon^2 \frac{\partial^2}{\partial \tilde{\eta}^2} \quad . \end{aligned}\right\} \tag{11.7}$$

The ϵ-expansions of $\Delta\phi$ and $\Delta\psi$ are now

$$\Delta\phi = \epsilon \Delta \phi_o + \epsilon^2(\Delta \phi_1 + 2\phi_{o\eta\tilde{\eta}}) + \epsilon^2(\Delta\phi_2 + 2\,\phi_{1\eta\tilde{\eta}} + \phi_{o\tilde{\eta}\tilde{\eta}}) + \ldots \;, \tag{11.8}$$

$$\Delta\psi = \epsilon \Delta\psi_o + \epsilon^2(\Delta\psi_1 + 2\,\psi_{o\eta\tilde{\eta}}) + \epsilon^3(\Delta\,\psi_2 + 2\,\psi_{1\eta\tilde{\eta}} + \psi_{o\tilde{\eta}\tilde{\eta}}) + \ldots \;. \tag{11.9}$$

Thus in the second and third order problems we account for the $\tilde{\eta}$-dependence of ϕ and ψ by making the changes

$$\left.\begin{aligned}
&\Delta\phi_1 \to \Delta\phi_1 + 2\phi_{o\eta\tilde{\eta}}\ , \quad \Delta\psi_1 \to \Delta\psi_1 + 2\psi_{o\eta\tilde{\eta}}\ ,\\
&\Delta\phi_2 \to \Delta\phi_2 + 2\phi_{1\eta\tilde{\eta}} + \phi_{o\tilde{\eta}\tilde{\eta}},\ \Delta\psi_2 \to \Delta\psi_2 + 2\psi_{1\eta\tilde{\eta}} + \psi_{o\tilde{\eta}\tilde{\eta}}\ ,\\
&\phi_{1\xi\eta} \to \phi_{1\xi\eta} + \phi_{o\xi\eta},\ \psi_{1\xi\eta} \to \psi_{1\xi\eta} + \psi_{o\xi\tilde{\eta}}\ ,\\
&\phi_{2\xi\eta} \to \phi_{2\xi\eta} + \phi_{1\xi\tilde{\eta}},\ \psi_{2\xi\eta} \to \psi_{2\xi\eta} + \psi_{1\xi\tilde{\eta}}
\end{aligned}\right\} \qquad (11.10)$$

On comparing with (10.14)-(10.17) we note that the first order problem is satisfied by any ϕ_o, ψ_o of the form

$$\phi_o = \sum_1^\infty B_n(\tilde{\eta}) \sin n(\zeta_2 - \tau)\ , \quad \psi_o = \sum_1^\infty C_n(\tilde{\eta}) \sin n(\zeta_1 - \tau) \qquad (11.11)$$

where

$$B_1(0) = \frac{2\cos a \sin a\,(\frac{1}{2} - \cos^2 a)}{(\frac{1}{2} - \cos^2 a)^2 + \sin a \cos^2 a \left(c_1^2/c_2^2 - \cos^2 a\right)^{\frac{1}{2}}}\ , \qquad (11.12)$$

$$C_1(0) = \frac{(\frac{1}{2} - \cos^2 a)^2 - \sin a \cos^2 a \left(c_2^2/c_1^2 - \cos^2 a\right)^{\frac{1}{2}}}{(\frac{1}{2} - \cos^2 a)^2 + \sin a \cos^2 a \left(c_2^2/c_1^2 - \cos^2 a\right)^{\frac{1}{2}}}\ , \qquad (11.13)$$

$$B_n(0) = C_n(0) = 0, \quad n = 2, 3, \ldots\ldots \quad . \qquad (11.14)$$

The sequences $\{B_n(\tilde{\eta})\}$, $\{C_n(\tilde{\eta})\}$ will be chosen so that, on substituting from (11.11) for ϕ_o and ψ_o in (9.21), (9.22), all terms proportional to $\cos n(\zeta_2 - \tau)$ on the right-hand side of (9.21) and all terms proportional to $\cos n(\zeta_1 - \tau)$ on the right-hand side of (9.22) vanish for $n = 1, 2, 3, \ldots$. This procedure will result in differential equations for the $B_n(\tilde{\eta})$, $C_n(\tilde{\eta})$.

On using the ϕ_o, ψ_o of (11.11) in (9.21)-(9.25) the second order problem takes the form

$$\eta<0:\ \phi_{1\tau\tau}-\frac{c_1^2}{c_2^2}\Delta\phi_1=\frac{1}{4}\left(\frac{c_1^2}{c_2^2}-1\right)\left\{\left[\sum_1^\infty n^2C_n(\tilde{\eta})\sin n(\zeta_1-\tau)\right]^2\right.$$

$$\left.-2\sin(\zeta-\tau)\sum_1^\infty n^2C_n(\tilde{\eta})\sin n(\zeta_1-\tau)\right\}$$

$$+\left(\frac{c_2^2}{c_1^2}\right)^2\left\{\tfrac{1}{2}\left(\frac{c_1^2}{c_2^2}-1\right)+2k_2\right\}\left[\sum_1^\infty n^2B_n(\tilde{\eta})\sin n(\zeta_2-\tau)\right]^2$$

$$-2\frac{c_1}{c_2}\sin\beta\sum_1^\infty n\,B_n'(\tilde{\eta})\cos n(\zeta_2-\tau)\ , \qquad (11.15)$$

$$\eta<0:\ \psi_{1\tau\tau}-\Delta\psi_1=\tfrac{1}{2}\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2}-1\right)\sum_1^\infty n^2\,B_n(\tilde{\eta})\sin n(\zeta_2-\tau)$$

$$\times\left[\sum_{k=1}^\infty k^2C_n(\tilde{\eta})\sin k(\zeta_1-\tau)-\sin(\zeta-\tau)\right]$$

$$-2\sin\alpha\sum_1^\infty n\,C_n'(\tilde{\eta})\cos n(\zeta_1-\tau)\quad , \qquad (11.16)$$

$$\eta=0:\ \tfrac{1}{2}\phi_{1\tau\tau}-\phi_{1\xi\xi}+\psi_{1\xi\eta}=\frac{1}{8}(\cos^2\alpha-\tfrac{1}{2})\cos 2(\xi\cos\alpha-\tau)-\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)$$

$$-\cos\alpha\sum_1^\infty n\,C_n'(0)\cos n(\xi\cos\alpha-\tau)\ , \qquad (11.17)$$

$$\eta=0:\ \tfrac{1}{2}\psi_{1\tau\tau}-\psi_{1\xi\xi}-\phi_{1\xi\eta}=\frac{1}{8}\sin\alpha\cos\alpha\cos 2(\xi\cos\alpha-\tau)$$

$$+\cos\alpha\sum_1^\infty n\,B_n'(0)\cos n(\xi\cos\alpha-\tau)\ . \qquad (11.18)$$

$\phi_{1\xi}-\psi_{1\eta}$ and $\phi_{1\eta}+\psi_{1\xi}$ periodic of period 2π in τ , (11.19)

where primes on the $B_n(\tilde{\eta})$, $C_n(\tilde{\eta})$ denote differentiation with respect to $\tilde{\eta}$.

Clearly we must have

$$C_n'(\tilde{\eta}) = 0\ ,\quad n=1,2,3,\ldots\ , \qquad (11.20)$$

in order that the right-hand side of (11.16) contain no terms proportional to $\cos n(\zeta_1-\tau)$. From this and (11.14) it follows that

$$C_1(\tilde{\eta}) \equiv C_1(0), \quad C_n(\tilde{\eta}) \equiv 0, \quad n = 2, 3, \dots \quad . \tag{11.21}$$

The sequence $\{B_n(\tilde{\eta})\}$ is to be determined from the requirement that

$$\left(\frac{c_2^2}{c_1^2}\right)^2 \left\{2k_2 + \tfrac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\right\}\left[\sum_1^\infty n^2 B_n(\tilde{\eta}) \sin n(\zeta_2-\tau)\right]^2 - 2\,\frac{c_1}{c_2} \sin\beta \sum_1^\infty n B_n'(\tilde{\eta}) \times \cos n(\zeta_2-\tau)$$

contain no terms proportional to $\cos n(\zeta_2-\tau)$, $n = 1, 2, \dots$.

Since

$$\left[\sum_1^\infty m^2 B_m(\tilde{\eta}) \sin m(\zeta_2-\tau)\right]^2 = \sum_{n=1}^\infty \left\{\sum_{k=1}^\infty k^2 (n+k)^2\, B_k(\tilde{\eta})\, B_{n+k}(\tilde{\eta}) - \tfrac{1}{2}\sum_{k=1}^{n} k^2 (n-k)^2 B_k(\tilde{\eta}) B_{n-k}(\tilde{\eta})\right\} \cos n(\zeta_2-\tau)$$
$$+ \tfrac{1}{2}\sum_1^\infty n^4\, B_n^2(\tilde{\eta}), \quad B_o(\tilde{\eta}) \equiv 0, \tag{11.22}$$

we have, on using (10.18) and setting

$$K_1 \equiv \frac{\left(\frac{c_2}{c_1}\right)^6 \left(4k_2 + \frac{c_2^2}{c_1^2} - 1\right)}{4\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}}}, \tag{11.23}$$

the following set of first order ordinary differential equations and initial conditions for the determination of the sequence $\{B_n(\tilde{\eta})\}$:

$$n B_n'(\tilde{\eta}) = K_1\left[\sum_{k=1}^{\infty} k^2(n+k)^2 B_k(\tilde{\eta}) B_{n+k}(\tilde{\eta}) - \tfrac{1}{2}\sum_{k=1}^{n} k^2(n-k)^2 B_k(\tilde{\eta}) B_{n-k}(\tilde{\eta})\right],$$

$$B_o(\tilde{\eta}) \equiv 0\,, \quad \eta < 0, \; n = 1, 2, 3, \ldots \;. \tag{11.24}$$

$$B_1(0) = \frac{2\cos a \sin a(\frac{1}{2} - \cos^2 a)}{(\frac{1}{2} - \cos^2 a)^2 + \sin a \cos^2 a(c_2^2/c_1^2 - \cos^2 a)^{\frac{1}{2}}}\,,$$

$$B_n(0) = 0, \; n = 2, 3, \ldots \;. \tag{11.25}$$

The next section is concerned with the construction of a solution of the system (11.24), (11.25).

§12. Determination of Depth-Dependent Amplitudes $B_n(\tilde{\eta})$

In this section we construct a solution to the problem described by equations (11.24), (11.25). Solution of the problem is facilitated by converting the system of ordinary differential equations into a single first order partial differential equation by means of a generating function.

Equation (11.24) can be written in a simpler form. Let

$$b_n(\tilde{\eta}) = \frac{n^2 B_n(\tilde{\eta})}{B_1(0)}\;. \tag{12.1}$$

Then (11.24) reads

$$\left[K_1 B_1(0)\right]^{-1} \frac{1}{n} b_n'(\tilde{\eta}) = \sum_{k=1}^{\infty} b_k(\tilde{\eta}) b_{n+k}(\tilde{\eta}) - \tfrac{1}{2}\sum_{k=1}^{n} b_k(\tilde{\eta}) b_{n-k}(\tilde{\eta})\,, \quad b_0(\tilde{\eta}) \equiv 0\,,$$

$$\tilde{\eta} < 0, \; n = 1, 2, 3, \ldots \;. \tag{12.2}$$

The initial condition (11.25) becomes

$$b_1(0) = 1, \ b_n(0), \quad n = 2, 3, \ldots . \tag{12.3}$$

The generating function referred to above is defined by

$$b(\theta, \tilde{\eta}) = \sum_{n=1}^{\infty} b_n(\tilde{\eta}) \sin n\,\theta \ , \quad 0 < \theta < \pi \ , \ \tilde{\eta} < 0 \ . \tag{12.4}$$

Since, by (12.1),

$$\frac{db_n}{d\tilde{\eta}} \sin n\,\theta = K_1 B_1(0)\, n \sin n\,\theta \left\{ \sum_{k=1}^{\infty} b_k(\tilde{\eta}) b_{n+k}(\tilde{\eta}) - \tfrac{1}{2} \sum_{k=1}^{n} b_k(\tilde{\eta})\, b_{n-k}(\tilde{\eta}) \right\},$$

$n = 1, 2, \ldots$, we have, by (12.4),

$$\left[K_1 B_1(0)\right]^{-1} \frac{\partial b}{\partial \tilde{\eta}} = -\frac{\partial}{\partial \theta} \left[\sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{\infty} b_k(\tilde{\eta}) b_{n+k}(\tilde{\eta}) - \tfrac{1}{2} \sum_{k=1}^{n} b_k(\tilde{\eta}) b_{n-k}(\tilde{\eta}) \right\} \cos \theta \right].$$

Since (11.22), (12.1) and (12.4) imply that

$$\sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{\infty} b_k(\tilde{\eta})\, b_{n+k}(\tilde{\eta}) - \tfrac{1}{2} \sum_{k=1}^{n} b_k(\tilde{\eta})\, b_{n-k}(\tilde{\eta}) \right\} \cos n\,\theta = b^2(\theta, \tilde{\eta}) - \tfrac{1}{2} \sum_{1}^{\infty} b_n^2(\tilde{\eta}) \ , \tag{12.5}$$

it follows that

$$[K_1 B_1(0)]^{-1} \frac{\partial b}{\partial \tilde{\eta}} = -\frac{\partial}{\partial \theta} \left[b^2(\theta, \tilde{\eta}) - \tfrac{1}{2} \sum_{1}^{\infty} b_n^2(\tilde{\eta}) \right] ,$$

that is,

$$[K_1 B_1(0)]^{-1} \frac{\partial b}{\partial \tilde{\eta}} + 2b \frac{\partial b}{\partial \theta} = 0, \ \tilde{\eta} < 0, \quad 0 < \theta < \pi \ . \tag{12.6}$$

From (12.3) the initial condition is

$$b(\theta, 0) = \sin \theta \ . \tag{12.7}$$

For the purpose of dealing with (12.6) in a more convenient domain in which both variables are positive we make the variable change $\eta \to \bar{\eta}$, where

$$\bar{\eta} = -\tilde{\eta} \quad . \tag{12.8}$$

The problem for $b(\theta, \bar{\eta})$ then reads

$$\frac{\partial b}{\partial \bar{\eta}} + K b \frac{\partial b}{\partial \theta} = 0 , \quad 0 < \bar{\eta} , \quad 0 < \theta < \pi \quad , \tag{12.9}$$

$$b(\theta, 0) = \sin \theta \quad , \quad 0 < \theta < \pi \quad , \tag{12.10}$$

where, by (11.23), (12.6), (12.8), (10.18) and the definition (11.12) of $B_1(0)$,

$$K = -2 K_1 B_1(0) = -\left(\frac{c_2}{c_1}\right)^6 \frac{\left\{2k_2 + \frac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\right\} . \, 2 \cos a \sin a \left(\frac{1}{2} - \cos^2 a\right)}{\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} \left\{ \left(\frac{1}{2} - \cos^2 a\right)^2 + \sin a \cos^2 a \left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} \right\}} \tag{12.11}$$

The characteristics of (12.9) are given by

$$\frac{d\theta}{d\bar{\eta}} = K b(\theta, \bar{\eta}) \quad . \tag{12.12}$$

Since $b(\theta, \bar{\eta})$ is constant along characteristics, (12.12) gives

$$\theta - K b(s, 0) \bar{\eta} = s \tag{12.13}$$

as the equation of the characteristic through $\theta = s$, $\bar{\eta} = 0$. Thus, by (12.10), for each s the straight line

$$\theta - (K \sin s)\bar{\eta} = s, \quad 0 \le \bar{\eta} \; , \quad 0 \le s \le \pi \; , \tag{12.14}$$

is the characteristic through $\theta = s$, $\bar{\eta} = 0$. We note that the lines $\theta = 0$ and $\theta = \pi$ are characteristics.

We now show that (12.14) has a unique solution $s(\theta, \bar{\eta})$ by proving that no two characteristics, issuing from distinct points on the line segment $\eta = 0$, $0 < \theta < \pi$, intersect in the region $0 < \bar{\eta}$, $0 < \theta < \pi$. We assume for the moment that K is positive (by (12.11), the sign of K is the opposite of that of $2 k_2 + \frac{1}{2}(c_1^2/c_2^2 - 1)$, where k_2 is a higher-order material constant. With K positive, all characteristics issuing from $\bar{\eta} = 0$, $\theta = s$ $(0 < s < \pi)$ are seen by (12.14) to have positive slope. Our proof consists of showing that the value $\bar{\eta}_\pi$ of $\bar{\eta}$ at which the characteristic issuing from $\bar{\eta} = 0$, $\theta = s$ intersects the line $\theta = \pi$ is a monotonically decreasing function of s for $0 < s < \pi$. We have, from (12.14),

$$\bar{\eta}_\pi(s) = \frac{\pi - s}{K \sin s} \quad . \tag{12.15}$$

The truth of our assertion then follows from the fact that $\frac{\pi - s}{\sin s}$ decreases monotonically from infinity to 1 as s increases from 0 to π. When K is negative, it can similarly be shown that no intersections occur by noting that all characteristics have negative slope and that the value of $\bar{\eta}$ at which the characteristic issuing from the point $\bar{\eta} = 0$, $\theta = s$ intersects the line $\theta = 0$ is a monotonically increasing function of s for $0 < s < \pi$.

Thus, given $(\theta, \bar{\eta})$, $0 < \theta < \pi$, $\bar{\eta} > 0$, equation (12.14) has a unique solution $s(\theta, \bar{\eta})$. The value of b at $(\theta, \bar{\eta})$ is given by

$$b(\theta, \bar{\eta}) = \sin s(\theta, \bar{\eta}) = \sin[\theta - K \bar{\eta} \sin s(\theta, \bar{\eta})] \quad , \tag{12.16}$$

i.e. $b(\theta, \bar{\eta})$ satisfies

$$b = \sin(\theta - K\bar{\eta}\, b) \quad . \tag{12.17}$$

From (12.15) we have, for positive K ,

$$\lim_{s \to \pi} \bar{\eta}_\pi(s) = \frac{1}{K} \quad , \tag{12.18}$$

from which it follows that

$$b(\pi, \bar{\eta}) = 0 \quad \text{for} \quad 0 \le \bar{\eta} \le \frac{1}{K} \quad . \tag{12.19}$$

For $\bar{\eta} > \frac{1}{K}$ we have

$$b(\pi, \bar{\eta}) = \sin[s_\pi(\bar{\eta})] \, , \pi\, \bar{\eta} > \frac{1}{K} \; , \tag{12.20}$$

where $s_\pi(\bar{\eta})$ is the unique root of

$$\frac{\sin s_\pi}{\pi - s_\pi} = \frac{1}{K\bar{\eta}} \quad , \quad 0 < s_\pi < \quad . \tag{12.21}$$

As $\bar{\eta}$ increases from $\frac{1}{K}$, $s_\pi(\bar{\eta})$ decreases from π to 0 , $b(\pi, \bar{\eta})$ increases from zero, reaches a maximum of 1 when $s_\pi(\bar{\eta}) = \pi/2$ (this occurs when $\bar{\eta} = \frac{1}{K}\frac{\pi}{2}$) , then decreases to 0 as $s_\pi(\bar{\eta})$ decreases from $\pi/2$ to 0 (i.e. as $\bar{\eta} \to \infty$). Thus the graph of $b(\pi, \bar{\eta})$ is as indicated in Figure 2.

The case $K < 0$ is similar. We have

$$\bar{\eta}_o(s) = -\frac{s}{K \sin s} \quad , \quad 0 < s < \pi \quad , \tag{12.22}$$

where $\bar{\eta}_o(s)$ is the $\bar{\eta}$-intercept of the characteristic starting from $\theta = s$, $\bar{\eta} = 0$. Since

$$\lim_{s \to 0} \bar{\eta}_o(s) = -\frac{1}{K} \tag{12.23}$$

it follows that $b(0, \bar{\eta}) = 0$ for $0 \leq \bar{\eta} \leq -\frac{1}{K}$ and that

$$b(0, \bar{\eta}) = \sin[s_o(\bar{\eta})] , \quad \bar{\eta} > -\frac{1}{K} , \tag{12.24}$$

where $s_o(\bar{\eta})$ is the unique root of

$$\frac{\sin s_o}{s_o} = -\frac{1}{K\bar{\eta}} , \quad 0 < s_o < \pi . \tag{12.25}$$

The graph of $b(0, \bar{\eta})$ $(K < 0)$ is identical with that of $b(\pi, \bar{\eta})$ $(K > 0)$.

The asymptotic behavior of $b(\theta, \bar{\eta})$ for large $\bar{\eta}$ is of prime importance since it determines the behavior of ϕ_o for large η. Let K be positive. From (12.14) it is clear that, for every fixed θ, $0 < \theta < \pi$, $s(\theta, \bar{\eta})$ tends to 0 as $\bar{\eta} \to \infty$. From (12.16) it then follows that, for every fixed θ, $0 < \theta < \pi$, $b(\theta, \bar{\eta})$ tends to zero as $\bar{\eta} \to \infty$. That the convergence of $s(\theta, \bar{\eta})$ to zero is uniform for $0 < \theta < \pi$ is seen from the following: by (12.14)

$$\frac{1}{K\bar{\eta}} = \frac{\sin s}{\theta - s} > \frac{\sin s}{\theta} , \quad \bar{\eta} > 0 , \quad 0 < s < \theta \leq \pi .$$

Thus

$$\sin s < \frac{\theta}{K\bar{\eta}} \leq \frac{\pi}{K\bar{\eta}} , \qquad \text{from which we have}$$

$$0 < s(\theta, \bar{\eta}) < \delta \text{ for all } \bar{\eta} > \frac{\pi}{K \sin \delta} , \quad 0 < \theta \leq \pi , \tag{12.26}$$

where δ is an arbitrary positive number.

From (12.16) and (12.17) it then follows that $b(\theta, \bar{\eta}) \sim \frac{\theta}{K\bar{\eta}}$, uniformly in θ, $0 < \theta \leq \pi$, as $\bar{\eta} \to \infty$. From this we conclude that

$$b_n(\tilde{\eta}) = \frac{2}{\pi} \int_0^\pi b(\theta,\bar{\eta}) \sin n\,\theta d\theta \sim \frac{2}{\pi K} \frac{1}{\bar{\eta}} \int_0^\pi \theta \sin n\,\theta d\theta \quad \text{as } \bar{\eta} \to \infty \tag{12.27}$$

i. e.

$$b(\bar{\eta}) \sim \frac{2}{K} \frac{(-1)^{n+1}}{n\,\tilde{\eta}} \quad \text{as } \tilde{\eta} \to \infty, \quad n = 1, 2, \ldots \quad . \tag{12.28}$$

In terms of $\tilde{\eta}$, (12.28) reads

$$b_n(\tilde{\eta}) \sim \frac{2}{K} \frac{(-1)^n}{n\,\tilde{\eta}} \quad \text{as } \tilde{\eta} \to -\infty, \; n = 1, 2, \ldots \quad . \tag{12.29}$$

When K is negative we find that

$$b(\theta, \bar{\eta}) \sim \frac{\theta - \pi}{K\bar{\eta}}, \text{ uniformly in } \theta, \; 0 \le \theta < \pi, \text{ as } \bar{\eta} \to \infty , \tag{12.30}$$

from which it follows that

$$b_n(\bar{\eta}) \sim -\frac{2}{Kn\bar{\eta}} \quad \text{as} \quad \bar{\eta} \to \infty, \; n = 1, 2, \ldots \tag{12.31}$$

or, in terms of $\tilde{\eta}$,

$$b_n(\tilde{\eta}) \sim \frac{2}{K\,n\,\tilde{\eta}} \quad \text{as } \tilde{\eta} \to -\infty , \quad n = 1, 2, \ldots \quad . \tag{12.32}$$

From (12.29), (12.32) and (12.1) it follows that

$$B_n(\tilde{\eta}) \sim \begin{cases} B_1(0) \; \dfrac{2}{K} \; \dfrac{(-1)^n}{n^3 \, \tilde{\eta}} \; , & K > 0 \\[2ex] B_1(0) \; \dfrac{2}{K} \; \dfrac{1}{n^3 \, \tilde{\eta}} \; , & K < 0 \end{cases} \tag{12.33}$$

as $\tilde{\eta} \to -\infty$, $n = 1, 2, 3, \ldots$. From (11.24), (11.25), (11.23) and (12.11) we have

$$B_n(\tilde{\eta}) \equiv 0, \; n = 2, 3, \ldots, \; B_1(\tilde{\eta}) \equiv B_1(0) \text{ when } K = 0 . \tag{12.34}$$

We have thus found a solution $\{B_n(\tilde{\eta})\}$ to the problem (11.24) - (11.25) viz.,

$$B_n(\tilde{\eta}) = \frac{B_1(0)}{n^2} \frac{2}{\pi} \int_0^{\pi} b(\theta, \tilde{\eta}) \sin n\theta d\theta , \quad \tilde{\eta} \le 0, \quad n = 1, 2, \ldots \qquad (12.35)$$

where, on making the variable change $\bar{\eta} \to \tilde{\eta} = -\bar{\eta}$ in (12.14), (12.16), $b(\theta, \tilde{\eta})$ is given in the form

$$b(\theta, \tilde{\eta}) = \sin s(\theta, \tilde{\eta}) , \qquad (12.36)$$

$s(\theta, \tilde{\eta})$ being the unique solution, in $0 < s < \pi$, of

$$\theta + \tilde{\eta} K \sin s = s . \qquad (12.37)$$

The asymptotic behavior of $B_n(\tilde{\eta})$, $n = 1, 2, \ldots$, for large $\tilde{\eta}$ is given by (12.33).

We have, in figure 3, sketched $b(\theta, \tilde{\eta})$ as a function of θ for various values of $\tilde{\eta}$, with $K > 0$.

§13. Modification of the Second Order Solution

We now write down the second order problem (11.15)-(11.19) again, having made the choices (12.35) and (11.21) for $\{B_n(\tilde{\eta})\}$ and $\{C_n(\tilde{\eta})\}$ respectively. We have thus ensured that no divergent oscillations will appear in the solution ϕ_1, ψ_1.

For brevity we set

$$C \equiv C_1(0) . \qquad (13.1)$$

Equations (11.15)-(11.19) now read

$$\eta<0: \ \phi_{1\tau\tau} - \frac{c_1^2}{c_2^2} \Delta\phi_1 = \frac{1}{4}\left(\frac{c_1^2}{c_2^2} - 1\right)\left\{-\frac{1}{2} C^2 \cos 2(\zeta_1 - \tau) + C \cos 2(\xi \cos a - \tau) - C\cos(2\eta \sin\alpha)\right\} + \frac{1}{2}\left(\frac{c_2^2}{c_1^2}\right)^2 \left[2k_2 + \frac{1}{2}\left(\frac{c_1^2}{c_2^2} - 1\right)\right] \sum_1^{\infty} n^4 B_n^2(\tilde{\eta}) + \frac{1}{8} C^2 \left(\frac{c_1^2}{c_2^2} - 1\right), \qquad (13.2)$$

$$\eta<0:\ \psi_{1\tau\tau}-\Delta\psi_1=\tfrac{1}{2}\,\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2}-1\right)\left[C\sin(\zeta_1-\tau)-\sin(\zeta-\tau)\right]\sum_1^\infty n^2B_n(\tilde{\eta})\sin n(\zeta_2-\tau)\ ,$$

(13.3)

$$\eta=0:\tfrac{1}{2}\phi_{1\tau\tau}-\phi_{1\xi\xi}+\psi_{1\xi\eta}=\frac{1}{8}(\cos^2a-\tfrac{1}{2})\cos 2(\xi\cos a-\tau)-\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)\ , \tag{13.4}$$

$$\eta=0:\tfrac{1}{2}\psi_{1\tau\tau}-\psi_{1\xi\xi}-\phi_{1\xi\eta}=\left(\frac{1}{8}\sin a+2B_2'(0)\right)\cos a\cos 2(\xi\cos a-\tau)\ , \tag{13.5}$$

$$\phi_{1\xi}-\psi_{1\eta}\ \text{and}\ \phi_{1\eta}+\psi_{1\xi}\ \text{periodic of period}\ 2\pi\ \text{in}\ \tau\ . \tag{13.6}$$

The right-hand side of (13.5) is obtained by noting that (11.24), (11.25) imply that

$$B_n'(0)=0,\ n=1\ \text{and}\ n=3,4,5,\ \ldots\ . \tag{13.7}$$

Further, by (11.23)-(11.25), (11.12), (10.18), (12.11),

$$B_2'(0)=\frac{1}{8}K\,B_1(0)=-\frac{\frac{1}{8}\left(\frac{c_2}{c_1}\right)^6\left\{2k_2+\frac{1}{2}\left(\frac{c_1^2}{c_2^2}-1\right)\right\}}{\left(\frac{c_2^2}{c_1^2}-\cos^2a\right)^{\frac{1}{2}}}\left[\frac{2\cos a\sin a(\frac{1}{2}-\cos^2a)}{\left(\frac{1}{2}-\cos^2a\right)^2+\sin a\cos^2a\left(\frac{c_1^2}{c_2^2}-\cos^2a\right)^{\frac{1}{2}}}\right]^2 .$$

(13.8)

We choose the following particular integrals $\phi_1^{(p)}$, $\psi_1^{(p)}$ of (13.2), (13.3) respectively:

$$\phi_1^{(p)}=-\frac{1}{4}\left(\frac{c_2}{c_1}\right)^6\left[2k_2+\tfrac{1}{2}\left(\frac{c_1^2}{c_2^2}-1\right)\right]\eta^2\sum_1^\infty n^4B_n^2(\tilde{\eta})+\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)C^2\tau^2$$

$$+\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)\left[-\tfrac{1}{2}C^2\frac{\cos 2(\zeta_1-\tau)}{c_1^2/c_2^2-1}+C\frac{\cos 2(\xi\cos a-\tau)}{c_1^2/c_2^2\cos^2a-1}-C\frac{\cos(2\eta\sin a)}{c_1^2/c_2^2\sin^2a}\right]$$

(13.9)

$$\psi_1^{(p)} = -\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)\left(1+C^2\right)\xi\eta$$

$$+\frac{1}{4}\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2}-1\right)\left[\sum_{n=-\infty}^{\infty} n^2 B_n(\tilde{\eta})\left\{C\frac{\cos[\zeta_1-n\zeta_2-(1-n)\tau]}{(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right.\right.,$$

$$\left.\left.-\frac{\cos[\zeta-n\zeta_2-(1-n)\tau]}{(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right\}\right], \qquad (13.10)$$

where

$$B_{-n}(\tilde{\eta}) = -B_n(\tilde{\eta}) \quad (n=1, 2, \ldots\ldots), \quad B_o(\tilde{\eta}) \equiv 0 \quad . \qquad (13.11)$$

We now add to the particular solutions $\phi_1^{(p)}$, $\psi_1^{(p)}$, solutions of the homogeneous equations associated with (13.2) and (13.3), respectively, which are sufficiently general to allow satisfaction of the boundary conditions (13.4), (13.5) and which permit the removal from the third order problem of all terms contributing divergent oscillations to ϕ_2, ψ_2. It turns out that we must seek ϕ_1, ψ_1 in the form

$$\phi_1 = \phi_1^{(p)}+\frac{1}{8}\left(\frac{c_1^2}{c_2^2}-1\right)\left(\frac{c_1^2}{c_2^2}\tau^2+\eta^2\right)A+\sum_1^{\infty} D_n(\tilde{\eta})\cos n(\zeta_2-\tau) , \qquad (13.12)$$

$$\psi_1 = \psi_1^{(p)}-\frac{1}{8}\left(\frac{c_1^2}{c_2^2}-1\right)\frac{c_1^2}{c_2^2}\xi\eta A+\sum_1^{\infty} E_n(\tilde{\eta})\cos n(\zeta_1-\tau)+\sum_1^{\infty} F_n(\tilde{\eta})\cos n(\zeta-\tau) . \qquad (13.13)$$

The constant A is, so far, arbitrary. The boundary conditions (13.4), (13.5) will place restrictions on the values $D_n(0)$, $E_n(0)$, $F_n(0)$ ($n = 1, 2, \ldots\ldots$) but the sequences $\{D_n(\tilde{\eta})\}$, $\{E_n(\tilde{\eta})\}$, $\{F_n(\tilde{\eta})\}$ are, at this stage, otherwise arbitrary.

On applying the boundary conditions (13.4), (13.5) to ϕ_1, ψ_1 of (13.12), (13.13) we obtain the following systems of equations for

$D_n(0)$, $E_n(0)$, $F_n(0)$ $(n = 1, 2, \ldots)$:

$$\left.\begin{aligned} &(\cos^2\alpha - \tfrac{1}{2}) D_n(0) + \sin\alpha\cos\alpha (E_n(0) - F_n(0)) = 0 , \\ &\qquad\qquad\qquad\qquad n=1,\ n = 3, 4, \ldots \\ &-\cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} D_n(0) + (\cos^2\alpha - \tfrac{1}{2})(E_n(0)+F_n(0)) = 0 , \end{aligned}\right\} \qquad (13.14)$$

$$\left.\begin{aligned} &(\cos^2\alpha - \tfrac{1}{2}) D_2(0) + \sin\alpha\cos\alpha (E_2(0) - F_2(0)) = V_1(\alpha) , \\ &-\cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} D_2(0) + (\cos^2\alpha - \tfrac{1}{2})(E_2(0)+F_2(0)) = V_2(\alpha) \end{aligned}\right\} \qquad (13.15)$$

where

$$V_1(\alpha) = \frac{1}{16}\,\frac{\frac{1}{2} - \cos^2\alpha}{c_1^2/c_2^2\,\cos^2\alpha - 1}\left[-\tfrac{1}{2}(1+C^2)\left(c_1^2/c_2^2\,\cos^2\alpha - 1\right) + C\left(c_1^2/c_2^2 - 1\right)\right]$$
$$+\frac{1}{8}\,\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2} - 1\right)\cos\alpha\left[C\,\frac{\sin\alpha + \frac{c_2}{c_1}\sin\beta}{\left(\sin\alpha + \frac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha} + \frac{\sin\alpha - \frac{c_2}{c_1}\sin\beta}{\left(\sin\alpha - \frac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha}\right], \qquad (13.16)$$

$$V_2(\alpha) = \frac{1}{4}\cos\alpha\left(\frac{1}{8}\sin\alpha + \frac{1}{4}KB_1(0)\right) - \frac{1}{32}C^2\sin\alpha\cos\alpha$$
$$+ B_1(0)\,\frac{1}{4}\,\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2} - 1\right)\left(\tfrac{1}{2} - \cos^2\alpha\right)\left[\frac{1}{\left(\sin\alpha - \frac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha} - \frac{C}{\left(\sin\alpha + \frac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha}\right] \qquad (13.17)$$

In the next section we consider the third order problem and determine A, $\{D_n(\tilde{\eta})\}$, $\{E_n(\tilde{\eta})\}$, $\{F_n(\tilde{\eta})\}$. The fact that, for each n, we have one degree of freedom in satisfying (13. 14), (13. 15) will later be utilized to eliminate undesirable terms from the expressions for $E_n(\tilde{\eta})$, $F_n(\tilde{\eta})$ $(n = 1, 2, \ldots)$.

§14. Determination of Depth-Dependent Amplitudes $D_n(\tilde{\eta})$

In this section we show that the quantities A, $D_n(\tilde{\eta})$, $E_n(\tilde{\eta})$, $F_n(\tilde{\eta})$ $(n = 1, 2, 3, \ldots)$ which appear in the expressions (13. 12), (13.13) for ϕ_1, ψ_1 are determined by the requirement that ϕ_2, ψ_2 be free of divergent oscillations.

We begin by writing down the third order problem (9.26)-(9.30), making the changes indicated by (11. 11) to account for the $\tilde{\eta}$-dependence of ϕ, ψ:

$$
\begin{aligned}
\eta<0:\ \phi_{2_{\tau\tau}} - \frac{c_1^2}{c_2^2}\,\Delta\phi_2 = {} & \frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right) - \frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)\cos 2(\zeta-\tau) - \frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)\phi_{o_{\tau\tau}} \\
& + \left(4k_2 + \frac{c_1^2}{c_2^2}-1\right)\Delta\phi_o\left[\Delta\phi_1 + 2\phi_{o_{\eta\tilde{\eta}}} + \frac{1}{8}\cos 2(\zeta-\tau)\right] + (k_2+2k_3)(\Delta\phi_o)^3 \\
& + \frac{1}{2}\left(\frac{c_1^2}{c_2^2}-1\right)\Delta\psi_1[\Delta\psi_o + \sin(\zeta-\tau)] + k_2\,\Delta\phi_o[\Delta\psi_o + \sin(\zeta-\tau)]^2 \\
& + \frac{c_1^2}{c_2^2}\left(2\phi_{1_{\eta\tilde{\eta}}} + \phi_{o_{\tilde{\eta}\tilde{\eta}}}\right),
\end{aligned} \tag{14.1}
$$

$$
\begin{aligned}
\eta<0:\ \psi_{2_{\tau\tau}} - \Delta\psi_2 = {} & -\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)\left[\psi_{o_{\tau\tau}} + \sin(\zeta-\tau)\right] + k_2(\Delta\phi_o)^2[\Delta\psi_o + \sin(\zeta-\tau)] \\
& + \frac{1}{2}\left(\frac{c_1^2}{c_2^2}-1\right)\Big[(\Delta\psi_o + \sin(\zeta-\tau))\Big\{\Delta{}_1 + 2\phi_{o_{\eta\tilde{\eta}}} + \frac{1}{8}\cos 2(\zeta-\tau) \\
& + \frac{1}{4}(\Delta\psi_o + \sin(\zeta-\tau))^2\Big\} + \Delta\phi_o\Delta\psi_1\Big] + 2\psi_{1_{\eta\tilde{\eta}}},
\end{aligned} \tag{14.2}
$$

$$\eta = 0: \tfrac{1}{2}\phi_{2_{\tau\tau}} - \phi_{2_{\xi\xi}} + \psi_{2_{\xi\eta}} = -\frac{1}{32}\left(\frac{c_1^2}{c_2^2} - 1\right)\phi_{o_{\tau\tau}} + \frac{1}{16}(\cos^2\alpha - \tfrac{1}{2})\cos 2(\xi\cos\alpha - \tau)$$

$$+ \frac{1}{32}\left(\frac{c_1^2}{c_2^2} - 1\right) \quad , \tag{14.3}$$

$$\eta = 0: \tfrac{1}{2}\psi_{2_{\tau\tau}} - \psi_{2_{\xi\xi}} - \phi_{2_{\xi\eta}} = -\frac{1}{32}\left(\frac{c_1^2}{c_2^2} - 1\right)\left[\psi_{o_{\tau\tau}} + \sin(\xi\cos\alpha - \tau)\right]$$

$$- \frac{1}{16}\sin\alpha\cos\alpha\cos 2(\xi\cos\alpha - \tau) + \phi_{1_{\xi\tilde{\eta}}} \quad , \tag{14.4}$$

$$\phi_{2_\xi} - \psi_{2_\eta} \text{ and } \phi_{2_\eta} + \psi_{2_\xi} \text{ periodic of period } 2\pi \text{ in } \tau \quad . \tag{14.5}$$

We have, in (14.1)-(14.5), omitted the terms $\psi_{o_{\eta\tilde{\eta}}}$ and $\psi_{o_{\xi\tilde{\eta}}}$ since, by (11.12), (11.21), these terms are zero.

We now identify those terms on the right-hand sides of (14.1), (14.2) which make divergent oscillatory contributions to ϕ_2, ψ_2. Clearly, by (11.12), (11.21), (13.9)-(13.13), each of the quantities $\phi_{o_{\tau\tau}}$, $\Delta\phi_o\, \Delta\phi_1$, $\Delta\phi_o$, $\phi_{o_{\eta\tilde{\eta}}}$, $(\Delta\phi_o)^3$, $\Delta\phi_o[\Delta\psi_o + \sin(\zeta - \tau)]^2$, $\phi_{1_{\eta\tilde{\eta}}}$, $\phi_{o_{\tilde{\eta}\tilde{\eta}}}$, appearing on the right-hand side of (14.1) makes such a contribution to ϕ_2. Those terms on the right-hand side of (14.2) which give rise to divergent oscillatory terms in ψ_2 are of two kinds, viz., those with arguments which are integer multiples of $\zeta_1 - \tau$ and of $\zeta - \tau$. Terms with argument proportional to $\zeta_1 - \tau$ arise from the presence of the quantities $\psi_{o_{\tau\tau}}$, $\Delta\psi_o\, \Delta\phi_1$, $(\Delta\psi_o)^3$, $\Delta\psi_o \sin^2(\zeta - \tau)$ $\Delta\psi_o(\Delta\phi_o)^2$, $\Delta\psi_o\, \phi_{o_{\eta\tilde{\eta}}}$, $\psi_{1_{\eta\tilde{\eta}}}$, while those with argument proportional to $\zeta - \tau$ are due to the presence of $(\Delta\phi_o)^2\sin(\zeta - \tau)$, $\Delta\phi_1 \sin(\zeta - \tau)$, and of $\sin(\zeta - \tau)$ alone.

On substituting for ϕ_1, ψ_1 from (13.12), (13.13) in (14.2) it is seen that the condition that the right-hand side of (14.2) contain no terms proportional to $\sin(\zeta - \tau)$, $\sin(\zeta_1 - \tau)$ is

$$2 \sin\alpha \sum_1^{\infty} n\{E_n'(\tilde{\eta}) \sin n(\zeta_1 - \tau) - F_n'(\tilde{\eta}) \sin n(\zeta - \tau)\} = M(\tilde{\eta})\{C \sin(\zeta_1 - \tau) - \sin(\zeta - \tau)\} \tag{14.6}$$

where

$$M(\tilde{\eta}) = \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)\left\{2\left(\frac{c_1^2}{c_2^2} - 1\right)A + C^2\right\} + \frac{1}{8}\left(\frac{c_2}{c_1}\right)^6\left[4k_2 - \left(\frac{c_1^2}{c_2^2} - 1\right)^2\right]\sum_1^{\infty} n^4 B_n^2(\tilde{\eta}) \ . \tag{14.7}$$

We satisfy (14.6) and the initial conditions (13.14) term by term as follows. For n = 3, 4, ... we choose

$$E_n(\tilde{\eta}) = F_n(\tilde{\eta}) = 0 \ , \qquad n = 3, 4, 5 \ldots . \tag{14.8}$$

For n = 1, the constant terms in $M(\tilde{\eta})$ contribute a term linear in $\tilde{\eta}$ to each of $E_1(\tilde{\eta})$, $F_1(\tilde{\eta})$. We eliminate these undesirable terms by choosing A so that the constant terms in $M(\tilde{\eta})$ vanish. We thus choose

$$A = - \frac{C^2}{2\left(\frac{c_1^2}{c_2^2} - 1\right)} \ , \tag{14.9}$$

so that $M(\tilde{\eta})$ is now given by

$$M(\tilde{\eta}) = \frac{1}{8}\left(\frac{c_2}{c_1}\right)^6 \left[4k_2 - \left(\frac{c_1^2}{c_2^2} - 1\right)^2\right]\sum_1^{\infty} n^4 B_n^2(\tilde{\eta}) \ . \tag{14.10}$$

By (14.6) ,

$$\left.\begin{aligned} E_1(\tilde{\eta}) &= E_1(0) - \frac{C}{2\sin\alpha}\int_{\tilde{\eta}}^{0} M(\sigma)\,d\sigma \quad, \\ F_1(\tilde{\eta}) &= F_1(0) - \frac{1}{2\sin\alpha}\int_{\tilde{\eta}}^{0} M(\sigma)\,d\sigma \quad; \end{aligned}\right\} \tag{14.11}$$

the integral appearing in (14.11) is bounded for all $\tilde{\eta}$ since, according to (12.33), (14.10) ,

$$M(\sigma) = O\left(\frac{1}{\sigma^2}\right) \text{ as } \sigma \to -\infty \quad . \tag{14.12}$$

The term $F_1(\tilde{\eta})\cos(\zeta-\tau)$ propagates in the same direction as the incident wave ψ^*. We avoid violation of the condition that there be no "incoming" reflected waves at $\tilde{\eta} = -\infty$ by requiring that $F_1(\tilde{\eta})$ tend to zero as $\tilde{\eta} \to -\infty$.

By (14.11), $F_1(\tilde{\eta}) \to 0$ as $\tilde{\eta} \to -\infty$ if and only if

$$F_1(0) = \frac{1}{2\sin\alpha}\int_{-\infty}^{0} M(\sigma)\,d\sigma \quad . \tag{14.13}$$

Having made a particular choice for $F_1(0)$, the initial values $D_1(0)$, $E_1(0)$ are now fully determined by (13.14). By (13.14), on recalling the definition (10.15) of C ,

$$E_1(0) + C\,F_1(0) = 0 \quad . \tag{14.14}$$

It then follows, from (14.13), that

$$E_1(0) = -\frac{C}{2\sin\alpha}\int_{-\infty}^{0} M(\sigma)\,d\sigma \quad . \tag{14.15}$$

Thus, by (14.11), (14.13), (14.15), we have

$$\left.\begin{aligned} E_1(\tilde{\eta}) &= -\frac{C}{2\sin\alpha}\left\{\int_{-\infty}^{0} M(\sigma)\,d\sigma + \int_{\tilde{\eta}}^{0} M(\sigma)\,d\sigma\right\}, \\ F_1(\tilde{\eta}) &= \frac{1}{2\sin\alpha}\int_{-\infty}^{\tilde{\eta}} M(\sigma)\,d\sigma \end{aligned}\right\} \tag{14.16}$$

where $M(\sigma)$ is given by (14.10). We now consider the terms in (14.6) corresponding to $n = 2$. Since, by (14.6), $E_2'(\tilde{\eta}) = F_2'(\tilde{\eta}) = 0$, it follows that both $E_2(\tilde{\eta})$ and $F_2(\tilde{\eta})$ are constant. The condition that reflected waves be outgoing at $\tilde{\eta} = -\infty$ requires that

$$F_2(\tilde{\eta}) \equiv 0 \quad . \tag{14.17}$$

From this and (13.15) it follows that

$$E_2(\tilde{\eta}) \equiv E_2(0) = \frac{(\cos^2\alpha - \frac{1}{2})V_2(\alpha) + \cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} V_1(\alpha)}{(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}, \tag{14.18}$$

where $V_1(\alpha)$, $V_2(\alpha)$ are given by (13.16), (13.17).

There remains the problem of determining the sequence $\{D_n(\tilde{\eta})\}$. As indicated earlier, determination of $\{D_n(\tilde{\eta})\}$ is a consequence of the requirement that the right-hand side of (14.1) be free of terms contributing divergent oscillations to ϕ_2. The undesirable contributions to ϕ_2 arise from the presence of the quantities $\phi_{o_{\tau\tau}}$, $\Delta\phi_o\Delta\phi_1$, $\Delta\phi_o\phi_{o_{\tilde{\eta}\tilde{\eta}}}$, $(\Delta\phi_o)^3$, $\Delta\phi_o[\psi_o+\sin(\zeta-\tau)]^2$, $\phi_{1_{\tilde{\eta}\tilde{\eta}}}$, and $\phi_{o_{\tilde{\eta}\tilde{\eta}}}$ on the right-hand side of (14.1). On substituting for ϕ_1 it turns out that the condition that there be no terms producing divergent oscillations in ϕ_2 is that the function of $\zeta_2-\tau$ and $\tilde{\eta}$ which appears on the right-hand side of (14.1) be identically zero, i.e. that

$$2\frac{c_1^2}{c_2^2}\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}}\sum_1^\infty n D_n'(\tilde{\eta})\sin n(\zeta_2-\tau)-\left(\frac{c_2}{c_1}\right)^2\left(4k_2+\frac{c_1^2}{c_2^2}-1\right)\Delta\phi_o\sum_1^\infty n^2 D_n(\tilde{\eta})\cos n(\zeta_2-\tau)$$

$$+\left[\tfrac{1}{2}k_2(1+C^2)-\frac{1}{16}\left(\frac{c_1}{c_2}\right)^2\left(\frac{c_1^2}{c_2^2}-1\right)-\left(4k_2+\frac{c_1^2}{c_2^2}-1\right)\left\{\frac{1}{8}C^2+\frac{1}{4}\left(\frac{c_2}{c_1}\right)^6\left(4k_2+\frac{c_1^2}{c_2^2}-1\right)\sum_1^\infty n^4 B_n^2(\tilde{\eta})\right\}\right]\Delta\phi_o$$

$$+2\left(4k_2+\frac{c_1^2}{c_2^2}-1\right)\Delta\phi_o\,\phi_{o_{\tilde{\eta}\tilde{\eta}}}+\left(\frac{c_1}{c_2}\right)^2\phi_{o_{\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}}}+(k_2+2k_3)(\Delta\phi_o)^3=0\quad. \qquad (14.19)$$

For convenience we set

$$\theta \equiv \zeta_2 - \tau \quad . \qquad (14.20)$$

We determine the sequence $\{D_n(\tilde{\eta})\}$ by regarding (14.19) as an equation for the generating function

$$d(\theta,\tilde{\eta}) \equiv \sum_1^\infty n\, D_n(\tilde{\eta}) \sin n\theta \quad ; \qquad (14.21)$$

the appropriate initial condition on $d(\theta,\tilde{\eta})$ will later be supplied. We begin by showing that $\Delta\phi_o$, $\phi_{o_{\tilde{\eta}\tilde{\eta}}}$ and $\phi_{o_{\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}}}$ can be written compactly in terms of the function $b(\theta,\tilde{\eta})$ (introduced in (12.4)) and members of the sequence $\{B_n(\tilde{\eta})\}$.

Recall that

$$\phi_o(\theta,\tilde{\eta}) = \sum_1^\infty B_n(\tilde{\eta}) \sin n\theta \quad . \qquad (14.22)$$

Thus, by (14.20, (12.1), (12.4) ,

$$\Delta\phi_o = -\left(\frac{c_2}{c_1}\right)^2\sum_1^\infty n^2 B_n(\tilde{\eta})\sin n\theta = -\left(\frac{c_2}{c_1}\right)^2 B_1(0)b(\theta,\tilde{\eta}) \,. \qquad (14.23)$$

Also, as can be shown by the definitions of ϕ_o, $b(\theta,\tilde{\eta})$ and repeated use of (12.6), (12.11),

$$\phi_{o_{\tilde{\eta}\tilde{\eta}}} = \begin{cases} \left(\dfrac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} \{\frac{1}{2} K\, B_1(0)\, b^2(\theta, \tilde{\eta}) - \sum_1^{\infty} n\, B_n'(\tilde{\eta})\} & \text{if } K > 0, \\ \left(\dfrac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} \{\frac{1}{2} K\, B_1(0)\, b^2(\theta, \tilde{\eta}) - \sum_1^{\infty} (-1)^n n B_n'(\tilde{\eta})\} & \text{if } K < 0, \end{cases} \tag{14.24}$$

$$\phi_{o_{\tilde{\eta}\tilde{\eta}\tilde{\eta}}} \begin{cases} -\frac{1}{3} K^2 B_1(0)[b^3(\theta, \tilde{\eta}) - \frac{\theta}{\pi} b^3(\pi, \tilde{\eta})] & \text{if } K > 0, \\ -\frac{1}{3} K^2 B_1(0)[b^3(\theta, \tilde{\eta}) - \frac{\pi-\theta}{\pi} b^3(\theta, \tilde{\eta})] & \text{if } K < 0. \end{cases} \tag{14.25}$$

In what follows it is assumed that K is positive. The case in which K is negative is similar. Using (14.21) and (14.23)-(14.25) in (14.19) we can now write down the following partial differential equation for $d(\theta, \tilde{\eta})$, for the case $K > 0$:

$$d_{\tilde{\eta}} - K b(\theta, \tilde{\eta})\, d_\theta = (a_o + a_1(\tilde{\eta}))\, b(\theta, \tilde{\eta}) + a_2[b^3(\theta, \tilde{\eta}) - \frac{\theta}{\pi} b^3(\pi, \tilde{\eta})] + a_3 b^3(\theta, \tilde{\eta}),$$

$$\tilde{\eta} < 0, \quad 0 < \theta < \pi. \tag{14.26}$$

The initial condition on $d(\theta, \tilde{\eta})$ is found to be

$$d(\theta, 0) = D_1(0) \sin\theta + 2D_2(0) \sin 2\theta, \quad 0 < \theta < \pi, \tag{14.27}$$

on using (14.8), (14.15), (14.13), (14.17), (14.18) in (13.14), (13.15) to show that

$$\left.\begin{aligned} D_n(0) &= 0, \quad n = 3, 4, 5, \ldots . \\ D_1(0) &= \frac{(1+C)\cos\alpha}{2(\cos^2\alpha - \frac{1}{2})} \int_{-\infty}^{0} M(\sigma)\, d\sigma, \\ D_2(0) &= \frac{(\cos^2\alpha - \frac{1}{2}) V_1(\alpha) - \sin\alpha \cos\alpha\, V_2(\alpha)}{(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha \cos^2\alpha \left(\dfrac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} \end{aligned}\right\} \tag{14.28}$$

where $V_1(\alpha)$, $V_2(\alpha)$, $M(\sigma)$ are given by (13.16), (13.17), (14.10) respectively.

The constants a_o, a_2, a_3 and the function $a_1(\tilde{\eta})$ appearing in (14.26) are given by

$$\left.\begin{aligned}
a_o &= \tfrac{1}{2}\left(\frac{c_2}{c_1}\right)^4\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{-\frac{1}{2}}\left[\tfrac{1}{2}k_2(1+C^2) - \frac{1}{16}\left(\frac{c_1}{c_2}\right)\left(\frac{c_1^2}{c_2^2} - 1\right) - \frac{1}{8}C^2\left(4k_2 + \frac{c_1^2}{c_2^2} - 1\right)\right]B_1(0),\\
a_1(\tilde{\eta}) &= \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\left(\frac{c_1}{c_2}\right)^2 K\left[-\tfrac{1}{2}K\sum_1^\infty n^4 B_n^2(\tilde{\eta}) + 2\sum_1^\infty n B_n'(\tilde{\eta})\right]\\
a_2 &= -\frac{1}{6}\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{-\frac{1}{2}} K^2 B_1(0).\\
a_3 &= \tfrac{1}{2}\left(\frac{c_2}{c_1}\right)^2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{-\frac{1}{2}}\left[\left(\frac{c_2}{c_1}\right)^6(k_2+2k_3)B_1^3(0) + \left(\frac{c_2}{c_1}\right)^2\left(4k_2 + \frac{c_1^2}{c_2^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right.\\
&\qquad\qquad \left.\times K B_1^2(0)\right].
\end{aligned}\right\} \qquad (14.29)$$

In contrast with the corresponding equation (12.6) for $b(\theta, \tilde{\eta})$, equation (14.26) is linear and nonhomogeneous. Both equations have characteristics given by

$$\frac{d\theta}{d\tilde{\eta}} = -K\, b(\theta, \tilde{\eta}) \qquad (14.30)$$

from which, by (12.7) and since b is constant along characteristics, it follows that

$$\theta = s - K\tilde{\eta}\sin s \qquad (14.31)$$

is the characteristic through the point $\theta = s$, $\tilde{\eta} = 0$.

We now obtain $d(\theta, \tilde{\eta})$ by integrating equation (14.26) along the straight line characteristic through $(\theta, \tilde{\eta})$ from $(s, 0)$ to $(\theta, \tilde{\eta})$. Since b is constant along characteristics we then find that

$$d(\theta,\tilde{\eta}) = d(s,0) + \left\{a_o \tilde{\eta} + \int_o^{\tilde{\eta}} a_1(\sigma)d\sigma\right\} b(\theta,\tilde{\eta}) + a_3\tilde{\eta} b^3(\theta,\tilde{\eta})$$
$$+ a_2\left\{\tilde{\eta}\, b^3(\theta,\tilde{\eta}) - \frac{1}{\pi}\int_o^{\tilde{\eta}} (s-\sigma K \sin s) b^3(\pi,\sigma)d\sigma\right\}, \qquad (14.32)$$

where we have, in the last integral, used (14.31). On using the initial conditions (12.7) and (14.27), (14.31) and the fact that

$$\sin s = b(s, 0) = b(\theta, \tilde{\eta}) ,$$

equation (14.32) reads

$$d(\theta,\tilde{\eta}) = \left[D_1(0) + a_o\tilde{\eta} + K\frac{a_2}{\pi}\int_{\tilde{\eta}}^{o} (\tilde{\eta}-\sigma)b^2(\pi,\sigma)d\sigma + A_1(\tilde{\eta})\right] b(\theta,\tilde{\eta})$$
$$+ 4D_2(0)b(\theta,\tilde{\eta})\{1-b^2(\theta,\tilde{\eta})\}^{\frac{1}{2}} + (a_2+a_3)\tilde{\eta}\, b^3(\theta,\tilde{\eta}) - \theta\int_{\tilde{\eta}}^{o} b^3(\pi,\sigma)d\sigma$$
$$(14.33)$$

where, by the second of (14.29),

$$A_1(\tilde{\eta}) \equiv -\int_{\tilde{\eta}}^{o} a_1(\sigma)d\sigma = \left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)\left(\frac{c_1}{c_2}\right) K\left[\frac{1}{2} K \int_{\tilde{\eta}}^{o} \sum_1^{\infty} n^4 B_n^2(\sigma)d\sigma\right.$$
$$\left. + 2\left\{\sum_1^{\infty} n B_n(\tilde{\eta}) - B_1(0)\right\}.\right] \qquad (14.34)$$

It should be noted that the last integral in (14.33) is zero when $-\frac{1}{K} = \tilde{\eta} \le 0$ since, as was seen in § 12,

$$b(\pi,\tilde{\eta}) = 0, \qquad -\frac{1}{K} \le \tilde{\eta} \le 0 , \qquad \text{when } K > 0 . \qquad (14.35)$$

We now examine the behavior of $d(\theta, \tilde{\eta})$ as $\tilde{\eta} \to -\infty$ with θ fixed, $0 < \theta < \pi$.

On using (12.33) and the fact that $b(\theta, \tilde{\eta}) \sim -\frac{\theta}{K\tilde{\eta}}$ as $\tilde{\eta} \to -\infty$, θ fixed, we find from (14.33) that

$$d(\theta, \tilde{\eta}) \sim A\theta \quad \text{as} \quad \eta \to -\infty, \tag{14.36}$$

uniformly in θ, $0 < \theta < \pi$, where

$$A = \frac{a_o}{K} - \left(1 + \frac{a_2}{\pi}\right) \int_{-\infty}^{o} b^3(\pi, \sigma) d\sigma, \tag{14.37}$$

and a_o, a_2 are given in (14.29).

Since the $D_n(\tilde{\eta})$ are derived from $d(\theta, \tilde{\eta})$ through

$$D_n(\tilde{\eta}) = \frac{2}{2\pi} \int_o^{\pi} d(\theta, \tilde{\eta}) \sin n\theta \, d\theta, \quad \tilde{\eta} < 0, \tag{14.38}$$

it follows from (14.36) that

$$D_n(\tilde{\eta}) \sim A \frac{(-1)^{n+1}}{n^2} \quad \text{as} \quad \eta \to -\infty, \quad n = 1, 2, \ldots . \tag{14.39}$$

In view of this, the series $\sum_{1}^{\infty} n^2 D_n(\tilde{\eta}) \sin n(\zeta_2 - \tau, \tilde{\eta})$ which will appear in our expressions for the Lagrange stresses must be interpreted as $d_\theta(\zeta_2 - \tau, \tilde{\eta})$.

We comment further on the functions $b(\theta, \tilde{\eta})$, $d(\theta, \tilde{\eta})$ in §20.

§15. Final Form of the First and Second Order Solutions

We present here a summary of the results of this chapter. To emphasize the fact that the solutions we present here are those of the first and second order problems for the potentials associated with the reflected wave, we agree here to restore the bars which were dropped, for reasons of convenience, from $\bar{\phi}$, $\bar{\psi}$ in §9.

Our solution $\bar{\phi}_o$, ψ_o to the first order problem for the reflected wave is

$$\left.\begin{aligned}\bar{\Phi}_o &= \sum_1^{\infty} B_n(\tilde{\eta}) \sin n(\zeta_2 - \tau) \;, \\ \bar{\Psi}_o &= C \sin(\zeta_1 - \tau)\end{aligned}\right\} \qquad (15.1)$$

where

$$\zeta_2 = \xi \cos\alpha - \eta \sin\alpha \;, \; \zeta_2 = \xi \cos\alpha - \eta \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} \;, \qquad (15.2)$$

and

$$C = \frac{\left(\cos^2\alpha - \frac{1}{2}\right)^2 - \sin\alpha \cos^2\alpha \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{\left(\cos^2\alpha - \frac{1}{2}\right) + \sin\alpha \cos^2\alpha \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} \;. \qquad (15.3)$$

The amplitudes $B_n(\tilde{\eta})$ $(n = 1, 2, \ldots)$ are obtained from

$$B_n(\tilde{\eta}) = \frac{2B_1(0)}{\pi n^2} \int_0^{\pi} \sin n\,\theta \; b(\theta, \tilde{\eta}) d\theta, \; \tilde{\eta} < 0, \; n = 1, 2, \ldots \;, \qquad (15.4)$$

where $b(\theta, \tilde{\eta})$ satisfies

$$b_{\tilde{\eta}} - K\, b\, b_{\theta} = 0, \quad \tilde{\eta} < 0, \quad 0 < \theta < \pi \;, \qquad (15.5)$$

$$b(\theta, 0) = \sin\theta \;, \quad 0 < \theta < \pi \;, \qquad (15.6)$$

and the constants $B_1(0)$ and K are given by

$$B_1(0) = \frac{2 \sin\alpha \cos\alpha \left(\frac{1}{2} - \cos^2\alpha\right)}{\left(\cos^2\alpha - \frac{1}{2}\right)^2 + \sin\alpha \cos^2\alpha \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} \;, \qquad (15.7)$$

$$K = - \frac{\left(\frac{c_2}{c_1}\right)^6 \left(4k_2 + \frac{c_1^2}{c_2^2} - 1\right) B_1(0)}{2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} \;. \qquad (15.8)$$

Our solution $\overline{\Phi}_1, \overline{\Psi}_1$ to the second order problem for the reflected wave is

$$\overline{\Phi}_1 = -\frac{1}{8}\left(\frac{c_2}{c_1}\right)^6 \left(4k_2 + \frac{c_1^2}{c_2^2} - 1\right)\eta^2 \sum_1^\infty n^4 B_n(\tilde{\eta}) - \frac{1}{16} C^2(\tau^2 + \eta^2)$$
$$+ \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)\left[-\frac{1}{2} C^2 \frac{\cos 2(\zeta_1 - \tau)}{\frac{c_1^2}{c_2^2} - 1} + C \frac{\cos 2(\xi\cos\alpha - \tau)}{\frac{c_1^2}{c_2^2}\cos^2\alpha - 1} - C \frac{\cos(2\eta \sin\alpha)}{\frac{c_1^2}{c_2^2}\sin^2\alpha}\right]$$
$$+ \sum_1^\infty D_n(\tilde{\eta}) \cos n(\zeta_2 - \tau) \quad , \qquad (15.9)$$

$$\overline{\Psi}_1 = -\frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1 - C^2\right)\xi\eta$$
$$+ \frac{1}{4}\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2} - 1\right)\left[\sum_{n=-\infty}^{\infty} n^2 B_n(\tilde{\eta})\left\{C \frac{\cos[\zeta_1 - n\zeta_2 - (1-n)\tau]}{(\sin\alpha - n\frac{c_2}{c_1}\sin\beta)^2 - (1-n)^2\sin^2\alpha}\right.\right.$$
$$\left.\left. - \frac{\cos[\zeta - n\,\zeta_2 - (1-n)\tau]}{(\sin\alpha + n\frac{c_2}{c_1}\sin\beta)^2 - (1-n)^2\sin^2\alpha}\right\}\right]$$
$$+ E_1(\tilde{\eta})\cos(\zeta_1 - \tau) + F_1(\tilde{\eta})\cos(\zeta - \tau) + E_2(0)\cos 2(\zeta_1 - \tau) \, . \qquad (15.10)$$

Here

$$B_{-n}(\tilde{\eta}) = -B_n(\tilde{\eta}) \quad (n = 1, 2, \ldots), \quad B_o(\tilde{\eta}) \equiv 0 \qquad (15.11)$$

$$\left.\begin{aligned} E_1(\tilde{\eta}) &= -\frac{C}{2\sin\alpha}\left\{\int_{-\infty}^{o} M(\sigma)d\sigma + \int_{\tilde{\eta}}^{o} M(\sigma)d\sigma\right\} \, , \\ F_1(\tilde{\eta}) &= \frac{1}{2\sin\alpha}\int_{-\infty}^{\tilde{\eta}} M(\sigma)d\sigma \end{aligned}\right\} \qquad (15.12)$$

where

$$M(\sigma) = \frac{1}{8}\left(\frac{c_2}{c_1}\right)^6\left[4\, k_2 - \left(\frac{c_1^2}{c_2^2} - 1\right)^2\right]\sum_1^\infty n^4\, B_n(\sigma) \quad , \tag{15.13}$$

k_2 being a material constant. The amplitudes $D_n(\tilde{\eta})(n = 1, 2, \ldots)$ are obtained as follows:

$$D_n(\tilde{\eta}) = \frac{2}{\pi\, n}\int_0^\pi d(\theta, \tilde{\eta})\, \sin n\,\theta\, d\theta, \; \tilde{\eta} < 0 \quad , \tag{15.14}$$

where, when $K > 0$, $d(\theta, \tilde{\eta})$ satisfies

$$d_{\tilde{\eta}} - Kb(\theta, \tilde{\eta})d_\theta = (a_o + a_1(\tilde{\eta}))\, b(\theta, \tilde{\eta}) + a_2\, [b^3(\theta, \tilde{\eta}) - \frac{\theta}{\pi}\, b^3(\pi, \eta)] + a_3 b^3(\theta, \tilde{\eta}),$$

$$\tilde{\eta} < 0, \; 0 < \theta < \pi \quad , \tag{15.15}$$

$$d(\theta, 0) = D_2(0)\, \sin\,\theta + 2\, D_2(0)\, \sin\, 2\,\theta, \; 0 < \theta < \pi \tag{15.16}$$

($a_o, a_1(\tilde{\eta}), a_2, a_3$ are given in (14.29)). When $K < 0$, a similar partial differential equation for $d(\theta, \tilde{\eta})$ is found while the initial condition (15.16) remains unchanged. A full discussion of the problems (15.5), (15.6) for $b(\theta, \tilde{\eta})$ and (15.15), (15.16) for $d(\theta, \tilde{\eta})$ is provided in §12 and § 14 respectively.

The constants β, $D_1(0)$, $E_2(0)$ are given by

$$\cos\beta = \frac{c_1}{c_2}\cos\alpha \quad , \qquad 0 < \beta < \alpha < \frac{\pi}{2} \quad , \tag{15.17}$$

$$D_1(0) = \frac{(1+C)\cos\alpha}{2(\cos^2\alpha - \frac{1}{2})}\int_{-\infty}^{0} M(\sigma)d\sigma \quad , \tag{15.18}$$

$$\left.\begin{aligned} D_2(0) &= \frac{(\cos^2\alpha - \frac{1}{2})\, V_1(\alpha) - \sin\alpha\, \cos\alpha\, V_2(\alpha)}{(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha\, \cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} \; , \\ E_2(0) &= \frac{(\cos^2\alpha - \frac{1}{2})\, V_2(\alpha) + \cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} V_1(\alpha)}{(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha\, \cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} \end{aligned}\right\} \tag{15.19}$$

where

$$
\left.
\begin{aligned}
V_1(\alpha) &= \frac{1}{16}\,\frac{\frac{1}{2}-\cos^2\alpha}{\frac{c_1^2}{c_2^2}\cos\alpha-1}\left[-\tfrac{1}{2}(1+C^2)\left(\frac{c_1^2}{c_2^2}\cos^2\alpha-1\right)+C\left(\frac{c_1^2}{c_2^2}-1\right)\right] \\
&\quad + B_1(0)\frac{1}{8}\left(\frac{c_2}{c_1}\right)^2\left(\frac{c_1^2}{c_2^2}-1\right)\cos\alpha\left[C\,\frac{\sin\alpha+\frac{c_2}{c_1}\sin\beta}{\left(\sin\alpha+\frac{c_2}{c_1}\sin\beta\right)^2-4\sin^2\alpha}\right. \\
&\qquad\qquad \left. + \frac{\sin\alpha-\frac{c_2}{c_1}\sin\beta}{\left(\sin\alpha-\frac{c_2}{c_1}\sin\beta\right)^2-4\sin^2\alpha}\right] \\
V_2(\alpha) &= \frac{1}{32}(1-C^2)\sin\alpha\cos\alpha+\frac{1}{16}\cos\alpha\, B_1(0)\, K \\
&\quad + B_1(0)\,\frac{1}{4}\,\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2}-1\right)\left(\tfrac{1}{2}-\cos^2\alpha\right)\left[\frac{1}{\left(\sin\alpha-\frac{c_2}{c_1}\sin\beta\right)^2-4\sin^2\alpha}\right. \\
&\qquad\qquad \left. - \frac{C}{\left(\sin\alpha+\frac{c_2}{c_1}\sin\beta\right)^2-4\sin^2\alpha}\right].
\end{aligned}
\right\}
\tag{15.20}
$$

CHAPTER V. RESULTS

§ 16. Summary of Results

In this chapter we derive the displacements and stresses corresponding to the solutions $\overline{\phi}_o$, $\overline{\psi}_o$ and $\overline{\phi}_1$, $\overline{\psi}_1$ of the first and second order problems for the reflected wave and discuss their properties. The nonlinear effects on the surface displacements are examined in detail. The particular case of normal incidence ($\alpha = \pi/2$) is also discussed.

The reader is referred to §15 for definition of the quantities C, $\{B_n(\tilde{\eta})\}$, $\{D_n(\tilde{\eta})\}$, $E_2(0)$, $E_1(\tilde{\eta})$, $F_1(\tilde{\eta})$, β, ζ, ζ_1, ζ_2 which appear in this chapter.

We begin by calculating the physical displacements $\overline{u}$, $\overline{v}$ associated with the reflected wave. By (6.17), (8.1), (8.8) and (9.1), they are given, correct to order ϵ^2, by

$$\frac{2\pi}{L}\,\overline{u} = \epsilon(\overline{\phi}_{o_\xi} - \overline{\psi}_{o_\eta}) + \epsilon^2(\overline{\phi}_{1_\xi} - \overline{\psi}_{1_\eta} - \overline{\psi}_{o_{\tilde{\eta}}}) + O(\epsilon^3)\,, \tag{16.1}$$

$$\frac{2\pi}{L}\,\overline{v} = \epsilon(\overline{\phi}_{o_\eta} + \overline{\psi}_{o_\xi}) + \epsilon^2(\overline{\phi}_{o_{\tilde{\eta}}} + \overline{\phi}_{1_\eta} + \overline{\psi}_{1_\xi}) + O(\epsilon^3)\;. \tag{16.2}$$

Substituting for $\overline{\phi}_o$, $\overline{\psi}_o$ from (15.1) and for $\overline{\phi}_1$, $\overline{\psi}_1$ from (15.9), (15.10)respectively we obtain to order ϵ^2, the following expressions for $\overline{u}$, $\overline{v}$:

$$\frac{2\pi}{L}\,\bar{u} = \epsilon\Big[\cos\alpha \sum_1^\infty n\,B_n(\tilde{\eta})\cos n(\zeta_2-\tau)+\sin\alpha\; C\cos(\zeta_1-\tau)\Big]$$

$$+\epsilon^2\Big[\frac{1}{16}\Big(\frac{c_1^2}{c_2^2}-1-C^2\Big)\xi-\frac{1}{8}\Big(\frac{c_1^2}{c_2^2}-1\Big)\frac{C\cos\alpha}{\frac{c_1^2}{c_2^2}\cos^2\alpha-1}\sin 2(\xi\cos\alpha-\tau)$$

$$-\frac{1}{4}\frac{c_2^2}{c_1^2}\Big(\frac{c_1^2}{c_2^2}-1\Big)\sum_{n=-\infty}^{\infty} n^2 B_n(\tilde{\eta})\left\{\frac{C(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)\sin[\zeta_1-n\zeta_2-(1-n)\tau]}{(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right.$$

$$\left.+\frac{(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)\sin[\zeta-n\zeta_2-(1-n)\tau]}{(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right\}$$

$$+\Big(\frac{1}{16}C^2\cos\alpha-2E_2(0)\sin\alpha\Big)\sin 2(\zeta_1-\tau)-\cos\alpha\sum_1^\infty n\,D_n(\tilde{\eta})\sin n(\zeta_2-\tau)$$

$$+\sin\alpha\{F_1(\tilde{\eta})\sin(\zeta-\tau)-E_1(\tilde{\eta})\sin(\zeta_1-\tau)\}\Big]+O(\epsilon^3)\quad, \tag{16.3}$$

$$\frac{2\pi}{L}\,\bar{v} = \epsilon\Big[\cos\alpha\; C\cos(\zeta_1-\tau)-\Big(\frac{c_2^2}{c_1^2}-\cos^2\alpha\Big)^{\frac{1}{2}}\sum_1^\infty n\,B_n(\tilde{\eta})\cos n(\zeta_2-\tau)\Big]$$

$$+\epsilon^2\Big[-\frac{1}{16}\Big(\frac{c_1^2}{c_2^2}-1+C^2\Big)\eta-\frac{1}{4}\Big(\frac{c_2}{c_1}\Big)^6\Big(4k_2+\frac{c_1^2}{c_2^2}-1\Big)\eta\sum_1^\infty n^4 B_n^2(\tilde{\eta})$$

$$-\frac{1}{4}\frac{c_2^2}{c_1^2}\Big(\frac{c_1^2}{c_2^2}-1\Big)\cos\alpha\sum_{n=-\infty}^{\infty} n^2(1-n)B_n(\tilde{\eta})\left\{\frac{C\sin[\zeta_1-n\zeta_2-(1-n)\tau]}{(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right.$$

$$\left.-\frac{\sin[\zeta-n\zeta_2-(1-n)\tau]}{(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right\}$$

$$+\sum_1^\infty\Big\{B_n'(\tilde{\eta})+\Big(\frac{c_2^2}{c_1^2}-\cos^2\alpha\Big)^{\frac{1}{2}} n\,D_n(\tilde{\eta})\Big\}\sin n(\zeta_2-\tau)$$

$$-(2E_2(0)\cos\alpha+\frac{1}{16}C^2\sin\alpha)\sin 2(\zeta_1-\tau)+\frac{1}{8}\Big(\frac{c_1^2}{c_2^2}-1\Big)\frac{c_2^2}{c_1^2}\,\frac{C\sin(2\eta\sin\alpha)}{\sin\alpha}$$

$$-\cos\alpha\{E_1(\tilde{\eta})\sin(\zeta_1-\tau)+F_1(\tilde{\eta})\sin(\zeta-\tau)\}\Big]+O(\epsilon^3)\quad. \tag{16.4}$$

The expressions (6.21) for the Lagrange stresses $q_{\alpha\beta}(\alpha, \beta=1,2)$ are linear in ϕ and ψ. We are thus able to write each of q_{11}, q_{12}, q_{21}, q_{22} as the sum of two terms, one contributed by the incident wave, the other by the reflected wave. From (6.21), (8.1), (8.8), (9.1) it then follows that the Lagrange stresses $\overline{q}_{11}$, $\overline{q}_{12}$, $\overline{q}_{21}$, $\overline{q}_{22}$ associated with the reflected wave are given by

$$
\begin{aligned}
\frac{\overline{q}_{11}}{2\mu} &= \epsilon\left(\tfrac{1}{2}\overline{\phi}_{o_{\tau\tau}} - \overline{\phi}_{o_{\eta\eta}} - \overline{\psi}_{o_{\xi\eta}}\right) + \epsilon^2\left(\tfrac{1}{2}\overline{\phi}_{1_{\tau\tau}} - \overline{\phi}_{1_{\eta\eta}} - \overline{\psi}_{1_{\xi\eta}} - 2\overline{\phi}_{o_{\eta\tilde{\eta}}} - \overline{\psi}_{o_{\xi\tilde{\eta}}}\right) \\
&\qquad + O(\epsilon^3) \ , \\
\frac{\overline{q}_{12}}{2\mu} &= -\epsilon\left(\tfrac{1}{2}\overline{\psi}_{o_{\tau\tau}} - \overline{\psi}_{o_{\xi\xi}} - \overline{\phi}_{o_{\xi\eta}}\right) - \epsilon^2\left(\tfrac{1}{2}\overline{\psi}_{1_{\tau\tau}} - \overline{\psi}_{1_{\xi\xi}} - \overline{\phi}_{1_{\xi\eta}} - \overline{\phi}_{o_{\xi\tilde{\eta}}}\right) + O(\epsilon^3) , \\
\frac{\overline{q}_{21}}{2\mu} &= \epsilon\left(\tfrac{1}{2}\overline{\psi}_{o_{\tau\tau}} - \overline{\psi}_{o_{\eta\eta}} + \overline{\phi}_{o_{\xi\eta}}\right) + \epsilon^2\left(\tfrac{1}{2}\overline{\psi}_{1_{\tau\tau}} - \overline{\psi}_{1_{\eta\eta}} - \overline{\phi}_{1_{\xi\eta}} - 2\overline{\psi}_{o_{\eta\tilde{\eta}}} + \overline{\phi}_{o_{\xi\tilde{\eta}}}\right) \\
&\qquad + O(\epsilon^2) \ , \\
\frac{\overline{q}_{22}}{2\mu} &= \epsilon\left(\tfrac{1}{2}\overline{\phi}_{o_{\tau\tau}} - \overline{\phi}_{o_{\xi\xi}} + \overline{\psi}_{o_{\xi\eta}}\right) + \epsilon^2\left(\tfrac{1}{2}\overline{\phi}_{1_{\tau\tau}} - \overline{\phi}_{1_{\xi\xi}} + \overline{\psi}_{1_{\xi\eta}} + \overline{\psi}_{o_{\xi\tilde{\eta}}}\right) + O(\epsilon^3) ,
\end{aligned} \tag{16.5}
$$

where μ is the Lamé shear modulus of the classical linear theory.

We now substitute for $\overline{\phi}_o$, $\overline{\psi}_o$, $\overline{\phi}_1$, $\overline{\psi}_1$ from §15 and obtain, from (16.5), the following expressions for $\overline{q}_{11}$, $\overline{q}_{12}$, $\overline{q}_{21}$, $\overline{q}_{22}$:

$$
\begin{aligned}
\frac{\bar{q}_{11}}{2\mu} = \epsilon & \left[\left(\frac{c_2^2}{c_1^2} - \tfrac{1}{2} - \cos^2\alpha \right) \sum_1^\infty n^2 B_n(\tilde{\eta}) \sin n(\zeta_2 - \tau) - \sin\alpha \cos\alpha \, C \sin(\zeta_1 - \tau) \right] \\
& + \epsilon^2 \Bigg[\frac{1}{16} \left(\frac{c_1^2}{c_2^2} - 1 \right) + \frac{1}{4} \left(\frac{c_2}{c_1} \right)^6 \left(4k_2 + \frac{c_1^2}{c_2^2} - 1 \right) \sum_1^\infty n^4 B_n^2(\tilde{\eta}) \\
& + \frac{1}{4} \frac{c_2^2}{c_1^2} \left(\frac{c_1^2}{c_2^2} - 1 \right) \cos\alpha \sum_{n=-\infty}^{\infty} n^2 (1-n) B_n(\tilde{\eta}) \\
& \times \left\{ \frac{C\left(\sin\alpha - n \frac{c_2}{c_1} \sin\beta\right) \cos\left[\zeta_1 - n\zeta_2 - (1-n)\tau\right]}{\left(\sin\alpha - n \frac{c_2}{c_1} \sin\beta\right)^2 - (1-n)^2 \sin^2\alpha} + \frac{\left(\sin\alpha + n \frac{c_2}{c_1} \sin\beta\right) \cos\left[\zeta - n\zeta_2 - (1-n)\tau\right]}{\left(\sin\alpha + n \frac{c_2}{c_1} \sin\beta\right)^2 - (1-n)^2 \sin^2\alpha} \right\} \\
& + \left\{ \frac{1}{8} C^2 \left(\tfrac{1}{2} - \sin^2\alpha\right) - 4 E_2(0) \sin\alpha \cos\alpha \right\} \cos 2(\zeta_1 - \tau) \\
& + \frac{1}{16} \left(\frac{c_1^2}{c_2^2} - 1 \right) C \left\{ 2 \frac{c_2^2}{c_1^2} \frac{\cos 2(\xi \cos\alpha - \tau)}{\frac{c_2^2}{c_1^2} - \cos^2\alpha} - 4 \frac{c_2^2}{c_1^2} \cos(2\eta \sin\alpha) \right\} \\
& + \sum_1^\infty \left\{ \left(\frac{c_2^2}{c_1^2} - \tfrac{1}{2} - \cos^2\alpha \right) n^2 D_n(\tilde{\eta}) + 2 \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha \right)^{\frac{1}{2}} n B_n'(\tilde{\eta}) \right\} \cos n(\zeta_2 - \tau) \\
& - \sin\alpha \cos\alpha \left\{ E_1(\tilde{\eta}) \cos(\zeta_1 - \tau) - F_1(\tilde{\eta}) \cos(\zeta - \tau) \right\} , \qquad (16.6)
\end{aligned}
$$

$$\frac{\overline{q}_{12}}{2\mu} = -\epsilon\Big[(\cos^2\alpha-\tfrac{1}{2})\,C\sin(\zeta_1-\tau) - \cos\alpha\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}}\sum_1^{\infty} n^2 B_n(\widetilde{\eta})\sin n(\zeta_2-\tau)\Big]$$

$$-\epsilon^2\Big[\frac{1}{4}\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2}-1\right)(\cos^2\alpha-\tfrac{1}{2})\sum_{n=-\infty}^{\infty} n^2(1-n)^2 B_n(\widetilde{\eta})\Big\{\frac{C\cos[\zeta_1-n\zeta_2-(1-n)\tau]}{(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}$$

$$+\{\tfrac{1}{8}C^2\sin\alpha\cos\alpha-4(\tfrac{1}{2}-\cos^2\alpha)E_2(0)\}\cos 2(\zeta_1-\tau)-\frac{\cos[\zeta-n\zeta_2-(1-n)\tau]}{(\sin\alpha+n\,c_2/c_1\sin\beta)^2-(1-n)^2\sin^2\alpha}\Big\}$$

$$-\cos\alpha\sum_1^{\infty}\Big\{\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}} n^2 D_n(\widetilde{\eta}) + n\,B_n'(\widetilde{\eta})\Big\}\cos n(\zeta_2-\tau)$$

$$-(\tfrac{1}{2}-\cos^2\alpha)\{E_1(\widetilde{\eta})\cos(\zeta_1-\tau)+F_1(\widetilde{\eta})\cos(\zeta-\tau)\}\Big]\,, \qquad (16.7)$$

$$\frac{\overline{q}_{21}}{2\mu} = \epsilon\Big[(\sin^2\alpha-\tfrac{1}{2})\,C\sin(\zeta_1-\tau) + \cos\alpha\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}}\sum_1^{\infty} n^2 B_n(\widetilde{\eta})\sin n(\zeta_2-\tau)\Big]$$

$$+\epsilon^2\Big[-\frac{1}{4}\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2}-1\right)\sum_{n=-\infty}^{\infty} n^2 B_n(\widetilde{\eta})\Big\{\frac{\Big[\tfrac{1}{2}(1-n)^2-(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)^2\Big]C\cos[\zeta_1-n\zeta_2-(1-n)\tau]}{(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}$$

$$-\frac{\Big[\tfrac{1}{2}(1-n)^2-(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)^2\Big]\cos[\zeta-n\zeta_2-(1-n)\tau]}{(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\Big\}$$

$$+\{\tfrac{1}{8}C^2\sin\alpha\cos\alpha-4(\tfrac{1}{2}-\sin^2\alpha)E_2(0)\}\cos 2(\zeta_1-\tau)$$

$$-\cos\alpha\sum_1^{\infty}\Big\{\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}} n^2 D_n(\widetilde{\eta}) + n\,B_n'(\widetilde{\eta})\Big\}\cos n(\zeta_2-\tau)$$

$$-(\tfrac{1}{2}-\sin^2\alpha)\{E_1(\widetilde{\eta})\cos(\zeta_1-\tau)+F_1(\widetilde{\eta})\cos(\zeta-\tau)\}\Big]\,,$$

(16.8)

$$\frac{\overline{q}_{22}}{2\mu} = \epsilon\left[\sin\alpha\,\cos\alpha\; C\sin(\zeta_1-\tau) - (\tfrac{1}{2}-\cos^2\alpha)\sum_1^\infty n^2 B_n(\tilde{\eta})\sin n(\zeta_2-\tau)\right]$$

$$+\epsilon^2\left[-\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)-\frac{1}{4}\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2}-1\right)\cos\alpha\sum_{n=-\infty}^{\infty} n^2(1-n)B_n(\tilde{\eta})\right]$$

$$\times\left\{\frac{C(\sin\alpha - n\frac{c_2}{c_1}\sin\beta)\cos[\zeta_1-n\zeta_2-(1-n)\tau]}{(\sin\alpha-n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right.$$

$$\left.+\frac{(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)\cos[\zeta-n\zeta_2-(1-n)\tau]}{(\sin\alpha+n\frac{c_2}{c_1}\sin\beta)^2-(1-n)^2\sin^2\alpha}\right\}$$

$$+\left\{\frac{1}{8}C^2(\tfrac{1}{2}-\cos^2\alpha)+4\,E_2(0)\sin\alpha\,\cos\alpha\right\}\cos 2(\zeta_1-\tau)$$

$$+\frac{1}{4}\left(\frac{c_1^2}{c_2^2}-1\right)\left(\tfrac{1}{2}-\cos^2\alpha\right)\frac{c_2^2}{c_1^2}\;\frac{\cos 2(\xi\cos\alpha-\tau)}{\frac{c_2^2}{c_1^2}-\cos^2\alpha}$$

$$-(\tfrac{1}{2}-\cos^2\alpha)\sum_1^\infty n^2\,D_n(\tilde{\eta})\cos n(\zeta_2-\tau)$$

$$+\sin\alpha\,\cos\alpha\,\{E_1(\tilde{\eta})\cos(\zeta_1-\tau)-F_1(\tilde{\eta})\cos(\zeta-\tau)\}\,. \qquad (16.9)$$

For comparison, we include here the ϵ-expansions of the Lagrange stresses $\overset{*}{q}_{11}, \overset{*}{q}_{12}, \overset{*}{q}_{21}, \overset{*}{q}_{22}$ associated with the incident wave. On using (8.1), (8.3), (8.4), (7.7) in (7.33) and expanding in powers of ϵ we find that

$$\frac{\overset{*}{q}_{11}}{2\mu} = -\epsilon\sin\alpha\,\cos\alpha\,\sin(\zeta-\tau)+\epsilon^2\left\{\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)+\frac{1}{8}(\tfrac{1}{2}-\sin^2\alpha)\cos 2(\zeta-\tau)\right\}$$

$$+O(\epsilon^3)\;, \qquad (16.10)$$

$$\frac{q^*_{12}}{2\mu} = -\epsilon(\tfrac{1}{2} - \cos^2\alpha)\sin(\zeta-\tau) + \epsilon^2 \sin\alpha\cos\alpha\,\frac{1}{8}\cos 2(\zeta-\tau) + O(\epsilon^3) , \qquad (16.11)$$

$$\frac{q^*_{21}}{2\mu} = \epsilon(\tfrac{1}{2}-\sin^2\alpha)\sin(\zeta-\tau) + \epsilon^2 \sin\alpha\cos\alpha\,\frac{1}{8}\cos 2(\zeta-\tau) + O(\epsilon^3) , \qquad (16.12)$$

$$\frac{q^*_{22}}{2\mu} = \epsilon\sin\alpha\cos\alpha\sin(\zeta-\tau) + \epsilon^2\left\{\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1\right)+\frac{1}{8}(\tfrac{1}{2}-\cos^2\alpha)\cos 2(\zeta-\tau)\right\} + O(\epsilon^3) \qquad (16.13)$$

One can verify, by using (13.14), (13.15), that $\bar{q}_{12}+q^*_{12} = 0$, $\bar{q}_{22}+q^*_{22}=0$ when $\eta = 0$, that is, that the condition that there be zero surface tractions is satisfied to order ϵ^2. Clearly, we also have

$$q^*_{13} = q^*_{23} = q^*_{31} = q^*_{32} = \bar{q}_{13} = \bar{q}_{23} = \bar{q}_{31} = \bar{q}_{32} = 0 . \qquad (16.14)$$

For the normal Lagrange stress q_{33} we have no expression linear in ϕ and ψ. We use (6.1), (8.1), (8.7), (8.8) in (6.16) to obtain the ϵ-expansion of the normal Lagrange stress q_{33} generated by the interaction of the incident and reflected waves. On expanding we find that

$$\begin{aligned}\frac{q_{33}}{2\mu} = {} & \epsilon\left[\tfrac{1}{2}\left(\frac{c_1^2}{c_2^2}-2\right)\left(\Delta\bar{\phi}_o + \phi^*_{o_{\zeta\zeta}}\right)\right] \\ & + \epsilon^2\Big[\tfrac{1}{2}\left(\frac{c_1^2}{c_2^2}-2\right)\left(\Delta\bar{\phi}_1 + 2\bar{\phi}_{o_{\eta\eta}} + \phi^*_{1_{\zeta\zeta}}\right) + \frac{1}{8}\left(\frac{c_1^2}{c_2^2}-2-\frac{b}{\mu}\right)\left(\Delta\bar{\psi}_o + \psi^*_{o_{\zeta\zeta}}\right)^2 \\ & + \frac{1}{4}\left(4k_2 + \frac{c_1^2}{c_2^2}-1\right)\left(\Delta\bar{\phi}_o + \phi^*_{o_{\zeta\zeta}}\right)^2 + \frac{b}{2\mu}\{[(\bar{\phi}_o+\phi^*_o)_{\xi\xi} - (\bar{\psi}_o+\psi^*_o)_{\xi\eta}][(\bar{\phi}_o+\phi^*_o)_{\eta\eta} + (\bar{\psi}_o+\psi^*_o)_{\xi\eta}] \\ & - [(\bar{\phi}_o+\phi^*_o)_{\xi\eta} + (\bar{\psi}_o+\psi^*_o)_{\xi\xi}][(\bar{\phi}_o+\phi^*_o)_{\xi\eta} - (\bar{\psi}_o+\psi^*_o)_{\eta\eta}]\}\Big] + O(\epsilon^3) ,\end{aligned}$$

(16.15)

where b is one of the two constants a and b which appear in the definition (4.11) of the strain energy density W and which are related by the second of (4.17).

On substituting for the starred quantities from (8.2), (8.3) and for the barred quantities from §15 we find that

$$
\begin{aligned}
\frac{q_{33}}{2\mu} = & -\epsilon\,\tfrac{1}{2}\left(\frac{c_1^2}{c_2^2}-2\right)\frac{c_2^2}{c_1^2}\sum_1^{\infty} n^2 B_n(\tilde{\eta})\sin n(\zeta_2-\tau) \\
& +\epsilon^2\Big[\tfrac{1}{2}\left(\frac{c_1^2}{c_2^2}-2\right)\Big\{-\tfrac{1}{8}C^2-\tfrac{1}{4}\left(\frac{c_2}{c_1}\right)^6\left(4k_2+\frac{c_1^2}{c_2^2}-1\right)\sum_1^{\infty} n^4 B_n^2(\tilde{\eta}) \\
& +\tfrac{1}{8}C^2\cos 2(\zeta_1-\tau)+\tfrac{1}{4}\left(\frac{c_1^2}{c_2^2}-1\right)\frac{c_2^2}{c_1^2}C\,\frac{\cos^2\alpha\,\cos 2(\xi\cos\alpha-\tau)}{\dfrac{c_2^2}{c_1^2}-\cos^2\alpha} \\
& +\tfrac{1}{4}\frac{c_2^2}{c_1^2}C\cos(2\eta\sin\alpha)+\tfrac{1}{8}\cos 2(\zeta-\tau) \\
& -\sum_1^{\infty}\left(\frac{c_2^2}{c_1^2}n^2 D_n(\tilde{\eta})+2\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}} n B_n'(\tilde{\eta})\right)\cos n(\zeta_2-\tau)\Big\} \\
& +\tfrac{1}{8}\left(\frac{c_1^2}{c_2^2}-2-\frac{b}{2\mu}\right)\{C\sin(\zeta_1-\tau)-\sin(\zeta-\tau)\}^2 \\
& +\tfrac{1}{4}\left(4k_2+\frac{c_1^2}{c_2^2}-1\right)\left(\frac{c_2}{c_1}\right)^4\Big\{\sum_1^{\infty} n^2 B_n(\tilde{\eta})\sin n(\zeta_2-\tau)\Big\}^2 \\
& +\frac{b}{2\mu}\Big\{-4\sin^2\alpha\cos^2\alpha\, C\sin(\zeta_1-\tau) \\
& +\Big\{\Big[\cos\alpha\sin\alpha\left(\frac{c_2^2}{c_1^2}-2\cos^2\alpha\right)-(\sin^2\alpha-\cos^2\alpha)\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}}\cos\alpha\Big]C\sin(\zeta_1-\tau) \\
& +\Big[\cos\alpha\sin\alpha\left(\frac{c_2^2}{c_1^2}-2\cos^2\alpha\right)+(\sin^2\alpha-\cos^2\alpha)\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}}\cos\alpha\Big]\sin(\zeta-\tau)\Big\} \\
& \times\sum_1^{\infty} n^2 B_n(\tilde{\eta})\sin n(\zeta_2-\tau)\Big\} + O(\epsilon^3). \qquad (16.16)
\end{aligned}
$$

The Lagrange stress $\overset{*}{q}_{33}$ corresponding to the incident wave alone has the ϵ-expansion

$$\frac{\overset{*}{q}_{33}}{2\mu} = \epsilon^2\left[\frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 2 - \frac{b}{\mu}\right) + \frac{b}{16\mu}\cos 2(\zeta - \tau)\right] + O(\epsilon^3) . \tag{16.17}$$

§17. Nonlinear Effects on the Reflection Pattern

In (16.3) and (16.4) the features most immediately visible are the infinity of propagation-directions and, respectively, the presence of terms proporational to ξ and η in the expressions for $\bar{u}$ and $\bar{v}$. A discussion of the propagation-directions is provided below. We begin by commenting on the presence of the terms linear in ξ and η, corresponding to uniform extensions in the X_1 and X_2 directions, respectively.

In §7 we made the observation concerning (7.33) that the Lagrange stresses $\overset{*}{q}_{\alpha\beta}(\alpha, \beta=1, 2)$, generated by the incident wave, are the sum

$$\overset{*}{q}_{\alpha\beta} = \hat{q}_{\alpha\beta} + 2\mu(G(r^*)-1)\delta_{\alpha\beta} , \quad (\alpha, \beta = 1, 2) \tag{17.1}$$

of a periodic Lagrange stress system $\hat{q}_{\alpha\beta}$ and a uniform hydrostatic stress of amount $2\mu(G(\overset{*}{r})-1)$. The ϵ-expansions (16.10)-(16.13) show that both $\frac{\overset{*}{q}_{11}}{2\mu}$ and $\frac{\overset{*}{q}_{22}}{2\mu}$ contain the constant term $\epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)$. The boundary condition (8.9) is a statement of the requirement that the reflected wave cancel the traction generated on the surface of the half-space by the incident wave. The condition of zero surface traction is that

$$\left.\begin{aligned} q_{12} &\equiv q^*_{12} + \bar{q}_{12} = 0 \\ q_{22} &\equiv q^*_{22} + \bar{q}_{22} = 0 \end{aligned}\right\} \text{ when } \eta = 0 . \tag{17.2}$$

It follows that $\frac{\bar{q}_{22}}{q\mu}$ must contain the constant term $-\epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)$, as indeed, by (16.9), it does. There is no requirement that the constant term $\epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)$ in $\frac{q^*_{11}}{2\mu}$ be cancelled. Our $\frac{\bar{q}_{11}}{2\mu}$ contains the term $\epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1\right)$ and so $q_{11} \equiv q^*_{11} + \bar{q}_{11}$ contains the term $\epsilon^2 \frac{1}{8}\left(\frac{c_1^2}{c_2^2} - 1\right)$. The resultant Lagrange stress component q_{11} thus has a term corresponding to a second order uniaxial tension of magnitude $\epsilon^2 \frac{1}{4}\mu\left(\frac{c_1^2}{c_2^2} - 1\right)$ in the X_1 direction, while q_{22} has no such term. From (16.17) it can be shown that the resultant Lagrange stress q_{33} contains the constant term of second order

$$\epsilon^2\left\{\frac{\mu}{8}\left(\frac{c_1^2}{c_2^2} - 2\right) - \frac{b}{16}(1 + C^2)\right\}$$

where b is a material constant (see (4.11), (4.17)). Whether the uniform stress in q_{33} is a tension or a compression depends on the value of the second order material constant b.

The presence of uniform stresses in the (infinite) body gives rise to displacements which are unbounded for large values of the space variables. This, as well as the fact that no displacement occurs in the X_3-direction, explains the presence of terms proportional to ξ, η in (16.3, (16.4) respectively.

We examine now the propagation-directions of the terms occurring in the expressions (16.3), (16.4) for the displacements $\bar{u}$, $\bar{v}$ associated with the reflected wave. We recall here that

$$\left.\begin{aligned} \zeta &= \xi \cos\alpha + \eta \sin\alpha \quad , \\ \zeta &= \xi \cos\alpha - \eta \sin\alpha \quad , \\ \zeta_2 &= \xi \cos\alpha - \eta \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha \right)^{\frac{1}{2}} \end{aligned}\right\} \qquad (17.3)$$

and that the incident wave propagates in the positive ζ-direction while the positive ζ_1- and ζ_2- directions are the directions of propagation of the shear and dilatation waves respectively of the classical linear theory.

Equations (16.3) and (16.4) show that both $\bar{u}$ and $\bar{v}$ contain terms propagating in the positive ζ, ζ_1 and ζ_2 directions. The term proportional to $\sin 2(\xi \cos\alpha - \tau)$ in (16.3) propagates in the positive X_1-direction while the terms proportional to $\eta \sum_1^\infty B_n^2(\tilde{\eta})$ and to $\sin(2\eta \sin\alpha)$ in (16.4) do not propagate. From (12.33) it follows that $\eta \sum_1^\infty n^4 B_n^2(\tilde{\eta})$ is bounded for large η. We showed earlier that

$$F_1(\tilde{\eta}) = 0\left(\frac{1}{\tilde{\eta}}\right) \text{ as } \tilde{\eta} \to -\infty \quad ,$$

so the terms proportional to $F_1(\tilde{\eta}) \sin(\zeta - \tau)$ do not represent incoming waves at $\tilde{\eta} = -\infty$.

We now examine the terms in (16.3), (16.4) with arguments $\zeta_1 - n\zeta_2 - (1-n)\tau$ and $\zeta - n\zeta_2 - (1-n)\tau$, $n = \pm 1, \pm 2, \pm 3, \ldots$. By (17.3) we have

$$\zeta_1 - n\zeta_2 - (1-n)\tau = \begin{cases} \left[\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} - \sin\alpha\right]\eta, & n = 1, \\ (1-n)\left[\xi\cos\alpha - \dfrac{\sin\alpha - n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{1-n}\eta - \tau\right], & n \neq 1, \end{cases} \tag{17.4}$$

$$\zeta - n\zeta_2 - (1-n)\tau = \begin{cases} \left[\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} + \sin\alpha\right]\eta, & n = 1, \\ (1-n)\left[\xi\cos\alpha - \dfrac{\sin\alpha + n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{n-1} - \tau\right], & n \neq 1 \end{cases} \tag{17.5}$$

We note that the following statements concerning the quantities

$$\frac{1}{1-n}\left\{\sin\alpha - n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}, \quad \frac{1}{n-1}\left\{\sin\alpha + n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\} \text{ hold:}$$

$\frac{1}{1-n}\left\{\sin\alpha - n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}$ increases from $\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}$ to $\frac{1}{2}\left[\sin\alpha + \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right]$ monotonically as n goes from $-\infty$ to -1, (17.6)

$\frac{1}{1-n}\left\{\sin\alpha - n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}$ increases from $2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} - \sin\alpha$ to $\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)$ monotonically as n goes from 2 to ∞, (17.7)

$\frac{1}{n-1}\left\{\sin\alpha + n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}$ decreases from $\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}$ to $-\frac{1}{2}\left\{\sin\alpha - \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}$ monotonically as n goes from $-\infty$ to -1, (17.8)

$$\left.\begin{array}{l}\dfrac{1}{n-1}\left\{\sin\alpha + n\left(\dfrac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\} \text{ decreases from } \sin\alpha + 2\left(\dfrac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} \\ \text{to } \left(\dfrac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} \text{ monotonically as } n \text{ goes from } 2 \text{ to } \infty.\end{array}\right\} \quad (17.9)$$

Thus, for functions with arguments $\zeta_1 - n\zeta_2 - (1-n)\tau$ and $\zeta - n\zeta_2 - (1-n)\tau$, n an integer not equal to 1 or 0, the respective propagation directions are those of the vectors.

$$\left(\cos\alpha, \ -\frac{1}{n-1}\left\{n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} - \sin\alpha\right\}\right), \left(\cos\alpha, -\frac{1}{n-1}\left\{n\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} + \sin\alpha\right\}\right).$$

The collection of all these vectors, where n takes on all integer values except 1 or 0, forms a fan, the two outermost rays of which have the directions of the vectors

$$\left(\cos\alpha, \ \max\left[\sin\alpha - 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}, \ \tfrac{1}{2}\left\{\sin\alpha - \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}\right]\right) \quad (17.10)$$

and

$$\left(\cos\alpha, \ -\left\{\sin\alpha + 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}\right) \ .$$

Equations (17.6) - (17.9) also show that, for sufficiently large $|n|$, the directions of these vectors are clustered arbitrarily close to that of the vector $\left(\cos\alpha, \ -\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right)$, that is, to the direction of propagation of the reflected dilatation wave in the linear theory.

In reference to (17.10), we have

$$\sin\alpha - 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} - \tfrac{1}{2}\left\{\sin\alpha - \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\} = \tfrac{1}{2}\left\{\sin\alpha - 3\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\} , \tag{17.11}$$

which can be shown to be a decreasing function of α, $0 < \alpha \leq \frac{\pi}{2}$. Since in this problem we consider only those values of α such that

$$\frac{\pi}{4} < \alpha_o < \alpha \leq \frac{\pi}{2} , \qquad \alpha_o = \arccos \frac{c_2}{c_1} ,$$

it follows that the right hand side of (17.11) lies between $\frac{1}{2}\sin\alpha_o$ and $\frac{1}{2}\left(1-3\frac{c_2}{c_1}\right)$. If

$$\frac{c_2}{c_1} < \frac{1}{3}$$

then, in (17.10), the maximum is $\sin\alpha - 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}$; if $\frac{c_2}{c_1} > \frac{1}{3}$ then there is a value of α_1 of α, $\frac{\pi}{4} < \alpha_o < \alpha_1 < \frac{\pi}{2}$, such that, if $\alpha_o < \alpha < \alpha_1$, then $\sin\alpha - 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}$ is the maximum while, if $\alpha_1 < \alpha \leq \frac{\pi}{2}$, then $\frac{1}{2}\left\{\sin\alpha - \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right\}$ is the maximum, with equality when $\alpha = \alpha_1$.

From (17.10), (16.3), (16.4) and the last paragraph it is clear that both $\bar{u}$ and $\bar{v}$ contain a finite number of terms which propagate in an "upward" direction, no matter what the values of α or of $\frac{c_2}{c_1}$. However, such terms do not represent incoming waves at $\eta = -\infty$ since each one of them is multiplied by some member of the sequence $\{B_n(\tilde{\eta})\}$ and, as we saw in §12, $B_n(\tilde{\eta}) = 0\left(\frac{1}{n^3\tilde{\eta}}\right)$, uniformly in n, as $\tilde{\eta} \to -\infty$.

The situation is illustrated by Figures 4(a), (b), (c) which show (to second order in ϵ) the directions of the various waves passing observers stationed near the surface, in the interior, and at depth of order $1/\epsilon^2$ in the half-space, respectively. Rays with two arrows indicate that both first and second order quantities are propagated in their directions. Rays with one arrow indicate that only second order quantities are propagated. The angles $\beta, \gamma, \delta, \theta$ are given by

$$\tan\beta = \frac{\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{\cos\alpha}, \quad \tan\gamma = \frac{\sin\alpha + \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{2\cos\alpha},$$

$$\tan\delta = \max\left[\frac{\sin\alpha - 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{\cos\alpha}, \frac{\sin\alpha - \left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{2\cos\alpha}\right]$$

$$\tan\theta = \frac{\sin\alpha + 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{\cos\alpha} < 3\tan\alpha .$$

The direction corresponding to the angle β is, of course, the direction of propagation of the reflected dilatation wave of the classical linear theory.

§18. Nonlinear Effects on the Surface Displacements

In this section we write down the total surface displacements and use them to obtain the qualitative features of the orbits described by the surface particles.

We first note that, by (17.3),

$$\zeta = \zeta_1 = \zeta_2 = \xi\cos\alpha \quad \text{when} \quad \eta = 0 , \tag{18.1}$$

and recall, from (11.14) and (15.11), that

$$B_n(0) \equiv - B_n(0) = 0, \quad n = 2, 3, \ldots \quad . \tag{18.2}$$

The total surface displacements are found by adding the displacements corresponding to the incident and reflected waves and by setting $\eta = 0$ in the resulting expressions. Let u^*, v^* be the displacements associated with the incident wave. From (8.1)-(8.3) we have

$$\left.\begin{aligned} u^* &= \frac{L}{2\pi}(\cos\alpha \ \phi^*_\zeta - \sin\alpha \ \psi^*_\zeta) \ , \\ v^* &= \frac{L}{2\pi}(\sin\alpha \ \phi^*_\zeta + \cos\alpha \ \psi^*_\zeta) \ , \end{aligned}\right\} \tag{18.3}$$

and u^*, v^* can be shown to have the ϵ-expansions

$$u^* = \epsilon \frac{L}{2\pi} \sin\alpha \{\cos(\zeta-\tau)-1\} + \epsilon^2 \frac{L}{2\pi} \frac{1}{16} \cos\alpha \sin 2(\zeta-\tau) + O(\epsilon^3) \ ,$$

$$v^* = -\epsilon \frac{L}{2\pi} \cos\alpha \{\cos(\zeta-\tau)-1\} + \epsilon^2 \frac{L}{2\pi} \frac{1}{16} \sin\alpha \sin 2(\zeta-\tau) + O(\epsilon^3) \ .$$

Since we have, up to now, consistently ignored the arbitrary rigid-body translations which have appeared at various stages in this work there is no loss of generality in taking

$$\frac{2\pi}{L} u^* = \epsilon \sin\alpha \cos(\zeta-\tau) + \epsilon^2 \frac{1}{16} \cos\alpha \sin 2(\zeta-\tau) + O(\epsilon^3) \ , \tag{18.4}$$

$$\frac{2\pi}{L} v^* = -\epsilon \cos\alpha \cos(\zeta-\tau) + \epsilon^2 \frac{1}{16} \sin\alpha \sin 2(\zeta-\tau) + O(\epsilon^3) \ . \tag{18.5}$$

We now obtain the total surface displacements by adding (18.4) and (16.3), adding (18.5) and (16.4), and then setting $\eta = 0$ in $u(\xi, \eta, \tilde{\eta}, \tau; \epsilon)$, $v(\xi, \eta, \tilde{\eta}, \tau; \epsilon)$. We find that

$$\frac{2\pi}{L}\, u(\xi, 0, 0, \tau;\epsilon) = \epsilon\, P \cos(\xi \cos\alpha - \tau) + \epsilon^2 \Big[R \sin(\xi \cos\alpha - \tau) + T \sin 2(\xi \cos\alpha - \tau) + \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1 - C^2\right)\xi \Big] + O(\epsilon^3), \quad (18.6)$$

$$\frac{2\pi}{L}\, v(\xi, 0, 0, \tau;\epsilon) = \epsilon\, Q \cos(\xi \cos\alpha - \tau) + \epsilon^2 \left[S \sin(\xi\cos\alpha - \tau) + U \sin 2(\xi \cos\alpha - \tau)\right] + O(\epsilon^3), \quad (18.7)$$

where

$$\left.\begin{aligned}
P &= -\tfrac{1}{2}\, \frac{\sin\alpha}{\cos^2\alpha - \tfrac{1}{2}}\,(C+1) \;, \\
Q &= -\frac{1}{2\cos\alpha}\,(1-C) \;, \\
R &= -\frac{1}{2\cos\alpha}\, D_1(0) \;, \\
S &= -\frac{\left(\dfrac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{2(\cos^2\alpha - \tfrac{1}{2})}\, D_1(0) \;, \\
T &= \frac{1}{8}\,\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2} - 1\right)\cos\alpha\, C + \frac{1}{16}(C^2+1)\cos\alpha - 2E_2(0)\sin\alpha - 2\cos\alpha\, D_2(0) \\
&\quad + \frac{1}{4}\,\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2} - 1\right) B_1(0) \left\{ \frac{C\left(\sin\alpha + \dfrac{c_2}{c_1}\sin\beta\right)}{\left(\sin\alpha + \dfrac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha} + \frac{\sin\alpha - \dfrac{c_2}{c_1}\sin\beta}{\left(\sin\alpha - \dfrac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha} \right\}, \\
U &= \frac{1}{8}\, B_1(0) K + 2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} D_2(0) - 2E_2(0)\cos\alpha - \frac{1}{16}(C^2-1)\sin\alpha \\
&\quad - \tfrac{1}{2}\,\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2} - 1\right)\cos\alpha\, B_1(0) \left\{ \frac{C}{\left(\sin\alpha + \dfrac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha} - \frac{1}{\left(\sin\alpha - \dfrac{c_2}{c_1}\sin\beta\right)^2 - 4\sin^2\alpha} \right\}
\end{aligned}\right\}$$

(18.8)

One can show, from (15.8), (15.13), (15.19), that

$$D_1(0) = -\frac{1}{4}\,\frac{\cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{\cos^2\alpha - \frac{1}{2}}\, B_1(0)(C+1)\,\frac{\left\{4k_2 - \left(\frac{c_1^2}{c_2^2} - 1\right)^2\right\}}{4k_2 + \frac{c_1^2}{c_2^2} - 1} ,$$

by deducing from (12.12), (12.13), (12.4), (12.1) that

$$\int_{-\infty}^{0} \sum_{1}^{\infty} k^4 B_k^2(\sigma)\, d\sigma = \frac{4}{|K|}\, B_1^2(0) .$$

According to the linear theory, each of the surface particles performs a harmonic oscillation in a straight line about the position it occupied in the undisturbed state. This straight line makes with the horizontal axis an angle whose tangent is

$$\frac{Q}{P} = \frac{\cos\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{\cos^2\alpha - \frac{1}{2}} ,$$

which is negative for all α lying in the range $\alpha_o < \alpha < \frac{\pi}{2}$ that we consider. This angle tends to zero as the angle of incidence approaches either $\frac{\pi}{2}$ or the critical value α_o. For each particle the amplitude of its oscillation is

$$\epsilon(P^2+Q^2)^{\frac{1}{2}} = \epsilon\,\frac{\sin\alpha\left\{\frac{1}{4} - \left(1 - \frac{c_2^2}{c_1^2}\right)\cos^2\alpha\right\}^{\frac{1}{2}}}{(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} .$$

We now consider the question of determining the orbits of surface particles to order ϵ^2. In the undeformed state let an arbitrary particle of the surface have coordinates $(X_o, 0) \equiv \left(\frac{L}{2\pi}\xi_o, 0\right)$ with respect to the frame X. Then, according to (18.6), (18.7), this particle has coordinates (x_o, y_o) with respect to X in the deformed state, where

$$x_o = X_o + \epsilon \frac{L}{2\pi} P \cos(\xi_o \cos\alpha - \tau) + \epsilon^2 \frac{L}{2\pi} \Big[R \sin(\xi_o \cos\alpha - \tau) + T \sin 2(\xi_o \cos\alpha - \tau) + \frac{1}{16}\Big(\frac{c_1^2}{c_2^2} - 1 - C^2\Big)\xi_o \Big] + O(\epsilon^3) , \qquad (18.9)$$

$$y_o = \epsilon \frac{L}{2\pi} Q \cos(\xi_o \cos\alpha - \tau) + \epsilon^2 \frac{L}{2\pi} [S \sin(\xi_o \cos\alpha - \tau) + U \sin 2(\xi_o \cos\alpha - \tau)] + O(\epsilon^3) . \qquad (18.10)$$

It proves convenient to introduce the rectangular Cartesian frame X' obtained from X by translating the origin to the point $\left(X_o\left\{1 + \epsilon^2 \frac{1}{16}\Big(\frac{c_1^2}{c_2^2} - 1 - C^2\Big)\right\}, 0\right)$ and then rotating the plane through the angle $\arctan \frac{Q}{P}$. The particle then has coordinates x_o', y_o' with respect to the frame X' where

$$x_o' = \epsilon a_1 \cos(\xi_o \cos\alpha - \tau) + \epsilon^2 \{a_2 \sin(\xi_o \cos\alpha - \tau) + a_3 \sin 2(\xi_o \cos\alpha - \tau)\} + O(\epsilon^3) , \qquad (18.11)$$

$$y_o' = \epsilon^2 a_4 \sin 2(\xi_o \cos\alpha - \tau) + O(\epsilon^3) , \qquad (18.12)$$

with

$$\left.\begin{aligned} a_1 &= \frac{L}{2\pi}(P^2+Q^2)^{\frac{1}{2}} , & a_2 &= \frac{L}{2\pi}(P^2+Q^2)^{-\frac{1}{2}}(PR+QS) , \\ a_3 &= \frac{L}{2\pi}(P^2+Q^2)^{-\frac{1}{2}}(PT+QU) , & a_4 &= \frac{L}{2\pi}(P^2+Q^2)^{-\frac{1}{2}}(PU-QT) . \end{aligned}\right\} \qquad (18.13)$$

A term proportional to $\epsilon^2 \sin(\xi_o \cos\alpha - \tau)$ does not appear in y_o' because, as can be seen from (18.5),

$$PS = QR = \frac{\frac{1}{2}\sin\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}} D_1(0)}{(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} . \tag{18.14}$$

Thus, by (18.12), (18.13), the particle orbit is, correct to order ϵ^2, a closed plane curve which intersects itself once if $a_4 \neq 0$. If $a_4 = 0$, then the particle oscillates along the straight line $y_o' = 0$, which is the line along which surface particles oscillate according to the classical linear theory. From (18.13), a_4 is zero if and only if PU - QT is zero.

The expression for TQ-UP is lengthy and does not seem to permit any significant simplification. One can show that k_2 is the only higher order material constant appearing in TQ, UP and that this constant appears in TQ and UP only through the additive terms

$$\frac{1}{16} B_1(0) K \quad \frac{\sin\alpha\,(\cos^2\alpha - \frac{1}{2})^2}{\left[(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right]^2}$$

and

$$-\frac{1}{16} B_1(0) K \quad \frac{\sin^2\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}}{\left[(\cos^2\alpha - \frac{1}{2})^2 + \sin\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}\right]^2}$$

respectively where, writing (15.8) again,

$$K = -\frac{\left(\frac{c_2}{c_1}\right)^6 \left(4\,k_2 + \frac{c_1^2}{c_2^2} - 1\right) B_1(0)}{2\left(\frac{c_2^2}{c_1^2} - \cos^2\alpha\right)^{\frac{1}{2}}} .$$

Thus k_2 appears in TQ-UP only through the additive term

$$-\frac{\left(\frac{c_2}{c_1}\right)^6\left(4\,k_2 + \frac{c_1^2}{c_2^2} - 1\right)\sin\alpha\;B_1^2(0)}{32\left[(\cos^2\alpha-\frac{1}{2})^2+\sin\alpha\cos^2\alpha\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}}\right]\left(\frac{c_2^2}{c_1^2}-\cos^2\alpha\right)^{\frac{1}{2}}}\,.$$

So, except in the unlikely instance that the value of k_2 is the same for all harmonic materials and is precisely that for which TQ-UP is identically zero, there are at least some materials for which a_4 is not identically zero. The same remark applies to a_3.

From (18.14), (18.15) and the expression for $D_1(0)$ following (18.8) it is clear that a_2 has

$$\frac{4\,k_2 - \left(\frac{c_1^2}{c_2^2} - 1\right)^2}{4\,k_2 + \frac{c_1^2}{c_2^2} - 1}$$

as a factor and hence indeed depends on the material under consideration.

The nonlinear effects on the orbits of surface particles are thus of two kinds. One is a translation of amount $\epsilon^2\frac{1}{16}\left(\frac{c_1^2}{c_2^2}-1-C^2\right)X_o$ in the horizontal direction where $(X_o, 0)$ is the position of the particle with respect to the frame X in the underformed state. The other nonlinear effect is that the particle describes the orbit given by (18.11), (18.12) where x_o', y_o' are the coordinates of the particle with respect to a rectangular Cartesian frame X'. This frame has its origin at the point $\left(\left\{1 + \epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 1 - C^2\right)\right\} X_o, 0\right)$ (with respect to X) and has its X_1', X_2' directions parallel and perpendicular to the line along which oscillation takes place in the linear theory. We have, in Figure 5, sketched the orbit for the case $a_3 > 0$, $a_4 > 0$, $\frac{a_2}{a_3} > 2$. The orbit intersects itself at the point O' whose position with respect to the frame X' is $(-\epsilon^2(a_2 - a_3), 0)$ correct to second order in ϵ. Other cases are similar.

§19. The Case of Normal Incidence $\alpha = \frac{\pi}{2}$

As a particular result, we now write out the displacements for the case of normal incidence, i.e. for the case $\alpha = \frac{\pi}{2}$.

From (18.4), (18.5) the displacements u^*, v^* associated with the incident wave are now

$$u^* = \epsilon \frac{L}{2\pi} \cos(\eta - \tau) , \tag{19.1}$$

$$v^* = -\epsilon^2 \frac{L}{2\pi} \frac{1}{16} \sin 2(\eta - \tau) + O(\epsilon^2) . \tag{19.2}$$

To order ϵ, these are the displacements of the shear wave of the classical linear theory.

Since, by (15.7), $B_1(0)$ is now equal to zero, we find from (15.4) that

$$B_n(\tilde{\eta}) \equiv 0 , \quad n = 1, 2, \ldots\ldots . \tag{19.3}$$

Equations (15.12), (15.13) and the first of (15.19) then imply that

$$E_1(\tilde{\eta}) = F_1(\tilde{\eta}) \equiv 0 . \tag{19.4}$$

By (15.18) and the first of (15.19),

$$D_1(0) = 0, \qquad D_2(0) = -2V_1(\tfrac{\pi}{2}) = \frac{1}{16}\frac{c_1^2}{c_2^2} \quad . \tag{19.5}$$

By (19.3) and the definition of $b(\theta, \tilde{\eta})$, we have

$$b(\theta, \tilde{\eta}) \equiv 0$$

when $\alpha = \frac{\pi}{2}$. On noting from (19.3), (15.8) that $K = 0$ when $\alpha = \frac{\pi}{2}$, the problem (15.15), (15.16) is seen to have the solution

$$d(\theta, \tilde{\eta}) = \frac{1}{8}\frac{c_1^2}{c_2^2} \sin 2\theta, \quad 0 < \theta < \pi, \ \tilde{\eta} < 0 \quad , \tag{19.6}$$

from which it follows, by (15.14), that

$$D_2(\tilde{\eta}) = \frac{1}{16}\frac{c_1^2}{c_2^2}, \qquad D_n(\tilde{\eta}) \equiv 0, \ n = 1, \ n=3, 4, 5, \ldots . \tag{19.7}$$

Finally, from (15.19), (15.20) we have

$$E_2(0) = -2V_2(\tfrac{\pi}{2}) = 0 \quad . \tag{19.8}$$

Using all of the above in (16.3), (16.4) we obtain the following expressions for $\bar{u}$, $\bar{v}$, the displacements associated with the reflected wave for the case of normal incidence:

$$\frac{2\pi}{L}\bar{u} = \epsilon\cos(\eta+\tau) + \epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 2\right)\xi + O(\epsilon^3) \ , \tag{19.9}$$

$$\frac{2\pi}{L}\bar{v} = \epsilon^2\left[-\frac{1}{16}\frac{c_1^2}{c_2^2}\eta - \frac{1}{8}\frac{c_1}{c_2}\sin 2\left(\frac{c_2}{c_1}\eta + \tau\right) + \frac{1}{16}\sin 2(\eta+\tau) + \frac{1}{8}\frac{c_2^2}{c_1^2}\left(\frac{c_1^2}{c_2^2} - 1\right)\sin 2\eta\right] + O(\epsilon^3) \ . \tag{19.10}$$

The total surface displacements $u(\xi, 0, 0, \tau;\epsilon)$, $v(\xi, 0, 0, \tau;\epsilon)$ are obtained by adding u^* and $\bar{u}$, v^* and $\bar{v}$ respectively and setting $\eta = 0$. On using (19.1), (19.2), (19.9), (19.10) or, alternatively, setting $\alpha = \frac{\pi}{2}$ in (18.6), (18.7) we find that, in the case of normal incidence,

$$\frac{2\pi}{L} u(\xi, 0, 0, \tau;\epsilon) = 2\epsilon \cos\tau + \epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 2\right)\xi + O(\epsilon^3), \tag{19.11}$$

$$\frac{2\pi}{L} v(\xi, 0, 0, \tau;\epsilon) = \epsilon^2 \frac{1}{16}\left(1 - \frac{1}{8}\frac{c_1}{c_2}\right) \sin 2\tau + O(\epsilon^3) \quad . \tag{19.12}$$

Equations (19.11), (19.12) show that, for the case of normal incidence, the paths described by the surface particles can be viewed as follows. If a particle occupies the point $(X_o, 0)$ with respect to the frame X in the undeformed state then, in the deformed state, it first suffers a translation of amount $\epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 2\right) X_o$. It then describes the orbit given by the parametric equations

$$x_o' = \epsilon \cdot \frac{L}{\pi} \cos\tau \,, \quad y_o' = \epsilon^2 \cdot \frac{1}{16}\left(1 - \frac{1}{8}\frac{c_1}{c_2}\right)\frac{L}{2\pi} \sin 2\tau \tag{19.13}$$

where x_o', y_o' are the coordinates of the particle, measured in the X_1, X_2 directions respectively, relative to the point $\left(\left\{1 + \epsilon^2 \frac{1}{16}\left(\frac{c_1^2}{c_2^2} - 2\right)\right\} X_o, 0\right)$. The orbit given by (19.13) is symmetric about this point which is the only point at which it intersects itself. The horizontal axis is an axis of symmetry of the orbit. Figure 6 illustrates the situation for the case $(1 - \frac{1}{8}\frac{c_1}{c_2}) > 0$.

§20. Nonlinear Effects on the Solution at Large Depth

In this section we comment further on the influence of the slowly varying amplitudes $B_n(\tilde{\eta}) = B_n(\epsilon\eta)$ and $D_n(\eta) = D_n(\epsilon\eta)$ upon the solutions ϕ_o and ϕ_1 in which they respectively appear (see (15.1) and (15.9)). According to (15.1), for example, the first approximation to the potential $\overline{\Phi}$ for the reflected wave is

$$\overline{\Phi}_o = \sum_{n=1}^{\infty} B_n(\tilde{\eta}) \sin n(\zeta_2 - \tau) \ . \tag{20.1}$$

To illustrate the influence of the B_n's more clearly, we consider for convenience the Laplacian of ϕ_o; $\Delta\overline{\Phi}_o$ contributes to the first approximations to stresses. Since

$$\zeta_2 = \xi \cos\alpha - \eta\sqrt{\frac{c_2^2}{c_1^2} - \cos^2\alpha} \ ,$$

we have from (20.1) that

$$\Delta\overline{\Phi}_o = -\frac{c_2^2}{c_1^2}\sum_{n=1}^{\infty} n^2 B_n(\tilde{\eta}) \sin n(\zeta_2 - \tau) \ . \tag{20.2}$$

By (12.1), (12.4) this can be written

$$\Delta\overline{\Phi}_o = -\frac{c_2^2}{c_1^2} B_1(0)\ b(\zeta_2 - \tau, \tilde{\eta}) \tag{20.3}$$

More explicitly,

$$\Delta\overline{\Phi}_o = -\frac{c_2^2}{c_1^2} B_1(0)\ b\left(\xi \cos\alpha - \eta\sqrt{\frac{c_2^2}{c_1^2} - \cos^2\alpha} - \tau, \ \epsilon\eta\right) \tag{20.4}$$

Thus the structure of $\Delta\overline{\Phi}_o$ may be thought of as a wave, travelling in the direction of the reflected dilatation wave according to linear theory, whose wave form is slowly altered because of the dependence of $b(\theta, \eta)$ on $\tilde{\eta}$. The way in which $b(\theta, \tilde{\eta})$ changes with $\tilde{\eta}$ is determined by the first order partial differential equation

$$b_{\tilde{\eta}} - K\, b\, b_{\theta} = 0 \tag{20.5}$$

This may be regarded as a first order wave equation with propagation speed Kb. We have earlier shown (Section (12)) that the solution b of (20.5) satisfying the "initial condition"

$$b(\theta, 0) = \sin\theta \tag{20.6}$$

at the free surface has the property that

$$b(\pi, \tilde{\eta}) = 0, \qquad -\frac{1}{K} < \tilde{\eta} < 0 \ . \tag{20.7}$$

(We consider only the case $K > 0$ for simplicity.) The evolution of $b(\theta, \tilde{\eta})$ as a function of θ for various $\tilde{\eta}$'s is shown in Fig. 3. Below the critical depth

$$\tilde{\eta} = -\frac{1}{K} \tag{20.8}$$

the function $b(\theta, \tilde{\eta})$, if continued periodically as a function of θ, sustains a jump discontinuity at $\theta = \pi$. See the curve drawn with solid lines in Fig. 7(c). This phenomenon is connected with the nonlinear nature of (20.5) and may be viewed in another way as follows. Equation (20.5) develops multiple valued solutions when characteristics issuing from distinct points on the initial line $\tilde{\eta} = 0$ intersect for $\tilde{\eta} < 0$. This corresponds to a "breaking" of the wave associated with b. Viewed as a multiple valued solution of (20.5), the graph of b would include the dashed portions of the graph in Fig. 7(c). The breaking effect occurs at a depth η in the half-space given by

$$\eta = \frac{\tilde{\eta}}{\epsilon} = -\frac{1}{\epsilon K} = \frac{\left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} \left[(\cos^2 a - \tfrac{1}{2})^2 + \sin a \cos^2 a \left(\frac{c_2^2}{c_1^2} - \cos^2 a\right)^{\frac{1}{2}} \right]}{\epsilon \left(\frac{c_2}{c_1}\right)^6 \left(4 k_2 + \frac{c_1^2}{c_2^2} - 1\right) \sin a \cos a (\tfrac{1}{2} - \cos^2 a)} \tag{20.9}$$

Since the amplitude ϵ of the incident wave is small, the critical depth given by (20.9) is large; it is infinite in the case $\epsilon \to 0$.

Thus an observer detecting reflected waves along the ray carrying the classical reflected dilatation wave would see a wave form as in Fig. (7a) if he were located near the free surface, as in Fig. 7(b) if he were at moderate depth in the half-space, and as in Fig. 7(c) if he were below the critical depth.

The depth (20.9) at which breaking appears to occur depends on the angle of incidence a and the second order material constant k_2, as well as on the amplitude ϵ on the incident wave. It may be noted that for normal incidence ($a = \pi/2$), the critical depth becomes infinite. The breaking thus does not occur for the case of normal incidence. (See the discussion in § 19.)

It may also be noted that the potential ψ does not share the breaking behavior associated with ϕ.

Finally, we comment on the behavior of the coefficients $D_n(\tilde{\eta})$ which contribute to the second order correction $\overline{\phi}_1$. (See Eq. (15.9).) It may be shown that the function d associated with the D_n's exhibits the same onset of breaking, and that this occurs at the same critical depth (20.9) as in b.

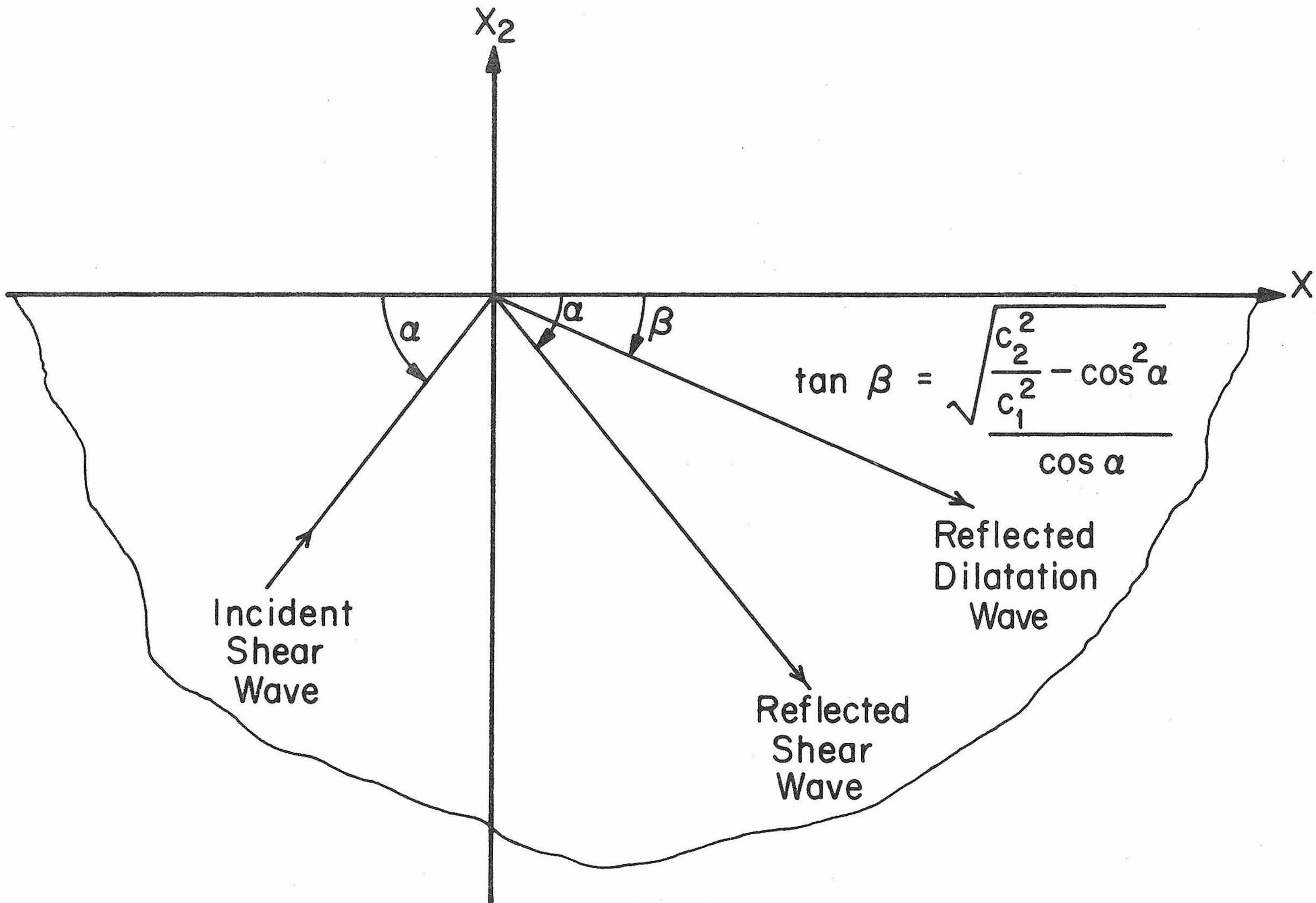
X2
X1
α
α
β
tan β = $\frac{\sqrt{\frac{c_2^2}{c_1^2} - \cos^2\alpha}}{\cos\alpha}$
Incident Shear Wave
Reflected Shear Wave
Reflected Dilatation Wave

Fig. 1

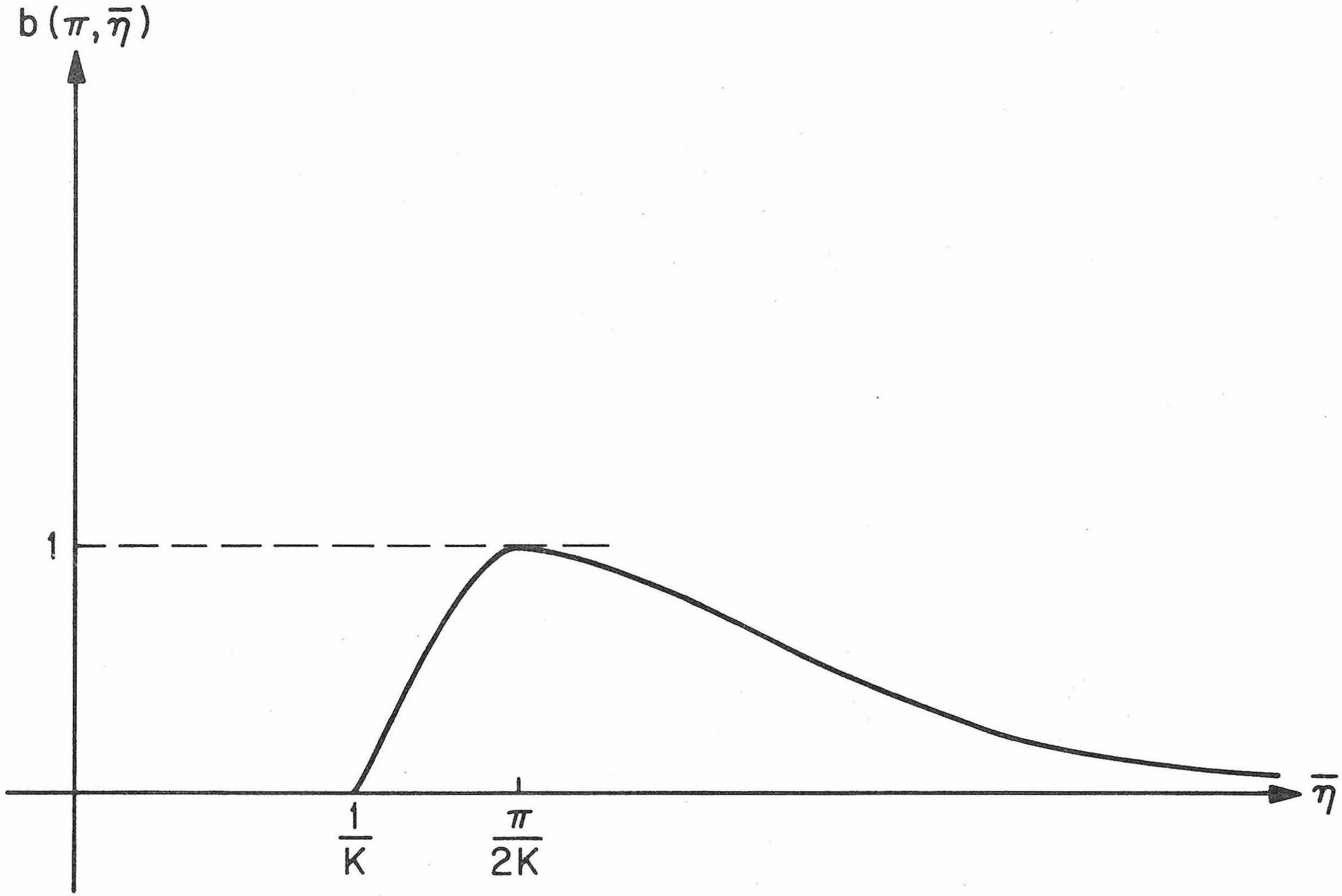

Fig. 2

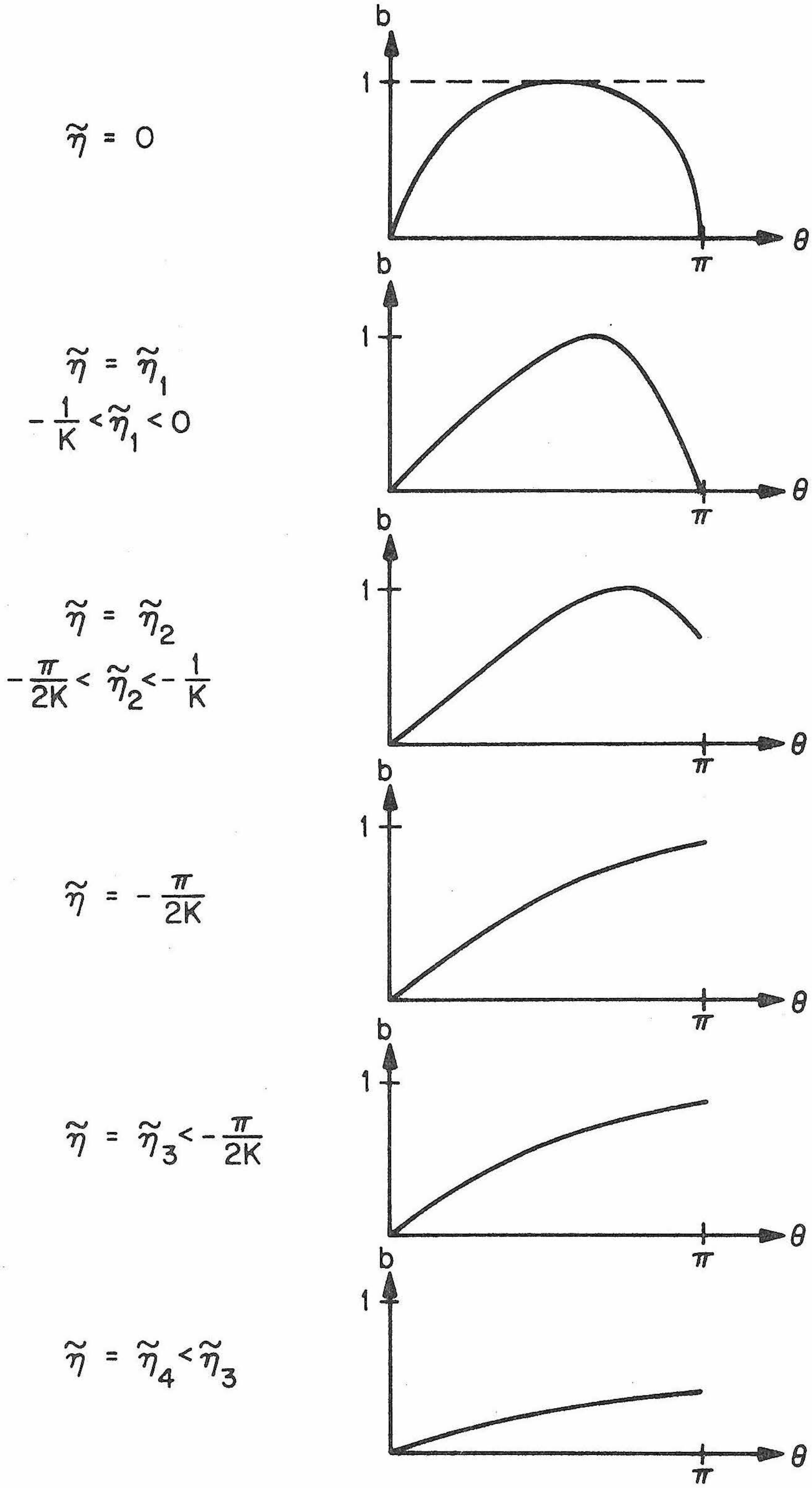

Fig. 3

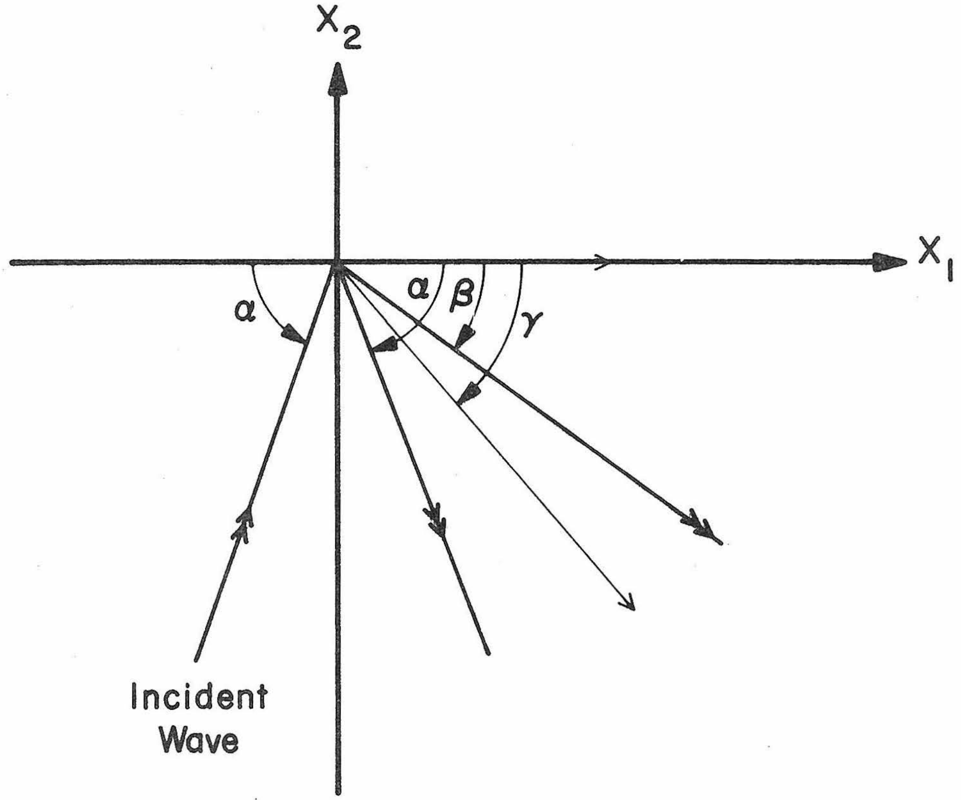

Fig. 4a

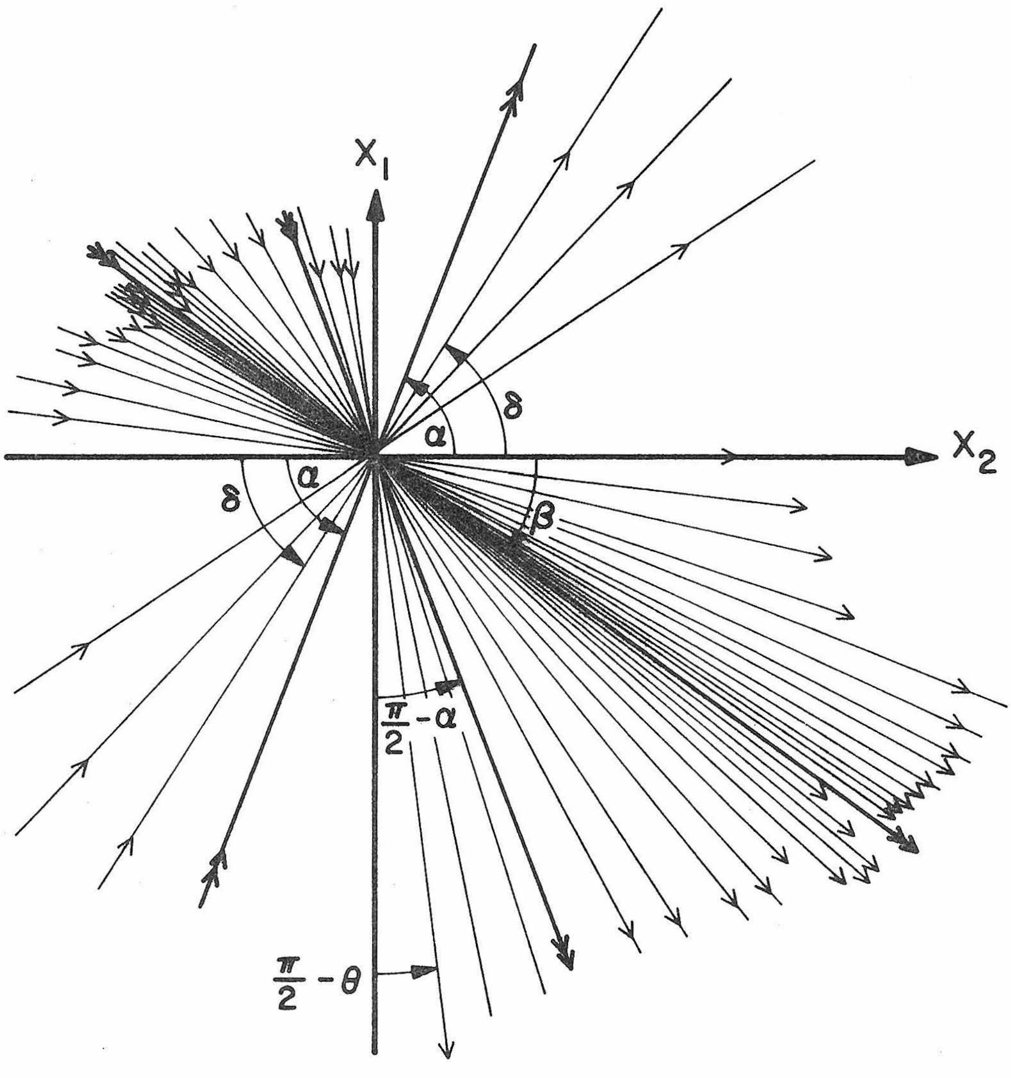

Fig. 4b

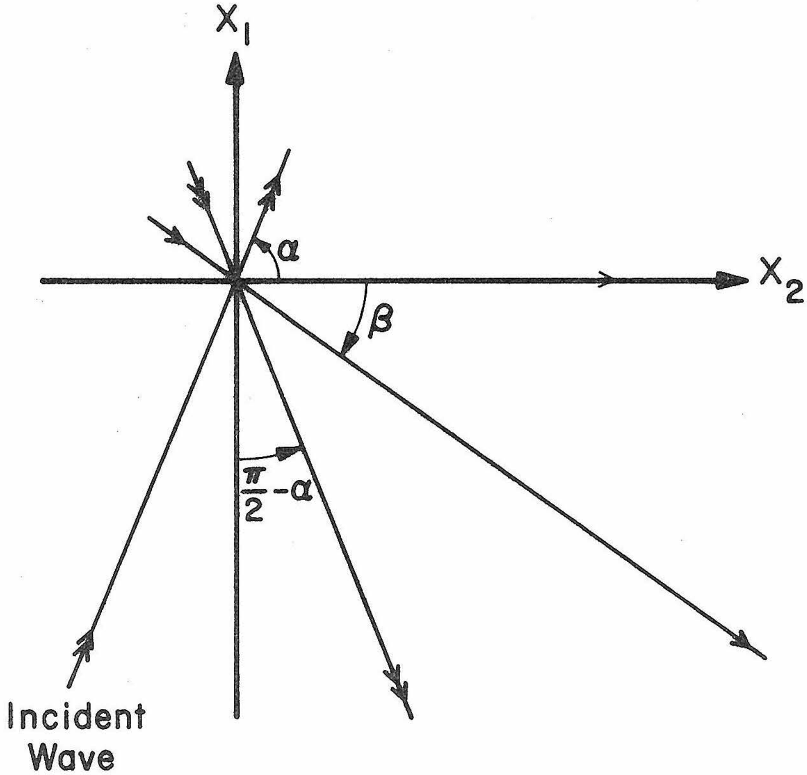

Fig. 4c

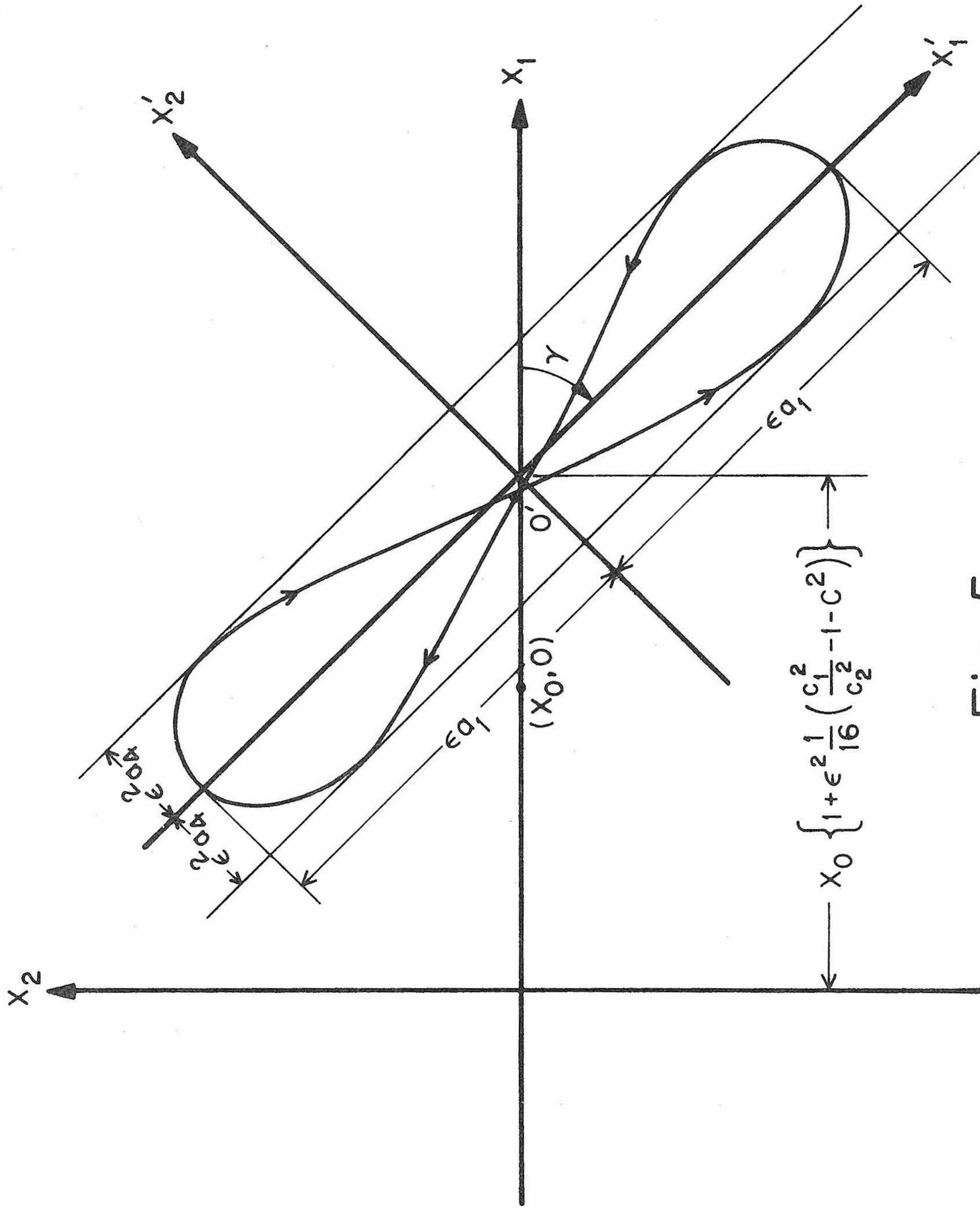

Fig. 5

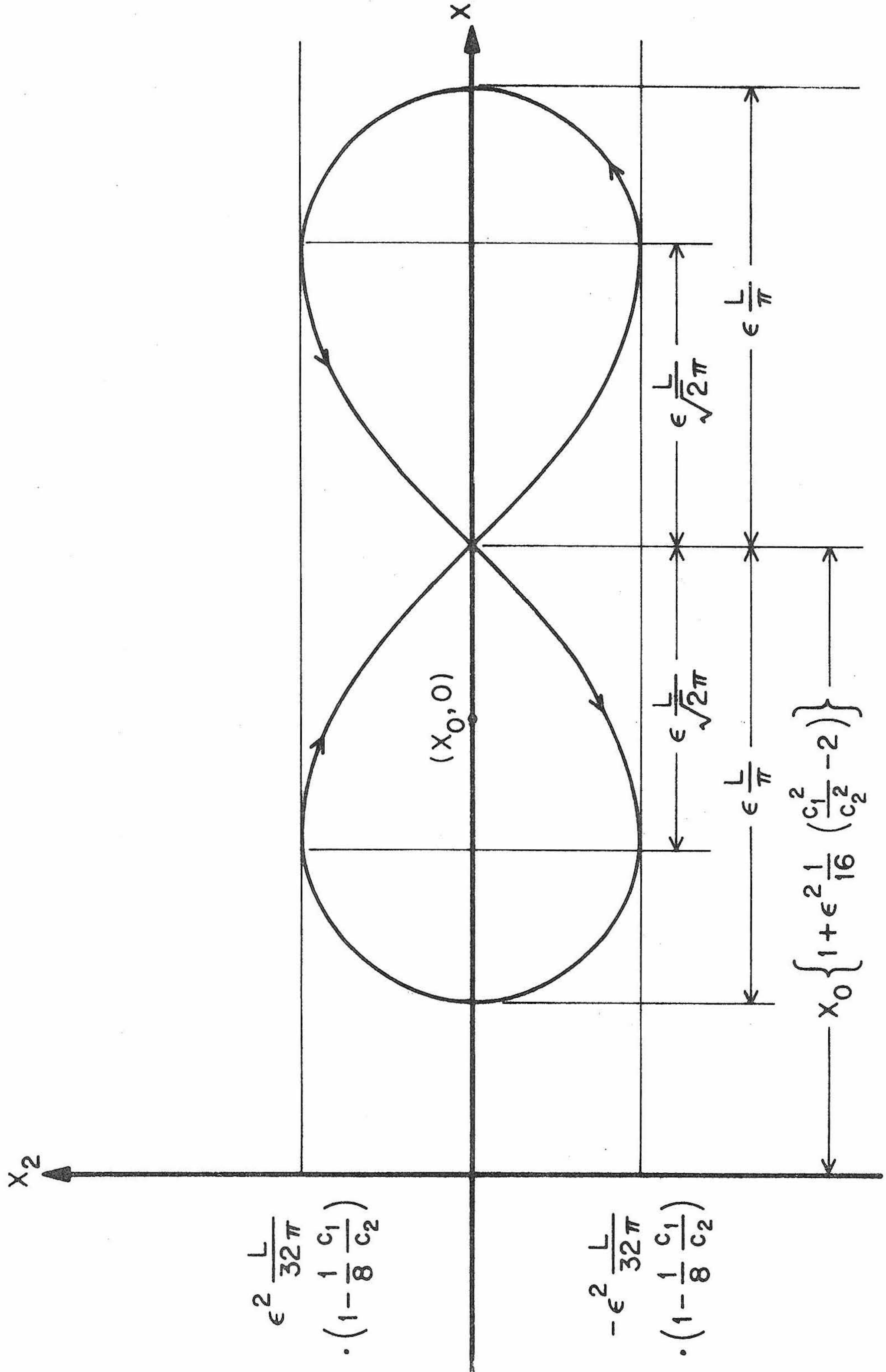
x1
x2
(x0, 0)
ε L/π
ε L/√2π
ε L/√2π
ε L/π
x0{1+ε2 1/16 (c1^2/c2^2 −2)}
ε2 L/32π
·(1 − 1/8 c1/c2)
−ε2 L/32π
·(1 − 1/8 c1/c2)

Fig. 6

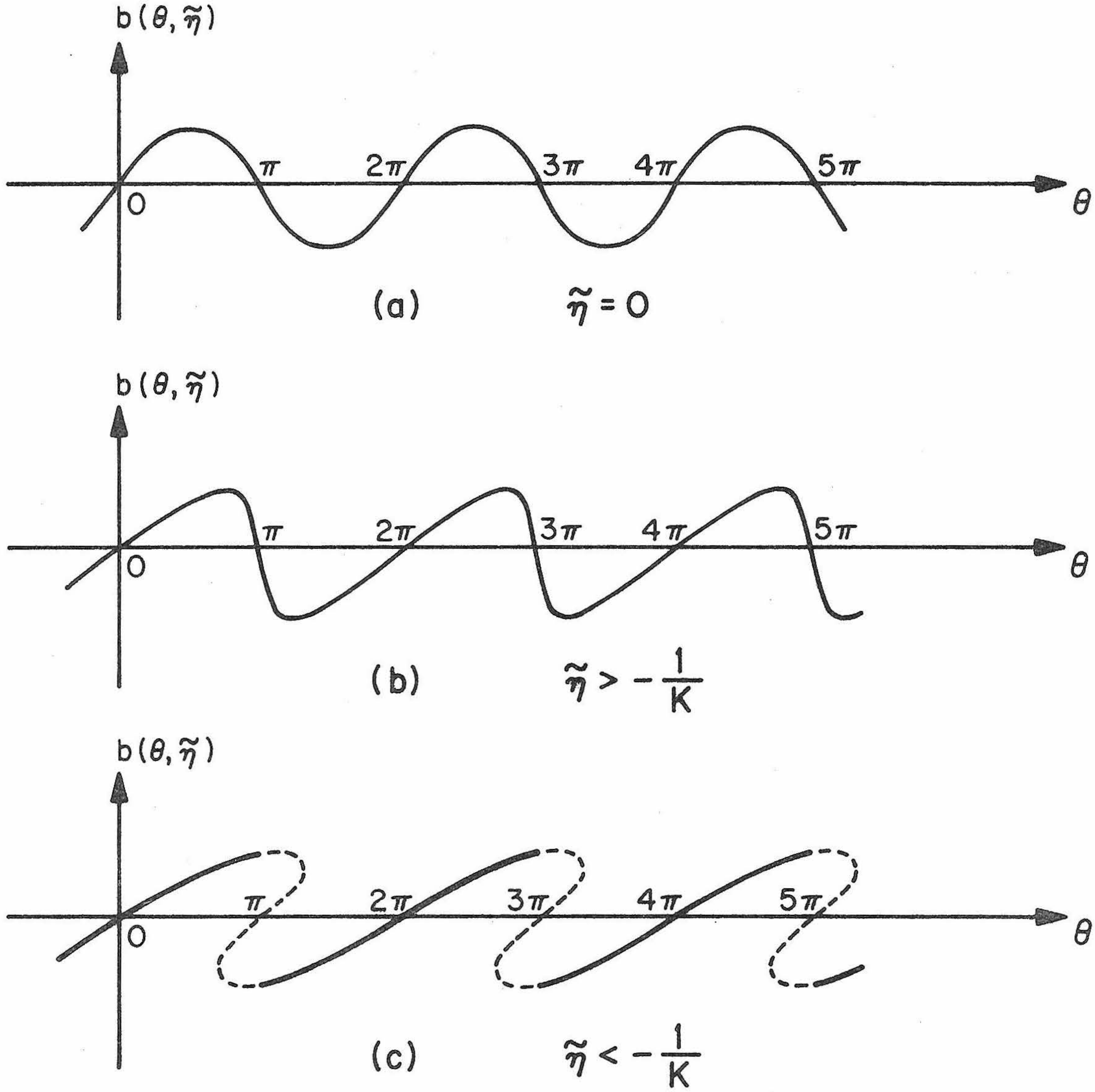
b(θ, η̃)
π
2π
3π
4π
5π
0
θ
(a)
η̃ = 0
b(θ, η̃)
π
2π
3π
4π
5π
0
θ
(b)
η̃ > − 1/K
b(θ, η̃)
π
2π
3π
4π
5π
0
θ
(c)
η̃ < − 1/K

Fig. 7

REFERENCES

[1] F. John, Plane strain problems for a perfectly elastic material of harmonic type. Comm. pure appl. Math. 13, 239-296 (1960).

[2] F. John, Plane elastic waves of finite amplitude. Hadamard materials and harmonic materials. Comm. pure appl. Math. 19, 309-341 (1966).

[3] E. Sternberg, On the integration of the Equations of Motion in the Classical Theory of Elasticity, Arch. Ratl. Mech. Anal., 5, 34-50 (1960).

[4] J. D. Cole, Perturbation Methods in Applied Mathematics. Blaisdell Publishing Company (1968).

[5] J. Hadamard, Leçons sur la Propagation des Ondes et les Equations de l'Hydrodynamique. Hermann, Paris (1903).

[6] C. Truesdell, General and Exact Theory of Waves in Finite Elastic Strain, J. Ratl. Mech. Anal., 8, 263-296 (1961).

[7] W. A. Green, The Growth of Plane Discontinuities Propagating into a Homogeneously Deformed Elastic Material. J. Ratl. Mech. Anal., 16, 79-88 (1964).

[8] E. Varley and J. Dunwoody, The effect of nonlinearity at an acceleration wave, J. Mech. Phys. Solids, 13, 17-28 (1965).

[9] R. A. Toupin and B. Bernstein, Sound waves in deformed perfectly elastic materials. Acoustoelastic effect. J. Acoust. Soc. Amer., 33, 216-225 (1961).

[10] M. Hayes and R. S. Rivlin, Propagation of a Plane Wave in an Isotropic Elastic Material Subjected to Pure Homogeneous Deformation, Arch. Ratl. Mech. Anal., 8, 15-22 (1961).

[11] D. R. Bland, Dilatation waves and shocks in large displacement isentropic elasticity, J. Mech. Phys. Solids, 12, 245-267 (1964).

[12] D. R. Bland, Plane Isentropic Large Displacement Simple Waves in a Compressible Elastic Solid, Z. angew. Math. Phys., 16, 752-769 (1965).

[13] D. R. Bland, On shock structure in a solid, J. Inst. Math. Applics., 1, 56-75 (1965).

References (Cont'd)

[14] B-T. Chu, Finite amplitude waves in incompressible perfectly elastic materials, J. Mech. Phys. Solids, 12, 45-57 (1964).

[15] W. D. Collins, One-dimensional nonlinear wave propagation in incompressible elastic materials. Q. J. Mech. Appl. Math., 19, 259-328 (1966).

[16] L. Davison, Propagation of plane waves of finite amplitude in elastic solids. J. Mech. Phys. Solids, 14, 249-270, (1966).

[17] L. Davison, Perturbation theory of nonlinear elastic wave propagation. Int. J. Solids Structures, 4, 301-322, (1968).

[18] E. Varley, Simple Waves in General Elastic Materials, Arch. Ratl. Mech. Anal., 20, 309-328 (1965).

[19] E. Varley, M. P. Mortell and A. Trowbridge, Modulated Simple Waves: An approach to Attenuated Finite Amplitude Waves, presented at ASME Winter Annual Meeting, Los Angeles, California, November 20, 1969.

[20] A. D. Fine, R. T. Shield, Second order effects in the propagation of elastic waves. Int. J. Solids Struct., 2, 605-620 (1966).

[21] A. E. Green, W. Zeena, Theoretical Elasticity, Oxford at the Clarendon Press (1954).

[22] W. M. Ewing, W. S. Jardetzky and F. Press, Elastic Waves in Layered Media, McGraw-Hill (1957).