METRIC THEORIES OF GRAVITY
AND THEIR ASTROPHYSICAL IMPLICATIONS

Thesis by
Wei-Tou Ni

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology

1973

(Submitted September 11, 1972)

ACKNOWLEDGEMENTS

I am especially indebted to Professor Kip S. Thorne, whose advice, encouragement and guidance made this thesis possible. I wish to express my deep gratitude to Professor Jon Mathews for encouragement and advice during the early stages of my graduate study. I would like to thank Clifford M. Will for helpful discussions. I am obliged to many friends, my family, and my high school advisor for their encouragements and understanding. The work reported here was performed in part while I was a Rand Corporation Fellow. I thank the National Science Foundation (GP-19887, GP-27304, GP-28027) and the National Aeronautics and Space Administration Caltech/-JPL Contract No. NAS 7-100 (188-41-54-02-01), and NGR 05-002-256 for financial support.

ABSTRACT

The increasing importance of relativistic gravity in astrophysics has led to the need for a detailed analysis of theories of gravity and their viability. Accordingly, in this thesis, metric theories of gravity are compiled, and are classified into four groups: (i) general relativity (ii) scalar-tensor theories (iii) conformally flat theories and (iv) stratified theories. The post-Newtonian limit of each theory is constructed and its Parametrized Post-Newtonian (PPN) values are obtained. These results, when combined with experimental data and with recent work by Nordtvedt and Will, show that, of all theories thus far examined by our group, the only currently viable ones are (i) general relativity, (ii) the Bergmann-Wagoner scalar-tensor theory and its special cases (Nordtvedt; Brans-Dicke-Jordan), (iii) recent, (as yet unpublished) vector-tensor theory by Nordtvedt, Hellings, and Will, and (iv) a new stratified theory by the author, which is presented for the first time in this thesis.

The PPN formalism is used to analyze stellar stability in any metric theory of gravity. This analysis enables one to infer, for any given gravitation theory, the extent to

which post-Newtonian effects induce instabilities in white dwarfs, in neutron stars, and in supermassive stars. It also reveals the extent to which our current <u>empirical</u> knowledge of post-Newtonian gravity (based on solar-system experiments) actually guarantees that relativistic instabilities exist. In particular, it shows that for "conservative theories of gravity", current solar-system experiments guarantee that relativistic corrections do induce dynamical instabilities in stars with adiabatic indices slightly greater than 4/3, while for "non-conservative theories", current experiments do not permit any firm conclusion.

TABLE OF CONTENTS

PART ONE

INTRODUCTION

Astronomical discoveries and observations in the past decade have forced astrophysicists to incorporate relativistic gravity into their theoretical model building -- models for the cosmic microwave radiation, QSO's, pulsars, and sources of gràvitational waves are all constrained by or involve relativistic gravity in a fundamental way. In building models, astrophysicists usually use general relativity as a working tool. But experiment has not yet told us which gravitational theory is correct -- general relativity, the Brans-Dicke theory, the Nordtvedt-Hellings-Will theory, or a theory which nobody has yet constructed.

Before the correct gravitational theory is found, it is desirable not to limit one's attention only to a particular theory. All currently viable theories of gravity should be catalogued and their astrophysical implications be looked into; by studying all currently viable theories of gravity, firmer astrophysical implications can be inferred; by looking into astrophysical implications, viable theories might be limited. This has motivated us to compile theories of gravity, to study their features and to examine their astrophysical implications.

In the course of compiling theories of gravity, our group has found that the currently viable ones are all metric theories. Hence in this thesis, we will restrict our attention to metric theories of gravity. The manner in which stability criteria for stars depend on the choice of gravitation theory is of considerable astrophysical import. Therefore, in addition to cataloguing metric theories, calculating their post-Newtonian limits, analyzing their viability, and proposing a new viable theory of gravity of our own, in this thesis we will also find criteria for stellar stability in any metric theory of gravity.

In Part Two, metric theories of gravity are compiled and classified according to the types of gravitational fields they contain, and the modes of interaction among those fields. The gravitation theories considered are classified as (i) general relativity (ii) scalar-tensor theories (iii) conformally flat theories and (iv) stratified theories with conformally flat space slices. The post-Newtonian limit of each theory is constructed and its PPN parameter values are obtained by comparing it with Will's version of the Parametrized Post-Newtonian formalism. Results obtained here, when combined with experimental data and with recent work by Nordtvedt and Will, show that, of all theories thus far examined by our group, the only currently viable ones are general relativity, the Bergmann-Wagoner scalar-tensor theory and its special cases (Nordtvedt; Brans-Dicke-Jordan), and new, unpublished theories by Nordtvedt, Hellings, and Will, and by the author.

Part Three presents our new, viable theory of gravity. This theory agrees with all experiments to date. It is a metric theory; it is Lagrangian-based; and it possesses a preferred frame with conformally flat space slices. With an appropriate choice of certain adjustable functions and parameters, this theory possesses precisely the same post-Newtonian limit as general relativity!

In Part Four, the "PPN formalism" -- which encompasses the post-Newtonian limit of nearly every metric theory of gravity-- is used to analyze stellar stability. This analysis enables one to infer, for any given gravitation theory, the extent to which post-Newtonian effects induce instabilities in white dwarfs, in neutron stars, and in supermassive stars. It also reveals the extent to which our current <u>empirical</u> knowledge of post-Newtonian gravity (based on solar-system experiments) actually guarantees that relativistic instabilities exist. In particular, it shows that: (i) for "conservative theories of gravity", current solar-system experiments guarantee that the critical adiabatic index, Γ_{crit}, for the stability of stars against radial pulsations exceeds the Newtonian value of 4/3:

$$\Gamma_{crit} = 4/3 + 2KM/R, \quad K \text{ positive and of order unity;}$$

(ii) for "nonconservative theories", current experiments do *not* permit any firm conclusion about the sign of $\Gamma_{crit} - 4/3$;

(iii) in the PPN approximation to <u>every</u> metric theory, the standard Schwarzschild criterion for convection is valid.

PART TWO
A COMPENDIUM OF METRIC THEORIES OF GRAVITY AND THEIR POST-NEWTONIAN LIMITS

(The Astrophysical Journal,
176:769-796,1972 September 15)

7

THE ASTROPHYSICAL JOURNAL, 176:769–796, 1972 September 15

THEORETICAL FRAMEWORKS FOR TESTING RELATIVISTIC GRAVITY. IV. A COMPENDIUM OF METRIC THEORIES OF GRAVITY AND THEIR POST-NEWTONIAN LIMITS*

WEI-TOU NI
California Institute of Technology
Received 1971 December 29; revised 1972 April 12

ABSTRACT

Metric theories of gravity are compiled and classified according to the types of gravitational fields they contain, and the modes of interaction among those fields. The gravitation theories considered are classified as (i) general relativity, (ii) scalar-tensor theories, (iii) conformally flat theories, and (iv) stratified theories with conformally flat space slices. The post-Newtonian limit of each theory is constructed and its Parametrized Post-Newtonian (PPN) values are obtained by comparing it with Will's version of the formalism.

Results obtained here, when combined with experimental data and with recent work by Nordtvedt and Will and by Ni, show that, of all theories thus far examined by our group, the only currently viable ones are general relativity, the Bergmann-Wagoner scalar-tensor theory and its special cases (Nordtvedt; Brans-Dicke-Jordan), and a recent, new vector-tensor theory by Nordtvedt, Hellings, and Will.

I. INTRODUCTION

In Paper I of this series, Thorne and Will (1971) described the theoretical and experimental foundation for "metric theories of gravity," and discussed qualitative aspects of the "Parametrized Post-Newtonian" (PPN) formalism. In Paper II, Will (1971*a*) used arbitrary metric parameters to generalize Chandrasekhar's (1965) post-Newtonian equations of hydrodynamics so that they encompass the post-Newtonian limits of most metric theories of gravity. The result was the Parametrized Post-Newtonian (PPN) formalism, which is based on earlier work by Eddington (1922), Robertson (1962), Schiff (1967), Baierlein (1967), and Nordtvedt (1968), and which was, in some sense, a fluid version of Nordtvedt's (1968) point-mass PPN formalism. As an application, in Paper II, Will used his PPN formalism to rederive the Nordtvedt effect (the breakdown in the Equivalence Principle for massive bodies), which was discovered earlier by Nordtvedt (1968, 1969) using his point-mass PPN formalism, and by Dicke (1969) in Brans-Dicke theory. In Paper III, Will (1971*b*) proved that metric theories which have post-Newtonian integral conservation laws for energy, momentum, angular momentum, and center-of-mass motion must satisfy a set of seven constraints on their PPN parameters. He also showed that the standard PPN form of the post-Newtonian metric of any theory of gravity is invariant under a post-Galilean transformation if and only if its PPN parameters Δ_1, Δ_2, β_1, and ζ satisfy a set of three constraints, which constitute a proper subset of the seven "conservation constraints." (See also Nordtvedt 1969.)

Our group at Caltech (Thorne, Will, Ni, and several new students) is now compiling a list of twentieth-century theories of gravity, both metric and nonmetric. Of each theory we ask the following questions: (i) *Is it self-consistent?* (It would not be self-consistent, for example, if a photon calculation predicted a different redshift from an electromag-

* Supported in part by the National Science Foundation [GP-28027 and GP-27304] and the National Aeronautics and Space Administration Caltech/JPL contract NAS 7-100 [188-41-54-02-01], and NGR 05-002-256.

netic-wave calculation.) (ii) *Is it complete?*—i.e., is it capable of analyzing from "first principles" the outcome of every experiment of interest? (Of course, a theory is not complete unless it meshes with and incorporates a consistent set of electromagnetic laws, quantum-mechanical laws, etc.) (iii) *Does it agree, to within several standard deviations, with all experiments performed to date?* We consider a theory of gravity *viable* only if it answers "yes" to all three questions.

When we convince ourselves that a theory is nonviable, we usually cease to examine it further. Thus, viability is a filter in our research. A review article (Thorne, Will, and Ni 1971) describes more precisely how this filter works, and lists a number of theories that have been caught in our filter and have been discarded.

The purpose of this paper (Paper IV in our series) is to examine in some detail every theory that has thus far passed successfully through our viability filter. It turns out that these "currently viable" theories are all metric theories and that one can analyze them by using the PPN formalism. For comparison this paper also analyzes metric theories that are self-consistent and complete but that fail to agree with experiment.

As a prerequisite to the analysis of explicit theories, § II of this paper briefly reviews the concept of "metric theory" and several related topics.

Section III analyzes the theories one by one. For each theory § III (i) states the fundamental postulates and equations, (ii) takes the theory's post-Newtonian limit and compares it with Will's version of the PPN formalism to extract the PPN parameters, (iii) compares the PPN parameters with current experimental limits to decide whether the theory is experimentally viable, and (iv) compares with the projected capabilities of future experiments to determine the prospects for testing the theory in the coming decade. The two theories in §§ III*D*(vii) and III*D*(viii) are new "stratified theories" invented recently by the author. Of all the theories catalogued in this paper, and of all other theories studied thus far by our group (see Thorne *et al.* 1971) the only viable ones are general relativity, scalar-tensor theories of the Dicke-Bergmann type, and a recent new vector-tensor theory by Nordtvedt, Hellings, and Will (cf. Will and Nordtvedt 1972). This conclusion is based on the results of this paper, combined with new experimental tests to be described in forthcoming papers by Nordtvedt and Will (1972) and by Ni (1972*a*).

Section IV summarizes the results of the analysis and makes a few concluding remarks.

An overview of all the results of Papers I–IV, and of detailed PPN analyses of various relativity experiments, is contained in the review paper (Thorne *et al.* 1971).

II. METRIC THEORIES—GENERAL REMARKS

A theory of gravity is a "metric theory" if and only if it can be given a mathematical representation in which two conditions hold (cf. Paper I).

Condition i. *There exists a metric of signature* -2, *which governs proper-length and proper-time measurements in the usual manner of special and general relativity:*

$$ds^2 = g_{ij}dx^i dx^j . \tag{1}$$

Condition ii. *Stressed matter and fields being acted upon by gravity respond in accordance with the equation*

$$\nabla \cdot T = 0 , \tag{2}$$

where T *is the total stress-energy tensor for all matter and nongravitational fields, and where* ∇ *is the covariant derivative with respect to the metric.*

Any metric theory of gravity can perfectly well be given a mathematical representation that violates conditions i and ii. For example, the Brans-Dicke theory, in the mathematical representation of Dicke (1962), does *not* satisfy either condition. Dicke's scalar field causes deviations from geodesic motion, and physical rods and clocks do not measure $ds^2 = g_{ij}dx^i dx^j$. However, in the original mathematical representation of Brans and Dicke (1961) the theory satisfies both conditions, so it is a metric theory.

Notice that, in the "canonical representation" of a metric theory, where conditions i and ii are satisfied, the metric is the only gravitational field which enters into the response equation $\nabla \cdot T = 0$. (The metric alone determines ∇, and T contains no gravitational fields.) This does not mean that the metric is the only gravitational field present. On the contrary, as in Brans-Dicke theory, there may be other fields. However, the role of the other fields can only be that of helping to generate the spacetime curvature associated with the metric. Matter may create other fields, and they plus matter may create the curvature, but they cannot act back directly on the matter. The matter responds only to the metric.

The metric theories in the compendium of § III are classified according to the types of gravitational fields they contain, and the mode of interaction among those fields. First (§ III*A*) comes the only theory that possesses just one gravitational field, the metric. Of course, this theory is *general relativity*. Then (§ III*B*) come *"scalar-tensor theories"*—theories in which the matter and nongravitational fields, acting via a wave equation in curved spacetime, generate a scalar gravitational field φ, and then φ acts together with the matter and fields, via a wave equation, to generate the metric. Among the scalar-tensor theories is that of Brans and Dicke (1961). Next (§ III*C*) come *conformally flat theories*—theories with a flat "background metric" $\mathfrak{n}$, a scalar field φ, and a conformally flat physical metric g. In these theories φ is generated by matter and nongravitational fields via a flat-space wave equation, and then φ and the flat-space metric $\mathfrak{n}$ generate the physical metric via the conformal relation

$$g = e^{-2f(\varphi)}\mathfrak{n} . \tag{3}$$

Conformally flat theories all predict zero light deflection and therefore disagree with experiment; they are included here primarily for their historical interest and as foils against which to compare other, viable theories. Section III*D* treats *stratified theories with conformally flat space slices*. Such theories, like Newtonian spacetime,[1] possesses a universal time coordinate t (which can be thought of as an *a priori* scalar field with timelike gradient), a flat-space background metric $\mathfrak{n}$, a scalar gravitational field φ, and the physical metric g. The field φ is generated by matter and nongravitational fields via a wave-type equation; and it then combines algebraically with t and $\mathfrak{n}$ to generate the physical metric g—e.g., by the equation

$$g = [e^{-2f(\varphi)} - e^{2f(\varphi)}]dt \otimes dt + e^{2f(\varphi)}\mathfrak{n} . \tag{4}$$

The 3-surfaces of constant t ("strata") are conformally flat in these theories. In general relativity and in scalar-tensor theories, the matter response equation follows from the field equations. But in conformally flat theories and in stratified theories with conformally flat space slices, the matter response equation does not follow from the field equations; it must be postulated separately.

Some of the theories analyzed in § III were incomplete in their original formulations. The author has completed them by making minor modifications. Completeness is achieved, for all theories analyzed in this paper, by invoking the equivalence principle: one postulates that in the local Lorentz frames of the physical metric all the nongravitational laws of physics take on their standard special relativistic forms. Of course, this guarantees that the two criteria for a metric theory are satisfied (atomic clocks and physical rods measure the proper time and length of g; and the nongravitational stress-energy tensor is divergence free); it also guarantees that test particles move along geodesics of g.

To calculate the PPN parameters for each theory in § III, we must first construct the theory's post-Newtonian limit, and then compare with the metric of the PPN formalism. (It turns out that the post-Newtonian limits of all these theories can be encompassed in

[1] See, e.g., chapter 12 of Misner, Thorne, and Wheeler (1972).

Will's version of the PPN formalism.) All the theories in §§ III*A*, *B*, and *C* satisfy the three constraints on PPN parameters for post-Galilean invariance; therefore, in the post-Newtonian calculations, the results do not depend upon which post-Galilean frame we have chosen. None of the theories in § III*D* are post-Galilean invariant; they possess a "preferred" frame. The post-Newtonian limits of these theories in the "preferred" frame fall into Will's (1971*a*) PPN formalism and hence by comparison one can extract their PPN parameters. Then one can use the procedure of Will and Nordtvedt (1972) to get the post-Newtonian limit in any other post-Galilean frame. For easy comparison and reference, we here write down Will's (1971*a*) PPN metric:

$$g_{00} = 1 - 2U + 2\beta U^2 - 4\Phi + \zeta\mathcal{A}\,, \quad g_{0\alpha} = \tfrac{7}{2}\Delta_1 V_\alpha + \tfrac{1}{2}\Delta_2 W_\alpha\,,$$

$$g_{\alpha\beta} = -(1 + 2\gamma U)\delta_{\alpha\beta}\,, \tag{5}$$

where

$$U(x, t) = \int \frac{\rho(x', t)}{|x - x'|}\,dx'\,, \quad \Phi(x, t) = \int \frac{\rho(x', t)\varphi(x', t)}{|x - x'|}\,dx'\,,$$

$$\varphi = \beta_1 v^2 + \beta_2 U + \tfrac{1}{2}\beta_3\Pi + \tfrac{3}{2}\beta_4 p/\rho\,, \quad \mathcal{A}(x, t) = \int \frac{\rho(x', t)[(x_\alpha - x'_\alpha)v_\alpha(x')]^2}{|x - x'|^3}\,dx'\,,$$

$$V_\alpha(x, t) = \int \frac{\rho(x', t)v_\alpha(x')}{|x - x'|}\,dx'\,,$$

$$W_\alpha(x, t) = \int \frac{\rho(x', t)v_\beta(x')(x_\beta - x'_\beta)(x_\alpha - x'_\alpha)}{|x - x'|^3}\,dx'\,. \tag{6}$$

Here ρ is rest-mass density, p is pressure, and Π is specific internal energy all measured in the matter's rest frame, and $v_\alpha = dx^\alpha/dt$ is the matter's coordinate velocity. The PPN parameters are γ, β, β_1, β_2, β_3, β_4, Δ_1, Δ_2, and ζ. Experiments to date place the following limits on the PPN parameters (see Thorne *et al.* 1971 for detailed discussion; see also Will and Nordtvedt 1972, Nordtvedt and Will 1972, and Ni 1972*a*):

$\gamma = 1.04 \pm 0.08$ (time delay and deflection experiments except that of Sramek 1971),

$\gamma = 0.80 \pm 0.10$ (Sramek's 1971 deflection experiment),

$\beta = 1.14(+0.2, -0.3)$ (perihelion shift plus time-delay experiments),

$|2\beta_1 - \beta_4 - 1 - \tfrac{1}{6}\zeta| \lesssim 0.4$ (Kreuzer measurement of $m_{\text{active}}/m_{\text{passive}}$),

$|\beta_3 - 1| \lesssim 0.05$ (Kreuzer measurement of $m_{\text{active}}/m_{\text{passive}}$),

$|7\Delta_1 + \Delta_2 - 4\gamma - 4| \lesssim 0.2$ (Earth rotation rate experiments [Nordtvedt and Will 1972]);

$|7\Delta_1 + \Delta_2 - 4\gamma - 4| \lesssim 10^{-4}$ (stability observations of white dwarfs [Ni 1972*a*]);[2]

$|\Delta_2 + \zeta - 1| \lesssim 0.03$ (Earth-tide measurements [Will 1971*b*]);

$|\Delta_2 + \zeta - 1| \lesssim 10^{-4}$ (stability observations of white dwarfs [Ni 1972*a*]);[3]

$|4\beta_1 - 2\gamma - 2 - \zeta| \lesssim 2 \times 10^{-5}$ (perihelion shift observations [Nordtvedt and Will 1972]);

$|4\beta_1 - 2\gamma - 2 - \zeta| \lesssim 10^{-4}$ (stability observations of white dwarfs [Ni 1972*a*]).[4] (7)

[2] The validity of these limits can be questioned. They rely on experimentally untested ideas about the forces which damp pulsations in white-dwarf stars; see Ni (1972*a*).

[3] See n. 2.

[4] See n. 2.

III. A PARTIAL CATALOG OF METRIC THEORIES OF GRAVITY

Conventions.—Throughout this paper, we use geometrized units and the sign conventions recommended by Misner, Thorne, and Wheeler (open letter to relativity theorists, September 1968), with signature -2.

Notation.—(i) $\boldsymbol{\eta}$ = a "background" Lorentz metric, whose existence is postulated by the theory. (ii) φ = a scalar gravitational field, which is generated by stress-energy, which helps to generate the physical metric, but which does not act back directly on matter or non-gravitational fields. (iii) t = a scalar field, which plays the role of a preferred, universal time coordinate. (iv) $\boldsymbol{g}$ = the physical metric, which governs clock rates and rod lengths, and to which matter responds via $\nabla \cdot \boldsymbol{T} = 0$. (v) $\boldsymbol{T}$ = the total stress-energy tensor for all matter and nongravitational fields. (vi) ρ^* = the gravitational source density which is defined either as $T_{ij}u^i u^j$, where $\boldsymbol{u}$ is the 4-velocity of the source, or as trace ($\boldsymbol{T}$).

A) General Relativity

a. Original formulation: Einstein (1916).

b. Principal subsequent references: Standard textbooks and references therein, e.g., Synge (1960), and Misner *et al.* (1972).

c. Gravitational fields present: $\boldsymbol{g}$.

d. Arbitrary parameters and functions: Cosmological constant Λ, which is known to be so small that it cannot be measured in the solar system; it will therefore be set to zero in this paper.

e. Field equations:

$$G_{ij} = 8\pi T_{ij} . \tag{8}$$

f. PPN parameters:

$$\gamma = \beta = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \Delta_1 = \Delta_2 = 1 , \quad \zeta = 0 .$$

g. Comparison with experiment: Agrees with all experiments to date.

h. Discussion: The post-Newtonian limit of general relativity was calculated by Chandrasekhar (1965), and the PPN parameters were designed by Will (1971*a*) in such a way that in general relativity they would each be zero or unity.

B) Scalar-Tensor Theories

i) *Bergmann-Wagoner Theory:* Bergmann's (1968) theory as modified and completed by Wagoner (1970)

a. Original formulation: Bergmann (1968); Wagoner (1970).

b. Gravitational fields present: φ, $\boldsymbol{g}$.

c. Arbitrary parameters and functions: Two arbitrary functions of φ, the "coupling function" $\omega(\varphi)$ and the cosmological function $\lambda(\varphi)$. The function $\lambda(\varphi)$ is known to be so small that it cannot be measured in the solar system. It will therefore be dropped in this paper. In the post-Newtonian limit (without cosmological function), there are two arbitrary parameters, ω = [value of $\omega(\varphi)$ far outside solar system], and Λ = [value of $(d\omega/d\varphi)(4 + 2\omega)^{-1}(3 + 2\omega)^{-2}$ far outside solar system].

d. Field equations in canonical representation of the theory (for a derivation of this representation from the noncanonical representations of Bergmann and Wagoner, see § *g* below):

$$R_{ij} - \tfrac{1}{2}Rg_{ij} - \lambda(\varphi)g_{ij} = \frac{8\pi}{\varphi}T_{ij} + \frac{\omega(\varphi)}{\varphi^2}(\varphi_{,i}\varphi_{,j} - \tfrac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}) + \varphi^{-1}(\varphi_{,i;j} - g_{ij}\Box\varphi) \tag{9}$$

$$\Box\varphi + \frac{2\varphi^2\lambda'(\varphi) - 2\varphi\lambda(\varphi)}{3 + 2\omega(\varphi)} = \frac{8\pi}{3 + 2\omega(\varphi)}T - \frac{\omega'(\varphi)}{3 + 2\omega(\varphi)}\varphi_{,i}\varphi^{,i} \tag{10}$$

e. PPN parameters:

$$\gamma = \frac{1+\omega}{2+\omega}, \quad \beta = 1 + \Lambda, \quad \beta_1 = \frac{3+2\omega}{4+2\omega}, \quad \beta_2 = \frac{1+2\omega}{4+2\omega} - \Lambda, \quad \beta_3 = 1,$$

$$\beta_4 = \frac{1+\omega}{2+\omega}, \quad \zeta = 0, \quad \Delta_1 = \frac{10+7\omega}{14+7\omega}, \quad \Delta_2 = 1,$$

where

$$\Lambda = \left[\frac{d\omega/d\varphi}{(4+2\omega)(3+2\omega)^2}\right]_{\text{far outside solar system}},$$

$$\omega = [\omega(\varphi)]_{\text{far outside solar system}}.$$

f. Comparison with experiment: Agrees with time-delay and deflection experiments except for Sramek (1971): to 1σ accuracy if $\omega > 23$; to 2σ accuracy if $\omega > 6$. Agrees with Sramek's deflection experiment: to 1σ accuracy if $8 > \omega > 4/3$; to 2σ accuracy if $\omega > \frac{1}{2}$. Agrees with perihelion-shift plus time-delay experiments: to 1σ accuracy if $-0.16 < \Lambda < 0.34$; to 2σ accuracy if $-0.46 < \Lambda < 0.64$. Agrees completely with gravimeter and Kreuzer experiments. Future experiments should concentrate on pushing ω toward ∞ and Λ toward 0 (general-relativity limit) or on determining ω and Λ. This is best done by experiments of highest precision—time delay, light deflection, and perihelion shift experiments.

g. Derivation and discussion of the above results: Bergmann's (1968) original paper dealt with an electromagnetic field F_{ij} and gravitational fields (metric $\hat{g}_{ij}$ and scalar field $\hat{\varphi}$); but it omitted all reference to matter. Bergmann assumed a Lagrangian density of the form

$$\mathcal{L} = (-\hat{g})^{1/2}[f_1(\hat{\varphi})\hat{R} + f_2(\hat{\varphi})M + f_3(\hat{\varphi})\hat{g}^{ij}\hat{\varphi}_{,i}\hat{\varphi}_{,j} + f_4(\hat{\varphi})], \tag{11}$$

where $\hat{R}$ is the curvature scalar formed from $\hat{g}_{ij}$, M is the Maxwell scalar formed from F_{ij}, and f_i's are arbitrary functions. The most straightforward way to insert matter and other fields into Bergmann's theory is to assume a Lagrangian density of the form

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_I. \tag{12}$$

Here

$$\mathcal{L}_G \equiv (-\hat{g})^{1/2}[f_1(\hat{\varphi})\hat{R} + f_3(\hat{\varphi})\hat{g}^{ij}\hat{\varphi}_{,i}\hat{\varphi}_{,j} + f_4(\hat{\varphi})], \tag{13}$$

is the vacuum gravitational Lagrangian; and $\mathcal{L}_I$ is the interaction Lagrangian, which includes the mutual coupling of the gravitational fields with all matter and nongravitational fields, and which is a function of $\hat{\varphi}$, $\hat{g}^{ij}$, F_{ij}, and all matter and other field variables.

The high-precision null results of the Eötvös-Dicke-Braginsky experiments make it plausible (and perhaps essential) to postulate that all matter and nongravitational fields couple to the gravitational field in the same way (Wagoner's 1970 principle of mutual coupling):

$$\mathcal{L}_I = \mathcal{L}_I[\hat{\psi}^2(\hat{\varphi})\hat{g}_{ij}, \quad \text{matter and nongravitational field variables}]. \tag{14}$$

Here $\hat{\psi}(\hat{\varphi})$ is a new arbitrary function of the scalar field $\hat{\varphi}$. This form of the Lagrangian guarantees that matter and nongravitational fields will satisfy the equation of motion

$$\nabla \cdot T = 0, \tag{15}$$

where ∇ is the covariant derivative with respect to the conformally transformed metric

$$g_{ij} \equiv \hat{\psi}^2(\hat{\varphi})\hat{g}_{ij}, \tag{16}$$

and where T is the stress-energy tensor obtained by varying the Lagrangian in the usual way

$$T^{ij} = \frac{1}{8\pi}(-g)^{-1/2}\frac{\partial \mathcal{L}_I}{\partial g_{ij}}. \tag{17}$$

(For proof, follow the general method in § 94 of Landau and Lifshitz 1962.) Consequently, if the Bergmann-Wagoner theory is a metric theory, the metric of its canonical representation must be g_{ij}, not $\hat{g}_{ij}$. To ensure that g_{ij} is indeed the physical metric, we shall postulate further that $\mathcal{L}_I(g_{ij}$, matter and nongravitational field variables) has the standard special relativistic form. Notice that this postulate is equivalent to the equivalence principle. A local Lorentz system in which the laws of special relativity and electromagnetism hold can then be constructed; and this implies that g_{ij} is the physical metric which governs proper-length and proper-time measurements. The derivation of these results is exactly the same as in general relativity and will be omitted here. With these assumptions, the resulting theory satisfies both postulates for metric theories and becomes a metric theory.

Before transforming to the canonical representation, let us notice that Wagoner's completion of the Bergmann theory is not merely a completion; it is also a specialization. To see this, rewrite Bergmann's Lagrangian (11) in Wagoner's form (12), (13):

$$\mathcal{L} = (-\hat{g})^{1/2}[f_1(\hat{\varphi})\hat{R} + f_3(\hat{\varphi})\hat{g}^{ij}\hat{\varphi}_{,i}\hat{\varphi}_{,j} + f_4(\hat{\varphi})] + \mathcal{L}_I \tag{18}$$

where

$$\mathcal{L}_I = +(-\hat{g})^{1/2}\hat{g}^{ik}\hat{g}^{jl}(A_{k,l} - A_{l,k})(A_{i,j} - A_{j,i})f_2(\hat{\varphi}). \tag{19}$$

This expression is invariant under conformal transformations $\tilde{g}_{ij} = \chi^2(\hat{\varphi})\hat{g}_{ij}$. Therefore, unless $f_2(\hat{\varphi})$ is a constant, we cannot put it into Wagoner's mutual coupling form (14). Hence Wagoner's completion of Bergmann's theory necessarily specializes the content of the theory. However, we see no other way to complete the theory and still be sure of satisfying the Eötvös-Dicke-Braginsky experiments. Hence, we shall adopt Wagoner's completion and call it the "Bergmann-Wagoner theory."

By employing a conformal transformation (change of mathematical representation), Wagoner (1970) put the theory into the form

$$\delta\int[(-\bar{g})^{1/2}(\bar{R} - n\bar{g}^{ij}\bar{\varphi}_{,i}\bar{\varphi}_{,j} + 2\lambda(\bar{\varphi})) + \mathcal{L}_I(\psi^2(\bar{\varphi})\bar{g}_{ij}, \text{ matter and nongravitational field variables})]d^4x = 0 \qquad (n = \pm 1). \tag{20}$$

The resulting Euler-Lagrange field equations are

$$\bar{R}^i{}_j - \tfrac{1}{2}\delta^i{}_j\bar{R} - n(\delta^k{}_j\bar{g}^{im} - \tfrac{1}{2}\delta^i{}_j\bar{g}^{km})\bar{\varphi}_{,k}\bar{\varphi}_{,m} - \lambda\delta^i{}_j = 8\pi\bar{T}^i{}_j, \tag{21}$$

$$(-\bar{g})^{-1/2}[(-\bar{g})^{1/2}\bar{g}^{ij}\varphi_{,i}]_{,j} + n\frac{d\lambda}{d\bar{\varphi}} = -8\pi n\psi^{-1}\frac{d\psi}{d\bar{\varphi}}T; \tag{22}$$

and the matter response equation, which follows from the field equations, is

$$(\bar{T}^m{}_i)_{;m} - (d\ln\psi/d\bar{\varphi})\bar{T}\bar{\varphi}_{,i} = 0. \tag{23}$$

This set of equations is not in the canonical form for a metric theory. To transform it into canonical form, we must notice that the metric element which governs proper-length and proper-time measurements is

$$ds^2 = g_{ij}dx^idx^j = [\psi(\bar{\varphi})]^2\bar{g}_{ij}dx^idx^j = [\hat{\psi}(\hat{\varphi})]^2\hat{g}_{ij}dx^idx^j. \tag{24}$$

Therefore, by a conformal transformation of the form

$$g_{ij} = [\psi(\bar{\varphi})]^2\bar{g}_{ij} = [\hat{\psi}(\hat{\varphi})]^2\hat{g}_{ij}\,, \tag{25}$$

we can put the theory into canonical form.

The variational principle then becomes

$$\delta\int\Big[\!\Big[(-g)^{1/2}\Big\{\psi^{-2}R + \Big[6\psi^{-4} - n\psi^{-2}\Big(\frac{1}{d\psi/d\bar{\varphi}}\Big)^2\Big]g^{ij}\psi_{,i}\psi_{,j} + \psi^{-2}\cdot 2\lambda\Big\} + \mathcal{L}_I(g_{ij},\ldots)\Big]\!\Big]d^4x = 0\,. \tag{26}$$

By letting $\varphi = \psi^{-2}(\bar{\varphi})$ and $\omega(\varphi) = -3/2 + n(d\ln\psi/d\bar{\varphi})^{-2}$, we can put this variational principle into the form

$$\delta\int\Big\{(-g)^{1/2}\Big[\varphi R - \frac{1}{\varphi}\,\omega(\varphi)g^{ij}\varphi_{,i}\varphi_{,j} + 2\varphi\lambda\Big] + \mathcal{L}_I(g_{ij},\ldots)\Big\}d^4x\,. \tag{27}$$

The resulting field equations are

$$R_{ij} - \tfrac{1}{2}Rg_{ij} - \lambda g_{ij} = \frac{8\pi}{\varphi}\,T_{ij} + \frac{\omega(\varphi)}{\varphi}\,(\varphi_{,i}\varphi_{,j} - \tfrac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}) + \varphi^{-1}(\varphi_{,i;j} - g_{ij}\Box\varphi)\,, \tag{28}$$

$$\Box\varphi + \frac{2\varphi^2\lambda' - 2\varphi\lambda}{3 + 2\omega(\varphi)} = \frac{8\pi}{3 + 2\omega(\varphi)}\,T - \frac{\omega'(\varphi)}{3 + 2\omega(\varphi)}\,\varphi_{,i}\varphi^{,i}\,. \tag{29}$$

In this representation, the canonical matter response equation follows from the field equations, or from the argument accompanying equation (15) above,

$$T^{ij}{}_{;i} = 0\,. \tag{30}$$

Will (1970) has calculated the PPN parameters for the Bergmann-Wagoner theory, and his results are shown in the résumé at the beginning of this section. But he did not give any details of the calculation, so we will give a few here. By setting $\lambda = 0$ (cosmological term unmeasurable in solar system), one notices that the field equations for Bergmann-Wagoner theory are exactly the same as those for Brans-Dicke theory except for the additional quadratic term $-\omega'(\varphi)[3 + 2\omega(\varphi)]^{-1}\varphi_{,i}\varphi^{,i}$ on the right-hand side of equation (10) and the φ dependence of ω. By following Nutku's (1969) approach in calculating the post-Newtonian limit of Brans-Dicke theory, one can obtain the post-Newtonian limit of Bergmann-Wagoner theory. The equations corresponding to equations (3), (18), (20), and (21) of Nutku (1969) are (note the difference in sign convention)

$$R_{ij} = \frac{8\pi}{\varphi}\Big[T_{ij} - \frac{1 + \omega(\varphi)}{3 + 2\omega(\varphi)}\,g_{ij}T\Big] + \frac{\omega(\varphi)}{\varphi^2}\,\varphi_{,i}\varphi_{,j} + \frac{1}{\varphi}\,\varphi_{,i;j} - \frac{1}{2}\,\frac{\omega'(\varphi)}{3 + 2\omega(\varphi)}\,\varphi_{,k}\varphi^{,k}\,\frac{1}{\varphi}\,g_{ij}\,, \tag{31}$$

$$\begin{aligned} R_{00} = \tfrac{1}{2}\nabla^2 h_{00} + \Big(\frac{2 + 2\omega}{2 + \omega}\,U\nabla^2 U + \frac{1}{2 + \omega}\,\frac{\partial^2 U}{\partial t^2} - \frac{3 + 2\omega}{2 + \omega}\,\nabla U\cdot\nabla U\Big) \\ = 4\pi\rho\Big(1 - \frac{1}{2 + \omega}\,U\Big)\Big(1 + \frac{3 + 2\omega}{2 + \omega}\,v^2 - 2U + \Pi + \frac{3 + 3\omega}{2 + \omega}\,\frac{p}{\rho}\Big) \\ + \Big(\frac{\partial^2 U}{\partial t^2} + \nabla U\cdot\nabla U\Big) + \frac{\omega'}{(3 + 2\omega)^2(2 + \omega)}\,\nabla U\cdot\nabla U \\ - 8\pi\rho\,\frac{\omega'}{(3 + 2\omega)^2(2 + \omega)}\,U\,, \end{aligned} \tag{32}$$

$$\phi = \frac{3+2\omega}{4+2\omega}v^2 + \left(\frac{1+2\omega}{4+2\omega} - \Lambda\right)U + \tfrac{1}{2}\Pi + \frac{3+3\omega}{4+2\omega}\frac{p}{\rho}\,, \tag{33}$$

$$h_{00} = -2U + [2(1+\Lambda)U^2 - 4\Phi] + O(c^{-6})\,. \tag{34}$$

In equations (32), (33), and (34)

$$\omega = [\omega(\varphi)]_{\text{far outside solar system}}\,, \tag{35}$$

$$\Lambda = \left[\frac{d\omega/d\varphi}{(4+2\omega)(3+2\omega)^2}\right]_{\text{far outside solar system}}\,. \tag{36}$$

All the other equations in §§ II and III of Nutku (1969) are unchanged. By comparing the metric obtained this way with the PPN metric (5), one obtains the PPN parameters listed at the beginning of this section. (See also § III*B* [ii] of this paper.)

In Paper III, Will (1971*b*) has shown that a metric theory has post-Newtonian integral conservation laws for energy, momentum, angular momentum, and center-of-mass motion if and only if its nine PPN parameters obey seven constraints. Thus, for "conservative theories" only two PPN parameters are freely specifiable. The Bergmann-Wagoner theory has two arbitrary parameters in the post-Newtonian limit, and it possesses post-Newtonian integral conservation laws. Hence it has the most general "conservative" PPN limit. The full theory also has integral conservation laws for energy, momentum, angular momentum, and center-of-mass motion (Ni 1972*b*).

ii) *Nordtvedt's Scalar-Tensor Theory*

a. Original formulation: Nordtvedt (1970).

b. Gravitational fields present: φ, g.

c. Arbitrary parameters and functions: One arbitrary function $\omega(\varphi)$ of φ; in the post-Newtonian limit, there are two arbitrary parameters ω = [value of $\omega(\varphi)$ far outside solar system]; Λ = [value of $(d\omega/d\varphi)(4+2\omega)^{-1}(3+2\omega)^{-2}$ far outside solar system].

d. Field equations:

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = \frac{8\pi}{\varphi}T_{ij} + \frac{\omega(\varphi)}{\varphi^2}(\varphi_{,i}\varphi_{,j} - \tfrac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}) + \varphi^{-1}(\varphi_{,i;j} - g_{ij}\Box\varphi)\,, \tag{37}$$

$$\Box\varphi = \frac{8\pi}{3+\omega(\varphi)}T - \frac{\omega'(\varphi)}{3+2\omega(\varphi)}\varphi_{,i}\varphi^{,i}\,. \tag{38}$$

e. PPN parameters: Same as Bergmann-Wagoner theory (see above).

f. Comparison with experiment: Same as Bergmann-Wagoner theory (see above).

g. Derivation and discussion of the above results: Direct comparison of the above equations with the canonical representation of the Bergmann-Wagoner theory (eqs. [9] and [10]) reveals the following theorem:[5] *Nordtvedt's scalar-tensor theory is equivalent to the Bergmann-Wagoner theory in the special case of zero cosmological term, i.e.,* $\lambda = 0$. (This theorem is far from obvious until the two theories have been transformed into the same representation.) From this theorem it follows that the two theories have the same PPN parameters (cosmological term unmeasurable in solar system).

The values of the PPN parameters can be either calculated directly (Will 1970) or inferred indirectly from Nordtvedt's work. The indirect route proceeds as follows: Nordtvedt (1968) developed a point-particle PPN formalism with seven parameters, which predates Will's nine-parameter fluid PPN formalism. After devising his scalar-tensor theory, Nordtvedt (1970) calculated its PPN parameters in his point-particle

[5] This theorem was discovered independently by C. M. Will.

formalism. Direct translation into Will's formalism using table 1 of Will (1971*a*) yields values for γ, β, β_1, β_2, Δ_1, Δ_2, and ζ. Since the parameters β_3 and β_4 are absent from a point-particle approximation, they must be inferred in some other manner. β_3 and β_4 cannot depend upon Λ, since $\Lambda \propto d\omega/d\varphi$ can appear in g_{ij} only in terms which are nonlinear in the mass source strengths; hence, β_3 and β_4 depend only on ω. But if ω is a constant, Nordtvedt's theory reduces to Brans-Dicke theory; hence, β_3 and β_4 must be the same in Nordtvedt's theory as in Brans-Dicke theory; they can be read directly off the Brans-Dicke results of the next section.

iii) *Brans-Dicke-Jordan Theory*

Jordan's (1948, 1955) theory, constructed independently by Thirry (1948), is a scalar-tensor theory in which the field equations depend explicitly on the matter Lagrangian, except for the case "$\eta = -1$." But the matter Lagrangian is not uniquely determined; there is freedom in adding the gradient of an arbitrary function. Therefore, to make the theory (except for the case "$\eta = -1$") complete and consistent, one must give rules for specifying the Lagrangian. Because such rules have never been spelled out, we shall confine our attention here to the case "$\eta = -1$." (Other justifications for ignoring the cases $\eta \neq -1$ are given by Brill 1962; see also the arguments around eq. [14] in § III*B*[i].) The special case $\eta = -1$ is equivalent to the theory of Brans and Dicke (1961). (For proof see, e.g., Dykla 1972.) It is this Brans-Dicke-Jordan theory which we examine here.

a. Original formulation: Jordan (1948, 1955), Thirry (1948), Brans and Dicke (1961).

b. Principal subsequent references: Brans (1962*a*, *b*), Dicke (1962), Dicke and Goldenberg (1967), Morganstern (1967), Morganstern and Chiu (1967), O'Connell and Salmona (1967), Salmona (1967), Shaviv and Bahcall (1967), Cocke and Cohen (1968), Dicke (1968), Freund and Nambu (1968), Greenstein (1968), Kaufmann (1968), Krogh and Baierlein (1968), Noerdlinger (1968), Toton (1968), Estabrook (1969), Janis, Robinson, and Winicour (1969), Nariai (1969*a*, *b*), Nutku (1969), Morganstern (1970), Bekenstein (1971), Hawking (1971), Mahanta and Reddy (1971), Morganstern (1971), Thorne and Dykla (1971), Dykla (1972).

c. Gravitational field present: φ, g.

d. Arbitrary parameters and functions: One arbitrary parameter, ω.

e. Field equations:

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = \frac{8\pi}{\varphi}T_{ij} + \frac{\omega}{\varphi^2}(\varphi_{,i}\varphi_{,j} - \tfrac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}) + \frac{1}{\varphi}(\varphi_{,i;j} - g_{ij}\Box\varphi)\,, \tag{39}$$

$$\Box\varphi = \frac{8\pi}{3+2\omega}T\,. \tag{40}$$

f. PPN parameters:

$$\gamma = \frac{1+\omega}{2+\omega}\,, \quad \beta = 1\,, \quad \beta_1 = \frac{3+2\omega}{4+2\omega}\,, \quad \beta_2 = \frac{1+2\omega}{4+2\omega}\,, \quad \beta_3 = 1\,,$$

$$\beta_4 = \frac{1+\omega}{2+\omega}\,, \quad \zeta = 0\,, \quad \Delta_1 = \frac{10+7\omega}{14+7\omega}\,, \quad \Delta_2 = 1\,.$$

g. Comparison with experiment: Agrees with time delay and deflection experiments except for Sramek (1971): to 1σ accuracy if $\omega > 23$; to 2σ accuracy if $\omega > 6$. Agrees with Sramek's deflection experiment: to 1σ accuracy if $8 > \omega > 4/3$; to 2σ accuracy if $\omega > 1/2$. Agrees completely with perihelion-shift measurements plus time-delay experiments, and gravimeter and Kreuzer experiments. Future experiments should concentrate on pushing ω toward ∞ or on determining ω. This is best done by time-delay and light-deflection experiments.

h. Derivation and discussion of the above results: Direct comparison of equations (39), (40), and (9), (10), and (37), (38) reveals that the Brans-Dicke-Jordan theory is the special case ω = const., $\lambda = 0$ of the Bergmann-Wagoner-Nordtvedt theories. Of course, this is because Bergmann and Nordtvedt devised their theories as generalizations of Brans-Dicke.

The post-Newtonian limit of the Brans-Dicke-Jordan theory was calculated by Nutku (1969). Its PPN parameters, as listed above, are readily obtained by comparing Nutku's post-Newtonian metric with the standard PPN metric of equation (5).

It is well known that Brans-Dicke-Jordan theory goes to general relativity as $\omega \rightarrow +\infty$ (Brans and Dicke 1961). It is also well known that Brans-Dicke-Jordan theory breaks down if $\omega = -3/2$. We will show in the following that, as $0 < \omega + 3/2 = \epsilon \lll 1$, Brans-Dicke-Jordan theory becomes approximately equivalent to a special case of the general conformally flat theory described in § III*C*(i) (below).

As $\omega \rightarrow -3/2$, and $0 < \omega + 3/2 = \epsilon \lll 1$ in the Brans-Dicke-Jordan theory, $\varphi = O(\epsilon^{-1})$ (cf. the field equation [40] for φ). By a conformal transformation of the metric

$$\bar{g}_{ij} = \varphi g_{ij} , \tag{41}$$

the field equation for g_{ij} is transformed into

$$\bar{G}_{ij} = \bar{R}_{ij} - \tfrac{1}{2}\bar{R}\bar{g}_{ij} = \frac{8\pi}{\varphi} T_{ij} + \tfrac{1}{2}(2\omega + 3)\varphi^{-2}[\varphi_{,i}\varphi_{,k} - \tfrac{1}{2}g_{ij}g^{lm}\varphi_{,l}\varphi_{,m}] . \tag{42a}$$

Therefore, $\bar{G}_{ij} = O(\epsilon)$ and $\bar{R}_{ij} = O(\epsilon)$, so as $\epsilon \rightarrow 0$, $\bar{R}_{ij} \rightarrow 0$. The solutions of $\bar{R}_{ij} = 0$ are gravitational waves. The source term for these waves is of $O(\epsilon)$; thus it is reasonable to assume that the waves themselves are also of order $O(\epsilon)$, i.e.,

$$\bar{R}_{ijkl} = O(\epsilon) . \tag{42b}$$

Thus, as $\epsilon \rightarrow 0$, $\bar{R}_{ijkl} \rightarrow 0$. But $\bar{R}_{ijkl} = 0$ is equivalent to saying that spacetime is flat in the metric $\bar{g}_{ij}$, i.e., $\bar{g}_{ij} = \eta_{ij}$. The Brans-Dicke-Jordan field equations (39) and (40), then assume the following limits, as $0 < \omega + 3/2 = \epsilon \lll 1$:

$$g_{ij} = \varphi^{-1}\eta_{ij} \tag{43}$$

and

$$\Box\varphi = \frac{8\pi}{\epsilon} T . \tag{44}$$

Let $\varphi = 1 + 2\psi$; equations (43) and (44) are then transformed to

$$g_{ij} = (1 + 2\psi)^{-1}\eta_{ij} , \tag{45}$$

$$\Box\psi = \frac{4\pi}{\epsilon} T . \tag{46}$$

With a redefinition of the gravitational constant,

$$G \equiv 1/\epsilon , \tag{47}$$

this becomes the special case

$$f(\psi) = \tfrac{1}{2} \ln (1 + 2\psi) = \psi - 2\psi^2 + \ldots , \tag{48}$$

$$k(\psi) = 1 , \tag{49}$$

$$p = 0 , \quad q = -2 , \tag{50}$$

of the conformally flat theory in § IIIC(i) below. Thus, we have shown that, when $0 < \omega + 3/2 = \epsilon \lll 1$, Brans-Dicke-Jordan theory becomes approximately equivalent to the above special case of the general conformally flat theory. There is a similar connection between the Bergmann-Wagoner-Nordtvedt theory and conformally flat theories. Therefore, in a certain sense, *scalar-tensor theories provide a continuous link between general relativity and conformally flat theories.*

C) Conformally Flat Theories

Conformally flat theories possess a global Lorentz metric η_{ij} ("prior geometry" in the language of Misner *et al.* [1972]; see also § III*D*) and a scalar field ψ, which generate the physical metric via the algebraic equation

$$g_{ij} = \psi\eta_{ij} \,. \tag{51}$$

The scalar field is generated by the matter via a wave-type equation,

$$W(\psi) = 4\pi\rho^* \,, \tag{52}$$

where ρ^* is the gravitational source density. There are two candidates for this gravitational source density:

$$\rho^* = T_{ij}u^i u^j \,, \tag{53}$$

where $\boldsymbol{u}$ is the 4-velocity of the source; and

$$\rho^* = \text{trace}\,(T) \,. \tag{54}$$

For a laboratory-sized object, the integrals of both these densities are equal to the total inertial mass of the object, and therefore both can yield a correct Newtonian limit. In the theories following with this alternative, they all have $\beta_4 = 0$ or -1, respectively; this, together with $\beta_1 = 0$ and $\zeta = 0$ for these theories, says that the choice $\rho^* = T_{ij}u^i u^j$, i.e., $\beta_4 = 0$, is in marginal disagreement with the Kreuzer experiment.

In some sense, a conformally flat theory is the simplest kind of theory one can design for gravity. But, since Maxwell equations are conformally invariant, such a theory must predict zero deflection of light. Thus, *all conformally flat theories disagree violently with the light-deflection experiments.*

In the following, we shall formulate the general case, and then catalog special cases.

i) General Conformally Flat Theory

a. Original formulation: This theory is due to the author and is published here for the first time. However, it is a very obvious generalization of the conformally flat theories described later in § III*C*.

b. Gravitational fields present: $\boldsymbol{\eta}$, φ, g.

c. Arbitrary parameters and functions: Two arbitrary functions of φ; $f(\varphi)$ and $k(\varphi)$. In the post-Newtonian limit there are two arbitrary parameters p and q defined below as coefficients in power series expansions of f and k.

d. Field equations:

$$g_{ij} = \exp\,[-2f(\varphi)]\eta_{ij} \,, \tag{55}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi k(\varphi)\rho^* \,, \tag{56}$$

where

$$f(\varphi) = \varphi + q\varphi^2 + \ldots \,, \tag{57}$$

$$k(\varphi) = 1 + p\varphi + \ldots \,, \tag{58}$$

and ρ^* is the gravitational source density defined either by equation (53) or equation (54).

e. PPN parameters:

$$\gamma = -1\,, \quad \beta = 1 - q\,, \quad \beta_1 = 0\,, \quad \beta_2 = \tfrac{1}{2}p\,, \quad \beta_3 = 1\,,$$

$$\beta_4 = 0 \text{ or } -1\,,^{6} \quad \zeta = 0 \quad \Delta_1 = -\tfrac{1}{7}\,, \quad \Delta_2 = 1\,.$$

f. Comparison with experiment: Predicts zero deflection and zero relativistic time delay. Disagrees violently with experiment.

g. Derivation and discussion of the above results: The most general set of equations of second order that one can construct for a conformally flat theory, if one demands (i) linearity in the second derivatives of the scalar field, (ii) spacetime inversion symmetry, and (iii) that the equation for the scalar field be of at most second degree, is this:

$$g_{ij} = \exp\left[-2\bar{f}(\bar{\varphi})\right]\eta_{ij}\,, \tag{59}$$

$$\eta^{ij}\bar{\varphi}_{,i,j} + h(\bar{\varphi})\bar{\varphi}_{,i}\bar{\varphi}_{,j}\eta^{ij} = 4\pi\bar{k}(\bar{\varphi})\rho^*\,. \tag{60}$$

But by an appropriate transformation of the form

$$\varphi = \varphi(\bar{\varphi})\,, \tag{61}$$

one can reduce these equations to the form (55), (56) of the above theory. Hence, this theory is the most general of its type that can be constructed.

For weak fields, the functions $f(\varphi)$ and $k(\varphi)$ can be expanded as follows:

$$f(\varphi) = a + b\varphi + q\varphi^2 + \ldots\,, \tag{62}$$

$$k(\varphi) = d + p\varphi + \ldots\,. \tag{63}$$

But by a change of units ($x^k \to \text{const.}\, x^k$), we can set $a = 0$, and thereby assure that

$$g_{ij} \to \eta_{ij} \quad \text{as} \quad \varphi \to 0\,. \tag{64}$$

By a rescaling of φ, we can set $b = 1$. Then, in order to have the Newtonian limit come out correctly, we must have $d = 1$. Thus equations (62) and (63) reduce to equations (57) and (58). Equations (55) and (56) together with equations (57) and (58) are the field equations of this general conformally flat theory.

To construct the post-Newtonian limit of this theory, we expand φ as

$$\varphi = \varphi_1 + \varphi_2 + O(6)\,, \tag{65}$$

where φ_1 is $O(2)$ and φ_2 is $O(4)$.[7] Correct to $O(4)$, the field equation (56) reads

$$\varphi_{1,00} - \nabla^2\varphi_1 - \nabla^2\varphi_2 = 4\pi\rho^*(1 + p\varphi)\,. \tag{66}$$

The $O(2)$ part is

$$\nabla^2\varphi_1 = -4\pi\rho\,; \tag{67}$$

i.e.,

$$\varphi_1 = U\,, \tag{68}$$

where U is the Newtonian potential. The $O(4)$ part is

$$\nabla^2\varphi_2 = U_{,00} - 4\pi\rho\left(pU + \frac{\rho^* - \rho}{\rho}\right). \tag{69}$$

[6] The value of β_4 depends on what generates the scalar field of the theory, i.e., $\rho^* = T_{ij}u^iu^j\,(\beta_4 = 0)$, or $\rho^* = \text{trace}\,(T)\ (\beta_4 = -1)$.

[7] By $O(n)$ we mean, in Chandrasekhar's (1965) language, $O(c^{-n})$; in Will's (1971*a*) language, $O(U/c^2) = O(p/\rho c^2) = O(v^2/c^2) = O(\Pi/c^2) = O(2)$.

Let χ be the solution of

$$\nabla^2\chi = -2U ; \tag{70}$$

i.e.,

$$\chi = -\int \rho |x - x'| dx' . \tag{71}$$

We can transform equation (69) to read

$$\nabla^2(\varphi_2 + \tfrac{1}{2}\chi_{,00}) = -4\pi\rho\left(pU + \frac{\rho^* - \rho}{\rho}\right). \tag{72}$$

Therefore,

$$\varphi_2 = -\tfrac{1}{2}\chi_{,00} + 2\Phi , \tag{73}$$

where

$$\nabla^2\Phi = -4\pi\rho\left(\tfrac{1}{2}pU + \frac{\rho^* - \rho}{\rho}\right). \tag{74}$$

Combining equations (68) and (73), we have

$$\varphi = U + 2\Phi - \tfrac{1}{2}\chi_{,00} + O(6) . \tag{75}$$

By equations (55) and (57), the physical metric is

$$g_{00} = 1 - 2U + 2(1 - q)U^2 - 4\Phi + \chi_{,00} + O(6) , \quad g_{0\alpha} = 0 ,$$

$$g_{\alpha\beta} = -(1 - 2U)\delta_{\alpha\beta} + O(4) . \tag{76}$$

Using the gauge transformation

$$x^{0\dagger} = x^0 + \tfrac{1}{2}\chi_{,0} , \quad x^{\alpha\dagger} = x^\alpha , \tag{77}$$

we can transform the metric (76) into

$$g_{00}{}^\dagger = g_{00} - \chi_{,00} + O(6) , \quad g_{0\alpha}{}^\dagger = g_{0\alpha} - \tfrac{1}{2}\chi_{,0\alpha} + O(5) , \quad g_{\alpha\beta}{}^\dagger = g_{\alpha\beta} . \tag{78}$$

Straightforward manipulations then give us

$$g_{00}{}^\dagger = 1 - 2U + 2(1 - q)U^2 - 4\Phi + O(6) ,$$

$$g_{0\alpha}{}^\dagger = -\tfrac{1}{2}V_\alpha + \tfrac{1}{2}W_\alpha + O(5) , \quad g_{\alpha\beta}{}^\dagger = -(1 - 2U)\delta_{\alpha\beta} + O(4) , \tag{79}$$

where V_α and W_α are defined by equations (6). By comparing this with the PPN metric (5), we obtain the PPN parameters listed at the beginning of this section.

A special case of this general theory can be derived from the variational principle:

$$\delta \int [\mathcal{L}_I(g_{ij}, \ldots) - 2f_1(\varphi)\varphi_{,i}\varphi_{,j}\eta^{ij}]d^4x = 0 , \tag{80}$$

where

$$g_{ij} = \exp[-2f_2(\varphi)]\eta_{ij} . \tag{81}$$

The Euler-Lagrangian field equations are

$$4f_1(\varphi)\eta^{ij}\varphi_{,i,j} - 2f_1'(\varphi)\varphi_{,i}\varphi_{,j}\eta^{ij} + 8\pi(\sqrt{-g})T^{ij}\frac{\partial g_{ij}}{\partial\varphi} = 0 , \tag{82}$$

where, as usual,

$$T^{ij} = \frac{1}{8\pi}(-g)^{-1/2}\frac{\partial\mathcal{L}_I}{\partial g_{ij}} . \tag{83}$$

Equation (82) is equivalent to

$$\eta^{ij}\varphi_{,i,j} - \frac{1}{2}\frac{f_1'(\varphi)}{f_1(\varphi)}\varphi_{,i}\varphi_{,j}\eta^{ij} = 4\pi\frac{f_2'(\varphi)}{f_1(\varphi)}\exp[-4f_2(\varphi)]T . \tag{84}$$

By performing a transformation of the form

$$\bar{\varphi} = \int \sqrt{[f_1(\varphi)]}d\varphi\ , \tag{85}$$

and then dropping bars and redefining $f_2(\varphi)$, we can reduce equations (80) and (81) to the following:

$$\delta \int [\mathcal{L}_I(g_{ij}, \ldots) - 2\varphi_{,i}\varphi_{,j}\eta^{ij}]d^4x = 0\ , \tag{86}$$

$$g_{ij} = \exp[-2f_2(\varphi)]\eta_{ij}\ . \tag{87}$$

Equation (84) then reduces to

$$\eta^{ij}\varphi_{,i,j} = 4\pi f_2'(\varphi)\exp[-4f_2(\varphi)]T\ . \tag{88}$$

This equation and equation (87) are in the form for our general conformally flat theory, with

$$f(\varphi) = f_2(\varphi) = \varphi + q\varphi^2 + \ldots\ , \tag{89}$$

$$k(\varphi) = f_2'(\varphi)\exp[-4f_2(\varphi)] = 1 + (2q - 4)\varphi + \ldots\ . \tag{90}$$

This is a special case of the general theory; in the post-Newtonian limit it has $p = 2q - 4$. In the post-Newtonian limit, it is also the special case that satisfies Will's constraints on the PPN parameters for theories having conservation laws. For this special case, the full theory also has integral conservations laws for energy, momentum, angular momentum, and center-of-mass motion (Ni 1972*b*). In general, the general theory cannot have all of these conservation laws because it violates some of Will's seven constraints.

ii) *The Whitrow-Morduch Conformally Flat Theory*

[This is a special case of the general conformally flat theory described in the last section, with $k(\varphi) = \exp[-4f(\varphi)]$.]

a. Original formulation: Whitrow and Morduch (1960, 1965).

b. Gravitational fields present: $\mathfrak{n}$, φ, g.

c. Arbitrary parameters and functions: One arbitrary function of φ; in the post-Newtonian limit, q is arbitrary, but $p = -4$.

d. Field equations:

$$g_{ij} = \exp[-2f(\varphi)]\eta_{ij}\ , \tag{91}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi\exp[-4f(\varphi)]\rho^*\ , \tag{92}$$

where

$$f(\varphi) = \varphi + q\varphi^2 + \ldots\ . \tag{93}$$

e. PPN parameters:

$$\gamma = -1\ , \quad \beta = 1 - q\ , \quad \beta_1 = 0\ , \quad \beta_2 = -2\ , \quad \beta_3 = 1\ ,$$

$$\beta_4 = 0 \text{ or } -1\ ,^8 \quad \zeta = 0\ , \quad \Delta_1 = -\tfrac{1}{7}\ , \quad \Delta_2 = 1\ .$$

f. Comparison with experiment: Disagrees violently with time-delay and light-deflection experiments.

g. Derivation and discussion of the above results: In the original formulation of Whitrow and Morduch (1960, 1965), the field equations were

$$g_{ij} = \exp[-2f(\varphi)]\eta_{ij}\ , \tag{94}$$

and

$$\eta^{ij}\varphi_{,i,j} = 4\pi\rho^*_{\text{flat}}\ , \tag{95}$$

[8] See n. 6.

where ρ^*_{flat} was the gravitational source density in flat spacetime. Whitrow and Morduch postulated that test particles move along geodesics of g_{ij}, but they said nothing about how to mesh this theory with the nongravitational laws of physics. To complete the theory, we postulate the equivalence principle: all nongravitational laws must take on their standard relativistic forms in the local Lorentz frames of g_{ij}. Then ρ^*_{flat} is related to ρ^* as follows:

$$\rho^*_{\text{flat}} = \exp\,[-4f(\varphi)]\rho^* \tag{96}$$

(this can be derived by making a conformal transformation of units); and the Whitrow-Morduch equations (94) and (95) reduce to the equations cited at the beginning of this section.

This theory is a special case of our general conformally flat theory (§ III*C*[i]) with

$$k(\varphi) = \exp\,[-4f(\varphi)] = 1 - 4\varphi + \ldots . \tag{97}$$

The PPN parameters can be read directly off those for the general theory.

iii) *The Littlewood-Bergmann Theory*

[This is a special case of the general conformally flat theory described in § III*C*(i) above, with $f(\varphi) = -\log(1-\varphi)$, $k(\varphi) = (1-\varphi)^4$.]

a. Original formulation: Littlewood (1953), Bergmann (1956).

b. Principal subsequent references: Whitrow and Morduch (1960, 1965).

c. Gravitational fields present: $\mathfrak{n}$, φ, g.

d. Arbitrary parameters and functions: None; in the post-Newtonian limit, $q = \frac{1}{2}$, $p = -4$.

e. Field equations:

$$g_{ij} = (1-\varphi)^2\eta_{ij} \tag{98}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi(1-\varphi)^4\rho^* \tag{99}$$

f. PPN parameters:

$$\gamma = -1\,, \quad \beta = \tfrac{1}{2}\,, \quad \beta_1 = 0\,, \quad \beta_2 = -2\,, \quad \beta_3 = 1$$

$$\beta_4 = 0 \text{ or } -1\,,^{9} \quad \zeta = 0\,, \quad \Delta_1 = -\tfrac{1}{7}\,, \quad \Delta_2 = 1\,.$$

g. Comparison with experiment: Disagrees violently with time delay and light deflection experiments.

h. Derivation and discussion of the above results: In the original formulation, the field equations were

$$g_{ij} = (1-\varphi)^2\eta_{ij} \tag{100}$$

and

$$\eta^{ij}\varphi_{,i,j} = 4\pi\rho^*_{\text{flat}}\,. \tag{101}$$

This is a special case of the Whitrow-Morduch theory (§ III*C*[ii] above) with $f(\varphi) = -\log(1-\varphi) = \varphi + \frac{1}{2}\varphi^2 + \ldots$; i.e., in the post-Newtonian limit $q = \frac{1}{2}$, $p = -4$. Therefore, after the same completion as was imposed on the Whitrow-Morduch theory, the above results follow from those of § III*C*(ii).

iv) *Nordström's First Theory*

[This is a special case of the general conformally flat theory described in § III*C*(i) above, with $f(\varphi) = \varphi$ and $k(\varphi) = e^{-4\varphi}$.]

a. Original formulation: Nordström (1912).

b. Principal subsequent references: Whitrow and Morduch (1960, 1965).

[9] See n. 6.

c. Gravitational fields present: $\mathfrak{n}$, φ, g.

d. Arbitrary parameters and functions: None; in the post-Newtonian limit, $q = 0$, $p = -4$.

e. Field equations:

$$g_{ij} = e^{-2\varphi}\eta_{ij} , \tag{102}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi e^{-4\varphi}\rho^* . \tag{103}$$

f. PPN parameters:

$$\gamma = -1 , \quad \beta = 1 , \quad \beta_1 = 0 , \quad \beta_2 = -2 , \quad \beta_3 = 1 ,$$

$$\beta_4 = 0 \text{ or } -1,^{10} \quad \zeta = 0 , \quad \Delta_1 = -\tfrac{1}{7} , \quad \Delta_2 = 1 .$$

g. Comparison with experiment: Disagrees violently with time-delay and light-deflection experiments.

h. Derivation and discussion of the above results: In the original formulation, the field equations were

$$g_{ij} = e^{-2\varphi}\eta_{ij} \tag{104}$$

and

$$\eta^{ij}\varphi_{,i,j} = 4\pi\rho^*_{\text{flat}} . \tag{105}$$

This is a special case of the Whitrow-Morduch theory (§ III*C*[ii] above) with $f(\varphi) = \varphi$; i.e., in the post-Newtonian limit, $q = \frac{1}{2}$, $p = -4$. Therefore, after completion in the manner of § III*C*(ii), the above results follow from those of § III*C*(ii).

v) *Nordström's Second Theory*

[This is a special case of the general conformally flat theory described in § III*C*(i) above, with $f(\varphi) = -\log(1 - \varphi)$ and $k(\varphi) = (1 - \varphi)^3$.]

a. Original formulation: Nordström (1913, 1914), Einstein and Fokker (1914).

b. Principal subsequent references: Whitrow and Morduch (1960, 1965).

c. Gravitational fields present: $\mathfrak{n}$, φ, g.

d. Arbitrary parameters and functions: None; in the post-Newtonian limit $q = \frac{1}{2}$, $p = -3$.

e. Field equations:

$$C_{ijkl} = 0 , \tag{106}$$

$$R = 24\pi T , \tag{107}$$

where C_{ijkl} is the Weyl conformal tensor and R is the curvature scalar, both constructed from g.

f. PPN parameters:

$$\gamma = -1 , \quad \beta = \tfrac{1}{2} , \quad \beta_1 = 0 , \quad \beta_2 = -3/2 , \quad \beta_3 = 1 ,$$

$$\beta_4 = -1 , \quad \zeta = 0 , \quad \Delta_1 = -\tfrac{1}{7} , \quad \Delta_2 = 1 .$$

g. Comparison with experiment: Disagrees violently with time-delay and light-deflection experiments.

h. Derivation and discussion of the above results: As in general relativity, the field equations (106) and (107) are geometric and make no reference to any gravitational fields except the physical metric g_{ij}. However, they guarantee the existence of a flat spacetime metric η_{ij} ("prior" geometry in the language of Misner *et al.* 1972) and a scalar field related to g_{ij} by

$$g_{ij} = \varphi^2\eta_{ij} ; \tag{108}$$

[10] See n. 6.

and they allow φ to be calculated from the variational principle

$$\delta\int[\mathcal{L}_I - \tfrac{1}{3}R(\sqrt{-g})]d^4x = 0 \,. \tag{109}$$

Expressed in terms of φ, the field equation (107) becomes

$$\eta^{ij}\varphi_{,i,j} = -4\pi T\varphi^3 \tag{110a}$$

or

$$\eta^{ij}\varphi_{,i,j}\varphi^{-1} = -4\pi T_{\text{flat}} \,. \tag{110b}$$

Equation (110) is Nordström's original field equation, while equation (107) is the Einstein-Fokker version.

By comparing equations (108) and (110) with equations (55) and (56), we conclude that Nordström's second theory is a special case of our general conformally flat theory (§ IIIC[i]) with

$$\varphi \to 1 - \varphi \,, \tag{111}$$

$$f(\varphi) = -\ln(1 - \varphi) = \varphi + \tfrac{1}{2}\varphi^2 + \ldots \,, \tag{112}$$

$$k(\varphi) = (1 - \varphi)^3 = 1 - 3\varphi + \ldots \,. \tag{113}$$

In the post-Newtonian limit, $q = \frac{1}{2}$, $p = -3$. Therefore, all the above results follow from those of § III*C*(i).

We notice further that

$$k(\varphi) = f'(\varphi)\exp[-4f(\varphi)] \tag{114}$$

holds (cf. eqs. [89] and [90]); therefore, Nordström's second theory is a special case of the special case of § III*C*(i) that can be given a Lagrangian formulation (cf. eq. [109]). Hence, the full theory has conservation laws for energy, momentum, angular momentum, and center-of-mass motion (Ni 1972*b*), and the PPN parameters satisfy Will's seven conservation constraints.

D) Stratified Theories with Conformally Flat Space Slices

From a certain viewpoint, the simplest kind of gravitational theory is the conformally flat type. However, all conformally flat theories predict zero light deflection and zero relativistic time delay. One way to remedy this is to postulate from the beginning that there exists a preferred Universal reference frame, determined perhaps by the Universe's large-scale distribution of matter; and to demand that the space slices ("strata") of this preferred reference frame are conformally flat, but the full spacetime is not. This section examines a particularly simple subclass of such stratified theories: a class in which the physical metric in the preferred reference frame has the form

$$ds^2 = e^{2f(\varphi)}dt^2 - e^{2g(\varphi)}(dx^2 + dy^2 + dz^2) \,, \tag{115}$$

where φ is a scalar field. In geometric, coordinate-free language such theories have (i) a background, flat metric $\boldsymbol{\eta}$; (ii) a Universal time coordinate t (scalar field) which is covariantly constant and has timelike gradients with respect to $\boldsymbol{\eta}$; (iii) a scalar gravitational field φ; and (iv) a physical metric $\boldsymbol{g}$ (in whose local Lorentz frames the special relativistic laws of physics are valid), constructed from $\boldsymbol{\eta}$, t, and φ by

$$\boldsymbol{g} = e^{2g(\varphi)}\boldsymbol{\eta} + [e^{2f(\varphi)} - e^{2g(\varphi)}]dt\otimes dt \,. \tag{116}$$

The theories differ from one another by their choice of the function $f(\varphi)$, $g(\varphi)$, and by their field equations for φ.

In the stratified theories, the background metric $\boldsymbol{\eta}$ and the Universal time coordinate (scalar field) t are aspects of the geometry of spacetime which are fixed immutably, i.e., which cannot be changed by changing the distribution of gravitating sources. Misner *et al.* (1972) give the name "prior geometry" to such geometric objects, and point out that one key element in Einstein's "principle of covariance" was the demand that spacetime be free of prior geometry. Whereas stratified theories and conformally flat theories always have prior geometry, general relativity and scalar-tensor theories have none.

For ease of description, our presentation of each theory below will be given solely in the preferred reference frame.

i) *Einstein's Theory with "Variable Velocity of Light"*

a. Original formulation: Einstein (1912).
b. Principal subsequent reference: Whitrow and Morduch (1965).
c. Gravitational fields present: $\boldsymbol{\eta}$, φ, t, g.
d. Arbitrary parameters and functions: None.
e. Metric and field equation in preferred reference frame:

$$ds^2 = (1 - 2\varphi)dt^2 - dx^2 - dy^2 - dz^2\,, \tag{117}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi\rho^*\,. \tag{118}$$

f. PPN parameters:

$$\gamma = 0\,, \quad \beta = 0\,, \quad \beta_1 = 0\,, \quad \beta_2 = 0\,, \quad \beta_3 = 1\,, \quad \beta_4 = 0 \text{ or } -1\,,$$ [11]

$$\zeta = 0\,, \quad \Delta_1 = -\tfrac{1}{7}\,, \quad \Delta_2 = 1\,.$$

g. Comparison with experiment: Disagrees violently with time delay, light deflection, and perihelion shift experiments.

h. Derivation and discussion of the above results: In the original formulation of Einstein (1912), the equation of motion for particles was derived from the variational principle

$$\delta\!\int ds = 0\,, \tag{119}$$

where

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2\,, \tag{120}$$

and where c is a scalar function which Einstein regarded as the velocity of light in the $\boldsymbol{\eta}$ metric. Einstein postulated that c depends on the scalar field φ in the following way:

$$c^2 = c_0^2 - 2\varphi\,, \tag{121}$$

and that φ is generated by ρ^* through the wave equation

$$\Box\varphi = 4\pi\rho^*_{\text{flat}} = 4\pi\rho^*\,. \tag{122}$$

By choosing suitable units, we can set $c_0 = 1$; and by postulating that Einstein's ds^2 is the "physical metric," we bring the theory into the form presented above.

The physical metric can always be transformed locally into the local Lorentz form

$$ds^2 = d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2\,, \tag{123}$$

where $d\bar{t}$ is the proper time interval and $dl = (d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2)^{1/2}$, the proper-length element. Since light trajectories all lie on null cones of this metric, the velocity of light as measured using the physical metric is always unity—as it must be for any theory that satisfies the equivalence principle.

The calculation of the post-Newtonian limit for this theory is similar to that for the general conformally flat theory in § IIIC(i), so we will not present it here.

ii) *The Whitrow-Morduch Theory with "Variable Velocity of Light"*

a. Original formulation: Whitrow and Morduch (1960, 1965).
b. Gravitational fields present: $\boldsymbol{\eta}$, φ, t, g.

[11] See n. 6.

c. Arbitrary parameters and functions: None.

d. Metric and field equation in preferred reference frame:

$$ds^2 = \exp\left(\frac{-2\varphi}{1-2\varphi}\right)\left(dt^2 - \frac{dx^2+dy^2+dz^2}{1-2\varphi}\right), \tag{124}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi\left(\frac{1}{1-2\varphi}\right)^2 \exp\left(-4\frac{\varphi}{1-2\varphi}\right)\rho^* \tag{125}$$

e. PPN parameters:

$$\gamma = 0\,, \quad \beta = -1\,, \quad \beta_1 = 0\,, \quad \beta_2 = 0\,, \quad \beta_3 = 1\,, \quad \beta_4 = 0 \text{ or } -1\,,$$[12]

$$\zeta = 0\,, \quad \Delta_1 = -\tfrac{1}{7}\,, \quad \Delta_2 = 1\,.$$

f. Comparison with experiment: Disagrees violently with time-delay and light-deflection experiments.

g. Derivation and discussion of the above results (cf. § III*D*[i]): Whitrow and Morduch write their physical line element in the form

$$ds^2 = \exp(-2\varphi/c^2)\left(dt^2 - \frac{dx^2+dy^2+dz^2}{c^2}\right), \tag{126}$$

so that $c \equiv (1-2\varphi)^{1/2}$ is the "velocity of light" in the flat $\mathfrak{n}$ metric. Their field equation for the scalar field is

$$\eta^{ij}\varphi_{,i,j} = 4\pi\rho^*_{\text{flat}} = 4\pi\left(\frac{1}{1-2\varphi}\right)^2 \exp\left(-4\frac{\varphi}{1-2\varphi}\right)\rho^*\,. \tag{127}$$

The analysis then proceeds as in § III*D*(i).

iii) *The Page-Tupper Theory with "Variable Velocity of Light"*

a. Original formulation: Page and Tupper (1968).

b. Gravitational fields present: $\mathfrak{n}$, φ, t, g.

c. Arbitrary parameters and functions: Two arbitrary functions of φ: $c(\varphi)$ and $f(\varphi/c^2)$. In the post-Newtonian limit, there are two arbitrary parameters a_1 and $b_1 = (2a_2 - R)$ defined below.

d. Metric and field equation in preferred reference frame:

$$ds^2 = f^2(\varphi/c^2)(c^2dt^2 - dx^2 - dy^2 - dz^2)\,, \tag{128}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi[f(\varphi/c^2)]^4\rho^*\,, \tag{129}$$

where

$$c^{-2} = 1 + Q\varphi + R\varphi^2 + O(\varphi^3)\,, \tag{130}$$

$$\begin{aligned} f(\varphi/c^2) &= 1 + a_1(\varphi/c^2) + a_2(\varphi/c^2)^2 + O(\varphi/c^2)^3 \\ &= 1 + a_1\varphi + (a_1Q + a_2)\varphi^2 + O(\varphi^3)\,; \end{aligned} \tag{131}$$

in order to have the correct Newtonian limit, we must have

$$Q = 2a_1 + 2\,. \tag{132}$$

(We will assume this in the following.)

[12] See n. 6.

e. PPN parameters:

$$\gamma = a_1 , \quad \beta = 5a_1{}^2 + 8a_1 + 4 + (2a_2 - R) , \quad \beta_1 = 0 ,$$

$$\beta_2 = 2a_1 , \quad \beta_3 = 1 , \quad \beta_4 = 0 \text{ or } -1 ,^{13} \quad \zeta = 0 , \quad \Delta_1 = -\tfrac{1}{7} , \quad \Delta_2 = 1 .$$

f. Comparison with experiment: Was thought, before 1972, to agree with all experiments if $a_1 \sim 1$ and $\beta_4 = -1$. Actually agrees with time-delay and light-deflection experiments only if $a_1 \sim 1$; but for this choice of a_1, disagrees violently with perihelion shift, with Earth rotation rate (Nordtvedt and Will 1972), and with white-dwarf stability observations[14] (Ni 1972*a*).

g. Derivation and discussion of the above results (cf. § III*D*[i]): Page and Tupper (1968) generalized the Whitrow-Morduch theory with "variable velocity of light" in order to give a theory in agreement with the experiments at that time. They write their physical line element in the form

$$ds^2 = f^2(\varphi/c^2)(c^2dt^2 - dx^2 - dy^2 - dz^2) \tag{133}$$

so that $c \equiv [1 + Q\varphi + R\varphi^2 + O(\varphi^3)]^{-1/2}$ is the "velocity of light" in the flat $\mathfrak{n}$ metric. They postulate that

$$\varphi = M/r , \tag{134}$$

where M is the mass of the central gravitating body and r is the distance from this body (i.e., $r^2 = x^2 + y^2 + z^2$). To give account of a general continuous system, we postulate the following field equation for the scalar field φ:

$$\eta^{ij}\varphi_{,i,j} = 4\pi\rho^*{}_{\text{flat}} = 4\pi[f(\varphi/c^2)]^4\rho^* . \tag{135}$$

The analysis then proceeds as in § III*D*(i). Note that the Whitrow-Morduch conformally flat theory is a special case of this theory with $c(\varphi) = 1$.

iv) *Modified Yilmaz Theory*

a. Original formulation: Yilmaz (1958, 1962).
b. Subsequent reference: Tupper and Page (1969).
c. Gravitational fields present: $\mathfrak{n}$, φ, t, g.
d. Arbitrary parameters and functions: None.
e. Metric and field equation in preferred reference frame:

$$ds^2 = e^{-2\varphi}dt^2 - e^{2\varphi}(dx^2 + dy^2 + dz^2) \tag{136}$$

$$\varphi^{,i}{}_{;i} = -4\pi \exp(-2\varphi)\rho^* . \tag{137}$$

f. PPN parameters:

$$\gamma = 1 , \quad \beta = 1 , \quad \beta_1 = 0 , \quad \beta_2 = 0 , \quad \beta_3 = 1 , \quad \beta_4 = 0 \text{ or } -1 ,^{15}$$

$$\zeta = 0 , \quad \Delta_1 = -\tfrac{1}{7} , \quad \Delta_2 = 1 .$$

g. Comparison with experiment: Was thought, before 1972, to agree with all experiments if $\beta_4 = -1$. Actually disagrees violently with perihelion shift, with Earth rotation rate (Nordtvedt and Will 1972), and with white-dwarf stability observations[16] (Ni 1972*a*).

h. Derivation and discussion of the above results: In his original paper, Yilmaz (1958)

[13] See n. 6.

[14] See n. 2.

[15] See n. 6.

[16] See n. 2.

did not give a complete account of nonstatic gravitational fields. For static fields, he gave the following set of equations:

$$ds^2 = \exp(-2\varphi)dt^2 - \exp(2\varphi)(dx^2 + dy^2 + dz^2) \tag{138}$$

and

$$\varphi^{,i}{}_{;i} = -4\pi \exp(-2\varphi)\Sigma_j M^j{}_0\delta(x - x^j) . \tag{139}$$

Changing from point-mass sources to continuous matter sources, we rewrite his field equation as

$$\varphi^{,i}{}_{;i} = -4\pi \exp(-2\varphi)\rho^* . \tag{140}$$

We then postulate that this set of equations, i.e., equations (138) and (140), apply to nonstatic fields as well as static fields.

The calculation of the post-Newtonian limit is similar to that in § III*C*(i).

Tupper and Page (1969) proposed alternative field equations for the Yilmaz theory; the modified theories are all special cases of their theory with "variable velocity of light" (cf. § III*D*[iii]).

v) *The Stratified Theory of Papapetrou I*

a. Original formulation: Papapetrou (1954*a*, *b*).

b. Gravitational fields present: **n**, φ, χ, *t*, *g*.

c. Arbitrary parameters and functions: None.

d. Metric and field equations in preferred reference frame:

$$ds^2 = e^{\chi}dt^2 - e^{\varphi}(dx^2 + dy^2 + dz^2) \tag{141}$$

and

$$e^{-\varphi}(\varphi_{,\alpha\alpha} + \chi_{,\alpha\alpha} + \tfrac{1}{4}\varphi_{,\alpha}\varphi_{,\alpha} + \tfrac{1}{2}\varphi_{,\alpha}\chi_{,\alpha} + \tfrac{1}{2}\chi_{,\alpha}\chi_{,\alpha})$$
$$- e^{-\chi}(3\varphi_{,00} + \tfrac{9}{4}\varphi_{,0}\varphi_{,0} - \tfrac{3}{2}\varphi_{,0}\chi_{,0}) = -8\pi T^i{}_i , \tag{142}$$

$$e^{-\varphi}(\varphi_{,\alpha\alpha} + \tfrac{1}{4}\varphi_{,\alpha}\varphi_{,\alpha}) - \tfrac{3}{4}e^{-\chi}\varphi_{,0}\varphi_{,0} = -8\pi T^0{}_0 . \tag{143}$$

e. PPN parameters:

$$\gamma = 1 , \quad \beta = 1 , \quad \beta_1 = 1 , \quad \beta_2 = 1 , \quad \beta_3 = 1 ,$$
$$\beta_4 = 1 , \quad \zeta = 0 , \quad \Delta_1 = \tfrac{3}{7} , \quad \Delta_2 = -3$$

f. Comparison with experiment: Was thought, until 1971, to agree with all experiments. Actually disagrees violently with the Earth-tide measurement.

g. Derivation and discussion of the above results: The field equations (142) and (143) can be derived from the Lagrangian (Papapetrou 1954*a*):

$$\mathcal{L} = \mathcal{L}_I(g_{ij}, \text{matter and nongravitational field variables}) + \mathcal{L}_G \tag{144}$$

where

$$\mathcal{L}_G = (\sqrt{-g})g^{ij}(\Gamma^m{}_{in}\Gamma^n{}_{jm} - \Gamma^m{}_{ij}\Gamma^n{}_{mn})$$
$$= e^{(3\varphi+\chi)/2}(-\tfrac{1}{2}e^{-\varphi}\varphi_{,\alpha}\varphi_{,\alpha} - e^{-\varphi}\varphi_{,\alpha}\chi_{,\alpha} + \tfrac{3}{2}e^{-\chi}\varphi_{,0}\varphi_{,0}) ; \tag{145}$$

they can also be written in the form (Papapetrou 1954*a*):

$$R = -8\pi T^i{}_i , \tag{146}$$

$$2R^0{}_0 = 8\pi(T^0{}_0 - T^\alpha{}_\alpha) , \tag{147}$$

where the Γ's and *R*'s are Christoffel symbols and Riemann tensors constructed from the physical metric *g*.

The calculation of the PPN parameters is analogous to that of § III*D*(vii) and will therefore be omitted. The results are listed above.

In this theory

$$|\Delta_2 + \zeta - 1| = 4 \gg 0.03 \,. \tag{148}$$

This disagrees violently with the gravimeter measurements.

vi) *The Stratified Theory of Papapetrou II*

a. Original formulation: Papapetrou (1954*c*).
b. Subsequent reference: Meister (1957).
c. Arbitrary parameters and functions: None.
d. Metric and field equations in preferred reference frame:

$$ds^2 = e^{-\varphi}dt^2 - e^{\varphi}(dx^2 + dy^2 + dz^2) \tag{149}$$

and

$$\varphi_{,\alpha\alpha} + 3e^{2\varphi}(\varphi_{,00} + \varphi_{,0}\varphi_{,0}) = -8\pi e^{\varphi}(T^0{}_0 - T^\alpha{}_\alpha) \,. \tag{150}$$

e. PPN parameters:

$$\gamma = 1\,, \quad \beta = 1\,, \quad \beta_1 = 1\,, \quad \beta_2 = 1\,, \quad \beta_3 = 1\,,$$
$$\beta_4 = 1\,, \quad \zeta = 0\,, \quad \Delta_1 = \tfrac{3}{7}\,, \quad \Delta_2 = -3\,.$$

f. Comparison with experiment: Was thought, until 1971, to agree with all experiments. Actually disagrees violently with the Earth-tide measurement.

g. Derivation and discussion of the above results: The field equations (149) and (150) can be derived from the Lagrangian (Papapetrou 1954*c*):

$$\mathcal{L} = \mathcal{L}_I(g_{ij}, \text{matter and nongravitational field variables}) + \mathcal{L}_G\,, \tag{151}$$

where

$$\mathcal{L}_G = (\sqrt{-g})g^{ij}(\Gamma^m{}_{in}\Gamma^n{}_{jm} - \Gamma^m{}_{ij}\Gamma^n{}_{mn}) = e^{\varphi}(\tfrac{1}{2}e^{-\varphi}\varphi_{,\alpha}\varphi_{,\alpha} + \tfrac{3}{2}e^{\varphi}\varphi_{,0}\varphi_{,0})\,. \tag{152}$$

The calculation of the PPN parameters is analogous to that of § III*D*(vii) and will therefore be omitted. Note that this theory has the same post-Newtonian limit as Papapetrou's first theory.

vii) *A New Lagrangian-Based, Stratified Theory*

a. Original formulation: This paper.
b. Gravitational fields present: **n**, φ, *t*, *g*.
c. Arbitrary parameters and functions: None.
d. Metric and field equation in preferred reference frame:

$$ds^2 = e^{-2\varphi}dt^2 - e^{2\varphi}(dx^2 + dy^2 + dz^2) \tag{153}$$

and

$$\delta\int[\mathcal{L}_I - 2(-g)^{1/2}\varphi_{,i}\varphi^{,i}]d^4x = 0\,; \tag{154}$$

i.e.,

$$\frac{\partial}{\partial x^i}\left[(-g)^{1/2}g^{ij}\frac{\partial\varphi}{\partial x^j}\right] + 2\pi(-g)^{1/2}T^{ij}\frac{\partial g_{ij}}{\partial\varphi} - \frac{1}{2}\frac{\partial(-g)^{1/2}g^{ij}}{\partial\varphi}\varphi_{,i}\varphi_{,j} = 0\,. \tag{155}$$

e. PPN parameters:

$$\gamma = 1\,, \quad \beta = 1\,, \quad \beta_1 = 1\,, \quad \beta_2 = 1\,, \quad \beta_3 = 1\,,$$
$$\beta_4 = 1\,, \quad \zeta = 0\,, \quad \Delta_1 = -\tfrac{1}{7}\,, \quad \Delta_2 = 1\,.$$

f. Comparison with experiment: Was invented in 1970 and was thought, until 1972, to agree with all experiments. Actually disagrees violently with Earth rotation rate (Nordtvedt and Will 1972) and with white-dwarf stability observations[17] (Ni 1972*a*).

[17] See n. 2.

g. Derivation and discussion of the above results: This theory has conserved integrals for energy, momentum, and angular momentum, but not for center-of-mass motion (Ni 1972*b*); it violates some of Will's (1971*b*) seven conservation constraints.

To obtain the post-Newtonian approximation, we proceed as follows. Let

$$\varphi = \varphi_1 + \varphi_2 + O(6)\,, \tag{156}$$

where $\varphi_1 = O(2)$ and $\varphi_2 = O(4)$. Correct to post-Newtonian order, the field equation is

$$\frac{\partial^2\varphi_1}{\partial t^2} - \nabla^2\varphi_1 - \nabla^2\varphi_2 = 4\pi(1 + 2\varphi_1)\rho\left(1 + \Pi + 3\frac{p}{\rho} + 2v^2\right). \tag{157}$$

The $O(2)$ part is

$$\nabla^2\varphi_1 = -4\pi\rho\,, \tag{158}$$

i.e.,

$$\varphi_1 = U\,, \tag{159}$$

where U is the Newtonian potential. The $O(4)$ part is

$$\nabla^2\varphi_2 = U_{,00} - 4\pi\rho(\Pi + 2U + 3p/\rho + 2v^2)\,. \tag{160}$$

Let χ be the solution of

$$\nabla^2\chi = -2U\,, \tag{161}$$

i.e.,

$$\chi = -\int\rho|x - x'|dx'\,. \tag{162}$$

We can transform equation (160) to

$$\nabla^2(\varphi_2 + \tfrac{1}{2}\chi_{,00}) = -4\pi\rho(\Pi + 2U + 3p/\rho + 2v^2)\,. \tag{163}$$

Therefore,

$$\varphi_2 = -\tfrac{1}{2}\chi_{,00} + 2\Phi\,, \tag{164}$$

where

$$\nabla^2\Phi = -4\pi\rho\left(\tfrac{1}{2}\Pi + U + \tfrac{3}{2}\frac{p}{\rho} + v^2\right). \tag{165}$$

Combining equations (159) and (164), we find

$$\varphi = U + 2\Phi - \tfrac{1}{2}\chi_{,00} + O(6)\,. \tag{166}$$

According to equations (153) and (166), the physical metric is

$$g_{00} = 1 - 2U + 2U^2 - 4\Phi + \chi_{,00}\,, \quad g_{0\alpha} = 0\,, \quad g_{\alpha\beta} = -\delta_{\alpha\beta}(1 + 2U)\,. \tag{167}$$

By using the gauge transformation

$$x^{0\dagger} = x^0 + \tfrac{1}{2}\chi_{,0}\,, \quad x^{\alpha\dagger} = x^\alpha\,, \tag{168}$$

we can transform the metric into the form

$$g_{00}{}^\dagger = 1 - 2U + 2U^2 - 4\Phi + O(6)\,, \quad g_{0\alpha}{}^\dagger = -\tfrac{1}{2}V_\alpha + \tfrac{1}{2}W_\alpha + O(5)\,,$$
$$g_{\alpha\beta}{}^\dagger = -(1 + 2U)\delta_{\alpha\beta} + O(4)\,, \tag{169}$$

where V_α and W_α are defined by equations (6). By comparing this with the PPN metric (5), we obtain the PPN parameter values listed at the beginning of this section.

viii) *A General Stratified Theory*

a. Original formulation: This paper.

b. Gravitational fields present: $\mathfrak{n}$, φ, t, g.

c. Arbitrary parameters and functions: Two arbitrary functions $f(\varphi)$ and $k(\varphi)$. In the post-Newtonian limit, there are two arbitrary parameters, p and q, which are coefficients in power-series expansions of $f(\varphi)$ and $k(\varphi)$ (see below).

d. Metric and field equation in preferred reference frame:

$$ds^2 = e^{-2f(\varphi)}dt^2 - e^{2f(\varphi)}[dx^2 + dy^2 + dz^2] \tag{170}$$

$$\eta^{ij}\varphi_{,i,j} = 4\pi\rho^* k(\varphi) \tag{171}$$

where

$$f(\varphi) = \varphi + q\varphi^2 + \ldots \tag{172}$$

$$k(\varphi) = 1 + p\varphi + \ldots . \tag{173}$$

e. PPN parameters:

$$\gamma = 1\,, \quad \beta = 1 - q\,, \quad \beta_1 = 0\,, \quad \beta_2 = \tfrac{1}{2}p\,, \quad \beta_3 = 1\,, \quad \beta_4 = 0 \text{ or } -1\,,$$ [18]

$$\zeta = 0\,, \quad \Delta_1 = -\tfrac{1}{7}\,, \quad \Delta_2 = 1\,.$$

(The derivation is similar to that in § IIIC[i].)

f. Comparison with experiments: Was invented in 1970 and was thought, until 1972, to agree with all experiments if $q \sim 0$ and $\beta_4 = -1$. Actually disagrees violently with perihelion-shift observations, with Earth rotation rate (Nordtvedt and Will 1972), and with white-dwarf stability observations[19] (Ni 1972*a*).

ix) *The Stratified Theory of Rosen*

a. Original formulation: Rosen (1971).

b. Gravitational fields present: $\mathfrak{n}$, Φ, Ψ, t, g.

c. Arbitrary parameters and functions: Two arbitrary parameters α and λ; in the post-Newtonian limit, one arbitrary parameter λ.

d. Metric and field equations in preferred reference frame:

$$ds^2 = \Phi^2 dt^2 - \Psi^2(dx^2 + dy^2 + dz^2)\,, \tag{174}$$

and

$$\alpha\left[\frac{1}{2}\frac{\Phi^2}{\Psi^4}\Psi_{,\alpha}\Psi_{,\alpha} + \frac{1}{2}\frac{1}{\Psi^2}(\Psi_{,0})^2\right] + \beta\left[\frac{\Phi_{,00}}{\Phi} + 3\frac{\Phi_{,0}\Psi_{,0}}{\Phi\Psi} - \frac{3}{2}\frac{(\Phi_{,0})^2}{\Phi^2} - \frac{\Phi_{,\alpha\alpha}\Phi}{\Psi^2} - \frac{\Phi_{,\alpha}\Psi_{,\alpha}\Phi}{\Psi^3} + \frac{1}{2}\frac{\Phi_{,\alpha}\Phi_{,\alpha}}{\Psi^2}\right] + \gamma\left[\frac{1}{2}\frac{\Psi_{,00}}{\Psi} + \frac{(\Psi_{,0})^2}{(\Psi)^2} - \frac{1}{2}\frac{\Psi_{,\alpha\alpha}\Phi^2}{\Psi^3}\right] = -8\pi T_{00}\,, \tag{175}$$

$$\alpha\left(-\frac{1}{2}\frac{\Psi_{,0}\Psi_{,0}}{\Phi^2} + \frac{\Psi\Psi_{,0}\Phi_{,0}}{\Phi^3} - \frac{\Psi\Psi_{,00}}{\Phi^2} - \frac{1}{2}\frac{\Psi_{,\alpha}\Psi_{,\alpha}}{\Psi^2} + \frac{\Psi_{,\alpha}\Phi_{,\alpha}}{\Psi\Phi} + \frac{\Psi_{,\alpha\alpha}}{\Psi}\right) + \beta\left(\frac{3}{2}\frac{\Phi_{,0}\Phi_{,0}\Psi^2}{\Phi^4} - \frac{1}{2}\frac{\Phi_{,\alpha}\Phi_{,\alpha}}{\Phi^2}\right) + \gamma\left(\frac{\Phi_{,0}{}^2\Psi^2}{\Phi^4} - \frac{1}{2}\frac{\Phi_{,00}\Psi^2}{\Phi^3} + \frac{1}{2}\frac{\Phi_{,\alpha\alpha}}{\Phi}\right) = -8\pi T_{\alpha\alpha}\,, \tag{176}$$

where

$$\gamma = 2\lambda\alpha \tag{177}$$

and

$$\beta = \lambda^2\alpha + 2\,. \tag{178}$$

[18] See n. 6.

[19] See n. 2.

e. PPN parameters:

$$\gamma = \lambda\,, \quad \beta = \tfrac{3}{4} + \tfrac{\lambda}{4}\,, \quad \beta_1 = \tfrac{1}{2} + \tfrac{\lambda}{2}\,, \quad \beta_2 = -\tfrac{1}{4} + \tfrac{5}{4}\lambda\,, \quad \beta_3 = 1\,,$$

$$\beta_4 = \lambda\,, \quad \Delta_1 = -\tfrac{1}{7}\,, \quad \Delta_2 = 1\,, \quad \zeta = 0\,.$$

f. Comparison with experiment: Was thought, until 1972, to agree with all experiments if $\lambda \sim 1$. Actually agrees with time-delay and deflection experiments only if $\lambda \sim 1$; but for this choice of λ, disagrees violently with Earth rotation rate (Nordtvedt and Will 1972) and with white-dwarf stability observations[20] (Ni 1972*a*).

g. Derivation and discussion of the above results: Instead of using only one scalar field in constructing the physical line element in the preferred coordinate system, Rosen (1971) uses two scalar fields. In his theory, the physical line element can be written in the form

$$ds^2 = \Phi^2 dt^2 - \Psi^2(dx^2 + dy^2 + dz^2)\,, \tag{179}$$

where Φ and Ψ are two scalar functions. The field equations can be derived from the Lagrangian:

$$\begin{aligned}\mathfrak{L} = {} & \mathfrak{L}_I(g_{ij}, \text{matter and nongravitational field variables}) \\ & + (-g)^{1/2}(\alpha g^{ij}\Psi_{,i}\Psi_{,j}/\Psi^2 + \beta g^{ij}\Phi_{,i}\Phi_{,j}/\Phi^2 + \gamma g^{ij}\Phi_{,i}\Psi_{,j}/\Phi\Psi)\,.\end{aligned} \tag{180}$$

Straightforward calculations yield the Euler-Lagrange equations (175) and (176). To obtain the correct Newtonian limit, we must place the following constraints on parameters α, β, and γ:

$$\gamma = 2\lambda\alpha\,, \quad \beta = \lambda^2\alpha + 2\,. \tag{181}$$

By proceeding as in § III*D*(vii), one can derive the PPN parameters listed above.

IV. CONCLUSIONS

This paper has given a partial catalog of metric theories of gravity; it has examined the post-Newtonian limits of each theory; and it has compared each theory with experiment. A subsequent paper will examine conserved integrals, gravitational radiation, and other aspects of each theory. Our catalog of metric theories will be expanded, and nonmetric theories will be studied closely in later papers.

The chief conclusion of this paper obtained by invoking results now in press by Nordtvedt and Will (Will and Nordtvedt 1972; Nordtvedt and Will 1972), and by Ni (1972*a*) is this: of all theories thus far examined by our group, the only currently viable ones are general relativity (§ III*A*); the Bergmann-Wagoner scalar-tensor theory and its special cases (Nordtvedt; Brans-Dicke-Jordan) (§ III*B*); and a recent new vector-tensor theory by Nordtvedt, Hellings, and Will (Will and Nordtvedt 1972). All these viable theories have general relativity as a limiting case, obtained as adjustable parameters tend toward certain limits. For example, scalar-tensor theories go over smoothly to general relativity in the limit as $\omega \rightarrow \infty$, $[d(\ln\omega)/d\varphi] \rightarrow 0$ and $\lambda \rightarrow$ constant (or, in post-Newtonian language, as $\omega \rightarrow \infty$ and $\Lambda \rightarrow 0$). Thus, the best tests of such theories are the relativity experiments of highest precision (light deflection, time delay, perihelion shift, Earth tide, and white-dwarf stability). To distinguish ever more clearly between general relativity and the currently viable theories, one must push these experiments to ever higher precision.

I would like to thank C. M. Will for giving me the results and details of his calculation of the PPN parameters of the scalar-tensor theories and for checking part of my

[20] See n. 2.

calculations. I thank K. Nordtvedt, Jr., and C. M. Will for helpful criticism and for permission to quote their unpublished results. I am deeply obliged to Kip S. Thorne, whose continued advice and support made this paper possible and whose critical comments and help about presentation made it readable.

REFERENCES

Baierlein, R. 1967, *Phys. Rev.*, **162**, 1275.
Bekenstein, J. D. 1971, "Identity of the Black Holes in the Scalar-Tensor Theory with Those of General Relativity" (preprint).
Bergmann, O. 1956, *Am. J. Phys.*, **24**, 39.
Bergmann, P. G. 1968, *Int. J. Theoret. Phys.*, **1**, 25.
Brans, C. H. 1962*a*, *Phys. Rev.*, **125**, 388.
———. 1962*b*, *ibid.*, p. 2194.
Brans, C. H., and Dicke, R. H. 1961, *Phys. Rev.*, **124**, 925.
Brill, D. R. 1962, in *Evidence for Gravitational Theories*, ed. C. Møller (New York: Academic Press).
Chandrasekhar, S. 1965, *Ap. J.*, **142**, 1488.
Cocke, W. J., and Cohen, J. M. 1968, *J. Math. Phys.*, **9**, 971.
Dicke, R. H. 1962, *Phys. Rev.*, **125**, 2163.
———. 1968, *Ap. J.*, **152**, 1.
———. 1969, *Gravitation and the Universe*, Jayne Lectures for 1969 (Philadelphia: American Philosophical Society).
Dicke, R. H., and Goldenberg, H. M. 1967, *Phys. Rev. Letters*, **18**, 313.
Dykla, J. J. 1972 (paper in preparation).
Eddington, A. S. 1922, *The Mathematical Theory of Relativity* (London: Cambridge University Press).
Einstein, A. 1912, *Ann. Phys.*, **38**, 443.
———. 1916, *ibid.*, **49**, 769.
Einstein, A., and Fokker, A. D. 1914, *Ann. Phys.*, **44**, 321.
Estabrook, F. E. 1969, *Ap. J.*, **158**, 81.
Freund, P. G. O., and Nambu, Y. 1968, *Phys. Rev.*, **174**, 1741.
Greenstein, G. S. 1968, *Ap. Letters*, **1**, 139.
Hawking, S. W. 1971, "Black Holes in the Brans-Dicke Theory of Gravitation" (preprint).
Janis, A. J., Robinson, D. C., and Winicour, J. 1969, *Phys. Rev.*, **186**, 1729.
Jordan, P. 1948, *Astr. Nach.*, **276**, 193.
———. 1955, *Schwerkraft und Weltall* (Braunschweig: Friedrich Vieweg and Son).
Kaufmann, W. J., III. 1968, *J. Math. Phys.*, **9**, 1053.
Krogh, C., and Baierlein, R. 1968, *Phys. Rev.*, **175**, 1576.
Landau, L. D., and Lifshitz, E. M. 1962, *The Classical Theory of Fields* (2d ed.; Reading, Mass.: Addison-Wesley Publishing Co.).
Littlewood, D. E. 1953, *Proc. Cambridge Phil. Soc.*, **49**, 90.
Mahanta, M. N., and Reddy, D. R. K. 1971, *J. Math. Phys.*, **12**, 929.
Meister, H. J. 1957, *Zs. Phys.*, **147**, 531.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1972, *Gravitation* (San Francisco: W. H. Freeman & Company).
Morganstern, R. E. 1967, *Phys. Rev.*, **163**, 1357.
———. 1970, *Phys. Rev. D*, **1**, 2969.
———. 1971, *ibid.*, **3**, 2946.
Morganstern, R. E., and Chiu, Hong-Yee. 1967, *Phys. Rev.*, **157**, 1228.
Nariai, H. 1969*a*, "On the Brans Solution in the Scalar-Tensor Theory of Gravitation" (report, Hiroshima University, Hiroshima).
———. 1969*b*, "The Optical Appearance of a Collapsing Star in Terms of the Scalar-Tensor Theory of Gravitation" (report, Hiroshima University, Hiroshima).
Ni, W.-T. 1972*a*, paper in preparation.
———. 1972*b*, paper in preparation.
Noerdlinger, P. D. 1968, *Phys. Rev.*, **170**, 1175.
Nordström, G. 1912, *Phys. Zeit.*, **13**, 1126.
———. 1913, *Ann. Phys.*, **42**, 533.
———. 1914, *ibid.*, **43**, 1101.
Nordtvedt, K., Jr. 1968, *Phys. Rev.*, **169**, 1017.
———. 1969, *ibid.*, **180**, 1293.
———. 1970, *Ap. J.*, **161**, 1059.
Nordtvedt, K., Jr., and Will, C. M. 1972, paper in preparation.
Nutku, Y. 1969, *Ap. J.*, **155**, 999.
O'Connell, R. F., and Salmona, A. 1967, *Phys. Rev.*, **160**, 1108.
Page, C., and Tupper, B. O. J. 1968, *M.N.R.A.S.*, **138**, 67.

Papapetrou, A. 1954*a*, *Math. Nach.*, **12**, 129.
———. 1954*b*, *ibid.*, **12**, 143.
———. 1954*c*, *Zs. Phys.*, **139**, 518.
Robertson, H. P. 1962, in *Space Age Astronomy*, ed. A. J. Deutsch and W. B. Klemperer (New York: Academic Press), p. 228.
Rosen, N. 1971, *Phys. Rev. D*, **3**, 2317.
Salmona, A. 1967, *Phys. Rev.*, **154**, 1218.
Schiff, L. J. 1967, in *Relativity Theory and Astrophysics. I. Relativity and Cosmology*, ed. J. Ehlers (Providence, R.I.: American Mathematical Society).
Shaviv, G., and Bahcall, J. N. 1967, *Pub. A.S.P.*, **79**, 438.
Sramek, R. A. 1971, *Ap. J. (Letters)*, **167**, 55.
Synge, J. L. 1960, *Relativity, the General Theory* (Amsterdam: North-Holland Publishing Company).
Thirry, Y. R. 1948, *C.R.*, **226**, 216.
Thorne, K. S., and Dykla, J. J. 1971, *Ap. J. (Letters)*, **166**, 135.
Thorne, K. S., and Will, C. M. 1971, *Ap. J.*, **163**, 595 (Paper I).
Thorne, K. S., Will, C. M., and Ni, W.-T. 1971, in *Proceedings of the Conference on Experimental Tests of Gravitational Theories*, ed. R. W. Davies (NASA-JPL Technical Memorandum 33-499).
Toton, E. 1968, *Phys. Rev. Letters*, **21**, 1401.
Tupper, B. O. J., and Page, C. 1969, *J. Phys. A, Proc. Phys. Soc. Ser. 2,2.*, p. 521.
Wagoner, R. V. 1970, *Phys. Rev. D*, **1**, 3209.
Whitrow, G. J., and Morduch, G. E. 1960, *Nature*, **188**, 790.
———. 1965, in *Vistas in Astronomy*, Vol. **6**, ed. A. Beer (Oxford: Pergamon Press).
Will, C. M. 1970 (private communication).
———. 1971*a*, *Ap. J.*, **163**, 611 (Paper II).
———. 1971*b*, *ibid.*, **169**, 1256 (Paper III).
Will, C. M., and Nordtvedt, K., Jr. 1972, paper in preparation.
Yilmaz, H. 1958, *Phys. Rev.*, **111**, 1417.
———. 1962, in *Evidence for Gravitational Theories*, ed. C. Møller (New York: Academic Press).

PART THREE
A NEW THEORY OF GRAVITY

(To be published in May 15 1973 issue of *Physical Review D*)

I. INTRODUCTION

Since 1970, the gravitation research group at Caltech has been analyzing the experimental foundations of relativistic theories of gravity. Our results to date are summarized in the "Varenna lecture notes" of Will.[1] Those results had led us to hope that current experiments were good enough to rule out all theories except (i) general relativity, and (ii) theories which reduce to general relativity when their adjustable parameters are appropriately adjusted (e.g., the Brans-Dicke-Jordan theory which reduces to general relativity as $\omega \to \infty$). We also had come to hope that general relativity could be distinguished from all other viable metric theories by the form of its post-Newtonian limit (PPN parameter values $\beta = \gamma = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$).

The purpose of this paper is to explode our hopes. More particularly, this paper will formulate a new theory of gravity which (for certain values of its adjustable parameters) has precisely the same post-Newtonian limit as general relativity, but can never reduce to general relativity in the full, non-linear case.

To distinguish experimentally between this new theory and general relativity, one will have to use non-post-Newtonian experiments. These could include: (i) gravitational-wave experiments, (ii) cosmological observations, and (iii) (in the distant future) post-post-Newtonian experiments. The present paper will not discuss such possibilities. Rather, it will merely present the new theory (§II) and compute its post-Newtonian limit (appendix).

II. PRESENTATION OF THE THEORY

We present the new theory using the notation and format of the author's recent "compendium of gravitation theories".[2] (In particular, note that we set $c = G = 1$.)

a. Gravitational fields present: A flat background metric $\boldsymbol{\eta} = \eta_{ij}\, \underset{\sim}{dx}^i \otimes \underset{\sim}{dx}^j$; scalar fields ϕ and t ; a one-form field $\underset{\sim}{\psi} = \psi_i\, \underset{\sim}{dx}^i$; and the physical metric $\boldsymbol{g} = g_{ij}\, \underset{\sim}{dx}^i \otimes \underset{\sim}{dx}^j$.

b. Arbitrary parameters and functions: Three arbitrary functions $f_1(\phi)$, $f_2(\phi)$, $f_3(\phi)$, and one arbitrary parameter e ; in the post-Newtonian limit, with appropriate choice of the cosmological model, there are four arbitrary parameters, a, b, d, and e (see below).

c. Prior geometry: The following constraints are imposed, à priori, on the geometrical relationships among the gravitational fields:

(i) flatness of the metric $\boldsymbol{\eta}$

$$(\text{Riemann tensor constructed from } \boldsymbol{\eta}) = 0 \quad ; \tag{1a}$$

(ii) "meshing constraints" on t, $\boldsymbol{\eta}$ and $\underset{\sim}{\psi}$

$$t_{|ij} = 0 \quad , \tag{1b}$$

$$t_{,i}\, t_{,j}\, \eta^{ij} = +1 \quad , \tag{1c}$$

(Here and below a slash denotes a covariant derivative with respect to $\boldsymbol{\eta}$, and η^{ij} is the inverse of η_{ij}.)

$$t_{,i}\, \psi_j\, \eta^{ij} = 0 \; ; \tag{1d}$$

(iii) algebraic equation for the physical metric in terms of the "auxiliary gravitational fields" $\boldsymbol{\eta}$, ϕ , t , $\underset{\sim}{\psi}$

$$\boldsymbol{g} = f_2(\phi)\boldsymbol{\eta} + [f_1(\phi) - f_2(\phi)]\underset{\sim}{dt} \otimes \underset{\sim}{dt} + \underset{\sim}{\psi} \otimes \underset{\sim}{dt} + \underset{\sim}{dt} \otimes \underset{\sim}{\psi} \; . \tag{1e}$$

d. Preferred coordinate system: The prior-geometric constraints (1) guarantee the existence of a preferred coordinate system in which (i) the time coordinate is equal to the scalar field t ; (ii) the components of $\boldsymbol{\eta}$ are Minkowskiian

$$\eta_{ij} = \text{diagonal } (1,-1,-1,-1) \quad ; \tag{2a}$$

(iii) $\underset{\sim}{\psi}$ is purely spatial

$$\psi_o = 0 \quad ; \tag{2b}$$

(iv) the physical line element $g_{ij}dx^i dx^j$ is

$$ds^2 = f_1(\phi)dt^2 - f_2(\phi)(dx^2+dy^2+dz^2)$$
$$+ 2\psi_1 dxdt + 2\psi_2 dydt + 2\psi_3 dzdt \quad . \tag{2c}$$

e. Lagrangian: The field equations are determined by an action principle

$$\delta \int \mathcal{L} \, d^4x = 0 \quad , \tag{3a}$$

where the Lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = L_I\sqrt{-g} + 2\{(1/e)\psi_{i|k}\psi_{j|\ell}\eta^{ij}\eta^{k\ell} - \phi_{,i}\phi_{,j}\eta^{ij}$$
$$+ [f_3(\phi)+1](\phi_{,i}t_{,j}\eta^{ij})^2\} \sqrt{-\eta} \tag{3b}$$

Here L_I is the standard "interaction Lagrangian" obtained by taking the standard Lagrangian for matter and nongravitational fields in flat spacetime, and replacing the Minkowskii metric by $\boldsymbol{g}$ (equivalence principle). The quantities g and η are the determinants of $\| g_{ij} \|$ and $\| \eta_{ij} \|$. In the action principle (3a) one is to vary the standard matter and nongravitational fields that appear in L_I, and the gravitational fields ϕ and $\underset{\sim}{\psi}$, while maintaining the prior-geometric constraints (1). In the preferred coordinate system (2) the Lagrangian density reduces to

$$\mathcal{L} = L_I\sqrt{-g} + (2/e)(\psi_{\alpha,\beta}\,\psi_{\alpha,\beta} - \psi_{\alpha,t}\,\psi_{\alpha,t}) + 2\phi_{,\alpha}\,\phi_{,\alpha} + 2f_3(\phi)\phi_{,t}\,\phi_{,t} \quad (4)$$

(Summation on repeated Greek indices).

f. Field equations: The nongravitational field equations derived from this action principle take on their standard general relativistic form ("equivalence printiple;" "comma-goes-to-semicolon rule"). The gravitational field equations derived from the action principle are

$$\psi_{i|j}{}^{|j} = 2\pi e(\sqrt{-g}/\sqrt{-\eta})\, T^{k\ell}(\partial g_{k\ell}/\partial\psi_j)(\eta_{ij} - t_{|i}\, t_{|j})$$

$$\phi_{|j}{}^{|j} - [f_3(\phi) + 1]\, \phi_{|i}{}^{|j}\, t^{|i}\, t_{|j} + \tfrac{1}{2} f_3{}'(\phi)(\phi_{|j}\, t^{|j})^2$$

$$= -2\pi(\sqrt{-g}/\sqrt{-\eta})\, T^{ij}(\partial g_{ij}/\partial\phi); \quad (5a)$$

in the preferred coordinate system, these equations reduce to

$$\psi_{\beta,\alpha\alpha} - \psi_{\beta,tt} = 4\pi e\sqrt{-g}\, T^{0\beta}$$

$$\phi_{,\alpha\alpha} + f_3(\phi)\phi_{,tt} - \tfrac{1}{2} f_3{}'(\phi)\phi_{,t}\,\phi_{,t} - 2\pi\sqrt{-g}\, T^{ij}(\partial g_{ij}/\partial\phi) = 0. \quad (5b)$$

Here the stress-energy tensor is the same as appears in the field equations of general relativity:

$$T_{ij} \equiv -\frac{1}{8\pi}\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\,L_I)}{\partial g^{ij}}\ ; \qquad T^{k\ell} \equiv g^{ik}g^{j\ell}T_{ij}\ . \quad (6)$$

g. Post-Newtonian limit: Expand the arbitrary functions $f_1(\phi)$, $f_2(\phi)$, and $f_3(\phi)$ in powers of ϕ. In order that the metric will become flat in the absence of gravity ($\phi = \psi = 0$), require $f_1(0) = f_2(0) = 1$. In order that the theory will reduce to Newton's theory in the weak-field, slow-motion limit, require $f_1(\phi) = 1 - 2\phi + \cdots$. Define a,b,d to be the coefficients of the first unconstrained terms ("post-Newtonian terms") in the expansions:

$$f_1(\phi) = 1 - 2\phi + 2b\phi^2 + \cdots , \tag{7a}$$

$$f_2(\phi) = 1 + 2a\phi + \cdots , \tag{7b}$$

$$f_3(\phi) = d + \cdots . \tag{7c}$$

Impose the "cosmological boundary conditions"

$\underset{\sim}{\psi} = \phi = 0$ far from the solar system (or whatever other system is being analyzed). (7d)

(Note: The values of $\underset{\sim}{\psi}$ and ϕ in interstellar space must actually be determined by the cosmological model. This paper makes no attempt at constructing cosmological models. However, it seems that, in order to exhibit large-scale homogeneity and isotropy as viewed from Earth, the cosmological model will have to set $\underset{\sim}{\psi} \simeq 0$ and $\phi \simeq \phi(t)$ in the neighborhood of the solar system. by a redefinition of ϕ and renormalization of constants, one can then set $\phi \simeq 0$ far from the solar system in the present epoch.) Then the post-Newtonian limit of the theory reduces to the Nordtvedt-Will[3] PPN formalism with PPN parameter values

$$\gamma = a , \quad \beta = b , \quad \alpha_1 = -2e - 4a - 4 , \quad \alpha_2 = -d - 1 ,$$
$$\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 \quad . \tag{8}$$

(The proof is given in an appendix.)

h. <u>Comparison with experiment</u>. By comparing the PPN-parameter values (8) with the list of experimental limits on PPN parameters as given by Ni,[4] one learns that <u>this theory agrees with all experiments to date</u> if

$$0.96 < a < 1.12 \text{ (time delay experiments)}$$
$$0.84 < b < 1.34 \text{ (perihelion shift plus time delay experiments)}$$
$$-1.03 < d < -0.97 \text{ (Earth-tide measurements)} \qquad (9)$$
$$-2.2 < e+2a < -1.8 \text{ (Earth rotation rate experiments).}$$

i. Comparison with general relativity. Notice that if

$$a = b = 1 \ , \quad d = -1 \ , \quad e = -4 \qquad (10)$$

then this theory has precisely the same post-Newtonian limit as general relativity! Thus, no post-Newtonian experiment can hope to make a clean distinction between this theory and general relativity.

j. Comparison with other Lagrangian-based theories. Will[5] and Ni[6] have shown that all Lagrangian-based metric theories whose post-Newtonian limits can be put into PPN form must satisfy the PPN parameter constraints

$$\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 \quad . \qquad (11)$$

Notice that the theory presented here has arbitrary values for all the remaining, unconstrained parameters. Thus, this theory possesses a most general post-Newtonian limit permitted for any Lagrangian-based metric theory.[7] This means that no post-Newtonian experiment can hope to make a clean distinction between this theory and any other Lagrangian based, metric theory which has PPN form post-Newtonian limit.

k. Special cases. When the arbitrary functions $f_1(\phi)$, $f_2(\phi)$, and $f_3(\phi)$ are suitably specialized, one obtains the following theories: "Papapetrou II" [see §§III.D.vi of Reference 2], and Ni's "Lagrangian-based, stratified theory" [see §III.D.vii of Reference 2].

1. <u>Conservation laws and gravitational radiation</u>. Global conservational laws and gravitational radiation for this theory will be discussed in a future paper.

III. CONCLUDING REMARKS

This theory requires considerable further study. Crucial items in testing it will be (i) its success or failure to produce cosmological models that agree with the large-scale features of our Universe, and (ii) the properties (polarization, intensity, and propagation speed) of its gravitational waves.

ACKNOWLEDGMENT

I am deeply obliged to Professor Kip S. Thorne for many valuable comments and for great assistance with the presentation.

APPENDIX

COMPUTATION OF THE POST-NEWTONIAN LIMIT

To obtain the post-Newtonian limit of our theory, we proceed as follows. For convenience, we shall work in the preferred coordinate system; and we shall set $\phi \to 0$ and $\underset{\sim}{\psi} \to 0$ as the "field point" $|\underset{\sim}{x}|$ goes to infinity [see remarks following Eq. (7d)]. Let

$$\begin{aligned} \phi &= \phi_1 + \phi_2 + O(6) \\ \psi_\beta &= \psi_{\beta 2} + O(5) \end{aligned} \tag{A1}$$

where $\phi_1 = O(2)$, $\phi_2 = O(4)$, and $\psi_{\beta 2} = O(3)$. [Here "$O(n)$" means of order "c^{-n}" in a post-Newtonian expansion.] Correct to post-Newtonian order, the field equations (5b) are

$$\begin{aligned} -d\frac{\partial^2\phi_1}{\partial t^2} - \nabla^2\phi_1 - \nabla^2\phi_2 &= 4\pi\rho[1 + (3a-1)U][1 + (2-2b)U \\ &\quad + v^2(1+a) + 3a\frac{p}{\rho} + \Pi] , \end{aligned} \tag{A2}$$

$$\nabla^2\psi_{\beta 2} = 4\pi e\,\rho\, v_\beta .$$

The $O(2)$ part of the field equations is

$$\nabla^2\phi_1 = -4\pi\rho , \tag{A3}$$

i.e.,

$$\phi_1 = U \tag{A4}$$

where U is the Newtonian potential. The $O(3)$ part is

$$\nabla^2\psi_{\beta 2} = 4\pi e\,\rho\, v_\beta , \tag{A5}$$

i.e.,

$$\psi_{\beta 2} = -eV_\beta = -e \int \frac{\rho(\underset{\sim}{x}',t)\; v_\beta(\underset{\sim}{x}',t)d\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \tag{A6}$$

The O(4) part is

$$\nabla^2\phi_2 = -dU_{,tt} - 4\pi\rho[(3a+1-2b)U + (1+a)v^2 + 3a\frac{p}{\rho} + \Pi] \tag{A7}$$

Let χ be the solution of

$$\nabla^2\chi = -2U \quad , \tag{A8}$$

i.e.,

$$\chi = -\int \rho|\underset{\sim}{x} - \underset{\sim}{x}'|d\underset{\sim}{x}' \quad . \tag{A9}$$

We can transform equation (A7) to

$$\nabla^2(\phi_2 - \frac{1}{2}d\cdot\chi_{,tt}) = -4\pi\rho[(3a+1-2b)U + (1+a)v^2 + 3a\frac{p}{\rho} + \Pi] \tag{A10}$$

Therefore

$$\phi_2 = \frac{1}{2}d\cdot\chi_{,tt} + 2\Phi \quad , \tag{A11}$$

where

$$\nabla^2\Phi = -4\pi\rho[\frac{1}{2}\Pi + \frac{1}{2}(3a+1-2b)U + \frac{1+a}{2}U + \frac{3}{2}a\frac{p}{\rho}] \quad . \tag{A12}$$

Combining equations (A4) and (A11), we find

$$\phi = U + 2\Phi + \frac{1}{2}d\chi_{,tt} + 0(6) \quad . \tag{A13}$$

According to equations (2c), (A6) and (A13), the physical metric is

$$g_{oo} = 1 - 2U + 2bU^2 - 4\Phi - d\chi_{,tt} + 0(6) \tag{A14}$$

$$g_{o\alpha} = -eV_\alpha \qquad \text{(A14)}$$

$$g_{\alpha\beta} = -\delta_{\alpha\beta}(1 + 2aU) \quad .$$

By using the gauge transformation

$$x^{o\dagger} = x^o - \frac{1}{2}\, d\chi_{,t}$$

$$x^{\alpha\dagger} = x^\alpha \quad , \qquad \text{(A15)}$$

we can transform the metric into the form

$$g_{oo}^{\dagger} = 1 - 2U + 2bU^2 - 4\Phi + O(6) \quad ,$$

$$g_{o\alpha}^{\dagger} = (\frac{1}{2}d - e)V_\alpha - \frac{1}{2}\, dW_\alpha + O(5) \quad , \qquad \text{(A16)}$$

$$g_{\alpha\beta}^{\dagger} = -(1 + 2aU) \quad ,$$

where

$$W_\alpha(\underset{\sim}{x},t) = \int \frac{\rho(\underset{\sim}{x}',t)v_\beta(\underset{\sim}{x}')(x_\beta - x_\beta')(x_\alpha - x_\alpha')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}\, dx' \qquad \text{(A17)}$$

By comparing this with the PPN metric as given by Will and Nordtvedt,[8] we obtain the PPN parameter values listed in Eq. (8).

REFERENCES

1 C. M. Will, Lectures presented at the International School of Physics "Enrico Fermi", Varenna, Italy, July 17 to July 29, 1972 (to be published in the Proceedings of the School).

2 W.-T. Ni, Astrophys. J. **176**, 769 (1972).

3 C. M. Will and K. Nordtvedt, Jr., Astrophys. J. **177**, 757 (1972).

4 Reference 2.

5 C. M. Will, Astrophys. J. **169**, 125 (1971).

6 W.-T. Ni, paper in preparation.

7 Exception: One could conceive of--but one has no examples of--Lagrangian-based, metric theories with post-Newtonian limits that are more complex than the Nordvedt-Will 9-parameter formalism. Our results do not apply to such theories.

8 Reference 3.

PART FOUR
RELATIVISTIC STELLAR STABILITY: AN EMPIRICAL APPROACH

(To be published in May 1973 issue
of *The Astrophysical Journal*)

I. INTRODUCTION AND SUMMARY

Relativistic corrections to Newtonian gravity should induce dynamical instabilities in stars with adiabatic indices slightly greater than 4/3. This fact was first discovered, within the framework of General Relativity (GR), by Chandrasekhar (1964a,b) and independently by Feynman [unpublished, but quoted in Fowler (1964)]. More recently Nutku (1969) has shown that the same type of instability is predicted by the Brans-Dicke theory of gravity (BDT), but that it is slightly weaker (stars are slightly more stable) than in GR. If the dynamical relativistic instability actually exists, as predicted by GR and BDT, then it plays a fundamental role in white dwarfs, in neutron stars, and in supermassive stars [see e.g. Thorne (1967) or Zel'dovich and Novikov (1971) for a review].

But it is conceivable that neither GR nor BDT is the correct relativistic theory of gravity. If so, might the relativistic instability not exist? Is it conceivable that relativistic effects would stabilize stars rather than destabilize them? William A. Fowler has asked this question of gravitation theorists so often at Caltech, that we have felt compelled to seek a firm answer. The most firm answer possible is one which relies heavily on experimental tests of relativistic gravitational effects, while assuming nothing (or almost nothing) about which relativistic theory of gravity is correct.

Of course, one cannot work in a complete theoretical vacuum. A minimal amount of theory is required to link the relativistic instability in stars to solar-system measurements of perihelion shift, light deflection, radar time delay, etc. That the amount of theory needed is small, however, one can see heuristically by noticing that both the perihelion shift and the relativistic instability are caused by a relativistic strengthening of Newtonian gravitational forces. [Stronger gravity than predicted by Newton when a star contracts means greater force to pull the star on inward, i.e. means less stability; stronger gravity than predicted by Newton when a planet approaches close to the sun (perihelion) means greater force to "whip" the planet around, and a resultant advance of its perihelion.]

The purpose of this paper is to derive a quantitative measure of the extent to which solar-system experiments imply the existence of the dynamical relativistic instabilities in stars. The "minimal amount of theory" to be used in the derivation is the Parametrized Post-Newtonian ("PPN") Framework of Nordtvedt and Will (Will and Nordtvedt 1972; Will 1971a; Nordtvedt 1968).

The PPN Framework is a post-Newtonian theory of gravity with adjustable parameters. In Will's fluid version, it has nine PPN parameters, γ, β, α_1, α_2, α_3, ζ_1, ζ_2, ζ_3, and ζ_4. The parameter γ measured curvature of the space-geometry; β measures the non-linearity of gravity; α_1, α_2, and α_3 measure "preferred-frame" effects; ζ_1, ζ_2, ζ_3, and ζ_4 measure the effects resulting from a breakdown of conservation laws. For theories which have no "preferred-frame" effects, all α's vanish (Nordtvedt and Will 1972; Will 1971b). For theories which have conservation laws for energy, momentum, angular momentum, and center-of-mass motion ("conservative thoeries"), all α's and ζ's vanish (Will 1971b). The post-Newtonian limit of every "metric theory of gravity"[1]

---

[1]Metric theories of gravity are a wide class of theories including (i) every theory that satisfies the equivalence principle (laws of physics in local Lorentz frames the same as in special relativity), and (ii) every

theory that the Caltech group has thus far examined and found to be complete, self-consistent, and in agreement with experiment. See Thorne, Will, and Ni (1971); Ni (1972) and Will (1972b) for full discussions.

---

known to us [except Whitehead's theory which is non-viable (Will 1971c] is a special case of the PPN Framework, corresponding to particular values of the PPN parameters. Ni (1972) has calculated the values of the parameters for a variety of theories, including general relativity, the scalar-tensor theories of Bergmann-Wagoner, Nordtvedt, and Brans-Dicke-Jordan, the conformally-flat theories of Witrow-Morduch, Littlewood-Bergmann and Nordström, and the stratified theories of Einstein, Witrow-Morduch, Page-Tupper, Yilmaz, Papapetrou and Rosen.

Experiments to date have placed the following limits on the PPN parameters [see Thorne, Will, and Ni (1971) or Will (1972b) for detailed discussion; see also Nordtvedt and Will (1972)]:

$\gamma = 1.04 \pm 0.08$ [time delay and deflection experiments except that of Sramek (1972)] (1)

$\gamma = 0.88 \pm 0.12$ [Sramek's (1972) deflection experiment] (2)

$\beta = 1.14 {}^{+0.2}_{-0.3}$ [perihelion shift plus time delay experiments] (3)

$|\zeta_4 - \frac{1}{3}\zeta_1 - \frac{1}{2}\alpha_3| \lesssim 0.4$ [Kreuzer (1966, 1968) measurement of $m_{active}/m_{passive}$][2] (4)

$|\zeta_3| \lesssim 0.05$ [Kreuzer (1966, 1968) measurement of $m_{active}/m_{passive}$][2] (5)

$|\alpha_1| \lesssim 0.2$ [Earth rotation rate experiments (Nordtvedt and Will 1972)] (6)

$|\alpha_2| \lesssim 0.03$ [Earth-tide measurements (Will 1971b)] (7)

$|\alpha_3| \lesssim 2 \times 10^{-5}$ [perihelion shift observations (Nordtvedt and Will 1972)] (8)

---

[2]Kreuzer's (1966) analysis of his data was completely correct, despite a recent claim to the contrary by Gilvarry and Muller (1972). Gilvarry and Muller err in making a quadratic fit to Kreuzer's data, rather than restricting themselves to a linear fit as did Kreuzer. Kreuzer measured the expansion of his liquid relative to teflon over a wide temperature range and thereby showed experimentally that the quadratic correction to the linear behavior must be negligibly small over the small temperature range used for the experiment. Moreover, the magnitude of the quadratic term obtained by Gilvarry and Muller using their least-squares fits is ridiculously large for any but pathological materials. We thank R. H. Dicke for a helpful discussion of these points.

In this paper it is shown that for conservative theories of gravity current experimental limits on the PPN parameters — based on perihelion shift, time delay, and deflection experiments — guarantee the existence of the dynamical relativistic instability in stars; while for non-conservative theories the present, experimentally undetermined state of the two PPN parameters ζ_2 and ζ_4 makes it uncertain whether relativistic effects will actually stabilize or destabilize stars. In quantitative terms, a non-rotating spherically symmetric star with adiabatic index

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln n}\right)_s = \frac{\rho + p}{p}\left(\frac{\partial p}{\partial \rho}\right)_s \qquad (9)$$

constant throughout its interior is unstable against adiabatic radial perturbations if and only if its radius R and geometrized mass (2M ≡ Schwarzschild radius) satisfy

$$\Gamma_1 - 4/3 \leq K(2M/R) \quad . \tag{10}$$

Here K is a constant that depends on the star's structure and upon the PPN parameters. If K is positive, there is a relativistic instability. If K is negative, relativity stabilizes the star. In the Newtonian limit K = 0. In GR and BDT K is positive and of order unity. Values of K for polytropic gas spheres, as evaluated in §IV of this paper, are tabulated in Tables 1 and 2.

Table 1 lists values of K for polytropic stars in the case of conservative theories of gravity. From the positive signs of the minimum values of K (column 3), we have the following conclusion: for conservative theories which are compatible with current solar-system experiments, relativistic corrections to Newtonian theory will always induce dynamical instabilities. It is interesting to note that γ has a positive contribution to K while β has a negative contribution; the same is true for the perihelion advance. This, together with the positivity of K, confirms the heuristic argument given at the beginning of this section.

Table 2 lists the values of K for the general PPN formalism and for several particular non-conservative theories. The third column gives minimum values of K corresponding to current experimental limits on the PPN parameters. If ζ_2 or ζ_4 (which are undetermined by experiments to date) were sufficiently negative, then K would be negative. For example, for the currently viable cases $\{\gamma = 0.76,\ \beta = 1.34,\ \zeta_2 = -0.5,\ \zeta_3 = \zeta_4 = 0\}$, and $\{\gamma = \beta = 1,\ \zeta_2 = -2.2,\ \zeta_3 = \zeta_4 = 0\}$ the value of K is negative.

Therefore we arrive at the following conclusion due to the lack of experimental information on ζ_2 and ζ_4, it is inconclusive whether relativistic effects will actually stabilize or destabilize stars. From the last three columns, one may notice that the Vector-Metric theory (Will and Nordtvedt 1972) and the Papapetrou (1954a,b,c) theories have the same K-values as general relativity, while K-values for the Modified Yilmaz theory (Ni 1972) are all negative.

Other aspects of dynamical stellar pulsations are also investigated in this paper. The Schwarzschild criterion is found to hold for the onset of dynamical instability against non-radial oscillations (convection). Sufficient conditions for self-adjointness of the linearized pulsation equations are derived. These conditions together with the condition $\zeta_1 = 0$ coincide with Will's conditions for the existence of ten post-Newtonian conserved integrals.

In §II the PPN formalism is summarized, the linearized pulsation equations are derived, and "preferred-frame terms" (which lead to vibrational-secular and other Machian-type instabilities) are separated out of the pulsation equations and reserved for study in a future paper. Section III derives a variational principle for dynamical stellar stability. Section IV derives the post-Newtonian conditions for the onset of a dynamical instability. Section V derives the Schwarzschild criterion for non-radial instabilities. Concluding remarks are make in §VI. An Appendix treats the problem of self-adjointness.

Throughout this paper, we follow closely the methods of Chandrasekhar (1965b), and we use geometrized units. The notations and conventions of this paper are the same as those of Chandrasekhar (1965b), and Will and

Nordtvedt (1972) — unless otherwise specified.

II. PPN FORMALISM AND EQUATIONS OF MOTION FOR SMALL OSCULLATIONS ABOUT EQUILIBRIUM

In the PPN formalism one describes the response of matter to gravity by the "local law of energy-momentum conservation"

$$\underset{\sim}{\nabla} \cdot \mathbf{T} = 0 \tag{11}$$

(where $\mathbf{T}$ is the stress-energy tensor, and $\underset{\sim}{\nabla}$ is the covariant derivative with respect to the PPN metric); and one describes the generation of gravity by matter in terms of the PPN metric (Will and Nordtvedt 1972):

$$\begin{aligned} g_{00} = {} & 1 - 2U + 2\beta U^2 - (2\gamma + 2 + \alpha_3 + \zeta_1)\Phi_1 + \zeta_1 \mathcal{A} \\ & - 2[(3\gamma - 2\beta + 1 + \zeta_2)\Phi_2 + (1 + \zeta_3)\Phi_3 + 3(\gamma + \zeta_4)\Phi_4] \\ & + (\alpha_1 - \alpha_2 - \alpha_3)w^2 U + \alpha_2 w^\alpha w^\beta U_{\alpha\beta} - (2\alpha_3 - \alpha_1)w^\alpha V_\alpha \quad , \end{aligned} \tag{12}$$

$$\begin{aligned} g_{0\alpha} = {} & \tfrac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)V_\alpha + \tfrac{1}{2}(1 + \alpha_2 - \zeta_1)W_\alpha \\ & + \tfrac{1}{2}(\alpha_1 - 2\alpha_2)w^\alpha U + \alpha_2 w^\beta U_{\alpha\beta} \quad , \end{aligned}$$

$$g_{\alpha\beta} = -(1 + 2\gamma U)\delta_{\alpha\beta} \quad .$$

Here $\underset{\sim}{w}$ is the velocity of the chosen coordinate frame relative to the "preferred-frame" of the Universe (if any); and

$$U(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' \quad , \tag{13}$$

$$\Phi_1(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t) v^2(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' , \quad (14)$$

$$\Phi_2(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t) U(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' , \quad (15)$$

$$\Phi_3(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t)\ \Pi(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' , \quad (16)$$

$$\Phi_4(\underset{\sim}{x}, t) = \int \frac{p(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' , \quad (17)$$

$$\mathcal{A}(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t)\ [\underset{\sim}{v}(\underset{\sim}{x}', t) \cdot (\underset{\sim}{x} - \underset{\sim}{x}')]^2}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} d\underset{\sim}{x}' , \quad (18)$$

$$V_\alpha(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t)\ v_\alpha(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' , \quad (19)$$

$$W_\alpha(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t)\ v_\beta(\underset{\sim}{x}', t)\ (x_\beta - x_\beta')\ (x_\alpha - x_\alpha')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} d\underset{\sim}{x}' , \quad (20)$$

$$U_{\alpha\beta}(\underset{\sim}{x}, t) = \int \frac{\rho(\underset{\sim}{x}', t)\ (x_\alpha - x_\alpha')(x_\beta - x_\beta')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} d\underset{\sim}{x}' . \quad (21)$$

Here ρ is rest-mass density, p is pressure, and Π is specific internal energy all measured in the matter's rest frame, and $v_\alpha = dx^\alpha/dt$ is the matter's coordinate velocity.

The equations of hydrodynamics governing a perfect fluid follow from equations (11), (12) and the form of the stress-energy tensor (Will 1972a):

$$\frac{\partial}{\partial t}(\sigma v^\alpha) + \frac{\partial}{\partial x^\beta}(\sigma v^\alpha v^\beta) - \rho \frac{\partial U}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}\Big\{p[1 + (3\gamma - 1)U]\Big\}$$

$$+ \rho \frac{d}{dt}[(5\gamma - 1)\ U\ v^\alpha - \frac{1}{2}(4\gamma + 4 + \alpha_1)\ V_\alpha - \frac{1}{2}\alpha_1\ U\ w^\alpha] \quad (22)$$

$$- \frac{1}{2}(\alpha_2 - \zeta_1 + 1)\,\rho\,\frac{\partial}{\partial t}(W_\alpha - V_\alpha)$$

$$+ \frac{1}{2}\,\rho[\,(4\gamma + 4 + \alpha_1)\,v^\beta + (\alpha_1 - 2\alpha_3)\,w^\beta]\,\frac{\partial V_\beta}{\partial x^\alpha}$$

$$- \rho\,\frac{\partial}{\partial x^\alpha}\,[2\Phi - \frac{1}{2}\,\zeta_1\,\mathcal{A} - \frac{1}{2}\,\alpha_2\,w^\alpha\,w^\beta\,U_{\alpha\beta} + \alpha_2\,w^\beta\,\chi_{,\beta 0}]$$

$$- \rho\,\frac{\partial U}{\partial x^\alpha}\,[2\bar{\phi} - \frac{1}{2}\,\alpha_1\,\underset{\sim}{w}\cdot\underset{\sim}{v} + \frac{1}{2}\,(\alpha_2 + \alpha_3 - \alpha_1)\,w^2] = 0 \qquad \text{(22 cont'd.)}$$

and

$$\frac{\partial\rho^*}{\partial t} + \frac{\partial(\rho^* v^\alpha)}{\partial x^\alpha} = 0 \qquad (23)$$

where

$$\sigma = \rho(1 + v^2 + 2U + \Pi + p/\rho) \qquad (24)$$

$$\phi = \frac{1}{4}(\alpha_3 + 2\gamma + 2 + \zeta_1)\,v^2 + \frac{1}{2}(3\gamma + 1 + \zeta_2 - 2\beta)\,U + \frac{1}{2}(1 + \zeta_3)\,\Pi + \frac{3}{2}(\gamma + \zeta_4)p/\rho \qquad (25)$$

$$\bar{\phi} = \frac{1}{2}(\gamma + 1)\,v^2 + \frac{1}{2}(3\gamma - 2\beta + 1)\,U + \frac{1}{2}\,\Pi + \frac{3}{2}\,\gamma\,p/\rho \qquad (26)$$

$$\rho^* = \rho(1 + \frac{1}{2}\,v^2 + 3\gamma U) \qquad (27)$$

$$\chi(\underset{\sim}{x}, t) = \int \rho(\underset{\sim}{x}', t)\,|\underset{\sim}{x} - \underset{\sim}{x}'|\,d\underset{\sim}{x}' \qquad (28)$$

$$\Phi(x, t) = \int \frac{\rho(\underset{\sim}{x}', t)\,\phi(\underset{\sim}{x}', t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\,d\underset{\sim}{x}' \quad . \qquad (29)$$

Consider an equilibrium spherically symmetric distribution of matter. The equation of hydrostatic equilibrium, which follows from (24), is

$$\frac{d}{dr}\Big\{[1 + (3\gamma - 1)\,U]\,p\Big\} = \rho[1 + (\frac{1}{3}\,\alpha_2 + \frac{1}{2}\,\alpha_3 - \frac{1}{2}\,\alpha_1)\,w^2]\,\frac{dU}{dr} + 2\rho\left(\frac{d\Phi}{dr} + \bar{\phi}\,\frac{dU}{dr}\right) \ . \qquad (30)$$

Let the equilibrium configuration be slightly perturbed and describe its perturbation by a Lagrangian displacement of the form

$$\underset{\sim}{\xi}(x)\, e^{i\Omega t} \quad . \tag{31}$$

The linearized form of the equations governing the perturbation, as derived by combining equations (22), (23), and (30), is

$$\Omega^2\left\{\sigma\xi^\alpha + \frac{1}{2}(\alpha_2 - \zeta_1 + 1)\,\rho(V_\alpha - W_\alpha) + [(5\gamma - 1)\,\xi^\alpha U - \frac{1}{2}(\alpha_1 + 4\gamma + 4)\,V_\alpha]\right\}$$

$$+ i\Omega\left[\frac{1}{2}\,\alpha_1\,\rho\Delta U w^\alpha - \frac{1}{2}\rho(\alpha_1 - 2\alpha_3)\,w^\beta\,\frac{\partial V_\beta}{\partial x^\alpha} + \alpha_2\rho\,\frac{\partial^2(\Delta\chi)}{\partial x^\alpha \partial x^\beta}\,w^\beta - \frac{1}{2}\,\alpha_1 w^\beta \xi^\beta \rho\,\frac{\partial U}{\partial x^\alpha}\right]$$

$$= \frac{\partial}{\partial x^\alpha}\left\{[1 + (3\gamma - 1)\,U]\,\Delta p + (3\gamma - 1)\,p\Delta U\right\} - \frac{\Delta\rho}{\rho}\,\frac{\partial}{\partial x^\alpha}\left\{[1 + \frac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1)\,w^2 + (3\gamma - 1)\,U]p\right\}$$

$$- \rho[1 + (\frac{1}{3}\,\alpha_2 + \frac{1}{2}\,\alpha_3 - \frac{1}{2}\,\alpha_1)\,w^2]\,\frac{\partial\Delta U}{\partial x^\alpha} - 2\rho(\Delta\bar{\phi}\,\frac{\partial U}{\partial x^\alpha} + \bar{\phi}\,\frac{\partial\Delta U}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}\,\Delta\Phi)$$

$$+ \frac{1}{2}\,\rho\alpha_2 w^\gamma w^\delta\,\frac{\partial}{\partial x^\alpha}\,(\Delta U_{\gamma\delta} - \frac{1}{3}\,\delta_{\gamma\delta}\,\Delta U) \quad , \tag{32}$$

and

$$\Delta\rho^* = -\,\rho^*\,\mathrm{div}\,\underset{\sim}{\xi} \quad . \tag{33}$$

Here and henceforth σ, ϕ, $\bar{\phi}$, and ρ^* are defined by equations (24), (25), (26), and (27), with all quadratic velocity terms (v^2 terms) omitted; and V_α and W_α are given by definitions (19) and (20) with $v_\alpha(\underset{\sim}{x}')$ replaced by $\xi^\alpha(\underset{\sim}{x}')$. [New V_α and W_α equal to $(1/i\Omega)$ times old V_α and W_α.] The symbol Δ denotes the Lagrangian change in the quantity that it qualifies.

Now we must evaluate the Lagrangian changes for various quantities explicitly in terms of $\underset{\sim}{\xi}$. From the definition of ρ^* and equation (33), it follows that

$$\Delta\rho = -\,\rho(\mathrm{div}\,\underset{\sim}{\xi} + 3\gamma\Delta U) \quad , \tag{34}$$

correct to post-Newtonian order. Similarly, the first law of thermodynamics and the definition of "the adiabatic index" Γ_1 lead to

$$\Delta\Pi = (p/\rho^2) \cdot \Delta\rho \ , \qquad \Delta p = \Gamma_1 (p/\rho)\, \Delta\rho \tag{35}$$

respectively, Therefore,

$$\Delta\phi = \frac{1}{2}(3\gamma + 1 + \zeta_2 - 2\beta)\,\Delta U - \frac{1}{2}\frac{p}{\rho}[(3\Gamma_1 - 1)(\gamma + \zeta_4) + 1 + \zeta_3)](\mathrm{div}\ \underset{\sim}{\xi} + 3\gamma\Delta U) \tag{36}$$

and

$$\Delta\bar{\phi} = \frac{1}{2}(3\gamma - 2\beta + 1)\,\Delta U - \frac{1}{2}\frac{p}{\rho}\,(3\Gamma_1\gamma - 3\gamma + 1)(\mathrm{div}\ \underset{\sim}{\xi} + 3\gamma\Delta U) \ . \tag{37}$$

Finally, the expressions for ΔU, $\Delta U_{\gamma\delta}$, ΔX, and $\Delta\Phi$ can be written down from equations (13), (21), (28), (29) as follows:

$$\Delta U = \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} U + \int_V \rho(\underset{\sim}{x}')\, \xi^\alpha(\underset{\sim}{x}')\, \frac{\partial}{\partial x^{\alpha'}} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}'$$
$$- 3\gamma \int_V \frac{\rho(\underset{\sim}{x}')\, \Delta U(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}' \tag{38}$$

$$\Delta U_{\gamma\delta} = \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} U_{\gamma\delta} + \int_V \rho(\underset{\sim}{x}')\, \underset{\sim}{\xi}^\alpha(\underset{\sim}{x}')\, \frac{\partial}{\partial x^{\alpha'}} \cdot \frac{(x_\gamma - x_\gamma')(x_\delta - x_\delta')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}\, d\underset{\sim}{x}'$$
$$- 3\gamma \int_V \rho(\underset{\sim}{x}')\, \Delta U(\underset{\sim}{x}')\, \frac{(x_\gamma - x_\gamma')(x_\delta - x_\delta')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}\, d\underset{\sim}{x}' \tag{39}$$

$$\Delta X = \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} X + \int_V \rho(\underset{\sim}{x}')\, \xi^\alpha(\underset{\sim}{x}')\, \frac{\partial}{\partial x^{\alpha'}} |\underset{\sim}{x} - \underset{\sim}{x}'|\, d\underset{\sim}{x}'$$
$$- 3\gamma \int_V \rho(\underset{\sim}{x}')\, \Delta U(\underset{\sim}{x}')\, |\underset{\sim}{x} - \underset{\sim}{x}'|\, d\underset{\sim}{x}' \tag{40}$$

and

$$\Delta\Phi = \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla}\Phi + \int_V \rho(\underset{\sim}{x}')\, \phi(\underset{\sim}{x}')\, \xi^\alpha(\underset{\sim}{x}')\, \frac{\partial}{\partial x^{\alpha'}} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}'$$
$$+ \int_V \frac{\rho(\underset{\sim}{x}')\, \Delta\phi(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}' \ . \tag{41}$$

The last two terms in equations (38), (39), (40), and (41) make up the Eulerian changes in U, $U_{\gamma\delta}$, χ and Φ, corresponding to their repsective Lagrangian changes.

Notice that the linearized pulsation equation (32) is not invariant under rotations. Terms linear in $\underset{\sim}{w}$ couple "ℓ-modes" (modes with spherical-harmonic index ℓ) to $(\ell - 1)$ and $(\ell + 1)$ modes; terms quadratic in $\underset{\sim}{w}$ couple ℓ-modes to $(\ell - 2)$, $(\ell - 1)$, $(\ell + 1)$, and $(\ell + 2)$ modes; all other terms are invariant under rotation. The terms linear in $\underset{\sim}{w}$ have imaginary coefficients; therefore they (like viscosity, energy generation, and radiative transport) contribute to the vibrational-secular stability of the star, but do not affect its dynamical stability. Terms quadratic in $\underset{\sim}{w}$ contribute to the dynamical stability and couple different angular modes. We will delay until a later paper all analyses of $\underset{\sim}{w}$-dependent terms ("preferred-frame terms") — including both the problem of vibrational-secular stability (linear in $\underset{\sim}{w}$) and preferred-frame influence on dynamical stability (quadratic in $\underset{\sim}{w}$). Thus, we shall set $\underset{\sim}{w} = 0$ throughout this paper.

III. THE VARIATIONAL PRINCIPLE

Equation (32), when supplemented by the expressions for the various Lagrangian changes in terms of $\underset{\sim}{\xi}$, becomes explicitly an equation for $\underset{\sim}{\xi}$. As boundary conditions, we shall demand that $\Delta p = 0$ at the surface of the star $(r = R)$, and that there be no physical singularity at the star's center $(r = 0)$. Equation (32) together with the boundary conditions then constitutes a characteristic value problem for Ω.

If a characteristic value problem is self-adjoint, the orthogonality relations for its characteristic functions hold and a variational base for determining Ω can be obtained. The stellar pulsation equations in the post-Newtonian limits of general relativity and Brans-Dicke theory are self-adjoint (Chandrasekhar 1956; Nutku 1969) as is the equation for radial oscillations in the full theory of general relativity (Chandrasekhar 1964b). In the Appendix, it is shown that the characteristic value problem in the PPN formalism is self-adjoint if and only if

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = \zeta_3 = \zeta_4 = 0 \quad . \qquad (42)$$

Although, in the general case, the characteristic value problem is not self-adjoint and the orthogonality relations do not hold, a variational integral can still be constructed in the following manner:

Take equation (32) with Ω replaced by the characteristic value $\Omega^{(i)}$ for the i-th normal mode, with $\tilde{\xi}$ replaced by the corresponding characteristic function $\tilde{\xi}^{(i)}$, and with $\tilde{w}$-terms deleted. Dot into this equation $\tilde{\xi}^{(j)}$, the characteristic function for the j-th mode, and integrate over the interior of the star. Thereby obtain

$$[\Omega^{(i)}]^2 \, Q^{(i,j)} = S^{(i,j)} + R^{(i,j)} \quad . \qquad (43)$$

Here $Q^{(i,j)}$ is expression (A.4) with 1 replaced by i, 2 replaced by j, and complex conjugations deleted; $S^{(i,j)}$ is the symmetric part of the right-hand side of (A.3), with similar replacements; and $R^{(i,j)}$ is (A.4) with similar replacements. Notice that $R^{(i,j)}$ is of post-Newtonian order:

$$R^{(i,j)} = O(2) \quad . \qquad (44)^3$$

---

[3]By O(n) we mean, in Chandrasekhar's (1965a) language, $O(c^{-n})$.

From equations (43) and (44), it follows immediately that the standard orthogonality relation for characteristic functions is valid to Newtonian order, i.e.

$$Q^{(i,j)} = O(2) \; [\Omega^{(i)} \neq \Omega^{(j)}] \quad . \qquad (45)$$

We assume, without proof, that the characteristic functions $\left\{ \underset{\sim}{\xi}^{(i)} \right\}$ from a complete set; and we normalize them to give

$$Q^{(I,I)} = 1 \qquad (46)$$

(no summation on capital letters).

Let ${}^{P}\underset{\sim}{\xi}^{(I)}$ be a solution which differs from $\underset{\sim}{\xi}^{(I)}$ by post-Newtonian order and has norm 1, i.e.

$${}^{P}\underset{\sim}{\xi}^{(I)} = \underset{\sim}{\xi}^{(I)} + O(2) \qquad (47)$$

$$Q\left({}^{P}\underset{\sim}{\xi}^{(I)}, {}^{P}\underset{\sim}{\xi}^{(I)}\right) = 1 \quad . \qquad (48)$$

Expand ${}^{P}\underset{\sim}{\xi}^{(I)}$ in terms of $\underset{\sim}{\xi}^{(j)}$:

$${}^{P}\underset{\sim}{\xi}^{(I)} = \sum_{j} C_{Ij} \, \underset{\sim}{\xi}^{(j)} \quad ,$$

and from equations (45), (46), (47), and (48), obtain

$$C_{Ij} = 1 + O(4), \quad (j = I) \qquad (49)$$

$$C_{Ij} = O(2), \quad (j \neq I) \quad . \qquad (50)$$

By combining equations (43), (45), (49), and (50), we obtain

$$S\left({}^{P}\underset{\sim}{\xi}^{(I)}, {}^{P}\underset{\sim}{\xi}^{(I)}\right) + R\left({}^{P}\underset{\sim}{\xi}^{(I)}, {}^{P}\underset{\sim}{\xi}^{(I)}\right) = \sum_{j,j'} [\Omega^{(j)}]^2 \, C_{Ij} \, C_{Ij'} \, Q^{(j,j')}$$

$$= [\Omega^{(I)}]^2 \, Q^{(I,J)} + O(4) \quad ; \qquad (51)$$

by combining equations (48), (49), and (50), we obtain

$$Q\left({}^{P}\underset{\sim}{\xi}^{(I)},\ {}^{P}\underset{\sim}{\xi}^{(I)}\right) = \sum_{j,j'} C_{Ij}\, C_{Ij'}\, Q^{(j,j')} = Q^{(I,I)} + O(4) \quad ; \qquad (52)$$

and by combining equations (51), (42), we finally conclude that

$$[\Omega^{(I)}]^2\, Q\left({}^{P}\underset{\sim}{\xi}^{(I)},\ {}^{P}\underset{\sim}{\xi}^{(I)}\right) = S\left({}^{P}\underset{\sim}{\xi}^{(I)},\ {}^{P}\underset{\sim}{\xi}^{(I)}\right) + R\left({}^{P}\underset{\sim}{\xi}^{(I)},\ {}^{P}\underset{\sim}{\xi}^{(I)}\right) + O(4) \ . \qquad (53)$$

Therefore we can use this equation, and any functions ${}^{P}\underset{\sim}{\xi}^{(I)}$ that agree with $\underset{\sim}{\xi}^{(I)}$ only at Newtonian order, to calculate $[\Omega^{(I)}]^2$ to post-Newtonian order. The Newtonian proper solutions $\left\{{}^{N}\underset{\sim}{\xi}^{(i)}\right\}$ are one set of such functions.

By suppressing the prefix "P" and superscript labels, by inserting from Appendix the values of Q, S, and R, and by performing some reductions, we bring our variational expression (53) into the form

$$\begin{aligned}
Q\Omega^2 = & \int_V \Gamma_1\, p[1 + (3\gamma - 1)\, U](\mathrm{div}\ \underset{\sim}{\xi})^2\, d\underset{\sim}{x} + \int_V (\mathrm{div}\ \underset{\sim}{\xi})\ \xi^\alpha \frac{\partial}{\partial x^\alpha}\left\{[1 + (3\gamma - 1)U]p\right\} d\underset{\sim}{x} \\
& + \int_V (3\Gamma_1\gamma - 3\gamma + 1)\ p\Delta U\ \mathrm{div}\ \underset{\sim}{\xi}\ d\underset{\sim}{x} + \int_V (2\beta - 1)\Delta U \xi^\alpha \frac{\partial p}{\partial x^\alpha}\, d\underset{\sim}{x} \\
& + \int_V (3\Gamma_1\gamma - 3\gamma + 1)\ p\ \mathrm{div}\ \underset{\sim}{\xi}\ \xi^\alpha \frac{\partial U}{\partial x^\alpha}\, d\underset{\sim}{x} - \int_V \rho\ \xi^\alpha \frac{\partial \Delta U}{\partial x^\alpha}\, d\underset{\sim}{x} \\
& - 2\int_V \rho\, \bar{\phi}\ \xi^\alpha \frac{\partial \Delta U}{\partial x^\alpha}\, d\underset{\sim}{x} - 2\int_V \rho \xi^\alpha \frac{\partial \Delta \Phi}{\partial x^\alpha}\, d\underset{\sim}{x} \ . \qquad (54)
\end{aligned}$$

Here

$$\begin{aligned}
Q = & \int_V \sigma |\underset{\sim}{\xi}|^2\, d\underset{\sim}{x} + \tfrac{1}{2}(\alpha_2 - \zeta_1 + 1) \int_V \int_V \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \frac{\underset{\sim}{\xi}(\underset{\sim}{x}) \cdot \underset{\sim}{\xi}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}\, d\underset{\sim}{x}' \\
& - \tfrac{1}{2}(\alpha_2 - \zeta_1 + 1) \int_V \int_V \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \frac{[\underset{\sim}{\xi}(\underset{\sim}{x}) \cdot (\underset{\sim}{x} - \underset{\sim}{x}')][\underset{\sim}{\xi}(\underset{\sim}{x}') \cdot (\underset{\sim}{x} - \underset{\sim}{x}')]}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}\, d\underset{\sim}{x}\, d\underset{\sim}{x}' \\
& + (5\gamma - 1) \int_V \int_V \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \frac{|\underset{\sim}{\xi}(\underset{\sim}{x})|^2}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}\, d\underset{\sim}{x}' - \tfrac{1}{2}(4\gamma + 4 + \alpha_1) \int_V \int_V \\
& \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \frac{\underset{\sim}{\xi}(\underset{\sim}{x}) \cdot \underset{\sim}{\xi}(\underset{\sim}{x}')}{|\underset{\sim}{x}' - \underset{\sim}{x}|}\, d\underset{\sim}{x}\, d\underset{\sim}{x}' \ , \qquad (55)
\end{aligned}$$

is a positive definite quantity in the post-Newtonian limit since the dominant, Newtonian part, $\int_V \sigma|\underset{\sim}{\xi}|^2 \, d\underset{\sim}{x}$, is positive definite. This equation can be used for a variational determination of Ω^2.

We shall now analyze the Lagrangian displacement $\underset{\sim}{\xi}$ into normal modes belonging to different vector spherical harmonics. (Since the pulsation equation without $\underset{\sim}{w}$-dependent terms is invariant under rotation, this procedure is justified.) Following the procedure of Chandrasekhar (1961, 1964c) and Lebovitz (1965), we define:

$$\xi_r = \frac{\psi(\gamma)}{r^2} \, Y_\ell^{\,m}(\Theta,\phi) \quad , \tag{56}$$

$$\xi_\Theta = \frac{1}{\ell(\ell+1)r} \, \frac{d\chi(r)}{d\, r} \, \frac{\partial Y_\ell^{\,m}(\Theta,\phi)}{\partial\Theta} \quad , \tag{57}$$

and

$$\xi_\phi = \frac{1}{\ell(\ell+1)\, r \sin\Theta} \, \frac{d\chi(r)}{d\, r} \, \frac{\partial Y_\ell^{\,m}(\Theta,\phi)}{\partial\phi} \tag{58}$$

(ξ_r, ξ_Θ, ξ_ϕ are physical components, not covariant components). After manipulations similar to those in §IV of Chandrasekhar (1965b), we obtain the following expression for the variational determination of Ω^2:

$$\begin{aligned}
Q\Omega^2 = {} & \int_0^R \Gamma_1 \, p[1 + (3\gamma - 1)U]\left[\frac{d}{dr}(\psi - x)\right]^2 \frac{dr}{r^2} \\
& + 2\int_0^R \frac{d}{dr}\left\{[1 + (3\gamma - 1)U]p\right\}\left(2\,\frac{\psi^2}{r} - \psi\,\frac{d\chi}{dr}\right)\frac{dr}{r^2} \\
& - \frac{4\pi}{2\ell+1}\int_0^R (1 + 2\phi + 2\bar{\phi})\left(J_\ell \, \frac{dK_\ell}{dr} - K_\ell \, \frac{dJ_\ell}{dr}\right)dr \\
& + (2\beta - 1 - \zeta_2)\int_0^R \rho[\Delta U(r)]^2 \, r^2 \, dr \\
& + (6\Gamma_1\gamma - 6\gamma + 2 + 3\Gamma_1\zeta_4 - 3\zeta_4 + \zeta_3)\int_0^R p\Delta U(r)\,\frac{d}{dr}(\psi - \chi)\, dr \quad .
\end{aligned} \tag{59}$$

where

$$J_\ell(r) = \int_0^r \rho(s)\; s^\ell \left[\ell\,\frac{\psi(s)}{s} + \frac{d\chi(s)}{ds}\right] ds \quad , \tag{60}$$

$$K_\ell(r) = \int_r^R \frac{\rho(s)}{s^{\ell+1}} \left[(\ell + 1)\,\frac{\psi(s)}{s} - \frac{d\chi(s)}{ds}\right] ds \quad , \tag{61}$$

and

$$\Delta U(r) = \frac{4\pi}{4\ell + 1}\left[\frac{J_\ell(r)}{r^{\ell+1}} - r^\ell\, K_\ell(r)\right] + \frac{\psi}{r^2}\,\frac{dU}{dr} \quad . \tag{62}$$

IV. THE POST-NEWTONIAN CONDITION FOR THE ONSET OF DYNAMICAL INSTABILITY

Consider the case of radial pulsations, i.e. pulsations with

$$\ell = 0 \text{ and } \chi = 0 \quad . \tag{63}$$

The substitutions

$$\psi = r^3\eta \text{ and } \xi_r = r\eta \quad , \tag{64}$$

reduce equation (59) to the form

$$\begin{aligned} Q\Omega^2 = {} & \int_0^R p[1 + (3\gamma - 1)\, U\,]\left[\Gamma_1\, r^4 \left(\frac{d\eta}{dr}\right)^2 + (3\Gamma_1 - 4)\,\frac{d}{dr}\,(r^3\eta^2)\right] dr \\ & + (2\beta - 1 - \zeta_2)\int_0^R \rho[\Delta U(r)]^2\, r^2\, dr \\ & + (6\Gamma_1\gamma - 6\gamma + 2 + 3\Gamma_1\zeta_4 - 3\zeta_4 + \zeta_3)\int_0^R p\Delta U(r)\,\frac{d}{dr}\,(r^3\eta)\, dr \\ & - (3\Gamma_1\zeta_4 - 3\zeta_4 + \zeta_3)\int_0^R p\,\frac{dU}{dr}\,\eta\,\frac{d(\eta r^3)}{dr}\, r\, dr \\ & + \zeta_2\int_0^R \frac{dp}{dr}\,\Delta U \eta r^3 dr \quad , \end{aligned} \tag{65}$$

where

$$\Delta U(r) = -4\pi \int_r^R \rho(s)\, \psi(s) \frac{ds}{s^2} + \frac{\psi}{r^2} \frac{dU}{dr} \quad . \tag{66}$$

Recall that p and ρ are the distributions of pressure and density in the equilibrium configuration in the post-Newtonian approximation, and they therefore include terms of O(2).

The condition for marginal stability (instability) follows from equation (65) by setting $\Omega^2 = 0$. In the particular case Γ_1 = const. — which implies η = const. at the point of onset of instability in the Newtonian limit, i.e. η = const. + O(2) — the condition for marginal instability (eq. [65] with $\Omega = 0$) involves the structure of the equilibrium configuration in the Newtonian approximation alone. Under these conditions the criterion for marginal instability becomes $\Gamma_1 = \Gamma_{crit}$, where

$$\Gamma_{crit} = \frac{4}{3} + \frac{1}{3W}\Bigg\{(2\beta - 1 - \zeta_2) \int_0^R [\Delta U(r)]^2\, dM(r) + 3[2\gamma + 2 + \zeta_3 + \zeta_4] \int_0^R \frac{p}{\rho} \Delta U(r)\, dM(r)$$
$$- 3(\zeta_3 + \zeta_4) \int_0^R \frac{p}{\rho} \left(\frac{dU}{dr} r\right) dM(r) + \zeta_2 \int \frac{\Delta U}{\rho} r \left(\frac{dp}{dr}\right) dM(r) \quad . \tag{67}$$

Here

$$W = -12\pi \int_0^R p r^2 dr \quad , \tag{68}$$

$$dM = 4\pi\rho r^2 dr \quad , \tag{69}$$

and

$$\Delta U(r) = -4\pi \int_r^R \rho s ds + r \frac{dU}{dr} \quad . \tag{70}$$

This result agrees with those in general relativity (Chandrasekhar 1965b) and in Brans-Dicke-Jordan theory (Nutku 1969) when specialized to the corresponding PPN values.

Criterion (67) for marginal instability may be reformulated as follows. A dynamical instability will set in if and only if the following inequality is satisfied:

$$\Gamma_1 \leq \Gamma_{crit} \equiv \frac{4}{3} + K\,\frac{2M}{R} \quad . \tag{71}$$

Here M is the mass and R is the radius of the configuration, and K is a constant (typically of order unity), depending on the Newtonian structure of the configuration. If K is positive, there is a relativistic instability; if K is negative, then relativistic effects stabilize the star.

For polytropes, an explicit expression for K can be obtained, from equations (67)-(71), in terms of Lane-Emden functions:

$$K = \frac{1}{2}\,(1 + \gamma)\,K_1 - (2\beta - 1)\,K_2 + (\zeta_3 + \zeta_4)(\tfrac{1}{4}\,K_1 - 3K_3) + \zeta_2(K_2 - K_4) \tag{72}$$

where

$$K_1 = \frac{12(5 - n)}{18(n + 1)\,\xi_1^{\,4}\,|\Theta_1'|^3} \cdot \int_0^{\xi_1} \left[\Theta + \xi_1 |\Theta_1'|\right] \Theta^{n+1}\,\xi^2\,d\xi \tag{73}$$

$$K_2 = \frac{(5 - n)}{18\xi_1^{\,4}\,|\Theta_1'|^3} \cdot \int_0^{\xi_1} \left[\Theta + \xi_1 |\Theta_1'|\right]^2 \Theta^{n}\,\xi^2\,d\xi \tag{74}$$

$$K_3 = \frac{5 - n}{18(n + 1)\,\xi_1^{\,4}\,|\Theta_1'|^3} \cdot \int_0^{\xi_1} \Theta^{n+1}\,\xi^3 \left|\frac{d\Theta}{d\xi}\right| d\xi \tag{75}$$

$$K_4 = \frac{5 - n}{18\,\xi_1^{\,4}\,|\Theta_1'|^3} \cdot \int_0^{\xi_1} \left[\Theta + \xi_1 |\Theta_1'|\right] \Theta^{n} \left|\frac{d\Theta}{d\xi}\right| \xi^3\,d\xi \tag{76}$$

and where n is the polytropic index, ξ_1 is the first zero of the Lane-Emden function Θ_n, and Θ_1' is the value of the derivative of Θ_n at ξ_1. The values of the constant K, evaluated with the aid of the foregoing formula for various values of n, are listed in Table 1 for conservative theories and in Table 2 for the general PPN formalism and for non-conservative theories. See §I for discussions of these tables and for the conclusions inferred from these tables.

V. NON-RADIAL OSCILLATIONS AND THE SCHWARZSCHILD CRITERION FOR CONVECTION

We shall now obtain the condition for the occurrence of a neutral mode of non-radial oscillation belonging to $\ell \geq 1$ in the general PPN framework. As in the last two sections, all "preferred-frame" effects ($\underset{\sim}{w}$-dependent terms) will be ignored. By setting $\Omega^2 = 0$ in equation (32), by following an analysis parallel to §VI of Chandrasekhar (1965b), and by using the result of §VIII of Chandrasekhar (1965b), one can show that, to O(2), a necessary and sufficient condition for the occurrence of a neutral mode of non-radial oscillation is that

$$S(r)\left[1 + \Pi \frac{\Gamma_3 - \Gamma_4}{\Gamma_3 - 1} \frac{\rho}{p} \frac{dp/dr}{d\rho/dr}\right] = 0 \;; \tag{77}$$

i.e. that

$$S(r) = 0 \tag{78}$$

over some finite interval of r. Here

$$S(r) \equiv \frac{dp}{dr} - \Gamma_1 \frac{p}{\rho} \frac{d\rho}{dr} \tag{79}$$

is the "Schwarzschild discriminant", and Γ_3 and Γ_4 are defined by

$$\Gamma_3 = 1 + \left[\frac{\partial(\log T)}{\partial(\log \rho)}\right]_s \;, \tag{80}$$

and

$$\Pi = \frac{1}{\Gamma_4 - 1} \frac{p}{\rho} \;. \tag{81}$$

By following a procedure similar to §VII of Chandrasekhar (1965b), one can also derive this condition from the variational principle (61).

The proportionality of the Newtonian and the post-Newtonian discriminants implies that the physical condition for convective instability remains the same in the PPN formalism as in general relativity and in Newtonian theory. Although for some PPN values, the characteristic value problem is not self-adjoint, the Schwarzschild criterion is still valid, and no new dynamical instabilities occur.

VI. CONCLUSIONS

In this paper, stability criteria for stellar pulsations were found using the general PPN formalism. These criteria are valid for almost all metric theories of gravity in the post-Newtonian approximation, when one ignores preferred-frame effects. As in general relativity, so also for conservative theories (conservative theories do not have preferred-frame effects), the relativistic corrections do actually induce dynamical instabilities in stars. But in the general case the present experimental uncertainty in the PPN parameters ξ_2, ξ_4 makes it inconclusive whether relativistic effects will actually stabilize or destabilize stars. As experimental tests are performed to higher precision, the answer may become definite. The differences in the dynamical stability criterion for various theories may affect the evolution of white dwarfs and supermassive stars; such effects are worth exploration. The relationship of non-self-adjointness to the non-existence of conservation laws is intriguing and should be examined further.

A subsequent paper will deal with the problem of Machian instabilities due to preferred-frame effects ($\underset{\sim}{w}$-dependent terms). Those instabilities, when combined with astronomical observations on white-dwarf pulsations, may lead to tight experimental limits on the "preferred-frame parameters" α_1, α_2, and α_3.

ACKNOWLEDGMENTS

I would like to thank S. Chandrasekhar for helpful discussions. I thank B. A. Zimmerman for assistance in the computer programming. Finally I wish to thank Kip S. Thorne, whose moral support and valuable advice made this paper possible and whose critical comments about presentation made it readable.

APPENDIX

SELF-ADJOINTNESS OF THE CHARACTERISTIC VALUE PROBLEM

We shall here derive the constraints which the PPN parameters must satisfy for the characteristic value problem [eq. (32)] to be self-adjoint. For this prupose we do not delete the $\underset{\sim}{w}$-dependent terms <u>ab initio</u> (cf. end of §II).

By bringing the right-hand side over to the left, write equation (32) in the form $\mathscr{L}\underset{\sim}{\xi} = 0$, where $\mathscr{L}$ is a linear operator. This equation is self-adjoint [or can be made so by multiplication with some weighting function $\mathscr{W}(\underset{\sim}{x})$] if and only if

$$\int_V \underset{\sim}{\xi}^{(1)*} \cdot \mathscr{W} \mathscr{L} \underset{\sim}{\xi}^{(2)} \, d\underset{\sim}{x} = \int_V \left[\underset{\sim}{\xi}^{(2)*} \cdot \mathscr{W} \mathscr{L} \underset{\sim}{\xi}^{(1)} \right]^* d\underset{\sim}{x} \quad , \qquad (A.1)$$

where the complex conjugation "*" is <u>not</u> permitted to act on the eigenvalue Ω (which is contained in $\mathscr{L}$). In this equation $\underset{\sim}{\xi}^{(1)}$ and $\underset{\sim}{\xi}^{(2)}$ are arbitrary functions satisfying the boundary conditions at $r = 0$ and $r = R$ (but <u>not</u> necessarily satisfying $\mathscr{L}\,\underset{\sim}{\xi}^{(1)} = 0$ or $\mathscr{L}\,\underset{\sim}{\xi}^{(2)} = 0$); V is the interior of the star; "*" denotes complex conjugation; and $d\underset{\sim}{x}$ denotes $dx\,dy\,dz = dx^1\,dx^2\,dx^3$. From this definition one readily verifies that (i) if the weighting function $\mathscr{W}$ is chosen real, then the $i\Omega$-terms prevent self-adjointness; (ii) if $\mathscr{W}$ has any imaginary part, then the Ω^2-terms prevent self-adjointness. It is possible to get rid of the $i\Omega$-terms by demanding $\alpha_1 = \alpha_2 = \alpha_3 = 0$; but it is not possible to get rid of the Ω^2-terms. Therefore, to have any hope of self-adjointness one must choose $\mathscr{W}(\underset{\sim}{x})$ real and

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \quad . \qquad (A.2)$$

Insist, then, that $\alpha_1 = \alpha_2 = \alpha_3 = 0$; and try, for the moment to prove self-adjointness with the trivial weighting function $\mathscr{W}(\underset{\sim}{x}) = 1$. Then the terms on the left-hand side of equation (32) give, when integrated,

$$[\Omega]^2 \Bigg\{ \int_V \sigma\, \xi^{(1)\alpha*} \xi^{(2)\alpha}\, d\underset{\sim}{x} + \frac{1}{2} \int_V \int_V \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \frac{\xi^{(1)\alpha*}(\underset{\sim}{x})\, \xi^{(2)\alpha}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}\, d\underset{\sim}{x}'$$

$$- \frac{1}{2}(\alpha_2 - \zeta_1 + 1) \int_V \int_V \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \xi^{(1)\alpha*}(\underset{\sim}{x})(x^\alpha - x'^\alpha)\, \xi^{(2)\beta}(\underset{\sim}{x}')(x^\beta - x'^\beta)$$

$$\frac{d\underset{\sim}{x}\, d\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}$$

$$+ (5\gamma - 1) \int_V \int_V \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \xi^{(1)\alpha*}(\underset{\sim}{x})\, \xi^{(2)\alpha}(\underset{\sim}{x})\, \frac{d\underset{\sim}{x}\, d\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

$$- \frac{1}{2}(\alpha_1 + 4\gamma + 4) \int_V \int_V \rho(\underset{\sim}{x})\, \rho(\underset{\sim}{x}')\, \xi^{(1)\alpha*}(\underset{\sim}{x})\, \xi^{(2)\alpha}(\underset{\sim}{x}')\, \frac{d\underset{\sim}{x}\, d\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \Bigg\} \equiv \Omega^2 Q^{(1,2)}. \qquad \text{(A.3)}$$

[For use in the text of the paper we retain all α terms except those that depend on $\underset{\sim}{w}$; but we keep in mind that henceforth in this appendix the α's vanish.] Notice that $Q^{(1,2)}$ is manifestly self-adjoint in the superscripts 1 and 2, i.e. $Q^{(1,2)} = Q^{(2,1)*}$. Reduction of the right-hand side of the integrated equation is less straightforward; it requires integrations by parts, followed by substitutions for various Lagrangian changes, simplifications, and rearrangements. After some work, the right-hand side is brought into the form

$$\Omega^2 Q^{(1,2)} = \int_V \Gamma_1 [1 + (3\gamma - 1)U]\, p\, \operatorname{div} \underset{\sim}{\xi}^{(1)*} \operatorname{div} \underset{\sim}{\xi}^{(2)}\, d\underset{\sim}{x}$$

$$+ \int_V \frac{1}{\rho} \frac{d\rho}{dr} \frac{d}{dr} \Bigg\{ [1 + (3\gamma - 1)U]\, p \Bigg\} \frac{[\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(1)*}][\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}]}{r^2}\, d\underset{\sim}{x} \qquad \text{(A.4)}$$

$$+ \int_V [1 + (3\gamma - 1)U] \frac{dp}{dr} \left[\frac{\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(1)*}}{r} \text{div } \underset{\sim}{\xi}^{(2)} + \frac{\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}}{r} \text{div } \underset{\sim}{\xi}^{(1)*}\right] d\underset{\sim}{x}$$

$$- \int_V \int_V \left\{1 + 2[\overline{\phi}(\underset{\sim}{x}) + \overline{\phi}(\underset{\sim}{x}')]\right\}$$

$$\frac{[\Delta\rho^{(1)*}(\underset{\sim}{x}) - \underset{\sim}{\xi}^{(1)*}(\underset{\sim}{x}) \cdot \underset{\sim}{\nabla}\rho(\underset{\sim}{x})][\Delta\rho^{(1)}(\underset{\sim}{x}') - \underset{\sim}{\xi}^{(2)}(\underset{\sim}{x}') \cdot \underset{\sim}{\nabla}\rho(\underset{\sim}{x}')]}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}d\underset{\sim}{x}'$$

$$+ \int_V \rho \frac{d}{dr} (3\gamma U + 2\phi) \left[\frac{\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(1)*}}{r} \Delta U^{(2)} + \frac{\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}}{r} \Delta U^{(1)*}\right] d\underset{\sim}{x}$$

$$- (3\gamma - 1) \int_V \rho \frac{dU}{dr} \left[\frac{\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(1)*}}{r} \Delta\Pi^{(2)} + \frac{\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}}{r} \Delta\Pi^{(1)*}\right] d\underset{\sim}{x}$$

$$- (3\gamma\Gamma_1 - 3\gamma + 1) \int_V \rho[\Delta U^{(1)*} \Delta\Pi^{(2)} + \Delta U^{(2)} \Delta\Pi^{(1)*}] d\underset{\sim}{x}$$

$$- 2 \int_V \left[\frac{dU}{dr} \frac{d\rho}{dr} (\phi - \overline{\phi}) + \frac{dU}{dr} \frac{d\phi}{dr} \rho\right] \frac{[\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(1)*}][\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}]}{r^2} d\underset{\sim}{x}$$

$$- (6\gamma + 1 + \zeta_2 - 2\beta) \int_V \rho\Delta U^{(1)*} \Delta U^{(2)} d\underset{\sim}{x} + R^{(1,2)} \qquad \text{(A.4 cont'd.)}$$

where

$$R^{(1,2)} = [- 3\zeta_4(\Gamma_1 - 1) - \zeta_3] \int_V \rho\Delta\Pi^{(1)*} \Delta U^{(2)} d\underset{\sim}{x}$$

$$+ 2 \int_V \rho\left(\Delta\phi^{(1)*} - \Delta\overline{\phi}^{(1)*} \frac{dU}{dr}\right) \frac{[\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}]}{r} d\underset{\sim}{x}$$

$$+ 2 \int_V \rho \frac{d}{dr} (\overline{\phi} - \phi) \frac{[\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}]}{r} \Delta U^{(1)*} d\underset{\sim}{x}$$

$$+ 2 \int_V (\phi - \overline{\phi})\left[- \Delta U^{(1)*} \Delta\rho^{(2)} + \Delta U^{(1)*} \frac{d\rho}{dr} \frac{[\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(2)}]}{r} + \Delta\rho^{(2)} \frac{dU}{dr} \frac{[\underset{\sim}{x} \cdot \underset{\sim}{\xi}^{(1)}]}{r}\right] d\underset{\sim}{x} \quad . \qquad \text{(A.5)}$$

Equation (A.4) reduces to equation (A.6) of Chandrasekhar (1965b), if we substitute in the PPN parameter values for general relativity. Aside from $R^{(1,2)}$, the terms on the right-hand side of equation (A.4), like $Q^{(1,2)}$ of the left-hand side, are manifestly self-adjoint in the superscripts 1 and 2. Therefore, the condition for equation (32) to be self-adjoint with weighting function $\mathcal{W}(\underset{\sim}{x}) = 1$ is

$$R^{(1,2)} = R^{(2,1)*} \tag{A.6}$$

or, equivalently (since one insists on this equality for <u>all</u> choices of $\underset{\sim}{\xi}^{(1)}$ and $\underset{\sim}{\xi}^{(2)}$):

$$\zeta_2 = \zeta_2 = \zeta_4 = 0 \ . \tag{A.7}$$

Might some other choice of weighting function aside from constant permit one to relax these constraints and still retain self-adjointness? No; because any other choice of $\mathcal{W}(\underset{\sim}{x})$ will destroy the self-adjointness of the left-hand side ($\Omega^2 \, Q^{(1,2)}$), and the arbitrariness of Ω^2 will prevent the non-self-adjoint terms thus created from always cancelling non-self-adjoint terms on the right-hand side.

In summary, equations (A.2) and (A.7) — i.e. $\alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = \zeta_3 = \zeta_4 = 0$ — are necessary and sufficient conditions for the self-adjointness of the linearized pulsation problem in the PPN formalism. These conditions, together with the condition that $\zeta_1 = 0$, are just Will's (1971b) conditions for theories of gravity to have post-Newtonian conserved integrals for energy, momentum, angular momentum, and center-of-mass motion.

TABLE 1

VALUES OF THE CONSTANT K FOR POLYTROPIC STARS

IN CONSERVATIVE THEORIES OF GRAVITY[a]

n	K				
	General Expression	Minimum Possible value[b]	General Relativity[c]	Brans-Dicke Theory[d] ($w = 6$)	Nordström's (1913) Second Theory
0	$0.4286(1 + \gamma) - 0.4047(2\beta - 1)$	0.0743	0.4524(0.452381)	0.3949(0.44047)	0
0.5	$0.4779(1 + \gamma) - 0.4527(2\beta - 1)$	0.0804	0.5030	0.4433	0
1.0	$0.5392(1 + \gamma) - 0.5131(2\beta - 1)$	0.0871	0.5654(0.565382)	0.4980(0.53964)	0
1.5	$0.6179(1 + \gamma) - 0.5908(2\beta - 1)$	0.0949	0.6450(0.645063)	0.5678	0
2.0	$0.7232(1 + \gamma) - 0.6949(2\beta - 1)$	0.1052	0.7514(0.751296)	0.6610(0.70257)	0
2.5	$0.8709(1 + \gamma) - 0.8414(2\beta - 1)$	0.1192	0.9004(0.900302)	0.7916	0
2.75	$0.9696(1 + \gamma) - 0.9394(2\beta - 1)$	0.1284	0.9999	0.8786	0
3.0	$1.0940(1 + \gamma) - 1.0628(2\beta - 1)$	0.1399	1.1252(1.12447)	0.9885(1.0294)	0
3.25	$1.2531(1 + \gamma) - 1.2210(2\beta - 1)$	0.1541	1.2852(1.28503)	1.1285	0
3.5	$1.4664(1 + \gamma) - 1.4332(2\beta - 1)$	0.1732	1.4997(1.49954)	1.3164	0
4.0	$2.2235(1 + \gamma) - 2.1869(2\beta - 1)$	0.2394	2.2602	1.9822	0
4.5	$4.5246(1 + \gamma) - 4.4826(2\beta - 1)$	0.4325	4.5666	4.0010	0

[a] By "conservative theory" we mean a theory that possesses conserved post-Newtonian integrals for energy, momentum angular momentum, and center-of-mass motion — i.e. a theory with $\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$.

[b] These are minimum values compatible with current experimental limits on β and γ: $\beta < 1.34$, $\gamma > 0.76$.

[c] The values in parentheses are those derived by Chandrasekhar (1965).

[d] The values in parentheses are those derived by Nutku (1969); the discrepancies are presumably due to errors in Nutku's numerical calculations.

VALUES OF THE CONSTANT K FOR POLYTROPIC STARS

IN THE GENERAL PPN FORMALISM

n	K: PPN Formalism	K: Minimum Possible Value[a]	K: Vector-Metric Theory[b] and Papapetrou Theories[c]	K: Modified Yilmaz Theory[d]
0	$0.4286(1+\gamma) - 0.4047(2\beta - 1) + 0.1429(\zeta_3 + \zeta_4) + 0.2143\,\zeta_2$	$0.0743 + 0.2143\,\zeta_2 + 0.1429(\zeta_4 - 0.05)$		-0.1191 or -0.2620
0.5	$0.4779(1+\gamma) - 0.4527(2\beta - 1) + 0.1522(\zeta_3 + \zeta_4) + 0.2427\,\zeta_2$	$0.0804 + 0.2427\,\zeta_2 + 0.1522(\zeta_4 - 0.05)$	SAME	-0.1346 or -0.2868
1.0	$0.5392(1+\gamma) - 0.5131(2\beta - 1) + 0.1667(\zeta_3 + \zeta_4) + 0.2778\,\zeta_2$	$0.0871 + 0.2778\,\zeta_2 + 0.1667(\zeta_4 - 0.05)$		-0.1569 or -0.3236
1.5	$0.6179(1+\gamma) - 0\ 5908(2\beta - 1) + 0.1871(\zeta_3 + \zeta_4) + 0.3225\,\zeta_2$	$0.0949 + 0.3225\,\zeta_2 + 0.1871(\zeta_4 - 0.05)$	as	-0.1871 or -0.3742
2.0	$0.7232(1+\gamma) - 0.6949(2\beta - 1) + 0.2155(\zeta_3 + \zeta_4) + 0.3821\,\zeta_2$	$0.1052 + 0.3821\,\zeta_2 + 0.2155(\zeta_4 - 0.05)$		-0.2283 or -0.4438
2.5	$0.8709(1+\gamma) - 0.8414(2\beta - 1) + 0.2583(\zeta_3 + \zeta_4) + 0.4657\,\zeta_2$	$0.1192 + 0.4657\,\zeta_2 + 0.2583(\zeta_4 - 0.05)$	in	-0.2893 or -0.5476
2.75	$0.9696(1+\gamma) - 0.9394(2\beta - 1) + 0.2841(\zeta_3 + \zeta_4) + 0.5216\,\zeta_2$	$0.1284 + 0.5216\,\zeta_2 + 0.2841(\zeta_4 - 0.05)$		-0.3274 or -0.6115
3.0	$1.0940(1+\gamma) - 1.0628(2\beta - 1) + 0.3190(\zeta_3 + \zeta_4) + 0.5917\,\zeta_2$	$0.1399 + 0.5917\,\zeta_2 + 0.3190(\zeta_4 - 0.05)$	General	-0.3772 or -0.6962
3.25	$1.2531(1+\gamma) - 1.2210(2\beta - 1) + 0.3637(\zeta_3 + \zeta_4) + 0.6821\,\zeta_2$	$0.1541 + 0.6821\,\zeta_2 + 0.3637(\zeta_4 - 0.05)$		-0.4427 or -0.8064
3.5	$1.4664(1+\gamma) - 1.4332(2\beta - 1) + 0.4242(\zeta_3 + \zeta_4) + 0.8030\,\zeta_2$	$0.1732 + 0.8030\,\zeta_2 + 0.4242(\zeta_4 - 0.05)$		-0.5309 or -0.9551
4.0	$2.2235(1+\gamma) - 2.1869(2\beta - 1) + 0.6395(\zeta_3 + \zeta_4) + 1.2326\,\zeta_2$	$0.2394 + 1.2326\,\zeta_2 + 0.6395(\zeta_4 - 0.05)$	Relativity	-0.8445 or -1.4840
4.5	$4.5246(1+\gamma) - 4.4826(2\beta - 1) + 1.2957(\zeta_3 + \zeta_4) + 2.5426\,\zeta_2$	$0.4325 + 2.5426\,\zeta_2 + 1.2957(\zeta_4 - 0.05)$		-1.8143 or -3.1100

FOOTNOTES FOR TABLE 2

[a]These are minimum values compatible with current experimental limits on β, γ, and ζ_3: $\beta < 1.34$, $\gamma > 0.76$, $\zeta_3 > -0.05$.

[b]Will and Nordtvedt (1972).

[c]Papapetrou (1954a,b,c).

[d]Ni (1972). The values of K depend on which "matter density" one chooses as source for the gravitational field: $\rho = T_{ij}\, u^i\, u^j$ = component of stress-energy tensor along four-velocity of matter [upper values]; or ρ = trace (T_{ij}) [lower values]. (cf. Ni 1972.)

REFERENCES

Chandrasekhar, S. 1961, Hydrodynamic and Hydromagnetic Stability (Oxford: Clarendon Press).

__________ 1964a, Phys. Rev. Letters, **12**, 114 and 437.

__________ 1964b, Ap. J., **140**, 417.

__________ 1964c, ibid., **139**, 664.

__________ 1965a, ibid., **142**, 1488.

__________ 1965b, ibid., **142**, 1519.

Fowler, W. A. 1964, Rev. Mod. Phys., **36**, 545.

Gilvarry, J. J., and Muller, P. M. 1972, Phys. Rev. Letters, **28**, 1665.

Kreuzer, L. B. 1966, Ph.D. thesis, Department of Physics, Princeton University (unpublished).

__________ 1968, Phys. Rev., **169**, 1007.

Lebovitz, N. R. 1965, Ap. J., **142**, 1257.

Ni. W.-T. 1972, Ap. J., **000**, 000.

Nordström, G. 1913, Ann. Physik, **42**, 533.

Nordtvedt, K., Jr. 1968, Phys. Rev., **169**, 1017.

Nordtvedt, K., Jr., and Will, C. M. 1972, Ap. J., in press.

Nutku, Y. 1969, Ap. J., **155**, 999.

Papapetron, A. 1954a, Math. Nachr., **12**, 129.

__________ 1954b, Math. Nachr., **12**, 143.

__________ 1954c, Z. Phys., **139**, 518.

Sramek, R. A. 1972, reported in the Fourth Cambridge Conference on Experimental Relativity .

Thorne, K. S. 1967, in High Energy Astrophysics, vol. 3, eds. C. DeWitt, P. Véron, E. Schatzman (Gordon and Breach: New York).

Thorne, K. S., Will, C. M., and Ni, W.-T. 1971, in Proceedings of the Conference on Experimental Tests of Gravitational Theories, ed. R. W. Davies (NASA-JPL Technical Memorandum 33-499).

Will, C. M. 1971a, Ap. J., 163, 611.

__________ 1971b, Ap. J., 169, 125.

__________ 1971c, Ap. J., 169, 141.

__________ 1972a, private communication.

__________ 1972b, Lectures, International School of Physics "Enrico Fermi" Varenna, Italy, July 17-July 29, 1972.

Will, C. M., and Nordtvedt, K., Jr. 1972, Ap. J., in press.

Zel'dovich, Ya. B., and Novikov, I. D. 1971, Relativistic Astrophysics Volume 1 (Chicago: University of Chicago Press).