THE GENERATION OF GRAVITATIONAL WAVES

Thesis by
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Materiem superabat opus.

Publius Ovidius Naso.

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ABSTRACT

This dissertation is written in three tracks. Track 1,(Pages ii-vi) is intended for those many many people who don't know any physics but who do know how to read English and a bit of Hungarian perhaps.

Track 2 is intended for those one hundred or so people in the whole world who do know classical physics and have serious interests in, or are experts in relativity theory. Papers 1, 2 and 3 dealing with relativity are entirely Track 2.

The remaining paper is Track 3 and is for the benefit of even fewer people who have a simultaneous interest in relativity and quantum field theory.

I will not discuss the abstracts of the individual papers here - since each paper is preceded by its own abstract. Suffice it to say that this dissertation is a collection of papers dealing with the theoretical aspects of how gravitational waves may (or may not) be generated by gravitational bremsstrahlung,and in Paper 4 Walter and I try to show that some classical relativity problems may be solved with much greater ease via a quantum approach.

A Limerick to

Gravitational Waves

There once was the theory of relativity
Which serious students pursue with humility,
Whether gravitational radiation
Is real, or just imagination
To them, became a matter of facility.

Oh you sly, sneaky, ancient gravity waves
You're putting experimentalists in their graves!
If you really exist
Why are you missed?
You even make theorists hide in their caves.

Some argue it's a matter of lucky detection,
Or perhaps of a different research direction.
But we'll all be elated
If waves are really generated,
'Cause then Nature will verify our prediction.

Einstein thought God was not malicious
I agree, but say Nature is capricious.
She will not simply reveal
Or make it easy to steal,
The secret treasures that make Her delicious.

When at last the darn radiation is found
New explanations and theories will abound.
In the zeal of celebration
Let's recall the quotation
"It's so sad that Einstein won't be around."

Pasadena April, 1977

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I. INTRODUCTION

During the course of the past several years while I've been a graduate student at Caltech, the hope that somewhere, somebody's gravitational wave detector will actually detect gravitational waves by the middle of the 1980's has improved considerably. (However, I should remark that the "... hope that gravitational waves will be detected within the next ten years ..." is a statement that has been heard for at least the past ten years!)

The work we've done in Papers 1, 2 and 3 was intended to provide an additional theoretical tool by which gravitational radiation sources may be analyzed. Any tool, regardless of how sophisticated it is or how much promise it offers, is worthless unless applied to solve a problem. We applied the mathematical tools we constructed to gain further insight into the bremsstrahlung problem, and had some success. The interesting results are contained in Paper 3, but we paid a price of some rather detailed and tedious preliminary work. It showed that the 'post-linear' formalism was usable and effective however it is safe to say that it did not provide a calculational shortcut to the final results. Since the contents of the papers speak for themselves, my remaining remarks in this section are intended to shed some light on how the work we did actually came to pass.

In 1972 K.S. Thorne (henceforth cited as Kip) and I decided to analyze in detail 'The Gravitational Bremsstrahlung Problem' with the hope that we could remove the previous restrictions on the ratio of the masses and any constraints on the velocities. Of course we only held out hope for the weak-field case. After about a year of probing (up several blind alleys) we realized that our initial approach using flat-space propagators was inadequate. We kept getting divergent volume integrals for the gravitational stress. The desire to solve the problem and find an answer (i.e. our personalities) forced us to return to the mathematical foundations of radiation problems and devise a scheme that would work. The result of all this hard work and fancy mathematics is Paper 1. The interesting physics was still to come.

By early 1975 Kip and I had the 'tools' to actually " Plug-In-

And-Grind " away at the physics problem we set out to solve. The results of this somewhat tedious task (usual polite understatement) are laid out in Paper 2 of the thesis - Paper III. of the published series. We never intended to have a Paper 2 just with formulas and a separate paper just with results - but the length and complexity of the details (and the page limitations of the Astrophisical Journal) forced us into it.

While all this was going on Walter De Logi (henceforth cited as Walter) and I started looking into the bremsstrahlung problem from the quantum point of view using diagrams. This approach was partially motivated by Richard Feynman's questions and suggestions during my oral candidacy exam in February 1975. Naturally our first results using diagrams had nothing to do with the problem we originally set out to solve. Walter and I digressed (after a seminar given at Caltech by Phil Peters), when we realized we could do plane wave scattering problems faster and easier using diagrams - with the added exciting bonus of being able to scatter waves from a scatterer with angular momentum - something that Peters could not do. The results of a few minutes of creative insight and several months of work is what turned out to be Paper 4.

Concurrent with the work with Walter, the analytical and computer work for Paper 3 was going on. I spent several months looking for a theoretical and/or programming mistake while trying to get the numerical results of our Paper 2 to agree with the results published by Peters - I finally concluded (correctly) that yes, there was a mistake somewhere. Peters had a previously undetected typographycal error in his paper and the discrepancy nicely vanished after a simple correction. At last, by the spring of 1977 we had the results to 'THE Gravitational Bremsstrahlung Problem' which is included here for the enjoyment of the reader. I think it contains the most interesting material in the whole thesis, and as it is written here is material which will ultimately be published in a joint paper with Kip.

For completeness, i.e. a desire to beat the bremsstrahlung problem to death from all directions, Walter and I tried, but could not find sufficient time to complete the problem using a quantum approach. Unfortunately that piece of work is not part of this thesis but will be published seperately, as soon as it's finished.

PAPER 1

THE ASTROPHYSICAL JOURNAL, 200: 245–262, 1975 September 1

THE GENERATION OF GRAVITATIONAL WAVES. I. WEAK-FIELD SOURCES*

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ABSTRACT

This paper derives (§§ II–IV) and summarizes (§ VI) a new "plug-in-and-grind" formalism (i.e., an algorithm) for calculating the gravitational waves emitted by any system with weak internal gravitational fields. If the internal fields have negligible influence on the system's motions, then the new formalism reduces to standard "linearized theory." Whether or not gravity affects the motions, if the motions are slow and internal stresses are weak, then the new formalism reduces to the standard "quadrupole-moment formalism" (§ V). In the general case the new formalism expresses the radiation in terms of a retarded Green's function for slightly curved spacetime, and then breaks the Green's function integral into five easily understood pieces: *direct radiation*, produced directly by the motions of the source; *whump radiation*, produced by the "gravitational stresses" of the source; *transition radiation*, produced by a time-changing time delay ("Shapiro effect") in the propagation of the nonradiative, $1/r$ field of the source; *focusing radiation*, produced when one portion of the source focuses, in a time-dependent way, the nonradiative field of another portion of the source; and *tail radiation*, produced by "back-scatter" of the nonradiative field in regions of focusing.

Subject headings: gravitation — relativity

I. INTRODUCTION

a) Introduction to This Series of Papers

Thanks to the pioneering work of Joseph Weber (1960, 1969), "gravitational-wave astronomy" may be a reality by 1980. Although Weber's "events" may turn out to be nongravitational in origin, second-generation detectors of the Weber "resonant-bar" type, with amplitude sensitivities roughly 100-fold better than today's bars, are now under construction (Braginsky 1974; Fairbank and Hamilton, as described in Boughn *et al.* 1974); and third-generation detectors are being discussed. The third generation should be able to detect and study the gravitational-wave bursts generated several times per year by supernovae in the Virgo cluster of galaxies. Detectors with other designs may succeed in detecting waves from pulsars (see, e.g., Braginsky and Nazarenko 1971) and from near-encounters of stars in dense star clusters (gravitational bremsstrahlung; see, e.g., Zel'dovich and Ponarev 1974). And, of course, totally unexpected sources may be detected. (For reviews of the prospects for gravitational-wave astronomy see Misner 1974; Rees 1974; and Press and Thorne 1972.)

In preparation for the era of gravitational-wave astronomy, our Caltech research group has embarked on a new project: We seek (1) to elucidate the realms of validity of the standard wave-generation formulae; (2) to devise new techniques for calculating gravitational-wave generation with new realms of validity; and (3) to calculate the waves generated by particular models of astrophysical systems. Throughout this project we shall confine ourselves to general relativity theory.

Most past calculations of gravitational-wave generation use one of three formalisms: (1) "linearized theory" or its quantum-theory analog; (2) the "quadrupole-moment formalism"; (3) "first-order perturbations of stationary, fully relativistic spacetimes."

"Linearized theory" is the formalism obtained by linearizing general relativity about flat spacetime (see, e.g., chapters 18 and 35 of Misner, Thorne, and Wheeler 1973—cited henceforth as MTW). It is also the unique linear spin-two field theory of gravitation in flat spacetime—and as such it has a simple quantum-theory formulation. (For references and overview see, in MTW, § 7.1, box 7.1, and part 5 of box 17.2). Linearized theory is typically used to calculate wave generation when the source's self-gravity has negligible influence on its motions (e.g., waves from spinning rods and from electromagnetic fields in a cavity). In this paper we shall devise a new wave-generation formalism valid for any system with small but nonnegligible self-gravity; and in Paper III (Kovács and Thorne 1975) we shall use that formalism to calculate the gravitational bremsstrahlung produced when two stars fly past each other with large impact parameter, but with arbitrary relative masses and velocities.

The "quadrupole-moment formalism" (in which the wave amplitude is proportional to the second time derivative of the source's mass quadrupole moment) dates back to Einstein (1918), and has been canonized by Landau

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and Lifshitz (1951). The derivations of this formalism which we find in the literature are valid only for systems with slow internal motions and weak (but nonnegligible) internal gravitational fields (see, e.g., the post$^{5/2}$-Newtonian derivation by Chandrasekhar and Esposito 1970, the matched-asymptotic-expansion derivation by Burke 1971, and the de Donder gauge derivation by Landau and Lifshitz 1951 as made more explicit in chapter 36 of MTW). However, a detailed analysis given in Paper II (Thorne 1975) shows that only the slow-motion assumption is needed: the quadrupole-moment formalism is valid for any slow-motion system, regardless of its internal field strengths. Paper II also extends that formalism to include the radiation produced by all of the source's other moments (both "mass" moments and "current" moments); and it derives formulae in terms of the moments for the near-zone fields, the radiation field, the radiation reaction, and the energy, momentum, and angular momentum carried off by the waves. In a forthcoming paper Thorne and Żytkow (1976) will use the extended formalism of Paper II to calculate the "current-quadrupole" gravitational waves produced by torsional oscillations of neutron stars.

"First-order perturbations of stationary, fully relativistic spacetimes" is a technique that has been used extensively in recent years to analyze waves from "fast-motion" oscillations of black holes and neutron stars, and from particles moving in the Schwarzschild and Kerr gravitational fields. (For reviews, see Press 1974, Ruffini 1973, and § 36.5 of MTW; see also the recent paper by Chung 1973.) It is not yet clear whether our project will delve into this technique.

b) Overview of This Paper

In this paper we confine attention to systems with weak internal gravitational fields. Section II rewrites the exact Einstein field equations in a non-covariant form ("de Donder form") that is amenable to weak-field approximations. Section III gives a systematic account of approximate, weak-field formalisms based on the exact de Donder form of the field equations—including the accuracy of the various formalisms and their relationships to each other. Section IV*a* applies the analysis of § III to astrophysical systems, and concludes that, when analyzing their structure and evolution, one must typically calculate the stress-energy tensor ${}_2T^{\mu\nu}$ and gravitational field ${}_1\bar{h}^{\mu\nu}$ with accuracies:

$$|(\text{error in } {}_2T^{\mu\nu})/{}_2T^{00}| \lesssim \epsilon^2\,, \qquad |(\text{error in } {}_1\bar{h}^{\mu\nu})/{}_1\bar{h}^{00}| \lesssim \epsilon\,,$$

$$\epsilon \equiv (\text{typical value of } {}_1\bar{h}^{00} \text{ inside source}) \sim (\text{mass of source})/(\text{size of source})\,.$$

Section IV*a* also concludes that the external gravitational field ${}_2\bar{h}^{\mu\nu}$ must typically be calculated to accuracy

$$|(\text{error in } {}_2\bar{h}^{\mu\nu})/{}_2\bar{h}^{00}| \lesssim \epsilon^2$$

if one desires reasonable accuracy in the radiative part of that field.

Section IV*b* presents a "postlinear" formalism for calculating a system's structure and evolution (${}_2T^{\mu\nu}$ and ${}_1\bar{h}^{\mu\nu}$) to the desired accuracy; and § IV*c* derives a formula for the higher-accuracy external field (${}_2\bar{h}^{\mu\nu}$), which contains the radiation. Section V shows how the resulting formalism, when applied to slow-motion systems, reduces to the standard "quadrupole-moment formalism."

We recommend that, before tackling the rest of this paper, the reader peruse § VI. That section summarizes our postlinear formalism and our formula for the external (radiation) field.

The "guts" of this paper, in terms of complex calculations, reside in the Green's function manipulations of § IV*c*. Our particular way of handling the Green's functions is motivated in Appendix A, and has been influenced by the following papers: DeWitt and Brehme (1960) (exact Green's functions for scalar and vector wave equations in curved spacetime); Robaschik (1963) (exact Green's function for tensor wave equation in curved spacetime); John (1973*a*, *b*), Bird (1974), and especially Peters (1966) (Green's functions in weakly curved spacetime). Although these papers had much influence on us, our specific manipulations are so different that we have found it impossible to trace the details of that influence in our writeup.

II. EXACT GENERAL RELATIVITY, REWRITTEN IN "WEAK-FIELD LANGUAGE"

We begin by writing the exact, nonlinear Einstein field equations in an arbitrary coordinate system in the form (§ 20.3 of MTW; § 100 of Landau and Lifshitz 1962)

$$H^{\mu\alpha\nu\beta}_{\text{L-L},\alpha\beta} = 16\pi(-g)(T^{\mu\nu} + t^{\mu\nu}_{\text{L-L}}) \tag{1}$$

where "L-L" means "Landau-Lifshitz," and where

$$H^{\mu\alpha\nu\beta}_{\text{L-L}} \equiv \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta}\,; \tag{2a}$$

$$\mathfrak{g}^{\mu\nu} \equiv (-g)^{1/2}g^{\mu\nu}\,, \qquad (-g) = -\det\|g_{\mu\nu}\| = -\det\|\mathfrak{g}^{\mu\nu}\|\,; \tag{2b}$$

$$\begin{aligned} t^{\alpha\beta}_{\text{L-L}} \equiv [16\pi(-g)]^{-1}\{&\mathfrak{g}^{\alpha\beta}{}_{,\lambda}\mathfrak{g}^{\lambda\mu}{}_{,\mu} - \mathfrak{g}^{\alpha\lambda}{}_{,\lambda}\mathfrak{g}^{\beta\mu}{}_{,\mu} + \tfrac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\mathfrak{g}^{\lambda\nu}{}_{,\rho}\mathfrak{g}^{\rho\mu}{}_{,\nu} \\ &- (g^{\alpha\lambda}g_{\mu\nu}\mathfrak{g}^{\beta\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda} + g^{\beta\lambda}g_{\mu\nu}\mathfrak{g}^{\alpha\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda}) + g_{\lambda\mu}g^{\nu\rho}\mathfrak{g}^{\alpha\lambda}{}_{,\nu}\mathfrak{g}^{\beta\mu}{}_{,\rho} \qquad \text{(2c)} \\ &+ \tfrac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\mathfrak{g}^{\nu\tau}{}_{,\lambda}\mathfrak{g}^{\rho\sigma}{}_{,\mu}\}\,. \end{aligned} \tag{2}$$

The equations of motion for the material stress-energy tensor $T^{\mu\nu}$ follow directly from the field equations (1) and can be written in the equivalent forms

$$T^{\alpha\beta}{}_{;\beta} = 0\,, \qquad [(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\rm L-L})]_{,\beta} = 0\,. \tag{3}$$

[Here and throughout this series of papers we use the notation and sign conventions of MTW; in particular $c = G = 1$; $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are the components of the metric; commas and ∂'s denote partial derivatives, $Y_{,\alpha} = \partial_\alpha Y = \partial Y/\partial x^\alpha$; semicolons denote covariant derivatives with respect to the metric $g_{\alpha\beta}$; and our signature is $(- + + +)$.]

Now, and henceforth in this paper, we impose three restrictions on our analysis: (1) We confine attention to systems with "weak internal gravity"—i.e., to systems throughout which one can introduce nearly Lorentz coordinates. (2) We confine ourselves to "isolated systems"—i.e., to systems that are surrounded by a region ("local wave zone"), much larger than a characteristic wavelength of the emitted waves, in which all waves are outgoing and in which external masses have negligible influence on the gravitational field. (3) We restrict our analysis to the interior of the source and its local wave zone, and throughout these regions we use nearly Lorentz, asymptotically flat coordinates, specialized to satisfy the de Donder gauge condition.

Mathematically, these restrictions state that the "gravitational field"

$$\bar{h}^{\mu\nu} \equiv -\mathfrak{g}^{\mu\nu} + \eta^{\mu\nu} \tag{4}$$

has the properties

$$|\bar{h}^{\mu\nu}| \ll 1 \quad \text{everywhere}\,, \tag{5a}$$

$$|\bar{h}^{\mu\nu}| \sim 1/r \quad \text{as} \quad r \to \infty\,, \qquad \text{where } r = (x^2 + y^2 + z^2)^{1/2}\,, \tag{5b}$$

$$\bar{h}^{\mu\nu} \text{ is devoid of incoming waves at } r \to \infty\,, \tag{5c}$$

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \quad \text{(de Donder condition)}\,. \tag{5d}$$

With these restrictions, the exact Einstein field equations (1) take on the form

$$\Box_s \bar{h}^{\mu\nu} = -16\pi(-g)^{1/2}(T^{\mu\nu} + t^{\mu\nu}_{\rm L-L}) - (-g)^{-1/2}\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}\,. \tag{6}$$

Here $\Box_s$ is the wave operator for *scalar* fields in the curved spacetime described by the metric $g_{\alpha\beta}$:

$$\Box_s \equiv (-g)^{-1/2}\partial_\alpha[(-g)^{1/2}g^{\alpha\beta}\partial_\beta]\,. \tag{7}$$

[Appendix A, which is best read after one has finished reading the rest of the paper, explains why we write the field equations in terms of $\Box_s$ rather than in terms of some other wave operator such as $\Box_f \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ (the flat-space wave operator) or $\Box_t$ (the curved-space wave operator for tensor fields).]

Equations (2b)–(7) are the exact, nonlinear equations of general relativity for any isolated, weak-field system—but they are written in a very special coordinate system rather than in generally covariant form.

Because $|\bar{h}^{\mu\nu}| \ll 1$, we can express each quantity in our formalism, except $T^{\mu\nu}$, as a power series in $\bar{h}^{\mu\nu}$. When writing down such a power series, it is convenient to raise and lower indices of $\bar{h}^{\mu\nu}$ with the Minkowski metric $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$:

$$\bar{h}_\alpha{}^\nu \equiv \eta_{\alpha\mu}\bar{h}^{\mu\nu}\,, \qquad \bar{h}_{\alpha\beta} \equiv \eta_{\alpha\mu}\eta_{\beta\nu}\bar{h}^{\mu\nu}\,, \qquad \bar{h} \equiv \bar{h}_\alpha{}^\alpha\,, \quad \text{etc.} \tag{8a}$$

It is also convenient to define a "trace-reversed" gravitational field $h^{\mu\nu}$ by

$$h^{\mu\nu} \equiv \bar{h}^{\mu\nu} - \tfrac{1}{2}\bar{h}\eta^{\mu\nu}\,, \tag{8b}$$

and to raise and lower its indices, like those of $\bar{h}^{\mu\nu}$, with the Minkowski metric. Note that equation (8b) implies

$$h \equiv h_\alpha{}^\alpha = -\bar{h}\,, \qquad \bar{h}^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2}h\eta^{\mu\nu}\,. \tag{8c}$$

To derive the explicit power series expansions for $\mathfrak{g}^{\mu\nu}$, $g^{\mu\nu}$, $g_{\mu\nu}$, etc., one can proceed as follows. Equation (4) is the desired expansion for $\mathfrak{g}^{\mu\nu}$. It contains only two terms:

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} - \bar{h}^{\mu\nu}\,. \tag{9a}$$

The expansion for the metric determinant $(-g)$ is obtained by inserting expression (9a) into the second of equations (2b):

$$(-g) = -\det\|\mathfrak{g}^{\mu\nu}\| = -\det\|\eta^{\mu\nu} - \bar{h}^{\mu\nu}\| = 1 - \bar{h} + \tfrac{1}{2}[(\bar{h})^2 - \bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}] + O[(\bar{h})^3]\,. \tag{9b}$$

The contravariant components of the metric are then obtained by inserting (9a, b) into the first of equations (2b):

$$g^{\mu\nu} = (-g)^{-1/2}\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} - (\bar{h}^{\mu\nu} - \tfrac{1}{2}\bar{h}\eta^{\mu\nu}) - \tfrac{1}{2}\bar{h}\bar{h}^{\mu\nu} + \tfrac{1}{8}\eta^{\mu\nu}[(\bar{h})^2 + 2\bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}] + O[(\bar{h})^3]\,; \tag{9c}$$

and the covariant components are obtained as the matrix inverse of these contravariant components:

$$g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \tfrac{1}{2}\bar{h}\eta_{\mu\nu} + \bar{h}_{\mu\alpha}\bar{h}^{\alpha}{}_{\nu} - \tfrac{1}{2}\bar{h}\bar{h}_{\mu\nu} + \tfrac{1}{8}\eta_{\mu\nu}(\bar{h}^2 - 2\bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}) + O[(\bar{h})^3] \,. \tag{9d}$$

The connection coefficients $\Gamma^{\mu}{}_{\alpha\beta}$, which appear in the usual expression $T^{\mu\nu}{}_{;\nu} = 0$ for the equations of motion, are obtained by inserting expansions (9c, d) into the standard formula

$$\Gamma^{\mu}{}_{\alpha\beta} = \tfrac{1}{2}g^{\mu\gamma}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}) = \tfrac{1}{2}(\bar{h}^{\mu}{}_{\alpha,\beta} + \bar{h}^{\mu}{}_{\beta,\alpha} - \bar{h}_{\alpha\beta}{}^{,\mu}) - \tfrac{1}{4}(\delta^{\mu}{}_{\alpha}\bar{h}_{,\beta} + \delta^{\mu}{}_{\beta}\bar{h}_{,\alpha} - \eta_{\alpha\beta}\bar{h}^{,\mu}) + O[(\bar{h})^2] \,. \tag{9e}$$

Similarly, the scalar-wave operator $\square_s$ is obtained by inserting expansions (9b, c) into equation (7), and using the de Donder gauge condition (5d) to simplify:

$$\square_s = \eta^{\alpha\beta}\partial_\alpha\partial_\beta - (\bar{h}^{\alpha\beta} - \tfrac{1}{2}\bar{h}\eta^{\alpha\beta})\partial_\alpha\partial_\beta + O[(\bar{h})^2] \,; \tag{9f}$$

and the components of the Landau-Lifshitz pseudotensor are obtained by inserting expansions (9a,b, c, d) and the de Donder condition (5d) into equation (2c):

$$\begin{aligned} t^{\alpha\beta}_{\text{L-L}} = (16\pi)^{-1}\{&\tfrac{1}{2}\eta^{\alpha\beta}\eta_{\lambda\mu}\bar{h}^{\lambda\nu}{}_{,\rho}\bar{h}^{\rho\mu}{}_{,\nu} + \eta_{\lambda\mu}\eta^{\nu\rho}\bar{h}^{\alpha\lambda}{}_{,\nu}\bar{h}^{\beta\mu}{}_{,\rho} - (\eta^{\alpha\lambda}\eta_{\mu\nu}\bar{h}^{\beta\nu}{}_{,\rho}\bar{h}^{\mu\rho}{}_{,\lambda} + \eta^{\beta\lambda}\eta_{\mu\nu}\bar{h}^{\alpha\nu}{}_{,\rho}\bar{h}^{\mu\rho}{}_{,\lambda}) \\ &+ \tfrac{1}{8}(2\eta^{\alpha\lambda}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\lambda\mu})(2\eta_{\nu\rho}\eta_{\sigma\tau} - \eta_{\rho\sigma}\eta_{\nu\tau})\bar{h}^{\nu\tau}{}_{,\lambda}\bar{h}^{\rho\sigma}{}_{,\mu}\} + O[(\bar{h})^3] \,. \end{aligned} \tag{9g}$$

Henceforth in this paper we shall regard $\mathfrak{g}^{\mu\nu}$, $(-g)$, $g^{\mu\nu}$, $g_{\mu\nu}$, $\Gamma^{\mu}{}_{\alpha\beta}$, $\square_s$, and $t^{\alpha\beta}_{\text{L-L}}$ as shorthand notation for the *infinite* power series expansions, whose first few terms are shown in equations (9). Given these expansions, the full content of general relativity is embodied in the equations of motion for the matter

$$T^{\mu\nu}{}_{,\nu} = -\Gamma^{\mu}{}_{\alpha\nu}T^{\alpha\nu} - \Gamma^{\nu}{}_{\alpha\nu}T^{\mu\alpha} \tag{10a}$$

and the Einstein field equations

$$\square_s\bar{h}^{\mu\nu} = -16\pi(-g)^{1/2}(T^{\mu\nu} + t^{\mu\nu}_{\text{L-L}}) - (-g)^{-1/2}\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha} \,. \tag{10b}$$

Henceforth we shall not impose the gauge conditions $\bar{h}^{\mu\nu}{}_{,\nu} = 0$; rather, we shall regard them as consequences of the equations of motion (10a) and the field equations (10b).

III. APPROXIMATION FORMALISMS FOR WEAK-FIELD SYSTEMS[1]

The formulation of general relativity embodied in equations (9) and (10) is an excellent starting point for derivations of weak-field approximation formalisms. To get a formalism of desired accuracy, one can simply truncate each infinite series appearing in equations (9) and (10) at the appropriate point.

We shall describe the accuracy of a formalism in terms of its "errors" (i.e., the deviations of its solutions from exact solutions of the exact equations [9] and [10]). In discussing errors, we shall use the small dimensionless parameter

$$\epsilon \equiv (\text{characteristic size of } \bar{h}^{\mu\nu} \text{ inside the system}) \,. \tag{11}$$

If the system is a dynamically changing lump of matter with mass M and linear size L (e.g., a pulsating star or an exploding atomic bomb), then

$$\epsilon \sim M/L \,.$$

If the system is several lumps with masses m and sizes l, separated by distances $b \gg l$ (e.g., a binary star system or two stars flying past each other), then

$\epsilon \sim m/b$ if one is interested only in the relative motions of the lumps,

$\epsilon \sim m/l$ if one is also interested in the internal structure and dynamics of the lumps.

We shall characterize every weak-field approximation formalism by two integers n_T and n_h. These "order indices" tell us the magnitude of the errors made by the formalism:[2]

$$|(\text{errors in } T^{\mu\nu})/T^{00}| \sim \epsilon^{n_T} \tag{12a}$$

$$|(\text{errors in } \bar{h}^{\mu\nu})/\bar{h}^{00}| \sim \epsilon^{n_h} \,. \tag{12b}$$

[1] This section is closely related to the Havas-Goldberg (1962) analysis of approximation formalisms for equations of motion of point masses.

[2] Note that all of the $|T^{\mu\nu}|$ are $\lesssim T^{00}$, and consequently all of the $|\bar{h}^{\mu\nu}|$ are $\lesssim \bar{h}^{00}$. This fact dictates the form of equations (12).

For example, a formalism of order $(n_T, n_h) = (1, 1)$ makes fractional errors of order ϵ in both the stress-energy tensor and the gravitational field, while a formalism of order (2, 1) makes fractional errors ϵ^2 in $T^{\mu\nu}$ and ϵ in $\bar{h}^{\mu\nu}$.

Errors in $\bar{h}^{\mu\nu}$, when fed into the equations of motion (10a), produce errors in $T^{\mu\nu}$; and similarly, errors in $T^{\mu\nu}$, when fed into the field equations (10b), produce errors in $\bar{h}^{\mu\nu}$. This feeding process places constraints on the order indices (n_T, n_h) of any self-consistent approximation formalism. The constraints are revealed explicitly by an order-of-magnitude analysis of equations (10a, b):

Consider a weak-field system with characteristic field strength ϵ and characteristic length-time scale l. Below each term of equations (10a, b), write the order of magnitude of that term:

$$T^{\mu\nu}{}_{,\nu} = -\Gamma^{\mu}{}_{\alpha\nu}T^{\alpha\nu} - \Gamma^{\nu}{}_{\alpha\nu}T^{\mu\alpha} \tag{13a}$$

$$(T^{00}/l) \qquad (\epsilon/l)(T^{00}) \qquad (\epsilon/l)(T^{00})$$

$$\Box_s \bar{h}^{\mu\nu} = -16\pi(-g)^{1/2}T^{\mu\nu} - 16\pi(-g)^{1/2}t^{\mu\nu}_{\mathrm{L-L}} - (-g)^{-1/2}\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}\,. \tag{13b}$$

$$(\epsilon/l^2) \qquad T^{00} \qquad (\epsilon^2/l^2) \qquad (\epsilon^2/l^2)$$

Equation (13a) shows that fractional errors ϵ^{n_h} in $\bar{h}^{\mu\nu}$ produce fractional errors ϵ^{n_h+1} in $T^{\mu\nu}$; i.e., $\epsilon^{n_T} \geq \epsilon^{n_h+1}$; i.e.,

$$n_T \leq n_h + 1\,. \tag{14a}$$

Equation (13b)—together with the order-of-magnitude field equation $T^{00} \sim \epsilon/l^2$—shows that fractional errors ϵ^{n_T} in $T^{\mu\nu}$ produce fractional errors ϵ^{n_T} in $\bar{h}^{\mu\nu}$; i.e., $\epsilon^{n_h} \geq \epsilon^{n_T}$; i.e.,

$$n_h \leq n_T\,. \tag{14b}$$

Equations (14a, b) can be restated as the following constraints on the order indices of any self-consistent approximation formalism:

$$n_h = n_T - 1 \quad \text{or} \quad n_h = n_T\,. \tag{15}$$

In other words, *the order (n_T, n_h) of any approximation formalism must be either $(n, n-1)$* or *(n, n) for some integer n.*

Suppose that a specific system has been analyzed using an approximation formalism of order $(n, n-1)$. Denote by ${}_nT^{\mu\nu}(x^\alpha)$ and ${}_{(n-1)}\bar{h}^{\mu\nu}(x^\alpha)$ the explicit expressions obtained in that analysis for the system's stress-energy tensor and gravitational field. From these expressions it is straightforward to generate an "improved" gravitational field ${}_n\bar{h}^{\mu\nu}(x^\alpha)$ with fractional errors ϵ^n. The key to doing this is the structure of the field equations (10b): In these field equations, fractional errors of order ϵ^{n-1} in $\bar{h}^{\mu\nu}$ produce fractional errors of order ϵ^n in both $\Box_s$ and the expression

$$-16\pi(-g)^{1/2}(T^{\mu\nu} + t^{\mu\nu}_{\mathrm{L-L}}) - (-g)^{-1/2}\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}\,.$$

Hence, ${}_n\bar{h}^{\mu\nu}(t, x)$ satisfies the differential equation

$${}_{(n,n-1)}\Box_s\,{}_n\bar{h}^{\mu\nu} = {}_{(n,n-1)}[-16\pi(-g)^{1/2}(T^{\mu\nu} + t^{\mu\nu}_{\mathrm{L-L}}) - (-g)^{-1/2}\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}]\,. \tag{16}$$

Here the prefix $(n, n-1)$ means that a quantity is to be calculated, with fractional error ϵ^n, using ${}_nT^{\mu\nu}$ and ${}_{(n-1)}\bar{h}^{\mu\nu}$. This inhomogeneous, linear wave equation for ${}_n\bar{h}^{\mu\nu}$ can be solved using the retarded scalar Green's function for curved spacetime (DeWitt and Brehme 1960):

$${}_{(n-1)}G(\mathscr{P}', \mathscr{P}) \equiv \begin{bmatrix}\text{(the retarded scalar Green's function for the curved spacetime with the metric } {}_{(n-1)}g_{\mu\nu} \\ \text{of the } (n, n-1) \text{ approximation—a Green's function with fractional errors } \epsilon^n\end{bmatrix}. \tag{17}$$

The result is

$${}_n\bar{h}^{\mu\nu}(\mathscr{P}) = \int {}_{(n,n-1)}[16\pi(-g)(T^{\mu\nu} + t^{\mu\nu}_{\mathrm{L-L}}) + \bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}]_{\mathscr{P}'}\,{}_{(n-1)}G(\mathscr{P}', \mathscr{P})d^4x'\,. \tag{18}$$

This paragraph can be summarized as follows: *Any approximation formalism of order $(n, n-1)$, when augmented by equation* (18) *for ${}_n\bar{h}^{\mu\nu}$, becomes an approximation formalism of order (n, n).*

Special relativity and linearized theory provide a simple example of the above remarks: Special relativity is the approximation formalism of order (1, 0) which one obtained by the extreme truncation process of setting $\bar{h}^{\mu\nu} = 0$ in equations (9) and (10):

$${}_0\bar{h}^{\mu\nu} = 0\,, \qquad {}_0t^{\mu\nu}_{\mathrm{L-L}} = 0\,, \qquad {}_0g_{\mu\nu} = \eta_{\mu\nu}\,, \qquad {}_1T^{\mu\nu}{}_{,\nu} = 0\,. \tag{19a}$$

The retarded scalar Green's function for a space with metric ${}_0g_{\mu\nu} = \eta_{\mu\nu}$ is

$$ {}_0G(\mathscr{P}', \mathscr{P}) = (4\pi)^{-1}\delta_{\rm ret}[\tfrac{1}{2}\eta_{\rho\sigma}(x^\rho - x^{\rho'})(x^\sigma - x^{\sigma'})] . \tag{19b} $$

(Here $\delta_{\rm ret}$ is zero if $\mathscr{P}$ lies in the causal past of $\mathscr{P}'$, and it is the Dirac delta function otherwise.) Hence, equation (18)—by which one must augment special relativity in order to obtain a formalism of order (1, 1)—has the form

$$ {}_1\bar{h}^{\mu\nu} = 4\int {}_1T^{\mu\nu}(\mathscr{P}')\delta_{\rm ret}[\tfrac{1}{2}\eta_{\rho\sigma}(x^\rho - x^{\rho'})(x^\sigma - x^{\sigma'})]d^4x' = 4\int \frac{{}_1T^{\mu\nu}(x^0 - |x - x'|, x')}{|x - x'|}\, d^3x' . \tag{20} $$

The resulting (1, 1) formalism (eqs. [19] augmented by eq. [20]) is the "linearized theory of gravity" (see, e.g., § 7.1, box 7.1, and chapter 18 of MTW).

Newtonian theory and the "quadrupole-moment formalism for wave generation" are another example. Newtonian theory is the weak-field formalism of order (2, 1) which one obtains by not only truncating each series that appears in equations (9) and (10), but by also imposing the slow-motion and small-stress assumptions

$$ v^2 \equiv |T^{0j}e_j/T^{00}|^2 \lesssim \epsilon , \qquad |T^{ij}/T^{00}| \lesssim \epsilon , \tag{21} $$

$$ \text{(size of system)/(characteristic time scale of changes)} \lesssim \epsilon^{1/2} . $$

Equation (18), by which one augments Newtonian theory in order to obtain a formalism of order (2, 2), has the form, when evaluated in the radiation zone

$$ {}_2\bar{h}_{ij}^{\rm TT}(t, x) = (2/r)\ddot{I}_{ij}^{\rm TT}(t - r) = \text{(gravitational radiation field)} . \tag{22} $$

Here I_{ij} is the reduced quadrupole moment of the source, and TT denotes "transverse-traceless" part. This is the standard wave-generation formula of the quadrupole-moment formalism; see chapter 36 of MTW.

IV. WAVE GENERATION BY A WEAK-FIELD SYSTEM

a) Motivation

Weak-field systems are of two types: those with negligible self-gravitational forces (rotating laboratory rods, microwave cavities, etc.), and those whose internal motions are significantly influenced by self-gravity (pulsating stars, binary star systems, etc.).

For a system with negligible self-gravity, special relativity gives a fairly accurate description of the internal motions; and, consequently, linearized theory [the (1, 1) formalism obtained by attaching eq. (18) or (20) onto special relativity] gives a fairly accurate description of gravitational-wave generation.

For most weak-field astrophysical systems, self-gravitational forces are important. In this case, when analyzing a system's internal motions, one must use a formalism of order (2, 1); and when calculating the waves those motions generate, one must augment the (2, 1) formalism by equation (18), thereby raising its order to (2, 2).[3] If the system has slow internal motions and weak internal stresses, Newtonian theory [order (2, 1)] will suffice for analyzing its motions, and the quadrupole-moment formalism [order (2, 2)] will suffice for wave generation. However, for analyzing fast-motion systems (e.g., two stars flying past each other with high velocity and deflecting each other slightly—the relativistic bremsstrahlung problem), one needs unrestricted (2, 1) and (2, 2) formalisms. The objective of the next two sections is to derive such formalisms.

b) The Postlinear Formalism

A weak-field formalism of order (2, 1), unrestricted by any constraints on velocities or stresses, can be obtained by truncating equations (9) and (10) at the appropriate order:

$$ {}_1\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} - {}_1\bar{h}^{\mu\nu} , \tag{23a} $$

$$ (-{}_1g) = 1 - {}_1\bar{h} = 1 + {}_1h , \tag{23b} $$

$$ {}_1g^{\mu\nu} = \eta^{\mu\nu} - {}_1h^{\mu\nu} \quad (\text{where } {}_1h^{\mu\nu} = {}_1\bar{h}^{\mu\nu} - \tfrac{1}{2}{}_1\bar{h}\eta^{\mu\nu}) , \tag{23c} $$

$$ {}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu} , \tag{23d} $$

$$ {}_1\Gamma^\mu{}_{\alpha\beta} = \tfrac{1}{2}({}_1h^\mu{}_{\alpha,\beta} + {}_1h^\mu{}_{\beta,\alpha} - {}_1h_{\alpha\beta}{}^{,\mu}) , \tag{23e} $$

$$ {}_1\Box_s = (\eta^{\alpha\beta} - {}_1h^{\alpha\beta})\partial_\alpha\partial_\beta , \tag{23f} $$

[3] In very special cases second-order gravitational forces may be as important, for the system's motions, as first-order forces. An example is a radially pulsating, weak-field star with adiabatic index very near 4/3 (Chandrasekhar 1964); see also the discussion accompanying equations (61) below. When analyzing such systems, one needs formalisms of order (3, 2) and (3, 3).

$$ {}_1t^{\alpha\beta}_{\mathrm{L-L}} = (16\pi)^{-1}\{\tfrac{1}{2}\eta^{\alpha\beta}\eta_{\lambda\mu}\,{}_1\bar{h}^{\lambda\nu}{}_{,\rho}\,{}_1\bar{h}^{\rho\mu}{}_{,\nu} + \eta_{\lambda\mu}\eta^{\nu\rho}\,{}_1\bar{h}^{\alpha\lambda}{}_{,\nu}\,{}_1\bar{h}^{\beta\mu}{}_{,\rho} - (\eta^{\alpha\lambda}\eta_{\mu\nu}\,{}_1\bar{h}^{\beta\nu}{}_{,\rho}\,{}_1\bar{h}^{\mu\rho}{}_{,\lambda} + \eta^{\beta\lambda}\eta_{\mu\nu}\,{}_1\bar{h}^{\alpha\nu}{}_{,\rho}\,{}_1\bar{h}^{\mu\rho}{}_{,\lambda}) + \tfrac{1}{8}(2\eta^{\alpha\lambda}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\lambda\mu})(2\eta_{\nu\rho}\eta_{\sigma\tau} - \eta_{\rho\sigma}\eta_{\nu\tau})\,{}_1\bar{h}^{\nu\tau}{}_{,\lambda}\,{}_1\bar{h}^{\rho\sigma}{}_{,\mu}\}\,, \tag{23g} $$

$$ {}_2T^{\mu\nu}{}_{,\nu} = -\,{}_1\Gamma^{\mu}{}_{\alpha\nu}\,{}_2T^{\alpha\nu} - {}_1\Gamma^{\nu}{}_{\alpha\nu}\,{}_2T^{\mu\alpha}\,, \tag{24a} $$

$$ \eta^{\alpha\beta}\,{}_1\bar{h}^{\mu\nu}{}_{,\alpha\beta} = -16\pi\,{}_2T^{\mu\nu}\,. \tag{24b} $$

We shall refer to the formalism described by these equations as the "postlinear formalism." To analyze a system using the postlinear formalism, one must first specify the functional dependence of the stress-energy tensor ${}_2T^{\mu\nu}$ on the system's nongravitational variables (e.g., density, pressure, velocities, electromagnetic field tensor, ...) and on the gravitational field ${}_1\bar{h}^{\mu\nu}$; and one must then solve equations (24a, b) simultaneously for the system's motions (${}_2T^{\mu\nu}$ accurate up to fractional errors $\sim\epsilon^2$) and for the gravitational field (${}_1\bar{h}^{\mu\nu}$ accurate up to fractional errors $\sim\epsilon$). Paper III will carry out such a calculation for the motion of two stars of arbitrary relative masses and velocities, which fly past each other with large impact parameter.

c) The Postlinear Wave-Generation Formalism

Having calculated a system's internal structure and motions using the postlinear formalism, one can then calculate the gravitational waves the system emits, ${}_2\bar{h}^{\mu\nu}$, by evaluating expression (18). In evaluating (18) one needs an explicit expression for the retarded Green's function ${}_1G(\mathscr{P}', \mathscr{P})$ associated with the metric ${}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}$. In the next subsection (§ IV*c*[i]) we derive ${}_1G(\mathscr{P}', \mathscr{P})$; then in § IV*c*(ii) we place constraints on our system which simplify ${}_1G(\mathscr{P}', \mathscr{P})$; and finally in § IV*c*(iii) we use ${}_1G(\mathscr{P}', \mathscr{P})$ to evaluate the wave field ${}_2\bar{h}^{\mu\nu}$.

i) *The Green's Function* ${}_1G(\mathscr{P}', \mathscr{P})$

We shall obtain ${}_1G(\mathscr{P}', \mathscr{P})$ by taking the weak-field limit of the exact Green's function $G(\mathscr{P}', \mathscr{P})$ for a space described by an exact metric $g_{\mu\nu}$. The exact Green's function is formally rather simple, so long as the congruence of geodesics that emanate from the source point $\mathscr{P}'$ does not get focused so strongly along the future light cone of $\mathscr{P}'$ that geodesics cross. Henceforth we shall assume "no crossing of geodesics on the light cone." Later (eqs. [48], [48′], [48″] below) we shall examine the constraints placed on the radiating system by this "no-crossing" assumption.

DeWitt and Brehme (1960) have derived the exact Green's function $G(\mathscr{P}', \mathscr{P})$ for the case of no crossing. Their Green's function consists of a "direct part" and a "tail"

$$ G(\mathscr{P}', \mathscr{P}) = G^{\mathrm{direct}} + G^{\mathrm{tail}}\,. \tag{25} $$

The direct part is nonzero only if $\mathscr{P}$ lies on the future light cone of $\mathscr{P}'$ [denoted $\mathscr{J}^+(\mathscr{P}')$]. By virtue of the "no-crossing" assumption, when $\mathscr{P}$ is near $\mathscr{J}^+(\mathscr{P}')$ there is a unique geodesic from $\mathscr{P}'$ to $\mathscr{P}$ with a unique squared length

$$ \begin{aligned} \Omega(\mathscr{P}', \mathscr{P}) &\equiv (\text{"World function," see Synge (1960)}) \\ &= \tfrac{1}{2}(-1 \text{ for timelike geodesic, } +1 \text{ for spacelike geodesic})(\text{proper distance along geodesic})^2 \\ &= \sigma \quad \text{in notation of DeWitt and Brehme (1960).} \end{aligned} \tag{26} $$

Because $\mathscr{J}^+(\mathscr{P}')$ is characterized by $\Omega = 0$, G^{direct} must have the form

$$ G^{\mathrm{direct}}(\mathscr{P}', \mathscr{P}) = (4\pi)^{-1}[\Delta(\mathscr{P}', \mathscr{P})]^{1/2}\delta_{\mathrm{ret}}[\Omega(\mathscr{P}', \mathscr{P})]\,, \tag{27} $$

where δ_{ret} is the Dirac delta function on and near $\mathscr{J}^+(\mathscr{P}')$, and is zero on and near the past light cone [$\mathscr{J}^-(\mathscr{P}')$]. The quantity $\Delta(\mathscr{P}', \mathscr{P})$ is an amplitude factor which would be unity in flat spacetime, but in curved spacetime is given by

$$ \Delta(\mathscr{P}', \mathscr{P}) = -\frac{\det\|\partial^2\Omega/\partial x^\alpha \partial x^{\beta'}\|}{|g(\mathscr{P})g(\mathscr{P}')|^{1/2}}\,. \tag{28} $$

We shall use an expression for the tail different from, but equivalent to, that given by DeWitt and Brehme. To derive our expression we insert equations (25) and (27) into the wave equation

$$ \begin{aligned} \Box_s G(\mathscr{P}', \mathscr{P}) &= -[g(\mathscr{P})g(\mathscr{P}')]^{-1/4}\delta(x^0 - x^{0\prime})\delta(x^1 - x^{1\prime})\delta(x^2 - x^{2\prime})\delta(x^3 - x^{3\prime}) \\ &\equiv -[g(\mathscr{P})g(\mathscr{P}')]^{-1/4}\delta_4(x - x')\,. \end{aligned} \tag{29} $$

The result is

$$ \Box_s G^{\mathrm{tail}} = -(4\pi)^{-1}\{(\Box_s\Delta^{1/2})\delta(\Omega) + [2\nabla_K\Delta^{1/2} + (\Box_s\Omega)\Delta^{1/2}]\delta'(\Omega) + (\nabla\Omega)^2\Delta^{1/2}\delta''(\Omega)\}\,, \tag{30a} $$

where δ' and δ'' are the first and second derivatives of the Dirac delta function for $\mathscr{P}$ on and near $\dot{J}^+(\mathscr{P}')$, and are zero for $\mathscr{P}$ on and near $\dot{J}^-(\mathscr{P}')$; ∇ is the 4-dimensional gradient operator; and ∇_K is the covariant derivative along the 4-vector

$$K \equiv \nabla\Omega\,. \tag{30b}$$

[Here and below we suppress the subscript "ret" on $\delta(\Omega)$.] We then manipulate expression (30a) using the relations

$$\Omega\delta''(\Omega) = -2\delta'(\Omega)\,, \qquad (\nabla\Omega)^2 = 2\Omega\,, \qquad \Box_s\Omega - 4 = \Box\Delta^{-1}\nabla_K\Delta\,. \tag{30c}$$

(The first of these is a standard identity for Dirac delta functions; the second and third are eqs. [1.11] and [1.63] of DeWitt and Brehme 1960.) The result is

$$\Box_s G^{\text{tail}} = -(4\pi)^{-1}(\Box_s\Delta^{1/2})\delta(\Omega)\,. \tag{31}$$

We then use relations (30c) and the relation $\Delta(\mathscr{P}, \mathscr{P}) = 1$ to rewrite this in the form

$$\Box_s[G^{\text{tail}} - (4\pi)^{-1}(1 - \Delta^{1/2})\delta(\Omega)] = +(4\pi)^{-1}(\nabla_K \ln\Delta)\delta'(\Omega)\,. \tag{32}$$

Equation (31) tells us that G^{tail} jumps from zero outside the light cone to a finite value inside the cone, without having any singularities on the cone. Equation (32) allows us to write (restoring the subscript "ret")

$$G^{\text{tail}}(\mathscr{P}', \mathscr{P}) = 0 \qquad \text{if } \mathscr{P}' \notin I^-(\mathscr{P}) \tag{33a}$$

$$= -(4\pi)^{-1}\int [\ln\Delta(\mathscr{P}', \mathscr{P}'')]_{,\alpha''}[\Omega(\mathscr{P}', \mathscr{P}'')]^{;\alpha''}\delta'_{\text{ret}}[\Omega(\mathscr{P}', \mathscr{P}'')]G(\mathscr{P}'', \mathscr{P})[-g(\mathscr{P}'')]^{1/2}d^4x'' \quad \text{if } \mathscr{P}' \in I^-(\mathscr{P})\,. \tag{33b}$$

Here $I^-(\mathscr{P})$ means "the interior of the past light cone of $\mathscr{P}$"; and condition (33a) suppresses the unwanted light-cone part of (33b) [i.e., suppresses $(4\pi)^{-1}(1 - \Delta^{1/2})\delta(\Omega)$].

Equation (33) is the form of the tail which we shall use. This form was suggested to us by the work of Peters (1966).

We now specialize the above equations for the retarded Green's function to the case of a weak gravitational field ${}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}$, beginning with equation (26) for the world function. Let λ be an affine parameter along the geodesic linking $\mathscr{P}'$ to $\mathscr{P}$:

$$\mathscr{C}(\lambda) \equiv \text{geodesic with coordinates } \xi^\alpha(\lambda)\,; \quad \mathscr{C}(0) = \mathscr{P}'\,, \quad \mathscr{C}(1) = \mathscr{P}\,, \quad 0 \le \lambda \le 1\,. \tag{34}$$

The equation (26) can be rewritten in the form (cf. Synge 1960, p. 47)

$$\Omega(\mathscr{P}', \mathscr{P}) = \int_{\mathscr{C}} \tfrac{1}{2}g_{\mu\nu}(d\xi^\mu/d\lambda)(d\xi^\nu/d\lambda)d\lambda\,. \tag{35}$$

The right-hand side is actually an action principle for the geodesic equation (cf. MTW, box 13.3). Therefore, if we evaluate the integral along the "straight line"

$${}_0\mathscr{C}(\lambda): \quad \xi^\alpha(\lambda) \equiv x^{\alpha'} + \lambda(x^\alpha - x^{\alpha'}) \tag{36}$$

(see Fig. 1), which differs by a fractional amount of $O(\epsilon)$ from the true geodesic $\mathscr{C}(\lambda)$, we will make fractional errors in Ω of $O(\epsilon^2)$. Such errors are acceptable in ${}_1G(\mathscr{P}', \mathscr{P})$, since its fractional errors are also $O(\epsilon^2)$; cf. equation (17). The result of integrating expression (35) along the slightly wrong curve ${}_0\mathscr{C}(\lambda)$ is

$${}_1\Omega(\mathscr{P}', \mathscr{P}) = {}_0\Omega(\mathscr{P}', \mathscr{P}) + \gamma(\mathscr{P}', \mathscr{P})\,, \tag{37}$$

where

$${}_0\Omega(\mathscr{P}', \mathscr{P}) \equiv \tfrac{1}{2}X^\alpha X^\beta\eta_{\alpha\beta}\,, \tag{38a}$$

$$\gamma(\mathscr{P}', \mathscr{P}) \equiv \tfrac{1}{2}X^\alpha X^\beta \int_{{}_0\mathscr{C}} {}_1h_{\alpha\beta}d\lambda\,, \tag{38b}$$

$$X^\alpha \equiv x^\alpha - x^{\alpha'}\,. \tag{38c}$$

Equation (37) is the desired expression for the world function. Turn next to the amplitude factor $\Delta(\mathscr{P}', \mathscr{P})$. Either by direct calculation from eqs. (28), (23b), (37), (38), and (B12), or by invoking equation (95) on page 63 of Synge (1960), one arrives at the expression

$${}_1\Delta(\mathscr{P}', \mathscr{P}) \equiv -\frac{\det\|{}_1\Omega_{,\alpha\beta'}\|}{|{}_1g\,{}_1g'|^{1/2}} = -(1 - \tfrac{1}{2}\,{}_1h - \tfrac{1}{2}\,{}_1h')\det\|{}_1\Omega_{,\alpha\beta'}\| = 1 + 2\alpha(\mathscr{P}', \mathscr{P})\,. \tag{39}$$

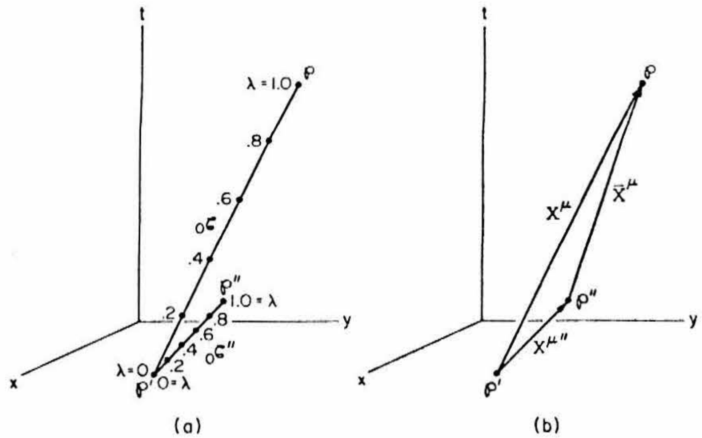

FIG. 1.—The points $\mathscr{P}$, $\mathscr{P}'$, $\mathscr{P}''$ used in evaluating the postlinear Green's function ${}_1G(\mathscr{P}', \mathscr{P})$ and in calculating the postlinear gravitational-wave field ${}_2\bar{h}^{\mu\nu}(\mathscr{P})$. Part (*a*) shows the parametrized straight-line curves ${}_0\mathscr{C}(\lambda)$ and ${}_0\mathscr{C}''(\lambda)$ linking $\mathscr{P}$, $\mathscr{P}'$, and $\mathscr{P}''$; part (*b*) shows the 4-vectors X^μ, $X^{\mu''}$; and $\bar{X}^\mu$ linking them.

Here

$$\alpha(\mathscr{P}', \mathscr{P}) \equiv \tfrac{1}{2}X^\alpha X^\beta \int_{{}_0\mathscr{C}} {}_1R_{\alpha\beta}\lambda(1 - \lambda)d\lambda\,, \tag{40a}$$

where ${}_1R_{\alpha\beta}$ is the Ricci tensor, accurate to first order in ${}_1h_{\mu\nu}$:

$$ {}_1R_{\alpha\beta} = -\tfrac{1}{2}\, {}_1h_{\alpha\beta,\rho}{}^{\rho}\,. \tag{40b}$$

In equation (39) we have simplified notation by using a prime to denote quantities evaluated at $\mathscr{P}'$, i.e., ${}_1h' \equiv {}_1h(\mathscr{P}')$ while ${}_1h \equiv {}_1h(\mathscr{P})$. Henceforth we shall reserve primes for this purpose—except that δ' and δ'' are still derivatives of Dirac delta functions.

Turn next to the "source term" $(\ln \Delta)_{,\alpha''}\Omega^{\cdot\alpha''}$ for the tail (eq. [33]). The tail itself is of $O(\epsilon)$ compared to the direct part of the Green's function; therefore we can permit fractional errors of $O(\epsilon)$ in the tail—which means we can use the zero-order value of $\Omega^{\cdot\alpha''}$ in the source of the tail:

$$[{}_0\Omega(\mathscr{P}', \mathscr{P}'')]^{\cdot\alpha''} = x^{\alpha''} - x^{\alpha'} \equiv X^{\alpha''}\,. \tag{41}$$

By combining this with equation (39) and by using equation (B7) of Appendix B, we bring the source of the tail into the form

$$[\ln {}_1\Delta(\mathscr{P}', \mathscr{P}'')]_{,\alpha''}[{}_0\Omega(\mathscr{P}', \mathscr{P}'')]^{\cdot\alpha''} = \beta(\mathscr{P}', \mathscr{P}'')\,, \tag{42}$$

where

$$\beta(\mathscr{P}', \mathscr{P}'') \equiv X^{\alpha''}X^{\beta''} \int_{{}_0\mathscr{C}''} {}_1R_{\alpha\beta}\lambda^2 d\lambda \tag{43a}$$

and ${}_0\mathscr{C}''$ is the "straight line" from $\mathscr{P}'$ to $\mathscr{P}''$ (see Fig. 1)

$$ {}_0\mathscr{C}'': \xi^\alpha = x^{\alpha'} + \lambda X^{\alpha''}\,. \tag{43b}$$

Turn, finally, to the propagator $G(\mathscr{P}'', \mathscr{P})$ and the volume element $(-g'')^{1/2}d^4x''$ which appear in equation (33) for the tail. Because the tail is of $O(\epsilon)$ compared to the direct part of the Green's function, we can ignore all curved space corrections in the amplitude of the propagator (but not its phase), and in the volume element:

$$G(\mathscr{P}'', \mathscr{P})(-g'')^{1/2}d^4x'' = (4\pi)^{-1}\delta_{\text{ret}}[{}_1\Omega(\mathscr{P}'', \mathscr{P})]d^4x'' \text{ in expression (33b) for } {}_1G^{\text{tail}}\,. \tag{44}$$

All of the pieces for the first-order Green's function are now at hand. By combining them (eqs. [25], [27], [33], [39], [42], and [44]) we obtain the following result:

$$ {}_1G(\mathscr{P}', \mathscr{P}) = {}_1G^{\text{direct}} + {}_1G^{\text{tail}}\,, \tag{45a}$$

$$ {}_1G^{\text{direct}} = (4\pi)^{-1}[1 + \alpha(\mathscr{P}', \mathscr{P})]\delta_{\text{ret}}[{}_1\Omega(\mathscr{P}', \mathscr{P})]\,, \tag{45b}$$

$$ \begin{aligned} {}_1G^{\text{tail}} &= 0 && \text{if } \mathscr{P}' \notin I^-(\mathscr{P}) \\ &= -(4\pi)^{-2}\int \beta(\mathscr{P}', \mathscr{P}'')\delta_{\text{ret}}[{}_1\Omega(\mathscr{P}', \mathscr{P}'')]\delta'_{\text{ret}}[{}_1\Omega(\mathscr{P}'', \mathscr{P})]d^4x'' && \text{if } \mathscr{P}' \in I^-(\mathscr{P})\,. \end{aligned} \tag{45c}$$

Here ${}_1\Omega(\mathscr{P}', \mathscr{P})$ [and similarly ${}_1\Omega(\mathscr{P}', \mathscr{P}'')$ and ${}_1\Omega(\mathscr{P}'', \mathscr{P})$] is given by equations (37) and (38), $\alpha(\mathscr{P}', \mathscr{P})$ (the "focusing function") is defined by expressions (40), and $\beta(\mathscr{P}', \mathscr{P}'')$ (the "tail generator") is defined by expressions (43).

ii) *Constraints Designed to Simplify the Green's Function*

Expression (45) for the Green's function is valid only if geodesics emanating from $\mathscr{P}'$ fail to cross on and near $J^+(\mathscr{P}')$. Crossing would be caused by gravitational focusing; and at any crossing point, the exact amplitude factor $\Delta(\mathscr{P}', \mathscr{P})$ would diverge. Thus, the criterion for no crossing is finiteness of Δ along $J^+(\mathscr{P}')$.

Consider our first-order expression (39), (40) for ${}_1\Delta$. Evaluate it in the mean rest-frame of the source, with coordinates centered on $\mathscr{P}'$ so that

$$X^\alpha = rn^\alpha\,, \quad n^0 \equiv 1\,, \quad \boldsymbol{n} = (\text{unit spatial vector pointing from } \mathscr{P}' \text{ to } \mathscr{P})\,,$$
$$r = (\text{spatial distance from } \mathscr{P}' \text{ to } \mathscr{P})\,, \tag{46}$$
$$\lambda = \bar{r}/r \equiv (\text{fractional distance from } \mathscr{P}' \text{ to } \mathscr{P})\,;$$

and invoke the first-order field equation ${}_1R_{\alpha\beta} = ({}_1T_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\,{}_1T)$. The result is

$${}_1\Delta(\mathscr{P}', \mathscr{P}) = 1 + 2\alpha\,, \qquad \alpha = \tfrac{1}{2}\int_0^r (n^\alpha n^\beta {}_1T_{\alpha\beta})\bar{r}(1 - \bar{r}/r)d\bar{r}\,. \tag{47}$$

This expression for ${}_1\Delta$ can *never* diverge if the source is bounded, because once the integration point $\bar{r}$ gets outside the source, ${}_1T_{\alpha\beta}$ vanishes and ${}_1\Delta$ stops increasing. However, if the focusing function α approaches unity inside the source, then second-order and higher effects will come into play. As one moves out into the vacuum beyond the source, those second-order effects will be essentially those of the "focusing" or "Raychaudhuri" equation; they will produce a divergence. Thus, the constraint

$$\text{CONSTRAINT:}\quad \alpha(\mathscr{P}', \mathscr{P}) \ll 1 \quad \text{for all} \quad \mathscr{P}' \quad \text{and} \quad \mathscr{P} \tag{48}$$

is necessary for the validity of the first-order analysis, and simultaneously protects us from "geodesic crossing."

For a system that is roughly homogeneous with mass M and linear size L, equation (47) gives

$$\alpha \sim (M/L) \sim \epsilon \ll 1\,; \tag{48'}$$

so there is no problem in satisfying the constraint (48). However, for a highly inhomogeneous system (lumps of mass m and size l, separated by distances $b \gg l$), and for rays originating in one lump and passing through another, equation (47) gives

$$\alpha \sim b(m/l^3)l \sim (b/l)(m/l)\,.$$

In this case the constraint (48) is significant: it says that to avoid too much ray focusing, the lumps must not be too far apart:

$$(b/l) \ll (l/m) \sim 10^6(l/R_\odot)(M_\odot/m)\,. \tag{48''}$$

The Green's function (45) would be much easier to use if, throughout it, we could replace the first-order world function ${}_1\Omega$ by its zero-order approximation ${}_0\Omega = \tfrac{1}{2}\eta_{\mu\nu}X^\mu X^\nu$. Let us examine ${}_1\Omega$ (eqs. [37] and [38]) in the rest frame of our source, for points $\mathscr{P}$ on or near $J_+(\mathscr{P}')$:

$${}_1\Omega(\mathscr{P}', \mathscr{P}) = \tfrac{1}{2}(X^0 + X)[-X^0 + X + 2\gamma(\mathscr{P}', \mathscr{P})/(X^0 + X)] \approx X(-X^0 + X + \Delta t_{\mathrm{S}})\,, \tag{49a}$$

where

$$X \equiv |\boldsymbol{X}| = (\text{distance from source to field point})\,, \tag{49b}$$

$$\Delta t_{\mathrm{S}} \equiv \gamma(\mathscr{P}', \mathscr{P})/X = (\text{"Shapiro time delay"})\,. \tag{49c}$$

For field points $\mathscr{P}$ far outside the source, the dominant contribution to the Shapiro time delay is the asymptotic "$1/r$" field of the source. It produces a huge delay of

$$\Lambda \equiv 2M\ln(X/L) = (\text{Shapiro time delay due to asymptotic field of source})\,,$$
$$M \equiv (\text{mass of source})\,, \tag{50}$$
$$L \equiv (\text{characteristic size of source})\,.$$

This delay is time independent and is independent of where inside the source $\mathscr{P}'$ is located (aside from a negligible piece of size $\sim 2ML/X$); therefore its only effect on the radiation is to delay the arrival time at a given radius. Henceforth, for ease of calculation, we shall remove this constant delay from the argument of our Green's function.

We can always reinsert it at the end of the calculation if we wish. With this constant delay removed, we can rewrite ${}_1\Omega$ as

$$ {}_1\Omega(\mathscr{P}', \mathscr{P}) \approx X[-X^0 + X + (\Delta t_S - \Lambda)_{\mathscr{P}'\mathscr{P}}] \approx {}_0\Omega(\mathscr{P}', \mathscr{P}) + [\gamma(\mathscr{P}', \mathscr{P}) + \Lambda X^\mu U_\mu] $$
$$ \text{for } \mathscr{P}' \text{ inside the source and } \mathscr{P} \text{ far outside it}, \tag{51} $$

where

$$ U_\mu = P_\mu/M = \text{4-velocity of source; } X^\mu U_\mu = -X. \tag{52} $$

The remaining "internally produced" delay between $\mathscr{P}'$ and $\mathscr{P}$, $\Delta t_S - \Lambda$, is of the same order of magnitude as the total delay between two internal points $\mathscr{P}'$ and $\mathscr{P}''$:

$$ (\Delta t_S - \Lambda)_{\mathscr{P}'\mathscr{P}} \sim (\Delta t_S - \Lambda)_{\mathscr{P}''\mathscr{P}} \sim (\Delta t_S)_{\mathscr{P}'\mathscr{P}''} \sim \int_{\text{across source}} {}_1h_{00}d\bar{r} \sim M \text{ for homogeneous source} $$
$$ \sim m \ln (b/l) \text{ for lumpy source}. \tag{53} $$

Henceforth we shall *assume* that this internal time delay is small compared with the characteristic time scale on which the source changes—i.e., small compared with the characteristic reduced wavelength $\lambda\!\!\!^{-}$ of the radiation emitted,

$$ \text{CONSTRAINT: } (\Delta t_S)_{\text{internal}} \sim m \ln (b/l) \ll \lambda\!\!\!^{-}. \tag{54} $$

[Example: If $\lambda\!\!\!^{-}$ is 100 times larger than the Schwarzschild radius, $2m$, of a lump, then b/l can be as large as $\exp(10) \sim 2 \times 10^4$ without causing problems. Another example: If $\lambda\!\!\!^{-} \gtrsim b$ (which is the case for bremsstrahlung), and if $l \gg m$ (which is required for fields to be weak), then the condition $b \gtrsim l$ (separation of lumps bigger than size of lumps) guarantees that constraint (54) is satisfied.]

The constraint (54) allows us to expand our delta functions $\delta({}_1\Omega)$ in powers of the internal time delay. Discarding terms that are quadratic and higher-order in $(\Delta t_S)_{\text{internal}}/\lambda\!\!\!^{-}$, we obtain for the Green's function (45)

$$ {}_1G(\mathscr{P}', \mathscr{P}) = {}_1G^{\text{direct}} + {}_1G^{\text{tail}}, \tag{55a} $$

$$ {}_1G^{\text{direct}}(\mathscr{P}', \mathscr{P}) = (4\pi)^{-1}\{\delta_{\text{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta}) + \alpha(\mathscr{P}', \mathscr{P})\delta_{\text{ret}}(\tfrac{1}{2}\bar{X}^\alpha X^\beta \eta_{\alpha\beta}) + [\gamma(\mathscr{P}', \mathscr{P}) + \Lambda X^\mu U_\mu]\delta'_{\text{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})\}; \tag{55b} $$

$$ {}_1G^{\text{tail}}(\mathscr{P}', \mathscr{P}) = 0 \quad \text{if } \mathscr{P}' \notin I^-(\mathscr{P}) $$
$$ = -(4\pi)^{-2}\int \beta(\mathscr{P}', \mathscr{P}'')\delta'_{\text{ret}}(\tfrac{1}{2}X^{\alpha''}X^{\beta''}\eta_{\alpha\beta})\delta_{\text{ret}}(\tfrac{1}{2}\bar{X}^\alpha \bar{X}_\beta \eta_{\alpha\beta})d^4x'' \quad \text{if } \mathscr{P}' \in I^-(\mathscr{P}). \tag{55c} $$

In these equations

$$ X^\alpha \equiv x^\alpha - x^{\alpha'}, \qquad \bar{X}^\alpha \equiv x^\alpha - x^{\alpha''}, \qquad X^{\alpha''} \equiv x^{\alpha''} - x^{\alpha'}; \tag{56} $$

see Figure 1.

Equations (55) are our final form for the scalar Green's function in a space with linearized metric ${}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}$. This Green's function has fractional errors

$$ |(\text{errors in } {}_1G)/{}_1G| \sim \text{Maximum of } \{\epsilon^2, \alpha\epsilon, [(\Delta t_S)_{\text{internal}}/\lambda\!\!\!^{-}]\epsilon\} \text{ in general,} $$
$$ \sim \epsilon^2 \text{ for most sources}; \tag{57} $$

and it has been stripped of its asymptotic time delay (eq. [50]).

iii) *The Gravitational-Wave Field* ${}_2\bar{h}^{\mu\nu}$

By inserting expressions (55) for ${}_1G(\mathscr{P}', \mathscr{P})$ into equation (18) we obtain the following expression for the gravitational field far outside a weak-field source:

$$ {}_2\bar{h}^{\mu\nu} = {}_2\bar{h}^{\mu\nu}_{\text{D}} + {}_2\bar{h}^{\mu\nu}_{\text{F}} + {}_2\bar{h}^{\mu\nu}_{\text{TR}} + {}_2\bar{h}^{\mu\nu}_{\text{W}} + {}_2\bar{h}^{\mu\nu}_{\text{TL}}; \tag{58a} $$

$$ {}_2\bar{h}^{\mu\nu}_{\text{D}} = 4\int \delta_{\text{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2T^{\mu\nu}(\mathscr{P}')[1 - {}_1\bar{h}(\mathscr{P}')]d^4x', \tag{58b} $$

$$ {}_2\bar{h}^{\mu\nu}_{\text{F}} = 4\int \alpha(\mathscr{P}', \mathscr{P})\delta_{\text{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2T^{\mu\nu}(\mathscr{P}')d^4x', \tag{58c} $$

$$ {}_2\bar{h}^{\mu\nu}_{\text{TR}} = 4\int [\gamma(\mathscr{P}', \mathscr{P}) + \Lambda X^\alpha U_\alpha]\delta'_{\text{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2T^{\mu\nu}(\mathscr{P}')d^4x', \tag{58d} $$

$$ {}_2\bar{h}^{\mu\nu}_{\text{W}} = 4\int \delta_{\text{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})[{}_1t^{\mu\nu}_{\text{L-L}} + (16\pi)^{-1}{}_1\bar{h}^{\mu\rho}{}_{,\sigma}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho}]_{\text{at }\mathscr{P}'}d^4x', \tag{58e} $$

$$ {}_2\bar{h}^{\mu\nu}_{\text{TL}} = (-1/\pi)\iint_{\mathscr{P}''\in I^-(\mathscr{P})} \beta(\mathscr{P}', \mathscr{P}'')\delta'_{\text{ret}}(\tfrac{1}{2}X^{\alpha''}X^{\beta''}\eta_{\alpha\beta})\delta_{\text{ret}}(\tfrac{1}{2}\bar{X}^\alpha \bar{X}^\beta \eta_{\alpha\beta})_2T^{\mu\nu}(\mathscr{P}')d^4x''d^4x'. \tag{58f} $$

Here $\mathscr{P}'$ and $\mathscr{P}''$ are source points with coordinates $x^{\alpha'}$ and $x^{\alpha''}$ (cf. Fig. 1); the field point $\mathscr{P}$ has coordinates x^{α}; ${}_2T^{\mu\nu}$, ${}_1t^{\mu\nu}_{\rm L-L}$, and ${}_1\bar{h}^{\mu\nu}$ are the stress-energy tensor, the pseudotensor, and the gravitational field obtained by a post-linear analysis (eqs. [23] and [24]); $\delta_{\rm ret}$ is the Dirac delta function on the future light cone of the source and zero on the past light cone; $\delta'_{\rm ret}$ is the derivative of $\delta_{\rm ret}$ with respect to its argument; X^{α}, $\bar{X}^{\alpha}$, and $X^{\alpha''}$ are

$$X^{\alpha} \equiv x^{\alpha} - x^{\alpha'}\,, \qquad \bar{X}^{\alpha} \equiv x^{\alpha} - x^{\alpha''}\,, \qquad X^{\alpha''} \equiv x^{\alpha''} - x^{\alpha'}\,; \tag{59a}$$

α, β, and γ are defined by integrals of the first-order Ricci tensor ${}_1R_{\mu\nu}$ and of the metric perturbation ${}_1\bar{h}^{\mu}{}_{\nu}$ along the straight line between two points

$$\alpha(\mathscr{P}', \mathscr{P}) = \tfrac{1}{2}X^{\alpha}X^{\beta}\int_0^1 {}_1R_{\alpha\beta}(x^{\mu'} + \lambda X^{\mu})\lambda(1-\lambda)d\lambda\,, \tag{59b}$$

$$\beta(\mathscr{P}', \mathscr{P}'') = X^{\alpha''}X^{\beta''}\int_0^1 {}_1R_{\alpha\beta}(x^{\mu'} + \lambda X^{\mu''})\lambda^2 d\lambda\,, \tag{59c}$$

$$\gamma(\mathscr{P}', \mathscr{P}) = \tfrac{1}{2}X^{\alpha}X^{\beta}\int_0^1 {}_1h_{\alpha\beta}(x^{\mu'} + \lambda X^{\mu})d\lambda\,; \tag{59d}$$

$-\Lambda X^{\alpha}U_{\alpha}$ is that portion of γ which is produced by the asymptotic, $1/r$, external field of the source

$$-\Lambda X^{\alpha}U_{\alpha} = \Lambda(-X^{\alpha}U_{\alpha}) = (\text{Shapiro time delay produced outside source}) \times (\text{distance from source point to field point}) \tag{59e}$$

(see § IVc[ii] above); and $\mathscr{P}' \in I^-(\mathscr{P})$ means that the integration (58f) is performed over field points $\mathscr{P}'$ that lie inside but not on the past light cone of $\mathscr{P}$.

Each piece of the distant gravitational field ${}_2\bar{h}^{\mu\nu}$ has its own physical origin and significance:

The first piece ${}_2\bar{h}^{\mu\nu}_{\rm D}$ is the "direct field." It is produced by the stress-energy ${}_2T^{\mu\nu}$ and propagates as though spacetime were flat. It includes the zero-order, nonradiative, "$1/r$" field of the source, and also that portion of the radiation produced "directly" by the source's motions. If the internal gravity of the source has negligible influence on the source's structure and evolution, then all other parts of ${}_2\bar{h}^{\mu\nu}$ will be negligible compared with the direct field ("linearized theory"; cf. eq. [20] and the associated discussion).

The second piece ${}_2\bar{h}^{\mu\nu}_{\rm F}$ is the "focusing field." It is the amount by which the direct field is augmented due to focusing as it passes through regions of nonzero Ricci curvature (nonzero stress-energy).

The third piece ${}_2\bar{h}^{\mu\nu}_{\rm TR}$ is the "transition field" (first discovered in the equations of general relativity by Chitre, Price, and Sandberg 1973, 1975; analog of "electromagnetic transition radiation," Ginzburg and Frank 1946). It is the amount by which the direct field changes due to Shapiro-type time delays within the time-varying source.

The fourth piece ${}_2\bar{h}^{\mu\nu}_{\rm W}$ is the "whump field." It is the field generated by "gravitational stresses" ${}_1t^{\mu\nu}_{\rm L-L} + (16\pi)^{-1}{}_1\bar{h}^{\mu\rho}{}_{,\sigma}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho}$. We have given it the name "whump" because in our minds we have a heuristic image of gravitational stresses linking various pieces of the source, and going "whump" (i.e., quickly rising in strength and then quickly falling) as the pieces of source move past each other.

The final piece ${}_2\bar{h}^{\mu\nu}_{\rm TL}$ is the "tail field." It is generated by the direct field in those regions where focusing has deformed the geometry of the direct wave fronts.

Although it is useful, heuristically and in calculations, to split ${}_2\bar{h}^{\mu\nu}$ into these five pieces, one should not attribute too much physical significance to each individual piece. For example, no individual piece satisfies the Einstein field equations or the de Donder gauge condition. However, the five individual pieces combine in such a way that their sum does satisfy the field equations and gauge condition; see Appendix C.

V. SLOW-MOTION LIMIT OF THE WAVE-GENERATION FORMULAE

Consider a weak-field system which has slow internal motions and weak internal stresses. Characterize it by the following parameters:

$$\begin{aligned} L &\equiv (\text{size of system}) \\ \lambdabar &\equiv (\text{characteristic time-scale of system}) = (\text{reduced wavelength of radiation}) \\ M &\equiv (\text{mass of system}) \\ v &\equiv (|T^{0j}|/T^{00})_{\max} = (\text{maximum internal velocity}) \\ S^2 &\equiv (|T^{ij}|/T^{00})_{\max} = \text{maximum of (stress)/(density)}\,. \end{aligned} \tag{60}$$

Chapter 36 of MTW derives the quadrupole-moment formalism for gravitational wave generation under the following assumptions (eqs. [36.18] of MTW)

$$L/\lambda\!\!\!^{-} \ll 1 \quad \text{which implies } v \ll 1\ ; \tag{61a}$$

$$M/L \ll L/\lambda\!\!\!^{-}\,, \quad S^2 \ll L/\lambda\!\!\!^{-}\,. \tag{61b}$$

Constraint (61a) is the standard slow-motion assumption—the only assumption truly necessary for validity of the quadrupole-moment formalism (see Paper II). Constraints (61b) say that the motion must not be *too slow* if a weak-field calculation is to yield the quadrupole-moment formalism. In terms of the characteristic frequency $\omega \equiv 1/\lambda\!\!\!^{-}$ this "not too slow" assumption says

$$\omega^2 \gg (M/L)(M/L^3)\,, \qquad \omega^2 \gg S^2(S/L)^2\,. \tag{61b'}$$

A violation of these assumptions occurs, in dynamical systems, only when the gravitational and stress forces counterbalance each other so precisely that second-order gravity, ${}_2h^{\mu\nu}$, can affect the motion significantly (cf. Chandrasekhar 1964). In this case an analysis based on the postlinear approximation cannot possibly give a correct description of the radiation.

It is instructive to see how the postlinear radiation formulae (58) of this paper yield the quadrupole-moment formalism, when applied to a system satisfying constraints (61).

We begin by combining the direct and whump fields (58b, e) and then breaking them up again, differently:

$${}_2\bar{h}^{\mu\nu}_{\rm D} + {}_2\bar{h}^{\mu\nu}_{\rm W} = {}_2\bar{h}^{\mu\nu}_{\rm DW1} + {}_2\bar{h}^{\mu\nu}_{\rm DW2}\ ; \tag{62a}$$

$${}_2\bar{h}^{\mu\nu}_{\rm DW1} \equiv 4\int \delta_{\rm ret}(\tfrac{1}{2}X^\alpha X^\beta\eta_{\alpha\beta})[(-{}_1g)({}_2T^{\mu\nu} + {}_1t^{\mu\nu}_{\rm L-L})]_{{\rm at}\,\mathscr{P}'}d^4x'\,, \tag{62b}$$

$${}_2\bar{h}^{\mu\nu}_{\rm DW2} \equiv (1/4\pi)\int \delta_{\rm ret}(\tfrac{1}{2}X^\alpha X^\beta\eta_{\alpha\beta})[{}_1\bar{h}^{\mu\rho}{}_{,\sigma}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho}]_{{\rm at}\,\mathscr{P}'}d^4x'\,. \tag{62c}$$

We then evaluate expression (62b) in the rest frame of the source

$${}_2\bar{h}^{\mu\nu}_{\rm DW1} = 4\int \frac{(-{}_1g)({}_2T^{\mu\nu} + {}_1t^{\mu\nu}_{\rm L-L})_{\rm ret}}{|\boldsymbol{x} - \boldsymbol{x}'|}\,d^3x'\ ; \tag{63}$$

and by carrying out the analysis of[4] MTW § 36.10, we bring the spatial transverse-traceless part of this field into the form

$$[{}_2h^{jk}_{\rm DW1}(t, \boldsymbol{x})]^{\rm TT} = (2/r)(d^2/dt^2)I\!\!\!^{-\rm TT}_{jk}(t - r) \sim (M/r)(L/\lambda\!\!\!^{-})^2\,. \tag{64}$$

Here $I\!\!\!^{-}_{jk}$ is the "reduced quadrupole moment" of the source, and $I\!\!\!^{-\rm TT}_{jk}$ is its transverse-traceless part. This is the standard quadrupole-moment formula for the radiation field.

An order-of-magnitude analysis shows that all other parts of our expression (58) for ${}_2\bar{h}_{jk}$ are negligible. In particular, by using the following relations valid for the source's interior

$${}_1\bar{h}^{00} \sim M/L\,, \quad {}_1\bar{h}^{0j} \sim Mv/L\,, \quad {}_1\bar{h}^{jk} \sim MS^2/L\,, \quad T^{jk} \sim MS^2/L^3\,, \quad {}_1\bar{h}^{\alpha\beta}{}_{,0} \sim {}_1\bar{h}^{\alpha\beta}/\lambda\!\!\!^{-}\,,$$
$${}_1\bar{h}^{\alpha\beta}{}_{,j} \sim {}_1\bar{h}^{\alpha\beta}/L\,, \quad \beta \sim M/L\,, \tag{65a}$$

as well as the relations

$$\alpha \sim M/L\,, \quad (\gamma + \Lambda X^\alpha U_\alpha) \sim rM\,, \tag{65b}$$

we obtain for the ratio of each other part to the "DW1" part (eq. [64]):

$$|{}_2\bar{h}^{jk}_{\rm DW2}/({}_2\bar{h}^{jk}_{\rm DW1})^{\rm TT}| \sim \left(\frac{M}{L}\right)\left(v + \frac{S^2}{L/\lambda\!\!\!^{-}}\right)^2 \ll 1\,, \tag{66a}$$

$$|{}_2\bar{h}^{jk}_{\rm F}/({}_2\bar{h}^{jk}_{\rm DW1})^{\rm TT}| \sim \left(\frac{M/L}{L/\lambda\!\!\!^{-}}\right)\left(\frac{S^2}{L/\lambda\!\!\!^{-}}\right) \ll 1\,, \tag{66b}$$

$$|{}_2\bar{h}^{jk}_{\rm TR}/({}_2\bar{h}^{jk}_{\rm DW1})^{\rm TT}| \sim |{}_2\bar{h}^{jk}_{\rm TL}/({}_2\bar{h}^{jk}_{\rm DW1})^{\rm TT}| \sim \left(\frac{M}{L}\right)\left(\frac{S^2}{L/\lambda\!\!\!^{-}}\right) \ll 1\,. \tag{66c}$$

[4] Note that $(-{}_1g)({}_2T^{\mu\nu} + {}_1t^{\mu\nu}_{\rm L-L})$ here plays the same role as $T^{\mu\nu} + t^{\mu\nu}$ in MTW. The key properties which they share are (i) vanishing coordinate divergence; (ii) same role in retarded integral for ${}_2\bar{h}^{\mu\nu}$.

VI. SUMMARY OF OUR "PLUG-IN-AND-GRIND" FORMALISM FOR WAVE GENERATION

Our postlinear formalism for wave generation can be summarized as follows:

Regime of validity. The formalism is valid for any system satisfying these constraints: (i) The gravitational field must be weak everywhere:

$$|\bar{h}^{\mu\nu}| \ll 1 \quad \text{everywhere}\,, \tag{67a}$$

and the source must be isolated (see discussion preceding eq. [4]). (ii) Gravitational and nongravitational forces must not balance each other so precisely as to enable second-order gravity to influence the system's motions significantly. (iii) The source must not focus substantially light rays emitted from within itself. Mathematically this constraint says

$$|\alpha(\mathscr{P}', \mathscr{P})| \ll 1 \text{ for } \mathscr{P}' \text{ any event inside the source,}$$
$$\mathscr{P} \text{ any event on the future light cone of } \mathscr{P}'\,, \tag{67b}$$

where α is defined by equation (59b). For further discussion of this constraint, see the first half of § IV*c*(ii). (iv) The "Shapiro time delay" for light propagation within the source must be small compared with the characteristic time-scale $\lambda\!\!\!^-$ for internal motions of the source. Mathematically this constraint says that in the mean rest-frame of the source

$$(\Delta t_S)_{\text{internal}} \equiv \gamma(\mathscr{P}', \mathscr{P})/|\boldsymbol{x} - \boldsymbol{x}'| \ll \lambda\!\!\!^-\,. \tag{67c}$$

Here $\boldsymbol{x}$ and $\boldsymbol{x}'$ are spatial locations of events $\mathscr{P}$ and $\mathscr{P}'$ that lie inside the source, $\mathscr{P}$ is on the future light cone of $\mathscr{P}'$, and γ is defined by equation (59d). For further discussion, see the second half of § IV*c*[ii].

Calculation of the system's motion. For a system satisfying these constraints one calculates the internal structure and dynamics by using the postlinear formalism of § IV*b* (eqs. [23] and [24]).

Calculation of the distant field. To calculate the gravitational field ${}_2\bar{h}^{\mu\nu}$ in the radiation zone, far from the source, one takes the result of the postlinear analysis, plugs it into equations (58) and (59), and grinds.

In Paper III we shall use this formalism to calculate gravitational bremsstrahlung radiation.

APPENDIX A

WHY USE THE CURVED-SPACE SCALAR-WAVE OPERATOR?

In laying the foundations of our analysis (in and near eq. [6]) we write the Einstein field equations in terms of the curved-space scalar wave operator $\Box_s$. We choose to do this because the obvious alternatives (the flat-space wave operator $\Box_f$ or the curved-space tensor wave operator $\Box_t$) would ultimately lead to complications or dangers in our analysis.

The flat-space operator $\Box_f$ treats the field propagation from the outset as though it were on flat-space characteristics (straight coordinate lines). Because the true characteristics suffer the Shapiro time delay which involves a logarithm of distance, the use of $\Box_f$ would lead to logarithmic divergences in the radiative field at large r. If one were sufficiently careful, one could remove those divergences without serious error—but that is a dangerous enterprise. Even if one succeeded, one would be left in the end with the interesting effects of focusing, time delay ("transition radiation"), and tail all lumped into the whump part of the field. We prefer to keep them separate.

Consider next the curved-space tensor wave operator

$$\Box_t \bar{h}^{\mu\nu} \equiv \bar{h}^{\mu\nu;\alpha}{}_{\alpha} + 2R_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu}\bar{h}^{\alpha\beta} - 2R_{\alpha}{}^{(\mu}\bar{h}^{\nu)\alpha} \tag{A1}$$

(cf. MTW eq. [35.64]). Because the true propagation equation for very weak gravitational waves on a curved background is $\Box_t \bar{h}^{\mu\nu} = 0$, it is tempting to formulate our analysis in terms of $\Box_t$ rather than $\Box_s$. By using $\Box_s$ we push into the "whump" part of ${}_2\bar{h}^{\mu\nu}$ an important physical effect: the curvature-induced rotation of polarization. In effect, part of our whump field corrects the error in our direct field's unrotated polarization. Had we used $\Box_t$ rather than $\Box_s$, polarization rotation would have shown up in § IV*c*(iii) as a separate piece of the radiation field.

The tensor wave operator has a disadvantage which, for our purposes, outweighs the above advantage. Suppose that one constructed a tensor Green's function for $\Box_t$

$$\Box_t G^{\mu\nu\alpha'\beta'}(\mathscr{P}', \mathscr{P}) = -\tfrac{1}{2}(g^{\mu\alpha'}g^{\nu\beta'} + g^{\mu\beta'}g^{\nu\alpha'})(gg')^{-1/4}\delta_4(x - x')\,, \tag{A2}$$

or for *any* other wave operator with the form

$$\Box_{\text{other}}\bar{h}^{\mu\nu} \equiv \bar{h}^{\mu\nu;\alpha}{}_{\alpha} + (\text{any "background" field})^{\mu\nu}{}_{\alpha\beta}\bar{h}^{\alpha\beta}\,. \tag{A3}$$

That Green's function would have a first-order tail ${}_1G^{\mu\nu\alpha'\beta'}_{\rm tail}(\mathscr{P}', \mathscr{P})$ with "sources" $\beta^{\mu\nu\alpha'\beta'}$ involving the Riemann tensor (cf. eqs. [43a] and [45c]). Such a tail would originate everywhere on the light cone of $\mathscr{P}'$, whereas the tail ${}_1G^{\rm tail}$ for our scalar Green's function originates only on rays that have passed through matter. In practical calculations involving lumpy sources—see, e.g., Paper III—that tail would be as difficult to calculate as the whump part of the field. We prefer our scalar tail because of its greater simplicity. By using $\Box_s$ we dump all serious calculational complexities, for lumpy sources, into the whump part of the field.

APPENDIX B

LINE-INTEGRAL IDENTITIES

The weak-field Green's function ${}_1G(\mathscr{P}', \mathscr{P})$ used in this paper involves three integrals α, β, γ along "straight lines." In this appendix we take the line of integration to be

$$ {}_0\mathscr{C}(\lambda):\ \xi^\alpha \equiv x^{\alpha'} + \lambda X^\alpha, \quad 0 \le \lambda \le 1, \quad X^\alpha \equiv x^\alpha - x^{\alpha'}. \tag{B1} $$

The three line integrals are

$$ \gamma \equiv \tfrac{1}{2} X^\mu X^\nu \int_0^1 {}_1h_{\mu\nu} d\lambda, \tag{B2} $$

$$ \alpha \equiv \tfrac{1}{2} X^\mu X^\nu \int_0^1 {}_1R_{\mu\nu}\lambda(1-\lambda) d\lambda, \tag{B3} $$

$$ \beta \equiv X^\mu X^\nu \int_0^1 {}_1R_{\mu\nu}\lambda^2 d\lambda, \tag{B4} $$

where ${}_1h_{\mu\nu}$ is assumed to satisfy the de Donder condition

$$ {}_1\bar{h}_{\mu\nu}{}^{,\nu} = {}_1h_{\mu\nu}{}^{,\nu} - \tfrac{1}{2}\, {}_1h_{,\mu} = 0 \tag{B5} $$

and the Ricci tensor is therefore given by

$$ {}_1R_{\mu\nu} = -\tfrac{1}{2}\, {}_1h_{\mu\nu,\rho}{}^{\rho}, \tag{B6} $$

and where the index notation used is that of a Lorentz frame in flat spacetime.

Below we list a number of useful identities linking the line integrals α, β, γ, their derivatives at point $\mathscr{P}$, and the values of ${}_1h_{\mu\nu}$ and ${}_1R_{\mu\nu}$ at $\mathscr{P}$:

$$ X^\rho \alpha_{,\rho} = \tfrac{1}{2}\beta, \tag{B7} $$

$$ X^\rho X\sigma \alpha_{,\rho\sigma} = -\beta + \tfrac{1}{2} X^\rho X^\sigma {}_1R_{\rho\sigma}, \tag{B8} $$

$$ X^\rho \gamma_{,\rho} = \gamma + \tfrac{1}{2} X^\rho X\sigma {}_1h_{\rho\sigma}, \tag{B9} $$

$$ X^\rho X^\sigma \gamma_{,\rho\sigma} = X^\rho X^\sigma {}_1h_{\rho\sigma} + \tfrac{1}{2} X^\rho X^\sigma X^\tau {}_1h_{\rho\sigma,\tau}, \tag{B10} $$

$$ \gamma_{,\rho}{}^{\rho} = -\beta + {}_1h. \tag{B11} $$

Similar identities involving derivatives at $\mathscr{P}'$ and mixed derivatives at $\mathscr{P}'$ and $\mathscr{P}$ can be derived fairly easily. For example,

$$ \gamma_{,\rho}{}^{\rho'} = -2\alpha - \tfrac{1}{2}\, {}_1h - \tfrac{1}{2}\, {}_1h', \tag{B12} $$

where ${}_1h \equiv {}_1h(\mathscr{P})$ and ${}_1h' \equiv {}_1h(\mathscr{P}')$.

The derivations of these identities are quite straightforward. The necessary techniques are illustrated by the following derivation of identity (B7): By differentiating definition (B3) and making use of equations (B1), we obtain

$$ \begin{aligned} \alpha_{,\rho} &= X^\mu \int_0^1 R_{\mu\rho}\lambda(1-\lambda)d\lambda + \tfrac{1}{2}X^\mu X^\nu \int_0^1 (\partial R_{\mu\nu}/\partial\xi^\sigma)(\partial\xi^\sigma/\partial x^\rho)\lambda(1-\lambda)d\lambda \\ &= X^\mu \int_0^1 R_{\mu\rho}\lambda(1-\lambda)d\lambda + \tfrac{1}{2}X^\mu X^\nu \int_0^1 (R_{\mu\nu,\sigma})(\lambda\delta^\sigma{}_\rho)\lambda(1-\lambda)d\lambda \\ &= X^\mu \int_0^1 R_{\mu\rho}\lambda(1-\lambda)d\lambda + \tfrac{1}{2}X^\mu X^\nu \int_0^1 R_{\mu\nu,\rho}\lambda^2(1-\lambda)d\lambda. \end{aligned} $$

Here $R_{\mu\nu,\sigma} = \partial R_{\mu\nu}/\partial\xi^\sigma$ is the derivative of $R_{\mu\nu}$ at the integration point $\mathscr{C}(\lambda)$. When multiplied by X^ρ this expression gives

$$X^\rho\alpha_{,\rho} = X^\mu X^\nu \int_0^1 R_{\mu\nu}\lambda(1-\lambda)d\lambda + \tfrac{1}{2}X^\mu X^\nu \int_0^1 (R_{\mu\nu,\rho}X^\rho)\lambda^2(1-\lambda)d\lambda$$

$$= X^\mu X^\nu \int_0^1 R_{\mu\nu}\lambda(1-\lambda)d\lambda + \tfrac{1}{2}X^\mu X^\nu \int_0^1 (dR_{\mu\nu}/d\lambda)\lambda^2(1-\lambda)d\lambda\,.$$

By integrating the last expression by parts we obtain

$$X^\rho\alpha_{,\rho} = X^\mu X^\nu \int_0^1 R_{\mu\nu}[\lambda(1-\lambda) - \tfrac{1}{2}(d/d\lambda)(\lambda^2-\lambda^3)]d\lambda = \tfrac{1}{2}X^\mu X^\nu \int_0^1 R_{\mu\nu}\lambda^2 d\lambda = \tfrac{1}{2}\beta\,; \qquad \text{QED}.$$

In this case the integration by parts gave no endpoint terms; but in other cases (eqs. [B8]–[B12]) nonzero endpoint terms are obtained.

In manipulations of our weak-field Green's function ${}_1G(\mathscr{P}', \mathscr{P})$ and of our second-order gravitational field ${}_2\bar{h}^{\mu\nu}$ (see, e.g., Appendix C) two other identities are useful:

$$\beta\delta' = (\alpha\delta)_{,\rho}{}^{\rho} - \alpha_{,\rho}{}^{\rho}\delta\,, \tag{B13}$$

$$(\gamma\delta' + \alpha\delta)_{,\rho}{}^{\rho} = \alpha_{,\rho}{}^{\rho}\delta + {}_1\bar{h}^{\rho\sigma}\delta_{,\rho\sigma}\,. \tag{B14}$$

Here δ is the flat-space propagator between $\mathscr{P}'$ and $\mathscr{P}$

$$\delta \equiv \delta_{\text{ret}}(\tfrac{1}{2}X^\rho X^\sigma\eta_{\rho\sigma})\,, \tag{B15a}$$

which is related to the 4-dimensional Dirac delta function

$$\delta_4(x - x') \equiv \delta(x^0 - x^{0'})\delta(x^1 - x^{1'})\delta(x^2 - x^{2'})\delta(x^3 - x^{3'}) \tag{B15b}$$

by

$$\delta_{,\rho}{}^{\rho} = -4\pi\delta_4(x - x')\,; \tag{B15c}$$

and δ' is the derivative of the propagator (B15a) with respect to its argument. The absence of primes on indices and on h's in (B13) and (B14) indicates that all derivatives and endpoint terms are taken at $\mathscr{P}$; none are at $\mathscr{P}'$. The identities (B13) and (B14) can be derived with some labor from the identities (B7)–(B11).

APPENDIX C

PROOF THAT THE "PLUG-IN-AND-GRIND" FORMULAE FOR ${}_2\bar{h}^{\mu\nu}$ SATISFY THE FIELD EQUATIONS AND GAUGE CONDITION

Here we briefly sketch the proof that our second-order gravitational field (eqs. [58]) satisfies the second-order Einstein field equation (eqs. [16] with $n = 2$) and the de Donder gauge condition ${}^2\bar{h}^{\mu\nu}{}_{,\nu} = 0$. As part of our proof we shall derive expressions for the amount by which each piece of ${}_2\bar{h}^{\mu\nu}$ fails, by itself, to satisfy the field equation and gauge condition.

A preliminary step in our proof is to rewrite the "tail" and "transition" fields (58f) and (58d) in new forms. Although expression (58f) for the tail seems optimal for practical radiation calculations, the restriction $\mathscr{P}' \in I^-(\mathscr{P})$ makes it nasty for formal manipulations. To get rid of this restriction we take expression (B13) for $\beta\delta'$, in it we replace $\mathscr{P}$ by $\mathscr{P}''$, and then we insert it into expression (58f). The result,

$${}_2\bar{h}^{\mu\nu}_{\text{TL}} = (1/\pi)\iint [\alpha(\mathscr{P}', \mathscr{P}'')]_{,\rho''}{}^{\rho''}\delta_{\text{ret}}(\tfrac{1}{2}X^{\alpha''}X^{\beta''}\eta_{\alpha\beta})\delta_{\text{ret}}(\tfrac{1}{2}\bar{X}_\alpha\bar{X}^\beta\eta_{\alpha\beta})_2T^{\mu\nu}(\mathscr{P}')d^4x''d^4x'\,, \tag{C1}$$

is an expression which gives the same value for ${}_2\bar{h}^{\mu\nu}_{\text{TL}}$ whether one imposes or omits the restriction $\mathscr{P}' \in I^-(\mathscr{P})$. One way to see that (C1) is oblivious to the restriction $\mathscr{P}' \in I^-(\mathscr{P})$ is this: Take the source equation (31) for the tail of the exact curved-space Green's function; calculate its lowest-order form

$$G^{\text{tail}}{}_{,\rho}{}^{\rho} = -(4\pi)^{-1}[\alpha(\mathscr{P}', \mathscr{P})]_{,\rho}{}^{\rho}\delta(\tfrac{1}{2}X^\alpha X^\beta\eta_{\alpha\beta})\,;$$

invert this using a flat-space propagator; use the resulting G^{tail} to calculate ${}_2\bar{h}^{\mu\nu}_{\text{TL}}$; the result will be expression (C1)—and nowhere in the derivation did one need to impose the restriction $\mathscr{P}' \in I^-(\mathscr{P})$.

Expression (58d) for the transition field involves a "time-delay function" $\gamma(\mathscr{P}', \mathscr{P})$ from which the logarithmic, "external time delay" $\Lambda(-X^\alpha U_\alpha)$ has been removed. A straightforward subtraction of the external time delay is well suited to practical calculations, but poorly suited to formal manipulations of ${}_2\bar{h}^{\mu\nu}$. In the formal manipulations of this appendix we shall perform the truncation in a "smoother" manner: We surround the source by a (hypothetical) cloud of negative-mass material, with total mass, $-M$, equal in magnitude to that of the source, $+M$. We put the cloud far enough from the source (e.g., at radius $\mathscr{L} \sim 100L$) that it is very diffuse, and thus contributes negligibly to the line integrals α and β; but near enough that the Shapiro time delay $2M \ln(\mathscr{L}/L)$ in going from source L to cloud $\mathscr{L}$ is small compared with the time scale λ of the source's internal motions. The cloud automatically removes the external Shapiro time delay; no artificial truncation of γ is needed. The second-order gravitational field is then given by equations (58) and (C1) *everywhere* (inside the source and out), except that we must remove the artificial truncation from (58d):

$${}_2\bar{h}^{\mu\nu}_{\mathrm{TR}} = 4\int \gamma(\mathscr{P}', \mathscr{P})\delta'_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2 T^{\mu\nu}(\mathscr{P}')d^4x' . \tag{C2}$$

Turn now to the proof that our second-order field satisfies the second-order Einstein field equation. We begin by applying the first-order wave operator

$${}_1\Box_s = (\eta^{\alpha\beta} - h^{\alpha\beta})\partial_\alpha\partial_\beta \tag{C3}$$

to each of the five pieces of our second-order field. By applying ${}_1\Box_s$ to the direct field (eq. [58b]) and by using equation (B15c) we obtain

$${}_1\Box_{s2}\bar{h}^{\mu\nu}_{\mathrm{D}} = -16\pi(1 - \tfrac{1}{2}\,{}_1\bar{h})_2 T^{\mu\nu} - 4\,{}_1\bar{h}^{\rho\sigma}\partial_\rho\partial_\sigma \int \delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2 T^{\mu\nu}(\mathscr{P}')d^4x' . \tag{C4a}$$

By applying ${}_1\Box_s$ to the whump field (eq. 58e) and by using (B15c) we obtain

$${}_1\Box_{s2}\bar{h}^{\mu\nu}_{\mathrm{W}} = -16\pi\,{}_1t^{\mu\nu}_{\mathrm{L-L}} - {}_1\bar{h}^{\mu\rho}{}_{,\sigma}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho} . \tag{C4b}$$

By applying ${}_1\Box_s$ to the tail field (eq. [C1]) and by using (B15c) we obtain

$${}_1\Box_{s2}\bar{h}^{\mu\nu}_{\mathrm{TL}} = -4\int [\alpha(\mathscr{P}', \mathscr{P})]_{,\rho}{}^{\rho}\delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2 T^{\mu\nu}(\mathscr{P}')d^4x' . \tag{C4c}$$

By applying ${}_1\Box_s$ to the focusing field (58c), and by using (B15c) and the relation $\alpha(\mathscr{P}, \mathscr{P}) = 0$ (cf. eq. [B3]) we obtain

$${}_1\Box_{s2}\bar{h}^{\mu\nu}_{\mathrm{F}} = 4\int [\alpha(\mathscr{P}', \mathscr{P})]_{,\rho}{}^{\rho}\delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2 T^{\mu\nu}(\mathscr{P}')d^4x' + 8\int [\alpha(\mathscr{P}', \mathscr{P})]_{,\rho}[\delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})]^{,\rho}{}_2 T^{\mu\nu}(\mathscr{P}')d^4x' . \tag{C4d}$$

By applying ${}_1\Box_s$ to the transition field (C2), and by using (B14), (B15c), and $\alpha(\mathscr{P}, \mathscr{P}) = 0$, we obtain

$${}_1\Box_{s2}\bar{h}^{\mu\nu}_{\mathrm{TR}} = 4\,{}_1\bar{h}^{\rho\sigma}\partial_\rho\partial_\sigma \int \delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})_2 T^{\mu\nu}(\mathscr{P}')d^4x' - 8\int [\alpha(\mathscr{P}', \mathscr{P})]_{,\rho}[\delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})]^{,\rho}{}_2 T^{\mu\nu}(\mathscr{P}')d^4x' . \tag{C4e}$$

By adding up all five pieces (C4a)–(C4e) we obtain

$${}_1\Box_{s2}\bar{h}^{\mu\nu} = -16\pi[(1 - \tfrac{1}{2}\,{}_1\bar{h})_2 T^{\mu\nu} + {}_1t^{\mu\nu}_{\mathrm{L-L}}] - {}_1\bar{h}^{\mu\rho}{}_{,\sigma}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho} , \tag{C5}$$

which is the second-order Einstein field equation (16).

Turn now to a proof that our field (58) satisfies the de Donder gauge condition ${}_2\bar{h}^{\mu\nu}{}_{,\nu} = 0$ except for fractional errors of $O(\epsilon^2)$. From (58b) and the relation

$$\partial_\nu \int \delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta}) f(\mathscr{P}')d^4x' = \int \delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})\partial_{\nu'} f(\mathscr{P}')d^4x' , \tag{C6}$$

valid for any function $f(\mathscr{P}')$, we obtain

$${}_2\bar{h}^{\mu\nu}_{\mathrm{D}}{}_{,\nu} = 4\int \delta_{\mathrm{ret}}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})\{{}_2T^{\mu\nu}(\mathscr{P}')[1 - {}_1\bar{h}(\mathscr{P}')]\}_{,\nu'}d^4x' . \tag{C7a}$$

From (58e), (C6), and ${}_1\bar{h}^{\rho\sigma}{}_{,\sigma} = 0$ we obtain

$$ {}_2\bar{h}^{\mu\nu}_{\rm W}{}_{,\nu} = 4\int \delta_{\rm ret}(\tfrac{1}{2}X^{\alpha}X^{\beta}\eta_{\alpha\beta})[{}_1t^{\mu\nu}_{\rm L-L}{}_{,\nu} + (16\pi)^{-1}{}_1\bar{h}^{\mu\rho}{}_{,\sigma\nu}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho}]_{{\rm at}\,\mathscr{P}'}d^4x' . \tag{C7b} $$

We now add (C7a) and (C7b) and use the postlinear equations of motion (24a) rewritten in the form

$$ [{}_2T^{\mu\nu}(1 - {}_1\bar{h}) + {}_1t^{\mu\nu}]_{,\nu} = 0 $$

(cf. eq. [3]) to obtain

$$ ({}_2\bar{h}^{\mu\nu}_{\rm D} + {}_2\bar{h}^{\mu\nu}_{\rm W})_{,\nu} = (4\pi)^{-1}\int \delta_{\rm ret}(\tfrac{1}{2}X^{\alpha}X^{\beta}\eta_{\alpha\beta})\,{}_1\bar{h}^{\mu\rho}(\mathscr{P}')_{,\sigma'\nu'}\,{}_1\bar{h}^{\nu\sigma}(\mathscr{P}')_{,\rho'}d^4x' . $$

We then use ${}_1h^{\mu\rho}(\mathscr{P}')_{,\rho'} = 0$ together with (C6) and a relabeling of indices to obtain

$$ ({}_2\bar{h}^{\mu\nu}_{\rm D} + {}_2\bar{h}^{\mu\nu}_{\rm X})_{,\nu} = (4\pi)^{-1}\partial_{\nu}\int \delta_{\rm ret}(\tfrac{1}{2}X^{\alpha}X^{\beta}\eta_{\alpha\beta})\,{}_1\bar{h}^{\mu\nu}(\mathscr{P}')_{,\sigma'\rho'}\,{}_1\bar{h}^{\rho\sigma}(\mathscr{P}')d^4x' . $$

We then give $\mathscr{P}'$ the new name $\mathscr{P}''$ and rewrite ${}_1\bar{h}^{\mu\nu}(\mathscr{P}'')$ as a retarded integral (the solution to eq. [24b]); the result is

$$ ({}_2\bar{h}^{\mu\nu}_{\rm D} + {}_2\bar{h}^{\mu\nu}_{\rm W})_{,\nu} = (1/\pi)\partial_{\nu}\iint \delta_{\rm ret}(\tfrac{1}{2}\bar{X}^{\alpha}\bar{X}^{\beta}\eta_{\alpha\beta})\,{}_1\bar{h}^{\rho\sigma}(\mathscr{P}'')[\delta_{\rm ret}(\tfrac{1}{2}X^{\alpha''}X^{\beta''}\eta_{\alpha\beta})]_{,\rho''\sigma''}\,{}_2T^{\mu\nu}(\mathscr{P}')d^4x'd^4x'' . \tag{C7c} $$

By applying ∂_{ν} to expression (C1), adding it onto (C7c), using identities (B14) and (B15c), and integrating by parts, we obtain

$$ \begin{aligned} ({}_2\bar{h}^{\mu\nu}_{\rm D} + {}_2\bar{h}^{\mu\nu}_{\rm W} + {}_2\bar{h}^{\mu\nu}_{\rm TL})_{,\nu} = &-4\partial_{\nu}\int \gamma(\mathscr{P}', \mathscr{P})\delta'_{\rm ret}(\tfrac{1}{2}X^{\alpha}X^{\beta}\eta_{\alpha\beta})\,{}_2T^{\mu\nu}(\mathscr{P}')d^4x' \\ &-4\partial_{\nu}\int \alpha(\mathscr{P}', \mathscr{P})\delta_{\rm ret}(\tfrac{1}{2}X^{\alpha}X^{\beta}\eta_{\alpha\beta})\,{}_2T^{\mu\nu}(\mathscr{P}')d^4x' . \end{aligned} \tag{C7d} $$

Comparison with expressions (C2) for ${}_2\bar{h}^{\mu\nu}_{\rm TR}$ and (58c) for ${}_2\bar{h}^{\mu\nu}_{\rm F}$ shows that

$$ ({}_2\bar{h}^{\mu\nu}_{\rm D} + {}_2\bar{h}^{\mu\nu}_{\rm W} + {}_2\bar{h}^{\mu\nu}_{\rm TL} + {}_2\bar{h}^{\mu\nu}_{\rm TR} + {}_2\bar{h}^{\mu\nu}_{\rm F})_{,\nu} = 0 ; \tag{C8} $$

i.e., our total second-order field *does* satisfy the de Donder gauge condition.

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PAPER 2.

THE GENERATION OF GRAVITATIONAL WAVES

III. DERIVATION OF BREMSSTRAHLUNG FORMULAE*

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ABSTRACT

Formulae are derived describing the gravitational waves produced by a stellar encounter of the following type: The two stars have stationary (i.e., non-pulsating), nearly Newtonian structures with arbitrary relative masses; they fly past each other with an arbitrary relative velocity; and their impact parameter is sufficiently large that they gravitationally deflect each other through an angle small compared to 90°.

I. INTRODUCTION AND MOTIVATION

The gravitational interaction of two stars flying past each other not only deflects their trajectories; it also produces gravitational waves. Those waves are the analogue of the electromagnetic bremsstrahlung radiation produced when an electron flies past an ion. For this reason they are called "gravitational bremsstrahlung radiation."

Formulae for gravitational bremsstrahlung radiation have been derived by many researchers in a variety of different contexts, using a variety of different methods and a variety of different approximations. Everybody agrees on the spectrum and angular distribution of the radiation for the rather trivial case of low stellar velocities and large impact parameters. However, at high velocities different methods give discrepant results (see Paper IV for details and references); and at no velocity, low or high, has there been a detailed study of the amplitude as a function of time, angle, and polarization. The time dependence of the amplitude is important because that is what would be measured by a broad-band gravitational-wave detector--for example by Doppler tracking of interplanetary spacecraft (Estabrook and Wahlquist 1975; Thorne and Braginsky 1976).

In this paper and its sequel (Kovàcs and Thorne 1977; "Paper IV") we shall attempt to give a definitive and complete treatment of the bremsstrahlung problem for the case of large impact parameters (i.e., for small deflections of the stellar trajectories). We shall place no constraints on the stellar velocities or on the ratio of the masses of the two stars.

This paper will be confined to a derivation of the amplitude as a function of time, angle, and polarization. Paper IV will examine the details of that amplitude, will use it to compute other features of the radiation including spectra, angular distribution of radiated energy, total energy radiated, low-velocity limit of various quantities, and high-velocity limit. Paper IV will also compare with the results of previous computations and will attempt to resolve their high-velocity discrepancies.

In computing gravitational bremsstrahlung at high velocities, $v \sim c$, one must not use either "linearized theory" or the "quadrupole-moment formalism." Linearized theory makes fractional errors of order unity at low velocities or high velocities; and the quadrupole-moment formalism makes fractional errors larger than unity at high velocities. (For discussion see, e.g., § IV.a of Paper I.) By contrast, the "post-linear formalism" constructed in Papers I and II (Thorne and Kovàcs 1975; Crowley and Thorne 1977) makes fractional errors of order (gravitational radii of stars)/(impact parameter) << 1, whether the encounter velocity is high or low. This paper will utilize the post-linear formalism.

This paper is divided into eight sections. Section II gives a post-linear analysis of the stars' trajectories and "first-order" gravitational fields ${}_1\bar{h}^{\mu\nu}$; §II also introduces the notation to be used throughout the paper (see its third paragraph). Section III introduces the parameters and functions in terms of which the radiation field will be expressed; and it presents a number of formulae and geometric relationships to be used in the derivation of the radiation field. Section IV presents formal post-linear expressions for the second-order gravitational field ${}_2\bar{h}^{\mu\nu}$, which contains the radiation. That field is broken up into five parts: "direct," "focussing," "transition," "tail," and "whump." Section V evaluates the direct field in terms of the parameters and functions of §III; §VI evaluates the sum of the focusing, transition, and tail fields; and §VII evaluates the whump field. Section VIII adds all the pieces together to yield the full second-order field ${}_2\bar{h}^{\mu\nu}$ (eqs. 8.1 and 8.2); and in the rest-frame of one of the stars it projects out the radiation field (transverse-traceless part of ${}_2\bar{h}_{jk}$). The final radiation field (eqs. 8.11 and 8.13) is expressed in terms of two time- and angle-dependent amplitudes for two orthogonal states of polarization.

II. POST-LINEAR ANALYSIS OF THE STELLAR ENCOUNTER

Consider two stars A and B with arbitrary masses m_A and m_B, which undergo a near encounter. Let the initial relative velocity of the stars (velocity of A as measured in rest frame of B, and velocity of B as measured in rest frame of A) be v ; and let the impact parameter of the encounter be b. Require that the stars' radii r_A and r_B be small compared with their impact parameter

$$r_A \ll b \ , \qquad r_B \ll b \quad , \tag{2.1a}$$

so that one can ignore tidal interactions between the stars and tidal gravitational radiation (Mashoon 1973). On the other hand, require that the stars be sufficiently large

$$r_A \gg m_A \ , \qquad r_B \gg m_B \tag{2.1b}$$

that their internal gravity is nonrelativistic, and the post-linear formalism is valid. Also require that

$$\zeta \equiv (m_A + m_B)/(bv^2) \ll 1 \tag{2.1c}$$

so that the angles of deflection of the stellar trajectories are very small.

We shall use the dimensionless quantity ζ as an expansion parameter in analyzing the encounter and the emitted radiation. In particular, we shall calculate the stars' trajectories, velocities, and accelerations only up to first order in ζ . This will be sufficient to give us fractional errors in the gravitational-wave field of order

$$\text{(error in wave field)/(wave field)} \sim \zeta\gamma^2 \quad , \tag{2.2}$$

where

$$\gamma \equiv (1 - v^2)^{-1/2} \tag{2.3}$$

(see D'Eath 1977). To obtain greater accuracy than (2.2) at high velocities one would have to perform a post-post-linear calculation or use an entirely different type of approximation scheme--e.g., that of D'Eath (1977).

We shall analyze the stars' motions using the post-linear formalism of Papers I and II. That formalism is couched in the language of flat Minkowskii spacetime, even though spacetime is actually curved. We shall perform the analysis using frame-independent Minkowskian language and notation. Greek indices will run from 0 to 3, Latin indices from 1 to 3; sans-serif letters (printed version) or letters with arrows over them (manuscript) will denote 4-vectors and 4-tensors; bold-face letters (printed version) or letters with squiggles under them (manuscript) will denote 3-vectors and 3-tensors; subscripted commas will denote partial derivatives; geometrized units ($c = G = 1$) will be used; and the signature of the metric will be $-+++$.

Each star carries with itself a clock that reads proper time--τ_A for star A and τ_B for star B. The clocks are synchronized to read $\tau_A = \tau_B = 0$ at the moment when they are nearest each other. We shall denote the world lines of the centers of the stars by

$$\mathcal{Z}_A{}^{\alpha}(\tau_A) = z_A{}^{\alpha}(\tau_A) + \mathfrak{z}_A{}^{\alpha}(\tau_A) \;, \quad \mathcal{Z}_B{}^{\alpha}(\tau_B) = z_B{}^{\alpha}(\tau_B) + \mathfrak{z}_B{}^{\alpha}(\tau_B) \;. \tag{2.4a}$$

Here $z_A{}^{\alpha}(\tau_A)$ and $z_B{}^{\alpha}(\tau_B)$ are the spacetime coordinates (in an arbitrary post-linear coordinate frame) of star A when its clock reads time τ_A, and of star B when its clock reads τ_B; $z_A{}^\alpha$ and $z_B{}^\alpha$ are the stars' unperturbed coordinates (coordinates in the limit $\zeta \to 0$); $\mathfrak{z}_A{}^\alpha$ and $\mathfrak{z}_B{}^\alpha$ are the first-order corrections to the unperturbed coordinates. We shall denote the 4-velocity of star K (=A or B) by

$$U_K{}^\alpha \equiv dZ_K{}^\alpha/d\tau_K = u_K{}^\alpha + w_K{}^\alpha \equiv dz_K{}^\alpha/d\tau_K + d\mathfrak{z}_K{}^\alpha/d\tau_K \quad ; \tag{2.4b}$$

and we shall denote its gravitationally-induced 4-acceleration by

$$a_K{}^\alpha \equiv dU_K{}^\alpha/d\tau_K = dw_K{}^\alpha/d\tau_K \quad . \tag{2.4c}$$

The unperturbed world lines $z_A{}^\alpha(\tau_A)$ and $z_B{}^\alpha(\tau_B)$ are straight lines in our (arbitrary) post-linear coordinate frame; see Figure 1. Define the "impact vector"

$$b^\alpha \equiv z_B{}^\alpha(\tau_B{=}0) - z_A{}^\alpha(\tau_A{=}0) = \left(\begin{array}{l}\text{vector separating the two stars}\\ \text{at their moment of closest approach}\\ \text{in the case of unperturbed world lines}\end{array}\right). \tag{2.5}$$

Of course, b^α is orthogonal to the unperturbed world lines of both stars

$$\vec{b}\cdot\vec{u}_B = \vec{b}\cdot\vec{u}_A = 0 \quad . \tag{2.6}$$

Call the center point of $\vec{b}$ the "collision event" and denote it $\vec{x}_c$. In terms of $\vec{x}_c$, $\vec{b}$, and the unperturbed 4-velocities $\vec{u}_A, \vec{u}_B$ the unperturbed world lines are described by

$$\vec{z}_K(\tau_K) = \vec{x}_c + \frac{1}{2}\,\varepsilon_K\,\vec{b} + \vec{u}_K\tau_K \;; \tag{2.7a}$$

$$\varepsilon_A \equiv -1 \;,\quad \varepsilon_B \equiv +1 \quad . \tag{2.7b}$$

The moving stars generate a post-linear gravitational field ${}_1\bar{h}_{\mu\nu}$. Its trace reversal

$$ {}_1h_{\mu\nu} \equiv {}_1\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,{}_1\bar{h} \tag{2.8a}$$

is related to the first-order metric of spacetime by

$$ {}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu} \quad . \tag{2.8b}$$

(See eqs. (I,23c,d)--i.e., eqs. (23c,d) of Paper I; and recall that in keeping with our flat-space notation all indices are raised and lowered using the Minkowskii metric $\eta_{\alpha\beta}$.) The gravitational field ${}_1\bar{h}^{\mu\nu}$ is generated by the post-linear stress-energy tensor ${}_2T^{\mu\nu} \equiv T^{\mu\nu}$ (we shall drop the prefix 2) through the flat-space wave equation (I,24b)

$$ \Box_f\,{}_1\bar{h}^{\mu\nu} \equiv \eta^{\alpha\beta}\,{}_1\bar{h}^{\mu\nu}{}_{,\alpha\beta} = -16\pi\, T^{\mu\nu} \quad . \tag{2.9a}$$

This equation has the solution

$$ {}_1\bar{h}^{\mu\nu}(x) = 16\pi \int {}_0G(x,x')\, T^{\mu\nu}(x')\, d^4x' \quad . \tag{2.9b}$$

Here x and x' denote field event and integration event; d^4x' is the flat-space 4-volume element

$$ d^4x' \equiv dx^{0'}\, dx^{1'}\, dx^{2'}\, dx^{3'} \quad , \tag{2.10}$$

and ${}_0G(x,x')$ is the flat-space propagator of equations (I,19b) and (II,6b)

$$ {}_0G(x,x') = (1/4\pi)\, \delta_{ret}[\tfrac{1}{2}(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'})\eta_{\alpha\beta}] \quad . \tag{2.11}$$

In this propagator δ_{ret} is the retarded Dirac delta function--which is zero for x in the causal past of x', and is the ordinary Dirac delta function otherwise.

Equation (2.9b) shows that ${}_1\bar{h}^{\mu\nu}$ obeys a linear superposition law. This allows us to split it into a field due to star A plus a field due to star B

$$ {}_1\bar{h}^{\mu\nu} = {}_1\bar{h}_A{}^{\mu\nu} + {}_1\bar{h}_B{}^{\mu\nu} \;;\; {}_1\bar{h}_K{}^{\mu\nu}(x) = 16\pi \int {}_0G(x,x')\, T_K{}^{\mu\nu}(x')d^4x' \;. \quad (2.12) $$

Here $T_K{}^{\mu\nu}$ is the post-linear stress-energy tensor of star K. This stress-energy tensor satisfies equation (I,24a)

$$ T_K{}^{\mu\nu}{}_{,\nu} = -{}_1\Gamma^{\mu}{}_{\alpha\nu}\, T_K{}^{\alpha\nu} - {}_1\Gamma^{\nu}{}_{\alpha\nu}\, T_K{}^{\mu\alpha} \;, \quad (2.13) $$

where ${}_1\Gamma^{\mu}{}_{\alpha\nu}$ are the first-order Christoffel symbols computed from the first-order metric (2.8b). Equations (2.12) and (2.13) are a coupled set of equations which determine: (i) the internal structures of the stars, (ii) the gravitational fields of the stars, (iii) the gravitational deflection of the stars' world lines, and (iv) the gravitational field of the system.

Because the stars are small compared to their impact parameter, their internal structures are influenced by their self-gravity, but not by each other's gravity--i.e., tidal interactions can be neglected. We shall take the internal structure of each star as given, and shall focus attention exclusively on the stars' relative motions.

Our neglect of tidal interactions implies that each star's self field has no influence on its trajectory. Hence, the motion of star A is governed entirely by the field of star B, and conversely. By using this fact and performing a 3-volume integral of equation (2.13) over the interior of star K, one can readily verify that the center of star B moves along a geodesic of the metric $g^{\mu\nu} = \eta^{\mu\nu} + h_A{}^{\mu\nu}$, and the center of star A moves along a geodesic of the metric $g^{\mu\nu} = \eta^{\mu\nu} + h_B{}^{\mu\nu}$. The proof is identical to that found in standard references—e.g., Exercise 39.15(d) of Misner, Thorne, and Wheeler (1973)--cited henceforth as MTW.] The resulting geodesic equation for star K--accurate to first order in the deflection parameter ζ--is

$$a_K{}^\alpha = -\frac{1}{2}({}_1h_{K'}{}^\alpha{}_{\beta,\gamma} + {}_1h_{K'}{}^\alpha{}_{\gamma,\beta} - {}_1h_{K'\beta\gamma}{}^{,\alpha})\, u_K{}^\beta\, u_K^\gamma \quad . \tag{2.14}$$

Here the symbol K' means "the other star", i.e., "not K"; and ${}_1h_{K'\alpha\beta}$ is the field of star K' at the location of K.

In calculating ${}_1h_{K'\alpha\beta}$ for use in this geodesic equation, we can treat star K' as a point source (tidal interactions negligible), which moves along the undeflected world line $z_{K'}{}^\alpha(\tau_{K'})$. The resulting fractional errors in $a_K{}^\alpha$ are of order ζ . The stress-energy tensor for star K' at event x' is then

$$T_{K'}{}^{\alpha\beta}(x') = \int m_{K'} u_{K'}{}^\alpha u_{K'}{}^\beta\, \delta_4[x' - z_{K'}(\tau_{K'})]\, d\tau_{K'} \quad , \tag{2.15}$$

where δ_4 is the 4-dimensional Dirac delta function

$$\delta_4(x) \equiv \delta(x^0)\, \delta(x^1)\, \delta(x^2)\, \delta(x^3) \quad . \tag{2.16}$$

By putting this stress-energy tensor into equation (2.12) for ${}_1\bar{h}_{K'}{}^{\mu\nu}$ and integrating over d^4x' we obtain

$${}_1\bar{h}_{K'}{}^{\mu\nu}(x) = 16\pi\, m_{K'}\, u_{K'}{}^\mu\, u_{K'}{}^\nu \int {}_0G[x, z_{K'}(\tau_{K'})]\, d\tau_{K'} \quad . \tag{2.17}$$

The integral over $\tau_{K'}$ can be performed using expression (2.11) for the propagator. The result is

$${}_1\bar{h}_{K'}{}^{\mu\nu}(x) = \frac{4m_{K'}\, u_{K'}{}^\mu\, u_{K'}{}^\nu}{\vec{\mathcal{K}}_{K'} \cdot \vec{u}_{K'}} \quad . \tag{2.18}$$

Here $\vec{\mathcal{K}}_{K'}$ is the unique past-directed null vector reaching from the field point x to the unperturbed world line of star K'. The corresponding trace-reversed gravitational field (metric perturbation) is

$$ {}_1h_{K'}{}^{\mu\nu}(x) = \frac{4m_{K'}(u_{K'}{}^{\mu}\, u_{K'}{}^{\nu} + \frac{1}{2}\eta^{\mu\nu})}{(\vec{\mathcal{K}}_{K'} \cdot \vec{u}_{K'})} \quad . \tag{2.19} $$

Here we have used the fact that the unperturbed 4-velocity $u_{K'}$ has unit length relative to the unperturbed metric: $u_{K'}{}^{\mu}\, u_{K'}{}^{\nu}\, \eta_{\mu\nu} = -1$.

In evaluating the 4-acceleration of star K we shall need the gradient of ${}_1h_{K'}{}^{\mu\nu}(x)$. We can evaluate the gradient by letting the field point $\vec{x}$ change by an arbitrary, small amount to $\vec{x} + \vec{\xi}$. The dependence of expression (2.19) on $\vec{x}$ is contained entirely in the null vector $\vec{\mathcal{K}}_{K'}$; thus the change in ${}_1\overleftrightarrow{h}_{K'}$ is

$$ \delta\, {}_1\overleftrightarrow{h}_{K'} = \vec{\xi} \cdot (\nabla\, {}_1\overleftrightarrow{h}_{K'}) = - \frac{{}_1\overleftrightarrow{h}_{K'}}{\vec{\mathcal{K}}_{K'} \cdot \vec{u}_{K'}} (\nabla_{\vec{\xi}}\, \vec{\mathcal{K}}_{K'}) \cdot \vec{u}_{K'} \quad , \tag{2.20a} $$

where $\nabla_{\vec{\xi}}$ denotes the gradient along $\vec{\xi}$ (notation of MTW). The quantity $\nabla_{\vec{\xi}}\, \vec{\mathcal{K}}_{K'}$ can be calculated using the relations (cf. eq. 2.7a)

$$ \begin{gathered} \vec{\mathcal{K}}_{K'} = \vec{z}_{K'} - \vec{x} = \vec{x}_c + \frac{1}{2} \varepsilon_{K'}\, \vec{b} + \vec{u}_{K'}\, \tau_{K'} - \vec{x} \quad , \\ (\vec{\mathcal{K}}_{K'})^2 = 0 \quad , \quad \mathcal{K}_{K'}{}^{0} < 0 \quad , \end{gathered} \tag{2.20b} $$

which determine $\mathcal{K}_{K'}$ and $\tau_{K'}$ in terms of the field point $\vec{x}$. In particular,

$$ \nabla_{\vec{\xi}}\, \vec{\mathcal{K}}_{K'} = -\nabla_{\vec{\xi}}\, \vec{x} + \vec{u}_{K'}\, \nabla_{\vec{\xi}}\, \tau_{K'} = -\vec{\xi} + \vec{u}_{K'}\, \nabla_{\vec{\xi}}\, \tau_{K'} \quad ; \tag{2.20c} $$

from which we can obtain

$$ 0 = \vec{\mathcal{K}}_{K'} \cdot (\nabla_{\vec{\xi}}\, \vec{\mathcal{K}}_{K'}) = -\vec{\mathcal{K}}_{K'} \cdot \vec{\xi} + (\vec{u}_{K'} \cdot \vec{\mathcal{K}}_{K'})\, \nabla_{\vec{\xi}}\, \tau_{K'} \quad . \tag{2.20d} $$

Solving (2.20d) for $\nabla_{\vec{\xi}}\, \tau_{K'}$ and inserting back into (2.20c) we find

$$\nabla_{\vec{\xi}}\, \vec{\mathcal{K}}_{K'} = -\vec{\xi} + \frac{\vec{\mathcal{K}}_{K'} \cdot \vec{\xi}}{\vec{\mathcal{K}}_{K'} \cdot \vec{u}_{K'}}\, \vec{u}_{K'} = \vec{\xi} \cdot \left(-\overleftrightarrow{\eta} + \frac{\vec{\mathcal{K}}_{K'} \otimes \vec{u}_{K'}}{\vec{\mathcal{K}}_{K'} \cdot \vec{u}_{K'}} \right) , \tag{2.20e}$$

where $\overleftrightarrow{\eta}$ is the Minkowskii metric. Combining this result with (2.20a) and recalling that $\vec{\xi}$ is arbitrary, we obtain

$${}_1h_{K'}{}^{\mu\nu}{}_{,\alpha} = \frac{4m_{K'}(u_{K'}{}^{\mu} u_{K'}{}^{\nu} + \frac{1}{2}\eta^{\mu\nu})}{(\vec{\mathcal{K}}_{K'} \cdot \vec{u}_{K'})^2} \left(u_{K'\alpha} + \frac{\mathcal{K}_{K'\alpha}}{\vec{\mathcal{K}}_{K'} \cdot \vec{u}_{K'}} \right) . \tag{2.21}$$

We are now in a position to evaluate the 4-acceleration of particle K. By inserting expression (2.21) for the gradient of $\overleftrightarrow{h}_{K'}$ into equation (2.14), we obtain

$$a_K{}^{\alpha} = \frac{m_{K'}}{(\vec{k}_{K'} \cdot \vec{u}_{K'})^2} \left[(2\gamma^2 - 1)\left(u_{K'}{}^{\alpha} + \frac{k_{K'}{}^{\alpha}}{\vec{k}_{K'} \cdot \vec{u}_{K'}} \right) + \left(\gamma - \frac{\vec{k}_{K'} \cdot \vec{u}_K}{\vec{k}_{K'} \cdot \vec{u}_{K'}} \right)(2u_K{}^{\alpha} - 4\gamma u_{K'}{}^{\alpha}) \right] . \tag{2.22}$$

Here

$$\gamma \equiv -\vec{u}_A \cdot \vec{u}_B = (1 - v^2)^{-1/2} \tag{2.23}$$

is the specific energy of star A as seen in the rest frame of B--and also the specific energy of B as seen in the rest frame of A. Also $\vec{k}_{K'}$ is $\vec{\mathcal{K}}_K$ evaluated at $x^{\alpha} = z_K{}^{\alpha}(\tau_K)$; i.e., it is the unique past-directed null vector reaching from the "field point" $x^{\alpha} = z_K{}^{\alpha}(\tau_K)$ to the world line of K'.

We now derive an expression for $\vec{k}_{K'}$ as an explicit function of τ_K, the proper time of the accelerated star. In our derivation we can treat both stars as though they moved along the straight lines $z_K(\tau_K)$ and $z_{K'}(\tau_{K'})$; by doing so we make fractional errors in $a_K{}^{\alpha}$ of order ζ. From equations (2.7) for the unperturbed world lines we have

$$\vec{k}_{K'} = [\mathcal{K}_{K'}]_{x=z_K(\tau_K)}$$
$$= \vec{z}_{K'}(\tau_{K'}) - \vec{z}_K(\tau_K) = \varepsilon_{K'}\vec{b} + \vec{u}_{K'}\tau_{K'} - \vec{u}_K\tau_K \quad . \tag{2.24a}$$

The time $\tau_{K'}$, at which the field must be "emitted" in order to reach star K at time τ_K, is determined by the condition that $\vec{k}_{K'}$ be null

$$0 = (\vec{k}_{K'})^2 = b^2 - \tau_K{}^2 - \tau_{K'}{}^2 + 2\gamma\, \tau_K\tau_{K'} \quad . \tag{2.24b}$$

Solving for $\tau_{K'}$ we obtain

$$\tau_{K'} = \gamma\, \tau_K - \ell(\tau_K) \quad , \tag{2.24c}$$

where ℓ is defined by

$$\ell(\tau_K) \equiv [b^2 + (\gamma^2 - 1)\tau_K{}^2]^{1/2} = [b^2 + v^2\gamma^2\, \tau_K{}^2]^{1/2}$$

$$= \left(\begin{array}{l}\text{distance from K' to K, as measured by star} \\ \text{K' at the moment, in his frame, when the} \\ \text{clock of K reads } \tau_K\end{array}\right) . \tag{2.25}$$

By combining expressions (3.24c) and (3.24a) we obtain the relation

$$\vec{k}_{K'} = \varepsilon_{K'}\vec{b} + [\gamma\, \tau_K - \ell(\tau_K)]\, \vec{u}_{K'} - \tau_K\vec{u}_K \; ; \tag{2.26}$$

and by inserting this into expression (2.22) for $a_K{}^\alpha$ and using the relations

$$\vec{u}_K{}^2 = \vec{u}_{K'}{}^2 = -1 \;, \quad \vec{u}_K \cdot \vec{u}_{K'} = -\gamma \;, \quad \vec{u}_K \cdot \vec{b} = \vec{u}_{K'} \cdot \vec{b} = 0 \;,$$

we obtain

$$\vec{a}_K = \frac{m_{K'}}{\ell^3(\tau_K)} \{\gamma^2(1+v^2)\varepsilon_{K'}\vec{b} + \tau_K[\gamma^3(1-3v^2)\, \vec{u}_{K'} - \vec{u}_K]\} \; . \tag{2.27}$$

The spatial part of this acceleration, as seen in the rest frame of K', is depicted in Figure 2.

The perturbation $\vec{w}_K$ of the 4-velocity of star K is the integral of $\vec{a}_K$ over time τ_K, with initial value $\vec{w}_K(-\infty) = 0$. Straightforward integration gives

$$\vec{w}_K = m_{K'} \left\{ \gamma^2(1+v^2)\ \varepsilon_{K'} \frac{\vec{b}}{b^2} \left[\frac{\tau_K}{\ell(\tau_K)} + \frac{1}{\gamma v}\right] - \frac{\gamma^3(1-3v^2)\ \vec{u}_{K'} - \vec{u}_K}{\gamma^2 v^2\ \ell(\tau_K)} \right\} . \quad (2.28)$$

Notice that the total deflection angle of K as seen in the rest frame of K' is

$$\Delta\theta_K = \frac{|\text{spatial part of } \vec{w}_K(\infty)|}{|\text{spatial part of } \vec{u}_K|} = \frac{2\gamma^2(1+v^2)\ m_{K'}/(\gamma v b)}{\gamma v} = \frac{2(1+v^2)}{v^2} \frac{m_{K'}}{b} , \quad (2.29)$$

which agrees with equation (25.49) of MTW--where the derivation required $m_K \ll m_{K'}$. (Our derivation is valid for arbitrary relative masses of K and K'.)

If we wished, we could integrate $\vec{w}_K(\tau_K)$ to obtain $\vec{\mathfrak{z}}_K(\tau_K)$, the perturbation in the world line of star K due to gravitational deflection. We shall not do so because our calculation of the bremsstrahlung will require no explicit knowledge of $\vec{\mathfrak{z}}_K(\tau_K)$.

The chief results of this section are: (i) expressions (2.4) which split into perturbed and unperturbed parts the world lines, 4-velocities, and 4-accelerations of the two stars; (ii) expressions (2.7) for the unperturbed world line of star K; (iii) expression (2.27) for the 4-acceleration of star K; (iv) expression (2.28) for the perturbation of the 4-velocity of star K; and (v) expressions (2.18), (2.19), and (2.21) for the first-order gravitational field of star K' at points far from the surface of K' (distances $r \gg r_{K'}$).

III. GEOMETRY AND NOTATION FOR THE BREMSSTRAHLUNG CALCULATIONS

We turn now from the details of the stellar encounter to a calculation of the bremsstrahlung it produces. In this section we introduce the notation and geometric relationships to be used in the calculation.

Most of our analysis will be frame-independent. It will often utilize an expansion of vectors and tensors in terms of four "basis vectors"

$$\vec{u}_A = (\text{unperturbed 4-velocity of star A}) \quad , \tag{3.1a}$$

$$\vec{u}_B = (\text{unperturbed 4-velocity of star B}) \quad , \tag{3.1b}$$

$$\vec{\not{b}} \equiv \vec{b}/b = \begin{pmatrix}\text{unit vector, orthogonal to } \vec{u}_A \text{ and } \vec{u}_B \\ \text{and pointed along impact direction}\end{pmatrix} \tag{3.1c}$$

$$\vec{q} \equiv \begin{pmatrix}\text{unit vector, orthogonal to } \vec{u}_A,\ \vec{u}_B, \text{ and } \vec{\not{b}}\ , \\ \text{and such that } \vec{u}_A,\ \vec{u}_B,\ \vec{\not{b}}\ ,\ \vec{q} \text{ form a right-handed} \\ \text{system, i.e., } \epsilon_{\alpha\beta\gamma\delta}\, u_A^{\ \alpha}\, u_B^{\ \beta}\, b^\gamma\, q^\delta > 0\end{pmatrix} \quad . \tag{3.1d}$$

The scalar products of these vectors are [cf. eqs. (2.6) and (2.23)]

$$\begin{gathered}\vec{u}_A^{\ 2} = \vec{u}_B^{\ 2} = -1 \ , \quad \vec{u}_A \cdot \vec{u}_B = -\gamma \ , \quad \vec{\not{b}}^2 = \vec{q}^2 = +1 \ , \\ \vec{u}_A \cdot \vec{\not{b}} = \vec{u}_A \cdot \vec{q} = \vec{u}_B \cdot \vec{\not{b}} = \vec{u}_B \cdot \vec{q} = \vec{\not{b}} \cdot \vec{q} = 0 \quad .\end{gathered} \tag{3.2}$$

The observer is located far from the region of space where the stars encounter each other. His separation from the encounter region is described by a past directed null vector

$$\vec{k} \equiv \begin{pmatrix}\text{past-directed null vector reaching from the} \\ \text{observer's world line to the "collision event" } \vec{x}_c\end{pmatrix} \quad . \tag{3.3}$$

The scalar products of $\vec{k}$ with the 4-velocities of the stars will be denoted by

$$R_A \equiv \vec{k} \cdot \vec{u}_A = \begin{pmatrix}\text{distance from encounter region to ob-} \\ \text{server as seen in rest frame of star A}\end{pmatrix} , \tag{3.4a}$$

$$R_B \equiv \vec{k} \cdot \vec{u}_B = \begin{pmatrix}\text{distance from encounter region to ob-} \\ \text{server as seen in rest frame of star B}\end{pmatrix} . \tag{3.4b}$$

We shall sometimes work in the rest frame of star A. In that frame we shall use Minkowskii coordinates with $\vec{\mathcal{P}}$ along the y direction, $\vec{q}$ along the z direction, and the motion of B along the x direction (Fig. 3):

$$\vec{u}_A = \vec{e}_0 \;, \quad \vec{u}_B = \gamma\vec{e}_0 + v\gamma\vec{e}_x \;, \quad \vec{\mathcal{P}} = \vec{e}_y \;, \quad \vec{q} = \vec{e}_z \;. \tag{3.5a}$$

In this frame the null vector $\vec{k}$ is

$$\vec{k} = -R_A(\vec{e}_0 + \vec{n}) \quad ; \tag{3.5b}$$

$$n \equiv \begin{pmatrix}\text{unit spatial vector pointing from}\\ \text{collision event toward distant}\\ \text{observer}\end{pmatrix} = \alpha\vec{e}_x + \beta\vec{e}_y + \delta\vec{e}_z \quad ; \tag{3.5c}$$

$$\alpha^2 + \beta^2 + \delta^2 = 1 \quad . \tag{3.5d}$$

We shall refer to (α,β,δ) as the "direction cosines" of the observer.

We shall also sometimes work in the rest frame of star B using coordinates $\bar{t},\bar{x},\bar{y},\bar{z}$ with

$$\vec{u}_A = \gamma\vec{e}_{\bar{0}} - v\gamma\vec{e}_{\bar{x}} \;, \quad \vec{u}_B = \vec{e}_{\bar{0}} \;, \quad \vec{\mathcal{P}} = \vec{e}_{\bar{y}} \;, \quad \vec{q} = \vec{e}_{\bar{z}} \quad ; \tag{3.6a}$$

$$\vec{k} = -R_B(\vec{e}_{\bar{0}} + \vec{\bar{n}}) \quad ; \tag{3.6b}$$

$$\vec{\bar{n}} = \bar{\alpha}\vec{e}_{\bar{x}} + \bar{\beta}\vec{e}_{\bar{y}} + \bar{\delta}\vec{e}_{\bar{z}} \quad ; \tag{3.6c}$$

$$\bar{\alpha}^2 + \bar{\beta}^2 + \bar{\delta}^2 = 1 \quad . \tag{3.6d}$$

Using equations (3.2)-(3.6) one can readily derive the following relationships among the quantities that refer to star A and those that refer to star B:

$$R_A = \gamma(1 + \bar{\alpha}v)R_B \;, \quad R_B = \gamma(1 - \alpha v)R_A \;, \tag{3.7a}$$

$$\alpha = \frac{\bar{\alpha} + v}{1 + \bar{\alpha}v} \;, \qquad \bar{\alpha} = \frac{\alpha - v}{1 - \alpha v} \;, \tag{3.7b}$$

$$\beta = \frac{\bar{\beta}}{\gamma(1 + \bar{\alpha}v)} \;, \qquad \bar{\beta} = \frac{\beta}{\gamma(1 - \alpha v)} \;, \tag{3.7c}$$

$$\delta = \frac{\bar{\delta}}{\gamma(1+\bar{\alpha}v)} \quad , \qquad \bar{\delta} = \frac{\delta}{\gamma(1-\alpha v)} \quad . \tag{3.7d}$$

Note that the scalar products of the observer's null vector $\vec{k}$ with the fundamental basis vectors $\vec{u}_A$, $\vec{u}_B$, $\vec{\not{b}}$, $\vec{q}$ are

$$\vec{k}\cdot\vec{u}_A = R_A \; , \quad \vec{k}\cdot\vec{u}_B = R_B \; , \quad \vec{k}\cdot\vec{\not{b}} = -\beta R_A = -\bar{\beta} R_B$$

$$\vec{k}\cdot\vec{q} = -\delta R_A = -\bar{\delta} R_B \quad . \tag{3.8}$$

As the observer moves along his own world line, he sees a time-changing gravitational field (radiation). We shall express the time dependence in terms of two parameters, T_A and T_B, which are functions of the observer's proper time τ_o . These parameters are defined as follows (see Fig. 4a): At proper time τ_o the observer is at a specific event $\vec{Z}_o(\tau_o)$ along his world line. The past light cone from $\vec{Z}_o(\tau_o)$ intersects the unperturbed world line of star K at a proper time $\tau_K = \tau_{Ko}$ along that world line. We shall refer to τ_{Ko} as the "retarded time for star K", and we shall define

$$T_K \equiv v\gamma\tau_{Ko} \quad . \tag{3.9}$$

The distance from K' to K at time τ_{Ko}, as measured in the rest frame of K' we shall denote by

$$\ell_K \equiv \ell(\tau_{Ko}) = (b^2 + T_K{}^2)^{1/2} \tag{3.10}$$

(cf. eq. 2.25). A simple calculation in the rest frame of either star A or star B yields the following relationships between the two retarded times:

$$\tau_{Ao} = (1 - \alpha v)\gamma\tau_{Bo} - \beta b \;, \qquad \tau_{Bo} = (1 + \bar{\alpha}v)\gamma\tau_{Ao} + \bar{\beta}b \;; \tag{3.11a}$$

or, equivalently,

$$T_A = \gamma[T_B - v(\alpha T_B + \beta b)] \;, \qquad T_B = \gamma[T_A + v(\bar{\alpha}T_A + \bar{\beta}b)] \;. \tag{3.11b}$$

Figure 4b shows several 4-vectors which will play important roles in our derivation of the radiation field. The first of these is the vector

$$\vec{S} \equiv \vec{u}_B\tau_{Bo} - \vec{u}_A\tau_{Ao} + \vec{b} \equiv \begin{pmatrix}\text{vector reaching from star A at} \\ \text{time } \tau_{Ao} \text{ to star B at time } \tau_{Bo}\end{pmatrix} \;. \tag{3.12a}$$

Note that the squared length of $\vec{S}$

$$S^2 \equiv \vec{S}^2 = (\bar{\alpha}b - \bar{\beta}T_A)^2 + \bar{\delta}^2\ell_A^{\;2} = (\alpha b - \beta T_B)^2 + \delta^2\ell_B^{\;2} \tag{3.12b}$$

is positive, except in the special case that one star is precisely in the "shadow" of the other [$\vec{S}$ parallel to $\vec{k}$ and therefore null; $\bar{\delta} = \delta = 0$, $\bar{\alpha}b - \bar{\beta}T_A = \alpha b - \beta T_B = 0$]; in that case S^2 vanishes. Note also that $\vec{S}$ has the following projections on the 4-velocities of the stars:

$$\begin{aligned} S_A &\equiv \vec{S}\cdot\vec{u}_A = \tau_{Ao} - \gamma\tau_{Bo} = -(\alpha T_B + \beta b) \;, \\ S_B &\equiv \vec{S}\cdot\vec{u}_B = \gamma\tau_{Ao} - \tau_{Bo} = -(\bar{\alpha}T_A + \bar{\beta}b) \;; \end{aligned} \tag{3.12c}$$

and that these projections satisfy

$$S^2 + S_A^{\;2} = \ell_B^{\;2} \;, \qquad S^2 + S_B^{\;2} = \ell_A^{\;2} \;. \tag{3.12d}$$

Note also that $\vec{S}$ is orthogonal to $\vec{k}$:

$$\vec{S}\cdot\vec{k} = 0 \;. \tag{3.12e}$$

The next vector of interest is

$$\vec{Y} \equiv \vec{S} + S_A \, \vec{u}_A = (\text{projection of } \vec{S} \text{ orthogonal to A})$$

$$= \begin{pmatrix} \text{vector perpendicular to } \vec{u}_A \text{ and reaching} \\ \text{from world line of A to } B \text{ at } \tau_{Bo} \end{pmatrix} . \tag{3.13a}$$

Note that

$$\vec{Y}^2 = \ell_B^{\,2} \quad , \qquad \vec{Y} \cdot \vec{k} = S_A \, R_A \quad . \tag{3.13b}$$

Another useful vector is

$$\vec{M} \equiv \vec{S} + (S_B - \ell_A) \, \vec{u}_B = \begin{pmatrix} \text{past-directed null vector reaching} \\ \text{from B at } \tau_{Bo} \text{ to world line of A} \end{pmatrix} . \tag{3.14a}$$

Its scalar product with $\vec{u}_A$,

$$M_A \equiv \vec{M} \cdot \vec{u}_A = S_A - \gamma S_B + \gamma \ell_A = \gamma \ell_A - v\gamma T_A \quad , \tag{3.14b}$$

is used in the construction of another vector

$$\vec{J} \equiv \vec{M} + M_A \, \vec{u}_A = \vec{Y} + (S_B - \ell_A)(\vec{u}_B - \gamma \vec{u}_A)$$

$$= \begin{pmatrix} \text{vector reaching orthogonally from} \\ \text{world line of A to tip of } \vec{M} \end{pmatrix} \tag{3.15a}$$

Note that

$$\vec{J}^2 = (M_A)^2 \quad , \qquad \vec{J} \cdot \vec{k} = (S_B - \ell_A) \, R_B + M_A \, R_A \quad . \tag{3.15b}$$

Note also that in the rest frame of A

$$Y^x = T_B \quad , \qquad Y^y = b \ , \qquad Y^z = 0 \ , \qquad |\underset{\sim}{Y}| = \ell_B \quad , \tag{3.16a}$$

$$J^x = \gamma(T_A - v\ell_A) \ , \quad J^y = b \ , \quad J^z = 0 \ , \quad |\underset{\sim}{J}| = M_A = \gamma(\ell_A - vT_A) . \tag{3.16b}$$

A useful check on our final answer for the radiation field is provided by the following demand: If we interchange stars A and B, and reverse the sign of $\vec{b}$ so it once again reaches from A to B at the moment of closest approach, and precisely reverse the spatial direction of the observer as seen in either frame A or frame B, then the observer should see precisely the same radiation field as before. In other words, the radiation field should be unchanged by the conversions

$$\begin{aligned}
&\vec{u}_A \to \vec{u}_B , \quad \vec{u}_B \to \vec{u}_A , \quad \vec{p} \to -\vec{p} , \quad \vec{q} \to -\vec{q} , \\
&R_A \to R_B , \quad R_B \to R_A , \\
&T_A \to T_B , \quad T_B \to T_A , \quad \ell_A \to \ell_B , \quad \ell_B \to \ell_A , \\
&\bar{\alpha} \to -\alpha , \quad \bar{\beta} \to -\beta , \quad \bar{\delta} \to -\delta \\
&\alpha \to -\bar{\alpha} , \quad \beta \to -\bar{\beta} , \quad \delta \to -\bar{\delta} .
\end{aligned} \tag{3.17}$$

IV. FORMAL EXPRESSIONS FOR GRAVITATIONAL FIELD IN WAVE ZONE

In Paper I we derived a "plug-in-and-grind" formula for the gravitational field ${}_2\bar{h}^{\mu\nu}$ far outside any post-linear system. This gravitational field satisfies the Lorentz-gauge condition ${}_2\bar{h}^{\mu\nu}{}_{,\nu} = 0$ and has fractional errors of $O(\varepsilon^2)$, where ε is the magnitude of the internal field of the source. By contrast, the ${}_1\bar{h}^{\mu\nu}$ of §II has fractional errors of $O(\varepsilon)$.

The formal expressions for ${}_2\bar{h}^{\mu\nu}$ (§IV.c.iii of Paper I, with minor changes in notation) involve the following biscalars--i.e., the following scalar functions of a source point x' and a field point x: (i) the flat-space propagator

$$ {}_0G(x,x') \equiv \frac{1}{4\pi}\,\delta_{\rm ret}[\tfrac{1}{2}\,(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'})\eta_{\alpha\beta}] \tag{4.1} $$

[cf. eq. (2.11) and associated discussion]; (ii) the derivative of the flat-space propagator with respect to the interval $\frac{1}{2}\,(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'})\eta_{\alpha\beta}$:

$$ {}_0G'(x,x') \equiv \frac{1}{4\pi}\,\delta'_{\rm ret}[\tfrac{1}{2}(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'})\eta_{\alpha\beta}] $$

$$ = -\frac{1}{(x^0 - x^{0'})}\,\frac{\partial}{\partial x^0}\,{}_0G(x,x') \quad ; \tag{4.2} $$

(iii) the focussing function (eqs. I,40a and I,59b)

$$ \alpha(x,x') = \tfrac{1}{2}(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'}) \int_0^1 {}_1R_{\alpha\beta}[x' + \lambda(x-x')]\;\lambda(1-\lambda)\,d\lambda \quad ; \tag{4.3a} $$

(iv) the tail generator (eqs. I,43a and I,59c)

$$ \beta(x,x') = (x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'}) \int_0^1 {}_1R_{\alpha\beta}[x' + \lambda(x-x')]\lambda^2\,d\lambda \quad ; \tag{4.3b} $$

and (v) the time-delay function (eqs. I,38b and I,59d)

$$ \gamma(x,x') = \tfrac{1}{2}\,(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'}) \int_0^1 {}_1h_{\alpha\beta}[x' + \lambda(x-x')]\,d\lambda \quad . \tag{4.3c} $$

Here ${}_1h_{\alpha\beta}$ is the post-linear metric perturbation of §II, ${}_1R_{\alpha\beta}$ is the Ricci curvature tensor calculated from the post-linear metric to first order in ${}_1h_{\alpha\beta}$, and λ is a flat-space affine parameter along the straight line reaching from x' to x (cf. eq. I,36). In place of the full time-delay function $\gamma(x,x')$ we shall be using a truncated version of it:

$$ \gamma^T(x,x') = \gamma(x,x') - \begin{pmatrix}\text{contribution produced by source's}\\ \text{stationary, external, "1/r" field}\end{pmatrix} \quad . \tag{4.3c'} $$

The truncation process is discussed at length in §IV.c.ii of Paper I; and we shall make it explicit in §VI below.

The external field ${}_2\bar{h}^{\mu\nu}$ consists of five parts (eqs. I,58):

a "direct field"

$$ {}_2\bar{h}^{\mu\nu}_D(x) = 16\pi \int {}_0G(x,x')\ T^{\mu\nu}(x')[1 - {}_1\bar{h}(x')]\ d^4x' \quad ; \tag{4.4a}$$

a "focusing field"

$$ {}_2\bar{h}^{\mu\nu}_F(x) = 16\pi \int \alpha(x,x')\ {}_0G(x,x')\ T^{\mu\nu}(x')\ d^4x' \quad ; \tag{4.4b}$$

a "transition field"

$$ {}_2\bar{h}^{\mu\nu}_{TR}(x) = 16\pi \int \gamma^T(x,x')\ {}_0G'(x,x')\ T^{\mu\nu}(x')\ d^4x' \quad ; \tag{4.4c}$$

a "tail field"

$$ {}_2\bar{h}^{\mu\nu}_{TL}(x) = -16\pi \iint {}_0G(x,x'')\ \beta(x'',x')\ {}_0G'(x'',x')\ T^{\mu\nu}(x')d^4x''\ d^4x' \ , \tag{4.4d}$$

where the integral is over source points x' inside and not on the past light cone of x; and a "whump field"

$$ {}_2\bar{h}^{\mu\nu}_W(x) = 16\pi \int {}_0G(x,x')\ W^{\mu\nu}(x')\ d^4x' \quad . \tag{4.4e}$$

The external field is the sum of these five parts

$$ {}_2\bar{h}^{\mu\nu} = {}_2\bar{h}^{\mu\nu}_D + {}_2\bar{h}^{\mu\nu}_F + {}_2\bar{h}^{\mu\nu}_{TR} + {}_2\bar{h}^{\mu\nu}_{TL} + {}_2\bar{h}^{\mu\nu}_W \quad . \tag{4.5}$$

The sources of the external field are the post-linear stress-energy tensor $T^{\mu\nu}$ and the gravitational "whump stresses" (eqs. I,58e and I,23g)

$$ W^{\mu\nu} \equiv M^{\mu\nu\alpha\beta\gamma\lambda\rho\sigma}\ {}_1\bar{h}_{\alpha\beta,\gamma}\ {}_1\bar{h}_{\lambda\rho,\sigma} \quad , \tag{4.6a}$$

where

$$ \begin{aligned} M^{\mu\nu\alpha\beta\gamma\lambda\rho\sigma} \equiv \frac{1}{16\pi} \{ & \tfrac{1}{2}\, \eta^{\mu\nu}\eta^{\alpha\rho}\eta^{\beta\sigma}\eta^{\gamma\lambda} + \eta^{\mu\alpha}\eta^{\nu\lambda}(\eta^{\gamma\sigma}\eta^{\beta\rho} + \eta^{\beta\sigma}\eta^{\gamma\rho}) \\ & - (\eta^{\mu\sigma}\eta^{\nu\alpha}\eta^{\beta\lambda}\eta^{\gamma\rho} + \eta^{\mu\alpha}\eta^{\nu\sigma}\eta^{\gamma\rho}\eta^{\beta\lambda}) \\ & + \tfrac{1}{8}\,(2\eta^{\mu\gamma}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\gamma\sigma})(2\eta^{\alpha\lambda}\eta^{\beta\rho} - \eta^{\lambda\rho}\eta^{\alpha\beta})\} \quad . \end{aligned} \tag{4.6b}$$

For some purposes it is useful to express the sum of the focusing, transition, and tail fields in the following form (eqs. II,36 and II,37c):

$$ {}_2\bar{h}^{\mu\nu}_{F}(x) + {}_2\bar{h}^{\mu\nu}_{TR}(x) + {}_2\bar{h}^{\mu\nu}_{TL}(x) \equiv {}_2\bar{h}^{\mu\nu}_{FTRTL}(x) $$

$$ = -\partial_\alpha \int {}_0G(x,x')\; {}_1\bar{h}^{\alpha\beta}(x')\; \partial_{\beta'}\; {}_1\bar{h}^{\mu\nu}(x')\; d^4x' \quad . \tag{4.7} $$

Here $\partial_\mu \equiv \partial/\partial x^\mu$ and $\partial_{\mu'} \equiv \partial/\partial x^{\mu'}$.

In order to use the above formalism, we need explicit expressions for the post-linear stress-energy tensor $T^{\mu\nu}$ and gravitational field ${}_1\bar{h}^{\mu\nu}$.

The stress-energy splits into parts $T_A{}^{\mu\nu}$ and $T_B{}^{\mu\nu}$ associated with the two stars

$$ T^{\mu\nu} = T_A{}^{\mu\nu} + T_B{}^{\mu\nu} \quad . \tag{4.8} $$

In its own instantaneous rest frame, each star is assumed to be a static Newtonian sphere[1] with stress-energy tensor that has physical components

[1]This assumption is needed for ease of exposition in §VI. However, Crowley and Thorne showed in Paper II that the bremsstrahlung radiation is independent of the internal structures of the stars. The stars need only satisfy the constraints of equations (2.1).

$$ T_K{}^{\hat{0}\hat{0}} = \rho_K(r) \;, \quad T_K{}^{\hat{0}\hat{j}} = 0 \;, \quad T_K{}^{\hat{j}\hat{k}} \ll \rho_K(r) \quad \text{for} \quad 0 < r < r_K $$
$$ T_K{}^{\alpha\beta} = 0 \quad \text{for} \quad r > r_K \quad ; \quad r \equiv (x^2 + y^2 + z^2)^{1/2} \quad . \tag{4.9} $$

The center of the star (origin of reference frame for eq. [4.9]) moves along the world line $Z_K^{\alpha}(\tau_K)$ which was computed in §II. At most stages of the calculation we can ignore the internal structures of the stars, and treat them as point masses with the following stress-energy tensor (valid in any post-linear reference frame):

$$T_K^{\mu\nu}(x') = \int m_K U_K^{\mu}(\tau_K)\, U_K^{\nu}(\tau_K)[1+\tfrac{1}{2}\,{}_1\bar{h}(x')]\,\delta_4[x'-Z_K(\tau_K)]d\tau_K \,. \quad (4.10)$$

Here $Z_K^{\alpha}(\tau_K)$ and $U_K^{\mu}(\tau_K)$ are the world line and 4-velocity of star K at time τ_K, as given by equations (2.4), (2.7), and (2.28); and $[1+\frac{1}{2}\,{}_1\bar{h}]$ is an approximation to $(-g)^{-1/2}$.

Like the stress-energy tensor, the post-linear gravitational field ${}_1\bar{h}^{\mu\nu}$ splits into two parts produced by the two stars

$$ {}_1\bar{h}^{\mu\nu} = {}_1\bar{h}_A^{\mu\nu} + {}_1\bar{h}_B^{\mu\nu} \quad (4.11a)$$

[cf. eq. (2.12)]. For field points far from star K we can write

$$ {}_1\bar{h}_K^{\mu\nu}(x) = \frac{4m_K u_K^{\mu}\, u_K^{\nu}}{\vec{\mathcal{K}}_K \cdot \vec{u}_K} \,, \quad (4.11b)$$

where $\vec{u}_K$ is the unperturbed 4-velocity of star K and $\vec{\mathcal{K}}_K$ is the unique past-directed null vector from the field point x to the unperturbed world line of K; see equation (2.18). This expression for ${}_1\bar{h}_K^{\mu\nu}(x')$ ignores the gravitational deflection of the stars' trajectories—i.e., it makes fractional errors of order ζ . It is easy to write an expression for ${}_1\bar{h}_K^{\mu\nu}(x')$ which takes account of the deflection of trajectories, and which also takes account of the internal structure of star K--but we shall not need such expressions.

The largest component of our external field is

$$ {}_2\bar{h}^{00} \sim m_A/R_A + m_B/R_B \quad . \tag{4.12a} $$

The radiative part of the field will be

$$ {}_2\bar{h}^{TT}_{jk} \sim \frac{m_A m_B}{R_A b} + \frac{m_A m_B}{R_B b} \sim {}_2\bar{h}^{00}\left(\frac{m_A + m_B}{b}\right) \equiv {}_2\bar{h}^{00}\,\varepsilon \quad . \tag{4.12b} $$

The fractional errors in our formulas for ${}_2\bar{h}^{\mu\nu}$ are $O(\varepsilon^2)$. Consequently, any terms which are cubic in the masses (e.g., $m_A{}^2 m_B/R_A b^2$) or of higher order can be shoved into the error and ignored. We shall do this ruthlessly throughout the computation. For example, in all parts of ${}_2\bar{h}^{\mu\nu}$ except the "direct" part, $T^{\mu\nu}$ appears multiplied by a quantity proportional to m_A or m_B (see eqs. [4.4b,c,d]). In such cases we need keep only those parts of $T^{\mu\nu}$ which are linear in m_A or m_B--which means that in place of equation (4.10) we can use the stress-energy tensor of an undeflected star

$$ T_K{}^{\mu\nu}(x') = m_K u_K{}^{\mu}\, u_K{}^{\nu} \int \delta_4[x' - z_K(\tau_K)]\, d\tau_K \quad . \tag{4.13} $$

V. DIRECT FIELD

When $T^{\mu\nu}$ and ${}_1\bar{h}^{\mu\nu}$ are split up into "A + B" contributions, expression (4.4a) for the direct field reads

$$ {}_2\bar{h}_D^{\mu\nu}(x) = 16\pi \int {}_0G(x,x')\, [T_A{}^{\mu\nu}(1 - {}_1\bar{h}_B) + T_B{}^{\mu\nu}(1 - {}_1\bar{h}_A)]_{\text{at } x'}\, d^4x' $$

$$ -16\pi \int {}_0G(x,x')\, [T_A{}^{\mu\nu}\, {}_1\bar{h}_A + T_B{}^{\mu\nu}\, {}_1\bar{h}_B]_{\text{at } x'}\, d^4x' \quad . \tag{5.1} $$

The second line is of quadratic order in the masses--and at quadratic order it is ignorant of the gravitational deflection of the stars' trajectories;

only the first-order (undeflected) $T_K^{\mu\nu}$ and ${}_1\bar{h}_K$ enter. As a result, the second line has nothing whatsoever to do with the radiation. It is simply a nonlinear contributor to the stationary self fields of stars A and B. Because we are not interested in the nonlinearities of the self fields, we shall drop the second line and write

$$ {}_2\bar{h}_D^{\mu\nu}(x) = \sum_K 16\pi \int {}_0G(x,x')\ T_K^{\mu\nu}(1 - {}_1\bar{h}_{K'})\ d^4x' \quad . \tag{5.2} $$

Here the summation is over $K = A$ and B, and K' means "not K".

Since the field point x is far outside the source, we can use the point-mass stress-energy tensor (4.10) in evaluating expression (5.2). By inserting it into (5.2), integrating over x' to remove the δ_4 of $T_K^{\mu\nu}$, integrating over τ_K to remove the δ_{ret} of ${}_0G$ (eq. 4.1), and omitting nonlinear self-fields as above, we obtain

$$ {}_2\bar{h}_D^{\mu\nu}(x) = \sum_K \left[\frac{4m_K\ U_K^{\mu}\ U_K^{\nu}}{\vec{\mathcal{K}}_K \cdot \vec{U}_K} (1 - \tfrac{1}{2}\ {}_1\bar{h}_{K'}) \right]_{ret} \quad . \tag{5.3} $$

Here $\vec{\mathcal{K}}_K$ is the unique past-directed null vector reaching from x to the perturbed world line of K, and "ret" means "evaluated at the tip of $\vec{\mathcal{K}}_K$." Because we are interested only in the "1/r" (radiative) part of the field, we can replace $\vec{\mathcal{K}}_K$ by the vector $\vec{k}$ of §III and Fig. 4; such a replacement produces time-dependent errors of $O(1/r^2)$. By making this replacement, by splitting $\vec{U}_K$ into its unperturbed and perturbed parts (eq. 2.4b), and by dropping terms cubic in the masses, we obtain

$$ \begin{aligned} {}_2\bar{h}_D^{\mu\nu}(x) = {} & \sum_K \left[\frac{4m_K\ u_K^{\mu}\ u_K^{\nu}}{\vec{k}\cdot\vec{u}_K} \right]_{ret} \\ & + \sum_K \left\{ \frac{4m_K}{\vec{k}\cdot\vec{u}_K} \left[2u_K^{(\mu}\ w_K^{\nu)} - u_K^{\mu}\ u_K^{\nu} \left(\frac{\vec{k}\cdot\vec{w}_K}{\vec{k}\cdot\vec{u}_K} + \tfrac{1}{2}\ {}_1\bar{h}_{K'} \right) \right] \right\}_{ret} \quad . \end{aligned} \tag{5.4} $$

The first line is the non-radiative "Coulomb" gravitational field of the source. It has nothing whatsoever to do with radiation, so we shall drop it. The second is the "direct" part of the radiation field. Because it is quadratic in the masses, we make a negligible error by evaluating it not at the <u>true</u> retarded time of the deflected trajectory, but rather at the retarded time τ_{Ko} of the unperturbed trajectory (cf. Fig. 4a):

$$ {}_2\bar{h}_D^{\mu\nu}(x) = \sum_K \left\{ \frac{4m_K}{\vec{k}\cdot\vec{u}_K} \left[2u_K^{(\mu} w_K^{\nu)} - u_K^{\mu} u_K^{\nu} \left(\frac{\vec{k}\cdot\vec{w}_K}{\vec{k}\cdot\vec{u}_K} + \frac{1}{2}\, {}_1\bar{h}_{K'} \right) \right] \right\}_{\text{at } z_K(\tau_{Ko})} . \tag{5.5} $$

The field ${}_1\bar{h}_{K'}[z_K(\tau_{Ko})]$ can be evaluated by taking the trace of expression (4.11b) and by invoking (2.24a)

$$ {}_1\bar{h}_{K'}[z_K(\tau_{Ko})] = \frac{-4m_{K'}}{\vec{k}_{K'}\cdot\vec{u}_{K'}} , $$

and by then using expression (2.26) to evaluate $\vec{k}_{K'}$. The result is

$$ {}_1\bar{h}_{K'}[z_K(\tau_{Ko})] = -4m_{K'}/\ell(\tau_{Ko}) = -4m_{K'}/\ell_K \tag{5.6} $$

[cf. eq. (3.10)]. By inserting this expression into (5.5), and by inserting expression (2.28) for $w_K(\tau_K)$, and by invoking equations (2.7b), (3.1c), (3.4), (3.8), (3.9), and (3.7a,c), we bring the direct field into the final form

$$ \begin{aligned} \frac{{}_2\bar{h}_D^{\mu\nu}}{4m_A m_B} = {}& u_A^{\mu} u_A^{\nu} \left[\frac{2(1-v^2) + \bar{\alpha}v(1+v^2)}{v^2(1+\bar{\alpha}v)R_A\ell_A} + \frac{\bar{\beta}(1+v^2)}{v(1+\bar{\alpha}v)R_A b} \left(\frac{T_A}{\ell_A} + 1 \right) \right] \\ & + u_B^{\mu} u_B^{\nu} \left[\frac{2(1-v^2) - \alpha v(1+v^2)}{v^2(1-\alpha v)R_B\ell_B} - \frac{\beta(1+v^2)}{v(1-\alpha v)R_B b} \left(\frac{T_B}{\ell_B} + 1 \right) \right] \\ & + u_A^{(\mu} u_B^{\nu)} \left[\frac{-2\gamma(1-3v^2)}{v^2} \left(\frac{1}{R_A\ell_A} + \frac{1}{R_B\ell_B} \right) \right] \end{aligned} $$

$$+ \not{b}^{(\mu}u_A^{\nu)}\left[\frac{2\gamma(1+v^2)}{vR_A b}\left(\frac{T_A}{\ell_A}+1\right)\right]$$

$$+ \not{b}^{(\mu}u_B^{\nu)}\left[\frac{-2\gamma(1+v^2)}{vR_B b}\left(\frac{T_B}{\ell_B}+1\right)\right] \quad . \tag{5.7}$$

As a check on this result, one can verify that it is invariant under the transformation (3.17).

VI. FOCUSING, TRANSITION, AND TAIL FIELDS

There are two methods for calculating the focusing, transition, and tail fields: that of Paper I (eqs. 4.4b,c,d above) and that of Paper II (eq. 4.7 above). Method I has the disadvantage that it requires a very special treatment of "shadow regions" (i.e., of observation events x from which one star is viewed as lying partially in front of the other). However, outside the shadow regions method I has the advantage that the Dirac delta functions in the integrands cause all volume integrals to collapse into line integrals. Method II, on the other hand, permits one to treat shadow regions on an equal footing with non-shadow regions; but its volume integrals do not collapse into line integrals. As indicated in Paper II, method II is powerful for proving theorems, while method I is powerful for carrying out explicit computations.

In this section we first use method II to show that the sum of the focusing, transition, and tail fields, ${}_2\bar{h}^{\mu\nu}_{FTRTL}(x)$, has a form in shadow regions which is the analytic extrapolation of its nonshadow form. We then use method I to calculate analytically the nonshadow form of ${}_2\bar{h}^{\mu\nu}_{FTRTL}$, and we assert on the basis of the method-II analysis that this analytic result is valid also in shadow regions.

For the method-II analysis pick a specific event Z_B on the world line of the center of star B, and construct the future light cone $\dot{J}_+(Z_B)$ in the rest frame of star A. Figure 5 is a spatial diagram of $\dot{J}_+(Z_B)$. In that figure the shadow region is the extrapolation to large distances of the cone marked "shadow region." The half-angle of the shadow cone, θ_s , is

$$\theta_s = (r_A + r_B)/b \ll 1 \quad . \tag{6.1}$$

In §V of Paper II Crowley and Thorne showed that throughout the method-II computation one can use a monopole, point-mass idealization of the stars' structures without producing any fractional errors larger than $\gamma r_A/b$ and $\gamma r_B/b$. This is equally true in the shadow regions and outside the shadow regions. This can only be true if the shadow has no influence on the form of $\bar{h}^{\mu\nu}_{FTRTL}$--i.e., if $\bar{h}^{\mu\nu}_{FTRTL}$ in the (very small) shadow regions is obtainable by analytic continuation of its form in the adjacent non-shadow regions.

Having established this important fact, we now compute the nonshadow form of $\bar{h}^{\mu\nu}_{FTRTL}$ using method I. In nonshadow regions the focussing field vanishes, so we need compute only the transition field and the tail field, and then add them together.

a) Transition Field

In expression (4.4c) for the transition field we can split the time-delay function γ^T and the stress-energy tensor $T^{\mu\nu}$ into parts due to each star. As with the direct field (§V), products involving the same star--i.e., $\gamma_A{}^T\, T_A{}^{\mu\nu}$ and $\gamma_B{}^T\, T_B{}^{\mu\nu}$--lead to nonradiative, nonlinear corrections to self-fields; they are of no interest to us, so we drop them. Expression (4.4c) then becomes

$$ {}_2\bar{h}^{\mu\nu}_{TR}(x) = \sum_K {}_2\bar{h}^{\mu\nu}_{TR,K}(x) \quad ; \qquad {}_2\bar{h}_{TR,K}(x) = 16\pi \int {}_0G'(x,x')\gamma_{K'}{}^{T}(x,x')T_K{}^{\mu\nu}(x')d^4x' \; . \tag{6.2} $$

Let us calculate $\gamma_A{}^T(x,x')$ in the rest frame of star A, for x the location of the distant observer and for x and x' such that the ray joining them does not pass through star A (no shadow at x); see Figure 6. We begin by calculating the untruncated time-delay function of equation (4.3c):

$$ \gamma_A(x,x') = \frac{1}{2}\,(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'}) \int_0^1 {}_1h_{A\alpha\beta}[x' + \lambda(x-x')]d\lambda \quad . \tag{6.3} $$

The post-linear field ${}_1h_{A\alpha\beta}$ produced by star A in its own rest frame (Fig. 6) is

$$ {}_1h_{A\alpha\beta}(x'') = \frac{2m_A}{|\underset{\sim}{x}''|}\,\delta_{\alpha\beta} \qquad ; \tag{6.4} $$

cf. eq. (2.19). Here $\underset{\sim}{x}''$ is the spatial part of the vector $x^{\alpha''}$ reaching from the origin (center of star A) to event x". This expression is valid not only at events far from star A, but also arbitrarily close to A's surface, because A is spherically symmetric (eq. 4.9 and footnote 1). Define

$$ X^\alpha \equiv x^\alpha - x^{\alpha'} \quad , \tag{6.5} $$

and combine with equations (6.3) and (6.4) to obtain

$$ \gamma_A(x,x') = m_A[(X^0)^2 + |\underset{\sim}{X}|^2]\int_0^1 \frac{d\lambda}{|\underset{\sim}{x}' + \lambda\underset{\sim}{X}|} \quad . \tag{6.6} $$

A straightforward integration yields

$$ \gamma_A(x,x') = \frac{m_A[(X^0)^2 + |\underset{\sim}{X}|^2]}{|\underset{\sim}{X}|}\,\ell n \left| \frac{|\underset{\sim}{x}|\,|\underset{\sim}{X}| + \underset{\sim}{x}\cdot\underset{\sim}{X}}{|\underset{\sim}{x}'|\,|\underset{\sim}{X}| + \underset{\sim}{x}'\cdot\underset{\sim}{X}} \right| \quad . \tag{6.7} $$

The truncation process, as described in §IV.c.ii of Paper I, consists of dropping the contribution to the delay function (6.6) which comes from outside the immediate neighborhood of the source--e.g., the contribution from outside a radius of ~10b. Performing this truncation, we get

$$\gamma_A{}^T(x,x') = \frac{m_A[(X^0)^2 + |\underset{\sim}{X}|^2]}{|\underset{\sim}{X}|} \ln\left(\frac{10b|\underset{\sim}{x}|}{|\underset{\sim}{x}'||\underset{\sim}{X}| + \underset{\sim}{x}'\cdot\underset{\sim}{X}}\right) . \qquad (6.8)$$

We now manipulate ${}_2\bar{h}^{\mu\nu}_{TR,B}(x)$ (eq. 6.2) keeping in mind equation (6.8) for $\gamma_A{}^T$ and using expression (4.2) for ${}_0G'$

$$\begin{aligned} {}_2\bar{h}^{\mu\nu}_{TR,B}(x) &= -16\pi \int \frac{1}{X^0} [\frac{\partial}{\partial x^0}\, {}_0G(x,x')]\, \gamma_A{}^T(x,x')\, T_B{}^{\mu\nu}(x')\, d^4x' \\ &= -16\pi \frac{\partial}{\partial x^0} \int {}_0G(x,x') \frac{1}{X^0} \gamma_A{}^T(x,x')\, T_B{}^{\mu\nu}(x')\, d^4x' \\ &\quad +16\pi \int {}_0G(x,x') \frac{\partial}{\partial x^0}\left[\frac{1}{X^0}\gamma_A{}^T(x,x')\right] T_B{}^{\mu\nu}(x')\, d^4x' \quad . \qquad (6.9) \end{aligned}$$

Because $[(1/X^0)\gamma_A{}^T(x,x')]$ depends on x^0 only through the quantity $X^0 = x^0 - x^{0'}$, and because the propagator forces $X^0 = |\underset{\sim}{X}| \simeq R_A$ = (distance to observer), the differentiation on the last line of (6.9) causes that line to be $\propto 1/R_A{}^2$ (after ${}_0G$ has been integrated out), rather than $\propto 1/R_A$ --i.e., it makes the last line nonradiative. For this reason we can drop the last line and write

$$ {}_2\bar{h}^{\mu\nu}_{TR,B}(x) = -16\pi \frac{\partial}{\partial x^0} \int {}_0G(x,x') \frac{1}{X^0} \gamma_A{}^T(x,x')\, T_B{}^{\mu\nu}(x')\, d^4x' \quad . \qquad (6.10)$$

By inserting expression (6.8) for $\gamma_A{}^T$, expression (4.13) for the "undeflected" $T_B{}^{\mu\nu}$, and expression (4.1) for ${}_0G(x,x')$, and then integrating out the delta functions we obtain

$$2\bar{h}^{\mu\nu}_{TR,B} = -\frac{\partial}{\partial x^0}\left\{\left[\frac{4m_A m_B u_B{}^\mu u_B{}^\nu}{\vec{k}\cdot\vec{u}_B}\,\frac{[(X^0)^2+|\underset{\sim}{X}|^2]}{X^0|\underset{\sim}{X}|}\,\ln\left|\frac{10b|\underset{\sim}{x}|}{|\underset{\sim}{x}'|\,|\underset{\sim}{X}| + \underset{\sim}{x}'\cdot\underset{\sim}{X}}\right|\right]_{\underset{\sim}{x}'=\underset{\sim}{z}_B(\tau_{Bo})}\right\} . \tag{6.11}$$

At the evaluation point, $\underset{\sim}{x}' = \underset{\sim}{z}_B(\tau_{Bo})$, X^μ is null with

$$X^0 = x^0 - z_B{}^0(\tau_{Bo}) = |\underset{\sim}{X}| = |\underset{\sim}{x}| - \underset{\sim}{n}\cdot\underset{\sim}{z}_B(\tau_{Bo});$$
$$\underset{\sim}{X}/|\underset{\sim}{X}| = \underset{\sim}{n} + 0(|\underset{\sim}{z}_B|/R_B); \qquad \underset{\sim}{n} \equiv \underset{\sim}{x}/|\underset{\sim}{x}| \quad . \tag{6.12}$$

Consequently, to order $1/R_B$ (radiative order)

$$2\bar{h}^{\mu\nu}_{TR,B} = \frac{8m_A m_B u_B{}^\mu\, u_B{}^\nu}{R_B}\left[\frac{(d\tau_{Bo}/dx^0)\ d/d\tau_B(|\underset{\sim}{z}_B| + \underset{\sim}{n}\cdot\underset{\sim}{z}_B)}{|\underset{\sim}{z}_B| + \underset{\sim}{n}\cdot\underset{\sim}{z}_B}\right]_{\tau_B=\tau_{Bo}} \tag{6.13}$$

here $d\tau_{Bo}/dx^0$ follows from eqs. (6.12), (2.4b), and Figure 3:

$$\frac{dx^0}{d\tau_{Bo}} = \left\{\frac{d}{d\tau_B}(z_B{}^0 - \underset{\sim}{n}\cdot\underset{\sim}{z}_B)\right\}_{\tau_{Bo}} = u_B{}^0 - \underset{\sim}{n}\cdot\underset{\sim}{u}_B = \gamma(1-\alpha v) \quad . \tag{6.14}$$

The quantities $|\underset{\sim}{z}_B|$ and $\underset{\sim}{n}\cdot\underset{\sim}{z}_B$ are conveniently evaluated by reference to Figure 4 and the associated discussion in §III: $\underset{\sim}{z}_B(\tau_{Bo})$ can be thought of as a 4-vector which is purely spatial in the rest frame of A. When thought of thus it is identical to the 4-vector $\vec{Y}$ of Figure 4. This fact, together with the relation $\underset{\sim}{n} = -\underset{\sim}{k}/R_A$ and equations (3.13b), implies that

$$\underset{\sim}{n}\cdot\underset{\sim}{z}_B(\tau_{Bo}) = -\vec{k}\cdot\vec{Y}/R_A = -S_A\ , \qquad |\underset{\sim}{z}_B(\tau_{Bo})| = |\vec{Y}| = \ell_B\ . \tag{6.15}$$

By combining this result with expression (3.12c) for S_A, with expression (3.10) for ℓ_B, and with expression (3.9) for T_B we obtain

$$[d/d\tau_B(|\underset{\sim}{z}_B| + \underset{\sim}{n}\cdot\underset{\sim}{z}_B)]_{\tau_{Bo}} = v\gamma(\alpha + T_B/\ell_B) \quad . \tag{6.16}$$

By combining equations (6.15), (3.12d), and (3.12c) we obtain

$$\left[\frac{1}{|\underset{\sim}{z}_B| + \underset{\sim}{n}\cdot\underset{\sim}{z}_B}\right]_{\tau_{Bo}} = \frac{1}{\ell_B - S_A} = \frac{\ell_B + S_A}{S^2} = \frac{\ell_B - \alpha T_B - \beta b}{S^2} \quad . \tag{6.17}$$

Finally, by inserting equations (6.14), (6.16), and (6.17) into (6.13) and invoking equations (3.5d) and (3.10) we obtain

$$_2\bar{h}^{\mu\nu}_{TR,B} = \left\{\frac{8m_A m_B v b(\alpha b - \beta T_B)}{R_B \ell_B S^2(1-\alpha v)} - \frac{8m_A m_B v[-\delta^2 T_B + \beta(\alpha b - \beta T_B)]}{R_B S^2(1-\alpha v)}\right\} u_B{}^\mu\, u_B{}^\nu . \tag{6.18}$$

The analogous expression for $_2\bar{h}^{\mu\nu}_{TR,A}$ can be derived either by an analogous calculation in the rest frame of B, or by invoking the invariance relation (3.17). The total transition field $(_2\bar{h}^{\mu\nu}_{TR,A} + {}_2\bar{h}^{\mu\nu}_{TR,B})$ is then

$$\frac{_2\bar{h}^{\mu\nu}_{TR}}{4m_A m_B} = u_A{}^\mu\, u_A{}^\nu \left\{-\frac{2vb(\bar{\alpha}b - \bar{\beta}T_A)}{R_A \ell_A S^2(1+\bar{\alpha}v)} - \frac{2v[-\bar{\delta}^2 T_A + \bar{\beta}(\bar{\alpha}b - \bar{\beta}T_A)]}{R_A S^2(1+\bar{\alpha}v)}\right\}$$
$$+ u_B{}^\mu\, u_B{}^\nu \left\{\frac{2vb(\alpha b - \beta T_B)}{R_B \ell_B S^2(1-\alpha v)} - \frac{2v[-\delta^2 T_B + \beta(\alpha b - \beta T_B)]}{R_B S^2(1-\alpha v)}\right\} \quad . \tag{6.19}$$

Note that this transition field diverges as $S^2 \to 0$ --i.e., as $\vec{S}$ becomes null--i.e., as $\vec{S}$ becomes parallel to $\vec{k}$ --i.e., as the observer moves into the shadow region; cf. Figure 4. It is easy to verify that the divergence has the form

$$_2\bar{h}^{\mu\nu}_{TR,K} \propto \frac{1}{b_{ray}}\frac{db_{ray}}{dx^0}\ , \quad b_{ray} \equiv \left(\begin{array}{l}\text{impact parameter of ray from star K,}\\ \text{as it passes star K}' \text{ on its way to}\\ \text{observer at } x\end{array}\right) \tag{6.20}$$

(cf. eq. 6.13). We will discuss the significance of this divergence in §VI.c below.

b) Tail Field

By discarding terms that represent nonlinear corrections to self fields, we can rewrite the tail field (4.4d) as

$$ {}_2\bar{h}^{\mu\nu}_{TL}(x) = \sum_K {}_2\bar{h}^{\mu\nu}_{TL,K}(x) \quad , \tag{6.21a}$$

$$ {}_2\bar{h}^{\mu\nu}_{TL,K}(x) = -16\pi \iint {}_0G(x,x'')\ \beta_{K'}(x'',x')\ {}_0G'(x'',x')\ T_K{}^{\mu\nu}(x')d^4x''d^4x' \ . \tag{6.21b}$$

Here the x' integral is over events x' inside and not on the past light cone of x.

Let us calculate $\beta_A(x'',x')$ in the rest frame of star A. In doing so we must take account of the internal structure of star A, because β_A is zero unless the straight line from x' to x'' passes through A. By combining expression (4.3b) for β with the post-linear field equation ${}_1R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2} T^\gamma{}_\gamma \eta_{\alpha\beta})$ and with expression (4.9) for $T_{A\alpha\beta}$, and by using spherical polar coordinates centered on the event x' (Fig. 7a), and by expressing the affine parameter in terms of radius r as $\lambda = r/r''$, we obtain

$$\beta_A(x'',x') = [(t'')^2 + (r'')^2] \int_0^{r''} 4\pi\rho_A(r/r'')^2\ dr/r'' \quad . \tag{6.22}$$

Let us now perform the integral over x'' in expression (6.21b), using expression (4.2) for ${}_0G'$:

$$\int {}_0G(x,x'')\beta_A(x'',x')\ {}_0G'(x'',x')d^4x''$$

$$= \int {}_0G(x,x'')\ \beta_A(x'',x')\Big[\frac{-1}{t''}\frac{\partial}{\partial t''}\ {}_0G(x'',x')\Big]\ d^4x'' \tag{6.23}$$

$$= + \int \frac{\partial}{\partial t''}\Big[\frac{1}{t''}\ {}_0G(x,x''_{CL}) \int \beta_A(x'',x')d\Omega''\Big]\ {}_0G(x''_{CL},x')dt''r''^2dr'' \ .$$

In the third line we have used integration by parts, plus the fact that, as long as the points x and x' are very far outside of star A and x" is in the shadow region where $\beta_A \neq 0$, ${}_0G(x,x'')$ and ${}_0G(x'',x')$ have negligible dependence on angle (θ'',ϕ''). This negligible angular dependence allowed us to pull the ${}_0G$'s out of the angular integral and evaluate them on the "center line" of the shadow

$$x''_{CL} = (t'',r'',0,0) \quad . \tag{6.24}$$

The angular integral in the third line of (6.23) can be performed using equation (6.22)

$$\int \beta_A(x'',x')d\Omega'' = [(t'')^2 + (r'')^2](r'')^{-3} \int_0^{r''} \int_{\substack{\text{all} \\ \text{angles}}} 4\pi\rho_A(r,\Omega'')r^2 d\Omega'' \, dr$$

$$= 4\pi m_A[(t'')^2 + (r'')^2](r'')^{-3} H(r''-r_s) \quad . \tag{6.25}$$

Here $H(\zeta)$ is the Heaviside step function

$$H(\zeta) = \int_{-\infty}^{\zeta} \delta(\zeta')d\zeta' = \begin{cases} 0 & \zeta < 0 \\ 1 & \zeta > 0 \end{cases} \tag{6.26}$$

and r_s is the distance from the center of star A to the source point x' (see Fig. 7a). By inserting expressions (6.25) and (4.1) into (6.23), by integrating first over t" and then using the fact that ${}_0G(x,x''_{CL})$ is a function of $x^0 - t''$, we obtain

$$\int {}_0G(x,x'')\,\beta_A(x'',x')\,{}_0G'(x'',x')\,d^4x''$$

$$= 4\pi m_A \int \frac{\partial}{\partial t''}\left\{\frac{(t'')^2+(r'')^2}{t''(r'')^3}\,H(r''-r_s)\,{}_0G(x,x''_{CL})\right\}\frac{1}{4\pi}\,\delta_{ret}\left(-\frac{1}{2}\,t''^2+\frac{1}{2}\,r''^2\right)dt''r''^2dr''$$

$$= m_A\int_0^\infty \frac{H(r''-r_s)}{(r'')^3}\left\{\left[1-\frac{(r'')^2}{(t'')^2}\right]{}_0G(x,x''_{CL})+\left[\frac{(t'')^2+(r'')^2}{t''}\right]\frac{\partial}{\partial t''}\,{}_0G(x,x''_{CL})\right\}_{r''=t''}r''dr''$$

$$= -2m_A\frac{\partial}{\partial x^0}\int_0^\infty \frac{H(r''-r_s)}{r''}\left[{}_0G(x,x''_{CL})\right]_{t''=r''}dr''$$

$$= -2m_A\frac{\partial}{\partial x^0}\int_{r_s}^\infty \left[\frac{{}_0G(x,x''_{CL})}{r''}\right]_{t''=r''}dr'' \quad . \qquad (6.27)$$

We now change our origin of spatial and temporal coordinates so it is centered on star A with $x^{0'} \neq 0$, and use the relations (Fig. 7b)

$$r_s = |\underset{\sim}{x}'| \ , \qquad r'' = |\underset{\sim}{x}'| + a \ , \qquad \underset{\sim}{x} = |\underset{\sim}{x}|\underset{\sim}{n} \qquad ,$$

$$x^{0''}_{CL} - x^{0'} = t'' = r'' = |\underset{\sim}{x}'| + a \ , \qquad \underset{\sim}{x}''_{CL} = -a\underset{\sim}{n}' \ , \qquad (6.28)$$

plus expression (4.1) to write

$$\int {}_0G(x,x'')\beta_A(x'',x')\,{}_0G'(x'',x')\;d^4x''$$

$$= -\frac{m_A}{2\pi}\frac{\partial}{\partial x^0}\int_0^\infty \frac{\delta_{ret}\left[-\frac{1}{2}(x^0-x^{0''}_{CL})^2+\frac{1}{2}\,(\underset{\sim}{x}-\underset{\sim}{x}''_{CL})^2\right]}{|\underset{\sim}{x}'|+a}\,da$$

$$= -\frac{m_A}{2\pi}\frac{\partial}{\partial x^0}\left\{\left[\frac{H[(x^0-x^{0'})-(|\underset{\sim}{x}'|+|\underset{\sim}{x}|)]}{(|\underset{\sim}{x}'|+a)\left|\frac{d}{da}[-\frac{1}{2}(x^0-x^{0''}_{CL})^2+\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{x}''_{CL})^2]\right|}\right]_{\substack{x^0-x^{0'}-|\underset{\sim}{x}'|-a\\ =|\underset{\sim}{x}|+a\underset{\sim}{n}\cdot\underset{\sim}{n}'}}\right\}. \qquad (6.29)$$

By invoking expressions (6.28), performing the differentiation with respect to a, setting

$$x^0 - |\underset{\sim}{x}| = (\text{retarded time as measured by A}) = \tau_{Ao} , \qquad (6.30)$$

and using $x^0 = |\underset{\sim}{x}| = R_A$ (aside from fractional errors $\sim 1/R_A$), we bring this into the form

$$\int {}_0G(x,x'')\beta_A(x'',x')\ {}_0G'(x'',x')d^4x''$$

$$= -\frac{m_A}{2\pi}\frac{\partial}{\partial\tau_{Ao}}\left\{\left[\frac{H[\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|]}{(|\underset{\sim}{x}'| + a)\ R_A(1 + \underset{\sim}{n}\cdot\underset{\sim}{n}')}\right]\ a(1 + \underset{\sim}{n}\cdot\underset{\sim}{n}') = \tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|\right\}$$

$$= -\frac{m_A}{2\pi}\frac{\partial}{\partial\tau_{Ao}}\left\{\frac{H[\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|]}{(\tau_{Ao} - x^{0'} + \underset{\sim}{x}'\cdot\underset{\sim}{n})\ R_A}\right\} . \qquad (6.31)$$

We now evaluate ${}_2\bar{h}^{\mu\nu}_{TL,B}$ by inserting (6.31) and (4.13) into (6.21b), and integrating out the $\delta_4(x''-z_B)$. The result is

$${}_2\bar{h}^{\mu\nu}_{TL,B} = + \frac{8m_A m_B\ u_B^{\ \mu}\ u_B^{\ \nu}}{R_A}\int_{-\infty}^{\infty}\frac{\partial}{\partial\tau_{Ao}}\left[\frac{H(\tau_{Ao} - z_B^{\ 0} - |\underset{\sim}{z}_B|)}{\tau_{Ao} - z_B^{\ 0} + \underset{\sim}{z}_B\cdot\underset{\sim}{n}}\right]d\tau_B . \qquad (6.32)$$

In the rest-frame of A the zero-order world line of B is (cf. Fig. 3 and eqs. 3.5)

$$z_B^{\ 0} = \gamma\tau_B , \quad \underset{\sim}{z}_B = v\gamma\tau_B\ \underset{\sim}{e}_x + \underset{\sim}{b} . \qquad (6.33)$$

By putting this into (6,32), using the geometric relationships of Figure 3 and integrating out τ_B, we obtain

$$2\bar{h}^{\mu\nu}_{TL,B} = \frac{8m_A m_B u_B^{\ \mu}\ u_B^{\ \nu}}{R_A}\ \frac{\partial}{\partial\tau_{Ao}}\left\{\left[\frac{\ell n\left|\tau_{Ao} - z_B^{\ 0} + \underset{\sim}{z}_B\cdot\underset{\sim}{n}\right|}{-\gamma(1-\alpha v)}\right]_{\tau_{Ao} - z_B^{\ 0} - |\underset{\sim}{z}_B| = 0}\right\}$$

$$= \frac{-8m_A m_B u_B^{\ \mu}\ u_B^{\ \nu}}{\gamma(1-\alpha v)R_A}\left[\frac{(d\tau_{B1}/dx^0)d/d\tau_B(|\underset{\sim}{z}_B| + \underset{\sim}{z}_B\cdot\underset{\sim}{n})}{|\underset{\sim}{z}_B| + \underset{\sim}{z}_B\cdot\underset{\sim}{n}}\right]_{\tau_B = \tau_{B1}} \tag{6.34}$$

where $\tau_{B1}(x^0)$ is defined by

$$\tau_{Ao} \equiv x^0 - |\underset{\sim}{x}| = z_B^{\ 0}(\tau_{B1}) + |\underset{\sim}{z}_B(\tau_{B1})| \quad . \tag{6.35}$$

Notice that expression (6.34) for the tail of B is identical to expression (6.13) for the transition of A --except that it is evaluated at a different time τ_B! Just as we evaluated (6.13) using 4-vector techniques, so we shall evaluate (6.34) using 4-vector techniques. We begin by noting that relation (6.35) says that to reach τ_{B1} on the world line of B, one should (i) begin at the field point x, (ii) go back along the past light cone of x until it hits the world line of A at $z_A(\tau_{Ao})$, (iii) go back along the past light cone of $z_A(\tau_{Ao})$ until it hits the world line of B at τ_{B1}. Comparison of this prescription with Figure 4 shows that τ_{B1} occurs at the tip of the vector $\vec{M}$ --and that therefore (since $\vec{J}$ is purely spatial in the rest frame of A)

$$\underset{\sim}{z}_B(\tau_{B1}) = \underset{\sim}{J} \quad . \tag{6.36}$$

By combining this relation with (6.33), (3.16), (3.5c), and (3.7a,b) we obtain

$$\left[\frac{d}{d\tau_B}\left(|\underset{\sim}{z}_B| + \underset{\sim}{z}_B\cdot\underset{\sim}{n}\right)\right]_{\tau_B=\tau_{B1}} = \frac{v^2\gamma^2\tau_B}{|\underset{\sim}{z}_B|} + \underset{\sim}{u}_B\cdot\underset{\sim}{n} = v\gamma\left(\frac{z_B^{\ x}}{|\underset{\sim}{z}_B|} + n^x\right)$$

$$= v\gamma\left(\frac{J^x}{|\underset{\sim}{J}|} + \alpha\right) = \frac{v\gamma}{M_A}\left[\gamma(1-\alpha v)T_A + \gamma(\alpha - v)\ell_A\right] = \frac{v\gamma R_B}{R_A}\left(\frac{T_A + \bar{\alpha}\ell_A}{M_A}\right) ; \tag{6.37}$$

and by combining with (6.35), (6.33), (3.16b), and (2.3) we obtain

$$\frac{dx^0}{d\tau_{B1}} = \frac{d\tau_{Ao}}{d\tau_{B1}} = \gamma\left(1 + \frac{vz_B{}^x}{|\underset{\sim}{z}_B|}\right) = \frac{\gamma(|\underset{\sim}{J}| + vJ^x)}{|\underset{\sim}{J}|} = \frac{\ell_A}{M_A} \quad . \tag{6.38}$$

In the rest frame of star A where $\vec{J}$ is purely spatial, equations (3.5b) and (3.12)-(3.15) imply

$$\frac{1}{|\underset{\sim}{z}_B| + \underset{\sim}{z}_B \cdot \underset{\sim}{n}} = \frac{1}{|\vec{J}| - \vec{k}\cdot\vec{J}/R_A} = \frac{R_A}{R_B(\ell_A - S_B)} = \frac{R_A(\ell_A + S_B)}{R_B S^2}$$

$$= \frac{R_A}{R_B}\left[\frac{\ell_A - (\bar{\alpha}T_A + \bar{\beta}b)}{S^2}\right] \quad . \tag{6.39}$$

By combining equations (6.37)-(6.39) with (6.34) we obtain

$$2\bar{h}^{\mu\nu}_{TL,B} = \left\{-\frac{8m_A m_B\, vb(\bar{\alpha}b - \bar{\beta}T_B)}{(1-\alpha v)\, R_A \ell_A\, S^2} + \frac{8m_A m_B v[-\bar{\delta}^2 T_A + \bar{\beta}(\bar{\alpha}b - \bar{\beta}T_A)]}{(1-\alpha v)\, R_A\, S^2}\right\} u_B{}^\mu\, u_B{}^\nu \,. \tag{6.40}$$

The analogous expression for $2\bar{h}^{\mu\nu}_{TL,A}$ can be derived either by an analogous calculation in the rest frame of B, or by invoking the invariance relation (3.17). The total tail field is then

$$\frac{2\bar{h}^{\mu\nu}_{TL}}{4m_A m_B} = u_A{}^\mu\, u_A{}^\nu \left\{+\frac{2vb(\alpha b - \beta T_B)}{(1+\bar{\alpha}v)R_B \ell_B S^2} + \frac{2v[-\delta^2 T_B + \beta(\alpha b - \beta T_B)]}{(1+\bar{\alpha}v)R_B\, S^2}\right\}$$
$$+ u_B{}^\mu\, u_B{}^\nu \left\{-\frac{2vb(\bar{\alpha}b - \bar{\beta}T_A)}{(1-\alpha v)R_A \ell_A S^2} + \frac{2v[-\bar{\delta}^2 T_A + \bar{\beta}(\bar{\alpha}b - \bar{\beta}T_A)]}{(1-\alpha v)R_A\, S^2}\right\} \,. \tag{6.41}$$

This tail field, like the transition field (eq. 6.19), diverges as the observer moves into the shadow region--and in just the same manner (eq. 6.20).

c) Sum of Focusing, Tail, and Transition Fields

Let us add the tail and transition fields (6.41) and (6.19). In doing so, by invoking equations (3.11b), (3.7), and (3.5d) we can show that the second term in each { } of (6.41) cancels the second term in each { } of (6.19). As a result, the sum of the tail field, transition field, and (vanishing) focusing field in non-shadow regions is

$$\frac{2\bar{h}^{\mu\nu}_{FTRTL}}{4m_A m_B} = \left(\frac{u_A{}^{\mu}\, u_A{}^{\nu}}{1+\bar{\alpha}v} + \frac{u_B{}^{\mu}\, u_B{}^{\nu}}{1-\alpha v}\right) 2vb \left(\frac{\alpha b - \beta T_B}{S^2 R_B \ell_B} - \frac{\bar{\alpha} b - \bar{\beta} T_A}{S^2 R_A \ell_A}\right) . \tag{6.42}$$

By the argument at the beginning of § VI, this formula is also valid in shadow regions.

Because of its validity in shadow regions, expression (6.42) must not contain any of the near-shadow divergence (eq. 6.20) that characterizes the transition and tail fields individually. The divergence in the tail must precisely cancel the divergence in the transition. That this is indeed true one can verify using expressions (3.7), (3.10), (3.11b), and (3.12b) in the neighborhood of the shadows--i.e., at

$$|\bar{\delta}| << 1, \quad |\bar{\alpha} b - \bar{\beta} T_A| << b \ ; \quad |\delta| << 1 \ , \quad |\alpha b - \beta T_B| << b. \tag{6.43}$$

This kind of cancellation between transition field and tail field has shown up previously in computations of electromagnetic radiation reaction for charged bodies moving in weak gravitational fields (Rudolf 1975). We suspect that the cancellation may be a rather general feature of computations with Green's functions in weakly curved spacetime. This strengthens the warning at the end of §IV of Paper I: One should not attribute great physical significance to the individual pieces of a weak-field Green's function or to any single one of the five pieces of our gravitational field ${}_2\bar{h}^{\mu\nu}$. Only the full Green's function and full radiation field are physically meaningful.

VII. WHUMP FIELD

a) Whump Field Expressed in Terms of Whump Integrals

The whump field (4.4e) is an integral of products of first-order fields. Only cross products (field of A times field of B) contribute to the radiation; "AA" products and "BB" products are nonlinear corrections to self fields and can therefore be ignored. Using this fact, we can rewrite the "whump stresses" (eq. 4.6a) as

$$W^{\mu\nu} = (M^{\mu\nu\alpha\beta\gamma\lambda\rho\sigma} + M^{\mu\nu\lambda\rho\sigma\alpha\beta\gamma})\ {}_1\bar{h}_{A\alpha\beta,\gamma}\ {}_1\bar{h}_{B\lambda\rho,\sigma} \quad . \tag{7.1}$$

By combining this expression with equation (4.11b) for ${}_1\bar{h}_{K\alpha\beta}$ and then inserting it into equation (4.4e) for the whump field, we obtain

$${}_2\bar{h}_W^{\mu\nu}(x) = 64 m_A m_B (M^{\mu\nu\alpha\beta\gamma\lambda\rho\sigma} + M^{\mu\nu\lambda\rho\sigma\alpha\beta\gamma})\ u_{A\alpha}\ u_{A\beta}\ u_{B\lambda}\ u_{B\rho}\ \mathcal{E}_{\gamma\sigma}(x) \ , \tag{7.2a}$$

$$\mathcal{E}_{\gamma\sigma}(x) \equiv \int 4\pi\ {}_0G(x,x') \left(\frac{1}{\mathcal{R}_A}\right)_{,\gamma'} \left(\frac{1}{\mathcal{R}_B}\right)_{,\sigma'} d^4x' \ , \tag{7.2b}$$

$$\mathcal{R}_K \equiv \mathcal{R}_K(x') \equiv \vec{\mathcal{X}}_K \cdot \vec{u}_K = \begin{pmatrix}\text{spatial distance from } x' \text{ to unperturbed} \\ \text{world line of K as measured in rest frame of K}\end{pmatrix} . \tag{7.2c}$$

By invoking expression (4.6b) for $M^{\mu\nu\alpha\beta\gamma\lambda\rho\sigma}$, performing the contraction of indices with the u's and using equations (3.2) for the scalar products of the u's, we bring (7.2a) into the form

$$\begin{aligned}{}_2\bar{h}_W^{\mu\nu}(x) = (4/\pi) m_A m_B \{&\tfrac{1}{2}(2\gamma^2 - 1)\ \mathcal{E}^{(\mu\nu)} + 2\gamma[u_A{}^{(\mu}(\vec{u}_B \cdot \overleftrightarrow{\mathcal{E}})^{\nu)} + (\overleftrightarrow{\mathcal{E}} \cdot \vec{u}_A)^{(\mu} u_B{}^{\nu)}] \\ &+ 2u_A{}^{(\mu} u_B{}^{\nu)}[\vec{u}_B \cdot \overleftrightarrow{\mathcal{E}} \cdot \vec{u}_A - \gamma \mathcal{E}^\rho{}_\rho] - \eta^{\mu\nu}[\gamma(\vec{u}_B \cdot \overleftrightarrow{\mathcal{E}} \cdot \vec{u}_A) + \tfrac{1}{4}(2\gamma^2 - 1)\mathcal{E}^\rho{}_\rho]\} \ .\end{aligned} \tag{7.3}$$

The "whump integrals" (7.2b) satisfy two important identities. Because, in the rest frame of star K ,

$$u_K{}^{\alpha}\left(\frac{1}{\mathcal{R}_K}\right)_{,\alpha'} = \frac{\partial}{\partial x^{0'}}\frac{1}{|\underset{\sim}{x}'|} = 0 \quad ,$$

they satisfy

$$\vec{u}_A \cdot \overleftrightarrow{\mathcal{E}} = \overleftrightarrow{\mathcal{E}} \cdot \vec{u}_B = 0 \quad . \tag{7.4}$$

Through much of the rest of this section we shall work in the rest frame of star A, where

$$u_A{}^0 = 1 , \quad \underset{\sim}{u}_A = 0 , \quad u_B{}^0 = \gamma , \quad \underset{\sim}{u}_B = \gamma v \underset{\sim}{e}_x \quad . \tag{7.5}$$

In this rest frame the identities (7.4) become

$$\mathcal{E}_{0\alpha} = 0 \quad , \quad \mathcal{E}_{\alpha 0} = -v\,\mathcal{E}_{\alpha x} \quad . \tag{7.6}$$

The basis vectors of this coordinate system are

$$\vec{e}_0 = \vec{u}_A , \quad \vec{e}_x = v^{-1}(\vec{u}_B/\gamma - \vec{u}_A) , \quad \vec{e}_y = \vec{\mathfrak{p}} , \quad \vec{e}_z = \vec{q} \quad . \tag{7.7}$$

By combining equations (7.4), (7.6) and (7.7) with $\overleftrightarrow{\mathcal{E}} = \mathcal{E}^{\alpha\beta}\,\vec{e}_\alpha \otimes \vec{e}_\beta$ we obtain the relations

$$\begin{aligned}
\mathcal{E}^{(\alpha\beta)} = {} & \frac{\mathcal{E}_{xx}}{v^2\gamma^2}\left(u_A{}^{\alpha} u_A{}^{\beta} + u_B{}^{\alpha} u_B{}^{\beta}\right) - \frac{(2-v^2)\,\mathcal{E}_{xx}}{v^2\gamma}\, u_A{}^{(\alpha} u_B{}^{\beta)} \\
& + \left(\frac{\mathcal{E}_{xy}+\mathcal{E}_{yx}}{v\gamma}\right)\mathfrak{p}^{(\alpha} u_B{}^{\beta)} + \left(\frac{\mathcal{E}_{xz}+\mathcal{E}_{zx}}{v\gamma}\right) q^{(\alpha} u_B{}^{\beta)} \\
& - \left(\frac{\mathcal{E}_{xy}}{v} + \frac{\mathcal{E}_{yx}}{v\gamma^2}\right)\mathfrak{p}^{(\alpha} u_A{}^{\beta)} - \left(\frac{\mathcal{E}_{xz}}{v} + \frac{\mathcal{E}_{zx}}{v\gamma^2}\right) q^{(\alpha} u_A{}^{\beta)} \\
& + \mathcal{E}_{yy}\mathfrak{p}^{\alpha}\mathfrak{p}^{\beta} + (\mathcal{E}_{yz} + \mathcal{E}_{zy})\mathfrak{p}^{(\alpha} q^{\beta)} + \mathcal{E}_{zz} q^{\alpha} q^{\beta} \quad ,
\end{aligned} \tag{7.8a}$$

$$(\overleftrightarrow{\mathcal{E}}\cdot\vec{u}_A)^\alpha = -\gamma^{-1}\,\mathcal{E}_{xx}u_B{}^\alpha + \mathcal{E}_{xx}u_A{}^\alpha - v\,\mathcal{E}_{yx}\mathfrak{b}^\alpha - v\,\mathcal{E}_{zx}q^\alpha \quad , \tag{7.8b}$$

$$(\vec{u}_B\cdot\overleftrightarrow{\mathcal{E}})^\beta = -\gamma^{-1}\,\mathcal{E}_{xx}u_A{}^\beta + \mathcal{E}_{xx}u_B{}^\beta + \gamma v\,\mathcal{E}_{xy}\mathfrak{b}^\beta + \gamma v\,\mathcal{E}_{xz}q^\beta \quad , \tag{7.8c}$$

$$\mathcal{E}^\alpha{}_\alpha = \mathcal{E}_{xx} + \mathcal{E}_{yy} + \mathcal{E}_{zz} \equiv \mathcal{E}_{jj} \quad . \tag{7.8d}$$

By inserting these expressions into (7.3) we obtain

$$\begin{aligned}
\frac{2\overline{h}_W^{\mu\nu}}{4m_Am_B} = {} & (u_A{}^\mu u_A{}^\nu + u_B{}^\mu u_B{}^\nu)\left[\frac{(1-3v^2)}{v^2}\frac{\mathcal{E}_{xx}}{2\pi}\right] \\
& + u_A{}^{(\mu}u_B{}^{\nu)}\left[\frac{-\gamma(2-v^2)(1-3v^2)}{v^2}\frac{\mathcal{E}_{xx}}{2\pi} - 4\gamma\frac{\mathcal{E}_{jj}}{2\pi}\right] \\
& + 2\mathfrak{b}^{(\mu}u_A{}^{\nu)}\left[-\frac{\gamma^2(2-3v^2-v^4)}{2v}\frac{\mathcal{E}_{yx}}{2\pi} - \frac{\gamma^2(1-3v^2)}{2v}\left(\frac{\mathcal{E}_{xy}-\mathcal{E}_{yx}}{2\pi}\right)\right] \\
& + 2\mathfrak{b}^{(\mu}u_B{}^{\nu)}\left[\frac{1}{v\gamma}\frac{\mathcal{E}_{yx}}{2\pi} + \frac{\gamma(1+v^2)}{2v}\left(\frac{\mathcal{E}_{xy}-\mathcal{E}_{yx}}{2\pi}\right)\right] \\
& + 2q^{(\mu}u_A{}^{\nu)}\left[-\frac{\gamma^2(2-3v^2-v^4)}{2v}\frac{\mathcal{E}_{zx}}{2\pi} - \frac{\gamma^2(1-3v^2)}{2v}\left(\frac{\mathcal{E}_{xz}-\mathcal{E}_{zx}}{2\pi}\right)\right] \\
& + 2q^{(\mu}u_B{}^{\nu)}\left[\frac{1}{v\gamma}\frac{\mathcal{E}_{zx}}{2\pi} + \frac{\gamma(1+v^2)}{2v}\left(\frac{\mathcal{E}_{xz}-\mathcal{E}_{zx}}{2\pi}\right)\right] \\
& + \gamma^2(1+v^2)\left[\mathfrak{b}^\mu\mathfrak{b}^\nu\frac{\mathcal{E}_{yy}}{2\pi} + \mathfrak{b}^{(\mu}q^{\nu)}\left(\frac{\mathcal{E}_{yz}+\mathcal{E}_{zy}}{2\pi}\right) + q^\mu q^\nu\frac{\mathcal{E}_{zz}}{2\pi}\right] \\
& + \eta^{\mu\nu}\left[2v^2\gamma^2\frac{\mathcal{E}_{xx}}{2\pi} - \frac{\gamma^2(1+v^2)}{2}\frac{\mathcal{E}_{jj}}{2\pi}\right] \quad .
\end{aligned} \tag{7.9}$$

b) Trick for Evaluating Whump Integrals

Our task now is to compute the spatial whump integrals $\mathcal{E}_{jk}$ and insert them into equation (7.9). In computing $\mathcal{E}_{jk}$ we shall use a trick suggested by the method of Peters (1970). We now derive the trick:

As in Papers I and II, define the Green's function for the spacetime with metric

$$ {}_1g_{\alpha\beta} = \eta_{\alpha\beta} + {}_1h_{A\alpha\beta} \tag{7.10} $$

by

$$ {}_1\Box_s\, G_A(x,x') \equiv (\eta^{\alpha\beta} - {}_1h_A{}^{\alpha\beta})G_{A,\alpha\beta} = -[1 - \tfrac{1}{2}\, {}_1h_A(x')]\delta_4(x-x') \ . \tag{7.11} $$

Split $G_A(x,x')$ into a flat-space part plus a perturbation due to the field ${}_1h_{A\alpha\beta}$

$$ G_A = {}_0G + \Delta G_A \quad , \tag{7.12} $$

and invoke the flat-space equation

$$ \eta^{\alpha\beta}\, {}_0G_{,\alpha\beta} = -\delta_4(x-x') $$

to bring (7.11) into the form

$$ {}_1\bar{h}_A{}^{\alpha\beta}(x)\ \partial_\alpha\partial_\beta\ {}_0G(x,x') = \eta^{\alpha\beta}\ \partial_\alpha\partial_\beta\ \Delta G_A(x,x') $$

$$ \equiv \Box_f\ \Delta G_A(x,x') \quad . \tag{7.13} $$

Now apply the differential operator $\partial_\gamma + \partial_{\gamma'}$ to both sides of this equation and use the fact that ${}_0G$ is a function of $x^\gamma - x^{\gamma'}$ (so $[\partial_\gamma + \partial_{\gamma'}]\ {}_0G = 0$) to obtain

$$ [{}_1\bar{h}_A{}^{\alpha\beta}(x)]_{,\gamma}\ \partial_\alpha\partial_\beta\ {}_0G(x,x') = \Box_f(\partial_\gamma + \partial_{\gamma'})\ \Delta G_A(x,x') \quad . \tag{7.14} $$

Next apply $-\partial_{\lambda'}$ to both sides and use $-\partial_{\lambda'}\ {}_0G = \partial_\lambda\ {}_0G$ to obtain

$$ [{}_1\bar{h}_A{}^{\alpha\beta}(x)]_{,\gamma}\ \partial_\alpha\partial_\beta\partial_\lambda\ {}_0G(x,x') = -\Box_f(\partial_\gamma + \partial_{\gamma'})\partial_{\lambda'}\ \Delta G_A(x,x') \quad . \tag{7.15} $$

Then multiply both sides by $16\pi\ T_B^{\ \mu\nu}(x')$ and integrate over x' using

$$ {}_1\bar{h}_B^{\ \mu\nu}(x) = 16\pi \int T_B^{\ \mu\nu}(x')\ {}_0G(x,x')\ d^4x' \tag{7.16} $$

to obtain

$$ {}_1\bar{h}_A^{\ \alpha\beta}{}_{,\gamma}\ {}_1\bar{h}_B^{\ \mu\nu}{}_{,\lambda\alpha\beta} = \Box_f\{-16\pi \int [(\partial_\gamma+\partial_{\gamma'})\partial_{\lambda'}\ \Delta G_A(x,x')]T_B^{\ \mu\nu}(x')d^4x'\}\ . \tag{7.17} $$

Then invert this equation using the flat-space Green's function to obtain

$$ \int [{}_1\bar{h}_A^{\ \alpha'\beta'}{}_{,\gamma'}\ {}_1\bar{h}_B^{\ \mu'\nu'}{}_{,\lambda'\alpha'\beta'}]\ {}_0G(x,x')d^4x' $$

$$ = 16\pi \int [(\partial_\gamma+\partial_{\gamma'})\partial_{\lambda'}\ \Delta G_A(x,x')]\ T_B^{\ \mu\nu}(x')\ d^4x' \quad . \tag{7.18} $$

Here the primes on the indices on the left-hand side indicate that the ${}_1\bar{h}$'s are evaluated at x', not x. Next rewrite the left-hand side using the gauge condition

$$ {}_1\bar{h}_A^{\ \alpha'\beta'}{}_{,\alpha'} = 0 + \text{(negligible terms quadratic in masses)} $$

and using integrations by parts and using $\partial_{\alpha'}\ {}_0G = -\partial_\alpha\ {}_0G$, to obtain

$$ \int {}_1\bar{h}_A^{\ \alpha'\beta'}{}_{,\gamma'}\ {}_1\bar{h}_B^{\ \mu'\nu'}{}_{,\lambda'\alpha'\beta'}\ {}_0G(x,x')d^4x' $$

$$ = \int ({}_1\bar{h}_A^{\ \alpha'\beta'}{}_{,\gamma'}\ {}_1\bar{h}_B^{\ \mu'\nu'}{}_{,\lambda'})_{,\alpha'\beta'}\ {}_0G(x,x')d^4x' $$

$$ = \int ({}_1\bar{h}_A^{\ \alpha'\beta'}{}_{,\gamma'}\ {}_1\bar{h}_B^{\ \mu'\nu'}{}_{,\lambda'})\ \partial_{\alpha'}\partial_{\beta'}\ {}_0G(x,x')\ d^4x' $$

$$ = \partial_\alpha\partial_\beta \int {}_1\bar{h}_A^{\ \alpha'\beta'}{}_{,\gamma'}\ {}_1\bar{h}_B^{\ \mu'\nu'}{}_{,\lambda'}\ {}_0G(x,x')\ d^4x' \quad . $$

Then invoke the relation (eqs. 4.11b and 7.2c)

$$ {}_1\bar{h}_K^{\alpha'\beta'} = 4m_K\, u_K^{\alpha}\, u_K^{\beta}/\mathcal{R}_K \tag{7.19} $$

and combine with equations (7.2b) and (7.18) to obtain

$$ \begin{aligned} &(4/\pi)m_A m_B (u_A^{\alpha}\, \partial_\alpha)(u_A^{\beta}\, \partial_\beta) u_B^{\mu}\, u_B^{\nu}\, \mathcal{E}_{\gamma\lambda} \\ &\quad = 16\pi \int [(\partial_\gamma + \partial_{\gamma'})\partial_{\lambda'}\, \Delta G_A(x,x')] T_B^{\mu\nu}(x')\, d^4x' \ . \end{aligned} \tag{7.20} $$

Then insert the stress-energy tensor for B (eq. 4.13) and integrate out the x' on the right-hand side to obtain

$$ m_A(u_A^{\alpha}\, \partial_\alpha)(u_A^{\beta}\, \partial_\beta)\, \mathcal{E}_{\gamma\lambda} = 4\pi^2 \int [(\partial_\gamma + \partial_{\gamma'})\partial_{\lambda'}\, \Delta G_A(x,x')]_{x'=z_B(\tau_B)} d\tau_B \ . \tag{7.21} $$

Finally, specialize to the rest frame of A and thereby obtain

$$ \mathcal{E}_{\gamma\lambda,00} = (4\pi^2/m_A) \int [(\partial_\gamma + \partial_{\gamma'})\partial_{\lambda'}\, \Delta G_A(x,x')]_{x'=z_B(\tau_B)}\, d\tau_B \ . \tag{7.22} $$

The perturbation in the Green's function is easily determined by comparing equations (7.12) and (4.1) with equations (55) and (56) of Paper I:

$$ \begin{aligned} \Delta G_A(x,x') &= \alpha_A(x,x')\, {}_0G(x,x') + \gamma_A^{T}(x,x')\, {}_0G'(x,x') \\ &\quad - \left\{ \int {}_0G(x,x'')\, \beta_A(x'',x')\, {}_0G'(x'',x')\, d^4x'' \right\} \cdot \begin{Bmatrix} 0 \text{ if } x' \notin I^-(x) \\ 1 \text{ if } x' \in I^-(x) \end{Bmatrix} . \end{aligned} \tag{7.23} $$

Here $x' \in I^-(x)$ means that x' lies inside and not on the past light cone of x. We shall restrict ourselves to field points x which are not in shadow regions. (As with the focusing-plus-transition-plus tail field (§VI), the analysis of Paper II guarantees that the whump field in shadow regions is the analytic extension of that in nearby nonshadow regions.) The restriction to nonshadow regions allows us to set $\alpha_A(x,x') = 0$, and to invoke equations (6.8), (4.2), (6.5), (6.31) and (6.30) to obtain

$$\frac{1}{m_A}\Delta G_A(x,x') = \left[\frac{(X^0)^2 + |\underset{\sim}{X}|^2}{|\underset{\sim}{X}|}\,\ell n\left(\frac{10b|\underset{\sim}{x}|}{|\underset{\sim}{x}'||\underset{\sim}{X}| + \underset{\sim}{x}'\cdot\underset{\sim}{X}}\right)\right]\left[-\frac{1}{X^0}\,\partial_0\ {}_0G(x,x')\right]$$
$$+\frac{1}{2\pi}\,\partial_0\left[\frac{H[\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|]}{(\tau_{Ao} - x^{0'} + \underset{\sim}{x}'\cdot\underset{\sim}{n})R_A}\right] \quad . \tag{7.24}$$

In the first line we can bring the differentiation all the way to the left by invoking the fact that

$$(1/X^0)\partial_0 X^0 = O(1/R_A) = \begin{pmatrix}\text{quantity that can be dropped without}\\ \text{changing radiative part of field}\end{pmatrix}. \tag{7.25}$$

We can also use the relations (cf. Fig. 6 and eqs. 4.1, 6.5, 6.30)

$${}_0G(x,x') = \frac{1}{4\pi}\,\delta_{ret}[-\tfrac{1}{2}(X^0)^2 + \tfrac{1}{2}|\underset{\sim}{X}|^2] = \frac{1}{2\pi(X^0 + |\underset{\sim}{X}|)}\,\delta(-X^0 + |\underset{\sim}{X}|) \ ,$$
$$X^0 - |\underset{\sim}{X}| = x^0 - |\underset{\sim}{x}| - x^{0'} + \underset{\sim}{x}'\cdot\underset{\sim}{n} = \tau_{Ao} - x^{0'} + \underset{\sim}{x}'\cdot\underset{\sim}{n} \ , \tag{7.26}$$
$$X^0 \simeq |\underset{\sim}{X}| \simeq |\underset{\sim}{x}| = R_A + (\text{fractional errors} \propto 1/R_A) \ ,$$
$$|\underset{\sim}{x}'||\underset{\sim}{X}| + \underset{\sim}{x}'\cdot\underset{\sim}{X} = R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot n) + (\text{fractional errors} \propto 1/R_A)$$

to bring (7.24) into the form

$$\frac{2\pi}{m_A}\Delta G_A(x,x') = -\partial_0\left\{\frac{\delta(\tau_{Ao} - x^{0'} + \underset{\sim}{x}'\cdot\underset{\sim}{n})}{R_A}\,\ell n\left|\frac{10b}{|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n}}\right| - \frac{H(\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|)}{(\tau_{Ao} - x^{0'} + \underset{\sim}{x}'\cdot\underset{\sim}{n})R_A}\right\} \ . \tag{7.27}$$

We can then use the fact that

$$(\partial_j + \partial_{j'})(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n}) = -n_j + n_j = 0$$

plus relations (7.26) to write

$$\frac{2\pi}{m_A}(\partial_j + \partial_{j'})\,\Delta G_A(x,x') = \partial_0 \left\{ \frac{\delta(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})}{R_A(|x'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})}(n^j + n^{j'}) \right.$$

$$\left. - \frac{\delta(\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|)}{R_A(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})}(n^j + n^{j'}) \right\} \quad ; \qquad (7.28)$$

and we can combine this with (7.22) to obtain

$$\left(\frac{\mathcal{E}_{j\lambda}}{2\pi}\right)_{,00} = \partial_0 \int \left\{ \partial_{\lambda'} \left[\frac{(n^j + n^{j'})\,\delta(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})} \right.\right.$$

$$\left.\left. - \frac{(n^j + n^{j'})\,\delta(\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|)}{R_A(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})} \right] \right\}_{x' = z_B(\tau_B)} d\tau_B . \qquad (7.29)$$

(Note that it does not matter whether we put our spatial indices up or down: $S^j = S_j$ for any $\underset{\sim}{S}$.)

This is our "trick equation" for evaluating the whump integrals. Its power lies in the fact that it involves only a single line integral along the world line of star B, whereas the original whump integrals (eq. 7.2b) involve extremely diffcult 3-dimensional integrals over the observer's past light cone.

c) Evaluation of Whump Integrals

For the special case of $\mathcal{E}_{jx} = -v^{-1}\,\mathcal{E}_{j0}$ we can use the fact that the [ ] quantity in eq. (7.29) is a function of $x^0 - x^{0'}$ to rewrite (7.29) as

$$\left(\frac{\mathcal{E}_{jx}}{2\pi}\right)_{,00} = \frac{1}{v}\,\partial_0\partial_0 \int \left\{ \frac{(n^j + n^{j'})\,\delta(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})} \right.$$

$$\left. - \frac{(n^j + n^{j'})\,\delta(\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|)}{R_A(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})} \right\}_{x' = z_B(\tau_B)} d\tau_B .$$

We can then integrate out the delta functions and wipe out the $\partial_0\partial_0$ on each side of the equation (the integration constant is zero because $\mathcal{E}_{jx}$ vanishes in the limit $\tau_{Ao} \to \infty$), and we can use the relations

$$[\tau_{Ao} = z_B^0 - \underset{\sim}{z}_B \cdot \underset{\sim}{n}] \Longleftrightarrow [\tau_B = \tau_{Bo}] \Longleftrightarrow \left[\begin{array}{l} \underset{\sim}{x}' = \underset{\sim}{z}_B \text{ lies at the} \\ \text{tip of } \vec{S} \text{ in Fig. 4} \end{array}\right], \quad (7.30a)$$

$$[\tau_{Ao} = z_B^0 + |\underset{\sim}{z}_B|] \Longleftrightarrow [\tau_B = \tau_{B1}] \Longleftrightarrow \left[\begin{array}{l} \underset{\sim}{x}' = \underset{\sim}{z}_B \text{ lies at the} \\ \text{tip of } \vec{M} \text{ in Fig. 4} \end{array}\right] \quad (7.30b)$$

(cf. eqs. 6.12, 6.30, and 6.35), thereby obtaining

$$\frac{\mathcal{E}_{jx}}{2\pi} = \left[\frac{n^j + n^{j'}}{vR_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})\, d\tau_{Ao}/d\tau_{Bo}}\right]_{x' = z_B(\tau_{Bo})}$$

$$- \left[\frac{n^j + n^{j'}}{vR_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})\, d\tau_{Ao}/d\tau_{B1}}\right]_{x'=z_B(\tau_{B1})} . \quad (7.31)$$

For the more general case, expression (7.29) reduces to

$$\left(\frac{\mathcal{E}_{jk}}{2\pi}\right)_{,00} = \partial_0\partial_0 \int \left\{ \frac{(n^j + n^{j'})\, n^k\, \delta(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})} + \frac{(n^j + n^{j'})\, n^{k'}\, \delta(\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|)}{R_A(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})} \right\}_{x'=z_B(\tau_B)} d\tau_B$$

$$+ \partial_0 \int \left\{ \frac{\delta(\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n})}{R_A} \frac{\partial}{\partial x^{k'}} \left[\frac{n^j + n^{j'}}{|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n}}\right] - \frac{\delta(\tau_{Ao} - x^{0'} - |\underset{\sim}{x}'|)}{R_A} \left[\frac{\partial}{\partial x^{k'}} + n^{k'} \frac{\partial}{\partial x^0}\right] \left[\frac{n^j + n^{j'}}{\tau_{Ao} - x^{0'} + \underset{\sim}{x}' \cdot \underset{\sim}{n}}\right] \right\}_{x'=z_B(\tau_B)} d\tau_B .$$

By integrating out the delta functions and wiping out one ∂_0 we bring this into the form

$$\left(\frac{\varepsilon_{jk}}{2\pi}\right)_{,0} = \partial_0 \left\{ \left[\frac{(n^j + n^{j'})n^k}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})\, d\tau_{Ao}/d\tau_{Bo}} \right]_{x'= z_B(\tau_{Bo})} \right.$$

$$\left. + \left[\frac{(n^j + n^{j'})n^{k'}}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})\, d\tau_{Ao}/d\tau_{B1}} \right]_{x'= z_B(\tau_{B1})} \right\}$$

$$+ \left\{ \left[\frac{\delta^{jk} - n^{j'} n^{k'}}{|\underset{\sim}{x}'|(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})} - \frac{(n^j + n^{j'})(n^k + n^{k'})}{(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})^2} \right] \left[\frac{1}{R_A d\tau_{Ao}/d\tau_{Bo}} \right] \right\}_{x'= z_B(\tau_{Bo})}$$

$$- \left\{ \left[\frac{\delta^{jk} - n^{j'} n^{k'}}{|\underset{\sim}{x}'|(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})} - \frac{(n^j + n^{j'})(n^k + n^{k'})}{(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})^2} \right] \left[\frac{1}{R_A d\tau_{Ao}/d\tau_{B1}} \right] \right\}_{x'= z_B(\tau_{B1})} . \quad (7.32)$$

Notice that the trace of this equation, after integration over time, reads

$$\frac{\varepsilon_{jj}}{2\pi} = \left[\frac{1}{R_A |\underset{\sim}{x}'| d\tau_{Ao}/d\tau_{Bo}} \right]_{x'= z_B(\tau_{Bo})}$$

$$+ \left[\frac{1}{R_A |\underset{\sim}{x}'| d\tau_{Ao}/d\tau_{B1}} \right]_{x'= z_B(\tau_{B1})} . \quad (7.33)$$

In equations (7.31), (7.32) and (7.33) it is helpful to note that, for $\tau_B = \tau_{Bo}$ or $\tau_B = \tau_{B1}$ and for $\underset{\sim}{x}' = \underset{\sim}{z}_B(\tau_B)$,

$$\frac{1}{d\tau_{Ao}/d\tau_B} \frac{n^x + n^{x'}}{(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})^2} = \frac{d\tau_B}{d\tau_{Ao}} \frac{d}{d\tau_B} \left[\frac{-1/v\gamma}{|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n}} \right]$$

$$= \partial_0 \left[\frac{-1/v\gamma}{|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n}} \right] \quad (7.34a)$$

(cf. eqs. 6.30 and 6.33), and that similarly

$$\frac{1}{d\tau_{Ao}/d\tau_B}\left[\frac{\delta^{xj}-n^{x'}n^{j'}}{|\underset{\sim}{x}'|(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})}-\frac{(n^x+n^{x'})(n^j+n^{j'})}{(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})^2}\right]$$

$$=\partial_0\left[\frac{(n^{j'}+n^j)/v\gamma}{|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n}}\right] \quad . \qquad (7.34b)$$

In particular, by combining (7.34b) with (7.32) and integrating over time, we obtain

$$\frac{\mathcal{E}_{xj}}{2\pi}=\left[\frac{n^j(n^x+n^{x'})}{R_A(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})\,d\tau_{Ao}/d\tau_{Bo}}+\frac{(n^{j'}+n^j)/v\gamma}{R_A(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})}\right]_{x'=z_B(\tau_{Bo})}$$

$$+\left[\frac{n^{j'}(n^x+n^{x'})}{R_A(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})\,d\tau_{Ao}/d\tau_{B1}}-\frac{(n^{j'}+n^j)/v\gamma}{R_A(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})}\right]_{x'=z_B(\tau_{B1})} \quad . \qquad (7.35)$$

In the first bracket

$$d\tau_{Ao}/d\tau_{Bo}=d(z_B{}^0-\underset{\sim}{z}_B\cdot\underset{\sim}{n})\,/\,d\tau_B=\gamma(1-vn^x) \quad ; \qquad (7.36a)$$

in the second

$$d\tau_{Ao}/d\tau_{B1}=d(z_B{}^0+|\underset{\sim}{z}_B|)/d\tau_B=\gamma(1+v^2\gamma\tau_B/|\underset{\sim}{z}_B|)=\gamma(1+vn^{x'}) \qquad (7.36b)$$

(cf. eqs. 7.30 and 6.33). Combining these relations with (7.35), we obtain

$$\frac{\mathcal{E}_{xj}}{2\pi}=\left[\frac{n^j(1+vn^{x'})+n^{j'}(1-vn^x)}{vR_A(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})\,d\tau_{Ao}/d\tau_{Bo}}\right]_{x'=z_B(\tau_{Bo})}$$

$$-\left[\frac{n^j(1+vn^{x'})+n^{j'}(1-vn^x)}{vR_A(|\underset{\sim}{x}'|+\underset{\sim}{x}'\cdot\underset{\sim}{n})\,d\tau_{Ao}/d\tau_{B1}}\right]_{x'=z_B(\tau_{B1})} \quad . \qquad (7.37)$$

Equations (7.31), (7.32), (7.33), and (7.37) are somewhat formal expressions for the whump integrals. We now evaluate them explicitly in terms of the same parameters ($\alpha,\beta,\delta,\bar{\alpha},\bar{\beta},\bar{\delta},v,\gamma,b,T_A,T_B,\ell_A,\ell_B,S^2,R_A,R_B$) as appear in equations (5.7) and (6.42) for the direct, focusing, transition, and tail fields.

The whump integrals $\mathcal{E}_{xx}$, $\mathcal{E}_{yx}$, and $\mathcal{E}_{zx}$ can be evaluated by inserting the vectors of Figure 4 into expression (7.31). The relevant quantities for insertion are

For $\tau_B = \tau_{Bo}$ --i.e., for $x' = z_B$ at tip of $\vec{S}$:

$$n^x = \alpha\ ,\quad n^y = \beta\ ,\quad n^z = \delta\quad ; \tag{7.38a}$$

$$x^{x'} = Y^x = T_B\ ,\quad x^{y'} = Y^y = b\ ,\quad x^{z'} = Y^z = 0\ ,\quad |\underset{\sim}{x}'| = |\vec{Y}| = \ell_B\ ; \tag{7.38b}$$

$$n^{x'} = T_B/\ell_B\ ,\quad n^{y'} = b/\ell_B\ ,\quad n^{z'} = 0\quad ; \tag{7.38c}$$

$$d\tau_{Ao}/d\tau_{Bo} = \gamma(1 - \alpha v)\quad ; \tag{7.38d}$$

cf. eqs. (3.5c), (3.16a), and (7.36a).

For $\tau_B = \tau_{B1}$ --i.e., for $x' = z_B$ at tip of $\vec{M}$:

$$n^x = \alpha\ ,\quad n^y = \beta\ ,\quad n^z = \delta\quad ; \tag{7.39a}$$

$$x^{x'} = J^x = \gamma(T_A - v\ell_A)\ ,\quad x^{y'} = J^y = b\ ,\quad x^{z'} = J^z = 0\ ,$$

$$|\underset{\sim}{x}'| = |\underset{\sim}{J}| = M_A = \gamma(\ell_A - vT_A)\quad ; \tag{7.39b}$$

$$n^{x'} = J^x/M_A\ ,\quad n^{y'} = J^y/M_A\ ,\quad n^{z'} = 0\quad ; \tag{7.39c}$$

$$d\tau_{Ao}/d\tau_{B1} = \gamma[1 + vn^{x'}] = \ell_A/M_A\quad ; \tag{7.39d}$$

cf. eqs. (3.5c), (3.16b), and (7.36b).

By inserting these quantities into expression (7.31) and then invoking eqs. (3.7), (3.10)-(3.12), and (3.16) we obtain the following:

$$\frac{\varepsilon_{xx}}{2\pi} = \frac{b(\alpha b-\beta T_B)}{vS^2 R_B\, \ell_B} - \frac{b(\bar{\alpha} b - \bar{\beta} T_A)}{vS^2\, R_A\, \ell_A} \quad , \tag{7.40}$$

$$\frac{\varepsilon_{yx}}{2\pi} = -\frac{T_B(\alpha b - \beta T_B)}{vS^2\, R_B\, \ell_B} - \frac{\gamma\{\bar{\beta}(1+\bar{\alpha}v)T_A{}^2 + [v\bar{\beta}^2 - \bar{\alpha}(1+\bar{\alpha}v)]bT_A - \bar{\alpha}\bar{\beta}vb^2\}}{vS^2 R_A\, \ell_A} \quad , \tag{7.41}$$

$$\frac{\varepsilon_{zx}}{2\pi} = \frac{\delta \ell_B}{vS^2 R_B} - \frac{\bar{\delta}\gamma}{vS^2 R_A \ell_A}\,[(1+\bar{\alpha}v)T_A{}^2 + v\bar{\beta}bT_A + b^2] \quad . \tag{7.42}$$

By combining relations (7.31) and (7.37) we obtain

$$\frac{\varepsilon_{xj} - \varepsilon_{jx}}{2\pi} = \left[\frac{n^{x'} n^{j} - n^{x} n^{j'}}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})\; d\tau_{Ao}/d\tau_{Bo}}\right]_{x'=z_B(\tau_{Bo})} - \left[\frac{n^{x'} n^{j} - n^{x} n^{j'}}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}' \cdot \underset{\sim}{n})\, d\tau_{Ao}/d\tau_{B1}}\right]_{x'=z_B(\tau_{B1})} \quad . \tag{7.43}$$

By then inserting expressions (7.38) and (7.39) into (7.43) and by invoking (3.7), (3.10)-(3.12), and (3.16) we obtain

$$\frac{\varepsilon_{xy} - \varepsilon_{yx}}{2\pi} = \frac{(\alpha b-\beta T_B)(\alpha T_B + \beta b)}{S^2 R_B\, \ell_B} + \frac{\gamma}{S^2 R_A \ell_A}\,[\bar{\beta}(\bar{\alpha}+v)T_A{}^2 + (\bar{\beta}^2 - \bar{\alpha}^2 - \bar{\alpha}v)bT_A - \bar{\alpha}\bar{\beta}\, b^2] \quad , \tag{7.44a}$$

$$\frac{\mathcal{E}_{xz}-\mathcal{E}_{zx}}{2\pi} = -\frac{\delta\tau_B(\alpha\tau_B+\beta b)}{S^2R_B\,\ell_B} + \frac{\bar{\delta}\gamma}{S^2R_A\,\ell_A}[(\bar{\alpha}+v)\tau_A{}^2 + \bar{\beta}b\tau_A + \dot{v}b^2] \,. \tag{7.44b}$$

We next insert expressions (7.38), (7.39) and (3.7a) into (7.33), obtaining

$$\frac{\mathcal{E}_{jj}}{2\pi} = \frac{\mathcal{E}_{xx}+\mathcal{E}_{yy}+\mathcal{E}_{zz}}{2\pi} = \frac{1}{R_A\ell_A} + \frac{1}{R_B\ell_B} \,. \tag{7.45}$$

We next turn to $\mathcal{E}_{zz}$. According to (7.32), with $n^{z'} = 0$ and

$dx^0/d\tau_{Ao} = 1$, it is given by

$$\frac{\mathcal{E}_{zz}}{2\pi} = \left[\frac{(n^z)^2}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})d\tau_{Ao}/d\tau_{Bo}}\right]_{x'=\, z_B(\tau_{Bo})}$$

$$- \int_{\tau_{B1}}^{\tau_{Bo}} \left[\frac{1-(n^z)^2 + \underset{\sim}{n}\cdot\underset{\sim}{n}'}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})^2}\right]_{x'=\, z_B(\tau_B)} d\tau_B \,. \tag{7.46}$$

The integral that appears here can be evaluated using $\underset{\sim}{n} = (\alpha,\beta,\delta)$ and $\underset{\sim}{x}' = (v\gamma\tau_B, b, 0)$:

$$\int \frac{1-(n^z)^2 + \underset{\sim}{n}\cdot\underset{\sim}{n}'}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})^2}\,d\tau_B = \int \frac{(1-\delta^2) + (\alpha v\gamma\tau_B + \beta b)(b^2+v^2\gamma^2\tau_B{}^2)^{-1/2}}{R_A[(b^2+v^2\gamma^2\tau_B{}^2)^{1/2} + \alpha v\gamma\tau_B + \beta b]^2}\,d\tau_B$$

$$= \frac{\beta v\gamma\tau_B - \alpha b}{R_A b v\gamma[(b^2+v^2\gamma^2\tau_B{}^2)^{1/2} + \alpha v\gamma\tau_B + \beta b]} = \frac{|x'|(n^y n^{x'} - n^x n^{y'})}{R_A b v\gamma(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})} \,. \tag{7.47}$$

By combining this with (7.46) and invoking (7.38), (7.39), (3.7), (3.10)-(3.12), and (3.16) we obtain

$$\frac{\mathcal{E}_{zz}}{2\pi} = \frac{\ell_B}{bvS^2R_B}\,[-(1-\alpha v)(\alpha b-\beta T_B) + \delta^2 bv]$$

$$+ \frac{\ell_A}{bvS^2R_A}\,[(1+\bar{\alpha} v)(\bar{\alpha} b-\bar{\beta} T_A) + \bar{\delta}^2 bv] \quad . \qquad (7.48)$$

By combining this result for $\mathcal{E}_{zz}$ with (7.45) for $\mathcal{E}_{xx}+\mathcal{E}_{yy}+\mathcal{E}_{zz}$, with (7.40) for $\mathcal{E}_{xx}$, and with (3.12b) for S^2 we obtain

$$\frac{\mathcal{E}_{yy}}{2\pi} = \frac{(\alpha b-\beta T_B)T_B[T_B - v(\alpha T_B+\beta b)]}{R_B\ell_B S^2 bv} \qquad (7.49)$$

$$- \frac{(\bar{\alpha} b-\bar{\beta} T_A)T_A[T_A + v(\bar{\alpha} T_A+\bar{\beta} b)]}{R_A\ell_A S^2 bv} \quad . \qquad (7.49)$$

We turn finally to $\mathcal{E}_{yz}+\mathcal{E}_{zy}$. From equation (7.32) with $n^{z'} = 0$ and $dx^0/d\tau_{Ao} = 1$ we have

$$\frac{\mathcal{E}_{yz}+\mathcal{E}_{zy}}{2\pi} = \left[\frac{n^z(2n^y+n^{y'})}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})d\tau_{Ao}/d\tau_{Bo}}\right]_{x'=z_B(\tau_{Bo})} \qquad (7.50)$$

$$+ \left[\frac{n^z n^{y'}}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})d\tau_{Ao}/d\tau_{B1}}\right]_{x'=z_B(\tau_{B1})} - \int_{\tau_{B1}}^{\tau_{Bo}}\left[\frac{2n^z(n^y+n^{y'})}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})^2}\right]_{x'=z_B(\tau_B)} d\tau_B .$$

The integral that appears here can be evaluated using $\underset{\sim}{n} = (\alpha,\beta,\delta)$ and $\underset{\sim}{x}' = (v\gamma\tau_B, b, 0)$:

$$\int \frac{2n^z(n^y+n^{y'})}{R_A(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})^2}\,d\tau_B = \int \frac{2\delta[\beta + b(b^2+v^2\gamma^2\tau_B^2)^{-1/2}]}{R_A[(b^2+v^2\gamma^2\tau_B^2)^{1/2} + \alpha v\gamma\tau_B + \beta b]^2}\,d\tau_B$$

$$= 2\ \frac{\alpha(b^2 + v^2\gamma^2\tau_B{}^2)^{1/2} + (1-\beta^2)v\gamma\tau_B + \alpha\beta b}{\delta R_A b v\gamma[(b^2 + v^2\gamma^2\tau_B{}^2)^{1/2} + \alpha v\gamma\tau_B + \beta b]}$$

$$= \frac{2|\underset{\sim}{x}'|[n^x + n^{x'} - n^{x'}(n^y)^2 + n^x n^y n^{y'}]}{R_A\ b v\gamma\ n^z(|\underset{\sim}{x}'| + \underset{\sim}{x}'\cdot\underset{\sim}{n})}\ . \tag{7.51}$$

By combining this with (7.50) and invoking (7.38), (7.39), (3.7), (3.10)-(3.12), and (3.16) we obtain

$$\frac{\varepsilon_{yz} + \varepsilon_{zy}}{2\pi} = \frac{\delta}{R_B S^2}\left\{-\frac{b(\alpha T_B + \beta b)}{\ell_B} - \frac{2\ell_B[T_B - v(\alpha T_B + \beta b)]}{bv}\right\}$$
$$+ \frac{\bar\delta}{R_A S^2}\left\{-\frac{b(\bar\alpha T_A + \bar\beta b)}{\ell_A} + \frac{2\ell_A[T_A + v(\bar\alpha T_A + \bar\beta b)]}{bv}\right\}. \tag{7.52}$$

This completes our evaluation of the whump integrals.

d) Final Answer for the Whump Field

By inserting expressions (7.40), (7.41), (7.42), (7.44), (7.45), (7.48), (7.49) and (7.52) for the whump integrals into equation (7.9) for the whump field, we obtain

$$\frac{2\bar h_W{}^{\mu\nu}}{4m_A m_B} = (u_A{}^\mu\ u_A{}^\nu + u_B{}^\mu\ u_B{}^\nu)\ \frac{(1-3v^2)}{v^2}\left\{-\frac{b(\bar\alpha b - \bar\beta T_A)}{vS^2 R_A\ \ell_A} + \frac{b(\alpha b - \beta T_B)}{vS^2 R_B \ell_B}\right\}$$
$$+ u_A{}^{(\mu}u_B{}^{\nu)}\left\{\frac{\gamma(2-v^2)(1-3v^2)}{v^2}\left[\frac{b(\bar\alpha b - \bar\beta T_A)}{vS^2 R_A \ell_A} - \frac{b(\alpha b - \beta T_B)}{vS^2\ R_B \ell_B}\right]\right.$$
$$\left. - 4\gamma\left[\frac{1}{R_A\ell_A} + \frac{1}{R_B\ell_B}\right]\right\}$$
$$+ \not{b}^{(\mu}u_A{}^{\nu)}\left\{\frac{\gamma}{v^2S^2R_A\ell_A}\ [\bar\beta T_A - \bar\alpha b][2(1-v^2)T_A + v(1+v^2)(\bar\alpha T_A + \bar\beta b)]\right.$$

$$- \frac{\gamma^2}{v^2 S^2 R_B \ell_B} [\beta T_B - \alpha b][(2-3v^2-v^4)T_B - v(1-3v^2)(\alpha T_B + \beta b)] \Big\}$$

$$+ \mathfrak{b}^{(\mu} u_B^{\ \nu)} \Big\{ - \frac{\gamma^2}{v^2 S^2 R_A \ell_A} [\bar{\beta} T_A - \bar{\alpha} b][(2-3v^2 - v^4)T_A + v(1-3v^2)(\bar{\alpha} T_A + \bar{\beta} b)]$$

$$+ \frac{\gamma}{v^2 S^2 R_B \ell_B} [\beta T_B - \alpha b][2(1-v^2)T_B - v(1+v^2)(\alpha T_B + \beta b)] \Big\}$$

$$q^{(\mu} u_A^{\ \nu)} \Big\{ \frac{\gamma \bar{\delta}}{v^2 S^2 R_A \ell_A} [2(1-v^2)\ell_A^{\ 2} + v(1+v^2) T_A (\bar{\alpha} T_A + \bar{\beta} b)]$$

$$- \frac{\gamma^2 \delta}{v^2 S^2 R_B \ell_B} [(2-3v^2-v^4)\ell_B^{\ 2} - v(1-3v^2) T_B (\alpha T_B + \beta b)] \Big\}$$

$$+ q^{(\mu} u_B^{\ \nu)} \Big\{ - \frac{\gamma^2 \bar{\delta}}{v^2 S^2 R_A \ell_A} [(2-3v^2-v^4)\ell_A^{\ 2} + v(1-3v^2) T_A (\bar{\alpha} T_A + \bar{\beta} b)]$$

$$+ \frac{\gamma \delta}{v^2 S^2 R_B \ell_B} [2(1-v^2)\ell_B^{\ 2} - v(1+v^2) T_B (\alpha T_B + \beta b)] \Big\} \qquad (7.53)$$

$$+ \mathfrak{b}^{\mu} \mathfrak{b}^{\nu} \gamma^2 (1+v^2) \Big\{ - \frac{(\bar{\alpha} b - \bar{\beta} T_A) T_A [T_A + v(\bar{\alpha} T_A + \bar{\beta} b)]}{R_A \ell_A S^2 bv}$$

$$+ \frac{(\alpha b - \beta T_B) T_B [T_B - v(\alpha T_B + \beta b)]}{R_B \ell_B S^2 bv} \Big\}$$

$$+ \mathfrak{b}^{(\mu} q^{\nu)} \gamma^2 (1+v^2) \Big\{ - \frac{\bar{\delta}}{R_A S^2} \left[\frac{b(\bar{\alpha} T_A + \bar{\beta} b)}{\ell_A} - \frac{2\ell_A [T_A + v(\bar{\alpha} T_A + \bar{\beta} b)]}{bv} \right]$$

$$- \frac{\delta}{R_B S^2} \left[\frac{b(\alpha T_B + \beta b)}{\ell_B} + \frac{2\ell_B [T_B - v(\alpha T_B + \beta b)]}{bv} \right] \Big\}$$

$$+ q^{\mu} q^{\nu} \gamma^2 (1+v^2) \Big\{ \frac{\ell_A}{bvS^2 R_A} [-(1+\bar{\alpha} v)(\bar{\beta} T_A - \bar{\alpha} b) + \bar{\delta}^2 bv]$$

$$+ \frac{\ell_B}{bvS^2 R_B} [(1-\alpha v)(\beta T_B - \alpha b) + \delta^2 bv] \Big\} .$$

$$+ \eta^{\mu\nu}\left\{2v^2\gamma^2\left[\frac{b(\bar{\beta}T_A-\bar{\alpha}b)}{vS^2R_A\ell_A} - \frac{b(\beta T_B-\alpha b)}{vS^2R_B\ell_B}\right]\right.$$

$$\left. - \frac{\gamma^2(1+v^2)}{2}\left[\frac{1}{R_A\ell_A} + \frac{1}{R_B\ell_B}\right]\right\} \quad .$$

VIII. TOTAL FIELD AND GRAVITATIONAL-WAVE AMPLITUDES

The total gravitational field ${}_2\bar{h}^{\mu\nu}$ in the radiation zone, with the non-radiative "coulomb" part removed, is obtained by adding together the direct field (5.7), the focusing-plus-transition-plus-tail field (6.42), and the whump field (7.53). The result is:

$${}_2\bar{h}^{\mu\nu} = 4m_Am_B\,[H_{AA}u_A{}^\mu u_A{}^\nu + H_{BB}u_B{}^\mu u_B{}^\nu + H_{AB}u_A{}^{(\mu}u_B{}^{\nu)}$$

$$+ H_{bA}\not{b}^{(\mu}u_A{}^{\nu)} + H_{bB}\not{b}^{(\mu}u_B{}^{\nu)} + H_{qA}q^{(\mu}u_A{}^{\nu)} + H_{qB}q^{(\mu}u_B{}^{\nu)}$$

$$+ H_{bb}\not{b}^\mu\not{b}^\nu + H_{bq}\not{b}^{(\mu}q^{\nu)} + H_{qq}q^\mu q^\nu + H_\eta\,\eta^{\mu\nu}] \quad . \tag{8.1}$$

The coefficients which appear here are as follows:

$$H_{AA} = \frac{2(1-v^2) + \bar{\alpha}v(1+v^2)}{v^2(1+\bar{\alpha}v)R_A\ell_A} + \frac{\bar{\beta}(1+v^2)}{v(1+\bar{\alpha}v)R_Ab}\left(\frac{T_A}{\ell_A}+1\right)$$

$$- \left[\frac{(1-3v^2)b}{v^3} + \frac{2vb}{1+\bar{\alpha}v}\right]\left[\frac{\bar{\alpha}b-\bar{\beta}T_A}{S^2R_A\ell_A} - \frac{\alpha b-\beta T_B}{S^2R_B\ell_B}\right] \quad , \tag{8.2a}$$

$$H_{BB} = \frac{2(1-v^2) - \alpha v(1+v^2)}{v^2(1-\alpha v)R_B\ell_B} - \frac{\beta(1+v^2)}{v(1-\alpha v)R_Bb}\left(\frac{T_B}{\ell_B}+1\right)$$

$$- \left[\frac{(1-3v^2)b}{v^3} + \frac{2vb}{1-\alpha v}\right]\left[\frac{\bar{\alpha}b-\bar{\beta}T_A}{S^2R_A\ell_A} - \frac{\alpha b-\beta T_B}{S^2R_B\ell_B}\right] \quad , \tag{8.2b}$$

$$H_{AB} = -\frac{2}{v^2\gamma}\left[\frac{1}{R_A\ell_A} + \frac{1}{R_B\ell_B}\right]$$

$$+ \frac{\gamma(2-v^2)(1-3v^2)b}{v^3}\left[\frac{\bar{\alpha}b-\bar{\beta}T_A}{S^2R_A\ell_A} - \frac{\alpha b-\beta T_B}{S^2R_B\ell_B}\right] \quad , \tag{8.2c}$$

$$H_{bA} = \frac{2\gamma(1+v^2)}{vR_Ab}\left(\frac{T_A}{\ell_A} + 1\right)$$

$$- \frac{\gamma}{v^2S^2R_A\ell_A}[\bar{\alpha}b-\bar{\beta}T_A][2(1-v^2)T_A + v(1+v^2)(\bar{\alpha}T_A+\bar{\beta}b)]$$

$$+ \frac{\gamma^2}{v^2S^2R_B\ell_B}[\alpha b-\beta T_B][(2-3v^2-v^4)T_B - v(1-3v^2)(\alpha T_B+\beta b)] \tag{8.2d}$$

$$H_{bB} = -\frac{2\gamma(1+v^2)}{vR_Bb}\left(\frac{T_B}{\ell_B} + 1\right)$$

$$+ \frac{\gamma^2}{v^2S^2R_A\ell_A}[\bar{\alpha}b-\bar{\beta}T_A][(2-3v^2-v^4)T_A + v(1-3v^2)(\bar{\alpha}T_A+\bar{\beta}b)]$$

$$- \frac{\gamma}{v^2S^2R_B\ell_B}[\alpha b-\beta T_B][2(1-v^2)T_B - v(1+v^2)(\alpha T_B+\beta b)] \quad , \tag{8.2e}$$

$$H_{qA} = \frac{\gamma\bar{\delta}}{v^2S^2R_A\ell_A}[2(1-v^2)\ell_A{}^2 + v(1+v^2)T_A(\bar{\alpha}T_A+\bar{\beta}b)]$$

$$- \frac{\gamma^2\delta}{v^2S^2R_B\ell_B}[(2-3v^2-v^4)\ell_B{}^2 - v(1-3v^2)T_B(\alpha T_B+\beta b)] \quad , \tag{8.2f}$$

$$H_{qB} = -\frac{\gamma^2\bar{\delta}}{v^2S^2R_A\ell_A}[(2-3v^2-v^4)\ell_A{}^2 + v(1-3v^2)T_A(\bar{\alpha}T_A+\bar{\beta}b)]$$

$$+ \frac{\gamma\delta}{v^2S^2R_B\ell_B}[2(1-v^2)\ell_B{}^2 - v(1+v^2)T_B(\alpha T_B+\beta b)] \quad , \tag{8.2g}$$

$$H_{bb} = \frac{\gamma^2(1+v^2)}{bv}\left\{-\frac{(\bar{\alpha}b-\bar{\beta}T_A)T_A}{S^2R_A\ell_A}[T_A+v(\bar{\alpha}T_A+\bar{\beta}b)]\right.$$

$$\left.+\frac{(\alpha b-\beta T_B)T_B}{S^2R_B\ell_B}[T_B-v(\alpha T_B+\beta b)]\right\} , \qquad (8.2h)$$

$$H_{bq} = -\frac{\gamma^2(1+v^2)\bar{\delta}}{S^2R_A}\left\{\frac{b(\bar{\alpha}T_A+\bar{\beta}b)}{\ell_A}-\frac{2\ell_A[T_A+v(\bar{\alpha}T_A+\bar{\beta}b)]}{bv}\right\}$$

$$-\frac{\gamma^2(1+v^2)\delta}{S^2R_B}\left\{\frac{b(\alpha T_B+\beta b)}{\ell_B}+\frac{2\ell_B[T_B-v(\alpha T_B+\beta b)]}{bv}\right\} , \qquad (8.2i)$$

$$H_{qq} = \frac{\gamma^2(1+v^2)\ell_A}{bvS^2R_A}[(1+\bar{\alpha}v)(\bar{\alpha}b-\bar{\beta}T_A)+\bar{\delta}^2bv]$$

$$-\frac{\gamma^2(1+v^2)\ell_B}{bvS^2R_B}[(1-\alpha v)(\alpha b-\beta T_B)-\delta^2bv] , \qquad (8.2j)$$

$$H_\eta = -2bv\gamma^2\left[\frac{\bar{\alpha}b-\bar{\beta}T_A}{S^2R_A\ell_A}-\frac{\alpha b-\beta T_B}{S^2R_B\ell_B}\right]-\frac{\gamma^2(1+v^2)}{2}\left[\frac{1}{R_A\ell_A}+\frac{1}{R_B\ell_B}\right] . \qquad (8.2k)$$

All of the quantities that appear in these coefficients are defined in §III.

One can easily verify that the gravitational field (8.1) possesses the star-interchange properties of equations (3.17).

In the radiation zone where this field exists, the post-linear gauge condition ${}_2\bar{h}^{\mu\nu}{}_{,\nu} = 0$ reduces to

$${}_2\dot{\bar{h}}^{\mu\nu}k_\nu = 0 , \qquad (8.3)$$

where $\vec{k}$ is the propagation vector of equations (3.5) and (3.6). It is straightforward but tedious to verify that our gravitational-wave field (8.1) satisfies this gauge condition. The verification utilizes equations (8.1), (8.2), (3.8), (3.7), (3.12b), and the following relationship [which is derivable from equation (3.5a) or (7.7), and $\overleftrightarrow{\eta} = -\vec{e}_0\otimes\vec{e}_0+\vec{e}_x\otimes\vec{e}_x+\vec{e}_y\otimes\vec{e}_y+\vec{e}_z\otimes\vec{e}_z$]:

$$\eta^{\mu\nu} = \frac{1}{v^2\gamma^2}\{u_B{}^\mu u_B{}^\nu + u_A{}^\mu u_A{}^\nu - 2\gamma u_A{}^{(\mu}u_B{}^{\nu)} + v^2\gamma^2(\not{b}^\mu \not{b}^\nu + q^\mu q^\nu)\} . \quad (8.4)$$

It is also straightforward, but tedious to take the slow-motion limit of expression (8.1) in the rest frame of star A and verify that its spatial components, ${}_2\bar{h}_{jk}$, agree with the slow-motion field derived in Paper IV (eq. 2.6). In taking the slow-motion limit the following approximate relationships [derivable from eqs. (3.5a), (3.7), and (3.10)-(3.12)] are helpful:

$$\vec{u}_A = \vec{e}_0 , \quad \vec{u}_B = \vec{e}_0 + v\vec{e}_x , \quad \vec{\not{b}} = \vec{e}_y , \quad \vec{q} = \vec{e}_z ,$$

$$\frac{1}{R_B} - \frac{1}{R_A} = \frac{\alpha v}{R} , \quad T_B - T_A = v(\alpha T + \beta b) , \quad \ell_B - \ell_A = \frac{vT(\alpha T+\beta b)}{\ell}$$

$$\bar{\alpha} - \alpha = -v(\beta^2+\delta^2) , \quad \bar{\beta} - \beta = v\alpha\beta , \quad \bar{\delta} - \delta = v\alpha\delta$$

$$S^2 = \delta^2(b^2+T^2) + (\alpha b-\beta T)^2 \quad (8.5)$$

Here the quantities on the right-hand side of each expression can be interpreted as

$$T = T_A = T_B , \quad \ell = \ell_A + \ell_B , \quad \alpha = \bar{\alpha} , \quad \beta = \bar{\beta} , \quad \delta = \bar{\delta} ,$$

with fractional errors of $O(v)$. (8.6)

In any Lorentz frame the radiation field is obtained by projecting out the transverse-traceless part of ${}_2\bar{h}_{jk}$ (see Box 35.1 of MTW). Let us perform such a projection in the rest frame of star A--which has basis vectors

$$\vec{e}_0 = \vec{u}_A , \quad \vec{e}_x = (\vec{u}_B - \gamma\vec{u}_A)/(v\gamma) , \quad \vec{e}_y = \vec{\not{b}} , \quad \vec{e}_z = \vec{q} \quad (8.7)$$

(cf. eq. 7.7). For the purpose of the projection we shall introduce spherical polar coordinates (r,θ,ϕ) centered on star A with the polar axis along the direction of motion of B (x direction), and with $(\theta,\phi) = (\pi/2,0)$

along the impact direction (y direction):

$$x = r\cos\theta\ ,\quad y = r\sin\theta\cos\phi\ ,\quad z = r\sin\theta\sin\phi\quad . \tag{8.8}$$

The orthonormal basis vectors associated with these spherical coordinates are

$$\vec{e}_{\hat{0}} = \vec{e}_0\quad , \tag{8.9a}$$

$$\vec{e}_{\hat{r}} \equiv \vec{n} = \vec{e}_x\cos\theta + \vec{e}_y\sin\theta\cos\phi + \vec{e}_z\sin\theta\sin\phi\ , \tag{8.9b}$$

$$\vec{e}_{\hat{\theta}} = -\vec{e}_x\sin\theta + \vec{e}_y\cos\theta\cos\phi + \vec{e}_z\cos\theta\sin\phi\quad , \tag{8.9c}$$

$$\vec{e}_{\hat{\phi}} = -\vec{e}_y\sin\phi + \vec{e}_z\cos\phi\quad . \tag{8.9d}$$

The observer, located in the $\vec{e}_{\hat{r}} = \vec{n}$ direction, uses the basis vectors $\vec{e}_{\hat{\theta}}$ and $\vec{e}_{\hat{\phi}}$ to characterize transverse directions; and he uses their tensor products

$$\overleftrightarrow{e}_+ \equiv \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\phi}}\ ,\qquad \overleftrightarrow{e}_\times = \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\phi}} + \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\theta}} \tag{8.10}$$

as the polarization basis states for the gravitational waves:

$${}_2\overleftrightarrow{\bar{h}}^{TT} = A_+ \overleftrightarrow{e}_+ + A_\times \overleftrightarrow{e}_\times\quad . \tag{8.11}$$

By applying the standard transverse-traceless projection operation to ${}_2\bar{h}_{jk}$ (Box 35.1 of MTW), we obtain the following expressions for the wave amplitudes A_+ and $A_\times$:

$$A_+ = \frac{1}{2}\left(\vec{e}_{\hat{\theta}} \cdot {}_2\overleftrightarrow{\bar{h}} \cdot \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \cdot {}_2\overleftrightarrow{\bar{h}} \cdot \vec{e}_{\hat{\phi}}\right)\quad , \tag{8.12a}$$

$$A_\times = \frac{1}{2}\left(\vec{e}_{\hat{\theta}} \cdot {}_2\overleftrightarrow{\bar{h}} \cdot \vec{e}_{\hat{\phi}} + \vec{e}_{\hat{\phi}} \cdot {}_2\overleftrightarrow{\bar{h}} \cdot \vec{e}_{\hat{\theta}}\right)\quad . \tag{8.12b}$$

These expressions are most easily evaluated in the rest frame of star A (basis vectors $\vec{e}_0$, $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$), using equations (8.1) and (8.7) for ${}_2\overleftrightarrow{h}$ and equations (8.9c,d) for $\vec{e}_{\hat{\theta}}$ and $\vec{e}_{\hat{\phi}}$. The result is

$$A_+ = \frac{4m_A m_B}{bR_A}\mathcal{A}_+ \quad , \qquad A_\times = \frac{4m_A m_B}{bR_A}\mathcal{A}_\times \quad , \tag{8.13a}$$

where R_A is the distance to the observer in the rest frame of star A (which we are using), and $\mathcal{A}_+$ and $\mathcal{A}_\times$ are dimensionless amplitudes given by

$$\begin{aligned}\mathcal{A}_+ = \frac{1}{2} bR_A[&\sin^2\theta(v^2\gamma^2 H_{BB} - H_{qq}) + (\sin^2\phi - \cos^2\theta\cos^2\phi)(H_{qq} - H_{bb}) \\ &- \cos\theta\sin\theta\cos\phi\, v\gamma H_{bB} - \cos\theta\sin\theta\sin\phi\, v\gamma H_{qB} \\ &+ (1+\cos^2\theta)\cos\phi\sin\phi\, H_{bq}]\end{aligned} \tag{8.13b}$$

$$\begin{aligned}\mathcal{A}_\times = \frac{1}{2} bR_A[&2\cos\theta\cos\phi\sin\phi(H_{qq} - H_{bb}) + \sin\theta\sin\phi\, v\gamma H_{bB} \\ &- \sin\theta\cos\phi\, v\gamma H_{qB} + \cos\theta(\cos^2\phi - \sin^2\phi)H_{bq}]\end{aligned} \tag{8.13c}$$

The dimensionless amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$ are functions of the velocity v and energy factor $\gamma = (1-v^2)^{-1/2}$ of star B, the impact parameter b, the observer's angular location (θ,ϕ), and the observer's proper time τ_0. One can express $\mathcal{A}_+$ and $\mathcal{A}_\times$ explicitly in terms of these parameters by making use of the direction-cosine relations

$$\alpha = \cos\theta \ , \quad \beta = \sin\theta\cos\phi \ , \quad \delta = \sin\theta\sin\phi \quad , \tag{8.14}$$

the equations of §III, and expressions (8.2) for the H's.

In Paper IV we shall examine the details of the radiation field (8.13) and its associated spectrum; and we shall compare it with the various special cases that have been computed by other researchers.

While carrying out this research we relied heavily on insights gained from close scrutiny of Peters (1970) bremsstrahlung calculation. Peters' formalism and approximation scheme were different from ours; but they were sufficiently similar to give us insight.

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FIGURE CAPTIONS

Figure 1. (a) Spatial diagram of unperturbed stellar encounter in center-of-mass frame. (b) Spacetime diagram of same encounter.

Figure 2. The spatial part of the 4-acceleration of star B, as seen in the rest frame of star A. The equation describing $\underset{\sim}{a}_B$ is [cf. eq. 2.27]

$$\underset{\sim}{a}_B = \frac{-m_A}{(b^2 + \gamma^2 v^2 \tau_B^2)^{3/2}} \left[(1 + 2\gamma^2 v^2) b \, \underset{\sim}{e}_y - \gamma v \tau_B \underset{\sim}{e}_x\right] .$$

Since $\gamma v \tau_B$ is distance travelled by B, the acceleration is directed toward the focal point shown in the figure. In the limit $v \ll 1$ the focal point coincides with the location of A; but for $\gamma \gg 1$ it is far on the other side of A. The vertical component of the acceleration is proportional to the inverse cube of the distance from A to B. Note that as $\gamma \to \infty$ the acceleration becomes more and more orthogonal to the trajectory of B.

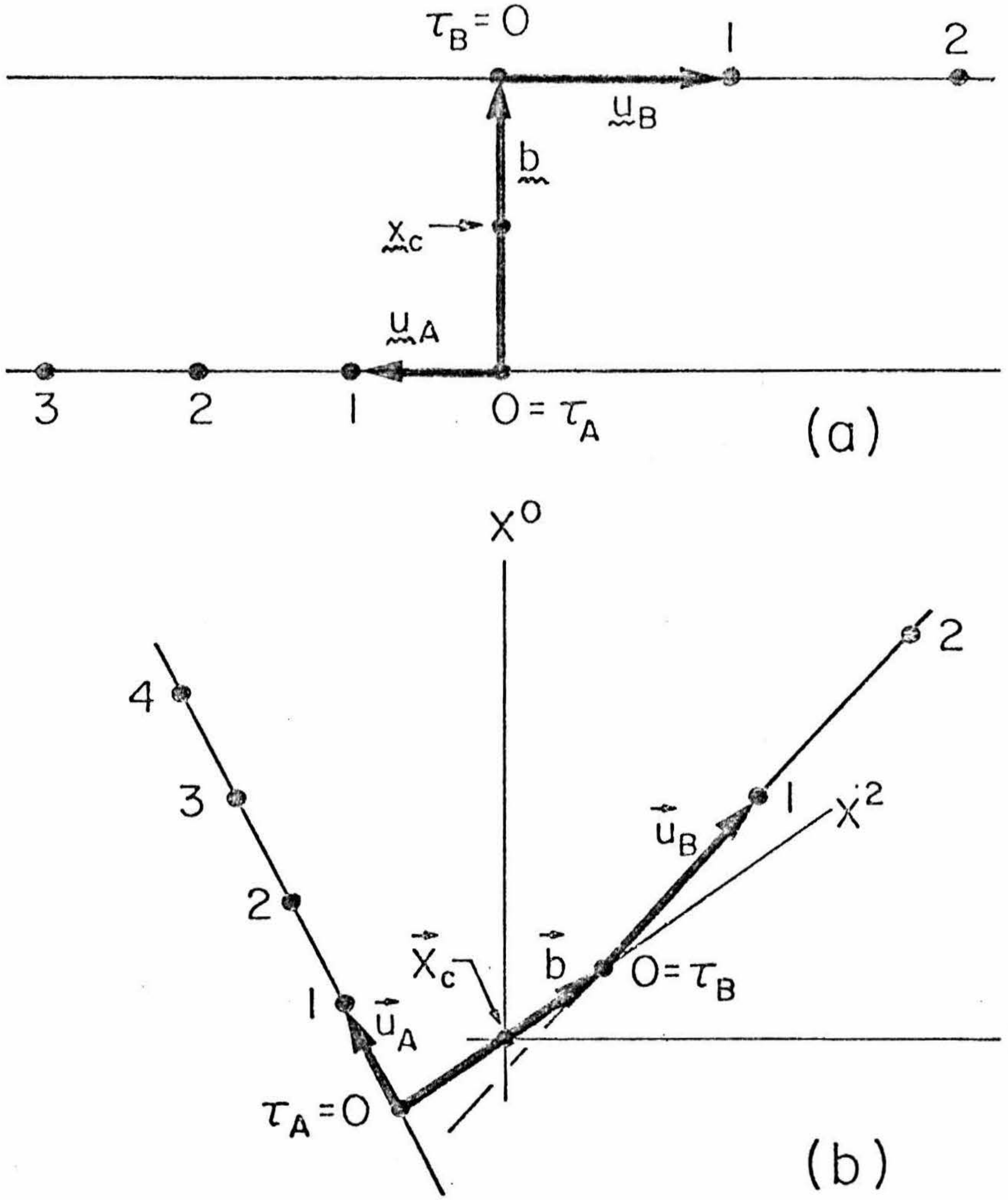
Figure 3. Spatial diagram in the rest frame of A showing the direction $\underset{\sim}{n}$ to the observer with its direction cosines α, β, δ; and showing the vector $\underset{\sim}{b}$ that reaches to the point of B's closest approach, and the spatial part of the velocity of B, $\underset{\sim}{u}_B$.

Figure 4. (a) Definition of retarded times τ_{Ao} and τ_{Bo}. (b) Definition of several vectors $\vec{S}$, $\vec{M}$, $\vec{J}$ to be used in the derivation of the radiation field.

Figure 5. A diagram showing the conical region of spacetime where star A shadows star B. This diagram is confined to the future light cone $\dot{J}_+(Z_B)$ of a specific event Z_B on the world line of B. In other words, the one spacetime direction suppressed from this diagram is the direction leading off $\dot{J}_+(Z_B)$. The coordinates x,y,z used in the diagram are the spatial coordinates of an inertial frame of star A.

Figure 6. Spatial diagram of the geometry used in computing the time-delay function $\gamma_A(x,x')$ in the rest frame of star A. The origin of the spatial coordinates is at the center of star A.

Figure 7. The spatial geometry used in evaluating $\beta_A(x'',x')$ and $\int {}_0G(x,x'')\ \beta_A(x'',x')\ {}_0G'(x'',x')d^4x''$. In equations (6.22)-(6.27) we use the coordinates and parameters of diagram (a). Thereafter we switch to the coordinates of diagram (b).



$\tau_B = 0$
1
2
u_B
b
X_c
u_A
3
2
1
$0 = \tau_A$
(a)
X^0
2
4
3
1
X^2
$\vec{u}_B$
2
$\vec{X}_c$
$\vec{b}$
$0 = \tau_B$
1
$\vec{u}_A$
X^1
$\tau_A = 0$
(b)

Fig. 1

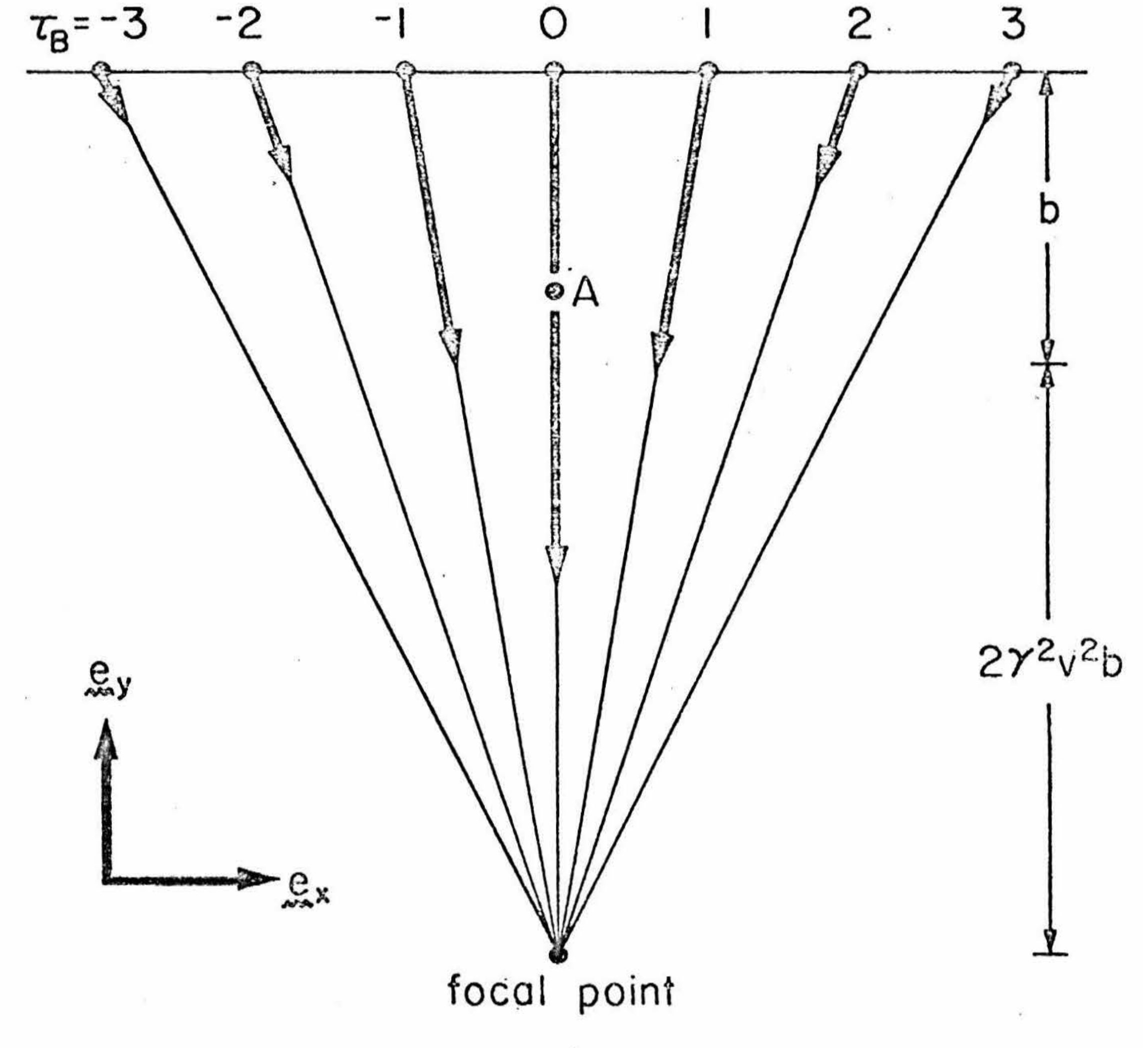


Fig. 2

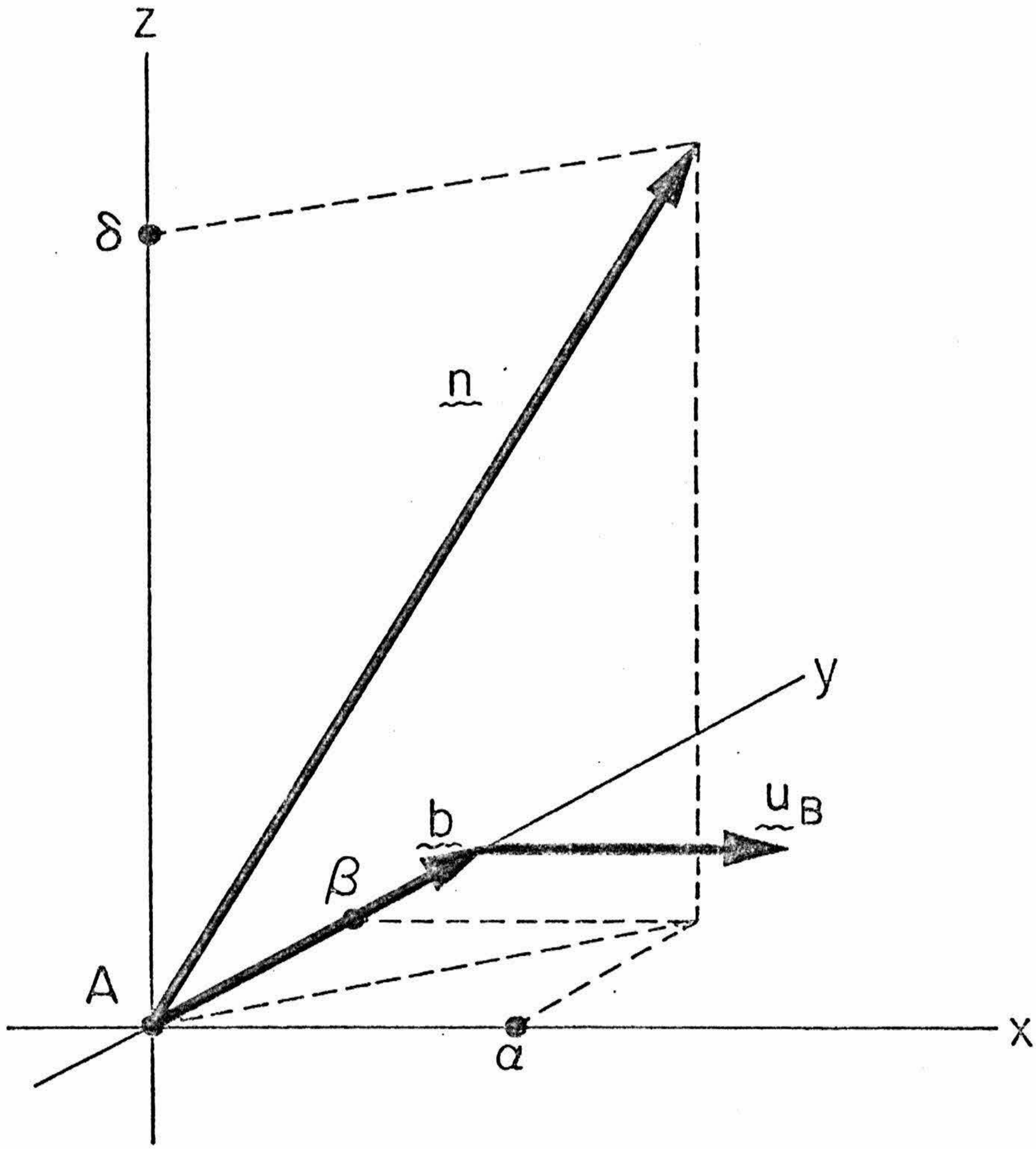


Fig. 3

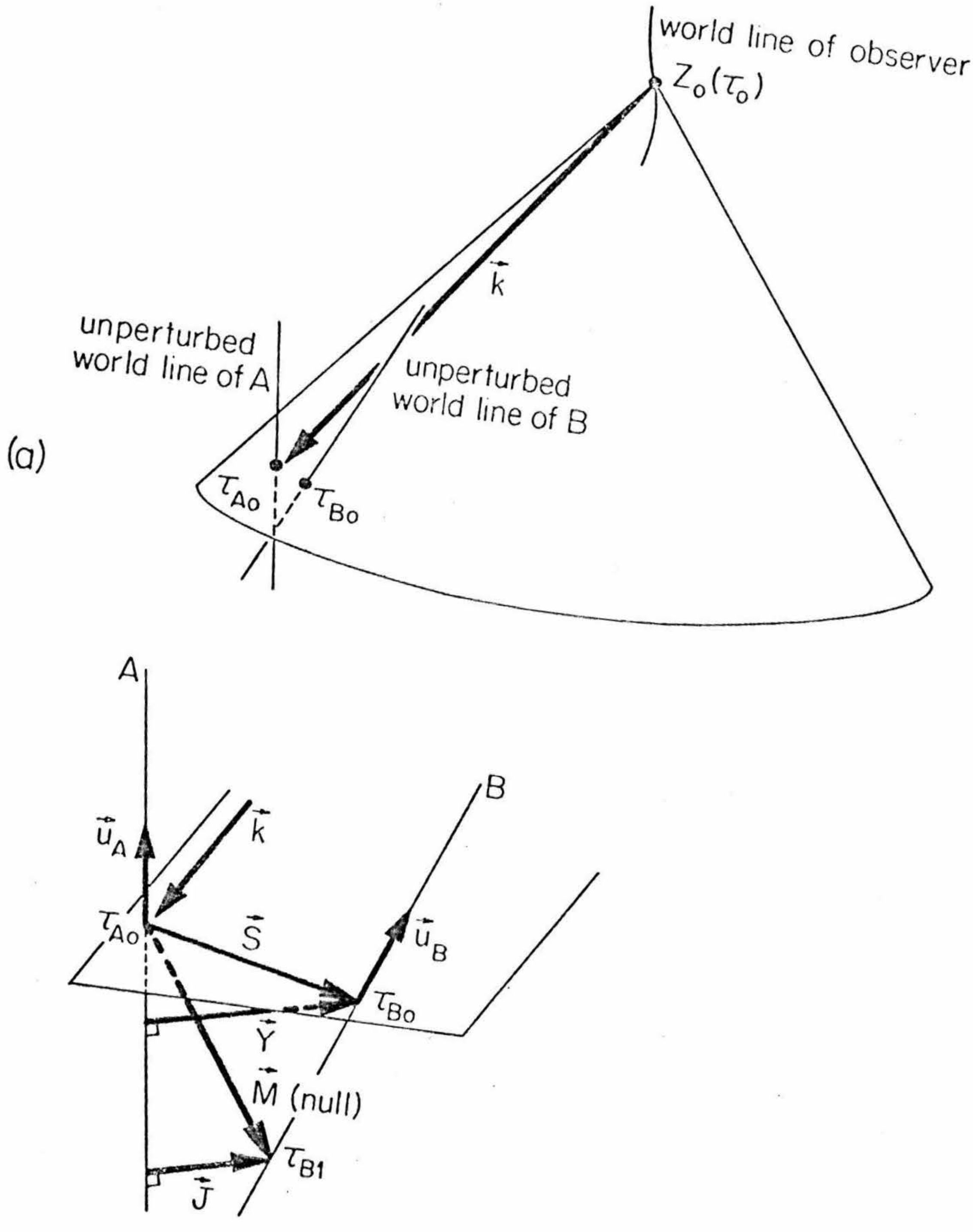

world line of observer
$Z_o(\tau_o)$
$\vec{k}$
unperturbed world line of A
unperturbed world line of B
(a)
τ_{Ao}
τ_{Bo}
A
B
$\vec{u}_A$
$\vec{k}$
τ_{Ao}
$\vec{S}$
$\vec{u}_B$
τ_{Bo}
$\vec{Y}$
$\vec{M}$ (null)
(b)
τ_{B1}
$\vec{J}$

Fig. 4

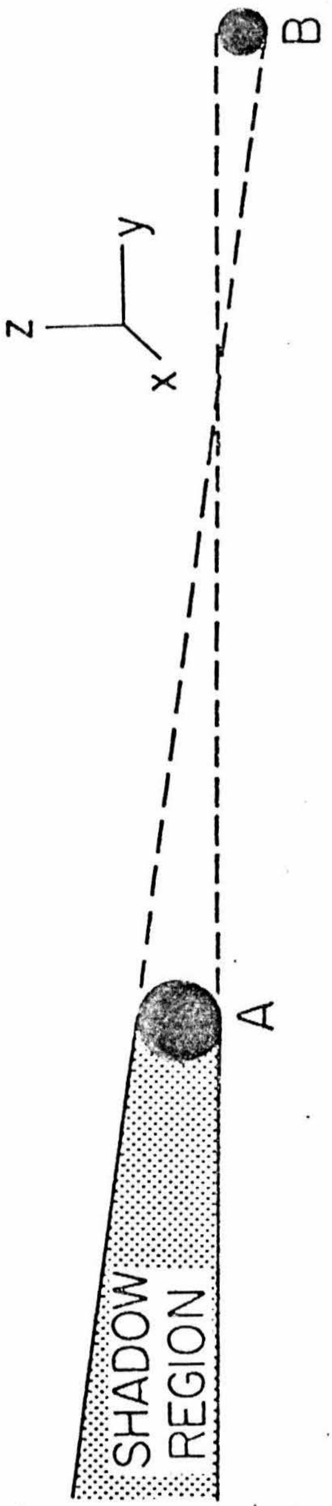


Fig. 5

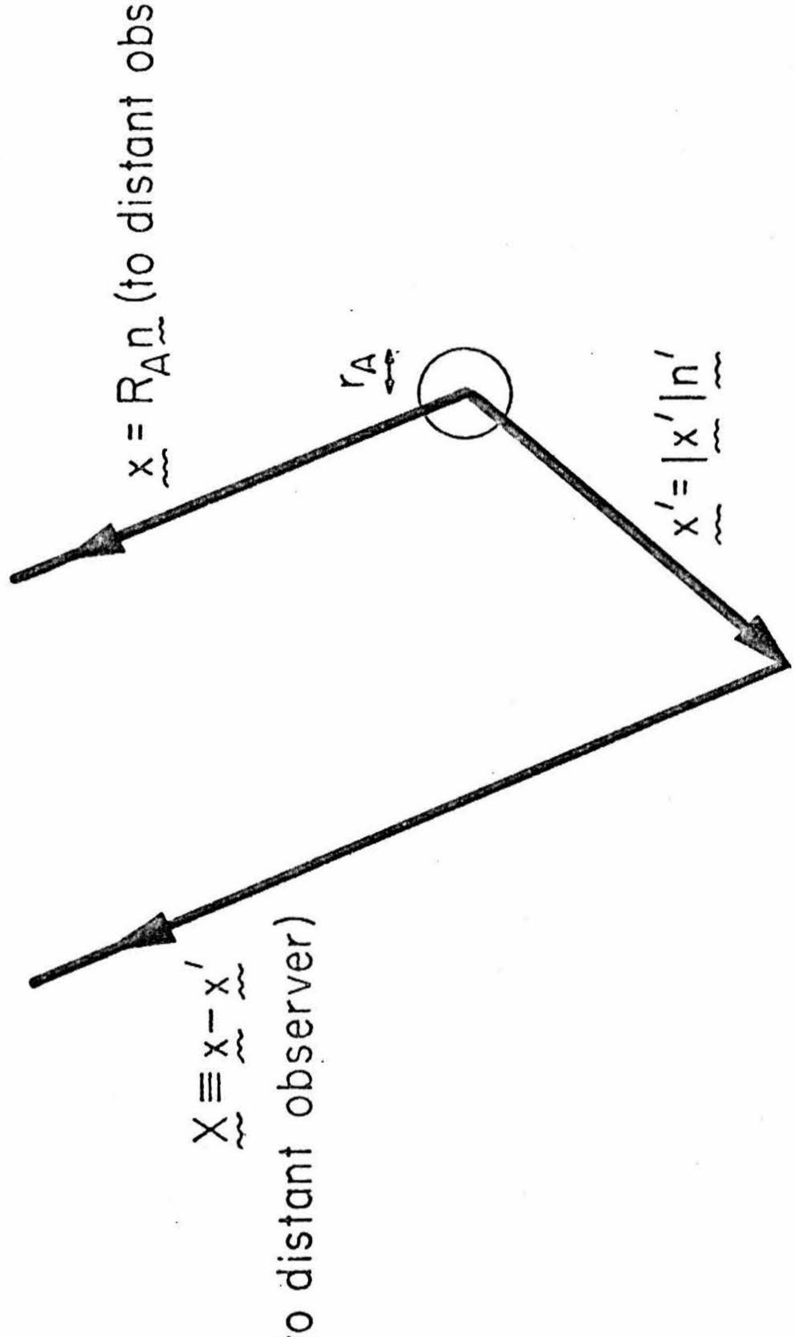


Fig. 6

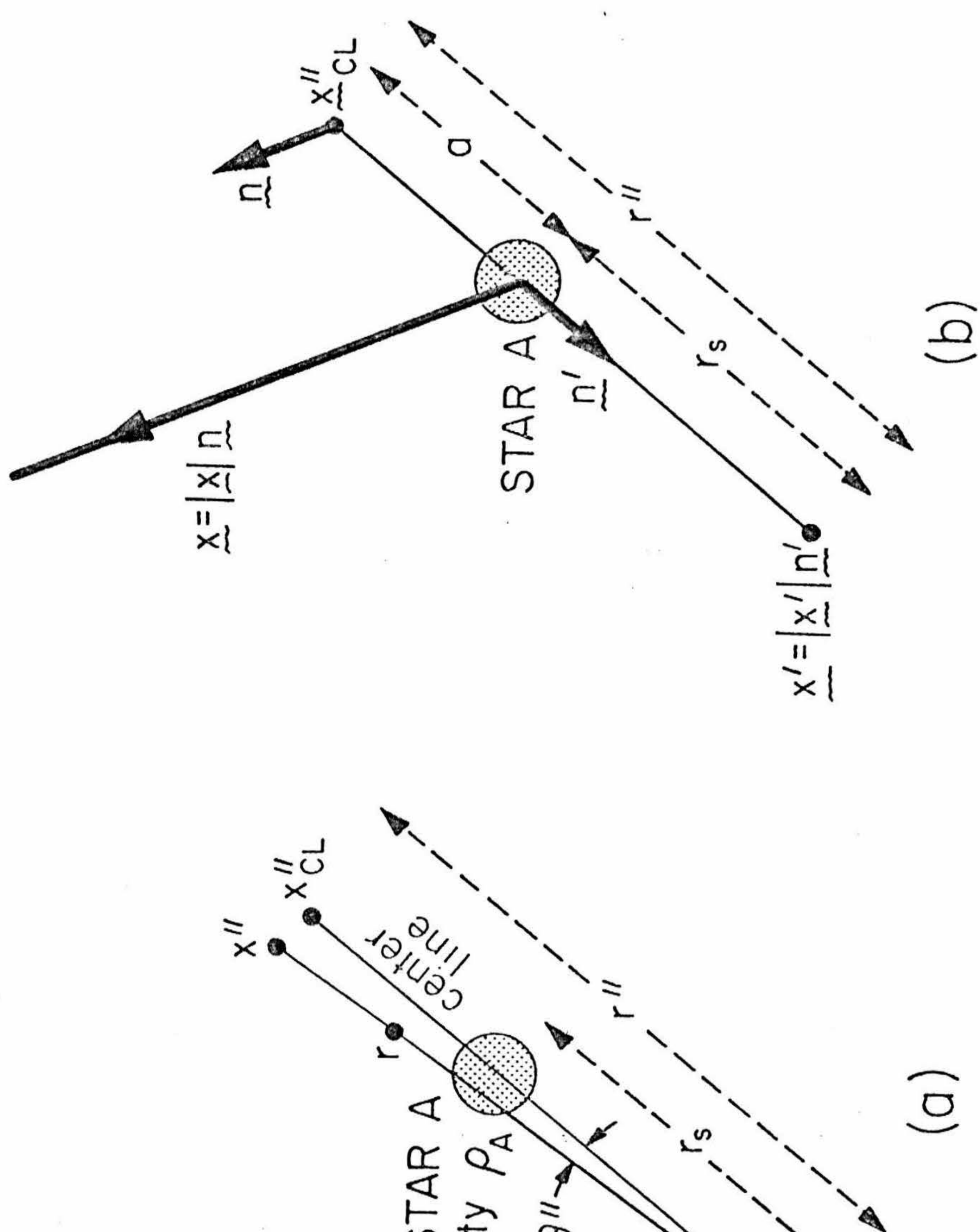

x = |x| n
STAR A
n'
n
x''_CL
a
r''
r_s
x' = |x'| n'
(b)
STAR A
density ρ_A
θ''
r
x''
x''_CL
center line
r''
r_s
x'
(a)

Fig. 7

PAPER 3.

THE GENERATION OF GRAVITATIONAL WAVES

IV. BREMSSTRAHLUNG*

This is material that will ultimately be a joint publication with Kip Thorne.

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ABSTRACT

This paper uses the mathematical formalism and equations derived in the first three papers of this series to give the results of our analysis of gravitational bremsstrahlung. We develop some new insights about the detailed structure of the waveforms and compare our results with those of other investigators. The key features of the analysis include a) the need to use not one, but two naturally occurring time scales, b) the symmetry property in the center-of-velocity frame, that reflection through the origin leaves the magnitude of the wave amplitudes unaltered, c) fairly simple analytic formulae for the amplitude as a function of time, direction, and polarization, d) proof that in the relativistic case there is no logarithmic dependence on γ in the total energy radiated, e) analytic formulae for the power spectrum in the extreme forward direction, valid for arbitrary velocities and f) expressions for the zero-frequency limit.

I. INTRODUCTION

We anticipate somewhat optimistically perhaps, that the detection of gravitational waves will become a reality by the mid 1980's. In this spirit we have embarked on this series of papers dealing with the generation of gravitational waves. Our aim is to elucidate the realms of validity of the present computational techniques and to try to devise new techniques valid in new realms. Our ultimate purpose is to be able to calculate the waves generated by models of astrophysical systems. The literature is replete with references by many researchers on the topic of gravitational radiation and a thorough review of computational techniques has been given by Thorne (1975).

The foundations for this paper have been laid down by the first three papers in this series (Thorne and Kovács 1975, "Paper I"; Crowley and Thorne 1977, "Paper II"; Kovács and Thorne 1977, "Paper III"). In this paper we give the results of our detailed analysis of gravitational bremsstrahlung.

The context in which aspects of this particular problem have been investigated previously has varied widely. The various approaches include the use of Newtonian theory, Turner (1977), Hansen (1972); post-Newtonian theory, Wagoner and Will (1976), Turner and Will (1977); the "quadrupole-moment formalism," Ruffini and Wheeler (1971); perturbation of the Schwarzschild geometry, Peters (1970), Misner (1972); quantum gravity, Barker and Gupta (1974), Smarr (1977), Feynman (1961); the method of virtual quanta, Matzner and Nutku (1974); the method of colliding plane waves, D'Eath (1977).

Nowhere before has there been a detailed analysis of the wave amplitude as a function of time, angle and polarization valid for arbitrary velocities and unrestricted mass ratios. Our analysis is valid for weak fields only and also requires that the angle of deflection be small compared to 90°. There is no hope of detecting gravitational radiation from any astrophysical system by surrounding the source with detectors covering 4π steradians. Therefore we focus little attention on the total gravitational wave luminosity L_{GW} or the total energy radiated per hertz $dE/d\nu$ — instead we concentrate on the structure (i.e., time, polarization and frame dependence) of the amplitudes since the amplitudes are the quantities that gravitational wave astronomers are seeking to detect (see MTW, chapt. 37).

This paper is divided into six sections. Section II gives the well known low-velocity limit and defines our notation and coordinate system. Section III discusses the frame dependence of the amplitudes, the symmetry properties in the center-of-velocity frame and the need to use two time parameters T_A, T_B to describe the radiation. Section III concludes by giving expressions for the wave amplitudes, [eqs. (3.17) and (3.18)] and their symmetry properties [eq. (3.20)], in the frame of star "A" as a function of time and angle. Section IV analyzes the high velocity limit of the amplitudes and breaks the radiation zone into three overlapping regions of interest: the "forward-region," the "intermediate-region" and the "backward-region" [eq. (4.1)]. Section V is devoted to the frame and time dependent features of the wave forms and includes a comparison between our work and that of Peters (1970).

Section VI contrasts our results with those of other authors and gives an expression for the specific flux [eq. (6.9)], valid for arbitrary velocities in the extreme forward direction. Section VI concludes by giving the zero-frequency-limit of the radiation.

II. LOW-VELOCITY LIMIT

Consider two objects (stars, planets, black holes...) which undergo a near encounter. Require that their interaction be entirely gravitational; and for specificity, call them "stars." Require that the stars' sizes r_A and r_B be small compared to their impact parameter b

$$r_A \ll b, \; r_B \ll b \tag{2.1a}$$

so that tidal interactions can be ignored. Also require that they move with low relative velocity v, and that their impact parameter be large enough to guarantee only small deflection of their orbits:

$$v \ll 1, \; \zeta \equiv (m_A + m_B)/(bv^2) \ll 1. \tag{2.1b}$$

The assumptions $v \ll 1$ and $b \gg (m_A + m_B)/v^2 \gg (m_A + m_B)$ ("slow motion" and "weak relative gravity") guarantee that the orbital motion can be analyzed with high accuracy using Newtonian gravitation theory, and (by virtue of 2.1a) using a point-particle description of the stars. This is true even if one or both of the "stars" is a black hole (D'Eath 1975). Moreover, these same assumptions guarantee that the gravitational waves emitted can be computed with high accuracy from the Newtonian motion using the standard "quadrupole-moment formula"

$$h_{jk}^{TT} = (2/r)\, \ddot{I}_{jk}^{TT}(t-r) \tag{2.2}$$

(eq. 36.20 of Misner, Thorne, and Wheeler 1973). Again this is true even in the case of black holes (Thorne 1977b).

The computation of the orbits and waves is so straightforward that we shall not give the details here, we give only the final result for the radiation field.

We shall describe the final result using Cartesian coordinates (t,x,y,z) in which star A is initially at rest at the origin; the moment of closest approach is time $t = 0$, the orbit of star B lies in the x-y plane with initial velocity in the x-direction, the unit spatial vector pointing from the origin toward the observer is

$$\mathbf{n} = \cos\theta \, \mathbf{e}_x + \sin\theta \cos\varphi \, \mathbf{e}_y + \sin\theta \sin\varphi \, \mathbf{e}_z , \tag{2.3}$$

and the observer is in the radiation zone, a distance r from the origin. (See Figure 1, and note our unconventional relationship between Cartesian and polar coordinates; a more conventional approach would make the notation change $\mathbf{e}_x \to \mathbf{e}_z$, $\mathbf{e}_y \to \mathbf{e}_x$, $\mathbf{e}_z \to \mathbf{e}_y$.) We shall decompose the radiation field into two orthogonal polarization states with basis tensors

$$\mathbf{e}^{+} = \mathbf{e}_\theta \otimes \mathbf{e}_\theta - \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi, \quad \mathbf{e}^{\times} = \mathbf{e}_\theta \otimes \mathbf{e}_\varphi + \mathbf{e}_\varphi \otimes \mathbf{e}_\theta \tag{2.4a}$$

where

$\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unit vectors in the θ and φ directions

$$\mathbf{e}_\theta = r^{-1}\partial/\partial\theta = - \mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta \cos\varphi + \mathbf{e}_z \cos\theta \sin\varphi , \tag{2.4b}$$

$$\mathbf{e}_\varphi = (r \sin\theta)^{-1}\partial/\partial\varphi = - \mathbf{e}_y \sin\varphi + \mathbf{e}_z \cos\varphi . \tag{2.4c}$$

In describing the time dependence of the radiation field we shall use retarded time $t - r$, measured in units of the time b/v for star B to travel a distance equal to the impact parameter:

$$T \equiv (t - r)\, v/b: \tag{2.5a}$$

and we shall use the function

$$\ell\,(T) \equiv (1 + T^2)^{1/2} , \tag{2.5b}$$

which equals the distance between the two stars at "time" T, measured in units of the impact parameter.

In terms of the above quantities, and to first order in the deflection parameter ζ, the gravitational-wave field (transverse-traceless part of the metric perturbation) is given by

$$h_{jk}^{TT}(t,r,\theta,\varphi) = A_+ \underset{\sim}{e}^+ + A_\times \underset{\sim}{e}^\times , \tag{2.6a}$$

$$A_+ = (4m_A m_B/br)\,\mathcal{A}_+,\quad A_\times = (4m_A m_B/br)\,\mathcal{A}_\times, \tag{2.6b}$$

$$\mathcal{A}_+ = \frac{1}{2}\left[\frac{1}{\ell^3} + \frac{1}{\ell}\right]\sin^2\theta + \frac{1}{2\ell^3}(\sin^2\varphi - \cos^2\theta\cos^2\varphi) + \left[\frac{T}{\ell^3} + \frac{T}{\ell} + 1\right]\cos\theta\sin\theta\cos\varphi , \tag{2.6c}$$

$$\mathcal{A}_\times = \frac{1}{\ell^3}\cos\theta\cos\varphi\sin\varphi - \left[\frac{T}{\ell^3} + \frac{T}{\ell} + 1\right]\sin\theta\sin\varphi . \tag{2.6d}$$

Here A_+ and $A_\times$ are the amplitudes of the two polarization states. Note that their magnitude is $4m_A m_B/br$, which has the physical significance and size

$$\frac{4m_A m_B/b}{r} = -4\,\frac{\left(\text{relative gravitational potential energy of stars at moment of closest approach}\right)}{(\text{distance from stars to observer})}$$

$$= -4\left(\text{Newtonian gravitational potential at observer's location produced by stars' relative potential energy}\right)$$

$$= (0.5 \times 10^{-22})\left(\frac{m_A}{M_\odot}\right)\left(\frac{m_B}{M_\odot}\right)\left(\frac{R_\odot}{b}\right)\left(\frac{10\ \text{kpc}}{r}\right) . \tag{2.7}$$

Here m_A and m_B are measured in solar masses, $M_\odot$: the impact parameter b is measured in solar radii $R_\odot$: and distance to the source r is measured in units of the distance to the center of our galaxy. Perhaps it is not totally hopeless to think of detecting such radiation in the 1980's (Zel'dovich and Polnarev 1974).

The time dependence and angular dependence of the radiation field are contained in the renormalized amplitudes $\mathcal{a}_+$ and $\mathcal{a}_\times$. They are of order unity for $|T| \lesssim 1$. Equations (2.6c,d) express them in terms of impact parameter, b, velocity times retarded time, $v(t - r) \equiv T$, distance between stars, $\ell \equiv (1 + T^2)^{1/2}$, and observer's angular location (θ,φ).

Notice that, the radiation pattern (2.5) is totally insensitive to the relative masses of stars A and B: it depends only on the product of their masses and the amplitude is not frame dependent in the low-velocity case. Moreover an observer at $\theta < \pi/2$ (toward whom star B moves in a frame where star A is at rest) sees precisely the same radiation features as an observer in the opposite direction (toward whom A moves in a frame where B is at rest). This fact is embodied in the parity properties of $\mathcal{a}_+$, $\mathcal{a}_\times$, and the polarization tensors:

$$\text{If } (\theta,\varphi) \to (\pi-\theta,\ \pi+\varphi): \begin{cases} \mathcal{a}_+ \to \mathcal{a}_+, \ \underset{\sim}{e}_+ \to \underset{\sim}{e}_+ \\ \mathcal{a}_\times \to \mathcal{a}_\times, \ \underset{\sim}{e}_\times \to -\underset{\sim}{e}_\times . \end{cases} \tag{2.8}$$

Figure 2 displays the time behavior of $\mathcal{a}_+$ and $\mathcal{a}_\times$ for several locations of the observer. Notice that, by choice of convention, the amplitudes begin at zero for $T \to -\infty$, but, except in special locations, they do not tend to zero for $T \to +\infty$! Since the displacement of a free-mass detector is directly proportional to the amplitude of the wave, bremsstrahlung radiation will produce a permanent change in the distance between the test masses of a free-mass detector. This angle dependent effect occurs because the stars' gravitational interaction permanently changes the magnitude of $\ddot{I}_{jk}$. No such effect occurs for radiation from sources (e.g., stellar collapse) which begin in a stationary state and end in a stationary state.

Free-mass (broad-band) detectors respond directly to the amplitude of a wave and have no mechanical restoring forces. On the other hand, resonant detectors (Weber-type bars, supercooled monocrystals of sapphire, etc.) can be tuned to yield information about the spectral content of the incoming waves. For our case, the spectral characteristics of the radiation are given by the Fourier transform of the renormalized amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$. These transforms are defined by;

$$\mathcal{A}_{\substack{+\omega \\ \times}} \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathcal{A}_{\substack{+ \\ \times}}\, e^{-i\omega t}\, dt = \frac{1}{(2\pi)^{1/2}} \frac{1}{v} \int_{-\infty}^{+\infty} \mathcal{A}_{\substack{+ \\ \times}}\, e^{-i(\omega/v)T}\, dT \ . \qquad (2.9a)$$

For the amplitude (2.6c,d) these Fourier transforms are

$$\mathcal{A}_{+\omega} = \frac{1}{(2\pi)^{1/2}} \left(\frac{b}{v}\right) \Big[-\{K_0 \sin^2\theta + uK_1(\sin^2\theta + \sin^2\varphi - \cos^2\theta \cos^2\varphi)\}$$

$$+ i\{2[K_1 + uK_0] \cos\theta \cos\theta \cos\varphi\} \Big] \ , \qquad (2.9b)$$

$$\mathcal{A}_{\times\omega} = \frac{-1}{(2\pi)^{1/2}} \left(\frac{2b}{v}\right) \Big[\{uK_1 \cos\theta \cos\varphi \sin\varphi\}$$

$$+ i\{(K_1 + uK_0) \sin\theta \sin\varphi\} \Big] \ , \qquad (2.9c)$$

where

$$u = \omega/vb, \ K_0 = K_0(u), \ K_1 = K_1(u) \ , \qquad (2.9d)$$

and where K_0 and K_1 are modified Bessel functions.

The specific flux carried by the waves in the two polarization states is (see p. 1027 of MTW)

$$\mathcal{F}_\nu^+ \left(\frac{\text{ergs}}{\text{cm}^2\,\text{Hz}}\right) = \frac{1}{8} \left(\frac{4m_A m_B}{br}\right)^2 |\omega\mathcal{A}_{+\omega}|^2$$

$$= \frac{1}{16} \left(\frac{4m_A m_B}{br}\right)^2 u^2 \Big[\{K_0 \sin^2\theta + uK_1(\sin^2\theta + \sin^2\varphi - \cos^2\theta \cos^2\varphi)\}^2 \Big] ,$$

$$(2.10a)$$

where $\nu \equiv \omega/2\pi$ is the frequency of the radiation. The total energy radiated per unit frequency is

$$\frac{dE}{d\nu} = \int (\mathcal{F}^{+}_{\nu} + \mathcal{F}^{\times}_{\nu}) r^2 d\Omega$$

$$= \frac{64}{5}\left(\frac{m_A m_B}{b}\right)^2 \left[(1/3 + u^2)u^2 K_0^{\ 2} + 3u^3 K_0 K_1 + (1 + u^2)u^2 K_1^{\ 2}\right]; \quad (2.11)$$

and the total energy radiated is

$$\Delta E = \int \frac{dE}{d\nu} d\nu = \frac{37\pi}{15}\left(\frac{m_A m_B}{b}\right)^2 \frac{v}{b} \, . \quad (2.12)$$

To gain insight into the low frequency domain of the radiation one should not concentrate on $a_{+\omega}$ or $a_{\times\omega}$, they diverge as $\omega \to 0$; rather, one must look at the real physical energy carried by the waves and consider $dE/d\nu$ as $\omega \to 0$. The zero frequency limit of $dE/d\nu$ may be read off from eq. (2.11) since the term in square brackets $\to 1$ as $\omega \to 0$. The result is,

$$\frac{dE}{d\nu} = \frac{64}{5}\left(\frac{m_A m_B}{b}\right)^2 \quad (2.13)$$

Alternatively it may be obtained by noting that for $\omega \to 0$

$$\omega a_{\substack{+\omega \\ \times}} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \dot{a}_{\substack{+ \\ \times}}(T) dT = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \frac{da_{\substack{+ \\ \times}}}{dT} \cdot dT = \frac{1}{(2\pi)^{1/2}} a_{\substack{+ \\ \times}} \Big|_{-\infty}^{+\infty}$$

$$= \frac{1}{(2\pi)^{1/2}} \Delta a_{\substack{+ \\ \times}} \quad (2.14)$$

where $\Delta a_{\substack{+ \\ \times}}$ denotes the change between the early and late limiting values of $a_{\substack{+ \\ \times}}$. Using eq. (2.6c, 2.6d) we can read off these values for $\Delta a_{\substack{+ \\ \times}}$;

$$\Delta a_{+} = 2 \cos\theta \sin\theta \cos\varphi \qquad \Delta a_{\times} = -2 \sin\theta \sin\varphi \quad (2.15)$$

and using (2.11) we get:

$$\frac{dE}{d\nu}\bigg|_{\omega=0} = \int(\mathcal{F}^{+}_{\nu} + \mathcal{F}^{\times}_{\nu})r^{2}d\Omega = 2\left(\frac{m_A m_B}{b}\right)^{2}\int\left[|\omega a_{+}|^{2} + |\omega a_{\times}|^{2}\right]d\Omega$$

$$= \frac{1}{\pi}\left(\frac{m_A m_B}{b}\right)^{2}\int\left[|\Delta a_{+}|^{2} + |\Delta a_{\times}|^{2}\right]d\Omega \qquad (2.16a)$$

which when integrated gives

$$\frac{dE}{d\nu}\bigg|_{\omega=0} = \frac{64}{5}\left(\frac{m_A m_B}{b}\right)^{2} . \qquad (2.16b)$$

As a closing remark in this section we note that either for the non-relativistic or the ultra-relativistic case the $\omega \to 0$ limit of $dE/d\nu$ may be attained by two methods — the tedious method of actually Fourier transforming the specific amplitudes then taking the $\omega \to 0$ limit or the rather trivial approach of looking at the net change in h_{jk}^{TT} before and after the "collision."

The above amplitudes and spectra have been previously derived by Ruffini and Wheeler (1971). Utilizing a quantum mechanical approach, Smarr (1977) obtained the same results for the $\omega = 0$ case. Turner (1977) used a purely Newtonian multipole expansion formalism to calculate the amplitudes and spectra for various bound and unbound orbits as a function of eccentricity.

III. ARBITRARY VELOCITIES

a) Assumptions

Turn now to the case of a stellar encounter with arbitrary relative velocities v: but still insist that the stars be small compared to the impact parameter b (negligible tidal interaction) and that the impact parameter be large enough to guarantee small deflection angle:

$$r_A \ll b,\ r_B \ll b,\ \zeta \equiv (m_A+m_B)/(bv^2) \ll \gamma^{-2} \equiv 1-v^2 . \tag{3.1}$$

In Paper III we used the post-linear wave-generation formalism of Papers I and II to compute the gravitational-wave field for such an encounter to within fractional accuracy

$$(\text{error in wave field})/(\text{wave field}) = O(\zeta\gamma^2) . \tag{3.2}$$

(Our analysis was not able to say whether the error was ζ, $\zeta\gamma$, $\zeta\gamma^2$, $\zeta\gamma^3$, ... The $\zeta\gamma^2$ error was deduced by D'Eath (1977) using a mathematical formalism very different from ours.) Unfortunately, our computation relied on the assumption that the stars are large enough

$$r_A \gg m_A,\ r_B \gg m_B \tag{3.3}$$

that their internal gravity is nonrelativistic. Thus, we have no guarantee that our results are valid if one or both of the objects are black holes or neutron stars. On the other hand, we know that the weak-interval-gravity assumption (3.3) is unnecessary in the low-velocity case $v \ll 1$, where the "quadrupole-moment formalism" is valid (Thorne 1977b), and also at high velocities $\gamma \gg 1$, where the "colliding plane-wave" formalism is valid (D'Eath 1977). In both these limits the waves are independent of the stars' internal structures so long as tidal forces can be neglected ($r_A \ll b$, $r_B \ll b$): black holes give the same radiation as normal stars. There is no apparent physical reason why structure-independence should break down at intermediate velocities. Thus, it seems reasonable to assume that our results are valid independently of the assumption (3.3).

b) Description of the Waves

Recently Thorne (1977a) has proposed that the results of gravitational wave calculations be expressed in terms of a complex scalar field

$$A \equiv A_+ + iA_\times \,, \tag{3.4a}$$

which has the property that at a fixed event in spacetime and in any reference frame the spatial, transverse, traceless part of the metric perturbation is

$$h_{jk}^{TT} = \text{Real}\,(A\bar{m}_j\bar{m}_k) = A_+\, e^+_{jk} + A_\times\, e^\times_{jk} \,. \tag{3.4b}$$

Here

$$\bar{\underset{\sim}{m}} \equiv \underset{\sim}{e}_1 - i\underset{\sim}{e}_2,\ \underset{\sim}{e}^+ \equiv \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 - \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2,\ \underset{\sim}{e}^\times \equiv \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1 \,, \tag{3.4c}$$

and $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ are vectors with the following properties: (i) they are purely spatial in the chosen reference frame; (ii) they are orthonormal and are orthogonal to the propagation 4-vector $\underset{\sim}{k}$ of the waves

$$\underset{\sim}{e}_a \cdot \underset{\sim}{e}_b = \zeta_{ab}, \qquad \underset{\sim}{e}_a \cdot \underset{\sim}{k} = 0 \,: \tag{3.4d}$$

(iii) together with any future-directed 4-vector $\underset{\sim}{e}_0$, the triad $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{k}$ forms a right-handed system

$$\underset{\sim}{\epsilon}\,(\underset{\sim}{e}_0, \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k}) = 0 \tag{3.4e}$$

(here $\underset{\sim}{\varepsilon}$ is the Levi-Civita tensor): (iv) $\underset{\sim}{e}_1$ lies in some chosen fiducial plane (2-space) which passes through the propagation vector $\underset{\sim}{k}$.

For our bremsstrahlung problem we choose our fiducial 2-space to be the one spanned by the propagation vector $\underset{\sim}{k}$ and the difference $\underset{\sim}{u}_B-\underset{\sim}{u}_A$ between the stellar 4-velocities

$$\text{fiducial plane} \qquad \underset{\sim}{k} \wedge (\underset{\sim}{u}_B - \underset{\sim}{u}_A) \; . \tag{3.5}$$

In writing down our formulae for the gravitational-wave field A we shall find it convenient to use three different reference frames: the rest frame of A (denoted $\mathcal{S}$), the rest frame of B (denoted $\bar{\mathcal{S}}$), and the center-of-velocity frame (denoted $\tilde{\mathcal{S}}$).

The rest frame of A is the same as that used in the low-velocity problem (Figure 1). In this frame the propagation vector is $\underset{\sim}{k} = \underset{\sim}{e}_0+\underset{\sim}{n}$, where $\underset{\sim}{n}$ is the unit radial vector pointing toward the observer (eq. 2.3); and $\underset{\sim}{u}_B-\underset{\sim}{u}_A = (\gamma-1)\, \underset{\sim}{e}_0 + v\gamma \underset{\sim}{e}_x$. Consequently the spatial part of the fiducial 2-surface is the $\underset{\sim}{n} \wedge \underset{\sim}{e}_x$ plane. The unit vector $\underset{\sim}{e}_1$ must lie in this plane and must be perpendicular to $\underset{\sim}{n}$; thus, up to sign (which is of no importance), $\underset{\sim}{e}_1$ is the unit vector pointing in the θ direction: $\underset{\sim}{e}_1 = \underset{\sim}{e}_\theta$. Conditions (3.4d,e) and the demand that $\underset{\sim}{e}_2$ be purely spatial then guarantees $\underset{\sim}{e}_2 = \underset{\sim}{e}_\varphi$:

$$\text{in frame } \mathcal{S} \text{ (rest frame of A) } \underset{\sim}{e}_1 = \underset{\sim}{e}_\theta,\; \underset{\sim}{e}_2 = \underset{\sim}{e}_\varphi \; . \tag{3.6}$$

Comparison of equations (3.4b,c) with (2.4a) and (2.5a) reveals that the above choice of fiducial 2-surface (eq. 3.5) leads to polarization base states which are the same as those used in our low-velocity analysis.

The rest frame of B (frame $\bar{S}$) is related to that of A by a Lorentz boost with velocity $+ v\underset{\sim}{e}_x$, and a translation $b\underset{\sim}{e}_y$ designed to place the origin of coordinates on star B:

$$\bar{t} = \gamma(t - vx),\ \bar{x} = \gamma(x-vt),\ \bar{y} = y - b,\ \bar{z} = z\ . \tag{3.7a}$$

$$\bar{r} = \gamma(1 - \alpha v)r,\ \bar{t} - \bar{r} = \frac{t-r+\beta b}{\gamma(1-\alpha v)}\ , \tag{3.7b}$$

$$\alpha \equiv \cos\theta = x/r, \qquad \beta \equiv \sin\theta\cos\varphi = y/r\ . \tag{3.7c}$$

Here $r \equiv (x^2 + y^2 + z^2)^{1/2}$ and $\bar{r} \equiv (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2}$ are distance from source to observer as measured in the two frames; $t-r$ and $\bar{t}-\bar{r}$ are retarded time; and we assume $|t-r| \ll r$, $|\bar{t}-\bar{r}| \ll \bar{r}$ (observer in radiation zone). It is straightforward to verify that in frame $\bar{S}$ the spatial part of the fiducial 2-surface is the $\underset{\sim}{\bar{n}} \wedge \underset{\sim}{e}_{\bar{x}}$ direction (where $\underset{\sim}{\bar{n}}$ is the unit spatial vector pointing toward the observer), and that consequently

$$\text{in frame } \bar{S} \text{ (rest frame of B) } \underset{\sim}{e}_1 = \underset{\sim}{e}_{\bar{\theta}}, \quad \underset{\sim}{e}_2 = \underset{\sim}{e}_{\bar{\varphi}} \tag{3.8}$$

where $(\underset{\sim}{e}_{\bar{\theta}}, \underset{\sim}{e}_{\bar{\varphi}})$ are related to $(\underset{\sim}{e}_{\bar{x}}, \underset{\sim}{e}_{\bar{y}}, \underset{\sim}{e}_{\bar{z}})$ in the same way as $(\underset{\sim}{e}_{\theta}, \underset{\sim}{e}_{\varphi})$ are related to $(\underset{\sim}{e}_x, \underset{\sim}{e}_y, \underset{\sim}{e}_z)$; see equations (2.4b,c) and Figure 1.

The center-of-velocity frame $\tilde{S}$ moves with speed $+ \tilde{v}\,\underset{\sim}{e}_x$ relative to S, and with speed $-\tilde{v}\,\underset{\sim}{e}_{\bar{x}}$ relative to $\bar{S}$, where

$$\tilde{v} = \gamma v/(\gamma+1), \qquad \tilde{\gamma} \equiv (1-\tilde{v}^2)^{-1/2} = [(\gamma+1)/2]^{1/2}\ . \tag{3.9}$$

The spatial origin of $\tilde{S}$ is situated half-way between the trajectories of the two stars. Thus

$$\tilde{t} = \tilde{\gamma}(t-\tilde{v}x), \quad x = \tilde{\gamma}(x-\tilde{v}t), \quad \tilde{y} = y - \frac{1}{2}b, \quad \tilde{z} = z\ ; \tag{3.10a}$$

$$\tilde{r} = \tilde{\gamma}(1-\alpha\tilde{v})r, \qquad \tilde{t}-\tilde{r} = \frac{t-r + 1/2\,\beta b}{\tilde{\gamma}(1-\alpha\tilde{v})}\ ; \tag{3.10b}$$

and similarly

$$\tilde{t} = \tilde{\gamma}(\bar{t}+\tilde{v}\bar{x}), \quad \tilde{x} = \tilde{\gamma}(\bar{x}+\tilde{v}\bar{t}), \quad \tilde{y} = \bar{y} + \frac{1}{2}b, \quad \tilde{z} = \bar{z}\ ; \tag{3.10c}$$

$$\tilde{r} = \tilde{\gamma}(1+\bar{\alpha}\tilde{v}), \quad \tilde{t}-\tilde{r} = \frac{\bar{t}-\bar{r} + 1/2\ \bar{\beta}b}{\tilde{\gamma}(1+\bar{\alpha}\tilde{v})}\ , \tag{3.10d}$$

$$\tilde{\alpha} = \cos\tilde{\theta} = \tilde{x}/\tilde{r}, \qquad \tilde{\beta} = \sin\tilde{\theta}\cos\tilde{\varphi} = \tilde{y}/\tilde{r}\ . \tag{3.10e}$$

In frame $\tilde{\mathcal{S}}$ the basis vectors for the polarization tensors are:

$$\underset{\sim}{e}_1 = \underset{\sim}{e}_{\tilde{\theta}}, \qquad \underset{\sim}{e}_2 = \underset{\sim}{e}_{\tilde{\varphi}} \tag{3.11}$$

Here $(\underset{\sim}{e}_{\tilde{\theta}}, \underset{\sim}{e}_{\tilde{\varphi}})$ are related to $(\underset{\sim}{e}_{\tilde{x}}, \underset{\sim}{e}_{\tilde{y}}, \underset{\sim}{e}_{\tilde{z}})$ in the same way as $(\underset{\sim}{e}_{\theta}, \underset{\sim}{e}_{\varphi})$ are related to $(\underset{\sim}{e}_x, \underset{\sim}{e}_y, \underset{\sim}{e}_z)$; see equations (2.4b,c) and Figure 1.

In Paper III we derived the full wave-zone gravitational field (the trace-reversed metric perturbation) $\bar{h}_{\mu\nu}$, in frame-independent language (equation III-8.1). From that result one can compute the scalar gravitational-wave field A by projecting out the spatial, transverse, traceless part of $\bar{h}_{\mu\nu}$ in any frame one wishes, and then equating it to expression (3.4b).

The gravitational waves of equation (III-8.1) have symmetry properties similar to those which we encountered in the low-velocity limit: The radiation field is totally insensitive to the relative masses of the stars A and B; it depends only on the product of their masses. Consequently, in the center-of-velocity frame, where the encounter would have forward-backward symmetry if the masses were identical, the radiation always has forward-backward symmetry:

$$A_+(\tilde{t},\tilde{r},\tilde{\theta},\tilde{\varphi}) = A_+(\tilde{t},\tilde{r},\pi - \tilde{\theta},\tilde{\varphi} + \pi) ,$$

$$A_\times(\tilde{t},\tilde{r},\tilde{\theta},\tilde{\varphi}) = - A_\times(\tilde{t},\tilde{r},\pi - \tilde{\theta},\tilde{\varphi} + \pi) . \qquad (3.12a)$$

This is the arbitrary-velocity analogue of equation (2.6); in the language of Paper III it is the symmetry property (III-3.17). Of course, the waves are also symmetric under reflection through the plane of the encounter

$$A_+(\tilde{t},\tilde{r},\tilde{\theta},\tilde{\varphi}) = A_+(\tilde{t},\tilde{r},\tilde{\theta}, - \tilde{\varphi}) ,$$

$$A_\times(\tilde{t},\tilde{r},\tilde{\theta},\tilde{\varphi}) = - A_\times(\tilde{t},\tilde{r},\tilde{\theta}, - \tilde{\varphi}) . \qquad (3.12b)$$

c) Analytic Expressions for the Amplitude

Previous computations of bremsstrahlung (Peters 1970; Matzner and Nutku 1974) have been carried out in the rest frame of one of the stars. To facilitate comparison with those results, we shall express our amplitudes in terms of the coordinates (t,r,θ,φ) of the rest frame of star A (Figure 1). In our expressions we shall use the direction cosines of the observer, as seen in the rest frame of A

$$\alpha = \cos\theta, \qquad \beta = \sin\theta\cos\varphi, \qquad \delta = \sin\theta\sin\varphi ; \qquad (3.13a)$$

and also the analogous direction cosines, as seen in the rest frame of B

$$\bar{\alpha} = \cos\bar{\theta} = \frac{\alpha - v}{1-\alpha v}, \qquad \bar{\beta} = \sin\bar{\theta}\cos\bar{\varphi} = \frac{\beta}{\gamma(1-\alpha v)} ,$$

$$\bar{\delta} = \sin\bar{\theta}\sin\bar{\varphi} = \frac{\delta}{\gamma(1-\alpha v)} . \qquad (3.13b)$$

Whereas in the low-velocity case there is only one characteristic timescale associated with the radiation, $\Delta t = b/v$ = (time for star B to move a distance equal to the impact parameter), in the general case there are two timescales. The first is the analog of $\Delta t = b/v$: Star B, as it passes star A, exerts a gravitational acceleration on A. Because of the "relativistic pancaking" of B's Coulomb gravitational field, A's acceleration has duration $\Delta t \sim b/(v\gamma)$ rather than $\Delta t \sim b/v$. Because the moment of closest approach is $t = 0$, this will produce radiation which is naturally described by the retarded-time parameter

$$T_A \equiv v\gamma(t-r)/b \ . \tag{3.14a}$$

The radiation should have a bump at time $T_A = 0$, of duration $\Delta T_A \sim 1$.

The second characteristic timescale is associated with the acceleration of star B. As seen in B's rest frame $\bar{S}$ its own acceleration has duration $\Delta\bar{t} = b/(v\gamma)$. The radiation produced by this acceleration is doppler shifted, as seen in the observer's rest frame (A's frame): there it has duration $\Delta t = \gamma(1-\alpha v)\ \Delta\bar{t}$. The strongest radiation is emitted at the moment of closest approach, $t = 0$, when star B is at $(x = 0,\ y = b,\ z = 0)$. The observer sees this radiation emanating from B at retarded time $t - r = -\beta b$. Thus, it is reasonable to introduce the time parameter

$$T_B \equiv \frac{t-r+\beta b}{\Delta t} = \frac{t-r+\beta b}{\gamma(1-\alpha v)\Delta\bar{t}} = \frac{v(t-r+\beta b)}{(1-\alpha v)b} = \frac{T_A+\beta v\gamma}{\gamma(1-\alpha v)} \ , \tag{3.14b}$$

and to expect the radiation to contain some sort of bump at time $T_B = 0$ with duration $\Delta T_B \sim 1$.

In writing down the radiation field it is convenient to introduce the following functions of our time parameters:

$$\ell_A \equiv (1 + T_A{}^2)^{1/2}, \quad \ell_B \equiv (1 + T_B{}^2)^{1/2} \ , \tag{3.15a}$$

$$S^2 \equiv (\alpha - \beta\, T_B)^2 + \delta^2\, \ell_B{}^2 = (\bar{\alpha} - \bar{\beta}\, T_A)^2 + \bar{\delta}^2 \ell_A{}^2 \,. \qquad (3.15b)$$

The two expressions for S^2 are equivalent by virtue of equations (3.13b) and (3.14b). The functions $1/\ell_A{}^3$ and $1/\ell_B{}^3$ describe the rise and fall of the gravitational accelerations that the stars exert on each other (eq. III-2.27), while S^2 is the squared length of the 4-vector which reaches from star A at the moment it crosses the past light cone of the observer to star B at its passage through the light cone (eq. III-3.12). Note that T_A, T_B, ℓ_A, ℓ_B, and S^2 as defined here are renormalized by a factor b from the same quantities in Paper III:

$$(T_A, T_B, \ell_A, \ell_B, S^2)_{\rm HERE} = (T_A/b, T_B/b, \ell_A/b, \ell_B/b, S^2/b^2)_{\rm III} \,. \qquad (3.16)$$

In terms of the above parameters the scalar gravitational-wave amplitudes are given by

$$A_+ = (4 m_A m_B/br)\mathcal{a}_+, \quad A_\times = (4 m_A m_B/br)\mathcal{a}_\times \,, \qquad (3.17a)$$

$$\mathcal{a}_+ = G_1 \sin^2\theta + G_2(\sin^2\varphi - \cos^2\theta \cos^2\varphi) + G_3 \cos\theta \sin\theta \cos\varphi$$
$$+ G_4 \cos\theta \sin\theta \sin\varphi + G_5(1 + \cos^2\theta) \cos\varphi \sin\varphi \,, \qquad (3.17b)$$

$$\mathcal{a}_\times = G_2 \cos\theta\, (2 \cos\varphi \sin\varphi) - G_3 \sin\theta \sin\varphi + G_4 \sin\theta \cos\varphi$$
$$+ G_5 \cos\theta\, (\cos^2\varphi - \sin^2\varphi) \,. \qquad (3.17c)$$

Here the functions $G_1, \ldots, G_5$ are

$$G_1 = \frac{\gamma[2(1-v^2) - \alpha v(1+v^2)]}{2(1-\alpha v)^2} \frac{1}{\ell_B} - \frac{\beta\gamma v(1+v^2)}{2(1-\alpha v)^2}\left(\frac{T_B}{\ell_B} + 1\right)$$
$$+ \left\{-\gamma^2\left[\frac{1-3v^2}{2v} + \frac{v^3}{1-\alpha v}\right] + \frac{\gamma(1+v^2)}{2v}\ell_A \ell_B\right\} \frac{1}{S^2}\left[\frac{\bar{\alpha}-\bar{\beta}T_A}{\ell_A} - \frac{\alpha-\beta T_B}{\gamma(1-\alpha v)\ell_B}\right] \,,$$
$$(3.18a)$$

$$G_2 = \frac{\gamma(1+v^2)}{2v}(T_A T_B - \ell_A \ell_B)\frac{1}{S^2}\left[\frac{\bar{\alpha}-\bar{\beta}T_A}{\ell_A} - \frac{\alpha-\beta T_B}{\gamma(1-\alpha v)\ell_B}\right], \qquad (3.18b)$$

$$G_3 = \frac{\gamma(1+v^2)}{1-\alpha v}\left(\frac{T_B}{\ell_B} + 1\right)$$
$$- \frac{\gamma^2}{2v}\left[2(1-v^2)T_B - v(1+v^2)(\alpha T_B + \beta)\right]\frac{1}{S^2}\left[\frac{\bar{\alpha}-\bar{\beta}T_A}{\ell_A} - \frac{\alpha-\beta T_B}{\gamma(1-\alpha v)\ell_B}\right], \qquad (3.18c)$$

$$G_4 = \frac{\gamma\delta}{2v(1-\alpha v)S^2}\left\{\frac{\gamma}{\ell_A}\left[(2-3v^2-v^4)\ell_A{}^2 + v(1-3v^2)T_A(\bar{\alpha}T_A+\bar{\beta})\right]\right.$$
$$\left. - \frac{1}{\ell_B}\left[2(1-v^2)\ell_B{}^2 - v(1+v^2)T_B(\alpha T_B+\beta)\right]\right\}, \qquad (3.18d)$$

$$G_5 = -\frac{\gamma(1+v^2)\delta}{2(1-\alpha v)S^2}\left\{\frac{\bar{\alpha}T_A+\bar{\beta}}{\ell_A} - \frac{2\ell_A T_B}{v\gamma} + \frac{\alpha T_B+\beta}{\ell_B} + \frac{2\ell_B T_A}{v\gamma}\right\}. \qquad (3.18e)$$

In expressions (3.17) we have introduced the same renormalized amplitudes $\mathcal{a}_+$ and $\mathcal{a}_\times$ as for the low-velocity case (eqs. 2.5a,b). (Note that because of the factor r in (3.17a), $\mathcal{a}_+$ and $\mathcal{a}_\times$ are not scalar fields.) Equations (3.17a,b,c) for $\mathcal{a}_+$ and $\mathcal{a}_\times$ are the same as equations (III-8.13b,c) and (III-8.2), with b replaced by unity because of our renormalization of T_A, T_B, ℓ_A, ℓ_B, S^2, and with some algebraic modifications that utilize equations (III-3.7) and (III-3.11b).

It is straightforward, using equations (III-8.5) with b = 1, to take the low-velocity limits of $\mathcal{a}_+$ and $\mathcal{a}_\times$ and show that they agree with equations (2.6c,d).

In expressions (3.18) the dependence on the function S^2 is somewhat delicate. For observation events at which one star is in the "shadow" of the other, S^2 vanishes. However, the quantities which multiply $1/S^2$

in $G_1,\ldots,G_5$ also vanish in the shadow — and they vanish in such a way that $G_1,\ldots,G_5$ change smoothly and not sharply as the observer moves into and then out of the shadow. For further discussion see § VIc of Paper III.

The forward-backward symmetry (3.12a) does not show up very clearly in expressions (3.17), (3.18) for A_+ and $A_\times$ because those expressions are written in the coordinates of A's rest frame. By transforming (3.12a) from center-of-velocity coordinates to A's coordinates (eqs. 3.10a and 3.9) we find that the amplitudes must be the same at the events $(t_1,\ x_1,\ y_1,\ z_1)$ and $(t_2,\ x_2,\ y_2,\ z_2)$ where

$$t_2 = \gamma(t_1 - vx_1), \quad x_2 = -\gamma(x_1 - vt_1), \quad y_2 = -y_1 + b, \quad z_2 = -z_1 \ . \tag{3.19}$$

In terms of spherical polar coordinates r, θ, φ and the retarded time parameter T_A (eq. 3.14a) this relationship says

$$A_+(T_{A2},\ r_2,\ \theta_2,\ \varphi_2) = A_+(T_{A1},\ r_1,\ \theta_1,\ \varphi_1) \ ,$$

$$A_\times(T_{A2},\ r_2,\ \theta_2,\ \varphi_2) = -A_\times(T_{A1},\ r_1,\ \theta_1,\ \varphi_1) \ , \tag{3.20a}$$

where

$$T_{A2} = \frac{T_{A1} + v\gamma\beta_1}{\gamma(1-\alpha_1 v)} \ , \quad r_2 = \gamma(1-\alpha_1 v) r_1, \ \cos\theta_2 = -\left(\frac{\cos\theta_1 - v}{1 - v\cos\theta_1}\right), \ \varphi_2 = \pi - \varphi \tag{3.20b}$$

Note that the direction cosines and A-B time parameters associated with these "equivalent" events are related by

$$\alpha_2 = -\bar{\alpha}_1, \quad \beta_2 = -\bar{\beta}_1, \quad \delta_2 = -\bar{\delta}_1, \quad T_{A2} = T_{B1} , \tag{3.20c}$$

$$\bar{\alpha}_2 = -\alpha_1, \quad \bar{\beta}_2 = -\beta_1, \quad \bar{\delta}_2 = -\delta_1, \quad T_{B2} = T_{A1} : \tag{3.20d}$$

cf. equation (3.13b), (3.14b), and (3.20b).

Before displaying graphs of the wave amplitudes and discussing their spectral features (§ V below), we shall derive high-velocity ($\gamma \gg 1$) formulae for them.

IV. HIGH-VELOCITY LIMIT

In the high-velocity limit, $\gamma \gg 1$, it is useful to split the radiation zone into three overlapping regions: the "forward region," the "intermediate region," and the "backward region." In terms of angles measured in the center-of-velocity frame $\tilde{S}$, these regions are

$$\begin{aligned} \text{forward} &: \tilde{\theta} \ll \pi/2 , \\ \text{intermediate} &: \tilde{\gamma}^{-1} \ll \tilde{\theta} ; \quad \tilde{\gamma}^{-1} \ll \pi - \tilde{\theta} , \\ \text{backward} &: \pi - \tilde{\theta} \ll \pi/2 . \end{aligned} \tag{4.1}$$

When one transforms to the rest frame of star A using the relations

$$\cos\theta = \frac{\cos\tilde{\theta} + \tilde{v}}{1 + \tilde{v}\cos\tilde{\theta}} , \quad \tilde{v} = \frac{\gamma v}{\gamma + 1} , \quad \tilde{\gamma} = \left(\frac{\gamma+1}{2}\right)^{1/2} , \quad \gamma \gg 1 , \tag{4.2}$$

one finds the following locations for these three regions:

$$\begin{aligned} \text{forward} &: \theta \ll \gamma^{-1/2} , \\ \text{intermediate} &: \gamma^{-1} \ll \theta \ll \pi/2 , \\ \text{backward} &: \theta \gg \gamma^{-1/2} . \end{aligned} \tag{4.3}$$

We shall now derive limiting forms for the gravitational-wave amplitudes (3.17), (3.18) in the forward, intermediate, and backward regions.

a) Forward Region

In the forward region we introduce D'Eath's (1977) angular variable

$$\psi \equiv \theta\gamma = \tilde{\theta}\gamma^{1/2} = \tilde{\theta}\tilde{\gamma}\sqrt{2} \tag{4.4a}$$

which has values of order unity at the angles of greatest interest, and which satisfies

$$\psi \ll \gamma^{1/2}, \quad \psi \ll \tilde{\gamma} \tag{4.4b}$$

throughout the forward region. We also introduce the notation

$$c \equiv \cos\varphi \equiv \cos\tilde{\varphi}, \quad s \equiv \sin\varphi \equiv \sin\tilde{\varphi} \; . \tag{4.4c}$$

The two time parameters T_A and T_B are related to each other and to retarded time (eqs. 3.14b and 3.13a) by

$$T_A = \frac{\gamma(t-r)}{b}, \qquad T_B = \frac{2\gamma(T_A + \psi\cos\varphi)}{1+\psi^2} \tag{4.5}$$

We expect the waves to show structure on the timescales $\Delta T_A \sim 1$ and $\Delta T_B \sim 1$, which correspond to times t as measured by the observer in A's rest frame

$$\Delta t \sim b/\gamma \text{ for } \Delta T_A \sim 1 \; ; \quad \Delta t \sim \frac{1}{2}(1+\psi^2)/\gamma^2 \text{ for } \Delta T_B \sim 1 \; . \tag{4.6}$$

Careful scrutiny of equations (3.17), (3.18) reveals that, when viewed on the slow timescale $\Delta T_A \sim 1$, A_+ and $A_\times$ have discontinuous time derivatives at the time $T_A = -\psi\cos\varphi$ (i.e. $T_B = 0$); but the values of

A_+ and $A_\times$ are continuous there. When viewed on the faster timescale $\Delta T_B \sim 1$ near $T_B = 0$, the time derivatives vary smoothly but there is no significant change in the amplitudes themselves. This behaviour permits us to focus attention exclusively on the slow timescale $\Delta T_A \sim 1$, but suggests that we shift the origin of our time parameter T_A to the moment $T_A = -\psi \cos\varphi$ of discontinuity:

$$\hat{T}_A \equiv T_A + \psi \cos\varphi = \frac{\gamma(t-r)}{b} + \psi \cos\varphi = \frac{\tilde{\gamma}(\tilde{t}-\tilde{r})}{b} + \frac{1}{2}\psi \cos\varphi \, . \qquad (4.7)$$

To derive the high-velocity limit of expressions (3.13)-(3.18) we now replace θ throughout those expressions by ψ/γ, we replace $\cos\varphi$ by c and $\sin\varphi$ by s, we replace T_A by $\hat{T}_A - \psi\cos\varphi$, and we expand in powers of $\varepsilon \equiv 1/\gamma$, keeping careful track of the magnitudes of the errors at each step. The results for various parameters appearing in A_+ and $A_\times$ are

$$\gamma \equiv 1/\varepsilon, \qquad v = 1 - \frac{1}{2}\varepsilon^2, \qquad \alpha = 1 - \frac{1}{2}\varepsilon^2\psi^2 \, , \qquad (4.8a)$$

$$\gamma(1-\alpha v) = \frac{1}{2}\varepsilon(1+\varphi^2), \qquad \beta = \varepsilon\psi c, \qquad \delta = \varepsilon\psi s \, , \qquad (4.8b)$$

$$\bar{\alpha} = \frac{1-\psi^2}{1+\psi^2}, \qquad \bar{\beta} = \frac{2\psi c}{1+\psi^2}, \qquad \bar{\delta} = \frac{2\psi s}{1+\psi^2} \, , \qquad (4.8c)$$

$$T_B = \frac{2\hat{T}_A}{\varepsilon(1+\psi^2)}, \qquad \ell_B = \frac{2|\hat{T}_A|}{\varepsilon(1+\psi^2)} \; ; \qquad (4.8d)$$

and also

$$\ell_A = \left[1 + (\hat{T}_A - \psi c)^2\right]^{1/2}, \qquad s^2 = 1 - \frac{4\psi c}{1+\psi^2}\hat{T}_A + \frac{4\psi^2}{(1+\psi^2)^2}\hat{T}_A^{\,2} \, . \qquad (4.9a)$$

In all these expressions the fractional errors are of order ε^2 - except v and α (eq. 4.8a), where they are of order ε^4. The result for A_+ and $A_\times$ is

$$A_+ + iA_\times = \frac{4m_A m_B}{br}(\alpha_+ + i\alpha_\times) = \frac{4m_A m_B}{\tilde{b}\tilde{r}\gamma}(\alpha_+ + i\alpha_\times) , \qquad (4.9b)$$

$$\alpha_+ = \frac{4\gamma^2}{(1+\psi^2)^2 s^2}\left\{\psi c s^2 + \left(\frac{1+\psi^4}{1+\psi^2}\right)|\hat{T}_A|\left(2c^2 - 1 - \frac{2\psi c}{1+\psi^2}\hat{T}_A\right)\right.$$

$$+ \frac{1}{\ell_A}\left[\frac{1}{2}\langle(1-2c^2) - (1+2c^2)\psi^2 - 2c^2\psi^4\rangle + \psi c\langle 1+2c^2+2(1+c^2)\psi^2\rangle\hat{T}_A\right.$$

$$\left.\left. + \langle(1-2c^2) - (1+4c^2)\psi^2\rangle\hat{T}_A^{\,2} + 2c\psi\hat{T}_A^{\,3}\right]\right\} , \qquad (4.9c)$$

$$\alpha_+ = \frac{4\gamma^2 s}{(1+\psi^2)^2 s^2}\left\{ - \psi(1+\psi^2)s^2 + 2(1-\psi^2)|\hat{T}_A|\left(-c + \frac{\psi}{1+\psi^2}\hat{T}_A\right)\right.$$

$$\left. + \frac{1}{\ell_A}\left[c(1+\psi^2) - 2\langle(1+c^2) + c^2\psi^2\rangle\psi\hat{T}_A + 2c(1+2\psi^2)\hat{T}_A^{\,2} - 2\psi\hat{T}_A^{\,3}\right]\right\}$$

$$(4.9d)$$

These formulae for A_+ and $A_\times$ have fractional errors of order maximum $(\gamma^{-1}, \psi\gamma^{-1/2})$. Notice the absolute-value signs on $\hat{T}_A$: they produce the discontinuities in the time derivatives of the amplitudes - discontinuities which would actually be smoothed out if one looked on the shorter timescale $\Delta\hat{T}_A \sim (1+\psi^2)/\gamma$ $[\Delta T_B \sim 1]$.

Equations (4.4), (4.7), and (4.9) are our final, forward-direction, high-velocity formulae for the gravitational waves in terms of either center-of-velocity coordinates $(\tilde{t},\tilde{r},\tilde{\theta},\tilde{\varphi})$ or rest-frame-of-A coordinates (t,r,θ,φ). We shall discuss these formulae in § V below.

b) Backward Region

Formulae valid for the backward region can be obtained from the above forward-region formulae by invoking the symmetry relations (3.12) - or, equivalently, (3.20) and (3.12b):

$$A_+ + iA_\times = \left(\frac{1+\psi^2}{2\gamma}\right)\frac{4m_A m_B}{br}(a_+ + ia_\times) = \frac{4m_A m_B}{b\tilde{r}\,\tilde{\gamma}}(a_+ + ia_\times), \qquad (4.10a)$$

$$a_+ = [\text{expression } 4.9c], \qquad a_\times = [\text{expression } 4.9d]\ , \qquad (4.10b)$$

where the quantities appearing in (4.9c,d) are given by

$$\psi = \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/2} = (\pi - \tilde{\theta})\tilde{\gamma}\sqrt{2}, \quad \tilde{\gamma} = (\gamma/2)^{1/2}\ , \qquad (4.10c)$$

$$c = -\cos\varphi = -\cos\tilde{\varphi}, \qquad s = +\sin\varphi = +\sin\tilde{\varphi}\ , \qquad (4.10d)$$

$$\hat{T}_A = \left(\frac{1+\psi^2}{2}\right)\left(\frac{t-r}{b}\right) = \frac{\tilde{\gamma}(\tilde{t}-\tilde{r})}{b} - \frac{1}{2}\psi c\ , \qquad (4.10e)$$

$$(\ell_A \text{ and } s^2) = (\text{expressions } 4.9a)\ . \qquad (4.10f)$$

c) Intermediate Region

In the intermediate region, $\gamma^{-1} \ll \theta \ll \pi/2$ ($\tilde{\gamma}^{-1} \ll \tilde{\theta}$, $\tilde{\gamma}^{-1} \ll \pi - \tilde{\theta}$), it is convenient to introduce the new angular variable

$$\chi = \theta\,(\gamma/2)^{1/2} = \theta\tilde{\gamma} = \left(\frac{1-\cos\tilde{\theta}}{1+\cos\tilde{\theta}}\right)^{1/2}, \qquad (4.11a)$$

which has values of order unity at the angles of greatest interest, and which satisfies

$$\gamma^{-1/2} \ll \chi \ll \gamma^{1/2}, \qquad \tilde{\gamma}^{-1} \ll \chi \ll \tilde{\gamma} \qquad (4.11b)$$

throughout the intermediate region. We also introduce the notation

$$c = \cos\varphi = \cos\tilde{\varphi}, \qquad s = \sin\varphi = \sin\tilde{\varphi} . \tag{4.11c}$$

As we have seen, in the forward (and also in the backward) direction, A_+ and $A_\times$ vary on the slower of the two timescales T_A and T_B. In the forward direction the slower timescale is $T_A = \hat{T}_A$ + constant; in the backward direction it is $T_B = \hat{T}_A$ + constant. In the intermediate region it turns out that there is a continuous transition from variation on timescale T_A to variation on timescale T_B. To make this transition apparent it is convenient to introduce a timescale defined by

$$\begin{aligned} \ddagger &\equiv \frac{1}{2}\left(\frac{1}{\chi} + \chi\right)\left(\frac{\tilde{t}-\tilde{r}}{2b}\right) = \left(\frac{\gamma}{2}\right)^{1/2}\left(\frac{t-r}{b\chi}\right) + \frac{c}{2} \\ &= \left(\frac{1}{2\gamma}\right)^{1/2}\frac{T_A}{\chi} + \frac{c}{2} = \left(\frac{1}{2\gamma}\right)^{1/2}\chi T_B - \frac{c}{2} . \end{aligned} \tag{4.12}$$

Here the relationships between the various timescales are derived by inserting equations (4.11) into (3.10b), (3.7c), and (3.14a,b), and by ignoring fractional corrections of order $(2/\gamma)^{1/2}$. Notice that at $\chi \to (2\gamma)^{-1/2}$ (the forward edge of the intermediate region), $\ddagger \to T_A + \frac{1}{2}c$; at $\chi \to (2\gamma)^{-1/2}$ (the backward edge of the intermediate region), $\ddagger \to T_B - \frac{1}{2}c$; and at $\chi = 1$ ($\tilde{\theta} = \pi/2$), $\ddagger = (t-r)/b$.

To derive high-velocity, intermediate-region formulae for A_+ and $A_\times$ we now replace θ throughout expressions (3.13)-(3.16) by $(2/\gamma)^{1/2}\chi$, we replace $\cos\varphi$ by c and $\sin\varphi$ by s; we replace T_A and T_B by $(2\gamma)^{1/2}\chi\,(\ddagger \mp c/2)$ (eqs. 4.12); and we expand in powers of $\eta \equiv (2/\gamma)^{1/2}$, keeping careful track of the magnitudes of the errors at each step. The results for various parameters appearing in A_+ and $A_\times$ are

$$\gamma = 2/\eta^2, \qquad v = 1 - \tfrac{1}{8}\eta^4, \qquad \alpha = 1 - \tfrac{1}{2}\eta^2\chi^2 , \tag{4.13a}$$

$$\gamma(1-\alpha v) = \chi^2, \qquad \beta = \eta\chi c, \qquad \delta = \eta\chi s , \tag{4.13b}$$

$$\bar{\alpha} = -(1-\tfrac{1}{2}\eta^2\chi^{-2}), \qquad \bar{\beta} = \eta\chi^{-1}c, \qquad \bar{\delta} = \eta\chi^{-1}s , \tag{4.13c}$$

$$T_A = 2\eta^{-1}\chi(\ddagger - \tfrac{1}{2}c), \qquad \ell_A = 2\eta^{-1}\chi \, |\ddagger - \tfrac{1}{2}c| , \tag{4.13d}$$

$$T_B = 2\eta^{-1}\chi^{-1}(\ddagger + \tfrac{1}{2}c), \qquad \ell_B = 2\eta^{-1}\chi^{-1} \, |\ddagger - \tfrac{1}{2}c| , \tag{4.13e}$$

$$\dot{s}^2 = s^2 + 4\ddagger^2 . \tag{4.13f}$$

In all these expressions the fractional errors are of order η^2 or $(\eta\chi)^2$ or $(\eta/\chi)^2$ except in v (fractional error of η^8), α (fractional error $\eta^4\chi^4$), and $\bar{\alpha}$ (fractional error $\eta^4\chi^{-4}$). The final results for A_+ and $A_\times$ are

$$A_+ + iA_\times = \frac{4m_A m_B}{br}(a_+ + ia_\times) = \left(\frac{1+\chi^2}{2\tilde{\gamma}}\right)\frac{4m_A m_B}{b\tilde{r}}(a_+ + ia_\times) , \tag{4.14a}$$

$$a_+ = -\frac{(2\gamma)^{3/2}}{\chi} \times \begin{cases} \dfrac{|\ddagger|}{s^2+4\ddagger^2} & \text{if } |\ddagger| > \tfrac{1}{2}|c| \\ \tfrac{1}{2}|c| & \text{if } |\ddagger| < \tfrac{1}{2}|c| , \end{cases} \tag{4.14b}$$

$$a_\times = -\frac{(2\gamma)^{3/2}s}{\chi} \times \begin{cases} (s^2+4\ddagger^2)^{-1} & \text{if } \ddagger < -\tfrac{1}{2}|c| \\ 1 & \text{if } -\tfrac{1}{2}|c| < \ddagger < \tfrac{1}{2}|c| \\ 2-(s^2+4\ddagger^2)^{-1} & \text{if } \tfrac{1}{2}|c| < \ddagger . \end{cases} \tag{4.14c}$$

As in the forward and backward directions, so also here, $\mathcal{a}_+$ and $\mathcal{a}_\times$ are continuous functions of time, but their time derivatives are discontinuous. The discontinuity at $\ddagger = -\frac{1}{2}c$ is caused by the passage of T_B through zero (eqs. 4.13d,e). If one were to look on the rapidly varying timescales T_A and T_B rather than the slower timescale T_A, one would see a smooth change of derivative at $\ddagger = \pm\frac{1}{2}c$.

Formulae (4.14) are embarrassingly simple considering the tortured route by which we have arrived at them. There must be some easy way to derive them; but we have not yet tried to find it.

V. DISCUSSION OF THE WAVE FORMS

The pioneering, and most definitive previous work on this problem, was that of Peters (1970). He studied the same type of encounter as we do, but with the added restriction that one star (say B) is a test particle with infinitesimal mass, and the other star is massive:

$$m_B << m_A \ . \tag{5.1}$$

His method (first-order perturbations of the Schwarzschild field of star A) was not amenable to the comparable-mass case. However, our analysis has shown that the radiation is independent of the mass ratio m_B/m_A.

By comparing Peters' equations (2.23) and (2.24) with our (3.4), (3.6), (3.14a), and (3.15a), one can derive an expression for the renormalized amplitudes $\mathcal{a}_+$ and $\mathcal{a}_\times$ in terms of Peters' metric functions A,B,C:

$$\frac{d\alpha_+}{dT_A} = \frac{1}{8v\gamma}\left[A \sin^2\theta + B(\cos^2\theta \cos^2\varphi - \sin^2\varphi) - 2C \cos\theta \sin\theta \cos\varphi\right] , \tag{5.2a}$$

$$\frac{d\alpha_\times}{dT_A} = \frac{1}{4v\gamma}\left[- B \cos\theta \cos\varphi \sin\varphi + C \sin\theta \sin\varphi\right] . \tag{5.2b}$$

Peters gives horrendously complicated expressions for his functions A,B,C in his Appendix A. The fundamental quantities which go into those expressions, rewritten in terms of our variables, are

$$\delta_{PETERS} = \beta_{US} = \sin\theta \cos\varphi, \qquad \epsilon_{PETERS} = \alpha_{US} = \cos\theta , \tag{5.2c}$$

$$\beta_{PETERS} = v_{US}, \qquad \gamma_{PETERS} = \gamma_{US}, \qquad \tau_{PETERS} = (T_A/v\gamma)_{US} . \tag{5.2d}$$

We did not have the fortitude to check analytically Peters' results against ours; so we checked them numerically instead. When we failed to get agreement, we asked Peters for help; he rechecked his formulae against his original 1969 computations and found a misprint in the paper; and after correction of the misprint, his formulae (5.2) and ours [analytic time derivatives of eqs. (5.15) and (5.16)] gave identical numerical results. Peters' misprint was the omission of a factor of $(1+\beta^2 y_2)$ in the denominator of the W_2 term in his expression for A.

Peters never discussed or graphed his wave forms, so we shall give a thorough discussion here:

Figure 3 shows the wave form $A_+(t)$ in the forward and backward direction ($\theta = 0$ and π, $\varphi = 0$) for various velocities ranging from $v = 0.0001$ to $v = 0.9999$. ($A_\times$ vanishes in the plane of the encounter.) Plotted vertically is

$$\frac{br\,A_+}{4m_A m_B \gamma^2} = \frac{\mathcal{A}_+}{\gamma^2} \text{ for } \theta = 0, \qquad \frac{br\,A_+}{4m_A m_B \gamma^3(1-v)} = \left(\frac{1+v}{1-v}\right)^{1/2} \frac{\mathcal{A}_+}{\gamma^2} \text{ for } \theta = \pi; \tag{5.3a}$$

cf. equations (3.18b). Note that in the rest frame of A, which we are using, at a fixed distance r from the source the backward amplitude is smaller than the forward amplitude by the factor $[(1-v)/(1+v)]^{1/2}$. Plotted horizontally is

$$T_A = \frac{v\gamma(t-r)}{b} \text{ for } \theta = 0, \qquad T_B = \frac{v(t-r)}{(1+v)b} \text{ for } \theta = \pi \tag{5.3b}$$

cf. equations (3.14a,b) and (3.20c,d). Note that the wave form in the backward direction changes more slowly, by the doppler factor $(1+v)\gamma = [(1+v)/(1-v)]^{1/2}$, than the wave form in the forward direction. The wave forms of Figure 3 were computed from equations (3.17), (3.18).

Figure 3 shows clearly the transition from the low-velocity regime [uppermost curve; eq. (2.5)] to the high-velocity regime [lowermost curve; eq. (4.9)]. Notice how, as v increases, the slope of the wave form becomes discontinuous at $T_A = T_B = 0$. Notice, moreover, that the discontinuity is smoothed out when one examines it on the "fast" timescale ($\Delta T_B \simeq 2\gamma T_A \sim 1$ in forward direction; $\Delta T_A \simeq 2\gamma \Delta T_B \sim 1$ in backward direction). These phenomena were discussed analytically in § IVa.

Figure 4 shows the wave forms $A_+(t)$ and $A_\times(t)$ in the directions $\tilde{\theta} = \pi/2$, $\tilde{\varphi} = 0$, $\pi/2$ (directions which are transverse in the center-of-velocity frame). Plotted vertically is

$$\frac{br A_{+ \text{or} \times}}{4m_A m_B \gamma^{3/2}} = \frac{b\tilde{r}\ \tilde{\gamma}\ A_{+\text{or}\times}}{4m_A m_B (2\tilde{\gamma}^2-1)^{3/2}} = \frac{\mathcal{a}_{+\text{or}\times}}{\gamma^{3/2}} ; \tag{5.4a}$$

cf. equations (3.9) and (3.10b). Plotted horizontally is

$$\ddagger \equiv \left(\frac{1}{2\gamma}\right)^{1/2} T_A + \frac{1}{2} = \left(\frac{1}{2\gamma}\right)^{1/2} T_B - \frac{1}{2}$$

$$= \left(\frac{\gamma}{2}\right)^{1/2} \frac{v(t-r)}{b} + \frac{1}{2} = \left(\frac{\gamma}{\gamma+1}\right)^{1/2} \frac{v(\tilde{t}-\tilde{r})}{b} . \tag{5.4b}$$

Figure 4 shows the transition from the low-velocity regime [uppermost curve of each set; eq. (2.6)] to the high-velocity regime [lowermost curve; eq. (4.14)]. Notice, again, the development of the discontinuities in slope at high velocities, as discussed in § IVc.

Figure 5 shows the wave form $A_+(t)$ at very high velocities, $\gamma \gg 1$, in the extreme forward region $\theta \lesssim \gamma^{-1}$ (or, equivalently, in the extreme backward direction $\pi - \theta \lesssim \pi/2$) and in the plane of the encounter, $\varphi = 0$. The curves are labeled by values of the angular parameter

$$\psi = \theta\gamma = \tilde{\theta}\tilde{\gamma}/2 \text{ in forward directions}$$

$$= \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/2} = (\pi - \tilde{\theta})\tilde{\gamma}/2 \text{ in backward directions} \tag{5.5a}$$

[cf. eqs. (4.4a) and (4.10c)]. Plotted vertically is

$$\frac{br A_+}{4m_A m_B \gamma^2} = \frac{b\tilde{r} A_+}{16 m_A m_B \tilde{\gamma}^3} = \frac{\mathcal{a}_+}{\gamma^2} \text{ in forward directions,} \tag{5.5b}$$

$$\frac{2}{1+\psi^2}\, \frac{br A_+}{4m_A m_B \gamma} = \frac{b\tilde{r} A_+}{16 m_A m_B \tilde{\gamma}^3} = \frac{\mathcal{a}_+}{\gamma^2} \text{ in backward directions} \tag{5.5c}$$

[cf. eqs. (4.9b) and (4.10a)]. Plotted horizontally is

$$\hat{T}_A + \psi = T_A = \frac{\gamma(t-r)}{b} = \frac{\tilde{\gamma}(\tilde{t}-\tilde{r})}{b} - \frac{1}{2}\psi \quad \text{in forward directions,} \tag{5.5d}$$

$$\hat{T}_A + \psi = T_B = \frac{1+\psi^2}{2}\,\frac{t-r}{b} + \psi = \frac{\tilde{\gamma}(\tilde{t}-\tilde{r})}{b} + \frac{1}{2}\psi \quad \text{in backward directions.} \tag{5.5e}$$

Notice how, for $\psi > 0.5$, the overall amplitude drops with decreasing ψ, and the characteristic timescale increases. As ψ gets somewhat larger than 1, this behavior is described by the intermediate-region formulae

$$a_+ \propto 1/\psi\ , \quad \Delta\hat{T}_A \propto \psi$$

[cf. eqs. (4.12) and (4.14)]. As ψ increases into the domain $\psi \gtrsim 1$, one can see the wave form beginning to approach the "inverted mesa" shape (Fig. 4a) characteristic of the intermediate region.

VI. SPECTRUM AND ENERGY

a) Total Energy Radiated

The total energy carried off by the waves, as seen in the rest frame of star A, is

$$\Delta E = \frac{1}{16\pi}\int_{-\infty}^{+\infty}\int_0^{\pi}\int_0^{2\pi}\left[(\partial_t A_+)^2+(\partial_t A_\times)^2\right] r^2 d\varphi\, d\theta\, dt; \tag{6.1}$$

see, e.g., equation (35.27) of MTW. Peters (1970) evaluated this triple integral numerically for various encounter velocities v. We have not repeated his computations, but we have confidence in them. From Peters' graph one can read off the high velocity limit

$$\Delta E = 20\ (m_A m_B/b^2)^2 b\gamma^3, \tag{6.2a}$$

where the factor of 20 is uncertain by a few tens of percent. Matzner and Nutku (1974) used the method of virtual quanta to compute ΔE in the high velocity limit, and obtained

$$\Delta E = 256\ (m_A m_B/b^2)^2\, b\gamma^3\, \ell n\ (4\gamma^2). \tag{6.2b}$$

Smarr (1977) evaluated the zero-frequency limit of the spectrum (see §c below) and, assuming (incorrectly) that $dE/d\omega d\Omega$ is roughly constant up to an angle-independent cutoff frequency $\omega_c = K\gamma/b$ with $K \sim 1$, he obtained

$$\Delta E = 10\ K\ (m_A m_B/b^2)^2 b\gamma^3\ \ell n\ (4\gamma^2) \tag{6.2c}$$

in agreement with Matzner and Nutku.

It is straightforward for us to compute ΔE at high velocities to within a factor of order unity: We first compute the energy radiated per unit solid angle, $dE/d\Omega$, in the extreme forward region $\theta \lesssim \gamma^{-1}$, in the intermediate region $\gamma^{-1} \ll \theta \ll \pi/2$, and in the extreme backward region $\theta \gtrsim \pi/2$. Then we integrate over solid angle.

In the extreme forward region the amplitude has magnitude

$$A_+ \sim A_\times \sim \frac{m_A m_B}{br}\,\gamma^2 \tag{6.3a}$$

[eq. (4.9)], and it varies on a timescale $\Delta\hat{T}_A \sim 1$ corresponding to

$$\Delta t \sim b/\gamma \tag{6.3b}$$

[eq. (4.7)]. Thus, in this region

$$\frac{dE}{d\Omega} \sim r^2 \left(\frac{A_+}{\Delta t}\right)^2 \Delta t \sim \left(\frac{m_A m_B}{b^2}\right)^2 \gamma^5\,. \tag{6.3c}$$

The region encompasses a solid angle $\Delta\Omega \sim \gamma^{-2}$, so it contributes an amount

$$(\Delta E)_{\text{extreme forward}} \sim \left(\frac{m_A m_B}{b^2}\right)^2 \gamma^3 \tag{6.3d}$$

to the total energy.

In the intermediate region (which Smarr presumed was responsible for the logarithmic term)

$$A_+ \sim A_\times \sim \frac{m_A m_B}{br} \, \frac{\gamma^{3/2}}{\chi} \sim \frac{m_A m_B}{br} \, \frac{\gamma}{\theta} \tag{6.4a}$$

[eqs. (4.14a) and (4.11a)]. These amplitudes agree with Smarr's zero-frequency values. The characteristic timescale in this region is $\Delta \ddagger \sim 1$, corresponding to

$$\Delta t \sim b\chi\gamma^{-1/2} \sim b\theta \tag{6.4b}$$

[eqs. (4.12) and (4.11a)]. Thus, in this region

$$\frac{dE}{d\Omega} \sim r^2 \left(\frac{A_+}{\Delta t}\right)^2 \Delta t \sim \left(\frac{m_A m_B}{b^2}\right)^2 \frac{\gamma^2}{\theta^3} \,. \tag{6.4c}$$

The region extends from near $\theta = \gamma^{-1}$ to near $\theta = \pi/2$, so it contains a total energy

$$(\Delta E)_{\text{intermediate}} = \int_{\gamma^{-1}}^{\pi/2} \frac{dE}{d\Omega} \, 2\pi \sin\theta \, d\theta \sim \left(\frac{m_A m_B}{b^2}\right)^2 \gamma^3 \,. \tag{6.4d}$$

Almost all of this energy is at the extreme forward edge of the region.

An analogous computation gives for the energy in the extreme backward region $\theta \gtrsim \pi/2$

$$(\Delta E)_{\text{backward}} \sim \left(\frac{m_A m_B}{b^2}\right)^2 \gamma^2 . \qquad (6.5)$$

Thus, the total energy radiated in the rest frame of "star" A [(6.3d) plus (6.4d) plus 6.5)] varies as γ^3, as found by Peters (1970), and not as $\gamma^3 \ \ell n\ (4\gamma^2)$ as found by Matzner and Nutku (1974) and inferred by Smarr (1977). While we understand why Smarr's method failed (incorrect assumption that the cut-off frequency is angle independent), we have not dug deeply enough into the Matzner-Nutku method of virtual quanta to understand its failure.

b) Spectra

To obtain expressions for the specific flux $\mathcal{F}_\nu^{+\times}$ or the energy radiated per unit frequency in the arbitrary velocity case we have to Fourier transform the wave amplitudes (3.17), as we did for the low velocity case [eqs. (2.8)-(2.11)]. Unfortunately, the Fourier transform of eq. (3.17) cannot be expressed in closed form for arbitrary values of (θ,φ). However, in the precisely forward direction $(\theta = \varphi = 0)$ and precisely backward direction $(\theta = \pi,\ \varphi = 0)$ the Fourier transform of the amplitude [eq. (3.17b)] can be done analytically.

Using eqs. (3.17b,c) with $\alpha = \overline{\alpha} = 1,\ \beta = \beta = \delta = \overline{\delta} = 0$, ($A_\times$ vanishes) we obtain for the "+" polarization

$$\mathcal{A}_+(\theta = 0, \varphi = 0) = \frac{\gamma^2(1 + v^2)}{2v}\left[\frac{(2T_B^{\ 2} + 1)}{\gamma \ell_B} - \frac{(1 + v)(2T_A^{\ 2} + 1)}{\ell_A}\right] . \qquad (6.6)$$

It suffices to discuss only the forward amplitude $\mathcal{A}_+(0,0)$ since the backward amplitude $\mathcal{A}_+(\pi,0)$ is obtained by using the Doppler shift relations (4.10); see also eq. (5.3a,b), i.e., $\mathcal{A}_+(\pi,0) = [(1 - v)/(1 + v)]^{1/2}\ \mathcal{A}_+(0,0)$.

The Fourier transform of (6.6) is defined by

$$\mathcal{a}_{+\omega} = \frac{b}{v\gamma(2\pi)^{1/2}} \int_{-\infty}^{\infty} \mathcal{a}_{+}(0,0)\, e^{-i(\omega b/v\gamma)T_A}\, dT_A , \qquad (6.7)$$

since $t - r = T_A b/v\gamma$. Performing the integral we obtain

$$\mathcal{a}_{+\omega} = \frac{\gamma(1+v^2)b}{v^2(2\pi)^{1/2}} \left[(1+v)\, K_2(x) - (1-v)\, K_2(y)\right] , \qquad (6.8)$$

where K_2 denotes a modified Bessel function with argument $x = \omega b/v\gamma$ or $y = \gamma(1-v)x$. Using equation (2.9a), we obtain for the specific flux in the forward direction for the case of arbitrary velocities

$$\mathcal{F}_{\nu}^{+} = \left(\frac{4m_A m_B}{br}\right)^2 \frac{\gamma^4(1+v^2)^2 x^2}{16\pi v^2} \left[(1+v)\, K_2(x) - (1-v)\, K_2(y)\right]^2 . \qquad (6.9)$$

The low-velocity limit of (6.9) may be obtained by setting $x = \omega b/v$, $\gamma \to 1$ and utilizing the recursion relations for the derivative of $K_2(x)$. The result is

$$\mathcal{F}_{\nu}^{+} = \left(\frac{4m_A m_B}{br}\right)^2 \frac{x^4}{16\pi} K_1^{\,2}(x) \qquad \text{for} \qquad v \ll 1. \qquad (6.10)$$

This agrees precisely with the low-velocity results of §II if we set $\theta = \varphi = 0$ in equation (2.9a).

Figure 6 shows the specific flux in the forward $(\theta = 0)$ and backward $(\theta = \pi)$ direction for velocities ranging from $v = 0.0001$ to $v = 0.9999999$. Plotted vertically is

$$\log_{10}\left[\frac{\mathcal{F}_\nu^+}{\gamma^4(4m_Am_B/rb)^2}\right] = \log_{10}\left[\frac{|\omega\mathcal{a}_{+\omega}|^2}{8\gamma^4}\right] \qquad \text{for } \theta = 0 \text{ or}$$

$$\log_{10}\left[\frac{\mathcal{F}_\nu^+}{\gamma^4(4m_Am_B/rb)^2}\left(\frac{1+v}{1-v}\right)\right] = \log_{10}\left[\frac{|\omega\mathcal{a}_{+\omega}|^2}{8\gamma^4}\left(\frac{1+v}{1-v}\right)\right] \qquad \text{for } \theta = \pi \tag{6.11a}$$

Plotted horizontally is

$$\log_{10}\left[\frac{\omega b}{v\gamma}\right] \quad \text{for} \quad \theta = 0, \quad \text{or} \quad \log_{10}\left[\frac{\omega b}{v\gamma}\left(\frac{1+v}{1-v}\right)^{1/2}\right] \qquad \text{for } \theta = \pi \tag{6.11b}$$

since for the backward direction the frequency is redshifted by the Doppler factor $1/\gamma(1+v) = [(1-v)/(1+v)]^{1/2}$.

The structures of the curves may be understood in terms of T_A, T_B and Figure 3. For low velocities only one timescale $T = T_A \simeq T_B$ is relevant, so the spectrum of Figure 6 shows only one characteristic frequency ($\omega \sim v/b$). At high velocities $\gamma \gg 1$, the two timescales T_A and T_B are no longer the same, so the spectrum shows two characteristic frequencies; that associated with its peak ($\omega \sim v\gamma/b$; $\Delta T_A \sim 1$) and that associated with the beginning of its exponential fall-off ($\omega \sim v\gamma^2/b$; $\Delta T_B \sim 1$). The $\mathcal{F}_\nu \propto \nu^{-2}$ behavior between peak and exponential fall-off due to the temporal discontinuity of A at high velocities in Figure 3. In the limit $v = 1$ ($\gamma = \infty$), a true discontinuity would exist in Figure 3, and the $\mathcal{F}_\nu \propto \nu^{-2}$ behavior in Figure 6 would continue forever. The behavior, $\mathcal{F}_\nu \propto \nu^{-2}$, is easily understood by noting that a discontinuity in $\partial A_+/\partial t$ produces a behavior $\mathcal{a}_{+\omega} \propto 1/\omega^2$ in the Fourier transform of $\mathcal{a}_+$, and therefore $\mathcal{F}_\nu \propto |\omega\mathcal{a}_{+\omega}|^2 \propto 1/\omega^2 \propto 1/\nu^2$.

c) Zero-Frequency Limit

To obtain formulae for the zero frequency limit we proceed by calculating the net change in $A_{\substack{+\\ \times}}$ as we did for the low-velocity case in §II. As T_A, $T_B \to -\infty$ the amplitudes $\mathcal{a}_+$ and $\mathcal{a}_\times \to 0$ since all the coefficients G_1 through $G_5 \to 0$, cf. equation (3.18). For T_A, $T_B \to +\infty$ only G_1 and G_3 are nonzero, and we can write

$$\Delta\mathcal{a}_+ = \frac{\gamma(1+v^2)\sin\theta\cos\varphi}{(1-v\cos\theta)^2}\left[2\cos\theta - v(1+\cos\theta)\right] , \qquad (6.12a)$$

$$\Delta\mathcal{a}_\times = \frac{-2\gamma(1+v^2)\sin\theta\sin\varphi}{(1-v\cos\theta)} , \qquad (6.12b)$$

where $\Delta\mathcal{a}_{\substack{+\\ \times}}$ denotes the change between the early and late limiting values of $\mathcal{a}_{\substack{+\\ \times}}$. The energy radiated per unit frequency interval per unit solid angle for either polarization is given by equations (2.13) and (2.14),

$$\left(\frac{dE}{d\omega d\Omega}\right)_{\substack{+\\ \times}} = \frac{r^2}{2\pi}\left(\mathcal{F}_{\nu}^{+\times}\right) = \frac{1}{32\pi^2}\left(\frac{4m_Am_B}{b}\right)^2\left(\Delta\mathcal{a}_{\substack{+\\ \times}}\right)^2 . \qquad (6.13)$$

Use of (6.12a) and (6.12b) in (6.13) gives us our final answer for the zero-frequency limit

$$\left(\frac{dE}{d\omega d\Omega}\right)_+ = \frac{1}{32\pi^2}\left(\frac{4m_Am_B}{b}\right)^2 \frac{\gamma^2(1+v^2)^2\sin^2\theta\cos^2\varphi}{(1-v\cos\theta)^4}\left[2\cos\theta - v(1+\cos\theta)\right]^2 \qquad (6.14a)$$

$$\left(\frac{dE}{d\omega d\Omega}\right)_\times = \frac{1}{8\pi^2}\left(\frac{4m_Am_B}{b}\right)^2 \frac{\gamma^2(1+v^2)^2\sin^2\theta\sin^2\varphi}{(1-v\cos\theta)^2} . \qquad (6.14b)$$

This result agrees with equation (2.12) of Smarr (1977) if we let $2m_A = r_{SCH}$, $m_B = m$, and $\delta = (1+v^2)r_{SCH}/bv$. (There is an overall factor of 1/2 difference due to different conventions.) We recover our previous results, cf. equation (2.13), if we integrate (6.14) over solid angle and take the low-velocity limit.

Because of the $\sin^2\theta$ term in equation (6.14) there is no zero frequency radiation emitted in the precisely forward ($\theta = 0$) direction. However, there is forward beaming due to the $(1 - v\cos\theta)$ factor in the denominator for nonzero values of θ within the usual "forward cone," i.e., for $0 < \theta \lesssim 1/\gamma$. To determine the high-velocity limit of (6.14) in the "forward region" we reintroduce the notation we used in §IVa, and make use of equations (4.4a), (4.4c), (4.8a,b). A careful expansion in powers of $\varepsilon \equiv 1/\gamma$ gives

$$\left(\frac{dE}{d\omega d\Omega}\right)_+ = \frac{2}{\pi^2}\left(\frac{4m_A m_B}{b}\right)^2 \frac{c^2\psi^2(1-\psi^2/2)^2}{\varepsilon^4(1+\psi^2)^4} \tag{6.15a}$$

$$\left(\frac{dE}{d\omega d\Omega}\right)_\times = \frac{2}{\pi^2}\left(\frac{4m_A m_B}{b}\right)^2 \frac{s^2\psi^2}{\varepsilon^4(1+\psi^2)^2} \quad . \tag{6.15b}$$

Note that these expressions for the energy radiated in the "forward region" have the same γ^4 dependence as equation (6.9) for energy radiated in the precisely forward ($\theta = 0$) direction. The symmetry properties discussed in §III tell us that $dE/d\omega d\Omega$ for the backward region (in frame A) is given by

$$\left(\frac{dE}{d\omega d\Omega}\right)^{\text{backward}}_{+\times} = \left(\frac{1-v}{1+v}\right)\left(\frac{dE}{d\omega d\Omega}\right)^{\text{forward}}_{+\times} \tag{6.16}$$

where $(dE/d\omega d\Omega)^{\text{forward}}$ is equation (6.15 a or b). To evaluate the energy radiated through the "forward cone" we integrate equations (6.15) over solid angle, where our angular variables are ψ and φ with $0 \le \psi \le 1$ and $0 \le \varphi \le 2\pi$. The expression for $dE/d\omega$ is

$$\left(\frac{dE}{d\omega}\right)_+ = \frac{4}{\pi}\left(\frac{m_A m_B}{b}\right)^2 \left[5/4 + \ell n(2)\right] \gamma^2 \tag{6.17a}$$

$$\left(\frac{dE}{d\omega}\right)_\times = \frac{16}{\pi}\left(\frac{m_A m_B}{b}\right)^2 \left[\ell n(2) - 1/2\right] \gamma^2 \,. \tag{6.17b}$$

We now turn our attention to the energy radiated at zero frequency in the "intermediate region." We use equations (4.11a), (4.13a), and (4.13b) in equation (6.14a) and (6.14b) to obtain

$$\left(\frac{dE}{d\omega d\Omega}\right)_+ = \frac{2}{\pi^2}\left(\frac{4m_A m_B}{b}\right)^2 \frac{c^2}{\chi^2 \eta^6}\left[1 - \frac{\eta^2}{2\chi^2}\right]^2 \tag{6.18a}$$

$$\left(\frac{dE}{d\omega d\Omega}\right)_\times = \frac{8}{\pi^2}\left(\frac{4m_A m_B}{b}\right)^2 \frac{s^2}{\chi^2 \eta^6} \quad . \tag{6.18b}$$

Note that since $2/\eta^2 = \gamma$ these expressions have the γ^3/χ^2 dependence we expect from equations (4.14b,c) and (6.13). To get an expression for the energy emitted into the "intermediate region" we integrate equations (6.18) over solid angle. The angular variables are χ and φ, with limits $(2\gamma)^{-1/2} \le \chi \le \pi/2$, $0 \le \varphi \le 2\pi$. The final expressions are given by

$$\left(\frac{dE}{d\omega}\right)_+ = \frac{2}{\pi}\left(\frac{4m_A m_B}{b}\right)^2 \frac{\ell n(2\gamma)}{\eta^4} \tag{6.19a}$$

$$\left(\frac{dE}{d\omega}\right)_\times = \frac{8}{\pi}\left(\frac{4m_A m_B}{b}\right)^2 \frac{\ell n(2\gamma)}{\eta^4} \tag{6.19b}$$

which are consistent with equation (2.13) of Smarr's (1977) zero frequency analysis. The agreement is more transparent when we note that

$$\ell n\left[\frac{1+v}{1-v}\right] = 2\ \ell n\left[\left(\frac{1+v}{1-v}\right)^{1/2}\right] = 2\ \ell n\left[(1+v)\gamma\right] \cong 2\ \ell n[2\gamma]$$

where we have neglected terms of order $1/\gamma^2$. To get an order of magnitude estimate of relative amounts of energy radiated into the "forward cone" vs. the "intermediate" region we use (6.17), (6.19), and get

$$\frac{(dE/d\omega)^{\text{forward}}_{+\text{ or }\times}}{(dE/d\omega)^{\text{intermediate}}_{+\text{ or }\times}} \sim \frac{1}{\ell n(2\gamma)}\ . \tag{6.20}$$

This shows that in the zero frequency limit, the energy radiated into the "intermediate region" dominates by a factor of $\sim \ell n(\gamma)$ the energy radiated into the "forward cone," in agreement with Smarr. Because the cutoff frequency of the spectrum is a function of angle (θ,φ), this effect is no longer valid when we consider the total energy radiated. (See remarks in §VIa.)

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FIGURE CAPTIONS

Fig. 1. Spatial diagram in the rest frame of A showing the trajectory of "star" B and the orientation of the unit vectors $\underset{\sim}{e}_\theta$ and $\underset{\sim}{e}_\varphi$. Note that the polar angle θ is measured from the x-axis.

Fig. 2. The time-evolution of the renormalized wave amplitudes for slow-motion bremsstrahlung in the rest frame of "star" A. The location of the observer is determined by the values of θ and φ. (See Fig. 1.) Note that for an observer in the plane of the encounter (the x-y plane), $\varphi = 0$ and the "x"-polarization vanishes.

Fig. 3. Plot of the renormalized wave amplitude $\mathcal{a}_+/\gamma^2$ (in frame A) in the forward ($\theta = 0$) direction for v = 0.0001, 0.1, 0.3, 0.5, 0.7, 0.96, 0.9999 (uppermost to lowermost). Note that as v increases the slope tends to become discontinuous at $T_A = T_B = 0$. This plot also describes the amplitude in the backward direction ($\theta = \pi$); see eq. (5.3).

Fig. 4. a) Plot of the renormalized wave amplitude $\mathcal{a}_+/\gamma^{3/2}$ for $\tilde{\theta} = \pi/2$, $\tilde{\varphi} = 0$ for velocities from v = 0.0001 (uppermost curve) to v = 0.999999 (lowermost curve). Note the discontinuity in the slope as v increases is symmetric about $\ddagger = 0$.

b) Plot of $\mathcal{a}_{+ \text{ or } \times}/\gamma^{3/2}$ for $\tilde{\theta} = \pi/2$, $\tilde{\varphi} = \pi/2$. The range of velocities is the same as in a) above. Note the development of a discontinuity in the slope of $\mathcal{a}_+$ and in the second derivative of $\mathcal{a}$ at $\ddagger = 0$ as v increases (lowermost curve in each set).

Fig. 5. Plot of α_+/γ^2 in the x-y plane for v = 0.9999 and for various values of the angular parameter $\psi = \theta\gamma$. For $\psi << 1$ we are in the extreme forward region and the curves are similar to the lowermost curve in Fig. 3. As ψ increases and we move away from the x-axis we enter the forward most part of the intermediate region and the curves begin to resemble the "inverted mesa" shape of Fig. 4a. These plots are also valid for backward directions; see eqs. (5.5).

Fig. 6. Plot of the specific flux eq. (6.11a) in the precisely forward direction for various velocities as a function of frequency. As ω increases the first "bump" is near $\dot{\omega} \sim v\gamma/b$ (for $\Delta T_A \sim 1$) followed by a $1/\omega^2$ dependence (which on the log scale gives a slope of -2) until at $\omega \sim v\gamma^2/b$ (for $\Delta T_B \sim 1$) the curve drops off exponentially.

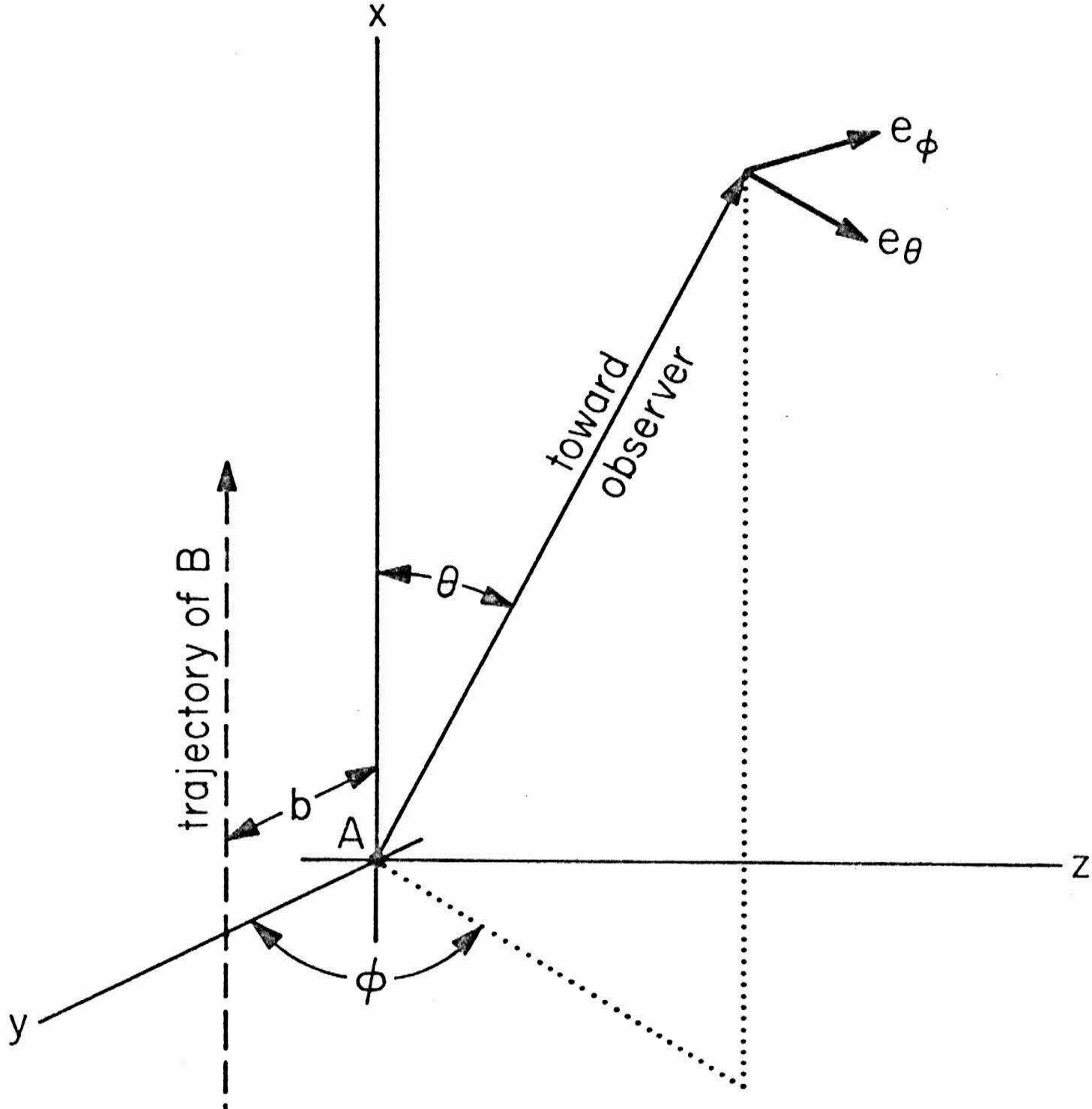
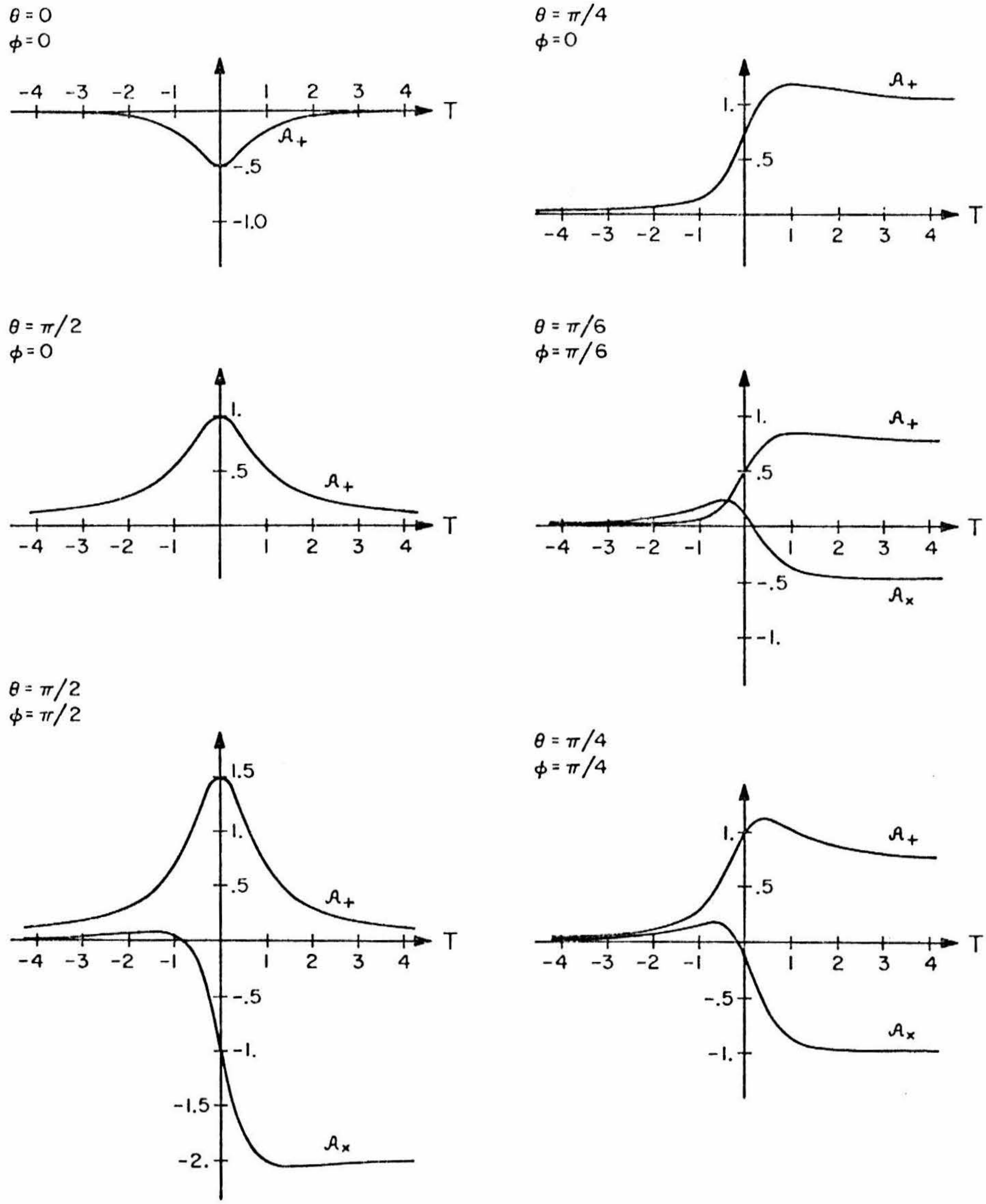

x
e_ϕ
e_θ
toward observer
trajectory of B
θ
b
A
z
ϕ
y

Fig. 1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φ = 0
A+
T
θ = π/4
φ = 0
A+
T
θ = π/2
φ = 0
A+
T
θ = π/6
φ = π/6
A+
Ax
T
θ = π/2
φ = π/2
A+
Ax
T
θ = π/4
φ = π/4
A+
Ax
T

Fig. 2

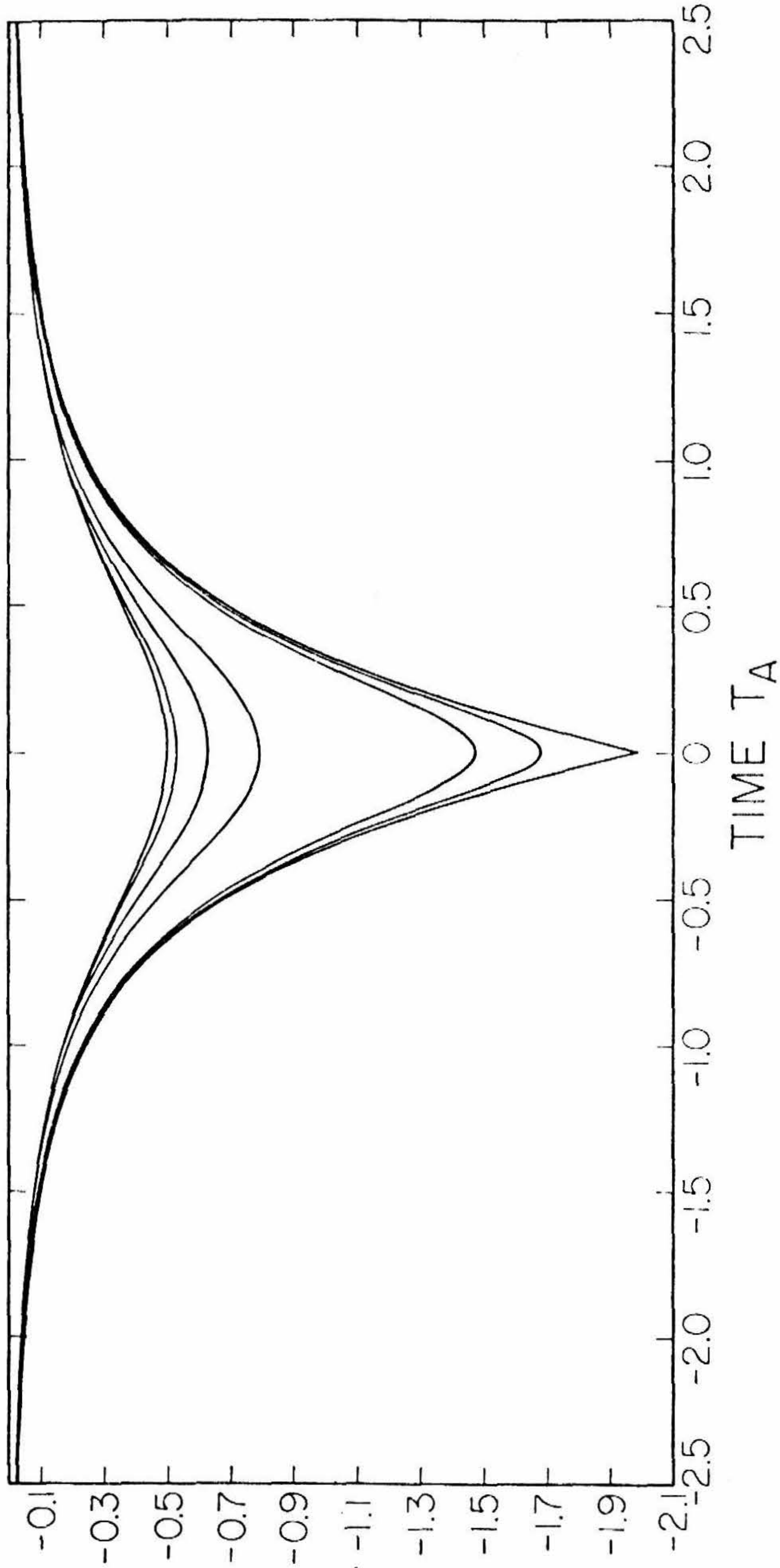

TIME T_A
-2.5
-2.0
-1.5
-1.0
-0.5
0
0.5
1.0
1.5
2.0
2.5
$\mathcal{A}_+/\gamma^2$
-0.1
-0.3
-0.5
-0.7
-0.9
-1.1
-1.3
-1.5
-1.7
-1.9
-2.1

Fig. 3

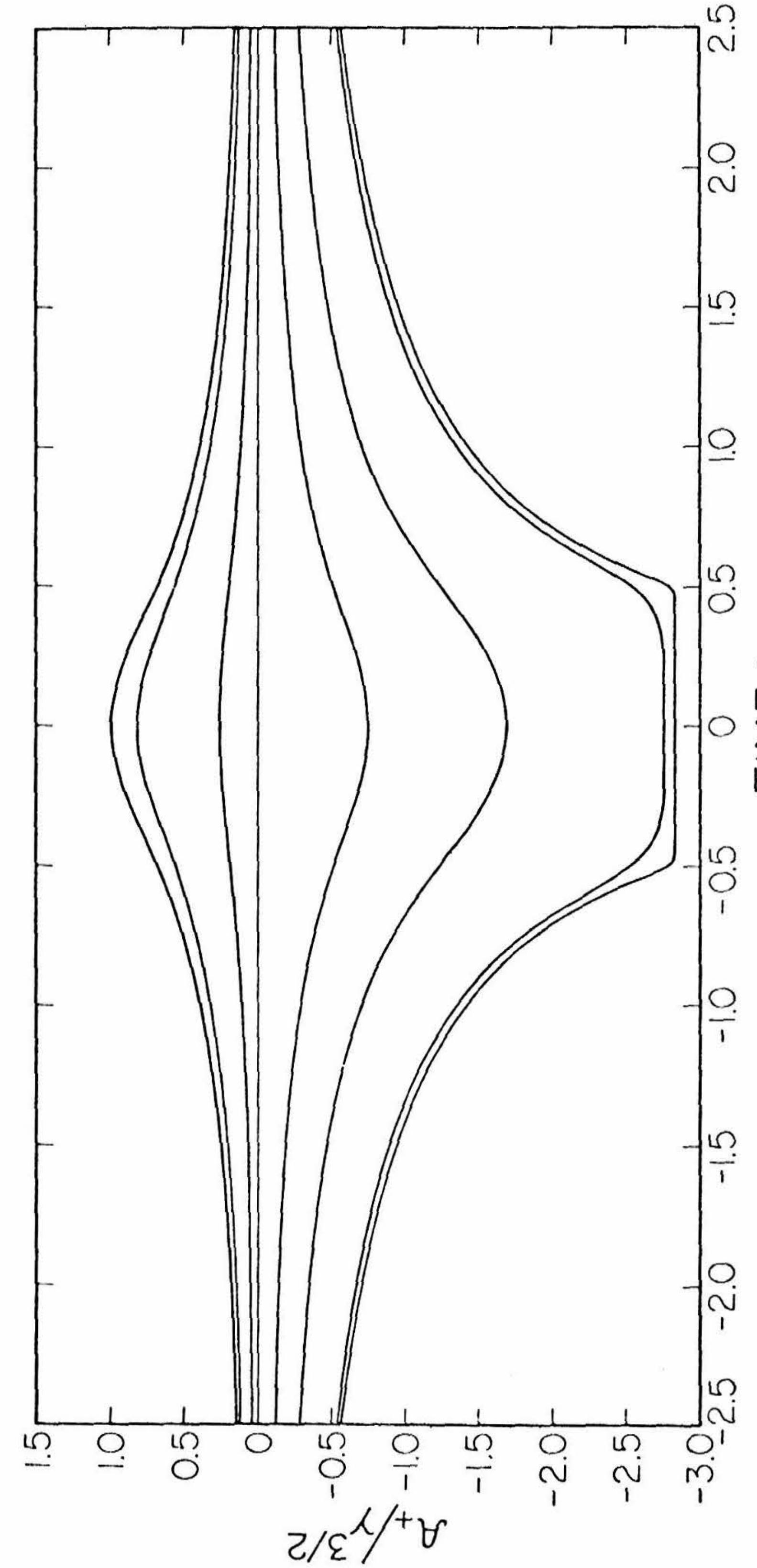
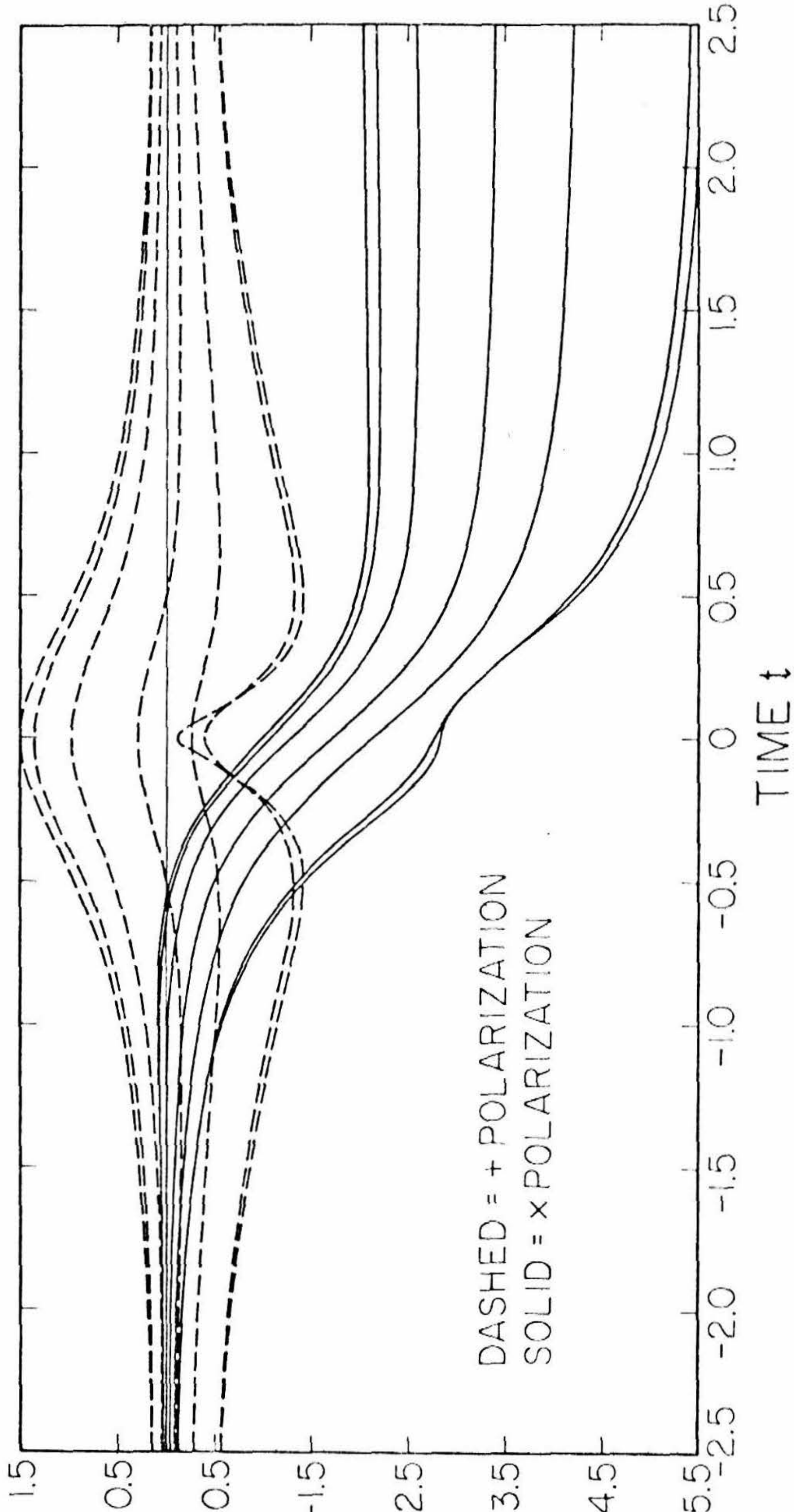


Fig. 4a



DASHED = + POLARIZATION
SOLID = × POLARIZATION
TIME t
A+/γ3/2
-2.5 -2.0 -1.5 -1.0 -0.5 0 0.5 1.0 1.5 2.0 2.5
1.5 0.5 -0.5 -1.5 -2.5 -3.5 -4.5 -5.5

Fig. 4b

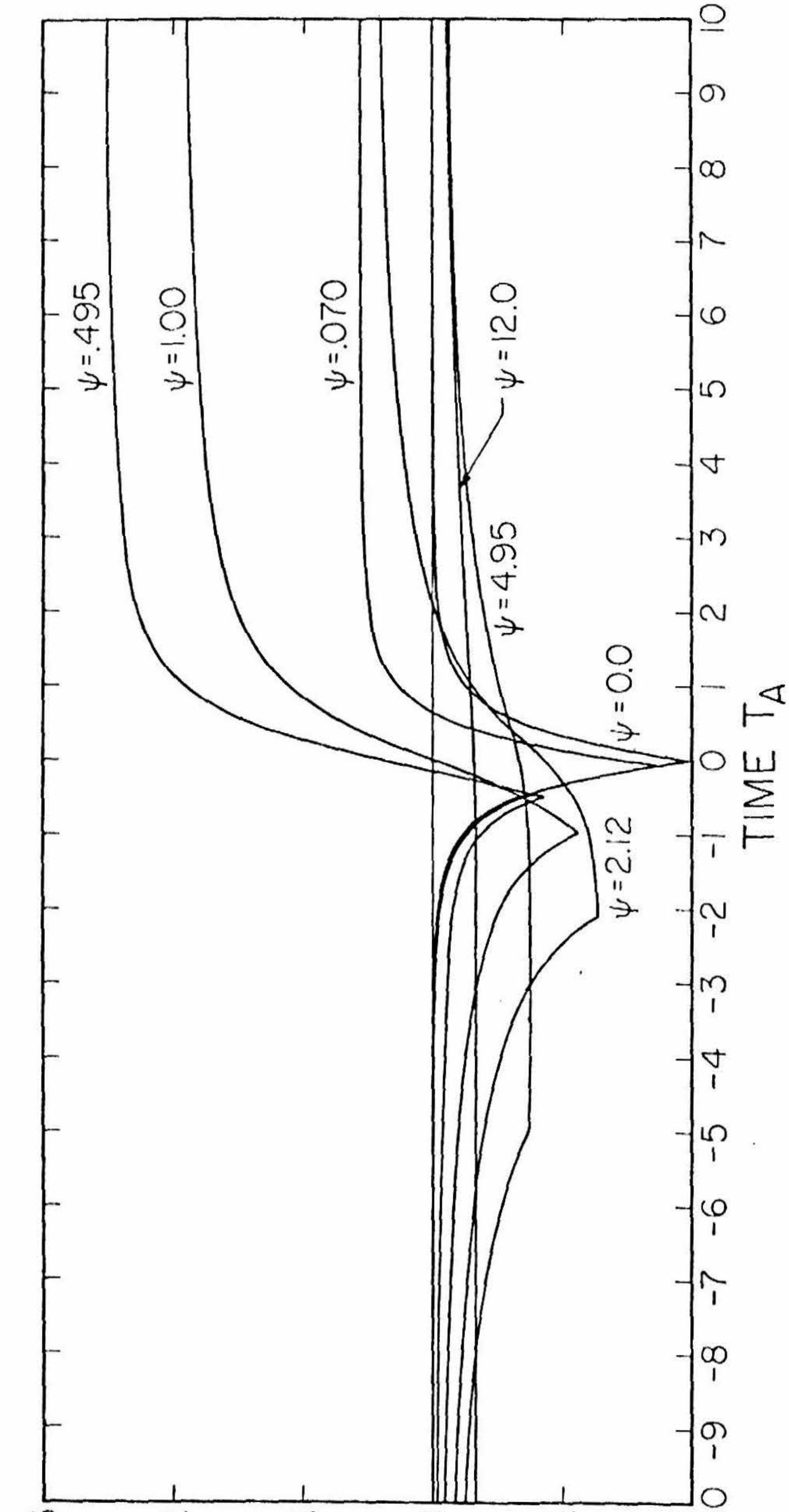

ψ=.495
ψ=1.00
ψ=.070
ψ=12.0
ψ=4.95
ψ=0.0
ψ=2.12
TIME T_A
A_+/γ^2
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
3 2 1 0 -1 -2

Fig. 5

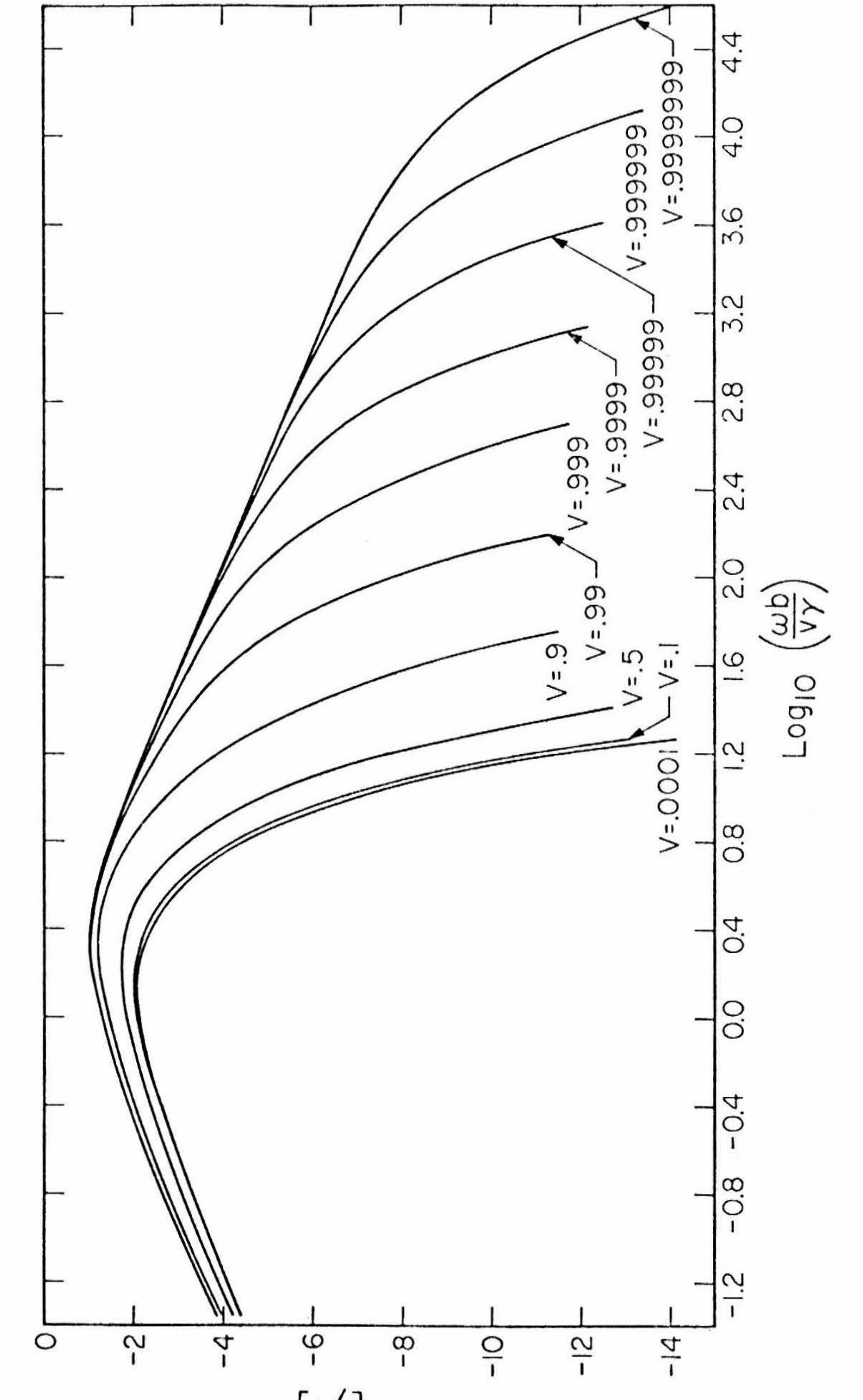

V=.0001
V=.1
V=.5
V=.9
V=.99
V=.999
V=.9999
V=.99999
V=.999999
V=.9999999
-1.2
-0.8
-0.4
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
4.0
4.4
0
-2
-4
-6
-8
-10
-12
-14

Fig. 6

PAPER 4.

That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance through a vacuum and without the mediation of anything else, by and through which this action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking, can ever fall into it.

Newton

The Gravitational Scattering of Zero-Rest-Mass Plane Waves*

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ABSTRACT

We have used the Feynman-diagram technique to calculate the differential cross sections $d\sigma/d\Omega$ for the scattering of zero-rest-mass plane waves of spin 0, 1, and 2 by linearized Schwarzschild and Kerr geometries in the long-wavelength, weak-field limit (wavelength of incident radiation $\gg$ radius of scatterer $\gg$ mass of scatterer). We find that the polarization of right (or left) circularly polarized electromagnetic waves is unaffected by the scattering process (i.e., helicity is conserved), and that the two helicity (polarization) states of the photon are scattered differently by the Kerr geometry. This coupling between the photon helicity and the angular momentum of the scatterer also leads to a partial polarization of unpolarized incident light. For gravitational waves, on the other hand, there is neither helicity conservation nor helicity-dependent scattering; and the angular momentum of the scatterer has no polarizing effect on incident, unpolarized gravitational waves.

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I. INTRODUCTION

Recent observations by Harwit et al.[1] have placed an upper limit on the difference of deflection between left and right circularly polarized radio beams passing near the limb of the sun. Whereas previous electromagnetic tests of general relativity (light bending near the sun, Shapiro time delay of radar signals, gravitational redshift[2]) probe only the geometric optics limit of electromagnetic-gravitational coupling, this experiment goes beyond geometric optics. The deflection is independent of polarization in the geometric optics limit; but for real, physical waves the helicity of the wave should couple to the angular momentum of the deflecting object ("magnetic-type" gravitational effect) to produce helicity-dependent deflection — helicity dependence which, for the sun, is below the accuracy of Harwit et al., but which should exist nevertheless.

A number of recent papers have used general relativity theory to investigate this helicity dependence and other aspects of the interaction between incoming waves and a gravitating body.[3-13] Gradually the full picture of such interactions is emerging; but there remain as yet a number of gaps in the picture. The purpose of this paper is to fill in one of those gaps: the full details of the long-wavelength limit for rotating and weakly gravitating bodies

$$\begin{aligned}(\text{wavelength}) \equiv 2\pi/\omega &\gg (\text{size of body}) \equiv L \\ &\gg (\text{gravitational radius}) \doteq M \qquad (1.1)\end{aligned}$$

for scalar and gravitational waves as well as electromagnetic.

In the regime $2\pi/\omega \gg L \gg M$ it is better to speak of a "scattering" of the waves than a "deflection"; and it is most useful to calculate the amplitude T_{fi} for scattering of an incoming plane wave $|i\rangle$ into an outgoing (final) plane wave $|f\rangle$. From this scattering amplitude one can derive everything of interest — the explicit form of the scattered wave; the differential scattering cross section $d\sigma/d\Omega$; the amount of focussing; the deflection angle in the regime where it has meaning, i.e., (wavelength) $\ll$ (impact parameter); etc.

We, like some others before us,[14-15] have found the Feynman-diagram technique to be far more powerful than partial-wave analyses for studying the long-wavelength limit of classical scattering. Historically the Feynman technique was first used in conjunction with quantum electrodynamical processes.[16-18] Its efficiency as a problem-solving tool soon led to its widespread use in many aspects of quantum interactions, including quantum gravity.[19-21] However, since classical scattering is the long-wavelength limit of quantum scattering, one can perfectly well use the technique to solve our type of classical problem.

Our paper is in six sections. Section II gives the Lagrangians, vertex rules and diagrams needed for each type of wave (scalar, electromagnetic, and gravitational), as well as the formula for the differential scattering cross section in terms of the transition amplitude. In sections III, IV, and V we treat the scattering of scalar, electromagnetic and gravitational waves, respectively. Section VI discusses and contrasts our results with those of other authors.

II. FEYNMAN DIAGRAMS FOR SCATTERING

The classical problem of the scattering of a massless field propagating in a slightly curved spacetime may be treated by quantizing both the gravitational background and the scattered field. In this scenario both fields evolve in a Minkowski spacetime and couple according to the Feynman vertex rules. This approach may be contrasted to the work of Peters,[13] in which the gravitational background is considered to be a passive nondynamical entity, whose influence on the propagating field is embodied in a curved spacetime Green function. In this section we summarize the relevant Feynman rules.

The wave equation for source-free scalar waves

$$\Box\psi - uR\psi = 0 \tag{2.1}$$

may be obtained from the Lagrangian density

$$\mathcal{L}_S = -\frac{\sqrt{-g}}{2}\left(g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} + uR\psi^2\right) \tag{2.2}$$

where u is a constant, R the curvature scalar, $g = \det \|g_{\alpha\beta}\|$ and $\Box \equiv (-g)^{-1/2}\,\partial_\alpha(g^{\alpha\beta}\sqrt{-g}\,\partial_\beta)$. For $u = 1/6$, ψ represents conformally invariant waves.

Following Feynman[19-21] and Gupta,[22] and since we require that $|\bar{h}^{\alpha\beta}| \ll 1$ everywhere, we expand the gravitational field about the flat Minkowski background:

$$\sqrt{-g}\, g^{\alpha\beta} \equiv \mathfrak{g}^{\alpha\beta} \equiv \eta^{\alpha\beta} - 2\lambda\bar{h}^{\alpha\beta} \tag{2.3}$$

where the gravitational coupling constant $\lambda = \sqrt{8\pi}$ and we use units in which $G = \hbar = c = 1$. Indices are raised and lowered using the Minkowski metric $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}\,(-1,1,1,1)$, commas denote partial derivatives, semicolons denote covariant derivatives with respect to the metric and $\bar{h}^{\alpha\beta}$ is the trace reversed metric perturbation.

The determinant factor $\sqrt{-g}$, $g^{\alpha\beta}$ and R now become infinite series in λ,

$$\sqrt{-g} = \sqrt{(-\det\|g_{\alpha\beta}\|)} = \sqrt{(-\det\|\mathfrak{g}^{\alpha\beta}\|)} = 1 - \lambda\bar{h} + O(\lambda^2) \qquad (2.4)$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} - 2\lambda(\bar{h}^{\alpha\beta} - \frac{1}{2}\bar{h}\eta^{\alpha\beta}) + O(\lambda^2) \qquad (2.5)$$

$$R = 2\lambda(\bar{h}^{\alpha\beta}{}_{,\alpha\beta} + \frac{1}{2}\bar{h}_{,\alpha}{}^{\alpha}) + O(\lambda^2) \qquad (2.6)$$

where the trace of the metric perturbation is denoted by $\bar{h} = \bar{h}_{\mu}{}^{\mu}$. Expanding (2.2) in powers of λ we find that:

$$\mathcal{L}_S = \sum_{n=0}^{\infty} \lambda^n \mathcal{L}_n \qquad (2.7)$$

where

$$\mathcal{L}_0 = -\frac{1}{2}\eta^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} \qquad (2.8)$$

$$\mathcal{L}_1 = \bar{h}^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - u(\bar{h}^{\alpha\beta}{}_{,\alpha\beta} + \frac{1}{2}\bar{h}_{,\alpha}{}^{\alpha})\psi^2 \ . \qquad (2.9)$$

The free (i.e., noninteraction) Lagrangian $\mathcal{L}_0$ describes the free propagation of the scalar field ψ in Minkowski space, whereas the terms proportional to λ, λ^2, etc. represent the interaction parts of $\mathcal{L}$, i.e., they determine how the gravitational field $\bar{h}^{\alpha\beta}$ couples to the scalar field ψ. In this formalism, quantization of the Lagrangian density is equivalent to treating $\bar{h}^{\alpha\beta}$ and ψ as quantum field operators.

From $\mathcal{L}_1$ we may derive the amplitude T_{21} for a transition of the scalar field from an initial plane wave state with wave-vector ("momentum") ${}^{1}k^{\alpha}$ to a final state with "momentum" ${}^{2}k^{\alpha}$ while absorbing a graviton with momentum q^{α} and polarization $\bar{e}^{\alpha\beta}$ (Fig. 1)

$$T_{21} = 2\lambda \bar{e}^{\alpha\beta}[{}^{1}k_{(\alpha}{}^{2}k_{\beta)} + u(q_{\alpha}q_{\beta} + \frac{1}{2}\eta_{\alpha\beta}q^{2})] \quad . \tag{2.10}$$

Here we have used the notation $A_{(\alpha}B_{\beta)} = \frac{1}{2}(A_{\alpha}B_{\beta} + B_{\alpha}A_{\beta})$ and the superscript 1 (2) denotes the initial (final) state. Conservation of 4-momentum requires that,

$${}^{2}\underline{k} = {}^{1}\underline{k} + \underline{q} \quad . \tag{2.11}$$

In this calculation we shall limit ourselves to interactions proportional to λ^{2}, (single graviton exchange); in other words, we shall calculate the scattering cross sections in the first Born approximation. In the classical limit for the scattering of waves with angular frequency ω by a mass M with angular momentum J, this corresponds to calculating at first order in the dimensionless quantities $M\omega$ and $J\omega^{2}$. Since our interest is restricted to a gravitational background geometry generated by classical energy-momentum distributions which are not affected appreciably by the scattering process, we may replace the virtual graviton by an external field.[23] In particular we consider only <u>static</u> fields; hence in the vertex rule (2.10) $\bar{e}^{\alpha\beta}$ stands for the 3-dimensional Fourier transform of $\bar{h}^{\alpha\beta}$ and the graviton 4-momentum is pure spacelike ($q^{0} = 0$).

The transition amplitude T_{21} above has been normalized by the definition

$$S_{21} = \delta_{21} + i(2\pi)^{4}\,\delta^{4}({}^{2}\underline{k} - {}^{1}\underline{k} - \underline{q})T_{21} \tag{2.12}$$

where S_{21} is the S-matrix connecting the initial to the final state. With this normalization for T_{21}, the differential cross section for the scattering of a zero-rest-mass wave with frequency ω into a solid angle $d\Omega$ is:

$$d\sigma = \frac{2\pi}{2\omega 2\omega} |T_{21}|^2 D \tag{2.13}$$

where D denotes the density of final states:

$$D = \frac{\omega^2}{(2\pi)^3} d\Omega \quad . \tag{2.14}$$

Turn now to the scattering of electromagnetic waves off a slightly curved background. The manifestly covariant photon Lagrangian density, obtained by minimal coupling to gravity, is:

$$\mathcal{L}_{EM} = -\frac{1}{4}\sqrt{-g}\,(g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}) \tag{2.15}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor computed from the Maxwell vector potential A_μ by:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad . \tag{2.16}$$

From (2.15) and (2.16) one obtains the field equations for the source-free electromagnetic field:

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \tag{2.17}$$

$$F_{\mu\nu;}{}^{\nu} = 0 \quad . \tag{2.18}$$

We expand the photon Lagrangian density in powers of λ according to (2.7) and obtain:

$$\mathcal{L}_0 = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \tag{2.19}$$

$$\mathcal{L}_1 = (\bar{h}^{\mu\alpha}\eta^{\nu\beta} - \frac{1}{4}\bar{h}\eta^{\mu\alpha}\eta^{\nu\beta})F_{\mu\nu}F_{\alpha\beta} \quad . \tag{2.20}$$

After proper permutation of the photon labels, $\mathcal{L}_1$ provides the graviton-photon-photon vertex rule (see Fig. 1)

$$T_{21} = 2\lambda\bar{e}^{\alpha\beta}\Big\{{}^1k_{(\alpha}{}^2k_{\beta)}\,({}^1\underline{\varepsilon}\cdot{}^2\underline{\varepsilon}^*) + {}^1\varepsilon_{(\alpha}{}^2\varepsilon^*{}_{\beta)}({}^1\underline{k}\cdot{}^2\underline{k}) - {}^1k_{(\alpha}{}^2\varepsilon^*{}_{\beta)}({}^2\underline{k}\cdot{}^1\underline{\varepsilon})$$
$$- {}^2k_{(\alpha}{}^1\varepsilon_{\beta)}({}^1k\cdot{}^2\varepsilon^*) - \frac{1}{2}\eta_{\alpha\beta}[({}^1\underline{k}\cdot{}^2\underline{k})({}^1\underline{\varepsilon}\cdot{}^2\underline{\varepsilon}^*)$$
$$- ({}^1\underline{k}\cdot{}^2\underline{\varepsilon}^*)({}^2\underline{k}\cdot{}^1\underline{\varepsilon})]\Big\} \quad . \tag{2.21}$$

Here ${}^1k^\alpha$ and ${}^1\varepsilon^\alpha$ are the 4-momentum and polarization vector of the ingoing photon, whereas ${}^2k^\alpha$ and ${}^2\varepsilon^\alpha$ denote the respective properties of the outgoing photon. In accordance with the external-field approximation $\bar{e}^{\alpha\beta}$ denotes the Fourier transform of $\bar{h}^{\alpha\beta}$. Note that the transition amplitude (2.21) is invariant under a gauge transformation of the form:

$${}^i\varepsilon_\alpha \to {}^i\varepsilon_\alpha + \gamma\,{}^ik_\alpha \qquad (i = 1,2) \tag{2.22}$$

where γ is an arbitrary scalar.

Finally we turn to the scattering of gravitational waves by the gravitational background. One arrives at the matter-free Einstein field equations

$$R_{\mu\nu} = 0 \tag{2.23}$$

by varying the Lagrangian density

$$\mathcal{L}_G = \frac{1}{2\lambda^2}\sqrt{-g}\,R \quad . \tag{2.24}$$

Taking for our basic fields $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ and $\mathfrak{g}_{\mu\nu} = g_{\mu\nu}/\sqrt{-g}$ rather than the metric itself, we can express the Einstein gravitational Lagrangian density (2.24) in the particularly convenient Goldberg[24] form:

$$\mathcal{L}_G = -\frac{1}{16\lambda^2}\left(2\mathfrak{g}^{\alpha\beta}\mathfrak{g}_{\sigma\mu}\mathfrak{g}_{\tau\nu} - \mathfrak{g}^{\alpha\beta}\mathfrak{g}_{\mu\tau}\mathfrak{g}_{\sigma\nu} - 4\,\eta^{\alpha}{}_{\sigma}\eta^{\beta}{}_{\tau}\mathfrak{g}_{\mu\nu}\right)\mathfrak{g}^{\mu\tau}{}_{,\alpha}\mathfrak{g}^{\sigma\nu}{}_{,\beta} \quad . \qquad (2.25)$$

After we expand (2.24) in powers of λ, the components of $\mathcal{L}_G$ become:

$$\mathcal{L}_0 = -\frac{1}{4}\left(2\bar{h}^{\alpha\beta,\mu}\bar{h}_{\alpha\beta,\mu} - \bar{h}^{,\mu}\bar{h}_{,\mu} - 4\,\bar{h}^{\alpha\beta,\mu}\bar{h}_{\mu\beta,\alpha}\right) \qquad (2.26)$$

$$\mathcal{L}_1 = -\bar{h}^{\mu\nu}\Big(\bar{h}_{\alpha\beta,\mu}\bar{h}^{\alpha\beta,}{}_{\nu} + 2\,\bar{h}_{\mu\alpha}{}^{,\beta}\bar{h}_{\nu\beta}{}^{,\alpha} - 2\,\bar{h}_{\mu\beta,\alpha}\bar{h}_{\nu}{}^{\beta,\alpha} + \bar{h}^{,\alpha}\bar{h}_{\mu\nu,\alpha} - \frac{1}{2}\bar{h}_{,\mu}\bar{h}_{,\nu}\Big) \quad . \qquad (2.27)$$

The interaction part $\mathcal{L}_1$, appropriately symmetrized with respect to the graviton labels, provides the expression for the three-graviton vertex (see Fig. 1):

$$\begin{aligned}
T_{21} = \lambda\bar{e}^{\mu\nu}\Big\{ & -2\big[{}^1\bar{\underline{\underline{e}}} : {}^2\bar{\underline{\underline{e}}}^{*}\,{}^1k_\mu\,{}^2k_\nu - {}^1\bar{e}_{\mu\nu}\,{}^2\bar{e}^{*\alpha\beta}q_\alpha\,{}^1k_\beta + {}^2\bar{e}^{*}_{\mu\nu}\,{}^1\bar{e}^{\alpha\beta}q_\alpha\,{}^2k_\beta\big] \\
& - 4\big[{}^1\bar{e}_{\mu\alpha}\,{}^2\bar{e}^{*}_{\nu\beta}\,{}^1k^\beta\,{}^2k^\alpha - q^\beta\big({}^2\bar{e}^{*}_{\mu\alpha}\,{}^1\bar{e}^{\alpha}{}_{\beta}\,{}^1k_\nu - {}^1\bar{e}_{\mu\alpha}\,{}^2\bar{e}^{*\alpha}{}_{\beta}\,{}^2k_\nu\big)\big] \\
& + 4\big[{}^1\underline{k}\cdot{}^2\underline{k}\,{}^1\bar{e}_{\mu\beta}\,{}^2\bar{e}^{*\beta}{}_{\nu} - \underline{q}\cdot{}^1\underline{k}\,{}^1\bar{e}_{\beta\nu}\,{}^2\bar{e}^{*\beta}{}_{\mu} + \underline{q}\cdot{}^2\underline{k}\,{}^2\bar{e}^{*}_{\beta\nu}\,{}^1\bar{e}^{\beta}{}_{\mu}\big] \\
& - \big[{}^1\underline{k}\cdot{}^2\underline{k}\big({}^1\bar{e}_{\mu\nu}\,{}^2\bar{e}^{*} + {}^2\bar{e}^{*}_{\mu\nu}\,{}^1\bar{e}\big) - \underline{q}\cdot{}^1\underline{k}\,{}^1\bar{e}\,{}^2\bar{e}^{*}_{\mu\nu} + \underline{q}\cdot{}^2\underline{k}\,{}^2\bar{e}^{*}\,{}^1\bar{e}_{\mu\nu} \\
& + \eta_{\mu\nu}\big(\underline{q}\cdot{}^2\underline{k} - \underline{q}\cdot{}^1\underline{k}\big)\,{}^1\bar{\underline{\underline{e}}} : {}^2\bar{\underline{\underline{e}}}^{*}\big] \\
& + \big[{}^1k_\mu\,{}^1k_\nu\,{}^1\bar{e}\,{}^2\bar{e}^{*} + \eta_{\mu\nu}\big(q_\alpha\,{}^2k_\beta\,{}^2\bar{e}^{*}\,{}^1\bar{e}^{\alpha\beta} - q_\alpha\,{}^1k_\beta\,{}^1\bar{e}\,{}^2\bar{e}^{*\alpha\beta}\big)\big]\Big\} \quad . \qquad (2.28)
\end{aligned}$$

where ${}^1k^\alpha$, ${}^1\bar{e}^{\alpha\beta}$; ${}^2k^\alpha$, ${}^2\bar{e}^{\alpha\beta}$; and q^α, $\bar{e}^{\alpha\beta}$ refer to the momenta and polarizations of the gravitons and ${}^1\bar{\underline{\underline{e}}} : {}^2\bar{\underline{\underline{e}}}$ denotes the tensor inner product. Unlike the graviton-photon-photon transition amplitude (2.21), the three-graviton transition amplitude is not invariant under the analogous gauge transformation, which in this instance is of the form;

$$ {}^{i}\bar{e}^{\alpha\beta} \to {}^{i}\bar{e}^{\alpha\beta} + {}^{i}k^{\alpha}\chi^{\beta} + {}^{i}k^{\beta}\chi^{\alpha} \qquad (i = 1,2) \tag{2.29} $$

where χ^{α} represents an arbitrary vector.

In general, the gauge invariance of the amplitudes is guaranteed by the Feynman-diagram formalism as long as all the diagrams of the same order in the coupling constant are included. Due to our ignorance of the propagator for an object of mass M and very high quantum mechanical spin, we omit all diagrams but the graviton-pole diagram. (This difficulty in formulating the quantum problem could probably be avoided by a classical analysis.) In the external-field approximation (no recoil of scatterer) the amplitude corresponding to this diagram is given by (2.28) where $\bar{e}^{\mu\nu}$ stands for the 3-dimensional Fourier transform of $\bar{h}^{\mu\nu}$. The external-field approximation serves to simplify the algebra but the effect of the omitted diagrams is to yield an amplitude (2.27) that is not gauge invariant, and is valid only for small scattering angles.

III. SCALAR WAVES

Since the waves have wavelength much larger than the scatterer, they cannot probe (at first order) either the scatterer's internal structure or the quadrupole and higher-order moments of its gravitational field. For this reason, and because we calculate only to lowest order in λ, we can approximate the scatterer's gravitational field by the linearized metric for the exterior of a spherical body endowed with angular momentum:

$$g_{oo} = -(1 - \frac{2M}{r}), \quad g_{oj} = g_{jo} = -\frac{2M}{r}(\underset{\sim}{a} \times \underset{\sim}{r})_j, \quad g_{jk} = (1 + \frac{2M}{r})\delta_{jk} . \quad (3.1)$$

Here M is the mass of the body and $M\underset{\sim}{a} = \underset{\sim}{J}$ is its angular momentum. The Fourier transforms of the $\overline{h}_{\alpha\beta}$ are given by,

$$\overline{e}_{oo} = \frac{\lambda M}{q^2}$$

$$\overline{e}_{oj} = \overline{e}_{jo} = \frac{i\lambda M}{2q^2}(\underset{\sim}{a} \times \underset{\sim}{q})_j$$

$$\overline{e}_{jk} = 0 \quad (3.2)$$

where $\underset{\sim}{q}$ is the (pure spacelike) momentum transfer $\underset{\sim}{q} = {}^2\underset{\sim}{k} - {}^1\underset{\sim}{k}$ $(q^o = 0)$. Permitting the angular momentum per unit mass $\underset{\sim}{a}$ to vanish in (3.1) or (3.2), we recover the linearized Schwarzschild geometry. Using Eqs. (2.10), (2.13), (2.14), and (3.2), the differential scattering cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{M^2}{\sin^4 \Theta/2}\left\{(1 - 2u \sin^2 \Theta/2)^2 + \omega^2\left(\underset{\sim}{a} \cdot ({}^1\hat{\underset{\sim}{k}} \times {}^2\hat{\underset{\sim}{k}})\right)^2\right\} . \quad (3.3)$$

In the above ω is the angular frequency of the scalar wave, ${}^1\hat{\underset{\sim}{k}}$ and ${}^2\hat{\underset{\sim}{k}}$ are unit 3-vectors along the propagation directions of the incident and scattered fields respectively, and Θ is the angle between ${}^1\hat{\underset{\sim}{k}}$ and and ${}^2\hat{\underset{\sim}{k}}$. Allowing $\underset{\sim}{a}$ to vanish (linearized Schwarzschild geometry) one recovers the result previously obtained by Peters:[13]

$$\left(\frac{d\sigma}{d\Omega}\right)_{SCHW.} = \frac{M^2}{\sin^4 \Theta/2}(1 - 2u \sin^2 \Theta/2)^2 . \quad (3.4)$$

Due to the r^{-1} dependence of the Newtonian potential, for the case of minimal coupling (u = 0), Eq. (3.5) reduces to the usual $1/\sin^4 \Theta/2$ Rutherford-type cross section. For non-minimal coupling ($u \neq 0$), the cross section still exhibits the Rutherford-type angular dependence, but only for $\Theta \ll 1$. This is not surprising, since it is the scalar curvature R which gives rise to u-dependent terms in the cross section. Considering that R is nonzero only along the worldline of the scatterer, we see that for large impact parameters (i.e., small scattering angles) the scalar curvature cannot significantly contribute to the differential cross section. One may rewrite the scattering cross section for rotating bodies (3.3) in the suggestive form:

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{SCHW.} + \frac{M^2 a^2 \omega^2}{\sin^4 \Theta/2} \sin^2\alpha \sin^2\Theta \sin^2\varphi \tag{3.5}$$

with α, Θ, and φ as shown in Fig. 2.

Equation (3.5) shows that the effect of angular momentum is to add a positive semi-definite term to $(d\sigma/d\Omega)_{SCHW}$. For small scattering angles this angular momentum term is negligible with respect to $(d\sigma/d\Omega)_{SCHW}$. This can be easily understood by noticing that for large impact parameters the r^{-1} dependence of the Newtonian potential $\bar{h}_{oo}$ dominates the r^{-2} dependence of the magnetic-type gravitational field $\bar{h}_{oi}$, which is the source of the angular momentum term. Another interesting feature of (3.5) is that the scattering in the backward direction is finite and independent of the angular momentum $\underset{\sim}{a}$:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\Theta=\pi} = M^2(1 - 2u)^2 \quad . \tag{3.6}$$

IV. ELECTROMAGNETIC WAVES

Theoretically more interesting and of possible observational importance is the gravitational scattering of electromagnetic waves. We choose the polarizations of the photons to be purely spacelike $[{}^1\underline{\varepsilon} = (0, {}^1\underset{\sim}{\varepsilon}),\ {}^2\underline{\varepsilon} = (0, {}^2\underset{\sim}{\varepsilon})]$ and use Eqs. (2.13), (2.14), and (2.21). The result for the scattering of electromagnetic waves with initial polarization ${}^1\underset{\sim}{\varepsilon}$ into some polarization ${}^2\underset{\sim}{\varepsilon}$ is:

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{M^2}{4\sin^4\Theta/2}\Bigg|(1+\cos\Theta)({}^1\underset{\sim}{\varepsilon}\cdot{}^2\underset{\sim}{\varepsilon}^*) - ({}^1\hat{\underset{\sim}{k}}\cdot{}^2\underset{\sim}{\varepsilon}^*)({}^2\hat{\underset{\sim}{k}}\cdot{}^1\underset{\sim}{\varepsilon})$$
$$+ i\omega\Big[2({}^1\hat{\underset{\sim}{k}}\times{}^2\hat{\underset{\sim}{k}})\cdot\underset{\sim}{a}({}^1\underset{\sim}{\varepsilon}\cdot{}^2\underset{\sim}{\varepsilon}^*) + \big(({}^2\hat{\underset{\sim}{k}}-{}^1\hat{\underset{\sim}{k}})\times{}^2\underset{\sim}{\varepsilon}^*\big)\cdot\underset{\sim}{a}({}^2\hat{\underset{\sim}{k}}\cdot{}^1\underset{\sim}{\varepsilon})$$
$$+ \big(({}^2\hat{\underset{\sim}{k}}-{}^1\hat{\underset{\sim}{k}})\times{}^1\underset{\sim}{\varepsilon}\big)\cdot\underset{\sim}{a}({}^1\hat{\underset{\sim}{k}}\cdot{}^2\underset{\sim}{\varepsilon}^*)\Big]\Bigg|^2 . \qquad (4.1)$$

For linear polarization (${}^1\underset{\sim}{\varepsilon}$ and ${}^2\underset{\sim}{\varepsilon}$ real) the contribution of the angular momentum $\underset{\sim}{a}$ to the cross section (4.1) will be proportional to $a^2\omega^2$, whereas for circular polarizations (${}^1\underset{\sim}{\varepsilon}$ and ${}^2\underset{\sim}{\varepsilon}$ complex) the contribution will include an $a\omega$-term. We first consider circular polarizations (i.e., pure helicity states) and we choose for the photon basis states:

$$ {}^1\underset{\sim}{\varepsilon}_L^R = \frac{1}{\sqrt{2}}(\hat{\underset{\sim}{e}}_x \pm i\,\hat{\underset{\sim}{e}}_y)$$
$$ {}^2\underset{\sim}{\varepsilon}_L^R = \frac{1}{\sqrt{2}}(\hat{\underset{\sim}{e}}_\Theta \pm i\,\hat{\underset{\sim}{e}}_\varphi) \qquad (4.2)$$

where $\hat{\underset{\sim}{e}}_x$, $\hat{\underset{\sim}{e}}_y$, $\hat{\underset{\sim}{e}}_\Theta$, $\hat{\underset{\sim}{e}}_\varphi$ are unit vectors in the x, y, Θ, and φ directions. After some algebraic manipulations (4.1) yields:

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = 0 \qquad (4.3)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\substack{RR\\LL}} = M^2 \left\{ [\cot g^2 \Theta/2 \pm 2a\omega \cos \Theta/2(\cos \alpha \cos \Theta/2 + \sin \alpha \sin \Theta/2 \cos \varphi)]^2 + 4\, a^2\omega^2 \sin^2 \alpha \cot g^2 \frac{\Theta}{2} \sin^2 \varphi \right\} \quad (4.4)$$

where the first (second) subscript denotes the initial (final) polarization and the upper (lower) sign in (4.4) refers to the RR (LL) case. For the linearized Schwarzschild geometry (4.4) reduces to recent results obtained by Peters:[13]

$$\left(\frac{d\sigma}{d\Omega}\right)^{SCHW.}_{RR} = \left(\frac{d\sigma}{d\Omega}\right)^{SCHW.}_{LL} = M^2 \cot g^4 \Theta/2 \quad . \qquad (4.5)$$

In the circular polarization basis the scattering matrix is diagonal, which explicitly shows that helicity is conserved by the scattering process. This is not restricted only to our situation, but rather is a general property of electromagnetic wave propagation in <u>any</u> orientable spacetime manifold.[11,25] Moreover, for the Schwarzschild geometry the scattering cross section is helicity independent, whereas for a rotating scatterer it is helicity dependent. This results in a differential gravitational deflection of right and left circularly polarized electromagnetic radiation by a rotating object. For a given impact parameter b of the incident beam, we define the angular splitting as:

$$\delta = \begin{pmatrix} \text{angle by which R helicity photon is scattered minus} \\ \text{angle by which L helicity photon is scattered} \end{pmatrix} \quad . \qquad (4.6)$$

We then solve the inverse scattering problem[26] and find, to lowest order in $a\omega$:

$$\delta = 2\, a\, \omega\, \cos\alpha \left(\frac{4M}{b}\right)^3 \left[\ell n\left(\frac{b}{2M}\right) - \frac{3}{4}\right] \quad . \qquad (4.7)$$

To obtain this result we have used the constraint that;

$$\delta \ll \frac{4M}{b} \ll 1 \quad . \tag{4.7}$$

It must be stressed that so far we have only discussed pure helicity states. For any linearly polarized or unpolarized incident wave the scattering cross section summed over final polarization states becomes:

$$\left(\frac{d\sigma}{d\Omega}\right)_{total} = M^2 \left\{ \begin{array}{l} \text{cotg}^4 \Theta/2 + 4\, a^2\, \omega^2 [\cos^2 \Theta/2 (\cos\alpha \cos\Theta/2 \\ + \sin\alpha \sin\Theta/2 \cos\varphi)^2 + (\sin\alpha\, \text{cotg}\, \Theta/2 \sin\varphi)^2] \end{array} \right\} \tag{4.8}$$

which for the Schwarzschild case reduces to (4.5). We therefore conclude that all linearly polarized incident beams are deflected through the same angle. However, since the diagonal elements of the scattering matrix in the circular polarization basis are unequal, linearly polarized incident waves become elliptically polarized. For an unpolarized wave packet, on the other hand, the paths of different helicity photons are split by an amount given by (4.7). In addition, the angular momentum $\tilde{a}$ induces a partial polarization of the scattered waves. We define the amount of this polarization by:

$$p = \left| \frac{\left(\frac{d\sigma}{d\Omega}\right)_{RR} - \left(\frac{d\sigma}{d\Omega}\right)_{LL}}{\left(\frac{d\sigma}{d\Omega}\right)_{RR} + \left(\frac{d\sigma}{d\Omega}\right)_{LL}} \right| \tag{4.9}$$

and we find to lowest order in a:

$$p = 4\, a\, \omega |\cos\alpha \cos\Theta/2 + \sin\alpha \sin\Theta/2 \cos\varphi| \sin\Theta/2\, \text{tg}\, \Theta/2 \quad . \tag{4.10}$$

In concluding this section we note that independent of $\tilde{a}$ or the initial polarization, the cross section for scattering in the backward direction vanishes.

V. CRAVITATIONAL WAVES

Using (2.13), (2.14), and (2.28) we compute the differential cross section for the scattering of gravitational waves from an initial polarization ${}^{1}\bar{\underset{\approx}{e}}$ into some final polarization ${}^{2}\bar{\underset{\approx}{e}}$:

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{M^2}{\sin^4 \Theta/2}\left(\cos^2 \Theta + \omega^2[({}^{1}\hat{\underset{\sim}{k}} \times {}^{2}\hat{\underset{\sim}{k}}) \cdot a]^2\right)\left|{}^{1}\bar{\underset{\approx}{e}} : {}^{2}\bar{\underset{\approx}{e}}^{*}\right|^2 \quad . \tag{5.1}$$

This result was derived in the transverse-traceless (TT) gauge.[2] Although the transition amplitude (2.28) is not gauge invariant by itself, (5.1) yields reliable results for small momentum transfers, i.e., for small scattering angles. By analogy with the photon case, we choose for the graviton basis states the circular polarizations given by:

$$ {}^{1}\bar{\underset{\approx}{e}}{}^{R}_{L} = \frac{1}{2}\,[\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_x - \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_y \pm i(\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_y + \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_x)]$$

$$ {}^{2}\bar{\underset{\approx}{e}}{}^{R}_{L} = \frac{1}{2}\,[\hat{\underset{\sim}{e}}_\Theta\hat{\underset{\sim}{e}}_\Theta - \hat{\underset{\sim}{e}}_\varphi\hat{\underset{\sim}{e}}_\varphi \pm i(\hat{\underset{\sim}{e}}_\Theta\hat{\underset{\sim}{e}}_\varphi + \hat{\underset{\sim}{e}}_\varphi\hat{\underset{\sim}{e}}_\Theta)] \quad . \tag{5.2}$$

Substitution of the initial and final states into (5.1) yields

$$\left(\frac{d\sigma}{d\Omega}\right)_{RL} = \left(\frac{d\sigma}{d\Omega}\right)_{LR} = \frac{M^2}{16 \sin^4 \Theta/2}\,(\cos^2\Theta + a^2\omega^2\sin^2\alpha \sin^2\Theta \sin^2\varphi)(1 - \cos\Theta)^4 \tag{5.3a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{RR} = \left(\frac{d\sigma}{d\Omega}\right)_{LL} = \frac{M^2}{16 \sin^4 \Theta/2}\,(\cos^2\Theta + a^2\omega^2\sin^2\alpha \sin^2\Theta \sin^2\varphi)(1 + \cos\Theta)^4 . \tag{5.3b}$$

The nonvanishing of (5.3a) clearly illustrates that here, unlike the electromagnetic case, helicity is not conserved. Moreover there is neither different scattering of opposite helicity states (cf. 5.3) nor partial polarization of unpolarized incident gravitational radiation.

The latter is easily seen by noting that the scattering cross section for <u>either</u> helicity state is given by [adding (5.3a) and (5.3b)]:

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \left(\frac{d\sigma}{d\Omega}\right)_L = \frac{M^2}{\sin^4 \theta/2} (\cos^2\theta + a^2\omega^2\sin^2 \alpha \sin^2\theta \sin^2\varphi)$$

$$(\cos^2\theta + \frac{1}{8} \sin^4\theta) \quad . \tag{5.4}$$

Similarly, for the scattering of orthogonal linear polarizations denoted by

$$\overset{1}{\underset{\approx}{e}}_{+} = \frac{1}{\sqrt{2}} (\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_x - \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_y)$$

$$\overset{1}{\underset{\approx}{e}}_{\times} = \frac{1}{\sqrt{2}} (\hat{\underset{\sim}{e}}_x\hat{\underset{\sim}{e}}_y + \hat{\underset{\sim}{e}}_y\hat{\underset{\sim}{e}}_x) \tag{5.5}$$

one finds, after summing over the final polarizations and use of (5.1),

$$\left(\frac{d\sigma}{d\Omega}\right)_+ = \frac{M^2}{\sin^4 \theta/2} (\cos^2\theta + a^2\omega^2\sin^2 \alpha \sin^2\theta \sin^2\varphi)$$

$$(\cos^2\theta + \frac{1}{4} \sin^4\theta \cos^2 2\varphi) \tag{5.6a}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_\times = \frac{M^2}{\sin^4 \theta/2} (\cos^2\theta + a^2\omega^2\sin^2 \alpha \sin^2\theta \sin^2\varphi)$$

$$(\cos^2\theta + \frac{1}{4} \sin^4\theta \sin^2 2\varphi) \quad . \tag{5.6b}$$

For unpolarized incident gravitational waves [i.e., averaging over φ in (5.6a) and (5.6b) and summing], the differential scattering cross section is given by (5.4). Allowing $\underset{\sim}{a} \to 0$, we recover Peters' results apart from a factor of $\cos^2\theta$:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{SCHW.}}_{\text{THIS paper}} = \cos^2\theta \left(\frac{d\sigma}{d\Omega}\right)^{\text{SCHW.}}_{\text{PETERS}} \quad . \tag{5.7}$$

For small angle scattering there is good agreement. One may recover Peters' result exactly by calculating the scattering of gravitational waves off a massive spin-0 meson. Inclusion of all the relevant Feynman diagrams then leads to a gauge invariant transition amplitude. Actually, for the choice of the TT-gauge only the graviton-pole and seagull diagrams survive, and one obtains Peters' results exactly, i.e.:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\mathrm{SCHW.}}_{{}^{1}\underset{\approx}{e} \to {}^{2}\underset{\approx}{e}} = \frac{M^2}{\sin^4 \Theta/2} \left| {}^{1}\underset{\approx}{e} : {}^{2}\underset{\approx}{e}^{*} \right|^2 \quad . \qquad (5.8)$$

As a concluding remark, we note that independent of the polarization of the incident gravitational wave and the angular momentum $\underset{\sim}{a}$, the cross section for backscatter is nonzero. Whereas the exact dependence of $(d\sigma/d\Omega)_{\Theta=\pi}$ on the angular momentum $\underset{\sim}{a}$ cannot be inferred from the cross sections derived above (they are valid only for small scattering angles), one finds from (5.8) that the gravitational backscatter in a linearized Schwarzschild geometry is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)^{\mathrm{SCHW.}}_{\Theta=\pi} = M^2 \quad . \qquad (5.9)$$

In addition, if the incident radiation is in a pure helicity state, the backscattered radiation must have the opposite helicity.

VI. SUMMARY AND CONCLUSIONS

The differential cross sections for the weak-field gravitational scattering of long-wavelength scalar, electromagnetic and gravitational waves have been calculated using Feynman perturbation methods.

For the linearized Schwarzschild geometry, we have recovered the results obtained by Peters,[13] although he used a Green function formalism. In particular, for electromagnetic waves helicity is conserved, whereas for gravitational waves it is not. Endowing the scatterer with an angular momentum $\underset{\sim}{a}$, leads to helicity-dependent effects in electromagnetic wave scattering. Although the photon helicity is still conserved, the coupling between this helicity and the angular momentum of the scatterer results in a) different scattering of right and left circularly polarized photons and b) partial polarization of unpolarized incident electromagnetic radiation. The high-frequency limits of these effects have been discussed before by Mashhoon.[5,11] Whereas in the high frequency limit ($\omega M \gg 1$), the angular split δ [defined by (4.6)], and polarization p [defined by (4.9)] are proportional to $a\omega^{-1}$, in the low-frequency limit ($\omega M \ll 1$) they are proportional to $a\omega$. This confirms the belief that the magnetic-type gravitational field of a rotating body distinguishes between the helicity states of a photon only in the diffraction limit, i.e., when the wavelength of the incident photon is of the same order as the Schwarzschild radius of the scatterer.

Gravitational waves do not exhibit any of these angular-momentum-induced effects.

As a final comment, we note that this method may easily be applied to the gravitational scattering of non-integer spin or massive fields.

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FIGURE CAPTIONS

Fig. 1. The graviton-zero rest mass field-zero rest mass field vertex. The wavy line represents a graviton. The solid lines represent either scalar, electromagnetic, or gravitational quanta.

Fig. 2. The spatial orientation of the angular momentum $\underset{\sim}{a}$ and the scattered direction ${}^{2}\hat{\underset{\sim}{k}}$ relative to the incident direction ${}^{1}\hat{\underset{\sim}{k}}$.

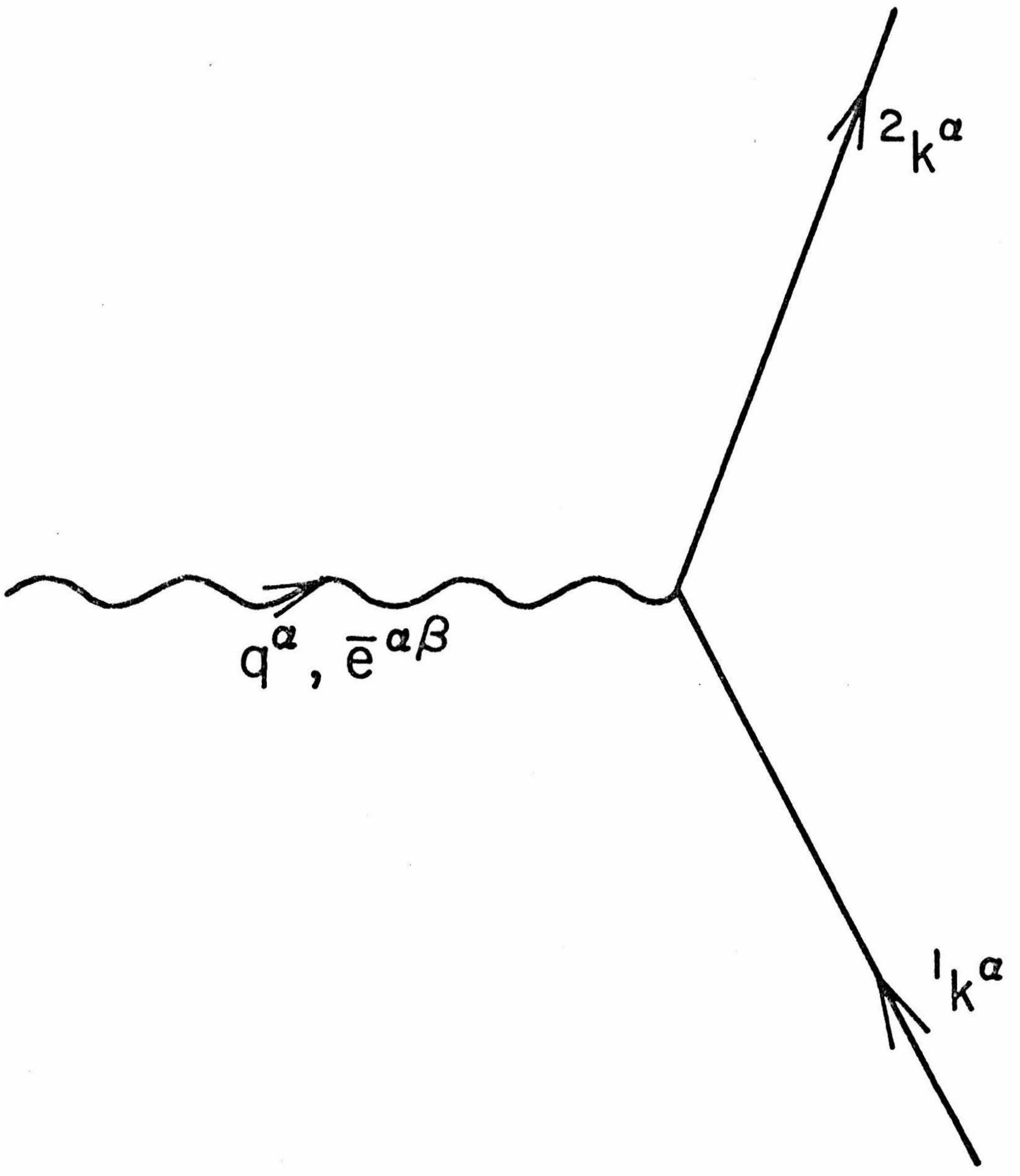

Fig. 1

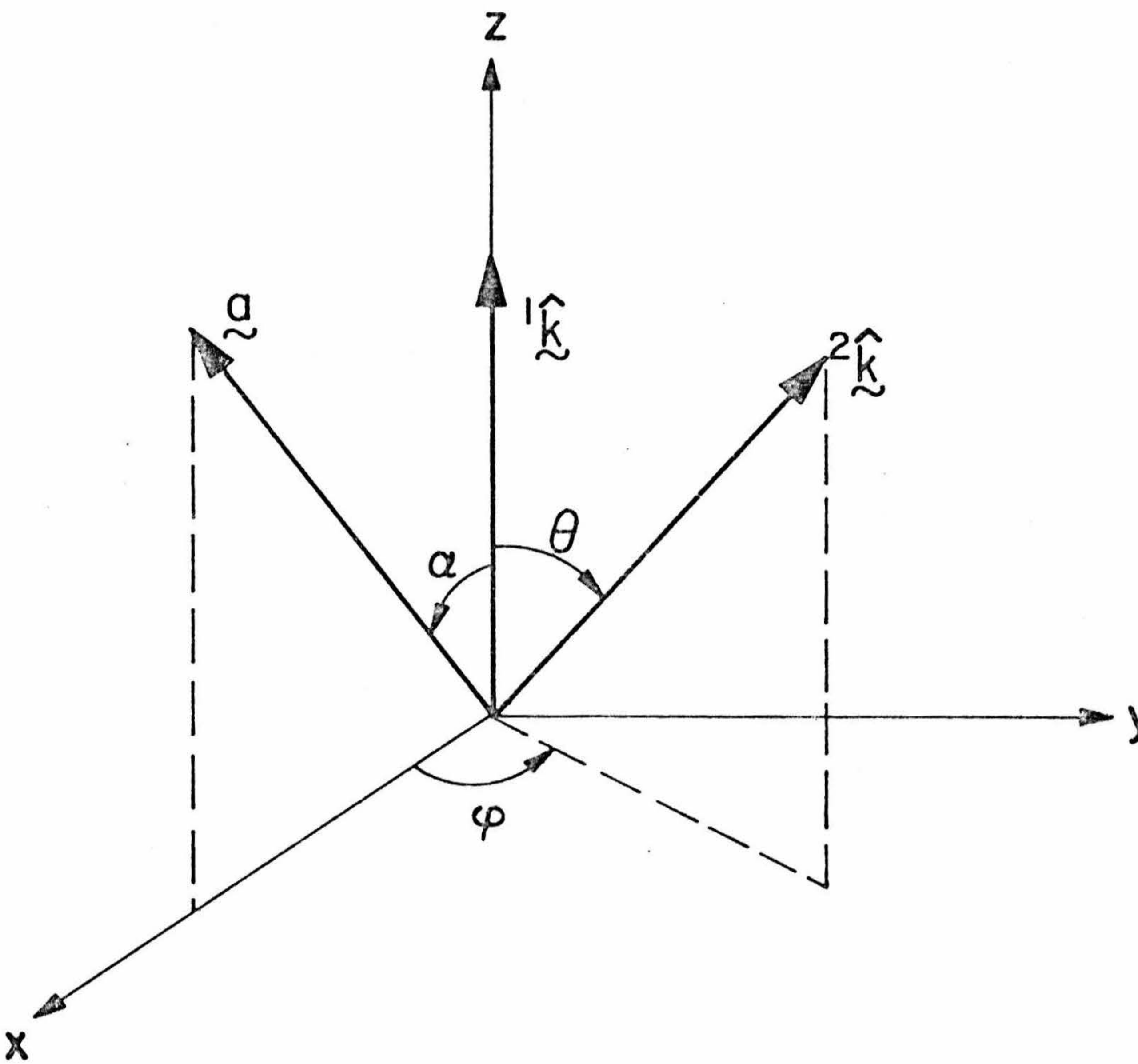

Fig. 2