# GEOMETRY OF FINITE DIMENSIONAL MOMENT SPACES AND APPLICATIONS TO ORTHOGONAL POLYNOMIALS

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# ABSTRACT

Various geometrical properties of the finite dimensional moment spaces generated by normalized distribution functions over $[0,\infty)$ and $(-\infty,\infty)$ are investigated. The moment spaces are found to be dual to the polynomial spaces. The structure of the latter is studied by means of this duality and of a representation theorem for positive polynomials. The extreme points of the polynomial spaces are associated with polynomials orthogonal with respect to the distributions generating the moment spaces. This correspondence is used in order to derive several properties of orthogonal polynomials.

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# CHAPTER I

## DISTRIBUTION FUNCTIONS AND MOMENT SPACES

Definition 1.1 $\varphi = \varphi(t)$ is a distribution function over $[0,\infty)$, symbolically $\varphi \in \Theta[0,\infty)$, if $\varphi$ takes the real line into itself and

(1.1a) $t_1 < t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$

(1.1b) $t < 0$ implies $\varphi(t) = 0$

(1.1c) $\lim_{t \to \infty} \varphi(t) = 1$

(1.1d) for all $t$ $\varphi(t + 0) = \varphi(t)$ .

Definition 1.2 $\varphi = \varphi(t)$ is a distribution function over $(-\infty.\infty)$, symbolically $\varphi \in \Theta(-\infty,\infty)$ if $\varphi$ takes the real line into itself and

(1.2a) same as (1.1a)

(1.2b) $\lim_{t \to -\infty} \varphi(t) = 0$

(1.2c) same as (1.1c)

(1.2d) same as (1.1d) .

Definition 1.3 The spectrum $\sigma(\varphi)$ of the distribution $\varphi$ is the set of points $t$ such that

$$\varphi(t + \delta) - \varphi(t - \delta) > 0$$

for every $\delta > 0$. If the number of points of $\mathfrak{S}(\varphi)$ is finite, it is denoted by $\bar{b}(\varphi)$.

Definition 1.4 A distribution $\varphi$ is said to have a step at $t_o$ of weight $\lambda > 0$ if

$$\varphi(t_o) - \varphi(t_o - 0) = \lambda > 0.$$

The unique distribution having a step of weight 1 at $t_o$ is denoted by $I_{t_o}$.

Definition 1.5 A distribution $\varphi$ is called arithmetic if it has a finite spectrum. The degree of $\varphi$, denoted by $b(\varphi)$, is then defined as follows:

if $\varphi \in \theta[0,\infty)$ $b(\varphi)$ = total number of steps, less $\frac{1}{2}$ if $\varphi$ has a step at 0.

if $\varphi \in \theta(-\infty,\infty)$ $b(\varphi) = \bar{b}(\varphi)$ = total number of steps.

Definition 1.6 The n-th moment $\mu_n$ of $\varphi$ is

$$\mu_n(\varphi) = \int_{-0}^{\infty} t^n \, d\varphi(t) \quad \text{if } \varphi \in \theta[0,\infty)$$

$$\mu_n(\varphi) = \int_{-\infty}^{\infty} t^n \, d\varphi(t) \quad \text{if } \varphi \in \theta(-\infty,\infty).$$

In both cases $\mu_o(\varphi) = 1$.

Definition 1.7 The n-th moment space $D^n[0,\infty)$, respectively $D^n(-\infty,\infty)$, is the set of points $x = (x_i) = (x_1,\dots,x_n)$ in $E^n$

whose coordinates are the moments $\mu_1(\varphi),\ldots,\mu_n(\varphi)$ of at least one distribution function $\varphi \in \Theta[0,\infty)$, respectively $\varphi \in \Theta(-\infty,\infty)$.

Definition 1.8 $x$ is an extreme point of a set $D \subset E^n$ if $x$ is not a convex combination of any two other points of $D$.

Definition 1.9 A pair $(\alpha_0,\alpha)$, where $\alpha_0$ is a number and $\alpha$ a set of $n$ numbers not all zero, is a separating plane between $x_0$ and $D$ if $\alpha_0 + \alpha \cdot x$ is non-positive for $x = x_0$ and non-negative for all $x \in D$. If $x_0 \in D$ then $(\alpha_0,\alpha)$ is called a supporting plane to $D$ at $x_0$.

Definition 1.10 $C^n[0,\infty)$ will denote the curve in $E^n$

$$x_i = t^i \qquad 0 \leq t < \infty \qquad (i = 1,2,\ldots,n).$$

$C^n(-\infty,\infty)$ will denote the curve in $E^n$

$$x_i = t^i \qquad -\infty < t < \infty \qquad (i = 1,2,\ldots,n).$$

Clearly $\quad C^n[0,\infty) \subset D^n[0,\infty)$

and $\quad C^n(-\infty,\infty) \subset D^n(-\infty,\infty)$,

since $\mu_n(I_t) = t^n$.

$D^n[0,\infty)$ and $D^n(-\infty,\infty)$ are convex regions.

Definition 1.11 Let $\underline{B}^n[0,\infty)$ denote the boundary of $D^n[0,\infty)$, i.e. $x \in \underline{B}^n[0,\infty)$ if $x \in D^n[0,\infty)$ and there exists a supporting plane to $D^n[0,\infty)$ at $x$. Define $\underline{B}^n(-\infty,\infty)$ similarly.

<u>Remark.</u> When clarity permits the notation $D^n$ will be used instead of either $D^n[0,\infty)$ or $D^n(-\infty,\infty)$. Similarly for $C^n$ and $\underline{B}^n$.

<u>Theorem 1.12</u> All extreme points of $D^n$ are in $C^n$.

<u>Proof</u> by contradiction. Let $x \in D^n - C^n$ and let $\varphi$ be a distribution such that

$$x_i = \int_{-\infty}^{\infty} t^i d\varphi(t).$$

Then the spectrum of $\varphi$ contains at least two points, say $t_1 < t_2$. If $a$ is chosen so that $t_1 < a < t_2$ then $0 < \varphi(a) < 1$. Let

$$\varphi_1(t) = \begin{cases} \dfrac{\varphi(t)}{\varphi(a)} & (t \leq a) \\ 1 & (t > a) \end{cases},$$

$$\varphi_2(t) = \begin{cases} 0 & (t \leq a) \\ \dfrac{\varphi(t) - \varphi(a)}{1 - \varphi(a)} & (t > a). \end{cases}$$

Then $\varphi_1$ and $\varphi_2$ are distributions and

$$x_i = \int_{-\infty}^{a} t^i d\varphi + \int_{a}^{\infty} t^i d\varphi = \varphi(a)\cdot\mu_i(\varphi_1) + [1 - \varphi(a)]\cdot\mu_i(\varphi_2)$$

is a convex combination of points of $D^n$, hence it is not an extreme point of $D^n$.

Theorem 1.13 If $n \geq 2$ all points of $C^n$ are extreme points.

Proof: Let $x_o = x(t_o)$. The plane

$$\begin{cases} \alpha_o = t_o^2 \\ \alpha = (-2\, t_o, 1, 0, 0, \ldots, 0) \end{cases}$$

contains $x_o$ but leaves all other points of $C^n$ on the same side and at a positive distance. Hence $x_o$ cannot be a convex combination of points of $C^n$.

Theorem 1.14 Every point $x \in \underline{B}^n$ is representable as a convex combination of points of $C^n$ in a unique way, and the corresponding distribution is arithmetic.

Proof: Given $x \in \underline{B}^n$ take any $\varphi$ such that

$$\mu_i(\varphi) \equiv \int t^i \, d\varphi(t) = x_i \qquad (i = 1,\ldots,n).$$

$x$ is then a convex combination of points of $C^n$. Since $x$ is in the boundary of $D^n$, every supporting plane $(\alpha_o, \alpha)$ to $D^n$ at $x$ must contain every point $(t^i)$ of $C^n$ such that $t \in \sigma(\varphi)$. The polynomial in $t$

$$\alpha_o + \alpha \cdot x(t) = \sum_{i=0}^{n} \alpha_i \, t^i$$

has at most $n$ distinct roots $t_j$ in the interval under consideration. But $t_o \in \sigma(\varphi)$ implies $\alpha_o + \alpha \cdot x(t_o) = 0$, hence $t_o \in \{t_j\}$. Hence $\sigma(\varphi)$ contains at most $n$ points, which do not depend on $\varphi$. Therefore $\varphi$ is an arithmetic distribution of degree n at most, with weights $\lambda_j$ which satisfy

$$x_i = \sum_{j=1}^{m} \lambda_j t_j^i \qquad (i = 1,\dots,n; \quad m = \bar{b}(\varphi) \leq n) \quad .$$

The determinant of the coefficients is a Vandermonde determinant, hence it is not zero, hence the system has at most one solution. However it must have one solution with positive $\lambda_j$ whose sum is 1. Hence $\varphi$ is uniquely determined and x is uniquely representable by

$$x = \sum_{j=1}^{m} \lambda_j \, x(t_j) \qquad (m \leq n) \quad .$$

Definition 1.15 $Co(C^n)$ is the set of all points which are finite convex combinations of points of $C^n$.

Theorem 1.16 Every point $x \in D^n$ is representable as a finite convex combination of points of $C^n$.

Proof: By Theorem 1.14 $bdD^n \subset Co(C^n)$, hence it is sufficient to show that int $D^n \subset Co(C^n)$.

It will first be shown that $\overline{Co(C^n)} \supset D^n$, where the bar denotes closure. In fact if there existed an x' such that $x' \in D^n$ and $x' \notin \overline{Co(C^n)}$, there would exist a separating plane $(\alpha_i)$ such that

$$\alpha . x \geq 0 \qquad x \in \overline{Co(C^n)}$$
$$\alpha . x' = \delta < 0$$

and a distribution $\varphi'$ such that

$$x_i' = \int t^i \, d\varphi'$$

For all $t, \sum \alpha_i t^i \geq 0$, hence

$$\int \sum \alpha_i t^i \, d\varphi' \geq 0.$$

But $\int \sum \alpha_i t^i \, d\varphi' = \sum \alpha_i x_i' = \alpha \cdot x' = \delta < 0$. Hence a contradiction. To show that $\text{int } D^n \subset Co(C^n)$ assume this statement false. Then for some $x$, $x \in \text{int } D^n$, $x \notin Co(C^n)$. By the previous result $x \in \overline{Co(C^n)}$ hence $x \in \text{bd } \overline{Co(C^n)}$. But since $x \in \text{int } D^n$ a full neighborhood of $x$ is in $D^n$, hence in $\overline{Co(C^n)}$, which contradicts $x \in \text{bd } \overline{Co(C^n)}$.

<u>Theorem 1.17</u> Any set of $k$ distinct points of $C^n (k \leq n)$ is linearly independent.

<u>Proof:</u> Let $(t_j^i)$ $(i = 1,\ldots,n;\ j = 1,\ldots,k)$ denote the coordinates of the $k$ points. Any determinant of order $k$ from the matrix $(t_j^i)$ is a Vandermonde determinant, hence it is $\neq 0$ and the points are linearly independent.

<u>Theorem 1.18</u> $D^n - \underline{B}^n$ is non-empty.

<u>Proof:</u> The moment of any non-arithmetic distribution defines an interior point of $D^n$. For example in the case $[0,\infty)$ choose $d\varphi(t) = e^{-t}dt$ whose moments are $\mu_i = i!$.

<u>Theorem 1.19</u> If $x \in D^n - \underline{B}^n$ then $x$ can be represented in infinitely many ways as a finite convex combination of points of $C^n$.

*Proof:* Let $x \in D^n - \underline{B}^n$. Choose an arbitrary point $x(t) \in C^n$ and draw a line through it and $x$. Since $x$ is interior there exists a point $y \in D^n$ lying on this line so that $x$ is between $y$ and $x(t)$. But by Theorem 1.16 $y$ is a finite combination of points of $C^n$, hence so is $x$. By varying $t$ one obtains infinitely many different representations.

# CHAPTER II

## COMPONENTS OF THE BOUNDARIES OF THE MOMENT SPACES

Definition 2.1 Let $L(x)$ denote the common intersection of all separating planes between $x$ and the closure $\bar{D}^n$ of $D^n$.* If there are no separating planes define $L(x) = E^n$. Let $a(x)$ denote the dimension of $L(x)$.

The empty set is defined to have dimension $-1$; hence $x \notin \bar{D}^n$ gives $a(x) = -1$.

Definition 2.2 Let $c(x)$ denote the dimension of $L(x) \cap \bar{D}^n$.

Definition 2.3 If $x \in D^n$ let $b(x)$, $\bar{b}(x)$ denote the minima of $b(\varphi)$, $\bar{b}(\varphi)$ over all $\varphi$ whose first $n$ moments are given by $x$.

By the finite representation theorem these minima are well defined. Thus $\bar{b}(x)$ represents the minimum number of points of $C^n$ which are used in a convex representation of $x$. Clearly if $b$ is an integer $\bar{b} = b$, if $b$ is a half integer $\bar{b} = b + \frac{1}{2}$.

Case $[0,\infty)$.

Theorem 2.4 $x \in \underline{B}^n$ if and only if $b(x) \leq \frac{n}{2}$.

Proof: Assume $b(x) \leq \frac{n}{2}$ and let $x$ be representable

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* $\bar{D}^n$ will denote the closure of $D^n$. However the upper bar will not, in general, denote closure (cf. Definition 2.8).

convexly by means of the points $x(t_j)$ where $t_j$ have the following values:

1. if $b$ is an integer $0 < t_1 < t_2 < \dots < t_b$

2. if $b$ is a half integer $0 = t_1 < t_2 \quad \dots < t_{\bar{b}}$.

In case 1 the polynomial $R_1 = \prod_{j=1}^{b} (t - t_j)^2$, and in case 2 the polynomial $R_2 = t \prod_{j=2}^{\bar{b}} (t - t_j)^2$, represents a plane in $E^n$, since each polynomial is of degree $\leq n$. In each case the polynomial in question contains the points $x(t_j)$ but leaves all other points of $C^n$ on the same side. Hence $R_k$ represents a supporting plane to $D^n$ containing $x$, therefore $x \in \underline{B}^n$. Conversely, if $x \in \underline{B}^n$ Theorem 1.14 assures the existence of a unique convex representation of $x$ by a finite number of points $x(t_j)$. Every supporting plane at $x$ must contain these points, hence the polynomial corresponding to this plane, which is of degree $\leq n$, must vanish at $t_j$ and be positive for all other values of $t$. Therefore this polynomial has at most $\frac{n}{2}$ roots, the possible root at 0 being counted $\frac{1}{2}$. Hence $b(x) \leq \frac{n}{2}$.

<u>Lemma 2.5</u> If $x \in \underline{B}^n$ then $a(x) = 2b(x) - 1$.

<u>Proof:</u> Let $x \in \underline{B}^n$ and let $(\alpha_0, \alpha)$ be any supporting plane at $x$. Then $P(t) = \sum_{i=0}^{n} \alpha_i t^i$ is non-negative over $[0, \infty)$ and vanishes at the points $t_j$ $(j = 1, \dots, \bar{b})$ used in the unique representation of $x$. Therefore $P(t)$ has the form

$$P(t) = Q(t)\ R_k(t)$$

where $R_k(t)$ is one of the polynomials of Theorem 2.4, depending on the values $t_j$. In each case the degree of $R_k(t)$ is exactly $2b(x)$, hence the degree of $Q(t)$ is at most $n - 2b(x)$. By definition $a(x)$ is the dimension of $L(x)$, the intersection of all supporting planes at $x$, hence $n - a(x)$ of these may be chosen linearly independent. Since the coefficients of the corresponding polynomials $Q(t)$ must also be independent, at least one of the $Q(t)$, say $Q_o(t)$, must be of degree $n-a(x)-1$ or higher. Hence

$$n-a(x)-1 \leq \text{degree } Q_o(t) \leq n-2b(x)$$

or

$$a(x) \geq 2b(x)-1.$$

On the other hand it is possible to choose $n-2b(x) + 1$ linearly independent values for $Q(t)$, for example $1, t, t^2, \ldots, t^{n-2b(x)}$, such that the corresponding polynomials $Q(t)R_k(t)$ represent linearly independent supporting planes at $x$. However, no more than $n-a(x)$ linearly independent supporting planes can be chosen through $x$. Hence $n-2b(x) + 1 \leq n-a(x)$ or $a(x) \leq 2b(x)-1$. Combining with the previous inequality this gives

$$a(x) = 2b(x) - 1.$$

<u>Definition 2.6</u> Given a non-negative integer $a \leq n$, $\underline{A}_a^n$ denotes the set of all points $x \in D^n$ for which $a(x) = a$.

Note that $a(x) = n$ if and only if $x \in \text{int } D^n$. Hence $\underline{A}^n_n$ is the interior of $D^n$.

If $a < n$, by Definition 2.1 there is at least one supporting plane at each point of $\underline{A}^n_a$, hence $\underline{A}^n_a \subset \underline{B}^n$. By Theorem 2.4 and Lemma 2.5 every point of $\underline{B}^n$ is in one of the sets $\underline{A}^n_a$. Each $\underline{A}^n_a$ is arcwise connected, since any two points $x$, $x'$ of $\underline{A}^n_a$ are convex combinations of $\bar{b}$ points of $C^n$ of the form

$$x = \sum_{j=1}^{\bar{b}} \lambda_j t^i_j , \quad x' = \sum_{j=1}^{\bar{b}} \lambda'_j t'^i_j ,$$

where $b = \frac{a+1}{2}$ and, as usual, if $b$ is a half integer $t_1 = t'_1 = 0$; these two points can be joined by varying the $\lambda_j$ and the $t_j$ continuously and without leaving $\underline{A}^n{}_a$. The number of independent parameters used in the representation of a point $x \in \underline{A}^n_a$ is precisely $a$. In fact the $\lambda_j$ are connected by the relation $\sum_{j=1}^{\bar{b}} \lambda_j = 1$, hence $\bar{b} - 1$ of them are independent. If a is odd, $b = \bar{b}$ is an integer and the number of variable $t_j$ is $b$, hence the total number of independent parameters is $2b - 1 = a$. If $a$ is even $b = \bar{b} - \frac{1}{2}$ is a half integer and only $\bar{b} - 1$ of the $t_j$ are variable, hence the total number of independent parameters is $2\bar{b} - 2 = 2b - 1 = a$. Thus $a$ is the dimension of $\underline{A}^n_a$. These results may be summarized in the following.

Theorem 2.7 $\underline{B}^n$ is the union of a collection of arcwise connected and mutually disjoint boundary components $\underline{A}_a^n$ of dimension a ($a = 0,1,\dots,n-1$).

Besides these proper boundary components, $D^n$ has a set of improper boundary components whose nature will now be investigated.

Definition 2.8 Let $\overline{B}^n = \overline{D}^n - D^n$.

Theorem 2.9 If $n > 1$ $\overline{B}^n$ is not empty; in fact there exist simplexes of points $x$ which are not in $D^n$, but such that in every neighborhood of $x$ there are points of $D^n$.

Proof: Consider a set of $b = \frac{n+1}{2}$ distinct values $t_j$. As usual, if $b$ is an integer this means $t_j \neq 0$, if $b$ is a half integer $t_1 = 0$, $t_j \neq 0$ ($j = 2,\dots,b+\frac{1}{2}$). Let K denote the simplex of all convex combinations of all the points $x(t_j)$. By Theorem 2.4 K is interior to $D^n$. Now if all the $t_j$ except $t_{\overline{b}}$ are made to approach finite and distinct values $\overline{t}_j$ and $t_{\overline{b}}$ is made to approach infinity, K will approach a simplex $K'$ which extends to infinity and is parallel to the $x_n$- axis. The polynomial $R_k$ defined in Theorem 2.4 is of degree $n+1$ and represents a plane in $E^{n+1}$ which supports $D^{n+1}$ along a simplex whose projection on $E^n$ is K. As $t_{\overline{b}} \to \infty$ $R_k$ becomes a polynomial of degree $n-1$, $R_k'$, hence it represents a supporting plane to $D^n$ which is parallel to the $x_n$-axis and contains $K'$.

Therefore the points of $K'$ cannot belong to $D^n$, since the only points of $R_k'$ which belong to $D^n$ are convex combinations of at most the first $b - 1 = \frac{n-1}{2}$ point $\bar{t}_j$, i.e. the points of the closure of $\underline{A}^n_{n-2}$.

The simplex $K'$ has dimension one higher than $\underline{A}^n_{n-2}$ and will be denoted by $\bar{A}^n_{n-1}$. A similar construction can be carried out by letting several of the $t_j$ approach infinity or by letting several of them approach the same values $\bar{t}_j$. This construction yields a set of simplexes $\bar{A}^n_a$ $(a = 1,\ldots,n-2)$ which together with $\bar{A}^n_{n-1}$ are contained in $\bar{B}^n$. The number $a$ denotes the dimension of $\bar{A}^n_a$. It will be shown later (Theorem 2.12) that $\overline{B^n}$ is actually the union of the $\bar{A}^n_a$ $(a = 1,\ldots,n-1)$.

Theorem 2.10 If $(\mu_1,\ldots,\mu_{n-1}) \in \operatorname{int} D^{n-1}$ and $\mu = (\mu_1,\ldots,\mu_n) \in D^n$ and if $\mu'_n > \mu_n$, then $\mu' = (\mu_1,\ldots,\mu_{n-1},\mu'_n) \in \operatorname{int} D^n$.

Proof: It is sufficient to show that $\mu'$ has positive distance from every supporting plane to $D^n$, for if $\mu'$ were not interior to $D^n$ it would have non-positive distance from at least one supporting plane. The distance of $\mu'$ from the supporting plane $(\alpha_0,\alpha_1,\ldots,\alpha_n)$ is given, up to a positive factor, by $\alpha_0 + \alpha_1\mu_1 + \ldots + \alpha_n\mu'_n$. If $\alpha_n = 0$ this distance is the same as that of $\mu$, which is positive since $(\mu_1,\ldots,\mu_{n-1}) \in \operatorname{int} D^{n-1}$. If $\alpha_n \neq 0$, $\alpha_n$ must be positive, since the polynomial $\sum_{j=o}^{n} \alpha_i t^i$ must be positive for $t \to \infty$. Hence from $\alpha_o + \alpha_1\mu_1 + \dots + \alpha_n \mu_n \geqslant 0$ it follows that $\alpha_o + \alpha_1\mu_1 + \dots + \alpha_n \mu'_n > 0$.

<u>Theorem 2.11</u> If $x \in \text{int } D^n$ then $x$ is representable in a unique way as a convex combination of $\frac{n+1}{2}$ points of $C^n$.

<u>Proof</u>: Let $x = (\mu_1, \dots \mu_n) \in \text{int } D^n$. Then by Theorem 1.16 $x \in Co(C^n)$ and there exists $T$ such that $x$ is a finite convex combination of $x(t_j)$ with $t_j < T$. By [1], there are two representations of $x$ involving $\frac{n+1}{2}$ points of $C^n$ with $t_j < T$. However in [1] one of the representations involves the point $x(T)$, which is given weight $\frac{1}{2}$ instead of 1 as it would in the present case. This representation must be ruled out, hence there is left exactly one representation involving $\frac{n+1}{2}$ points of $C^n$ with $t_j < T$. To prove that it is unique and independent of $T$, let it be given by

$$\mu_i = \sum_{j=1}^{\bar{b}} \lambda_j t_j^i \qquad (b = \frac{n+1}{2}, \ i = 1, \dots, n)$$

and let $\varphi$ be a distribution having these moments. Setting $\mu_{n+1} = \sum_{j=1}^{\bar{b}} \lambda_j \, t_j^{n+1}$, the point $(\mu_1, \dots, \mu_{n+1})$ is in $\underline{A}_n^{n+1}$ since its representation involves $\frac{n+1}{2}$ points of $C^{n+1}$. If there existed another representation of $x$ involving $\frac{n+1}{2}$ points of $C^n$, there should exist a distribution $\psi$ having the same first $n$ moments as $\varphi$ and with $(\mu_1,\dots,\mu_n,\mu_{n+1}(\psi)) \in \underline{A}_n^{n+1}$. For this representation to be different from the one previously obtained, the condition $\mu_{n+1}(\psi) \neq \mu_{n+1}(\varphi)$ must be satisfied, since any point of $\underline{A}_n^{n+1}$ has a unique representation. However, if $\mu_{n+1}(\psi) < \mu_{n+1}(\varphi)$ it can be easily seen by considering the supporting plane at $(\mu_1,\dots,\mu_n, \mu_{n+1}(\varphi))$ that $(\mu_1,\dots,\mu_n,\mu_{n+1}(\psi))$ would lie outside $D^{n+1}$, while if $\mu_{n+1}(\psi) > \mu_{n+1}(\varphi)$, by Theorem 2.9 the point $(\mu_1,\dots,\mu_n,\mu_{n+1}(\psi))$ would be interior to $D^{n+1}$. Both assumptions lead to a contradiction, hence the representation of $x$ in term of $\frac{n+1}{2}$ points of $C^n$ is unique.

Theorems 1.14, 1.19, 2.4 and 2.11 show that while a point of $\underline{B}^n$ admits only one convex representation in terms of points of $C^n$ and this representation involves at most $\frac{n}{2}$ points of $C^n$, a point of int $D^n$ admits infinitely many representations, but only one minimal one, i.e. one involving $\frac{n+1}{2}$ points of $C^n$.

<u>Theorem 2.12</u> $\overline{B}^n$ is the union of the simplexes $\overline{A}_a^n (a = 1, \dots, n-1)$.

<u>Proof:</u> Let $x \in \overline{B}^n$ be the limit of a sequence $x(k) (k=1,2,\dots)$ of points of $D^n$ and let

$$x_i(k) = \sum_{j=1}^{\overline{b}} \lambda_j(k)\ t_j^i\ (k)$$

be the minimal convex representations of the points $x(k)$. Although the numbers $\lambda_j(k)$ and $t_j(k)$ need not approach unique limits as $k \to \infty$, there exists a subsequence $k_s$ such that all the $\lambda_j(k_s)$ and $t_j(k_s)$ approach unique limits $\lambda_j$, $t_j$ as $s \to \infty$. Some of the $t_j$ may have the value $\infty$. If all the $t_j$ are finite then $x \in D^n$. If some have the value $\infty$, it may be seen by Theorem 2.9 that the sequence of simplexes spanned by the $t_j(k_s)$ approaches one of the simplexes $\bar{A}_a^n$, hence the point $x$ lies in one of the $\bar{A}_a^n$.

Using Theorem 2.11 it is possible to extend Lemma 2.5 to all points of $D^n$ and state

Theorem 2.13 If $x \in D^n$ then $a(x) = 2b(x) - 1$.

Proof: For $x \in \underline{B}^n$ the proof is given in Lemma 2.5. For $x \in \text{int } D^n$, $a(x) = n$ by Definition 2.1 and $b(x) = \frac{n+1}{2}$ by Theorem 2.11.

Theorem 2.14 If $x \in \underline{B}^n$ then $c(x) = \bar{b}(x) - 1$.

Proof: Given $x$, the polynomial $R_k$ defined in Theorem 2.4 represents a supporting plane at $x$ which contains exactly those points of $C^n$ which are used in the convex representation of $x$. The convex set $S(x)$ in which this plane intersects $D^n$ has dimension $\bar{b}(x) - 1$, since it is spanned by the $\bar{b}(x)$ linearly independent points $x(t_j)$. But since every supporting plane at $x$ must contain $S(x)$, $c(x)$ is by definition the dimension of $S(x)$. Hence $c(x) = \bar{b}(x) - 1$.

The indices $b(x)$, $\bar{b}(x)$ have so far been defined only for $x \in D^n$. If $x \in \bar{B}^n$ there is no distribution having $x$ as its moment point, hence it would not make sense to define $b(x)$ as $\min b(\varphi)$. However, the definition could be extended by considering the points of $\overline{B}^n$ as convex combinations of some (finite) points of $C^n$ and the point at infinity of $C^n$, and taking for $b(x)$ the number of points used in this convex representation of $x$, counting the origin as well as the point at infinity as $\frac{1}{2}$. $\overline{b}(x)$ would simply be defined as the number of points used in the representation of $x$, each counted with weight 1. With these definitions the formulas contained in Theorems 2.13 and 2.14 can be proved also for points of $\overline{B}^n$.

<u>Case $(-\infty,\infty)$</u>

The essential difference between the previous case and the present one is that now there is no point which plays the special role played by the origin in the case $[0,\infty)$. In other words all the points used in a convex representation are given weight 1, hence $b(x)$ is always an integer and $b(x) = \overline{b}(x)$. This involves several modifications in the statements and the proofs of the theorems which are valid in the previous case.

Theorem 2.4 still holds, but only case 1 may occur, hence the statement may be sharpened to read: $x \in \underline{B}^n$ if and only if $b(x) \leq \left[\frac{n}{2}\right]$.

If $x \in \underline{B}^n$ the statement $a(x) = 2b(x) - 1$ of Lemma 2.5 holds if $n$ is even, the proof being the same as in the case

$[0,\infty)$. If $n$ is odd then $a(x) = 2b(x)$, because in this case $Q(t)$ must be positive over $(-\infty,\infty)$, hence it must be of even degree, hence its degree must be $\leq n - 2b - 1$, therefore it is possible to choose only $n - 2b(x)$ linearly independent values for $Q(t)$.

The simplexes $\underline{A}_a^n$ may be defined as in Definition 2.6. However, for the discussion of the boundary components the cases when $n$ is even or odd must be treated separately.

If $n$ is even and $x \in \underline{B}^n$, $a(x)$ must be odd, since $a(x) = 2b(x) - 1$. Furthermore since $x$ is represented in terms of the $2b - 1$ independent parameters $t_j$, $\lambda_j(\sum_{j=1}^{b} \lambda_j = 1)$, it follows that $\underline{A}_a^n$ is a simplex of dimension $a$. Hence the boundary components of $\underline{B}^n$ are odd dimensional. As for the improper boundaries, they can be obtained as in the case $[0,\infty)$. Since $n$ is even, as some of the $t_j$ approach $\pm\infty$ the points $x(t_j)$ approach the $+\infty$ direction of the $x_n$-axis. Hence the improper boundaries are, as before, ruled surfaces containing half lines. Since the improper boundary components are projections, from the point at infinity of $C^n$, of proper boundaries of odd dimension which are not parallel to the $x_n$-axis, they are even dimensional.

If $n$ is odd the final results are the same but the argument is slightly different. In fact if $x \in \underline{B}^n$, $a(x)$ must now be even because $a(x) = 2b(x)$. However $\underline{A}_a^n$ is now a combination of $2b - 1 = a - 1$ independent parameters, hence $\underline{A}_a^n$ has dimension $a - 1$. Since in this case all the supporting planes to $D^n$ are parallel to the $x_n$-axis, the linear manifold $L(x)$ will contain, in addition to the simplex of convex combinations of the points $x(t_j)$ used in representing $x$, the entire lines projecting the points of this simplex from the point at infinity of $C^n$. In each of these lines only the point of intersection with the simplex of the $x(t_j)$ belongs to $D^n$, in fact to $\underline{A}_a^n$; the other points of each line belong to an improper boundary component which has dimension $a$. Hence when $n$ is odd $D^n$ has a set of odd dimensional proper boundary components and a set of even dimensional improper boundary components.

Theorem 2.10 holds again if $n$ is even, the proof being the same as for the case $[0,\infty)$. If $n$ is odd a stronger result holds, because then for any supporting plane $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ the coefficient $\alpha_n$ must vanish, hence given $(\mu_1, \ldots, \mu_{n-1}) \in \text{int } D^n$ and any $\mu_n$, $(\mu_1, \ldots, \mu_n) \in \text{int } D^n$.

Theorem 2.11 holds if $n$ is odd and may be proved by applying the results of [1] to an interval $[-T, T]$, since the representation there given does not involve the end points of the interval. If $n$ is even Theorem 2.11 does not hold, because in this case it is possible, given $(\mu_1, \ldots, \mu_n) \in \text{int } D^n$, to choose $\mu_{n+1}$ arbitrarily and, by the statement above, to determine a unique representation of $(\mu_1,\ldots,\mu_{n+1}) \in \text{int } D^{n+1}$ in terms of $\frac{n+2}{2}$ points of $C^{n+1}$, which gives also a representation of $(\mu_1,\ldots,\mu_n)$ in terms of $\frac{n+2}{2}$ points of $C^n$. Since $\mu_{n+1}$ is arbitrary and different points of $D^{n+1}$ have different representations, there is a one-parameter family of representations for $(\mu_1,\ldots,\mu_n)$.

Theorem 2.13 holds, for $x$ int $D^n$, only if $n$ is odd, in which case $a(x) = n$ and $b(x) = \frac{n+1}{2}$, hence $a = 2b - 1$. If $n$ is even and $x \in \text{int } D^n$, then $a(x) = n$ and $b(x) = \frac{n+2}{2}$, hence $a = 2b - 2$. Combining this with the remark about Lemma 2.5, the following four cases are obtained

| | n odd | n even |
|---|---|---|
| $x \in \underline{B}^n$ | $a = 2b$ | $a = 2b - 1$ |
| $x \in \text{int } D^n$ | $a = 2b - 1$ | $a = 2b - 2$ |

Theorem 2.14 still holds, with the same proof as in the case $[0,\infty)$.

If the index $b(x)$ is defined for $x$ in the improper boundary as the number of points used in its convex representation, counting also the point at infinity, then the formulas $a(x) = 2b(x)$ (n odd), $a(x) = 2b(x) - 1$ (n even) and $c(x) = b(x) - 1$ may be easily verified also for improper boundary points.

## Examples to Chapters I and II

It is convenient at times to introduce the moment cone $D^n$ which is defined as the set in $E^{n+1}$ of all points whose coordinates are the moments $(\mu_0, \mu_1, \ldots, \mu_n)$ of some distribution $\varphi(t)$ satisfying all the conditions of Definitions 1.1 and 1.2 except (1.1c) or (1.2c) instead of which only $\mu_0 > 0$ is required. The moment space $D^n$ may then be considered as the section obtained by cutting the moment cone with the plane $\mu_0 = 1$, and the moment cone as the projection of the moment space from the origin of $E^{n+1}$. (The same notation is used for the moment cone and the moment space; the context will always be such as to avoid ambiguity.) The advantage of this point of view lies in the fact that it is possible to impose other normalizing conditions than $\mu_0 = 1$, which give rise to a simpler graphical representation of the moment space. In fact while in the plane $\mu_0 = 1$ the moment space is a region which extends to infinity, it will be shown later that it is often possible to choose a normalizing plane such that its section of the moment cone lies in a bounded region.

Figure 2.1 represents the moment cone $D^1[0,\infty)$ and shows that while the section $\mu_0 = 1$ is a half line, the section $\mu_0 + \mu_1 = 1$ is a line segment (including one extreme).

Figures 2.2 and 2.3 show the moment space $D^2[0,\infty)$ and the section obtained by normalizing the moment cone with $\mu_0 + \mu_2 = 1$.

Figures 2.4 and 2.5 show the same sections in the case $(-\infty,\infty)$. In the case $[0,\infty)$ the moment cone is bounded by the surface

$$(*) \quad \begin{cases} \mu_0 = u & u \geq 0 \\ \mu_1 = ut & t \geq 0 \\ \mu_2 = ut^2 & \end{cases}$$

and $\mu_1 = 0$, in the case $(-\infty,\infty)$ by the surface

$$(**) \quad \begin{cases} \mu_0 = u & \\ \mu_1 = ut & u \geq 0 \, . \\ \mu_2 = ut^2 & \end{cases}$$

The surfaces (*) and (**) are the same as

$$\mu_0 \mu_2 = \mu_1^2 \qquad \mu_0 \geq 0 \quad \mu_1 \geq 0$$
$$\mu_0 \mu_2 = \mu_1^2 \qquad \mu_0 \geq 0$$

respectively. Using the transformation

$$\mu_0 + \mu_2 = 2\xi$$
$$-\mu_0 + \mu_2 = 2\eta - 1$$

it can be seen that the plane $\mu_0 + \mu_2 = 1$ cuts these surfaces along the curves

$$\mu_1^2 + (\eta - \tfrac{1}{2})^2 = \tfrac{1}{4} \qquad \mu_1 \geq 0$$
$$\mu_1^2 + (\eta - \tfrac{1}{2})^2 = \tfrac{1}{4}$$

respectively.

Figure 2.6 represents the space $D^3[0,\infty)$. According to the previous discussion the proper boundary $\underline{B}^3$, consists of $\underline{A}_2^3$, $\underline{A}_1^3$, and $\underline{A}_o^3$ where $\underline{A}_2^3$ is obtained by joining the origin with the points of the curve $C^3$, $\underline{A}_1^3$ is the curve $C^3$ itself and $\underline{A}_o^3$ is the origin. The improper boundary $\overline{B}^3$ consists of $\overline{A}_2^3$ and $\overline{A}_1^3$ where $\overline{A}_2^3$ is obtained by joining the points of $C^3$ with the point at infinity of $C^3$ and $\overline{A}_1^3$ by joining the origin with the point at infinity of $C^3$. In the figure $\underline{A}_2^3$ is marked by heavy lines and $\overline{A}_2^3$ by dashed lines.

Figure 2.7 represents $D^3(-\infty,\infty)$. This space consists of the points which lie "inside" the surface $\mu_2 = \mu_1^2$, i.e. the points for which $\mu_2 > \mu_1^2$, plus the points of $C^3$. The curve $C^3$ lies on the surface in question. The dashed line denotes the intersection of this surface with the plane $\mu_3 = 0$.

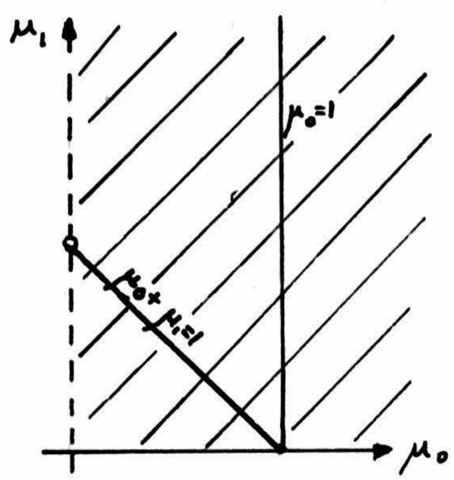


Figure 2.1

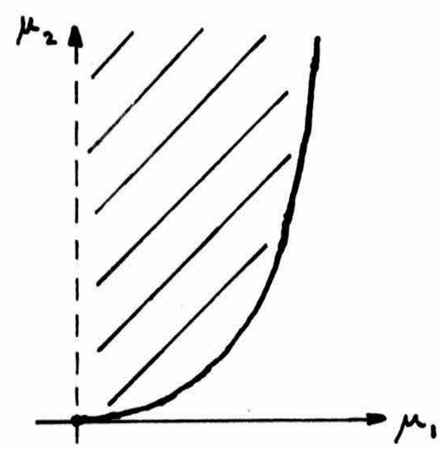


Figure 2.2

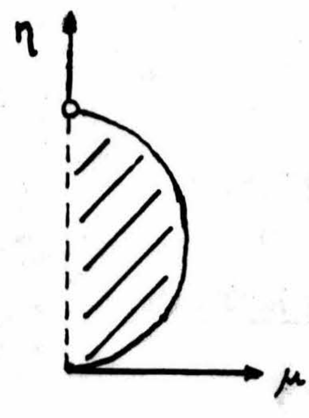


Figure 2.3

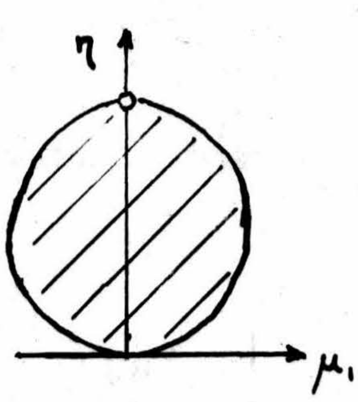


Figure 2.4

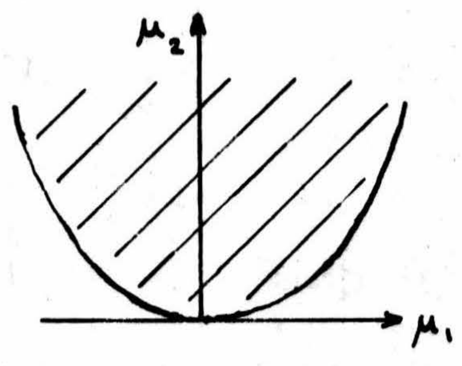


Figure 2.5

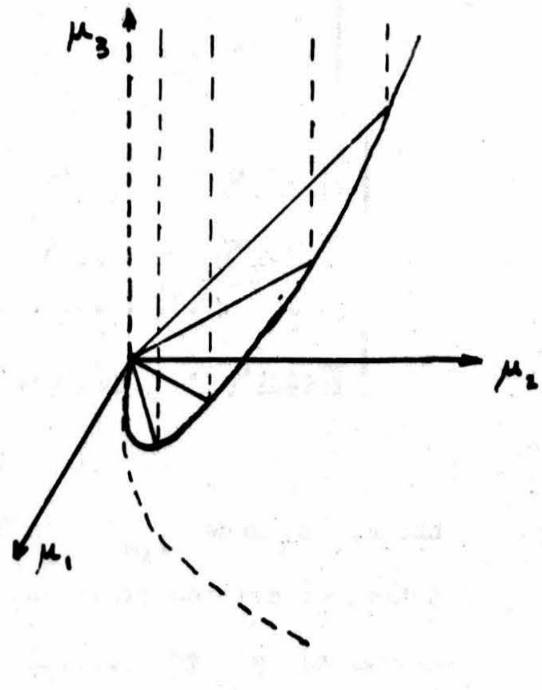


Figure 2.6

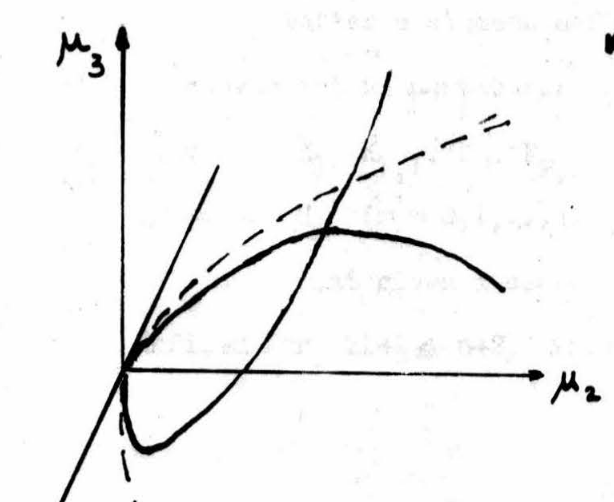


Figure 2.7

# CHAPTER III

## DETERMINANT CRITERIA FOR MOMENT SEQUENCES

Definition 3.1 Let

$$H_{2i-2} = K_i = \begin{vmatrix} \mu_o & \mu_1 & \cdots & \mu_{i-1} \\ \mu_1 & \mu_2 & \cdots & \mu_i \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{i-1} & \mu_i & \cdots & \mu_{2i-2} \end{vmatrix}$$

and

$$H_{2i-2,l} = K_{i,l} = \begin{vmatrix} \mu_l & \mu_{l+1} & \cdots & \mu_{i-1+l} \\ \mu_{l+1} & \mu_{l+2} & \cdots & \mu_{i+l} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{i-1+l} & \mu_{i+l} & \cdots & \mu_{2i-2+l} \end{vmatrix} .$$

Let also $H_{2i-1,l} = H_{2i-2,l+1}$

The latter statement defines $H_{m,l}$ when $m$ is odd. It is convenient in particular because it permits to denote the sequence $K_1, K_{1,1}, K_2, K_{2,1}, K_3, \ldots,$ by the simpler expression $H_m$ $(m = 0,1,\ldots)$.

Note that given a sequence $(\mu_o, \mu_1, \ldots, \mu_n)$ $K_{i,l}$ is defined for $2i+l \leq n+2$, while $H_{m,l}$ is defined for $m + l \leq n$.

<u>Definition 3.2</u> Let

$$V_i(t_1,\dots,t_i) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_i \\ \dots & \dots & \dots & \dots \\ t_1^{i-1} & t_2^{i-1} & \dots & t_i^{i-1} \end{vmatrix} .$$

<u>Theorem 3.3</u> If $\mu_s = \sum_{j=1}^{\bar{b}} \lambda_j t_j^s \quad (s = 0,1,\dots,n)$

then for $2i + l \leq n + 2$

$$K_{i,l} = \begin{cases} \sum\limits_{(j_1,\dots,j_i)} \lambda_{j_1}\cdots\lambda_{j_i} \cdot t_{j_1}^{l} \cdots t_{j_i}^{l} \cdot [V_i(t_{j_1},\dots,t_{j_i})]^2 & i \leq \bar{b} \\ 0 & i > \bar{b} \end{cases}$$

where $\sum\limits_{(j_1,\dots,j_i)}$ is the sum taken when $(j_1,\dots,j_i)$ varies over all combinations of the numbers $(1,\dots,\bar{b})$ taken $i$ at a time.

<u>Proof:</u> By Definition 3.1

$$K_{i,l} = \begin{vmatrix} \sum \lambda_j t_j^{l} & \sum \lambda_j t_j^{l+1} & \dots & \sum \lambda_j t_j^{l+i-1} \\ \dots & \dots & \dots & \dots \\ \sum \lambda_j t_j^{l+i-1} & \sum \lambda_j t_j^{l+i} & \dots & \sum \lambda_j t_j^{l+2i-2} \end{vmatrix}$$

which is equal to the sum of the $\bar{b}^i$ determinants of the form

$$\begin{vmatrix} \lambda_{j_1} t_{j_1}^{l} & \lambda_{j_2} t_{j_2}^{l+1} & \dots & \lambda_{j_i} t_{j_i}^{l+i-1} \\ \dots & \dots & \dots & \dots \\ \lambda_{j_1} t_{j_1}^{l+i-1} & \lambda_{j_2} t_{j_2}^{l+i} & \dots & \lambda_{j_i} t_{j_i}^{l+2i-2} \end{vmatrix}$$

$$= \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_i}\cdot t_{j_1}^{l}\, t_{j_2}^{l+1}\cdots\, t_{j_i}^{l+i-1}\cdot\, V_i(t_{j_1},\ldots,t_{j_i})$$

where the set $(j_1,\ldots,j_i)$ is made to vary over all dispositions with repetition of the numbers $(1,\ldots,\bar{b})$ taken $i$ at a time. If $i > \bar{b}$ all dispositions $(j_1,\ldots,j_i)$ involve some repetition. Since any repetition in the set $(j_1,\ldots,j_i)$ would cause the corresponding $V_i(t_{j_1},\ldots,t_{j_i})$ to vanish, if $i > \bar{b}$ all the $V_i$ vanish hence $K_{i,l} = 0$. If $i \leq \bar{b}$ it suffices to extend the summation to the dispositions without repetition. All these dispositions may be obtained by first varying $(j_1,\ldots,j_i)$ over all combinations of $(1,2,\ldots,\bar{b})$ taken $i$ at a time and then taking with each combination all the permutation of its elements. Let $[s_1,\ldots,s_i]$ denote a permutation of $(j_1,\ldots,j_i)$, let $p$ denote its parity and observe that a permutation of the quantities $(t_{j_1},\ldots,t_{j_i})$ affects only the algebraic sign of $V_i(t_{j_1},\ldots,t_{j_i})$. Then

$$K_{i,l} = \sum_{(j_1,\ldots,j_i)}\ \sum_{[s_1,\ldots,s_i]} \lambda_{s_1}\cdots\lambda_{s_i}\, t_{s_1}^{l}\; t_{s_2}^{l+1}\cdots t_{s_i}^{l+i-1}\cdot V_i(t_{s_1},\ldots,t_{s_i})$$

$$= \sum_{(j_1,\ldots,j_i)} \Big\{ \lambda_{j_1}\cdots\lambda_{j_i}\, t_{j_1}^{l}\cdots t_{j_i}^{l}\; V_i(t_{j_1},\ldots,t_{j_i})\cdot \sum_{[s_1,\ldots,s_i]} (-1)^p\, t_{s_1}^{o}\, t_{s_2}^{1}\cdots t_{s_i}^{i-1} \Big\} .$$

The inside summation is nothing but the expansion of $V_i(t_{j_1},\dots,t_{j_i})$ hence

$$K_{i,1} = \sum_{(j_1,\dots,j_i)} \lambda_{j_1}\cdots\lambda_{j_i}\, t^1_{j_1}\cdots t^1_{j_i}\cdot\left[V_i(t_{j_1},\dots,t_{j_i})\right]^2.$$

Case $[0,\infty)$

Theorem 3.4 If $\mu \in D^n$ and $b(\mu) = b$, then

(3.4a) $H_l > 0$ $\quad (l = 0,1,\dots,2b-1)$

(3.4b) $H_{2b,m} = 0$ $\quad (m = 0,1,\dots,n-2b)$ .

Remarks. By Theorems 2.4 and 2.11, if $\mu \in D^n$ then $b(\mu) \le \frac{n+1}{2}$. If $b = \frac{n+1}{2}$ the second set of conditions is vacuous.

Proof: Stated in terms of $K_{i,m}$ the conclusion of the theorem reads:

if $b$ is an integer

(3.4c) $K_i > 0$ $\quad (i = 1,\dots,\bar{b})$

(3.4d) $K_{i,1} > 0$ $\quad (i = 1,\dots,b)$

(3.4e) $K_{b+1,m} = 0$ $\quad (m = 0,\dots,n-2b)$ ,

if $b$ is a half integer

(3.4c) $K_i > 0$ $\quad (i = 1,\dots,\bar{b})$

(3.4f) $K_{i,1} > 0$ $\quad (i = 1,\dots,\bar{b}-1)$

(3.4g) $K_{\bar{b},m+1} = 0$ $\quad (m = 0,\dots,n-2b)$ .

By Theorem 3.3 $K_{i,1} = 0$ for $i > \bar{b}$. Hence in particular (3.4e) holds. If $i \le \bar{b}$, then

$$V_i(t_{j_1},\dots,t_{j_i}) = \prod_{r<s}(t_{j_r} - t_{j_s}) \neq 0$$

hence $[V_i(t_{j_1},\dots,t_{j_i})]^2 > 0$. All the $\lambda_j$ are positive. If furthermore b is an integer all the $t_j$ are positive, hence $K_{i,l} > 0$ being a sum of positive terms. In particular (3.4c) and (3.4d) are true.

If b is a half integer $t_1 = 0$, $t_j > 0 (j \neq 1)$. If $i \leq \bar{b}$ then $K_{i,l} > 0$ if $l = 0$, hence (3.4c) is valid. Also $K_{i,l} > 0$ if $i < \bar{b}$, because then some of the combinations $(t_{j_1},\dots,t_{j_i})$ do not contain $t_1$, hence some of the terms in the sum are positive; hence (3.4f). However if $l \neq 0$ and $i = \bar{b}$, then the sum giving $K_{\bar{b},l}$ consists of only one term and this term contains the factor $t_1 = 0$. Hence in this case $K_{\bar{b},l} = 0$ and (3.4g) follows.

Theorem 3.5 (Converse of 3.4). If $\mu = (\mu_0,\dots,\mu_n)$ is any sequence, if b is an integer or a half-integer $(\frac{b}{2} \leq b \leq \frac{n+1}{2})$ and if (3.5a) $H_l > 0$ $(l = 0,1,\dots,2b-1)$

(3.5b) $H_{2b,m} = 0$ $(m = 0,1,\dots,n-2b)$

then $\mu \in D^n$ and $b(\mu) = b$.

Proof: The theorem will be proved first for $b = \frac{n+1}{2}$, then for $b = \frac{n}{2}$, finally for b arbitrary.

The proof for $b = \frac{n+1}{2}$ is by induction on n. By Chapter II, $\mu \in \operatorname{int} D^n$ if and only if $\mu \in D^n$ and $b(\mu) = \frac{n+1}{2}$. Hence it is sufficient to show that $H_l > 0$ $(l = 0,\dots,n)$

implies $\mu \in \text{int } D^n$. Clearly if $H_o \equiv \mu_o > 0$ and $H_1 \equiv \mu_1 > 0$ then $(\mu_o,\mu_1) \in \text{int } D^1$. Let now $s \geq 2$ and assume

a) $H_l > 0$ $(l = 0,\ldots,s-1)$ implies $(\mu_o,\ldots,\mu_{s-1}) \in \text{int } D^{s-1}$

b) $H_l > 0$ $(l = 0,\ldots,s)$.

By the induction assumption a) and according to Theorem 2.11

$$\mu_i = \sum_{j=1}^{\bar{b}} \lambda_j t_j^i \qquad (i = 0,\ldots,s-1; \quad b = \tfrac{s}{2}).$$

Let
$$\underline{\mu}_s = \sum_{j=1}^{\bar{b}} \lambda_j t_j^s$$

and let $\underline{\mu} = (\mu_o,\ldots,\mu_{s-1},\underline{\mu}_s)$. Then $b(\underline{\mu}) = \frac{s}{2}$ and by Theorem 3.4, $H_s(\underline{\mu}) = 0$. Since $H_{s-2} > 0$, $H_s \equiv H_s(\mu) > H_s(\underline{\mu})$ implies $\mu_s > \underline{\mu}_s$. By Theorem 2.4 and Lemma 2.5, $\underline{\mu} \in \underline{A}^s_{s-1}$, hence by Theorem 2.10 $\mu_s > \underline{\mu}_s$ implies $(\mu_o,\ldots,\mu_s) \in \text{int } D^s$.

Proof for $b = \frac{n}{2}$. By the previous part, $(\mu_o,\ldots,\mu_{n-1}) \in \text{int } D^{n-1}$, hence

$$\mu_i = \sum_{j=1}^{\bar{b}} \lambda_j t_j^i \qquad (i = 0,\ldots,n-1; \; b = \tfrac{n}{2}) .$$

The condition $H_n = 0$ is a linear equation in $\mu_n$, the coefficient of $\mu_n$ being $H_{n-2}$, which is by assumption $\neq 0$ (except if $b = \frac{1}{2}$, in which case the truth of the statement to be proved may be verified directly). Hence $H_n = 0$ admits a unique solution. But by Theorem 3.4

$$\mu_n = \sum_{j=1}^{\bar{b}} \lambda_j t_j^n$$

is a solution. Hence $\mu_n$ must have this value, therefore $\mu \in D^n$ and $b(\mu) = \frac{n}{2}$.

The proof for $b < \frac{n}{2}$ is by induction. The statement "$H_l > 0$ $(l = 0,\ldots,2b-1)$ and $H_{2b,m} = 0$ $(m = 0,\ldots,k)$ imply $\mu_i = \sum_{j=1}^{\bar{b}} \lambda_j t_j^i$ $(i = 0,\ldots,2b+k)$" has just been proved for $k = 0$. Assume that it holds for a given value of $k$ and that

$$H_l > 0 \quad (l = 0,\ldots,2b-1)$$

$$H_{2b,m} = 0 \ (m = 0,\ldots,k+1).$$

It will then be shown that

$$\mu_i = \sum_{j=1}^{\bar{b}} \lambda_j t_j^i \qquad (i = 0,\ldots,2b+k+1).$$

Consider in fact the condition $H_{2b,k+1} = 0$. If $b$ is an integer it reads

$$H_{2b,k+1} \equiv \begin{vmatrix} \mu_{k+1} \cdots\cdots \mu_{b+k+1} \\ \cdots\cdots\cdots \\ \mu_{b+k+1} \cdots\cdots \mu_{2b+k+1} \end{vmatrix} = 0$$

while if $b$ is a half integer it reads

$$H_{2b,k+1} \equiv H_{2\bar{b}-2,k+2} \equiv \begin{vmatrix} \mu_{k+2} \cdots\cdots \mu_{\bar{b}+k+1} \\ \cdots\cdots\cdots\cdots \\ \mu_{\bar{b}+k+1} \cdots\cdots \mu_{2b+k+1} \end{vmatrix} = 0 \quad .$$

In both cases it is a linear equation in $\mu_{2b+k+1}$ in which the coefficient of this quantity has the value $H_{2(b-1),k+1}$. The order of the latter determinant is $b$ and $\bar{b} - 1$

respectively hence, by the argument given in the proof of Theorem 3.4, this determinant is $\neq 0$. Thus $H_{2b,k+1} = 0$ must have a solution in $\mu_{2b+k+1}$, but by Theorem 3.4

$$\mu_{2b+k+1} = \sum_{j=1}^{\bar{b}} \lambda_j t_j^{2b+k+1}$$

is the solution. This completes the induction proof.

Theorems 3.4 and 3.5 give a complete characterization of the proper boundary components $\underline{A}_a^n$ $(a = 0,\ldots,n-1)$ of $D^n$, and of $\underline{A}_n^n \equiv \text{int } D^n$. In fact by Theorem 2.13 b is constant for all the points in the same component and $b = \frac{a+1}{2}$.

The classical conditions that an infinite sequence $(\mu_o,\mu_1,\ldots)$ be a moment sequence are usually stated in the form

$$K_i \geq 0, \quad K_{i,1} \geq 0 \qquad (i = 1,2,\ldots)$$

and do not involve the determinants $K_{i,m}$ $(m > 1)$.* It will be shown next that a finite number of these conditions are sufficient to assure that $(\mu_o,\ldots,\mu_n)$ be in $D^n$. The conditions that will be obtained have the disadvantage, over those given in Theorem 3.5, that when they are written down they seem to involve moments of order higher than n. It will be apparent from the proofs, however, that they do not actually involve such moments, since the corresponding cofactors in the determinants are always vanishing.

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* See [2] p. 6.

<u>Lemma 3.6</u> Given a sequence $a_o, a_1, \ldots, a_{2n}$ let

$$A_{i+1} = \begin{vmatrix} a_o a_1 & \ldots & a_i \\ a_1 a_2 & \ldots & a_{i+1} \\ \ldots & \ldots & \ldots \\ a_i a_{i+1} & \ldots & a_{2i} \end{vmatrix} .$$

Then if $A_{n+1} = A_n = 0$ all the minors of order $n$ not containing $a_{2n}$ must vanish.

This is a well known result (see [4] p.370) which holds in fact for any symmetric determinant.

<u>Lemma 3.7</u> Let

$$A_{i+1,l} = \begin{vmatrix} a_l & a_{l+1} & \ldots & a_{i+l} \\ a_{l+1} & a_{l+2} & \ldots & a_{i+l+1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i+l} & a_{i+l+1} & \ldots & a_{2i+l} \end{vmatrix} .$$

If $A_{n+1} = A_n = \ldots = A_{n+1-k} = 0$ then $A_{n+1-k,l} = 0$ $(l = 0,1,\ldots,k)$.

<u>Proof</u> by induction. Lemma 3.6 gives in particular that $A_{n,o} = A_{n,1} = 0$. The statement to be proved is clearly true for $k = 0$, (and by Lemma 3.6 also for $k = 1$). Assume that it holds for a fixed value of $k$ and that $A_{n+1} = A_n = \ldots = A_{n-k} = 0$. By the induction assumption $A_{n+1-k,l} = 0$. By Lemma 3.6 $A_{n+1-k,o} = 0$ and $A_{n-k,o} = 0$ imply $A_{n-k,1} = 0$; $A_{n+1-k,1} = 0$ and $A_{n-k,1} = 0$ imply $A_{n-k,2} = 0$; etc.; $A_{n+1-k,k} = 0$ and

$A_{n-k,k} = 0$ imply $A_{n-k,k+1} = 0$. Hence the statement holds for $k + 1$ and the induction proof is completed.

Lemma 3.8 If $H_{m,l} = H_{m-2,l} = 0$ then $H_{m-1,l} = 0$.

Proof: If $m = 2i$ the lemma states that if $K_{i+1,l} = K_{i,l} = 0$ then $K_{i,l+1} = 0$. This follows immediately from Lemma 3.6 since $K_{i,l+1}$ is a minor of order $i$ of $K_{i+1,l}$ not containing $\mu_{2i+l}$. If $l = 2i-1$ the lemma states that if $K_{i,l+1} = K_{i-1,l+1} = 0$ then $K_{i,l} = 0$. This follows also from Lemma 3.6. In fact if $K_{i,l+1} = K_{i-1,l+1} = 0$, Lemma 3.6 implies that all the minors of order $i - 1$ of $K_{i,l+1}$ not containing the last row of $K_{i,l+1}$ must vanish. But these minors are identical with those obtained from the last $i - 1$ rows of $K_{i,l}$. Since they vanish, $K_{i,l} = 0$.

Lemma 3.9 Let $\mu_l = \sum_{j=1}^{\bar{b}} \lambda_j t_j^l$ and let $L_i$ be any minor of order $i$ from the matrix of $H_n$. Then if $i > \bar{b}$, $L_i = 0$. If $b$ is a half integer and if $L_{\bar{b}}$ does not contain $\mu_o$, $L_{\bar{b}} = 0$.

The proof is similar to part of that of Theorem 3.3 and is omitted.

Theorem 3.10 If $\mu = (\mu_o, \ldots, \mu_n)$ $D^n$ and $b(\mu) = b$ then

$$(3.10a) \qquad H_l > 0 \qquad (0 \leq l \leq 2b-1)$$

$$(3.10b) \qquad H_l = 0 \qquad (2b \leq l \leq 2n-2b) \; .$$

(If $b = \frac{n+1}{2}$ the second set of conditions is vacuous.)

Proof: The conditions (3.10a) are the same as (3.4a) which were proved previously. The conditions (3.10b) are equivalent to the following two statements:

A) The principal minors of order $i$ $(\bar{b} + 1 \leq i \leq n - \bar{b} + 1)$ obtained by deleting the last $n - \bar{b} + 1 - i$ rows and columns of the matrix

$$\begin{pmatrix} \mu_0 & \mu_1 \cdots \mu_{n-\bar{b}} \\ \mu_1 & \mu_2 \cdots \mu_{n-\bar{b}+1} \\ \cdots & \cdots \\ \mu_{n-\bar{b}} & \mu_{n-\bar{b}+1} \cdots \mu_{2n-2\bar{b}} \end{pmatrix}$$

are equal to zero.

B) The principal minors of order $i$ $(\bar{b} + 1 \leq i \leq n - b)$ obtained by deleting the last $n - \bar{b} + 1 - i$ rows and columns of the matrix

$$\begin{pmatrix} \mu_1 & \mu_2 & \cdots \mu_{n-\bar{b}} \\ \mu_2 & \mu_3 & \cdots \mu_{n-\bar{b}+1} \\ \cdots & \cdots & \cdots \\ \mu_{n-\bar{b}+1} & \mu_{n-\bar{b}+2} & \cdots \mu_{2n-2b} \end{pmatrix}$$

are equal to zero, and if $b$ is a half integer also the ones of order $\bar{b}$ and $n - \bar{b} + 1$ are equal to zero.

Proof of A). By Lemma 3.9 and the assumption that

$$\mu_i = \sum_{j=1}^{\bar{b}} \lambda_j t_j^i \qquad (i = 0,\ldots,n)$$

all the minors of order $\bar{b}+1$ formed from the first $\bar{b}+1$ rows vanish. Hence the result follows immediately.

Proof of B). The first part of the statement follows by the same argument, since the minors of order $\bar{b}+1$ formed from the first $\bar{b}+1$ rows of the matrix obtained from (3.10d) by canceling the last row and column involve only the first $n$ moments. If $b$ is a half integer, by Lemma 3.9 the minors of order $\bar{b}$ obtained from the first $\bar{b}$ rows of (3.10d) vanish, hence the principal minor of order $\bar{b}$ and that of order $n-\bar{b}+1$ (i.e. the determinant of the matrix (3.10d)) also vanish.

<u>Theorem 3.11</u> If $\mu = (\mu_o,\dots,\mu_n)$ is any sequence of numbers and if for some integer or half integer $b$ $(\frac{1}{2} \leq b \leq \frac{n+1}{2})$

(3.11a) $H_l > 0$ $\qquad (0 \leq l \leq 2b-1)$

(3.11b) $H_{2b} = 0$ $\qquad$ provided $b < \frac{n+1}{2}$

and either set of conditions

(3.11c) $H_l = 0$ $\qquad (2b < l \leq 2n-2b,\ l \text{ odd})$

(3.11d) $H_l = 0$ $\qquad (2b < l \leq 2n-2b,\ l \text{ even})$

then $\mu \in D^n$, $b(\mu) = b$ and the other set also holds. (Note that if $b = \frac{n+1}{2}$ conditions (3.11b, c, d) are vacuous. If $b = \frac{n}{2}$ then (3.11c, d) are vacuous.)

<u>Proof:</u> The third part of the conclusion follows from Theorem 3.10 once the first two have been established. If $b$

is an integer, (3.11b) and (3.11d) become

$$H_l = 0 \quad (l = 2b,\ 2b+2,\ 2b+4,\ \ldots,\ 2n-2b)$$

and (3.11c) becomes

$$H_l = 0 \quad (l = 2b+1, 2b+3,\ \ldots,\ 2n-2b-1)$$

hence, by Lemma 3.8, (3.11b) and (3.11d) imply (3.11c). It may be seen similarly that if $b$ is a half integer then (3.11b) and (3.11c) imply (3.11d). Hence it suffices to prove the theorem under assumptions (3.11 a,b,c) if $b$ is an integer and (3.11 a,b,d) if $b$ is a half integer. In either case if $b = \frac{n}{2}$ or $\frac{n+1}{2}$ the statement of the theorem coincides with that of Theorem 3.5. If $b < \frac{n}{2}$ then (3.11a) still coincides with (3.5a). Hence it is sufficient to prove the following two statements for $b < \frac{n}{2}$:

I. If $b$ is an integer, (3.11 b,c) imply (3.5b).

II. If $b$ is a half integer (3.11b,d) imply (3.5b).

Proof of I. Restated in terms of the $K_{i,l}$ this proposition becomes:

$$(3.11e) \quad \begin{aligned} &K_{b+1} = 0 \\ &K_{b+1,1} = K_{b+2,1} = \cdots = K_{n-b,1} = 0 \end{aligned}$$

implies

$$(3.11f) \quad \begin{aligned} &K_{b+1} = 0 \\ &K_{b+1,m} \qquad (m = 1, \ldots, n-2b) \end{aligned} \quad .$$

By Lemma 3.7 (3.11e) implies (3.11f).

Proof of II. In this case the proposition becomes:

$$K_{\bar{b},1} = 0$$

$$K_{\bar{b}+1} = K_{\bar{b}+2} = \dots = K_{n-\bar{b}+1} = 0$$

implies

$$K_{\bar{b},1} = 0$$

$$K_{\bar{b},m} = 0 \qquad (m = 1,\dots,n-2\bar{b}+1).$$

This follows again from Lemma 3.7.

A simple necessary and sufficient condition may be obtained if $\mu$ is assumed to be a moment point:

Theorem 3.12 If $\mu = (\mu_o,\dots,\mu_n) \in D^n$ then $b(\mu) = b$ if and only if

$$H_l > 0 \qquad (l = 0,\dots,2b-1),$$

$$H_l = 0 \qquad (l = 2b,\dots,n).$$

Proof: The necessity of the condition is proved in Theorem 3.10. The sufficiency follows from the fact that the determinant conditions stated are, for different values of b, mutually exclusive.

Implied in the proof of the above theorems is that of the following.

Theorem 3.13 If $(\mu_o,\dots,\mu_{n-1}) \in \text{int } D^{n-1}$, then the minimum value which $\mu_n$ can assume if $(\mu_o,\dots,\mu_n) \in D^n$ is obtained by solving $H_n = 0$ for $\mu_n$.

## Case $(-\infty,\infty)$

Of the two necessary and sufficient conditions that a finite sequence be a moment sequence (Theorems 3.4 and 3.5 and Theorems 3.10 and 3.11) only the latter may be extended by analogy to the present case.

The following necessary condition, which is the analogue of Theorem 3.4, is valid and may be easily proved by means of Theorem 3.3.

If $\mu \in D^n$ and $b(\mu) = b$ then

$$K_i > 0 \qquad (i = 1,\dots,b),$$

$$K_{b+1,m} = 0 \qquad (m = 0,\dots,n-2b).$$

This condition however is not sufficient in general (e.g. $n = 3$, $b = 1$, $\mu_0 = 1$, $\mu_1 = \mu_2 = 0$, $\mu_3 \neq 0$ is not a moment point).

Using the methods of the previous proofs it is possible to prove the following necessary and sufficient conditions:

$\mu \in D^n$ and $b(\mu) = b$ if and only if

$$K_i > 0 \qquad (i = 1,\dots,b)$$

$$\begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{b+m} \\ \mu_1 & \mu_2 & \dots & \mu_{b+1+m} \\ \mu_b & \mu_{b+1} & \dots & \mu_{2b+m} \end{vmatrix} = 0 \qquad (m = 0,\dots,n-2b).$$

(the second part is vacuous if $b = \left[\frac{n}{2}\right] + 1$).

The analogue of Theorems 3.10 and 3.11 may be stated thus:

$\mu \in D^n$ and $b(\mu) = b$ if and only if

$$K_i > 0 \qquad (i = 1,\dots,b)$$

$$K_i = 0 \qquad (i = b+1,\dots, n-b+1) \;.$$

The necessity of these conditions is easy to verify. The proof of their sufficiency is more complicated and is here omitted.

The analogue of Theorem 3.12 is:

If $\mu \in D^n$ then $b(\mu) = b$ if and only if

$$K_i > 0 \qquad (i = 1,\dots,b),$$

$$K_i = 0 \qquad (i = b+1,\dots, \left[\tfrac{n}{2}\right]).$$

Theorem 3.13 is still valid when $n$ is even. If $n$ is odd cf. remark about Theorem 2.10.

# CHAPTER IV

# THE POLYNOMIAL SPACES

It has been pointed out in Chapters I and II that to any plane in $E^n$ (or, equivalently, to any plane in $E^{n+1}$ passing through the origin) there may be associated a polynomial of degree $\leq n$. This fact has been used in the proofs of some of the theorems stated in those chapters. To investigate some of the consequences of this correspondence between planes and polynomials, the polynomial spaces will now be defined and studied both in themselves and in their relations to the moment spaces.

Definition 4.1 The set of points $(a_0, a_1, \ldots, a_n)$ of $E^{n+1}$ such that $\sum_{i=0}^{n} a_i t^i \geq 0$ for $0 \leq t < \infty$ (or $-\infty < t < \infty$) is called the n-th polynomial cone over $[0,\infty)$ (or $(-\infty,\infty)$) and is denoted by $P^n[0,\infty)$ (or $P^n(-\infty,\infty)$). If the coordinates $a_i$ are subject to the normalizing condition

$$\sum_{i=0}^{n} i!\, a_i = 1$$

the region obtained is called the n-th polynomial space and is also denoted by $P^n[0,\infty)$ (or $P^n(-\infty,\infty)$).

The n-th polynomial cone is clearly an (n+1)-dimensional region while the n-th polynomial space is an n-dimensional region. It is easy to verify that the polynomial space is convex and, by comparison with the space $P^n[0,1]$, that it is bounded.

<u>Extreme points of the polynomial space.</u>

To say that a polynomial of degree $n$ has $k$ roots at $\infty$ shall mean that its degree is $n-k$. To say that it has no roots at $\infty$ shall mean that its degree is exactly $n$.

<u>Theorem 4.2</u> A point of the space $P^n[0,\infty)$ (i.e. a normalized polynomial positive over $[0,\infty)$) is:

A) an extreme point if and only if it has $n$ roots in $[0,\infty]$

B) a boundary point if and only if it has some roots in $[0,\infty]$

C) an interior point if and only if it has no roots in $[0,\infty]$

A similar result holds for $P^n(-\infty,\infty)$.

<u>Proof of A)</u> Assume $P(t)$ has degree $n$ and all its roots in $[0,\infty]$. Then either

$$P(t) = \alpha\, t \prod_j (t - t_j)^2 \qquad 0 \leq t_j$$

$$\text{or } P(t) = \beta \prod_j (t - t_j)^2 \qquad 0 \leq t_j \quad .$$

Assume $P(t)$ is not extreme. Then $P(t) = \lambda Q(t) + (1-\lambda)R(t)$ $(0 < \lambda < 1)$ and $Q \not\equiv R$. $Q$ and $R$ must have as roots all the roots of $P$. Hence,

$$Q(t) = P(t)\, Q_1(t) \qquad Q_1(t) \geq 0 \quad (0 \leq t < \infty)$$

$$R(t) = P(t)\, R_1(t) \qquad R_1(t) \geq 0 \quad (0 \leq t < \infty) \; .$$

By a theorem of Lukács (see [3] p. 4)

$$\text{(4.2a)} \qquad \begin{aligned} Q_1(t) &= q_1^2 + q_2^2 + t(q_3^2 + q_4^2) \\ R_1(t) &= r_1^2 + r_2^2 + t(r_3^2 + r_4^2) \end{aligned} \; .$$

Now the relation $\lambda Q_1 + (1 - \lambda) R_1 = 1$ must hold; but this is impossible if $Q_1$, $R_1$ have the form (4.2a) unless $Q_1 = 1$, $R_1 = 1$, in which case $Q = R$. The contradiction implies that P is indeed extreme.

Conversely, if $P(t)$ has a negative root $t_o$ then

$$P(t) = (t - t_o)\, P_1(t) = (t - t_o - a)\, P_1(t) + a\, P_1(t)$$

$$P_1(t) \geq 0 \qquad (0 \leq t < \infty)$$

and if $0 < a < -t_o$, $P(t)$ is represented as a convex combination of two points of $P^n\,[0,\infty)$, hence it is not extreme.

If $P(t)$ has a pair of complex roots then

$$P(t) = (t - a - ib)(-a + ib)\, P_1(t) = (t - a)^2\, P_1(t) + b^2\, P_1(t)$$

$$P_1(t) \geq 0 \qquad (0 \leq t < \infty).$$

<u>Proof of B) and C)</u> If $P(t)$ is strictly positive over $[0,\infty)$ and of exact degree $n$, any small perturbation of the coefficients, in particular any one satisfying the normalization condition, leads to a small variation of the roots, which are all complex or negative, hence it leads to a polynomial which is again strictly positive. If $P(t)$ has some roots in $[0,\infty]$ and if some of them are finite, a small displacement of one of them while the others are kept fixed leads to a polynomial which is negative over the interval of displacement. If some of the roots are infinite, i.e. if $P(t)$ is of degree $< n$, then the addition of a term $-\epsilon t^n$ leads to a polynomial

which is negative for large values of $t$. In either case there are points in the neighborhood of $P(t)$ which are not in $P^n[0,\infty)$. Since both the hypotheses and the conclusions of B and C are mutually exclusive, the proof is complete.

In the case $(-\infty,\infty)$ the proofs follow the same arguments. In part A the representation

$$Q_1(t) = q_1^2 + q_2^2$$

is used.

It is noteworthy that when $n$ is odd, since every polynomial in $P^n(-\infty,\infty)$ has at least one infinite root, the space $P^n(-\infty,\infty)$ has no interior points.

Representation theorems for positive polynomials.

The polynomials to be studied are assumed positive over $[0,\infty)$ and $(-\infty,\infty)$ respectively. All these polynomials have positive leading coefficients which shall be normalized to have the value 1.

Lemma 4.3 Given $P_{2n}(t) > 0$ in $[0,\infty)$ and any number $t_n > 0$, there exists a unique polynomial $P_n{}^2 = \prod_{j=1}^{n}(t - t_j)^2$ $(0 < t_1 < t_2 < \ldots < t_n)$ and a number $a > 0$ for which $P_{2n}(t) \geq \frac{1}{a} P_n^2(t)$ $(0 \leq t \leq t_n)$ with equality holding at $t = 0$ and once in each interval $(t_j, t_{j+1})$.

The proof of both existence and uniqueness is exactly the same as that given in [1] for a polynomial positive over $[0,1]$, although in this case the extreme $t_n$ is a double root of $P_n^2$. Quantities $a_j$ can be defined by

$$a_j = \max_{t_j \le t \le t_{j+1}} \frac{P_n^2(t)}{P_{2n}(t)} \qquad (j = 0,\dots,n-1;\ t_o = 0)$$

and it can be shown that there exists a unique set $(t_1,\dots,t_{n-1})$ such that $a_j = a$ $(j = 0,\dots,n-1)$. This set $t_j$ and the number $a$ obviously satisfy the lemma.

Lemma 4.4 $a$ is a continuous increasing function of $t_n$, and for $t_n$ sufficiently large $a$ becomes arbitrarily large.

Proof of continuity. The following propositions will first be proved:

A) If $t_n \le T$, there exists $C$ such that $a \le C$;

B) If $t_n \ge T$, there exists $c$ such that $a \ge c$.

To prove A): let $t_n \le T$ and let $p = \min_{0 \le t \le T} P_{2n}(t)$. Then $a_j \le \frac{1}{p} \max_{t_j \le t \le t_{j+1}} P_n^2(t)$.

By varying the $t_j$ over the compact set $0 \le t_1 \le t_2 \dots \le t_n \le T$ each $a_j$ assumes a maximum value $A_j$. Take $c = \max_j A_j$.

To prove B): let $p' = \max_{0 \le t \le T} P_{2n}(t)$. For at least one $j$, $t_{j+1} - t_j \ge \frac{T}{n}$ ; hence

$$a_j = \max_{t_j \le t \le t_{j+1}} \frac{P_n^{\,2}}{P_{2n}} \ge \frac{1}{p'} \max_{t_j \le t \le t_{j+1}} \prod_{i=1}^{n} (t-t_j)^2$$

$$\ge \frac{1}{p'}\left(\frac{t_{j+1} - t_j}{2}\right)^{2n} \ge \frac{1}{p'} \left(\frac{T}{2n}\right)^{2n} .$$

Hence $$a \ge \frac{1}{p'} \left(\frac{T}{2n}\right)^{2n} .$$

It will be proved next that all the $t_j$ $(j = 0,\dots,n-1)$ of Lemma 4.3 are continuous functions of $t_n$. Assume the contrary. There would then exist a sequence $t_n^k \to t_n$ $(k = 1, 2, \dots)$ such that at least one of the corresponding sequences $t_j^k$ defined by Lemma 3.3 does not converge to $t_j$. Then a subsequence $t_n^{k_i}$ could be selected such that all the subsequences $t_j^{k_i}$ converge to values $t_j'$ with $t_j' \neq t_j$ for at least one $j$, and the sequence $a^{k_i} \to a' > 0$. The polynomial $(P_n')^2 = \prod_{j=1}^{n} (t - t_j')^2$ and the number $a'$ would then satisfy the same conditions as the polynomial $P_n^2$ and the number $a$ of Lemma 4.3, which contradicts the uniqueness of $P_n^2$. Now if $t_n$ is changed by a small amount, the $t_j$ change also by a small amount, and so does $a$ which is the common value of the $a_j$.

<u>Proof</u> that $a \to \infty$ as $t_n \to \infty$. Let $C > 0$ be given and assume that there exist arbitrarily large values of $t_n$ for which $a < C$. For these values

$$C \ P_{2n}(t) > P_n^2(t)$$

for $0 \leq t \leq t_n$, in particular

$$(4.4a) \qquad C \ P_{2n}(i) > P_n^2(i) \qquad (i = 1,2,\dots,n+1).$$

But if $t_n$ is arbitrarily large, the values of $P_n^2$ at points other than its roots become also arbitrarily large, thus contradicting (4.4a).

Proof of monotonicity. Consider $a$ as a function of $t_n$ and assume $a(t_n) = a(t'_n)$ for $t_n \neq t'_n$. Then the polynomials $\frac{1}{a} P_n^2$ and $\frac{1}{a} P'^2_n$ have $2n$ common roots; and their difference, which is a polynomial of degree $2n - 1$ with $2n$ roots, must vanish identically.

Theorem 4.5 There exists a unique representation of a polynomial $P_{2n}(t) > 0 \quad (0 \leq t < \infty)$ as a sum

$$P_{2n}(t) = \prod_{j=1}^{n} (t - t_j)^2 + \alpha\, t \prod_{j=1}^{n-1} (t - \bar{t}_j)^2 \qquad \alpha > 0$$

$$(0 < t_1 < \bar{t}_1 < t_2 < \cdots < \bar{t}_{n-1} < t_n) .$$

By Lemma 4.4, since $a \to 0$ as $t_n \to 0$, there exists $t_n$ such that $a(t_n) = 1$. Hence the corresponding unique polynomial $P_n^2 = \prod_{j=1}^{n} (t - t_j)^2$ is such that the difference $P_{2n} - P_n^2$ is a polynomial of degree $2n - 1$. By Lemma 4.3 this difference polynomial is non-negative for $0 \leq t \leq t_n$ and vanishes at $0$ and at values $\bar{t}_j$ $(t_j < \bar{t}_j < t_{j+1})$. Hence this polynomial is of the form

$$\alpha\, t \prod_{j=1}^{n} (t - t_j)^2 \qquad \alpha > 0.$$

By a similar argument it is possible to obtain the following

Theorem 4.6 Given a polynomial $P_{2n+1}(t)$ there exists a unique representation

$$P_{2n+1}(t) = t \prod_{j=1}^{n} (t - t_j)^2 + \alpha \prod_{j=1}^{n} (t - \bar{t}_j)^2 \qquad \alpha > 0$$

$$(0 < \bar{t}_1 < t_1 < \bar{t}_2 < \ldots < \bar{t}_n < t_n).$$

Lemma 4.7 Given any polynomial $P_{2n}(t) > 0 \quad (-\infty,\infty)$ and any $t_1$, there exists a unique polynomial

$$P_n^{\,2}(t) = \prod_{j=1}^{n} (t - t_j)^2 \qquad (t_1 < t_2 < \ldots < t_n)$$

for which $\quad P_{2n}(t) \geqslant P_n^{\,2}(t) \qquad (t_1 < t < t_n)$

with equality holding once in each interval $(t_j, t_{j+1})$.

The proof is similar to that of Lemmas 4.3 and 4.4 and Theorem 4.6, after transferring the origin to the point $t_1$, the only difference being that the left hand point $t_1$ of the interval under consideration is also a double root of $P_n^{\,2}$.

Lemma 4.8 $t_j$ $(j = 2, \ldots, n)$ as determined by Lemma 4.7 are continuous functions of $t_1$. Also $t_n$ is a monoton increasing function of $t_1$.

Proof of monotonicity. After defining $a_j$ as above, it can be shown like in Lemma 4.4 that for any fixed $t_1$, $a = a\,(t_1, t_n)$ is a continuous increasing function of $t_n$. Similarly for any fixed $t_n$ $a$ is a continuous decreasing function of $t_1$. Conversely by the continuity and monotonicity $t_n$ is a continuous increasing function of $a$. Now assuming $a(t_1, t_n) = 1$, if $t_1$ is given an increment $\epsilon > 0$, $a(t_1 + \epsilon, t_n) = 1 - \delta$ for some $\delta > 0$. Hence

$a(t_1 + \epsilon, t_n + \eta) = 1$ can have a solution only for $\eta > 0$. But by the existence theorem such $\eta$ does exist.

Proof of continuity. It can be easily seen that if $t_1$ varies in a closed bounded interval, so does $t_j$. If $t_j$ were not a continuous function of $t_1$, there would exist a sequence $t_1^{\,k}$ such that the corresponding $t_j^{\,k}$ does not converge to $t_j$. A subsequence $t_j^{\,k_i}$ would then converge to $t_j'$ with $t_j' \neq t_j$ for at least one $j$. But then the uniqueness of the polynomial $P_n^{\,2} = \prod_{j=1}^{n} (t - t_j)$ would be contradicted.

Theorem 4.9 Given any $P_{2n}(t) > 0$ $(-\infty, \infty)$ there exists a unique representation

$$(4.9a) \qquad P_{2n}(t) = \prod_{j=1}^{n} (t - t_j)^2 + \alpha \prod_{j=1}^{n-1} (t - \bar{t}_j)$$

$$(\alpha > 0; \quad t_1 < \bar{t}_1 < t_2 < \bar{t}_2 < \dots < \bar{t}_{n-1} < t_n) .$$

Proof of uniqueness. The polynomial $P_n^{\,2}(t)$ of Lemma 4.7 depends on the choice of $t_1$. If for a certain choice of $t_1$, $P_{2n} - P_n^{\,2} > 0$ for all $t$ and (4.9a) holds, then (4.9a) holds for a unique $\alpha > 0$ and a unique set $t_j$. If $t_1' \neq t_1$ could be chosen with $P_{2n} - P_n'^{\,2} > 0$, the polynomial $P_n^{\,2} - P_n'^{\,2}$ would have $2n - 1$ zeros. In fact assume for simplicity that $t_1 < t_1'$, that the $t_j$ and the $t_j'$ are all distinct and that the $\bar{t}_j$ and $\bar{t}_j'$ are also all distinct. Then

$$P_n^{\,2}(t_j) = 0 < P_n'^{\,2}(t_j) \qquad (j = 1,\ 2,\ \ldots,\ n)$$

$$P_n^{\,2}(\bar{t}_j) = P_{2n}(\bar{t}_j) > P_n^{\,2}(t_j) \qquad (j = 1,\ 2,\ \ldots,\ n\text{-}1)$$

$P_n^{\,2}(t_n') < P_{2n}(t'_n) = P_n'^{\,2}(t_n')$ since by Lemma 4.8 $t_n < t_n'$. Hence $P_n^{\,2} - P_n^{\,2}$ would have $2n - 1$ sign changes, therefore $2n - 1$ zeros. If some of the $t_j$ and $t'_j$ coincided, or if some of the $\bar{t}_j$ and $\bar{t}_j'$ coincided, the number of zeros would decrease, but their multiplicity would increase, and it can be easily seen that the total number of zeros of $P_n^{\,2} - P_n'^{\,2}$, counting multiplicity, would still be $2n - 1$. Hence the polynomials $\alpha \prod_{j=1}^{n-1} (t - \bar{t}_j)^2$ and $\alpha' \prod_{j=1}^{n-1} (t - \bar{t}_j')^2$ would also cross $2n - 1$ times. But then these polynomials of degree $2n - 2$ would be identically equal, and so would $P_n^{\,2}$ and $P_n'^{\,2}$. This establishes the uniqueness of the representation.

To prove the existence construct a $t_1$ such that for the corresponding $P_n^{\,2}$

$$P_{2n} - P_n^{\,2} \geq 0 \qquad \text{for all } t.$$

If the latter relation did not hold, in addition to the roots $t_j$ of combined multiplicity $2n - 2$, the polynomial $P_{2n} - P_n^{\,2}$ would have only one simple root, say $t^*$. Either $t^* > t_n$ or $t^* < t_1$. Since by Lemma 4.8 $P_n^{\,2}$ depends continuously on $t_1$, and since the root $t^*$ is simple, the set $t_1$ such that $t^*$ exists is open. Furthermore it can be easily seen that by choosing $t_1$ sufficiently small $t^*$ exists and $t^* > t_n$, while by choosing $t_1$ large, $t^* < t_1$. Since the set of $t_1$ such that $t^* > t_n$ and the set of $t_1$ such that $t^* < t_1$ are disjoint and each is open, there must exist a point $T$ between them. Choosing $t_1 = T$ we have $P_{2n} - P_n^2$ 0. In other words define

$$T = \sup t_1 \text{ such that } t^* \text{ exists and } t^* > t_n;$$

the set is non-void and bounded above, hence $T$ exists.

The representation theorems are susceptible of a simple geometrical interpretation in terms of the normalized polynomial spaces $P^n$. By varying the roots of an extremal polynomial of the form $\prod_j (t - t_j)^2$ or $t \prod_j (t - t_j)^2$ and if necessary by letting some of them approach $\infty$ it is possible to connect arcwise any two extremal polynomials by means of extremal polynomials if and only if they are both of even degree or both of odd degree. Hence if $n \geq 1$ the space $P^n$ $[0,\infty)$ has two disjoint extremal components such that any point of $P^n$ $[0,\infty)$ is a convex combination of one point of each component.

The space $P^n(-\infty,\infty)$ has only one extremal component.

# CHAPTER V

# RELATIONS BETWEEN THE MOMENT SPACES AND THE POLYNOMIAL SPACES

The notion of conjugate convex cones has been investigated in [1]. Some of the pertinent definitions and theorems will be listed here.

The following statements are valid whether $C$ is a closed set or not:

(5.0a) Define the convex cone $C \subset E^{n+1}$ as the set of points $x = (x_0,\ldots,x_n)$ such that $x, x' \in C$ implies $\lambda x + \lambda' x' \in C$ for all positive $\lambda, \lambda'$.

(5.0b) Define the conjugate cone $C^*$ to a given $C$ as the set of points $y$ such that

$$\sum_{i=0}^{n} x_i y_i \equiv (x,y) \geq 0 \qquad \text{for all} \quad x \in C$$

(5.0c) $C^*$ is closed and convex.

(5.0d) $C^*$ is the set of planes of support to $C$. A plane which supports along an element of $C$ other than the origin is a point in the boundary of $C^*$. (The converse is not true if $C$ is not closed).

(5.0e) Any two non-zero points in the same line through the origin represent the same plane. If $C$ is proper, i.e. it does not contain a complete line through the origin, it is possible to normalize its non-zero points by means of a plane parallel to an interior point of $C^*$. The resulting cross section, K, is a bounded convex set.

(5.0f) Let $K$, $K^*$ be bounded cross sections of proper cones $C$, $C^*$. Then

interior points of $K$ are planes not meeting $K^*$,

boundary points of $K$ are planes of support to $K^*$,

exterior points of $K$ are planes cutting through $K^*$.

(5.0g) If $x$ is in the boundary of a cone $C$ and if $y$ is interior to the convex set of points in which $x$ meets $C^*$, then $y$ is said to be conjugate to $x$.

(5.0h) If $x \in C$ is conjugate to $y \in C^*$ and if $y$ is conjugate to any $z \in C$, then $y$ is conjugate to $x$.

(5.0i) The indices $a(x)$, $c(x)$ of a point lying in a proper, (n+1)-dimensional cone $C$ are defined in the plane of any bounded n-dimensional cross section containing $x$ as in Definitions 2.1 and 2.2. They are independent of the cross section used.

(5.0j) If $y \in C^*$ is conjugate to $x \in C$ then

$$a(x) + c(y) = n - 1.$$

If $C$ is closed then $C$ is conjugate to $C^*$, hence $C$ and $C^*$ are dual to each other. In this case the following additional statements hold.

(5.0k) $C^{**} = C$.

(5.0l) The converse of the second statement in (5.0d) holds, i.e. a point is in the boundary of $C^*$ if and only if it is a supporting plane to $C$ at a non-zero point.

(5.0m) If $x \in C$ is conjugate to $y \in C^*$ or if $y$ is conjugate to $x$ then

$$a(x) + c(y) = n - 1 .$$

In connection with (5.0j) and (5.0m) it is to be noted that if $x \in C$, then every supporting plane to $C$ at $x$ is a supporting plane to $\overline{C}$ at $x$ and vice versa. Hence if $x \in C$ is conjugate to $y \in C^*$ with respect to $\overline{C}$ it is also conjugate with respect to $C$, although if $x$ is conjugate to $y$ with respect to $C$ it need not be conjugate with respect to $\overline{C}$. Thus $a_C(x) = a_{\overline{C}}(x)$ although $c_C(x)$ and $c_{\overline{C}}(x)$ need not be equal.

The moment cone $D^n$ introduced in Chapter II, its closure $\overline{D}^n$ and the polynomial cone $P^n$ are obviously convex cones.

<u>Theorem 5.1</u> The cone $P^n$ is conjugate to the cone $D^n$. The cones $P^n$ and $\overline{D}^n$ are mutually conjugate.

<u>Proof:</u> Let $(\alpha_o, \alpha)$ denote a plane not intersecting $D^n$. Then for all $\mu = (\mu_i)$

$$(\alpha, \mu) \geq 0.$$

In particular for $\mu_i = t^i \quad (i = 0,1,\ldots,n)$

$$P(t) = \sum_{i=o}^{n} \alpha_i \, t^i \geq 0,$$

hence $(\alpha_o, \alpha)$ is an element of $P^n$.

Conversely let $(\alpha_o, \alpha)$ denote a polynomial $P^n \geq 0$. Then since any moment point $\mu$ is in $Co(C^n)$ and since $\sum \alpha_i t^i \geq 0$,

$$(\alpha, \mu) \geq 0.$$

The second part of the theorem follows from (5.0k) and the fact that $\overline{D^n}$ is closed.

The first part of the theorem together with (5.0c) implies that $P^n$ is closed.

As far as $P^n(-\infty,\infty)$ is concerned, the following discussion will be limited to the case of $n$ even, which is the most interesting since $P^{2m+1}(-\infty,\infty) = P^{2m}(-\infty,\infty)$.

Lemma 5.2 $P$ is conjugate to $x = \sum_{j=1}^{b} \lambda_j x(t_j)$ if and only if the only roots of $P$ in $[0,\infty]$ (or $[-\infty,\infty]$ ) are roots of minimal order at $t_j$, i.e. double roots except for a simple root at $t_1$ if $t_1 = 0$ in the case $[0,\infty.]$

Proof: By the definition of conjugateness $P$ is conjugate to $x$ if and only if $P$ is a plane interior to the set of supporting planes to $D^n$ at $x$. The planes of this set have the form

$$P(t) = Q(t) \prod_{j=1}^{b} (t - t_j)^2$$

where $Q(t)$ is any polynomial non-negative in the interval under consideration. For this it is necessary and sufficient (cf. Theorem 4.2c) that $Q(t)$ be exactly of degree $n - 2b(x)$ and strictly positive. But this is the case if and only if the only roots of $P$ are roots of minimal order at $t_j$.

Theorem 5.3 Let $P \in \text{bd } P^n$.

A) If $P$ is conjugate to $x \in D^n$ then $x$ is conjugate to $P$.

B) If $P$ is not conjugate to any $x \in D^n$ then $P$ has roots of order greater than the minimal or roots at $\infty$.

Proof of A) By Lemma 5.2 the only roots of $P$ are at the points $t_j$ used in the convex representation of $x$. Hence the set of points which the supporting plane $P$ has in common with $D^n$ is precisely $L(x) \cap D^n$. Since $x$ is interior to this set $x$ is conjugate to $P$.

Proof of B) Since $P \in \text{bd } P^n$, by Theorem 4.2B $P$ must have some real roots (infinity being considered a real root). The result follows then from Lemma 5.2 by contradiction.

Theorem 5.4 If $x \in D^n$ is conjugate to $P$ then just one of two alternatives holds:

1. $P$ is conjugate to $x$
2. $P$ has multiple roots or roots at $\infty$.

Proof: By (5.0h) and Theorem 5.3B.

Theorem 5.5 If $P \in P^n$ is conjugate to $x \in D^n$, or vice versa, then $P(t_j) = 0$ if and only if $x(t_j)$ is used in the unique representation of $x$.

Proof: By Theorem 5.3B it is sufficient to prove the statement under the assumption that $x$ is conjugate to $P$.

Since $x$ is interior to the simplex which $P$ has in common with $D^n$ and since the representation of $x$ is unique, all the vertices of the simplex are used. These vertices correspond precisely to the roots of $P(t)$ in the fundamental interval.

The results stated so far illustrate the relations between $D^n$ and $P^n$. To study the geometry of $P^n$ it is convenient to investigate the relations between $\overline{D^n}$ and $P^n$. The points of $\overline{D}^n - D^n$ are convex combinations of points of $C^n$ and the point at infinity of $C^n$ in the sense explained in Chapter II. If $x$ is such a point, any supporting plane $P$ at $x$ is parallel to the $x_n$-axis, hence it is a polynomial of degree $n-1$ at most. Hence the set of roots of minimal order of Lemma 5.2 must be extended to include a possible simple root at $\infty$ in the case $[0,\infty)$ and a possible double root in the case $(-\infty,\infty)$. If $x \sim \{t_j\}$ denotes the fact that $x$ is a convex combination of the $t_j$ $(j = 1,\ldots,\overline{b})$, where possibly $t_{\overline{b}} = \infty$, then Lemma 5.2 may be restated as follows: $P$ is conjugate to $x \sim \{t_j\}$ if and only if the only roots of $P$ in $[0,\infty]$ (or $[-\infty,\infty]$ ) are roots of minimal order at $t_j$. Theorem 5.3, 5.4 and 5.5 hold also for $x \in \overline{D}^n$ with the proviso that a simple root at $\infty$ in the case $[0,\infty]$ and a double root at $\infty$ in the case $(-\infty,\infty)$ are to be considered of minimal order. The proofs are similar to those given above.

Definition 5.6 Let $\bar{b}(P)$ denote the number of distinct roots of $P \in P^n$ in $[0,\infty]$ (or $[-\infty,\infty]$) and let $b(P)$ denote the same except for a half count given to the roots at 0 or $\infty$ in the case of $P^n[0,\infty)$.

Theorem 5.7 If $P \in P^n$ is conjugate to $x \in \overline{D^n}$, or vice versa, then

(5.7a) $b(P) = b(x)$

(5.7b) $\bar{b}(P) = \bar{b}(x)$

(5.7c) $2b(P) = n - c(P)$

(5.7d) $\bar{b}(P) = n - a(P)$ .

Proof: The first two relations follow from Theorem 5.5. (5.7c) is proved by means of Theorems 2.13, (5.0m) and (5.7a):

$$2b(P) = 2b(x) = a(x) + 1 = n - c(P).$$

(5.7d) is proved by means of Theorems 2.14, (5.0m) and (5.7b).

Theorem 5.8 A) For every $P \in \text{bd } P^n$ there exists an $x \in \text{bd } \overline{D}^n$ which is conjugate to it. (This statement is true for closed convex cones in general.)

B) Every point $x$ is conjugate to some $P$.

C) All points conjugate to a given $P$ are in the same boundary component of $\overline{D}^n$.

Proof of A) Given $x \in \text{bd } \overline{D}^n$ choose a plane $P$ interior to the set of planes supporting $\overline{D}^n$ at $x$. Then $P$ is conjugate to $x$.

Proof of B) It follows from part A and Theorem 5.3A.

Proof of C) If $x$ and $x'$ are both conjugate to $P$, Theorem 5.5 implies that they are both combinations of the same points $t_j$, hence the statement follows immediately.

Theorem 5.9 The boundary of $P^n[0,\infty)$ may be partitioned into disjoint, individually connected components $\overline{Q}^n_c$, $\underline{Q}^n_c$ $(c = 0,1,\dots,n-1)$ such that $c(P) = c$ if $P \in \overline{Q}^n_c$ or $\underline{Q}^n_c$ and such that the upper bar denotes a root at $\infty$ and the lower bar denotes no roots at $\infty$. The closures of $\overline{Q}^n_o$ and $\underline{Q}^n_o$ are the extremal components of $P^n$. Moreover $a(P)$ is constant over each component and satisfies the relation

$$n + c(P) - 2 \leq 2\underline{a}(P) \leq n + c(P).$$

Proof: Let for $c = 0, 1, \dots, n-1$

$$\overline{Q}^n_c = \{ P \mid \exists\, x \in \overline{A}^n_{n-1-c} ;\ x \text{ conjugate } P \},$$

$$\underline{Q}^n_c = \{ P \mid \exists\, x \in \underline{A}^n_{n-1-c} ;\ x \text{ conjugate } P \}.$$

Then Theorems 2.7, 2.12 and 5.8 imply that the partition of the boundary of $P^n$ is exhaustive and that the components are mutually disjoint. That they are individually connected follows from the connectedness of each $\overline{A}^n_a$, $\underline{A}^n_a$. That $c(P) = c$ if $P \in \overline{Q}^n_c$ or $\underline{Q}^n_c$ follows from the definition of $\overline{Q}^n_c$ and $\underline{Q}^n_c$ and from (5.0m). That the upper bar denotes a root of $P$ at $\infty$ and the lower bar denotes no roots at $\infty$ follows from Theorem 5.5 and from the fact that the representation of $x$

involves or not the point at $\infty$ according as $x$ lies in some $\bar{A}_a^n$ or in some $\underline{A}_a^n$. According to Chapter IV the extreme points of $P^n$ are positive polynomials of degrees $n$ and $n-1$ having $n$ and $n-1$ roots respectively in $[0,\infty]$. These polynomials are limits of polynomials having $n$ and $n-1$ roots respectively in $[0,\infty)$, $\bar{b}$ of the roots being distinct, with $b = \frac{n}{2}$ and $\frac{n-1}{2}$ respectively. The latter polynomials have as their conjugates the points of $\underline{A}_{n-1}^n$ and $\bar{A}_{n-1}^n$ respectively, hence they lie in $\underline{Q}_o^n$ and $\bar{Q}_o^n$ respectively. The fact that $a(P)$ is constant in each boundary component of $P^n$ follows from (5.0m) and from the fact that $c(x)$ is constant in each boundary component of $\bar{D}^n$. Finally the relation

$$n + c(P) - 2 \leq 2a(P) \leq n + c(P)$$

is deduced from Theorems 2.13, 2.14 and 5.7 and from the relation

$$b(x) \leq \bar{b}(x) \leq b(x) + 1.$$

In the case of $P^n(-\infty,\infty)$ it must be noted that the point at infinity does not play a special part as far as the boundary components of $\bar{D}^n$ are concerned. In fact if $\bar{D}^n$ is normalized by setting $\mu_o + \mu_n = 1$, the curve $C^n$ is given parametrically by

$$x_i = \frac{t_i}{1 + t^n}$$

which is a closed curve regular at $t = \infty$. Hence the improper boundary component $\underline{A}_{2b-2}$ of $D^n$, which is obtained by joining all convex combinations of b-1 points of $C^n$ with its point at $\infty$ becomes part of the proper boundary component $\underline{A}^n_{2b-1}$ obtained by taking all convex combinations of b points of $C^n$. Thus all boundary components of $\overline{D}^n$ are odd dimensional. Hence Theorem 5.9 becomes:

The boundary of $P^n(-\infty,\infty)$ may be partitioned into disjoint, individually connected components $Q^n_c$ $(c = 0,2,\ldots,n-2)$ such that $c(P) = c$ if $P \in Q^n_c$. The closure of $Q^n_o$ is the extremal component of $P^n(-\infty,\infty)$. Moreover $a(P)$ is constant on each component and $2a(P) = n + c(P)$.

The proof is similar to that for the case $0,\infty)$, with

$$Q^n_c = \left\{ P \,\middle|\, \exists\ x\ \underline{A}^n_{n-1-c}\ ;\ x \text{ conjugate } P \right\}.$$

The fact that the greatest value of c is n-2 means that the boundary of $P^n(-\infty,\infty)$ does not contain any "flat" pieces, i.e. any linear simplexes of dimension n-1.

<u>Examples of polynomial spaces.</u>

The extreme points of $P^2\,[0,\infty)$ are of the form $P(t) = \alpha\,(t-u)^2$ where $0 \leq u \leq \infty$ and $P(t) = \beta\,t$. The normalizing condition $\sum i!\,a_i = 1$ gives

$$\alpha = \frac{1}{1 - 2u + 2u^2}\ , \quad \beta = 1.$$

Making the orthogonal transformation

$$\begin{cases} a_o - a_1 & = 1 + x \\ a_o + a_1 - a_2 & = 1 + y \\ a_o + a_1 + 2a_2 & = z \end{cases}$$

the points of $\underline{Q}_o^n$ are given, in the normalizing plane $z = 1$, by the parametric equations

$$\begin{cases} x = \dfrac{4u - 2}{2u^2 - 2u + 1} \\ \\ y = \dfrac{-3}{2u^2 - 2u + 1} \end{cases} \qquad (0 \leq u < \infty) .$$

$\overline{Q}_o^n$ consists of the point $(2,0)$.

Similarly the extreme points of $P^2(-\infty,\infty)$ are of the form $P(t) = \alpha (t-u)^2$ where $-\infty \leq u \leq \infty$. In the plane $z = 1$ $\underline{Q}_o^2$ is given by the same equations as before with $-\infty \leq u \leq \infty$.

Figure 5.1 shows schematically the three spaces $P^2(-\infty,\infty) \subset P^2[0,\infty) \subset P^2[0,1]$. The polynomials shown in the figure have been written without the normalizing constants.

Figure 5.2 represents the spaces $\overline{D}^2[0,\infty)$ normalized by $\mu_o + \mu_2 = 1$ (cf. Figure 2.3) and $P^2[0,\infty)$. The polynomials 1 and $t^2$ are not conjugate to any point of $\overline{D}^2$ since they have double roots at $\infty$ and 0 respectively. The polynomial $t$ is not conjugate to any point of $D^2$ since it has a root at $\infty$ but it is conjugate to the point $\overline{A}_o^2 \subset \overline{D}^2$. The point $\underline{A}_o^2$ is conjugate to $t$ with respect to $D^2$ but not with respect to $\overline{D}^2$.

Figure 5.3 represents the space $P^3[0,\infty)$. Its extreme points have the form

$$\begin{array}{ll} 1) & \alpha(t-u)^2 \\ 2) & \beta t(t-u)^2 \end{array} \qquad 0 \leq u \leq \infty$$

Set 1) has already been discussed. Set 2) is dealt with by means of the orthogonal transformation

$$\begin{cases} a_o - a_1 & = 1 + x \\ a_o + a_1 - a_2 & = 1 + y \\ a_o + a_1 + 2a_2 - a_3 & = 1 + z \\ a_o + a_1 + 2a_2 + 6a_3 & = w \end{cases},$$

which gives

$$\begin{cases} x = \dfrac{u^2}{u^2 - 4u + 6} \\ y = \dfrac{-u^2 - 2u}{u^2 - 4u + 6} \\ z = \dfrac{-u^2 + 4u - 1}{u^2 - 4u + 6} \\ w = 1 \qquad \text{(normalization)} \end{cases}.$$

The boundary component $\underline{Q}_1^3$ is obtained by joining points of $\overline{Q}_o^3$ and $\underline{Q}_o^3$ corresponding to the same values of u.

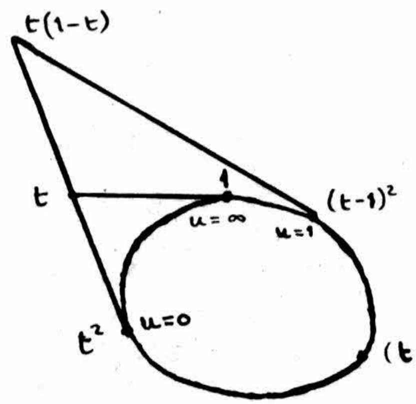


Figure 5.1

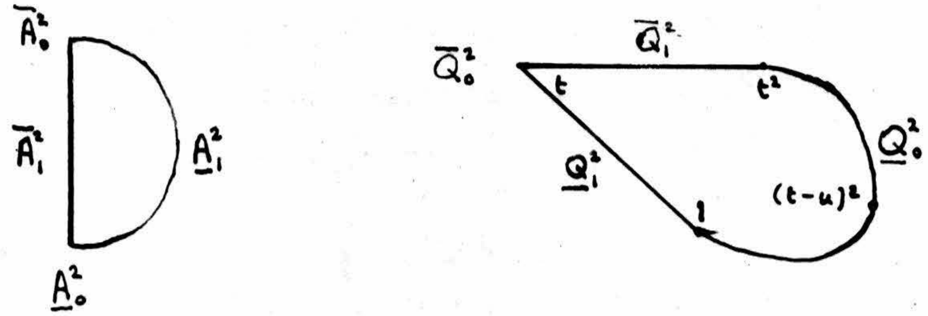


Figure 5.2

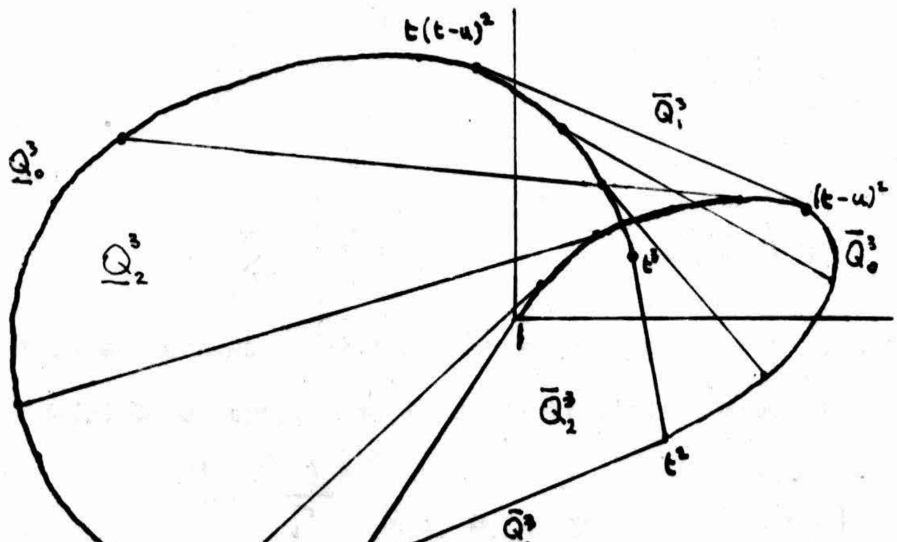


Figure 5.3

The fact that $P^n$ is conjugate to $D^n$ may be used to obtain a new proof of Theorem 3.12. In fact the statement that $P^n$ is conjugate to $D^n$ may be reworded as follows: if $\mu = (\mu_0,\ldots,\mu_n)$ is a reduced moment sequence then

$$P(t) \equiv \sum_{i=0}^{n} a_i t^i \geqslant 0$$

implies

$$(\mu,P) \equiv \sum_{i=0}^{n} a_i \mu_i \geqslant 0 \, .$$

The condition "$P \in P^n$ implies $(\mu,P) \geqslant 0$" is equivalent to the same condition applied to a subset of $P^n$ containing all the extreme points of $P^n$.

Case $[0,\infty)$.

The subset chosen in this case is that of the polynomials of the form

$$\left(\sum_{i=0}^{l} a_i t^i\right)^2 \qquad l = \left[\tfrac{n}{2}\right]$$

and

$$t\left(\sum_{i=0}^{l} a_i t^i\right)^2 \qquad l = \left[\tfrac{n-1}{2}\right] \; ;$$

this set contains all the extreme points of $P^n$. The condition $(\mu,P) \geqslant 0$ applied to these polynomials becomes

$$\sum_{i,j=0}^{l} a_i a_j \mu_{i+j} \geqslant 0 \qquad l = \left[\tfrac{n}{2}\right]$$

and

$$\sum_{i,j=o}^{l} a_i a_j \mu_{i+j+1} \geq 0 \qquad l = \left[\frac{n-1}{2}\right]$$

respectively and these conditions in turn are equivalent to

$$K_i \geq 0 \qquad (i = 1,\ldots,\left[\tfrac{n}{2}\right])$$

and $$K_{i,1} \geq 0 \qquad (i = 1,\ldots,\left[\tfrac{n-1}{2}\right])$$

which may be written together as

$$H_l \geq 0 \qquad (l = 0,\ldots,n).$$

If now $b(\mu) = b$ is an integer, any supporting plane at $\mu$ must contain the points $t_j$ used in the representation of $\mu$, hence it must have degree $2b$ at least. For any supporting $P$ of degree $< 2b$, $(\mu,P) > 0$, therefore

$$\sum_{i,j=o}^{b-1} a_i a_j \mu_{i+j} \quad \text{and} \quad \sum_{i,j=o}^{b-1} a_i a_j \mu_{i+j+1}$$

are positive definite, while

$$\sum_{i,j=o}^{b} a_i a_j \mu_{i+j}$$

is positive semidefinite and there exists a set $(a_o,\ldots,a_b)$, unique up to a positive multiple, such that

$$\sum_{i,j=o}^{b} a_i a_j \mu_{i+j} = 0.$$

Hence

$$K_i > 0 \quad (i = 1,\dots,b), \quad K_i = 0 \quad (i = b+1,\dots, \left[\tfrac{n}{2}\right] ),$$

$$K_{i,1} > 0 \quad (i = 1,\dots,b), \quad K_{i,1} = 0 \quad (i = b+1,\dots, \left[\tfrac{n-1}{2}\right] ).$$

If $b$ is a half integer it is found similarly that

$$\sum_{i,j=o}^{\bar{b}-1} a_i a_j \mu_{i+j} \quad \text{and} \quad \sum_{i,j=o}^{\bar{b}-2} a_i a_j \mu_{i+j+1}$$

are positive definite while

$$\sum_{i,j=o}^{\bar{b}-1} a_i a_j \mu_{i+j+1}$$

is positive semidefinite. Hence

$$K_i > 0 \quad (i = 1,\dots,\bar{b}), \quad K_i = 0 \quad (i = \bar{b}+1,\dots, \left[\tfrac{n}{2}\right] ),$$

$$K_{i,1} > 0 \quad (i = 1,\dots,\bar{b}-1), \quad K_{i,1} = 0 \quad (i = \bar{b},\dots, \left[\tfrac{n-1}{2}\right] ).$$

These results coincide with Theorem 3.12.

Definition 5.10 Let

$$P_n(t) = \begin{vmatrix} \mu_o & \mu_1 & \dots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \dots & \mu_n & t \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n-1} & t^n \end{vmatrix}$$

and

$$p_n(t) = \begin{vmatrix} \mu_1 & \mu_2 & \dots & \mu_n & 1 \\ \mu_2 & \mu_3 & \dots & \mu_{n+1} & t \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{n+1} & \mu_{n+2} & \dots & \mu_{2n} & t^n \end{vmatrix}.$$

<u>Theorem 5.11</u> Let $(\mu_0,\dots,\mu_{n-1}) \in \text{int } D^{n-1}$, $\underline{\mu}_n = \min \mu_n$ for $(\mu_0,\dots,\mu_n)\ D^n$, and $\underline{\mu} = (\mu_0,\dots,\mu_{n-1},\underline{\mu}_n)$. Then:

A) If $n = 2m$ the unique supporting plane at $\underline{\mu}$ is $[P_m(t)]^2$ and the spectrum of the unique distribution $\underline{\varphi}$ whose moment point is $\underline{\mu}$ consists of the roots of $P_m(t)$.

B) If $n = 2m+1$ the unique supporting plane at $\underline{\mu}$ is $t\,[p_m(t)]^2$ and the spectrum of $\underline{\varphi}$ consists of $0$ and the roots of $p_m(t)$.

<u>Proof:</u> In case A since $b(\underline{\mu}) = m$,

$$K_i > 0 \quad (i = 1,\dots,m), \quad K_{m+1} = 0$$

hence the system of equations

$$\sum_{i=0}^{n} a_i \mu_{i+j} = 0 \quad (j = 0,\dots,n)$$

has a solution. Since $K_m > 0$ the matrix of the system has rank $m$, therefore the roots are proportional to the cofactors of the last column of $K_{m+1}$. Hence $\sum_{i=0}^{m} a_i t^i$ is proportional to $P_m(t)$ and the first part of the statement follows. Since the supporting plane at $\underline{\mu}$ vanishes precisely at the points used in the representation of $\underline{\mu}$, these points and no others belong to the spectrum of $\underline{\varphi}$. The proof of B is carried out similarly.

<u>Theorem 5.12</u> Given $\mu = (\mu_0,\dots,\mu_n)$ and a distribution $\varphi(t)$ such that $\mu(\varphi) =$ , the polynomials $P_i(t)$ $(0 \le i \le [\frac{n}{2}])$ and $p_i(t)$ $(0 \le i \le [\frac{n-1}{2}])$ form orthogonal systems over the

interval $[0,\infty)$ with respect to $d\varphi(t)$ and $td\varphi(t)$ respectively. The normalization constants of $P_i$ and $p_i$ are $\sqrt{K_i K_{i+1}}$ and $\sqrt{K_{i,1}\ K_{i+1,1}}$ respectively.

Proof: If $0 \leq j < i \leq \left[\frac{n}{2}\right]$ then

$$\int_0^\infty t^j\ P_i(t)d\varphi(t) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{i-1} & \mu_j \\ \mu_1 & \mu_2 & \cdots & \mu_i & \mu_{j+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_i & \mu_{i+1} & \cdots & \mu_{2i-1} & \mu_{j+i} \end{vmatrix} = 0,$$

since two columns of the determinant are equal. Furthermore

$$\int_0^\infty P_i^2\ (t)\ d\varphi(t) = K_i \int_0^\infty t^i P_i(t)d\ \varphi(t)$$

$$= K_i \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{i-1} & \mu_i \\ \mu_1 & \mu_2 & \cdots & \mu_i & \mu_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_i & \mu_{i+1} & \cdots & \mu_{2i-1} & \mu_{2i} \end{vmatrix} = K_i K_{i+1},$$

hence the normalizing constant of $P_i$ is $\sqrt{K_i K_{i+1}}$. The part of the theorem concerning $p_i$ is proved similarly.

Definition 5.13 A function of bounded variation $\varphi$ (t) is said to have a sign change at $t_2$ if there exist numbers $t_1$ and $t_2$ $(t_1 \leq t_2)$ and a number $\delta_1 > 0$ such that

(5.13a) $\varphi(t) = 0 \qquad (t_1 < t < t_2)$

(5.13b) $\varphi(t_1 - \ )\cdot\varphi(t_2 + \ ) < 0 \quad (0 < \delta < \delta_1)$.

Theorem 5.14 If $\varphi$ and $\psi$ are distinct distributions having the same first n moments then $\varphi - \psi$ has at least n sign changes.

Proof by contradiction. If $\varphi - \psi$ has sign changes at $t_j (j = 1,\dots,n' < n)$ let $Q(t) = \prod_{j=1}^{n'} (t - t_j)$. Then

$$P(t) = \int_0^t Q(t)dt$$

is a polynomial of degree n at most, hence

$$\int_{0-}^{\infty} P(t)d(\varphi - \psi) = 0.$$

Integrating by parts and using the fact that $(\varphi - \psi)(0-) = 0$,

$$0 = \int_{0-}^{\infty} P\, d(\varphi - \psi) = \int_{0-}^{A} P\, d(\varphi - \psi) + \int_{A}^{\infty} P\, d(\varphi - \psi)$$

(5.14a)

$$= (\varphi - \psi)(A)\cdot P(A) - \int_0^A (\varphi - \psi)\, Q\, dt + \int_A^{\infty} P\, d(\varphi - \psi).$$

Given $\epsilon > 0$ there exist $A_1$ and C such that $A > A_1$ implies $P(A) < CA^n$. Also, since $\varphi$ has n moments, there exists $A_2$ such that $A > A_2$ implies

$$A^n [1 - \varphi(A)] \leq \int_A^{\infty} t^n\, d\varphi(t) < \frac{\epsilon}{C}$$

(the first inequality holds for any A). Similarly there exists $A_3$ such that $A > A_3$ implies

$$A^n [1 - \psi(A)] < \frac{\epsilon}{C}.$$

Hence if $A > A_1, A_2, A_3$

$$\left|(\varphi - \psi)(A)\cdot P(A)\right| = \left|1 - \psi(A) - 1 + \varphi(A)\right| \cdot \left|P(A)\right| < \frac{2\epsilon}{CA^n} CA^n = 2\epsilon .$$

Also if A is sufficiently large

$$\left|\int_A^{\infty} P\, d(\varphi - \psi)\right| < \epsilon .$$

But $\left|\int_0^A (\varphi - \psi)\, Q\, dt\right|$ is positive and non decreasing as A increases, because $\varphi - \psi \not\equiv 0$, and $\varphi - \psi$ and Q have sign changes at the same points, hence their product is always nonpositive or nonnegative. Thus a contradiction to (5.14a) is obtained.

Lemma 5.15 If $\mu \in \text{int } D^{2m}$ the roots $t_j$ of $P_m$ and the roots $u_j$ of $p_m$ interlock according to the pattern

(5.15a) $$0 < t_1 < u_1 < \dots < t_m < u_m .$$

If $\mu \in \text{int } D^{2m-1}$ the roots of $P_m$ and $p_{m-1}$ interlock according to

(5.15b) $$0 < t_1 < u_1 < \dots < u_{m-1} < t_m .$$

Proof. That the roots are all distinct, real and positive is a consequence of the geometrical interpretation of polynomials as supporting planes. If $n = 2m$ let $\underline{\varphi}$ and $\underline{\psi}$ be distributions corresponding to the points $(\mu_1, \dots, \mu_{2m-1}, \underline{\mu}_{2m})$ and $(\mu_1, \dots, \mu_{2m}, \underline{\mu}_{2m+1})$. Since $\overline{b}(\underline{\mu}) = m$ and $b(\underline{\psi}) = m + 1$, $\underline{\varphi}$ and $\underline{\psi}$ have altogether $2m + 1$ saltus points, $(0, t_j, u_j)$ $(j = 1, \dots, m)$ and are constant elsewhere. By Theorem 5.13 $\underline{\varphi} - \underline{\psi}$ must have $2m - 1$ sign changes. 0 is not a sign change since $\underline{\varphi}(0-) = \underline{\psi}(0-) = 0$; nor is the greatest point in the set $(0, t_j, u_j)$, say v, a sign change since $\underline{\varphi}(v+) = \underline{\psi}(v+) = 1$. Hence all the other saltus points must be sign changes, therefore the $t_j$ and the $u_j$ must alternate and since $\varphi(0+) = 0$ and $\psi(0+) > 0$ the smallest saltus point is $t_1$. Hence the $t_j$ and $u_j$ satisfy the inequalities (5.15a). The points $x(t_j)$ and the points $x(0)$ and $x(u_j)$ are involved in the unique representations of $\underline{\mu}$ and $\mu$ respectively, hence by Theorem 5.11 $P_n$ and $p_n$ vanish precisely at $t_j$ and $u_j$ respectively. The second part of the theorem is proved similarly.

<u>Theorem 5.16</u> If $n = 2m$ there exists a 1 : 1 : 1 continuous correspondence between the interior points of the space $D^{2m}$, the ordered pairs of polynomials $(P_m, p_m)$ and the open simplex of strictly interlocking roots

$$0 = u_0 < t_1 < u_1 < \cdots < t_m < u_m .$$

If $n = 2m - 1$ there exists a 1 : 1 : 1 correspondence between the interior points of the space $D^{2m-1}$, the ordered pairs of polynomials $(P_m, p_{m-1})$ and the open simplex of strictly interlocking roots

$$0 = u_0 < t_1 < u_1 < \cdots < u_{m-1} < t_m.$$

<u>Proof:</u> If $n = 2m$, by Lemma 5.14 the point $\mu$ determines uniquely the interlocking sets $t_j$, and $u_j$, and by the representation theorem it determines uniquely the positive numbers $\lambda_j$ $(j = 1,\ldots,m)$ and $\lambda'_j$ $(j = 0,\ldots,m)$ such that

$$\text{(5.16a)} \qquad \mu_i = \sum_{j=1}^{m} \lambda_j t_j^i = \sum_{j=0}^{m} \lambda'_j u_j^i \quad (i = 0,\ldots,2m-1),$$

$$\mu_{2m} = \sum_{j=0}^{m} \lambda'_j u_j^{2m} .$$

Conversely it will be shown that given any interlocking sets $t_j$ and $u_j$ the point $\mu$ is uniquely determined and satisfies (5.16a). In order that the sets $t_j$, $u_j$ represent a point $\mu$ the conditions

$$(5.16b) \quad \sum_{j=1}^{m} \lambda_j t_j^i - \sum_{j=0}^{m} \lambda_j' u_j^i = 0 \qquad (i = 0,\ldots,2m-1)$$

must hold, where $u_o^{\,o} = 0^o = 1$. Letting

$$(5.16c) \quad \alpha_{2j} = \lambda_j', \quad \alpha_{2j-1} = -\lambda_j, \quad w_{2j} = u_j, \quad w_{2j-1} = t_j$$

(5.16b) becomes

$$(5.16d) \quad \sum_{j=0}^{2m} \alpha_j w_j^i = 0 \qquad (i = 0,\ldots,2m-1) .$$

The $2m \times 2m + 1$ matrix of the coefficients of (5.16d) is

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & t_1 & u_1 & \ldots & u_m \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & t_1^{2m-1} & u_1^{2m-1} & \ldots & u_m^{2m-1} \end{pmatrix} .$$

All the minors of order $2m$ of this matrix are $\neq 0$ since they are Vandermondians with distinct elements. Hence the roots $\alpha_j$ of (5.16c) are proportional to these minors taken with alternating signs. The condition $\sum_{j=1}^{m} \lambda_j = 1$ fixes the proportionality constant and (5.16c) implies that the $\lambda_j$ and the $\lambda_j'$ are positive. Thus there exists a unique solution for the $\lambda_j$, $\lambda'_j$ and hence a unique point which satisfies (5.16a). This proves that the correspondence between int $D^{2m}$ and the simplex of interlocking roots is 1:1. That the correspondence between $D^{2m}$ and the ordered pairs $(P_m, p_m)$ is 1:1 is obvious from Definition 5.10. The continuity of the correspondence between $D^{2m}$ and the pairs $(P_m, p_m)$ is also an immediate consequence of Definition 5.10, and the continuity of the correspondence between the pairs $(P_m, p_m)$ and the simplex of the roots is obvious. The proof for $n = 2m - 1$ is obtained by a similar argument.

Geometrically if $n = 2m$ the point $\mu$ may be interpreted as the unique intersection in $D^{2m}$ of the convex simplex spanned by the points $x(u_j)$ and the projection from the point at infinity in the $x_{2m}$ direction of the convex simplex spanned by the points $x(t_j)$. The u-simplex determines the unique supporting plane to $D^{2m+1}$ at $(\mu_1, \ldots, \mu_{2m}, \mu_{2m+1})$ while the t-simplex determines the unique supporting plane to $D^{2m}$ at $(\mu_1, \ldots, \mu_{2m-1}, \mu_{2m})$.

Lemma 5.14 and Theorem 4.5 imply that the extreme points of the space $P^{2m}$ are, except for the normalization constant, the polynomials $P_i$ $(0 \leq i \leq m)$ and $tp_i^2$ $(0 \leq i < m-1)$ corresponding to $\mu \in$ int $D^{2m-1}$, and the extreme points of the space $P^{2m+1}$ are the polynomial $P_i^2(0 \leq i \leq m)$ and $tp_i^2$ $(0 \leq i \leq m)$ corresponding to $\mu \in$ int $D^{2m}$. This fact may be used to prove the following.

Theorem 5.17 There exists a homeomorphism between the interior of the space $P^{2m}$ and the interior of $D^{2m}$ such that polynomials of the form

$$\lambda P_m^2 + (1-\lambda)\ tp_{m-1}^2 \qquad (0 < \lambda < 1), \tag{5.17a}$$

where $P_m$ and $p_{m-1}$ are fixed, correspond to moment points having the same first 2m-1 moments. A similar correspondence exists between $P^{2m+1}$ and $D^{2m+1}$.

The proof is given for the case $n = 2m$. To a pair $(P_m, p_{m-1})$ there may be associated on one hand the straight line segment (5.17a) consisting of interior points of $P^{2m}$, on the other hand the half line consisting of interior points of $D^{2m}$ obtained by taking the point $(\mu_1, \ldots, \mu_{2m-1}) \in \text{int } D^{2m-1}$ corresponding to $(P_m, p_{m-1})$ and letting $\underline{\mu}_{2m} < \mu_{2m} < \infty$.* To obtain a pointwise correspondence let $l_1$ denote the line through $P^{2m}$

$$\lambda P_m^2 + (1-\lambda)\ tp_{m-1}^2 \quad , \qquad \lambda \text{ arbitrary}$$

and $l_2$ the line through $D^{2m}$

$$\mu_1, \ldots, \mu_{2m-1} \text{ fixed}, \quad \mu_{2m} \text{ arbitrary}.$$

To a point P on $l_1$ interior to $P^{2m}$ there corresponds a

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* If $D^{2m}$ is normalized by a condition which makes it bounded the half line in question becomes also a straight line segment.

plane exterior to $D^{2m}$ and not parallel to the $x_{2m}$-axis which intersects $l_2$ in a point $\mu'$. The harmonic conjugate $\mu$ of $\mu'$ with respect to $\underline{\mu}$ and the point at infinity of $l_2$ is the point interior to $D^{2m}$ corresponding to P. An alternate way of obtaining $\mu$ is to take first the harmonic conjugate $P'$ of P with respect to $P_m^2$ and $tp_{m-1}^2$, which is exterior to $P^{2m}$ and represents a plane through $D^{2m}$ whose intersection with $l_2$ gives $\mu$. Still another way consists in considering the points $(\mu_1, \ldots, \mu_{2m-1}, \underline{\mu}_{2m})$ and $(\mu_1, \ldots, \mu_{2m-1}, \infty)$ as supporting planes to $P^{2m}$ at $P_m^2$ and $t\,p_{m-1}^2$ respectively, then taking the plane through their intersection and through the point P and finally constructing its harmonic conjugate with respect to the two supporting planes; the plane thus constructed corresponds to the point $\mu \in \text{int } D^{2m}$. All these procedures are equivalent since the cross ratio is invariant under stellar duality as well as under projections and sections. They are also reversible, hence the mapping is $1:1$. Furthermore the mapping is clearly exhaustive and continuous both ways.

<u>Case $(-\infty, \infty)$</u>

Many of the results just proved can be extended to the case $(-\infty, \infty)$. The theorems will be stated here but the proof will be omitted or only sketched.

The proof of Theorem 3.12 is the case $(-\infty,\infty)$ is obtained by considering only the polynomials of the form

$$\left(\sum_{i=0}^{l} a_i t^i\right)^2 \qquad l = \left[\frac{n}{2}\right]$$

and the corresponding quadratic form

$$\sum_{i,j=0}^{l} a_i a_j \mu_{i+j} .$$

Theorem 5.11A still holds and the same proof is valid.

Theorem 5.12 may be restated thus: given $\mu = (\mu_0, \ldots, \mu_{2m})$ and a distribution $\varphi(t)$ such that $\mu(\varphi) = \mu$, the polynomials $P_i(t)$ $(0 \leq i \leq m)$ form an orthogonal system over the interval $(-\infty,\infty)$ with respect to $d\varphi(t)$. The normalizing constant of $P_i$ is $\sqrt{K_i K_{i+1}}$.

Theorem 5.14 still holds but the proof is modified by setting

$$P(t) = \int_a^t Q(t)\, dt$$

where $a < \min_j t_j$.

Lemma 5.15 and Theorem 5.16 are modified in a very important respect, since the $p_m$ do not play any role in the present case. Lemma 5.15 becomes: if $\mu \in \operatorname{int} D^{2m-1}$ the roots $t_j$ of $P_m$ and the roots $u_j$ of $P_{m-1}$ interlock according to the pattern

$$t_1 < u_1 < t_2 < \ldots < u_{m-1} < t_m .$$

The proof makes use of the distributions $\varphi$ and $\psi$ giving rise to the points $(\mu_1, \ldots, \mu_{2m-3}, \mu_{2m-2})$ and $(\mu_1, \ldots, \mu_{2m-1}, \mu_{2m})$ and of the fact that $\varphi - \psi$ must have at least 2m-3 sign changes.

Theorem 5.16 becomes: there exists a 1:1:1 continuous correspondence between the interior points of the space $D^{2m-1}$, the ordered pairs of polynomials $(P_m, p_{m-1})$ and the simplexes of strictly interlocking roots

$$t_1 < u_1 < t_2 < \ldots < u_{m-1} < t_m .$$

Theorem 5.17 becomes: there exists a homeomorphism between the interior of $P^{2m}$ and the interior of $D^{2m}$ such that polynomials of the form

$$\lambda P_m^2 + (1 - \lambda) \quad P_{m-1}^2 \qquad (0 < \lambda < 1),$$

where $P_m$ and $P_{m-1}$ are fixed, correspond to moment points having the same first 2m-1 moments.

# CHAPTER VI

## SOME PROPERTIES OF THE POLYNOMIALS $P_n$ AND $p_n$ .

Several properties of the polynomials $P_n$ and $p_n$ may be derived from the theorems proved in the previous chapter.

Theorem 6.1 Let $\varphi$ be any distribution associated with a point $\mu \in \text{int } D^{2m}\,[0,\infty)$ and let $\underline{\varphi}$ be the distribution associated with $\underline{\mu}$ and $t_j$ $(j = 1,\ldots,m)$ the saltus points of $\underline{\varphi}$. Then

(6.1a) $\underline{\varphi}(t_j-) < \varphi(t_j-) \le \varphi(t_j+) < \underline{\varphi}(t_j+)$ .

Proof: By Theorem 5.14 the number of sign changes of $\underline{\varphi} - \varphi$ is $2m-1$. Since $\varphi$ is nondecreasing and since $\underline{\varphi}(t) = 0$ for $t < t_1$ and $\underline{\varphi}(t) = 1$ for $t \ge t_m$, one sign change must occur at each $t_j$ and one between any two consecutive $t_j$. Hence (6.1a) must hold.

Remark. A similar statement holds for $\mu \in D^{2m+1}\,[0,\infty)$ and for $\mu \in D^{2m+1}(-\infty,\infty)$.

Theorem 6.2 If $\varphi$ is any distribution associated with $\mu \in \text{int } D^{2m}$ and the Christoffel numbers are defined (cf. [3] p. 47) by

$$(6.2a) \qquad \xi_j = \int \frac{P_m(t)}{P_m'(t_j)(t-t_j)}\, d\varphi(t)$$

where $t_j$ are the roots of $P_m$, and if the weights of $\underline{\varphi}$ are denoted by $\lambda_j$, then $\xi_j = \lambda_j$ $(j = 1,\ldots,m)$.

Proof: For any polynomial $P(t)$ of degree $\leq 2m-1$ the Gauss-Jacobi quadrature formula gives

(6.2b) $$\int P(t)d\varphi(t) = \sum_{j=1}^{m} \xi_j P(t_j) .$$

Since $\mu_i(\varphi) = \mu_i(\underline{\varphi})$ $(i = 1,\ldots,2m-1)$, (6.2b) still holds if $\varphi$ is replaced by $\underline{\varphi}$. Hence

$$\xi_j = \int \frac{P_m}{P_m'(t_j)(t-t_j)} \, d\underline{\varphi}(t) = \underline{\varphi}(t_j+) - \underline{\varphi}(t_j-) = \lambda_j .$$

An immediate consequence of Theorems 6.1 and 6.2 is the separation theorem for Christoffel numbers:

Theorem 6.3 Given any distribution $\varphi$ associated with $\mu \in \operatorname{int} D^{2m}$ there exist numbers $x_j$ $(t_j < x_j < t_{j+1})$ such that

$$\varphi(t_j) < \varphi(x_j) = \sum_{i=1}^{j} \xi_i < \varphi(t_{j+1}) .$$

It may be shown that the polynomials $P_i$ and $p_i$ are related by means of certain recursion formulas:

Theorem 6.4 If $\mu \in \operatorname{int} D^{2m-1}$ $0,\infty)$ and $1 \leq i \leq m$ then

(6.4a) $$P_i = \alpha_i p_i + \beta_i p_{i-1}$$

(6.4b) $$tp_{i-1} = \frac{1}{\beta_{i-1}} P_i + \frac{1}{\alpha_i} P_{i-1}$$

where

(6.4c) $$\alpha_i = \frac{K_i}{K_{i,1}} , \quad \beta_i = \frac{K_{i+1}}{K_{i,1}} .$$

Proof: Let $\pi_i$ denote an unspecified polynomial of degree i. If

$$\alpha_i = \frac{K_i}{K_{i,1}}$$

then

$$P_i - \alpha_i p_i = \pi_{i-1} = \sum_{j=0}^{n-1} b_j\, p_j \quad .$$

Therefore if $h \leq i - 2$

$$0 = \int_0^\infty t p_h (P_i - \alpha_i p_i)\, d\varphi = \sum_{j=0}^{i-1} b_j \int_0^\infty t p_h p_j d\varphi = b_h\, K_{h,1} K_{h+1,1}$$

and since $K_{h,1} \neq 0$ and $K_{h+1,1} \neq 0$, $b_h = 0$.

If $h = i - 1$

$$\int_0^\infty t\, p_{i-1}(P_i - \alpha_i p_i) d\varphi = K_{i-1,1} \int_0^\infty t^i p_i d\varphi = K_{i-1,1} K_{i+1}$$

$$= b_{i-1}\, K_{i-1,1}\, K_{i,1} \quad ,$$

hence

$$\beta_i = b_{i-1} = \frac{K_{i+1}}{K_{i,1}}$$

which proves (6.4a).

(6.4b) is proved similarly:

$$t p_{i-1} - \frac{1}{\beta_{i-1}} P_i = \pi_{i-1} = \sum_{j=0}^{i-1} a_j\, P_j \quad .$$

Therefore if $h \leq i - 2$

$$0 = \int_0^\infty P_h (t p_{i-1} - \frac{1}{\beta_{i-1}} P_i)\, d\varphi = \sum_{j=0}^{i-1} a_j \int_0^\infty P_h\, P_j\, d\varphi$$

$$= a_h\, K_h\, K_{h+1}$$

hence $a_h = 0$.

If $h = i - 1$

$$\int_0^\infty p_{i-1}(t\, p_{i-1} - \frac{1}{\beta_{i-1}} p_i)\, d\varphi = K_{i-1} \int_0^\infty t \cdot t^{i-1}\, p_{i-1} d\varphi$$

$$= K_{i-1}\, K_{i,1} = a_{i-1}\, K_{i-1}\, K_i$$

hence $\quad a_{i-1} = \dfrac{K_{i,1}}{K_i} = \dfrac{1}{\alpha_i}$ .

An extremal property of the polynomials $p_m$ may be proved by a simple argument:

<u>Theorem 6.5</u> Let $\varphi$ be a distribution associated with a moment point $\mu \in \text{int } D^{2m}\, [0,\infty)$. Among all polynomials $P = \sum_{i=0}^{2m} a_i t^i \in P^{2m}\, [0,\infty)$ for which

(6.5a) $\displaystyle\int_0^\infty P\, d\varphi = 1$ ,

$P(0) = \max$ for $P = p_m^2$ .

<u>Proof</u>: $\mu \in \text{int } D^{2m}\, [0,\infty)$ implies that $\mu_i = \sum_{j=0}^{m} \lambda_j t_j^i$ $(i = 0,\dots,2m)$ where $t_o^o = 0^o = 1$. Hence (6.5a) gives

$$1 = \sum_{i=0}^{2m} a_i \mu_i = \sum_{i=0}^{2m} \sum_{j=0}^{m} a_i \lambda_j t_j^i = \sum_{j=0}^{m} \lambda_j \sum_{i=0}^{2m} a_i t_j^i$$

$$= \sum_{j=0}^{m} \lambda_j\, P(t_j) = \lambda_o P(0) + \sum_{j=1}^{m} \lambda_j\, P(t_j).$$

Since the $\lambda_j$ are fixed and positive, max $P(0)$ is reached for the nonnegative polynomial which vanishes at $t_j$, which is precisely $p_m^2$ .

# CHAPTER VII

## RELATIONS BETWEEN $D^n[0,a]$, $D^n[0,\infty)$ AND $D^n(-\infty,\infty)$.

It is possible to obtain the spaces $D^{2m}[0,\infty)$, $D^{2m-1}[0,\infty)$ and $D^{2m-1}(-\infty,\infty)$ by means of a passage to the limit from $D^{2m}[0,a]$, $D^{2m-1}[0,a]$ and $D^{2m-1}[-a,a]$ respectively. This limiting process preserves certain properties of $D^{2m}[0,a]$, $D^{2m-1}[0,a]$ and $D^{2m-1}[-a,a]$ in a sense which will now be explained. The results will be stated and proved only for the case $D^{2m-1}[0,a]$ although similar results are valid in the other cases as well.

It is shown in [1] that if $(\mu_1,\ldots,\mu_{2m-1}) \in \operatorname{int} D^{2m-1}[0,1]$ and if $P_m(t)$ is defined as in Definition 5.10 and $\overline{p}_{m-1}(t)$ is defined by

$$\overline{p}_{m-1}(t) = \begin{vmatrix} \mu_1-\mu_2 & \mu_2-\mu_3 & \cdots\mu_{m-1}-\mu_m & 1 \\ \mu_2-\mu_3 & \mu_3-\mu_4 & \cdots\mu_m-\mu_{m+1} & t \\ \cdots & \cdots & \cdots & \cdots \\ \mu_m-\mu_{m+1} & \mu_{m+1}-\mu_{m+2} & \cdots\mu_{2m-2}-\mu_{2m-1} & t^{m-1} \end{vmatrix}$$

then $P_m(t)$ and $\overline{p}_{m-1}(t)$ are the m-th and (m-1)-th orthogonal polynomial over $[0,1]$ with respect to $d\varphi$ and $t(1-t)d\varphi$ respectively, where $\varphi$ is any distribution over $[0,1]$ associated with $\mu$. Furthermore it is shown (cf. Theorem 5.16) that there exists a 1:1:1 continuous correspondence between the points of int $D^{2m-1}$, the ordered pairs of polynomials $(P_m, \overline{p}_{m-1})$ and the open simplex

$$0 < t_1 < w_1 < t_2 < \dots < w_{m-1} < t_m < 1$$

there $t_j$ are the roots of $P_m$ and $w_j$ those of $\overline{p}_{m-1}$. Also the unique supporting planes at $\underline{\mu} = (\mu_1, \dots, \mu_{2m-1}, \underline{\mu}_{2m})$ and $\overline{\mu} = (\mu_1, \dots, \mu_{2m-1}, \overline{\mu}_{2m})$ are the polynomials $P_m^2$ and $t(1-t)\overline{p}_{m-1}^2$ respectively.

These results may be extended to an interval $[0,a]$ with $a > 0$. In fact define

$$\overline{p}_{m-1}(t;a) = \begin{vmatrix} \mu_1 - \frac{\mu_2}{a} & \dots & \mu_{m-1} - \frac{\mu_m}{a} & 1 \\ \mu_2 - \frac{\mu_3}{a} & \dots & \mu_m - \frac{\mu_{m+1}}{a} & t \\ \cdot & \cdot & \cdot & \cdot \\ \mu_m - \frac{\mu_{m+1}}{a} & \dots & \mu_{2m-2} - \frac{\mu_{2m-1}}{a} & t^{m-1} \end{vmatrix}$$

and let $\varphi_a(t)$ be a distribution such that $\varphi_a(0-) = 0$, $\varphi_a(a) = 1$ and $(\mu_1, \dots, \mu_{2m-1}) \in \text{int } D^{2m-1}[0,a]$. Then $\varphi_a(\frac{t}{a}) \equiv \varphi_1(\tau)$ is a distribution over $[0,1]$ and $(\frac{\mu_1}{a}, \dots, \frac{\mu_{2m-1}}{a^{2m-1}}) \in \text{int } D^{2m-1}[0,1]$. The unique supporting planes to $D^{2m-1}[0,1]$ at $(\frac{\underline{\mu}_i}{a^i})$ and $(\frac{\overline{\mu}_i}{a^i})$ are

$$\begin{vmatrix} \mu_0 & \dots & \frac{\mu_{m-1}}{a^{m-1}} & 1 \\ \frac{\mu_1}{a} & \dots & \frac{\mu_m}{a^m} & t \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\mu_m}{a^m} & \dots & \frac{\mu_{2m-1}}{a^{2m-1}} & t^m \end{vmatrix}^2$$

and

$$t(1-t)\begin{vmatrix} \frac{\mu_1}{a}-\frac{\mu_2}{a^2} & \cdots & \frac{\mu_{m-1}}{a^{m-1}}-\frac{\mu_m}{a^m} & 1 \\ \frac{\mu_2}{a^2}-\frac{\mu_3}{a^3} & \cdots & \frac{\mu_m}{a^m}-\frac{\mu_{m+1}}{a^{m+1}} & t \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\mu_m}{a^m}-\frac{\mu_{m+1}}{a^{m+1}} & \cdots & \frac{\mu_{2m-2}}{a^{2m-2}}-\frac{\mu_{2m-1}}{a^{2m-1}} & t^{m-1} \end{vmatrix}^2$$

respectively. Hence the supporting planes to $D^{2m}[0,a]$ at $\mu$ and $\overline{\mu}$, obtained by setting $\tau = at$ and then factoring out the powers $-(1+2+\ldots+m) - (1+2+\ldots+m-1) = -m^2$ and $-(1+2+\ldots+m-1) - (1+2+\ldots+m-1) = m(m-1)$ of $a$, are $a^{-m^2}P_m(\tau)$ and $a^{m(m-1)-1}t(1-\frac{t}{a})\,\overline{p}^{\,2}_{m-1}(\tau;a)$ respectively. The variable used is of course immaterial to the positions of the planes. So are the constant factors, which are important only for the computation of the normalization constants.

If $a \to \infty$ the planes $P_m^2(t)$ and $t(1-\frac{t}{a})\overline{p}^{\,2}_{m-1}(t;a)$ approach the planes $P_m^2(t)$ and $tp^2_{m-1}(t)$ (the former in fact does not change). Hence the homeomorphism between int $D^{2m-1}[0,a]$, $(P_m(t), \overline{p}_{m-1}(t;a))$ and the open simplex

$$0 < t_1 < w_1 < \ldots < w_{m-1} < t_m < a$$

approaches the homeomorphism between int $D^{2m-1}[0,\infty)$, $(P_m(t), p_{m-1}(t))$ and the open simplex

$$0 < t_1 < w_1 < \ldots < u_{m-1} < t_m < \infty \;.$$

Using the representation theorem this may be stated more precisely as follows:

<u>Theorem 7.1</u> Given $\mu = (\mu_1,\ldots,\mu_{2m-1}) \in$ int $D^{2m-1}$ $0,\infty)$ there exists $A > 0$ such that for $a > A$, $\mu \in$ int $D^{2m-1}[0,a]$.

$\mu$ determines uniquely:

1) the roots

$$0 < t_1 < u_1 < \ldots < u_{m-1} < t_m \leq A$$

of the supporting planes to $D^{2m}[0,\infty)$ at $(\mu_1,\ldots,\mu_{2m-1}, \underline{\mu}_{2m})$ and $(\mu_1,\ldots,\underline{\mu}_{2m-1})$

2) for any given $a > A$, the root

$$0 < t_1 < w_1(a) < \ldots < w_{m-1}(a) < t_m \leq A$$

of the supporting planes to $D^{2m}[0,a]$ at $(\mu_1,\ldots,\mu_{2m-1}, \underline{\mu}_{2m})$ and $(\mu_1,\ldots,\mu_{2m-1}, \overline{\mu}_{2m})$. Then

$$\lim_{a \to \infty} w_j(a) = u_j \qquad (j = 1,\ldots,m-1) .$$

# CHAPTER VIII

## SOME RELATIONS BETWEEN THE MOMENTS, THE WEIGHTS AND THE ROOTS.

An inequality relating the moments of a distribution with the weights and the roots of the associated polynomials is obtained by considering the following problem: given $(\mu_1,\ldots,\mu_{2m-1}) \in \text{int } D^{2m-1}(-\infty,\infty)$ determine under what conditions it is possible to assign a root $t_o = t_o^{(m+1)}$ and a weight $\lambda_o = \lambda_o^{(m+1)}$ $(0 < \lambda_o < 1)$ such that there exists a point $(\mu_1,\ldots,\mu_{2m-1},\mu_{2m},\mu_{2m+1}) \in \text{int } D^{2m+1}(-\infty,\infty)$ whose unique convex representation uses the point $x(t_o)$ with weight $\lambda_o$.

This problem may be solved by obtaining a solution of the $2m+2$ equations

$$(8.1a) \qquad \mu_i = \sum_{j=0}^{m} \lambda_j t_j^i \qquad (i = 0,\ldots,2m+1)$$

in the $2m+2$ unknowns $\lambda_j$, $t_j$ $(j = 1,\ldots,m)$, $\mu_{2m}$ and $\mu_{2m+1}$. However if it is possible to obtain $\mu_{2m}$ and $\mu_{2m+1}$ from $(\mu_1,\ldots,\mu_{2m-1})$, $\lambda_o$ and $t_o$, then the existence of a solution for $\lambda_j$ and $t_j$ $(j = 1,\ldots,m)$ is assured by the unique representation theorem.

The point $t_o$ is a root of $P_{m+1}$, hence it must be different from the roots $u_j$ of $P_m$, which are uniquely determined by $\mu_1, \ldots, \mu_{2m-1}$, because the roots of $P_m$ and $P_{m+1}$ must interlock.

If $t_o$ is a root of $P_{m+1}$ and $\varphi(t)$ is a distribution associated with the point $(\mu_1, \ldots, \mu_{2m+1})$, then

$$\int_{-\infty}^{\infty} \frac{P_{m+1}(t)\, P_m(t)}{t - t_o}\, d\varphi(t) = K_{n+1} \int_{-\infty}^{\infty} (t^m + \ldots)\, P_m(t) d\varphi(t)$$

$$= K_{m+1}^2,$$

and by the mechanical quadrature formula

$$\int_{-\infty}^{\infty} \frac{P_{m+1}(t)\, P_m(t)}{t - t_o}\, d\varphi(t) = \sum_{j=0}^{m} \lambda_j \frac{P_{m+1}(t_j)\, P_m(t_j)}{t_j - t_o}$$

$$= \lambda_o^{(m+1)}\, P'_{m+1}(t_o)\, P_m(t_o),$$

since the $t_j$ are the roots of $P_{m+1}$. Hence

$$\lambda_o^{(m+1)} = \frac{K_{m+1}^2}{P'_{m+1}(t_o)\, P_m(t_o)} = \frac{K_{m+1}^2}{P'_{m+1}(t_o)P_m(t_o) - P_{m+1}(t_o)P'_m(t_o)}$$

which by [3] eq. (3.2.4) reduces to

$$\text{(8.1b)} \qquad \lambda_o^{(m+1)} = \frac{1}{\sum_{i=0}^{m} \frac{P_i^2(t_o)}{K_i K_{i+1}}}$$

where $K_o = 1$. Let $\nu = \mu_{2m} - \underline{\mu}_{2m}$. Then the only term in the denominator in (8.1b) which contains $\nu$ is

$$K_{m+1} = \begin{vmatrix} \mu_o & \mu_1 & \cdots & \mu_m \\ \mu_1 & \mu_2 & \cdots & \mu_{m+1} \\ \cdot & \cdot & \cdots & \cdot \\ \mu_m & \mu_{m+1} & \cdots & \mu_{2m} \end{vmatrix} + \begin{vmatrix} \mu_o & \mu_1 & \cdots \mu_{m-1} & 0 \\ \mu_1 & \mu_2 & \cdots \mu_m & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \mu_m & \mu_{m+1} & \cdots \mu_{2m-1} & \nu \end{vmatrix} = K_m \nu .$$

Hence

$$(8.1c) \quad \lambda_o^{(m+1)} = \frac{1}{\sum_{i=0}^{m-1} \frac{P_i^2(t_o)}{K_i K_{i+1}} + \frac{P_m^2(t_o)}{K_m^2 \nu}} = \frac{1}{\frac{1}{\lambda_o^{(m)}} + \frac{P_m^2(t_o)}{K_m^2 \nu}}$$

which gives

$$(8.1d) \quad \nu = \frac{\alpha_m \lambda_o^{(m+1)}}{1 - \frac{\lambda_o^{(m+1)}}{\lambda_o^{(m)}}} \quad \text{where} \quad \alpha_m = \frac{P_m^2(t_o)}{K_m^2} \quad .$$

$\lambda_o^{(m)}$ is a known function of $\mu_1, \ldots, \mu_{2m-2}$ and $t_o$, and

$$\frac{1}{\lambda_o^{(m)}} = 1 + \sum_{i=1}^{m-1} \frac{P_i^2(t_o)}{K_i K_{i+1}} \geq 1 \quad ;$$

if $t_o \neq u_j$ the equal sign holds only if $m = 1$. Furthermore $\alpha_m \geq 0$ and the inequality is strict if $t_o \neq u_j$ . Hence, if $0 < \lambda_o^{(m+1)} < \lambda_o^{(m)}$ and $t_o \neq u_j$, (8.1d) gives

$$\frac{d\nu}{d\lambda_o^{(m+1)}} = \frac{\alpha_m}{\left(1 - \frac{\lambda_o^{(m+1)}}{\lambda_o^{(m)}}\right)^2} > 0 \, ,$$

therefore $\nu$ is a monoton increasing function of $\lambda_o^{(m+1)}$

which takes all the values between $0$ and $\infty$ as $\lambda_o^{(m+1)}$ varies between $0$ and $\lambda_o^{(m)}$. If $\lambda_o^{(m+1)} > \lambda_o^{(m)}$ then $\nu < 0$, hence $(\mu_1, \ldots, \mu_{2m})$ is not a moment sequence. If $t_o$ equals one of the $u_j$ then $\nu$ is not defined and (8.1c) shows that $\lambda_o^{(m+1)}$ does not depend on $\nu$ and equals $\lambda_o^{(m)}$. These results are illustrated in Figure 8.1, which shows a $t_o - \nu - \lambda_o^{(m+1)}$ diagram.

The value of $\mu_{2m+1}$ may be obtained from the fact that

$$P_{m+1}(t_o) \quad \begin{vmatrix} \mu_o & \cdots & \mu_m & 1 \\ \mu_1 & \cdots & \mu_{m+1} & t_o \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{m+1} & \cdots & \mu_{2m+1} & t_o^{m+1} \end{vmatrix} \equiv \Pi_{m+1}(t_o) - \mu_{2m+1} P_m(t_o) = 0.$$

where $\Pi_{m+1}(t_o)$ is a polynomial whose coefficient do not depend on $\mu_{2m+1}$. Solving for $\mu_{2m+1}$ this gives

$$(8.1e) \quad \mu_{2m+1} = \frac{\Pi_{m+1}(t_o)}{P_m(t_o)} \quad .$$

The graph of $\mu_{2m+1}$ as a function of $t_o$ is shown in Figure 8.2. In fact for any given value of $\mu_{2m+1}$, $\Pi_{m+1}(t_o) - \mu_{2m+1} P_m(t_o)$ has $m+1$ simple roots which interlock with those of $P_m(t)$. Hence $\frac{\Pi_{m+1}(t_o)}{P_m(t_o)}$ has simple poles at the points $u_j$. Furthermore the leading coefficients of $\Pi_{m+1}$ and $P_m$ are $K_{m+1}$ and $K_m$ respectively, which are both positive.

These results may be summarized in the following

<u>Theorem 8.1</u> Given $(\mu_1,\ldots,\mu_{2m-1}) \in \text{int } D^{2m-1}$ and any number $t_o$ define

$$\lambda_o^{(m)} = \lambda_o^{(m)}(\mu_1,\ldots,\mu_{2m-2}, t_o) = \frac{1}{\sum_{i=0}^{m} \frac{P_i^2(t_o)}{K_i K_{i+1}}} .$$

Denote the roots of $P_m$ by $u_j$. Then there exist numbers $\mu_{2m}$ and $\mu_{2m+1}$ such that $(\mu_1,\ldots,\mu_{2m+1})$ is a point of int $D^{2m+1}$ having a root at $t_o$ of given weight $\lambda_o^{(m+1)}$ if and only if $t_o \neq u_j$ and $0 < \lambda_o^{(m+1)} < \lambda_o^{(m)}$. The values of $\mu_{2m} = \underline{\mu}_{2m} + \nu$ and $\mu_{2m+1}$ are then uniquely determined by (8.1d) and (8.1e).

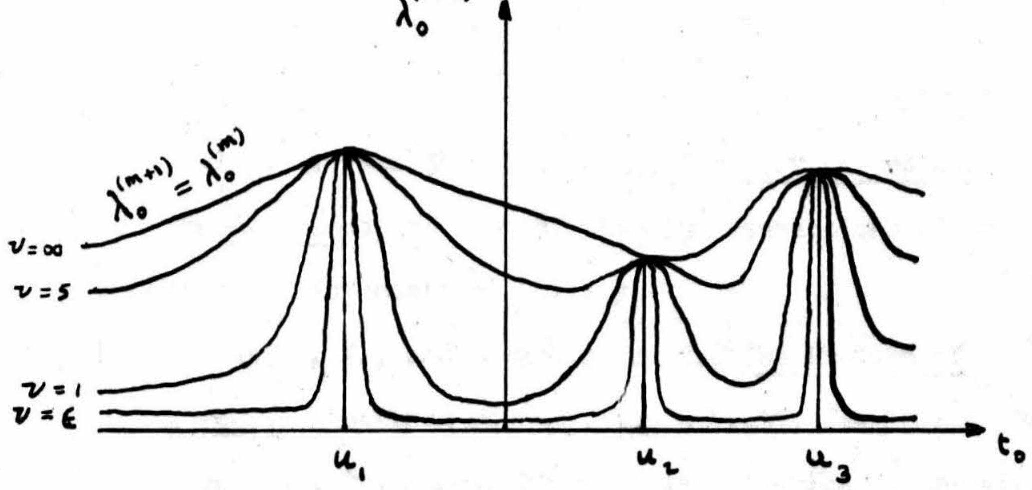


Figure 8.1

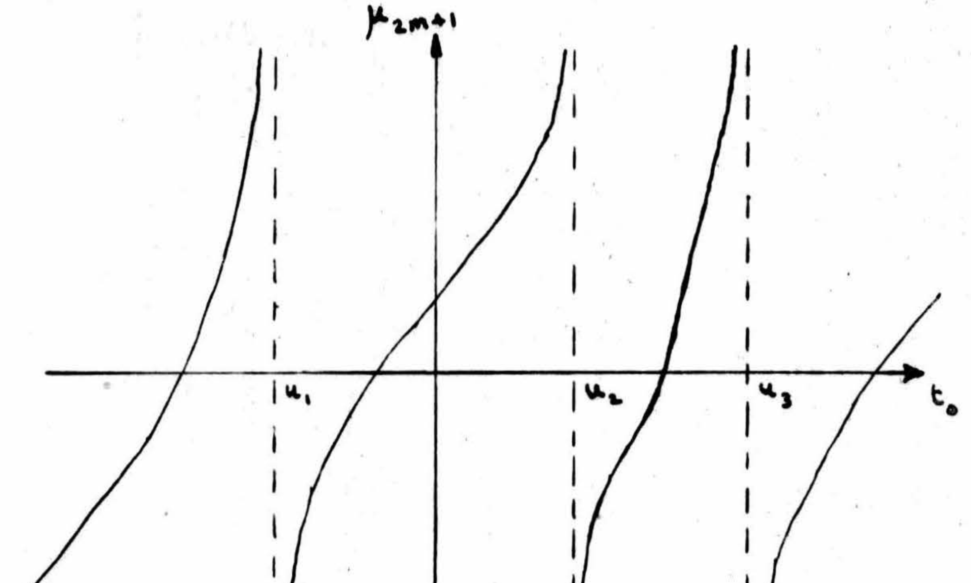


Figure 8.2

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