POLARIZATION PROPERTIES OF ASTROPHYSICAL MASERS

Thesis by

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To my Parents

and Mary

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ABSTRACT

Observed interstellar OH and H_2O maser lines at 18 cm and 1.35 cm exhibit unusual polarization properties. The OH emitters with the highest brightness temperatures,usually associated with H II regions, almost always show a high degree of circular polarization. The H_2O maser line, on the other hand, is rarely polarized, and then only linearly polarized.

The preference for circular polarization in the brightest OH sources was attributed by Litvak to the mechanism of parametric down-conversion. In this process the higher-frequency components of a Zeeman split maser line are down-converted to lower-frequency components and to an electron cyclotron wave. This mechanism is shown to be too weak to be of importance in astrophysical masers.

The polarization properties of the OH and H_2O masers are related to the physical conditions in the maser clouds. It is found that the magnetic field, the plasma, and trapped infrared lines in maser sources play an important role in determining the polarization of the emitted radiation.

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SUMMARY

Maser emission has been detected from two interstellar molecules, water and hydroxyl. The water line is rarely polarized, but in several sources it shows a considerable degree of linear polarization. The hydroxyl lines, from both the ground and the excited rotational levels, are often strongly polarized. In particular, many of the sources associated with HII regions exhibit a high degree of circular polarization. The aim of this thesis is to understand these observed maser polarization properties.

The thesis consists of three parts. In part I the mechanism of parametric down-conversion proposed by Litvak to explain the observed preference for circular polarization in OH maser emission is shown to be too weak to be important.

In part II, the propagation of maser radiation in the presence of a magnetic field and plasma is studied. It is found that the magnetic field and plasma strongly influence the polarizations of the emitted radiation. The character of the polarization depends upon the relative sizes of the following parameters: The decay rate of the maser levels, Γ, the stimulated emission rate, R, the Zeeman splitting, $g\Omega$, and the bandwidth of the maser radiation, $\Delta\omega$. A simple example of a maser operating between upper and lower states of total angular momenta $F_a = 1$ and $F_b = 0$ is investigated. A summary of the polarization properties of the maser radiation which propagates at an angle θ to the magnetic field is presented in Table 1.

TABLE 1

POLARIZATION PROPERTIES OF MASER RADIATION PROPAGATING AT AN ANGLE θ TO THE MAGNETIC FIELD FOR VARIOUS RANGES OF THE VALUE OF $g\Omega$

	Unsaturated Maser $\Gamma > R$	Saturated Maser $\Gamma < R$
$g\Omega > \Delta\omega$	σ components dominate if $\sin^2\theta < 2/3$ π component dominates if $\sin^2\theta > 2/3$	Zeeman pattern
$\Delta\omega > g\Omega > R$	Unpolarized	$\frac{Q}{I} = \frac{3\sin^2\theta - 2}{3\sin^2\theta}$ if $\sin^2\theta > 1/3$ $= -1$ if $\sin^2\theta < 1/3$
$R > g\Omega > \Gamma$ $(g\Omega \sin\theta)^2 > R\Gamma$		$\frac{Q}{I} = 1/3$
$(g\Omega \sin\theta)^2 < R\Gamma$	Unpolarized	Unpolarized

Infrared line radiation trapped between a maser level and other rotational levels produces a rapid relaxation of population among the degenerate sublevels of the maser level. Part III takes account of the effect that this cross-relaxation has on the polarization of the maser radiation. The effect is pronounced when the cross-relaxation rate, γ, is greater than the stimulated emission rate. This result is illustrated in Table 2.

TABLE 2

MASER POLARIZATION PROPERTIES IN THE CASE OF $\gamma > R$

	Unsaturated Maser $\Gamma > R$	Saturated Maser $\Gamma < R$
$g\Omega > \Delta\omega$	σ components dominate if $\sin^2\theta < 2/3$ π component dominates if $\sin^2\theta > 2/3$	σ components dominate if $\sin^2\theta < 2/3$ π component dominates if $\sin^2\theta > 2/3$
$\Delta\omega > g\Omega > R$	Unpolarized	Unpolarized
$R > g\Omega > \Gamma$ $(g\Omega \sin\theta)^2 > R\Gamma$		Unpolarized
$(g\Omega \sin\theta)^2 < R\Gamma$	Unpolarized	Unpolarized

Faraday rotation caused by a plasma also affects the polarization of the maser radiation. Results in Tables 1 and 2 indicate that unsaturated masers emit polarized radiation for $g\Omega > \Delta\omega$. If the Faraday rotation per gain length is large, the π component of the Zeeman pattern is unpolarized and the σ components are 100% circularly polarized. For saturated masers, the amount of Faraday rotation need only be large over the region of saturated amplification in order to affect the maser polarization. This effect is indicated in Table 3.

Observed H_2O masers are rarely polarized, and then only linearly polarized. This property can be readily understood in relation to the theoretical results presented in Tables 1 - 3.

TABLE 3

MASER POLARIZATION PROPERTIES IN THE CASE OF $\gamma > R$ AND LARGE FARADAY ROTATION

	Unsaturated Maser $\Gamma > R$	Saturated Maser $\Gamma < R$
$g\Omega > \Delta\omega$	σ components dominate if $\sin^2\theta < 8/9$ π component dominates if $\sin^2\theta > 8/9$	σ components dominate if $\sin^2\theta < 8/9$ π component dominates if $\sin^2\theta > 8/9$
$\Delta\omega > g\Omega > R$	Unpolarized	Unpolarized

Because the g values of the upper and lower states of the microwave water transition are about 8×10^{-4}, the Zeeman splitting is smaller than the bandwidth ($\sim 10^5$ Hz) of the maser line for magnetic fields below 40 Gauss. For $R < g\Omega < \Delta\omega$, the stable polarization is linear. However, the growth of linear polarization is suppressed unless the stimulated emission rate exceeds the cross-relaxation rate which is about a few times per second.

The g values of the levels involved in those hydroxyl transitions which have been observed as masers are all of order unity except for the levels involved in the $\Pi_{\frac{1}{2}}$, $J = \frac{1}{2}$, $F = 1 \rightarrow 0$ transition for which the g values are very much smaller. No circular polarization has been observed in this transition. It seems very probable that the stimulated emission rates in OH masers are smaller than the cross-relaxation rate. In this case no polarization of the OH maser would be observed if $g\Omega < \Delta\omega$. The presence of 100% circular polarization in many OH sources associated with H II regions suggests that the case of $g\Omega > \Delta\omega$, $\gamma > R$ and large Faraday rotation is the most frequently realized one. However the absence of obvious Zeeman patterns and the sometimes prevalence of one type of circular polarization seem to indicate that there is an additional mechanism giving rise to a competition between the two circular polarizations.

PART I

ON PARAMETRIC DOWN-CONVERSION IN ASTROPHYSICAL MASERS

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ON PARAMETRIC DOWN-CONVERSION IN ASTROPHYSICAL MASERS

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ABSTRACT

The mechanism of parametric down-conversion proposed by Litvak cannot explain the observed preference for circular polarization in OH maser emission because the nonlinear interaction between oppositely circularly polarized microwaves is too weak.

I. INTRODUCTION

Maser emission from many OH sources exhibits strong circular polarization. Litvak (1970) has suggested that the process of parametric down-conversion (Bloembergen 1965) is responsible for the circular polarization in these sources. His proposal involves the coupling of two microwaves with an electron cyclotron wave through the nonlinear polarization which they induce in the plasma. The higher-frequency microwave is down-converted to the lower-frequency microwave and the electron cyclotron wave. In the presence of a static magnetic field, the OH microwave lines are split into Zeeman components whose separations are on the order of the electron cyclotron frequency. Parametric down-conversion would reduce the intensities of the higher-frequency components and enhance the intensities of the lower-frequency ones. Thus, this process might account for the preference for one circular polarization over the other which is observed in some sources.

Based on his calculation of the magnitude of this effect, Litvak claimed that it is important in OH maser sources. Our calculation of the parametric gain coefficient yields a value which is much smaller than Litvak's. Litvak (1971) has kindly informed us that there is an error in his expressions for the propagator of the electron cyclotron wave which accounts for the major difference between his result and ours. The error is due to the use of one factor of n_q in place of the correct factor $\partial[\omega(q)n_q]/\partial\omega(q)$ in the denominators of the propagators in his equations (21) and (21′). Litvak and we now agree that this mechanism of parametric down-conversion is unimportant in astrophysical OH masers.

In § II we derive expressions for the nonlinear current densities driven by three monochromatic waves in the absence of damping. We treat the damping of the electron cyclotron wave by electron collisions in § III. We derive the parametric gain coefficient for interacting monochromatic waves in § IV and extend this result to broad-band signals in § V. Finally, in § VI we apply the theory to OH maser sources.

II. THE SOURCE OF THE NONLINEAR INTERACTION

In this section we derive expressions describing the nonlinear interaction of three monochromatic waves in the absence of damping. The electric fields of the waves are

$$E_q(r, t) = \text{Re}\,\{\mathfrak{E}_q(r, t)\}$$
$$\mathfrak{E}_q(r, t) = \hat{e}_q A_q(r) \exp\,[i(k_q\cdot r - \omega_q t)]\,, \qquad q = 1, 2, 3 \quad (1)$$

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where the subscripts 1 and 2 denote the higher- and lower-frequency microwaves and the subscript 3 denotes the electron cyclotron wave. In the absence of nonlinear interactions, the complex amplitudes, A_q, are constant and the electric fields satisfy

$$\nabla \times \nabla \times \mathfrak{E}_q(r, t) - \omega_q^2 c^{-2}[I + 4\pi\chi(\omega_q)]\cdot\mathfrak{E}_q(r, t) = 0 , \tag{2}$$

where the susceptibility tensor

$$\chi(\omega) = \frac{\omega_p^2}{4\pi(\Omega_e^2 - \omega^2)\omega} \begin{vmatrix} \omega & -i\Omega_e & 0 \\ i\Omega_e & \omega & 0 \\ 0 & 0 & -(\Omega_e^2 - \omega^2)/\omega \end{vmatrix} . \tag{3}$$

In writing equation (3) we have taken the z-axis to lie along the direction of B_0. We have used the cold-plasma approximation for χ and, in addition, neglected ion motions. The parameters Ω_e and ω_p are the electron gyro and plasma frequencies. In typical maser sources they are thought to be of order 10^4 s^{-1}, well below microwave frequencies. Consequently, the microwaves are only weakly affected by the magnetoplasma. Except for propagation almost exactly orthogonal to the magnetic field, the microwave modes are transverse and circularly polarized. On the other hand, the electric field of the electron cyclotron wave is almost parallel to its wave vector. Thus the energy of this wave flows nearly perpendicular to the wave vector. This gives rise, in the presence of nonlinear interaction, to the rapid growth of the electron cyclotron wave with distance traversed in the direction of its propagation vector. However, the effective increase in the energy flow path also enhances the damping of the wave, as we shall show in § III.

We assume perfect phase matching of the three waves. Thus $k_3(\omega_3) = k_1(\omega_1) - k_2(\omega_2)$. This assumption will be relaxed in § V where we treat broad-band signals.

The nonlinear terms in the current density give rise to the interaction among the three waves. These terms are due to perturbations in both the electron velocity and the electron number density. The electron equation of motion reads

$$\frac{dv}{dt}(r, t) = -\frac{e}{m}\left\{E(r, t) + \frac{v}{c}(r, t) \times [B_0 + B(r, t)]\right\} , \tag{4}$$

where

$$E(r, t) = \sum_{q=1}^{3} E_q(r, t) ; \quad B(r, t) = \sum_{q=1}^{3} B_q(r, t) .$$

The time rate of change of the electron velocity at a fixed point is

$$\frac{\partial v}{\partial t}(r, t) = \frac{dv}{dt}(r, t) - [v(r, t)\cdot\nabla]v(r, t) . \tag{5}$$

The electron number density $n(r, t)$ consists of a mean value n_0 and a perturbed part $\delta n(r, t)$ due to the waves. From the continuity equation it follows that

$$\frac{\partial}{\partial t}\delta n(r, t) \approx -n_0\nabla\cdot v(r, t) . \tag{6}$$

The current density is

$$J(r, t) = -en(r, t)v(r, t) . \tag{7}$$

We wish to derive the lowest order nonlinear terms in $J(r, t)$. They are of second order in the amplitudes, A_q, of the three waves and arise in two ways. Some of them come from the product of the first-order number-density perturbation and the first-order velocity. The other terms result from the second-order velocity perturbation. The velocity, up to terms of second order in the amplitudes, follows from equations (1), (4), and (5). The first-order number-density perturbation is obtained from the first-order velocity by use of equation (6).

Even to second order in the amplitudes, the complete expression for the nonlinear current density is too cumbersome to be presented here. Fortunately, a great simplification may be achieved by retaining only the lowest-order terms in an expansion in powers of $\omega_3/\omega_1 \ll 1$. Thus we find

$$J_1^{NL}(r, t) = -\frac{e}{2m\omega_2}\mathrm{Re}\,\{[k_3 \cdot \chi(\omega_3) \cdot \hat{e}_3]\hat{e}_2 A_2(r) A_3(r) \exp[i(k_1 \cdot r - \omega_1 t)]\}$$

$$J_2^{NL}(r, t) = \frac{e}{2m\omega_1}\mathrm{Re}\,\{[k_3 \cdot \chi(\omega_3) \cdot \hat{e}_3]^* \hat{e}_1 A_1(r) A_3^*(r) \exp[i(k_2 \cdot r - \omega_2 t)]\}$$

$$J_3^{NL}(r, t) = \frac{e\omega_3}{2m\omega_1\omega_2}\mathrm{Re}\,\{[\hat{e}_1 \cdot \hat{e}_2^*]\chi(\omega_3) \cdot k_3 A_1(r) A_2^*(r) \exp[i(k_3 \cdot r - \omega_3 t)]\}\ . \quad (8)$$

The terms in $J_1^{NL}(r, t)$ and $J_2^{NL}(r, t)$ arise from the perturbation in the electron number density. They were omitted by Litvak, who kept only the contribution from the $v \times B$ term in the Lorentz force which is smaller than the dominant term by a factor ω_3/ω_1. The leading contributions to $J_3^{NL}(r, t)$ come from the second-order electron velocity. Our expression for $J_3^{NL}(r, t)$ differs from Litvak's because he used $dv(r, t)/dt$ instead of $\partial v(r, t)/\partial t$ in calculating $\partial J(r, t)/\partial t$.

It is clear from equation (8), and the fact that $\chi(\omega)$ is self-adjoint, that the following equalities hold:

$$\sum_{q=1}^{3} \langle E_q(r, t) \cdot J_q^{NL}(r, t)\rangle = 0\ , \quad (9)$$

$$\frac{\langle E_1(r, t) \cdot J_1^{NL}(r, t)\rangle}{\omega_1} + \frac{\langle E_2(r, t) \cdot J_2^{NL}(r, t)\rangle}{\omega_2} = 0\ ,$$

$$\frac{\langle E_1(r, t) \cdot J_1^{NL}(r, t)\rangle}{\omega_1} + \frac{\langle E_3(r, t) \cdot J_3^{NL}(r, t)\rangle}{\omega_3} = 0\ , \quad (10)$$

where the angular brackets denote time average. Equation (9) is the statement of energy conservation for the three waves. Equations (10) are the Manley-Rowe relations (Armstrong *et al.* 1962) and express the fact that for each photon of frequency ω_1 which is absorbed, one photon of frequency ω_2 and another of frequency ω_3 are emitted. We have verified that terms of the next higher order in ω_3/ω_1 also satisfy equations (9) and (10). We note that Litvak's expressions for $J_q^{NL}(r, t)$ satisfy equation (9) but not equations (10).

We shall need the complex forms of the nonlinear current densities in § IV. They are given by the expressions on the right-hand sides of equations (8) before the real parts are taken. We shall denote them by $\mathfrak{J}_q(r, t)$.

III. DAMPING OF THE ELECTRON CYCLOTRON WAVE

We consider damping of the electron cyclotron wave due to electron collisions. We neglect damping of the microwaves since it is less important.

The effect of collisions may be introduced into the electron equation of motion by adding a term $-\gamma v(r, t)$ to the right-hand side of equation (4). The parameter γ is the electron collision frequency. The addition of the collision term modifies the susceptibility tensor and thus the dispersion relation. Consequently, the wave vector of the electron cyclotron wave becomes complex. A general expression for the wave vector may be obtained by solving the homogeneous wave equation or, equivalently, by use of the

Appleton-Hartree formula (Budden 1961). A particularly simple form holds in the limit $\omega_p > \Omega_e$. In this case

$$k_3^2(\omega_3) \approx \frac{\omega_p^2 \omega_c}{c^2[(\omega_c - \omega_3) - i\gamma]}, \tag{11}$$

where

$$\omega_c \simeq \Omega_e |\cos\theta| .$$

Here θ is the angle between $\boldsymbol{k}_3$ and $\boldsymbol{B}_0$. The real and imaginary parts of k_3, k_{3r} and k_{3i}, follow directly:

$$k_{3r}^2 = \frac{\omega_p^2 \omega_c}{2c^2} \frac{\{(\omega_c - \omega_3) + [(\omega_c - \omega_3)^2 + \gamma^2]^{1/2}\}}{(\omega_c - \omega_3)^2 + \gamma^2},$$

$$\frac{k_{3i}}{k_{3r}} = \frac{\gamma}{(\omega_c - \omega_3) + [(\omega_c - \omega_3)^2 + \gamma^2]^{1/2}}. \tag{12}$$

For $\Omega_e \gtrsim \omega_p$ somewhat more complicated formulae are needed to express ω_3 and k_3. However, none of our conclusions depends sensitively on the restriction $\omega_p > \Omega_e$.

We observe from equation (12) that k_{3r} cannot be made arbitrarily large. The maximum value of k_{3r} is approximately

$$k_{3r}^{\mathrm{MAX}} = \frac{\omega_p}{c}\left(\frac{\Omega_e}{\gamma}|\cos\theta|\right)^{1/2}. \tag{13}$$

Thus, for perfect phase matching, $\boldsymbol{k}_{3r} = \boldsymbol{k}_1 - \boldsymbol{k}_2$, the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ is restricted to values less than

$$\phi^{\mathrm{MAX}} \approx k_{3r}^{\mathrm{MAX}}/k_1 . \tag{14}$$

Equation (14) assumes $\phi^{\mathrm{MAX}} < \pi$.

The strength of the interaction among the three waves is proportional to k_{3r}. Equation (12) shows that k_{3r} increases as $\omega_3 \rightarrow \omega_c$. However, k_{3i}/k_{3r} also increases as $\omega_3 \rightarrow \omega_c$. At resonance $k_{3i} \approx k_{3r}$ and the damping length is comparable to one wavelength. The energy of the electron cyclotron wave flows almost perpendicular to the wave vector. It is pertinent to obtain the total distance the energy travels as it advances a distance k_{3i}^{-1} in the direction of the wave vector. This distance is

$$d = \frac{1}{k_{3i}\cos\alpha}, \tag{15}$$

where α is the angle between the wave vector and the Poynting vector. The expression for $\cos\alpha$ follows directly from the homogeneous wave equation or the Appleton-Hartree formulae. For $\omega_p > \Omega_e$,

$$\cos\alpha = \frac{2\omega_p^2}{c^2}\frac{|\cot\theta|}{|k_3|^2}. \tag{16}$$

From equations (12), (15) and (16), it is clear that d in typical OH sources is very much smaller than the source dimensions.

We note that other forms of damping such as Landau damping and cyclotron resonance damping may be important. Thus collisional damping is only a minimum estimate.

IV. THE PARAMETRIC GAIN COEFFICIENT

Maxwell's wave equation for the electron cyclotron wave, including the nonlinear interaction, reads

$$\left\{\nabla \times \nabla \times I - \frac{\omega_3^2}{c^2} I - \frac{4\pi}{c^2}\omega_3(\omega_3 + i\gamma)\chi(\omega_3 + i\gamma)\right\}\cdot \mathfrak{E}_3(r, t) = \frac{i4\pi\omega_3}{c^2}\mathfrak{J}_3^{\mathrm{NL}}(r, t) . \tag{17}$$

Since the electron cyclotron wave is driven by the nonlinear current density, the direction of its electric field differs slightly from that of a free wave. An approximate method of obtaining the equation governing the amplitude $A_3(r)$ is to take the scalar product of equation (17) with $\hat{e}_3^*$ (Armstrong *et al.* 1962). The conditions in OH maser sources are such that the damping length is much smaller than the distance over which the values of $A_1(r)$ and $A_2(r)$ change. Thus $A_3(r)$ achieves a steady-state value which depends on the local value of $\mathfrak{J}_3^{\mathrm{NL}}(r, t)$. We obtain

$$A_3(r) \approx -\frac{4\pi\omega_3\hat{e}_3^*\cdot\mathfrak{J}_3^{\mathrm{NL}}(r,t)}{c^2 k_{3r}k_{3i}\cos^2\alpha}\exp\left[-i(k_{3r}\cdot r-\omega_3 t)\right], \tag{18}$$

for $k_{3i}/k_{3r} < 1$.[1] By use of equation (8) for $\mathfrak{J}_3^{\mathrm{NL}}(r, t)$, with k_3 replaced by k_{3r}, we find

$$A_3(r) \approx \frac{ek_{3r}\Omega_e^3\sin^2\theta|\cos\theta|}{4m\omega_1\omega_2\omega_p^2\gamma}(\hat{e}_1\cdot\hat{e}_2^*)A_1(r)A_2^*(r), \tag{19}$$

where we have made use of equations (12) and (16).

By solving equations similar to equation (17) we can obtain expressions governing the growth of the microwave amplitudes. Neglecting damping, we have

$$\begin{aligned}(k_1\cdot\nabla)|A_1(r)|^2 &= -\frac{\Omega_e^3\sin^2\theta}{8\omega_p^2\gamma}|\cos\theta|\,|\hat{e}_1^*\cdot\hat{e}_2|^2\left(\frac{e|A_2(r)|}{mc\omega_2}\right)^2 k_{3r}^2|A_1(r)|^2,\\ (k_2\cdot\nabla)|A_2(r)|^2 &= \frac{\Omega_e^3\sin^2\theta}{8\omega_p^2\gamma}|\cos\theta|\,|\hat{e}_1^*\cdot\hat{e}_2|^2\left(\frac{e|A_1(r)|}{mc\omega_1}\right)^2 k_{3r}^2|A_2(r)|^2,\end{aligned} \tag{20}$$

where $A_3(r)$ has been replaced by the expression in equation (19). The parametric gain coefficient is

$$K = \frac{1}{A_2(r)}(\hat{k}_2\cdot\nabla)A_2(r) = \frac{\Omega_e^3\sin^2\theta|\cos\theta|}{16k_2\omega_p^2\gamma}|\hat{e}_1^*\cdot\hat{e}_2|^2\left(\frac{e|A_1(r)|}{mc\omega_1}\right)^2 k_{3r}^2. \tag{21}$$

V. BROAD-BAND MICROWAVES

In order to apply parametric down-conversion to cosmic OH sources we must extend our previous results to take into account the finite bandwidth of the microwave signals. We consider two broad-band microwaves propagating in directions $\hat{k}_1$ and $\hat{k}_2$. Each microwave is assumed to be the sum of identically polarized components having different frequencies. We assume that the phases and amplitudes of the individual frequency components are uncorrelated. This assumption may not be correct. For example, the signals in the maser sources may be in the form of pulses.

We assume that the frequency separation of the two microwaves is comparable to their individual bandwidths and to the electron cyclotron frequency. Each frequency component of one microwave interacts with all the frequency components of the other; however, only a small fraction of these interactions are significant. It is evident from equation (12) that a small fractional change in ω_3 results in a much larger fractional change in k_{3r}. For each frequency component of the higher-frequency microwave there is at most one frequency component of the lower-frequency microwave with which a

[1] A more accurate derivation of equation (18) must take into account the difference in the directions of the electric field vectors of the forced and the free electron cyclotron waves. This distinction is important because

$$\cos^2\alpha = -2\hat{e}_3^*\cdot[\hat{k}_3\times(\hat{k}_3\times\hat{e}_3)] \ll 1.$$

A more accurate method of obtaining the steady-state value of $A_3(r)$ is to invert the operator in equation (17) which acts on $\mathfrak{E}_3(r, t)$. This procedure yields an expression for $A_3(r)$ which differs from that given by equation (18) in having the additional factor of $||k_3|^2/(k_3k_{3r})|^2$. This factor is essentially unity for $k_{3i}/k_{3r} < 1$.

perfect phase match is achieved. For a phase mismatch of an amount $\Delta \boldsymbol{k} = \boldsymbol{k}_1(\omega_1) - \boldsymbol{k}_2(\omega_2) - \boldsymbol{k}_{3r}(\omega_3)$ the parametric gain coefficient is reduced below the value given by equation (21) for perfect phase matching by a factor $k_{3i}^2/(k_{3i}^2 + \Delta k^2)$. Thus only couplings giving rise to a phase mismatch $|\Delta k| < k_{3i}$ are important. From the relations given by equations (12) it is clear that significant interaction occurs for pairs whose difference frequency is confined to an interval of width γ about the value needed for perfect phase matching. For broad-band microwaves, $\Delta\omega > \gamma$, this effect reduces the parametric gain coefficient by a factor $\gamma/\Delta\omega$ relative to its value for equally intense monochromatic waves. Thus for broad-band signals

$$K = \frac{\Omega_e^3 \sin^2\theta |\cos\theta|}{16 k_2 \omega_p^2 \Delta\omega} |\hat{e}_1^* \cdot \hat{e}_2|^2 \left(\frac{e|A_1(r)|}{m c \omega_1}\right)^2 k_{3r}^2 . \tag{22}$$

Although the calculations are not given here, we have also derived the expression for K in the limit $\omega_p < \Omega_e$. In this case

$$K = \frac{\omega_p^2 \sin^2\theta}{16 k_2 \Omega_e \Delta\omega} |\hat{e}_1^* \cdot \hat{e}_2|^2 \left(\frac{e|A_1(r)|}{m c \omega_1}\right)^2 k_{3r}^2 . \tag{22'}$$

Here

$$k_3^2(\omega_3) \approx \frac{\omega_p^2(1 + \cos^2\theta)\Omega_e}{2c^2[(\omega_c' - \omega_3) - i\gamma]} , \tag{11'}$$

where

$$\omega_c' \approx (\Omega_e^2 + \omega_p^2 \sin^2\theta)^{1/2} .$$

VI. APPLICATION TO OH MASER SOURCES

We are interested in the competition between oppositely circularly polarized modes in OH sources. Thus $|\hat{e}_1^* \cdot \hat{e}_2| \approx (\phi/2)^2$ where $\cos\phi = \hat{k}_1 \cdot \hat{k}_2$. Furthermore, $k_{3r} \simeq k\phi$ with $k = \omega/c \sim k_1 \sim k_2$. Both of these relations assume $\phi < 1$. With these approximations equation (22) is transformed into

$$K = \frac{\Omega_e^3 \sin^2\theta |\cos\theta|}{2^8 \omega_p^2 \Delta\omega} \left(\frac{e|A_1(r)|}{m c \omega}\right)^2 \phi^6 k . \tag{23}$$

We note from equations (13) and (14) that damping due to electron collisions restricts ϕ to be less than

$$\phi^{\text{MAX}} \approx \frac{\omega_p}{\omega}\left(\frac{\Omega_e}{\gamma} |\cos\theta|\right)^{1/2} . \tag{24}$$

A typical set of numerical parameters for a maser OH source might be $\gamma = 10^{-3}\ \text{s}^{-1}$, $\Omega_e = 2 \times 10^4\ \text{s}^{-1}$, $\omega_p = 6 \times 10^4\ \text{s}^{-1}$, $\omega = 10^{10}\ \text{s}^{-1}$, and $\Delta\omega \simeq 2 \times 10^4\ \text{s}^{-1}$. We take a value of $|A_1| \simeq 10^{-6}$ esu. This value applies to a source of 100 f.u. at a distance of 10 kpc having a diameter of 100 a.u. With these parameters and $\theta = \pi/4$,

$$\phi^{\text{MAX}} \approx 2.3 \times 10^{-2} \tag{25}$$

and the corresponding

$$K^{\text{MAX}} \approx 3.0 \times 10^{-32}\ \text{cm}^{-1} . \tag{26}$$

The greatest uncertainty in determining K^{MAX} arises from the factor $(\phi^{\text{MAX}})^6$, which depends on the parameters ω_p, Ω_e, and γ we have chosen. However, even for a conservative estimate of $\phi^{\text{MAX}} \approx 1$ our result indicates that the parametric down-conversion resulting from nonlinear interactions due to the magnetoplasma is unimportant in OH masers. This differs from the conclusion reached by Litvak. There are three major sources for this disagreement. Two of these sources act to diminish Litvak's value for K^{MAX}. The first factor is due to the limitation that damping by electron collisions places on ϕ^{MAX}. The second factor arises from Litvak's error in the propagator of the electron cyclotron wave. The third source of the difference between our result and Litvak's is due to his neglect of the electron number-density perturbation associated with the electron cyclotron wave, as discussed following equation (8). By itself, this oversight would have led him to underestimate the size of the parametric gain coefficient. However, its effect is much smaller than those due to the first two factors. Thus, Litvak overestimated the importance of his mechanism for parametric down-conversion.

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PART II

MAGNETIC FIELD, PLASMA, AND MASER POLARIZATION

ABSTRACT

The equations governing the transfer of polarized radiation in astrophysical masers are derived. It is found that the magnetic field and the plasma in maser sources play a central role in determining the polarization of the emitted radiation. The character of the polarization depends upon the relative sizes of the decay constant of the maser levels, Γ, the stimulated emission rate, R, the Zeeman splitting, $g\Omega$, and the bandwidth of the amplified radiation, $\Delta\omega$.

Unsaturated masers ($R < \Gamma$) emit unpolarized radiation unless $g\Omega \gtrsim \Delta\omega$. For $g\Omega \gtrsim \Delta\omega$ they amplify the Zeeman pattern if the Faraday rotation per gain length in the source is small. If the Faraday rotation per gain length is large, the σ components of the Zeeman pattern are 100% circularly polarized and the π component is unpolarized.

Saturated masers ($R > \Gamma$) emit unpolarized radiation unless $g\Omega \gtrsim (R\Gamma)^{1/2}$. If the Faraday rotation across the region of saturated amplification is small, the emitted radiation is partially linearly polarized for $(R\Gamma)^{1/2} < g\Omega << \Delta\omega$ whereas for $g\Omega \gtrsim \Delta\omega$ it is just the amplified Zeeman pattern. If the Faraday rotation across the saturated region is large, all linear polarization is destroyed. For $g\Omega > \Delta\omega$, the σ components of the Zeeman pattern are again 100% circularly polarized.

I. INTRODUCTION

Maser emission has been detected from two interstellar molecules, water and hydroxyl. The water line is most commonly unpolarized, but in several sources exhibits a considerable degree of linear polarization (Buhl, Synder, Schwartz and Barrett 1969; Sullivan 1971). The hydroxyl lines, from both the ground and the excited rotational levels, are often strongly polarized. The high degree of circular polarization which is typical of lines from sources associated with HII regions is especially striking (Palmer and Zuckerman 1967; Ball and Meeks 1968; Robinson, Goss and Manchester 1970). The aim of this paper is to relate the observed polarizations to the physical conditions in the maser clouds.

The specific problem investigated here is the transfer of radiation in a maser operating between upper and lower states of total angular momenta $F_a = 1$ and $F_b = 0$, respectively. Although states of higher angular momenta are involved in the observed interstellar masers, all of the important physics is illustrated by an investigation of this simple example.

There is one respect in which the model analyzed here does not faithfully represent conditions in astrophysical masers. This is due to the neglect of trapped infrared line radiation. In real cosmic masers this trapped line radiation relaxes population differences among the magnetic sublevels of the $F_a = 1$ state. A comprehensive treatment of the effects of trapped infrared lines on astrophysical maser emission will be given in part III.

The equations describing the radiation field are developed in section II. In section III the equation of motion of the molecular density matrix is set up and the macroscopic polarization induced by the radiation field in the active medium is related to its off-diagonal components. The density matrix equations of motion are solved in a variety of limiting cases in section IV. The results derived in sections II and IV are then combined in section V to provide expressions governing the transfer of maser radiation in these limiting cases. Finally, applications of the theory to the observed polarizations in cosmic masers are presented in section VI.

II. THE ELECTROMAGNETIC FIELD

The propagation of radiation in cosmic masers is affected by the active molecules and by free electrons. The polarization density $\underset{\sim}{P}$ and the current density $\underset{\sim}{J}$ will be used to describe the molecular polarization density and the electron current density. The radiation field is treated classically and approximated locally by a plane wave. Partial justification for this approximation is provided by the results obtained in Goldreich and Keeley (1972). There it was shown that the radiation near the outer edge of a saturated cosmic maser is directed in a small solid angle about any point. Maxwell's equations in gaussian units then read

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{D} = 4\pi\rho \qquad \underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \qquad \underset{\sim}{\nabla} \times \underset{\sim}{B} = 4\pi \underset{\sim}{J} + \frac{1}{c}\frac{\partial \underset{\sim}{D}}{\partial t} \tag{1}$$

$$\underset{\sim}{D} = \underset{\sim}{E} + 4\pi \underset{\sim}{P} \quad .$$

The wave equation for a plane wave travelling in the + z direction takes the form

$$\frac{1}{c^2}\frac{\partial^2 \underset{\approx}{E}}{\partial t^2} + \hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{k}} \times \frac{\partial^2 \underset{\approx}{E}}{\partial z^2} = -\frac{4\pi}{c^2}\frac{\partial^2 \underset{\approx}{P}}{\partial t^2} - \frac{4\pi}{c}\frac{\partial \underset{\approx}{J}}{\partial t}, \qquad (2)$$

where $\hat{\underset{\approx}{k}} = \underset{\sim}{k}/|\underset{\sim}{k}|$ is the unit propagation vector.

The transverse part of the electric field is decomposed into its circularly polarized components and reads

$$\underset{\sim}{E}(z,t) = \mathrm{Re}\left\{E^+(z,t)\hat{\underset{\sim}{e}}^+ + E^-(z,t)\hat{\underset{\sim}{e}}^-\right\},$$

where

$$E^{\pm}(z,t) = \mathscr{E}^{\pm}(z,t)\exp\left\{-i\left[\omega_0(t - z/c) + \phi^{\pm}(z,t)\right]\right\} \qquad (3)$$

and ω_0 is the resonant frequency of the maser transition. The amplitudes, $\mathscr{E}^{\pm}(z,t)$, and the phases, $\phi^{\pm}(z,t)$, are real, slowly varying, functions of space and time. That is, $|\partial\mathscr{E}^{\pm}/\partial t| \ll \omega_0|\mathscr{E}^{\pm}|$, $|\partial\mathscr{E}^{\pm}/\partial z| \ll k|\mathscr{E}^{\pm}|$, $|\partial\phi^{\pm}/\partial t| < \omega_0$ and $|\partial\phi^{\pm}/\partial z| \ll k$. A similar decomposition of the polarization and current densities yields

$$\underset{\sim}{P}(z,t) = \mathrm{Re}\left\{P^+(z,t)\hat{\underset{\sim}{e}}^+ + P^-(z,t)\hat{\underset{\sim}{e}}^-\right\}$$

$$\underset{\sim}{J}(z,t) = \mathrm{Re}\left\{J^+(z,t)\hat{\underset{\sim}{e}}^+ + J^-(z,t)\hat{\underset{\sim}{e}}^-\right\}, \qquad (4)$$

$$P^{\pm}(z,t) = \mathscr{P}^{\pm}(z,t)\exp\left\{-i\left[\omega_0(t - z/c) + \phi^{\pm}(z,t)\right]\right\} \qquad (5)$$

$$J^{\pm}(z,t) = \mathscr{J}^{\pm}(z,t)\exp\left\{-i\left[\omega_0(t - z/c) + \phi^{\pm}(z,t)\right]\right\}.$$

However, unlike $\mathscr{E}^{\pm}$ which are real, $\mathscr{P}^{\pm}$ and $\mathscr{J}^{\pm}$ are complex.

If equations (3), (4) and (5) are substituted into equation (2) and the transverse components are projected out, the result is

$$\frac{D}{Dz}\left[\mathscr{E}^{\pm} \exp(-\,i\phi^{\pm})\right] = 2\pi\left[\frac{i\omega_0}{c}\mathscr{P}^{\pm} - \mathscr{J}^{\pm}\right]\exp(-\,i\phi^{\pm}) \quad (6)$$

where

$$\frac{D}{Dz} = \frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} \quad .$$

It follows immediately from equation (6) that

$$\frac{D\mathscr{E}^{\pm 2}}{Dz} = 2\pi i\,\frac{\omega_0}{c}\left[\mathscr{P}^{\pm}\,\mathscr{E}^{\pm} - \mathscr{P}^{\pm *}\,\mathscr{E}^{\pm}\right] - 2\pi\left[\mathscr{J}^{\pm}\,\mathscr{E}^{\pm} + \mathscr{J}^{\pm *}\,\mathscr{E}^{\pm}\right] \quad (7)$$

$$\frac{D}{Dz}\left[\mathscr{E}^{+}\mathscr{E}^{-}\exp(-\,i\Delta\phi)\right] = 2\pi i\,\frac{\omega_0}{c}\left[\mathscr{P}^{+}\mathscr{E}^{-} - \mathscr{P}^{-*}\,\mathscr{E}^{+}\right]\exp(-i\Delta\phi)$$

$$- \, 2\pi\left[\mathscr{J}^{+}\mathscr{E}^{-} + \mathscr{J}^{-*}\,\mathscr{E}^{+}\right]\exp(-i\Delta\phi) \quad (8)$$

where $\Delta\phi = \phi^{+} - \phi^{-}$. Equations (7) and (8) are the equations which govern the transfer of polarized maser radiation. The next two sections are devoted to evaluating the source terms in these equations.

The fluctuation spectrum of the radiation field plays an essential role in the analysis to be presented in this paper. The central assumption made concerning the statistical behavior of the radiation field is that it is stationary. This assumption implies that expectation values are equivalent to time averages. If the electric field is written as a Fourier integral

$$E^{\mu}(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E^{\mu}(\omega) e^{-i\omega(t - z/c)} d\omega, \qquad \mu = + \text{ or } - \qquad (9)$$

then the assumption of stationary statistics implies that

$$\langle E^{\mu}(\omega) E^{\nu *}(\omega')\rangle = 2\pi F^{\mu\nu}(\omega)\delta(\omega - \omega') ,$$

where the triangular brackets denote expectation value. In writing equation (9) the small effect of the medium (i.e., the active molecules and the magnetoplasma) on the phase velocity of the electromagnetic waves has been ignored. A justification of the assumption of stationary statistics and several results which it implies are derived in Appendix A. A few of the more important consequences are stated here without proof.

The spectral energy distribution in the line is taken to be gaussian so that

$$F^{\mu\nu}(\omega) = \frac{\langle E^{\mu}(z,t) E^{\nu *}(z,t)\rangle}{\sqrt{2\pi}\ \Delta\omega} \exp\left[- \frac{(\omega - \omega_r)^2}{2\Delta\omega^2}\right] . \qquad (10)$$

It then follows immediately that

$$\langle E^{\mu}(z,t)E^{\nu^*}(z',t')\rangle = \langle E^{\mu}(z,t)E^{\nu^*}(z,t)\rangle \exp\left\{-\left[(t-t')-\frac{(z-z')}{c}\right]^2\frac{\Delta\omega^2}{2}\right\}$$

$$\times \exp\left\{-i\omega_r\left[(t-t')-\frac{(z-z')}{c}\right]\right\}. \qquad (11)$$

This result is extensively used in section IV.

The maser medium is assumed to be homogeneous and permeated by a static magnetic field $\underset{\sim}{B}_0$. The direction of the magnetic field, which makes an angle θ with the propagation vector (or + z axis), defines the z' axis of the primed coordinate system. The common x and x' axes are perpendicular to the plane formed by the z and z' axes. The circularly polarized unit vectors are defined by $\hat{\underset{\sim}{e}}^{\pm} = (\hat{\underset{\sim}{e}}_x \pm i\hat{\underset{\sim}{e}}_y)/\sqrt{2}$. The equations of radiative transfer are most conveniently described in terms of the Stokes parameters which are defined as (Chandrasekhar 1950)

$$\begin{aligned}
I &= \frac{c}{8\pi}\langle \mathcal{E}^{+2} + \mathcal{E}^{-2}\rangle \\
V &= \frac{c}{8\pi}\langle \mathcal{E}^{+2} - \mathcal{E}^{-2}\rangle = I_R - I_L \\
Q &= \frac{c}{8\pi}\langle 2\,\mathcal{E}^{+}\mathcal{E}^{-}\cos\Delta\phi\rangle = I_x - I_y \qquad (12)\\
U &= \frac{c}{8\pi}\langle 2\,\mathcal{E}^{+}\mathcal{E}^{-}\sin\Delta\phi\rangle \; .
\end{aligned}$$

The relations between the Stokes parameters and the linearly polarized intensities follow directly from the definitions of $\hat{\underset{\sim}{e}}^{\pm}$.

III. THE DENSITY MATRIX EQUATION OF MOTION

The behavior of the active molecules is most easily described in terms of a density matrix. The development of the density matrix equation of motion presented here follows in most respects that given by Sargent, Lamb and Fork (1967). However, the method used to obtain approximate solutions of these equations is entirely different from theirs.

The equation of motion satisfied by the density matrix is

$$\frac{\partial \rho}{\partial t} = - i \left[H\rho - \rho H \right] - \Gamma\rho + \Lambda, \qquad (13)$$

where Γ and Λ are diagonal matrices. Actually, a single density matrix can only describe molecules at a fixed position and time (z,t) moving at a specific velocity v along z since the Hamiltonian matrix, $\hbar H$, is a function of all three variables. The v dependence arises from the doppler shift. The phenomenological decay constants which appear in the matrix Γ include the effects of transitions induced by infrared and harder photons as well as honest collisions with atoms and molecules. Collisions which destroy phase memory but do not induce transitions are unimportant at microwave frequencies and are ignored. In the interests of simplicity, it will be assumed that both levels a and b have the same decay constant. The matrix Λ has for its components the rates of excitation per unit volume per unit velocity interval into the magnetic sublevels of states a and b. It is assumed that the excitation is isotropic and thus independent of the magnetic quantum number within a given level.

The macroscopic polarization induced by the radiation field in the active molecules is given by

$$\underset{\sim}{P}(z,t) = e\int_{-\infty}^{\infty} dv \ \mathrm{tr}\left[\rho(z,t;v)\underset{\sim}{r}\right] , \qquad (14)$$

where $\underset{\sim}{r}$ is the matrix of the position vector.

The solution of equation (13) will be carried out in a variety of limiting cases in the next section. The character of the solution in each case will be determined by the relative values of four frequencies. They are: the radian bandwidth of the electromagnetic waves, $\Delta\omega$, the damping frequency, Γ, the Zeeman splitting, $g\Omega$, and the stimulated emission rate R. Here g is the Landé g value appropriate to the upper state and $\Omega = eB_0/mc$ is the radian gyrofrequency. The stimulated emission rate is $R \sim (p\mathcal{E}/\hbar)^2/\Delta\omega$ where p is the reduced dipole matrix element of the maser transition.

There are two natural directions to choose as the axis of quantization. For cases in which $g\Omega >> R$, the magnetic field direction is the most convenient quantization axis, whereas for $g\Omega << R$, greater simplicity is achieved by quantizing about the axis of propagation. The form of the Hamiltonian, $\hbar H$, depends upon this choice of axis. For quantization along the magnetic axis

$$H_{a'_m a'_n} = (\omega_0 + \frac{g\Omega m}{2})\delta_{mn}$$

$$H_{bb} = 0 \qquad (15)$$

$$H_{a'_m b} = V_{a'_m b} \; .$$

For quantization about the propagation direction

$$H_{a_m a_n} = \omega_0 \delta_{mn} + \frac{g\Omega}{2\sqrt{2}} \begin{vmatrix} \sqrt{2}\cos\theta, & i\sin\theta, & 0 \\ -i\sin\theta, & 0, & i\sin\theta \\ 0, & -i\sin\theta, & -\sqrt{2}\cos\theta \end{vmatrix}$$

$$H_{bb} = 0 \qquad (16)$$

$$H_{a_m b} = V_{a_m b} \; .$$

In equations (15) and (16) the subscripts on a' and a take on the values 1, 0, − 1. The primes attached to the a in equations (15) denote quantization about the magnetic (z') axis. The matrix elements of the time-dependent perturbation energy associated with the microwave electric field are given by

$$\hbar V_{a'_m b}(z,t) = p\left\{ \underset{\sim}{E}\cdot\frac{(\hat{\underset{\sim}{i}} - i\hat{\underset{\sim}{j}}')}{\sqrt{2}}\,\delta_{m,+1} - \underset{\sim}{E}\cdot\hat{\underset{\sim}{k}}'\,\delta_{m,0} - \underset{\sim}{E}\cdot\frac{(\hat{\underset{\sim}{i}} + i\hat{\underset{\sim}{j}}')}{\sqrt{2}}\,\delta_{m,-1}\right\} \tag{17}$$

and

$$\hbar V_{a_m b}(z,t) = p\left\{ \underset{\sim}{E}\cdot\frac{(\hat{\underset{\sim}{i}} - i\hat{\underset{\sim}{j}})}{\sqrt{2}}\,\delta_{m,+1} - \underset{\sim}{E}\cdot\frac{(\hat{\underset{\sim}{i}} + i\hat{\underset{\sim}{j}})}{\sqrt{2}}\,\delta_{m,-1}\right\}, \tag{18}$$

(Sargent, Lamb and Fork 1967). Here $\hat{\underset{\sim}{i}}'$, $\hat{\underset{\sim}{j}}'$, $\hat{\underset{\sim}{k}}'$ and $\hat{\underset{\sim}{i}}$, $\hat{\underset{\sim}{j}}$, $\hat{\underset{\sim}{k}}$ are the unit vectors associated with the primed and un-primed coordinate systems. In terms of the unit polarization vectors

$$\frac{\hat{\underset{\sim}{i}} \pm i\hat{\underset{\sim}{j}}'}{\sqrt{2}} = \frac{(1 \pm \cos\theta)}{2}\hat{\underset{\sim}{e}}^{+} + \frac{(1 \mp \cos\theta)}{2}\hat{\underset{\sim}{e}}^{-} \mp \frac{i\sin\theta}{\sqrt{2}}\hat{\underset{\sim}{k}}$$

$$\hat{\underset{\sim}{k}}' = \cos\theta\,\hat{\underset{\sim}{k}} - \frac{i\sin\theta}{\sqrt{2}}(\hat{\underset{\sim}{e}}^{+} - \hat{\underset{\sim}{e}}^{-}). \tag{19}$$

In practice the rotating wave approximation (Lamb 1964) will always be used in solving the density matrix equation of motion. Thus only the negative frequency parts of $\underset{\sim}{E}$ are used in the expressions for $V_{a'_m b}$ and $V_{a_m b}$.

From here on the practice of attaching primes to the quantities referred to the magnetic axis basis will be dropped. It should always be clear which axis is being used as the quantization axis.

IV. APPROXIMATE SOLUTIONS OF THE DENSITY MATRIX EQUATION OF MOTION

An approximation method will be used in solving the density matrix equation of motion. It is based on neglecting the temporal fluctuations of the density matrix elements within a single level. That is, it is assumed that $\rho_{a_m a_n}$ and ρ_{bb} are constants. This procedure is commonly used in the derivation of the ordinary rate equations. However, it is important to bear in mind that it is an approximation. Since all of the results derived in this paper depend upon this approximation some effort will be expended on its justification. Unfortunately, the authors have not been able to produce a rigorous defense of this crucial approximation. Only plausibility arguments in favor of its adoption are presented here.

In all cosmic masers observed to date, the bandwidth of the maser radiation, $\Delta\omega$, is much larger than the stimulated emission rate R. Thus the auto-correlation time of the electric field, which is essentially $\Delta\omega^{-1}$ (c f. eq.[11]), is much shorter than the time between successive absorptions and stimulated emissions. In this limit one might guess that the fluctuations of $\rho_{a_m a_m} - \rho_{bb}$ and $\rho_{a_m a_n}(m \neq n)$ would be smaller than their expectation values. Furthermore, the power spectrum of the fluctuations might be expected to peak near $\Delta\omega$. The actual behavior of $\rho_{a_m a_m} - \rho_{bb}$ and $\rho_{a_m a_n}$ ($m \neq n$) is entirely different. The fundamental reason is that the molecules, being resonant systems, do not respond equally to all the frequency components of the electric field. Instead, they effectively filter the signal and respond most strongly to those frequency components which are closest to their resonant frequency. The molecular response is governed by only a limited spectral slice of the whole signal. Since the autocorrelation time of the electric field associated with this spectral slice is greater than $\Delta\omega$, the fluctuations in $\rho_{a_m a_m} - \rho_{bb}$ and $\rho_{a_m a_n}$ ($m \neq n$) are both larger and slower than one might at first have guessed. A mathematical description of these fluctuations is contained in Appendix B. It is shown there that the fluctuations of $\rho_{a_m a_m} - \rho_{bb}$ and $\rho_{a_m a_n}$ ($m \neq n$) are comparable to their expectation values in cases of saturation $R > \Gamma$.

In spite of the obvious risk in ignoring the fluctuations in $\rho_{a_m a_m} - \rho_{bb}$ and $\rho_{a_m a_n}$ ($m \neq n$), there is persuasive evidence that this simplification does not lead to any significant errors in calculating the polarization properties of maser radiation. This evidence is based on the results of perturbation theory calculations in which the density matrix is derived to third order in the electric field.

These calculations can be carried through without neglecting the fluctuations in $\rho_{a_m a_n}$ and ρ_{bb}. It is then found that the final results for the transfer of polarized radiation are independent of whether or not these matrix elements are assumed to be constants.

For the remainder of this paper the approximation that $\rho_{a_m a_m} - \rho_{bb}$ and $\rho_{a_m a_n}$ $(m \neq n)$ are constants will be adopted. Only when the final theoretical results are compared with observations will the question of its validity be reopened.

A) Magnetic Axis Quantization

With $\rho_{a_m a_n}$ and ρ_{bb} assumed constant, the components of the density matrix equation of motion can be manipulated to read

$$\lambda_a = \Gamma \rho_{a_m a_m} + i(V_{a_m b} \rho_{b a_m} - \rho_{a_m b} V_{b a_m}) \qquad (a)$$

$$0 = \left[\Gamma + i \frac{g\Omega}{2} (m-n)\right] \rho_{a_m a_n} + i(V_{a_m b} \rho_{b a_n} - \rho_{a_m b} V_{b a_n}) \quad n \neq m \qquad (b) \qquad (20)$$

$$3\lambda_a + \lambda_b = \Gamma \left(\sum_m \rho_{a_m a_m} + \rho_{bb} \right) \qquad (c)$$

$$\rho_{a_m b} = i \sum_n \rho_{a_m a_n} \int_{-\infty}^{t} \exp\left\{-\left[\Gamma + i(\omega_0 + \frac{g\Omega}{2} m)\right](t - t')\right\} V'_{a_n b} \, dt'$$

$$- i \rho_{bb} \int_{-\infty}^{t} \exp\left\{-\left[\Gamma + i(\omega_0 + \frac{g\Omega}{2} m)\right](t - t')\right\} V'_{a_m b} \, dt' . \qquad (21)$$

In equations (20) and (21) $\rho = \rho(z,t;v)$, $V = V(z,t)$ and $\lambda = \lambda(v)$. A prime attached to V indicates that its argument is (z',t') where $z' = z - v(t-t')$.

When equations (21) are substituted into equations (20) and the expectation values of the resulting expressions are taken one obtains

$$\lambda_a = 2\mathrm{Re}\left\{\begin{aligned}&\sum_n \rho_{a_m a_n}\int_{-\infty}^{t}\exp\left\{-\left[\Gamma + i\left(\omega_0 + \frac{g\Omega}{2}m\right)\right](t-t')\right\}\langle V'_{a_n b}V_{ba_m}\rangle dt'\\ &-\rho_{bb}\int_{-\infty}^{t}\exp\left\{-\left[\Gamma + i\left(\omega_0 + \frac{g\Omega}{2}m\right)\right](t-t')\right\}\langle V'_{a_m b}V_{ba_m}\rangle dt'\\ &+\frac{\Gamma}{2}\rho_{a_m a_m}\end{aligned}\right\} \quad \text{(a)}$$

$$0 = \left\{\begin{aligned}&\sum_k \rho_{a_m a_k}\int_{-\infty}^{t}\exp\left\{-\left[\Gamma + i\left(\omega_0 + \frac{g\Omega}{2}m\right)\right](t-t')\right\}\langle V'_{a_k b}V_{ba_n}\rangle\, dt'\\ &+\sum_k \rho_{a_k a_n}\int_{-\infty}^{t}\exp\left\{-\left[\Gamma - i\left(\omega_0 + \frac{g\Omega}{2}n\right)\right](t-t')\right\}\langle V'_{ba_k}V_{a_m b}\rangle\, dt'\\ &-\rho_{bb}\int_{-\infty}^{t}\exp\left\{-\left[\Gamma + i\left(\omega_0 + \frac{g\Omega}{2}m\right)\right](t-t')\right\}\langle V'_{a_m b}V_{ba_n}\rangle\, dt'\\ &-\rho_{bb}\int_{-\infty}^{t}\exp\left\{-\left[\Gamma - i\left(\omega_0 + \frac{g\Omega}{2}n\right)\right](t-t')\right\}\langle V'_{ba_n}V_{a_m b}\rangle\, dt'\\ &+\left[\Gamma + i\frac{g\Omega}{2}(m-n)\right]\rho_{a_m a_n}\end{aligned}\right\} \quad \text{(b)} \quad (22)$$

$$3\lambda_a + \lambda_b = \Gamma\left(\sum_m \rho_{a_m a_m} + \rho_{bb}\right). \qquad \text{(c)}$$

Equations (22) are ten linear equations in the ten density matrix elements $\rho_{a_m a_n}$ and ρ_{bb}. The solution of these equations will now be carried out in a number of limiting cases.

Case 1 - $g\Omega >> \Delta\omega$

In this case the maser will amplify radiation in three narrow bands of width $\Delta\omega$ centered on the resonant frequencies $\omega_{a_m b} = \omega_0 + g\Omega m/2$. The radiation in each band will interact strongly with only a single magnetic sublevel of the upper state. Thus the most important terms in the integrands in equations (22) are those which contain the expectation values of the product of two electric fields in the radiation band which resonates with the frequency in the exponential term. If only these dominant terms are retained, equations (22) may be rewritten as

$$\lambda_a = \Gamma \rho_{a_m a_m} + U_{a_m a_m} (q + q^*)(\rho_{a_m a_m} - \rho_{bb}) \qquad \text{(a)}$$

$$0 = \left[\Gamma + i \frac{g\Omega}{2} (m - n)\right] \rho_{a_m a_n} \qquad \text{(b)} \quad (23)$$

$$3\lambda_a + \lambda_b = \Gamma\left(\sum_m \rho_{a_m a_m} + \rho_{bb}\right), \qquad \text{(c)}$$

where

$$U_{a_m a_m} = \langle \mathcal{V}_{a_m b} \mathcal{V}_{b a_m} \rangle \tag{24}$$

and

$$q(v) = \int_0^\infty dx \, \exp\left\{ - \left[\Gamma + i \frac{v}{c} \omega_r \right] x - \frac{(\Delta\omega x)^2}{2} \right\} . \tag{25}$$

Since $\Gamma \ll \Delta\omega$ in all cases of interest

$$\mathrm{Re}\, q = \frac{1}{\Delta\omega} \sqrt{\frac{\pi}{2}} \, \exp\left[- \frac{1}{2} \left(\frac{v}{c} \frac{\omega_r}{\Delta\omega} \right)^2 \right] . \tag{26}$$

The term $\mathcal{V}_{a_m b}$ in equation (24) includes only those contributions to $V_{a_m b}$ which are due to that band of radiation which is centered on the frequency $\omega_r = \omega_0 + g\Omega m/2$. In deriving equation (25) for q, it has been assumed that $v/c \ll 1$.

Equation (23) may now be solved in two limiting cases.

Case 1a - $g\Omega \gg \Delta\omega$ $\quad$ $R \gg \Gamma$

In this case the maser is saturated and

$$\rho_{a_m a_m} - \rho_{bb} = \frac{\lambda_a - \lambda_b}{8U_{a_m a_m} \, \mathrm{Re}\, q} . \tag{27}$$

To the same order $\rho_{a_m a_n} = 0$ for $m \neq n$. Thus

$$\rho_{a_m b} = i(\rho_{a_m a_m} - \rho_{bb}) \int_{-\infty}^{t} \exp\left\{ - \left[\Gamma + i(\omega_0 + \frac{g\Omega}{2} m) \right] (t - t') \right\} V'_{a_m b} \, dt' . \tag{28}$$

Case 1b - $g\Omega \gg \Delta\omega \qquad R \ll \Gamma$

Here

$$\rho_{a_m a_m} - \rho_{bb} = \frac{\lambda_a - \lambda_b}{\Gamma} \tag{29}$$

while, again, to the same order $\rho_{a_m a_n} = 0$ for $m \neq n$ and

$$\rho_{a_m b} = i(\rho_{a_m a_m} - \rho_{bb}) \int_{-\infty}^{t} \exp\left\{-\left[\Gamma + i(\omega_0 + \frac{g\Omega}{2} m)\right](t-t')\right\} V'_{a_m b}\, dt'. \tag{30}$$

Case 2 - $R \ll g\Omega \ll \Delta\omega$

In this case the terms $g\Omega m/2$ in the arguments of the exponentials in the integrands of equations (22) may be neglected. These equations may then be cast in the form

$$\lambda_a = 2\mathrm{Re}\left\{\frac{\Gamma}{2}\rho_{a_m a_m} + q^* \sum_n U_{a_m a_n}\rho_{a_n a_m} - q^* U_{a_m a_m}\rho_{bb}\right\} \qquad \text{(a)}$$

$$0 = \left[\Gamma + i\frac{g\Omega}{2}(m-n)\right]\rho_{a_m a_n} + q^* \sum_k U_{a_m a_k}\rho_{a_k a_n} + q \sum_k U_{a_k a_n}\rho_{a_m a_k}$$

$$- (q + q^*) U_{a_m a_n}\rho_{bb} \qquad \text{(b)} \quad (31)$$

$$3\lambda_a + \lambda_b = \Gamma(\sum_m \rho_{a_m a_m} + \rho_{bb}).$$

Here

$$U_{a_m a_n} = \langle V_{a_m b} V_{b a_n}\rangle = U^*_{a_n a_m}. \tag{32}$$

The definition of q is identical to that given by equation (25).

Equations (31) will be solved in both the limit of strong saturation $R \gg \Gamma$ and in the limit of negligible saturation $R \ll \Gamma$. In both of these limits it follows from equations (31b) that $\rho_{a_m a_n}$ for $m \neq n$ are much smaller than the population differences and may be neglected. In the case $R \gg \Gamma$ they are smaller by a factor of order $R/g\Omega$, while in the case $R \ll \Gamma$ the factor is of order the smaller of $R/g\Omega$ and R/Γ. When the $\rho_{a_m a_n}$ for $m \neq n$ are neglected, equations (31) become formally identical to equations (23). The results quoted in cases 2a and 2b then follow directly from those derived for cases 1a and 1b.

Case 2a - $R \ll g\Omega \ll \Delta\omega \qquad \Gamma \ll R$

$$\rho_{a_m a_m} - \rho_{bb} = \frac{\lambda_a - \lambda_b}{8U_{a_m a_m} \operatorname{Re} q} . \tag{33}$$

$$\rho_{a_m b} = i(\rho_{a_m a_m} - \rho_{bb}) \int_{-\infty}^{t} \exp\left\{-\left[\Gamma + i\omega_0\right](t-t')\right\} V'_{a_m b}\, dt' . \tag{34}$$

Case 2b - $R \ll g\Omega \ll \Delta\omega \qquad R \ll \Gamma$

$$\rho_{a_m a_m} - \rho_{bb} = \frac{\lambda_a - \lambda_b}{\Gamma} . \tag{35}$$

$$\rho_{a_m b} = i(\rho_{a_m a_m} - \rho_{bb}) \int_{-\infty}^{t} \exp\left\{-\left[\Gamma + i\omega_0\right](t-t')\right\} V'_{a_m b}\, dt'. \qquad (36)$$

B) Radiation Axis Quantization

Case 3 - $g\Omega \ll R \ll \Delta\omega$

The choice of the radiation direction as the axis of quantization is convenient in cases where $g\Omega \ll R \ll \Delta\omega$. In these cases the density matrix equation of motion may be cast into a form similar to that given in equation (31). One simplification which arises from quantizing along the radiation direction is that the radiation field does not produce any transitions between the $F_a = 1$ $m_a = 0$ sublevel and the ground state $F_b = 0$. However, the portion of the Hamiltonian due to the magnetic field is no longer diagonal and this fact produces some compensating difficulties.

In component form the density matrix equations now read

$$\lambda_a = 2\mathrm{Re}\left\{\frac{\Gamma}{2}\rho_{a_{\pm1}a_{\pm1}} + q^*U_{a_{\pm1}a_{\pm1}}(\rho_{a_{\pm1}a_{\pm1}} - \rho_{bb}) + q^*U_{a_{\pm1}a_{\mp1}}\rho_{a_{\mp1}a_{\pm1}} \mp \frac{g\Omega\sin\Theta}{2\sqrt{2}}\rho_{a_0 a_{\pm1}}\right\} \qquad (a)$$

$$\lambda_a = 2\mathrm{Re}\left\{\frac{\Gamma}{2}\rho_{a_0 a_0} + \frac{g\Omega\sin\Theta}{2\sqrt{2}}(\rho_{a_1 a_0} - \rho_{a_{-1}a_0})\right\} \qquad (b)$$

$$3\lambda_a + \lambda_b = \Gamma\left\{\sum_m \rho_{a_m a_m} + \rho_{bb}\right\} \qquad \text{(c)}$$

$$0 = (\Gamma + q^*U_{a_{\pm1}a_{\pm1}})\rho_{a_{\pm1}a_0} + q^*U_{a_{\pm1}a_{\mp1}}\rho_{a_{\mp1}a_0} \pm i\,\frac{g\Omega\cos\Theta}{2}\,\rho_{a_{\pm1}a_0}$$

$$\mp \frac{g\Omega\sin\Theta}{2\sqrt{2}}\left\{\rho_{a_0a_0} - \rho_{a_{\pm1}a_{\pm1}} + \rho_{a_{\pm1}a_{\mp1}}\right\} \qquad \text{(d)}$$

(37)

$$0 = (\Gamma + q^*U_{a_1a_1} + qU_{a_{-1}a_{-1}})\rho_{a_1a_{-1}} + q^*U_{a_1a_{-1}}\rho_{a_{-1}a_{-1}} + qU_{a_1a_{-1}}\rho_{a_1a_1}$$

$$-(q+q^*)U_{a_1a_{-1}}\rho_{bb} + ig\Omega\cos\Theta\rho_{a_1a_{-1}} - \frac{g\Omega\sin\Theta}{2\sqrt{2}}(\rho_{a_0a_{-1}} - \rho_{a_1a_0}),$$

(e)

where the definition of the $U_{a_ma_n}$ is again given by equation (32).

Case 3a - $g\Omega << R << \Delta\omega$ $\quad\Gamma << R$

Because the stimulated emission rate is greater than the magnetic precession rate, the mixing of the magnetic sublevels of the upper state will not occur between successive absorptions and stimulated emissions. However, a diffusive transfer of population between the $m_a = \pm 1$ and the $m_a = 0$ sublevels might be expected to occur on a time scale of order $R/(g\Omega\sin\Theta)^2$. Such a process would be of importance if $\Gamma << (g\Omega\sin\Theta)^2/R$.

The general solution of equations (37) for the elements of the density matrix is exceedingly complicated. However, an enormous simplification can be achieved with the help of a few approximations. The most drastic of these is the assumption that the radiation field is <u>not</u> circularly polarized. There is good reason to believe that circular polarization will not be produced in this case since it is proved in the next section that circular polarization does not arise even for somewhat stronger magnetic fields (i.e. for $R \ll g\Omega \ll \Delta\omega$ as shown in section V under the heading case 2a). The restriction to zero circular polarization implies that $U_{a_1a_1} = U_{a_{-1}a_{-1}}$. By symmetry considerations, it is clear that any linear polarization which is produced in this case will be aligned either parallel or perpendicular to the projection of the magnetic field on the plane orthogonal to the propagation direction. Thus $U_{a_1a_{-1}}$ is real. In order to simplify the notation, $U_{a_1a_1} = U_{a_{-1}a_{-1}}$ will be replaced by S and $U_{a_1a_{-1}}$ by T in the remainder of this section.

It is convenient to define new parameters

$$G = \frac{(g\Omega\sin\Theta)^2}{4\Gamma S}\ , \tag{38}$$

and

$$G' = \frac{G(q + q^*)}{|q|^2}\ . \tag{39}$$

In view of the discussion presented in the preceeding paragraphs it would be expected that

$$\rho_{a_{\pm 1}a_{\pm 1}} - \rho_{bb} = O\left(\frac{\lambda_a - \lambda_b}{R}\right)$$

$$\rho_{a_0 a_0} - \rho_{bb} = O\left(\frac{\lambda_a - \lambda_b}{\Gamma(1 + G')}\right) \cdot \tag{40}$$

In spite of the simplifying assumptions it still takes a somewhat lengthy calculation to prove that

$$\rho_{a_{\pm 1}a_{\pm 1}} - \rho_{bb} = \frac{(\lambda_a - \lambda_b)S}{2(q + q^*)(S^2 - T^2)} \left\{\frac{2(S - T) + G'[3S + T]}{3(S - T) + 4G'S}\right\} \tag{41}$$

and

$$\rho_{a_0 a_0} - \rho_{bb} = \frac{(\lambda_a - \lambda_b)(S - T)}{\Gamma[3(S - T) + 4G'S]} \quad . \tag{42}$$

In deriving equations (41) and (42) the following relations have been used:

$$\rho_{a_{\pm 1}a_0} = \pm \frac{g\Omega\sin\theta}{2\sqrt{2}} \frac{(\rho_{a_0a_0} - \rho_{bb})}{q^*(S - T)} \quad \text{(a)}$$

(43)

$$\rho_{a_1a_{-1}} = -\frac{T}{S}(\rho_{a_1a_1} - \rho_{bb}) - \frac{G\Gamma}{2|q|^2} \frac{(\rho_{a_0a_0} - \rho_{bb})}{(S - T)} \cdot \text{(b)}$$

The expressions for $\rho_{a_m b}$ follow from equation (21) when account is taken of the fact that $V_{a_0 b} = 0$ and the $g\Omega m/2$ terms in the arguments of the exponentials are negligible. They will not be written out explicitly here.

Case 3b- $g\Omega \ll R \ll \Delta\omega \qquad R \ll \Gamma$

The appropriate expressions for the density matrix elements in this case are identical to those given previously in Case 2b.

V. THE EQUATIONS OF RADIATIVE TRANSFER

In this section the results derived in the previous section will be used to evaluate the molecular source terms in the equations of radiative transfer. This is a two-step procedure. First, the contribution to the source terms from molecules moving at a specific velocity v is calculated. Then the resulting expressions are integrated over the Maxwellian velocity distribution of the molecules. The velocity dependence of the excitation parameters, λ_a and λ_b, is assumed to reflect the molecular velocity distribution and thus is given by

$$\lambda_{a,b} = \Lambda_{a,b} \frac{e^{-v^2/2u^2}}{\sqrt{2\pi}\, u} , \qquad (44)$$

where Λ_a and Λ_b are the total excitation rates per unit volume.

The contribution to the source terms due to the magnetoplasma is easily calculated. It is

$$2\pi\left[\mathcal{J}^{\pm}\mathcal{E}^{\pm} + \mathcal{J}^{\pm *}\mathcal{E}^{\pm}\right] = 0$$

$$(45)$$

$$2\pi\left[\mathcal{J}^{+}\mathcal{E}^{-} + \mathcal{J}^{-*}\mathcal{E}^{+}\right]\exp(-i\Delta\phi) = + i \frac{\omega_p^2 \Omega \cos\theta\, \mathcal{E}^{+}\mathcal{E}^{-}\exp(-i\Delta\phi)}{\omega_c^2} ,$$

where ω_p is the radian plasma frequency.

A) Transfer on-resonance

In this part of section V the radiation is assumed to be centered on the resonant frequencies of the active molecules. The case numbers refer to those defined previously in section IV.

Case 1a - $g\Omega >> \Delta\omega \qquad R >> \Gamma$

From equations (3), (12), (17), (19) and (24) it follows that

$$U_{a_{\pm1}a_{\pm1}} = \frac{\pi}{2c}\left(\frac{p}{\hbar}\right)^2 \left\{(1 + \cos^2\theta)I_{\pm1} \pm 2\cos\theta\, V_{\pm1} + \sin^2\theta\, Q_{\pm1}\right\} \qquad (a)$$

(46)

$$U_{a_0a_0} = \frac{\pi}{c}\left(\frac{p}{\hbar}\right)^2 \sin^2\theta \left\{I_0 - Q_0\right\} , \qquad (b)$$

where the subscripts on the Stokes parameters distinguish among the three radiation bands by indicating the magnetic sublevel of the upper state to which each couples. Again, using these same equations and, in addition, equations (7), (8), (14), (27), (28), (44) and (45) one obtains

$$\frac{DI_{\pm1}}{Dz} = \frac{1}{4}\,\hbar\omega(\Lambda_a - \Lambda_b)$$

$$\frac{DV_{\pm 1}}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\left\{\frac{(1+\cos^2\Theta)V_{\pm 1} \pm 2\cos\Theta\, I_{\pm 1}}{(1+\cos^2\Theta)I_{\pm 1} \pm 2\cos\Theta\, V_{\pm 1} + \sin^2\Theta\, Q_{\pm 1}}\right\}$$

$$\frac{DQ_{\pm 1}}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\left\{\frac{\sin^2\Theta\, I_{\pm 1} + (1+\cos^2\Theta)\, Q_{\pm 1}}{(1+\cos^2\Theta)\, I_{\pm 1} \pm 2\cos\Theta\, V_{\pm 1} + \sin^2\Theta\, Q_{\pm 1}}\right\}$$

$$- \frac{\omega_p^{\,2}}{\omega^2}\,\frac{\Omega\cos\Theta\, U_{\pm 1}}{c}$$

$$\frac{DU_{\pm 1}}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\left\{\frac{(1+\cos^2\Theta)\, U_{\pm 1}}{(1+\cos^2\Theta)I_{\pm 1} \pm 2\cos\Theta\, V_{\pm 1} + \sin^2\Theta\, Q_{\pm 1}}\right\}$$

$$+ \frac{\omega_p^{\,2}}{\omega^2}\,\frac{\Omega\cos\Theta\, Q_{\pm}}{c} \tag{47}$$

$$\frac{DI_0}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)$$

$$\frac{DV_0}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\,\frac{V_0}{I_0 - Q_0}$$

$$\frac{DQ_0}{Dz} = -\frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b) - \frac{\omega_p^2}{\omega^2}\frac{\Omega\cos\theta\, U_0}{c}$$

$$\frac{DU_0}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\frac{U_0}{I_0 - Q_0} + \frac{\omega_p^2}{\omega^2}\frac{\Omega\cos\theta\, Q_0}{c} .$$

Equations (47) are the equations of radiative transfer for maser radiation expressed in terms of the Stokes parameters. In calculations of the limiting polarizations reached in the saturated regime, it is convenient to use the variables

$$X = V/I, \qquad Y = Q/I, \qquad Z = U/I.$$

Then equations (47) transform into

$$\frac{DX_{\pm1}}{D\ln I_{\pm1}} = -X_{\pm1} + \frac{(1+\cos^2\theta)X_{\pm1} \pm 2\cos\theta}{(1+\cos^2\theta) \pm 2\cos\theta\, X_{\pm1} + \sin^2\theta\, Y_{\pm1}}$$

$$\frac{DY_{\pm1}}{D\ln I_{\pm1}} = -Y_{\pm1} + \frac{\sin^2\theta + (1+\cos^2\theta)Y_{\pm1}}{(1+\cos^2\theta) \pm 2\cos\theta\, X_{\pm1} + \sin^2\theta\, Y_{\pm1}} - \beta I_{\pm1}\cos\theta Z_{\pm1}$$

(48)

$$\frac{DZ_{\pm1}}{D\ln I_{\pm1}} = -Z_{\pm1} + \frac{(1+\cos^2\theta)Z_{\pm1}}{(1+\cos^2\theta) \pm 2\cos\theta\, X_{\pm1} + \sin^2\theta\, Y_{\pm1}} + \beta I_{\pm1}\cos\theta Y_{\pm1}$$

$$\frac{DX_0}{D\ln I_0} = - X_0 + \frac{X_0}{1 - Y_0}$$

$$\frac{DY_0}{D\ln I_0} = - (1 + Y_0) - \beta I_0 \cos\theta \, Z_0$$

$$\frac{DZ_0}{D\ln I_0} = - Z_0 + \frac{Z_0}{1 - Y_0} + \beta I_0 \cos\theta \, Y_0 \quad .$$

Here

$$\beta = \frac{4\omega_p^2 \Omega}{\hbar\omega^3 c(\Lambda_a - \Lambda_b)} \quad . \tag{49}$$

The limiting polarizations in the absence of significant Faraday rotation ($\beta I << 1$) follow from equations (48) with X, Y, Z set equal to constants. They are:

$$X_{\pm 1} = \pm \frac{2\cos\theta}{1 + \cos^2\theta} , \qquad Y_{\pm 1} = \frac{\sin^2\theta}{1 + \cos^2\theta} , \qquad Z_{\pm 1} = 0$$

$$X_0 = 0 , \qquad Y_0 = - 1 , \qquad Z_0 = 0 \quad . \tag{50}$$

Since $X_{\pm 1}^2 + Y_{\pm 1}^2 = 1$, these lines are 100% elliptically polarized.

For large Faraday rotation

$$X_{\pm 1} = \pm 1 , \qquad Y_{\pm 1} = 0 , \qquad Z_{\pm 1} = 0$$

$$\frac{DX_0}{Dz} = 0 , \qquad Y_0 = 0 , \qquad Z_0 = 0 . \tag{51}$$

Thus for large Faraday rotation the σ components are completely circularly polarized.

Case 1b - $g\Omega >> \Delta\omega$ $R << \Gamma$

The only difference between this case and the previous one is the replacement of $8U_{a_m a_m}\mathrm{Re}\,q$ by Γ in the denominators of the expressions for the population differences (cf. eqs. [27] and [29]). The equations of radiative transfer now read

$$\frac{DI_{\pm 1}}{Dz} = \alpha\left\{(1 + \cos^2\theta)I_{\pm 1} \pm 2\cos\theta\, V_{\pm 1} + \sin^2\theta\, Q_{\pm 1}\right\}$$

$$\frac{DV_{\pm 1}}{Dz} = \alpha\left\{(1 + \cos^2\theta)V_{\pm 1} \pm 2\cos\theta\, I_{\pm 1}\right\}$$

$$\frac{DQ_{\pm 1}}{Dz} = \alpha \left\{ \sin^2\theta \, I_{\pm 1} + (1 + \cos^2\theta) Q_{\pm 1} \right\} - \frac{\omega_p^2}{\omega^2} \frac{\Omega \cos\theta \, U_{\pm 1}}{c}$$

$$\frac{DU_{\pm 1}}{Dz} = \alpha \left\{ (1 + \cos^2\theta) U_{\pm 1} \right\} + \frac{\omega_p^2}{\omega^2} \frac{\Omega \cos\theta \, Q_{\pm 1}}{c} \qquad (52)$$

$$\frac{DI_0}{Dz} = 2\alpha \sin^2\theta \left\{ I_0 - Q_0 \right\}$$

$$\frac{DV_0}{Dz} = 2\alpha \sin^2\theta \, V_0$$

$$\frac{DQ_0}{Dz} = 2\alpha \sin^2\theta \left\{ Q_0 - I_0 \right\} - \frac{\omega_p^2}{\omega^2} \frac{\Omega \cos\theta \, U_0}{c}$$

$$\frac{DU_0}{Dz} = 2\alpha \sin^2\theta \, U_0 + \frac{\omega_p^2}{\omega^2} \frac{\Omega \cos\theta \, Q_0}{c}$$

where

$$\alpha = \frac{(2\pi)^{3/2}}{4} \frac{\hbar\omega}{c} \frac{(\Lambda_a - \Lambda_b)}{\Gamma} \left(\frac{p}{\hbar}\right)^2 \frac{1}{[\Delta\omega^2 + (\omega u/c)^2]^{1/2}} \; . \qquad (53)$$

The limiting polarizations are easily derived in this case because the transfer equations are linear in the Stokes parameters. If the Stokes parameters for each line are set equal to constants multiplied by $e^{\lambda z}$ a linear eigenvalue problem is obtained. The limiting polarization is given by the eigenvector which corresponds to the eigenvalue having the largest positive real part.

In the absence of Faraday rotation this procedure yields

$$\lambda_{\pm 1}^{max} = 2\alpha(1 + \cos^2\theta)$$
$$X_{\pm 1} = \frac{\pm 2\cos\theta}{1 + \cos^2\theta}, \qquad Y_{\pm 1} = \frac{\sin^2\theta}{1 + \cos^2\theta}, \qquad Z_{\pm 1} = 0 \tag{54}$$

$$\lambda_{0}^{max} = 4\alpha \sin^2\theta$$
$$X_0 = 0, \qquad Y_0 = -1, \qquad Z_0 = 0. \tag{55}$$

In the limit of large Faraday rotation

$$\lambda_{\pm 1}^{max} = \alpha(1 + |\cos\theta|)^2$$
$$X_{\pm 1} = \pm 1, \qquad Y_{\pm 1} = 0, \qquad Z_{\pm 1} = 0 \tag{56}$$

$$\mathrm{Re}\lambda_0^{max} = 2\alpha \sin^2\theta$$

$$X_0 = f , \qquad Y_0 = 0 , \qquad Z_0 = 0 , \tag{57}$$

where f is a constant with magnitude less than unity but is otherwise indeterminate.

Case 2a - $R \ll g\Omega \ll \Delta\omega \quad R \gg \Gamma$

In this case the radiation field consists of a single line centered at the resonant frequency ω_0. The slight separation in frequency of the different Zeeman components may be ignored. The equations of transfer are derived in a manner similar to that outlined in case 1a. The expressions for $U_{a_m a_m}$ are identical to those given by equations (46) for case 1a except that now there are no subscripts on the Stokes parameters. The transfer equations then read

$$\frac{DI}{Dz} = \frac{3}{4}\hbar\omega(\Lambda_a - \Lambda_b)$$

$$\frac{DV}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\left\{ \frac{(1+\cos^2\theta)V + 2\cos\theta\, I}{(1+\cos^2\theta)I + 2\cos\theta V + \sin^2\theta Q} + \frac{V}{I - Q} + \frac{(1+\cos^2\theta)V - 2\cos\theta\, I}{(1+\cos^2\theta)I - 2\cos\theta V + \sin^2\theta\, Q} \right\}$$

$$\frac{DQ}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\left\{\frac{\sin^2\theta I + (1+\cos^2\theta)Q}{(1+\cos^2\theta)I + 2\cos\theta V + \sin^2\theta Q} - 1 + \frac{\sin^2\theta I + (1+\cos^2\theta)Q}{(1+\cos^2\theta)I - 2\cos\theta V + \sin^2\theta Q}\right\}$$

$$- \frac{\omega_p^2}{\omega^2}\,\frac{\Omega\cos\theta U}{c} \tag{58}$$

$$\frac{DU}{Dz} = \frac{1}{4}\hbar\omega(\Lambda_a - \Lambda_b)\left\{\frac{(1+\cos^2\theta)U}{(1+\cos^2\theta)I + 2\cos\theta V + \sin^2\theta Q} + \frac{U}{I - Q} + \frac{(1+\cos^2\theta)U}{(1+\cos^2\theta)I - 2\cos\theta V + \sin^2\theta Q}\right\}$$

$$+ \frac{\omega_p^2}{\omega^2}\,\frac{\Omega\cos\theta Q}{c} \quad .$$

As in case 1a it proves convenient here to reexpress equations (58) in terms of the variables X, Y, and Z. This yields

$$\frac{3DX}{D\ln I} = \left\{\frac{3Y - 2}{1 - Y} + \frac{2\sin^2\theta\left[(1 + \cos^2\theta)Y + \sin^2\theta\right]}{\left[(1 + \cos^2\theta) + \sin^2\theta Y\right]^2 - 4\cos^2\theta X^2}\right\}X$$

$$\frac{3DY}{D\ln I} = -(3Y + 1) + \frac{2\left[(1 + \cos^2\theta) + \sin^2\theta Y\right]\left[(1 + \cos^2\theta)Y + \sin^2\theta\right]}{\left[(1 + \cos^2\theta) + \sin^2\theta Y\right]^2 - 4\cos^2\theta X^2}$$

$$- \beta I\cos\theta Z \qquad (59)$$

$$\frac{3DZ}{D\ln I} = \left\{\frac{3Y - 2}{1 - Y} + \frac{2(1 + \cos^2\theta)\left[(1 + \cos^2\theta) + \sin^2\theta Y\right]}{\left[(1 + \cos^2\theta) + \sin^2\theta Y\right]^2 - 4\cos^2\theta X^2}\right\}Z$$

$$+ \beta I\cos\theta Y.$$

In circumstances where Faraday rotation is unimportant (i.e., $\beta I \ll 1$) the limiting polarizations may be found by setting the left hand sides of equations (59) equal to zero. Some straightforward algebra shows that there are two independent solutions for the limiting polarization. One solution is

$$X = 0\,, \qquad Y = -1\,, \qquad Z = 0, \quad \text{for } \sin^2\theta \le \frac{1}{3}\,,$$

(60)

$$X = 0\,, \qquad Y = \frac{3\sin^2\theta - 2}{3\sin^2\theta}\,, \qquad Z = K, \quad \text{for } \sin^2\theta \ge \frac{1}{3}\,,$$

where K is a constant restricted to the range $K^2 \le 1 - Y^2$. The other solution reads

$$X = \pm\frac{2}{3}\,\frac{\left[2 + 3\cos^2\theta\right]^{1/2}}{\left[1 + \cos^2\theta\right]}\,, \qquad Y = \frac{1 + 3\cos^2\theta}{3(1 + \cos^2\theta)}\,, \qquad Z = 0.$$

(61)

The stability of these two solutions may be tested by substituting $X = \overline{X} + \epsilon$, $Y = \overline{Y} + \delta$ and $Z = \overline{Z} + \gamma$ back into equations (59) and collecting the terms which are linear in the small perturbations ϵ, δ, and γ. Here $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ refer to either of the equilibrium solutions. The substitutions $\epsilon = \hat{\epsilon} I^s$, $\delta = \hat{\delta} I^s$ and $\gamma = \hat{\gamma} I^s$, where $\hat{\epsilon}$, $\hat{\delta}$ and $\hat{\gamma}$ are constants, yield a linear eigenvalue problem for s. The resulting eigenvalues are:

$$s_1 = -\frac{(2+3\cos^2\theta)}{6\cos^2\theta}, \quad s_2 = -\frac{(1-3\sin^2\theta)}{3\cos^2\theta}, \quad s_3 = -\frac{(1-3\sin^2\theta)}{6\cos^2\theta},$$

$$\text{if } \sin^2\theta \le \frac{1}{3}\,.$$

(62)

$$s_1 = -\frac{3\cos^2\theta}{2}, \quad s_2 = -\frac{(3\sin^2\theta - 1)}{2}, \quad s_3 = 0, \quad \text{if } \sin^2\theta \geq \frac{1}{3}$$

for the equilibrium solution given by equation (60) and

$$s_1 = 3\cos^2\theta, \quad s_2 = -1 - \frac{3\cos^2\theta}{2}, \quad s_3 = \frac{3\cos^2\theta}{2}, \tag{63}$$

for the equilibrium solution given by equation (61). Thus the first of the two equilibrium solutions is stable for all θ and the second solution is unstable for all θ.

Under conditions for which the Faraday rotation across the saturated portion of the source is large, it is obvious from equations (59) that Y and Z approach zero. The equation for X then admits the equilibrium values $X = 0$ and $X^2=1$. It is easily shown that the former value is stable but the latter is not.

The general nature of the limiting polarization given by equation (60) may be elucidated with the help of a simple physical argument. Consider first the case of propagation at right angles to the magnetic field. A molecule which is in either the $m_a = 1$ or the $m_a = -1$ sublevel may be stimulated to emit by photons polarized perpendicular to the magnetic field but not by those polarized parallel to the field. On the other hand, a molecule which is in the $m_a = 0$ sublevel may be stimulated to emit by photons polarized parallel to the field but not by those polarized perpendicular to the field. For isotropic pumping, the rate of excitation of all three magnetic sublevels is the same. For saturated amplification, it then follows that for each photon which is emitted polarized parallel to the field, two photons are produced polarized perpendicular to the field. Thus the reason why the fractional linear polarization $Y = 1/3$ at $\theta = \pi/2$ is clear.

As θ decreases from $\pi/2$ to 0, Y decreases monotonically crossing 0 at $\sin^2\theta = 2/3$ and reaching -1 at $\sin^2\theta = 1/3$. A qualitative understanding of the variation of Y with θ is easy to achieve. A photon propagating at an angle θ to $\underset{\sim}{B}_0$ and polarized along the x axis ($\perp$ to $\underset{\sim}{B}_0$) can stimulate emission from molecules which are in the $m_a = \pm 1$ substates but not from those which are in the $m_a = 0$ substate. On the other hand, a photon propagating in the same direction but polarized along the y axis (along $\hat{\underset{\sim}{j}}$) can stimulate emission from molecules in any of the magnetic sublevels $m_a = \pm 1$ or $m_a = 0$. The ratio of the transition probabilities for stimulated emission from the $m_a = \pm 1$ levels by photons polarized along y to those polarized along x is $\cos^2\theta$. Thus, as θ goes from $\pi/2$ towards 0(or π) the relative amplification of photons polarized along y to those polarized along x increases. Since only photons polarized along y can stimulate molecules which are in the $m_a = 0$ substate, the fractional linear polarization is along the y axis for small $\sin^2\theta$.

Case 2b - $R \ll g\Omega \ll \Delta\omega \qquad R \ll \Gamma$

In this case the equations for the Stokes parameters are especially simple. In fact

$$\frac{DS}{Dz} = 4\alpha S \tag{64}$$

where S stands for any of the Stokes parameters and α is again given by equation (53). The terms due to Faraday rotation could easily be included if desired, but then, the transfer equations would have to be explicitly written out for each Stokes parameter.

In this case the presence of the magnetic field has no observable consequences.

Case 3a - $g\Omega \ll R \ll \Delta\omega \qquad R \gg \Gamma$

The derivation of the equations of radiative transfer from equations (40)-(43) is straightforward and yields

$$\frac{DI}{Dz} = (\Lambda_a - \Lambda_b)\hbar\omega \frac{1}{\sqrt{2\pi}\,u} \int_{-\infty}^{\infty} dv \left\{ \frac{2(I + Q) + 3G'I}{3(I + Q) + 4G'I} \right\} e^{-v^2/2u^2} \tag{a}$$

$$\frac{DQ}{Dz} = (\Lambda_a - \Lambda_b)\hbar\omega \frac{1}{\sqrt{2\pi}\, u} \int_{-\infty}^{\infty} dv \left\{ \frac{G'I}{3(I + Q) + 4G'I} \right\} e^{-v^2/2u^2} . \qquad \text{(b) (65)}$$

The integral over v cannot be explicitly performed because of the complicated v dependence of G'. The dependence of G' on v reflects the fact that the rate of transfer of population between the $m_a = 0$ and the $m_a = \pm 1$ sublevels is a function of the molecular velocity. This result is not surprising since the diffusion of population depends upon the degree of saturation which in turn is a function of velocity. In fact, a complete treatment of the equations of radiative transfer in this case would reveal that the linear polarization varies with the frequency difference from the line center. Fortunately, the transfer equations possess simple forms in the two limits $G' \rightarrow 0$ and $G' \rightarrow \infty$. They are:

$$\frac{DI}{Dz} = \frac{2(\Lambda_a - \Lambda_b)\hbar\omega}{3} \qquad \text{(a)}$$

$$\text{(66)}$$

$$\frac{DQ}{Dz} = 0 \qquad \text{(b)}$$

for $G' \ll 1$ and

$$\frac{DI}{Dz} = \frac{3(\Lambda_a - \Lambda_b)\hbar\omega}{4} \qquad \text{(a)}$$

$$\text{(67)}$$

$$\frac{DQ}{Dz} = \frac{(\Lambda_a - \Lambda_b)\hbar\omega}{4} \qquad \text{(b)}$$

for $G' \gg 1$. In the first limit ($G' \ll 1$) the magnetic field is too weak to produce any discernible effect on the polarization. In the second limit ($G' \gg 1$) the presence of the field produces a net linear polarization of $Y = Q/I = 1/3$. Furthermore, the intensity grows at 9/8 the previous rate since now the $m_a = 0$ sublevel is contributing to the emitted power.

The results derived in this case depend crucially upon the assumption that the radiation field is unidirectional. If the radiation is beamed into a cone of opening angle γ, the rate of stimulated emission from the $m_a = 0$ sublevel to the ground state is approximately $R\gamma^2$. Clearly, the results obtained here must be modified if $R\gamma^2 > (g\Omega\sin\Theta)^2/4R$ or $2R\gamma > g\Omega\sin\Theta$.

Case 3b - $\quad g\Omega \ll R \ll \Delta\omega \qquad R \ll \Gamma$

The equations of radiative transfer in this case reduce to those given in case 2b.

B) Transfer off-resonance

The velocity gradients inferred to exist in cosmic masers imply that the emitted radiation must frequently traverse regions in which it is appreciably off-resonance. A particularly simple example, which illustrates some of the features that arise in the off-resonance transfer of polarized radiation, is worked out briefly. More complicated cases are left to the reader.

The example treated here is the transfer of monochromatic radiation at a frequency ω such that $|\omega - \omega_0| >> g\Omega$ and $u\omega_0/c$, where ω_0 is the resonance frequency of the π component of the Zeeman multiplet of a molecule at rest. In addition, it is assumed that the population differences $\rho_{a_m a_m} - \rho_{bb} = (\lambda_a - \lambda_b)/\Gamma$ for $m = 0, \pm 1$. This assumption would be violated if there were also directional radiation at the resonance frequency of sufficient intensity to saturate the molecules. Under the conditions stated above, the principal effects that the molecules have on the propagation of the radiation arise from the contribution they make to the index of refraction. The terms in the equations of transfer which give rise to the absorption and emission of radiation are smaller than the refractive terms by a factor of order $\Gamma/(\omega - \omega_0) << 1$ and may be neglected.

Because the off-resonance radiation does not contribute. to the saturation, the transfer problem is linear. For this reason, and also, because the index of refraction is a function of $\omega - \omega_0$, it is most convenient to work out the transfer equations for a monochromatic wave. Since the transition is unsaturated, equations (22) imply that $\rho_{a_m a_n} << (\lambda_a - \lambda_b)/\Gamma$ for $m \neq n$. Thus equations (21) yield

$$\rho_{a_m b} = + i \frac{(\lambda_a - \lambda_b)}{\Gamma} \frac{V_{a_m b}}{\Gamma + i\left[\omega_0 + \frac{g\Omega}{2} m - \omega(1 - \frac{v}{c})\right]} . \qquad (68)$$

The derivation of the equations of transfer is similar to that outlined in previous cases and is not repeated here. The final results are

$$\frac{DI}{Dz} = 0$$

$$\frac{DV}{Dz} = \frac{\pi}{2} \frac{\hbar\omega}{c} \frac{(\Lambda_a - \Lambda_b)}{\Gamma} \left(\frac{p}{\hbar}\right)^2 \frac{(g\Omega \sin\Theta)^2}{(\omega - \omega_0)^3} U$$

$$\frac{DQ}{Dz} = 2\pi \frac{\hbar\omega}{c} \frac{(\Lambda_a - \Lambda_b)}{\Gamma} \left(\frac{p}{\hbar}\right)^2 \frac{g\Omega \cos\Theta}{(\omega - \omega_0)^2} U \qquad (69)$$

$$\frac{DU}{Dz} = -2\pi \frac{\hbar\omega}{c} \frac{(\Lambda_a - \Lambda_b)}{\Gamma} \left(\frac{p}{\hbar}\right)^2 \frac{g\Omega \cos\Theta}{(\omega - \omega_0)^2} Q$$

$$- \frac{\pi}{2} \frac{\hbar\omega}{c} \frac{(\Lambda_a - \Lambda_b)}{\Gamma} \left(\frac{p}{\hbar}\right)^2 \frac{(g\Omega \sin\Theta)^2}{(\omega - \omega_0)^3} V .$$

In writing equations (69), only the lowest order terms in $g\Omega/(\omega - \omega_0)$ and $u\omega/(\omega - \omega_0)c$ were retained. The first order terms in $g\Omega/(\omega - \omega_0)$ describe the magnetorotation of the linear polarization due to the off-resonance index of refraction. Note the dependence of the rate of rotation on $(\omega - \omega_0)$. A glance at equations (52), (53) and (64) shows that the magnetorotation per on-resonance optical depth (or gain length if $\Lambda_a - \Lambda_b > 0$) is approximately

$$\Delta\Phi = \left(\frac{u\omega}{c}\right) \frac{g\Omega\cos\theta}{(\omega - \omega_0)^2} \ . \tag{70}$$

Magnetorotation is a well known phenomenon (Mitchell and Zemansky 1934) and has been measured in the laboratory. It seems likely that it has also been observed in OH maser sources (cf. section VI).

VI. COMPARISON OF THEORY AND OBSERVATION

The most important of the theoretical results obtained in the previous section are collected here. This summary is intended to aid later comparisons between theory and observation. It is well to bear in mind that the theoretical results are strictly applicable to the amplification and propagation of plane wave maser radiation in homogeneous media. Thus any effects due to velocity and magnetic field gradients in the maser clouds must be taken into account separately.

For unsaturated amplification, the ambient magnetic field is important only if $g\Omega \gtrsim \Delta\omega$. If $g\Omega \gtrsim \Delta\omega$, and if the Faraday rotation per gain length is small, the maser will amplify the Zeeman pattern. If $g\Omega \gtrsim \Delta\omega$, but the Faraday rotation per gain length is large, the σ components are circularly polarized and the π component is unpolarized. In both cases the relative amplification of the σ and π components depends upon the angle between the propagation direction and the magnetic field.

For saturated amplification, the magnetic field affects the polarization if $(g\Omega\sin\theta)^2 > R\Gamma$. Faraday rotation is important if it amounts to a radian or more across the region of saturated amplification. In the absence of Faraday rotation, the fractional linear polarization ranges from 0 to 1/3 as $(g\Omega\sin\Theta)^2/R\Gamma$ varies from $0 \rightarrow \infty$ (note $g\Omega < R$). For $R < g\Omega < \Delta\omega$ partial linear polarization is also produced. The fractional linear polarization in this case is given by $Q/I = -1$ for $\sin^2\Theta < 1/3$ and $Q/I = (3\sin^2\Theta - 2)/3\sin^2\Theta$ for $\sin^2\Theta > 1/3$. For $g\Omega > \Delta\omega$, the Zeeman pattern is amplified. The σ and π components have similar intensities for all Θ, unlike the corresponding case of unsaturated amplification. If Faraday rotation is important the linear polarization is destroyed. For $g\Omega > \Delta\omega$, the σ components are then circularly polarized and the π component is unpolarized.

For off-resonance propagation there is a rotation of the plane of linear polarization which depends on the frequency offset from resonance. This effect is most important where the molecular transition is unsaturated.

The g values of the upper and lower states of the microwave water transition are about 8×10^{-4}. Thus, unless the magnetic field exceeds 40G, the Zeeman splitting is smaller than the bandwidth of the maser line which is typically of order 10^5Hz. For this reason, the fact that circular polarization has never been detected in water masers is easy to understand. The high degree of linear polarization that has been observed in some water masers, such as those in Orion A, suggests both that these masers are saturated and that they possess sizeable magnetic fields . Since a reasonable choice for R is of order 50 sec^{-1} (Goldreich and Keeley 1972) the magnetic fields must be at least of order 10^{-2}G.

The g values of the levels involved in those hydroxyl transitions which have been observed as masers are all of order unity except for the levels involved in the $\Pi_{1/2}$, J = 1/2, F = 1 → 0 transition for which the g values are very much smaller. The observations of circular polarization in the ground state maser lines imply the presence of magnetic fields of order 10^{-3}G or larger in the sources (typical line-widths are of order 3 x 10^3Hz). Somewhat larger fields are suggested by the circular polarization observed in the higher frequency lines associated with the excited states. The interpretation of the circular polarization in terms of Zeeman splitting is consistent with the fact that no circular polarization has been observed in the $\Pi_{1/2}$, J = 1/2, F = 1→0 line. In general, it is difficult to group the maser lines into Zeeman patterns but some plausible candidates have been put forth (Zuckerman, Yen, Gottlieb and Palmer 1972).

In many sources which show large amounts of circular polarization the linear polarization is very low. Some lines in these sources are very nearly 100% circularly polarized. Both the absence of linear polarization and the presence of 100% circular polarization suggest that Faraday rotation is important in these sources. For unsaturated amplification this would require a minimum of several radians of rotation per gain length. However, in saturated sources the requirement is weaker and amounts to several radians of rotation across the saturated region. Since the high brightness temperature OH masers are probably at least partially saturated (Goldreich and Keeley 1972) the weaker condition is likely to be the relevant one. Taking 10^2AU as the length of the region of saturated amplification and $B = 10^{-2}$G, an electron density $N_e = 3\text{cm}^{-3}$ is required to give a radian of rotation across the saturated region (of a ground state OH maser). For sources which show appreciable linear as well as circular polarization, N_e is presumably somewhat smaller.

An interesting feature of the linear polarization in OH maser sources is that its position angle often varies rapidly across even narrow lines. As one conclusion of their survey of OH sources, Manchester, Robinson and Goss (1970) comment that "the profiles for the linear Stokes parameters Q and U often have very narrow features which are unresolved by the 1 kHz filters, although the circular polarization profiles are adequately resolved." They cite G 305.4 + 0.2, NGC6334A, NGC6334B and W33A as examples of this behavior. Magnetorotation, which occurs during off-resonance propagation, provides an explanation of these observations. The following numerial example illustrates the sort of parameters that are required. For $g = 1$, $\cos\Theta = 1$, $B = 10^{-2}G$, $u/c = 2 \times 10^{-6}$, $f_0 = 1.7 \times 10^9$ Hz, $f - f_0 = 2 \times 10^4$ Hz (corresponding to a shift of 4 km/s from resonance) equation (70) yields

$$|\Delta\Phi| = 2.5 \times 10^{-1} \frac{\text{radians}}{\text{optical depth}}$$

$$\left|\frac{d\Delta\Phi}{df}\right| = 2.5 \times 10^{-2} \frac{\text{radians}}{\text{optical depth - kHz}}$$

Thus a differential rotation of one radian per kHz is produced over 4.0×10^1 optical depths.

Up to this point, the discussion has been concentrated on those features of the observations which are easy to rationalize in terms of the theory. Unfortunately, there are some observational facts which do not find ready explanations in the theory. The first of these is the absence of obvious Zeeman patterns in those sources which show appreciable circular polarization. There is no explanation for this fact in the theory. A plausible explanation, which relies on magnetic field and velocity gradients in the sources, has been proposed by Cook (1966). This paper offers nothing better. The second fact which is hard to explain is the absence of OH maser sources which are predominately linearly polarized. The existence of such sources might be expected since the theory predicts linear polarization when $\Gamma << R << g\Omega << \Delta\omega$ (Case 2a). If the OH masers are unsaturated in cases for which $R << g\Omega << \Delta\omega$ the problem would be solved. However, this is not a very satisfactory solution. There is one source, W42, in which linear polarization dominates (Robinson, Goss and Manchester 1970) and it may be an example of Case 2a but one source of this kind is hardly enough. One interesting possibility for future observations would be to check if there is a systematic variation in brightness temperature between those OH maser sources which are circularly polarized and those which are not. If the unpolarized sources are unsaturated, they should have lower brightness temperatures.

The absence of obvious Zeeman patterns together with the scarcity of linearly polarized lines in OH maser sources may be an indication of the failure of the theoretical derivation of the equations of radiative transfer. However, a substantial resolution of these difficulties is achieved when the effects due to trapped infrared line radiation are included in the transfer equations (cf. part III). For example, the relaxation of population differences among the magnetic sublevels of the $F_a = 1$ state produced by the trapped radiation reduces the rate of growth of linear polarization described in cases 2a and 3a.

Previous theoretical work on the polarization of cosmic maser radiation has largely been discredited, a fate we hope this paper will avoid. Heer (1966) suggested that the circular polarization observed in OH masers was due to the non-linear competition between oppositely circularly polarized modes which has been observed in laboratory lasers. Studies of the growth of circular polarization in lasers do show that saturation effects can lead to the spontaneous growth of circular polarization, at least for $\Delta F = 0$ transitions. However, the application of these results to astrophysical masers by Heer (1966) and by Heer and Settles (1967) has been criticized by Bender (1967) and by Litvak (1970b). Their criticism is centered on the fact that the results obtained by Heer (1966) and Heer and Settles (1966) are valid only for a monochromatic signal. Bender (1967) showed that where perturbation theory is valid ($R < \Gamma$) and the signal is broadband ($\Delta\omega > \Gamma$) circular polarization is suppressed (cf. also pg. 2112 of Litvak 1970 which corrects a technical error in Bender 1967). The authors' investigation of this process, which is not limited to perturbation theory, confirms Bender's conclusions. However, it also indicates that results similar to those obtained for monochromatic signals would be valid for broadband signals if $R > \Delta\omega$. This is an interesting conclusion but it is not applicable to astrophysical masers since they all have $R \ll \Delta\omega$.

Litvak (1970a) suggested that parametric down-conversion was responsible for the preference of one circular polarization over the other which is observed in some OH sources. This process involves the coupling of two microwaves with an electron cyclotron wave through the non-linear polarization they induce in the magnetoplasma. The higher-frequency microwave is down-converted into the lower-frequency microwave and the electron cyclotron wave. Although this mechanism seemed attractive at first, it is now known that it is much too weak to be of importance in astrophysical masers (Goldreich and Kwan 1972).

APPENDIX A

The electric field at any point in the amplifying medium is a superposition of waves travelling in different directions. The phase velocity of one Fourier component of one polarization mode travelling in a particular direction depends on the polarization induced in the medium by the total electric field at the point considered. In order for the field at one point to have non-stationary statistical properties, definite phase relations must be maintained among the Fourier components of the field.

For radiation travelling all in one direction, a pulse may propagate between two points separated by many wavelengths only if the phase velocities at different frequencies are the same. This condition is encountered in laboratory lasers. If however there are many plane waves travelling in different directions, then even at a given frequency there will be relative phase shifts of order

$$\delta = \frac{2\pi d\theta^2}{\lambda} \tag{A1}$$

when the propagation distance is d and $\theta << 1$ is the angle between the plane waves. In equation (A1) it is assumed that the phase velocity is isotropic. Even if it is not, it is impossible that a physical situation can exist in which the anisotropy can exactly compensate for the directional effect. From the dispersion relations between the real and imaginary parts of the refractive index of the amplifying medium, it may be proved that the maximum phase shift which can be produced over a distance of the order of the gain length is about one radian. It is shown below that the phase shifts to be expected because of the spread in propagation directions are orders of magnitude larger; hence the formation and propagation of pulses seem impossible.

It is expected that even if the observed sources are amplified background "point" sources (which may therefore have very narrow radiation beams in the maser cloud) rather than amplified spontaneous emission from the cloud itself, the dominant contribution to the electric field at any point in the cloud will be from the amplified spontaneous emission. In this case it seems very improbable that pulses can develop in the beam of the background source unless they can develop in the noise radiation field also. For the OH and H_2O masers the length d is probably at least 10^{11} cm, while $\lambda \approx 1 - 20$ cm. The angular spread of the noise radiation is expected to be much larger than $\Theta \approx 10^{-5}$ which is required to keep δ as small as π (Goldreich and Keeley 1972). Thus it seems highly unlikely that pulsing can occur.

APPENDIX B

If two atoms are excited to a pure quantum state at time t_0 and $t_0 + \tau$, and thereafter interact with a wave-train having bandwidth $\Delta\omega >> R^{-1}$, it is not true that the states of the two atoms at time $t > t_0 + \tau$ are uncorrelated if $\Delta\omega\tau > 1$. This fact may be demonstrated readily for a two-level atom.

The density matrix equations for an ensemble of two level atoms excited to states a or b at any time in the past are

$$\dot{\Delta} = \lambda_a - \lambda_b - \Gamma\Delta + 2\mathrm{Re}(iV_{ba}\rho_{ab})$$
$$\dot{\rho}_{ab} = -(i\omega_0 + \Gamma)\rho_{ab} + i\,\Delta\,V_{ab} \quad , \tag{B1}$$

where $\Delta \equiv \rho_{aa} - \rho_{bb}$, ω_0 is the resonant frequency of the atoms, and λ_a and λ_b are the usual excitation rates. The expectation value $\langle\Delta(t)\Delta(t + \tau)\rangle$ might be expected to show a decrease for $\tau\Delta\omega \gtrsim 1$, but in fact the relevant time-scale is shown below to be Γ^{-1}.

Assume

$$V_{ab} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\omega)\exp(-i\omega t)d\omega \quad , \tag{B2}$$

where $U(\omega) \equiv U^*(-\omega)$. It is assumed that $U(\omega)$ has a bandwidth $\Delta\omega$ about the resonant frequency ω_0. Then the second-order perturbation theory solution for Δ is

$$\Delta_2(t) = \Delta_0\left\{1 - \frac{1}{\pi}\,\mathrm{Re}\int_{-\infty}^{\infty} d\omega' d\omega'' \left[\frac{U(\omega')U^*(\omega'')}{\Gamma + i(\omega_0 - \omega')}\right.\right.$$

$$\left.\left.\times\ \frac{\exp\left[i(\omega'' - \omega')t\right]}{\Gamma + i(\omega'' - \omega')}\right]\right\}, \qquad (B3)$$

where $\Delta_0 \equiv (\lambda_a - \lambda_b)/\Gamma$ is the zeroth-order solution. It is convenient to introduce the Fourier transform

$$D_2(\nu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Delta_2(t)\exp(i\nu t)dt, \qquad (B4)$$

which is found to be

$$D_2(\nu) = \Delta_0\left\{\sqrt{2\pi}\ \delta(\nu) - \frac{1}{\sqrt{2\pi}}\,\frac{1}{\Gamma - i\nu}\int_{-\infty}^{\infty} U(\omega')U^*(\omega' - \nu)d\omega'\right.$$

$$\left.\left[\frac{1}{\Gamma + i(\omega_0 - \omega')} + \frac{1}{\Gamma - i(\omega_0 + \omega')}\right]\right\}. \qquad (B5)$$

The required expectation value is

$$\langle \Delta_2(t)\ \Delta_2(t + \tau)\rangle \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\nu' \left\{\langle D_2(\nu)D_2{}^*(\nu')\rangle\right.$$

$$\left.\exp\left(i\nu'(t + \tau) - i\nu t\right)\right\}. \qquad (B6)$$

$$D_2(\nu)D_2^*(\nu') = \Delta_0^{\,2}\Bigg\{2\pi\ \delta(\nu)\delta(\nu')$$

$$-\ \frac{\delta(\nu)}{\Gamma-i\nu'}\int_{-\infty}^{\infty} d\omega' U^*(\omega')U(\omega'-\nu')\left[\frac{1}{\Gamma-i(\omega_0-\omega')} + \frac{1}{\Gamma+i(\omega_0+\omega')}\right]$$

$$-\ \frac{\delta(\nu')}{\Gamma-i\nu}\int_{-\infty}^{\infty} d\omega' U(\omega')U^*(\omega'-\nu)\left[\frac{1}{\Gamma+i(\omega_0-\omega')} + \frac{1}{\Gamma-i(\omega_0+\omega')}\right]$$

$$+\ \frac{1}{2\pi}\ \frac{1}{\Gamma-i\nu}\ \frac{1}{\Gamma+i\nu'}\int_{-\infty}^{\infty} d\omega' d\omega'' \Bigg(U(\omega')U^*(\omega'-\nu)U^*(\omega'')U(\omega''-\nu')$$

$$\times\left[\frac{1}{\Gamma+i(\omega_0-\omega')} + \frac{1}{\Gamma-i(\omega_0+\omega')}\right]\left[\frac{1}{\Gamma-i(\omega_0-\omega'')} + \frac{1}{\Gamma+i(\omega_0+\omega'')}\right]\Bigg)\Bigg\}. \tag{B7}$$

Now assume

$$\langle U(\omega)U^*(\omega')\rangle = f(\omega)\ \delta(\omega-\omega'). \tag{B8}$$

Then it follows that

$$\langle D_2(\nu)D_2^*(\nu')\rangle \simeq 2\pi\Delta_0^{\,2}\Bigg\{\delta(\nu)\delta(\nu')\left[1 - \frac{2f(\omega_0)}{\Gamma} + \frac{f^2(\omega_0)}{\Gamma^2}\right]$$

$$+\ \frac{1}{2\pi}\ \frac{\delta(\nu-\nu')\ f^2(\omega_0)}{\Gamma(\Gamma^2+\nu^2)}\Bigg\}. \tag{B9}$$

The terms in $\delta(\nu)\delta(\nu')$ are just $\langle D_2(\nu)\rangle\langle D_2(\nu')\rangle$. It is clear that a fourth-order perturbation theory solution for Δ would in general contribute other terms in $f^2(\omega_0)$ to $\langle D(\nu)D^*(\nu')\rangle$. These terms would arise from the product of zeroth-order terms in the expansion of $D(\nu)$ with fourth-order terms from $D^*(\nu')$ and vice-versa; thus all such terms are multiplied by $\delta(\nu)$ or $\delta(\nu')$. It may be shown that they are smaller than the similar terms in equation (B9) by a factor of at least $\Delta\omega/\Gamma$. Other small terms have also been omitted. Finally,

$$\langle\Delta_2(t)\Delta_2(t + \tau)\rangle \simeq \Delta_0^{\,2}\left[\left(1 - \frac{f(\omega_0)}{\Gamma}\right)^2 + \frac{1}{2}\,\frac{f^2(\omega_0)}{\Gamma^2}\,e^{-\Gamma\tau}\right] \tag{B10}$$

The term multiplying $\delta(\nu - \nu')$ in equation (B9), and the corresponding term in equation (B10) show the effect of correlations in the fluctuations in $\Delta_2(t)$. This may be seen more directly if equation (B10) is derived directly from equation (B3). (Note however that equation [B9] cannot be derived from equation [B10].) Equation (B10) shows that such correlations are unimportant only for lag times $\tau \gtrsim \Gamma^{-1}$. This upper limit on τ is physically reasonable because for $\tau > \Gamma^{-1}$, almost all the atoms contributing to $\Delta_2(t)$ will have decayed, and so no correlation with $\Delta_2(t+\tau)$

is expected. Equation (B9) shows that it is the low frequency Fourier components of $\Delta(t)$ which are responsible for the correlation.

The above analysis is not valid if the stimulated emission rate R exceeds the decay rate Γ, because the perturbation series does not converge. It is expected that if $R \gg \Gamma$, the correlation time appearing in equation (B10) will be essentially R^{-1} rather than Γ^{-1}.

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PART III

TRAPPED INFRARED LINES, CROSS-RELAXATION AND MASER POLARIZATION

ABSTRACT

Infrared line radiation trapped between a maser level and other rotational levels produces a rapid relaxation among the degenerate substates of the maser level. The rate of this relaxation is comparable to the spontaneous decay rates of the infrared transitions. This cross-relaxation has important effects on the apparent source sizes and the polarization properties of interstellar masers. It also affects the relative amplification of the components of a hyperfine-split maser line.

The effect on apparent source size is pronounced when the cross-relaxation rate γ exceeds the decay rate of the maser levels. In this limit, cross-relaxation enables maser radiation directed in a narrow solid angle to saturate the population excess in all magnetic sublevels. This property is essential to the arguments which suggest that the apparent sizes of interstellar OH and H_2O masers are much smaller than their physical sizes.

Cross-relaxation has an important effect on the polarization of radiation emitted by saturated masers if the relaxation rate γ is greater than the stimulated emission rate R. For cases in which the Zeeman splitting $g\Omega$ is greater than the maser linewidth $\Delta\omega$ the maser amplifies the Zeeman pattern. In the presence of rapid cross-relaxation $(\gamma > R)$, the rates of amplification of the σ and π components of the Zeeman pattern are unequal and depend upon the angle between the propagation direction and the magnetic field. For $R < g\Omega < \Delta\omega$, the limiting maser polarization is linear. However, cross-relaxation suppresses the growth of linear polarization until and unless the stimulated emission rate becomes as large as the relaxation rate.

I. INTRODUCTION

In deriving maser transfer equations, it is important to take into account any process which affects the distribution of population among the degenerate sublevels of the maser states. This is because maser photons travelling in different directions or polarized differently compete for the population excess in the various sublevels. Any process which causes a systematic transfer of population among the sublevels could have an important effect on the physical properties of the maser radiation.

Strong resonance radiation trapping between the maser levels and other rotational states gives rise to a rapid relaxation of population among the degenerate sublevels (Litvak 1970). The rate of this relaxation is on the order of the spontaneous emission rate of the resonance photons. Collisions between the maser molecules and other atoms and molecules provide additional relaxation but at a rate which is on the order of the decay rate of the maser levels. Thus this latter contribution to the relaxation rate is never of importance and may be neglected.

The process of cross-relaxation of population among degenerate sublevels has little effect in unsaturated masers because of the absence of gain competition between maser photons. Thus only saturated masers are studied here. In §II the rate of cross-relaxation due to resonance radiation trapping is derived. The effects that this cross-relaxation has on maser source size and polarization are investigated in §III and §IV respectively. In §V, the influence of trapped resonance radiation on the relative amplification of the individual hyperfine components of the 1.35 cm H_2O line is discussed. The theoretical results are compared with observations in §VI.

II. RESONANCE RADIATION TRAPPING AND CROSS-RELAXATION

For simplicity it is assumed that the maser operates between upper and lower states of total angular momenta $F_a = 1$ and $F_b = 0$ respectively. A third state having $F_c = 0$ is assumed to lie above the upper maser level and to be coupled to it by an electric dipole transition. Although the level structures of OH and H_2O molecules are much richer than assumed here, this simple model suffices to illustrate all of the important physics.

The radiation which is trapped between states c and a shall be referred to as trapped infrared radiation. For the OH and H_2O cosmic masers, it would lie in the far-infrared if state c was a rotational state of the ground vibrational level and in the near-infrared if state c was an excited vibration-rotational state. In either case, it shall always be assumed that the Doppler width of the trapped infrared radiation exceeds the Zeeman splitting of the $F_a = 1$ state.

In this section the coupling between states c and a is considered. In most respects the treatment of this two-state system follows closely that developed by Goldreich, Keeley, and Kwan (1972) (hereafter called paper II).

The infrared photons propagate in all directions. The radiation propagating in any direction may be approximated by a plane wave which obeys stationary statistics as described in §II of paper II. Plane waves propagating in different directions are uncorrelated. The electric field of the radiation which propagates along $\hat{\underset{\sim}{k}}$ may be decomposed into its circularly polarized components as

$$\underset{\sim}{E}(\ell,t) = \mathrm{Re}\{E^{+}(\ell,t)\,\hat{\underset{\sim}{e}}^{+} + E^{-}(\ell,t)\,\hat{\underset{\sim}{e}}^{-}\} \quad , \tag{1}$$

where

$$E^{\pm}(\ell,t) = \mathscr{E}^{\pm}(\ell,t)\,\exp\{-i[\omega_o(t-\ell/c) + \phi^{\pm}(\ell,t)]\} \quad , \tag{2}$$

and $\ell = \hat{\underset{\sim}{k}}\cdot\underset{\sim}{r}$. Here ω_o is the resonant frequency of the infrared transition. The unit circular-polarization vectors $\hat{\underset{\sim}{e}}^{\pm}$ are equal to $(\hat{\underset{\sim}{i}} \pm i\hat{\underset{\sim}{j}})/\sqrt{2}$, where $\hat{\underset{\sim}{i}}$ and $\hat{\underset{\sim}{j}}$ are two real unit vectors normal to $\hat{\underset{\sim}{k}}$. The amplitudes $\mathscr{E}^{\pm}$ and the phases $\phi^{\pm}$ are real functions of space and time and they vary only slightly over distances of order a wavelength and times of order a wave period.

The effect of the molecules on the infrared radiation field is conveniently described in terms of the polarization vector. Because waves travelling in different directions are uncorrelated, each wave is affected only by that component of the polarization vector which has the same wave vector. This component reads

$$\underset{\sim}{P}(\ell,t) = \mathrm{Re}\{P^{+}(\ell,t)\,\hat{\underset{\sim}{e}}^{+} + P^{-}(\ell,t)\,\hat{\underset{\sim}{e}}^{-}\} \quad , \tag{3}$$

where

$$P^{\pm}(\ell,t) = \mathscr{P}^{\pm}(\ell,t)\,\exp\{-i[\omega_o(t-\ell/c) + \phi^{\pm}(\ell,t)]\} \quad . \tag{4}$$

The transfer equations for radiation travelling in direction $\hat{\underset{\sim}{k}}$ are

$$\frac{D\mathscr{E}^{\pm 2}}{D\ell} = 2\pi i\,\frac{\omega_o}{c}\,[\mathscr{P}^{\pm}\mathscr{E}^{\pm} - \mathscr{P}^{\pm *}\mathscr{E}^{\pm}] \quad , \tag{5}$$

and

$$\frac{D}{D\ell}\,[\mathscr{E}^{+}\mathscr{E}^{-}\exp(-i\Delta\phi)] = 2\pi i\,\frac{\omega_o}{c}\,[\mathscr{P}^{+}\mathscr{E}^{-} - \mathscr{P}^{-*}\mathscr{E}^{+}]\exp(-i\Delta\phi) \quad , \tag{6}$$

where

$$\frac{D}{D\ell} = \frac{\partial}{\partial \ell} + \frac{1}{c}\frac{\partial}{\partial t} ,$$

and $\Delta\phi = \phi^{+} - \phi^{-}$. The transfer equations will henceforth be written in terms of the Stokes parameters whose definitions are (Chandrasekhar 1950)

$$I = \frac{c}{8\pi} < \mathscr{E}^{+2} + \mathscr{E}^{-2} > , \qquad V = \frac{c}{8\pi} < \mathscr{E}^{+2} - \mathscr{E}^{-2} >$$

$$Q = \frac{c}{8\pi} < 2\mathscr{E}^{+} \mathscr{E}^{-} \cos \Delta\phi > , \qquad U = \frac{c}{8\pi} < 2\mathscr{E}^{+} \mathscr{E}^{-} \sin \Delta\phi > , \tag{7}$$

where the angular brackets denote expectation values.

The behavior of the molecules is described in terms of a density matrix $\rho(\underset{\sim}{r},t)$. Actually, a single density matrix can only describe molecules at a fixed position and time, moving with a unique velocity, since the molecular Hamiltonian is a function of position, time and velocity. The explicit dependence of $\rho(\underset{\sim}{r},t)$ on the velocity will not be included here. A detailed treatment of this point was given in paper II. In this section only the two-level system (comprised of levels c and a) is studied. Thus, only a submatrix of the entire density matrix is needed.

The macroscopic polarization vector is

$$\underset{\sim}{P}(\underset{\sim}{r},t) = e \, \mathrm{tr}[\rho(\underset{\sim}{r},t) \, \underset{\sim}{r}] , \tag{8}$$

where $\underset{\sim}{r}$ is the matrix of the position vector. The components of the total polarization vector orthogonal to $\hat{\underset{\sim}{k}}$ are

$$\underset{\sim}{P}(\ell,t) = 2p \, \mathrm{Re} \left\{ \begin{array}{l} [\rho_{ca_+} \dfrac{(1-\cos\theta)e^{-i\alpha}}{2} - \rho_{ca_o} \dfrac{\sin\theta}{\sqrt{2}} + \rho_{ca_-} \dfrac{(1+\cos\theta)e^{i\alpha}}{2}] \hat{\underset{\sim}{e}}^{+} \\ +[-\rho_{ca_+} \dfrac{(1+\cos\theta)e^{-i\alpha}}{2} - \rho_{ca_o} \dfrac{\sin\theta}{\sqrt{2}} - \rho_{ca_-} \dfrac{(1-\cos\theta)e^{i\alpha}}{2}] \hat{\underset{\sim}{e}}^{-} \end{array} \right\} \tag{9}$$

where p is the reduced dipole matrix element of the infrared transition. The angles θ and α are spherical polar coordinates which specify the orientation of the wave direction $\hat{\underset{\sim}{k}}$ relative to the quantization axis (z-axis) and an arbitrary x-axis.

The contribution to the Hamiltonian matrix by the radiation which propagates in direction $\hat{\underset{\sim}{k}}$ is

$$V_{ca_{\pm}}(\ell,t) = pe^{\pm i\alpha}\,[-(1 \mp \cos\theta)E^{+} + (1 \pm \cos\theta)E^{-}]\,/\,4$$

$$V_{ca_{o}}(\ell,t) = p\sin\theta[E^{+} + E^{-}]\,/\,2\sqrt{2} \qquad (10)$$

In equations (10) the rotating wave approximation (Lamb 1964) has been used. Thus only the negative frequency parts of $\underset{\sim}{E}$ appear in the expressions for V_{ca_m}.

The solution of the density matrix equation of motion may be carried out in a manner similar to that described in part A of §IV of paper II and is not repeated here. As before, the crucial step in obtaining an approximate solution is to treat $\rho_{a_m a_n}$ and ρ_{cc} as constants. The resulting expressions for the off-diagonal density matrix elements connecting levels c and a are

$$\rho_{ca_m} = \frac{i}{\hbar}\rho_{cc}\int_{-\infty}^{t}\exp\{-[\Gamma + i\omega_o](t-t')\}\,V_{ca_m}(\ell,t')dt'$$

$$-\frac{i}{\hbar}\sum_{n}\rho_{a_n a_m}\int_{-\infty}^{t}\exp\{-[\Gamma + i\omega_o](t-t')\}\,V_{ca_n}(\ell,t')dt' + \cdots, \qquad (11)$$

where only the contributions explicitly due to $V_{ca_m}(\ell,t)$ have been written out.

It is always possible to choose a basis in which $\rho_{a_m a_n} = 0$ for $m \neq n$ since the density matrix is Hermitian. Such bases were used in paper II in cases 1a and 2a. In case 3a, the $\rho_{a_m a_n}$, $m \neq n$ did not vanish. However, they were very much smaller than $\rho_{a_o a_o} - \rho_{a_\pm a_\pm}$. Consequently, in all three cases the $\rho_{a_m a_n}$, $m \neq n$ are unimportant in determining the properties of the trapped infrared radiation and may be neglected in equation (11). It is worth noting, and easily proved, that the polarized and anisotropic infrared radiation that is present when the sublevel populations $\rho_{a_m a_m}$ are not all equal does not itself generate the off-diagonal matrix elements $\rho_{a_m a_n}$, $m \neq n$.

The transfer equations for the infrared radiation follow from equations (5)-(11) and have the form

$$\frac{DI}{D\ell} = -\beta \left\{ \begin{array}{l} (\rho_{a_+ a_+} - \rho_{cc})[(1+\cos^2\theta)I - 2\cos\theta V - \sin^2\theta Q] \\ \\ +(\rho_{a_o a_o} - \rho_{cc})\ 2\sin^2\theta[I+Q] \\ \\ +(\rho_{a_- a_-} - \rho_{cc})[(1+\cos^2\theta)I + 2\cos\theta V - \sin^2\theta Q] \end{array} \right\} + \varepsilon\,\rho_{cc} \qquad (12a)$$

$$\frac{DV}{D\ell} = -\beta \left\{ \begin{array}{l} (\rho_{a_+ a_+} - \rho_{cc})[-2\cos\theta\ I + (1+\cos^2\theta)V] \\ \\ +(\rho_{a_o a_o} - \rho_{cc})\ 2\sin^2\theta\ V \\ \\ +(\rho_{a_- a_-} - \rho_{cc})[2\cos\theta\ I + (1+\cos^2\theta)V] \end{array} \right\} \qquad (12b)$$

$$\frac{DQ}{D\ell} = -\beta \left\{ \begin{array}{l} (\rho_{a_+a_+} - \rho_{cc})[-\sin^2\theta\ I + (1+\cos^2\theta)Q] \\ +(\rho_{a_oa_o} - \rho_{cc})\ 2\ \sin^2\theta\ [I+Q] \\ +(\rho_{a_-a_-} - \rho_{cc})[-\sin^2\theta\ I + (1+\cos^2\theta)Q] \end{array} \right\} \tag{12c}$$

$$\frac{DU}{D\ell} = -\beta \left\{ \begin{array}{l} (\rho_{a_+a_+} - \rho_{cc})\ (1+\cos^2\theta)U \\ +(\rho_{a_oa_o} - \rho_{cc})\ 2\ \sin^2\theta\ U \\ +(\rho_{a_-a_-} - \rho_{cc})(1+\cos^2\theta)U \end{array} \right\} \tag{12d}$$

where $\beta = \dfrac{\pi^{3/2}|p|^2\omega_o}{2^{1/2}\hbar c\ \Delta\omega}$ and $\varepsilon = \dfrac{\hbar\omega_o}{4\pi} A$.

The symbol A denotes the spontaneous emission rate of state c and $\Delta\omega$ is the radian bandwidth of the infrared line. The last term in equation (10a) describes the contribution of spontaneous emission to the growth of the total intensity. It has been introduced classically since a formal quantum-mechanical approach would require quantization of the radiation field. It is the authors' belief that the same final expressions would be obtained by the more formal method of treating spontaneous emission, albeit in a less obvious way.

Under conditions appropriate to interstellar masers $(kT \sim h\nu_{IR})$ it is expected that $\rho_{a_ma_m} - \rho_{cc}$ will be much larger than $\rho_{a_ma_m} - \rho_{a_na_n}$. Hence a perturbation solution of equations (12) for the steady-state values of the Stokes parameters in powers of $(\rho_{a_ma_m} - \overline{\rho}_{aa}) / (\overline{\rho}_{aa} - \rho_{cc})$ (where

$3\bar{\rho}_{aa} = \rho_{a_+a_+} + \rho_{a_oa_o} + \rho_{a_-a_-})$ is appropriate. To first order it yields

$$I(\hat{k}) = I_o - \frac{1}{4}\left(\frac{\bar{\rho}_{aa} - \rho_{a_oa_o}}{\bar{\rho}_{aa} - \rho_{cc}}\right)(3\cos^2\theta - 1)I_o \quad , \tag{13a}$$

$$V(\hat{k}) = \frac{1}{2}\left(\frac{\rho_{a_+a_+} - \rho_{a_-a_-}}{\bar{\rho}_{aa} - \rho_{cc}}\right)\cos\theta\, I_o \quad , \tag{13b}$$

$$Q(\hat{k}) = \frac{3}{4}\left(\frac{\bar{\rho}_{aa} - \rho_{a_oa_o}}{\bar{\rho}_{aa} - \rho_{cc}}\right)\sin^2\theta\, I_o \tag{13c}$$

$$U(\hat{k}) = 0 \quad , \tag{13d}$$

where $I_o = \dfrac{\varepsilon\rho_{cc}}{4\beta(\bar{\rho}_{aa} - \rho_{cc})}$.

It is evident from equations (13) that the infrared radiation which propagates in direction $\hat{k}$ will be slightly polarized if the sublevel populations are unequal.

The infrared radiation field perturbs the sublevel populations of the upper maser state. If only these perturbation terms are written out explicitly, the equations of motion for the diagonal matrix elements of the upper maser level are

$$\frac{\partial\rho_{a_\pm a_\pm}}{\partial t} = \frac{-\beta}{\hbar\omega_o}(\rho_{a_\pm a_\pm} - \rho_{cc})\int[(1+\cos^2\theta)I(\hat{k}) \mp 2\cos\theta\, V(\hat{k}) - \sin^2\theta\, Q(\hat{k})]\, d\Omega + \frac{A}{3}\rho_{cc} + \cdots \tag{14a}$$

$$\frac{\partial \rho_{a_o a_o}}{\partial t} = \frac{-\beta}{\hbar\omega_o}(\rho_{a_o a_o} - \rho_{cc}) \int 2 \sin^2\theta \, [I(\hat{\underset{\sim}{k}}) + Q(\hat{\underset{\sim}{k}})] \, d\Omega + \frac{A}{3}\rho_{cc} + \cdots \qquad (14b)$$

With the help of equations (12) and (13), equations (14) become

$$\begin{aligned} \frac{\partial \rho_{a_\pm a_\pm}}{\partial t} &= -\gamma[2(\rho_{a_\pm a_\pm} - \rho_{a_\mp a_\mp}) + (\rho_{a_\pm a_\pm} - \rho_{a_o a_o})] + \cdots \\ \frac{\partial \rho_{a_o a_o}}{\partial t} &= -\gamma[(\rho_{a_o a_o} - \rho_{a_+ a_+}) + (\rho_{a_o a_o} - \rho_{a_- a_-})] + \cdots \quad , \end{aligned} \qquad (15)$$

where the cross-relaxation rate

$$\gamma = \frac{A\,\rho_{cc}}{30(\bar{\rho}_{aa} - \rho_{cc})} \quad . \qquad (16)$$

In the following two sections the effects of cross-relaxation on the apparent size and polarization properties of interstellar masers are studied.

III. CROSS-RELAXATION AND MASER SOURCE SIZE AND SATURATION

Maser radiation travelling along the z (quantization) axis does not interact with the $F_a = 1$, $m_a = 0$ sublevel. However, if the maser radiation is sufficiently intense to depopulate the $m_a = \pm 1$ sublevels, cross-relaxation will transfer population from the $m_a = 0$ sublevel into the $m_a = \pm 1$ sublevels. This transfer increases the maser gain in the $\hat{\underset{\sim}{z}}$ direction and reduces the gain in directions orthogonal to $\hat{\underset{\sim}{z}}$. This effect of cross-relaxation may be illustrated by solving the rate equations which govern the sublevel populations. These equations read

$$\frac{d}{dt}\rho_{a_\pm a_\pm} = \Lambda_a - BJ_\pm(\rho_{a_\pm a_\pm} - \rho_{bb}) - \Gamma\rho_{a_\pm a_\pm} - 2\gamma(\rho_{a_\pm a_\pm} - \rho_{a_\mp a_\mp}) - \gamma(\rho_{a_\pm a_\pm} - \rho_{a_o a_o})$$

$$\frac{d}{dt}\rho_{a_o a_o} = \Lambda_a - \Gamma\rho_{a_o a_o} - \gamma(2\rho_{a_o a_o} - \rho_{a_+ a_+} - \rho_{a_- a_-}) \tag{17}$$

$$3\Lambda_a + \Lambda_b = \Gamma(\rho_{a_+ a_+} + \rho_{a_- a_-} + \rho_{a_o a_o} + \rho_{bb}) \quad .$$

In equations (17), Λ_a and Λ_b are the pump rates per magnetic sublevel into states a and b ; B is the Einstein coefficient for induced emission; J_+ and J_- are the right and left-circularly polarized specific intensities averaged over both the absorption profile and directions in space. It has been assumed that the specific intensity of the maser radiation is substantial only within a small solid angle about $\hat{\underset{\sim}{z}}$. The decay rate Γ has been assumed to be the same for both levels.

For unpolarized maser radiation, the steady-state solutions of the rate equations take the form:

Case 1. $\Gamma > \gamma$, $BJ > \Gamma$

$$\rho_{a_\pm a_\pm} - \rho_{bb} = \frac{2}{3}\frac{(\Lambda_a - \Lambda_b)}{BJ} \quad , \quad \rho_{a_o a_o} - \rho_{bb} = \frac{(\Lambda_a - \Lambda_b)}{3\Gamma} \quad ; \tag{18}$$

Case 2. $\Gamma < \gamma$, $BJ > \Gamma$

$$\rho_{a_\pm a_\pm} - \rho_{bb} = \frac{3}{4}\frac{(\Lambda_a - \Lambda_b)}{BJ} \quad , \quad \rho_{a_o a_o} - \rho_{bb} = \frac{(\Lambda_a - \Lambda_b)(6\gamma + BJ)}{8\gamma\ BJ} \tag{19}$$

A comparison of the expressions for $\rho_{a_\pm a_\pm} - \rho_{bb}$ in cases 1 and 2 shows that in the latter case the maser radiation grows faster (by a factor 9/8) since for $\gamma > \Gamma$ the $m_a = 0$ sublevel contributes to the power.

The relation between the apparent and physical sizes of saturated masers derived by Goldreich and Keeley (1972) depends upon the ability of maser radiation travelling in one direction to deplete all the population excess in the upper maser sublevels. By adopting a scalar atom model, they implicitly assumed that this condition was satisfied in interstellar masers. Their investigation showed that maser radiation travelled nearly radially in the outer regions of spherical saturated maser clouds. For a real maser (such as one operating between levels $F_a = 1$ and $F_b = 0$) quantized about the radial direction, the $\Delta m = 0$ transitions would not be saturated by the radially directed maser photons. If the $\Delta m = 0$ population excess were maintained it would provide a large gain along chords which traverse the outer portion of the maser cloud. In this circumstance, the apparent size of a saturated maser source would be nearly as large as its physical size.

Clearly, rapid population relaxation across the magnetic sublevels of the maser states $\gamma >> \Gamma$ would decrease the population excess in the $\Delta m = 0$ transitions. Under this condition, the scalar atom model used by Goldreich and Keeley is a good approximation.

A magnetic field which is not along the $\hat{\underset{\sim}{z}}$ direction would also cause a transfer of population from the $m_a = 0$ sublevel to the other two sublevels. However, this population transfer cannot be simply described in terms of a cross-relaxation rate as was possible for the transfer due to trapped resonance radiation. The results derived in case 3a of paper II show that the population excess in the $m_a = 0$ level can contribute to the maser power along the quantization axis if the magnetic field is of a strength such that $(g\Omega \sin \theta)^2 > R\Gamma$. Here $g\Omega$ is the Zeeman splitting of the upper maser level, R is the stimulated emission rate and θ is the angle between the quantization axis and the magnetic field.

IV. CROSS-RELAXATION AND THE POLARIZATION OF MASER RADIATION

The polarization properties of maser radiation in the presence of a magnetic field were investigated in paper II. In this section, the presence of trapped infrared radiation is taken into account and its effect on polarization is studied. It is straightforward to introduce the cross-relaxation terms given by equations (15) into the density matrix equation of motion and to derive the appropriate modifications of the results obtained in paper II for the different limiting cases. Only the final results for the polarization of the maser radiation in saturated masers are presented here, together with physical arguments which elucidate their nature. The case numbers and notation are the same as in paper II.

Case 1a. $R \ll \Delta\omega \ll g\Omega \quad , \quad \Gamma \ll R$

In this case the Zeeman splitting exceeds the maser bandwidth and the maser radiation consists of three separate lines. In the absence of cross-relaxation, the maser amplifies the Zeeman pattern and the σ and π components grow at the same rate. Although the intensities of the σ and π components are the same, the populations in the $m_a = \pm 1$ and $m_a = 0$ sublevels are not equal because the relative rates of stimulated emission from these levels depend upon the angle between the magnetic field and the propagation direction.

In the presence of a cross-relaxation process, there is a transfer of population from the less to the more depleted sublevels. This transfer results in unequal rates of growth for the σ and π components. The difference in the rates of growth is large when the cross-relaxation rate exceeds the stimulated emission rate. The mathematical basis for these deductions is clearly displayed by explicitly writing out the ratios of the transfer equations for the total intensities of the three Zeeman components. These ratios have the form

$$\frac{DI_+}{DI_-} = \frac{(\rho_{a_+a_+} - \rho_{bb})[(1+\cos^2\theta)I_+ + 2\cos\theta\, V_+ + \sin^2\theta\, Q_+]}{(\rho_{a_-a_-} - \rho_{bb})[(1+\cos^2\theta)I_- - 2\cos\theta\, V_- + \sin^2\theta Q_-]}$$

and (20)

$$\frac{DI_+}{DI_o} = \frac{(\rho_{a_+a_+} - \rho_{bb})[(1+\cos^2\theta)I_+ + 2\cos\theta V_+ + \sin^2\theta\, Q_+]}{(\rho_{a_oa_o} - \rho_{bb})\; 2\sin^2\theta\; [I_o - Q_o]} \quad ,$$

where the subscripts on the Stokes parameters denote the three radiation bands by indicating the magnetic sublevel of the upper maser state to which each couples. The variable θ denotes the angle between the magnetic field and the radiation axis.

If cross-relaxation is unimportant, the numerators and denominators of the right-hand sides of equations (20) are equal and the σ and π components grow at the same rate. In the limit that the cross-relaxation rate is much greater than the stimulated emission rates, it is the populations in the magnetic sublevels that are equal. Then the values of the right-hand sides of equations (20) are in general different from unity and depend upon the values of the Stokes parameters for each Zeeman component and the value of θ . In the absence of significant Faraday rotation the limiting values of the Stokes parameters are

$$V_\pm = \pm\frac{2\cos\theta}{(1+\cos^2\theta)} I_\pm \quad , \quad Q_\pm = \frac{\sin^2\theta}{(1+\cos^2\theta)} I_\pm \quad , \quad U_\pm = 0 \quad ,$$

(21)

$$V_o = 0 \quad , \quad Q_o = -I_o \quad , \quad U_o = 0 \quad .$$

For large Faraday rotation, they are

$$V_{\pm} = \pm I_o \quad , \qquad Q_{\pm} = 0 \quad , \qquad U_{\pm} = 0 \quad ,$$
$$V_o = \kappa I_o \quad , \qquad Q_o = 0 \quad , \qquad U_o = 0 \quad , \tag{22}$$

where $-1 < \kappa \leq 1$. When the first set of Stokes parameters is substituted into equations (20), these equations are transformed into

$$\frac{DI_+}{DI_-} = \frac{I_+}{I_-} \quad , \qquad \frac{DI_+}{DI_o} = \frac{(1 + \cos^2\theta)I_+}{2 \sin^2\theta \; I_o} \quad . \tag{23}$$

Hence the σ components dominate for $\sin^2\theta < 2/3$ and the π component grows faster for $\sin^2\theta > 2/3$. In addition, the σ_+ and σ_- components grow at rates proportional to their individual intensities. Thus if the intensity of one of the σ components ever became larger than that of the other, this imbalance would be preserved as they grew.

For large Faraday rotation, the second set of Stokes parameters is appropriate and equations (20) become

$$\frac{DI_+}{DI_-} = \frac{I_+}{I_-} \quad , \qquad \frac{DI_+}{DI_o} = \frac{(1 + |\cos\theta|)^2}{2 \sin^2\theta} \frac{I_+}{I_o} \quad . \tag{24}$$

The σ components now dominate for $\sin^2\theta < 8/9$ or $8\pi/3$ steradians.

Case 2a. $R \ll g\Omega \ll \Delta\omega$, $\Gamma \ll R$

In the absence of any cross-relaxation, it so happens that for the stable polarization in this limiting case, the populations in the magnetic sublevels of the upper maser state are equal. Therefore, the presence of cross-relaxation among the sublevels does not change the polarization which is given by

$$V = 0 \quad , \quad Q = -I \quad , \quad U = 0 \quad , \quad \text{for } \sin^2\theta \leq 1/3$$

$$V = 0 \quad , \quad Q = \frac{(3\sin^2\theta - 2)}{3\sin^2\theta} I \quad , \quad U = \kappa I \quad , \quad \text{for } \sin^2\theta \geq 1/3 \tag{25}$$

where κ is a constant in the range $\kappa^2 \leq 1 - (Q/I)^2$.

The cross-relaxation of population among the sublevels does have one important effect, which is that it reduces the rate of growth toward the stable polarization. As a simple example to elucidate this point, consider the case of maser radiation, initially unpolarized, propagating at right angles to the magnetic field. Molecules in the $m_a = 0$ and $m_a = \pm 1$ sublevels are stimulated to emit by photons polarized respectively parallel and perpendicular to the magnetic field. If the photon intensities in the two polarization modes are equal, the stimulated emission rate from the $m_a = 0$ sublevel is greater than that from the $m_a = \pm 1$ sublevels. As a result, the $m_a = 0$ sublevel is relatively under-populated compared to the $m_a = \pm 1$ sublevels. The presence of a cross-relaxation process produces a transfer of population from the $m_a = \pm 1$ sublevels to the $m_a = 0$ sublevel. If the relaxation rate is faster than the stimulated emission rate from the $m_a = \pm 1$ sublevels, the population transfer out of these levels would considerably slow the growth of linear polarization perpendicular to the magnetic field and thus delay the approach to the stable polarization which in this case is $Q = I/3$. For arbitrary directions of propagation, the equation governing the growth of linear polarization is

$$\frac{3D\ Y}{D\ \ell n\ I} = \frac{\eta I \sin^2\theta(1 - Y^2)[(3\sin^2\theta - 2) - 3\sin^2\theta\ Y]}{\eta I \sin^2\theta(1 - Y)\ [(1 + \cos^2\theta) + \sin^2\theta\ Y] + 2\gamma} , \tag{26}$$

where

$$\eta = \frac{\pi^{3/2}\ |p|^2}{2^{1/2}\ \hbar^2\ c\Delta\omega} . \tag{27}$$

Unless the stimulated emission rate $R \sim \eta I$ is greater than the cross-relaxation rate γ, the growth rate of linear polarization is small.

Case 3a. $g\Omega << R << \Delta\omega$, $\Gamma << R$

In this case, for which the magnetic precession rate is slower than the microwave stimulated emission rate, it is most convenient to choose the quantization axis along the direction of propagation of the maser radiation. As viewed with this choice of axis, the population excess in the $m_a = \pm 1$ sublevels is thoroughly depleted by stimulated emission but there is no stimulated emission from the $m_a = 0$ sublevel. However, a diffusive transfer of population from the $m_a = 0$ sublevel into the $m_a = \pm 1$ sublevels is produced by the magnetic field. In the limit $(g\Omega \sin\theta)^2 >> \Gamma R$, this population transfer results in a limiting polarization $Y = 1/3$.

The presence of cross-relaxation due to trapped infrared radiation imposes a further condition $(g\Omega \sin\theta)^2 >> \gamma R$ which must be satisfied if linear polarization is to arise. The reason for this extra condition is easy to understand. Magnetic diffusion transfers population from the $m_a = 0$ sublevel into a coherent superposition of the $m_a = \pm 1$ substates which interacts with photons polarized along $\hat{z} \times \tilde{B}$. Thus, population transfer by magnetic diffusion produces maser radiation which is polarized along $\hat{z} \times \tilde{B}$. On the other hand, population transfer due to cross-relaxation does not favor any particular polarization. Thus, if the cross-relaxation rate exceeds the diffusive transfer rate, the $m_a = 0$ sublevel still contributes to the maser power but no net polarization arises.

V. RESONANCE RADIATION TRAPPING AND HYPERFINE SELECTION IN H_2O MASERS

The upper and lower levels of the 1.35 cm H_2O transition are split into three hyperfine components having total angular momenta $F_a = 7,6,5$ and $F_b = 6,5,4$. Thus there are six allowed hyperfine-split transitions between the upper and lower levels. Of these, the $F_a = 6 \rightarrow F_b = 6$, $F_a = 5 \rightarrow F_b = 6$, and $F_a = 5 \rightarrow F_b = 5$ transitions are unimportant since their line strengths are two orders of magnitude smaller than those of the three remaining transitions $F_a = 7 \rightarrow F_b = 6$, $F_a = 6 \rightarrow F_b = 5$, and $F_a = 5 \rightarrow F_b = 4$ (Sullivan 1971). The transfer equations for these hyperfine components neglecting overlap read

$$\frac{dI_i}{d\ell} = \frac{\hbar\omega}{2\Delta\omega} B_i I_i \, g_{a_i} (n_{a_i} - n_{b_i}) \quad , \tag{28}$$

where the subscript i takes on the values 1, 2 and 3 for the $F_a = 7 \rightarrow F_b = 6$, $F_a = 6 \rightarrow F_b = 5$, and $F_a = 5 \rightarrow F_b = 4$ transitions respectively. The g_{a_i} are the degeneracies of the upper sublevels for each transition, the B_i are the Einstein coefficients for stimulated emission, and the n_{a_i} and n_{b_i} are the populations per magnetic substate (per unit volume) for the upper and lower hyperfine states. In equation (28) the spontaneous emission source term has been neglected.

Trapped resonance radiation may have an important effect on the relative growth of the hyperfine components because it tends to equalize the population per degenerate sublevel within a given rotational state. If the rate of this relaxation is greater than the stimulated emission rates from the three upper hyperfine levels, the population inversions $(n_{a_i} - n_{b_i})$,

$i = 1,2,3$ are close to equal. In this circumstance, the $F_a = 7 \rightarrow F_b = 6$ hyperfine line would grow fastest since the relative values of $g_{a_1} B_1$: $g_{a_2} B_2$: $g_{a_3} B_3$ are 1: 91/108 : 117/165 .

The rate of relaxation of population among the hyperfine states is readily deduced in a manner analogous to that described in §2 for the relaxation among magnetic sublevels. For the specific case in which a maser state is coupled to a higher rotational state, the relaxation rate is

$$\gamma \simeq \frac{1}{3} \frac{AN}{(n-N)} \quad , \qquad (29)$$

where n,N are respectively the populations per sublevel in the maser state and in the higher rotational state, and A is the spontaneous emission rate per sublevel of the upper rotational state.

VI. CONCLUSIONS

In cosmic OH and H_2O masers, the maser levels are coupled to other rotational levels of the ground and excited vibrational states by far and near infrared resonance radiation. The observed brightness temperatures require a path of at least twenty exponential gain lengths. The relative line strengths of the maser and infrared transitions imply that unless the sources are very thin in one direction, the optical depths in the infrared lines are much greater than unity. The resulting trapping of infrared radiation gives rise to a rapid relaxation of population among the magnetic sublevels of a given rotational state. It seems likely that in interstellar masers $kT \gtrsim h\nu_{FIR}$ where ν_{FIR} is a typical far infrared frequency. The relaxation rate due to trapped far infrared radiation in our simple model is $\gamma \sim kT\, A/(30\, h\nu_{FIR})$ (cf. eq. [16]), where a typical value for the Einstein A is $\sim 1\ sec^{-1}$. For OH and H_2O masers, each sublevel of a maser state is coupled to a few sublevels of several rotational states. Thus the total relaxation rate is probably several times per second. Because the line strengths for rotation-vibration transitions are much weaker than those for pure rotational transitions, the trapping of near infrared radiation is less important for temperatures below 2000^oK.

Application of the theoretical results derived in this paper to interstellar masers requires knowledge of both the stimulated emission rates and the decay rates. Unfortunately, present data permit only rough estimates of these parameters. Typical brightness temperatures for OH and H_2O sources associated with HII regions are $10^{12}\ {}^oK$ and $10^{14}\ {}^oK$ respectively. If the model values for the ratios of apparent to physical source size obtained by

Goldreich and Keeley (1972) are adopted, it follows that the stimulated emission rates corresponding to the above brightness temperatures are 0.2 s^{-1} and 6 s^{-1} in the outer regions of the OH and H_2O maser clouds. It is important to bear in mind that the actual stimulated emission rates may differ from these estimated values by an order of magnitude or more. Nevertheless, it seems quite likely that the stimulated emission rates in OH masers are smaller than the cross-relaxation rates, whereas in H_2O masers the two rates are probably comparable. Available evidence suggests that OH sources near HII regions are saturated. The best argument for saturation is based on the fact that usually two or even three of the four ground-state hyperfine components are observed to have comparable intensities even though they have quite different line strengths. The saturation of the OH sources implies that the decay rates of the maser levels are smaller than the stimulated emission rates. There is no direct evidence that the H_2O sources are saturated. However, the high observed brightness temperatures taken together with theoretical estimates of the maximum possible gain for unsaturated masers (Goldreich and Keeley 1972) suggest that at least the stronger H_2O sources are saturated. Again, this indicates that the decay rates of the maser levels are smaller than, or at most comparable to, the stimulated emission rates.

Because the decay rates in OH masers are quite a bit smaller than the cross-relaxation rates, the model calculations of Goldreich and Keeley (1972) which relate apparent source size to saturation are applicable. For H_2O masers, the decay rates may also be smaller than the cross-relaxation rates, but the evidence favoring this view is not compelling. However, because the maser levels have large angular momenta, directional maser radiation can deplete most of the population excess even in the absence of cross-relaxation. For example, saturation by directional maser radiation, which couples upper and lower levels of total angular momenta F_a and $F_b = F_a - 1$, reduces the inversion of the $\Delta m = 0$ transitions by a factor of $4F_a^2 - 1$. This factor equals 195 for the strongest hyperfine component $(F_a = 7 - F_b = 6)$ of the 1.35 cm H_2O line. The small residual inversion of the $\Delta m = 0$ transitions would not provide a very large gain along chords in the outer regions of saturated maser clouds.

The scarcity of linear polarization in OH maser emission can be nicely explained by the combined effects of Faraday rotation and resonance radiation trapping. If the cross-relaxation rate is greater than the stimulated emission rate, which seems quite probable, the growth of linear polarization is suppressed for $R < g\Omega < \Delta\omega$, even in the absence of significant Faraday rotation. For $g\Omega > \Delta\omega$, and no Faraday rotation, the elliptically polarized σ components dominate for propagation at angles θ with respect to the magnetic field which satisfy $\sin^2\theta < 2/3$. At larger values of θ, the linearly polarized π component is stronger. If the Faraday rotation across the region of amplification is large, the σ components are 100 percent circularly polarized and dominate for $\sin^2\theta < 8/9$. The π component is unpolarized and stronger for $\sin^2\theta > 8/9$. This latter case, $g\Omega > \Delta\omega$ and large Faraday rotation, appears to be the most frequently realized one in OH masers associated with HII regions.

The H_2O maser sources show little polarization and then only linear polarization. Because the Landé g values of the maser levels in H_2O are very small, the Zeeman splitting of the maser line is smaller than its bandwidth for magnetic fields below 40G. For $R < g\Omega < \Delta\omega$, the stable polarization is linear. However, the growth of linear polarization is suppressed unless the stimulated emission rate exceeds the cross-relaxation rate. In connection with this point, it is of interest to note that Sullivan (1971) has remarked that the fractional linear polarizations (where present) of the features in Orion A vary in the same sense as their intensities. Sullivan's observations suggest that the stimulated emission rates in these sources are at least as large as the cross-relaxation rates but not very much larger. From this information and the observed brightness temperatures, the ratios of the physical to the apparent sizes of the maser sources can be deduced. Based on an observed brightness temperature of $T_B \sim 5 \times 10^{13}$ ^oK (Moran et al 1971) and an assumed stimulated emission rate of 5 s^{-1}, this procedure yields a value of 40 for the ratio of physical to apparent sizes for the +3 and +9 km/sec features in Orion A.

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