Three Essays on Mechanism Design

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ABSTRACT

This thesis addresses mechanism design problems in three different contexts.

Chapter 2 compares two widely used student assignment mechanisms, the deferred-acceptance algorithm (DA) and the Boston algorithm (BA), in the context of the Chinese College Admission System. Two features of this system separate the study in this chapter from previous studies. First, the maximal number of schools that a student can apply to is fixed, and is significantly smaller than the total number of schools nationwide. Second, schools' preferences over applicants are not publicly observed. Under further assumptions, which include that applicants have the same preferences over schools and schools rank applicants by a common standard, I find that students are more likely to compete for seats at top schools under DA than BA. Furthermore, there are cases in which students' over-competition of top schools under DA results in a less efficient outcome compared to BA.

Chapter 3 studies the mechanism design problem in a market where buyers have type-dependent outside options. Previous literature usually assumes that buyers obtain a fixed value if they do not participate in a sale. This chapter focuses on scenarios in which the value of the option outside of a particular sale varies across different types of buyers. In such a scenario, an optimal mechanism for selling a private-valued item to unit-demand buyers is a second-price auction, with either a reserve price or a fixed show-up fee. This mechanism induces segregation of the market: buyers with a type which values the item high enough will exercise their outside option.

Chapter 4 analyzes grant-issuing processes in a mechanism design framework. Applicants submit their proposals for projects that may not be carried out without external funds. The grant issuer makes a selection from the proposals and decides the amount to award each selected project within a budget. This chapter characterizes optimal mechanisms to efficiently allocate the grant-issuer's budget. The optimal mechanism overcomes the problem of mis-allocation of the current merit-based mechanism. However, the problem of crowding-out private funds still stands. This chapter also shows how the specific form of institutional constraints — the flexibility of the budget constraint, and whether an applicant can reject a grant after being rewarded — affects the form of the optimal grant-issuing mechanism.

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Chapter 1

INTRODUCTION

In this thesis, I address three mechanism design problems in the contexts of school choice, private-value auctions, and grant issuing.

Chapter 2 is inspired by the Chinese college admission process. From 2005, the Chinese College Admission System started adopting the deferred-acceptance algorithm (DA) to replace the Boston algorithm (BA). Chapter 2 theoretically examines the impact of this transition.

Comparisons between DA and BA have been made in a variety of scenarios in previous studies. For example, (Abdulkadiroglu and Sonmez, 2003) pointed out that DA outperforms BA in both strategy-proofness and efficiency in the context of public school choice. Nonetheless, there are two features special to the Chinese College Admission System that have not received a thorough discussion. The first feature is that students can apply to no more than a fixed number k of colleges. The second feature is that students do not have complete information about the colleges' preferences over their competitors.

I incorporate these features in a model where applicants have the same preferences over schools, and schools rank applicants by a common standard. In this model, the advantages of DA over BA in strategy-proofness and efficiency no longer hold. I find that students under DA are more likely to use the truth-telling strategy — ranking the top k schools in truthful order on their preference lists. This strategy results in fierce competition among students for seats at the top k schools. Consequently, the seats at lower-ranked schools are left empty and a significant fraction of students are left unmatched. This leads to my second finding: under certain conditions, BA outperforms DA in terms of efficiency, as measured by the sum of students' and schools' expected utilities.

Chapter 3 characterizes an optimal mechanism in a market where buyers have options outside of the mechanism with various values. Most studies on optimal auction design assume that the seller is a monopoly, and hence all buyers face an outside option valued at zero. However, this is no longer the case nowadays, since similar items are available through different channels. For example, when a cellphone seller on eBay selects her selling mechanism, she must account for the availability of cellphones of the same model on Amazon. In this case, different buyers value the cellphone differently, and they could value the outside option (a cellphone on Amazon) differently as well.

I consider a market where a seller sells an item to risk-neutral buyers with unit demand. Similar items are available to all buyers at a fixed price somewhere else. Buyers have privately-observed types that determine their item valuation. They make participation decisions after observing their types. In analogy to the competition between Amazon and eBay, I assume that the items corresponding to the outside option have weakly higher qualities, which are publicly observed. In this market, I characterize an optimal mechanism for the seller: a second-price auction conducted among buyers who participate in the seller's sale, with either a reserve price or a show-up fee. This optimal mechanism induces market segregation: low-value buyers exit the market, medium-value buyers participate in the seller's auction, while buyers with high values exercise their outside option right away.

Chapter 4 focuses on grant-issuing mechanisms. Grants are an important source of funding for various types of projects. Despite the large size and common use of grants in practice, there are few theoretical studies on how to efficiently allocate a grant-issuer's budget. Towards that end, I consider the following model. To apply for a grant, applicants need to simultaneously submit their project proposals to the grant issuer. Each proposed project, if carried out, generates a value composed of two parts: a public value and a private value. The public value, or the "merit" of a proposal, is a benefit received by the whole society. The private value is a net benefit received exclusively by each proposer. The grant issuer can assess the public value of each project perfectly, but has little knowledge about its private value besides the amount reported by the applicant in the proposal. Each proposer's utility comes from the private value of his project and the payment from the grant if he is awarded. In particular, the public value does not concern proposers. In contrast, the grant issuer cares about both the public value and the private value, i.e. the total welfare. The goal of the grant issuer is to design proper rules to select which project should be supported by the grant, and how much to pay for the selected projects, such that the total welfare is maximized.

Two sets of institutional constraints are considered: the budget constraint and the constraint of individual rationality. Depending on specific scenarios, I consider variations of each constraint: the ex ante and ex post budget constraints represent a flexible and a strict bugdet respectively; the constraints of ex post and interim individual rationality reflect whether or not a proposer can turn down a grant at no cost. It turns out that if the grant issuer faces an ex ante budget constraint, the optimal mechanism does not depend on which of the two specifications of individual rationality is imposed. The mechanism can be implemented by take-it-or-leave-it offers made from the grant issuer to proposers. Each offer specifies a maximal amount of payment to each proposer. Intuitively, this offered payment increases as the budget increases. Furthermore, the higher public value a project generates, the larger the amount offered by the grant issuer. If the grant issuer faces an ex post budget constraint, which is a stronger constraint, the form of optimal mechanism differs with different specifications of individual rationality. If the optimal mechanism is constrained by interim individual rationality, in the optimal mechanism, a proposal is selected only when the reported private value exceeds a cutoff. The payment depends on other proposers' private values. This mechanism generates the same welfare for the grant issuer as in the ex ante budget scenario. If the optimal mechanism is constrained by ex post individual rationality, the cutoff selection rule is no longer optimal. In other words, whether a proposer is selected can depend on the reported private value of other proposers.

Chapter 2

CENTRALIZED COLLEGE ADMISSIONS UNDER APPLICATION CONSTRAINTS AND INCOMPLETE INFORMATION

2.1 Introduction

Every year, hundreds of thousands of Chinese high school graduate students in each province are matched to over a thousand colleges throughout the country via a centralized student assignment mechanism — the Chinese College Admission System. From 2005, provinces started adopting the deferred-acceptance algorithm (DA) to replace the Boston algorithm (BA) that had been used in the system. By 2017, 30 out of 31 provinces in China have adopted the deferred-acceptance algorithm,[1] covering 99% of high school graduates.[2] This transition was triggered by the work of Abdulkadiroglu and Sonmez, 2003, in which the authors pointed out DA's advantages over BA in stability, strategy-proofness, and efficiency. Nonetheless, there are two features special to the Chinese College Admission System that are not incorporated in their work. The first feature is that students can apply to no more than a fixed number of colleges. The second feature is that students usually do not have complete information about the colleges' preferences over their competitors. The goal of this paper is to theoretically compare the student assignment outcomes induced by DA and BA with the presence of these two features.

In both DA and BA, each student submits a list demonstrating her preference over schools to the centralized system. Each student is first considered by the school that is ranked on top of her list. If rejected, she is considered by her next preferred school according to her preference list. The procedure goes on until each student is accepted or her preference list is exhausted. Among students who are considered by a school, the school accepts the most preferred applicants up to its capacity. In DA, a school forms a temporary match with students: if a more preferred student shows up after the school's capacity is fulfilled, the school will reject the least preferred student in order to accept the more preferred one. In BA, however, once a school accepts a student, the school cannot reject her upon the arrival of more preferred

[1] The exception is Qinghai. Data sources: goo.gl/4LE4m7; http://www.gaosan.com/gaokao/123716.html; http://www.gaosan.com/gaokao/147431.html

[2] Data source: http://edu.sina.com.cn/gaokao/2015bm/

students. This indicates that under BA, the students who rank a school higher on their preference lists have a higher priority of getting accepted by that school, even if the school prefers other students who rank it lower.

The properties of DA have been extensively explored in two-sided matching theory.[3] Among them, the properties of strategy-proofness and stability are especially important for school choice problems. First, in DA, students' dominant strategy is to truthfully report their preferences over schools. Therefore, DA guarantees that students suffer no loss if they do not strategize. Second, DA generates stable outcomes. In other words, when the student assignment outcome is realized, there are no students and schools who would prefer each other to the assignment outcome. In contrast, as pointed out by Abdulkadiroglu and Sonmez, 2003, BA does not have either of these properties.

DA's advantages over BA, however, stand under fairly strict conditions: (1) there are no restrictions on the number of schools that a student can apply to, and (2) both schools and students have complete information about each other's true preferences. In the Chinese College Admission System, however, neither of these two conditions is satisfied. First, students are allowed to apply to no more than a fixed number of colleges, and this number is far smaller than the total number of colleges.[4] Second, incomplete information is inevitable. On the preference list, students rank pairs composed of a university and a desired major within that university. The capacity for each major is fixed. Even if students know their standing among all students, there is lack of information about their direct competitors — students who share their interest in a particular major. Consider a student who is interested in computer science. In order to study computer science in college, she will directly compete for the seats of this major in each college with other students who have the same interest. The problem is that the information about a student's interest in majors is usually private.

With the presence of restrictions on applications and incomplete information, DA is no longer strategy-proof and can generate ex-post unstable and inefficient outcomes. In this paper, I compare DA and BA in a simplified environment. I consider one given major. The capacity and quality of each school are given with regard to this

[3]See Roth and Sotomayor, 1992 for a review.

[4]The fixed number varies across provinces. For example, in Henan province, the province with the largest number of high school graduates, students can apply to up to nine universities, while in Guangdong province, the second largest province, students can only apply up to seven universities. The number in either case is much smaller than a rough estimation of the total number of 1000 colleges in China.

major. Furthermore, each student has two potential types: interested in the given major or not. If a student is not interested in the major, she will apply to other majors and hence does not compete for seats of the given major. If she is interested, she will apply to the schools with the given major and get a utility equal to the quality of the school that she is ultimately admitted to. A school's utility from accepting a student equals the student's grade (GPA). Capacities and qualities of schools are publicly observed. Students also have complete information about all students' GPAs. However, they do not know other students' types except for the distribution of types.

Given the set of students who are interested in the given major, there is a unique stable matching in this environment: students with the highest GPAs match with the school of the highest quality up to its capacity; among the remaining students, students with the highest GPAs match with the school of the second highest quality up to its capacity, etc.

When students' types are private information, and students can each apply to only a limited number k of schools, the stable matching will not be achieved by either DA or BA. I show that students under DA are more likely to use the truth-telling strategy — incorporating the top k schools in their preference lists and ranking them in truthful order. This strategy results in a fierce competition among students for seats at top k schools. Consequently, seats of schools with lower ranks will be empty and a significant fraction of students will be left unmatched. This leads to my second finding: under certain conditions, BA outperforms DA in terms of efficiency, as measured by the sum of students' and schools' expected utilities.

My work is related to several strands of literature. First, it follows the long history of work directed at the school choice problems. Roth and Sotomayor, 1992 reviewed work in which the school choice problem is modeled as a many-to-one matching procedure. After Abdulkadiroglu and Sonmez, 2003, much work has followed discussing the relative merits of different school choice mechanisms in terms of strategyproofness (P. A. Pathak and Sönmez, 2011; Abdulkadiroglu, P. Pathak, et al., 2006; Kesten, 2012) and efficiency (Erdil and Ergin, 2008; Kesten, 2010; Abdulkadiroglu, P. A. Pathak, and Roth, 2009). Among them, the most related is the study of school choices with a restriction on the number of applied schools. Haeringer and Klijn, 2009 compared BA, DA, and Top Trading Cycles algorithm (TTC) in a complete information setting with limited number of applications. They showed that BA can generate stable outcomes under conditions where DA and TTC fail. However, in experiments, more stable outcomes are achieved under DA than both BA and TTC (Calsamiglia, Haeringer, and Klijn, 2010). My paper focuses on analyzing efficiency instead of stability of BA and DA. This is because with the incorporation of incomplete information into the model, neither BA nor DA generates ex-post stable outcomes.

This paper also closely relates to the discussion of matching under incomplete information about preferences. Roth, 1982 showed that there is no stable matching mechanism that has truth-telling as a dominant strategy for every individual. This result was later generalized to the case of incomplete information about individual preferences (Roth and Rothblum, 1999). Ehlers and Massó, 2004 further showed that truth-telling is an equilibrium induced by a stable mechanism if and only if there exists a unique stable matching based on the common belief of preferences. Featherstone and Niederle, 2011 studied BA and DA in an environment that guarantees the existence of truth-telling equilibrium under a stable mechanism (Eeckhout, 2000). They identified conditions under which BA yields more efficient outcomes than DA, and confirmed their theoretical result in experiments. Abdulkadiroglu, Che, and Yasuda, 2011 also discovered situations where BA outperforms DA in efficiency in a different context. Compared to their work, my model additionally includes the constraint on applications. Under this constraint, both BA and DA are unstable, and truth-telling equilibrium is no longer guaranteed.

Lastly, this paper is related to literature on decentralized college admissions. Studies on centralized student assignment mechanisms are usually based on the context of public school choice. Schools' welfare in this context is uaually ignored.[5] In contrast, in the context of college admission, the quality of enrolled students affects the welfare of colleges from a lot of aspects, such as potential alumni donation, research funding, and the schools' reputation. As a result, schools' utilities also consist an important part in the model of college admission problems (Chade, Lewis, and Smith, 2003, Chade and Smith, 2006, and Che and Koh, 2016, among others). In this paper, I follow their work by taking the schools' benefit into consideration.

The rest of this paper is organized as follows: Section 2 sets up the model. Section 3 discusses equilibrium strategies of students under both BA and DA. Section 4 compares the welfare generated by the equilibrium strategies and presents an example in which BA yields a more efficient outcome than DA. Section 5 concludes and points out possible extensions. All proofs are relegated to the appendix.

[5] See Erdil and Ergin, 2008 for an example.

2.2 Model

This section contains three parts: the setup of the environment, the introduction of two student assignment mechanisms, and the formal description of two main assumptions underlying the papers' results.

Environment

There are m students in total, denoted by $M = \{1, 2, \ldots, m\}$. One major is exogenously given, and schools with this major are denoted by $N = \{S_1, S_2, \ldots, S_n\}$.

Let $Q = \{q_1, \ldots, q_n\}$ denote schools' qualities regards to the major. Without loss of generality, $q_1 > q_2 > \cdots > q_n > 0$.[6] For the tractability of the model, assume further that schools' qualities have a constant ratio, i.e. $\frac{q_{i+1}}{q_i} = c \in (0, 1)$ for $i = 1, \ldots, n-1$, and normalize $q_1 = 1$. A smaller c represents a larger difference in schools' qualities. Each school has a fixed capacity of the given major. Assume that capacities and qualities of schools are common knowledge.

For an arbitrary student $i \in M$, let g_i denote student i's GPA. Assume further that $g_1 > g_2 > \ldots > g_m > 0$. One obvious caveat of this assumption is that it abstracts away cases in which two students have the same grade. I make this assumption out of a twofold consideration: theoretically, this assumption largely simplifies the analysis in students' equilibrium strategies, as will be shown in the next section; empirically, for each student who participated in the Chinese College Entrance Examination, the probability of there being another student having the same score is very small.[7] So how the mechanism assigns two students with the same score does not significantly affect each student's strategy.

Assume that all students' GPAs are common knowledge, and school j's utility from accepting student i is i's GPA g_i.

There are two types of students: students who are interested in the given major and students who are not. If a student is interested in the major, assume that her utility from admission to school j is school j's quality q_j. If the student is not interested

[6] It is without loss of generality to assume no two schools have the same quality, since students can see schools of the same quality as one school with larger capacity.

[7] According to most college recruiting policies in China, if two students from the same province have the same total score, their rankings depend on a sequence of comparisons between several disciplines individually. The details of this comparison differ for different provinces, but they share a similar spirit. For example, colleges in Shanxi Province compare students' scores in math (with maximal score of 150), verbal (maximal score of 150), and science (maximal score of 300). From a student's point of view, given her score, the probability of there being another student having the exact same score is $\frac{1}{150^2 * 300}$, where the denominator is larger than the number of students registered for the exam in any province. http://gaokao.chsi.com.cn/gkxx/zc/ss/201705/20170526/1607986501-6.html

in the major, she receives a negative utility from studying this major in any schools, and hence will not apply. The probability of a student being interest is denoted by $\pi \in (0, 1)$. Furthermore, students' interests are independent from one another.[8] If a student ends up getting rejected by all schools on her preference list, she gets a utility valued 0. Similarly, if a school has a seat that is not assigned to any student, the school receives a utility valued 0 for that seat.

Two student assignment mechanisms

This paper focuses on two student assignment mechanisms: the deferred-acceptance algorithm (DA) and the Boston algorithm (BA).

Both mechanisms start with students reporting their preferences over schools. I assume that reported preferences are strict. In the first round, students' application files go to the schools on top of their preference lists. Schools accept their most preferred students until they reach their capacity and reject the rest. In the second round, students' files go to the second school on their preference lists if they are rejected by their reported favorite schools. BA and DA depart from each other at how schools consider the applicants from the second and following rounds. Under DA, schools' acceptance of students is tentative in each round. Specifically, schools rank applicants of the current round together with those who they have accepted in all previous rounds. Then they accept their most preferred students up to their capacity and reject the rest. Under BA, in contrast, schools' acceptance of students is permanent. In other words, schools only rank the applicants of the current round and accept their most preferred ones up to the remaining capacity accounting for the seats taken in all previous rounds. Both mechanisms continue until each student is either accepted by a school or rejected by all schools on her preference list.

Constraints on applications and incomplete information

Given the set of students who are interested in the major, there is a unique stable matching — a positive assortative matching. In this match, the school with the highest quality admits students of top grades up to the school's capacity; the school with the second highest quality admits students with the highest grades among the rest, and so on. This match is also utilitarian efficient.

This stable and efficient match can be achieved by both DA and BA if there are

[8]These assumptions are only for notational simplicity. The results in Section 3 can be easily generalized to cases where different students have different probabilities $\pi_1, \ldots, \pi_m$, or where the distributions of students' types are not independent.

no restrictions on applications and there is complete information about students' types. In what follows, I analyze the matching outcome under two additional assumptions: first, each student can apply to no more than k schools, where $k < n$; second, students' types are private knowledge. Each student only knows her own type and the probability of other students being interested in the major. In other words, although students know their GPA standings among all students, they have incomplete information about their direct competitors. In particular, they do not know the schools' preferences among other applicants.

2.3 Equilibrium strategies

In this section, I discuss equilibrium strategies of students who are interested in the major. First, I simplify the strategy space by ruling out the "order-reversal strategies", strategies in which S_j is reported preferable to S_i while S_i is actually more preferred. Formally,

Definition 1. *A strategy is an* order-reversal *strategy, if there exist i, j such that $S_j \hat{P} S_i$ in a student's reported preference list among her reported acceptable schools, while $S_i P S_j$ in the true preference.*

Lemma 1. *Order-reversal strategies are weakly dominated under DA.*

To show this result, suppose there is a student that uses an order-reversal strategy $\hat{P}$. Then there must exist two adjacent schools S_i and S_j on the student's preference list, such that their reported order $S_j \hat{P} S_i$ is a reversal of the true preference $S_i P S_j$. Now consider a new strategy $\hat{P}'$, in which the order of S_i and S_j is swapped to $S_i \hat{P}' S_j$ while everything else remains the same as $\hat{P}$. We can show that under DA, strategy $\hat{P}'$ generates a weakly better outcome for the student than $\hat{P}$ in all possible cases.

Fist, if the student is accepted by a school that is reported preferable to both S_i and S_j under $\hat{P}$, nothing changes if we swap S_i and S_j to the truthful order. Second, if the student is accepted by a school reported less preferable than both S_i and S_j, or unmatched, it indicates that the capacities of both S_i and S_j are fulfilled with students of higher GPAs, so swapping the order will not change the outcome. Third, if the student is accepted by S_i under $\hat{P}$, it implies that there are not enough students with higher GPAs to fulfill the capacity of S_i. In this case, the student will still end up with S_i if we swap the order to $S_i \hat{P}' S_j$. At last, if the student is accepted by S_j under $\hat{P}$, like previous case, it means there are not enough students with higher GPAs to fulfill the capacity of S_j. Under $\hat{P}'$ the student can guarantee a seat at S_j.

What's different is that now S_i is ranked higher than S_j. It gives the student a chance of getting admitted to S_i before her file goes to S_j. Therefore, by reporting $S_i\hat{P}'S_j$, the student will end up with either S_i or S_j, a better outcome than S_j.

Remark 1. The above argument does not hold for BA. Under BA, the student can be accepted by S_j under $\hat{P}$ but get rejected by both S_i and S_j under $\hat{P}'$. This is because BA endows students with a higher priority to get accepted by a school if they rank that school higher on their preference lists. By moving S_j one place down, the student can lose the seat of S_j to someone who ranks S_j higher.

Remark 2. If the schools are allowed to be indifferent between students, and ties are broken randomly, the argument for Lemma 1 is no longer applicable. This is because with indifference, each student's matching outcome is uncertain, even when all competitors' GPAs and their strategies are given. In fact, under DA, the order-reversal strategies are no longer dominated.

The intuition is that by using an order-reversal strategy, a student can avoid competing directly with other applicants for the same school in the same round. She only needs to compete with those tentatively accepted students, and this can lead to a larger probability of getting admitted. To see this more clearly, consider an example with three schools and four students. Suppose each school has one seat and all schools are indifferent among all students. Assume that each student can apply to up to two schools. When all the other three students report $S_1\hat{P}S_2$, if the fourth student does the same, her probability of getting accepted by S_1 is $\frac{1}{4}$. In contrast, if she reports $S_2\hat{P}S_1$, her probability of getting accepted by S_1 becomes $\frac{1}{3}$.[9]

The above discussion shows that by focusing on strict preferences, we are able to simplify the strategy space and avoid technical complications.

If a student lists top k schools in truthful orders, she is in fact truthfully revealing her preference subject to the application restriction. In what follows, I call this strategy "truth-telling." Formally,

Definition 2. *A strategy is* truth-telling *if the reported preference is* $S_1\hat{P}S_2\ldots\hat{P}S_k$.

First note that under BA, if all students but one use the truth-telling strategy, it is without loss of generality to rule out order-reversal strategies for the remaining

[9] By reporting $S_2\hat{P}S_1$, the probability of the student getting accepted by S_2 is $\frac{1}{3}$. If she is not accepted by S_2, her file goes to S_1, and she will compete with the student who was tentatively accepted by S_1 in the first round. So her probability of getting accepted by S_1 is $\left(1-\frac{1}{3}\right)\frac{1}{2}=\frac{1}{3}$. A complete and detailed analysis of the example can be found in the appendix.

student.

Lemma 2. *Given that all other students use the truth-telling strategy, order-reversal strategies are weakly dominated for a student under BA.*

The argument for this result is simple: given that all other students' strategies are truth-telling, if the student is matched with S_i under an order-reversal strategy, she can achieve a (weakly) better outcome by reporting S_{i-1} if $i > 1$ or S_i if $i = 1$ as her favorite school, leaving the rest place blank on her preference list. This strategy is not an order-reversal strategy.

Lemmas 1 and 2 together show that under both mechanisms, if all students but one are truthful, the remaining student will not rank a less preferred school on top of a more preferred school. In particular, the truth-telling strategy is not ruled out. It is interesting to see when this student also reports truthfully. The results are presented in the following proposition.

Proposition 1. *Given the distribution of students' types π, for mechanism s, where $s \in \{BA, DA\}$, there exists $\bar{c}^s \in (0, 1)$ such that the truth-telling strategy is an equilibrium strategy for all applicants under mechanism s if and only if $0 < c \leq \bar{c}^s$. Furthermore, $\bar{c}^{BA} \leq \bar{c}^{DA}$.*

The first part of the result is quite intuitive: when reporting her preferences, a student is making a trade-off between less preferred but safe choices and more preferred but risky choices. The larger the difference between schools' qualities, the more risk a student is willing to bear by applying to better schools. In the extreme case, when c is close to 0, no school other than the top school is worth attending. As a result, it is an equilibrium for all applicants to use the truth-telling strategy.

For the second part of the result in Proposition 1, recall that under BA, the higher a school is ranked on a student's preference list, the higher priority of the student to get accepted by that school. In particular, if a student ranks a school higher than anyone else, she is guaranteed a seat at that school. In contrast, under DA, even if a student reports a school as her favorite school, she still needs to compete with others who rank the school lower on their lists. As a result, by misreporting her preference, a student can hope for a better outcome under BA than DA. It should be expected that the truth-telling strategy is less appealing for students under BA.

By Proposition 1, when $c \in (\bar{c}^{BA}, \bar{c}^{DA})$, students under DA will all apply to the top k schools. In this equilibrium, all schools worse than S_k will end up empty.

This is not the case for BA and schools worse than S_k will have applicants. It is clear that the matching outcome under DA is not a Pareto improvement over BA. However, further analysis is required to determine which mechanism generates a more efficient outcome in terms of the total social welfare.

2.4 Efficiency

In this section, I provide an example in which BA is more efficient than DA in terms of total welfare received by both sides.

Example 1. *Let $m = n = 3$. The maximal number of schools that a student can apply is $k = 2$. The school quality ratio $c = 0.6$. Set the probability of each student applying to these schools $\pi = \frac{1}{2}$. Suppose that each school has only one seat. When g_3 is large enough, BA generates a higher social welfare than DA.*

To show the result in this example, first note that the equilibrium strategy under DA is truth-telling for all students, should they apply to these schools, whereas this is not the case under BA.

Lemma 3. *Under DA, it is an equilibrium that all three students report $S_1\hat{P}S_2$.*[10]

Lemma 4. *Under BA, it is an equilibrium that student 1 reports $S_1\hat{P}S_2$, student 2 reports $S_1\hat{P}S_2$ with probability $\frac{2}{3}$ and $S_1\hat{P}S_3$ with probability $\frac{1}{3}$, and student 3 reports $S_1\hat{P}S_2$ with probability $\frac{1}{5}$ and $S_2\hat{P}S_3$ with probability $\frac{4}{5}$.*

It is worth noting that the different procedures of DA and BA make it different to verify the equilibrium strategies under each mechanism. Under DA, from a student's perspective, only the strategies of students with higher grades affect her decision. For example, whether student 3 reports $S_1\hat{P}S_2$ or $S_2\hat{P}S_3$ does not affect the expected utility of student 2. In contrast, under BA, the strategies of all students affect outcomes. The expected utility of student 2 from reporting $S_1\hat{P}S_2$ decreases if student 3 reports $S_2\hat{P}S_3$ instead of $S_1\hat{P}S_2$. This fact of BA leads to the mixed strategy equilibrium. If student 3 reports $S_1\hat{P}S_2$, student 2's best response is the truth-telling strategy as well. If student 2 reports truthfully, student 3's best response is $S_2\hat{P}S_3$, so that she can guarantee a spot in school 2. Given student 3's strategy

[10] Note that the top student is always indifferent between strategies $S_1\hat{P}S_2$, $S_1\hat{P}S_3$, and S_1. Furthermore, the specific strategy used by the top student does not affect the equilibrium strategy derivation for the other students, nor does it affect the utility of schools. As a result, although there are multiple equilibria, they are equivalent in terms of the utilities of schools and students in these equilibria.

$S_2 \hat{P} S_3$, student 2's best response is no longer the truth-telling strategy, since that way student 2 can end up unmatched. Instead, the best response for student 2 is $S_1 \hat{P} S_3$. Student 3's best response to this strategy is the truth-telling strategy, since this way student 3 can end up with at least school 2. It can be seen that no two strategies are mutual best responses. So there are no pure strategy equilibria.

Given the equilibrium strategy profile, we can compare the expected utility of schools and students generated by the equilibrium strategy.

First, in the identified equilibrium, the top school prefers the outcome under DA, while the bottom school prefers the outcome under BA. This is because under the truth-telling strategy induced by DA, all applicants apply to the top school. Meanwhile no students apply to school 3. In contrast, under BA, it is likely that student 3 gives up school 1 and turns to her safety choice schools 2 and 3. Student 2 also may also give up school 2 and applly to school 3 instead. These changes in strategies make it possible for school 3 to fulfill its capacity, and make school 1 less competitive. Furthermore, school 2 also prefers DA. Compared to DA, under BA, school 2 is more likely ending up either with student 3 or empty.

Second, on the student side, the bottom student is better off under BA. To see this, note that by reporting $S_2 \hat{P} S_3$, student 3 is guaranteed a seat at school 2. Since the student mixes between $S_2 \hat{P} S_3$ and $S_1 \hat{P} S_2$, her utility from the two strategies must be equal. So under BA her utility is school 2's quality c. Comparatively, under DA, the student has a small probability of getting admitted by school 1 or school 2. Meanwhile, it is possible for her to end up unmatched. Her expected utility is lower than the utility of a guaranteed seat at school 2. At last, the middle student suffers under BA. Since under DA she can at least be admitted by school 2, but under BA, she may end up in school 3 or even unmatched.

For the total welfare, when all three students apply, the matching outcome under DA is always assortative for the top 2 schools and top 2 students. But there is an efficiency loss due to school 3 and student 3 never being matched. Differently, under BA, the matching outcome is more complicated and not necessarily assortative, but the bottom school and bottom student will not be left unmatched. Under BA, efficiency loss is largely attributed to the possibility of the middle student being unmatched. When g_3 is large enough, it means the utility of any school from admitting student 3 is large enough. In this case, under BA, the efficiency gain from guaranteeing student 3 a seat in a school can compensate the efficiency loss due to the possibility of missing student 2.

2.5 Conclusion and Discussion

The design of a student assignment mechanism is a key issue in school choice problems. In the context of the Chinese College Admission System, this paper compares two widely used mechanisms: the deferred-acceptance algorithm (DA) and the Boston algorithm (BA). In this context, there is a limit on the number of schools that a student can apply to, and students have incomplete information about schools' preferences. There are two main findings:

First, under both mechanisms, the truth-telling strategy is an equilibrium strategy of all applicants if the difference in school qualities is large enough. Furthermore, the threshold of this difference under BA is larger than DA. In other words, when schools' qualities are not very different from one another, the truth-telling strategy is more appealing to applicants under DA than under BA.

Second, though a conclusion has not been reached about efficiency in general, there are cases where BA, compared to DA, benefits bottom schools and students at the expense of the welfare of top schools and top students. With certain parameter values, BA can be more efficient than DA in the sense that BA yields a higher total welfare.

One natural extension would be to characterize equilibria under BA and DA when truth-telling does not constitute part of an equilibrium under either mechanism. This is the case when school qualities are close to one another. Intuitively, students have more incentive to misreport under BA. As a result, the equilibrium strategy profiles under BA are expected to be "further" from truth-telling than under DA. Haeringer and Halaburda, 2013 provided a measure of the distance between a strategy to truth-telling. Their measure may be utilized for the analysis in this direction.

Another possible extension would allow for heterogeneity in student and school preferences. Although this paper considers students with two types of preferences, all students who participate in the application process have aligned preferences. In reality, except for its quality, schools usually have features that different students value differently, such as the location, size, alumni, etc. In addition, in many student assignment systems other than the Chinese College Admission System, different schools may have different preferences over students.[11] However, this extension would require nontrivial changes of the model in this paper. In particular, the characterization of students' types can be multi-dimensional. This will tremendously complicate the analysis if students are assumed to only know their own types.

[11] Che and Koh, 2016

2.6 Appendix

More about the example in Remark 2, Section 3

In the example there are three schools and four students. Each school has only one seat and each student can only apply to up to two schools. All schools are indifferent over all four students. Under DA, if the other three students all report $S_1\hat{P}S_2$, then for the fourth student, she has four options after eliminating dominated strategies: $S_1\hat{P}S_2$, $S_1\hat{P}S_3$, $S_2\hat{P}S_3$, and $S_2\hat{P}S_1$. Note that the last strategy is an order-reversal strategy. Now we compare the expected utilities from these strategies:

- If the student reports $S_1\hat{P}S_2$ like the others, she can be accepted by S_1 with probability $\frac{1}{4}$. If she is rejected by S_1, she will be accepted by S_2 with probability $\frac{1}{3}$. As such, her expected utility is
$$\frac{1}{4} + \left(1 - \frac{1}{4}\right)\frac{1}{3}c \tag{2.1}$$
- If the student reports $S_1\hat{P}S_3$, she will be accepted by S_1 with probability $\frac{1}{4}$. If she is rejected by S_1, she will be accepted by S_3 for sure since no other students apply for S_3. As a result, her expected utility becomes
$$\frac{1}{4} + \frac{3}{4}c^2 \tag{2.2}$$
- If the student reports $S_2\hat{P}S_3$, she will compete with other two students who were rejected by S_1. Her probability of getting accepted by S_2 is $\frac{1}{3}$. If she is rejected by S_2, she will go to S_3 for sure. So her expected utility is
$$\frac{1}{3}c + \frac{2}{3}c^2 \tag{2.3}$$
- If the student reports $S_2\hat{P}S_1$, she will compete with two student who are rejected by S_1. She will be accepted by S_2 with probability $\frac{1}{3}$. If she is rejected by S_2, she will compete with the student who was tentatively accepted by S_1. Her probability of getting accepted by S_1 is $\frac{1}{2}$. As a result, her expected utility from this order-reversal strategy is
$$\frac{1}{3}c + \left(1 - \frac{1}{3}\right)\frac{1}{2} \tag{2.4}$$

It can be seen that when c is small enough ($c < \frac{2+\sqrt{13}}{9}$), the expected utility under the order-reversal strategy is the highest. In other words, the order-reversal strategy is not dominated.

Proof of Proposition 1

Proof. Given that all other students are truth-telling, under the constant ratio c of schools' qualities, a student of GPA g's expected utility is denoted by $EU_c^g(a')$, where a' stands for the student's strategy. We start the proof with the following lemma:

Lemma 5. *Under BA, given the probability π of each student applying to the schools, if the truth-telling strategy is an equilibrium strategy for all applicants under some $\bar{c} \in (0,1)$, then for $c \in (0,\bar{c}]$, it is also an equilibrium strategy for all applicants.*

Proof. Let a denote the truth-telling strategy. It suffices to show that

$$\min_{a' \neq a}\{EU_c^g(a) - EU_c^g(a')\} \tag{2.5}$$

is either non-increasing in c or nonnegative.

Let p_i^g denote the probability of the student of GPA g being matched with S_i if all students are truth-telling. In what follows, since we only discuss the strategy of the student with GPA g, we drop the sup-script g for notational simplicity. Under BA, given that all other students are truth-telling, we can apply Lemma 2 to calculate the highest utility that a student can get by deviating from the truth-telling strategy.

$$\begin{aligned}&\max_{a' \neq a} EU_c^g(a') = \\ &\max\left\{\max_{1\le l\le k-1}\sum_{i=1}^{l} p_i q_i + \left(1-\sum_{i=1}^{l} p_i\right) q_{l+1}, q_2\right\}\end{aligned} \tag{2.6}$$

Recall that

$$q_i = c^{i-1} \tag{2.7}$$

Substitute (2.6) and (2.7) into (2.5) and rearrange to obtain

$$\begin{aligned}&\min_{a' \neq a}\{EU_c^g(a) - EU_c^g(a')\} = \\ &\min\left\{\min_{1\le l\le k-1}\left(\sum_{i=l+1}^{k} p_i c^{i-1} - (1-\sum_{i=1}^{l} p_i)c^{l+1}\right), \sum_{i=1}^{k} p_i c^{i-1} - c\right\}\end{aligned} \tag{2.8}$$

Suppose (2.8) is increasing in c at $c = c_1 \in (0,1)$. It suffices to show that $\min_{a' \neq a}\{EU_{c_1}^g(a) - EU_{c_1}^g(a')\} \geq 0$. Since the right-hand side of (2.8) is continuous in c, there exists a $\delta > 0$ such that for all $c_2 \in (c_1, c_1+\delta) \cap (0,1)$

$$\min_{a' \neq a}\{EU_{c_1}^g(a) - EU_{c_1}^g(a')\} - \min_{a' \neq a}\{EU_{c_2}^g(a) - EU_{c_2}^g(a')\} < 0 \tag{2.9}$$

Consider the following cases:

Case 1: $\min_{a'\neq a}\{EU^g_{c_1}(a) - EU^g_{c_1}(a')\} = \sum_{i=1}^{k} p_i c_1^{i-1} - c_1$

By definition

$$\min_{a'\neq a}\{EU^g_{c_2}(a) - EU^g_{c_2}(a')\} \leq \sum_{i=1}^{k} p_i c_2^{i-1} - c_2$$

As a result,

$$\begin{aligned}
0 > \min_{a'\neq a}\{EU^g_{c_1}(a) - EU^g_{c_1}(a')\} - \min_{a'\neq a}\{EU^g_{c_2}(a) - EU^g_{c_2}(a')\} \\
\geq \sum_{i=1}^{k} p_i c_1^{i-1} - c_1 - \left(\sum_{i=1}^{k} p_i c_2^{i-1} - c_2\right) \\
\Rightarrow \sum_{i=1}^{k} p_i \frac{c_1^{i-1} - c_2^{i-1}}{c_1 - c_2} > 1 \\
\Leftrightarrow \sum_{i=2}^{k} p_i c_1^{i-2} \frac{1 - (\frac{c_2}{c_1})^{i-1}}{1 - \frac{c_2}{c_1}} > 1
\end{aligned}$$

Since the last Equation holds for all $c_2 \in (c_1, c_1 + \delta) \cap (0, 1)$, we can let c_2 approach c_1 and have

$$\begin{aligned}
\sum_{i=2}^{k} p_i c_1^{i-2} \lim_{c_2 \to c_1+} \frac{1 - (\frac{c_2}{c_1})^{i-1}}{1 - \frac{c_2}{c_1}} \geq 1 \\
\Leftrightarrow \sum_{i=2}^{k} p_i c_1^{i-2}(i-1) - 1 \geq 0
\end{aligned}$$

This means

$$\frac{\partial\left(\sum_{i=1}^{k} p_i c_1^{i-1} - c_1\right)}{\partial c_1} \geq 0$$

Therefore,

$$\sum_{i=1}^{k} p_i c_1^{i-1} - c_1 \geq \sum_{i=1}^{k} p_i \cdot 0^{i-1} - 0 = 0$$

i.e. $\min_{a'\neq a}\{EU^g_{c_1}(a) - EU^g_{c_1}(a')\} \geq 0$.

Case 2: $\min_{a'\neq a}\{EU^g_{c_1}(a) - EU^g_{c_1}(a')\} = \sum_{i=l^*+1}^{k} p_i c_1^{i-1} - (1 - \sum_{i=1}^{l} p_i) c_1^{l^*+1}$

By the same logic as in Case 1, we have

$$\sum_{i=l^*+1}^{k} p_i c_1^{i-1} - (1-\sum_{i=1}^{l^*} p_i) c_1^{l^*+1} - \left(\sum_{i=l^*+1}^{k} p_i c_2^{i-1} - (1-\sum_{i=1}^{l^*} p_i) c_2^{l^*+1}\right) < 0$$

$$\Leftrightarrow 1-\sum_{i=1}^{l^*} p_i < \frac{\sum_{i=l^*+1}^{k} p_i (c_1^{i-1} - c_2^{i-1})}{c_1^{l^*+1} - c_2^{l^*+1}} = p_{l^*+1}(c_1^{l^*} - c_2^{l^*})$$

$$+ \sum_{i=l^*+2}^{k} p_i c_1^{i-2-l^*} \frac{1-(\frac{c_2}{c_1})^{i-1}}{1-(\frac{c_2}{c_1})^{l^*+1}}$$

$$\Rightarrow 1-\sum_{i=1}^{l^*} p_i \leq \lim_{c_2 \to c_1+} \left(p_{l^*+1}(c_1^{l^*} - c_2^{l^*}) + \sum_{i=l^*+2}^{k} p_i c_1^{i-2-l^*} \frac{1-(\frac{c_2}{c_1})^{i-1}}{1-(\frac{c_2}{c_1})^{l^*+1}}\right)$$

$$= \sum_{i=l^*+2}^{k} p_i c_1^{i-2-l^*} \frac{i-1}{l^*+1}$$

$$\Leftrightarrow \sum_{i=l^*+2}^{k} p_i c_1^{i-2}(i-1) - (l^*+1)(1-\sum_{i=1}^{l^*} p_i) c_1^{l^*} \geq 0$$

$$\Rightarrow \frac{\partial \left(\sum_{i=l^*+1}^{k} p_i c_1^{i-2} - (1-\sum_{i=1}^{l^*} p_i) c_1^{l_1^*+1}\right)}{\partial c_1}$$

$$\Rightarrow \sum_{i=l^*+1}^{k} p_i c_1^{i-2} - (1-\sum_{i=1}^{l^*} p_i) c_1^{l_1^*+1} \geq 0$$

i.e. $\min_{a' \neq a} \{EU_{c_1}^g(a) - EU_{c_1}^g(a')\} \geq 0$. □

Next, for DA, similar result holds:

Lemma 6. *Under DA, given the probability π of each student applying to the schools, if the truth-telling strategy is an equilibrium strategy for all applicants under some $\bar{c} \in (0,1)$, then for $c \in (0, \bar{c}]$, it is also an equilibrium strategy for all applicants.*

Proof. Similar to the proof of Lemma 5, it suffices to show that

$$\min_{a' \neq a} \{EU_c^g(a) - EU_c^g(a')\} \tag{2.10}$$

is either non-increasing in c or nonnegative.

Under DA, applying Lemma 1, the highest utility of a student by deviating from truth-telling is

$$\max_{a'} EU_c^g(a') = \max\left\{\max_{1 \leq l \leq k-1} \left(\sum_{i=1}^{l} p_i q_i + \sum_{i=l+2}^{k+1} p_i' q_i\right), \sum_{i=2}^{k+1} p_i' q_i\right\} \tag{2.11}$$

where p_i and p'_i are the probabilities of the student getting admitted by S_i under the truth-telling strategy and a deviation of truth-telling strategy, respectively. It is straightforward to see that $p'_i \geq p_i$ for $l+2 \leq i \leq k$ and $p'_2 \geq p_2$. Furthermore, $\sum_{i=l+2}^{k+1} p'_i = 1 - \sum_{i=1}^{l} p_i$ and $\sum_{i=2}^{k+1} p'_i = 1$.

Substitute (2.11) into (2.10) to get

$$\min_{a' \neq a}\{EU_c^g(a) - EU_c^g(a')\} = \min_{0 \leq l \leq k-1} \left(\sum_{i=l+1}^{k} p_i c^{i-1} - \sum_{i=l+2}^{k+1} p'_i c^{i-1} \right) \tag{2.12}$$

Suppose (2.10) is increasing in c at $c = c_1 \in (0,1)$. It suffices to show that $\min_{a' \neq a}\{EU_{c_1}^g(a) - EU_{c_1}^g(a')\}$ is nonnegative. Since the right-hand side of (2.10) is continuous in c, there exists a $\delta > 0$ such that for all $c_2 \in (c_1, c_1 + \delta) \cap (0,1)$,

$$\min_{a' \neq a}\{EU_{c_1}^g(a) - EU_{c_1}^g(a')\} - \min_{a' \neq a}\{EU_{c_2}^g(a) - EU_{c_2}^g(a')\} < 0 \tag{2.13}$$

Let $i^* \in arg \min_{1 \leq i \leq k} \left(p_i c_1^{i-1} - (1 - \sum_{j \neq i} p_j) c^k \right)$. If $i^* \geq 1$, then

$$\begin{aligned}
0 &> \min_{0 \leq l \leq k-1} \left(\sum_{i=l+1}^{k} p_i c_1^{i-1} - \sum_{i=l+2}^{k+1} p'_i c_1^{i-1} \right) - \min_{0 \leq l \leq k-1} \left(\sum_{i=l+1}^{k} p_i c_2^{i-1} - \sum_{i=l+2}^{k+1} p'_i c_2^{i-1} \right) \\
&\geq \left(\sum_{i=i^*+1}^{k} p_i c_1^{i-1} - \sum_{i=i^*+2}^{k+1} p'_i c_1^{i-1} \right) - \left(\sum_{i=i^*+1}^{k} p_i c_2^{i-1} - \sum_{i=i^*+2}^{k+1} p'_i c_2^{i-1} \right) \\
\Leftrightarrow & \frac{\sum_{i=i^*+1}^{k} p_i c_1^{i-1} (1 - (\frac{c_2}{c_1})^{i-1})}{\sum_{i=i^*+2}^{k+1} p_i c_1^{i-1} (1 - (\frac{c_2}{c_1})^{i-1})} \geq 1 \\
\Rightarrow & \frac{\sum_{i=i^*+1}^{k} p_i c_1^{i-1} \lim_{c_2 \to c_1+} (1 - (\frac{c_2}{c_1})^{i-1})}{\sum_{i=i^*+2}^{k+1} p_i c_1^{i-1} (1 - \lim_{c_2 \to c_1+} (\frac{c_2}{c_1})^{i-1})} = \frac{\sum_{i=i^*+1}^{k} p_i c_1^{i-2} (i-1)}{\sum_{i=i^*+2}^{k+1} p_i c_1^{i-2} (i-1)} \geq 1 \\
\Rightarrow & \frac{\partial \left(\sum_{i=i^*+1}^{k} p_i c_1^{i-1} - \sum_{i=i^*+2}^{k+1} p'_i c_1^{i-1} \right)}{\partial c_1} \geq 0 \\
\Rightarrow & \sum_{i=i^*+1}^{k} p_i c_1^{i-1} - \sum_{i=i^*+2}^{k+1} p'_i c_1^{i-1} \geq 0
\end{aligned}$$

If $i^* = 0$, by same argument we have

$$\frac{\sum_{i=i^*+2}^{k} p_i c_1^{i-2}(i-1)}{\sum_{i=i^*+2}^{k+1} p_i c_1^{i-2}(i-1)} \geq 1$$

$$\frac{\partial \left(\sum_{i=i^*+1}^{k} p_i c_1^{i-1} - \sum_{i=i^*+2}^{k+1} p_i' c_1^{i-1} \right)}{\partial c_1} \geq 0$$

$$\Rightarrow \sum_{i=i^*+1}^{k} p_i c_1^{i-1} - \sum_{i=i^*+2}^{k+1} p_i' c_1^{i-1} \geq 0$$

□

Next, we prove $\bar{c}^{BA} \leq \bar{c}^{DA}$. Note that

$$\sum_{i=l+2}^{k+1} p_i' q_i \leq \left(1 - \sum_{i=1}^{l} p_i'\right) q_{l+1}$$

and

$$\sum_{i=2}^{k+1} p_i' q_i \leq q_2$$

Compare (2.6) and (2.11) to obtain

$$\max_{a' \neq a} EU_c^g(a')^{BA} \geq \max_{a' \neq a} EU_c^g(a')^{DA}$$

$$\Rightarrow \left(EU_c^g(a) - \max_{a' \neq a} EU_c^g(a')\right)^{BA} \leq \left(EU_c^g(a) - \max_{a' \neq a} EU_c^g(a')\right)^{DA}$$

It shows that if the expected utility from the strategy of truth-telling is the highest under BA, it must also be the highest under DA. So if the truth-telling strategy is an equilibrium strategy under BA, it must also be an equilibrium strategy under DA.

At last, we show that there exist $\bar{c}^{BA}$ and $\bar{c}^{DA}$ such that the truth-telling strategy is an equilibrium strategy for all applicants. This is quite straightforward: first note that the right-hand side of (2.8) is positive as $c \to 0+$ as long as $p_i > 0$ for some $i \in \{1, \ldots, k\}$. Since $\pi \in (0, 1)$, $p_i \geq (1 - \pi)^{m-1} > 0$. So $\bar{c}^{BA} \in (0, 1)$ must exist. We show previously that $\bar{c}^{DA} \geq \bar{c}^{BA}$, as a result, $\bar{c}^{DA} \in (0, 1)$ also exists. □

Proof of Lemma 3

Proof. By Lemma 1, it suffices to consider strategies $S_1\hat{P}S_2$, $S_1\hat{P}S_3$, $S_2\hat{P}S_3$, and S_3.

It is straightforward to see that reporting $S_1\hat{P}S_2$ is a dominant strategy for students of g_1 and g_2 under DA. Given their strategies, the expected utilities of the student of g_3 from these strategies are

$$\begin{aligned} EU^{g_3}(S_1\hat{P}S_2)^{DA} &= 2\pi(1-\pi)c + (1-\pi)^2 \\ EU^{g_3}(S_1\hat{P}S_3)^{DA} &= \pi^2c^2 + 2\pi(1-\pi)c^2 + (1-\pi)^2 \\ EU^{g_3}(S_2\hat{P}S_3)^{DA} &= \pi^2c^2 + 2\pi(1-\pi)c + (1-\pi)^2c \\ EU^{g_3}(S_3)^{DA} &= c^2 \end{aligned}$$

Plug in $\pi = 0.5$ and $c = 0.6$ to see that $S_1\hat{P}S_2$ brings the highest expected utility. So it is an equilibrium strategy of the student of g_3 to report $S_1\hat{P}S_2$ as well. $\square$

Proof of Lemma 4

Proof. We need to verify that the strategy in Lemma 4 is indeed an equilibrium strategy.

Given the students of g_1 and g_3's strategies, for the student of g_2:

$$\begin{aligned} EU^{g_2}(S_1\hat{P}S_2)^{BA} &= \delta(1-\pi+\pi c) + (1-\delta)(1-\pi+(1-\pi)\pi c) \\ EU^{g_2}(S_1\hat{P}S_3)^{BA} &= 1-\pi+\pi c^2 \\ EU^{g_2}(S_2\hat{P}S_3)^{BA} &= EU^{g_2}(S_2\hat{P}S_1)^{BA} = EU^{g_2}(S_2)^{BA} = c \\ EU^{g_2}(S_3)^{BA} &= EU^{g_2}(S_3\hat{P}S_1)^{BA} = EU^{g_2}(S_3\hat{P}S_2)^{BA} = c^2 \end{aligned}$$

where δ is the probability of the student with g_3 reporting $S_1\hat{P}S_2$. Plug in $\delta = \frac{1}{5}, \pi = \frac{1}{2}$, and $\delta = 0.6$ to get $EU^{g_2}(S_1\hat{P}S_2)^{BA} = EU^{g_2}(S_1\hat{P}S_3)^{BA} > EU^{g_2}(S_2\hat{P}S_3)^{BA} > EU^{g_2}(S_3)^{BA}$.

Given the students of g_1 and g_2's strategies, for the student of g_3:

$$\begin{aligned} EU^{g_3}(S_1\hat{P}S_2)^{BA} &= \theta((1-\pi)^2 + 2\pi(1-\pi)c) + (1-\theta)((1-\pi)^2 + (1-(1-\pi)^2)c) \\ EU^{g_3}(S_1\hat{P}S_3)^{BA} &= \theta((1-\pi)^2 + (1-(1-\pi)^2)c^2) + (1-\theta)(2\pi(1-\pi)c^2 + (1-\pi)^2) \\ EU^{g_3}(S_2\hat{P}S_3)^{BA} &= EU^{g_3}(S_2\hat{P}S_1)^{BA} = EU^{g_3}(S_2)^{BA} = c \\ EU^{g_3}(S_3)^{BA} &= EU^{g_3}(S_3\hat{P}S_1)^{BA} = EU^{g_3}(S_3\hat{P}S_2)^{BA} = c^2 \end{aligned}$$

where θ is the probability of the student of g_2 reporting $S_1\hat{P}S_2$. Plug in $\theta = \frac{2}{3}$, $\pi = \frac{1}{2}$, and $\delta = 0.6$ to get $EU^{g_3}(S_1\hat{P}S_2)^{BA} = EU^{g_3}(S_2\hat{P}S_3)^{BA} > EU^{g_3}(S_1\hat{P}S_3)^{BA} > EU^{g_3}(S_3)^{BA}$.

Note that in this example, strategies $S_2\hat{P}S_3$, $S_2\hat{P}S_1$, and S_2 yield the same expected utility for student 3. In addition, student 2's utility remains the same no matter which of the three strategies is employed by student 3. As a result, there are multiple equilibria, but they are equivalent in terms of the utilities generated from the equilibrium strategies. □

Proof of Example 1

Proof. Given the equilibrium strategies characterized in Lemma 3, under DA, school 1's expected utility is

$$EU_{DA}^{S_1} = \pi g_1 + (1-\pi)\pi g_2 + (1-\pi)^2\pi g_3 \tag{2.14}$$

The first term on the right-hand side shows that if student 1 is interested in the major, she will list S_1 as her top choice and S_1 receives g_1 from accepting the student. The second term represents the utility S_1 receives if student 1 is not interested but student 2 applies to the schools. The third term corresponds to the case when neither student 1 or 2 is interested in the major but student 3 is interested and hence applies to the schools.

Following the same logic, other schools' expected utilities are

$$EU_{DA}^{S_2} = \pi^2 g_2 + 2(1-\pi)\pi^2 g_3$$
$$EU_{DA}^{S_3} = 0$$

Similar calculations can be made given the equilibrium strategies under BA (Lemma 4).

$$EU_{BA}^{S_1} = \pi g_1 + (1-\pi)\pi g_2 + (1-\pi)^2\pi\frac{1}{5}g_3$$
$$EU_{BA}^{S_2} = \frac{4}{5}\pi g_3 + \frac{1}{5}\pi\left(2\pi(1-\pi)g_3 + \pi^2\left(\frac{2}{3}g_2 + \frac{1}{3}g_3\right)\right) + (1-\pi)\pi^2\frac{2}{3}g_2$$
$$EU_{BA}^{S_3} = \pi^2\frac{1}{3}g_2$$

It can be immediately seen that $EU_{DA}^{S_1} > EU_{BA}^{S_1}$, and $EU_{DA}^{S_3} < EU_{BA}^{S_3}$. In other words, the top school prefers DA to BA, while the bottom school prefers BA. In addition, when $\pi = 0.5$ and $c = 0.6$,

$$EU_{DA}^{S_1} + EU_{DA}^{S_2} + EU_{DA}^{S_3} - EU_{BA}^{S_1} - EU_{BA}^{S_2} - EU_{BA}^{S_3} = \frac{1}{15}g_2 - \frac{13}{120}g_3$$

Since $\frac{1}{15} < \frac{13}{120}$, when g_3 is close to g_2, the school side suffers a loss if the mechanism switches from DA to BA.

For students, under DA

$$\begin{aligned} EU_{DA}^{g_1} &= 1 \\ EU_{DA}^{g_2} &= \pi c + (1-\pi) \\ EU_{DA}^{g_3} &= 2\pi(1-\pi)c + (1-\pi)^2 \end{aligned}$$

Under BA

$$\begin{aligned} EU_{BA}^{g_1} &= 1 \\ EU_{BA}^{g_2} &= \pi c^2 + (1-\pi) \\ EU_{BA}^{g_3} &= c \end{aligned}$$

It can be seen that $EU_{DA}^{g_1} = EU_{BA}^{g_1}$, $EU_{DA}^{g_2} > EU_{BA}^{g_2}$, and $EU_{DA}^{g_3} < EU_{BA}^{g_3}$. In other words, the top student is indifferent between two mechanisms, the median student benefits from DA, while the bottom student suffers under DA. In addition, when $\pi = 0.5$ and $c = 0.6$,

$$EU_{DA}^{g_1} + EU_{DA}^{g_2} + EU_{DA}^{g_3} - EU_{BA}^{g_1} - EU_{BA}^{g_2} - EU_{BA}^{g_3} = 0.07$$

So DA is beneficial to students overall, but it can hurt the school side when all students have relatively high grades. In particular, when $\frac{13}{120}g_3 - \frac{1}{15}g_2 > 0.07$, BA yields a higher social welfare than DA. $\square$

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Chapter 3

OPTIMAL MECHANISMS WITH TYPE-DEPENDENT OUTSIDE OPTIONS

3.1 Introduction

Most studies on optimal mechanism design assume that the seller is a monopoly, and hence all buyers face an outside option valued at zero.[1] However, this is no longer the case in today's differentiated marketplace. For example, when a cellphone seller on eBay selects her selling mechanism, she must account for the availability of cellphones of the same model on Amazon. In this case, different buyers value the cellphone differently, and in particular, they value the outside option (a cellphone on Amazon) differently as well. It is natural to ask: what is the optimal mechanism for a seller when potential buyers have heterogeneously valued outside option?

I address this question by considering a market where a seller sells an item to risk-neutral buyers with unit demand. Similar items are available to all buyers at a fixed price somewhere else. Buyers have privately observed types that determine their item valuation. They can either visit the seller or exercise the outside option, but not both.[2] Buyers make their participation decision after observing their types.[3] In analogy to the competition between Amazon and eBay, I assume that the items of the outside option have weakly higher qualities, and that qualities are publicly observed.

I characterize an optimal mechanism for the seller: a second-price auction conducted among buyers who participate in the seller's sale, with either a reserve price or a show-up fee. In addition, this optimal mechanism induces market segregation: low-value buyers exit the market, medium-value buyers participate in the seller's auction, and buyers with high values exercise their outside option right away.

This form of the optimal mechanism is surprisingly simple considering the complexity of the set of mechanisms the seller could consider. In my model, because of

[1] See Klemperer, 1999 for a review.

[2] We can also interpret the outside option as a sale conducted by a second seller, and we assume the buyer does not approach both simultaneously, which is confirmed empirically by (Haruvy et al., 2008). For cases without assuming exclusion, see Gerding et al., 2008.

[3] This setup is in line with McAfee, 1993, Burguet and Sákovics, 1999, etc. For scenarios where buyers learn their value after visiting sellers, see Peters and Severinov, 1997 and Damianov, 2012.

the endogeneity of buyers' participation decisions (Jullien, 2000), the seller's mechanism is a menu of allocation and payment rules for *each possible set of visitors*. Nonetheless, I show that the optimal mechanism's allocation and payment rules are *independent of the set of visitors*.

Related literature

My work is closest to Krishna and Perry, 1998 and Jullien, 2000. Both papers consider a general form of type-dependent outside options. Krishna and Perry, 1998 derive the optimal mechanism within the class of mechanisms which require full participation, i.e. no buyer exercises her outside option. I relax this requirement in my model. Jullien, 2000 considers partial participation in a single principal-agent model, while my model resides in a more complicated context, since there are multiple buyers competing for a single item and each buyer's surplus depends on other buyers' decisions.

The literature on the feature of "buy-it-now" in online markets has a similar way of modeling the outside option as my work. This strand of literature is motivated by the prevalence of a combined use of fixed price and an auction as a pricing mechanism in practice (Mathews, 2004). As a result, studies of this topic usually focus on analyzing the benefit brought by the "buy-it-now" feature to sellers, while regards the form of the auction as given. (See Ockenfels, Reiley, and Sadrieh, 2006 and Haruvy et al., 2008 for surveys.)

Another strand of literature that deals with type-dependent outside options is in the context of competing mechanisms. Initiated by McAfee, 1993, the optimal mechanism design problem has been studied in a framework where there are multiple sellers competing with each other for potential customers. This framework allows for more complex features of outside options. Besides type-dependence, a buyer's outside option of visiting one seller is endogenously determined by the mechanisms offered by other sellers and the visiting choices of other buyers. The optimal mechanism is in general intractable because of the intricate interactions between sellers and buyers. For tractability, researchers mainly focus on large markets (McAfee, 1993, Peters and Severinov, 1997), a restricted space of sellers' mechanisms (Burguet and Sákovics, 1999, Hernando-Veciana, 2005, Ashlagi, Monderer, and Tennenholtz, 2011, Pai, 2010, Ivanova-Stenzel and Salmon, 2011), or a restricted space of buyers' strategies (Virág, 2010). In the current paper, I explore the optimal mechanism given a specific setting of outside options without imposing any of the above restrictions.

There are studies that consider endogenous participation in auctions from perspectives other than type-dependent outside options. Palfrey and Pevnitskaya, 2008 studied the self-selection problem regarding to participants' risk attitude when bidders make participation choices before knowing their private types. Merlob, Plott, and Zhang, 2012 introduced an entry fee in a procurement auction. The entry fee is paid from the bidders to the auctioneer, and aims to screen out bidders with high costs. In both papers, since all bidders face a fixed outside option, their participation decisions in equilibrium show a different pattern from my paper.

The rest of this paper is organized as follows: Section 2 sets up the model and states the main result. Section 3 provides a sketch of the proof. Section 4 concludes and points out possible extensions. The proofs are relegated to the appendix.

3.2 Model

There are $N+1$ risk-neutral buyers with unit demand. Let $\mathcal{N} = \{1, \ldots, N+1\}$ denote the set of buyers. Each buyer has a private type $v \in [0,1]$. Buyers' types are drawn independently and identically from a distribution $F(v)$ with associated density $f(v)$. Let $g(\mathbf{v})$ and $G(\mathbf{v})$ denote the density and cumulative distribution function of the joint distribution respectively, where $\mathbf{v} = (v_1, \ldots, v_{N+1})$. From independence, $g(\mathbf{v}) = \Pi_{i \in \mathcal{N}} f(v_i)$ and $G(\mathbf{v}) = \Pi_{i \in \mathcal{N}} F(v_i)$. Let $\mathcal{V} = [0,1]^{N+1}$ denote the space of buyers' types. The types are observed privately, while their distribution is common knowledge.

There is one seller selling one unit of an item of quality $k \in (0,1]$. If a buyer does not participate in the seller's sale, he can buy one unit of the item of quality 1 at a fixed price p somewhere else. Both p and qualities are exogenously given and publicly observed.

The value to a buyer of receiving one unit of the item is the product of the item's quality and the buyer's type. The corresponding surplus of the buyer, $S(v)$, is this value net of her payment. If the buyer refrains from entering the market altogether, utilizing neither the seller nor the outside option, I say that the buyer has *exited the market*. In that case, a buyer gets a surplus of zero.

I focus on the optimization of the seller's revenue. The seller specifies how to sell her item by announcing an incentive compatible direct mechanism. Here, by "direct mechanism" I mean the allocation and payment of the item depend on participants' reported types. By "incentive compatible" I mean *in equilibrium*, if a buyer participates in the sale, he cannot achieve higher surplus by misreporting his type, given that all other participants report truthfully. According to the revelation principle, it is without loss of generality to focus on incentive compatible direct mechanisms.

Formally, let $\mathcal{I} \subset \mathcal{N}$ denote the set of buyers who visit the seller. The seller's mechanism can be expressed as

$$\{Q^{\mathcal{I}}, M^{\mathcal{I}}\}_{\mathcal{I} \subset \mathcal{N}} \tag{3.1}$$

In (3.1), $Q^{\mathcal{I}} : [0,1]^{|\mathcal{I}|} \mapsto \Delta^{|\mathcal{I}|}$ determines the probability of each visitor in $\mathcal{I} \subset \mathcal{N}$ getting the item based on the reported type, where $\Delta^{|\mathcal{I}|}$ stands for the standard simplex with dimension $|\mathcal{I}|$. $M^{\mathcal{I}} : [0,1]^{|\mathcal{I}|} \mapsto \mathbb{R}^{|\mathcal{I}|}$ determines the payment made from each visitor in $\mathcal{I}$ to the seller. The seller can make payments to buyers, i.e. payments can be negative.

I only consider *anonymous mechanisms* in this model. An anonymous mechanism does not distinguish buyers by their identities. In particular, the allocation and payment rules of such a mechanism only depend on the *number* of visitors.

As an illustrative example, think of a market with two buyers $\mathcal{N} = \{1, 2\}$. The set $\mathcal{I}$ of buyers who visit the seller can be $\emptyset$, $\{1\}$, $\{2\}$, or $\{1, 2\}$. The seller should therefore announce four allocation rules (and corresponding payment rules), one for each case: $(Q_1^{\emptyset}, Q_2^{\emptyset})$, $(Q_1^{\{1\}}(x_1), Q_2^{\{1\}}(x_1))$, $(Q_1^{\{2\}}(x_2), Q_2^{\{2\}}(x_2))$, and $(Q_1^{\{1,2\}}(x_1, x_2)$, $Q_2^{\{1,2\}}(x_1, x_2))$, where x_i is buyer i's reported type. For anonymous mechanisms, there are two restrictions of the mechanisms: (a) $Q_1^{\{1\}}(x_1) = Q_2^{\{2\}}(x_2)$ whenever $x_1 = x_2$; and (b) $Q_1^{\{1,2\}}(x_1, x_2) = Q_2^{\{1,2\}}(x_2, x_1)$.

Facing any given mechanism, each buyer needs to make two decisions: (1) to participate in the seller's sale or to exercise the outside option (participation strategy), and (2) if participate, which type to report to the seller (reporting strategy). The equilibrium concept of the buyers' game is the *perfect Bayesian symmetric equilibirum*. In an equilibrium, no buyer can achieve a higher expected surplus by deviating either in the participation strategy or the reporting strategy. In addition, buyers with identical values employ the same strategy. In incentive compatible direct mechanisms, the equilibrium reporting strategy is reporting truthfully, hence we only need to consider buyers' participation strategy.

At last, I follow the convention that if a buyer expects to receive a zero surplus regardless of his participation decision, he exits the market.

My main result characterizes an optimal mechanism:[4]

Theorem. *Suppose the distribution of private types has a full support on* $[0, 1]$ *with a non-decreasing hazard rate* $\frac{f(v)}{1-F(v)}$ *and a non-increasing density function. An optimal mechanism conducts a second-price auction among the seller's visitors with either a reserve price or a show-up fee.*

I provide a sketch of the proof for this result in the next section.

3.3 Solve for optimal mechanisms

I follow two steps to prove the theorem: I first simplify the mechanism space and then derive an optimal mechanism within the simplified mechanism space.

Mechanism space simplification

My first result shows that the symmetric equilibrium participation strategy is a cutoff strategy:

Proposition 1. *Given any incentive compatible direct mechanism of the seller, in any symmetric equilibrium of the buyers, there exist* $\underline{v} \in [0, p]$ *and* $\bar{v} \in [p, 1]$*, such that buyers with type* $v \in (\underline{v}, \bar{v})$ *visit the seller,* $v \in (\bar{v}, 1)$ *buy the item from the outside option, and* $v \in [0, \underline{v}]$ *exit the market.*

To see why buyers always use a cut-off strategy in equilibrium, first note that buyers with private value below p, the fixed price asked in the outside option, have a zero value for the outside option. If a buyer with value $v_0 < p$ chooses to visit the seller, it means this buyer has a positive surplus by visiting the seller. Incentive compatibility of the seller's mechanism requires buyers' surplus from participating in the sale increases with their value. As a result, any buyer with value above v_0 and below p should also visit the seller. This gives the cut-off strategy for buyers with value below p.

For buyers with value above p, although their surplus by visiting the seller increases with their values, the value of the outside option grows more. This result is mainly driven by the assumption that the seller has only one unit of the item with weakly lower quality. More specifically, suppose a buyer with value $u_0 > p$ exercises the

[4] Note that this is not the unique optimal mechanism. In fact, any variation of the allocation rule and the payment rule on a zero-measure set does not affect the seller's surplus. Besides, there are mechanisms that do not fall in the class incentive compatible direct mechanisms and they can generate the same surplus for the seller.

outside option in equilibrium. Now think about a buyer with a slightly higher value $u_0 + \Delta$. Compared to the previous buyer, the surplus of this buyer from visiting the seller increases by roughly $q(u_0)k\Delta$, where $q(\cdot) \leq 1$ is the probability of a buyer winning the item from the seller. Meanwhile, the increase in the value of the outside option for this buyer is Δ. Since $k \leq 1$ by assumption, the increase in the surplus from visiting the seller is smaller. If the buyer with value u_0 finds it more beneficial to exercise the outside option, the buyer with value $u_0 + \Delta$ must find even more so. Therefore, the buyers with value above p also uses a cut-off strategy.

With type-dependent outside options and no requirement of full participation, the major complication of the problem is the contingency of mechanisms on the number of visitors. Proposition 2 further simplifies the mechanism space.

Proposition 2. *Any symmetric equilibrium outcome of a mechanism can be implemented by an incentive compatible direct mechanism, which is in addition independent of the number of visitors.*

Proof. Denote by $r(\cdot)$ buyers' equilibrium participation strategy. By Proposition 1, $r(v)$ is either 1 or 0 for $v \in [0, 1]$ almost everywhere.

Given the original mechanism $(\hat{Q}, \hat{M})$ and any realization of private values $\mathbf{v}$, let $\mathcal{J}^* \subset \mathcal{N}$ denote the set of buyers who visit the seller in equilibrium. Formally, $\Pi_{i \in \mathcal{J}^*} r(v_i) = 1$. For any $\mathcal{I} \subset \mathcal{N}$ and for any $i \in \mathcal{I}$, let

$$\tilde{Q}_i^{\mathcal{I}}(\mathbf{v}^{\mathcal{I}}) = \begin{cases} \hat{Q}_i^{\mathcal{J}^* \cup \{i\}}(\mathbf{v}^{\mathcal{J}^* \cup \{i\}}) & r(v_i) = 1 \\ 0 & \text{Otherwise} \end{cases} \tag{3.2}$$

$$\tilde{M}_i^{\mathcal{I}}(\mathbf{v}^{\mathcal{I}}) = \hat{M}_i^{\mathcal{J}^* \cup \{i\}}(\mathbf{v}^{\mathcal{J}^* \cup \{i\}}) \tag{3.3}$$

for all $\mathcal{J} \subset \mathcal{N} \backslash \{i\}$.

It can be verified that (1) $\tilde{Q}_j^{\mathcal{I}}(\mathbf{v}^{\mathcal{I}}) \in [0, 1]$ and $\sum_{j \in \mathcal{I}} \tilde{Q}_j^{\mathcal{I}}(\mathbf{v}^{\mathcal{J}}) \leq 1$; (2) $\tilde{Q}$ is positive responsive; (3) $\tilde{Q}$ is incentive compatible; and (4) $r(v)$ is still an equilibrium under the mechanism $(\tilde{Q}, \tilde{M})$. This completes the proof. □

In addition, the constructed mechanism generates a unique equilibrium participation strategy, and this makes the optimality problem well-defined.

Lemma 1. *The symmetric equilibrium outcome under a mechanism in Proposition 2 is unique up to a zero-measure set of types.*

The proof is rather technical and is relegated to the appendix.

Since the mechanism does not depend on the set of visitors, we can denote the allocation rule and the payment rule simply by $(Q(\mathbf{v}), M(\mathbf{v}))$. In other words, it suffices to study mechanisms which emulate the seller observing all buyers' values and allocating the item among them.

Optimal mechanisms

An optimal mechanism is an allocation rule and a payment rule which maximize $\sum_{i=1}^{N+1} \mathbb{E}(kr(v)m(v))$, where $r(v)$ is the probability of a buyer with value v visiting the seller in equilibrium. Since buyers' equilibrium strategy is a cutoff strategy by Proposition 1, there exist thresholds $\underline{v}$ and $\bar{v}$ such that $r(v) = 1$ if $v \in (\underline{v}, \bar{v})$ and $r(v) = 0$ otherwise. As an alternative method of calculating $\underline{v}$ and $\bar{v}$ given the seller's mechanism, we can regard the thresholds as the seller's choice, and impose the requirement that they are the equilibrium thresholds for buyers. This requirement is referred to as the participation constraints.

We can now solve for the optimal mechanism using standard techniques: we link the payment scheme to the allocation rule through the incentive compatibility constraints, and account for the feasibility constraint $\sum_{i \in \mathcal{N}} Q_i(\mathbf{v}) \leq 1$.

Lemma 2. *The solution to the following optimization problem provides an optimal mechanism, where the objective is*

$$\max_{Q, m(0), \bar{v}, \underline{v}; \lambda} \int_{\mathcal{V}} g(\mathbf{v}) \sum_{i \in \mathcal{N}} (\psi(v_i) Q_i(v_i, \mathbf{v}_{-i})) d\mathbf{v} + (N+1)km(0)(F(\bar{v}) - F(\underline{v}) - \lambda) - (N+1)\lambda(\bar{v} - p) \tag{3.4}$$

subject to the second order incentive compatibility constraint, participation constraints, and feasibility constraints. $\lambda \geq 0$ *is the Lagrangian multiplier for the participation constraints.* $\psi(\cdot)$ *is the "adjusted virtual value":*

$$\psi(v) = v - \frac{1 - F(v)}{f(v)} + \frac{1 - F(\bar{v}) + \lambda}{f(v)} \tag{3.5}$$

The objective function in (3.4) intuitively illustrates the structure of the expected revenue. The term $\psi(v_i) Q_i(v_i, \mathbf{v}_{-i})$ in the integral shows one source of the revenue: the seller get a buyer's adjusted virtual value by giving him the item. The adjusted virtual value for buyers is Myerson's virtual value adjusted to the endogeneity of participation.

The first constant term represents the revenue loss from paying a show-up fee $-km(0)$ to participants. In Myerson, 1981 this number is 0. Now with endogenous participation, the seller may have incentive to pay a strictly positive show-up fee to encourage buyers with higher values to participate in the sale. But the payment is made to all participants regardless of their types.

The last term shows the cost that the seller needs to pay if she would like to keep buyers with value above p into her sale. For these buyers, their values for the outside option are positive and increase with their types. To keep them in the sale, the seller needs to make extra sacrifice, such that these buyers' loss of giving up the outside option is well compensated.

Note that $\underline{v}$ can be interpret as the reserve price, which is set by the seller to screen out buyers with low values. An incentive compatible mechanism can have either a positive reserve price, or a positive show-up fee, but not both. Otherwise buyers with value below the reserve price will over-report their types to get the show-up fee.

From (3.4), it is optimal for the seller to assign the item to a buyer with the highest adjusted virtual value. It is routine to verify this allocation rule satisfies the incentive compatibility constraints when the distribution of values has an increasing hazard rate and a nonincreasing density function. Given the allocation rule, it is straightforward to derive the payment rule. It turns out that it has the same form as a second-price auction with either a reserve price or a show-up fee. The theorem then follows.

Remarks

Despite their similarity, there are significant differences between the optimal mechanism identified here and in Myerson, 1981.

The first difference is technical. Note that the condition for the second-price auction to be optimal is more strict here. This is because an increasing hazard rate is not sufficient to guarantee the fulfillment of the second order incentive compatibility constraint, which requires the item to go to the buyer with the highest private value. Due to an additional term $\frac{1-F(\bar{v})+\lambda}{f(v_i)}$ in the adjusted virtual value function, a non-creasing hazard rate together with a non-increasing density function, such as a uniform distribution or an exponential distribution, can guarantee that the adjusted virtual value is monotonic in v. To assign the item to the buyer with the highest reported value is then in line with assigning the item to the buyer with the highest adjusted virtual value.

Second, there is, in general, no closed form solution for the optimal reserve price or show-up fee. This is because a higher reserve price, or a lower show-up fee, on one hand increases the seller's revenue directly, but on the other hand decreases the revenue by generating less participation in terms of $\bar{v} - \underline{v}$. The resolution of the tension between "price" and "quantity" depends on specific forms of the distribution of private values, as well as the value of parameters k and p. One general feature of the reserve price is that it never exceeds Myerson's optimal reserve price.

Corollary 1. *The optimal reserve price satisfies* $\underline{v}^* \leq \phi^{-1}(0)$*, where* $\phi(v) = v - \frac{1-F(v)}{f(v)}$.

This result directly follows the Theorem and is intuitive. With the type-dependent outside option, a low reserve price has the merit of increasing the turnout of the sale. Roughly speaking, with the competition of another seller, the seller of interest must decrease the price in order to attract more potential buyers.

For a large market, we can obtain a more detailed characterization.

Proposition 3. *Let* $N \to \infty$*, then the optimal show-up fee is 0, and the optimal reserve price* $\underline{v}$ *satisfies* $\underline{v} - \frac{(F(p)-F(\underline{v}))f(p)}{f(\underline{v})} = 0$.

When $N \to \infty$, only buyers with values $v \leq p$ visit the seller. The highest bid is practically the fixed price of the outside option. Since the participants of the sale all have a zero-valued outside option, the optimal reserve price hence coincides with the Myerson's optimal reserve price, given that buyers' private values are distributed according to the original distribution truncated at p.

3.4 Conclusion and discussion

In this paper I study an optimal mechanism design problem for a seller when buyers have a type-dependent outside option. I model the outside option as the opportunity to buy items of the same or higher quality at a fixed price from somewhere else. I relax the full participation requirement imposed in Krishna and Perry, 1998, so that the seller's mechanism specifies the allocation and payment rules in all possible cases of who show up in her sale. Despite the complication in the seller's mechanism, I derive an optimal mechanism that is fairly simple to characterize: a second price auction conducted among buyers who participate in the seller sale, with either a reserve price or a fixed show-up fee. In addition, the optimal mechanism leads to a segregation of the market: low value buyers exit the market, medium value buyers visit the seller, in hope of getting the item at a price lower than the fixed price of the outside option, and buyers with high value exercise their outside option.

Reiss, 2008 and Kirchkamp, Poen, and Reiss, 2009 consider the outside option from a different perspective. In their models, the outside option is a payoff received by a buyer if he fails in the auction. This idea can be incorporated into my model by keeping the outside option available after the seller's sale. This additional trading opportunity for the buyers benefits and hurts the seller at the same time. On the one hand, buyers are more willing to participate in the seller's sale, since they face less opportunity costs by doing so. On the other hand, the additional payment that buyers can get when they fail the auction provides an incentive of underbidding (Kirchkamp, Poen, and Reiss, 2009). To induce truth-telling among buyers, the seller has to offer more as the information rent than Myerson, 1981. Furthermore, the new informational rent is increasing in buyers' types. These effects complicate the optimal mechanism design problem substantially, since the seller's revenue is not necessarily maximized by assigning the item to a buyer with the highest bid. It calls for more future work to get a detailed characterization of the optimal mechanism in this setup.

3.5 Appendix

Proof of Proposition 1

Proof. Let $r_i(v)$ denote the probability of Buyer i visiting the seller when his private value is v. Since we only consider symmetric equilibrium, $r_i(v) = r(v)$ for all $i \in \mathcal{N}$. Given the participation strategy, the probability of buyers in a set $\mathcal{J} \subset \mathcal{N}$ visiting Seller 2 is

$$R^{\mathcal{J}}(\mathbf{v}) = \Pi_{j\in\mathcal{J}} r(v_j) \Pi_{l\in\mathcal{N}\setminus\mathcal{J}} (1 - r(v_l)) \tag{3.6}$$

By definition, $0 \leq R^{\mathcal{J}}(\mathbf{v}) \leq 1$, and for any $\mathbf{v} \in \mathcal{V}$,

$$\sum_{j=0}^{N+1} \sum_{\mathcal{J}\subset\mathcal{N}, |\mathcal{J}|=j} R^{\mathcal{J}}(\mathbf{v}) = 1 \tag{3.7}$$

First consider buyers with value $v > p$. It suffices to show that in equilibrium (a) it is not the case that buyers with values on a nonzero measure set are indifferent between visiting the seller and the outside option; and (b) if there exists a $v_0 > p$ such that the surplus of a buyer with v_0 is larger by visiting the seller, then it is also the case for buyers with value $v \in (p, v_0)$.

Suppose $r(\cdot)$ is an equilibrium participation strategy for buyers under an incentive compatible direct mechanism $\{\hat{Q}^{\mathcal{I}}, \hat{M}^{\mathcal{I}}\}_{\mathcal{I}\subset\mathcal{N}}$. If $r(v) = 0$ for $v \in (p, 1)$ almost everywhere, which means in equilibrium almost no buyers with value above p visit the seller, set $\bar{v} = p$ and we are done. Otherwise, by (3.6), $R^{\emptyset}(\mathbf{v}_{-i}) < 1$ for $\mathbf{v}_{-i}$ in a nonzero measure set.

The surplus of a buyer from visiting the seller is

$$S(v_i) = k\int_{\mathcal{V}_{-i}} \sum_{j=0}^{N} \sum_{\mathcal{J}\subset\mathcal{N}\setminus\{i\},|\mathcal{J}|=j} \left(Q_i^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}})v_i - M_i^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}})\right) R^{\mathcal{J}}(\mathbf{v}_{-i})\mathbf{g}(\mathbf{v}_{-i})d\mathbf{v}_{-i} \quad (3.8)$$

For notational simplicity, let

$$\rho(v_i) = \int_{\mathcal{V}_{-i}} \sum_{j=0}^{N} \sum_{\mathcal{J}\subset\mathcal{N}\setminus\{i\},|\mathcal{J}|=j} Q_i^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}})R^{\mathcal{J}}(\mathbf{v}_{-i})\mathbf{g}(\mathbf{v}_{-i})d\mathbf{v}_{-i} \quad (3.9)$$

$$t(v_i) = \int_{\mathcal{V}_{-i}} \sum_{j=0}^{N} \sum_{\mathcal{J}\subset\mathcal{N}\setminus\{i\},|\mathcal{J}|=j} M_i^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}})R^{\mathcal{J}}(\mathbf{v}_{-i})\mathbf{g}(\mathbf{v}_{-i})d\mathbf{v}_{-i} \quad (3.10)$$

By incentive compatibility, $S(v)$ is increasing in v for $v \in \{v : r(v) > 0\}$. In addition, for any $v, v' \in \{v : r(v) > 0\}$

$$(\rho(v) - \rho(v'))v \geq t(v) - t(v') \geq (\rho(v) - \rho(v'))v'$$

Now let's take a closer look at $\rho(v)$, which is the probability of a buyer with value v winning the item from the seller if he visits the seller.

Lemma 3. $0 \leq \rho(v) \leq 1$. *In addition, if* $R^{\emptyset}(\mathbf{v}_{-i}) < 1$ *for* $\mathbf{v}_{-i}$ *in a nonzero measure set,* $\rho(v) = 1$ *only for* v *in a zero-measure subset of* $[0, 1]$.

Proof. By (3.7) and the fact that $0 \leq Q_i \leq 1$, we have $0 \leq \rho(v) \leq 1$.

To prove the second part, suppose towards a contradiction that $\rho(v) = 1$ for v on a nonzero measure set $\mathcal{S} \subset [0, 1]$. By (3.9) and (3.6), $\rho(v) = 1$ only if $Q_i^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}}) = 1$ for some $\mathcal{J}$ and for $\mathbf{v}^{\mathcal{J}} \in [0, 1]^{|\mathcal{J}|}$ almost everywhere. So $Q_i^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}}) = 1$ for $\mathbf{v} \in \mathcal{S}^{|\mathcal{J}|+1} \subset [0, 1]^{|\mathcal{J}|+1}$ almost everywhere as well. Since $R^{\emptyset}(\mathbf{v}_{-i}) < 1$, we must have a $\mathcal{J}$ such that $Q_i^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}}) = 1$ for $\mathbf{v}^{\mathcal{J}\cup\{i\}} \in \mathcal{S}^{|\mathcal{J}|+1}$ almost everywhere and $|\mathcal{J}| > 0$. By anonymity, for any $j \in \mathcal{J}$,$Q_j^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}}) = 1$ also holds for $\mathbf{v} \in \mathcal{S}^{|\mathcal{J}|+1}$. Then $\sum_{l\in\mathcal{J}} Q_j^{\mathcal{J}\cup\{i\}}(\mathbf{v}^{\mathcal{J}\cup\{i\}}) = |\mathcal{J}| + 1 > 1$, violating the requirement for Q to be an allocation rule. $\square$

From Lemma 3 we can get another important result: in equilibrium buyers with almost all values use a pure participation strategy:

Lemma 4. $r(v) \in \{0, 1\}$ *for* $v \in [p, 1]$ *almost everywhere.*

Proof. By assumption, $r(v) > 0$ for v in a nonzero measure subset of $[0, 1]$. If a buyer uses a mixed participation strategy, i.e. $r(v) \in (0, 1)$, she must be indifferent between visiting the seller and the outside option. In other words, $S(v) = v - p$. This implies on a non-zero measure set, given any pair of $v, v + \epsilon$ where $\epsilon > 0$, we need $v + \epsilon - S(v + \epsilon) = v - S(v)$. Substitute and rearrange, and we need

$$(1 - k\rho(v + \epsilon))\epsilon + k(t(v + \epsilon) - t(v)) - k(\rho(v + \epsilon) - \rho(v))v = 0 \tag{3.11}$$

Note that $t(v + \epsilon) - t(v) \geq (\rho(v + \epsilon) - \rho(v))v$, (3.11) implies

$$(1 - k\rho(v + \epsilon))\epsilon \leq 0 \tag{3.12}$$

Since $k \leq 1$ and $\rho(v) = 1$ only on a zero-measure set, (3.12) cannot hold on a non-zero set of v. $\square$

Suppose a buyer with value v_0 does not participate in equilibrium. Now we show that any buyer with value $v > v_0$ should also exercise their outside option. Suppose not, i.e. there exists a $v' > v_0$ such that the buyer with value v' participate in the auction. It means

$$v' - p \leq \max_{\hat{v}} k(\rho(\hat{v})v' - t(\hat{v})) = k(\rho(v')v' - t(v'))$$

But for v_0 we have

$$v_0 - p \geq \max_{\hat{v}} k(\rho(\hat{v})v_0 - t(\hat{v})) \geq k(\rho(v')v_0 - t(v'))$$

By assumption, $v' - p > v_0 - p$. Therefore,

$$v' - v_0 \leq k(\rho(v')v' - t(v')) - k(\rho(v')v_0 - t(v')) = k\rho(v')(v' - v_0)$$

Since $k\rho(v') \leq 1$ and the equality holds only on a zero-measure set, we reached a contradiction. As a result, the equilibrium participation strategy must be a threshold strategy.

At last we consider the equilibrium participation strategy for buyers with value below p. Since we apply the convention that if a buyer receives a zero surplus both from the seller and the outside option, she exits the market, then we only need to prove that if there exists a $u_0 < p$ such that the surplus of a buyer with u_0 is bigger by visiting the seller, then it is still the case for buyers with value $v \in (u_0, p)$. This follows directly from $S(v) \geq k(\rho(u_0)v - t(u_0)) \geq k(\rho(u_0)v - t(u_0)) \geq 0$. So we complete the proof. $\square$

Proof of Lemma 1

Proof. The uniqueness holds for a more general set of mechanisms.

Definition 1 (Positive responsiveness)**.** *Let* $\mathbf{v}^\Lambda = (v_k)_{k\in\Lambda}$*. Call an allocation rule* positively responsive *if both of the following conditions are satisfied:*

1. *for any* $\mathcal{J}, \mathcal{I} \subset \mathcal{N}$*,* $\mathcal{J} \subset \mathcal{I}$ *implies* $Q_i^{\mathcal{J}}(\mathbf{v}^{\mathcal{J}}) \geq Q_i^{\mathcal{I}}(\mathbf{v}^{\mathcal{I}})$ *for any* $i \in \mathcal{I} \cap \mathcal{J}$*, and for any* $\mathbf{v} \in \mathcal{V}$*;*

2. *for any* $\mathcal{J}, \mathcal{I} \subset \mathcal{N}$ *satisfying* $\mathcal{J} \backslash (\mathcal{J} \cap \mathcal{I}) \neq \emptyset$*,* $\mathcal{I} \backslash (\mathcal{J} \cap \mathcal{I}) \neq \emptyset$*, and* $\mathcal{J} \cap \mathcal{I} \neq \emptyset$*, if* $v_k < v_l$ *for any* $k \in \mathcal{J} \backslash (\mathcal{J} \cap \mathcal{I})$ *and* $l \in \mathcal{I} \backslash (\mathcal{J} \cap \mathcal{I})$*, then* $Q_i^{\mathcal{J}}(\mathbf{v}^{\mathcal{J}}) \geq Q_i^{\mathcal{I}}(\mathbf{v}^{\mathcal{I}})$ *for any* $i \in \mathcal{I} \cap \mathcal{J}$*, and for any* $\mathbf{v} \in \mathcal{V}$

A direct mechanism is positively responsive *if the allocation rule of the mechanism is positively responsive.*

The positive responsiveness of a mechanism restricts the dependence of the mechanism and the number of buyers who show up. The first condition requires a buyer's probability of winning the item to be nonincreasing with the number of buyers who participate. And the second condition requires a buyer's probability of winning to be nonincreasing when the "average value" of visitors is higher. In particular, the mechanism we construct in the proof of Proposition 2 is positively responsive.

Now we prove the uniqueness of equilibrium for positive responsive mechanisms.

By Proposition 1, the only way of having multiple equilibria is that there are different thresholds. Suppose towards a contradiction that both $(\underline{u}_1, \bar{u}_1)$ and $(\underline{u}_2, \bar{u}_2)$ are equilibria thresholds and $(\underline{u}_1, \bar{u}_1) \neq (\underline{u}_2, \bar{u}_2)$. Then it can be exactly one of the following six cases:

1. $\underline{u}_1 < \underline{u}_2$, $\bar{u}_1 < \bar{u}_2$;

2. $\underline{u}_1 \geq \underline{u}_2$, $\bar{u}_1 < \bar{u}_2$;

3. $\underline{u}_1 < \underline{u}_2$, $\bar{u}_1 = \bar{u}_2$;

4. $\underline{u}_1 > \underline{u}_2, \bar{u}_1 > \bar{u}_2$;

5. $\underline{u}_1 \leq \underline{u}_2, \bar{u}_1 > \bar{u}_2$;

6. $\underline{u}_1 > \underline{u}_2, \bar{u}_1 = \bar{u}_2$;

Note that Cases 4-6 can be obtained from Cases 1-3 by switching labels, so we only need to consider the first three cases.

Figure 3.1 illustrates some of the cases:

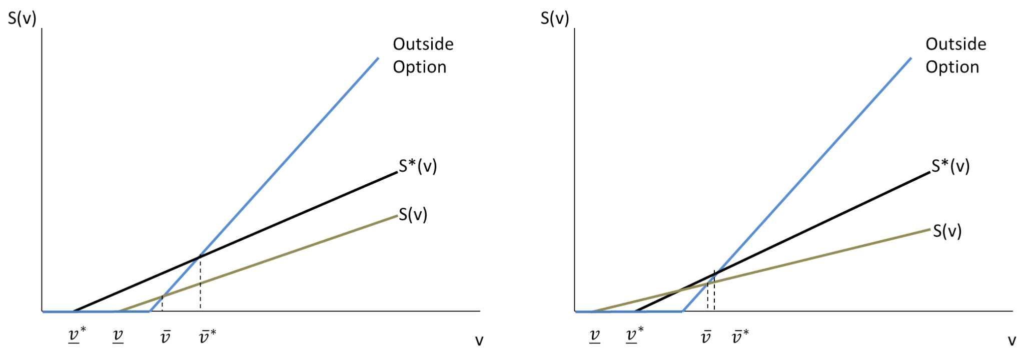

Figure 3.1: Uniqueness of equilibrium participation strategy

Denote buyers' surplus from visiting the seller in each equilibrium as $S^{(1)}$ and $S^{(2)}$, and the set of buyers visiting the seller given a realized value $\mathbf{v}$ is $\mathcal{J}_1(\mathbf{v})$ and $\mathcal{J}_2(\mathbf{v})$.

For Case 1, there must exist buyers with value $v \in (\underline{u}_2, \bar{u}_1)$ such that $S^{(1)} < S^{(2)}$. On the other hand, given $\mathbf{v}$, in equilibrium it must be $v_k < v_l$ for any $k \in \mathcal{J}_1(\mathbf{v}) \backslash (\mathcal{J}_1(\mathbf{v}) \cap \mathcal{J}_2(\mathbf{v}))$, and for any $l \in \mathcal{J}_2(\mathbf{v}) \backslash (\mathcal{J}_1(\mathbf{v}) \cap \mathcal{J}_2(\mathbf{v}))$. So by Definition 1, we must have $Q_i^{\mathcal{J}_1(\mathbf{v})}(\mathbf{v}^{\mathcal{J}_1(\mathbf{v})}) \geq Q_i^{\mathcal{J}_2(\mathbf{v})}(\mathbf{v}^{\mathcal{J}_2(\mathbf{v})})$. By (3.9), this implies $S^{(1)}(v) \geq S^{(2)}(v)$. This is a contradiction.

For Case 2, buyers with value $v \in (\underline{u}_1, \bar{u}_1)$ must have $S^{(1)} < S^{(2)}$. However, since $\underline{u}_1 \geq \underline{u}_2$ and $\bar{u}_1 < \bar{u}_2$, the set of buyers visiting Seller 2 in equilibrium must be $\mathcal{J}_1(\mathbf{v}) \subset \mathcal{J}_2(\mathbf{v})$ for any $\mathbf{v}$. By the definition of positive responsiveness, we have $Q_i^{\mathcal{J}_1(\mathbf{v})}(\mathbf{v}^{\mathcal{J}_1(\mathbf{v})}) \geq Q_i^{\mathcal{J}_2(\mathbf{v})}(\mathbf{v}^{\mathcal{J}_2(\mathbf{v})})$, and this implies that $S^{(1)} \geq S^{(2)}$. So we have reached a contradiction again.

For Case 3, the argument is similar to Case 2, only in this case we should have $S^{(1)} > S^{(2)}$ for buyers with value $v \in (\underline{u}_2, \bar{u}_2)$, but the positive responsiveness requires $S^{(1)} \leq S^{(2)}$.

As a result, none of the cases can happen, so we must have $(\underline{u}_1, \bar{u}_1) = (\underline{u}_2, \bar{u}_2)$. This completes the proof of uniqueness. □

Proof of Lemma 2

Proof. First we introduce the notation of the *reduced forms* of allocation and payment rules:

$$q_i(v) = \int_{\mathcal{V}_{-i}} Q_i(v, \mathbf{v}_{-i}) g_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \tag{3.13}$$

$$m_i(v) = \int_{\mathcal{V}_{-i}} M_i(v, \mathbf{v}_{-i}) g_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \tag{3.14}$$

Since the distribution of buyers' values are identical and the mechanism does not depend on identities, we have $q_i(v) = q(v)$, and $m_i(v) = m(v)$ for all $i \in \mathcal{N}$.

For incentive compatibility to be satisfied, we need it to be optimal for a buyer to report her type truthfully if she visits Seller 2. Let $\hat{v}$ denote the reported type, and then a buyer's surplus from visiting the seller is

$$S(v) = \max_{\hat{v}} k(q(\hat{v})v - m(\hat{v})) \tag{3.15}$$

$$= k(q(v)v - m(v)) \tag{3.16}$$

Lemma 5. *$S(v)$ is continuous and convex in v. In addition,*

$$S(v) = S(0) + \int_0^v kq(x)dt \tag{3.17}$$

This lemma simply follows from the requirement of incentive compatibility and the envelope theorem.

To guarantee the convexity of $S(v)$, we need that $S'(v)$ is nondecreasing in v. So

Condition (SOIC)**.** *The reduced forms of the allocation rule $q(v)$ satisfies the second order incentive compatibility (SOIC) constraint if $q(v)$ is nondecreasing in v.*

By (3.15) and (3.17), we have

$$km(v) = kvq(v) - S(0) - \int_0^v kq(x)dx \tag{3.18}$$

By Proposition 1, the equilibrium participation strategy $r(v) = 1$ if and only if $v \in [\underline{v}, \bar{v}]$. Then

$$\mathbb{E}(kr(v)m(v)) = -S(0)\int_{\underline{v}}^{\bar{v}} f(v)dv + k\int_{\underline{v}}^{\bar{v}} q(v)f(v)\phi(v)dv \tag{3.19}$$

where

$$\phi(v) = v - \frac{1-F(v)}{f(v)} + \frac{1-F(\bar{v})}{f(v)} \tag{3.20}$$

By (3.15) we have $S(0) = -km(0)$, and plug in (3.13) to get

$$\begin{aligned}\sum_{i\in\mathcal{N}} \mathbb{E}(km_i(v_i)) = (N+1)km(0)(F(\bar{v}) - F(\underline{v})) \\ + \int_{\mathcal{V}} g(\mathbf{v})\sum_{i\in\mathcal{N}}(kQ_i(v_i, v_{-i})\phi(v_i))d\mathbf{v}\end{aligned} \tag{3.21}$$

To guarantee that $\underline{v}$ and $\bar{v}$ define an equilibrium participation strategy, we need the buyer with value $\bar{v}$ to be indifferent between visiting the seller and the outside option when $\bar{v} < 1$. When $\bar{v} = 1$, we need that visiting the seller makes the buyer with the highest value weakly better off comparing to buying from the outside option. Formally, we need

$$-km(0) + k\int_{\underline{v}}^{\bar{v}} q(x)dx = \bar{v} - p \qquad \text{if } \bar{v} < 1 \tag{3.22}$$

$$-km(0) + k\int_{\underline{v}}^{\bar{v}} q(x)dx \geq \bar{v} - p \qquad \text{if } \bar{v} = 1 \tag{3.23}$$

Similarly, if $\underline{v}$, the buyers with $v = \underline{v}$ must be indifferent between visiting the seller and the outside option if $\underline{v} > 0$. If $\underline{v} = 0$, we need to have the buyer with the lowest value to be weakly better off to visit the seller. Since buyers with value below $\underline{v}$ have no chance to win the item, we have

$$S(\underline{v}) = S(0) + \int_0^{\underline{v}} q(x)dx = S(0) = -km(0)$$

The participation constraint for buyers with value $\underline{v}$ is simply

$$-km(0) = 0 \qquad \text{if } \underline{v} > 0 \tag{3.24}$$

$$-km(0) \geq 0 \qquad \text{if } \underline{v} = 0 \tag{3.25}$$

And at last

$$0 \le \underline{v} \le p \le \bar{v} \le 1 \tag{3.26}$$

To summarize, the objective is

$$\max_{Q,m(0),\bar{v},\underline{v}} (N+1)km(0)(F(\bar{v}) - F(\underline{v}))$$
$$+ \int_{\mathcal{V}} g(\mathbf{v}) \sum_{i \in \mathcal{N}} (kQ_i(v_i, v_{-i})\phi(v_i) d\mathbf{v}$$

subject to the participation constraints in (3.22) to (3.26), the second order incentive compatibility constraint, and the feasibility constraints which guarantees that $Q(\cdot)$ is indeed an allocation rule

$$Q_i(\mathbf{v}) \ge 0 \quad \forall i \in \mathcal{N}, \quad \forall \mathbf{v} \in \mathcal{V} \tag{3.27}$$
$$\sum_{i \in \mathcal{N}} Q_i(\mathbf{v}) \le 1 \quad \forall \mathbf{v} \in \mathcal{V} \tag{3.28}$$

We focus on the participation constraints and ignore other conditions for now, and we can write down the Lagrangian for the problem:

$$\begin{aligned}\mathcal{L}(Q, m(0), \bar{v}, \underline{v}; \lambda) = {} & (N+1)km(0)(F(\bar{v}) - F(\underline{v})) \\ & + \int_{\mathcal{V}} g(\mathbf{v}) \sum_{i \in \mathcal{N}} (kQ_i(v_i, v_{-i})\phi(v_i)) d\mathbf{v} \\ & + \lambda \sum_{i \in \mathcal{N}} \left(-km(0) + k \int_{\underline{v}}^{\bar{v}} q_i(x) dx - (\bar{v} - p) \right)\end{aligned}$$

Rearrange to get

$$\begin{aligned}\mathcal{L}(Q, m(0), \bar{v}, \underline{v}; \lambda) = {} & (N+1)km(0)(F(\bar{v}) - F(\underline{v}) - \lambda) - (N+1)\lambda(\bar{v} - p) \\ & + \int_{\mathcal{V}} g(\mathbf{v}) \sum_{i \in \mathcal{N}} (kQ_i(v_i, v_{-i})\psi(v_i)) d\mathbf{v}\end{aligned}$$

This is the objective function in (3.4). □

Proof of Proposition 3.2

Proof. It is straightforward to see from the objective function that when the distribution of private types has a non-decreasing hazard rate and a non-increasing density function, the optimal allocation rule is to assign the item with a visitor with the highest private value. We only need to drive the payment rule.

By (3.18), we have

$$\int_{\mathcal{V}_{-i}} kM_i(v_i, \mathbf{v}_{-i}) g_{-i}(\mathbf{v}_{-}i) d\mathbf{v}_{-i} = \int_{\mathcal{V}_{-i}} \left(km(0) + kv_i Q_i - \int_0^{v_i} kQ_i(x, \mathbf{v}_{-i}) dx \right) g_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}$$

A simple way to have this equality is to set

$$kM_i(v_i, \mathbf{v}_{-i}) = km(0) + kv_i Q_i - \int_0^{v_i} kQ_i(x, \mathbf{v}_{-i}) dx \tag{3.29}$$

According to the allocation rule, if a buyer with value v loses, she still loses the item when she has a value below v; while if a buyer wins, she still wins as long as her value is above both the second highest value and having a positive adjusted virtual value. As a result, from (3.29) we can derive

$$M_i(v_i, \mathbf{v}_{-i}) = \begin{cases} m(0) + \max\{v^{(2)}, \psi^{-1}(0)\} & Q_i = 1 \\ m(0) & Q_i = 0 \end{cases}$$

where $v^{(2)}$ denotes the second highest value among the seller's visitors.

We can interpret $m(0)$ as a show-up fee and $\psi^{-1}(0)$ as the reserve price. Furthermore, $\underline{v} = \psi^{-1}(0)$. From the proof of Lemma 2 we can see that only when $\underline{v} = 0$, $m(0)$ can be strictly negative. As a result the optimal mechanism is a second price auction with either a reserve price or a show-up fee, but not both. $\square$

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Chapter 4

OPTIMAL GRANT-ISSUING MECHANISMS

4.1 Introduction

Grants are an important source of both private- and government-based funding for various types of projects (science, education innovation, support for developing societies, etc.). More than five trillion dollars of US Federal grants have been awarded since 2008, accounting for about 20% of total government spending.[1] Private foundations also award billions of dollars in grants each year in areas such as health and education to improve social welfare.[2] Despite the large size and common use of grants in practice, there are few theoretical studies on how to efficiently allocate a grant-issuer's budget. This paper aims to derive an optimal grant issuing method under a mechanism design framework.

To capture the essence of the grant issuing process, this paper considers the following model. To apply for a grant, applicants need to simultaneously submit their project proposals to the grant issuer. Each proposed project, if carried out, generates a value composed of two parts: a public value and a private value. The public value, or the "merit" of a proposal, is a benefit received by the whole society. The private value is a net benefit received exclusively by each proposer. It can be interpreted as the personal gain net of the personal cost. For example, a computer scientist who receives a grant from the National Science Foundation (NSF) can conduct a project which improves the performance of a widely used algorithm (public value). In addition, the receipt of the grant per se is an honor to the scientist (personal gain). On the other hand, to conduct the project, the scientist needs to bear its cost on equipment and overhead (personal loss). For each project, the grant issuer can assess its public value perfectly, but has little knowledge about its private value besides the amount reported by the applicant in the proposal. Each proposer's utility comes from the private value of his project and the payment of the grant if he is awarded. In particular, the public value does not concern proposers. In contrast, the grant issuer cares about both the public value and private value, i.e. the total

[1] Other types of spending include contracts, loans, direct payments to individuals, insurance payments, etc. Data source: https://www.usaspending.gov/Pages/TextView.aspx?data=OverviewOfAwardsByFiscalYearTextView

[2] http://data.foundationcenter.org/#/fc1000/subject:all/all/total/list/2012

welfare. The goal of this paper is to design proper rules that determine which project are supported by the grant and how much to pay for the selected projects, such that the total welfare is maximized.

Before tackling that problem, we need to discuss two sets of institutional constraints: the budget constraint and the constraint of individual rationality. In different scenarios, there are different specifications of these constraints.

Oftentimes, the grant issuer faces a budget. For example, governmental funding agencies may face a budget by the congress appropriation. Similarly, private foundations may be bound by donations. In principle, having some flexibility over the total amount of expenditures could be beneficial. Such flexibility would allow the grant issuer to adjust the overall spending according to the quality of the submitted proposals. Specifically, if the grant issuer is allowed to smooth her budget across budget sessions, when there is very few high quality proposals in a session, she can save her budget for sessions when a lot of proposals show high quality. In contrast, if the budget is fixed within each session, even proposals of high quality could lose the opportunity to get funded. This is the case when many other high quality proposals happen to be submitted in the same session. Intuitively, variable funding amounts over time and flexible application deadlines suggest a potentially more flexible budget constraint, such as the Justfilms grant of the Ford Foundation and Grants for Art and Culture of Japan Foundation.[3,4] In many other cases, the budget is fixed and a deficit is not allowed. For example, for the grant supporting studies on human trafficking in the Asia-Pacific region, the fixed number of rewards and the fixed amount of payment for each selected project indicate a strict of budget of 35,000,000.[5] In this paper, both forms of budget are discussed.

Another set of constraints is about how the grant issuer can manage to have all proposers apply for the grant. For each proposer, if he expects the benefit is less from applying for the grant than what he could have obtained on his own, he will not apply. The grant issuing process needs to avoid such situations in order to encourage as many applications as possible. This constraint is usually called the constraint of individual rationality in literature. We should note that in the context of grant application, it is not obvious how to calculate what a proposer can get from applying to the grant. In this paper I consider two different scenarios.

[3] http://www.fordfoundation.org/work/our-grants/justfilms/

[4] http://www.jfny.org/arts_and_culture/smallgrant.html

[5] goo.gl/yVXYss

In the first scenario, there are no contracts or legal obligations for the grant applicant to conduct the project upon the receipt of the grant. In other words, if a proposer is awarded the grant, but realizes the size of the grant is insufficient to cover the cost of the project, he can reject the grant at no cost. For the grant issuer, a rejection of a grant is the same as, if not worse than, the proposer does not apply for the grant. To avoid such rejection, the grant issuing process needs to guarantee enough payment to cover the costs of selected proposals. This case is referred to as ex post individual rationality.

The second scenario is that contracts, legal obligations, or potential value of the grant forbid the proposer from turning down a grant if he gets rewarded. For example, the receipt of a grant from NSF provides a shining entry on the proposer's resume, as it can be used as an indicator to the proposer's credibility for future funding opportunities. As a result, even if the grant is insufficient to cover cost of the project, the proposer is still willing to accept the grant and conduct the project. In this case, the grant issuer does not need to worry about the grant being turned down. To encourage applications, she only needs to make sure the proposers are better off in expectation by applying to the grant. This is referred to as interim individual rationality.

In general, any outcome of a mechanism subject to a fixed budget can be achieved under a flexible budget of the same size. Therefore, the grant issuer should obtain a higher social welfare when the budget is flexible. It turns out that whether the flexible budget provides a strictly better outcome depends on the forms of individual rationality: for interim individual rationality, the results under fixed and flexible budget constraints are the same; for ex post individual rationality, the welfare under a flexible budget is strictly higher.

Specifically, if the grant issuer faces a flexible budget, the optimal mechanism does not depend on the specification of individual rationality. The mechanism can be implemented by take-it-or-leave-it offers made from the grant issuer to proposers. Each offer specifies a maximum amount of payment to each proposer. Intuitively, this offered payment increases as the budget increases. Furthermore, the higher public value a project generates, the larger amount offered by the grant issuer.

If the grant issuer faces a fixed budget, the form of optimal mechanisms differs with different specifications of individual rationality. If interim individual rationality is considered, in the optimal mechanism, a proposal is selected only when the reported private value exceeds a cutoff. The payment depends on other proposers' private values. This mechanism generates the same welfare for the grant issuer if she faces a flexible budget of the same size. If ex post individual rationality is considered, the cutoff selection rule is no longer optimal. In other words, whether a proposer is selected can depend on the reported private value of other proposers.

The framework of this paper is closely related to the frameworks used in the research of auctions and procurements. However, some features special to the context of grant issuing process depart this paper from studies of those areas. In auctions, the total number of objects is usually exogenously given.[6] In contrast, the total number of awarded grants can be contingent on the number and quality of proposed projects. In procurements, the quantity of delivered objects is usually assumed to be a non-negative real number, but the outcome of a proposal is binary: either it is carried out successfully, or it fails to deliver.[7] These differences largely simplify the analysis and result in a tractable optimal mechanism even when there is asymmetry between grant proposers, who serve the role of bidders in the current paper.[8]

This paper also contributes to the literature comparing and connecting different specifications of the budget constraint and individual rationality in various contexts (Kosmopoulou, 1999). The relation of interim and ex post individual rationality has been discussed extensively in the bilateral trading literature (Myerson and Satterthwaite, 1983, Gresik, 1991, Flesch, Schröder, and Vermeulen, 2016, etc.). In contrast, as is pointed out by Severinov and Song, 2010, it has not been sufficiently explored to connect two types of individual rationality constraints in the context involving multiple participants. For this context, Makowski and Mezzetti, 1994 established the equivalence between ex ante budget balance with ex post individual rationality and ex post budget balance with interim individual rationality. This paper further discusses the case with ex post budget constraint and ex post individual rationality.

For the literature on grants, Ensthaler and Giebe, 2014 reviewed empirical studies that analyze the effects of grants on innovation activities. Comparatively, there are few papers theoretically analyzing the grant issuing process. Giebe, Grebe, and Wolfstetter, 2006 point out the inefficiency of the widely applied merit-based mechanism. They emulated grant applications in a first-price auction setting in a

[6] See Krishna, 2009 for a review.

[7] See Laffont and Tirole, 1993 for a review on studies of procurements. Hainz and Hakenes, 2012 considers different success possibilities of projects in a different context.

[8] The introduction of asymmetric bidders in auctions generally brings a lot of complications. See for example Hubbard, Kirkegaard, et al., 2015

lab experiment and obtained an improvement in efficiency. Their work was later extended by Ensthaler and Giebe, 2014. They propose an incentive compatible auction to further improve the efficiency of grant allocation. This paper studies the grant issuing process in a more general setting and focus optimal mechanisms.

The rest of this paper is organized as follows. In Section 2 I describe the formal model. Sections 3 and 4 discuss the derivation of an optimal mechanism under flexible and fixed budget constraints, respectively. Section 5 concludes. All technical results and details of proofs are relegated to the appendix.

4.2 Model

There are N proposals for a grant. Each proposer, if his proposal is selected by the grant issuer, will conduct the project and generate a value composed of two parts: a public value and a private value. The public value, or the "merit" of a proposal, is the benefit to the whole society and is assumed to be perfectly evaluated by the grant issuer. Let $\mathbf{s} = (s_1, \ldots, s_N)$ denote the vector of public values. The private value is a net benefit received exclusively by each proposer. It can be interpreted as the personal gain net of the cost. The personal gain can be in forms such as reputation improvement or a sense of achievement, brought by the project to its proposer; the cost is the opportunity cost of each proposer to conduct the proposed project. Let $\mathbf{v} = (v_1, \ldots, v_N)$ denote the vector of private values. Each v_i is privately known to proposer i. I assume each private value is distributed according to a CDF $F_i(\cdot)$ associated with a PDF $f_i(\cdot)$ with full support on $\mathcal{V}_i = [\underline{\nu}_i, \bar{\nu}_i]$. The distributions of private values are independent. Denote by $F(\cdot)$ the joint CDF of private values, and $\mathcal{V} = \times_{i=1}^{N} \mathcal{V}_i$ the support of the joint distribution. I assume that the distributions of private values are common knowledge.

There is heterogeneity over two aspects of the model: the various public values of participants and the different distributions of private values. In particular, I allow for the distribution of a proposal's private value to be depend on its public value.

Each proposer can decide whether or not to apply for the grant after observing his private value. The proposers who apply for the grant submit proposals simultaneously. If a proposer is not selected by the grant issuer, I assume that he can choose whether or not to carry out the project on his own. If he does, he will generate the same public value and private value, but he receives no funds from the grant; if not, no value will be generated. The equilibrium concept I use is the Bayesian Nash Equilibrium.

By the revelation principle, it is without loss of generality to focus on incentive compatible direct mechanisms. For direct mechanisms, the action that a proposer can take is to report his private value $\hat{v}_i \in \mathcal{V}_i$. The selection and payment results are determined by these reports. Formally, $\mathbf{q}(\hat{\mathbf{v}}) = (q_i(\hat{\mathbf{v}}))_{i=1,\ldots,N}$, and $\mathbf{t}(\hat{\mathbf{v}}) = (t_i(\hat{\mathbf{v}}))_{i=1,\ldots,N}$. Here, q_i is the probability of proposer i to be selected, and t_i is the payment awarded to proposer i, *given that he is selected.* If proposer i is not selected, it is not without loss of generality to set $t_i = 0$

Each proposer cares only about his private value. So if proposer i reports $\hat{v}_i$ while all other participants report their private values truthfully, his utility given his actual private value v_i is

$$U_i(\hat{v}_i; v_i) = \int_{\mathcal{V}_{-i}} (q_i(\hat{v}_i, \mathbf{v}_{-i})(v_i + t_i(\hat{v}_i, \mathbf{v}_{-i})) + (1 - q_i(\hat{v}_i,\mathbf{v}_{-i})) \max(v_i, 0)) \; dF_{-i}(\mathbf{v}_{-i}) \tag{4.1}$$

Considering the integrand, the first term is the benefit that proposer i can get if he is selected. By assumption, once the proposal is selected, the proposer has to conduct the project and realize a private value of v_i in addition to the payment specified by the mechanism $t_i(\hat{v}_i, \mathbf{v}_{-i})$. The second term is the benefit he can get if not selected. The proposer will decide whether or not to conduct the project without funds from the grant. Since the proposer only cares about the private value v_i, he will not carry out the project unless the private value is non-negative.

Incentive compatibility requires that no proposer has an incentive to misreport given truthful reports of all other proposers. Formally,

Definition 2. *A direct mechanism is* incentive compatible *if*

$$v_i \in \arg\max_{\hat{v}_i} U_i(\hat{v}_i; v_i), \tag{4.2}$$

where $U_i(\hat{v}_i; v_i)$ is defined in (4.1).

In addition, individual rationality deems it unacceptable for proposers to obtain less by applying to the grant than what they could have obtained on their own. In order to encourage as many applications as possible, the grant mechanism needs to satisfy the constraint of individual rationality, i.e. to guarantee that no proposer suffers a loss from applying for the grant. Since the calculation of a proposer's utility from applying to the grant varies in different scenarios, different forms of the constraint of individual rationality need to be considered.

In the first scenario, proposers can turn down a grant after getting rewarded, i.e. to freely walk away from the mechanism after the realization of selection and payment outcome. To avoid losing proposers in the end, the mechanism needs to satisfy *ex post individual rationality*. In this case, all proposers should be willing to accept *every possible* outcome of the mechanism, compared to what they can get outside of the mechanism. Formally,

Definition 3. *An incentive compatible mechanism* $(\mathbf{q}, \mathbf{t})$ *satisfies ex post individual rationality if and only if*

$$q_i(\mathbf{v})(v_i + t_i(\mathbf{v})) + (1 - q_i(\mathbf{v}) \max\{v_i, 0\}) \geq \max\{v_i, 0\} \tag{4.3}$$

If the proposers face an extremely high cost of rejecting the grant once they are awarded it, to encourage applications, the grant issuing mechanism needs to satisfy *interim individual rationality*. Under this constraint, the utility each proposer can get outside of the mechanism does not exceed his expected utility from applying for the grant.

Definition 4. *A direct incentive compatible mechanism satisfies* interim individual rationality *if*

$$\mathbb{E}_{\mathbf{v}_{-i}} U_i(v_i; v_i) \geq \max\{v_i, 0\} \tag{4.4}$$

for all i *and* $v_i \in \mathcal{V}_i$.

The grant issuer cares about both the public value and the private value. If all proposers report truthfully their private values, the utility function of the grant issuer is

$$W(\mathbf{q}) = \int_{\mathcal{V}} \sum_{i=1}^{N} \left(q_i(\mathbf{v})(s_i + v_i) + (1 - q_i(\mathbf{v}))\mathbb{I}(v_i \geq 0)(s_i + v_i)\right) dF(\mathbf{v}) \tag{4.5}$$

Inside the summation is the total value, public and private, generated by each proposer: the first term shows that if proposer i is selected, he has to carry out the project and hence realize both types of values. The second term comes from the fact that if a proposer is not selected, he only generates a total value of $s_i + v_i$ if the private value v_i is no less than zero.

For simplicity, assume $s_i > -\underline{\nu}_i$ for all i, so that all proposals are desirable for the grant issuer.

The grant issuer faces an exogenously given budget. For example, governmental funding agencies may face a budget by the congress appropriation. Similarly, private foundations may be bound by donations. To represent different flexibility of the budget constraints, I consider two scenarios.

In the first scenario, the budget can be over- or under- spent at the end of the grant issuing process, but before receiving proposals, the grant issuer needs to make sure the *expected* total payment does not exceed the budget B. This budget constraint is referred to as the ex ante budget constraint. Formally,

Definition 5. *A mechanism* $(\mathbf{x}, \mathbf{t})$ *satisfies the* ex ante *budget constraint if*

$$\mathbb{E}_{\mathbf{v}} \sum_{i=1}^{N} q_i(\mathbf{v}) t_i(\mathbf{v}) \leq B \tag{4.6}$$

In contrast, the ex post budget constraint requires the total payment not exceeding the budget for *every possible payment realization*. In particular, no over-spending is allowed. Formally,

Definition 6. *A mechanism* $(\mathbf{x}, \mathbf{t})$ *satisfies the* ex post *budget constraint if*

$$\sum_{i \in \{j : q_j(\mathbf{v}) > 0\}} t_i(\mathbf{v}) \leq B \tag{4.7}$$

4.3 Grants with ex ante budget constraint

In this section, I derive optimal mechanisms when the grant issuer only needs to make sure the expected total payment does not exceed the budget. Depending on whether the proposers can forgo the grant based on the associated payment, we need to solve the following two problems:

Problem 1.

$$\max_{\mathbf{q}, \mathbf{t}} \int_{\mathcal{V}} \sum_{i=1}^{N} \left(q_i(\mathbf{v})(s_i + v_i) + (1 - q_i(\mathbf{v}))\mathbb{I}(v_i \geq 0)(s_i + v_i) \right) dF(\mathbf{v})$$

subject to incentive compatibility (4.2), ex post individual rationality (4.3), and ex ante budget constraint (4.6).

and

Problem 2.

$$\max_{\mathbf{q}, \mathbf{t}} \int_{\mathcal{V}} \sum_{i=1}^{N} \left(q_i(\mathbf{v})(s_i + v_i) + (1 - q_i(\mathbf{v}))\mathbb{I}(v_i \geq 0)(s_i + v_i) \right) dF(\mathbf{v})$$

subject to interim incentive compatibility (4.2), interim individual rationality (4.4), and ex ante budget constraint (4.6).

It turns out that there is a mechanism that solves both problems,[9] as characterized in Proposition 4:

Proposition 4. *Suppose all private value distributions have an increasing hazard rate $\frac{f_i(v_i)}{1-F_i(v_i)}$, then there exist thresholds $\pi = (\pi_1, \dots, \pi_N)$ such that $\pi_i \leq 0$ and the following mechanism $(\mathbf{q}^*, \mathbf{t}^*)$ solves both Problems 1 and 2:*

$$q_i^*(v_i, \mathbf{v}_{-i}) = \begin{cases} 1 & v_i \geq \pi_i \\ 0 & v_i < \pi_i \end{cases} \tag{4.8}$$

$$t_i^*(v_i, \mathbf{v}_{-i}) = \begin{cases} -\pi_i & v_i \geq \pi_i \\ 0 & v_i < \pi_i \end{cases}. \tag{4.9}$$

Proposition 4 narrows down the candidates for the optimal mechanisms to mechanisms with a "cutoff selection rule" and a "fixed price payment rule." In this class of mechanisms, the probability of selecting a proposal i does not depend on the proposer's reported value, as long as it is above the threshold π_i. In addition, the selected proposer receives a fixed amount from the mechanism designer $-\pi_i$. Based on Proposition 4, to find an optimal mechanism, it suffices to find a set of optimal thresholds π such that the mechanism designer's objective function is maximized.

To show this result, we first consider Problem 2. In this case, we can apply the techniques used in Myerson, 1981 and use the Lagrangian multiplier to incorporate constraints of budget and individual rationality into the objective function. It turns out that the optimal mechanism should select proposals with a positive "virtual value." This value is an adjustment of each proposal's private value due to the budget constraint and the sacrifice made by the grant issuer to induce a truthful report. Like in Myerson, 1981, an increasing hazard rate of the distribution of private values guarantees a positive correlation between v_i and the virtual value. As a result, we can simply implement a cutoff selection rule for each participant, in which a proposal will be selected if and only if it generates a non-negative virtual value. It is straightforward to see this selection rule is incentive compatible. In

[9] The solutions to Problems 1 and 2 are not unique. If a mechanism solves one of the problems, any variation of the mechanism on a zero-measure set of private values also solves that problem. This paper characterizes only one of the multiple optimal mechanisms.

addition, it can be verified that the solution also satisfies the ex post individual rationality and hence is also a solution to Problem 1.

Note in particular that $-\pi_i$ is paid to all applicants with private value above π_i, including proposers with $v_i > 0$. This result demonstrates the problem of "crowding-out private funds" in grant issuing process. (Wallsten, 2000) Ideally, grants should be only awarded to projects in need of external financial aids. Proposition 4 shows that when projects can be carried out without grants, they will still apply to the grant and use public funds as a substitute of private funds. This result is led by the constraint of incentive compatibility. To induce applicants report truthfully about their private values, the grant issuer has to pay them more than what is needed to support their projects as "information rents." Otherwise the applicants will extract the information rents in forms of misreporting their private values.

An interesting feature of the cutoff selection rule and the fixed price payment rule is their implementation: instead of collecting all participants-reported private values, the mechanism designer can simply provide each participant a take-it-or-leave-it offer of a payment of $-\pi_i$. This way, a proposer will accept the offer if and only if his private value is above π_i, and the optimal selection would be realized automatically.

To take a closer look at the optimal vector of thresholds π, first note that if $B \geq \sum_{i=1}^{N}(-\underline{\nu}_i)$, i.e. when the budget is large enough to support all proposals, the optimal mechanism can simply set $\pi_i = \underline{\nu}_i$ so that all proposals are selected and paid at the maximal cost. If the budget is relatively small, the derivation of π_i's depends on specific forms of private type distributions and hence becomes much more intricate. In what follows, I provide a more detailed characterization.

Proposition 5. *Suppose $\bar{\nu}_i > 0$, $\underline{\nu}_i < 0$ for all i, and $f_i'(v_i) \geq 0$ for $v_i \in [\underline{\nu}_i, \bar{\nu}_i]$, then for any i, the optimal threshold π_i is weakly increasing with the public value s_i, weakly decreasing in the public value of other projects s_j for $j \neq i$, and weakly decreasing in the budget B.*

Note that $f_i'(v_i)$ implies an increasing hazard rate. So the results in Proposition 5 applies to a smaller set of distributions than Proposition 4.

The impact of public values on the optimal threshold is very intuitive. Since the grant issuer cares about the total of both public and private values, a higher public value of a project can compensate for its low private value. As a result, the grant issuer is more willing to select and pay more for projects with higher public values. On the other hand, since the budget is fixed, a higher payment to one project will result in a lower payment to other projects. Therefore, a higher public value of the other project will lead to a lower chance of selecting any particular project.

It does not mean that the budget should always go to the project with the highest public value and then move on to the project with the second highest public value, etc. until exhausted. This type of mechanisms is often referred to as the merit-based mechanism (Giebe, Grebe, and Wolfstetter, 2006). One problem of the merit-based mechanism is that it does not take the private value of projects into consideration. Setting a high threshold in the selection rule helps screening out projects with low private values. If the grant issuer always supports the project with the highest public value, regardless of its reported private value, when the project's private value is very low, the total welfare generated by this project is no longer the highest among all projects. This is especially true when the difference between public values is not very large. In addition, paying too much money for the project with the highest public value diminishes the funding opportunity for other projects.

For an example, let $\underline{\nu} = -1$ and $\bar{\nu} = 1$. Suppose the budget is $B = 1.5$ and there are only two proposals. Table 4.1 provides optimal thresholds corresponding to different values of s_1 and s_2. By assumption, Project 1 has a higher public value. If it is optimal to always select Project 1, we should see $\pi_1 = -1$. According to table 4.1, this is not the case if the difference between public values of two projects is small.

s_1	s_2	π_1	π_2
2	1	-1	-0.6
1.9	1	-1	-0.6
1.8	1	-1	-0.6
1.7	1	-1	-0.6
1.6	1	-1	-0.6
1.5	1	-1	-0.6
1.4	1	-1	-0.6
1.3	1	-0.9	-0.7
1.2	1	-0.9	-0.7
1.1	1	-0.8	-0.8

Table 4.1: Optimal thresholds of two proposals, $\underline{\nu} = -1$, $\bar{\nu} = 1$, $B = 1.5$

Next we take a closer look at the impact of the budget on the optimal thresholds. Proposition 5 states that if the budget increases, every threshold π_i weakly decreases.

In other words, given the realized private values of projects, an increase in the budget can lead to the selection of some projects that are not selected under a smaller budget. Furthermore, those projects which were selected under the smaller budget will still be selected. Therefore, no projects will be hurt by the budget increase.

Note that the assumption of $\bar{\nu}_i > 0$ plays a crucial rule in this result. Otherwise the increase in budget may no longer yield a Pareto improvement. Suppose there is a project i with extremely high public value compared to all other projects, and $\bar{\nu}_i < 0$. Since the highest possible private value is negative, proposer i will never carry out this project without external funds. If $B < -\bar{\nu}_i$, the budget is not enough to fund it completely, leaving no chance for the project to be carried out. When the budget increases such that $B > -\bar{\nu}_i$, it is possible for the grant issuer to support this extremely beneficial project. To maximize the total welfare, the budget should be entirely used on this project, resulting in all previously selected projects losing their funding.

It should be pointed out that the payment rule in Proposition 4 does not satisfy the ex post budget constraint. For example, if all proposals have private values of exactly π_i, the total payments exceed the budget since $\sum_i^N(-\pi_i) > \sum_i^N(1-F_i(\pi_i)(-\pi_i)) = B$.

In fact, the ex ante budget constraint is a very weak constraint. To see this, consider the following problem:

Problem 3.

$$\max_{\mathbf{q},\mathbf{t}} \int_{\mathcal{V}} \sum_{i=1}^{N} \left(q_i(\mathbf{v})(s_i + v_i) - \lambda q_i(\mathbf{v}) t_i(\mathbf{v})\right) dF(\mathbf{v})$$

subject to incentive compatibility (4.2) and interim individual rationality (4.4), where $\lambda > 0$ is an exogenous constant.

In this problem, the budget constraint is substituted with an additional term in the objective function $-\lambda \sum_i^N q_i(\mathbf{v}) t_i(\mathbf{v})$. In other words, instead of facing a budget constraint, the grant issuer can spend as much as she wants on supporting the proposed projects, but suffers from a cost proportional to her expenditure. The value of λ reflects how sensitive the grant issuer is to the total expenditure.

The objective function in Problem 3 and the Lagrangian of Problem 2 that incorporates the budget constraint are very similar. If we set λ the to be the value of the Lagrangian multiplier that solves Problem 2, or set the budget in Problem 2 equal to the optimal total payment in Problem 3, the two problems are equivalent.

4.4 Grants with ex post budget constraint

In this section, we discuss the mechanism design problem when the grant issuer has a budget that cannot be exceeded after the selection and payment outcome is realized. Like in the previous section, two forms of individual rationality constraints are considered. First, if the proposers can freely exit the mechanism only before the announcement of the selection and payment outcome, we need to solve the following problem:

Problem 4.

$$\max_{\mathbf{q},\mathbf{t}} \int_{\mathcal{V}} \sum_{i=1}^{N} \left(q_i(\mathbf{v})(s_i+v_i) + (1-q_i(\mathbf{v}))\mathbb{I}(v_i \geq 0)(s_i+v_i)\right) dF(\mathbf{v})$$

subject to interim incentive compatibility (4.2), interim individual rationality (4.4), and ex post budget constraint (4.7).

An optimal selection rule to Problem 4 also turns out to be a cutoff rule. Different from Problem 2, the corresponding payment rule becomes more involved.

Proposition 6. *Suppose the distribution function has an increasing hazard rate. Then there exists* $\pi = (\pi_1, \ldots, \pi_N)$ *such that* $\pi_i \leq 0$ *for all* i *and the following mechanism* $(\mathbf{q}^*, \mathbf{t}^*)$ *solves Problem 4:*

$$q_i^*(v_i, \mathbf{v}_{-i}) = \begin{cases} 1 & v_i \geq \pi_i \\ 0 & v_i < \pi_i \end{cases} \tag{4.10}$$

$$t_i^*(v_i, v_{-i}) = \begin{cases} -\pi_i + \frac{1}{N-1}\sum_{j\neq i}\left(-\pi_j(1-F_j(\pi_j)) - T_j^*(v_j)\right) & v_i \geq \pi_i \\ 0 & v_i < \pi_i \end{cases}, \tag{4.11}$$

where $T_i^*(v_i)$ *is the expected payment and satisfies:* $T_i^*(v_i) = -\pi_i$ *for* $v_i \geq \pi_i$*,* $T_i^*(v_i) = 0$ *for* $v_i < \pi_i$*, and* $\sum_{i=1}^{N} -\pi_i(1-F_i(\pi_i)) \leq B$*.*

To derive the results in Proposition 6, we first consider a weaker form of ex post budget constraint that is easier to deal with:

Definition 7. *A mechanism* $(\mathbf{q}, \mathbf{t})$ *satisfies the* weak ex post budget constraint *if*

$$\sum_{i=1}^{N} q_i(\mathbf{v}) t_i(\mathbf{v}) \leq B \tag{4.12}$$

for all $\mathbf{v} \in [\underline{v}, \bar{v}]^N$.

Note that if the selection rule is such that a grant is either issued with probability 1 or not at all, the weak ex post budget constraint coincides with the ex post budget constraint.

Following the technique of d'Aspremont and Gérard-Varet, 1979, we can show an equivalence between the weak ex post budget constraint and the ex ante budget constraint:

Lemma 6. *If an incentive compatible mechanism $(\mathbf{q}, \mathbf{t})$ satisfies the ex ante budget constraint, then there exists an incentive compatible mechanism $(\mathbf{q}, \mathbf{t}^*)$ such that $T_i(v_i) = \int_{\mathcal{V}_{-i}} q_i(\mathbf{v}) t_i^*(\mathbf{v}) dF_{-i}(\mathbf{v}_{-i})$ for all i and $(\mathbf{q}, \mathbf{t}^*)$ satisfies the weak ex post budget constraint.*

The logic behind the result of Lemma 6 is as follows. If a mechanism satisfies the ex ante budget constraint, it means the expected total payment does not exceed the budget. We only need to construct a new payment rule, without touching the selection rule. Towards that end, the constructed ex post payment rule preserves the expected payment, but the ex post payment is adjusted by adding terms with zero expectation in order to satisfy the ex post budget constraint. If the adjustment part for each proposer does not depend on his own report, incentive compatibility still holds under the new payment rule.

With Lemma 6, we can first focus on solving the problem with interim individual rationality and ex ante budget constraint, which is solved in Proposition 4. Then we apply the selection rule in Proposition 4 and adjust the payment rule to satisfy the weak ex post budget constraint.

Note that the optimal selection rule in Proposition 4 is binary. So the ex post budget constraint is equivalent to weak ex post budget constraint. Therefore the above mechanism is also optimal for Problem 4.

In Proposition 6, the selection rule is still a cutoff rule, but the payment rule no longer corresponds to a fixed price. Instead, it depends on other proposers' reported types and cannot be implemented by take-it-or-leave-it offers. In particular, the mechanism in Proposition 6 requires all thresholds π_i's to be announced to all participants. This is different from the last section, where proposers only need to know their own threshold.

Furthermore, the mechanism in Proposition 6 does not satisfy ex post individual rationality. Compared to Proposition 4, the payment rule here has an additional term. This term helps keep budget balance when many proposers are selected at the same time. The more proposers are selected, the less each of them will receive. In the extreme case, when all proposers report exactly the thresholds, the payment for each proposer is $-\pi_i + \frac{1}{N-1}\sum_{j\neq i}(F_j(\pi_j)\pi_j) < -\pi_i$, and this violates ex post individual rationality.

Next we discuss the case where the proposers can reject a grant if their proposals are selected. Combined with the fixed budget constraint, the optimization problem becomes more intricate. In particular, the techniques used to solve the previous problems are no longer applicable, since a cutoff selection rule is no longer optimal, as demonstrated in the following example.

Example 2. *Suppose there are two proposers with s_1 and s_2. Each proposer has two potential private values: v_H and v_L. Assume $v_L < v_H < 0$ and each proposer has value v_H with probability p and v_L with probability $1-p$. In addition, assume $-v_L \leq B < -2v_L$ and $s_1 + v_L < s_2 + v_H < s_1 + v_H$. Then it is not an optimal mechanism to have a cutoff selection rule in the above setting.*

In the above example, the budget can be spent on at most one project. In addition, the difference between public values s_1 and s_2 is quite small. Consider the following mechanism, where the selection rule is:

$$q_1(v_H,\cdot) = q_1(v_L,v_L) = 1 \quad q_1(v_L,v_H) = 0$$
$$q_2(\cdot,v_L) = q_2(v_H,v_H) = 0 \quad q_2(v_L,v_H) = 1.$$

The corresponding payment rule is:

$$t_1(v_H,\cdot) = t_1(v_L,v_L) = -v_L \quad t_1(v_L,v_H) = 0$$
$$t_2(\cdot,v_L) = t_2(v_H,v_H) = 0 \quad t_2(v_L,v_H) = -v_L.$$

It is straightforward to see that this mechanism satisfies ex post budget constraint and ex post individual rationality. To verify incentive compatibility, first note that no proposer has an incentive to misreport if they have a private value v_H. Since proposers with v_H will always conduct the project, they are better off getting selected and paid. Reporting v_L will decrease the possibility of getting selected and hence decrease the expected payment. So any proposer with v_H will avoid reporting v_L. On the other hand, proposers with v_L have no incentive to over-report. This is because regardless of their reports, they always receive a utility of 0. Therefore, incentive compatibility holds.

The selection rule in the above mechanism is no longer a cutoff rule, since whether a participant is selected depends on the other participant's report. In addition, this mechanism generates a higher welfare than any cutoff selection rules that satisfy both ex post individual rationality and the ex post budget constraint.

It is worth noting that in this example, the form of optimal mechanisms depends highly on public values. Suppose, instead, that the difference between s_1 and s_2 is very large ($s_1 - s_2 > v_H - v_L$), then the first project should always be selected by setting the threshold at v_L. The corresponding optimal mechanism takes the form of a cutoff selection rule. As a result, even if the proposers have no information about public values, they can make inferences from the form of the mechanism announced by the grant issuer.

4.5 Conclusion and discussion

This paper discusses optimal mechanism design problems of the grant issuing procedure under different environments. If the grant issuer faces a flexible budget, the optimal selection rule is a cutoff rule: each proposal is selected only when the reported private value is higher than a proposer-specific threshold. The threshold for each proposer decreases in the budget, decreases in the public value that can be generated by the proposer's project, and increases with the public value generated by other proposers' projects. If there is an extremely high cost for the selected proposers to turn down the grant, the optimal selection rule has the same form even when the grant issuer faces a fixed budget. In contrast, if the grant issuer faces a fixed budget and selected proposers can turn down the grant freely, whether a proposer can be selected depends on the reported private values of other proposers, and hence a cutoff selection rule is not necessarily optimal.

The above result shows the value of maintaining a flexible budget and imposing a high cost of rejecting a grant to prevent proposers from backing out after the grant is awarded. The absence of both factors not only brings a loss in social welfare, but also makes the implementation of optimal mechanism very complicated.

The mechanisms proposed in this paper overcome the problem of misallocation brought by the merit-based mechanisms (Giebe, Grebe, and Wolfstetter, 2006). By setting a proposer-specific threshold, my mechanism screens out the projects with low private values, which is neglected in the merit-based mechanisms.

However, the inefficiency caused by the problem of crowding-out private funds (Wallsten, 2000) still stands in my mechanism. To induce applicants to report truthfully about their private values, the grant issuer has to pay them more than what is needed to support their projects as "information rents." As a result, projects which can be carried out without external financial support will use grants as a substitute for private funds.

One aspect that is not modeled in this paper is moral hazard. In reality, the selected proposers may take much less effort into the project after receiving the grant and hence provide lower merit than the initial assessment. This case can be ruled out when both of the following conditions are satisfied: first, there is no noise in measuring the quality of the project; and two, the cost of breaking their promises is large enough so that no proposers are willing to take the risk. For grants provided by institutions like the National Institutes of Health (NIH), the delivery in the end of the grant period is usually very specific. In addition, reputation helps proposers maintain a long relationship with the grant issuer. In this case moral hazard is rare and the model is applicable. For other grants especially grants for basic sciences, it is difficult to tell whether the failure of a project comes from the lack of effort or the risk of the project itself. In these cases, moral hazard is possible and future work is needed.

4.6 Appendix

Constraints simplification

In this section, I show some technical results to simplify the constraints and these results will be used for further proof of the main results of this paper.

Let $u_i(v_i) = U_i(v_i; v_i)$. For notational simplicity, let

$$Q_i(v_i) = \int_{\mathcal{V}_{-i}} q_i(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i}) \tag{4.13}$$

$$T_i(v_i) = \int_{\mathcal{V}_{-i}} q_i(v_i, \mathbf{v}_{-i}) t_i(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i}) \tag{4.14}$$

Intuitively, $Q_i(v_i)$ and $T_i(v_i)$ respectively represent proposer i's probability of getting selected and his payment in expectation of other proposers' private values. In what follows I refer to $Q_i(v_i)$ and $T_i(v_i)$ as the expected selection and payment rules respectively.[10]

For incentive compatible mechanisms, the participants' utility functions can be characterized in the following lemma:

[10]In previous literature, Q_i and T_i are usually referred to as the "reduced forms" of q_i and t_i (Border, 1991 for example). However, in this paper $\mathbf{q}$ is not required to be a simplex. So I use different terms to avoid confusion.

Lemma 7. *A direct mechanism is interim incentive compatible if and only if*

$$u_i(v_i) = \begin{cases} \underline{u}_i + \int_{\underline{\nu}_i}^{v_i} Q_i(z)dz & v_i \leq 0 \\ \underline{u}_i + \int_{\underline{\nu}_i}^{0} Q_i(z)dz + v_i & v_i > 0 \end{cases} \tag{4.15}$$

where $\underline{u}_i$ is a constant and $Q_i(v_i)$ is non-decreasing in v_i.

The proof is omitted since it follows standard techniques used in Krishna, 2009.

Note that proposers with $v_i > 0$ will realize their private value regardless of the selection result, so they have incentive to under-report (over-report) if the utility from participating in the mechanism is growing at a rate lower (higher) than 1. As a result, in Lemma 7, the mechanism has to provide these proposers a utility function growing exactly at 1. For the rest proposers, the selection outcome matters more, since if not selected, they will receive a utility valued at 0. In line with Myerson, 1981, it turns out these participants have no incentive to misreport if and only if the utility grows at a rate which is exactly their probability of getting selected, and the expected selection rule has to be monotonic.

By substituting the utility function of Lemma 7 into Definition 8, the constraint of interim individual rationality can be simplified.

Lemma 8. *An incentive compatible mechanism $(\mathbf{q}, \mathbf{t})$ satisfies the interim individual rationality if and only if*

$$\underline{u}_i \geq 0 \tag{4.16}$$

Proof. Plug (4.15) into (4.16) to get

$$u_i(v_i) \geq \max\{v_i, 0\} \Leftrightarrow \begin{cases} \underline{u}_i + \int_{\underline{\nu}_i}^{v_i} Q_i(z)dz \geq 0 & v_i \leq 0 \\ \underline{u}_i + \int_{\underline{\nu}_i}^{0} Q_i(z)dz + v_i \geq v_i & v_i > 0 \end{cases} \Leftrightarrow \underline{u}_i \geq 0$$

□

Note that $\underline{u}_i = u_i(\underline{\nu}_i)$. So the result in Lemma 8 means that as long as the proposer with the lowest possible private value is willing to participate in the mechanism, proposers with higher private values must be willing to participate as well. In particular, although the proposers with private value above zero has better outside option (v_i as opposed to 0 for participants with private value below zero), it is still enough to guarantee their participation just by providing the lowest private value participant a non-negative utility.

At last, the ex ante budget constraint can also be simplified as the following:

Lemma 9. *A mechanism* $(\mathbf{x}, \mathbf{t})$ *satisfies the ex ante budget constraint if and only if*

$$\sum_{i=1}^{N}\left(\underline{u}_i + \int_{\underline{\nu}_i}^{0}\left(\frac{1-F_i(v_i)}{f_i(v_i)} - v_i\right) Q_i(v_i) dF_i(v_i)\right) \leq B \tag{4.17}$$

The proof is omitted since the result follows directly from previous lemmas.

One interesting feature of (4.17) is that the left-hand side shows as if the payment is only made to participants with private value below zero. However, this does not mean no payment will be made to participants with positive private values. Plugging (4.15) into (4.1), it is easy to see that the payment for participants with $v_i > 0$ is $\int_{\underline{\nu}_i}^{0} Q_i(z)dz$, which does not depend on v_i. This means the information rent to induce truthful reports from participants with positive private value is a constant. On the other hand, different from Myerson, 1981, the information rent for participants with value below 0 is $\frac{F_i(0)-F_i(v_i)}{f_i(v_i)}$, instead of $\frac{1-F_i(v_i)}{f_i(v_i)}$. The term $\frac{1-F_i(v_i)}{f_i(v_i)}$ in (4.17) contains both the payment to participants with value $v_i \leq 0$ and $v_i > 0$.

Proof of Proposition 4

Proof. We first solve problem 2:

By Lemmas 7, 8 and 9, Problem 2 can be simplified as the following:

$$\begin{aligned}
\max_{\mathbf{Q},\underline{\mathbf{u}}} \sum_{i=1}^{N} \int_{\mathcal{V}_i} & \left(Q_i(\mathbf{v})(s_i + v_i) + (1 - Q_i(\mathbf{v}))\mathbb{I}(v_i \geq 0)(s_i + v_i)\right) dF_i(v_i) \\
\text{s.t. } & Q_i(v_i) \text{ non-decreasing in } v_i \text{ for all } i \\
& \underline{u}_i \geq 0 \\
& \sum_{i=1}^{N}\left(\underline{u}_i + \int_{\underline{\nu}_i}^{0}\left(\frac{1-F_i(v_i)}{f_i(v)_i} - v_i\right) Q_i(v_i) dF_i(v_i)\right) \leq B
\end{aligned}$$

We ignore the monotonicity constraint on Q_i for now and write down the Lagrangian

$$\begin{aligned}\mathcal{L}(\mathbf{Q},\underline{\mathbf{u}};\mu,\gamma) =& \mu B + \sum_{i=1}^{N}(\gamma_i - \mu)\underline{u}_i \\ &+ \sum_{i=1}^{N}\left(\int_{\underline{\nu}_i}^{0} Q_i(v_i)\left(s_i + (1+\mu)v_i - \mu\frac{1-F_i(v_i)}{f_i(v_i)}\right)dF_i(v_i)\right) \\ &+ \sum_{i=1}^{N}\int_{0}^{\bar{\nu}_i}(s_i+v_i)dF_i(v_i) \end{aligned} \tag{4.18}$$

First note that it is without loss of generality to focus on mechanisms with $\underline{u}_i = 0$. This is because if in a mechanism there exist some $\underline{u}_i > 0$, the value objective function won't change if we set $\underline{u}_i^* = 0$ and keep the allocation rule Q_i the same as before. In addition, the new mechanism also satisfies compatibility, individual rationality, and budget constraint.

Second, the last term (4.18) does not depend on the value of $\mathbf{Q}$ for $v_i > 0$. So we only need to focus on the selection of $\mathbf{Q}$ for $v_i \in [\underline{\nu}_i, 0]$.

For notational simplicity, let $\phi_i(v_i) = s_i + (1+\mu)v_i - \mu\frac{1-F_i(v_i)}{f_i(v_i)}$. It is straightforward to see that it is a pointwise optimal solution if for $v_i \in [\underline{\nu}, 0]$,

$$Q_i^*(v_i) = \begin{cases} 1 & \phi_i(v_i) \geq 0 \\ 0 & \phi_i(v_i) < 0 \end{cases}$$

When the distribution function has an increasing hazard rate, $\phi_i(v_i)$ is an increasing function of v_i. So there exists $\pi_i \in [\underline{\nu}_i, 0]$ such that $\phi_i(v_i) \geq 0$ for $0 \geq v_i > \pi_i$ and $\phi_i(v_i) < 0$ for $v_i \leq \pi_i$.

Correspondingly, the payment rule

$$T_i^* = \begin{cases} -\pi_i & v_i \geq \pi_i \\ 0 & v_i < \pi_i \end{cases}$$

By the constraint of incentive compatibility, $Q_i(v_i)$ has to be non-decreasing in v_i. If $\pi_i < 0$, $Q_i(0) = 1$, then we need $Q_i(v_i) = 1$ and $T_i(v_i) = -\pi_i$ for $v_i > 0$. If $\pi_i = 0$,then $Q_i^*(v_i) = 0$, and $T_i^*(v_i) = 0$ for all $v_i \leq 0$. In this case, for $v_i > 0$, set $Q_i^*(v_i) = 1$ and $T_i^*(v_i) = 0 = -\pi_i$, the objective function reaches maximum and all constraints are still satisfied. As a result, $Q_i^*(v_i) = 1$ and $T_i^*(v_i) = -\pi_i$ for all $v_i > 0$.

At last, it is straightforward to check that $(\mathbf{Q}^*, \mathbf{T}^*)$ can be implemented by $(\mathbf{q}^*, \mathbf{t}^*)$ given in the proposition.

Note that the cutoff selection rule and fixed price payment rule also satisfy ex post individual rationality (4.3), which is a stronger constraint than the interim individual rationality. So the optimal mechanism in Proposition 4 is also optimal subject to ex post individual rationality. $\square$

Proof of Proposition 5

Proof. First of all, if $B \geq \sum_{i=1}^{N}(-\underline{\nu}_i)$, the optimal solution is to set $\pi_i = \underline{\nu_i}$, which is independent of s_i and B, so Proposition 5 holds naturally. If $B < \sum_{i=1}^{N}(-\underline{\nu}_i)$, we start the proof with Lemma 10:

Lemma 10. *If $B < \sum_{i=1}^{N}(-\underline{\nu}_i)$, there exists $\lambda > 0$ such that each optimal threshold π_i satisfy exactly one of the three conditions:*

1. $\pi_i = 0$ *and* $\xi(\pi_i; \lambda) \leq 0$
2. $\pi_i = \underline{\nu}_i$ *and* $\xi(\pi_i; \lambda) \geq 0$
3. $\pi_i \in (\underline{\nu}_i, 0)$ *and* $\xi(\pi_i; \lambda) = 0$

and $\sum_{i=1}^{N}(1 - F_i(\pi_i))(-\pi_i) = B$*, where* $\xi(v_i; \lambda) = s_i + (1 + \lambda)v_i - \lambda\frac{1-F_i(v_i)}{f_i(v_i)}$ *is the adjusted virtual value.*

Proof. To solve for the optimal thresholds, by Proposition 6, substitute the cutoff selection rule and fixed payment rule into the grant issuer's utility function (4.5) and ex ante budget constraint (4.17), and the problem becomes

$$\max_{\pi} \sum_{i=1}^{N} \int_{\pi_i}^{\bar{\nu}_i} (s_i + v_i) f_i(v_i) dv_i \tag{4.19}$$

$$\text{s.t.} \sum_{i=1}^{N} (1 - F_i(\pi_i))(-\pi_i) \leq B \tag{4.20}$$

Incorporate the budget constraint into the objective function by applying the Lagrangian multiplier, and we have

$$\mathcal{L}(\pi; \lambda) = \sum_{i=1}^{N} \int_{\pi_i}^{\bar{\nu}_i} (s_i + v_i) f_i(v_i) dv_i + \lambda B - \lambda \sum_{i=1}^{N} (1 - F_i(\pi_i))(-\pi_i) \tag{4.21}$$

The first order derivative

$$\frac{\partial \mathcal{L}}{\partial \pi_i} = \lambda - (s_i + \pi_i(1+\lambda))f_i(\pi_i) - \lambda F_i(\pi_i) \tag{4.22}$$

And the second order derivative

$$\frac{\partial^2 \mathcal{L}}{\partial \pi_i^2} = -(1+2\lambda)f_i(\pi_i) - (s_i + (1+\lambda)\pi_i)f_i'(\pi_i) \tag{4.23}$$

It is straightforward to see that when $f_i'(v_i) \geq 0$, $\frac{\partial^2 \mathcal{L}}{\partial \pi_i^2} < 0$ so that (4.21) is concave in π_i and hence the optimization problem has a unique solution.

Note that $\pi_i \in [\underline{\nu}, 0]$, so the maximizer π_i must satisfy either $\frac{\partial \mathcal{L}}{\partial \pi_i}|_{\pi_i=0} \geq 0$, $\frac{\partial \mathcal{L}}{\partial \pi_i}|_{\pi_i=\underline{\nu}} \leq 0$, or $\frac{\partial \mathcal{L}}{\partial \pi_i}|_{\pi_i \in (\underline{\nu},0)} = 0$. Substitute these conditions into (4.22) to obtain the result in Lemma 10. □

Lemma 11. *The optimal thresholds π_i's are nondecreasing in λ.*

Proof. First, suppose $\pi_i \in (\underline{\nu_i}, 0)$. By 10, we have

$$\lambda - (s_i + \pi_i(1+\lambda))f_i(\pi_i) - \lambda F_i(\pi_i) = 0$$

Take derivative of λ to obtain

$$\frac{\partial \pi_i}{\partial \lambda}\left((1+2\lambda)f_i(\pi_i) + (1+s_i+\lambda)f_i'(\pi_i)\pi_i\right) = 1 - F_i(\pi_i) - \pi_i f_i(\pi_i) \tag{4.24}$$

The multiplier of $\frac{\partial \pi_i}{\partial \lambda}$ on the left-hand-side of (4.24) is positive since $\lambda > 0$ and by assumption $f_i'(\pi_i) \geq 0$.

To tell the sign of the right-hand side of (4.24), let $g(x) = 1 - F_i(x) - xf_i(x)$. Then $g'(x) = -2f_i(x) - xf_i'(x) < 0$ on $(-\underline{\nu}_i, 0)$. Since $g(0-) = 1 - F_i(0) \geq 0$, $g(\pi_i) > 0$ for $\pi_i \in (-\underline{\nu}_i, 0)$. So the right-hand-side of (4.24) is positive.

As a result, $\frac{\partial \pi_i}{\partial \lambda} > 0$ for $\pi_i \in (-\underline{\nu}_i, 0)$.

Now suppose $\pi_i = 0$, then by Lemma 10,

$$(1 - F_i(\pi_i) - \pi_i f_i(\pi_i))\lambda - (s_i + \pi_i)f_i(\pi_i) \geq 0$$

When λ increases, keeping π_i the same, the left-hand side increases as well, since by previous discussion $(1 - F_i(\pi_i) - \pi_i f_i(\pi_i)) \geq 0$. So the above condition still holds and the updated π_i^* should still be 0. So π_i is nondecreasing in λ

At last, suppose $\pi_i = \underline{\nu}_i$. Then by Lemma 10,

$$(1 - F_i(\pi_i) - \pi_i f_i(\pi_i))\lambda - (s_i + \pi_i)f_i(\pi_i) < 0$$

When λ increases, keeping π_i the same, the left-hand side also increases, and the above condition may fail. If it still holds, the updated π_i^* does not change. Otherwise, the new π_i^* will increase. Any of these cases should happen, π_i is nondecreasing in λ.

This completes the proof of the lemma. □

Now we consider the changes of π_i with s_i and s_j. If s_i increases with everything else held the same, including λ, π_i decreases and the budget is in short. To keep budget balance, by Lemma 11, λ must increase so that all thresholds π_j for $j \neq i$ increase. However, the increase in λ must be small enough so that the decrease in π_i due to the increase in s_i is not offset by the increase in λ, otherwise all thresholds are higher and the there is extra budget that can be used to improve the objective function. As a result, the increase in s_i will cause decrease in π_i and increase in π_j for $j \neq i$.

When the budget B increases, λ must decrease, otherwise the increase in λ will lead π_i's increase by Lemma 11 and there will be extra budget, making the thresholds no longer optimal. As a result, the increase in B will cause the decrease in π_i for all i.

This completes the proof of Proposition 5. □

Proof of Lemma 6

Proof. For notational simplicity, let

$$A_i(v_i) = \begin{cases} \int_{\underline{v}}^{v_i} Q_i(z)dz - Q_i(v_i)v_i & v_i \leq 0 \\ \int_{\underline{v}}^{0} Q_i(z)dz & v_i > 0 \end{cases} \tag{4.25}$$

Construct

$$t_i(\mathbf{v}) = \alpha_i + A_i(v_i) - \frac{1}{N-1}\sum_{j \neq i} A_j(v_j) \tag{4.26}$$

Then

$$\begin{aligned} T_i(v_i) &= \int_{\mathcal{V}_{-i}} t_i(\mathbf{v}) dF_{-i}(\mathbf{v}_{-i}) \\ &= \alpha_i - \frac{1}{N-1}\int_{\mathcal{V}_{-i}} \sum_{j \neq i} A_j(v_j) dF_{-i}(\mathbf{v}_{-i}) + A_i(v_i) \end{aligned} \tag{4.27}$$

Since $T_i(v_i) = \underline{u}_i + A_i(v_i)$, compare terms with (4.27), and we have

$$\underline{u}_i = \alpha_i - \frac{1}{N-1}\int_{\mathcal{V}_{-i}} \sum_{j \neq i} A_j(v_j) dF_{-i}(\mathbf{v}_{-i}) \tag{4.28}$$

Then

$$\begin{aligned}
\sum_{i=1}^{N} t_i(\mathbf{v}) &= \sum_{i=1}^{N} \alpha_i \\
&= \sum_{i=1}^{N} \underline{u}_i + \sum_{i=1}^{N} \int_{\underline{\nu}_i}^{\bar{\nu}_i} A_i(v_i) dF_i(v_i) \\
&= \sum_{i=1}^{N} \int_{\underline{\nu}_i}^{\bar{\nu}_i} T_i(v) dF_i(v) \leq B
\end{aligned}$$

This completes the proof. □

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