ELECTROMAGNETIC WAVE GENERATION AND PROPAGATION IN GRAVITATIONAL FIELDS

Thesis by

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ABSTRACT

We use Feynman perturbation techniques to analyze some aspects of electromagnetic wave generation and propagation in weak gravitational fields.

In the first part of this report we calculate differential cross sections $d\sigma/d\Omega$ for the scattering of plane electromagnetic waves by weakly gravitating and rotating bodies in the long-wavelength limit (wavelength of incident radiation >> radius of scatterer >> mass of scatterer). We find that the polarization of right (or left) circularly polarized electromagnetic waves is unaffected by the scattering process (i.e., helicity is conserved), and that the two helicity states of the photon are scattered differently by a rotating body. This coupling between the photon helicity and the angular momentum of the scatterer also leads to a partial polarization of unpolarized incident light.

For the sake of comparison, we also compute the differential cross sections for the gravitational scattering of scalar and gravitational waves. For the latter there is neither helicity conservation nor helicity-dependent scattering; and the angular momentum has no polarizing effect on incident, unpolarized gravitational waves.

In the second part of this report, we analyze the conversion of gravitational waves into electromagnetic waves (and vice versa) under the "catalytic" action of a static electromagnetic background field. Closed-form differential cross sections are presented for conversion in the Coulomb field of a point charge, electric and

magnetic dipole fields, and uniform electrostatic and magnetostatic fields. Using the model calculation of conversion in a Coulomb field, we discuss the problems that we must face when calculating non-gauge-invariant transition amplitudes, as is frequently done in the literature.

We conclude this report by pointing out how charged-particle beams may be used (in principle) as direction-sensitive gravitational-wave detectors.

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Le secret d'ennuyer est celui de tout dire.

Voltaire.

I. INTRODUCTION

Ever since Einstein's celebrated derivation of lightbending by a spherical mass,[1] wave propagation in curved spacetime has been the subject of study by physicists, mathematicians and an occasional engineer. Einstein's calculations are based on the conventional methods of ray optics and completely forego the wave nature of light. A more acceptable approach is to start from field-theoretic considerations, i.e., from the Maxwell equations in a Riemannian space:[2]

$$F^{\mu\nu}{}_{;\nu} = j^{\mu} \quad , \tag{1}$$

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \quad . \tag{2}$$

Here $F_{\mu\nu}$ is the antisymmetric electromagnetic-field tensor and j^{μ} is the 4-current in Lorentz-Heaviside (rationalized) units.*

In the "ε-μ formalism" of Volkov et al.[3] Eqs. (1) and (2) are recast into the form of Maxwell equations in an inhomogeneous, bi-anisotropic medium embedded in flat spacetime. A gravitational

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* In the following we shall use natural units ($G = \hbar = c = 1$) and a metric $g_{\alpha\beta}$ with signature +2. The determinant of $g_{\alpha\beta}$ is denoted by g. Minkowski spacetime is described by the metric $\eta_{\alpha\beta}$ = diag (-1,1,1,1). Covariant derivatives relative to $g_{\alpha\beta}$ are denoted by semicolons and partial derivatives by commas or the symbol ∂. Greek indices run from 0 to 3, Latin indices from 1 to 3. We shall also use the abbreviations $\underline{a}\cdot\underline{b} = \eta_{\alpha\beta}a^{\alpha}b^{\beta}$ and $\tilde{a}\cdot\tilde{b} \equiv \eta_{ij}a^{i}b^{j}$. Symmetrization of indices is denoted by round brackets, i.e., $a_{(\mu}b_{\nu)} = \frac{1}{2}(a_{\mu}b_{\nu} + a_{\nu}b_{\mu})$.

field thus endows the vacuum with permittivity and permeability properties. In complete analogy with the theory of electrodynamics in continuous media, we may then solve the problem of electromagnetic wave propagation (and generation) in a gravitational field, by identifying the fictitious polarization currents and computing the electromagnetic fields which they generate. This we shall do in a systematic way with the help of Feynman diagrams.

A convenient starting point is the source-free Maxwell equation $F^{\mu\nu}{}_{;\nu} = 0$, which can also be written as

$$(-g)^{-\frac{1}{2}}\, \partial_\nu \left(\sqrt{-g}\, g^{\mu\alpha} g^{\nu\beta}\, F_{\alpha\beta}\right) = 0\,. \tag{3}$$

The effects of gravity are most easily seen if we expand the gravitational background about Minkowski spacetime. For this we put

$$\sqrt{-g}\, g^{\alpha\beta} \equiv \mathfrak{g}^{\alpha\beta} \equiv \eta^{\alpha\beta} - 2\lambda \bar{h}^{\alpha\beta}\,, \tag{4}$$

where $\lambda = \sqrt{8\pi(G)}$ is the gravitational coupling constant. The indices of the trace-reversed metric perturbation $\bar{h}^{\alpha\beta}$ are lowered, by definition, with the Minkowski metric $\eta_{\alpha\beta}$. From (4) we obtain

$$\sqrt{-g} = \sqrt{-\det||g_{\alpha\beta}||} = \sqrt{-\det||\mathfrak{g}^{\alpha\beta}||} = 1 - \lambda\bar{h} + O(\lambda^2), \tag{5}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} - 2\lambda(\bar{h}^{\alpha\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\bar{h}) + O(\lambda^2)\,, \tag{6}$$

where $\bar{h} = \bar{h}^{\mu}{}_{\mu}$.

With the aid of (3), (4), (6) we find

$$F^{\mu\nu}{}_{,\nu} = 2\lambda \left[(\eta^{\mu\alpha}\bar{h}^{\nu\beta} + \bar{h}^{\mu\alpha}\eta^{\nu\beta} - \tfrac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\bar{h})F_{\alpha\beta}\right]_{,\nu} + O(\lambda^2). \quad (7)$$

This equation now is a flat-space equation and the gravitational field $\bar{h}^{\alpha\beta}$ is just another field, like $F_{\alpha\beta}$, propagating in Minkowski space-time.

From (7) it follows that to the lowest order in λ, the effect of the gravitational field on the dynamics of the electromagnetic field can be embodied in a fictitious polarization current density

$$j^{\mu}_{pol} = 2\lambda \left[(\eta^{\mu\alpha}\bar{h}^{\nu\beta} + \bar{h}^{\mu\alpha}\eta^{\nu\beta} - \tfrac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\bar{h})F_{\alpha\beta}\right]_{,\nu} . \quad (8)$$

This polarization current is distributed throughout space. It is present even in the absence of physical charges and may be looked upon as a vacuum polarization stemming from the interaction of two neutral fields $\bar{h}^{\alpha\beta}$ and $F_{\alpha\beta}$. Note that the polarization current is linear both in the gravitational field and the electromagnetic field. Through Eq. (7) this polarization current acts as the source of an electromagnetic field, which (in a perturbation calculation) must be considered as a correction to the flat-space electromagnetic field, satisfying $F^{\mu\nu}{}_{,\nu} = 0$. In diagram language: an F-line (dashed line) joins an h-line (wavy line) in a vertex and gives rise to another F-line (Fig. 1).

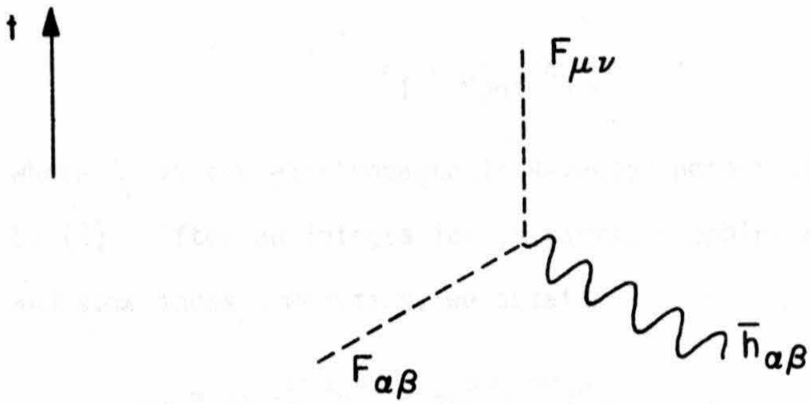

Fig. 1 The 2-photon-graviton vertex. The interaction between the electromagnetic field $F_{\alpha\beta}$ and the gravitational field $\bar{h}_{\alpha\beta}$ generates a fictitious polarization current. This current acts as the source of an additional electromagnetic field $F_{\mu\nu}$.

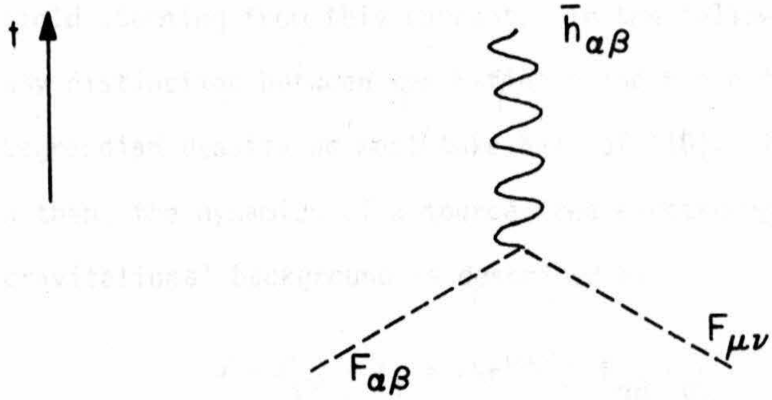

Fig. 2 The 2-photon-graviton vertex revisited. The electromagnetic field tensors $F_{\alpha\beta}$ and $F_{\mu\nu}$ beat against one another to produce a stress-energy distribution, which then acts as the source of a gravitational field.

This interaction is described by the Lagrangian density

$$\mathcal{L}_I = j^{\mu}_{pol} A_{\mu} , \tag{9}$$

where A_{μ} is the electromagnetic 4-vector potential and j^{μ}_{pol} is given by (8). After an integration by parts, dropping a pure divergence and some index gymnastics, we obtain

$$\mathcal{L}_I = 2\lambda \, (\overline{h}^{\mu\alpha}\eta^{\nu\beta} - \tfrac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}\overline{h})F_{\alpha\beta} \, F_{\mu\nu} , \tag{10}$$

where

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} . \tag{11}$$

In (10) $F_{\alpha\beta}$ is the electromagnetic field which by its interaction with $\overline{h}^{\mu\alpha}$ generates the polarization current, and $F_{\mu\nu}$ is the electromagnetic field stemming from this current. In the following we shall not make any distinction between the F-fields and hence for the interaction Lagrangian density we must take half of (10). To the lowest order in λ then, the dynamics of a source-free electromagnetic field in a gravitational background is described by

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_I = -\tfrac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta} \, F_{\alpha\beta} \, F_{\mu\nu}$$

$$+ \lambda \, (\overline{h}^{\mu\alpha}\eta^{\nu\beta} - \tfrac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}\overline{h})F_{\alpha\beta} \, F_{\mu\nu} , \tag{12}$$

with $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. The first term $\mathcal{L}_{EM}$ is the familiar Lagrangian density for a free electromagnetic field in flat spacetime. The second term $\mathcal{L}_I$ accounts for the interaction between the electromagnetic field and the induced polarization current.

It is easily checked that an infinitesimal variation of the action $\mathcal{S} = \int \mathcal{L} d^4x$ with respect to A_μ results in the Maxwell equation (7).

There is another way of arriving at the interaction Lagrangian density $\mathcal{L}_I$. In Einstein's linearized theory, $\bar{h}^{\mu\nu}$ couples to the stress-energy tensor $T_{\mu\nu}$ of the electromagnetic field. This $T_{\mu\nu}$ is quadratic in the fields $F_{\mu\nu}$. In diagram language: 2 F-lines join in a vertex to produce an h-line (Fig. 2).

The interaction Lagrangian density is[4,5]

$$\mathcal{L}_I = \lambda \, h^{\mu\nu} \, T_{\mu\nu} \quad , \tag{13}$$

with

$$h^{\mu\nu} = \bar{h}^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\bar{h} \quad , \tag{14}$$

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\ \alpha} - \tfrac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}\,F^{\alpha\beta} \quad . \tag{15}$$

Substituting (14) and (15) into (13) results in the interaction Lagrangian density $\mathcal{L}_I$, obtained above.

Now we turn Fig. 2 around and obtain Fig. 1: an h-line joins an F-line to produce another F-line. But the source of an F-line can be called a current and so we conclude again that a polarization current is induced, which is bilinear in h and F.

The 2-photon-graviton vertex will play a central role in this report. E.g., Fig. 1 may describe

i) Scattering of electromagnetic waves by a stationary gravitational field—light ray bending by a massive body, gravitational focusing, coupling of photon helicity to the angular momentum of a rotating massive body, may all be deduced from a careful analysis of diagram 1.

ii) Or the ingoing F-line may stem from a static charge and the h-line may be a gravitational wave. Fig. 1 then describes how a gravitational wave "shakes loose" the Coulomb field of a charge and causes it to radiate electromagnetically, very much in the same way as a dielectric wave incident on a static electromagnetic field generates electromagnetic waves. (This process is called transition scattering.)

iii) Or the F-lines may represent electromagnetic cavity modes and h an incident gravitational wave. Fig. 1 then describes how a gravitational disturbance feeds photons from one cavity mode into another.

iiii) Or the ingoing F-line may stem from a charge moving through the h-field of a massive body. This is gravitationally induced electromagnetic bremsstrahlung.

Similarly Fig. 2 describes:

i) How an electromagnetic wave propagating in a static electromagnetic background is gradually transformed into a gravitational wave;

ii) How two electromagnetic cavity modes beat against one another to generate a gravitational wave;

iii) How two charges moving in each other's electromagnetic field generate gravitational waves: electromagnetically induced gravitational bremsstrahlung.

To calculate all these effects we may of course work with the Maxwell (and Einstein) field equations. E.g., one may solve the problem of electromagnetic wave scattering by a weak gravitational field in the following way:[6] compute the induced current density from (8), identify the electric and magnetic dipole moments per unit volume that generate this polarization current, and calculate the radiation field of these dipole distributions with the standard techniques of flat-space electrodynamics.

We shall find it much easier to start directly from the Lagrangian and to use Feynman perturbation techniques. It must be stressed, however, that all of the processes described in this report are classical processes. Only for reasons of ease and straightforwardness shall we borrow from the techniques of relativistic quantum mechanics.

This report is divided in three parts. Chapter II discusses the scattering of long-wavelength, zero-restmass plane waves by weak gravitational fields. Though we are mainly interested in electromagnetic waves, we shall also investigate scalar and gravitational wave scattering. This can be done with little additional effort and the results exhibit some interesting dissimilarities with electromagnetic wave scattering.

While Chapter II discusses the behaviour of an electromagnetic wave in a stationary gravitational field, Chapter III investigates the behaviour of a static electromagnetic field in a gravitational wave background. We shall see that the gravitational wave is gradually converted into an electromagnetic wave. The inverse process is also discussed: the generation of gravitational waves, due to electromagnetic wave propagation in a static electromagnetic background.

Finally, in Chapter IV we show how the conversion mechanism allows in principle the use of charged particle-beams as gravitational-wave detectors.

II. GRAVITATIONAL SCATTERING OF ZERO-RESTMASS PLANE WAVES

1. The Raison d'être of this Chapter

With the advent of black-hole physics, wave propagation on Riemannian manifolds has become a fashionable topic, as gravitational scattering of electromagnetic and gravitational radiation provides possible means (though not very promising at present) to detect collapsed objects. Recently, a number of papers have used general relativity theory to analyze the scattering and absorption of scalar, electromagnetic and gravitational waves by a fully nonlinear Schwarzschild black hole. Hildreth[7] and Matzner[8] have studied the scattering and absorption of scalar waves, using a partial-wave analysis. Vishveshwara[9] has used the Regge-Wheeler formalism and a partial-wave expansion to study the interaction with gravitational waves. Mashhoon[10] and Fabbri[11] have studied the electromagnetic wave problem using partial-wave expansions and the ε-μ formalism of Volkov et al.[3]

The mathematically more tractable problem of scattering by weakly gravitating, nonrotating spherically symmetric bodies has been studied also. Einstein[1] discussed the deflection of electromagnetic waves by a spherical, nonrotating body in the geometric optics limit and for large impact parameters. Mo and Papas[12] used a combination of Debye and ε-μ formalisms to restudy the same problem as Einstein and discover an increase in electromagnetic wave intensity due to gravitational focusing. Westervelt[6] used flat-space wave equations to calculate the scattering of plane electromagnetic and gravitational waves by the Newtonian field of a point mass.

We take special note of a paper by Peters,[13] who calculated cross sections for the scattering of long-wavelength, plane scalar, electromagnetic and gravitational waves by a weak Schwarzschild scatterer. His method utilized Green functions in a weakly curved spacetime. Peters' paper was motivated by a question raised in a conversation with J. A. Wheeler: "Is the scattered radiation sensitive as to whether the scatterer is a black hole or some other spherical body with the same mass?" In the high-frequency limit, the impinging waves are certainly able to probe the internal structure of the scatterer,and hence the scattering cross sections for black holes and condensed bodies should not agree. However, one did not expect a different behaviour for the two kinds of scatterers in the long-wavelength limit. Peters' results shattered this belief. A comparison of his weak-field results with the corresponding black-hole results explicitly shows that even when the wavelength is much larger than the radius of the black hole or the condensed body, there is a disagreement between their cross sections.

Motivated by a talk on these problems by Peters at Caltech in the spring of 1976, we set out to check whether the lowest-order quantum perturbation calculations agreed with his classical results. As our calculations were so much simpler than his, we were able to add angular momentum to the scatterer and to investigate its influence on the scattering cross sections. This bonus is especially exciting as recent observations by Harwit _et al._[14] have placed an upper limit on the difference of deflection between left and right circularly polarized radio-beams passing near the limb of the sun. Whereas previous electromagnetic tests of general relativity (light bending and quasar radio-wave bending near the sun, Shapiro time-delay of radar signals, gravitational redshift in the earth's gravitational field)[2] probe only the geometric optics limit of electromagnetic-gravitational coupling, this experiment goes beyond geometric optics. The deflection is independent of polarization in the geometric optics limit (the ray follows a null geodesic regardless of its polarization state); but for "full-blooded" waves the helicity of the wave should couple to the angular momentum of the deflecting object ("magnetic-type" gravitational effect) to produce helicity-dependent deflection--helicity dependence which, for the sun, is below the accuracy of Harwit _et al._, but which should exist nevertheless.

A number of recent papers have used general relativity theory to investigate this helicity dependence and other aspects of the interaction between incoming waves and a rotating, gravitating body.[15-17] Gradually, the full picture of such interactions is emerging; but there remain as yet a number of gaps in the picture. The purpose of this part of the report is to fill in one of those gaps: the full details of the long-wavelength limit for rotating and weakly gravitating bodies

$$(\text{wavelength}) \equiv 2\pi/\omega \gg (\text{size of body}) \equiv L$$
$$\gg (\text{gravitational radius}) \equiv M \qquad (1.1)$$

for scalar and gravitational waves as well as electromagnetic.

In the regime $2\pi/\omega \gg L \gg M$ it is better to speak of a "scattering" of the waves than a "deflection"; and it is most useful to calculate the amplitude T_{fi} for scattering of an incoming plane wave $|i\rangle$ into an outgoing (final) plane wave $|f\rangle$. From this scattering amplitude one can derive everything of interest--the explicit form of the scattered wave; the differential scattering cross section $d\sigma/d\Omega$; the amount of focusing; the deflection angle in the regime where it has meaning, i.e. (wavelength) << (impact parameter); etc.

We, like some others before us ,[18-19] have found the Feynman-diagram technique to be extremely powerful for studying long-wavelength scattering. In section 2 we give the Lagrangians and the Feynman vertex rules needed for each type of wave (scalar, electromagnetic, and gravitational), as well as the formula for the differential scattering cross section in terms of the transition amplitude. In sections 3, 4, and 5 we treat the scattering of scalar, electromagnetic and gravitational waves, respectively. Section 6 discusses and contrasts our results with those of other authors.

2. Feynman Diagrams for Scattering

The classical problem of the scattering of a massless field propagating in a slightly curved spacetime may be treated by quantizing both the linearized gravitational background and the scattered field. In this scenario both fields evolve in a Minkowski spacetime and couple according to the Feynman vertex rules. This approach may be contrasted to the work of Peters,[13] in which the gravitational background is considered to be a passive nondynamical entity, whose influence on the propagating field is embodied in a curved-spacetime Green function. In this section we summarize the relevant Feynman rules. For this we need the interaction parts of the Lagrangian densities. To obtain these interaction parts, we could of course follow the line of argument developed in the introduction. We choose to follow a more elegant route. We shall start from manifestly covariant Lagrangians in curved spacetime (which contain the interaction parts of all orders in the gravitational coupling constant λ) and shall expand them about flat spacetime.

The wave equation for source-free scalar waves

$$\Box\Psi - uR\Psi = 0 \tag{2.1}$$

may be obtained from the Lagrangian density

$$\mathcal{L}_S = -\frac{\sqrt{-g}}{2}\,(g^{\alpha\beta}\Psi_{,\alpha}\Psi_{,\beta} + uR\Psi^2), \tag{2.2}$$

where u is a constant, R the curvature scalar and $\Box \equiv (-g)^{-\frac{1}{2}}\partial_\alpha(g^{\alpha\beta}\sqrt{-g}\,\partial_\beta)$. For $u = 1/6$, Ψ represents conformally invariant waves.

Following Feynman[4-5] and Gupta,[20] and since we require that

$|h^{\alpha\beta}| \ll 1$ everywhere, we expand the gravitational field about the flat Minkowski background:*

$$\sqrt{-g}\, g^{\alpha\beta} \equiv \mathfrak{g}^{\alpha\beta} \equiv \eta^{\alpha\beta} - 2\lambda\bar{h}^{\alpha\beta} , \tag{2.3}$$

where the gravitational coupling constant $\lambda = \sqrt{8\pi}$ and $\bar{h}^{\alpha\beta}$ is the trace-reversed metric perturbation.

The determinant factor $\sqrt{-g}$, $g^{\alpha\beta}$ and R now become infinite series in λ:

$$\sqrt{-g} = \sqrt{-\det||g_{\alpha\beta}||} = \sqrt{-\det||\mathfrak{g}^{\alpha\beta}||} = 1 - \lambda\bar{h} + O(\lambda^2), \tag{2.4}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} - 2\lambda(\bar{h}^{\alpha\beta} - \frac{1}{2}\bar{h}\eta^{\alpha\beta}) + O(\lambda^2), \tag{2.5}$$

$$R = 2\lambda(\bar{h}^{\alpha\beta}{}_{,\alpha\beta} + \frac{1}{2}\bar{h}_{,\alpha}{}^{\alpha}) + O(\lambda^2), \tag{2.6}$$

where the trace of the metric perturbation is denoted by $\bar{h} = \bar{h}_{\mu}{}^{\mu}$. Expanding (2.2) in powers of λ we find that

$$\mathcal{L}_S = \sum_{n=0}^{\infty} \lambda^n \mathcal{L}_n , \tag{2.7}$$

where

$$\mathcal{L}_0 = -\frac{1}{2}\eta^{\alpha\beta}\,\Psi_{,\alpha}\Psi_{,\beta} , \tag{2.8}$$

$$\mathcal{L}_1 = \bar{h}^{\alpha\beta}\,\Psi_{,\alpha}\Psi_{,\beta} - u(\bar{h}^{\alpha\beta}{}_{,\alpha\beta} + \frac{1}{2}\bar{h}_{,\alpha}{}^{\alpha})\Psi^2 . \tag{2.9}$$

The free (i.e., noninteraction) Lagrangian $\mathcal{L}_0$ describes the free propagation of the scalar field Ψ in Minkowski space, whereas the

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* The expansion procedure, based on (2.3) may seem rather arbitrary. We have, however, verified that a slightly different expansion procedure, based on $g_{\alpha\beta} \equiv \eta_{\alpha\beta} + 2\lambda h_{\alpha\beta}$, with $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h}$, leads to the same results.

terms proportional to λ, λ^2, etc. represent the interaction parts of $\mathcal{L}$, i.e., they determine how the gravitational field $\overline{h}^{\alpha\beta}$ couples to the scalar field Ψ.

From $\mathcal{L}_1$ we may derive the amplitude T_{21} for a transition of the scalar field from an initial plane-wave state with wave-vector ("momentum") ${}^1k^\alpha$ to a final state with "momentum" ${}^2k^\alpha$ while absorbing a graviton with "momentum" q^α and polarization $\overline{e}^{\alpha\beta}$ (Fig. 3):

$$T_{21} = 2\lambda\overline{e}^{\alpha\beta}[{}^1k_{(\alpha}{}^2k_{\beta)} + u(q_\alpha q_\beta + \frac{1}{2}\eta_{\alpha\beta}q^2)] . \qquad (2.10)$$

Here the superscript 1(2) denotes the initial (final) state. Conservation of 4-momentum requires that

$$\underline{{}^2k} = \underline{{}^1k} + \underline{q} . \qquad (2.11)$$

In this calculation we shall limit ourselves to interactions proportional to λ^2, (single-graviton exchange); in other words, we shall calculate the scattering cross sections in the first Born approximation. In the classical limit for the scattering of waves with angular frequency ω by a mass M with angular momentum J, this corresponds to calculating at first order in the dimensionless quantities $M\omega$ and $J\omega^2$. Since our interest is restricted to a gravitational background geometry generated by classical energy-momentum distributions which are not affected appreciably by the scattering process, we may replace the virtual graviton by an external field.[21] In particular we consider only static fields; hence in the vertex rule (2.10) $\overline{e}^{\alpha\beta}$ stands for the 3-dimensional Fourier transform of $\overline{h}^{\alpha\beta}$ and the graviton 4-momentum is pure spacelike ($q^0 = 0$).

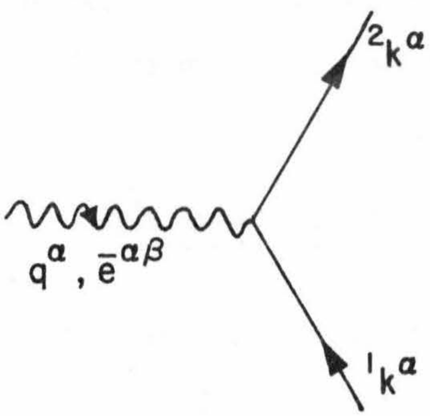

Fig. 3 The graviton-zero restmass field-zero restmass field vertex. The wavy line represents a graviton. The solid lines represent either scalar, electromagnetic, or gravitational quanta.

The transition amplitude T_{21} above has been normalized according to the definition

$$S_{21} = \delta_{21} + i(2\pi)^4 \, \delta^4({}^2\underline{k} - {}^1\underline{k} - \underline{q}) T_{21} \, , \tag{2.12}$$

where S_{21} is the S-matrix connecting the initial to the final state. With this normalization for T_{21}, the differential cross section for the scattering of a zero-restmass wave with frequency ω into a solid angle $d\Omega$ is

$$d\sigma = \frac{2\pi}{2\omega 2\omega} |T_{21}|^2 \, D \, , \tag{2.13}$$

where D denotes the density of final states,

$$D = \frac{\omega^2}{(2\pi)^3} \, d\Omega \quad . \tag{2.14}$$

Turn now to the scattering of electromagnetic waves off a slightly curved background. The manifestly covariant photon Lagrangian density, obtained by minimal coupling to gravity, is

$$\mathcal{L}_{EM} = - \frac{1}{4} \sqrt{-g} \, (g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}) \, , \tag{2.15}$$

where $F_{\mu\nu}$ is the electromagnetic-field tensor computed from the Maxwell vector potential A_μ by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} . \tag{2.16}$$

From (2.15) and (2.16) one obtains the field equations for the source-free electromagnetic field:

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \,, \tag{2.17}$$

$$F_{\mu\nu}{}^{;\nu} = 0 \,. \tag{2.18}$$

We expand the photon Lagrangian density in powers of λ according to (2.7) and obtain

$$\mathcal{L}_0 = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \,, \tag{2.19}$$

$$\mathcal{L}_1 = (\bar{h}^{\mu\alpha}\eta^{\nu\beta} - \frac{1}{4}\bar{h}\eta^{\mu\alpha}\eta^{\nu\beta})F_{\mu\nu}F_{\alpha\beta} \,. \tag{2.20}$$

Note that $\mathcal{L}_1$ agrees with the interaction Lagrangian density that was derived in Chapter I in a different, more intuitive way. After proper permutation of the photon labels, $\mathcal{L}_1$ provides the graviton-photon-photon vertex rule (see Fig. 3)

$$\begin{aligned} T_{21} = 2\lambda\bar{e}^{\alpha\beta}\Big\{&{}^1k_{(\alpha}{}^2k_{\beta)}\,({}^1\underline{\varepsilon}\cdot{}^2\underline{\varepsilon}^*) + {}^1\varepsilon_{(\alpha}{}^2\varepsilon^*{}_{\beta)}\,({}^1\underline{k}\cdot{}^2\underline{k}) - {}^1k_{(\alpha}{}^2\varepsilon^*{}_{\beta)}({}^2\underline{k}\cdot{}^1\underline{\varepsilon}) \\ &-{}^2k_{(\alpha}{}^1\varepsilon_{\beta)}({}^1\underline{k}\cdot{}^2\underline{\varepsilon}^*) - \frac{1}{2}\eta_{\alpha\beta}[({}^1\underline{k}\cdot{}^2\underline{k})({}^1\underline{\varepsilon}\cdot{}^2\underline{\varepsilon}^*) \\ &-({}^1\underline{k}\cdot{}^2\underline{\varepsilon}^*)({}^2\underline{k}\cdot{}^1\underline{\varepsilon})]\Big\} \,. \end{aligned} \tag{2.21}$$

Here ${}^1k^\alpha$ and ${}^1\varepsilon^\alpha$ are the 4-momentum and polarization vector of the ingoing photon, whereas ${}^2k^\alpha$ and ${}^2\varepsilon^\alpha$ denote the respective properties of the outgoing photon. In accordance with the external-field approximation $\bar{e}^{\alpha\beta}$ denotes the Fourier transform of $\bar{h}^{\alpha\beta}$. Note that the transition amplitude (2.21) is invariant under a gauge transformation of the form

$${}^i\varepsilon_\alpha \rightarrow {}^i\varepsilon_\alpha + \gamma\,{}^ik_\alpha \qquad (i = 1,2) \,, \tag{2.22}$$

where γ is an arbitrary scalar.

Finally, turn to the scattering of gravitational waves by a gravitational background. One arrives at the matter-free Einstein field equations

$$R_{\mu\nu} = 0 \tag{2.23}$$

by varying the Lagrangian density

$$\mathcal{L}_G = \frac{1}{2\lambda^2} \sqrt{-g}\, R \quad . \tag{2.24}$$

We take now for our basic fields $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ and $\mathfrak{g}_{\mu\nu} = g_{\mu\nu}/\sqrt{-g}$ rather than the metric itself. After performing some integrations by parts and dropping a pure divergence, we can express the Einstein gravitational Lagrangian density (2.24) in the particularly convenient Goldberg[22] form, which contains no derivatives of $g_{\mu\nu}$ higher than the first:

$$\mathcal{L}_G = - \frac{1}{16\lambda^2} (2\mathfrak{g}^{\alpha\beta}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\tau\nu} - \mathfrak{g}^{\alpha\beta}\mathfrak{g}_{\mu\tau}\mathfrak{g}_{\sigma\nu} - 4\, \eta^{\alpha}{}_{\sigma}\eta^{\beta}{}_{\tau}\mathfrak{g}_{\mu\nu})\mathfrak{g}^{\mu\tau}{}_{,\alpha}\mathfrak{g}^{\sigma\nu}{}_{,\beta} . \tag{2.25}$$

After we expand (2.25) in powers of λ, the components of $\mathcal{L}_G$ become

$$\mathcal{L}_0 = - \frac{1}{4} (2\bar{h}^{\alpha\beta,\mu}\bar{h}_{\alpha\beta,\mu} - \bar{h}^{,\mu}\bar{h}_{,\mu} - 4\, \bar{h}^{\alpha\beta,\mu}\bar{h}_{\mu\beta,\alpha}) , \tag{2.26}$$

$$\mathcal{L}_1 = -\bar{h}^{\mu\nu}(\bar{h}_{\alpha\beta,\mu}\bar{h}^{\alpha\beta}{}_{,\nu} + 2\, \bar{h}_{\mu\alpha}{}^{,\beta}\bar{h}_{\nu\beta}{}^{,\alpha} - 2\, \bar{h}_{\mu\beta,\alpha}\bar{h}_{\nu}{}^{\beta,\alpha}$$
$$+ \bar{h}^{,\alpha}\bar{h}_{\mu\nu,\alpha} - \frac{1}{2}\bar{h}_{,\mu}\bar{h}_{,\nu}) \quad . \tag{2.27}$$

The interaction part $\mathcal{L}_1$, appropriately symmetrized with respect to the graviton labels, provides the expression for the three-graviton vertex (see Fig. 3):

$$
\begin{aligned}
T_{21} = \lambda \bar{e}^{\mu\nu}\Big\{ & -2\big[{}^1\underline{\underline{\bar{e}}} : {}^2\underline{\underline{\bar{e}}}^{*}\,{}^1k_\mu\,{}^2k_\nu - {}^1\bar{e}_{\mu\nu}\,{}^2\bar{e}^{*\alpha\beta} q_\alpha\,{}^1k_\beta + {}^2\bar{e}^{*}{}_{\mu\nu}\,{}^1\bar{e}^{\alpha\beta} q_\alpha\,{}^2k_\beta\big] \\
& - 4\big[{}^1\bar{e}_{\mu\alpha}\,{}^2\bar{e}^{*}{}_{\nu\beta}\,{}^1k^\beta\,{}^2k^\alpha - q^\beta({}^2\bar{e}^{*}{}_{\mu\alpha}\,{}^1\bar{e}^\alpha{}_\beta\,{}^1k_\nu - {}^1\bar{e}_{\mu\alpha}\,{}^2\bar{e}^{*\alpha}{}_\beta\,{}^2k_\nu)\big] \\
& + 4\big[{}^1\underline{k}\cdot{}^2\underline{k}\,{}^1\bar{e}_{\mu\beta}\,{}^2\bar{e}^{*}{}_\nu{}^\beta - \underline{q}\cdot{}^1\underline{k}\,{}^1\bar{e}_{\beta\nu}\,{}^2\bar{e}^{*}{}_\mu{}^\beta + \underline{q}\cdot{}^2\underline{k}\,{}^2\bar{e}^{*}{}_{\beta\nu}\,{}^1\bar{e}_\mu{}^\beta\big] \\
& - \big[{}^1\underline{k}\cdot{}^2\underline{k}({}^1\bar{e}_{\mu\nu}\,{}^2\bar{e}^{*} + {}^2\bar{e}^{*}{}_{\mu\nu}\,{}^1\bar{e}) - \underline{q}\cdot{}^1\underline{k}\,{}^1\bar{e}\,{}^2\bar{e}^{*}{}_{\mu\nu} + \underline{q}\cdot{}^2\underline{k}\,{}^2\bar{e}^{*}\,{}^1\bar{e}_{\mu\nu} \\
& + \eta_{\mu\nu}(\underline{q}\cdot{}^2\underline{k} - \underline{q}\cdot{}^1\underline{k})\,{}^1\underline{\underline{\bar{e}}} : {}^2\underline{\underline{\bar{e}}}^{*}\big] \\
& + \big[{}^1k_\mu\,{}^1k_\nu\,{}^1\bar{e}\,{}^2\bar{e}^{*} + \eta_{\mu\nu}(q_\alpha\,{}^2k_\beta\,{}^2\bar{e}^{*}\,{}^1\bar{e}^{\alpha\beta} - q_\alpha\,{}^1k_\beta\,{}^1\bar{e}\,{}^2\bar{e}^{*\alpha\beta})\big]\Big\} \,, \qquad (2.28)
\end{aligned}
$$

where ${}^1k^\alpha$, ${}^1\bar{e}^{\alpha\beta}$; ${}^2k^\alpha$, ${}^2\bar{e}^{\alpha\beta}$; and q^α, $\bar{e}^{\alpha\beta}$ refer to the momenta and polarizations of the gravitons and ${}^1\underline{\underline{\bar{e}}}:{}^2\underline{\underline{\bar{e}}}$ denotes the tensor inner product. Unlike the graviton-photon-photon transition amplitude (2.21), the three-graviton transition amplitude is not invariant under the analogous gauge transformation, which in this instance is of the form

$$
{}^i\bar{e}^{\alpha\beta} \to {}^i\bar{e}^{\alpha\beta} + {}^ik^\alpha\chi^\beta + {}^ik^\beta\chi^\alpha - \eta^{\alpha\beta}\,{}^i\underline{k}\cdot\underline{\chi} \quad (i = 1,2)\,, \qquad (2.29)
$$

where χ^α represents an arbitrary vector. In general, the gauge invariance of the amplitudes is guaranteed by the Feynman-diagram formalism as long as all the diagrams of the same order in the coupling constant are included. Owing to our ignorance of the propagator for an object of mass M and very high quantum-mechanical spin, we omit all diagrams but the graviton-pole diagram. (This difficulty in formulating the quantum problem could probably be avoided by a classical analysis.) In the external-field approximation (no recoil of scatterer) the amplitude corresponding to this diagram is given by (2.28) where $\overline{e}^{\mu\nu}$ stands for the 3-dimensional Fourier transform of $\overline{h}^{\mu\nu}$. The external-field approximation serves to simplify the algebra but the effect of the omitted diagrams is to yield an amplitude (2.28) that is not gauge invariant, and is valid only for small scattering angles.

3. Scalar Waves

Since the waves have wavelengths much larger than the scatterer, they cannot probe (at first order) either the scatterer's internal structure or the quadrupole and higher-order moments of its gravitational field. For this reason, and because we calculate only to lowest order in λ, we can approximate the scatterer's gravitational field by the linearized metric for the exterior of a spherical body endowed with angular momentum:

$$g_{oo} = -(1 - \frac{2M}{r}), \quad g_{oj} = g_{jo} = -\frac{2M}{r^3}(\underset{\sim}{a} \times \underset{\sim}{r})_j, \quad g_{jk} = (1 + \frac{2M}{r})\eta_{jk}. \tag{3.1}$$

Here M is the mass of the body and $M\underset{\sim}{a} = \underset{\sim}{J}$ is its angular momentum. The Fourier transforms of the $\overline{h}_{\alpha\beta}$ are given by

$$\begin{aligned} \overline{e}_{oo} &= \frac{\lambda M}{q^2} \quad, \\ \overline{e}_{oj} = \overline{e}_{jo} &= \frac{i\lambda M}{2q^2}(\underset{\sim}{a} \times \underset{\sim}{q})_j \quad, \\ \overline{e}_{jk} &= 0 \quad, \end{aligned} \tag{3.2}$$

where $\underset{\sim}{q}$ is the (pure spacelike) momentum transfer $\underset{\sim}{q} = {}^2\underset{\sim}{k} - {}^1\underset{\sim}{k}$ $(q^o = 0)$. Permitting the angular momentum per unit mass $\underset{\sim}{a}$ to vanish in (3.1) or (3.2), we recover the linearized Schwarzschild geometry. Using Eqs. (2.10), (2.13), (2.14), and (3.2), the differential scattering cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{M^2}{\sin^4 \theta/2} \left\{(1 - 2u \sin^2 \theta/2)^2 + \omega^2(\underset{\sim}{a} \cdot ({}^1\hat{\underset{\sim}{k}} \times {}^2\hat{\underset{\sim}{k}}))^2\right\} . \qquad (3.3)$$

In the above ω is the angular frequency of the scalar wave, ${}^1\hat{\underset{\sim}{k}}$ and ${}^2\hat{\underset{\sim}{k}}$ are unit 3-vectors along the propagation directions of the incident and scattered fields respectively, and θ is the angle between ${}^1\hat{\underset{\sim}{k}}$ and ${}^2\hat{\underset{\sim}{k}}$. Allowing $\underset{\sim}{a}$ to vanish (linearized Schwarzschild geometry) one recovers the result previously obtained by Peters[13] through a classical first-order Born analysis:

$$\left(\frac{d\sigma}{d\Omega}\right)_{SCHW} = \frac{M^2}{\sin^4 \theta/2} (1 - 2u \sin^2 \theta/2)^2 . \qquad (3.4)$$

Due to the r^{-1} dependence of the Newtonian potential, Eq. (3.4) reduces to the usual $1/\sin^4(\theta/2)$ Rutherford-type cross section for the case of minimal coupling ($u = 0$). For non-minimal coupling ($u \neq 0$), the cross section still exhibits the Rutherford-type angular dependence, but only for $\theta \ll 1$. This is not surprising, since it is the scalar curvature R which gives rise to u-dependent terms in the cross section. Considering that R is nonzero only along the worldline of the scatterer, we see that for large impact parameters (i.e., small scattering angles) the scalar curvature cannot significantly contribute to the differential cross section.

That the lowest-order classical perturbation result of Peters agrees with our lowest-order quantum-mechanical perturbation result does not come as a surprise in the light of past experience with scattering of charged particles in a Coulomb potential (although the gravitational scattering problem is somewhat more complicated owing to the presence of a tensor potential). This is not the end of the story however. For (nonrelativistic) Coulomb scattering it is a well known fact that the quantum-mechanical first-order Born approximation reproduces the classically derived Rutherford formula. And this Rutherford formula is <u>exact</u> in the nonrelativistic limit, both in classical and quantum mechanics. Does this mean that the higher-order corrections in the Coulomb scattering problem do not give any contribution? Dalitz[23] has analyzed the problem and has found that (in the nonrelativistic limit) the higher-order corrections do not affect the first-order Born approximation apart from endowing the scattering amplitude with an overall phase factor. It follows that the first-order Born approximation gives the exact cross section.

If we wish to make similar higher-order calculations in the gravitational case, we must take the nonlinear corrections to the Newtonian potential into account. And these nonlinearities do affect the first-order Born scattering amplitude in a nontrivial way, as is evident from a comparison between our weak-field result and the black-hole result.

The most obvious disagreement, as noted by Peters, is the appearance of a u-dependent term in (3.4). For scattering off a black hole the cross section cannot depend on u , as $R = 0$ everywhere. And even when we set $u = 0$, we do not achieve agreement as the cross section for scattering low-frequency scalar waves off black holes shows a logarithmic dependence on the frequency.[8] Moreover, for black-hole scattering there is a nonzero absorption cross section,[24] unlike

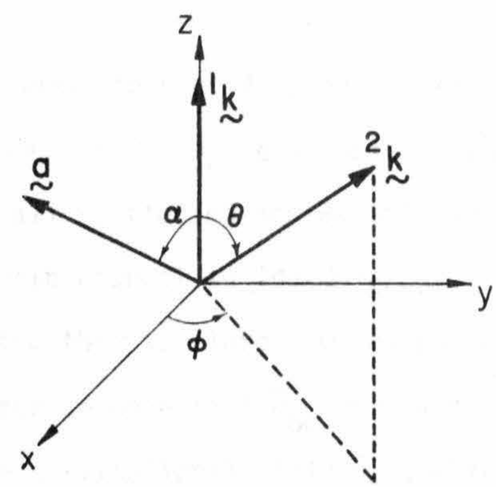

Fig. 4 The spatial orientation of the angular momentum $\underset{\sim}{a}$ and the scattered direction $^2\underset{\sim}{k}$ relative to the incident direction $^1\underset{\sim}{k}$.

the result for weak—field scattering.

One may rewrite the scattering cross section for rotating bodies (3.3) in the suggestive form

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{SCHW} + \frac{M^2 a^2 \omega^2}{\sin^4 \theta/2} \sin^2\alpha \sin^2\theta \sin^2\phi , \qquad (3.5)$$

with α, θ, and ϕ as shown in Fig. 4. Equation (3.5) shows that the effect of angular momentum is to add a positive semi-definite term to $(d\sigma/d\Omega)_{SCHW}$. For small scattering angles this angular-momentum term is neglibible with respect to $(d\sigma/d\Omega)_{SCHW}$. This can be easily understood by noticing that for large impact parameters the r^{-1} dependence of the Newtonian potential $\bar{h}_{00}$ dominates the r^{-2} dependence of the magnetic-type gravitational field $\bar{h}_{0j}$, which is the source of the angular-momentum term.

Another interesting feature of (3.5) is that the scattering in the backward direction is finite and independent of the angular momentum $\underset{\sim}{a}$:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\theta=\pi} = M^2(1 - 2u)^2 \quad . \qquad (3.6)$$

4. Electromagnetic Waves

Theoretically more interesting and of possible observational importance is the gravitational scattering of electromagnetic waves. We choose the polarizations of the photons to be purely spacelike $[{}^1\underline{\varepsilon} = (0, {}^1\underset{\sim}{\varepsilon}),\ {}^2\underline{\varepsilon} = (0, {}^2\underset{\sim}{\varepsilon})]$ and use Eqs. (2.13), (2.14), and (2.21). The result for the scattering of electromagnetic waves with initial polarization ${}^1\underset{\sim}{\varepsilon}$ into some polarization ${}^2\underset{\sim}{\varepsilon}$ is

$$\frac{d\sigma}{d\Omega} = \frac{M^2}{4\sin^4\theta/2} \Big| (1+\cos\theta)({}^1\underset{\sim}{\varepsilon} \cdot {}^2\underset{\sim}{\varepsilon}^*) - ({}^1\hat{\underset{\sim}{k}} \cdot {}^2\underset{\sim}{\varepsilon}^*)({}^2\hat{\underset{\sim}{k}} \cdot {}^1\underset{\sim}{\varepsilon})$$

$$+ i\omega[2({}^1\hat{\underset{\sim}{k}} \times {}^2\hat{\underset{\sim}{k}}) \cdot \underset{\sim}{a}({}^1\underset{\sim}{\varepsilon} \cdot {}^2\underset{\sim}{\varepsilon}^*) + (({}^2\hat{\underset{\sim}{k}} - {}^1\hat{\underset{\sim}{k}}) \times {}^2\underset{\sim}{\varepsilon}^*) \cdot \underset{\sim}{a}({}^2\hat{\underset{\sim}{k}} \cdot {}^1\underset{\sim}{\varepsilon})$$

$$+ (({}^2\hat{\underset{\sim}{k}} - {}^1\hat{\underset{\sim}{k}}) \times {}^1\underset{\sim}{\varepsilon}) \cdot \underset{\sim}{a}({}^1\hat{\underset{\sim}{k}} \cdot {}^2\underset{\sim}{\varepsilon}^*)]\Big|^2 . \tag{4.1}$$

For linear polarizations (${}^1\underset{\sim}{\varepsilon}$ and ${}^2\underset{\sim}{\varepsilon}$ real) the contribution of the angular momentum $\underset{\sim}{a}$ to the cross section (4.1) will be proportional to $a^2\,\omega^2$, whereas for circular polarizations (${}^1\underset{\sim}{\varepsilon}$ and ${}^2\underset{\sim}{\varepsilon}$ complex) the contribution will include an $a\omega$-term. We first consider circular polarizations (i.e., pure-helicity states) and we choose for the photon basis states

$${}^1\underset{\sim}{\varepsilon}_{L}^{R} = \frac{1}{\sqrt{2}}(\hat{\underset{\sim}{e}}_x \pm i\,\hat{\underset{\sim}{e}}_y) ,$$

$${}^2\underset{\sim}{\varepsilon}_{L}^{R} = \frac{1}{\sqrt{2}}(\hat{\underset{\sim}{e}}_\theta \pm i\,\hat{\underset{\sim}{e}}_\phi) , \tag{4.2}$$

where $\hat{\underset{\sim}{e}}_x$, $\hat{\underset{\sim}{e}}_y$, $\hat{\underset{\sim}{e}}_\theta$, $\hat{\underset{\sim}{e}}_\phi$ are unit vectors in the x, y, θ, and ϕ directions. After some algebraic manipulations (4.1) yields

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = 0 , \tag{4.3}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\substack{RR\\LL}} = M^2 \left\{ \begin{aligned} &[\mathrm{cotg}^2\theta/2 \pm 2a\omega\cos\theta/2(\cos\alpha\cos\theta/2 \\ &+ \sin\alpha\sin\theta/2\cos\phi)]^2 + 4\,a^2\omega^2\sin^2\alpha\,\mathrm{cotg}^2\tfrac{\theta}{2}\sin^2\phi \end{aligned} \right\} , \tag{4.4}$$

where the first (second) subscript denotes the initial (final) polarization and the upper (lower) sign in (4.4) refers to the RR (LL) case. For the linearized Schwarzschild geometry (4.4) reduces to recent results obtained by Peters:[13]

$$\left(\frac{d\sigma}{d\Omega}\right)^{SCHW}_{RR} = \left(\frac{d\sigma}{d\Omega}\right)^{SCHW}_{LL} = M^2 \cot g^4 \theta/2 \quad . \tag{4.5}$$

In the circular polarization basis the scattering matrix is diagonal, which explicitly shows that helicity is conserved by the scattering process. This is not restricted only to our situation, but rather is a general property of electromagnetic wave propagation in any orientable spacetime manifold.[16,25] In particular, it holds also for the fully nonlinear Schwarzschild and Kerr geometries. Moreover, for a nonrotating scatterer the scattering cross section is helicity independent whereas for a rotating scatterer it is helicity dependent. This results in a differential gravitational deflection of right and left circularly polarized electromagnetic wave-packets by a rotating object. For a given impact parameter b of the incident beam, we define the angular splitting as

$$\delta = \begin{pmatrix}\text{angle by which R helicity photon is deflected}\\ \text{angle by which L helicity photon is deflected}\end{pmatrix} \text{minus} \tag{4.6}$$

We then solve the inverse scattering problem[26] and find, to lowest order in $a\omega$,

$$\delta = 2\, a\, \omega \cos\alpha \left(\frac{4M}{b}\right)^3 \left[\ln\left(\frac{b}{2M}\right) - \frac{3}{4}\right] \quad . \tag{4.7}$$

To obtain this result we have used the constraint that $\delta \ll \frac{4M}{b} \ll 1$. Note that to the first order in $\tilde{a}$, there is no differential deflection when the direction of incidence is orthogonal to the angular momentum.

It must be stressed that so far we have only discussed pure-helicity states. For any linearly polarized or unpolarized incident wave the scattering cross section summed over final polarization states becomes

$$\frac{d\sigma}{d\Omega} = M^2 \left\{ \begin{aligned} &\text{cotg}^4\theta/2 + 4\, a^2\, \omega^2[\cos^2 \theta/2(\cos\alpha \cos\theta/2 \\ &+ \sin\alpha \sin\theta/2 \cos\phi)^2 + (\sin\alpha\, \text{cotg}\, \theta/2 \sin\phi)^2] \end{aligned} \right\} . \qquad (4.8)$$

We therefore conclude that all linearly polarized incident beams are deflected through the same angle so that a null-test is possible in this case.[14] However, since the diagonal elements of the scattering matrix in the circular polarization basis are unequal when $\underset{\sim}{a} \neq 0$, linearly polarized incident waves become elliptically polarized when incident on a rotating mass. For an unpolarized wave-packet, on the other hand, the paths of different-helicity photons are split by an amount given by (4.7). In addition, the angular momentum $\underset{\sim}{a}$ induces a partial polarization of the scattered waves. We define the degree of polarization by

$$p = \frac{(\frac{d\sigma}{d\Omega})_{RR} - (\frac{d\sigma}{d\Omega})_{LL}}{(\frac{d\sigma}{d\Omega})_{RR} + (\frac{d\sigma}{d\Omega})_{LL}} , \qquad (4.9)$$

and we find to lowest order in $a\omega$,

$$p = 4\, a\, \omega(\cos\alpha \cos\theta/2 + \sin\alpha \sin\theta/2 \cos\phi)\sin\theta/2\, \text{tg}\, \theta/2 . \qquad (4.10)$$

In concluding this section we note that independent of $\underset{\sim}{a}$, the initial polarization and the direction of incidence, the cross section for scattering in the backward direction vanishes. This property has been noticed before by Mashhoon[10] for a nonrotating scatterer and for a rotating scatterer when the waves are incident along the rotation axis. Indeed, a theorem[26] in electromagnetic theory states that if a scatterer is axially symmetric about the axis of incidence of a plane wave, then the off-diagonal scattering-matrix elements (in the circular polarization basis) vanish in the forward direction, while the diagonal elements vanish in the backward direction. This theorem is immediately applicable to gravitational scattering of electromagnetic waves, as in the ε-μ formalism the gravitational field may be replaced by an equivalent bi-anisotropic medium, embedded in flat space. Hence the backscattered photon--if present--must have the opposite helicity of the incident circularly polarized photon. This contradicts helicity conservation (see Eq. (4.3)) and hence backscatter is absent.*

The above argument is valid also for black-hole scattering and seems to be at variance with the "glory effect".[2] However, this effect has been shown to be absent when interference between the backscattered waves is taken into account.[10]

5. Gravitational Waves

Using (2.13), (2.14), and (2.28) we compute the differential cross section for the scattering of gravitational waves from an initial polarization ${}^1\bar{\underset{\approx}{e}}$ into some final polarization ${}^2\bar{\underset{\approx}{e}}$:

$$\frac{d\sigma}{d\Omega} = \frac{M^2}{\sin^4 \theta/2} \{\cos^2 \theta + \omega^2[({}^1\hat{\underset{\sim}{k}} \times {}^2\hat{\underset{\sim}{k}}) \cdot \underset{\sim}{a}]^2\} |{}^1\bar{\underset{\approx}{e}}:{}^2\bar{\underset{\approx}{e}}^*|^2 \ . \qquad (5.1)$$

---

* Note that these arguments hold irrespective of the wavelength of the electromagnetic waves and that they contradict the result by Nordtvedt[27], that vector waves <u>are</u> backscattered in the Newtonian field of a point mass.

This result was derived in the transverse-traceless (TT) gauge.[2] Although the transition amplitude (2.28) is not gauge invariant by itself, (5.1) yields reliable results for small momentum-transfers, i.e., for small scattering angles. By analogy with the photon case, we choose for the graviton basis states the circular polarizations given by

$$\underset{\approx L}{\overset{1-R}{e}} = \frac{1}{2}\,[\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_x - \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_y \pm i(\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_y + \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_x)]\ ,$$

$$\underset{\approx L}{\overset{2-R}{e}} = \frac{1}{2}\,[\hat{\underset{\sim}{e}}_\theta\hat{\underset{\sim}{e}}_\theta - \hat{\underset{\sim}{e}}_\phi\hat{\underset{\sim}{e}}_\phi \pm i(\hat{\underset{\sim}{e}}_\theta\hat{\underset{\sim}{e}}_\phi + \hat{\underset{\sim}{e}}_\phi\hat{\underset{\sim}{e}}_\theta)]\ . \qquad (5.2)$$

Substitution of the initial and final states into (5.1) yields

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{M^2}{16\sin^4\theta/2}\,(\cos^2\theta + a^2\omega^2\sin^2\alpha\sin^2\theta\sin^2\phi)(1-\cos\theta)^4\ , \qquad (5.3a)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{M^2}{16\sin^4\theta/2}\,(\cos^2\theta + a^2\omega^2\sin^2\alpha\sin^2\theta\sin^2\phi)(1+\cos\theta)^4\ . \qquad (5.3b)$$

The nonvanishing of (5.3a) clearly illustrates that here, unlike the electromagnetic case, helicity is <u>not</u> conserved. Moreover, there is neither different scattering of opposite helicity states nor partial polarization of unpolarized incident gravitational radiation. This is easily seen by noting that the scattering cross section for <u>either</u> helicity state is given by [adding (5.3a) and (5.3b)]

$$\left(\frac{d\sigma}{d\Omega}\right)_{R} = \left(\frac{d\sigma}{d\Omega}\right)_{L} = \frac{M^2}{\sin^4\theta/2}\,(\cos^2\theta + a^2\omega^2\sin^2\alpha\sin^2\theta\sin^2\phi)(\cos^2\theta + \frac{1}{8}\sin^4\theta)\ . \qquad (5.4)$$

Similarly, for the scattering of orthogonal linear polarizations denoted by

$$^{1}\bar{\underset{\approx}{e}}_{+} = \frac{1}{\sqrt{2}} (\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_x - \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_y),$$

$$^{1}\bar{\underset{\approx}{e}}_{\times} = \frac{1}{\sqrt{2}} (\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_y + \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_x), \tag{5.5}$$

one finds, after summing over the final polarizations and use of (5.1),

$$(\frac{d\sigma}{d\Omega})_{+} = \frac{M^2}{\sin^4 \theta/2} (\cos^2\theta + a^2\omega^2\sin^2\alpha\sin^2\theta\sin^2\phi)(\cos^2\theta + \frac{1}{4}\sin^4\theta\cos^2 2\phi), \tag{5.6a}$$

$$(\frac{d\sigma}{d\Omega})_{\times} = \frac{M^2}{\sin^4 \theta/2} (\cos^2\theta + a^2\omega^2\sin^2\alpha\sin^2\theta\sin^2\phi)(\cos^2\theta + \frac{1}{4}\sin^4\theta\sin^2 2\phi). \tag{5.6b}$$

For unpolarized incident gravitational waves the differential scattering cross section is given by (5.4). Allowing $\underset{\sim}{a} \to 0$, we recover Peters' results apart from a factor of $\cos^2\theta$:

$$(\frac{d\sigma}{d\Omega})^{SCHW}_{THIS\ report} = \cos^2\theta(\frac{d\sigma}{d\Omega})^{SCHW}_{PETERS} . \tag{5.7}$$

For small-angle scattering there is good agreement. One may recover Peters' result exactly by calculating the scattering of gravitational waves off a massive spin-0 meson.[18] Inclusion of all the relevant Feynman diagrams then leads to a gauge invariant transition amplitude. Actually, for the choice of the TT gauge only the t-channel graviton-pole diagram and the seagull diagram survive, and one obtains Peters' results exactly, i.e.,

$$(\frac{d\sigma}{d\Omega})^{SCHW}_{^{1}\bar{\underset{\approx}{e}}\to{}^{2}\bar{\underset{\approx}{e}}} = \frac{M^2}{\sin^4 \theta/2} \left|{}^{1}\bar{\underset{\approx}{e}}:{}^{2}\bar{\underset{\approx}{e}}^{*}\right|^2 . \tag{5.8}$$

As a concluding remark, we note that independent of the polarization of the incident gravitational wave and the angular momentum $\underset{\sim}{a}$, the cross section for backscatter is nonzero. Whereas the exact dependence of $(d\sigma/d\Omega)_{\theta=\pi}$ on the angular momentum $\underset{\sim}{a}$ cannot be inferred from the cross sections derived above (they are valid only for small scattering angles), one finds from (5.8) that the gravitational backscatter in a linearized Schwarzschild geometry is given by

$$\left(\frac{d\sigma}{d\Omega}\right)^{SCHW}_{\theta=\pi} = M^2. \tag{5.9}$$

In addition, if the incident radiation is in a pure-helicity state, the backscattered radiation must have the opposite helicity.

6. Summary and Conclusions

The differential cross sections for the weak-field gravitational scattering of long-wavelength scalar, electromagnetic and gravitational waves have been calculated using Feynman perturbation methods.

For the linearized Schwarzschild geometry, we have recovered the results obtained by Peters,[13] although he used a classical Green function formalism. In particular, for electromagnetic waves helicity is conserved, whereas for gravitational waves it is not. Endowing the scatterer with an angular momentum $\underset{\sim}{a}$, leads to helicity-dependent effects in electromagnetic wave scattering. Although the photon helicity is still conserved, the coupling between this helicity and the angular momentum of the scatterer results in i) different scattering of right and left circularly polarized photons and ii) partial polarization of unpolarized incident electromagnetic radiation. The high-frequency limits of these effects have been discussed before by Mashhoon.[10,16] Whereas in the high-frequency limit ($\omega M >> 1$), the angular split δ [defined by (4.6)], and polarization p [defined by (4.9)] are proportional to $a\omega^{-1}$, in the low-frequency limit ($\omega M << 1$) they are proportional to $a\omega$. This confirms the belief that the magnetic-type gravitational field of a rotating body clearly distinguishes between the helicity states of a photon only in the diffraction limit, i.e., when the wavelength of the incident photon is of the same order as the Schwarzschild radius of the scatterer.

Gravitational waves do not exhibit any of these angular-momentum-induced effects.

As a final comment, we note that this method may easily be applied to the gravitational scattering of non-integer spin* or massive fields.

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* In formulating the scattering problem for neutrinos and electrons one should use the vierbein formalism.

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III. ELECTROGRAVITATIONAL CONVERSION IN STATIC ELECTROMAGNETIC FIELDS

1. Introduction

Recent work in general relativity theory indicates that in any spacetime permeated by an electromagnetic background, a nontrivial coupling exists between electromagnetic and gravitational perturbations. Whereas the total energy in these perturbations is conserved, photon and graviton numbers individually are not. This implies the existence of conversion cross sections, expressing the fact that a static electromagnetic field may serve as a "catalyst" for converting electromagnetic waves into gravitational waves and vice versa.

The machinery for these conversion processes is easily discussed in terms of the picture developed in Chapter I. We have seen that the interaction of a gravitational field with an electromagnetic field induces a polarization current, which acts as the source of an additional electromagnetic field. I.e., a gravitational field acts on an electromagnetic field by changing the dielectric permittivity ε and magnetic permeability μ of the vacuum. Thus, when a gravitational wave propagates through a static electromagnetic background, the electromagnetic fieldlines will be alternately stretched and compressed owing to the changes in ε and μ; and this alternating field-configuration will then act as a source of electromagnetic waves. Somewhat picturesquely it can be said that the virtual photons of the static electromagnetic field are "shaken loose" by the bumps of spacetime and as a result become real electromagnetic quanta.

This process is well-known in conventional electromagnetic theory and is called transition radiation.[1] Fig. 1 gives a diagram describing this process.

The reverse process is also possible. Just as a nonstationary state of matter can generate gravitational waves, an alternating electromagnetic field can (under certain conditions) generate gravitational waves. Consider, e.g., an electromagnetic wave F, which propagates through a magnetostatic field F_0. The stress-energy tensor of the total electromagnetic field is the sum of three terms: a term proportional to ${F_0}^2$, a term proportional to F^2 and an interference term proportional to F_0F. The first two terms do not act as a source of gravitational waves, but the interference term does. This process is represented in Fig. 2.

Electrogravitational conversion was known to Whittaker[2] as early as 1947. Gertsenshteĭn,[3] however, was the first to actually calculate a conversion efficiency. In 1961 he used Einstein's linearized theory to consider the resonance of electromagnetic waves and gravitational waves in a strong uniform magnetostatic field. Weber and Hinds[4] investigated similar conversion processes by employing the Hamiltonian formulation of general relativity theory. The problem of the electromagnetic response of a capacitor to an incident gravitational wave has been investigated by Lupanov.[5] We take special note of a series of papers by an Italian research group,[6-10] in which various conversion mechanisms are studied. Both a Lagrangian-based quantum theory of gravity and classical general relativity theory are used. Their conclusions include possible astrophysical consequences and suggestions for gravitational-wave experiments. Papini and Valluri[11] used a Lagrangian-based quantum theory of gravity as well to study the role of conversion scattering in pulsars. Ginzburg and Tsytovich[12] recently calculated conversion cross sections by using the formal analogy between conversion scattering and dielectric wave-induced transition radiation.

It was hoped that electrogravitational resonance near a Reissner-Nordstrøm (charged, nonrotating) black hole would have observationally detectable consequences. Insight into the details of electrogravitational resonance in the neighbourhood of a charged black hole has been provided by Gerlach,[13] who originally found the coupled electromagnetic-gravitational perturbation equations in the JWKB limit. The Newman-Penrose formalism was used by Chitre et al.[14] to obtain the wave equations for mixed gravitational and electromagnetic perturbations in the neighbourhood of a slightly charged black hole ($Q/M \ll 1$). However, numerical studies[15-16] have shown that the electrogravitational interconversion can become efficient only when the charge-to-mass ratio Q/M of the black hole is near unity. Black holes with such an extreme Q/M ratio are unlikely to exist. Nevertheless, the problem of coupled electromagnetic and gravitational perturbations in the vicinity of a Reissner-Nordstrøm black hole remains interesting in principle, and Matzner[17] has recently calculated the conversion cross sections in the long-wavelength limit for quadrupole waves.

We shall not address ourselves to the strong-field (black hole) problem, which requires the use of the full mathematical apparatus of general relativity theory. Rather we undertake this study with conversion processes in a hot magnetic universe, pulsars, interstellar magnetic fields, etc. in mind. Therefore, we study the simplified case of Minkowski spacetime, permeated by various static electromagnetic backgrounds. The conversion efficiencies are extremely small, of course, but one may not do away a priori with these conversion processes if they are allowed to act on astrophysical distance and time scales.

In the following we shall use Feynman perturbation techniques to derive conversion cross sections in closed form and to analyze in detail their dependence on the polarization of the incident wave. Many of our results have been obtained before by the use of some other method. The reader is invited to compare the ease with which results can be obtained by the Feynman perturbation technique as opposed to the calculations hitherto used.

This chapter is in eight sections. Section 2 summarizes the relevant Feynman rules. Section 3 treats interactions with a non-spinning test charge. In sections 4 and 5 we calculate conversion cross sections in electric and magnetic dipole fields. Sections 6 and 7 are devoted to conversion in uniform magnetostatic and electrostatic fields. Finally, in section 8 we discuss our results in the light of previous investigations and make remarks about some inaccuracies in the literature.

2. The Feynman Rules

We review here the Feynman rules which will be relevant for our purposes. The Lagrangian density describing the interaction of a charged massive scalar field (a pion say) and a photon field in a Minkowski background is

$$\mathcal{L} = - \{\eta^{\mu\nu}[(\partial_\mu + ieA_\mu)\psi^*][(\partial_\nu - ieA_\nu)\psi] + M^2\psi^*\psi$$

$$+ \tfrac{1}{4}\, \eta^{\mu\alpha}\eta^{\nu\beta}\, F_{\mu\nu}F_{\alpha\beta}\} \;. \qquad (2.1)$$

Here ψ is the scalar field, M is the scalar field's mass and e is its charge in Lorentz-Heaviside (rationalized) units (for an electronic charge $\frac{e^2}{4\pi} \simeq \frac{1}{137}$). A_μ is the Maxwell 4- potential and $F_{\mu\nu}$ is the electromagnetic-field tensor computed from A_μ by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \;. \qquad (2.2)$$

Through minimal substitution we obtain from (2.1) the corresponding manifestly covariant Lagrangian density in a curved background:

$$\mathcal{L} = - \sqrt{-g}\, \{g^{\mu\nu}[(\partial_\mu + ieA_\mu)\psi^*][(\partial_\nu - ieA_\nu)\psi] + M^2\psi^*\psi$$

$$+ \tfrac{1}{4}\, g^{\mu\alpha}g^{\nu\beta}\, F_{\mu\nu}F_{\alpha\beta}\} \;. \qquad (2.3)$$

An infinitesimal variation of ψ^* in the action $\mathcal{S} = \int \mathcal{L} d^4x$, yields the field equation for ψ

$$\frac{1}{\sqrt{-g}} [(\partial_\mu - ieA_\mu)\, \sqrt{-g}\, g^{\mu\nu}(\partial_\nu - ieA_\nu)]\psi - M^2\psi = 0 \;. \qquad (2.4)$$

Similarly, varying the action $\mathcal{S}$ with respect to A_μ provides a set of Maxwell equations

$$F^{\mu\nu}{}_{;\nu} = ej^\mu \quad , \qquad (2.5)$$

where the current j^μ is defined as

$$j^{\mu} = i\, g^{\mu\nu}[\psi(\partial_{\nu}+ieA_{\nu})\psi^{*}-\psi^{*}(\partial_{\nu}-ieA_{\nu})\psi] \quad . \tag{2.6}$$

By invoking the field equation (2.4) one can show that j^{μ} satisfies the conservation law

$$j^{\mu}{}_{;\mu} = 0 \quad . \tag{2.7}$$

The other Maxwell equations

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \tag{2.8}$$

follow from (2.2).

As in Chapter II we define the gravitational field as the deviation from Minkowski spacetime:

$$\sqrt{-g}\, g^{\alpha\beta} \equiv \mathfrak{g}^{\alpha\beta} \equiv \eta^{\alpha\beta}-2\lambda\bar{h}^{\alpha\beta} \quad , \tag{2.9}$$

and expand the Lagrangian density (2.3) in powers of λ. We find

$$\mathcal{L} = \mathcal{L}_{S} + \mathcal{L}_{EM} + \mathcal{L}_{I} \quad ,$$

with

$$\mathcal{L}_{S} = -\,(\eta^{\mu\nu}\psi^{*}{}_{,\mu}\psi_{,\nu} + M^{2}\psi^{*}\psi) \quad , \tag{2.10a}$$

$$\mathcal{L}_{EM} = -\,\tfrac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \quad , \tag{2.10b}$$

$$\begin{aligned}\mathcal{L}_{I} = & -ie\eta^{\mu\nu}(A_{(\mu}\psi_{,\nu)}\psi^{*}-A_{(\mu}\psi^{*}{}_{,\nu)}\psi) \\ & + 2\lambda\, \bar{h}^{\mu\nu}\,(\psi^{*}{}_{,(\mu}\psi_{,\nu)}+\tfrac{1}{2}\eta_{\mu\nu}M^{2}\psi^{*}\psi) \\ & + 2ie\lambda\bar{h}^{\mu\nu}(A_{(\mu}\psi_{,\nu)}\psi^{*}-A_{(\mu}\psi^{*}{}_{,\nu)}\psi) \\ & + \lambda(\bar{h}^{\mu\alpha}\eta^{\nu\beta}-\tfrac{1}{4}\bar{h}\,\eta^{\mu\alpha}\eta^{\nu\beta})F_{\mu\nu}F_{\alpha\beta} + O\,(e^{2},\lambda^{2}) .\end{aligned} \tag{2.10c}$$

$\mathcal{L}_S$ and $\mathcal{L}_{EM}$ describe the propagation in Minkowski space of the free massive scalar and photon fields and allow us to deduce their propagators (in momentum space):

$$D^S(p) = \frac{1}{p^2+M^2-i\varepsilon}\,, \qquad (2.11a)$$

$$D^{EM}_{\mu\nu}(k) = \frac{\eta_{\mu\nu}}{k^2-i\varepsilon}\,. \qquad (2.11b)$$

Here ε is a small real positive number.

The Lagrangian density $\mathcal{L}_I$ describes the mutual interaction of the scalar, photon and graviton fields and yields the Feynman vertex functions (see Fig. 5):

(a) The scalar particle-scalar particle-photon vertex:

$$T = e({}^1p+{}^2p)_\mu\ \varepsilon^\mu \quad , \qquad (2.12a)$$

(b) The scalar particle-scalar particle-graviton vertex:

$$T = 2\lambda\ ({}^1p_{(\mu}\,{}^2p_{\nu)} + \tfrac{1}{2}M^2\eta_{\mu\nu})\bar{e}^{\ \mu\nu} \quad , \qquad (2.12b)$$

(c) The scalar particle-scalar particle-graviton-photon vertex:

$$T = -2e\lambda\ ({}^1p+{}^2p)_{(\mu}\ \varepsilon_{\nu)}\bar{e}^{\ \mu\nu} \quad , \qquad (2.12c)$$

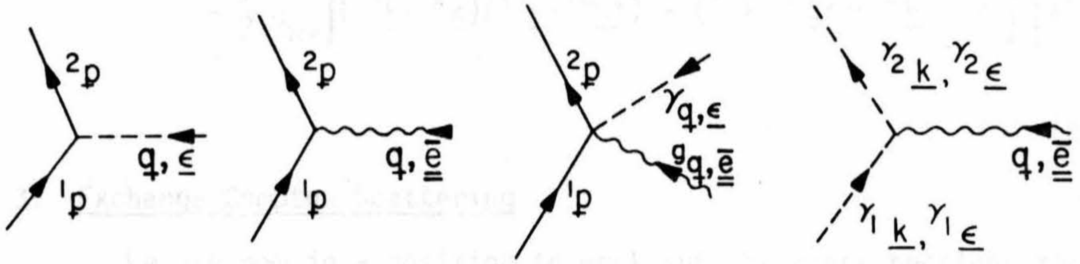

Fig. 5. The Feynman vertices. The solid lines represent scalar quanta, the dashed lines represent photons, the wavy lines represent gravitons.

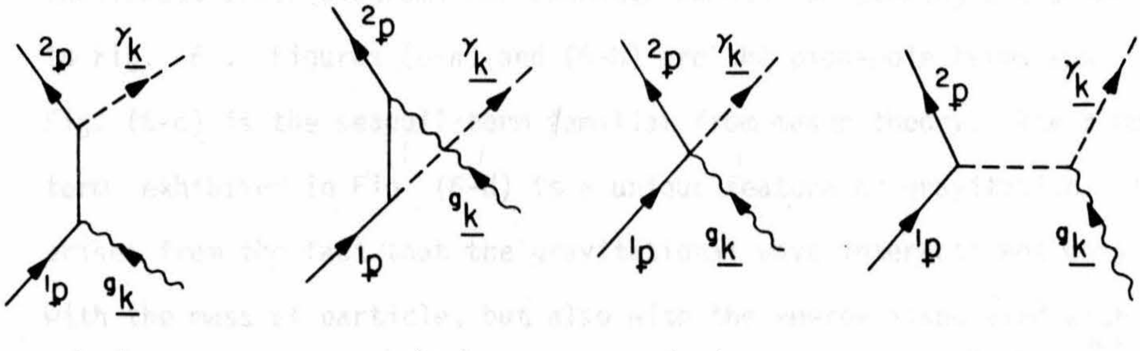

Fig. 6. Feynman graphs for exchange Compton scattering (graviton → photon) by a charged spin-0 meson

(d) The graviton-photon-photon vertex:

$$T = 2\lambda \Big\{ {}^{\gamma 1}k_{(\mu}\,{}^{\gamma 2}k_{\nu)}\,{}^{\gamma 1}\underline{\varepsilon}\cdot{}^{\gamma 2}\underline{\varepsilon}^* + {}^{\gamma 1}\varepsilon_{(\mu}\,{}^{\gamma 2}\varepsilon^*_{\nu)}\,{}^{\gamma 1}\underline{k}\cdot{}^{\gamma 2}\underline{k}$$

$$- {}^{\gamma 1}k_{(\mu}\,{}^{\gamma 2}\varepsilon^*_{\nu)}\,{}^{\gamma 2}\underline{k}\cdot{}^{\gamma 1}\underline{\varepsilon} - {}^{\gamma 2}k_{(\mu}\,{}^{\gamma 1}\varepsilon_{\nu)}\,{}^{\gamma 1}\underline{k}\cdot{}^{\gamma 2}\underline{\varepsilon}^*$$

$$- \frac{1}{2}\eta_{\mu\nu}\Big[({}^{\gamma 1}\underline{k}\cdot{}^{\gamma 2}\underline{k})({}^{\gamma 1}\underline{\varepsilon}\cdot{}^{\gamma 2}\underline{\varepsilon}^*) - ({}^{\gamma 1}\underline{k}\cdot{}^{\gamma 2}\underline{\varepsilon}^*)({}^{\gamma 2}\underline{k}\cdot{}^{\gamma 1}\underline{\varepsilon})\Big]\Big\}\bar{e}^{\mu\nu} \quad . \tag{2.12d}$$

3. Exchange Compton Scattering

We are now in a position to work out the cross sections for the conversion of gravitational waves into electromagnetic waves in the electrostatic field of a charged scalar particle.

Let the initial and final 4-momenta of the scalar particle be ${}^1\underline{p} = ({}^1E, {}^1\tilde{p})$ and ${}^2\underline{p} = ({}^2E, {}^2\tilde{p})$ and those of the incident graviton and scattered photon ${}^g\underline{k} = ({}^g\omega, {}^g\tilde{k})$ and ${}^\gamma\underline{k} = ({}^\gamma\omega, {}^\gamma\tilde{k})$ respectively. The polarizations of the graviton and photon are denoted by $\bar{e}^{\mu\nu}$ and ε^μ. The lowest-order diagrams for exchange Compton scattering are shown in Fig. 6 . Figures (6-a) and (6-b) are the pion-pole terms and Fig. (6-c) is the seagull term familiar from meson theory. The t-pole term exhibited in Fig. (6-d) is a unique feature of gravitation. It arises from the fact that the gravitational wave interacts not only with the mass of particle, but also with the energy associated with its long-range electrostatic field.

A straightforward application of the Feynman rules summarized in Sec. 2 yields for the individual contributions of the separate graphs

$$T_a = 2\lambda e({}^1p_\mu \bar{e}^{\mu\nu}\,{}^1p_\nu + \tfrac{1}{2}M^2\bar{e})\,{}^2\underline{p}\cdot\underline{\varepsilon}^*({}^1\underline{p}\cdot{}^g\underline{k})^{-1} \ , \tag{3.1a}$$

$$T_b = -2\lambda e({}^2p_\mu \bar{e}^{\mu\nu}\,{}^2p_\nu + \tfrac{1}{2}M^2\bar{e})\,{}^1\underline{p}\cdot\underline{\varepsilon}^*({}^1\underline{p}\cdot{}^\gamma\underline{k})^{-1} , \tag{3.1b}$$

$$T_c = -2\lambda e\ \bar{e}^{\mu\nu}({}^1p+{}^2p)_{(\mu}\,\varepsilon^*_{\nu)} \ , \tag{3.1c}$$

$$\begin{aligned} T_d = -\lambda e\{&{}^\gamma k_\mu \bar{e}^{\mu\nu}\,{}^\gamma k_\nu({}^1\underline{p}+{}^2\underline{p})\cdot\underline{\varepsilon}^* - \bar{e}^{\mu\nu}({}^1p+{}^2p)_{(\mu}\,\varepsilon^*_{\nu)}\,{}^g\underline{k}\cdot{}^\gamma\underline{k} \\ &- \bar{e}^{\mu\nu}\,{}^\gamma k_{(\mu}\,\varepsilon^*_{\nu)}\,{}^\gamma\underline{k}\cdot({}^1\underline{p}+{}^2\underline{p}) + \bar{e}^{\mu\nu}\,{}^\gamma k_{(\mu}({}^1p+{}^2p)_{\nu)}\,{}^g\underline{k}\cdot\underline{\varepsilon}^* \\ &+ \tfrac{1}{2}\bar{e}[{}^g\underline{k}\cdot{}^\gamma\underline{k}({}^1\underline{p}+{}^2\underline{p})\cdot\underline{\varepsilon}^* - {}^g\underline{k}\cdot\underline{\varepsilon}^*\ {}^\gamma\underline{k}\cdot({}^1\underline{p}+{}^2\underline{p})]\}\ ({}^g\underline{k}\cdot{}^\gamma\underline{k})^{-1} . \end{aligned} \tag{3.1d}$$

Here

$$\bar{e} = \bar{e}_\mu{}^\mu \ . \tag{3.2}$$

To obtain the above, we have used

$$\begin{aligned} {}^1\underline{p}\cdot{}^1\underline{p} &= {}^2\underline{p}\cdot{}^2\underline{p} = -M^2 \ , \\ {}^g\underline{k}\cdot{}^g\underline{k} &= {}^\gamma\underline{k}\cdot{}^\gamma\underline{k} = 0 \ , \\ {}^g k_\mu \bar{e}^{\mu\nu} &= {}^g k_\nu \bar{e}^{\mu\nu} = {}^\gamma k_\mu \varepsilon^\mu = 0 \ . \end{aligned} \tag{3.3a-c}$$

To investigate the gauge invariance of the scattering amplitude, we consider the transformations

$$\begin{aligned} \bar{e}^{\mu\nu} &\to \bar{e}^{\mu\nu} + {}^g k^{(\mu}\chi^{\nu)} - \frac{1}{2}\eta^{\mu\nu}\,{}^g\underline{k}\cdot\underline{\chi} \ , \\ \varepsilon^\mu &\to \varepsilon^\mu + f\,{}^\gamma k^\mu \ , \end{aligned} \tag{3.4a-b}$$

where f and χ^μ are arbitrary functions. It can readily be shown that the individual terms of the scattering amplitude are not gauge invariant though their sum is. Indeed, the fact that the sum turns out to be invariant under gauge transformations (which in this instance takes the form that the scattering amplitude vanishes under the substitutions $\bar{e}^{\mu\nu} \to {}^{g}k^{(\mu}\chi^{\nu)} - \frac{1}{2}\eta^{\mu\nu}\, {}^{g}\underline{k}\cdot\underline{\chi}$, $\varepsilon^{\mu} \to {}^{\gamma}k^{\mu}$), is a strong test which assures us that no algebraic errors have entered into the calculation.

In our expressions for the cross sections we shall use the laboratory frame, in which

$$
\begin{aligned}
&{}^{1}\underset{\sim}{p} = 0 \ , \ {}^{1}E = M, \\
&{}^{g}\underset{\sim}{k} = {}^{2}\underset{\sim}{p} + {}^{\gamma}\underset{\sim}{k} \ , \qquad\qquad (3.5) \\
&{}^{g}\omega + M = {}^{\gamma}\omega + {}^{2}E \quad .
\end{aligned}
$$

We remove the gauge freedom for the electromagnetic field by choosing the photon polarization ε^{μ} to be purely spacelike ($\varepsilon^{0} = 0$). The gravitational gauge freedom is specified by choosing the transverse-traceless (TT) gauge ($\bar{e}^{\mu o} = \bar{e}^{o\mu} = 0$; $\bar{e} = 0$).

We then see that the contributions of the diagrams (a) and (b) vanish and the remaining terms take a much simpler form:

$$T_c = 2\lambda e\, \bar{e}^{ij}\, {}^{\gamma}k_{(i}\, \varepsilon^{*}_{j)} \ , \qquad (3.6a)$$

$$T_d = 2\lambda e\, \frac{{}^{\gamma}\omega}{{}^{g}\omega - {}^{\gamma}\omega}\, \bar{e}^{ij}\, {}^{\gamma}k_{(i}\, \varepsilon^{*}_{j)} \ , \qquad (3.6b)$$

where we have used

$$
\begin{aligned}
&{}^{2}\underset{\sim}{p}\cdot\underset{\sim}{\varepsilon} = {}^{g}\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon} \,, \\
&\bar{e}^{ij}\, {}^{2}p_{(i}\, \varepsilon_{j)} = -\bar{e}^{ij}\, {}^{\gamma}k_{(i}\, \varepsilon_{j)} \,, \\
&\bar{e}^{ij}\, {}^{\gamma}k_{(i}\, {}^{2}p_{j)} = -\bar{e}^{ij}\, {}^{\gamma}k_{(i}\, {}^{\gamma}k_{j)} \,, \qquad (3.7a\text{-}e) \\
&{}^{g}\underline{k}\cdot{}^{\gamma}\underline{k} = M({}^{\gamma}\omega - {}^{g}\omega) \,, \\
&{}^{\gamma}\underline{k}\cdot({}^{1}\underline{p} + {}^{2}\underline{p}) = -M({}^{\gamma}\omega + {}^{g}\omega) \,.
\end{aligned}
$$

The above relations follow from conservation of energy-momentum and from the transverse nature of the photon and graviton.

The frequency of the outgoing photon is related to the frequency of the incident graviton through the Compton relation

$$
{}^{\gamma}\omega = \frac{{}^{g}\omega}{1 + 2\frac{{}^{g}\omega}{M}\sin^2\frac{\theta}{2}} \,, \qquad (3.8)
$$

where θ is the angle between ${}^{g}\underset{\sim}{k}$ and ${}^{\gamma}\underset{\sim}{k}$.

The differential cross section for converting a graviton with frequency ${}^{g}\omega$ and polarization $\bar{e}^{\mu\nu}$ into a photon with frequency ${}^{\gamma}\omega$ and polarization ε^{μ} is

$$
d\sigma = \frac{2\pi}{2M\,2{}^{g}\omega\; 2{}^{2}E\; 2{}^{\gamma}\omega} \,|T_c + T_d|^2\, D, \qquad (3.9)
$$

where D denotes the density of final states

$$
D = \frac{1}{(2\pi)^3}\, \frac{{}^{2}E\,{}^{\gamma}\omega^3}{M\,{}^{g}\omega}\, d\Omega. \qquad (3.10)
$$

Substitution of (3.6) and (3.10) and in (3.9) and use of (3.8) yields

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{8\pi \sin^4 \frac{\theta}{2}} \left(\frac{1}{1 + \frac{2 g_\omega}{M} \sin^2 \frac{\theta}{2}} \right)^2 |\bar{e}^{ij}\, \gamma\hat{k}_{(i}\, \varepsilon^*_{j)}|^2 \,. \qquad (3.11)$$

(laboratory frame; valid for all g_ω/M)

Here $\gamma\hat{\underset{\sim}{k}}$ is a unit vector in the direction of the outgoing photon.

In the nonrelativistic region, i.e., for $g_\omega \ll M$, there is negligible recoil of the scatterer and (3.11) reduces to

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{8\pi \sin^4 \frac{\theta}{2}} |\bar{e}^{ij}\, \gamma\hat{k}_{(i}\, \varepsilon^*_{j)}|^2 \,. \qquad (3.12)$$

(nonrelativistic limit,* in laboratory frame)

It is easily seen that the cross section (3.12) is solely due to the contribution of the t-pole diagram. We therefore conclude that although the t-pole diagram is not invariant by itself with respect to gravitational gauge transformations, it yields the correct nonrelativistic scattering amplitude in the laboratory frame, <u>but only if one chooses the TT gauge for the graviton</u>. One is free to choose the photon gauge, as the t-pole term is invariant with respect to photon gauge transformations.

When $g_\omega \ll M$, the source of the electromagnetic background field is not appreciably affected by the incident graviton. This justifies the use of the external-field approximation[18] in the nonrelativistic limit. In this approximation the differential conversion cross section is given by

---

*This formula is also valid for small scattering angles ($\sin^2 \theta/2 \ll M/2g_\omega$) for any g_ω.

$$d\sigma = \frac{2\pi}{2g_\omega \, 2\gamma_\omega} |T|^2 \, D \, , \tag{3.13}$$

with $g_\omega = \gamma_\omega$ and $D = \frac{\gamma_\omega^2}{(2\pi)^3} d\Omega$. (3.14)

The transition amplitude to be used in (3.13) is given by (2.12d), where one of the photon polarizations that appear in it stands for the 3-dimensional Fourier transform of the Coulomb potential

$$A_\mu = \frac{e}{4\pi r} \eta_{\mu o} \, , \tag{3.15}$$

i.e.,

$$\gamma \, {}^1\varepsilon_\mu \equiv \mathcal{F}\{A_\mu\} = \frac{e}{q^2} \eta_{\mu o} \, . \tag{3.16}$$

Here $\underline{q}$ is pure spacelike (no recoil of the scatterer). It is readily checked that the external-field approximation leads to the nonrelativistic cross section (3.12).

From now on we shall restrict our attention to this more realistic case of nonrelativistic scattering (unless otherwise stated). The relativistic (R) cross sections can be obtained from the nonrelativistic (NR) cross sections by multiplication by the appropriate factor:

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \left(\frac{\gamma_\omega}{g_\omega}\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_{NR} = \left(\frac{1}{1+ \frac{2g_\omega}{M} \sin^2 \frac{\theta}{2}}\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_{NR} \, . \tag{3.17}$$

Choose now for the basis states of the incident graviton and the outgoing photon the circular polarizations

$$\bar{\underset{\approx}{e}}\begin{smallmatrix}R\\L\end{smallmatrix} = \tfrac{1}{2}\,[\hat{e}_x\hat{e}_x - \hat{e}_y\hat{e}_y \pm i(\hat{e}_x\hat{e}_y + \hat{e}_y\hat{e}_x)] \,, \tag{3.18a}$$

$$\underset{\sim}{\varepsilon}\begin{smallmatrix}R\\L\end{smallmatrix} = \frac{1}{\sqrt{2}}\,(\hat{e}_\theta \pm i\hat{e}_\phi) \,, \tag{3.18b}$$

where $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_\theta$ and $\hat{e}_\phi$ are the unit vectors in the x, y, θ and ϕ directions (the z axis is the polar axis; ϕ is measured in the x-y plane from the x axis) and where the +(-) signs refer to the R(L) circularly polarized waves. Substituting (3.18) into (3.12), we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{e^2}{16\pi}\cot^2\frac{\theta}{2}\,(1-\cos\theta)^2 \,, \tag{3.19a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{e^2}{16\pi}\cot^2\frac{\theta}{2}\,(1+\cos\theta)^2 \,, \tag{3.19b}$$

where the first (second) subscript denotes the graviton (photon) polarization. The cross section for converting circularly polarized gravitons into photons (of any polarization)is then

$$\left(\frac{d\sigma}{d\Omega}\right)_{R} = \left(\frac{d\sigma}{d\Omega}\right)_{L} = \frac{e^2}{8\pi}\cot^2\frac{\theta}{2}\,(1+\cos^2\theta) \,. \tag{3.20}$$

For any angle $\theta \neq 0$ the outgoing electromagnetic radiation is not circularly polarized anymore, but elliptically polarized. In the forward direction, however, the outgoing photon is circularly polarized and, moreover, has the same helicity as the incident graviton. We also note that there is no backscatter:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\theta=\pi} = 0 \,. \tag{3.21}$$

It is worthwhile to compare these conversion cross sections with the Compton-scattering cross sections for photons and gravitons. The photon Compton-scattering cross section (in the nonrelativistic limit) is the familiar Thomson cross section

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{2}\left(\frac{e^2}{4\pi M}\right)^2 (1+\cos^2\theta) \ , \tag{3.22}$$

where M is the mass of the scatterer. Unlike Thomson scattering, the conversion cross sections (3.19b) and (3.20) exhibit a Rutherford peak in the forward direction.* This feature is entirely due to the t-pole term and is also present in the cross section for graviton scattering

$$\frac{d\sigma}{d\Omega} = \frac{M^2}{\sin^4\frac{\theta}{2}} \left(\cos^8\frac{\theta}{2} + \sin^8\frac{\theta}{2}\right) \ . \tag{3.23}$$

Turn now to linear polarizations. We choose for the graviton basis states

$$\bar{\underset{\sim}{e}}_+ = \frac{1}{\sqrt{2}} (\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_x - \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_y) \ , \tag{3.24a}$$

$$\bar{\underset{\sim}{e}}_x = \frac{1}{\sqrt{2}} (\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_y + \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_x) \ . \tag{3.24b}$$

Substituting (3.24) into (3.12) and summing over the polarizations of the outgoing photon we find

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* It must be remarked that the Rutherford peak is suppressed if the charge is embedded in a dielectric medium. In this case the forward travelling electromagnetic wave is slower than the gravitational wave and eventually will get out of phase with it, i.e., a medium reduces the coherence length of the process. This results in a finite value for the differential cross section in the forward direction.[1,12]

$$\left(\frac{d\sigma}{d\Omega}\right)_+ = \frac{e^2}{4\pi} \cot^2\frac{\theta}{2}\,(\sin^2 2\phi + \cos^2\theta\,\cos^2 2\phi)\ , \tag{3.25a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_\times = \frac{e^2}{4\pi} \cot^2\frac{\theta}{2}\,(\cos^2 2\phi + \cos^2\theta\,\sin^2 2\phi)\ . \tag{3.25b}$$

From (3.19) it follows that the outgoing photon is also linearly polarized. For unpolarized incident waves we must average over the incident polarization states and we recover (3.20).

The relativistic differential cross sections can also be expressed in terms of the frequency ${}^{\gamma}\omega$ of the outgoing photon instead of in terms of the scattering angle θ. Using (3.8) and (3.11), we find, after integration over ϕ,

$$\frac{d\sigma}{d^{\gamma}\omega} = \frac{e^2}{2^g\omega}\cdot\left(\frac{M}{{}^g\omega}\right)\left\{\frac{{}^g\omega}{M}\left(\frac{{}^g\omega}{{}^{\gamma}\omega}-1\right)^{-1} - \tfrac{1}{2}\right\}\left\{1 + \left[1 - \frac{M}{{}^g\omega}\left(\frac{{}^g\omega}{{}^{\gamma}\omega}-1\right)\right]^2\right\}\ , \tag{3.26}$$

where the range of ${}^{\gamma}\omega$ is

$$\frac{{}^g\omega}{1+\frac{2^g\omega}{M}} \le {}^{\gamma}\omega \le {}^g\omega\ . \tag{3.27}$$

The total conversion cross section obtained by integrating (3.20) diverges because of the long-range character of the Coulomb field. This divergence may be avoided by Debye shielding if the scattering takes place in a plasma. In the nonrelativistic limit the interaction of the gravitational wave with the fixed charge is now assumed to take place through a screened Coulomb potential

$$A_\mu = e\,\frac{\exp(-r/\lambda_D)}{4\pi r}\,\eta_{\mu o}\ . \tag{3.28}$$

Here the Debye screening length λ_D is given in terms of the electron thermal velocity

$$v_{T_e} = \sqrt{\frac{2kT_e}{M_e}}\ , \tag{3.29}$$

and the plasma frequency

$$\omega_{pe} = \sqrt{\frac{Ne^2}{M_e}} \tag{3.30}$$

as

$$\lambda_D = \frac{v_{T_e}}{\sqrt{2}\,\omega_{pe}}\ . \tag{3.31}$$

In the above, k, T_e, M_e and N are respectively the Boltzmann constant, the electron temperature, the electron mass, and the electron number density. The screened Coulomb potential has a spatial Fourier transform

$$^{\gamma 1}\varepsilon_\mu = \frac{e}{\underset{\sim}{q}^2 + q^2_{sc}}\,\eta_{\mu o}\ , \tag{3.32}$$

where $\underset{\sim}{q}$ is the momentum of the spacelike photon mediating the Coulomb interaction and $q_{sc} = 1/\lambda_D$.

Using (3.32) and (2.12d) we find that when shielding occurs, (3.20) must be replaced by

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpolarized} = \left(\frac{d\sigma}{d\Omega}\right)_R = \left(\frac{d\sigma}{d\Omega}\right)_L = \frac{e^2}{32\pi}\cdot\frac{\sin^2\theta(1+\cos^2\theta)}{\left[\sin^2\frac{\theta}{2} + \left(\frac{q_{sc}}{2\omega}\right)^2\right]^2}\ . \tag{3.33}$$

For linearly polarized incident gravitational waves the $\frac{1}{2}(1+\cos^2\theta)$ must be replaced by $(\sin^2 2\phi + \cos^2\theta\cos^2 2\phi)$ or $(\cos^2 2\phi + \cos^2\theta \sin^2 2\phi)$ for the + and x polarizations respectively.

The total cross section now becomes finite and is given by

$$\sigma = \int_0^{\pi}\int_0^{2\pi} \frac{d\sigma}{d\Omega} \sin\theta d\theta d\phi = 2e^2 [\ln(2\omega\lambda_D) - \frac{4}{3}] \quad . \qquad (3.34)$$

Note that this result is only valid for tenuous plasmas, i.e., for $\omega\lambda_D >> 1$. For dense plasmas the electromagnetic index of refraction

$$n(\omega) = \sqrt{1 - \omega_{pe}^2 / \omega^2} \qquad (3.35)$$

will not allow the electromagnetic wave to travel with the same phase velocity as the gravitational wave and therefore the conversion cross section will be reduced to a value which is considerably smaller than (3.34). (See also footnote on p.52.)

If the scattering does not take place in a plasma but the incident gravitational wave-front has a width D, the Rutherford forward-scattering peak is again suppressed and (3.34) applies approximately with the Debye screening length λ_D being replaced by the width D.

Finally, note that the formulas derived above for a point charge are also valid (to the lowest order in ω) for a charge distribution confined to the coherency volume

$$V_c << (\frac{2\pi}{\omega})^3 \quad . \qquad (3.36)$$

All of the previous formulas applied to conversion of gravitational waves into electromagnetic waves. The inverse process

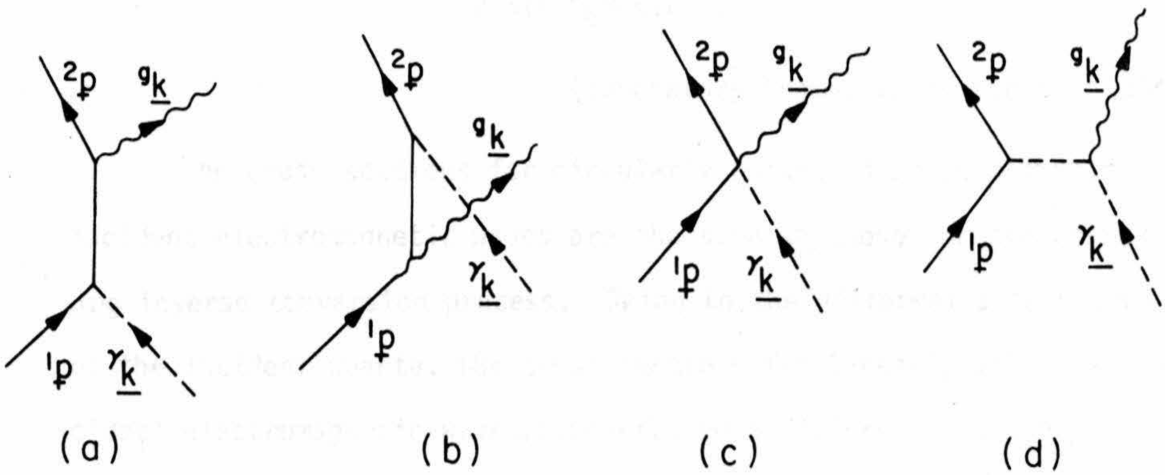

Fig. 7. Feynman graphs for exchange Compton scattering (photon → graviton) by a charged spin-0 meson.

(electromagnetic wave to gravitational wave conversion) is also possible and is described by the Feynman diagrams in Fig. 7. Straightforward calculations similar to the ones above, lead to the differential cross section for converting an electromagnetic wave with frequency ${}^{\gamma}\omega$, polarization $\underset{\sim}{\varepsilon}$ and propagation direction ${}^{\gamma}\hat{\underset{\sim}{k}}$ into a gravitational wave with frequency ${}^{g}\omega$ and polarization $\underset{\approx}{\bar{e}}$

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{8\pi \sin^4\frac{\theta}{2}} \left(\frac{1}{1+\frac{2{}^{\gamma}\omega}{M}\sin^2\frac{\theta}{2}} \right)^2 |\bar{e}^{ij*}\, {}^{\gamma}\hat{k}_{(i}\, \varepsilon_{j)}|^2 . \quad (3.37)$$

(laboratory frame; valid for all ${}^{\gamma}\omega/M$)

The cross sections for circularly polarized or unpolarized incident electromagnetic waves are the same as those for the corresponding inverse conversion process. Owing to the different spin nature of the incident quanta, the cross sections for linearly polarized incident electromagnetic waves, however, show a different ϕ dependence than the corresponding inverse-process results. Specifically, for electromagnetic wave polarizations along the x and y axes we find (in the nonrelativistic approximation)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\underset{\sim}{\hat{e}}_x} = \frac{e^2}{4\pi}\cot^2\frac{\theta}{2}\,(\sin^2\phi + \cos^2\theta\cos^2\phi), \quad (3.38a)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\underset{\sim}{\hat{e}}_y} = \frac{e^2}{4\pi}\cot^2\frac{\theta}{2}\,(\cos^2\phi + \cos^2\theta\sin^2\phi). \quad (3.38b)$$

The conversion cross sections derived above, are exceedingly small. Discarding the slowly varying logarithmic function we can write (3.34) as

$$\sigma \sim 2(\frac{G}{c^4})e^2 = 8\pi\ell_p^2\,\alpha \qquad (3.39)$$

where $\ell_p = \sqrt{G\hbar/c^3} \simeq 1.6 \times 10^{-33}$cm is the Planck distance and (in the case of an electronic charge) $\alpha = e^2/(4\pi\hbar c) \simeq 1/137$ is the fine structure constant. We find $\sigma \sim 10^{-67}\text{cm}^2$.

Papini and Valluri[11] have estimated the gravitational radiation that is generated by the interaction of photons with the space-charge in the magnetosphere of a pulsar. For NP0532 (the Crab pulsar) and in the radio-frequency range 10^8 to 10^9Hz they find that gravitational radiation due to this process is emitted at the rate $\sim 10^{-21}$erg/sec!

4. Conversion in an Electric Dipole Field

Turn now to electrogravitational conversion with an electric dipole field acting as a catalyst, and utilize the external field approximation. The electric dipole field[19,*]

$$\underset{\sim}{E} = \left[\frac{3\underset{\sim}{r}(\underset{\sim}{p}\cdot\underset{\sim}{r}) - \underset{\sim}{p}(\underset{\sim}{r}\cdot\underset{\sim}{r})}{4\pi r^5} \right]' - \frac{1}{3}\underset{\sim}{p}\,\delta^3(\underset{\sim}{r}) \tag{4.1}$$

is obtained by applying (2.2) to the Maxwell 4-potential

$$A_o = -\frac{\underset{\sim}{p}\cdot\underset{\sim}{r}}{4\pi r^3} \quad , \tag{4.2a}$$

$$A_j = 0 \quad . \tag{4.2b}$$

Here $\underset{\sim}{p}$ is the electric dipole moment. The Fourier transforms of the A_μ are given by**

$$\sigma_o = i\,\frac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{q^2} \quad , \tag{4.3a}$$

$$\sigma_j = 0 \quad , \tag{4.3b}$$

where $\underset{\sim}{q}$ is the pure spacelike momentum-transfer (no recoil of the dipole).

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*The prime []' in (4.1) indicates the following prescription: When []' occurs in any integral over position space, replace []' by zero for $r < \varepsilon$, evaluate the integral and then take the limit $\varepsilon \to 0$. With this prescription and with the help of a convergence factor $e^{-\lambda r}$ (λ is an arbitrarily small positive number) one can, for example, show that $\int \underset{\sim}{E} d^3x = -1/3\ \underset{\sim}{p}$.

**The Fourier transform (4.3a) is valid only when $|\underset{\sim}{q}| \neq 0$. Indeed, from (4.3a) one deduces the Fourier transform of (4.1): $\mathcal{F}\{\underset{\sim}{E}\} = \mathcal{F}\{-\nabla A^o\} = -i\underset{\sim}{q}\mathcal{F}\{A^o\} = -\underset{\sim}{q}(\underset{\sim}{p}\cdot\underset{\sim}{q})/q^2$. This expression is a function of the direction but not the magnitude of $\underset{\sim}{q}$ and has no unique limit for $\underset{\sim}{q} \to 0$. This strongly hints that one cannot use (4.3a) to calculate

We choose a pure spacelike photon polarization and the TT gauge for the graviton and use equations (2.12d), (3.13), (3.14) and (4.3). The result for converting a graviton with polarization $\underset{\approx}{\bar{e}}$, frequency ω, and propagation direction ${}^{g}\underset{\sim}{k}$ into a photon with polarization $\underset{\sim}{\varepsilon}$, frequency ω and propagation direction ${}^{\gamma}\underset{\sim}{\hat{k}}$ is

$$\frac{d\sigma}{d\Omega} = \frac{\omega^2 p^2}{8\pi \sin^4 \frac{\theta}{2}} \cdot [\sin\alpha \sin\theta \cos\phi - \cos\alpha(1 - \cos\theta)]^2 \times \left| \bar{e}^{ij}\, {}^{\gamma}\hat{k}_{(i}\varepsilon^*_{j)} \right|^2 \quad . \tag{4.4}$$

The angles θ, ϕ and α are defined in Fig. 8 . It must be stressed that (4.4) is not valid for $\theta = 0$.

Consider now circular polarizations. After some algebraic manipulations we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{\omega^2 p^2}{16\pi} [\sin\alpha (1 + \cos\theta)\cos\phi - \cos\alpha \sin\theta]^2 \times (1 + \cos\theta)^2 \quad , \tag{4.5a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{\omega^2 p^2}{16\pi} [\sin\alpha (1 + \cos\theta)\cos\phi - \cos\alpha \sin\theta]^2 \times (1 - \cos\theta)^2 \quad , \tag{4.5b}$$

(continued) $\mathcal{F}\{\underset{\sim}{E}\}$ at $g = 0$. A careful evaluation of the Fourier transform of (4.1) (taking the prescriptions of the above footnote into account) reveals that

$$\mathcal{F}\{\underset{\sim}{E}\} = \begin{cases} -1/3\underset{\sim}{p} & \text{for } |\underset{\sim}{q}| = 0 \\ -\underset{\sim}{q}\,\dfrac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{q^2} & \text{otherwise} \end{cases}$$

Note that $-1/3\underset{\sim}{p}$ is the angular average of $-\underset{\sim}{q}\,\dfrac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{q^2}$.

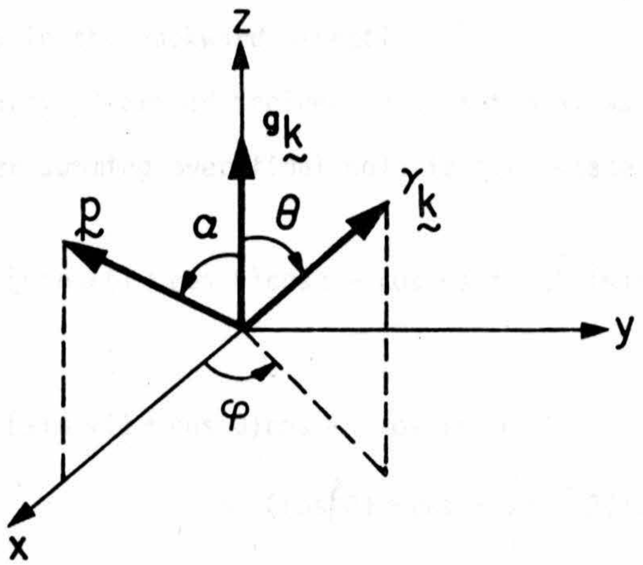

Fig. 8. The spatial orientation of the electric dipole $\underset{\sim}{p}$ and the direction $^\gamma\underset{\sim}{k}$ of the outgoing photon relative to the direction $^g\underset{\sim}{k}$ of the incident graviton.

where the first (second) subscript denotes the graviton (photon) polarization. Summing over final polarizations, we obtain the conversion cross sections for circularly polarized (or unpolarized) waves

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \left(\frac{d\sigma}{d\Omega}\right)_L = \frac{\omega^2 p^2}{8\pi}\left[\sin\alpha(1+\cos\theta)\cos\phi - \cos\alpha\sin\theta\right]^2 (1+\cos^2\theta). \tag{4.6}$$

Note that the outgoing wave is elliptically polarized and that the cross section vanishes in the backward direction.

For linearly polarized incident gravitational waves, one finds from (4.4), after summing over final polarization states,

$$\left(\frac{d\sigma}{d\Omega}\right)_+ = \frac{\omega^2 p^2}{4\pi}\left[\sin\alpha(1+\cos\theta)\cos\phi - \cos\alpha\sin\theta\right]^2 (\sin^2 2\phi + \cos^2\theta\cos^2 2\phi), \tag{4.7a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_\times = \frac{\omega^2 p^2}{4\pi}\left[\sin\alpha(1+\cos\theta)\cos\phi - \cos\alpha\sin\theta\right]^2 \times (\cos^2 2\phi + \cos^2\theta\sin^2 2\phi). \tag{4.7b}$$

As can be seen from (4.5), the outgoing electromagnetic wave is also linearly polarized.

An obvious feature of the above cross sections is the absence of a Rutherford peak in the forward direction, a manifestation of the fact that the dipole field falls off faster than r^{-1}. This yields a finite total cross section*

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*The total cross section (4.8) is obtained by integrating the above differential cross sections, which are valid for all scattering angles θ but the forward direction ($\theta = 0$). This total cross section is correct as the differential cross section does not have a delta function-like singularity for $\theta = 0$. In fact, using $\mathcal{F}\{\tilde{E}\} = -1/3\, \tilde{p}$ for $\tilde{q} = 0$, one easily shows: $(d\sigma/d\Omega)_{\theta=0} = [(\omega^2 p^2)/9\pi]\sin^2\alpha$, regardless of the incident polarization state.

$$\sigma = \frac{7}{15}\,\omega^2 p^2 (1 - \frac{1}{7}\cos^2\alpha), \text{ for any incident polarization} \quad . \qquad (4.8)$$

Maximal conversion occurs when the direction of incidence is orthogonal to the dipole moment.

Electromagnetic-to-gravitational wave conversion is also described by (4.4) with the following substitutions: $\bar{\underset{\approx}{e}} \to \bar{\underset{\approx}{e}}^*$, $\underset{\sim}{\varepsilon}^* \to \underset{\sim}{\varepsilon}$. For circularly polarized or unpolarized incident electromagnetic waves the conversion cross sections are the same as the cross sections for the corresponding inverse process. For linearly polarized incident electromagnetic waves, on the other hand, the cross sections exhibit a different ϕ dependence when compared with (4.7):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\hat{\underset{\sim}{e}}_x} = \frac{\omega^2 p^2}{4\pi}\,[\sin\alpha(1+\cos\theta)\cos\phi - \cos\alpha\sin\theta]^2(\sin^2\phi + \cos^2\theta\cos^2\phi), \qquad (4.9a)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\hat{\underset{\sim}{e}}_y} = \frac{\omega^2 p^2}{4\pi}\,[\sin\alpha(1+\cos\theta)\cos\phi - \cos\alpha\sin\theta]^2(\cos^2\phi + \cos^2\theta\sin^2\phi). \qquad (4.9b)$$

The corresponding total cross sections are

$$\sigma_{\hat{\underset{\sim}{e}}_x} = \frac{2}{5}\,\omega^2 p^2 (1 - \frac{1}{12}\sin^2\alpha) \quad , \qquad (4.10a)$$

$$\sigma_{\hat{\underset{\sim}{e}}_y} = \frac{17}{30}\,\omega^2 p^2 (1 - \frac{5}{17}\cos^2\alpha) \quad . \qquad (4.10b)$$

5. Conversion in a Magnetic Dipole Field

The magnetic dipole field[19]

$$\underset{\sim}{B} = \left[\frac{3\underset{\sim}{r}(\underset{\sim}{m}\cdot\underset{\sim}{r}) - \underset{\sim}{m}(\underset{\sim}{r}\cdot\underset{\sim}{r})}{4\pi r^5}\right]' + \frac{2}{3}\underset{\sim}{m}\,\delta^3(\underset{\sim}{r}) \qquad (5.1)$$

may be obtained by applying (2.2) to the Maxwell 4-potential

$$A_o = 0 \quad , \qquad (5.2a)$$

$$A_j = \left(\frac{\underset{\sim}{m} \times \underset{\sim}{r}}{4\pi r^3}\right)_j \quad . \qquad (5.2b)$$

Here $\underset{\sim}{m}$ is the magnetic dipole moment. The Fourier transforms of the A_μ are given by*

$$\sigma_o = 0 \quad , \qquad (5.3a)$$

$$\sigma_j = -i\left(\frac{\underset{\sim}{m} \times \underset{\sim}{q}}{q^2}\right)_j \quad . \qquad (5.3b)$$

Again we choose a pure spacelike polarization for the photon and the TT gauge for the graviton. With the aid of (5.3), (2.12d), (3.13) and (3.14) we find the differential cross section for converting a graviton with polarization $\overline{\underset{\approx}{e}}$, frequency ω and propagation direction ${}^{g}\hat{\underset{\sim}{k}}$ into a photon with polarization $\underset{\sim}{\varepsilon}$, frequency ω and propagation direction ${}^{\gamma}\hat{\underset{\sim}{k}}$

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*Again the Fourier transform (5.3b) is only valid as long as $|\underset{\sim}{q}| \neq 0$. Invoking (5.3b) one finds $\mathcal{F}\{\underset{\sim}{B}\} = \mathcal{F}\{\nabla \times \underset{\sim}{A}\} = i\underset{\sim}{q} \times \mathcal{F}\{\underset{\sim}{A}\} = \underset{\sim}{q} \times \frac{(\underset{\sim}{m} \times \underset{\sim}{q})}{q^2}$ $= -\underset{\sim}{q}\,\frac{\underset{\sim}{m}\cdot\underset{\sim}{q}}{q^2} + \underset{\sim}{m}$, for $|\underset{\sim}{q}| \neq 0$. Taking the footnotes on p.59 into account, one finds that $\mathcal{F}\{\underset{\sim}{B}\} = 2/3\,\underset{\sim}{m}$, for $|\underset{\sim}{q}| = 0$.

$$\frac{d\sigma}{d\Omega} = \frac{\omega^2 m^2}{8\pi \sin^4\frac{\theta}{2}} \left| \bar{e}^{ij} \left\{ {}^{\gamma}\hat{k}_{(i}{}^{\gamma}\hat{k}_{j)} \underset{\sim}{\varepsilon}^{*}\cdot[\underset{\sim}{m} \times ({}^{\gamma}\hat{\underset{\sim}{k}} - {}^{g}\hat{\underset{\sim}{k}})] + \varepsilon^{*}_{(i}[\hat{\underset{\sim}{m}} \times ({}^{\gamma}\hat{\underset{\sim}{k}} - {}^{g}\hat{\underset{\sim}{k}})]_{j)} (1 - {}^{\gamma}\hat{\underset{\sim}{k}}\cdot{}^{g}\hat{\underset{\sim}{k}}) \right.\right.$$

$$\left.\left. + \; {}^{\gamma}\hat{k}_{(i}\varepsilon^{*}_{j)} \; {}^{\gamma}\hat{\underset{\sim}{k}}\cdot(\hat{\underset{\sim}{m}} \times {}^{g}\hat{\underset{\sim}{k}}) + {}^{\gamma}\hat{k}_{(i}[\hat{\underset{\sim}{m}} \times ({}^{\gamma}\hat{\underset{\sim}{k}} - {}^{g}\hat{\underset{\sim}{k}})]_{j)} \; {}^{g}\hat{\underset{\sim}{k}}\cdot\underset{\sim}{\varepsilon}^{*} \right\} \right|^2 . \tag{5.4}$$

In the above $\hat{\underset{\sim}{m}}$ is a unit vector along the direction of $\underset{\sim}{m}$.

Using the notations of the previous sections, we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{\omega^2 m^2}{16\pi} \{[\cos\alpha \sin\theta + \sin\alpha(1 - \cos\theta)\cos\phi]^2 + [2\sin\alpha \sin\phi]^2\} (1 + \cos\theta)^2 , \tag{5.5a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{\omega^2 m^2}{16\pi} \{[\cos\alpha \sin\theta + \sin\alpha(1 - \cos\theta)\cos\phi]^2 + [2\sin\alpha \sin\phi]^2\} (1 - \cos\theta)^2 , \tag{5.5b}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{R} = \left(\frac{d\sigma}{d\Omega}\right)_{L} = \frac{\omega^2 m^2}{8\pi} \{[\cos\alpha \sin\theta + \sin\alpha(1 - \cos\theta)\cos\phi]^2 + [2\sin\alpha \sin\phi]^2\} (1 + \cos^2\theta) , \tag{5.6}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{+} = \frac{\omega^2 m^2}{4\pi} \{[(\cos\alpha \sin\theta + \sin\alpha(1 - \cos\theta)\cos\phi)\sin 2\phi - 2\sin\alpha \sin\phi \cos 2\phi]^2 + \cos^2\theta[(\cos\alpha \sin\theta + \sin\alpha(1 - \cos\theta)\cos\phi)\cos 2\phi + 2\sin\alpha \sin\phi \sin 2\phi]^2\} , \tag{5.7a}$$

$$(\frac{d\sigma}{d\Omega})_{x} = \frac{\omega^2 m^2}{4\pi} \{[(\cos\alpha\sin\theta + \sin\alpha(1-\cos\theta)\cos\phi)\cos 2\phi$$
$$+ 2\sin\alpha\sin\phi\sin 2\phi]^2$$
$$+ \cos^2\theta[(\cos\alpha\sin\theta + \sin\alpha(1-\cos\theta)\cos\phi)\sin 2\phi$$
$$- 2\sin\alpha\sin\phi\cos 2\phi]^2\} \quad . \qquad (5.7b)$$

Unlike conversion in an electric dipole field, these cross sections do not vanish in the backward direction:

$$(\frac{d\sigma}{d\Omega})_{\theta=\pi} = \pi^{-1}\,\omega^2 m^2 \sin^2\alpha \quad . \qquad (5.8)$$

The total cross section is finite and is given by*

$$\sigma = \frac{9}{5}\,\omega^2 m^2 (1 - \frac{7}{9}\cos^2\alpha) \text{ , for any polarization} \quad . \qquad (5.9)$$

As before, electromagnetic-into-gravitational wave conversion is described by (5.4), modulo the substitutions $\bar{\underset{\approx}{e}} \to \bar{\underset{\approx}{e}}^*$, $\underset{\sim}{\varepsilon}^* \to \underset{\sim}{\varepsilon}$. Formulas (5.5)-(5.6) remain the same, but (5.7) must be replaced by

$$(\frac{d\sigma}{d\Omega})_{\hat{\underset{\sim}{e}}_x} = \frac{\omega^2 m^2}{4\pi} \{[\cos\alpha\sin\theta + \sin\alpha(1-\cos\theta)\cos\phi]^2 + [2\sin\alpha\sin\phi]^2\}$$
$$\times\ (\sin^2\phi + \cos^2\theta\cos^2\phi) \quad , \qquad (5.10a)$$

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*Using $\mathcal{F}\{\underset{\sim}{B}\} = 2/3\ \underset{\sim}{m}$ for $\underset{\sim}{q} = 0$, one shows that $(d\sigma/d\Omega)_{\theta=0} = \frac{4\omega^2 m^2}{9\pi}\sin^2\alpha$, and hence the footnote on p. 62 is applicable.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\hat{\underset{\sim}{e}}_y} = \frac{\omega^2 m^2}{4\pi} \{[\cos\alpha \sin\theta + \sin\alpha (1 - \cos\theta)\cos\phi]^2 + [2 \sin\alpha \sin\phi]^2\}$$

$$\times \ (\cos^2\phi + \cos^2\theta \sin^2\phi) \quad . \tag{5.10b}$$

The corresponding total cross sections are

$$\sigma_{\hat{\underset{\sim}{e}}_x} = \frac{61}{30}\,\omega^2 m^2 (1 - \frac{49}{61}\cos^2\alpha) \quad , \tag{5.11a}$$

$$\sigma_{\hat{\underset{\sim}{e}}_y} = \frac{47}{30}\,\omega^2 m^2 (1 - \frac{35}{47}\cos^2\alpha) \quad . \tag{5.11b}$$

The above results were derived for a pointlike dipole, but they yield estimates of the proper order of magnitude for any magnetic scatterer whose characteristic dimensions are considerably less than the wavelength of the incident radiation. For a uniformly magnetized sphere with radius a and internal magnetic induction $\underset{\sim}{B}_{in}$ we have $\underset{\sim}{m} = 2\pi a^3 \underset{\sim}{B}_{in}$ and we obtain from (5.9)

$$\sigma \sim 2 \times 10^{-46} [1/4\pi]\, B_{in}^2 \ a^4 (\frac{a}{\lambda})^2 \quad , \tag{5.12}$$

where $\lambda = 2\pi/\omega$. The quantity between square brackets must be included if one expresses B_{in} in Gaussian units instead of Heaviside-Lorentz units.

The parameters for the pulsar NP0532 are: $a \sim 10^6$ cm, $B_{in} \sim 10^{12}$ gauss. For $\lambda = 10a$, (5.12) yields $\sigma \sim .2\ cm^2$. There is no observable conversion of long-wavelength radiation by magnetic stars.[10,12]

6. Conversion in a Uniform Magnetostatic Field

We first study the inverse Gertsenshteǐn process,[3,8] i.e., conversion of a gravitational wave into an electromagnetic wave in a homogeneous magnetostatic background. Consider a plane gravitational wave

$$\bar{h}^{\mu\alpha} = \bar{e}^{\mu\alpha} e^{i\,{}^{g}\underline{k}\cdot\underline{x}} \quad , \tag{6.1}$$

propagating along the z axis and incident on a uniform magnetostatic field $\underset{\sim}{B}$ (see Fig. 9). This magnetic background is confined to the region between the planes $z = -\ell/2$ and $z = \ell/2$ and makes an angle α with ${}^{g}\underset{\sim}{k}$:

$$\underset{\sim}{B} = B\,\mathrm{rect}(\tfrac{z}{\ell})(\sin\alpha\,\hat{\underset{\sim}{e}}_x + \cos\alpha\,\hat{\underset{\sim}{e}}_z) \quad , \tag{6.2}$$

where the <u>rectangle</u> function is defined by

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{6.3}$$

In the TT gauge the conversion process is described by the 2-photon-graviton interaction functional (cf. eq. (2.10c))

$$\mathcal{S}_I = \lambda \int \bar{h}^{ij}\,\eta^{\nu\beta}\,F_{i\nu}F_{j\beta}\,d^4x \quad , \tag{6.4}$$

where $F_{i\nu}$ and $F_{j\beta}$ stand both for the outgoing electromagnetic wave

$$F_{\mu\nu} = -i({}^{\gamma}k_\mu\,\varepsilon^*_\nu - {}^{\gamma}k_\nu\,\varepsilon^*_\mu)\,e^{-i\,{}^{\gamma}\underline{k}\cdot\underline{x}} \tag{6.5}$$

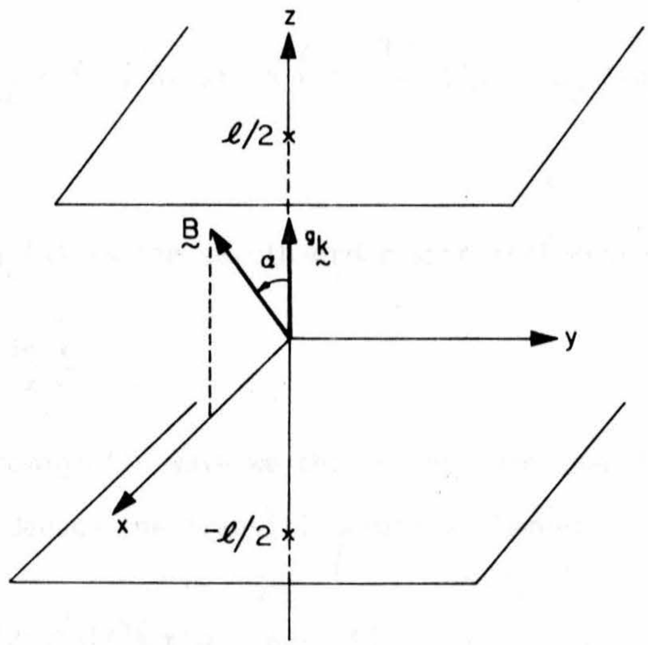

Fig. 9. The spatial orientation of the uniform magnetic background $\underset{\sim}{B}$ relative to the direction ${}^{g}\underset{\sim}{k}$ of the incident graviton.

and the magnetic background

$$F_{o\nu} = -F_{\nu o} = 0$$

$$F_{13} = -F_{31} = 0$$

$$F_{12} = -F_{21} = (2\pi)^3 B\ell \cos\alpha \int j_o(\frac{q_3\ell}{2})\delta(q_1)\delta(q_2)\delta(q_o)e^{i\underline{q}\cdot\underline{x}}\frac{d^4q}{(2\pi)^4} ,$$

$$F_{23} = -F_{32} = (2\pi)^3 B\ell \sin\alpha \int j_o(\frac{q_3\ell}{2})\delta(q_1)\delta(q_2)\delta(q_o)e^{i\underline{q}\cdot\underline{x}}\frac{d^4q}{(2\pi)^4} . \tag{6.6}$$

In the above $j_o(x)$ is the zeroth-order spherical Bessel function

$$j_o(x) = \frac{\sin x}{x} . \tag{6.7}$$

For the electromagnetic wave we choose the pure spacelike gauge ($\varepsilon_o = 0$). From (6.4) we deduce the transition-matrix element

$$T_{fi} = i2(2\pi)^2\lambda[({}^{\gamma}\hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{B}}) \cdot \underset{\approx}{\overline{e}} \cdot \underset{\sim}{\varepsilon}^*]$$

$$\times B\ell\, j_o\left[\frac{({}^{\gamma}k_3 - {}^{g}k_3)\ell}{2}\right] {}^{\gamma}k_3\, \delta({}^{\gamma}k_1 - {}^{g}k_1)\delta({}^{\gamma}k_2 - {}^{g}k_2) . \tag{6.8}$$

The presence of δ-functions in (6.8) means, among other things, that the electromagnetic wave is constrained to travel along the $\pm z$ directions.

To obtain the transition probability per second we must square (6.8) and substitute into the "golden rule"

$$\frac{\text{transition probability}}{\text{second}} = \int \frac{2\pi}{2{}^{\gamma}k^{o}\, 2{}^{g}k^{o}} |T_{fi}|^2 \, D \tag{6.9}$$

where D is the density of final states

$$D = \frac{d^3\, {}^{\gamma}k}{(2\pi)^3} \delta({}^{\gamma}k^{o} - {}^{g}k^{o}) \quad . \tag{6.10}$$

This transition probability exhibits quadratic dependence on $\delta({}^{\gamma}k_1 - {}^{g}k_1)$ and $\delta({}^{\gamma}k_2 - {}^{g}k_2)$. Following the usual procedure, we put

$$|\delta({}^{\gamma}k_1 - {}^{g}k_1)|^2 = \frac{L}{2\pi} \delta({}^{\gamma}k_1 - {}^{g}k_1) \quad , \tag{6.11a}$$

$$|\delta({}^{\gamma}k_2 - {}^{g}k_2)|^2 = \frac{L}{2\pi} \delta({}^{\gamma}k_2 - {}^{g}k_2) \quad , \tag{6.11b}$$

with L an arbitrarily large but finite length. Ill-defined mathematical expressions containing squares of δ-functions can be avoided if one uses wave packets to represent the ingoing and outgoing waves. The infinities in $|T_{fi}|^2$ arise as a consequence of the infinite extent of the interaction region (infinitely wide wavefronts propagating in a magnetic background, which itself is infinitely extended in the transverse directions).

Therefore we calculate the transition rate per unit area

$$\Gamma \equiv \left(\frac{\text{transition probability}}{\text{cm}^2 \text{ second}}\right) = \int \frac{2\pi}{2{}^{\gamma}k^{o}\, 2{}^{g}k^{o}} \frac{|T_{fi}|^2}{L^2} D \tag{6.12}$$

and we obtain, after using (6.8), (6.10) and (6.11),

$$\Gamma_{\text{forward}} = 8\pi B^2 \ell^2 \sum_{\underset{\sim}{\varepsilon}} |({}^{\gamma}\hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{B}}) \cdot \overline{\underset{\approx}{e}} \cdot \underset{\sim}{\varepsilon}^*|^2 \quad , \tag{6.13a}$$

$$\Gamma_{\text{backward}} = 8\pi B^2 \ell^2 \; j_o{}^2\,(\omega\ell) \sum_{\underset{\sim}{\varepsilon}} |({}^{\gamma}\hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{B}}) \cdot \overline{\underset{\approx}{e}} \cdot \underset{\sim}{\varepsilon}^*|^2 \; . \tag{6.13b}$$

Here $\sum_{\underset{\sim}{\varepsilon}}$ denotes summation over the final photon polarizations and $\omega \equiv {}^{g}\underset{\sim}{k}{}^{o} = {}^{\gamma}k^{o}$. Evaluating $P \equiv |({}^{\gamma}\hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{B}}) \cdot \overline{\underset{\approx}{e}} \cdot \underset{\sim}{\varepsilon}^*|^2$ for different choices of initial and final polarizations, we arrive at

$$P_{+\rightarrow \hat{\underset{\sim}{e}}_x} = P_{\times\rightarrow \hat{\underset{\sim}{e}}_y} = 0 \quad , \tag{6.14a}$$

$$P_{\times\rightarrow \hat{\underset{\sim}{e}}_x} = P_{+\rightarrow \hat{\underset{\sim}{e}}_y} = \frac{1}{2}\sin^2\alpha \quad , \tag{6.14b}$$

$$P_{RL} = P_{LR} = 0 \quad , \tag{6.14c}$$

$$P_{RR} = P_{LL} = \frac{1}{2}\sin^2\alpha \quad , \tag{6.14d}$$

i.e., linearly polarized gravitons generate linearly polarized photons, whereas circularly polarized gravitons generate circularly polarized photons with the same helicity.

Note that these transition probabilities have been computed for an incident number flux $= \dfrac{1 \text{ particle}}{\text{cm}^2 \text{second}}$. It follows that

$$T^{03}_{EMW} = (\pm)\Gamma\; T^{03}_{GW} = \begin{pmatrix} +1 \\ \\ -j_o^2(\omega\ell) \end{pmatrix} 4\pi B^2 \sin^2\alpha\; \ell^2 T^{03}_{GW} \quad , \tag{6.15}$$

where T^{03}_{EMW} and T^{03}_{GW} are the power flux of the electromagnetic wave and gravitational wave respectively and where the upper (lower) sign refers to forward (backward) outgoing electromagnetic radiation. The electromagnetic power flux in the backward direction is smaller by a factor $j_0^2(\omega\ell)$ as compared with the flux in the forward direction and vanishes if the condition $\ell = n\frac{\lambda}{2}$ $(n = 1,2,\cdots;\lambda = \frac{2\pi}{\omega})$ is met. The fact that the conversion efficiency Γ is quadratic in ℓ depends critically on the equality of the propagation velocities of the electromagnetic and gravitational waves. If we introduce a medium with a dielectric constant $\neq 1$, we destroy the coherence between the gravitational and electromagnetic perturbations and thereby put a limit on the useful length ℓ. Note also that for propagation along the field lines of $\underset{\sim}{B}$ resonant conversion does not occur.

A magnetic field with finite transverse directions $\sim L$ $(L >> \frac{2\pi}{\omega})$ has a conversion cross section of the order

$$\begin{aligned} \sigma &\sim 4\pi B^2 L^2 \ell^2 \sin^2\alpha \\ &\sim 4\pi B^2 Vt \sin^2\alpha \quad , \end{aligned} \tag{6.16}$$

where V is the volume of the magnetic field region, and t is the travel time of the gravitational perturbation through the magnetostatic background. The propagation direction of the outgoing electromagnetic wave is not confined to only (±) the direction of the incident gravitational wave, but can be within a cone (with half-angle $\sim 1/\omega L$) centered about this direction of incidence.

For the Gertsenshteǐn process (conversion of electromagnetic waves into gravitational waves) all of the formulas above apply, allowing the substitution $\underset{\sim}{\varepsilon}^* \to \underset{\sim}{\varepsilon}$, $\underset{\approx}{\bar{e}} \to \underset{\approx}{\bar{e}}^*$ in (6.8), (6.13).

There is an extensive literature on the Gertsenshtein emission of very-high-frequency gravitational waves in astrophysical situations. The following list (which is by no means exhaustive) serves as an illustration of the near-impossibility of imagining astrophysical scenarios in which the Gertsenshtein process is of practical interest.

a) <u>Laboratory</u>

$B = 10^5$ gauss, $\ell = 10^3$ cm, $\Gamma = 10^{-33}$

b) <u>Interstellar magnetic fields</u>

If the magnetic background is chaotic with an ordered structure on some scale $\ell_c \gg 2\pi/\omega$ (ℓ_c stands for correlation length), the gravitational waves generated in different cells are incoherent. One must therefore add their energies and one obtains for the conversion efficiency,

$$\Gamma \sim 2\pi \langle B^2 \rangle \ell_c t \quad , \tag{6.17}$$

where t is the time of passage of the electromagnetic wave through the magnetic background. With $B \sim 10^{-5}$ gauss, $\ell_c \sim 10$ lt-yr, $t \sim 10^7$ yr, one finds $\Gamma \sim 10^{-15}$ (Ref. 3).

c) <u>Cosmological field</u>

Zel'dovich[20] has pointed out that an observable effect could exist in a universe with a homogeneous magnetic field that varies according to the "freezing-in" law $B = B_0(1+z)^2$, where B_0 is the magnetic field at the present moment and z is the redshift. For $B_0 \sim 10^{-6}$ gauss, $z \sim 10^3$ (time of recombination of the primordial plasma) he finds

$\Gamma \sim 0.1$. This would lead to a reduction by 10% of the cosmic microwave background in a wide belt perpendicular to the cosmological field. No such effect has been observed. Zel'dovich attributes this to the presence of atoms and free charges which scatter the photons and thereby reduce the coherence length of the process. The conversion process can only be important in an empty hot magnetic universe.

d) Quasars

Estimates of the gravitational power emitted by 3C273 through the Gertsenshteĭn process have been given by Papini and Valluri[11] for various spectral regions. The graviton yield peaks at infra-red frequencies with an upper limit $\sim 10^{30}$ erg/sec. The corresponding flux at Earth, assuming the distance to 3C273 to be 500 Mpc, is $\sim 10^{-25}$ erg/cm^2 sec.* This is negligible compared with the fluxes $\sim 10^{-12}$ erg/cm^2 sec of the broad-band bursts that originate in huge explosions in distant quasars, as conjectured by Ozernoi[21] and Press and Thorne.[22]

e) Pulsars

Papini and Valluri[11] have estimated the graviton emission in various frequency regions due to the Gertsenshteĭn process in NP0532. They find a gravitational luminosity $\sim 10^{30}$ erg/sec, with a peak in the soft x-ray range. The corresponding flux at Earth is $\sim 10^{-14}$ erg/cm^{2}sec.* A comparable flux (in an entirely different frequency range!) would be generated through the quadrupole-moment radiation mechanism for a value of the ellipticity $\varepsilon \sim 10^{-7}$ (Ref. 22,23).

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*Note that Papini and Valluri have based their calculations on a cross section which is too large by a factor $\sim$600. The numerical results that we cite take the correction to the cross section into account.

7. Conversion in a Uniform Electrostatic Field

Finally, turn to the Lupanov process[5,8] (and its inverse), i.e., gravitational-to-electromagnetic wave conversion (and *vice versa*) in a homogeneous electrostatic field. Choose the same geometrical configuration as in Fig. 9, with $\underset{\sim}{B}$ being replaced by $\underset{\sim}{E}$:

$$\underset{\sim}{E} = E\ \mathrm{rect}(z/\ell)(\sin\alpha\ \hat{\underset{\sim}{e}}_x + \cos\alpha\ \hat{\underset{\sim}{e}}_z) \quad . \tag{7.1}$$

The conversion processes are again described by (6.4) where the electrostatic background is now

$$\begin{aligned} F_{10} &= -F_{01} = (2\pi)^3 E\ell \sin\alpha \int j_o\left(\frac{q_3\ell}{2}\right)\delta(q_1)\delta(q_2)\delta(q_o)\ e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} \frac{d^4q}{(2\pi)^4} , \\ F_{30} &= -F_{03} = (2\pi)^3 E\ell \cos\alpha \int j_o\left(\frac{q_3\ell}{2}\right)\delta(q_1)\delta(q_2)\delta(q_o)\ e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} \frac{d^4q}{(2\pi)^4} , \\ &\text{All other } F_{\mu\nu} = 0 \quad . \end{aligned} \tag{7.2}$$

The transition amplitude is given by (6.8) with B being replaced by E. The conversion efficiencies and cross sections of Sec. 6 are applicable *tutti quanti*, if we substitute B by E.

8. Conclusions and Comparisons with Previous Results

We have computed electrogravitational conversion cross sections using Feynman perturbation methods for various electromagnetic backgrounds. For reasons of ease and straightforwardness, a quantum approach has been used to calculate a process which is classical in se (the conversion efficiencies do not depend on $\hbar$).

For the exchange Compton scattering various authors have obtained conflicting results. Papini and Valluri[11] and Matzner[17] obtained erroneously finite total cross sections. Our results confirm the findings of Ginzburg and Tsytovich,[12] who exploited the formal analogy with electromagnetic transition radiation and obtained exactly the nonrelativistic limit of our (non-integrable) differential cross section. The divergence is avoided only after either introducing Debye screening or by limiting the spatial extent of the incident wavefronts. Boughn[24] also arrived at a divergent cross section in the form of a multipole series. The quadrupole term in this series is the most important one, but the higher multipole terms do not fall off fast enough to ensure convergence of the series. For this reason one may not limit oneself to quadrupole waves in computing the total cross section for unscreened charges, as Matzner does. Screening, however, imposes a cut-off on the multipole series at some maximum value of the angular-momentum eigenvalue, and in this situation Matzner's result is essentially correct.

It must be stressed that we have calculated a gauge invariant transition-matrix element. We have also shown that in the nonrelativistic regime ($\omega \ll M$) one can still obtain the correct transition-matrix element by limiting one's attention to the t-pole diagram <u>if one chooses the TT gauge for the gravitational wave</u>. This is what Ginzburg and Tsytovich, and Boughn have done. If one were to choose a non-TT gauge for the gravitational wave, the t-pole term becomes (in the nonrelativistic limit)

$$T = \frac{2\lambda e}{(\gamma_{\underset{\sim}{k}} - g_{\underset{\sim}{k}})^2} \{\omega \, \bar{e}^{ij} \, \gamma_{k(i}\varepsilon^*_{j)} - \omega \, \bar{e}^{ij} \, g_{k(i}\varepsilon^*_{j)} + \frac{1}{2} \omega \, \bar{e} \, g_{\underset{\sim}{k}} \cdot \underset{\sim}{\varepsilon}^*$$

$$- \frac{1}{2} \bar{e}^{io} \, \varepsilon^*_i (\gamma_{\underset{\sim}{k}} - g_{\underset{\sim}{k}})^2 - \bar{e}^{\mu o} \, \gamma_{k_\mu} \, g_{\underset{\sim}{k}} \cdot \underset{\sim}{\varepsilon}^* \} \quad . \tag{8.1}$$

For notations see Sec. 3. The transition-matrix element (8.1) was calculated for a pure spacelike photon gauge. (The t-pole term is independent of the photon gauge.) In the TT gauge only the first term in (8.1) survives. Note that for a non-TT gauge the backscatter is nonzero:

$$T_{(\gamma_{\underset{\sim}{k}} = -g_{\underset{\sim}{k}})} = -2\lambda e \, \bar{e}^{jo} \, \varepsilon^*_j \quad . \tag{8.2}$$

If we choose to calculate in the TT gauge, however, we find $T_{(\gamma_{\underset{\sim}{k}} = -g_{\underset{\sim}{k}})} = 0$. This glaringly illustrates the ambiguities we must face if we calculate a transition-matrix element which is not gauge invariant. The best we can hope for is that for an appropriate choice of gauge, the effect of the omitted diagrams is negligible. The gauge to choose for this problem is the TT gauge.

Finally, note that we have studied exchange Compton scattering only for spinless particles. For spin-1/2 fermions the calculations are similar but more complicated, due to the extra spin degrees of freedom.* In the nonrelativistic limit, however, the results for scalar particles are valid for spin-1/2 fermions as well.

Conversion scattering in the field of dipoles has received attention from Ginzburg and Tsytovich, and Papini and Valluri. Ginzburg and Tsytovich give differential cross sections that are integrated over ϕ. Our differential cross sections for an electric dipole, when integrated over ϕ, agree with the results of Ginzburg and Tsytovich.** For magnetic dipoles, however, Ginzburg and Tsytovich find the same results as for electric dipoles, whereas ours are different (unless $\alpha = 0$). This is because they do not use the correct field for a magnetic dipole.***

The Gertsenshteĭn and Lupanov resonant processes (and their inverses) have been analyzed rigorously by Boccaletti _et al._[8] For electro-

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* One should actually use the vierbein formalism in formulating this problem.

** Ginzburg and Tsytovich omitted a term in their equation (29): The expression $\frac{1}{2}\sin^2\theta_0(1+\cos^2\theta)$ should be replaced by $\frac{1}{2}\sin^2\theta_0(1+\cos\theta)^2$ The term $\sin^2\theta_0\cos\theta$ was left out, as it does not contribute to the total cross section (Ginzburg and Tsytovich, private communication).

*** Ginzburg and Tsytovich start from the magnetic _scalar_ potential $\sigma^0 = -i\,\frac{\underset{\sim}{q}\cdot\underset{\sim}{m}}{q^2}$, from which they find $\mathcal{F}\{\underset{\sim}{B}\} = -\underset{\sim}{q}\,\frac{\underset{\sim}{q}\cdot\underset{\sim}{m}}{q^2}$. This is _not_ the Fourier transform of the expression (5.1) for the magnetic dipole field, but rather of $\underset{\sim}{B} = \frac{3\underset{\sim}{r}(\underset{\sim}{m}\cdot\underset{\sim}{r}) - \underset{\sim}{m}(\underset{\sim}{r}\cdot\underset{\sim}{r})}{4\pi r^5} - \frac{1}{3}\underset{\sim}{m}\,\delta^3(\underset{\sim}{r})$. As the current density does not vanish everywhere, one should really use the magnetic _vector_ potential $\underset{\sim}{\sigma} = -i\,\frac{\underset{\sim}{m}\times\underset{\sim}{q}}{q^2}$. From this one finds $\mathcal{F}\{\underset{\sim}{B}\} = -\underset{\sim}{q}\,\frac{\underset{\sim}{q}\cdot\underset{\sim}{m}}{q^2} + \underset{\sim}{m}$, which is the Fourier transform of (5.1). The term m (which was neglected by Ginzburg and Tsytovich) is responsible for the different behaviour of magnetic and electric dipoles. If one uses the magnetic vector potential, the method employed by Ginzburg and Tsytovich leads to our results (Ginzburg and Tsytovich, private communication).

magnetic-to-gravitational wave conversion our results are identical with theirs; only the transverse components of the background field contribute to conversion, the converted wave propagates only in the same or in the opposite direction of the incident wave, the converted wave propagating in the backward direction is weaker than the converted wave propagating forwards and may be absent completely, and the conversion efficiency depends quadratically on the travel time of the perturbation through the background. We also confirm their numerical correction to Gertsenshteĭn's original results.

There is some disagreement with the results of Boccaletti _et al._ for gravitational-into-electromagnetic wave conversion in a homogeneous background. These authors find a backward travelling electromagnetic wave if the incident gravitational wave propagates _along_ the field lines of the background. This erroneous feature (which destroys the symmetry between gravitational-into-electromagnetic wave conversion and the corresponding inverse process) is due to their choice of a gravitational gauge which is not TT. If one chooses to use the TT gauge, the method used by Boccaletti _et al._ reproduces our results.

The conversion efficiencies are forbiddingly small. Even in astrophysical objects with strong magnetic fields and large photon fluxes, the very-high-frequency gravitational luminosities are meager. The prospects for detection with current or foreseeable technology are bleak. Certainly mechanical detectors would hardly be suitable.

Conversion scattering may, however, play a role in the laboratory generation[25] and detection[26,27] of very-high-frequency gravitational radiation ("Hertz-type" experiment). In the laboratory one may compensate for the smallness of the effects by exploiting resonance and coherence. This would be achieved by using an electromagnetic resonator to generate coherently highly monochromatic gravitational waves with a known phase. These gravitational waves would subsequently be detected by a second electromagnetic resonator with a set of eigenfrequencies which are tuned to the wave. Resonant reception occurs when the frequency of the gravitational wave is equal to the difference (or sum) of two resonator eigenfrequencies.

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IV. CHARGED-PARTICLE BEAMS AS GRAVITOELECTRIC ANTENNAS

"My reason for now attacking this question [is]: Because I can."

S. Weinberg

1. Overview

When gravitational waves propagate through matter they induce displacements and motions in it. Mechanical gravitational-wave antennas exploit these interactions. But gravitational waves do not only interact with matter. They couple to the stress-energy tensor of all fields, including the electromagnetic field, and this forms the operating basis of gravitoelectric antennas. Many different types of gravitoelectric antennas have been devised. For a quick entry into the literature see the review articles by Press and Thorne[1], and by Pisarev[2]. Of particular relevance to this chapter is the work of Pargamanik and Dimanshteĭn[3], and Dimanshteĭn[4]. They point out that the electromagnetic radiation field of an accelerated charge is altered by a passing gravitational wave, and that monitoring the synchrotron radiation in electron accelerators constitutes a possible scheme (at least in principle) to detect gravitational waves. From Chapter III it must be clear however that a charge need not be accelerated in order to serve as a gravitoelectric antenna. (Even a charge at rest radiates photons in a characteristic way when interacting with a gravitational wave.)

In the following we shall show that a uniformly moving charged-particle beam acts as a direction-sensitive gravitoelectric antenna; i.e., a beam moving along the propagation direction of a gravitational wave has a different radiation pattern and a different radiation intensity from a beam moving orthogonal to the propagation direction of the gravitational wave or colliding head-on with it. In Section 2 we give the differential conversion cross sections that are valid for any velocity of the charge and for any incident direction of the gravitational wave. We discuss also the salient features of the radiation patterns. In section 3 we calculate the total electromagnetic power that is radiated. Section 4 contains our conclusions.

2. The Differential Cross Sections

In Chapter III we have computed the gauge invariant transition amplitude for the conversion of gravitational waves into electromagnetic waves in the Coulomb field of a charged scalar particle. In calculating the differential cross section we have subsequently used the restframe of the charge. This part of the report is concerned with the electromagnetic radiation that is emitted by a uniformly moving charge in a gravitational wave background.

Let the initial 4-momentum of the particle be ${}^1\underline{p} = ({}^1E, {}^1\tilde{p})$, where ${}^1E = \gamma M$, ${}^1\tilde{p} = \gamma M\tilde{v}$ with $\gamma = (1-v^2)^{-1/2}$. The final 4-momentum of the particle is denoted by ${}^2\underline{p} = ({}^2E, {}^2\tilde{p})$. The 4-momenta of the incident graviton and outgoing photon are denoted by ${}^g\underline{k} = ({}^g\omega, {}^g\tilde{k})$ and ${}^\gamma\underline{k} = ({}^\gamma\omega, {}^\gamma\tilde{k})$ respectively. The differential cross section for converting a graviton with angular frequency ${}^g\omega$ and polarization $\underline{\underline{e}}$ into a photon with angular frequency ${}^\gamma\omega$, polarization $\underline{\varepsilon}$ and propagation direction lying within the solid angle $d\Omega$ is then

$$d\sigma = \frac{2\pi}{2\,{}^{g}\omega\; 2\gamma M 2\,{}^{\gamma}\omega\; 2\,{}^{2}E}\;|T|^2\; D \quad , \tag{2.1}$$

where D is the density of final states[5]

$$D = \frac{{}^{2}E\,{}^{\gamma}\omega^3}{(2\pi)^3}\cdot\frac{d\Omega}{\gamma M\,{}^{g}\omega(1 - v\cos\theta_1)} \quad , \tag{2.2}$$

and T is the transition amplitude [Chapter III, Eqs. (3.1a-d)].

In the above θ_1 is the angle between $\underset{\sim}{v}$ and ${}^{g}\underset{\sim}{k}$ (see Fig. 10).

The angular frequency of the outgoing photon is given by

$$^{\gamma}\omega = \frac{1 - v\cos\theta_1}{1 - v\cos\theta_2 + \frac{^{g}\omega}{\gamma M}(1 - \cos\theta)}\;{}^{g}\omega \quad , \tag{2.3}$$

where θ_2 and θ are the angles between $\underset{\sim}{v}$ and ${}^{\gamma}\underset{\sim}{k}$, and between ${}^{g}\underset{\sim}{k}$ and ${}^{\gamma}\underset{\sim}{k}$ respectively. The angle θ may be expressed as $\cos\theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi$, where ϕ is the angle between the planes formed by ${}^{g}\underset{\sim}{k}$, $\underset{\sim}{v}$, and ${}^{\gamma}\underset{\sim}{k}, \underset{\sim}{v}$. In the nonrelativistic limit, i.e., for ${}^{g}\omega \ll \gamma^{-1}(1-v\cos\theta_1)^{-1} M$, (2.3) reduces to

$$^{\gamma}\omega = \frac{1-v\cos\theta_1}{1-v\cos\theta_2}\;{}^{g}\omega \quad , \tag{2.4}$$

and there is no recoil of the scatterer (${}^{2}E = {}^{1}E = \gamma M$).

As the transition amplitude T is a Lorentz scalar, it may be evaluated in any Lorentz frame at our convenience. In the charge's restframe and for the purely spacelike photon gauge and the TT graviton gauge, the

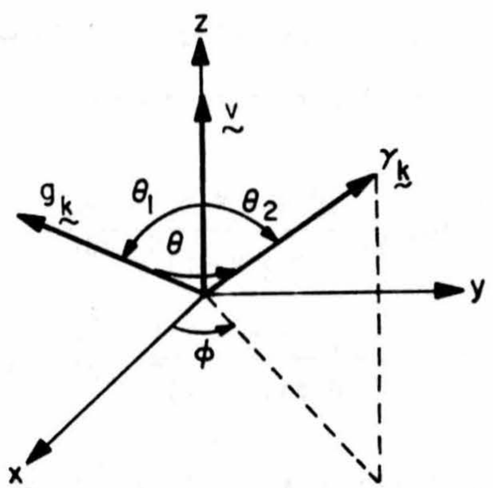

Fig. 10 Exchange Compton scattering of a graviton (momentum ${}^{g}\underset{\sim}{k}$) into a photon (momentum ${}^{\gamma}\underset{\sim}{k}$) by a moving scalar charge (momentum $\gamma M\underset{\sim}{v}$).

transition amplitude is given by Eqs. (3.6a,b) of Chapter III. With the aid of (2.1)-(2.3) we then find

$$\frac{d\sigma}{d\Omega} = \frac{e^2}{8\pi \sin^4 \frac{\theta'}{2}} \frac{1 - v \cos \theta_1}{\gamma^2[1 - v \cos \theta_2 + \frac{g_\omega}{\gamma M}(1 - \cos \theta)]^2} |\bar{e}^{ij'} \gamma \hat{k}'_{(i} \varepsilon'^*_{j)}|^2 , \tag{2.5}$$

where a prime labels quantities that are measured in the charge's rest-frame.* In the nonrelativistic regime we may omit the term $\frac{g_\omega}{\gamma M}(1 - \cos \theta)$ in the denominator, which we shall do from now on.

Choose now circular polarization basis states for the incident gravitons and the outgoing photons and find

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{e^2}{16\pi} \frac{1 - v \cos \theta_1}{\gamma^2(1 - v \cos \theta_2)^2} \operatorname{cotg}^2 \frac{\theta'}{2} (1 + \cos \theta')^2 , \tag{2.6a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{e^2}{16\pi} \frac{1 - v \cos \theta_1}{\gamma^2(1 - v \cos \theta_2)^2} \operatorname{cotg}^2 \frac{\theta'}{2} (1 - \cos \theta')^2 . \tag{2.6b}$$

Using $\cos \theta' = \cos \theta_1' \cos \theta_2' + \sin \theta_1' \sin \theta_2' \cos \phi'$, where

$$\phi' = \phi \tag{2.7a}$$

$$\cos \theta_i' = \frac{\cos \theta_i - v}{1 - v \cos \theta_i} \qquad (i = 1,2) \quad , \tag{2.7b}$$

$$\sin \theta_i' = \frac{1}{\gamma} \cdot \frac{\sin \theta_i}{1 - v \cos \theta_i} \qquad (i = 1,2) \quad , \tag{2.7c}$$

we put the differential cross sections (2.6a,b) into their final form

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*The cross section (2.5) is easily put into the form $\frac{d\sigma}{d\Omega} = \gamma^{-2}(1 - v \cos\theta_2)^{-2} \times (1 - v \cos\theta_1)(\frac{d\sigma}{d\Omega})'$. This [and hence also (2.5)] could have been derived at once from $\frac{d\sigma}{d\Omega} d\Omega = (1 - v \cos\theta_1)(\frac{d\sigma}{d\Omega})' d\Omega'$, with $d\Omega' = \gamma^{-2}(1 - v \cos\theta_2)^{-2} d\Omega$.

$$(\frac{d\sigma}{d\Omega})_{RR} = (\frac{d\sigma}{d\Omega})_{LL} = \frac{e^2}{16\pi} \cdot \frac{1 - v \cos\theta_1}{\gamma^2(1 - v\cos\theta_2)^2}$$

$$\cdot \left[1 + \frac{(\cos\theta_1 - v)(\cos\theta_2 - v) + \gamma^{-2}\sin\theta_1 \sin\theta_2 \cos\phi}{(1 - v\cos\theta_1)(1 - v\cos\theta_2)} \right]^3$$

$$\cdot \left[1 - \frac{(\cos\theta_1 - v)(\cos\theta_2 - v) + \gamma^{-2}\sin\theta_1 \sin\theta_2 \cos\phi}{(1 - v\cos\theta_1)(1 - v\cos\theta_2)} \right]^{-1}, \qquad (2.8a)$$

$$(\frac{d\sigma}{d\Omega})_{RL} = (\frac{d\sigma}{d\Omega})_{LR} = \frac{e^2}{16\pi} \cdot \frac{1 - v \cos\theta_1}{\gamma^2(1 - v\cos\theta_2)^2}$$

$$\cdot \left[1 + \frac{(\cos\theta_1 - v)(\cos\theta_2 - v) + \gamma^{-2}\sin\theta_1 \sin\theta_2 \cos\phi}{(1 - v\cos\theta_1)(1 - v\cos\theta_2)} \right]$$

$$\cdot \left[1 - \frac{(\cos\theta_1 - v)(\cos\theta_2 - v) + \gamma^{-2}\sin\theta_1 \sin\theta_2 \cos\phi}{(1 - v\cos\theta_1)(1 - v\cos\theta_2)} \right] . \qquad (2.8b)$$

The differential cross sections simplify considerably for $\theta_1 = 0, \frac{\pi}{2}, \pi$:

a) $\theta_1 = 0$

$$(\frac{d\sigma}{d\Omega})_{RR} = (\frac{d\sigma}{d\Omega})_{LL} = \frac{e^2}{16\pi} \cdot \frac{1}{(1+v)^5\gamma^{10}(1-v\cos\theta_2)^4} \cdot \frac{(1+\cos\theta_2)^3}{1-\cos\theta_2}, \qquad (2.9a)$$

$$(\frac{d\sigma}{d\Omega})_{RL} = (\frac{d\sigma}{d\Omega})_{LR} = \frac{e^2}{16\pi} \cdot \frac{\sin^2\theta_2}{(1+v)\,\gamma^6(1-v\cos\theta_2)^4}, \qquad (2.9b)$$

b) $\theta_1 = \frac{\pi}{2}$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{e^2}{16\pi}\cdot\frac{1}{(1-v\cos\theta_2)^4}\cdot\frac{(1+v^2-2v\cos\theta_2+\gamma^{-2}\sin\theta_2\cos\phi)^3}{1-\sin\theta_2\cos\phi}, \tag{2.9c}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{e^2}{16\pi}\cdot\frac{1}{\gamma^4(1-v\cos\theta_2)^4}$$

$$\cdot\,(1+v^2-2v\cos\theta_2+\gamma^{-2}\sin\theta_2\cos\phi)(1-\sin\theta_2\cos\phi)\ , \tag{2.9d}$$

c) $\theta_1 = \pi$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{e^2}{16\pi}\cdot\frac{(1+v)^5}{(1-v\cos\theta_2)^4}\cdot\frac{(1-\cos\theta_2)^3}{1+\cos\theta_2}\ , \tag{2.9e}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{e^2}{16\pi}\cdot\frac{(1+v)\sin^2\theta_2}{\gamma^4(1-v\cos\theta_2)^4}\ . \tag{2.9f}$$

In Fig. 11 we have plotted the differential cross sections (2.8a,b) for various values of the angle of incidence θ_1. A quick glance reveals at once that the radiation patterns are sensitive to the direction of incidence of the gravitational wave: a relativistic beam colliding head-on with or moving across the gravitational wave radiates more easily than a beam chasing the gravitational wave from behind. A few structural features are worth mentioning:

i) Rutherford peak

The photon radiation patterns have a Rutherford peak in the direction of the incident gravitational wave ($\theta_2 = \theta_1$); and in this direc-

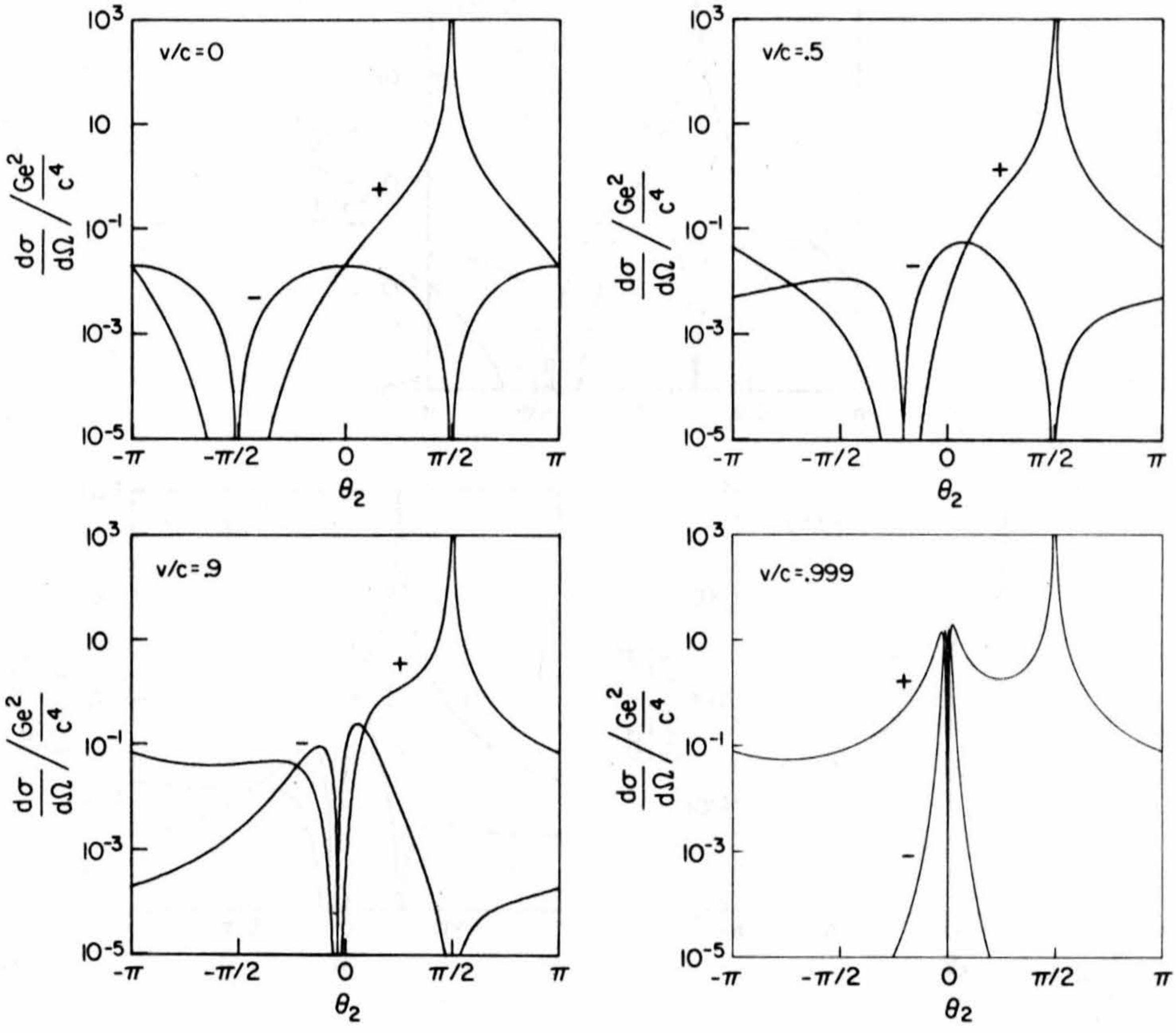

Fig. 11a Differential conversion cross sections in the plane $\phi = 0$ for $\theta_1 = \pi/2$. The + sign refers to the RR(LL) case; the - sign to the RL(LR) case.

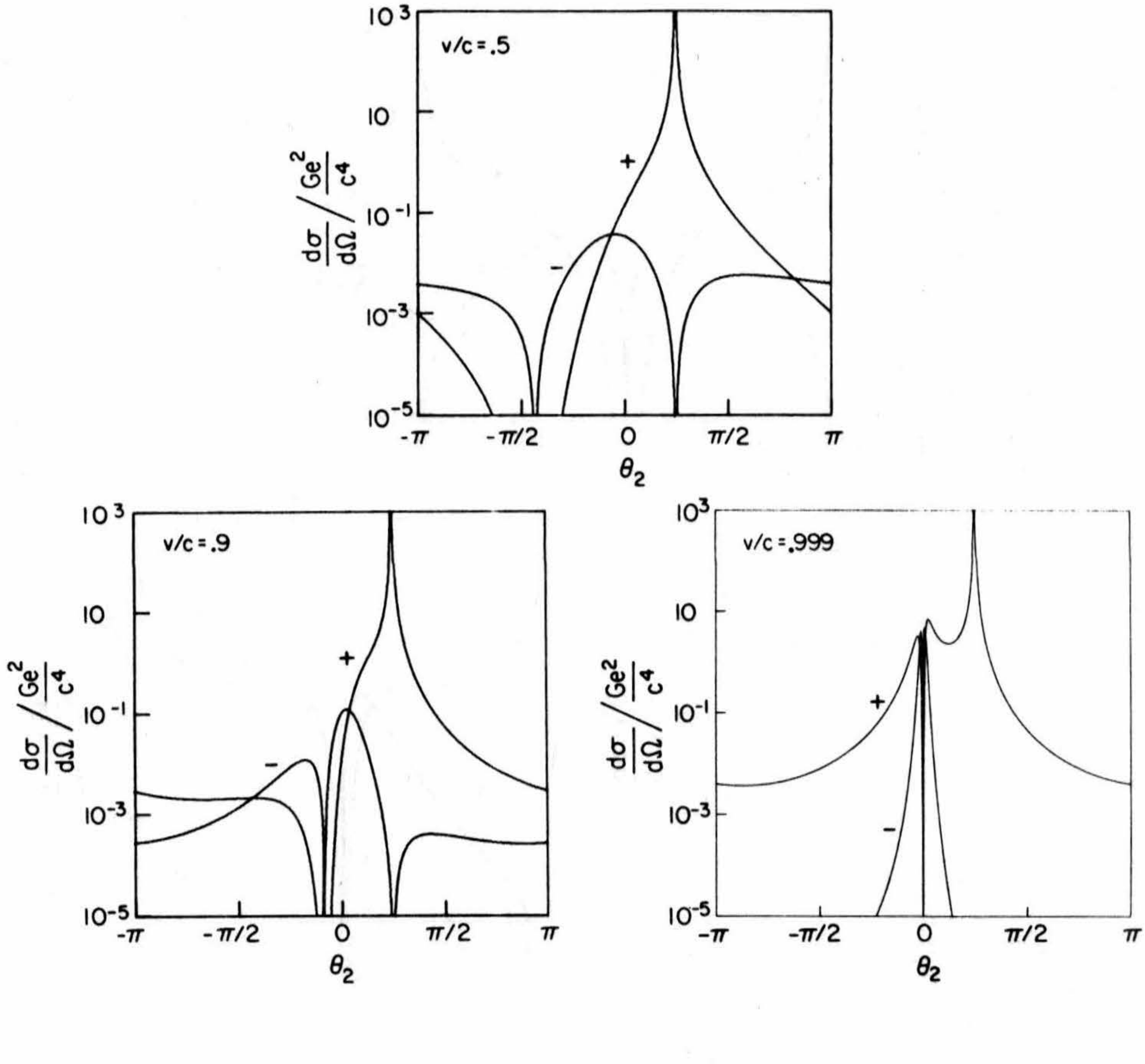

Fig. 11b Differential conversion cross sections in the plane $\phi = 0$ for $\theta_1 = \pi/4$.

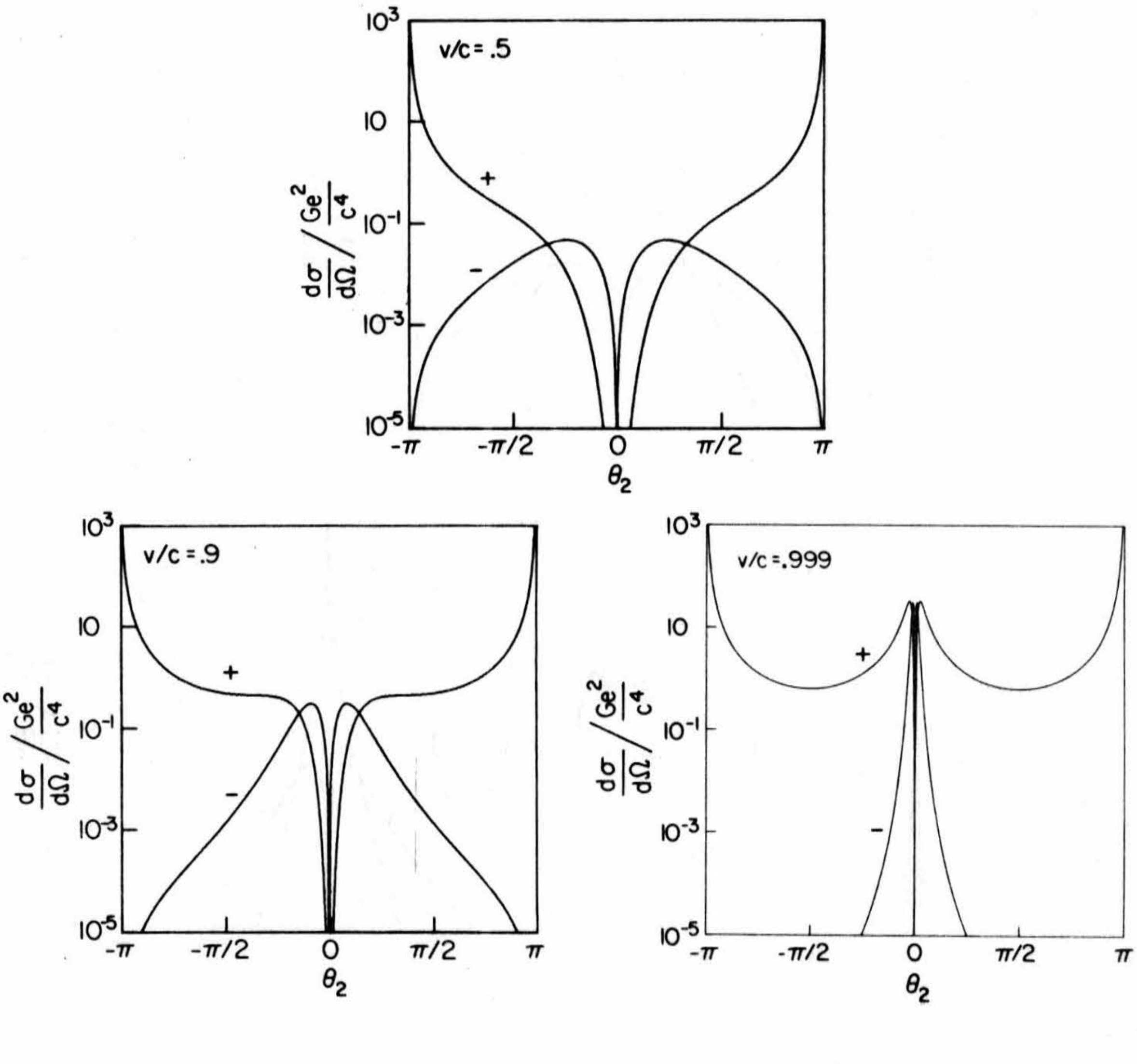

Fig. 11c Differential conversion cross sections for $\theta_1 = \pi$.

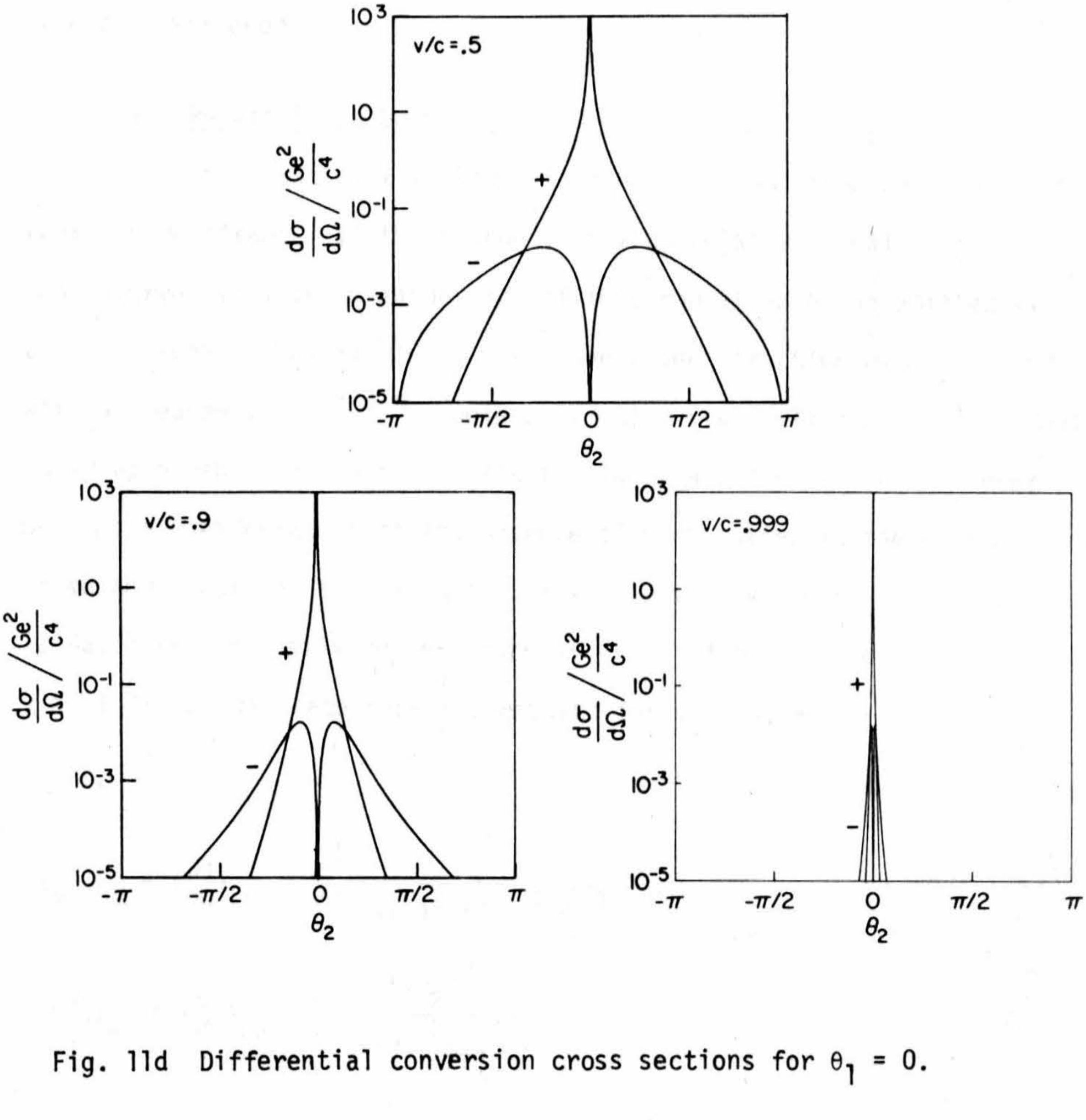

Fig. 11d Differential conversion cross sections for $\theta_1 = 0$.

tion the outgoing photons have the same circular polarization and the same wavelength as the incident gravitons. In reality the radiation pattern will turn over at scattering angles $\theta = |\theta_2 - \theta_1| \sim ({}^g\omega\, \lambda_D)^{-1}$, where λ_D is the distance (in the charge's restframe) beyond which the Coulomb field is screened.

ii) Relativistic beaming

The first plot in Fig. 11a shows the radiation pattern in the charge's restframe. If the charge is moving relativistically ($\gamma >> 1$), the observer will see a large part of this radiation being emitted within a narrow cone of half-angle $\theta_2 \sim \gamma^{-1}$ about the direction of motion and with a frequency ${}^\gamma\omega \sim \gamma^2\, {}^g\omega$ (unless $\theta_1 \simeq 0$). Figure 11 shows the gradual build-up of this "head-light" effect. The sharp dip within the forward beam is a manifestation of the absence of backscatter in the charge's restframe and occurs at angles $\sin\theta_2 = \gamma^{-2}(1+v^2 - 2v\cos\theta_1)^{-1}\sin\theta_1$. From (2.8a,b) we can easily derive expressions valid for the regime $\gamma >> 1 >> \theta_2$; we state here the results for $\theta_1 = 0, \pi/2, \pi$:

a) $\theta_1 = 0$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{e^2}{2\pi}\cdot\frac{1}{(\gamma\theta_2)^2\,(1+\gamma^2\theta_2^2)^4}\quad, \tag{2.10a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{e^2}{2\pi}\cdot\frac{(\gamma\theta_2)^2}{(1+\gamma^2\theta_2^2)^4}\quad. \tag{2.10b}$$

b) $\theta_1 = \frac{\pi}{2}$

$$\left(\frac{d\Omega}{d\sigma}\right)_{RR} = \left(\frac{d\Omega}{d\sigma}\right)_{LL} = \frac{e^2}{\pi} \cdot \frac{\gamma^2}{(1+\gamma^2\theta_2^2)^4}\left(\frac{1}{4}\gamma^{-2} + \gamma^2\theta_2^2 + \theta_2\cos\phi\right)^3 \quad , \tag{2.10c}$$

$$\left(\frac{d\Omega}{d\sigma}\right)_{RL} = \left(\frac{d\Omega}{d\sigma}\right)_{LR} = \frac{e^2}{\pi} \cdot \frac{\gamma^2}{(1+\gamma^2\theta_2^2)^4}\left(\frac{1}{4}\gamma^{-2} + \gamma^2\theta_2^2 + \theta_2\cos\phi\right) \quad , \tag{2.10d}$$

c) $\theta_1 = \pi$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{2e^2}{\pi} \cdot \frac{\gamma^2(\gamma\theta_2)^6}{(1+\gamma^2\theta_2^2)^4} \quad , \tag{2.10e}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{2e^2}{\pi} \cdot \frac{\gamma^2(\gamma\theta_2)^2}{(1+\gamma^2\theta_2^2)^4} \quad . \tag{2.10f}$$

3. Total Radiated Electromagnetic Power

To obtain the total radiated power we must integrate the differential cross sections while taking the angular distribution of the transformed frequency into account. It is simpler, however (and more elucidating!), to pattern the computation of the radiated power after the classic Feenberg-Primakoff[6] treatment of inverse Compton scattering.

Consider a charged particle passing with velocity v through a swarm of gravitons with a number flux density $n({}^g\omega, \theta_1)$. These gravitons all have the same angular frequency ${}^g\omega$ and their propagation direction ${}^g\hat{\underset{\sim}{k}}$ makes an angle θ_1 with $\underset{\sim}{v}$. Labelling with a prime the quantities that are measured in the charge's restframe, we find for the rate at which gravitons are converted into plutons

$$\frac{dN}{dt} = \gamma^{-1} \frac{dN}{dt'} = \gamma^{-1} n'({}^{g}\omega', \theta_1') \sigma'({}^{g}\omega') \quad , \tag{3.1}$$

where $\sigma'({}^{g}\omega')$ is the total cross section for converting a graviton with angular frequency ${}^{g}\omega' = \gamma {}^{g}\omega(1-v \cos \theta_1)$ by a charge at rest. When the inequality ${}^{g}\omega' << M$ is satisfied (nonrelativistic limit) and when many wavelengths $(\frac{2\pi}{{}^{g}\omega'})$ are contained within the screening radius λ_D, one may use the cross section (3.34) of Chapter III. Invoke now the relativistic transformation formula $n'({}^{g}\omega', \theta_1') = \gamma n({}^{g}\omega, \theta_1)(1-v \cos \theta_1)$, and write (3.1) as

$$\frac{dN}{dt} = n({}^{g}\omega, \theta_1)(1-v \cos \theta_1) \sigma'({}^{g}\omega') \quad . \tag{3.2}$$

In the charge's restframe the conversion photons have the same frequency as the incident gravitons and are emitted at angles θ_2' with respect to the direction of motion of the charge. According to the observer these photons have frequencies

$${}^{\gamma}\omega = \gamma(1+v \cos \theta_2') {}^{\gamma}\omega' = \gamma^2 (1-v \cos \theta_1)(1+v \cos \theta_2') {}^{g}\omega \ , \tag{3.3}$$

and the electromagnetic power radiated by the charge is

$$\frac{dW}{dt} = \int n({}^{g}\omega, \theta_1)(1-v \cos \theta_1) \left(\frac{d\sigma}{d\Omega}\right)' {}^{\gamma}\omega \, d\Omega' \quad . \tag{3.4}$$

Here $(\frac{d\sigma}{d\Omega})'$ is the differential conversion cross section evaluated in the restframe of the charge and $d\Omega'$ is the solid angle element corresponding to the scattering angle θ'. By virtue of (3.3) we may reduce (3.4) to the form

$$\frac{dW}{dt} = [n({}^{g}\omega, \theta_1) {}^{g}\omega] \gamma^2 (1-v \cos \theta_1)^2 \int \left(\frac{d\sigma}{d\Omega}\right)' (1+v \cos \theta_2') \, d\Omega' \quad . \tag{3.5}$$

The term in square brackets is just the graviton power flux in the observer's restframe. Hence we conclude that the total radiation cross section for a moving charge is

$$\chi = \gamma^2(1-v\cos\theta_1)^2 \int \left(\frac{d\sigma}{d\Omega}\right)' (1+v\cos\theta_2')\, d\Omega' \quad . \tag{3.6}$$

The factor $\gamma^2(1-v\cos\theta_1)^2$ is a consequence of i) the transformation of the graviton number flux density, which introduces a factor $(1-v\cos\theta_1)$; ii) the frequency shift of ${}^{\gamma}\omega$ with respect to ${}^{g}\omega$, which introduces a factor $\gamma^2(1-v\cos\theta_1)$.

Use now

$$\left(\frac{d\sigma}{d\Omega}\right)' = \frac{e^2}{8\pi} \frac{\sin^2\theta'(1+\cos^2\theta')}{[1-\cos\theta' + 2(2\,{}^{g}\omega'\,\lambda_D)^{-2}]^2} \quad , \tag{3.7}$$

and

$$\cos\theta_2' = \cos\theta_1' \cos\theta' + \sin\theta_1' \sin\theta' \cos\alpha' \quad , \tag{3.8}$$

where α' is the angle between the $({}^{g}\underset{\sim}{k}', \underset{\sim}{v})$ plane and the $({}^{g}\underset{\sim}{k}', {}^{\gamma}\underset{\sim}{k}')$ plane, and where $\sin\theta_1'$ and $\cos\theta_1'$ are given by (2.7b,c), and find

$$\chi = 2e^2(1-v\cos\theta_1)\left\{\frac{1}{3} v\gamma^2(v-\cos\theta_1) + \ln[2\gamma\,{}^{g}\omega\,\lambda_D(1-v\cos\theta_1)] - \frac{4}{3}\right\} \quad . \tag{3.9}$$

The cross section (3.9) is valid for any angle of incidence θ_1 and any velocity v. The only assumptions made in its derivation are i) weak screening, i.e., ${}^{g}\omega'\lambda_D = \gamma(1-v\cos\theta_1)\,{}^{g}\omega\,\lambda_D \gg 1$; ii) nonrelativistic scattering (no recoil), i.e., ${}^{g}\omega' = \gamma(1-v\cos\theta_1)\,{}^{g}\omega \ll M$. If either of these conditions is not fulfilled, the resulting radiation cross section is even smaller than (3.9). Note that the logarithmic term is multiplied only by the graviton number flux transformation factor $(1-v\cos\theta_1)$, and not by the frequency shift factor $\gamma^2(1-v\cos\theta_1)$, as the Rutherford-peak photons have the same frequency as the incident gravitons. For a highly relativistic charge ($v \simeq 1$, $\gamma >> 1$) and for $\theta_1 \neq 0$, the forward beam gains in relative importance as compared to the Rutherford peak and the radiation cross section becomes insensitive to the cut-off:

$$\chi \simeq \frac{2}{3} e^2 \gamma^2 (1 - \cos\theta_1)^2 \quad , \qquad \text{for } \theta_1 \neq 0 \quad . \tag{3.10}$$

On the other hand, if the relativistic charge is chasing the gravitational wave from behind, the Rutherford peak is the whole story and the radiation cross section is even smaller than for a charge at rest:

$$\chi \simeq \frac{e^2}{\gamma^2} [\ell n(\frac{g_\omega \lambda_D}{\gamma}) - \frac{3}{2}] \; , \qquad \text{for } \theta_1 = 0 \quad . \tag{3.11}$$

In the limit $\gamma \to \infty$ there is no conversion at all. The charge is trying to keep up with the gravitational wave and in doing so it does not experience a time-varying permittivity and permeability, which is the conditio sine qua non for conversion scattering.

4. Conclusion

Charged-particle beams act as direction-sensitive gravitoelectric antennas for very-high-frequency gravitational waves. (Many wavelengths of the gravitational wave should fit within the screening radius λ_D.) A relativistic charge may radiate either more ($\theta_1 \neq 0$) or less ($\theta_1 = 0$) than a charge at rest, but even in the best case the conversion cross sections are discouragingly small. However fascinating these processes may be, they are probably of no practical interest. The importance of these calculations is that they provide yet another clue that there is no radiationless trajectory for a charge in a gravitational field region,[7] in contrast with the situation in flat-space electrodynamics.

All of the above formulas also apply to the generation of gravitational waves due to the uniform motion (both relativistic and non-relativistic)of a charged particle in an electromagnetic wave background.

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So [said the doctor]. Now vee may perhaps to begin Yes?

Philip Roth in "Portnoy's Complaint".