Radiation Reaction in Binary Systems in General Relativity

Thesis by

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0.2 Abstract

This thesis is concerned with current problems in, and historical aspects of, the problem of radiation reaction in stellar binary systems in general relativity. Part I addresses current issues in the orbital evolution due to gravitational radiation damping of compact binaries. A particular focus is on the inspiral of small bodies orbiting large black holes, employing a perturbation formalism. In addition, the merger, at the end of the insprial, of comparable mass compact binaries, such as neutron star binaries is also discussed. The emphasis of Part I is on providing detailed descriptions of sources and signals with a view to optimising signal analysis in gravitational wave detectors, whether ground- or space-based interferometers, or resonant mass detectors.

Part II of the thesis examines the historical controversies surrounding the problem of gravitational waves, and gravitational radiation damping in stellar binaries. In particular, it focuses on debates in the mid 20th-century on whether binary star systems would really exhibit this type of damping and emit gravitational waves, and on the "quadrupole formula controversy" of the 1970s and 1980s, on the question whether the standard formular describing energy loss due to emission of gravitational waves was correctly derived for such systems. The study sheds light on the role of analogy in science, especially where its use is controversial, on the importance of style in physics and on the problem of identity in science, as the use of history as a rhetorical device in controversial debate is examined. The concept of the *Theoretician's Regress* is introduced to explain the difficulty encountered by relativists in closing debate in this controversy, which persisted in one form or another for several decades.

Part I: Gravitational Waves from Coalescing Binary Systems in General Relativity

Contents

Chapter 1

Introduction

1.1 Nature of the Thesis

This Ph.D. thesis is cross-disciplinary. It contains two conceptually interrelated parts, one in the field of physics (Part I), the other in the history of science (Part II).

The physics portion deals with the influence of gravitational radiation reaction on the orbits of inspiraling and colliding compact binary systems (binaries made of black holes and neutron stars). It also deals with the gravitational waveforms that such binaries emit, and information that can be extracted from those waveforms. This research is a foundation for planned observations by the LIGO/VIRGO network of ground-based gravitational-wave detectors, and the planned low-frequency, space-based LISA gravitational-wave detector.

The history of science portion is a historical study of a decades-long controversy that has surrounded the issue of gravitational radiation reaction in general relativity. In essence, this is a history of the antecedents of the physics research in Part I.

The California Institute of Technology offers a Ph.D. in physics, but none in the history of science; therefore, this thesis is the basis for the author's physics Ph.D., with a minor in the Science, Ethics and Society program. Nevertheless, the author and his thesis committee - which contains both historians and physicists - regard the work accomplished in this thesis as satisfying the customary requirements for a joint Ph.D. in physics and the history of science.

1.2 Motivation

Construction is currently underway on a new generation of sophisticated and sensitive gravitational wave detectors, including the American LIGO [1], the French-Italian VIRGO [2], and others are planned, both of interferometric and resonant mass designs. One senses a real prospect that gravitational waves will be detected by several instruments in the early years of the next decade. If useful information is to be extracted from the observed signals concerning the source systems from which they originated, a great deal of theoretical input will be required. Much work remains to be done so that the theoretical information available will be sufficiently detailed when detections do occur to make optimal use of the observational data right away. While the first largely undisputed observational evidence of the existence of gravitational waves, from the orbital decay of the binary pulsar PSR1913+16 [3], caught theorists somewhat by surprise, there will be no excuse in this instance if they are not fully prepared.

Part I of this thesis presents several contributions to the theoretical understanding of one class of potential sources for gravitational wave detectors, binary systems consisting of neutron stars and black holes. Such binaries are often referred to as *compact binaries*. Part I is principally concerned with the evolution of the orbits of such systems due to *radiation reaction*, that is the inspiral of the orbit due to loss of orbital energy and angular momentum to the emission of the gravitational waves. This inspiral, which causes the orbital motion to become faster and faster as the two bodies approach each other, results in a monotonic increase in the frequency and amplitude of the waves emitted. Finally the two bodies slam into each other, merging quickly and violently. This gives the search from such binaries a distinctive "chirp" pattern, which it is hoped will aid considerably in extracting such signals from detector noise. However, this will only be the case if one knows beforehand exactly how the waveform will evolve, since otherwise one's signal filters will quickly fall out of phase with the rapidly oscillating signal [4]. The radiation reaction problem will thus be a critical one for the detection of signals from such sources.

Since the evolution of the inspiral chirp depends fairly critically on the masses of the binary components, a successful signal extraction should allow one to accurately estimate the masses of the bodies (especially the product $\mu^{3/5}M^{2/5}$, where μ is the binary's reduced mass and M its total mass, known as the "chirp mass"). However, information about the size and internal composition of the two bodies will only become significant in the waveform from the merger at the end of the inspiral. This is the least well understood part of the evolution, and presents technical difficulties to detection. One chapter of Part I of this thesis is devoted to a scheme to learn something about this part of the inspiral by proxy.

Part II of the thesis is devoted to a historical study of gravitational waves, focusing on the problem of radiation reaction in binary systems. This was long a controversial topic in general relativity, with some prominent researchers at one time expressing the view that such systems would not emit gravitational Waves. How the situation has evolved to a stage where very large scale experimental efforts to detect such signals are underway, with strong theoretical backing, is one of the principal questions asked in that study.

1.3 Estimation of neutron-star equation of state data by gravitational wave detectors [Overview of Chapter 2]

A gravitational wave with a *memory* refers to one which, on passing through a system of particles and moving them relative to each other, leaves them with different relative separations after its passage than they had initially. It was thought at one time that only waves emitted by sources, some of whose constituent elements were gravitationally unbound either before or after the emission, had memories. The Christodoulou memory, however, is a non-linear portion of the gravitational wave emitted by all sources, including bound systems, such as binary stars [5]. The Christodoulou memory can be viewed as the variable gravitational field produced at the detector *by the energy flux of gravitational radiation emerging from the source itself* [5, 6, 7]. The energy flux can be thought of as the unbound portion of the system whose departure results in a permanent change in the source's Coulomb gravitational field.

The Christodoulou memory is thus, in some sense, the wave of a wave, or the gravitational field of a wave, and as such is a decidedly non-linear effect. It thus might seem surpising that in the wave from a coalesing binary system the memory may be as much as a tenth the size of the amplitude of the primary wave. This raises the hope that this interesting effect might be detectable by proposed gravitational wave detectors such as LIGO. Although the memory is a DC effect, Braginsky and Thorne [8] have shown how, by integrating on a timescale reflecting the most sensitive frequency of the detector, one can optimise the memory signal during the period of the passage of the primary waves past the detector, while the memory is building up. For neutron star and black hole binaries observable by LIGO, the greatest quantity of gravitational radiation emitted is in the last hundredth of a second before the two bodies merge. Since LIGO's optimum frequency is near 100 Hz, it is thus well suited to measuring this memory, in frequency terms.

The possibility of detecting the Christodoulou memory may be of particular interest, since the primary wave during this last burst before merger is too high in frequency for detection by LIGO in, for instance, binaries containing neutron stars. This is unfortunate, since although the inspiral waveform, when detected by LIGO or similar intruments, should permit excellent estimates of the masses of the binary components, only the coalesence waveform will contain much information about the size of composition of the components. At present there are large uncertainties in our knowledge of the size and the equation of state of neutron stars. Since the Christodoulou memory continues to grow until the two bodies hit each other and merge together to form an axisymmetric body, it seems clear that the size of the memory would depend quite strongly on the size of the binary components. Thus, detection of the memory from coalescing binaries containing neutron stars would be one possible way in which gravitational wave detectors of the LIGO type could be employed to estimate neutron star radii.

Such a method of looking for evidence of neutron star equations of state is discussed in Chapter 2, and the prospects are found, in the end, to be unpromising. It seems unlikely that LIGO will be capable of detecting the Christodoulou memory at all, based on current astrophysical estimates of compact binary merger frequencies, except by a serendipitous event. Space based detectors, such as the proposed LISA [9], may be able to detect the memory from mergers of systems involving super-massive black holes, due to their sensitivity to lower frequencies than ground-based detectors, which allows them to integrate over the more gradual growth of the memory from such large systems. Such measurements may teach us much about black hole physics, but little about neutron-star radii.

At present, the most promising method of estimating equation of state data from neutron star binaries using gravitational wave detectors involves the construction of a xylophone of specially tuned narrow band detectors. This arrangement could consist of either large spherical resonant mass detectors [10] or special dual-recycled interferometers contained in the LIGO housing [11]), staggered in frequency across the kilohertz range which the coalescence waveform is expected to traverse. For such detectors, the power spectrum of the coalescence waveform (its energy per unit frequency as a function of frequency) would be the principal quantity of measurement. Therefore theoretical estimates of this function for merging binaries are badly needed in order to plan detection and signal analysis strategies. The best models of such mergers at present are smooth particle hydrodynamic simulations involving only Newtonian gravity with radiation reaction included by applying quadrupole formula energy losses to the system [12, 13, 14, 15]. Estimates of the power spectrum based on such models are promising, in that they show that the power drops sharply above a certain frequency enabling an unambiguous estimate of the cut-off point on the basis of yes-no responses from the xylophone of detectors [14, 16] (an additional advantage for these detectors would be that the broadband LIGO detector would presumably see the inspiral waveform, indicating the presence of the coalesence waveform at the higher frequency).

However, these models, whose primary motivation has not been the estimation of gravitational wave effects hitherto, do not take into account the strong relativistic gravity of the coalescence phase of the inspiral. Post-Newtonian corrections to the spectra indicate that the actual spectra, while still potentially providing highly useful information, will be much more complicated than the Newtonian estimates would suggest [16]. Efforts have recently been made to simulate actual relativistic effects of neutron star mergers [17], with unexpected results [18], but it is widely suspected that the simpifications employed in this simulation compromise the results [19, 20]. It remains for other groups, such as the Centrella group, to follow with new relativistic simulations for comparison. At present, it is still too early to make a definitive statement about the prospects for gravitational wave detectors providing strong evidence on neutron star equations of state.

1.4 Radiation reaction effects in extreme-mass-ratio binaries [Overview of Chapters 3-5]

In order to extract information about the nature of signals and sources from observations by LIGO-type detectors, sophisticated filtering techniques, designed to increase the signal-to-noise ratio in the output, will be required. In the case of coalescing compact binaries, this will require accurate theoretical templates of the expected signal [4]. The monotonic increase in frequency and amplitude of the signal from these sources (the "chirp") depends crucially on the back reaction effect of the wave emission on the source, which governs the inspiral (the orbital decay due to the loss of energy and angular momentum to the waves). For this reason, it may be important in some cases, to calculate not only the energy and angular momentum lost "to infinity" by the source, but also, for instance, the same quantities which may be carried to the horizon of one or more black holes in the source system itself. For this reason, the flux of energy carried down the hole is calculated in chapters 3 and 4 dealing with gravitational waves produced by a (relatively) small particle orbiting a central black hole.

While it was long expected that binary star systems would undergo orbital decay if they lost energy to the emission of gravitational waves (see part II for a historical discussion of doubts on this score), the first papers to make a particular study of orbital evolution under radiation damping were those of Peters and Mathews [21] and Peters [22] in the mid-sixties. The latter paper showed, in the Newtonian limit of weak gravitational fields and small velocities (which is to say, large orbital radii), that circular Keplerian orbits would remain circular as they decreased in radius due to the damping, and furthermore, that non-circular, eccentric orbits would tend to become more circular (less eccentric) under the influence of the back reaction effect. Such an orbital circularization is known in other dissipation contexts, such as satellites falling to earth as the result of drag in the upper atmosphere. The dissipating effect has a tendancy to drive the orbit down in the effective gravitational potential, towards the potential minimum which defines circular orbits.

For this reason, one might have naively expected that the radiation-damping-induced circularization demonstrated by Peters would be exhibited also in the strong field fast motion regime of small orbital radii. This is a difficult issue to test, because no exact solution to the two-body problem of orbital mechanics is known in general relativity, the theory which is thought to govern such systems. However, well known solutions to the static or stationary gravitational field produced by a single massive body exist in general relativity, the Schwarzschild (non-rotating) and Kerr (rotating) solutions. These solutions represent, in the point-mass limit, black holes. This permits, in principle, the use of perturbation theory to describe the field of a relatively small body orbiting around a central black hole. Since one expects such a perturbed field to contain gravitational waves far from the source, such a method can accurately describe the flux of energy and angular momentum carried from the binary system, without the slow-motion or weak gravity approximations associated with equal mass binary calculations.

Perturbation formalisms, such as that of Teukolsky [23], permit the investigation of a conjecture due to Amos Ori, that the Peters' effect of decreasing eccentricity for radiation damped orbits does not entirely hold in the strong field region. Ori's argument was that, as the particle approached the *innermost stable circular orbit* (ISCO), after which, for highly relativistic gravity, an orbiting particle loses all dynamical stability and plunges into the central body, the alteration in the shape of the effective potential (in which the minimum defining stable circular orbits would be about to dissapear), would lead to orbital eccentricity increasing rather than decreasing. This could be explained in the following terms. Although the particle would continue to move towards the potential minimum, and therefore decrease in eccentricity (due the narrowing of the potential walls at the bottom), the potential itself would be broadening and shallowing as the ISCO approached, and the minimum prepared to turn into a saddle point. This broadening effect would at some point overcome the particle's dropping towards the minimum, and cause its eccentricity to increase. In order to examine the details of this conjecture, it would be necessary to solve the perturbation equations for a Schwarzschild black hole, and find, if it existed, the actual point (the *critical radius*), at which the change in behaviour occured.

The results presented in chapter 3, show that such a critical point does exist in the Schwarzschild case, quite close to the ISCO ($r = 6M$ in Schwarzschild), at $r = 6.6792M$. Down to this radius r_c slightly eccentric orbits continue to become more circular, but as the insprial continues beyond this point, they begin becoming more eccentric. The results of chapter 4, dealing with slightly eccentric equatorial orbits in Kerr, appear to conclusively confirm the relationship between critical radius and ISCO. The position of the ISCO for orbits around a rotating black hole depends greatly on the sense of rotation of the orbit relative to the black hole spin, and on the rapidity of the black hole's spin. For all cases, except one, the critical point at which orbits start becoming less circular occurs within $1M$ or less of the ISCO. The exceptional case is for an extreme Kerr black hole (which is rotating as fast as the theory permits), with a particle in prograde orbit. In addition, the results of chapter 4 show that for prograde orbits, the critical radius draws closer and closer to the ISCO in terms of the Boyer-Lindquist radial coordinate as the black hole's rate of spin increases.

The role played by energy and angular momentum lost to the black hole in the calculations of chapter 3 and 4 should briefly be mentioned. In the Schwarzschild case, the effect is never of any great importance, being entirely negligible for large radii, and constituting on the order of 1% of the evolutionary effects near the ISCO.

For prograde orbits in the Kerr case, the orbit can continue until much closer to the central body, and not surprisingly, the contributions due to the waves interacting with the black hole are more significant. What one finds in this case is that the particle actually absorbs more energy from outgoing waves heading away from the black hole then it loses to the waves headed down the black hole. This phenomenon is known as *superradiance* [24]. The contributions for energy lost down the hole are not peaked to the same harmonics as those from waves sent to infinity. Therefore it turns out that for certain (less-significant) frequencies or harmonics of emitted radiation, and for a particle in a prograde orbit, close in ($r \sim 2M$) to a black hole with large spin (a/M close to 1, where a is the spin) the particle actually gains orbital energy and angular momentum. If one sums over all the frequencies however, it continues to lose energy and to decay in its orbit (that is decrease its orbital radius).

Two obvious ways in which this work for slightly eccentric orbits can be generalized are to the case of generally eccentric orbits around a non-rotating black hole, and to the case of general, non-equatorial orbits in Kerr (not confined to the equatorial plane of the spinning black hole). In chapter 5 results for the former case are presented. Again, it is shown that orbits tend to lose eccentricity up until a point shortly before the onset of instability (which is defined for eccentric orbits by orbits with a *semi-latus rectum* of $p = 6 + 2e$, where e is the eccentricity). For orbits with an eccentricity e approaching 1 (the limit at which the orbit becomes unbound), the critical semi-latus rectum p (at which eccentricity begins increasing) approaches arbitrarily closely the value p at which dynamical orbital instability sets in (the equivalent of the ISCO for non-circular orbits).

The case of non-equatorial orbits in Kerr is one for which difficulties are encountered in the formalism employed in chapters 3, 4 and 5. The constants of the motion represented by orbital energy and angular momentum are, with the mass of the particle, sufficient to describe equatorial orbits uniquely. In the non-equatorial Kerr case, another constant, known as the *Carter constant* is encountered. While mathematically this constant of the motion is on the same footing as the others already discussed (it reflects a Killing-tensor symmetry in the underlying Kerr geometry), it does not have a clear Newtonian analogue, which might have aided one in equating a quantity in the far field flux with a quantity describing the orbital motion. Recently, some progress and promising methods have been put forward to deal with this problem [25, 26, 27] . In the meantime, in chapter 6 is presented an argument which avoids discussing the details of the radiation reaction force on the particle and thus the issue of how to describe the alteration in its orbit due to a changing Carter constant caused by wave emission. It does this by showing that the symmetries of circular orbits around a Kerr black hole (defined as an orbit of constant Boyer-Linquist radius, sometimes referred to as a "quasi-circular" orbit) ensure that such an orbit remains circular under radiation reaction. This extends this long standing result to the general Kerr case in a particularly useful way, since we can expect most black holes to rotate, and many or most bodies orbiting them to have non-equatorial orbits.

The results discussed in chapters 3,4,5 and 6 may have some application to LIGO and other ground-based detectors, in the case of binaries with small mass ratios, but not so small that their gravitational wave frequencies never enter the LIGO bandwidth. In addition the results from such calculations are important since they are not limited to slow velocities and weak fields for the systems they describe, unlike the post-Newtonian approximation schemes typically used in the case of comparable mass binaries. It is still not known for sure how much accuracy will be lost in signal extraction by the use of post-Newtonian templates in signal extraction in LIGO, assuming, as seems possible, that exact numerical solutions for equal mass systems will be unavailable when LIGO goes online. By applying these post-Newtonian estimates to the extreme mass ratio limit, and comparing them with exact strong field calculations such as those presented here, it is possible to make useful estimates of the likely loss in signal-to-noise ratio involved in employing post-Newtonian signal templates[28].

Because of their sensitivity to low frequencies, space-based detectors such as LISA should be able to detect gravitational waves from solar massed size objects spiralling into supermassive black holes of 10^6 to 10^9 solar masses. Such systems would be well modelled by these results. Assuming supermassive black holes are sufficiently common in the universe, which is thought quite likely, and assuming that a space-based detector is launched early in the next century, as is hoped by its proponents, the type of perturbation analysis used in this thesis may play an important role in signal analysis.

1.5 Overview of Part II

Part II of this thesis is concerned with a historical study of the radiation reaction problem in general relativity. Chapter 1 presents a discussion of the central historical issues raised by the study, such as the role of analogy in general relativity, the culture and practice of theory, and the concept of the *theoretician's regress.* Chapter 2 deals with the prehistory of the gravitational radiation reaction problem, before the birth of the general theory of relativity. Chapters 3-5 deal with the development of gravitational wave theory from the origins of general relativity to the second world war, focusing in chapter 5 on Einstein's abortive attempt to disprove the existence of gravitational waves in 1936.

Chapters 6-8 deal with the post-war controversy over whether gravitational radiation damping existed for binary star systems, focusing on the sceptics who felt that such systems did not emit gravitational radiation. Chapter 9 deals briefly with post-war sources of funding for work in this field. Chapters 10-12 deal with the period of the 1960s, when general relativity began to emerge into the mainstream of theoretical physics, and great strides were made in the understanding of gravitational waves.

Chapters 13-16 discuss the quadrupole formula controversy of the 1970s and 1980s, including the further theoretical developments, the impact of experimental evidence, and the role of history in the debates over the validity of the quadrupole formula for radiation reaction. Chapters 17-20 discuss various issues arising out of the study, from the role of style in physics, to the place of relativity in contemporary physics, to the technical issues confronting researchers in the back reaction problem in general relativity.

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Chapter 2

Prospects for detecting the Christodoulou memory of gravitational waves from a coalescing compact binary

Abstract

A coalescing compact binary, during its last tenth of a second of life, emits a burst of gravitational waves consisting of a high-frequency "chirp", with frequencies much greater than 100 Hz, superimposed on a gradually growing memory, known as the Christodoulou memory. Most of the memory's growth occurs over the last few hundredths of a second, so its signal has strong Fourier components at $f \sim 100$ Hz. The planned LIGO/VIRGO broadband gravitational-wave detectors have optimal performance at frequencies around 100 Hz and should be well suited, in terms of

frequencies, to detect the growth of the memory amidst the chirp. If one or both of the binary's components is a neutron star (the other being either a neutron star or a black hole), then the growth of the memory will be cut off by the star's tidal disruption. The larger the neutron star's radius the sooner the cutoff and correspondingly the weaker the total memory. Therefore, from a LIGO/VIRGO measurement of the memory's strength, one could hope to infer the neutron-star radius. The prospects for such measurements to succeed are evaluated quantitatively and found to be poor because of the weakness of the memory. Even under optimistic circumstances the memory is so weak that only for a black-hole/black-hole binary is there much chance of detecting it, and then the prospects are only marginal.

2.1 Introduction

The continuing efforts to improve the sensitivity of gravitational-wave detectors, and the commencement of ambitious programs to build the kilometer-scale LIGO [1] and VIRGO [2] network of detectors, have encouraged attempts to predict the behaviour of potential sources and types of gravitational waves. One important type are gravitational waves with memory. These are waves which leave a system of free masses with a permanent relative displacement following their passage. Braginsky and Thorne [3] have discussed the optimal experimental strategy for detecting memories, and have estimated the sensitivities of a variety of detectors to waves with memory.

Until recently it was thought that waves with memory are restricted to sources whose constituent components are not gravitationally bound to each other, either initially or finally, or both. However, in 1991 Christodoulou [4] showed that strongly radiating bound systems, such as coalescing compact binary stars, have a significant memory created by nonlinearities in the wave-emission mechanism. This Christodoulou memory is actually the tranverse-traceless part of the gravitational field produced by the stress-energy tensor of the gravitational wave itself. [4, 5, 6]. Because, for a coalescing binary, it can be as large as a tenth the size of the primary wave, the Christodoulou memory may be detectable by the planned network of gravitational-wave detectors, including LIGO [1] and VIRGO [2].

Coalescing compact binaries (whether NS/NS, NS/BH or BH/BH, where NS and BH mean neutron star and black hole) are expected to be among the strongest sources of gravitational waves for LIGO/VIRGO. Amongst the most interesting information that the experimenters might hope to extract from these binaries' waves is the neutron star mass-radius relation, since from it one can deduce the equation of state of matter at densities from nuclear to about ten times nuclear [7], which is little understood at present. Unfortunately, although waves from such a binary, in the frequency band of good expected LIGO/VIRGO performance (roughly 10 to 300 Hz), depend strongly on the binary's masses and might therefore allow fairly accurate mass measurements [8, 9, 10], they are insensitive to the radii of the binary's constituents. Strong dependance on the radii occurs only near the end of the inspiral, when the two objects are interacting tidally, merging and/or disrupting, and the frequencies of the primary waves are around a kilohertz. At these high frequencies the LIGO/VIRGO detectors will have relatively poor performance because of serious photon shot noise. In view of this, two different methods have been suggested for determining the radii [8]. One involves measurements of the kilohertz primary waves using specially configured narrow-band detectors. The other uses measurements of the primary waves' Christodoulou memory, detected by the same LIGO/VIRGO broad-band detectors which will attempt the mass measurements. This paper and a companion one will evaluate these two methods, the memory method here and the narrow-band detector method in the companion paper [11]. As we shall see, the memory method is not very promising. In contrast, the narrow-band approach shows considerable promise.

The method discussed here relies on the following properties of the memory. The memory grows most strongly on timescales of the order of a hundredth of a second, and therefore has its strongest Fourier components around the hundred Hertz region where the broadband detectors perform best. The primary waves and the growth of the memory are both cut off when the binary's neutron star or stars are tidally disrupted. The larger the radius of the neutron star, the sooner this occurs. As a result, the strength of the memory, and therefore the strength of the optimally filtered signal in the detectors, is quite sensitive to the neutron-star radius. Specifically, the memory's strength is of the order of (distance to earth)$^{-1}\times$ (the energy carried off by the primary wave burst). The energy carried off is of the order of the binary's gravitational binding energy at tidal disruption, that is $\sim \mu M/2R$, where μ and M are the binary's reduced and total masses and R is the neutron-star radius. Therefore, the memory behaves like $h \propto 1/R$. Unfortunately, as we will see in section IV, for the case of binaries containing a neutron star, the memory is likely to be too weak for either detection or measurement, even when fairly optimistic assumptions are made concerning event rate and detector sensitivities. For some two-black-hole binaries, however, the memory might be just detectable.

The paper is organized as follows. In section II, I sketch a derivation, based on the quadropole moment formalism, of the time evolution of the wave's memory. The final formula for the memory, Eq. (2.5), is somewhat innaccurate, because of post-Newtonian and higher-order relativistic effects. The magnitude and sign of the errors are discussed at the end of section IV, and are seen not to change any of the paper's conclusions. In section III, I discuss the method of optimal signal processing that would be used, with a broadband LIGO/VIRGO detector, to search for the memory and measure its size. I also write down the formulas for the memory's signal-to-noise ratio and for the detector noise spectrum (assuming an "advanced" LIGO detector [1]) which goes into the signal-to-noise formula. In section IV, I describe two different calculations of the signal-to-noise ratio, which I have carried out, one based largely on numerical integrations, the other on analytical approximations. I give analytical formulae and a graph from which one can infer the S/N for any desired binary. In section V I apply my results to specific examples of NS/NS, NS/BH and BH/BH binaries. In section VI I discuss the implications of my results. Throughout the paper I use units in which Newton's constant of gravitation and the speed of light are unity, i.e. $G = c = 1$.

2.2 The form of the Christodoulou memory

Thorne gives an expression for the net Christodoulou memory, when it has ceased growing, in terms of the total energy per unit solid angle, $dE/d\Omega'$, carried off by the primary waves [5]. Since the memory, at any moment of retarded time during its growth, is produced by the stress-energy of all the waves emitted up until then, one can obtain an expression for the time-evolving memory $h(t)$ by replacing $dE/d\Omega'$ in

Thorne's formula with $\int_{-\infty}^{t}(d^2E/d\Omega' dt')dt'$. The result is

$$h(t) = \frac{2}{r}\int_{-\infty}^{t}\int \frac{d^2E}{dt'd\Omega'}(1+\cos\theta')e^{2i\phi'}d\Omega' dt'. \tag{2.1}$$

The mathematical and geometric conventions are as follows: $h(t)$ is a complex gravitational-wave field at Earth, equal to $h_+ + ih_\times$, with the transverse $+$ axes chosen arbitrarily and the $\times$ axes at 45° to it; the direction from source to Earth is the z'-axis (the polar axis) and the $+$ axes are taken to be the x' and y' axes; θ' and ϕ' are the polar co-ordinates that correspond to this source-based Cartesian system ($x' = r\sin\theta'\cos\phi'$, $y' = r\sin\theta'\sin\phi'$, $z' = r\cos\theta'$); the angular integral is over the direction, in terms of θ' and ϕ', of emission of the primary waves and r is the distance from source to Earth. The power radiated by the binary into unit solid angle has been evaluated by many researchers, for instance Peters and Mathews [12], using the quadropole moment formalism. It is, after averaging over one complete orbit,

$$\frac{d^2E}{d\Omega' dt} = \frac{1}{2\pi}\frac{\mu^2M^3}{a^5}(1+6\cos^2\theta+\cos^4\theta), \tag{2.2}$$

where a, the orbital radius, shrinks due to radiation reaction in a manner given by

$$a = \left(\frac{256}{5}\mu M^2 t\right)^{\frac{1}{4}} \tag{2.3}$$

where t is the time until final coalescence, assuming the system consists of two idealised point masses. In these equations, $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass and $M = m_1 + m_2$ is the total mass of the system and θ (not to be confused with θ') is the angle that the primary direction of emission (θ',ϕ') makes with the binary's rotation axis.

To simplify the evaluation of the angular integral in Eq. (2.1), orient the x',y' axes so that the binary's rotation axis lie in the x'-z' plane, and denote by ι the angle between the rotation axis and the direction to earth (the z' direction). Then

$$\cos\theta = \sin\iota\sin\theta'\cos\phi' + \cos\iota\cos\theta'. \tag{2.4}$$

Inserting Eqs. (2.2), (2.3) and (2.4) into Eq. (2.1) and evaluating the angular integral, one obtains the following expression for the memory.

$$h(t) = \frac{3}{32}\Big(\frac{5\mu^3 M^2}{r^4 t}\Big)^{1/4}\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota \tag{2.5}$$

This expression agrees, except for a factor of two, with the similar equation derived by Wiseman and Will [6].

Note that $h(t) \equiv h_+ + ih_\times$ is real, so the simplifying choice of orientation for the observer's x',y' axes has made $h_\times$ vanish and left $h(t) = h_+(t)$. Note also that the memory increases as the time to coalescence t decreases. The objects involved are not point masses, however. At some point, say at time t_k, tidal forces disrupt them, they begin to merge, and the abrupt reduction of energy emitted in the burst, which is likely to occur within about one orbital period [13, 14, 15], causes the memory to stop growing. Thereafter, it retains the amplitude it had at $t = t_k$.

The waveform in Eq. (2.5), with its growth terminated at $t = t_k$, is plotted in Fig. 2.1. Because $h_\times = 0$, the signal sensed by the detector is $h_d = F_+ h_+$, where $F_+ \leq 1$ is the detector's quadropolar antenna beam pattern function, which depends on the orientation of the detector to the incoming wave (Eq. (104a) and Fig. 9.9 of Ref[16]). If the detector happens to be so oriented as to maximise h_d then $F_+ = 1$ and therefore $h_d = h(t)$.

For comparison with the time evolution $h(t)$ of the memory, Eq. (2.5), it will be important to know the time evolution of the frequency, f_p of the primary waves, which is twice the orbital frequency [16]. Therefore

$$\begin{aligned} f_p &= \frac{1}{\pi M}\Big(\frac{5M}{256\mu}\Big)^{3/8}\Big(\frac{M}{t}\Big)^{3/8} \\ &= 3.387\times 10^3\mathrm{Hz}\Big(\frac{M}{4\mu}\Big)^{3/8}\Big(\frac{M_\odot}{M}\Big)^{5/8}\Big(\frac{1\mathrm{ms}}{t}\Big)^{3/8}. \end{aligned} \tag{2.6}$$

Recall that for a binary, $M/4\mu \geq 1$.

2.3 Analysing the signal in the detector

In order to estimate the ability of a detector, such as LIGO, to detect the memory, it is necessary to calculate the signal-to-noise ratio of the signal in the detector. An experimenter, knowing the detector's noise spectrum, constructs a filter which is designed to let through the signal, while blocking out as much of the noise as possible. The Wiener optimal filter for a signal $h(t)$, seen in a detector with a one-sided noise spectrum $S_h(f)$, is a function $k(t)$, whose Fourier transform is related to the Fourier transform of the signal by

$$\tilde{k}(f) = \frac{\tilde{h}(f)}{S_h(f)}. \tag{2.7}$$

The filter is therefore a function similar to the signal, except that those frequencies which are noisy in the detector are suppressed. This is illustrated by the numerically derived graph of the filter function, $k(t)$, in Fig. 2.2. The signal-to-noise ratio after optimal filtering, and taking account of the detector's beam pattern, is given by [17]

$$\Big(\frac{S}{N}\Big)^2 = 2F_+^2\int_{-\infty}^{\infty} k(t)h(t)dt \tag{2.8}$$

or

$$\Big(\frac{S}{N}\Big)^2 = 4F_+^2\int_0^{\infty} \frac{|\tilde{h}(f)|^2}{S_h(f)}df. \tag{2.9}$$

In evaluating S/N for the Christodoulou memory, I have used the noise spectrum projected for the "advanced" detectors in LIGO, which may be in operation by the middle of the next decade. I have approximated their noise spectrum in the following way.

$$S_h(f) = \begin{cases} (h_m^2/f_m)(f/f_m)^2, & f \geq f_m \\ (h_m^2/f_m)(f/f_m)^{-4}, & f < f_m \end{cases}, \tag{2.10}$$

where $h_m = 1.0 \times 10^{-23}$ and $f_m = 70\text{Hz}$ [1].

This approximation to $S_h(f)$ ignores a seismic cutoff (in which S_h rises very rapidly below 10Hz) because it turns out to have a negligible influence on the signal-to-noise ratio. (Without the cutoff, $S_h \propto f^{-2}$ and $|\tilde{h}|^2 \propto f^{-4}$ at small f [Eq. (2.15) below], so the cutoff produces a correction $\sim (10\text{Hz}/70\text{Hz})^3$, which is much less than one per cent.) Between the seismic cutoff and 70Hz (the optimal frequency) the noise is principally due to thermal noise in the test mass suspension, and above 70Hz to photon shot noise in the interferometer beam [1]. The frequency of the primary gravitational waves from the coalescing binary, which is approaching the kilohertz range in the last 1/10 second of the inspiral, will fall outside of the LIGO detection window at that time. Although the Christodoulou memory is a DC signal, and therefore unobservable because of the low frequency noise, its growth is detectable by observing it over the last $\sim 1/10$s of the burst. At the very end of the signal the rate of change is quick enough, so that a great part of the signal can be seen on the timescale of LIGO's optimal frequency, which is an important advantage for detection, as pointed out by Braginsky and Thorne [3]. To the observers, the primary wave burst is a precursor to the memory it generates, because the two are detectable by them at different times.

Another type of detector which may go into operation in the next century is a space based interferometer, such as the recently proposed LISA [18]. In the absence of seismic noise, such detectors would be much more sensitive to low frequency waves, and could therefore see more of the growth of the memory than any earth-based detector. Its noise spectrum can be modeled in the following way [18]:

$$S_h(f) = \begin{cases} (h_m^2/f_1)(f/f_2), & f \geq f_2 \\ (h_m^2/f_1), & f_1 \leq f < f_2 \\ (h_m^2/f_1)(f/f_1)^{-2}, & f < f_1 \end{cases}, \tag{2.11}$$

where $h_m = 3 \times 10^{-23}$, $f_1 = 10^{-3}$ Hz and $f_2 = 10^{-1}$ Hz. Up to f_1 the noise is due to residual effects which perturb the spacecraft's inertial motion. Above f_1 it is due to photon shot noise, and the disimprovement beyond f_2 occurs because of the waves becoming shorter than the interferometer arm.

The binary coalescences which the LIGO/VIRGO detectors will be searching for are thought to be very rare: a few per year to a few per day at the strengths detectable by these instruments. In order to be certain, with 99% confidence, that an observed primary-wave signal is not due to the detector's internal Gaussian noise, one must require that a signal-to-noise ratio of 6 or better be registered in two independant detectors simultaneously [8]. However, once the primary waves have been discovered, one can predict from them, to within an accuracy of about 0.001s or less, the time at which the memory will register most strongly in the detector. With this knowledge, there are no free parameters to be solved for in the memory measurement, and the detector's optimally filtered output will therefore be a single number. The noise component of this number should be Gaussianly distributed, so the usual Gaussian criteria for detection apply. For measurement by a single detector, an observed signal with $S/N = 1$ is real with 68% confidence, which is increased to 95% confidence if $S/N = 2$ and to $99\frac{3}{4}$% confidence if $S/N = 3$. More to the point, if both LIGO detectors register $S/N = 2$ then one can be $99\frac{3}{4}\%$ confident that the memory was detected. Therefore, in the following section, I shall regard $S/N = 2$ as a reasonable criterion for detectability by LIGO. In the case of a space-based detector, where only one instrument would be in operation, $S/N \geq 3$ would be required, without a coincident detection by another system.

2.4 The signal to noise ratios

I have computed the signal-to-noise ratio S/N for the Christodoulou memory produced by NS/BH, NS/NS and BH/BH binaries, under a variety of assumptions about the time at which tidal disruption or coalescence terminates the primary waves. In order to compute the S/N, one needs to find the Fourier transform of the signal $\tilde{h}(f)$. One method I have used is to calculate this function and the signal to noise itself numerically, using a Fast Fourier Transform algorithm taken from Numerical Recipes [19]. In an idealized case, with the cut off time $t_k \leq 0.1$ms, so that $2\pi f_m t_k$ (the detector's optimal angular frequency times the cutoff time) can be regarded as a small parameter, I have been able to do the whole calculation analytically, in the following way.

Write the signal in the form

$$h(t) = \begin{cases} h_k (t_k/t)^{\frac{1}{4}}, & t \geq t_k \\ h_k, & t \leq t_k \end{cases}, \tag{2.12}$$

where $h_k = h(t_k)$ is given by Eq. (2.5):

$$\begin{aligned} h_k &= \frac{3}{32}\Big(\frac{5\mu^3 M^2}{r^4 t_k}\Big)^{1/4}\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota \\ &= 2.56 \times 10^{-24}\Big(\frac{4\mu}{M}\Big)^{3/4}\Big(\frac{M}{M_\odot}\Big)^{5/4}\frac{1}{x^{1/4}} \end{aligned}$$

$$\times\Big(\frac{200\mathrm{Mpc}}{r}\Big)\Big(1-\frac{\sin^2\iota}{18}\Big)\sin^2\iota. \qquad (2.13)$$

Here $x = 2\pi f_m t_k$, so that t_k is rescaled by LIGO's optimal frequency, f_m. The distance r is measured in units of 200Mpc for reasons that will become evident below. Then one can reexpress Eq. (2.12) as

$$h(t) = h_k(t_k/t)^{\frac{1}{4}}\Theta(t-t_k) + h_k\Theta(t_k-t), \qquad (2.14)$$

where Θ is the step function, $\Theta(\xi) = 1$ for $\xi > 0$ and 0 for $\xi \leq 0$. The Fourier transforms of the step function and the power law $1/t^{1/4}$ are well known (see, for instance, the Bateman papers [20]). Using them and the convolution theorem it is possible to compute the Fourier transform of Eq. (2.14). When the convolution is evaluated as a power series in $2\pi f t_k$, the result is,

$$\begin{aligned}\tilde{h}(f) = -\frac{ih_k}{2\pi f}\Big(1 &+ i\alpha(2\pi f t_k)^{1/4} + i(1-\frac{4}{3}\gamma)2\pi f t_k \\ &- \frac{1}{2}(1-\frac{8}{7}\gamma)(2\pi f t_k)^2 + \ldots\Big)\end{aligned} \qquad (2.15)$$

where $\alpha = \rho + i\sigma$ with $\rho = 0.469$ and $\sigma = 1.13$, and where $\gamma = 6.28$, and terms which are higher order in ft_k are neglected. See the appendix where the derivation of this equation is explained in more detail, and the exact values of α and γ are given.

Using Eqs. (2.9) and (2.15) and the noise spectrum defined in Eq. (2.10), it is straightforward to calculate the signal-to-noise ratio for the memory, seen in an advanced LIGO detector. The result is

$$\begin{aligned}\Big(\frac{S}{N}\Big)^2 &\equiv \frac{2}{3}F_+^2\Big(\frac{h_k}{\pi h_m}\Big)^2\Sigma^2 = \frac{2}{3}F_+^2\Big(\frac{h_k}{\pi h_m}\Big)^2 \\ &\times\Big[1-\frac{288}{143}\sigma x^{1/4} + \frac{36}{35}|\alpha|^2 x^{1/2}\end{aligned}$$

$$+\frac{288}{119}\rho(1-\frac{4}{3}\gamma)x^{5/4}$$
$$-\frac{96}{35}\gamma(1-\frac{7}{6}\gamma)x^2+\ldots\Big]. \tag{2.16}$$

Again, higher order terms are neglected. The coefficients in this power series in $x = 2\pi f_m t_k$ are mostly of order unity, except for the x^2 term. Therefore, for an error of less than 10% to order x^2, $x = 2\pi f_m t_k$ must be ≤ 0.04, which means (since $f_m = 70$Hz), that t_k must be $\leq 10^{-4}$sec. The next large coefficient term is the x^4 term, but at that order high frequency contributions from the discontinuous derivative at $t = t_k$ introduce infinities into the series. So one cannot improve the range of validity of the analytic approximation by simply going to higher order in the expansion, without drastically altering and complicating the calculation. In the numerical case, however, it is straightforward to round off the sharp edge at the cutoff (which is unphysical), and so the numerical results are preferred for $t_k \geq .1$ms. For a comparison of the numerical and analytical results, and a graph of Σ, as defined in Eq. (2.16), see Fig. 2.3.

Since the detector's minimum noise level is $h_m = 10^{-23}$, and since the memory's full strength h_k is given by Eq. (2.13), in the limit $2\pi f_m t_k \ll 1$ the signal-to-noise ratio (2.16) becomes

$$\begin{aligned}\frac{S}{N} &= \Big(\frac{2}{3}\Big)^{1/2}\Big(\frac{h_k}{\pi h_m}\Big)F_+ \\ &= 0.0665F_+\Big(\frac{4\mu}{M}\Big)^{3/4}\Big(\frac{M}{M_\odot}\Big)^{5/4}\Big(\frac{200\mathrm{Mpc}/r}{(2\pi f_m t_k)^{1/4}}\Big) \\ &\quad\times\Big(1-\frac{\sin^2\iota}{18}\Big)\sin^2\iota. \end{aligned}\tag{2.17}$$

This S/N increases monotonically with decreasing cutoff time t_k, as one would expect. The same calculation can be made in the case of the space-based detector, using the noise spectrum in Eq. (2.11). The result is

$$\begin{aligned}\left(\frac{S}{N}\right)^2 &= 2\left(\frac{F_+h_k}{\pi h_m}\right)^2\Big[1-\frac{32}{15}\sigma x_1^{1/4}+\frac{4}{3}|\alpha|^2 x_1^{1/2}\\ &-\frac{32}{9}\rho(1-\frac{4}{3}\gamma)x_1^{5/4}+\frac{32}{63}\gamma(1-\frac{7}{6}\gamma)x_1^2+\ldots\\ &-\frac{f_1}{f_2}\Big(\frac{1}{4}-\frac{16}{21}\sigma x_2^{1/4}+\frac{2}{3}|\alpha|^2 x_2^{1/2}\\ &-\frac{16}{3}\rho(1+\frac{4}{3}\gamma)x_2^{5/4}\\ &-\frac{16}{21}(\ln x_2-1)\gamma(1-\frac{7}{6}\gamma)x_2^2+\ldots\Big)\Big].\end{aligned} \tag{2.18}$$

Here, the expansion is in terms of $x_1 = 2\pi f_1 t_k$ and $x_2 = 2\pi f_2 t_k$.

The cut-off time, t_k, is constrained in three different ways. One is by the total amount of energy radiated in the primary waves, which I shall refer to as $E_{\rm GW}$. The relation is

$$\left(\frac{t_k}{M}\right)^{1/4} = \frac{1}{32}\frac{\mu}{E_{\rm GW}}\left(\frac{5M}{\mu}\right)^{1/4}. \tag{2.19}$$

Since $E_{\rm GW}$ cannot exceed μ and will typically be less, and since μ cannot exceed $M/4$, then

$$\begin{aligned} t_k &= 9.4\times10^{-11}\sec\left(\frac{M}{M_\odot}\right)\left(\frac{\mu}{E_{\rm GW}}\right)^4\frac{M}{4\mu}\\ &\geq 9.4\times10^{-11}\sec\left(\frac{M}{M_\odot}\right).\end{aligned} \tag{2.20}$$

A second constraint on t_k comes from the total number of orbits left until $t=0$, if there were no cutoff, which is

$$N_{\rm orb} = \frac{4}{5}f_p t_k = \frac{4}{5\pi}\left(\frac{5M}{256\mu}\right)^{3/8}\left(\frac{t_k}{M}\right)^{5/8}. \tag{2.21}$$

This number must exceed unity if the analysis is to make any sense, since the quadropole formalism requires averaging over an orbit. This constraint says that

$$t_k = \left(\frac{5\pi}{4}\right)^{8/5}\left(\frac{64}{5}\right)^{3/5}\left(\frac{4\mu}{M}\right)^{3/5}N_{\rm orb}^{8/5}M$$

$$> \; 2 \times 10^{-4}\mathrm{sec}\Big(\frac{4\mu}{M}\Big)^{3/5}\Big(\frac{M}{M_\odot}\Big). \tag{2.22}$$

This constraint (2.22) is more severe than (2.20) for binaries with realistic mass ratios, $4\mu/M > 2.77\times10^{-11}$ (i.e. $m_2/m_1 > 6.93\times10^{-12}$), while (2.20) is more severe in the unrealistic regime of very extreme mass ratios, where $4\mu/M > 2.77 \times 10^{-11}$ (i.e. $m_2/m_1 < 6.93 \times 10^{-12}$). Therefore, in cases of interest to us, (2.22) is always the active constraint.

The third restriction is the actual relation between t_k and the size of the binary's components, which depends on the onset of tidal disruption. If disruption begins as the two bodies first touch, then

$$\begin{aligned} t_k &= \frac{5}{64}\Big(\frac{M}{4\mu}\Big)\Big(\frac{r_1+r_2}{M}\Big)^4 M \\ &= 3.85 \times 10^{-7}\mathrm{sec}\Big(\frac{M}{4\mu}\Big)\Big(\frac{r_1+r_2}{M}\Big)^4\Big(\frac{M}{M_\odot}\Big), \end{aligned} \tag{2.23}$$

where r_1 and r_2 are the radii of the two bodies [12]. In the most extreme conceivable case, where $r_1+r_2 = M$ (so the sum of the bodies' physical radii is equal to the sum of half their Schwarzschild radii, recall that a rapidly spinning hole has a radius equal to half of the Schwarzschild radius), this gives $t_k \geq 3.85 \times 10^{-7}(M/4\mu)(M/M_\odot)$.

In the LIGO/VIRGO frequency band one deals with binaries for which $4\mu \geq 1M_\odot$, $M \leq 300M_\odot$, so (2.22) is generally the active constraint on t_k. For LISA the mass ratio can be much more extreme, so either (2.22) or (2.23) can be the relevant constraint.

2.5 Examples and applications

Narayan, Piran and Shemi [21] and Phinney [22] have estimated a coalescence rate of a few per year for neutron star/black hole binaries at a distance of 200 Mpc from

Earth. To model this type of source I use $m_1 = 10M_\odot$, $m_2 = 1.4M_\odot$. One can derive an estimate of the total energy radiated in the primary wave from the energy of the last stable circular orbit of the neutron star (treated as a point mass) about the Black Hole, which is assumed to be maximally rotating (in order to maximise the S/N). This gives $E_{\rm GW} = 0.4828\mu$ [23], which implies (from Eqs. (2.19) and (2.16)) that the signal to noise would be,

$$\frac{S}{N} = 2.0\Big(\frac{200\mathrm{Mpc}}{r}\Big)F_+\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota. \tag{2.24}$$

The cut-off time in this case is unrealistically small ($t_k \approx 10\mu s$, which is much less than an orbital period), because of the inadequecy of the quadropole approximation. Using my numerical calculation of S/N (see Fig. 2.3) and still taking a deliberately exaggerated case, if the two bodies begin to merge as they touch and the black hole is non-rotating, then the memory should stop growing at $a \approx 40$km (in fact, the last stable circular orbit in this case is at $a \approx 90$km). At this stage the frequency of the primary waves would have reached about 1500 Hz, and $t_k = .00036$s. In this extreme case the signal-to-noise in an advanced LIGO detector would be

$$\frac{S}{N} = 0.45\Big(\frac{200\mathrm{Mpc}}{r}\Big)F_+\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota. \tag{2.25}$$

If the black hole is near maximally rotating, as seems likely for one which is in a binary system, where it would be spun up by infalling debris, then its horizon is at half the Schwarzschild radius. Still assuming a radius of 10 km for the neutron star, merger could commence no later than when they are touching at $a \approx 25$km. The signal-to-noise then amounts to

$$\frac{S}{N} = 1.1\Big(\frac{200\mathrm{Mpc}}{r}\Big)F_+\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota. \tag{2.26}$$

Since $F_+ \leq 1$, and since $r \approx 200$Mpc is actually a rough lower bound for how far one must look to see several NS/BH coalescences per year [21, 22], and since all of these very optimistic examples give S/N of near or less than two, the prospects are rather poor for advanced LIGO/VIRGO detectors to see the Christodoulou memory from NS/BH binaries.

If a sizeable fraction of the main-sequence progenitors of NS/NS binaries actually make such binaries when they die, rather than disrupting during a supernova outburst or dying via some other route, then the NS/NS coalescence rate could be several per year at distances as close as $\sim$ 30Mpc [22, 24]. With $m_1 = m_2 = 1.4 M_\odot$, and following Rasio and Shapiro [13] in supposing a dramatic reduction in power radiated at $a \approx 20$km then

$$\frac{S}{N} = 0.75\Big(\frac{30\text{Mpc}}{r}\Big)F_+\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota. \tag{2.27}$$

Thus, even under these most optimistic assumptions, the prospects for seeing NS/NS memories are dim.

In the case of two $10 M_\odot$ black holes, one can set an upper bound on the energy radiated in the primary wave by demanding that the total surface area of the holes be conserved. For two non-rotating holes this sets $E_{GW} = 0.293M$, where M is the total mass. The memory generated by this much energy in the primary wave would produce a signal to noise in the detector (derived analytically) of

$$\frac{S}{N} = 14.0\Big(\frac{200\text{Mpc}}{r}\Big)F_+\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota. \tag{2.28}$$

The cut-off time in this case, $t_k < 1\mu$s, is even more unrealistic then in the first cited NS/BH case above. In an effort to estimate S/N with a more realistic coalescence time take $t_k = .0006$s, for which the (numerical) result is

$$\frac{S}{N} = 1.3\Big(\frac{200\text{Mpc}}{r}\Big)F_+\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota. \tag{2.29}$$

Since 200Mpc is a lower bound on how far one must look to see several BH/BH coalescences per year [22, 24], the prospects are modestly hopeful (but only modestly) that LIGO/VIRGO might one day detect the memory of a BH/BH coalescence.

As was discussed earlier, space-based interferometers are more sensitive to very low frequency waves, and might be expected, for that reason, to be better able to detect the Christodoulou memory than ground-based detectors. However, they have no particular advantage in the case of neutron-star binaries, where the bulk of the growth in the memory takes place over a timescale favourable to LIGO. In Eq. (2.18), the dominant terms in the series expansion depend on $x_1 = 2\pi f_1 t_k$, where $f_1 = 10^{-3}$ Hz. If $t_k < .16s$, so $x_1 \leq 10^{-3}$, then all terms except the first three are negligable and can be ignored. Assuming the constraint in Eq. (2.22) applies to the cut-off time, and letting $N_{\text{orb}} = 1$, so that merger is presumed to take place at the latest possible time (within the framework of the approximation scheme), then $t_k = (5\pi/4)^{8/5}(256/5)^{3/5}M_c$ and so

$$t_k = 4.66 \times 10^{-4}\frac{M_c}{M_\odot} \tag{2.30}$$

where $M_c = \mu^{3/5}M^{2/5}$ is the chirp mass of the binary. In this way, a formula can be derived relating the signal to noise generated by a binary with chirp mass M_c at a distance r from the detector, to these two quantities, assuming that merger takes place when $N_{\text{orb}} = 1$. The approximation should hold good up to $M_c/M_\odot \approx 300$.

$$\begin{aligned}\frac{S}{N} &\approx .15\frac{M_c/M_\odot}{r/200\text{Mpc}}F_+\Big(1 - \frac{\sin^2\iota}{18}\Big)\sin^2\iota \\ &\quad\times\Big[1 - .0997\Big(\frac{M_c}{M_\odot}\Big)^{1/4}\end{aligned}$$

$$+.003417\left(\frac{M_c}{M_\odot}\right)^{1/2}\Big]^{1/2}. \tag{2.31}$$

In the case of a $10M_\odot$ black hole and a neutron star at $r = 200$Mpc, then $M_c = 3M_\odot$ and so $S/N \leq .5$. A NS/NS binary has a chirp mass of $M_c = 1.219M_\odot$, and at $r = 200$Mpc its signal-to-noise would be $S/N \leq 1.1$. Two $10M_\odot$ black holes have a chirp mass of $M_c = 8.7M_\odot$. At a range of 200Mpc, their signal-to-noise would be $S/N \leq 1.2$. Finally, a binary with chirp mass of $100M_\odot$ at a distance of 1Gpc would produce a signal-to-noise ratio of $S/N \leq 2.5$. This last example, begins to approach the sort of large mass systems which are of special interest to space-based detectors like LISA, but these systems are of no interest from the point of view of measuring neutron-star radii. Obviously LISA would be no more use than LIGO for estimating neutron-star radii, but it might well be capable of detecting the memory produced by very massive binaries, if, for instance, mergers between super-massive Black Holes are sufficiently common within a few Gpc.

In this paper I have made use of a simplistic quadropole-moment (i.e. slow-motion) calculation of the memory. Would a more relativistic approach increase the memory, thereby increasing the odds of detection? Finn [25] has made a detailed numerical calculation of the power emitted by systems consisting of low mass bodies in equatorial orbits around massive rotating black holes. As it happens, his figures indicate that the quadropole approximation consistently overestimates the power emitted in the burst by a modest factor, and therefore the memory would, in reality, be modestly weaker than I have made it. More specifically, in the case of a small body in a prograde orbit around a nearly maximally rotating Kerr black hole, Finn finds that for an orbital radius of $10M$, where M is the mass of the hole, the loss of energy due to gravitational-wave emission is roughly 90% of its value as derived by the quadropole formula. When the particle reaches the last stable circular orbit at $r = M$, his result has fallen to about 50% of the quadropole value. The nearly Newtonian approximation can therefore be taken as at least a rough guide to the results of a more realistic calculation.

2.6 Conclusions

In all of the cases considered above involving neutron stars, the signal-to-noise ratio fell on or below the threshold for detection ($S/N = 2$). In addition, one has to remember that the indication is that the true memory would be somewhat smaller, because of the quadropole formula's overestimation of the energy emitted by the binary in the last stages of coalescence. In fact, since detection depends largely on the rate of growth of the memory in the last split second of the inspiral, it seems likely that the signal to noise, in practice, could be as low as close to half this paper's estimated value, in the case of a rapidly spinning black hole. Therefore the chances of even an advanced LIGO interferometer detecting the Christodoulou memory, except serendipitously, from sources like these appears to be small. Certainly, there is little chance of using the signal to estimate neutron-star radii, unless coalescence rates have been drastically underestimated.

Fortunately another, more promising, method of measuring neutron-star radii has been proposed [8] . This scheme involves the use of several narrow band detectors with optimal frequencies staggered around 1kHz, which would register the strength of the primary signal (if any) as it passes through their frequency. The waves' cut-off frequency can then be estimated from their responses, and therefore the neutron-star radii can be deduced [8]. A companion paper [11] gives a quantitative description and evaluation of this method.

The results given above indicate (in agreement with a cruder estimate by Thorne [5]), that the memory from coalescing binaries consisting of two large black holes within ≈ 200Mpc of earth might be detectable by LIGO. But the lack of any detailed understanding of the behaviour of BH/BH binaries makes any prediction uncertain. My results merely demonstrate that the memory from such systems *may* be strong enough to be seen by very sensitive detectors. It seems likely that at least the existence of the memory might be confirmed by a particularly strong event.

Acknowledgements

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Appendix

Let $H(t) = h'(t)\Theta(t - t_k)$, where $h'(t) = c/t^{\frac{1}{4}}$ $(c = h_k t_k^{\frac{1}{4}})$ for $t > 0$ and $= 0$ for $t < 0$. This is the first term in Eq. (2.14), the Fourier transform of which is needed to derive Eq. (2.15). The Fourier transform of $H(t)$ is given by

$$\tilde{H}(\omega) = \int_{-\infty}^{\infty} \tilde{h}'(\omega')\tilde{\Theta}(\omega - \omega')d\omega' \tag{2.32}$$

(where $\omega = 2\pi f$), by the convolution theorem. From the Bateman papers [20], $\tilde{h}'(\omega) = \alpha/\omega^{3/4}$ for $\omega > 0$ and $= \alpha^*/|\omega|^{3/4}$ for $\omega < 0$, where

$$\alpha = \frac{\pi}{2}\frac{1}{\Gamma(\frac{1}{4})}\sec\frac{\pi}{8} + i\Gamma(\frac{3}{4})\cos\frac{\pi}{8}. \tag{2.33}$$

Therefore

$$\begin{aligned}\tilde{H}(\omega) &= ic\alpha\int_0^\infty \frac{e^{i(\omega-\omega')t_k}}{\omega-\omega'}\frac{d\omega'}{\omega'^{3/4}} \\ &\quad +ic\alpha^*\int_{-\infty}^0 \frac{e^{i(\omega-\omega')t_k}}{\omega-\omega'}\frac{d\omega'}{|\omega'|^{3/4}}\end{aligned} \tag{2.34}$$

which implies that

$$\tilde{G}(\omega) \equiv \frac{d}{dt_k}\tilde{H}(\omega) = -\frac{\gamma c}{t_k^{1/4}}e^{i\omega t_k} \tag{2.35}$$

where $\beta = \int_0^\infty e^{-ix}dx/x^{3/4}$ and $\gamma = \alpha\beta + \alpha^*\beta^* \approx 6.28$. From this it follows that

$$\tilde{H}(\omega) = \int \tilde{G}(\omega)dt_k + \tilde{F}(\omega), \tag{2.36}$$

where $\tilde{F}(\omega)$ is independant of t_k. Now,

$$\begin{aligned}\int \tilde{G}(\omega)dt_k &= -\frac{c\gamma}{\omega^{3/4}}\int \frac{e^{i\omega t_k}}{(\omega t_k)^{1/4}}d(\omega t_k) \\ &= -\frac{c\gamma}{\omega^{3/4}}\Big[\frac{4}{3}(\omega t_k)^{3/4} \\ &\quad +\frac{4}{7}i(\omega t_k)^{7/4} + \ldots\Big],\end{aligned} \tag{2.37}$$

which is zero as $t_k \to 0$. Therefore, $\tilde{F}(\omega) = \tilde{H}(\omega, t_k = 0) = \tilde{h}'(\omega)$, since $H(t, t_k = 0)$ is simply $h'(t)$. From this, and Eqs. (2.36) and (2.37), and the above expression for $\tilde{h}'(\omega)$, together with the Fourier transform of the second term in Eq. (2.14), one derives Eq. (2.15).

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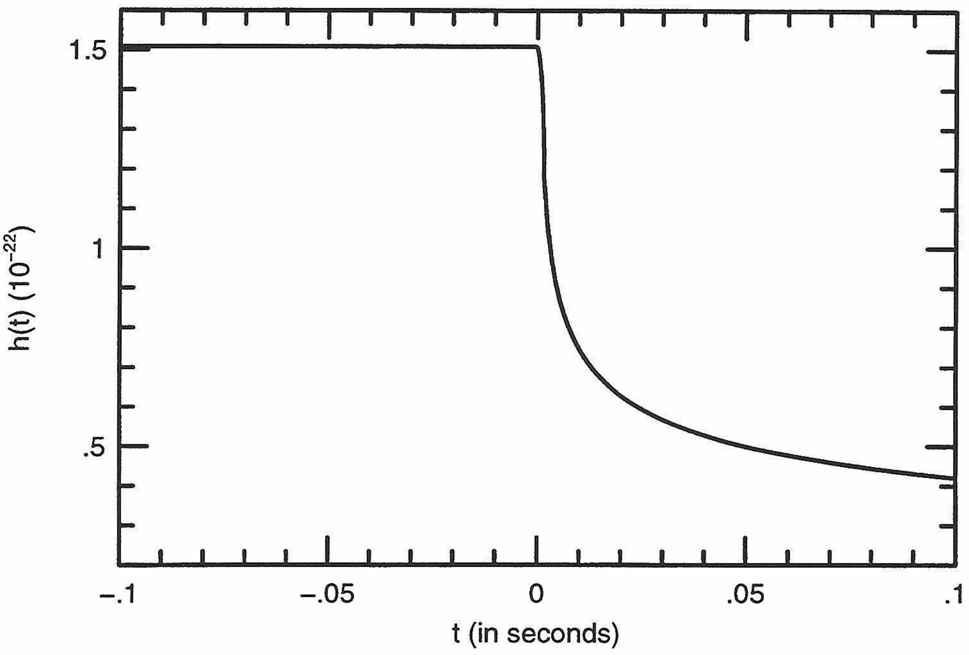

Figure 2.1: A graph of the data train representing the memory. The dependance of $h(t)$ is $t^{-1/4}$ for $t \geq t_k$ and it is constant after this. (Recall that t is the time *until* final coalescence in the ideal, Newtonian, point mass limit.) The discontinuity at t_k is rounded off by an elliptical function in the numerical calculation , in order to model the actual turn off of the wave, which takes place over at least an orbital period of the motion. In this figure, the round-off occurs over 1/500th of a second. Both this and the figure below illustrate the case of two $10M_\odot$ black holes at a distance of 200Mpc, with $t_k = .0006$s.

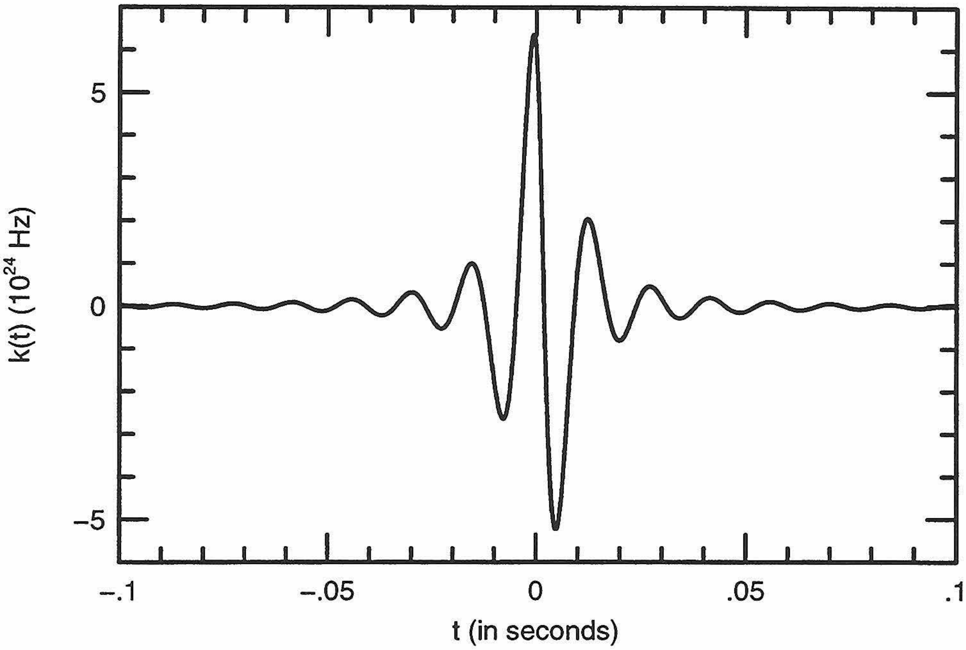

Figure 2.2: A graph of the filter for the signal in Fig. 2.1, derived numerically, using a fast Fourier transform. Notice that this function, the integral of whose product with $h(t)$ gives the signal-to-noise, is appreciably non-zero only in the .01s or so around t_k. Thus the signal-to-noise ratio derived from this filter depends almost entirely on the part of the signal where the rate of change is about 100 Hz, near the peak of the LIGO sensitivity. The units of k(t) are in Hz.

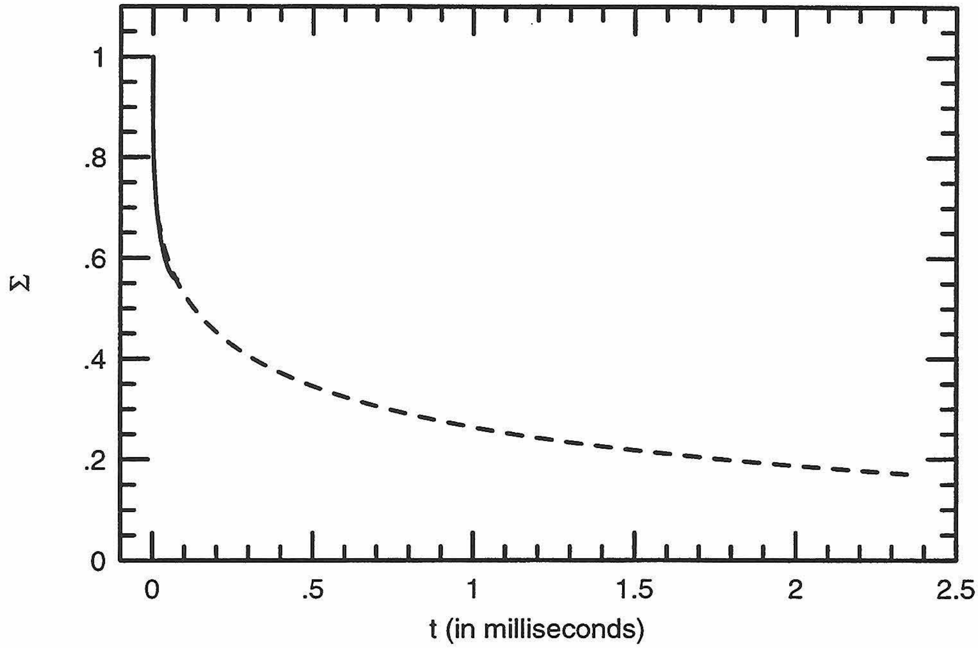

Figure 2.3: A graph of Σ (defined in Eq. (2.16)), for an advanced LIGO detector with the noise spectrum given in Eq. (2.10), versus cut-off time t_k. Note that it is a monotonically decreasing function of the cut-off time. The solid line in the graph was derived analytically, from the series in Eq. (2.16), and the dotted line represents data which were calculated numerically, by means of a fast Fourier transform, and then integrating the product of the signal and filter functions (such as those shown in Figs. 2.1 and 2.2), in each case.

Chapter 3

Gravitational radiation from a particle in circular orbit around a black hole: Stability of circular orbits under radiation reaction

with Theocharis Apostolatos, Amos Ori and Eric Poisson

Abstract

We use the Teukolsky perturbation formalism to show that: (i) a particle in circular motion around a nonrotating black hole remains on a circular orbit under the influ-

ence of radiation reaction; and (ii) circular orbits are stable only if the orbital radius is greater than a critical radius $r_c \simeq 6.6792M$, where M is the mass of the black hole. A circular orbit is stable if, when slightly perturbed so that it acquires a small eccentricity, radiation reaction decreases the eccentricity; a circular orbit is unstable if radiation reaction increases the eccentricity. Our analysis is restricted by four major assumptions: (i) the black hole is nonrotating, (ii) the eccentricity is always small, (iii) the gravitational perturbations are linear, and (iv) the adiabatic approximation (that radiation reaction takes place over a timescale much larger than the orbital period) is valid. On the other hand, our analysis is not limited to weak-field, slow-motion situations; it is valid for particle motion in strong gravitational fields.

3.1 Introduction and summary

3.1.1 Motivation

A particle of mass μ, which interacts with the gravitational field of an isolated object of mass M, does not, in general, move on a spacetime geodesic. This is due to the fact that the combined system emits gravitational waves; the problem of radiation reaction—to determine the influence of this emission on the motion of the particle—is a difficult one in general relativity.

Gravitational radiation reaction has a well-known electromagnetic analogue: A charged particle, accelerated by an electric field, does not move according to the

Lorentz equations of the motion, because of the emission of electromagnetic waves. There are difficult conceptual problems associated with radiation reaction in electromagnetism [1]; however, these conceptual problems are not a serious impediment to computations, at least when radiation reaction is a small effect. The use of half retarded minus half advanced potentials, together with the rejection of runaway solutions on physical grounds, provide a well-defined calculational basis for most applications [2].

In contrast with the electromagnetic case, the problem of gravitational radiation reaction is plagued with conceptual *and* calculational difficulties, which are mostly due to the non-local character of the problem. Non-locality enters in essentially two different ways: (i) As a consequence of the principle of equivalence, a gravitational wave can be identified as such only in a region of spacetime larger than several wavelengths [3], which precludes the construction of a local radiative field. And (ii) because gravitational waves are in general scattered by the curvature of spacetime, waves emitted at one time may influence the motion of the particle at some later time [4]; these *tails* in the waves can produce noticeable effects, most especially if curvature is large.

In order to gain insight into the general problem of gravitational radiation reaction, it is important to look at simple special cases for which the above problems can be addressed. To study such a simple case is the main purpose of this paper.

The question of radiation reaction is most pressing in the context of the late evolution of compact binary systems [5], since the waves generated by such systems are the most promising for detection by kilometer-size interferometric detectors [6]. Extraction of the information encoded in the waves will require an accurate calculation of the expected waveforms [7]; these theoretical waveforms are used as matched filters through which the detected signal is processed [8]. Radiation reaction governs the rate at which the wave frequency increases with time, as the compact objects spiral together toward coalescence. During the last stages of evolution, when the waves are most interesting for detection, the wave frequency sweeps from approximately 10 Hz to several hundred Hz in just a few minutes, during which the waves oscillate about 10^4 times. It is therefore essential to incorporate radiation reaction, to a fractional accuracy of at least 10^{-4}, into the calculation of the theoretical waveforms [7]. Thus, the practical importance of radiation reaction in the evolution of compact binary systems provides more motivation for the work presented here.

Also of interest are the last stages of evolution, under radiation reaction, of a solar-mass compact object orbiting a galactic, supermassive black hole. Such a binary system could be observed with an eventual space-based interferometric detector, which would operate between 10^{-4} Hz and 10^{-1} Hz [9]. Because we consider small mass ratios (see subsection B below), the results presented in this paper are directly relevant to these sources.

Most of the work devoted so far to gravitational radiation reaction, in particular for the two-body problem, has been restricted to weak-field, slow-motion situations

[3, 10, 11, 12, 13]. Lincoln and Will [12] have calculated, using post-Newtonian theory, the orbital motion of a binary system at post$^{5/2}$-Newtonian order, which only incorporates radiation reaction at leading order. These calculations are not accurate enough for the purpose of constructing matched filters for interferometric detectors [7]. Higher-order corrections to the post-Newtonian, radiation-reaction force have recently been calculated by Iyer and Will [13].

By comparison, very little has been done for strong-field situations. Gal'tsov [14] has laid the foundations for strong-field radiation-reaction calculations in the case of particle motion in the field of a Kerr black hole. His formalism is based on the notion of a local, gauge-dependent radiation-reaction force. However, Gal'tsov's only explicit calculation of this force was also restricted to weak-field, slow-motion situations. Finn, Ori, and Thorne [15] have studied the strong-field transition between inspiral and plunge motion in Kerr; however, their analysis does not require a detailed knowledge of radiation-reaction effects. In this paper, we present concrete results on radiation reaction in strong-field situations.

3.1.2 The problem, method of solution, and approximations

We study the effects of radiation reaction on the bound motion of a particle of mass μ in the geometry of a Schwarzschild black hole of mass M. Two quantities are of fundamental interest: the orbit's averaged radius r_0, and the orbit's eccentricity ε. The radius r_0 denotes the averaged value of the orbit's Schwarzschild radial coordinate; the maximal value of the orbital radius defines the eccentricity: $r_{\max} = r_0(1+\varepsilon)$. More precise definitions of r_0 and ε will be given in Sec. II. We shall suppose that both the eccentricity ε and the mass ratio μ/M are much smaller than unity. However, we do not suppose that the radius r_0 is large, so our analysis includes strong-field situations.

We adopt the Teukolsky perturbation formalism [16], and consider the linear gravitational perturbations associated with the motion of the particle. The perturbations are described in terms of the complex Weyl scalar Ψ_4, which becomes radiative at large distances from the source. The rates of loss of orbital energy E, and orbital angular momentum L, due to gravitational radiation, can be calculated by solving the Teukolsky equation.

The *secular* evolution (the evolution over timescales much larger than the orbital period) of r_0 and ε can be determined from the knowledge of $\dot{E}$ and $\dot{L}$, where an overdot denotes time differentiation followed by an average over several orbital periods. In particular, the following relations can be derived (Sec. II): $\dot{r_0} = \dot{r_0}(r_0, \dot{L})$, and $\dot{\varepsilon} = \dot{\varepsilon}(\varepsilon, r_0, \dot{E}, \dot{L})$. We shall use the perturbation formalism to calculate the rates of loss of energy and angular momentum. These calculations are performed (i) analytically, for the special case of weak fields and slow motions; and (ii) numerically, for the general case.

Our calculations are restricted to small eccentricities, $\varepsilon \ll 1$. The work presented in this paper can therefore be interpreted as a stability analysis: A circular orbit with radius r_0 is slightly perturbed and acquires a small eccentricity ε. The orbit evolves because of radiation reaction; the sign of $\dot{\varepsilon}$ determines whether the perturbed orbit is driven more circular, or more eccentric. Circular orbits are thus *stable* if $\dot{\varepsilon} < 0$, and are *unstable* if $\dot{\varepsilon} > 0$. Previous studies have shown that circular orbits are always stable in weak-field, slow-motion situations [11, 12]; our own study confirms this, and also determines whether this remains true in strong-field situations.

Recently, and independently of us, Tanaka *et al.* [17] numerically calculated the gravitational waveforms, and the fluxes of energy and angular momentum at infinity, for orbits with *arbitrary* eccentricities. The differences between their analysis and ours are significant. Tanaka *et al.* are mostly concerned with what can be observed at infinity, and are not much concerned with radiation reaction. In particular, they do not calculate the fluxes of energy and angular momentum at the black-hole horizon, which we do here, and which is important for radiation reaction. We have become aware of the work by Tanaka *et al.* very shortly before submitting this paper for publication.

Our calculations are also restricted to small mass ratios. This condition comes from two requirements: (i) that the gravitational perturbations be small enough to be linear, which implies $\mu/M \ll 1$, and (ii) that the *adiabatic approximation* be valid. The adiabatic approximation supposes that radiation reaction takes place over a timescale which is much larger than the orbital period. We shall show below (Sec. IV F) that this implies a restriction on μ/M; this restriction is not severe at large distances, but becomes $\mu/M \ll (1 - 6M/r_0)^{3/2}$ for r_0 approaching $6M$. The adiabatic approximation must therefore break down at $r_0 = 6M$, where, even without radiation reaction, circular orbits become unstable.

The adiabatic approximation is a fundamental feature of our analysis. It allows us to suppose that, over timescales comparable to the orbital period, the motion of the particle is, in fact, geodesic; non-geodesic behavior becomes noticeable only over much larger timescales. Moreover, the motion is also strictly periodic, and, consequently, the gravitational waves have a well-resolved frequency spectrum; the waves' frequencies change appreciably only over timescales much larger than the orbital period. Our problem is therefore one for which we first determine the rates of loss of energy and angular momentum for the slightly eccentric, geodesic motion of a particle around a Schwarzschild black hole, and then use these rates to infer the slow evolution of the orbit.

3.1.3 The results

Our analysis first allows us to prove that, if the particle's orbit is strictly circular ($\varepsilon = 0$), then radiation reaction produces a strictly circular evolution. In other words, *circular orbits remain circular under radiation reaction.* Previous proofs of this statement were restricted to weak-field, slow-motion situations [11, 12]; our proof is valid both for weak and strong fields.

If the eccentricity is small but not identically zero, our analysis shows that radiation reaction (i) *decreases* the eccentricity if r_0 is larger than a certain critical value r_c, and (ii) *increases* the eccentricity if r_0 is smaller than r_c. Thus circular orbits are *stable* if $r_0 > r_c$, and *unstable* if $r_0 < r_c$. The point $r_0 = r_c$ corresponds to $\dot{\varepsilon}$ changing sign; we have estimated numerically that

$$r_c/M \simeq 6.6792. \tag{3.1}$$

Our results are most conveniently presented in terms of the dimensionless quantity $c(r_0)$, defined as

$$c(r_0) = \frac{r_0\,\dot{\varepsilon}}{\dot{r_0}\,\varepsilon} = \frac{d\ln\varepsilon}{d\ln r_0}, \tag{3.2}$$

and which can be interpreted as the ratio of the *inspiral timescale* $r_0/|\dot{r_0}|$ (the timescale over which the orbital radius r_0 changes appreciably) over the *circularization timescale* $\varepsilon/|\dot{\varepsilon}|$ (the timescale over which the eccentricity changes appreciably). By virtue of the adiabatic approximation, both timescales are much larger than the orbital period. A plot of $c(r_0)$, obtained numerically, is given in Fig. 1.

For large r_0 (weak-field, slow-motion), $c(r_0)$ can be calculated analytically (Sec. V A), and takes the form

$$c(r_0) = \frac{19}{12}\Big[1 - \frac{3215}{3192}v^2 + \frac{377}{152}\pi v^3 + O(v^4)\Big], \tag{3.3}$$

where $v = (M/r_0)^{1/2} \ll 1$ acts as a post-Newtonian expansion parameter. The leading-order term of Eq. (3.3) corresponds to a Newtonian calculation of the orbit, together with the use of the quadrupole formula to determine $\dot{r_0}$ and $\dot{\varepsilon}$ [11]. The first-order correction (at post-Newtonian, v^2, order) corresponds to post-Newtonian corrections to the orbital motion. The second-order correction (at post$^{3/2}$-Newtonian, v^3, order) corresponds to effects due to the propagation of the gravitational waves in the field of the black hole—effects associated with the tails of the waves.

For values of r_0 approaching $6M$ (highly relativistic situation; Sec. V B), $c(r_0)$ behaves according to

$$c(r_0 \to 6M) \sim -\frac{1}{4}(1 - 6M/r_0)^{-1}, \tag{3.4}$$

and therefore grows to arbitrarily large, negative values. This behavior is a consequence of the fact that circular orbits, even without radiation reaction, become unstable at $r_0 = 6M$. We recall that the limit $r_0 \to 6M$ must be taken with care, in view of the adiabatic approximation; orbits arbitrarily close to $r_0 = 6M$ can be considered at the price of taking μ/M sufficiently small.

Eqs. (3.3) and (3.4) are derived analytically, and imply that $c(r_0)$ must change sign at some radius $r_0 = r_c$. We have therefore provided an *analytical proof* that circular orbits are stable in the range $r_0 > r_c > 6M$ only. However, a numerical calculation is necessary to show that $c(r_0)$ changes sign only once, and to determine the value of r_c, Eq. (3.1).

The complete evolution of the eccentricity, so long as it remains small, can be obtained by integrating Eq. (3.2). It is most convenient to parametrize the evolution with r_0, and to express the eccentricity in terms of the function $\gamma(r_0; r_i)$, defined as

$$\gamma(r_0; r_i) = \ln \frac{\varepsilon(r_0)}{\varepsilon(r_i)} = \int_{r_i}^{r_0} \frac{c(r_0')}{r_0'} dr_0', \tag{3.5}$$

where r_i is some initial radius. If r_0 and r_i are both much larger than $6M$, then Eqs. (3.3), (3.5) imply

$$\gamma(r_0; r_i) \simeq \alpha(r_0/M) - \alpha(r_i/M), \tag{3.6}$$

where

$$\alpha(x) = \frac{19}{12}\Big[\ln x + \frac{3215}{3192}x^{-1} - \frac{377}{1288}\pi x^{-3/2}\Big]. \tag{3.7}$$

If, on the other hand, r_0 is very close to $6M$, but $r_i \gg 6M$, then Eqs. (3.4), (3.5) imply

$$\gamma(r_0; r_i) \simeq -\frac{1}{4}\ln(r_0/6M - 1). \tag{3.8}$$

The behavior of $\gamma(r_0; r_i)$, for r_i arbitrarily fixed to $100M$, is depicted in Fig. 2. From this curve one can easily infer the corresponding $\gamma(r_0; r_i)$ for $r_i < 100M$.

As one sees from Fig. 2, during the weak-field, slow-motion phase of the orbital evolution, the eccentricity is reduced by many orders of magnitude—the orbit becomes essentially circular. The eccentricity reaches a minimum value when $r_0 = r_c$, and then starts increasing. Eventually, if the mass ratio μ/M is arbitrarily small and the adiabatic approximation holds, the orbit shrinks to a radius r_0 for which the eccentricity becomes equal to its initial value; in general this occurs very close to $6M$, as is indicated on the graph. For reasonable mass ratios, however, the eccentricity has not increased by much by the time the adiabatic approximation breaks down. As an example, consider a solar-mass object spiraling around a $10^6 M_\odot$ galactic black hole; this example is particularly relevant to space-based gravitational-wave detectors [9]. For $\mu/M = 10^{-6}$, the adiabatic approximation becomes invalid in the vicinity of $r_0 = r_1$, where $r_1/M = 6.002$; our numerical results then imply $\varepsilon(r_1)/\varepsilon(r_c) \simeq 4.0$. For such binary systems, the inspiral time from $r_0 = r_c$ to $r_0 = r_1$ is of the order of two years. For $\mu/M = 10^{-8}$, the ratio of the eccentricities is only increased by a factor of two.

3.1.4 Organization of the paper

The remainder of this paper is devoted to deriving the results quoted in the preceding subsection. We begin with a precise formulation of the problem in Sec. II. We first provide definitions for the quantities r_0 and ε, and then derive the evolution equations $\dot{r_0} = \dot{r_0}(r_0, \dot{L})$, $\dot{\varepsilon} = \dot{\varepsilon}(\varepsilon, r_0, \dot{E}, \dot{L})$. Two conditions which ensure that $\dot{\varepsilon} \propto \varepsilon$ are imposed, and are justified in later sections. The first condition is that, for circular motion, gravitational waves carry energy and angular momentum in such a way that $\dot{E}/\dot{L} = \Omega = (M/{r_0}^3)^{1/2}$; the second condition is that corrections to $\dot{E}$ and $\dot{L}$, due to nonvanishing eccentricity, are second order in ε. The fact that $\dot{\varepsilon} \propto \varepsilon$ implies that circular orbits remain circular under radiation reaction; the stability of circular orbits depends on the sign of the proportionality factor.

We present a brief summary of the Teukolsky perturbation formalism [16] in Sec. III. First, the inhomogeneous Teukolsky equation, and its formal solution, are described in detail. Then we explain the method for extracting, from the solution, the gravitational waveforms, and the rates at which the waves carry energy and angular momentum. The section is concluded with a proof, valid for arbitrarily strong fields, that $\dot{E}/\dot{L} = \Omega$ for circular orbits.

The calculations relevant for slightly eccentric motion are presented in Sec. IV. The first step consists of integrating the radial and azimuthal geodesic equations; the integration is carried out to second order in the eccentricity. This calculation is presented in subsection A, and subsection B offers an overview of the remaining steps. The form of the results obtained for $r(t)$ and $\phi(t)$ allows us, in subsection C, to (i) identify the frequency spectrum of the gravitational waves, (ii) witness important simplifications, and (iii) prove that corrections to $\dot{E}$ and $\dot{L}$ are second order in the eccentricity. All of this may be achieved without performing detailed calculations; instead, all computations are kept at a schematic level. These schematic calculations are pushed even further, in subsection D, to derive expressions for $\dot{r_0}$ and $\mu\dot{\varepsilon}/\varepsilon$; this allows us to witness more cancellations, which greatly simplify the problem. The detail of the remaining calculations are presented in subsection E. Conditions on μ/M, which ensure the validity of the adiabatic approximation, are formulated in subsection F.

We present our analytical and numerical results in Sec. V. We first consider the weak-field, slow-motion ($r_0 \gg 6M$) limit of our formalism, and derive post-Newtonian expansions for the quantities of interest. This analysis yields Eq. (3.3) above. We then consider the highly relativistic ($r_0 \to 6M$) limit of the formalism, which is also tractable analytically. This analysis yields Eq. (3.4) above. In situations where r_0 is neither very large nor very close to $6M$, our equations must be integrated numerically, which we describe next. Our numerical analysis yields Eq. (3.1) above, as well as the graphs presented in the Figures.

We conclude in Sec. VI with a recapitulation of our fundamental results, and a discussion of our approximations.

Throughout the paper we use geometrized units in which the speed of light and the gravitational constant are set equal to unity. Most of the paper is essentially self-contained, except for Sec. V, which relies heavily on previous papers in this series. These previous papers are concerned with purely circular orbits; paper I [18] is devoted to analytical methods, while paper II [19] is devoted to numerical methods. Both analytical and numerical methods are utilized in this paper.

3.2 Formulation of the problem

3.2.1 Definition of r_0 and ε

Timelike geodesics in the Schwarzschild geometry obey the following equations:

$$\begin{aligned} dt/d\tau &= \tilde{E}/f, \\ d\phi/d\tau &= \tilde{L}/r^2, \\ (dr/d\tau)^2 + V(\tilde{L}, r) &= \tilde{E}^2, \end{aligned} \tag{3.9}$$

where τ is the particle's proper time; $\tilde{E} = E/\mu$ and $\tilde{L} = L/\mu$ are, respectively, the specific orbital energy and angular momentum. We have also introduced $f =$

$1 - 2M/r$, and $V(\tilde{L}, r)$ is the effective potential for radial motion,

$$V(\tilde{L}, r) = f(1 + \tilde{L}^2/r^2). \tag{3.10}$$

We suppose that the motion takes place in the equatorial plane, $\theta = \pi/2$, and near a minimum of the potential $V(\tilde{L}, r)$. We *define* the radius $r = r_0$ to be the position of this minimum; since $\partial V/\partial r|_{r=r_0} = 0$, we have

$$\tilde{L}^2 = M^2[v^2(1 - 3v^2)]^{-1}, \tag{3.11}$$

where $v = (M/r_0)^{1/2}$. Radial motion corresponds to small oscillations about $r = r_0$.

We *define* the eccentricity ε so that $r = r_0(1 + \varepsilon)$ is a turning point of the radial motion, at which $\tilde{E}^2 = V(\tilde{L}, r)$. This equation can be expanded in powers of ε, which yields

$$\begin{aligned}(1 - 3v^2)\tilde{E}^2 &= (1 - 2v^2)^2 + v^2(1 - 6v^2)\varepsilon^2 - \\ &\quad - 2v^2(1 - 7v^2)\varepsilon^3 + O(\varepsilon^4).\end{aligned} \tag{3.12}$$

Eq. (3.11) implies $r_0 = r_0(\tilde{L})$, while Eq. (3.12) implies $\varepsilon = \varepsilon(\tilde{L}, \tilde{E})$.

3.2.2 Radiation reaction—evolution of r_0 and ε

The results of subsection A imply that the knowledge of the rates of loss of energy and angular momentum, due to gravitational radiation, is sufficient to determine the evolution of both r_0 and ε. We are interested in the *secular* evolution of these quantities—the evolution over timescales much larger than the orbital period. The secular evolution is well defined, and can be unambiguously calculated. In contrast, the short-term evolution is not so well defined, because gravitational waves cannot be localized in a region of spacetime smaller than a few wavelengths [3]. To perform a time averaging over several orbital periods is therefore a fundamental feature of our calculations. We shall henceforth denote by an overdot the operation of time differentiation followed by an average over several orbital periods; thus $\dot{\psi} = \langle d\psi/dt \rangle$, for any quantity ψ.

An evolution equation for r_0 is obtained by using Eq. (3.11) to calculate $\mu \dot{r_0} = (dr_0/d\tilde{L})\dot{L}$, which yields

$$\mu \dot{r_0} = 2(1-3v^2)^{3/2}[v(1-6v^2)]^{-1}\dot{L}. \tag{3.13}$$

Similarly, one may use Eqs. (3.11) and (3.12) to calculate $\mu\dot{\varepsilon} = (\partial\varepsilon/\partial\tilde{E})\dot{E} + (\partial\varepsilon/\partial\tilde{L})\dot{L}$, which yields

$$\begin{aligned} \mu\dot{\varepsilon} &= \frac{1}{\varepsilon}\frac{(1-2v^2)(1-3v^2)^{1/2}}{v^2(1-6v^2)}\Big\{\Big[1+\frac{v^2(1-6v^2)}{2(1-2v^2)^2}\varepsilon^2 + O(\varepsilon^3)\Big]\dot{E} \\ &- \Big[1-\frac{1-12v^2+18v^4}{(1-2v^2)(1-6v^2)}\varepsilon^2 + O(\varepsilon^3)\Big]\Omega\dot{L}\Big\}, \end{aligned} \tag{3.14}$$

where $\Omega = v/r_0 = (M/{r_0}^3)^{1/2}$.

The rate of loss of orbital energy is equal to minus the rate at which gravitational waves carry energy. We therefore write $\dot{E} = -\dot{E}^{(\mathrm{GW})}$, and expand $\dot{E}^{(\mathrm{GW})}$ in powers of the eccentricity:

$$\dot{E}^{(\mathrm{GW})} = \dot{E}^{(0)} + \varepsilon\dot{E}^{(1)} + \varepsilon^2\dot{E}^{(2)} + O(\varepsilon^3); \tag{3.15}$$

$\dot{E}^{(0)}$ corresponds to circular motion. Similarly, we write $\dot{L} = -\dot{L}^{(\mathrm{GW})}$, and

$$\dot{L}^{(\mathrm{GW})} = \dot{L}^{(0)} + \varepsilon\dot{L}^{(1)} + \varepsilon^2\dot{L}^{(2)} + O(\varepsilon^3). \tag{3.16}$$

In Secs. III C and IV C below, we will show that

$$\dot{E}^{(0)} = \Omega\dot{L}^{(0)}, \quad \dot{E}^{(1)} = \dot{L}^{(1)} = 0, \tag{3.17}$$

which implies that the lowest-order corrections to $\dot{E}^{(\mathrm{GW})}$ and $\dot{L}^{(\mathrm{GW})}$ are second order in the eccentricity.

Substitution of Eqs. (3.16) and (3.17) into (3.13) implies

$$\mu\dot{r}_0 = -2M(1-3v^2)^{3/2}[v^4(1-6v^2)]^{-1}\dot{E}^{(0)} + O(\varepsilon^2); \tag{3.18}$$

the evolution of r_0 is therefore dominated by the circular limit of Eq. (3.13), and corrections due to the small eccentricity can be ignored.

Substitution of Eqs. (3.15), (3.16), and (3.17) into (3.14) yields important cancellations, and the final answer is

$$\begin{aligned} \mu\dot{\varepsilon} &= -\varepsilon(1-2v^2)(1-3v^2)^{1/2}[v^2(1-6v^2)]^{-1} \\ &\quad \times[g(v)\dot{E}^{(0)} + \dot{E}^{(2)} - \Omega\dot{L}^{(2)}] + O(\varepsilon^2), \end{aligned} \tag{3.19}$$

where

$$g(v) = \frac{2-27v^2+72v^4-36v^6}{2(1-2v^2)^2(1-6v^2)}. \tag{3.20}$$

Thus the calculation of $\mu\dot{\varepsilon}$ requires the computation of $\dot{E}^{(\mathrm{GW})}$ and $\dot{L}^{(\mathrm{GW})}$, accurately to second order in the eccentricity. Due to the crucial relations (3.17), $\mu\dot{\varepsilon}$ is itself *linear* in the eccentricity.

Eqs. (3.17) are therefore the key to the proof that circular orbits remain circular under radiation reaction, since Eq. (3.19) implies $\dot{\varepsilon}(\varepsilon = 0) = 0$. The problem of determining the evolution of r_0 and ε is now equivalent to that of calculating $\dot{E}^{(0)}$, and the pieces of $\dot{E}^{(2)}$ and $\dot{L}^{(2)}$ which do not cancel out when the combination $\dot{E}^{(2)} - \Omega\dot{L}^{(2)}$ is constructed.

3.3 The perturbation formalism

This section contains a brief summary of the relevant equations. More detail can be found in paper I [18], and in the references quoted herein.

3.3.1 The Teukolsky equation

The stress-energy tensor associated with the motion of a particle perturbs the gravitational field of a Schwarzschild black hole. The gravitational perturbations are described by the Weyl scalar $\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$, where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $n^\alpha = \frac{1}{2}(1, -f, 0, 0)$, and $\bar{m}^\alpha = (0, 0, 1, -i \csc\theta)/\sqrt{2}r$; throughout we denote complex conjugation with an overbar. At large distances, Ψ_4 describes outgoing gravitational waves; at the black-hole horizon, Ψ_4 describes ingoing waves.

The Weyl scalar can be decomposed into Fourier-harmonic components according to

$$r^4\Psi_4 = \int_{-\infty}^{\infty} d\omega \sum_{\ell m} R_{\omega\ell m}(r)\,{}_{-2}Y_{\ell m}(\theta,\phi)e^{-i\omega t}, \tag{3.21}$$

where ${}_sY_{\ell m}(\theta,\phi)$ are spin-weighted spherical harmonics [20]; the sums over ℓ and m are restricted to $-\ell \leq m \leq \ell$ and $\ell \geq 2$. The radial function $R_{\omega\ell m}(r)$ satisfies the inhomogeneous Teukolsky equation [16],

$$\left[r^2 f\frac{d^2}{dr^2} - 2(r-M)\frac{d}{dr} + U(r)\right]R_{\omega\ell m}(r) = -T_{\omega\ell m}(r), \tag{3.22}$$

with

$$U(r) = f^{-1}\left[(\omega r)^2 - 4i\omega(r-3M)\right] - \lambda, \tag{3.23}$$

where $\lambda = (\ell-1)(\ell+2)$.

The source term in Eq. (3.22) is calculated from the particle's stress-energy tensor $T^{\alpha\beta}(x) = \mu\int d\tau\, u^\alpha u^\beta \delta^{(4)}[x - z(\tau)]$, where x is the spacetime point, $z(\tau)$ the particle's trajectory with tangent $u^\alpha = dz^\alpha/d\tau$, and τ is the particle's proper time. The first step is to construct the projections ${}_0T = T_{\alpha\beta}n^\alpha n^\beta$, ${}_{-1}T = T_{\alpha\beta}n^\alpha \bar{m}^\beta$, and ${}_{-2}T = T_{\alpha\beta}\bar{m}^\alpha \bar{m}^\beta$. Then one calculates the Fourier-harmonic components ${}_sT_{\omega\ell m}(r)$ according to

$${}_sT_{\omega\ell m}(r) = \frac{1}{2\pi}\int dt d\Omega\, {}_sT\, {}_s\bar{Y}_{\ell m}(\theta,\phi)e^{i\omega t}, \tag{3.24}$$

where $d\Omega$ is the element of solid angle. The source is

$$\begin{aligned} T_{\omega\ell m}(r) &= 2\pi\{2[\lambda(\lambda+2)]^{1/2} r^4\, {}_0T_{\omega\ell m}(r) + \\ &\quad + 2(2\lambda)^{1/2} r^2 f \mathcal{L}\, r^3 f^{-1}\, {}_{-1}T_{\omega\ell m}(r) + \\ &\quad + r f \mathcal{L} r^4 f^{-1} \mathcal{L}\, r\, {}_{-2}T_{\omega\ell m}(r)\}, \end{aligned} \tag{3.25}$$

where $\mathcal{L} = fd/dr + i\omega$.

The inhomogeneous Teukolsky equation (3.22) can be integrated by means of a Green's function [21]. The solution at large radii is

$$R_{\omega\ell m}(r \to \infty) \sim \mu\, \omega^2 Z^H_{\omega\ell m} r^3 e^{i\omega r^*}, \tag{3.26}$$

where $r^* = r + 2M\ln(r/2M - 1)$, and the solution near the black-hole horizon is

$$R_{\omega\ell m}(r \to 2M) \sim \mu\, \omega^3 Z^\infty_{\omega\ell m} r^4 f^2 e^{-i\omega r^*}. \tag{3.27}$$

The amplitudes $Z^{H,\infty}_{\omega\ell m}$ are defined by

$$Z^{H,\infty}_{\omega\ell m} = \frac{1}{2i\mu\, \omega^2 Q^{\rm in}_{\omega\ell}} \int_{2M}^{\infty} dr\, \frac{R^{H,\infty}_{\omega\ell}(r) T_{\omega\ell m}(r)}{r^4 f^2}, \tag{3.28}$$

where the functions $R^H_{\omega\ell}(r)$ and $R^\infty_{\omega\ell}(r)$ are solutions of the *homogeneous* Teukolsky equation. $R^H_{\omega\ell}(r)$ is the solution with boundary conditions corresponding to ingoing waves at the black-hole horizon, $R^H_{\omega\ell}(r \to 2M) \sim (\omega r)^4 f^2 e^{-i\omega r^*}$; $R^H_{\omega\ell}(r)$ represents a superpostion of ingoing and outgoing waves at large radii, $R^H_{\omega\ell}(r \to \infty) \sim Q^{\rm in}_{\omega\ell}(\omega r)^{-1} e^{-i\omega r^*} + Q^{\rm out}_{\omega\ell}(\omega r)^3 e^{i\omega r^*}$. $R^\infty_{\omega\ell}(r)$ is the solution with boundary conditions corresponding to outgoing waves at infinity, $R^\infty_{\omega\ell}(r \to \infty) \sim (\omega r)^3 e^{i\omega r^*}$; $R^\infty_{\omega\ell}(r)$ represents a superposition of ingoing and outgoing waves at the horizon.

The amplitudes $Z^{H,\infty}_{\omega\ell m}$ satisfy the identities

$$\bar{Z}^{H,\infty}_{-\omega,\ell,-m} = (-1)^\ell Z^{H,\infty}_{\omega\ell m}, \tag{3.29}$$

which we now derive. We use the fact that $u^\theta = 0$, which implies ${}_s\bar{T} = (-1)^s {}_sT$; substitution into Eq. (3.24), using ${}_sY_{\ell,-m}(\theta,\phi) = (-1)^{s+\ell} {}_s\bar{Y}_{\ell m}(\theta,\phi)$, then yields

${}_s\bar{T}_{-\omega,\ell,-m}(r) = (-1)^\ell {}_sT_{\omega\ell m}(r)$. It follows from this and Eq. (3.25) that $\bar{T}_{-\omega,\ell,-m}(r) = (-1)^\ell T_{\omega\ell m}(r)$. The homogeneous Teukolsky equation is invariant under complex conjugation followed by $\omega \to -\omega$, so $\bar{R}^{H,\infty}_{-\omega,\ell}(r) = R^{H,\infty}_{\omega\ell}(r)$ and $\bar{Q}^{\rm in}_{-\omega,\ell} = Q^{\rm in}_{\omega\ell}$. Eq. (3.29) finally follows from Eq. (3.28).

3.3.2 Waveforms; energy and angular momentum fluxes

At large distances, the two fundamental polarizations of the gravitational waves, h_+ and $h_\times$, can be obtained from Eqs. (3.21) and (3.26); they are

$$h_+ - ih_\times = \frac{2\mu}{r}\sum_{\ell m} {}_{-2}Y_{\ell m}\int_{-\infty}^{\infty} d\omega\, Z^H_{\omega\ell m}e^{-i\omega u}, \tag{3.30}$$

where $u = t - r^*$ represents retarded time. The transverse traceless gravitational-wave tensor is

$$h^{\rm TT}_{ab} = (h_+ - ih_\times)m_a m_b + (h_+ + ih_\times)\bar{m}_a\bar{m}_b. \tag{3.31}$$

The rates at which gravitational waves carry energy and angular momentum to infinity can be calculated from the Isaacson stress-energy tensor [22], which is constructed from $h^{\rm TT}_{ab}$. An alternative but equivalent method involves reading off the multipole moments of the radiative field, as defined by Thorne [23], and using the relevant equations of Ref. [23] to calculate $\dot{E}^\infty$ and $\dot{L}^\infty$. To present the results, we now specialize to the case considered in this paper, in which the frequency spectrum of the waves is characterized by a discrete set of distinct frequencies ω_k. Then

$$Z^H_{\omega\ell m} = \sum_k Z^{Hk}_{\ell m}\delta(\omega - \omega_k), \tag{3.32}$$

and

$$\dot{E}^{\infty} = \frac{\mu^2}{4\pi}\sum_{\ell mk} {\omega_k}^2 |Z^{Hk}_{\ell m}|^2, \tag{3.33}$$

$$\dot{L}^{\infty} = \frac{\mu^2}{4\pi}\sum_{\ell mk} \frac{m}{\omega_k} {\omega_k}^2 |Z^{Hk}_{\ell m}|^2. \tag{3.34}$$

The rates at which the black hole absorbs energy and angular momentum can be calculated along similar lines [24]. From $\Psi_4(r \to 2M)$ one recovers the gravitational-wave tensor, from which the Isaacson stress-energy tensor is calculated. The calculation of the fluxes then reproduces the results of Teukolsky and Press [25], which were derived in a completely different manner:

$$\dot{E}^{H} = \frac{\mu^2}{4\pi M^2}\sum_{\ell mk} \alpha^k_\ell |Z^{\infty k}_{\ell m}|^2, \tag{3.35}$$

$$\dot{L}^{H} = \frac{\mu^2}{4\pi M^2}\sum_{\ell mk} \frac{m}{\omega_k} \alpha^k_\ell |Z^{\infty k}_{\ell m}|^2, \tag{3.36}$$

for

$$Z^{\infty}_{\omega\ell m} = \sum_k Z^{\infty k}_{\ell m}\,\delta(\omega - \omega_k). \tag{3.37}$$

We have introduced

$$\alpha^k_\ell = \frac{2^{12}[1 + 4(M\omega_k)^2][1 + 16(M\omega_k)^2]}{[\lambda(\lambda + 2)]^2 + 144(M\omega_k)^2}\,(M\omega_k)^8. \tag{3.38}$$

The total rates of loss of energy and angular momentum are then $\dot{E}^{(\mathrm{GW})} = \dot{E}^{\infty} + \dot{E}^{H}$, and $\dot{L}^{(\mathrm{GW})} = \dot{L}^{\infty} + \dot{L}^{H}$.

3.3.3 Proof that $\dot{E}^{(0)} = \Omega \dot{L}^{(0)}$

For circular motion, the particle's stress-energy tensor is proportional to $\delta(\phi - \Omega t)$. Eqs. (3.24) and (3.25) then imply $T_{\omega \ell m} \propto \delta(\omega - m\Omega)$—the wave frequency ω is a harmonic of the orbital frequency Ω. Eq. (3.28) further implies $Z^{H,\infty}_{\omega \ell m} \propto \delta(\omega - m\Omega)$, so that we can write

$$Z^{H,\infty}_{\omega \ell m} = A^{H,\infty}_{\ell m} \delta(\omega - m\Omega), \tag{3.39}$$

which is a special case of Eqs. (3.32) and (3.37), with $\omega_k = m\Omega$. Eqs. (3.33)–(3.36) then yield

$$\dot{E}^{\infty} = \Omega \dot{L}^{\infty} = \frac{\mu^2}{4\pi} \sum_{\ell m} (m\Omega)^2 |A^H_{\ell m}|^2, \tag{3.40}$$

and

$$\dot{E}^{H} = \Omega \dot{L}^{H} = \frac{\mu^2}{4\pi M^2} \sum_{\ell m} \alpha_\ell |A^{\infty}_{\ell m}|^2, \tag{3.41}$$

where $\alpha_\ell = \alpha^k_\ell(\omega_k = m\Omega)$. Finally, Eqs. (3.40) and (3.41) imply $\dot{E}^{(0)} = \Omega \dot{L}^{(0)}$. Notice that the proof does not require the explicit calculation of $A^{H,\infty}_{\ell m}$. The key to the proof is the observation that for a mode of given m and ω_k, $\dot{E}^{\infty,H} / \dot{L}^{\infty,H} = \omega_k / m$. This property is very general and holds for arbitrary fields; cf. Ref. [26].

3.4 Gravitational waves from slightly eccentric motion

3.4.1 First step—slightly eccentric motion

The first step of the calculation consists of solving the geodesic equations for slightly eccentric orbits. We begin with the radial equation. Eqs. (3.9) imply

$$(dr/dt)^2 + U(\tilde{E}, \tilde{L}, r) = 0, \tag{3.42}$$

where

$$U(\tilde{E}, \tilde{L}, r) = (f/\tilde{E})^2[V(\tilde{L}, r) - \tilde{E}^2]. \tag{3.43}$$

Our strategy is to expand $r(t)$ according to

$$r(t) = r_0[1 + \varepsilon\xi^{(1)}(t) + \varepsilon^2\xi^{(2)}(t) + O(\varepsilon^3)], \tag{3.44}$$

and to similarly expand $U(\tilde{E}, \tilde{L}, r)$, using Eqs. (3.11) and (3.12). Collecting terms of equal order in ε yields (i) a differential equation for $\xi^{(1)}(t)$,

$$(d\xi^{(1)}/dt)^2 = {\Omega_r}^2(1 - \xi^{(1)\,2}), \tag{3.45}$$

where

$$\Omega_r = \Omega(1 - 6v^2)^{1/2} \tag{3.46}$$

is the *radial frequency*—the fundamental frequency of radial motion; and (ii) a linear differential equation for $\xi^{(2)}(t)$,

$$\frac{1}{{\Omega_r}^2}\frac{d\xi^{(1)}}{dt}\frac{d\xi^{(2)}}{dt} + \xi^{(1)}\xi^{(2)} = -\frac{1-7v^2}{1-6v^2} + \frac{2v^2}{1-2v^2}\xi^{(1)} + \frac{1-11v^2+26v^4}{(1-2v^2)(1-6v^2)}\xi^{(1)\,3}. \tag{3.47}$$

Eq. (3.45) can be integrated to give

$$\xi^{(1)}(t) = \cos \Omega_r t, \tag{3.48}$$

where the time origin is chosen so that $r(t = 0) = r_0(1 + \varepsilon)$. Substitution of Eq. (3.48) into (3.47) then yields, after integration,

$$\xi^{(2)}(t) = q_1(v)(1 - \cos \Omega_r t) + q_2(v)(1 - \cos 2\Omega_r t), \tag{3.49}$$

where $q_1(v) = (1-7v^2)(1-6v^2)^{-1}$, and $q_2(v) = (1-11v^2+26v^4)[2(1-2v^2)(1-6v^2)]^{-1}$.

Integration of the azimuthal equation proceeds along similar lines. Eqs. (3.9) imply

$$d\phi/dt = (\tilde{L}/\tilde{E})(f/r^2), \tag{3.50}$$

which may be expanded in powers of ε using Eqs. (3.11), (3.12), (3.44), (3.48), and (3.49). Integration then yields

$$\begin{aligned} \phi(t) &= \Omega_\phi t - \varepsilon p_1(v) \sin \Omega_r t + \varepsilon^2 p_2(v) \sin \Omega_r t + \\ &\quad + \varepsilon^2 p_3(v) \sin 2\Omega_r t + O(\varepsilon^3), \end{aligned} \tag{3.51}$$

where $p_1(v) = 2(1 - 3v^2)[(1 - 2v^2)(1 - 6v^2)^{1/2}]^{-1}$, $p_2(v) = 2(1 - 3v^2)(1 - 7v^2)[(1 - 2v^2)(1 - 6v^2)^{3/2}]^{-1}$, $p_3(v) = (5 - 64v^2 + 250v^4 - 300v^6)[4(1 - 2v^2)^2(1 - 6v^2)^{3/2}]^{-1}$; and

$$\Omega_\phi = \left[1 - \frac{3(1 - 3v^2)(1 - 8v^2)}{2(1 - 2v^2)(1 - 6v^2)}\varepsilon^2\right]\Omega \tag{3.52}$$

is the *azimuthal frequency*—the fundamental frequency of azimuthal motion. That $\Omega_\phi \neq \Omega_r$ reflects the fact that eccentric orbits in Schwarzschild are not closed.

3.4.2 The remaining steps—an overview

The next steps of the calculation consist of (i) substituting the results of the preceding subsection into the expression for the particle's stress-energy tensor,

$$T^{\alpha\beta} = \mu \frac{u^\alpha u^\beta}{r^2 u^t} \delta[r - r(t)] \delta(\cos\theta) \delta[\phi - \phi(t)]; \tag{3.53}$$

(ii) constructing the projections ${}_sT$, and (iii) expanding to second order in the eccentricity. In particular, we must expand $r - r(t)$ about $r - r_0$, thereby introducing derivatives of the radial δ-function; and expand $\phi - \phi(t)$ about $\phi - \Omega_\phi t$, which introduces derivatives of the azimuthal δ-function.

The next task is to obtain the Fourier-harmonic components ${}_sT_{\omega\ell m}(r)$, using Eq. (3.24). The integration over ϕ implies that the derivatives of $\delta(\phi - \Omega_\phi t)$ are integrated by parts, and the n-th derivative of $\delta(\phi - \Omega_\phi t)$ is therefore equivalent to $(im)^n \delta(\phi - \Omega_\phi t)$.

Once the source to the Teukolsky equation has been evaluated using Eq. (3.25), we calculate $Z^{H,\infty}_{\omega\ell m}$ using Eq. (3.28). Since the source has support only at $r = r_0$, the integral can be performed analytically, and involves several integrations by parts. As a result, $Z^{H,\infty}_{\omega\ell m}$ can be expressed as a function of (i) r_0, (ii) the functions $R^{H,\infty}_{\omega\ell}(r)$ and their derivatives at $r = r_0$, and (iii) the coefficient $Q^{\text{in}}_{\omega\ell}$.

In weak-field, slow-motion situations (r_0 large), the analytical techniques developed in paper I [18] may be used to calculate, approximately, $R^H_{\omega\ell}(r)$ and $Q^{\text{in}}_{\omega\ell}$. The result is an analytical expression for $Z^H_{\omega\ell m}$, valid for $r_0 \gg 6M$. Since $\dot{E}^H/\dot{E}^\infty$ and

$\dot{L}^H/\dot{L}^\infty$ are of order v^8 and hence very small [14, 27], the weak-field, slow-motion calculation does not require the computation of $Z^\infty_{\omega\ell m}$.

In a strong-field situation, $R^{H,\infty}_{\omega\ell}(r)$ and $Q^{\rm in}_{\omega\ell}$ must be obtained, for a given value of r_0, by numerically integrating the homogeneous Teukolsky equation. The result is then a numerical expression for $Z^{H,\infty}_{\omega\ell m}$, valid for that value of r_0.

Once $Z^{H,\infty}_{\omega\ell m}$ has been obtained, we observe that the continuous sum over ω reduces to a discrete sum, as in Eqs. (3.32) and (3.37). We then calculate $\dot{E}^{(\rm GW)}$ and $\dot{L}^{(\rm GW)}$ with the help of Eqs. (3.33)–(3.36). Finally, Eqs. (3.18) and (3.19) are used to calculate $\dot{r_0}$ and $\dot{\varepsilon}/\varepsilon$.

3.4.3 Frequency spectrum, simplifications, and proof that $\dot{E}^{(1)} = \dot{L}^{(1)} = 0$

Each step of the calculation, as outlined in the preceding subsection, would require an extremely long and tedious computation if some remarkable simplifications did not occur along the way. These simplifications arise because: (i) The gravitational waves possess a frequency spectrum characterized by a discrete set of frequencies. As in the circular case, the waves have frequencies equal to the harmonics of the azimuthal frequency, $\omega = m\Omega_\phi$. However, a small eccentricity also implies the existence of *side bands* [28], at $\omega = m\Omega_\phi \pm \Omega_r$, and $\omega = m\Omega_\phi \pm 2\Omega_r$. (ii) The calculation of $\dot{E}^{(\rm GW)}$ and $\dot{L}^{(\rm GW)}$ includes a time averaging, which causes a large number of terms to vanish. In particular, all $O(\varepsilon)$ terms average out, as do most

$O(\varepsilon^2)$ terms. And (iii) the calculation of $\dot{\varepsilon}/\varepsilon$ only requires the computation of $\dot{E}^{(2)} - \Omega \dot{L}^{(2)}$, which also generates important cancellations.

We now look more closely into the nature of the waves' frequency spectrum. The calculation of ${}_sT_{\omega\ell m}(r)$ was outlined in subsection B. After the angular integration has been performed, it is clear from Eqs. (3.48), (3.49), (3.51), and (3.53) that the next step is to integrate over time terms which are proportional to: (i) $e^{i(\omega - m\Omega_\phi)t}$; (ii) $e^{\pm i\Omega_r t} e^{i(\omega - m\Omega_\phi)t}$; and (iii) $e^{\pm 2i\Omega_r t} e^{i(\omega - m\Omega_\phi)t}$. It is also clear that the terms with dependence (i) are dominantly $O(\varepsilon^0)$, while the terms with dependence (ii) are dominantly $O(\varepsilon)$, and the terms with dependence (iii) are dominantly $O(\varepsilon^2)$. Correspondingly, time integration yields terms which are proportional to: (i) $\delta(\phi - m\Omega_\phi)$, with magnitude $O(\varepsilon^0)$; (ii) $\delta(\phi - m\Omega_\phi \pm \Omega_r)$, with magnitude $O(\varepsilon)$; and (iii) $\delta(\phi - m\Omega_\phi \pm 2\Omega_r)$, with magnitude $O(\varepsilon^2)$. Finally, Eqs. (3.25), (3.28), and (3.30) imply that the gravitational waves possess the frequency spectrum described previously.

Our schematic considerations can be pushed further. It is indeed clear from the results obtained thus far that $Z^{H,\infty}_{\omega\ell m}$ must have the following structure (we momentarily remove the H, ∞ subscripts for the sake of clarity):

$$\begin{aligned} Z_{\omega\ell m} &= A_{\ell m}\delta(\omega - \omega_m) - \tfrac{1}{2}B^-_{\ell m}\delta(\omega - \omega_-)\varepsilon - \tfrac{1}{2}B^+_{\ell m}\delta(\omega - \omega_+)\varepsilon \\ &+ C_{\ell m}\delta(\omega - \omega_m)\varepsilon^2 + D^-_{\ell m}\delta(\omega - \omega_-)\varepsilon^2 + D^+_{\ell m}\delta(\omega - \omega_+)\varepsilon^2 \\ &+ E^{-2}_{\ell m}\delta(\omega - \omega_{-2})\varepsilon^2 + E^{+2}_{\ell m}\delta(\omega - \omega_{+2})\varepsilon^2 + O(\varepsilon^3), \end{aligned} \tag{3.54}$$

where $\omega_m = m\Omega_\phi$, $\omega_\pm = m\Omega_\phi \pm \Omega_r$, and $\omega_{\pm 2} = m\Omega_\phi \pm 2\Omega_r$. The various coefficients of the δ-functions are expected to be complicated functions of (i) r_0, (ii) $R^{H,\infty}_{\omega\ell}(r)$ and their derivatives at $r = r_0$, and (iii) $Q^{\rm in}_{\omega\ell}$. All these coefficients *can* be calculated with the help of the equations presented in this and the preceding section; however, we shall now show that only a small number actually *need* be calculated.

Substitution of Eq. (3.54) into (3.33)–(3.36), using (3.32) and (3.37), yields

$$\begin{aligned}
\dot{E}^\infty &= \frac{\mu^2}{4\pi}\sum_{\ell m}{\omega_m}^2\Big[|A^H_{\ell m}+\varepsilon^2 C^H_{\ell m}|^2+\Big(\frac{\omega_-}{\omega_m}\Big)^2|\tfrac{1}{2}B^{H-}_{\ell m}|^2\varepsilon^2\\
&+ \Big(\frac{\omega_+}{\omega_m}\Big)^2|\tfrac{1}{2}B^{H+}_{\ell m}|^2\varepsilon^2+O(\varepsilon^3)\Big], \qquad (3.55)\\
\Omega_\phi\dot{L}^\infty &= \frac{\mu^2}{4\pi}\sum_{\ell m}{\omega_m}^2\Big[|A^H_{\ell m}+\varepsilon^2 C^H_{\ell m}|^2+\frac{\omega_-}{\omega_m}|\tfrac{1}{2}B^{H-}_{\ell m}|^2\varepsilon^2\\
&+ \frac{\omega_+}{\omega_m}|\tfrac{1}{2}B^{H+}_{\ell m}|^2\varepsilon^2+O(\varepsilon^3)\Big], \qquad (3.56)\\
\dot{E}^H &= \frac{\mu^2}{4\pi M^2}\sum_{\ell m}\alpha_\ell\Big[|A^\infty_{\ell m}+\varepsilon^2 C^\infty_{\ell m}|^2+\frac{\alpha_\ell^-}{\alpha_\ell}|\tfrac{1}{2}B^{\infty-}_{\ell m}|^2\varepsilon^2\\
&+ \frac{\alpha_\ell^+}{\alpha_\ell}|\tfrac{1}{2}B^{\infty+}_{\ell m}|^2\varepsilon^2+O(\varepsilon^3)\Big], \qquad (3.57)\\
\Omega_\phi\dot{L}^H &= \frac{\mu^2}{4\pi M^2}\sum_{\ell m}\alpha_\ell\Big[|A^\infty_{\ell m}+\varepsilon^2 C^\infty_{\ell m}|^2+\frac{\alpha_\ell^-}{\alpha_\ell}\frac{\omega_m}{\omega_-}|\tfrac{1}{2}B^{\infty-}_{\ell m}|^2\varepsilon^2\\
&+ \frac{\alpha_\ell^+}{\alpha_\ell}\frac{\omega_m}{\omega_+}|\tfrac{1}{2}B^{\infty+}_{\ell m}|^2\varepsilon^2+O(\varepsilon^3)\Big], \qquad (3.58)
\end{aligned}$$

where $\alpha_\ell = \alpha^k_\ell(\omega_k = \omega_m)$ and $\alpha^\pm_\ell = \alpha^k_\ell(\omega_k = \omega_\pm)$. These results teach us that the coefficients $D^{H,\infty\pm}_{\ell m}$ and $E^{H,\infty\pm 2}_{\ell m}$ are irrelevant to our calculation; their contributions vanish after the time averaging has been carried out. More simplifications arise below.

Eqs. (3.55)–(3.58) imply that corrections to $\dot{E}^{(\rm GW)}$ and $\dot{L}^{(\rm GW)}$, due to nonvanishing eccentricity, are second-order in ε. Thus $\dot{E}^{(1)} = \dot{L}^{(1)} = 0$, as was first written in

Eq. (3.17). The proof that circular orbits remain circular under radiation reaction is now complete.

3.4.4 Calculation of $\dot{r_0}$ and $\mu\dot{\varepsilon}/\varepsilon$

The calculation of $\dot{r_0}$ is almost complete. Explicit expressions for $A_{\ell m}^{H,\infty}$ will be given in subsection E; these may be used together with Eqs. (3.40) and (3.41) to calculate $\dot{E}^{(0)}$, which is then substituted in Eq. (3.18).

The calculation of $\mu\dot{\varepsilon}/\varepsilon$ requires the computation of $\dot{E}^{(0)}$ and $\dot{E}^{(2)} - \Omega\dot{L}^{(2)}$. In Eqs. (3.55)–(3.58), a number of terms are *explicitly* second-order in the eccentricity; others are $O(\varepsilon^2)$ only *implicitly*, by virtue of the fact that $\Omega_\phi = \Omega(1-\Delta\Omega\varepsilon^2)$, where $\Delta\Omega$ can be read off from Eq. (3.52). To make all dependence on ε explicit, we now adapt our notation so that $\omega_\pm = m\Omega \pm \Omega_r$, and write ${\omega_m}^2|A_{\ell m}^{H,\infty}|^2 = (m\Omega)^2|A_{\ell m}^{H,\infty}|^2 + O(\varepsilon^2)$. It follows that the quantity $\dot{E}^{(2)} - \Omega\dot{L}^{(2)} + \Delta\Omega\dot{E}^{(0)}$ only requires the calculation of the coefficients $B_{\ell m}^{H,\infty\pm}$. With the help of Eq. (3.19), we finally obtain

$$\mu\dot{\varepsilon}/\varepsilon = -\frac{(1-2v^2)(1-3v^2)^{1/2}}{v^2(1-6v^2)}[\Gamma - h(v)\dot{E}^{(0)}], \tag{3.59}$$

where $\Gamma = \Gamma^\infty + \Gamma^H$, with

$$\Gamma^\infty = \frac{\mu^2}{16\pi}\Omega_r \sum_{\ell m}\Big(\omega_+|B_{\ell m}^{H+}|^2 - \omega_-|B_{\ell m}^{H-}|^2\Big), \tag{3.60}$$

where $\omega_\pm = m\Omega \pm \Omega_r$; and

$$\Gamma^H = \frac{\mu^2}{16\pi M^2}\Omega_r \sum_{\ell m}\Big(\frac{\alpha_\ell^+}{\omega_+}|B_{\ell m}^{\infty+}|^2 - \frac{\alpha_\ell^-}{\omega_-}|B_{\ell m}^{\infty-}|^2\Big). \tag{3.61}$$

We also have

$$h(v)=\frac{1-12v^2+66v^4-108v^6}{2(1-2v^2)^2(1-6v^2)}, \tag{3.62}$$

with $v=(M/r_0)^{1/2}$.

Eqs. (3.59) – (3.62) imply that the calculation of $\mu\dot{\varepsilon}/\varepsilon$ is much simpler than the individual computations, to second order in the eccentricity, of $\dot{E}^{(\mathrm{GW})}$ and $\dot{L}^{(\mathrm{GW})}$. Because of the occurrence of important cancellations, the calculation only requires the computation of $B^{H,\infty\pm}_{\ell m}$, and the leading-order part of $A^{H,\infty}_{\ell m}$. Computation of all other coefficients, as well as the $O(\varepsilon^2)$ part of $A^{H,\infty}_{\ell m}$, is superfluous.

Because of those various cancellations, the calculation of $\mu\dot{\varepsilon}/\varepsilon$ may now proceed in complete ignorance of the $O(\varepsilon^2)$ corrections to the motion of the particle. The only essential correction, the $O(\varepsilon^2)$ part of Ω_ϕ, has already been incorporated into Eq. (3.59). The computation of $B^{H,\infty\pm}_{\ell m}$ only requires a calculation accurate to first order in the eccentricity.

3.4.5 Calculation of $A^{H,\infty}_{\ell m}$ and $B^{H,\infty\pm}_{\ell m}$

The calculation follows the lines of subsection B above. We find

$$A^{H,\infty}_{\ell m}=\frac{\pi}{i(\omega_m r_0)^2 Q^{\mathrm{in}}_{\omega_m \ell}}\Big({}_0A^{H,\infty}_{\ell m}+{}_{-1}A^{H,\infty}_{\ell m}+{}_{-2}A^{H,\infty}_{\ell m}\Big), \tag{3.63}$$

where (we momentarily remove all unnecessary indices for the sake of clarity)

$$\begin{aligned}
{}_0A &= {}_0a f_0 R,\\
{}_{-1}A &= {}_{-1}a f_0[(2f_0+i\omega_m r_0)R-f_0 r_0 R'],
\end{aligned} \tag{3.64}$$

$$
{}_{-2}A = {}_{-2}af_0[i\omega_m r_0(2-2v^2+i\omega_m r_0)R - 2(f_0+i\omega_m r_0)f_0 r_0 R' + (f_0 r_0)^2 R''].
$$

Here, $\omega_m = m\Omega$, $f_0 = 1-2M/r_0 = 1-2v^2$, $R = R^{H,\infty}_{\omega_m \ell}(r_0)$, and a prime denotes differentiation with respect to r_0. Also

$$
B^{H,\infty\pm}_{\ell m} = \frac{\pi}{i(\omega_\pm r_0)^2 Q^{\text{in}}_{\omega_\pm \ell}}\Big({}_0B^{H,\infty\pm}_{\ell m} + {}_{-1}B^{H,\infty\pm}_{\ell m} + {}_{-2}B^{H,\infty\pm}_{\ell m}\Big), \tag{3.65}
$$

where

$$
\begin{aligned}
{}_0B^\pm &= {}_0c^\pm R_\pm - {}_0a(f_0 r_0 R'_\pm - 4v^2 R_\pm),\\
{}_{-1}B^\pm &= {}_{-1}c^\pm[(2-4v^2+i\omega_\pm r_0)R_\pm - f_0 r_0 R'_\pm] +\\
&\quad + {}_{-1}a[(4v^2-8v^4+6iM\omega_\pm - i\omega_\pm r_0)R_\pm\\
&\quad - f_0(1+i\omega_\pm r_0)r_0 R'_\pm + (f_0 r_0)^2 R''_\pm],\\
{}_{-2}B^\pm &= {}_{-2}c_\pm[(f_0 r_0)^2 R''_\pm - 2f_0(1-2v^2+i\omega_\pm r_0)r_0 R'_\pm\\
&\quad + i\omega_\pm r_0(2-2v^2+i\omega_\pm r_0)R_\pm] -\\
&\quad - {}_{-2}a[(f_0 r_0)^3 R'''_\pm - 2i\omega_\pm r_0 (f_0 r_0)^2 R''_\pm\\
&\quad - F_0(2-8v^2+8v^4-10iM\omega_\pm + 2i\omega_\pm r_0 + {\omega_\pm}^2 {r_0}^2)r_0 R'_\pm +\\
&\qquad + 2i\omega_\pm r_0(1-6v^2+4v^4-4iM\omega_\pm + i\omega_\pm r_0)R_\pm],
\end{aligned} \tag{3.66}
$$

with $\omega_\pm = m\Omega \pm \Omega_r$ and $R_\pm = R^{H,\infty}_{\omega_\pm \ell}(r_0)$. We have introduced

$$
\begin{aligned}
{}_0a_{\ell m} &= [\lambda(\lambda+2)]^{1/2}\,{}_0Y_{\ell m}(\tfrac{\pi}{2},0)\,[2(1-2v^2)(1-3v^2)^{1/2}]^{-1},\\
{}_{-1}a_{\ell m} &= i\lambda^{1/2}\,{}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0)\,v[(1-2v^2)^2(1-3v^2)^{1/2}]^{-1},\\
{}_{-2}a_{\ell m} &= -{}_{-2}Y_{\ell m}(\tfrac{\pi}{2},0)\,v^2[2(1-2v^2)^3(1-3v^2)^{1/2}]^{-1},
\end{aligned} \tag{3.67}
$$

where $\lambda = (\ell-1)(\ell+2)$, and

$$ {}_sc^{\pm}_{\ell m} = {}_sa_{\ell m}\Big[2-s-2(3-s)v^2 \pm i(2+s)v(1-6v^2)^{1/2} \pm 2m(1-3v^2)(1-6v^2)^{-1/2}\Big]. \quad (3.68)$$

The previous equations imply the following symmetry properties: $\bar{A}^{H,\infty}_{\ell,-m} = (-1)^\ell A^{H,\infty}_{\ell m}$; $\bar{B}^{H,\infty\mp}_{\ell,-m} = (-1)^\ell B^{H,\infty\pm}_{\ell m}$, and for $m=0$, $\bar{B}^{H,\infty-}_{\ell,0} = B^{H,\infty+}_{\ell,0}$.

3.4.6 The adiabatic approximation

We conclude this section by formulating the conditions under which the adiabatic approximation holds. The results of this subsection were summarized in Sec. I B.

We require that the *inspiral timescale* $r_0/|\dot{r_0}|$ always be much smaller than the *orbital period* $2\pi/\Omega_r$. Using Eq. (3.18), this requirement becomes

$$\mu/M \ll \frac{1}{4\pi}\frac{v^5(1-6v^2)^{3/2}}{(1-3v^2)^{3/2}}\frac{1}{(M/\mu)^2\dot{E}^{(0)}}. \quad (3.69)$$

At large radii, $r_0 \gg 6M$, $(M/\mu)^2\dot{E}^{(0)} \simeq 32v^{10}/5$ and the adiabatic condition (3.69) becomes $\mu/M \ll (5/128\pi)v^{-5}$. This is superseded by a wide margin by the condition $\mu/M \ll 1$, which ensures that the gravitational perturbations are linear. Near $r_0 = 6M$, we may use the numerical results of Sec. V C and put $(M/\mu)^2\dot{E}^{(0)} \simeq 9\times10^{-4}$, and Eq. (3.69) becomes $\mu/M \ll 2.8(1-6v^2)^{3/2}$. This condition is far more restrictive than $\mu/M \ll 1$.

3.5 Analytical and numerical results

3.5.1 Weak-field, slow-motion case

For $r_0 \gg 6M$ and $v = (M/r_0)^{1/2} \ll 1$, the analytical techniques developed in paper I may be used to calculate, approximately, $R^H_{\omega\pm\ell}(r)$ and $Q^{\rm in}_{\omega\pm\ell}$. The expressions for these quantities may then be substituted into the equations of Sec. IV D and E, to obtain $\mu\dot{\varepsilon}/\varepsilon$ in the form of a post-Newtonian expansion. As was mentioned previously, there is no need to calculate $\dot{E}^H$ and $\dot{L}^H$, because they contribute only at order v^8 to the post-Newtonian expansion [27]. The calculations are straightforward and will be presented without much detail.

The calculation of $\mu\dot{\varepsilon}/\varepsilon$ up through order v^3 beyond Newtonian requires the computation of $B^{H\pm}_{\ell m}$ for $\ell = 2$ and $\ell = 3$. We may use the symmetry properties of $B^{H\pm}_{\ell m}$ and only consider nonnegative values of m; for $m = 0$, only $B^{H+}_{\ell,0}$ is required. We find

$$
\begin{aligned}
B^{H+}_{2,2} &= (\pi/5)^{1/2} v^2[-18 + 27v^2 - 54\pi v^3 + O(iv^3, v^4)], \\
B^{H-}_{2,2} &= (\pi/5)^{1/2} v^2[6 + \tfrac{221}{7}v^2 + 6\pi v^3 + O(iv^3, v^4)], \\
B^{H+}_{2,1} &= (\pi/5)^{1/2} v^2[-\tfrac{16}{3}iv + \tfrac{176}{21}iv^3 + O(v^4)], \\
B^{H-}_{2,1} &= O(v^6), \\
B^{H+}_{2,0} &= (\pi/30)^{1/2} v^2[-4 + \tfrac{206}{7}v^2 - 4\pi v^3 + O(iv^3, v^4)], \\
B^{H+}_{3,3} &= (\pi/42)^{1/2} v^2[64iv + O(v^3)],
\end{aligned}
$$

$$B^{H-}_{3,3} = (\pi/42)^{1/2}v^2[-24iv + O(v^3)],$$

$$B^{H+}_{3,1} = (\pi/70)^{1/2}v^2[\tfrac{8}{3}iv + O(v^3)],$$

$$B^{H-}_{3,1} = O(v^6), \tag{3.70}$$

and $B^{H\pm}_{3,m} = O(v^5)$ for $m = \{0, 2\}$. In the above, the notation $O(iv^3)$ signifies that those terms of order v^3, which are purely imaginary, do not contribute, at order v^3, to $|B^{H\pm}_{\ell m}|^2$. That the coefficients $B^{H-}_{\ell,1}$ are so small is due to the fact that, for $m = 1$, $\omega_- = \Omega - \Omega(1-6v^2)^{1/2} = 3v^2\Omega + O(v^4)$; since ω_- is suppressed by a factor v^2 with respect to ω_+, the resulting $B^{H-}_{\ell,1}$ is much smaller than $B^{H+}_{\ell,1}$.

We now substitute Eqs. (3.70) into (3.60) and (3.59), and use the post-Newtonian expansion

$$\dot{E}^{(0)} = \dot{E}_N\Big[1 - \frac{1247}{336}v^2 + 4\pi v^3 + O(v^4)\Big] \tag{3.71}$$

derived in paper I [$\dot{E}_N = \frac{32}{5}(\mu/M)^2 v^{10}$ is the leading-order, Newtonian expression]; this yields

$$\dot{\varepsilon} = \dot{\varepsilon}_N\Big[1 - \frac{6849}{2128}v^2 + \frac{985}{152}\pi v^3 + O(v^4)\Big], \tag{3.72}$$

where $\dot{\varepsilon}_N$ is the leading-order, Newtonian expression,

$$\mu\dot{\varepsilon}_N/\varepsilon = -\tfrac{304}{15}(\mu/M)^2 v^8. \tag{3.73}$$

Throughout the post-Newtonian regime, $v \ll 1$, $\dot{\varepsilon}$ is negative—radiation reaction therefore reduces the eccentricity.

Substitution of Eq. (3.71) into (3.18), and use of Eqs. (3.72) and (3.73) yields Eq. (3.4).

3.5.2 Highly relativistic case

Analytical calculations may also be carried out in the case where r_0 approaches $6M$. Because $h(v)$ diverges when $v^2 \to 1/6$, cf. Eq. (3.62), and because both $\dot{E}^{(0)}$ and Γ have well-defined limits when $r_0 \to 6M$, $\mu\dot{\varepsilon}/\varepsilon$ is dominated by the second term on the right-hand side of Eq. (3.59).

Our claim that $\dot{E}^{(0)}$ is well behaved in the vicinity of $r_0 = 6M$ can be substantiated by (i) an inspection of the perturbation formalism, which shows no sign of a singularity at $r_0 = 6M$; in particular, $R^{H,\infty}_{\omega\ell}(r)$ and $Q^{\rm in}_{\omega\ell}$, for $M\omega = mM\Omega = 6^{-3/2}m$, are well behaved. And (ii) with numerical calculations, which confirm the proper behavior of $\dot{E}^{(0)}$ in the vicinity of $r_0 = 6M$.

The proper behavior of Γ can be established as follows. Writing $\delta = (1 - 6M/r_0)^{1/2} \ll 1$, we first infer the various δ-dependence of the relevant quantities. Using the equations of Sec. IV E, we find that the ${}_sa_{\ell m}$ are independent of δ, while ${}_sc^{\pm}_{\ell m} = \pm m\, {}_sa_{\ell m}\delta^{-1} + O(\delta^0)$. Using the fact that $R^{H,\infty}_{\omega_\pm \ell}(r)$ and $Q^{\rm in}_{\omega_\pm \ell}$ are properly behaved, Eq. (3.65) then implies $B^{\pm}_{\ell m} = \pm k_{\ell m}\delta^{-1} + k^{\pm}_{\ell m} + O(\delta)$, where $k_{\ell m}$ and $k^{\pm}_{\ell m}$ are independent of δ. The fact that, at leading order in δ, $B^{+}_{\ell m}$ and $B^{-}_{\ell m}$ differ only by a sign is an important aspect of this discussion. [The case $m = 0$ requires special thought, since then $\omega_\pm = \pm\delta\Omega$, and Eq. (3.65) suggests that $B^{\pm}_{\ell,0}$ might be more singular than $O(\delta^{-1})$. However, a careful study of the Teukolsky equation reveals that this does not happen.] The final step is to substitute our result for $B^{\pm}_{\ell m}$ into Eq. (3.60), and notice a remarkable cancellation of the leading-order, $O(\delta^{-2})$ terms.

Multiplication by $\Omega_r = \delta\Omega$ then ensures that each term in the sum over ℓ and m is $O(\delta^0)$. That Γ has a well-defined limit follows from the fact that the sum converges for every $r_0 \geq 6M$, which was verified numerically.

Having established that Γ and $\dot{E}^{(0)}$ have well-defined limits when r_0 approaches $6M$, Eq. (3.59) reduces to

$$\mu\dot{\varepsilon}/\varepsilon \sim \frac{3}{2\sqrt{2}}\dot{E}^{(0)}|_{r_0=6M}(1-6M/r_0)^{-2}, \tag{3.74}$$

for $r_0 \to 6M$.

Substitution of Eq. (3.74) and (3.18) into (3.2) yields Eq. (3.4).

3.5.3 General case—numerical integration

When r_0 is neither very large nor very close to $6M$, $R^{H,\infty}_{\omega\pm\ell}(r)$ and $Q^{\rm in}_{\omega\pm\ell}$ must be calculated numerically. By performing the integration for a wide range of orbital radii, we obtain $\mu\dot{\varepsilon}/\varepsilon$ as a function of r_0. The numerical results may then be checked against the limiting cases (3.72) and (3.74).

We have carried out the numerical integration using a straightforward generalization of the algorithm presented in paper II [19] (we shall not repeat the discussion of paper II here). We have constructed our integrator upon the Bulirsh-Stoer method, using fortran subroutines given in Ref. [29]; all operations were performed with double precision. We have verified that our numerical results are in agreement with the limiting cases of subsections A and B; this agreement gives us great confidence in our results, which are summarized in Fig. 1.

It is easy to obtain high numerical accuracy by adjusting the tolerance of our integrator to a very small value; we have typically chosen a tolerance of 10^{-6}. Although it is hard to *prove*, we *believe* our numbers to be accurate to at least six significant digits. Consequently, our estimate of the critical radius r_c (at which $\dot{\varepsilon}$ changes sign) should be accurate to six significant digits; we have chosen to quote only five digits in Eq. (3.1).

The accuracy of our numerical results is also subject to errors of non-numerical origin, which arise because the infinite sum over ℓ must be truncated. The magnitude of the error thus induced can be controlled by requiring that the terms ignored contribute to a fractional error no greater than a certain value ζ. Since a multipole of order ℓ contributes a fractional amount of order $(M/r_0)^{\ell-2}$ to $\dot{E}$ and $\dot{L}$ [18], we arrive at the following criterion on the maximal value of ℓ which needs be included in the sum,

$$\ell_{\text{max}} \geq 2 - \log\zeta / \log(r_0/M). \tag{3.75}$$

For example, choosing $\zeta = 10^{-6}$ yields $\ell_{\text{max}} = 10$ for $r_0/M = 6$, and $\ell_{\text{max}} = 3$ for $r_0/M = 10^6$.

The graph of Fig. 2 was obtained by numerically integrating Eq. (3.5), in the range between $r_0/M = 6 + 10^{-8}$ and $r_0/M = 100$. The integration was performed using the extended trapezoidal rule, which is accurate enough for our purposes.

3.6 Conclusion

We have established in this paper that a particle in circular motion around a nonrotating black hole remains on a circular orbit under the influence of radiation reaction. Furthermore, we have shown that circular orbits are *stable* only if the orbital radius is greater than a critical radius $r_c \simeq 6.6792M$, where M is the mass of the black hole.

Also, our analysis permits us to follow the evolution, under radiation reaction, of an orbit's eccentricity, so long as it remains small. We find that the eccentricity is reduced by many orders of magnitude during the post-Newtonian phase of the inspiral, but that it starts increasing once the orbit's radius is smaller than r_c. For reasonable values of μ/M, the eccentricity increases by at most an order of magnitude before the adiabatic approximation breaks down and the particle begins its plunge toward the black hole.

Our analysis is restricted by four major assumptions: (i) the black hole is nonrotating, (ii) the eccentricity is always small, (iii) the gravitational perturbations are linear, and (iv) the adiabatic approximation is valid. On the other hand, our analysis is not limited to weak-field, slow-motion situations; it is valid for particle motion in strong gravitational fields.

We now examine whether any of our four assumptions could be relaxed, and at what cost, in future work.

Assumption (i) could be removed without much effort, that is, our analysis could be extended to the case of a rotating black hole, if and only if the orbit lies in the hole's equatorial plane. In the more general and more interesting situation of non-equatorial orbits, the formulation of the problem of radiation reaction would take a significantly different form. In such cases, the motion possesses a non-vanishing value of the Carter constant, whose rate of change cannot be simply (if at all) related to the rates of change of energy and (vectorial) angular momentum. The general analysis would therefore require techniques more sophisticated than the ones utilized here; for example, a numerical implementation of Gal'tsov's formalism [14].

Assumption (ii) is one of simplicity, and could be removed without introducing additional conceptual difficulties. For example, a calculation valid to higher order in the eccentricity could be carried out, at the price of a modest effort. A calculation valid to all orders in ε could also be performed by numerical integration of the geodesic equations; see Ref. [17].

Assumption (iii) cannot be removed easily. Strong-field analyses valid for arbitrary mass ratios would require either the formulation of a higher-order perturbation theory, or the complete numerical solution of Einstein's equations for the two-body problem. Both approaches are still a long way into the future. A recent analysis by Kidder, Will, and Wiseman [30] suggests that the value of the critical radius r_c should increase with the mass ratio μ/M.

Assumption (iv) could be removed (at least partially) by incorporating, at the very beginning, radiation-reaction effects into the motion of the particle. Thus the motion would be non-geodesic to begin with, and higher-order radiation-reaction effects could then be calculated. These higher-order effects would be quite small at large orbital radii; but for a given mass ratio, there exists an orbital radius r_0 at which the adiabatic approximation breaks down, and at which higher-order effects would become important. The breakdown of the adiabatic approximation, and the transition from slow inspiral to fast plunge, is discussed in Ref. [15].

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the gravitational power comes from the modes $\ell = |m| = 2$; we therefore have $\dot{E}^H/\dot{E}^\infty = v^8[1 + O(v^2)]$. This behavior is confirmed by our numerical results.

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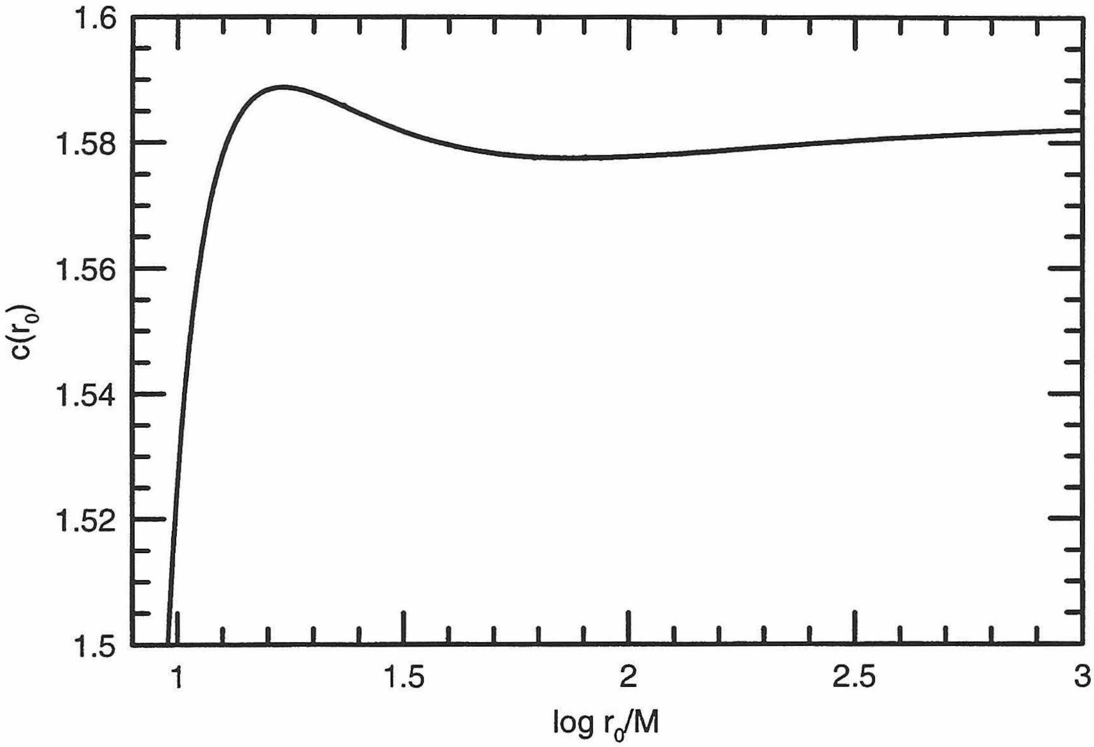

Figure 3.1: A plot of $c(r_0)$, as defined in Eq. (3.2), as a function of $\log r_0/M$. Shown is the range $10 < r_0/M < 1000$, in which $c(r_0)$ has the most interesting behavior. For $r_0/M > 1000$, $c(r_0)$ is well approximated by Eq. (3.3), and approaches the value $19/12 \simeq 1.5833$ as r_0 tends to infinity. The function $c(r_0)$ changes sign at $r_0 = r_c \simeq 6.6792M$, and approaches minus infinity when $r_0 \to 6M$, in a way well described by Eq. (3.4).

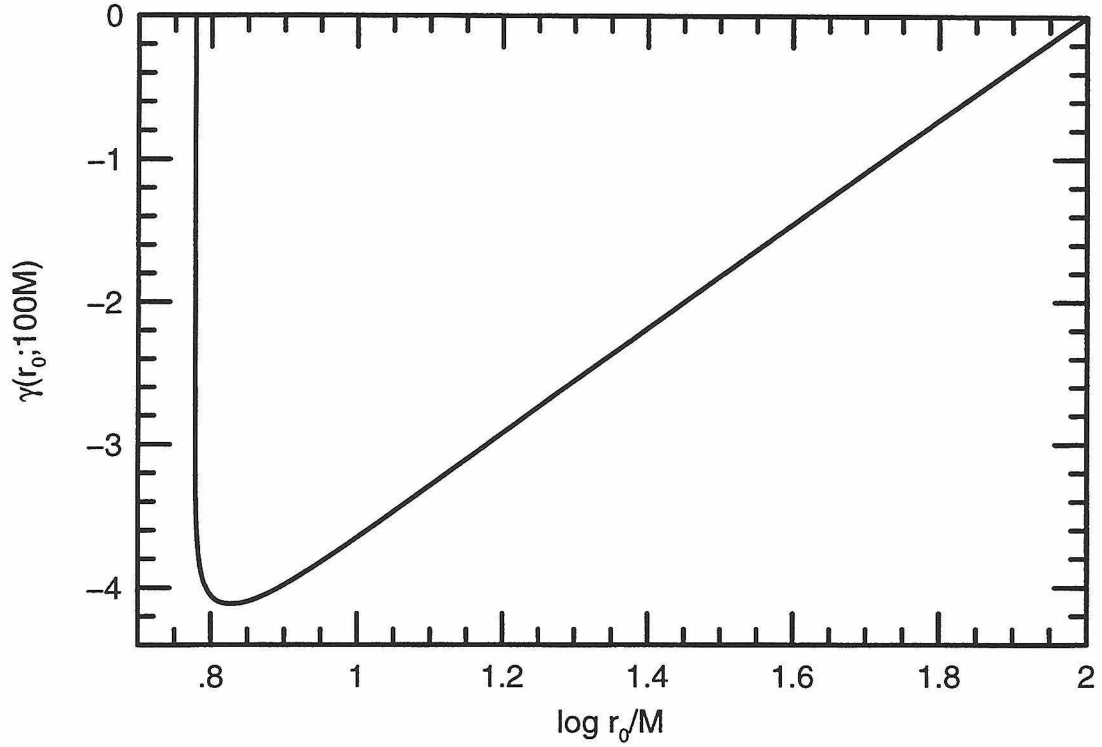

Figure 3.2: A plot of $\gamma(r_0, r_i)$, as defined in Eq. (3.5), for $r_i = 100M$, as a function of $\log r_0/M$. The curve may be continued, both to the left and to the right, using the analytical estimates (1.6) – (1.8). For example, $\gamma(1000M; 100M) \simeq 3.6392$. The function $\gamma(r_0; r_i)$ has a minimum at $r_0 = r_c \simeq 6.6792M$, and grows to plus infinity when $r_0 \to 6M$. Horizontal lines intersect the curve at two distinct points ($r_0 = r_1$ and $r_0 = r_2$) for which the eccentricity is equal, $\varepsilon(r_1) = \varepsilon(r_2)$.

Chapter 4

Stability under radiation reaction of circular equatorial orbits around Kerr black holes

Abstract

We examine the evolution, under gravitational radiation reaction, of slightly eccentric equatorial orbits of small objects around Kerr black holes. Our method involves numerical integration of the Sasaki-Nakamura equation. It is discovered that such orbits decrease in eccentricity throughout most of the inspiral, until shortly before the innermost stable circular orbit (ISCO), when a critical radius r_c, is reached beyond which the inspiralling orbits increase in eccentricity. It is shown that the

number of orbits remaining in this last (eccentricity increasing) phase of the inspiral is an order of magnitude less for prograde orbits around rapidly spinning black holes than for retrograde orbits. In the extreme limit of a Kerr black hole with spin parameter $a = 1$, the critical radius may dissapear altogether.

4.1 Introduction

Gravitational waves emitted by solar-mass-size compact bodies orbiting massive ($10^6 M_\odot$ and greater) black holes (and spiralling towards them as they lose energy and angular momentum to the emitted radiation) are a favoured source for gravitational wave detectors sensitive to low-frequency radiation, such as proposed space-based detectors like the Laser Interferometer Space Antenna (LISA) [1]. Systems of this type lend themselves to theoretical analysis via perturbation theory, because of the extreme mass ratio between the two bodies. In recent years, the Teukolsky perturbation formalism for black holes has been employed successfully to describe orbital decay of small bodies orbiting a large Schwarzschild (i.e. non-rotating) black hole [4, 13, 14]. One result of this work has been to modify the long-standing result [12] that, under radiation reaction, orbits tend to become more circular as they slowly decay. In fact, inside a critical radius, which is $r_c = 6.6792M$ for nearly circular orbits in the Schwarzschild geometry, non-circular orbits tend to become more, rather than less, eccentric [4]. Although a precisely circular orbit would remain circular inside the critical radius, its circularity is longer stable to small perturbations as the orbital decay continues.

Despite their intrinsic interest, these results may prove of limited usefulness for any future low-frequency gravitational wave detectors, since there is no reason to expect that large black holes should typically have no spin at all. Just the opposite (that they should exhibit strong rotation) is perhaps more to be expected [2]. Therefore it is of great interest to extend this type of analysis to the case of rotating, or Kerr black holes. This presents no difficulty for the Teukolsky perturbation formalism itself, which was developed for the Kerr metric, but a problem does arise in dealing with an additional constant of the motion which governs orbits around spinning black holes. Unlike the energy and angular momentum, whose flux can easily be determined from the waves far from the source, until very recently there was no clear understanding of how to calculate the amount of "Carter constant" carried away by the emitted radiation. In spite of this, it has been shown recently, for general orbits in Kerr, that circular orbits (defined as orbits of constant Boyer-Lindquist radius, and sometimes referred to as "quasi-circular") remain circular under radiation reaction [3, 15, 16]. While progress continues in developing techniques for dealing with general orbits in Kerr [6, 7, 27], it now seems worthwhile to investigate the case of nearly-circular, equatorial orbits around rotating black holes [5]. Equatorial orbits in the Kerr spacetime, like orbits in Schwarzschild, can be said to have zero "Carter constant", which remains unchanged during orbital decay. Looking at these orbits can tell us if the behaviour previously observed for slightly-eccentric orbits in Schwarzschild is also seen in the Kerr metric for all values of the Kerr spin parameter $0 \leq a \leq 1$.

It is shown in this paper that, for equatorial orbits, it is generally true that a critical radius, r_c exists beyond which slightly eccentric orbits become less circular due to radiation reaction, and that this radius is encountered shortly before the radius of the innermost stable circular orbit (ISCO). This is best illustrated by examining the behavior of the parameter $c = r_o\dot{e}/e\dot{r_o}$, where e is the orbital eccentricity, and r_o the mean radius, and an overdot differentiation by time. This parameter is negative for orbits evolving with increasing eccentricities, and positive for decreasing eccentricity. Near the ISCO one can show, as in Sec. 9 below, that c diverges towards negative infinity near the ISCO, for nearly all values of a. There is an apparent exception to this behaviour in the limiting case of a maximally rotating Kerr black hole with $a = M$. In that case, the horizon and the ISCO are both located at $r = M$ in Boyer-Lindquist co-ordinates, although they are still separate in terms of proper radial distance. As one approaches $r = M$, for the case of a prograde orbit around an extreme Kerr black hole, c is both postive and finite, approaching the limit of $3/2$ at $r = M$. Not surprisingly therefore, for prograde orbits around black holes with very large $a > .99M$, the transition to eccentricity-increasing inspiral takes place only shortly before the onset of dynamical instability at the ISCO in terms of the Boyer-Linquist radial co-ordinate. The number of orbits remaining at this point is an order of magnitude fewer for such cases than it is for retrograde orbits in the same geometry.

Since the radius of the ISCO is much smaller for prograde than for retrograde orbits (with $a = M$, $r_{\rm ISCO} = M$ for prograde orbits and $r_{\rm ISCO} = 9M$ for retrograde orbits), the critical radius is also much smaller for prograde orbits. These results demonstrate than the onset of "back reaction instability" for circular orbits precedes, and is intimately connected with, the onset of dynamical instability signified by the ISCO. It seems reasonable to conjecture that the alteration in the shape of the radial protential as the ISCO approaches, at which point the minimum of the effective potential vanishes, is reponsible for the gain in eccentricity.

The organisation of the paper is as follows. In section 2, the orbital equations for geodesic motion (.i.e. not including radiation reaction) are solved analytically for slightly eccentric, equatorial orbits. In section 3 the Tuekolsky perturbation formalism is described, and section 4 shows how to calculate the fluxes of energy and angular momentum carried away from the system using this formalism. In section 5 the Sasaki-Nakamura equation, which is actually solved rather than the Tuekolsky radial equation for numerical reasons, is presented. In section 6 the Teukolsky source function is calculated for a perturbing particle following the orbits of section 2, and the results of both of these sections come together in section 7 to yield the rate of change of orbital eccentricity due to radiation damping. This orbital evolution is described under the assumption of adiabaticity (that the orbital evolution is much slower than the orbital period), which introduces constraints which are discussed in section 8. Finally, in section 9, the analytic and numerical results are presented, followed by a discussion of their significance in section 10. A guide to the essential points of the arguments is given at the end of section 7.

4.2 Description of the orbit

Since the perturbation of the Kerr metric producing the gravitational waves is that of a small particle orbiting the black hole, it will be necessary to solve the orbital equations for a particle in orbit around a rotating black hole. We require expressions for $r(t)$, $\phi(t)$ and $\theta(t)$ to describe the orbit in Boyer-Lindquist co-ordinates. Since we restrict ourselves to equatorial orbits, the solution for the θ motion is trivial, $\theta = \pi/2$ is a constant throughout. The equatorial orbital equations for a particle in the Kerr spacetime, in these co-ordinates (leaving aside the trivial $d\theta/d\tau = 0$), are well known [17]

$$\mu\Sigma^2 dr/d\tau = [(E(r^2 + a^2) - aL_z)^2 - \Delta(\mu^2 r^2 + (L_z - aE)^2)]^{\frac{1}{2}} \equiv \sqrt{R} \quad (4.1)$$

$$\mu\Sigma^2 d\phi/d\tau = -(aE - L_z/\sin^2\theta) + (a/\Delta)(E(r^2 + a^2) - aL_z) \equiv \Phi \quad (4.2)$$

$$\mu\Sigma^2 dt/d\tau = -a(aE\sin^2\theta - L_z) + \frac{(r^2 + a^2)}{\Delta}(E(r^2 + a^2) - aL_z) \equiv T, \quad (4.3)$$

where τ is proper time, $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, and the black hole's spin parameter a is defined for convenience as $a = \vec{J}\cdot\hat{L}/M$, with $\vec{J}$ the spin angular momentum vector of the black hole, and $\hat{L}$ a unit vector pointing in the direction of the particle's orbital angular momentum vector. For prograde orbits (in which the particle orbits in the same sense as the black hole's spin) a is positive, and for retrograde orbits (in which the particle rotates in the opposite sense to the hole), a is negative. Recall that we restrict attention to equatorial orbits only, so that $\vec{J}$ and $\hat{L}$ are either parallel or anti-parallel. In addition, M and μ are the system's total and reduced masses respectively. It is a condition of the perturbation scheme that $\mu/M \ll 1$, where M is the mass of the central black hole and μ the mass of the orbiting particle. Finally, E and L_z are the particle's orbital energy and angular momentum, respectively.

We now consider slightly eccentric orbits, and define a mean radius r_0, so that $\partial(R/r_0^4)/\partial r|_{r=r_0} = 0$. The eccentricity e is defined so that $R(r = r_0(1+e)) = 0$. These definitions are chosen so that as $e \to 0$, r_0 reduces to the constant radius of a circular orbit, and so that e corresponds, when $e \ll 1$ and in the appropriate limits, to definitions of the eccentricity of an orbit commonly used in the Schwarzschild geometry and in Newtonian mechanics [4]. These defining equations for r_0 and e permit us to write the orbital energy and angular momentum in terms of these two quantities. Since we assume throughout that e is a small quantity, it is convenient to expand E and L_z in terms of it,

$$E(r_0, e) = E_0(r_0) + eE_1(r_0) + e^2E_2(r_0) + e^3E_3(r_0) + O(e^4) \tag{4.4}$$

$$L_z(r_0, e) = L_0(r_0) + eL_1(r_0) + r^2L_2(r_0) + e^3L_3(r_0) + O(e^4). \tag{4.5}$$

Using our two equations in r_0 and e, it is easy to show that

$$
\begin{aligned}
E_0 &= \mu \frac{1-2v^2+qv^3}{(1-3v^2+2qv^3)^{\frac{1}{2}}} && (4.6)\\
E_1 &= 0 && (4.7)\\
E_2 &= \mu \frac{v^2(1-3v^2+qv^3+q^2v^4)(1-6v^2+8qv^3-3q^2v^4)}{2(1-3v^2+2qv^3)^{\frac{3}{2}}(1-2v^2+q^2v^4)} && (4.8)\\
E_3 &= -\mu \frac{v^2(1-3v^2+qv^3+q^2v^4)(1-7v^2+10qv^3-4q^2v^4)}{(1-3v^2+2qv^3)^{\frac{3}{2}}(1-2v^2+q^2v^4)} && (4.9)\\
L_0 &= \mu \frac{r_0 v(1-2qv^3+q^2v^4)}{(1-3v^2+2qv^3)^{\frac{1}{2}}} && (4.10)\\
L_1 &= 0 && (4.11)\\
L_2 &= \mu \frac{qr_0v^5(q-3v+qv^2+q^2v^3)(1-6v^2+8qv^3-3q^2v^4)}{2(1-3v^2+2qv^3)^{\frac{3}{2}}(1-2v^2+q^2v^4)} && (4.12)\\
L_3 &= -\mu \frac{qr_0v^5(q-3v+qv^2+q^2v^3)(1-7v^2+10qv^3-4q^2v^4)}{(1-3v^2+2qv^3)^{\frac{3}{2}}(1-2v^2+q^2v^4)}. && (4.13)
\end{aligned}
$$

Here $v = \sqrt{M/r_0}$ and $q = a/M$. These results, up to order e^2 are given in Ref. [5].

We wish to write the change in the eccentricity brought about by the loss of orbital angular momentum and energy, in terms of the rates of loss of those two quantities. Since we have E and L_z as functions of r_0 and e, we use the chain rule for differentiation to write

$$
\begin{aligned}
\dot{E} &= -dE_{GW}/dt = \frac{\partial E}{\partial e}\dot{e} + \frac{\partial E}{\partial r_0}\dot{r}_0 && (4.14)\\
\dot{L}_z &= -dL_{GW}/dt = \frac{\partial L_z}{\partial e}\dot{e} + \frac{\partial L_z}{\partial r_0}\dot{r}_0, && (4.15)
\end{aligned}
$$

where dE_{GW}/dt and dL_{GW}/dt are the total energy and angular momentum carried towards infinity and the black hole horizon per unit time by the gravitational waves, averaged over several wavelengths. We will write these quantities also in terms of e

and r_0,

$$\frac{dE_{GW}}{dt} = \dot{E}_0 + e\dot{E}_1 + e^2\dot{E}_2 + O(e^3) \tag{4.16}$$

$$\frac{dL_{GW}}{dt} = \dot{L}_0 + e\dot{L}_1 + e^2\dot{L}_2 + O(e^3). \tag{4.17}$$

As we shall find later, $\dot{E}_1 = \dot{L}_1 = 0$. Eliminating r_0 from Eq. (4.15), we derive

$$\dot{e} = [-\frac{dE_{GW}}{dt}L_z' + \frac{dL_{GW}}{dt}E']/[\frac{\partial E}{\partial e}L_z' - \frac{\partial L_z}{\partial e}E'], \tag{4.18}$$

where $' \equiv \partial/\partial r_0$.

Substiting Eqs. (4.17) and (4.5) into Eq. (4.18), we find, keeping terms up to order e^2,

$$\dot{e} = \frac{-L_0'(\dot{E}_0 - \frac{E_0'}{L_0'}\dot{L}_0) - e^2 L_0'(\dot{E}_2 - \frac{E_0'}{L_0'}\dot{L}_2) - e^2 L_2'(\dot{E}_0 - \frac{E_2'}{L_2'}\dot{L}_0)}{2e(E_2 L_0' - L_2 E_0')}. \tag{4.19}$$

Now, from Eqs. (4.13), we see that

$$\frac{E_0'}{L_0'} = \frac{\sqrt{M}}{r_0^{\frac{3}{2}} + a\sqrt{M}} = \Omega, \tag{4.20}$$

where Ω is the angular frequency of a circular orbit of radius r_0. It follows from an interesting (and quite general [22]) characteristic of circular orbits, and will be shown later in this case that, the circular (i.e. zeroth order in the eccentricity) rates of loss of energy and angular momentum are related by

$$\dot{E}_0 = \Omega\dot{L}_0. \tag{4.21}$$

Therefore

$$\mu\dot{e} = -ej(v)[g(v)\dot{E}_0 + \dot{E}_2 - \Omega\dot{L}_2], \tag{4.22}$$

where

$$j(v)=\frac{\mu}{E_2-\Omega L_2}=\frac{(1+qv^3)(1-2v^2+q^2v^4)(1-3v^2+2qv^3)^{\frac{1}{2}}}{v^2(1-6v^2+8qv^3-3q^2v^4)} \tag{4.23}$$

and

$$g(v)=\frac{L_2'}{L_0'}-\frac{E_2'}{E_0'}=\frac{\mathcal{G}(v)}{2(1+qv^3)(1-6v^2+8qv^3-3q^2v^4)(1-2v^2+q^2v^4)^2}, \tag{4.24}$$

where

$$\begin{aligned}\mathcal{G}(v)=2 &- 27v^2+72v^4-36v^6+38qv^3-17q^2v^4-144qv^5+86q^2v^6\\ &+ 4q^3v^7+72qv^7-12q^4v^8-36q^2v^8-23q^4v^{10}+30q^5v^{11}\\ &- 9q^6v^{12}\end{aligned} \tag{4.25}$$

Since $\dot{e}$ is proportional to e in this equation, it is plain that a precisely circular orbit (one with $e=0$), will remain circular under radiation reaction, provided that we can indeed show below that $\dot{E}_0=\Omega\dot{L}_0$ and $\dot{E}_1=\dot{L}_1=0$. It is also plain that the question of the stability of an orbit's circularity will be determined by the sign of Eq. (4.22), which requires us to calculate the loss of orbital energy and angular momentum up to second order in e.

Similarly we can solve for $\dot{r}_0$, the rate of change of the orbital radius, which tells us that to leading order $\dot{r}_0=-\dot{E}_0/E_0'$, which implies that

$$\mu\dot{r}_0/r_0=-\frac{2(1-3v^2+2qv^3)^{3/2}}{v^2(1-6v^2+8qv^3-3q^2v^4)}\dot{E}_0. \tag{4.26}$$

With this in hand it is possible to proceed to the solution of the geodesic equations [Eqs. (4.3)]. We expand $r(t)$ about the mean radius r_0 in terms of the small eccentricity e, so that

$$r(t) = r_0[1 + er_1(t) + e^2 r_2(t) + O(e^3)]. \tag{4.27}$$

Making use of the expansions of E, L_z and $r(t)$ in terms of e, we expand out the equation $(dr/dt)^2 = R/T^2$, and collect terms of order e^2 and e^3 (note that the e^3 term in $r(t)$ does not contribute until $O(e^4)$ in R/T^2), giving us two differential equations,

$$(dr_1/dt)^2 = \Omega_r^2(1 - r_1^2), \tag{4.28}$$

where we define a radial frequency,

$$\Omega_r = \Omega(1 - 6v^2 + 8qv^3 - 3q^2v^4)^{\frac{1}{2}} \tag{4.29}$$

and

$$\frac{1}{\Omega_r^2}\frac{dr_1}{dt}\frac{dr_2}{dt} + r_1 r_2 = f_1(v) + f_2(v)r_1 + f_3(v)r_1^3, \tag{4.30}$$

where

$$f_1(v) = -\frac{1 - 7v^2 + 10qv^3 - 4q^2v^4}{1 - 6v^2 + 8qv^3 - 3q^2v^4} \tag{4.31}$$

$$f_2(v) = \frac{2v^2(1 - 2qv^3 + q^2v^4)}{(1 + qv^3)(1 - 2v^2 + q^2v^4)} \tag{4.32}$$

$$f_3(v) = \frac{\mathcal{F}_3(v)}{(1 + qv^3)(1 - 2v^2 + q^2v^4)(1 - 6v^2 + 8qv^3 - 3q^2v^4)}, \tag{4.33}$$

and

$$\begin{aligned}\mathcal{F}_3(v) = 1 &- 11v^2 + 26v^4 + 11qv^3 - 3q^2v^4 - 41qv^5 + 15q^2v^6 \\ &- 10qv^7 + 7q^3v^7 + 24q^2v^8 - 4q^4v^8 - 27q^3v^9 + 16q^4v^{10} \\ &- 4q^5v^{11}.\end{aligned} \tag{4.34}$$

Integrating these equations in order, we find,

$$r_1(t) = \cos(\Omega_r t) \tag{4.35}$$

$$r_2(t) = -f_1(v)(1-\cos(\Omega_r t)) + \frac{1}{2}f_3(v)(1-\cos(2\Omega_r t)). \tag{4.36}$$

It remains to solve for the ϕ-motion. Again we expand out the geodesic equation $d\phi/dt = \Phi/T$, integration of which yields

$$\phi(t) = \Omega_\phi t - ep(v)\sin(\Omega_r t) + O(e^2), \tag{4.37}$$

where

$$p(v) = \frac{2(1-3v^2+2qv^3)}{[(1+qv^3)(1-2v^2+q^2v^4)(1-6v^2+8qv^3-3q^2v^4)^{1/2}]} \tag{4.38}$$

and

$$\begin{aligned}
\Omega_\phi &= \Omega\Big[1 \\
&- \frac{3(1-11v^2+24v^4+13qv^3-4q^2v^4-46qv^5+25q^2v^6+q^3v^7-3q^4v^8)}{2(1+qv^3)(1-2v^2+q^2v^4)(1-6v^2+8qv^3-3q^2v^4)}e^2 \\
&+ O(e^3)\Big] \\
&\equiv \Omega[1-\Delta\Omega e^2 + O(e^3)]
\end{aligned} \tag{4.39}$$

is the azimuthal angular frequency. The $O(e^2)$ part of $\phi(t)$ which is proportional to $\sin(\Omega_r t)$ is not given, as neither it nor the $O(e^2)$ part of $r(t)$ contribute to the final result for $\dot{e}$, for reasons which will become clear later. Only the $O(e^2)$ part of Ω_ϕ (i.e. $\Delta\Omega$) is required, although it is necessary to know $r_2(t)$ to derive $\Delta\Omega$.

4.3 The Teukolsky formalism

We employ a scheme previously used in the Schwarzschild case to investigate the evolution of slightly eccentric orbits under radiation reaction [4]. This scheme is based on the Teukolsky formalism for perturbations of the Kerr metric. In this formalism one can decompose the Weyl scalar ψ_4 (which describes gravitational wave fluxes near infinity for such a system) as follows,

$$\psi_4 = \frac{1}{(r - ia\cos\theta)^4}\int_{-\infty}^{+\infty}\sum_{lm} R_{lm\omega}(r)_{-2}S^{a\omega}_{lm}(\theta)e^{im\theta}e^{i\omega t}d\omega, \tag{4.40}$$

where ${}_{-2}S^{a\omega}_{lm}$ is the spheroidal harmonic function of spin weight $s = -2$. The normalization used here for these functions is $\int_0^\pi |{}_{-2}S^{a\omega}_{lm}(\theta)|^2 \sin\theta d\theta = 1/2\pi$. The radial function $R_{lm\omega}(r)$ obeys the Teukolsky equation,

$$\Delta^2 \frac{d}{dr}\Big(\frac{1}{\Delta}\frac{dR_{lm\omega}}{dr}\Big) - V(r)R_{lm\omega}(r) = T_{lm\omega}(r), \tag{4.41}$$

where $T_{lm\omega}$ is the Teukolsky source function, to be evaluated below. The Teukolsky potential is defined by

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \tag{4.42}$$

where $K = (r^2 + a^2)\omega - ma$ and λ is the eigenvalue associated with the appropriate spheroidal harmonic ${}_{-2}S^{a\omega}_{lm}$.

We can define two solutions to the homogeneous Teukolsky equation, $R^H_{lm\omega}(r)$ and $R^\infty_{lm\omega}(r)$, with the following boundary conditions,

$$R^H_{lm\omega} \sim \Delta^2 e^{ikr^*}, \text{as} \quad r \to r_+ \tag{4.43}$$

$$R^H_{lm\omega} \sim r^3 B^{\rm out}_{lm\omega} e^{i\omega r^*} + \frac{1}{r} B^{\rm in}_{lm\omega} e^{-i\omega r^*}, \text{as} \quad r \to \infty \tag{4.44}$$

and

$$R^\infty_{lm\omega} \sim D^{\rm out} e^{ikr^*} + \Delta^2 D^{\rm in} e^{-ikr^*}, \text{as} \quad r \to r_+ \tag{4.45}$$

$$R^\infty_{lm\omega} \sim r^3 e^{-i\omega r^*}, \text{as} \quad r \to \infty, \tag{4.46}$$

where $k = \omega - ma/(2Mr_+)$, $r_+ = M + \sqrt{M^2 - a^2}$ is the radius of the black hole horizon, and r^*, the tortoise co-ordinate, is defined as

$$r^* = r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \tag{4.47}$$

where $r_- = M - \sqrt{M^2 - a^2}$.

From Ref. [11], the solution of the Teukolsky equation (solved via a retarded Green's function) is

$$R_{lm\omega}(r) = R^\infty_{lm\omega}(r) Z^H(r) + R^H_{lm\omega}(r) Z^\infty(r), \tag{4.48}$$

where

$$Z^H(r) = \frac{1}{2i\omega B^{\rm in}_{lm\omega}} \int_{r_+}^{r} \frac{R^H_{lm\omega}(r) T_{lm\omega}(r)}{\Delta^2} dr \tag{4.49}$$

and

$$Z^\infty(r) = \frac{1}{2i\omega B^{\rm in}_{lm\omega}} \int_{r}^{\infty} \frac{R^\infty_{lm\omega}(r) T_{lm\omega}(r)}{\Delta^2} dr. \tag{4.50}$$

For convenience, we will write $Z^H_{lm\omega} = Z^H(r \to \infty)$ and $Z^\infty_{lm\omega} = Z^\infty(r \to r_+)$, and therefore our two solutions can be written as

$$R_{lm\omega}(r \to \infty) \sim Z^H_{lm\omega} r^3 e^{i\omega r^*} \tag{4.51}$$

and

$$R_{lm\omega}(r \to r_+) \sim Z^{\infty}_{lm\omega}\Delta^2 e^{-ikr^*}. \tag{4.52}$$

4.4 Energy and angular momentum fluxes

Towards infinity, the Weyl scalar can be related to the two fundamental polarizations of gravitational waves by

$$\psi_4 = \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times). \tag{4.53}$$

From this and Eq. (4.40) above, we can determine the averaged energy and angular momentum fluxes at infinity, employing the Isaacson stress-energy tensor to define the energy flux in the wave [28], as

$$\langle\frac{dE_{\rm GW}}{dt}\rangle = \dot{E}^{\infty} = \sum_{lmk} \frac{|Z^H_{lmk}|^2}{4\pi\omega_k^2} \tag{4.54}$$

and

$$\langle\frac{dL_{\rm GW}}{dt}\rangle = \dot{L}^{\infty}_z = \sum_{lmk} \frac{m|Z^H_{lmk}|^2}{4\pi\omega_k^3}, \tag{4.55}$$

where the amplitude coefficient is decomposed into a discrete set of frequencies based on the particle's orbital motion,

$$Z^H_{lm\omega} = \sum_k Z^H_{lmk}\delta(\omega - \omega_k). \tag{4.56}$$

Energy and angular momentum are also lost by radiation through the horizon of the central black hole. Again, ψ_4 completely describes the waves as $r^* \to -\infty$, and following the same procedure as with the waves at infinity [24, 8] we find

$$\dot{E}^H = \sum_{lmk} \alpha_l^k \frac{|Z_{lmk}^\infty|^2}{4\pi\omega_k^2} \tag{4.57}$$

and

$$\dot{L}_z^H = \sum_{lmk} \alpha_l^k \frac{m|Z_{lmk}^\infty|^2}{4\pi\omega_k^3}, \tag{4.58}$$

with an identical decompostion of $Z_{lm\omega}^\infty$ as with $Z_{lm\omega}^H$, and where

$$\alpha_l^k = \frac{2^8 w_k^5 k(k^2+4\epsilon^2)(k^2+16\epsilon^2)(2Mr_+)^5}{|C|^2} \tag{4.59}$$

and $\epsilon = \sqrt{M^2-a^2}/4Mr_+$ and

$$\begin{aligned} |C|^2 &= [(\lambda+2)^2 + 4a\omega m - 4a^2\omega^2](\lambda^2 + 36a\omega m - 36a^2\omega^2) \\ &+ (2\lambda+3)(96a^2\omega^2 - 48a\omega m) + 144\omega^2(M^2-a^2). \end{aligned} \tag{4.60}$$

This result for energy and angular momentum lost down the hole is in agreement with that of Teukolsky and Press [8]. The total rates of loss of energy and angular momentum by the system are $\dot{E}^H + \dot{E}^\infty$ and $\dot{L}_z^H + \dot{L}_z^\infty$.

4.5 The Sasaki-Nakamura equation

The preceding section makes it clear that our chief task is to calculate the amplitudes $Z_{lmk}^{H,\infty}$, and it is apparent from Eqs. (4.49) and (4.50) that this will entail solving the Teukolsky equation to find the amplitude of the in-going waves at infinity, $B_{lm\omega}^{\text{in}}$ from Eq. (4.44). Numerically this presents a problem, however, since the ingoing waves for this solution are completely swamped by the outoing waves at large radii [compare amplitudes of $B^{\rm out}_{lm\omega}r^3$ and $B^{\rm in}_{lm\omega}/r$ as $r \to \infty$ in Eq. (4.44)]. In the Schwarzschild case this problem is typically avoided by solving instead the Regge-Wheeler equation, and transforming its solution to that of the Teukolsky equation via the Chandrasekhar transformation [25]. The virtue of this is that, in the Regge-Wheeler formalism, with its short-range potential, the ingoing and outgoing waves near infinity have the same order of magnitude.

In the Kerr case, Sasaki and Nakamura have found an equation with the same useful properties as the Regge-Wheeler equation in Schwarzschild which, moreover, reduces to the latter equation when $a \to 0$ [10]. The Sasaki-Nakamura equation is written as follows

$$\frac{d^2X_{lm\omega}}{dr^2} - F(r)\frac{dX_{lm\omega}}{dr} - U(r)X_{lm\omega} = 0. \tag{4.61}$$

The functions $F(r)$ and $U(r)$ are given in the appendix. The equivalents to our two solutions to the Teukolsky equation are

$$X^H_{lm\omega} \sim A^{\rm out}_{lm\omega}e^{i\omega r^*} + A^{\rm in}_{lm\omega}e^{-i\omega r^*}, \text{as} \quad r \to \infty \tag{4.62}$$

$$X^H_{lm\omega} \sim e^{-ikr^*}, \text{as} \quad r \to r_+ \tag{4.63}$$

and

$$X^\infty_{lm\omega} \sim e^{i\omega r^*}, \text{as} \quad r \to \infty \tag{4.64}$$

$$X^\infty_{lm\omega} \sim D^{\rm out}e^{ikr^*} + D^{\rm in}e^{-ikr^*}, \text{as} \quad r \to r_+. \tag{4.65}$$

The transformations between the quantities we require are

$$R^{H}_{lm\omega} = \frac{1}{\eta}[(\alpha + \frac{\beta_{,r}}{\Delta})\chi^{H}_{lm\omega} - \frac{\beta}{\Delta}\chi^{H}_{lm\omega,r}], \tag{4.66}$$

$$R^{\infty}_{lm\omega} = -\frac{c_0}{4\omega^2\eta}[(\alpha + \frac{\beta_{,r}}{\Delta})\chi^{\infty}_{lm\omega} - \frac{\beta}{\Delta}\chi^{\infty}_{lm\omega,r}], \tag{4.67}$$

and

$$B^{\text{in}}_{lm\omega} = -\frac{1}{4\omega^2}A^{\text{in}}_{lm\omega}, \tag{4.68}$$

where $\chi^{H,\infty}_{lm\omega} = X^{H,\infty}_{lm\omega}/\sqrt{r^2+a^2}$, and c_0, α, β and η are given in the appendix.

4.6 The source term

The Teukolsky source term is given by [18]

$$T_{lm\omega} = 4\int \rho^{-5}\bar{\rho}^{-1}(B'_2 + B'^{*}_2)e^{-im\phi+i\omega t}{}_{-2}S^{a\omega}_{lm}d\Omega dt, \tag{4.69}$$

where

$$\begin{aligned} B'_2 &= -\frac{1}{2}\rho^8\bar{\rho}L_{-1}[\rho^{-4}L_0(\rho^{-2}\bar{\rho}^{-1}T_{nn})] \\ &- \frac{1}{2\sqrt{2}}\rho^8\bar{\rho}\Delta^2 L_{-1}[\rho^{-4}\bar{\rho}^2 J_+(\rho^{-2}\bar{\rho}^{-2}\Delta^{-1}T_{\bar{m}n})], \end{aligned} \tag{4.70}$$

$$\begin{aligned} B'^{*}_2 &= -\frac{1}{4}\rho^8\bar{\rho}J_+[\rho^{-4}J_+(\rho^{-2}\bar{\rho}T_{\bar{m}\bar{m}}] \\ &- \frac{1}{2\sqrt{2}}\rho^8\bar{\rho}\Delta^2 J_+[\rho^{-4}\bar{\rho}^2\Delta^{-1}L_{-1}(\rho^{-2}\bar{\rho}^{-2}T_{\bar{m}n})], \end{aligned} \tag{4.71}$$

and $\rho = (r - ia\cos\theta)^{-1}$, with $\bar{\rho}$ its complex conjugate. The operators L_s and J_+ are defined as

$$L_s = \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + s\cot\theta \tag{4.72}$$

and

$$J_+ = \partial_r + i\frac{K}{\Delta}. \tag{4.73}$$

The tetrad components of the particle's energy momentum tensor can be written

$$\begin{aligned}
T_{nn} &= \frac{C_{nn}}{\sin\theta}\delta(r - r(t))\delta(\theta - \pi/2)\delta(\phi - \phi(t)), &(4.74)\\
T_{\bar{m}n} &= \frac{C_{\bar{m}n}}{\sin\theta}\delta(r - r(t))\delta(\theta - \pi/2)\delta(\phi - \phi(t)), &(4.75)\\
T_{\bar{m}\bar{m}} &= \frac{C_{\bar{m}\bar{m}}}{\sin\theta}\delta(r - r(t))\delta(\theta - \pi/2)\delta(\phi - \phi(t)), &(4.76)
\end{aligned}$$

where

$$\begin{aligned}
C_{nn} &= C_{nn}^{(0)} + C_{nn}^{(1)}\frac{dr}{dt} + C_{nn}^{(2)}(\frac{dr}{dt})^2\\
&= \frac{\mu}{4\Sigma^3\dot{t}}[E(r^2 + a^2) - aL_z]^2 + \frac{\mu}{2\Sigma^2}[E(r^2 + a^2) - aL_z]\frac{dr}{dt}\\
&+ \frac{\mu\dot{t}}{4\Sigma}(\frac{dr}{dt})^2 &(4.77)\\
C_{\bar{m}n} &= C_{\bar{m}n}^{(0)} + C_{\bar{m}n}^{(1)}\frac{dr}{dt}\\
&= \frac{\mu\rho}{2\sqrt{2}\Sigma^2\dot{t}}[E(r^2 + a^2) - aL_z][i\sin\theta(aE - \frac{L_z}{\sin^2\theta})]\\
&- \frac{\mu\rho}{2\sqrt{2}\Sigma}[i\sin\theta(aE - \frac{L_z}{\sin^2\theta}]\frac{dr}{dt} &(4.78)\\
C_{\bar{m}\bar{m}} &= \frac{\mu\rho^2}{2\Sigma\dot{t}}[i\sin\theta(aE - \frac{L_z}{\sin^2\theta})]^2 &(4.79)
\end{aligned}$$

and $\dot{t} = dt/d\tau$.

Integrating by parts, and making use of the adjoint operator $L_s^\dagger = \partial_\theta - m/\sin\theta + a\omega\sin\theta + s\cot\theta = \partial_\theta + f(\theta)$, which bears the following useful relation to the operator L_s defined above:

$$\int_0^\pi h(\theta)L_s[g(\theta)]\sin\theta d\theta = -\int_0^\pi g(\theta)L_{1-s}^\dagger[h(\theta)]\sin\theta d\theta, \tag{4.80}$$

with $g(\theta)$ and $h(\theta)$ arbitrary functions [5], we find that

$$T_{lm\omega} = \int_{-\infty}^{\infty}\int_{0}^{2\pi} \Delta^2[\{(A_{nn0} + A_{\bar{m}n0} + A_{\bar{m}\bar{m}0})\delta(r - r(t)\} \tag{4.81}$$

$$+ \{(A_{\bar{m}n1} + A_{\bar{m}\bar{m}1})\delta(r - r(t))\}_{,r} nonumber \tag{4.82}$$

$$+ \{A_{\bar{m}\bar{m}2}\delta(r - r(t))\}_{,rr}]\delta(\phi - \phi(t))e^{i\omega t - im\phi} d\phi dt. \tag{4.83}$$

The A's are all functions of r only, and in each case $A = A^{(0)} + A^{(1)}(dr/dt) + A^{(2)}(dr/dt)^2$, where (writing ${}_{-2}S^{a\omega}_{lm}$ simply as S hereafter for simplicity)

$$A^{(i)}_{nn0} = -\frac{2}{\Delta^2}C^{(i)}_{nn}r^3(rS_{,\theta\theta} - 2iaS_{,\theta} + 2rf(\pi/2)S_{,\theta}$$

$$- 2iaf(\pi/2)S + rS(f(\pi/2)^2 - 2), \tag{4.84}$$

$$A^{(i)}_{\bar{m}n0} = \frac{2\sqrt{2}}{\Delta}C^{(i)}_{\bar{m}n}r^3(S_{,\theta} + f(\pi/2)S)(i\frac{K}{\Delta} + \frac{2}{r}), \tag{4.85}$$

$$A^{(0)}_{\bar{m}\bar{m}0} = -r^2C_{\bar{m}\bar{m}}S(-i(\frac{K}{\Delta})_{,r} - (\frac{K}{\Delta})^2 + \frac{2i}{r}\frac{K}{\Delta}), \tag{4.86}$$

$$A^{(i)}_{\bar{m}n1} = \frac{2\sqrt{2}}{\Delta}r^3C^{(i)}_{\bar{m}n}(S_{,\theta} + f(\pi/2)S), \tag{4.87}$$

$$A^{(0)}_{\bar{m}\bar{m}1} = -2r^2C_{\bar{m}\bar{m}}S(i\frac{K}{\Delta} + \frac{1}{r}), \tag{4.88}$$

$$A^{(0)}_{\bar{m}\bar{m}2} = -r^2C_{\bar{m}\bar{m}}S, \tag{4.89}$$

$$A^{(2)}_{\bar{m}n0} = A^{(2)}_{\bar{m}n1} = A^{(1)}_{\bar{m}\bar{m}0} = A^{(2)}_{\bar{m}\bar{m}0} = A^{(1)}_{\bar{m}\bar{m}1} = A^{(2)}_{\bar{m}\bar{m}1} = A^{(1)}_{\bar{m}\bar{m}2} = A^{(2)}_{\bar{m}\bar{m}2} = 0 \tag{4.90}$$

In every case the spheroidal harmonic function (S) and its derivatives are evaluated at $\theta = \pi/2$.

It is now easy to show, from Eqs. (4.49) and (4.50) and using integration by parts (keeping in mind that we are interested only in closed orbits, for which $r_+ < r < \infty$

always holds strictly), that

$$Z^{H,\infty}_{lm\omega} = \frac{1}{2i\omega B^{\rm in}_{lm\omega}} \int_{r_+}^{\infty} \int_{-\infty}^{\infty} \int_0^{2\pi} I^{H,\infty}_{lm\omega}(r)\delta(r - r(t))\delta(\phi - \phi(t)) d\phi dt dr, \qquad (4.91)$$

for which $I^{H,\infty}_{lm\omega}(r) = I^{(0)}_{lm\omega}(r) + I^{(1)}_{lm\omega}(r)(dr/dt) + I^{(2)}_{lm\omega}(r)(dr/dt)^2$, where

$$I^{(i)}_{lm\omega}(r) = R^{H,\infty}_{lm\omega}(A^{(i)}_{nn0} + A^{(i)}_{\bar{m}n0} + A^{(i)}_{\bar{m}\bar{m}0}) - \frac{dR^{H,\infty}_{lm\omega}}{dr}(A^{(i)}_{\bar{m}n1} + A^{(i)}_{\bar{m}\bar{m}1}) + \frac{d^2 R^{H,\infty}_{lm\omega}}{dr^2} A^{(i)}_{\bar{m}\bar{m}2}. \qquad (4.92)$$

It is necessary to expand $Z^{H,\infty}_{lm\omega}$ in terms of the eccentricity e, keeping in mind that we wish, as shown in section 2 above, to find $\dot{E}^{H,\infty}$ and $\dot{L}^{H,\infty}_z$ to second order in e, and that each of these is proportional to $|Z^{H,\infty}_{lm\omega}|^2$. However, it transpires that only terms up to order e in the integrand of Eq. (4.91) contribute to order e^2 in $\dot{e}$, the rate of change of eccentricity derived from $\dot{E}^{H,\infty}$ and $\dot{L}^{H,\infty}_z$. The reasons for this emerge as we proceed to expand $I^{H,\infty}_{lm\omega}(r)$, $\delta(r - r(t))$ and $\delta(\phi - \phi(t))$ in powers of e.

Employing the expansions of $r(t)$ and $\phi(t)$ derived above [Eqs. (4.27) and (4.37)], we can write the product of delta functions in Eq. (4.91) as a product of two Taylor expansions in the small parameter e, about the points $r - r_0$ and $\phi - \Omega_\phi t$.

$$\begin{aligned} \delta(r - r(t))\delta(\phi - \phi(t)) &= \delta(r - r_0)\delta(\phi - \Omega_\phi t) - e r_0 \cos\Omega_r t \delta'(r - r_0)\delta(\phi - \Omega_\phi t) \\ &- e p(r_0) \sin\Omega_r t \delta'(\phi - \Omega_\phi t)\delta(r - r_0) + O(e^2), \qquad (4.93) \end{aligned}$$

where the prime denotes differentiation with respect to the function's argument. We can integrate by parts in Eq. (4.91) to integrate terms containing derivatives of delta functions, and this will simply mean that $\delta'(\phi - \Omega_\phi t)$ will be replaced by $im\delta(\phi - \Omega_\phi t)$, since $e^{-im\phi}$ is the only other part of the integrand which depends on

ϕ. Completing the ϕ integration thus leaves us with the overall factor $e^{i(\omega - m\Omega_\phi)t}$, and some terms depending on $\cos\Omega_r t$, $\sin\Omega_r t$ and, in the $O(e^2)$ part, on $\cos 2\Omega_r t$ and $\sin 2\Omega_r t$. Following the time integration, then, we will have a series of delta functions of the type $\delta(\omega - m\Omega_\phi)$ [at all orders, except $O(e)$], $\delta(\omega - m\Omega_\phi \pm \Omega_r)$ (at all orders from $O(e)$ up) and, in general, $\delta(\omega - m\Omega_\phi \pm k\Omega_r)$ at $O(e^k)$ and above.

These delta functions, after integration over ω to derive ψ_4 [Eq. (4.40)], produce terms representing energy and angular momentum emitted at the fundamental (circular motion) frequency $w_m = m\Omega_\phi$, and at a series of discrete sidebands, $w_\pm = m\Omega_\phi \pm \Omega_r$ and $w_{\pm k} = m\Omega_\phi \pm k\Omega_r$. The occurrence of these delta functions also justifies the decomposition of $Z^{H,\infty}_{lm\omega}$ referred to earlier [Eq. (4.56) above].

It is, of course, $|Z^{H,\infty}_{lm\omega}|^2$ which is integrated in Eq. (4.40). Therefore, up to order e^2, only those $O(e^2)$ terms in $Z^{H,\infty}_{lm\omega}$ which cross multiply with $O(e^0)$ terms will contribute. Since the frequency must be single valued for any given term, only the circular harmonic (w_m) term in $O(e^2)$ survives the Fourier transform which produces the Weyl scalar, all other terms being annihilated. The $O(e)$ terms in Z have no circular harmonic term, as mentioned before, so these terms only contribute to loss of energy and engular momentum at $O(e^2)$.

As seen from Eq. (4.22) above, it is the difference $\dot{E}_2 - \Omega\dot{L}_2$ on which $\dot{e}$ actually depends at leading order. Eqs. (4.54),(4.55),(4.57) and (4.58) show that

$$\dot{E}_n - \Omega\dot{L}_n \propto 1 - \frac{m\Omega}{\omega_k}, \quad \text{atorder} \quad e^n \tag{4.94}$$

which is zero to leading order if $\omega_k = \omega_m = m\Omega_\phi$. This means not only that the $O(e^2)$ terms in $Z^{H,\infty}_{lm\omega}$ do not contribute at all to $\dot{e}$ below $O(e^3)$, but also that $\dot{E}_0 - \Omega \dot{L}_0$ is also zero to leading order, as noted above [Eq. (4.21)]. In fact, since the eccentric correction to the azimuthal frequency Ω_ϕ is itself of $O(e^2)$, the circular losses of energy and angular momentum contribute to $\dot{e}$ at $O(e^2)$ to leading order, like the first order terms in Z. Therefore there is no loss of E and L_z at $O(e)$, and so $\dot{E}_1 = 0$ and $\dot{L}_1 = 0$ as claimed in section 2.

This proves that a precisely circular equatorial orbit in Kerr will always remain circular under radiation reaction (as long as the adiabatic approximation still holds). Furthermore it means that to find the leading order correction to this condition for slightly eccentric orbits, and thus establish the stability of circularity, we need only examine the $O(e)$ terms in Eq. (4.40), and can drop all $O(e^2)$ corrections to the motion, except for the $\Delta\Omega$ part of Ω_ϕ. This also means, of course, that only contributions to the loss of energy and angular momentum from the first pair of sidebands ($\omega = \omega_\pm$) need be included with the circular harmonic (ω_m) in calculating $\dot{e}$ to leading order.

4.7 Calculation of rate of change of eccentricity

As a final step before integration of Eq. (4.91), the function $I^{H,\infty}_{lm\omega}(r)$ must be expanded up to first order in e. It contains terms which depend on dr/dt which, by Eq.

(4.27) above, is $O(e)$ at leading order, $dr/dt = -er_0\Omega_r \sin\Omega_r t + O(e^2)$. Therefore we will write

$$I^{H,\infty}_{lm\omega}(r) = I^{(0)}_{lm\omega}(r) - eI^{(1)}_{lm\omega}(r) r_0 \Omega_r \sin\Omega_r t + O(e^2). \tag{4.95}$$

Thus, doing a final integration by parts in the integral over r in Eq. (4.91), we find

$$Z^{H,\infty}_{lm\omega} = -\frac{\pi}{i\omega B^{\rm in}_{lm\omega}}[I^{(0)}_{lm\omega}(r_0)\delta(\omega - m\Omega_\phi) - eB^+_{lm}\delta(\omega-\omega_+) - eB^-_{lm}\delta(\omega-\omega_-) + O(e^2)], \tag{4.96}$$

where

$$B^\pm_{lm} = \frac{1}{2}(r_0 \frac{dI^{(0)}_{lm\omega}}{dr}|_{r=r_0} \pm mp(r_0) I^{(0)}_{lm\omega}(r_0) \mp I^{(1)}_{lm\omega}(r_0) r_0 \Omega_r). \tag{4.97}$$

The argument of the preceding section shows that, in order to calculate the quantity $\dot{E}_2 - \Omega\dot{L}_2 + \Delta\Omega\dot{E}_0$, we need only evaluate the co-efficients $B^\pm_{lm}$ in Z_{lm}. Therefore, returning to Eq. (4.22), we have

$$\mu\dot{e}/e = -j(v)[\Gamma - h(v)\dot{E}_0] \tag{4.98}$$

where

$$\begin{aligned} \Gamma &= \dot{E}_2 - \Omega\dot{L}_2 + \Delta\Omega\dot{E}_0 & (4.99) \\ &= \frac{\Omega_r}{4\pi}\sum_{lm}\Big(\frac{|B^{H+}_{lm}|^2}{\omega_+^3} - \frac{|B^{H-}_{lm}|^2}{\omega_-^3}\Big) \\ &+ \frac{\Omega_r}{4\pi}\sum_{lm}\Big(\frac{|B^{\infty+}_{lm}|^2}{\omega_+^3}\alpha_l^+ - \frac{|B^{\infty-}_{lm}|^2}{\omega_-^3}\alpha_l^-\Big) & (4.100) \end{aligned}$$

and

$$h(v) = \Delta\Omega - g(v) = \frac{\mathcal{H}(v)}{2(1+qv^3)(1-2v^2+q^2v^4)^2(1-6v^2+8qv^3-3q^2v^4)}, \tag{4.101}$$

with

$$
\begin{aligned}
\mathcal{H}(v) = 1 \quad &- \quad 12v^2 + 66v^4 - 108v^6 + qv^3 + 8q^2v^4 - 72qv^5 - 20q^2v^6 \\
&+ \quad 204qv^7 + 38q^3v^7 - 42q^2v^8 - 9q^4v^8 - 144q^3v^9 + 116q^4v^{10} \\
&- \quad 27q^5v^{11}.
\end{aligned}
\tag{4.102}
$$

In summary, Eq. (4.98) is the equation which allows us to compute the change in eccentricity for an inspiralling orbit, and Eq. (4.26) defines the rate of inspiral. Eq. (4.100), Eq. (4.97 and Eq. (4.92), for Γ, and Eqs. (4.54) and (4.57) with the $O(e^0)$ part of Eq. (4.96) for $\dot{E}_0$, allow us to express $\dot{e}$ in terms of the solution of the radial Tuekolsky equation $R^{H,\infty}_{lm\omega}$, and its derivatives, as well as the incoming wave amplitude $B^{in}_{lm\omega}$. These quantities are in turn derived numerically by solving the Sasaki-Nakamura equation as described below in section 9, and employing the transformations given in Eqs. (4.66), (4.67) and (4.68). The important functions $j(v)$, $h(v)$ and $\Delta\Omega$ in Eq. (4.98) are all derived in solving the equations of geodesic motion for the orbiting body in section 2.

4.8 Adiabatic condition

The whole preceding argument depends on an adiabatic condition on the motion which says that the inspiral timescale $r_0/|\dot{r}_0|$ is much greater than the orbital period of the motion $2\pi/\Omega_r$. The necessity for this condition is most noticeable in the approximation which describes the evolution of the particle's motion under back reaction as passing through a series of geodesic orbits, each defined as if no back reaction were taking place during that orbit. Once the inspiral proceeds on a timescale which is about as short as the time to complete an orbit, this approximation loses all validity. Using Eq. (4.26), we find that the adiabatic condition can be written,

$$\frac{\mu}{M} \ll \frac{v^5}{2\pi} \frac{(1-6v^2+8qv^3-3q^2v^4)^{3/2}}{(1-3v^2+2qv^3)^{3/2}(1+qv^3)} \frac{1}{(M/\mu)^2 \dot{E}_0}. \tag{4.103}$$

Just as the inspiral timescale must be greater than an orbital period, so too must the circularization timescale $e/\dot{e}$. However, this quantity is almost invariably less than the inspiral timescale, so Eq. (4.103) is the key condition. For very large radii, in the Newtonian limit, $(M/\mu)^2\dot{E}_0 \simeq 32v^{10}/5$ (for a discussion of this limit see Ref. [4]) and the condition is simply $\mu/M \ll (5/128\pi)v^{-5}$, which is very much less restrictive than the linear perturbation condition $\mu/M \ll 1$, upon which the Teukolsky formalism rests. Approaching the ISCO however, where the numerical results tell us that $\dot{E}_0$ remains finite and of the same order as its Newtonian value, we see that the adiabatic limit on μ/M is proportional to $(1-6v^2+8qv^3-3q^2v^4)^{3/2}$, which becomes vanishingly small as the ISCO nears. Therefore, near the ISCO the adiabatic condition supercedes the linear perturbation condition, as the leading constraint on μ/M. Only by imagining a test particle which has vanishingly small mass can we apply the results of our calculation all the way to the ISCO, but no doubt there exist real physical systems, with $\mu/M \leq 10^{-6}$ for instance, which are correctly described for almost all of the inspiral by this approximation (recalling that our calculations presume that the particle is a point mass as a further simplification). This issue will be discussed more quantitatively in Ref. [29].

4.9 Results

With the results of section 7, it only remains to calculate $R^{H,\infty}_{lm\omega}$, $B^{\rm in}_{lm\omega}$ [Eqs. (4.43) and (4.44)] and ${}_{-2}S^{a\omega}_{lm}(\pi/2)$ [Eq. (4.40)] numerically to find $\dot{e}/e$. To find the solutions to the radial equation [Eq. (4.41)] one actually solves the Sasaki-Nakamura equation [Eq. (4.61)] for $X^{H,\infty}_{lm\omega}$ and $A^{\rm in}_{lm\omega}$ [Eqs. (4.62) and (4.63)]. These solutions are very smooth, apart from a singularity at the horizon r_+, and so Bulirsch-Stoer integration works very well in integrating them. The singularity is avoided by starting the integration from a point just outside the horizon (typically at $r_+ + 10^{-8}$). The solutions are insensitive to variations by several orders of magnitude of this small increment. Richardson polynomial extrapolation is used to evaluate $A^{\rm in}_{lm\omega}$ as $r \to \infty$, since it can be expressed as the first term in a polynomial in $1/\omega r$ defining the amplitude of the ingoing wave at large r in Eq. (4.62) [13]. This amplitude is evaluated for several endpoints of integration, doubling the endpoint radius at each trial, allowing the extrapolator to evaluate the limit of the amplitude as $r \to \infty$, which is $A^{\rm in}_{lm\omega}$.

The Spheroidal harmonic functions are calculated by expressing them as a linear combination of spherical harmonics of equal m, summed over all available values in

l (truncating the series after 30 terms in practice). Substituting this series into the second-order ODE defining the spheroidal harmonics gives us a 5-term recurrence relation for the co-efficients of the expansion. This can be solved using matrix eigenvalue routines which, like the Bulirsch-Stoer integrator and the polynomial extrapolator, are found in Ref. [9]. The derivative of each spheroidal harmonic is also expressible as a combination of spherical harmonics of different spin-weight values by use of the edth operator [20]. This procedure, for the scalar case only, is found in [19]. A more detailed description is given in the appendix to the thesis. Useful checks for the numerical results are found in the Schwarzschild limit, in [4] and in the circular limit, in [21]. Analytically the results of sections 2 and 7 reduce to those of [4] in the Schwarzschild limit and those of section 2 to the results of [5] in the post-Newtonian limit.

The accuracy of the numerical results is limited by several factors. The relative accuracies of the Bulirsch-Stoer integrator and the Richardson extrapolator can be increased easily, at some loss in computing speed. For these calculations they were set to 10^{-6} and 10^{-5} respectively. The solution of the eigenvalue problem has very good accuracy, but the approximation of the spheroidal harmonics as a combination of spherical harmonics begins to lose accuracy seriously when $a\omega$ becomes much larger than order unity. However, this only occurs for very high ($m > 20$) harmonics of the motion for small radii, and these contributions are not required at the accuracy used here. The chief limit on accuracy is, in fact, the number of harmonics in l and

m which are calculated. Invariably, for small eccentricity orbits, the leading order contribution is for $l = 2$, $m = 2$, and the significance of the contribution decreases sharply (but less so for small radii) with increasing l and m. A simple estimate, used in Ref. [4], enables one to reliably estimate the inaccuracy involved in truncating the calculation at $l = l_{\max}$. It tells us that, for a relative error (in estimates of the loss of energy and angular momentum) no greater than η, with a mean orbital radius r_0, then $l_{\max} \geq \log \eta / \log(M/r_0) + 3$. Taking all of these factors into account, we can generally estimate the accuracy of the numerical results at 10^{-5}, and certainly the relative errors should be no greater than 10^{-4} in most cases.

A useful parameter with which to investigate the orbital evolution is c, which represents a ratio of the inspiral timescale to the circularization timescale, or

$$c = \frac{r_0}{e} \frac{de/dt}{dr_0/dt}. \tag{4.104}$$

Again, c is positive when radiation reaction circularizes the orbit, and negative when it drives the orbit more eccentric. In order to see analytically the behaviour of c as the ISCO approaches, recall Eq. (4.98) and write

$$c = -\frac{r_0}{\mu \dot{r}_0} j(v)[\Gamma - h(v)\dot{E}_0]. \tag{4.105}$$

As $r_0 \to r_{\mathrm{ISCO}}$, the radius of the innermost stable circular orbit, the function $h(v)$ [Eq. (4.101)] diverges, since $r_{\mathrm{ISCO}}^2 - 6Mr_{\mathrm{ISCO}} + 8a\sqrt{Mr_{\mathrm{ISCO}}} - 3a^2 = 0$. Since the numerical results show that Γ and $\dot{E}_0$ remain finite in all cases, it is apparent that Γ (which is otherwise dominant), contributes negligibly near r_{ISCO}. Therefore, making use of the expression for $\dot{r}_0$ from Eq. (4.26), we find for r_0 near $r_{\rm ISCO}$,

$$c \sim -\frac{\mathcal{H}}{4(1-2v^2+qv^3)(1-3v^2+2qv^3)(1-6v^2+8qv^3-3q^2v^4)}. \tag{4.106}$$

Again, $1-6v^2+8qv^3-3q^2v^4 \to 0$ as $r \to r_{\rm ISCO}$, so c diverges at the ISCO. However, its sign as this point approaches depends on the function $\mathcal{H}$ [Eq. (4.102)], since the expressions in the denominator are all positive for $r > r_{\rm ISCO}$. It is obvious that for large r, $\mathcal{H}$ is always positive, but for small values of r, which can be achieved by prograde orbits around rapidly spinning black holes ($a > .95M$), $\mathcal{H}$ can become negative. However, it always becomes positive again before the ISCO, so that $c \to -\infty$ at the ISCO, in all cases except one.

The exceptional case is the extreme one of $a \to M$. At this unique point, $\mathcal{H}$, and all expressions in the denominator of Eq. (4.106) go to zero. Setting $q = 1$ in Eq. (4.106), and canceling factors of $(v-1)$ from both numerator and denominator, one finds that

$$\lim_{q=1, v\to 1} c = 3/2, \tag{4.107}$$

which is both positive and finite, in contrast to the usual behaviour at the ISCO.

However, as Fig. 1 shows, the curves describing the critical radius and the ISCO do approach each other in terms of the Boyer-Lindquist radial coordinate as $a \to M$, as our analysis of c might suggest. Therefore it is interesting to investigate the consequences of this for massive particles inspiraling around near extreme Kerr black holes. A useful measure here is the number of orbits left in the inspiral once the particle reaches the critical radius, that is, the number of orbits it will take the particle to reach the ISCO. Defining t_c as the inspiral time between r_c and $r_{\rm ISCO}$, and referring to Eq. (4.26) for the rate of inspiral, we have

$$t_c = -\int_{r_c}^{r_{\rm ISCO}} \frac{v^2(1-6v^2+8qv^3-3q^2v^4)}{2\dot{E}_0(1-3v^2+2qv^3)}\frac{\mu dr_0}{r_0}. \tag{4.108}$$

To a rough approximation, we can take $\dot{E}_0$ as constant in this region, and therefore

$$t_c \approx \frac{\mu}{\dot{E}_0}|\sqrt{1-3v^2+2qv^3}(-\frac{1}{2}+\frac{v^2-1}{2(1-3v^2+2qv^3)})|_{v_c}^{v_{\rm ISCO}}. \tag{4.109}$$

Approximately, the number of orbits left in this time will be

$$N_c \approx \frac{t_c}{T} \approx \frac{t_c\Omega}{2\pi} \approx \frac{\mu}{M}\frac{v_c^3}{2\pi\dot{E}_0}\frac{1}{1+qv^3}|\sqrt{1-3v^2+2qv^3}(-\frac{1}{2}+\frac{v^2-1}{2(1-3v^2+2qv^3)})|_{v_c}^{v_{\rm ISCO}}. \tag{4.110}$$

Note that $\dot{E}_0 \propto (\mu/M)^2$, so that N_c is inversely proportional to μ/M. In the test particle limit $\mu/M \to 0$, $N_c \to \infty$.

For $a = -.9M$, we find that $N_c \approx .035M/\mu$, while for $a = .99M$, $N_c \approx .0025M/\mu$. Note that the rate of energy loss is similar in these two cases (retrograde orbits radiate more energy for an orbit of given radius than do prograde orbits), but the distance between r_c and $r_{\rm ISCO}$ is much smaller in the latter case. The condition of Eq. (4.103) at the critical radius for $a = .99M$ is $\mu/M \ll .01$, so these estimates are still applicable to systems with extreme mass ratios, such as compact solar-mass-size objects spiralling into rapidly rotating supermassive black holes. For such a system, a prograde orbit spends an order of magnitude or more fewer orbits in the eccentricity increasing phase than does a retrograde orbit. Furthermore, the orbital periods for these two cases (a prograde orbit with $r_0 \sim 1.5M$, and a retrograde orbit with $r_0 \sim 9.5M$) are also very different, with the period of the retrograde orbit an order of magnitude longer. The retrograde orbit therefore spends a factor of hundreds more time gaining eccentricity than the prograde orbit.

Fig. 1 illustrates the positions of the horizon, ISCO and the critical radius for prograde and retrograde orbits around black holes of all spins. Fig. 2 illustrates the behaviour of c for Schwarzschild orbits ($a = 0$) and for prograde and retrograde orbits around a Kerr black hole with $a = .9M$. The dramatic plunge in c towards negative values as the ISCO approaches is seen in all three cases.

4.10 Conclusions

The results of this paper broadly confirm the experience of the non-rotating case, in that radiation reaction tends to reduce orbital eccentricity until near the the ISCO, when the onset of dynamical instability is prefigured by a period of decircularization of the inspiralling orbit. It seems reasonable to suppose that this effect is induced by alterations in the shape of the radial potential R as the ISCO approaches, since at the ISCO, the minimum which defines the particle's circular orbit dissapears. The tendancy of prograde orbits around rapidly rotating black holes to begin increasing in eccentricity only very shortly before the plunge into the black hole (at $r_{\rm ISCO}$) suggests that massive bodies in such orbits will not experience much increase in eccentricity at the end of their inspiral, in comparison with bodies in retrograde orbits, or the non-rotating case. In the case of prograde orbits around an extreme Kerr black hole, the fact that c is positive arbitrarily close to $r = M$, suggests that there is no critical radius in this exceptional case.

Another effect of the back reaction force on the orbit is one which tends to alter the inclination angle, which measures the maximum departure of the orbit from the equatorial plane. Ryan [23] has shown that nearly equatorial prograde orbits tend to increase their inclination angle under radiation reaction, thus moving away from being equatorial, although the effect is not very pronounced. Retrograde orbits, on the other hand, tend to decrease their inclination angle (since the spin-orbit interaction is attractive for retrograde orbits). Therefore, by the late stages of inspiral, one might not expect prograde orbits to have remained very close to the equatorial plane. This illustrates the need for a more general calculation of orbital evolution in the Kerr geometry, which deals with the issue of the Carter constant.

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Appendix

The potential functions $F(r)$ and $U(r)$ of the Sasaki-Nakamura equation (4.61) are given in this appendix.

$$F(r) = \frac{\eta_{,r}}{\eta} \frac{\Delta}{r^2 + a^2} \tag{4.111}$$

where

$$\eta = c_0 + c_1/r + c_r/r^2 + c_3/r^3 + c_4/r^4 \tag{4.112}$$

and

$$c_0 = -12i\omega M + \lambda(\lambda + 2) - 12a\omega(a\omega - m) \tag{4.113}$$

$$c_1 = 8ia[3a\omega - \lambda(a\omega - m)] \tag{4.114}$$

$$c_2 = -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2] \tag{4.115}$$

$$c_3 = 24ia^3(a\omega - m) - 24Ma^2 \tag{4.116}$$

$$c_4 = 12a^4. \tag{4.117}$$

$$U(r) = \frac{\Delta U_1}{(r^2 + a^2)^2} + G^2 + \frac{\Delta G_{,r}}{r^2 + a^2} - FG \tag{4.118}$$

where

$$
\begin{aligned}
G &= -\frac{2(r-M)}{r^2+a^2} + \frac{r\Delta}{(r^2+a^2)^2} && (4.119)\\
U_1 &= V + \frac{\Delta^2}{\beta}[(2\alpha + \frac{\beta_{,r}}{\Delta})_{,r} - \frac{\eta_{,r}}{\eta}(\alpha + \frac{\beta_{,r}}{\Delta})] && (4.120)\\
\alpha &= -i\frac{K\beta}{\Delta^2} + 3iK_{,r} + \lambda + \frac{6\Delta}{r^2} && (4.121)\\
\beta &= 2\Delta(-iK + r - M - \frac{2\Delta}{r}). && (4.122)
\end{aligned}
$$

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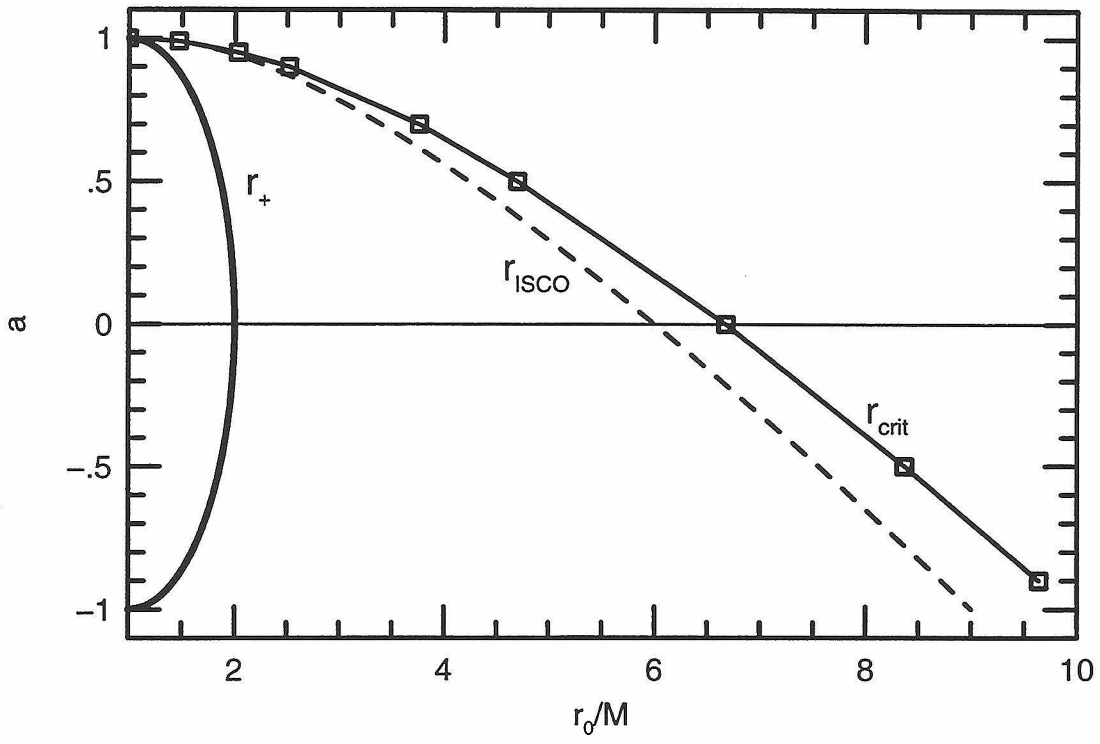

Figure 4.1: Graphs showing the positions of the horizon (r_+), innermost stable circular orbit ($r_{\rm ISCO}$) and critical radius ($r_{\rm crit}$) in terms of the mean orbital radius r_0 for all black hole spins ($a \le M$). Positive a indicates a prograde orbit, and negaive a a retrograde orbit.

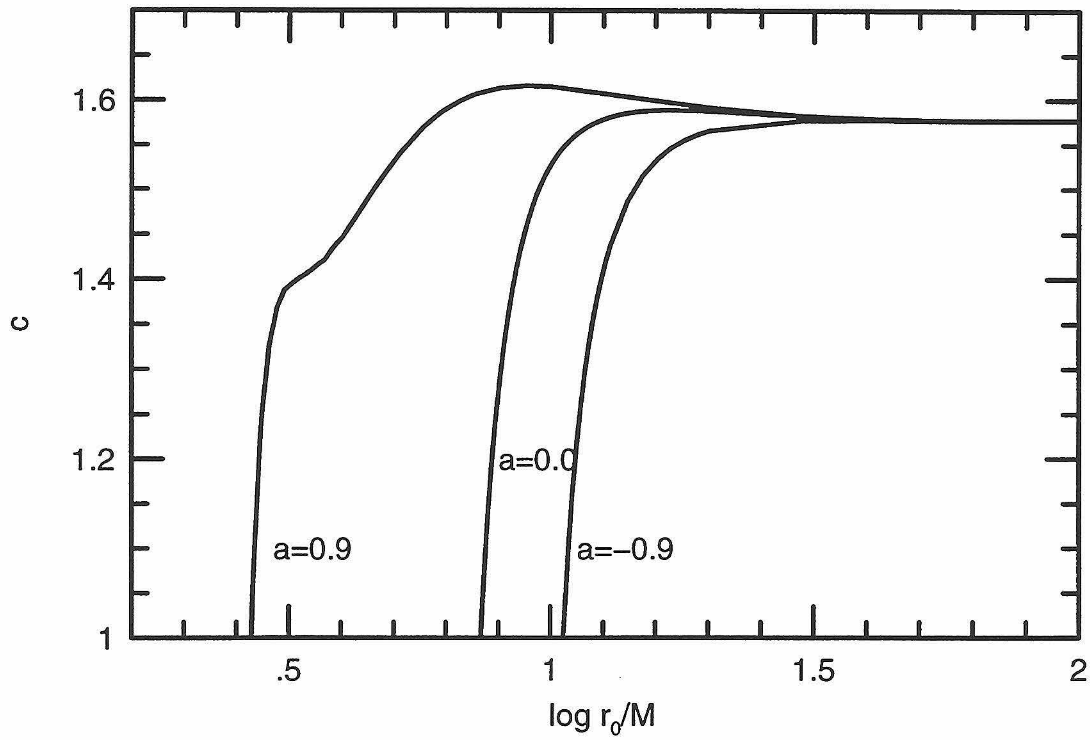

Figure 4.2: Curves showing the evolution of the parameter c, defined in Eq. (4.104), as the mean orbital radius r_0 decreases, for three different types of orbit. For a black hole with spin $a = .9M$, both the prograde ($a = 0.9$) and retrograde orbits ($a = -0.9$) are shown. Also shown is the case of a Schwarzschild black hole ($a = 0.0$). In each case c begins to fall quickly towards zero as the innermost stable cirular orbit approaches. The strange kink in the $a = 0.9$ curve near the bottom left hand corner of the graph may be due to the function $\mathcal{H}$ [Eq. (4.102)], upon which c depends strongly near the ISCO [Eq. (4.106)], dropping near to zero in value as r_0 decreases.

Chapter 5

Gravitational radiation reaction for bound motion around a Schwarzschild black hole

with Curt Cutler and Eric Poisson

Abstract

A particle of mass μ moves, in the absence of external forces, in the geometry of a nonrotating black hole of mass M. The system (black hole plus particle) emits gravitational waves, and the particle's orbit evolves under radiation reaction. The aim of this paper is to calculate this evolution. Our calculations are carried out under

the assumptions that $\mu/M \ll 1$, that the orbit is bound, and that radiation reaction takes place over a time scale much longer than the orbital period. The bound orbits of the Schwarzschild spacetime can be fully characterized, apart from initial conditions, by two orbital parameters: the semi-latus rectum p, and the eccentricity e. These parameters are so defined that the turning points of the radial motion (the values of the Schwarzschild radial coordinate at which the radial component of the four-velocity vanishes) are given by $r_1 = pM/(1+e)$ and $r_2 = pM/(1-e)$. The units are such that $G = c = 1$. We use the Teukolsky perturbation formalism to calculate the rates at which the gravitational perturbations generated by the orbiting particle remove energy and angular momentum from the system. These are then related to the rates of change of p and e, which determine the orbital evolution. We find that the radiation reaction continually decreases p, in such a way that the particle eventually plunges inside the black hole. Plunging occurs when p becomes smaller than $6 + 2e$. (Orbits for which $p < 6 + 2e$ do not have a turning point at $r = r_1$.) For weak-field, slow-motion orbits (which are characterized by large values of p), the radiation reaction decreases e also. However, for strong-field, fast-motion orbits (small values of p), the radiation reaction *increases* the eccentricity if p is sufficiently close to its minimum value $6 + 2e$. The change of sign of de/dt can be interpreted as a precursor effect to the eventual plunging of the orbit.

5.1 Introduction and summary

5.1.1 The problem

A particle of mass μ moves, in the absence of external forces, in the gravitational field of a nonrotating black hole of mass M. It is assumed that the motion is bound, and that $\mu \ll M$, but no restriction is put on the strength of the gravitational field at the particle's location: the field is arbitrarily strong, and the motion arbitrarily fast. The system (black hole plus particle) possesses a time-varying mass distribution, and therefore emits gravitational waves. These waves remove energy and angular momentum from the system. The question we intend to tackle in this paper is the following: How does the system react to the emission of gravitational waves? Or more precisely: What is the orbital evolution under the influence of gravitational radiation reaction?

The present work complements and generalizes two previous analyses: one by Apostolatos, Kennefick, Ori, and Poisson [1]; the other by Tanaka, Shibata, Sasaki, Tagoshi, and Nakamura [2].

Apostolatos *et al.* [1] considered the evolution, under gravitational radiation reaction, of slightly eccentric orbits in the Schwarzschild spacetime. This paper generalizes that work by considering the evolution of *any* bound orbit. In the language to be introduced in subsection D, the results of Apostolatos *et al.* can be recovered by taking the $e \to 0$ limit of those presented here.

Tanaka *et al.* [2] have also considered a wide class of bound orbits, and have therefore contributed significantly to the problem treated here. This paper extends and complements their work by providing: (i) a useful language in which to describe the results (the p-e plane, to be introduced in subsection D); (ii) analytical results which apply, approximately, to some interesting regions of the p-e plane; and (iii) a discussion which is focused entirely on the radiation reaction, rather than on the fluxes of energy and angular momentum at infinity, which is the main focus of Tanaka *et al.*

5.1.2 Motivation

The chief motivation for this work comes from the desire to achieve a deeper understanding of gravitational radiation reaction in the relativistic two-body problem, especially in situations where the gravitational field is strong, and the motion fast. (For an overview of the two-body problem in general relativity, see Ref. [3] and references therein.)

For weak fields and slow motions, the physics of gravitational radiation reaction is well understood [3, 4, 5, 6, 7, 8, 9]. In this context, calculations are carried out using post-Newtonian theory [10], and the equations of motion are derived accurately to some order in v/c. (Here, v is the orbital velocity, and c the speed of light.) To leading order in post-Newtonian theory [11], the radiation reaction is taken into account by adding a piece $\Phi_{\rm rr}$ to the Newtonian potential [4, 5]. Post-Newtonian corrections to the radiation reaction force have recently been calculated by Blanchet [8], and by Iyer and Will [9].

Because the post-Newtonian expansion is presumably only asymptotic, and not convergent [3, 10], it is not clear that post-Newtonian theory will ever succeed in providing an accurate description of the radiation reaction in situations where v/c is not small. To understand the strong-field effects, it is therefore useful to employ an alternative approach. Although limited to the case of orbital motion around a nonrotating black hole, this paper presents concrete results on radiation reaction in strong fields.

Because of the restriction $\mu \ll M$, the results presented in this paper are not directly applicable to the inspiral, and final coalescence, of a compact binary system of two comparable masses [12]. This problem will have to be solved using either post-Newtonian theory, or the techniques of numerical relativity which are currently under intense development [13]. However, it is conceivable that certain features of the small-mass-ratio orbital evolution will also be present in the more general case. (One such feature, the increase of the orbital eccentricity during the last stages of the inspiral, will be discussed below.) We may therefore hope that the results presented here will eventually be useful for interpretation purposes, when the evolution of binary systems with large mass ratios is better understood. In the mean time, our results will provide useful ways to check other methods of analysis, including post-Newtonian theory and numerical relativity.

Additional motivation for our work comes from the possibility that gravitational waves generated by the capture of solar-mass compact stars by supermassive black holes residing in galactic nuclei may be observed by eventual space-borne detectors, such as the proposed LISA (Laser Interferometer Space Antenna) project [14]. Such detectors are designed to operate in the frequency band between 10^{-3} Hz and 10^{-1} Hz, so that the waves emitted during the last stages of the capture are observable only if the central black hole has a mass ranging from $10^4 M_\odot$ to $10^6 M_\odot$. To avoid tidal disruption by the black hole [15], the captured star must be compact (a white dwarf, a neutron star, or a black hole).

Stars, normal or compact, are continually injected, by N-body processes, toward the vicinity of the central black hole, where they lose orbital energy and angular momentum to gravitational waves [16]. Eventually the star interacts solely with the black hole, and the orbital evolution becomes dominated by gravitational radiation reaction. Because these systems have small mass ratios, and because it can be expected that the stars move on highly eccentric orbits following their capture [16], the results presented in this paper are directly relevant to these sources of gravitational waves.

The application of our work to the capture of solar-mass compact objects by supermassive black holes will be the subject of a separate publication [17].

5.1.3 Method of solution

There currently exists no prescription to calculate the radiation reaction force acting on a (pointlike or extended) particle moving in a given background gravitational field, excluding the well-understood case of weak fields and slow motions. (We will return to this point, and discuss Gal'tsov's proposal for such a prescription [18], in subsection F.) Nevertheless, the problem considered in this paper can be tackled using a rather simple-minded approach, which we now describe.

A particle of mass μ moves, in the absence of external forces, in the geometry of a nonrotating black hole, and slightly perturbs the hole's gravitational field. The total field can be calculated by solving Einstein's equations perturbatively about the Schwarzschild solution [19]. The resulting equations take the form of linear wave equations for the perturbations, with the particle's stress-energy tensor acting as a source. The perturbations propagate away from the source as gravitational waves, and carry with them energy and angular momentum. Solving the perturbation equations allows us to calculate the rates at which energy and angular momentum are removed from the system (black hole plus particle).

The timelike geodesics of the Schwarzschild spacetime (Sec. II A) can be fully characterized, apart from initial conditions, by two orbital parameters, $\tilde{E}$ the orbital energy per unit mass, and $\tilde{L}$ the orbital angular momentum per unit mass. (Here, the mass is that of the particle.) The rates at which these quantities change with time are obtained from the solutions to the perturbation equations, and the orbital evolution under radiation reaction is determined.

Such a calculation can easily be carried out if the two following conditions hold.

First, the gravitational perturbations produced by the orbiting particle must have small amplitudes. This is to ensure that the nonlinearities of the perturbation fields can, to sufficient accuracy, be ignored. This will be the case if the inequality $\mu/M \ll 1$ is enforced.

Second, we must require that the orbits change very little over time scales which are comparable to the orbital period. This is because the source term in the wave equations — the stress-energy tensor, which depends on the particle's world line — must be specified before the equations are integrated. The motion of the particle must therefore be specified during the time interval over which the wave equations are integrated. And because the orbital motion is essentially periodic (Sec. II D), this time interval can be set equal to the orbital period. This procedure is self-consistent only if radiation reaction occurs over a time scale much longer than the orbital period, which shall be assumed here. This *adiabatic approximation* can be imposed by formulating additional constraints on the size of μ/M. These constraints will be derived in Sec. IV D.

When the adiabatic approximation is valid, the calculation proceeds as follows. We begin by assuming that the motion is strictly geodesic over several orbital periods, and we evaluate the particle's stress-energy tensor. We then compute, by integrating the wave equations, $\langle d\tilde{E}/dt \rangle$ and $\langle d\tilde{L}/dt \rangle$, the *time-averaged* rates of change of the orbital parameters. (The average is taken over several orbital periods.) Finally, we infer from these quantities the slow, secular evolution of the orbit. Provided that μ/M is suitably constrained, the results are compatible with the initial assumption, and the calculation is self-consistent.

In this paper, the gravitational perturbations are described using the Teukolsky formalism [20], in which all information about the perturbations is contained in the complex-valued function Ψ_4, a particular component of the Weyl tensor. In this formalism, a single wave equation needs to be solved, and the rates at which energy and angular momentum are carried away can easily be obtained from the solution. The Teukolsky formalism will be reviewed in Sec. III B.

5.1.4 Orbital parameters

The evolution, under radiation reaction, of the bound orbits of the Schwarzschild spacetime can best be described in terms of a set of orbital parameters which is different from the set $\{\tilde{E}, \tilde{L}\}$. For this purpose we introduce p, the orbit's semi-latus rectum, and e, its eccentricity. Both p and e are dimensionless, and are regular functions of $\tilde{E}$ and $\tilde{L}$. (See Sec. II B below, which contains a more detailed presentation.)

The new orbital parameters are defined as follows. For bound orbits, the radial motion (the evolution of the Schwarzschild radial coordinate r as a function of proper time τ) takes place between two turning points (the values of r at which $dr/d\tau = 0$).

We denote the periastron by r_1, and the apastron by r_2, so that $r_1 \leq r_2$. We define p and e such that $r_1/M = p/(1+e)$ and $r_2/M = p/(1-e)$, using units in which $G = c = 1$. The semi-latus rectum therefore measures the size of the orbit, while the eccentricity measures its degree of non-circularity.

The bound orbits of the Schwarzschild spacetime can be represented by those points in the p-e plane (Fig. 1) which satisfy the inequalities $0 \leq e < 1$, $p \geq 6 + 2e$. Points for which $p < 6 + 2e$ represent plunging orbits (these do not have a turning point at $r = r_1$). The boundary $p = 6 + 2e$ will be referred to as the *separatrix.* Points on the p-axis represent *stable* circular orbits, which have vanishing eccentricity. Points on the separatrix represent *unstable* circular orbits, for which $e \neq 0$.

In the absence of radiation reaction, p and e are constants of the motion. In the presence of radiation reaction, p and e evolve slowly, over a time scale long compared with the orbital period. The evolution of a given orbit therefore traces a trajectory in the p-e plane. (The p-e plane can be regarded as a phase space, and the trajectories as phase curves.) Our goal in this paper is to calculate the radiation-reaction trajectories.

5.1.5 The results

It is most convenient to represent the radiation-reaction trajectories, or phase curves, in terms of a phase diagram, in which the tangent vectors $(\dot{p}, \dot{e})$ — the phase velocity field — are plotted. (Here and throughout, a dot denotes differentiation with respect to time followed by an average over several orbital periods.) Such a representation is given in Fig. 1. The results can also be expressed in terms of the function $c(p, e)$, where

$$c(p, e) = \frac{d \ln e}{d \ln p}. \tag{5.1}$$

In Fig. 2 we have provided a three-dimensional plot of $c(p, e)$ for the most interesting range of orbital parameters.

Before we proceed with a summary of our main results, we must first recall one of the main conclusions of Ref. [1]: If an orbit has a vanishing eccentricity initially, then the radiation reaction does not change the value of the eccentricity. In other words, circular orbits remain circular under radiation reaction. [This statement follows directly from Eq. (5.89) below, which implies that $\dot{e} \propto e$ for small eccentricities.] In such circumstances, the value of p slowly decreases until $p = 6$ is reached, at which point the particle plunges inside the black hole.

We now discuss the more general case of orbits possessing nonvanishing eccentricities, first describing the results which were obtained using analytical methods.

We begin with a discussion of weak-field situations (Sec. IV A), that is, orbits with large values of p. In this case we find that $\dot{p} < 0$, $\dot{e} < 0$, and

$$c(p \to \infty, e) \sim \frac{19}{12}\left(1 + \tfrac{7}{8}e^2\right)^{-1}\left(1 + \tfrac{121}{304}e^2\right), \tag{5.2}$$

which is valid up to fractional corrections of order p^{-1}. These conclusions recover the well-known result that weak-field radiation reaction decreases both the size of the orbit and its eccentricity [21, 22]. In Ref. [1], the first post-Newtonian corrections to Eq. (5.2) were calculated for the case of small eccentricities. For completeness, we quote this result here:

$$c(p \to \infty, e \ll 1) = \frac{19}{12}\Big[1 - \tfrac{3215}{3192}p^{-1} + \tfrac{377}{152}\pi p^{-3/2} + O(p^{-2}, e^2)\Big]. \tag{5.3}$$

We next turn to cases for which the gravitational field is extremely strong. More precisely, we now consider points in the p-e plane which are very close to the separatrix $p = 6 + 2e$. (In this region, the validity of the adiabatic approximation implies severe restrictions on μ/M; see Sec. IV D.) It is also possible, for such orbits, to calculate the radiation reaction analytically (Sec. IV B). We find that when the inequality $p - 6 - 2e \ll \min(1, 4e)$ is satisfied [a more precise version of this condition is given by Eqs. (5.28) and (5.35) below], then $\dot{p} < 0$, $\dot{e} > 0$, and

$$c(p \to 6 + 2e, e \gg \varepsilon/4) \sim -\frac{1-e}{e}. \tag{5.4}$$

Here, $\varepsilon \equiv p - 6 - 2e$, and Eq. (5.4) is valid up to fractional corrections of order $(\varepsilon/4e)\ln(\varepsilon/4e)$.

Equation (5.4) is valid for small eccentricities *provided* that $\varepsilon \ll 4e$. This amounts to approaching the point $(p, e) = (6, 0)$ along a path which lies very close to the separatrix. The result is different if we approach the point $(6, 0)$ in a different direction. For example, if we choose a path which lies very close to the p-axis

(Sec. IV C), so that $e \ll p-6$, then we find that $\dot{p} < 0$, $\dot{e} > 0$, and

$$c(p \to 6, e \ll p-6) \sim -\frac{3}{2}(p-6)^{-1}, \tag{5.5}$$

which is valid up to fractional corrections of order $\max[p-6, e^2/(p-6)^2]$. Equation (5.5) was first derived in Ref. [1]. Equations (5.4) and (5.5) imply that the point $(6,0)$ is, in some sense, a singularity of the p-e plane. Not only does $c(p,e)$ diverge at that point, but its degree of divergence depends on the direction of approach.

We remark at the end of Sec. IV C that Eqs. (5.4) and (5.5), but not Eqs. (5.2) and (5.3), are in fact valid for *any* type of radiation field.

Equations (5.4) and (5.5) both imply that near the separatrix, radiation reaction *increases* the eccentricity: $\dot{e} > 0$ everywhere near $p = 6 + 2e$ [23]. This is in marked contrast with weak-field situations, for which the eccentricity always decreases. This result, that gravitational radiation reaction increases the eccentricity if p is sufficiently close to $6+2e$, is the main conclusion of this paper. (This discovery was first made by Tanaka *et al.* [2]. Our contribution is the analytical proof that this occurs for *any* eccentricity.)

The asymptotic expressions for $c(p,e)$, Eqs. (5.2)–(5.5), taken together, imply the existence of a *critical curve* in the p-e plane, along which $de/dp = 0$. Equation (5.4) further implies that the critical curve meets with the separatrix at $e = 1$. A portion of the critical curve is displayed in Fig. 3.

The evolution of an orbit under gravitational radiation reaction typically proceeds as follows (Fig. 1). Suppose that the orbit lies initially in the weak-field region, so that $p \gg 6$. Radiation reaction slowly decreases both p and e, until the orbit crosses the critical curve and the eccentricity reaches its minimum. From then on, the radiation reaction continues to decrease p, but now increases e. Finally, the orbit reaches the separatrix, and the particle plunges inside the black hole.

Because the critical curve lies relatively close to the separatrix, the change of sign of $\dot{e}$ — a genuine strong-field effect — can be interpreted as a precursor effect to the eventual plunging of the orbit.

The results represented in Figs. 1–3 were obtained numerically. We will describe our numerical methods in Sec. V below.

5.1.6 Future work

The techniques used in this paper could readily be extended to the case of a particle moving in the equatorial plane of a Kerr black hole. This is because the equatorial orbits of the Kerr spacetime can also be fully characterized by two orbital parameters. The radiation reaction can therefore be calculated in the same way.

The same cannot be said of orbits in Kerr which lie outside the equatorial plane. These orbits are characterized by three orbital parameters: orbital energy, orbital angular momentum, and the Carter constant [24]. There is no known relation — and it is not even clear whether one exists — between the rate of change of the Carter constant and the fluxes of energy and (vectorial) angular momentum carried by the gravitational waves. It is therefore unlikely that this most general problem will be solved before the elaboration of a robust formalism for strong-field, fast-motion radiation reaction [25].

Conceivably, such a formalism might be constructed along the lines of DeWitt and Brehme's derivation [26] of the curved spacetime version of the Lorentz-Dirac equation [27]. The problem to be solved is a generalization of the one examined in this paper. A particle of mass μ moves in the (arbitrary but known) gravitational field $g_{\alpha\beta}$ of an isolated mass M. (The prototype metric is the Kerr solution, but the problem may be formulated more generally.) To first order in μ/M, which is assumed small, what are the equations that the motion of the particle satisfies? (To zeroth order, the particle follows a geodesic of $g_{\alpha\beta}$; to first order, the system emits gravitational waves and radiation reaction takes place.)

This problem appears tractable, because the small perturbations produced by the particle obey linear wave equations in the background field $g_{\alpha\beta}$, and these wave equations can be formally integrated with the help of retarded Green's functions [26]. Because the Green's functions have support both on and inside the light cone, the resulting radiation-reaction force will depend both on the instantaneous state of the particle, and on its entire past history.

Gal'tsov has already proposed [18], for the special case of the Kerr metric, a radiation-reaction formalism based on solutions to the Teukolsky equation [20] of the half retarded minus half advanced type. But because of the causal structure of the Green's functions, the radiation-reaction force constructed in this way depends not only on the particle's past history, but also on its future history. (This is because the advanced Green's function has support inside the future light cone of the field point.) It is therefore not clear whether the Gal'tsov formalism is suitable for calculating the evolution of the nonequatorial orbits [28]. However, it should be quite adequate for the special case of *periodic* orbits [28], for which the past and future histories are identical (apart from the slow evolution due to radiation reaction).

5.1.7 Organization of the paper

The rest of the paper is devoted to the derivation of the results summarized in subsection E.

We begin in Sec. II with a detailed study of the bound orbits of the Schwarzschild spacetime. Most of the material presented in this section is not new, but for convenience the discussion is essentially self-contained. The geodesic equations are written, and the orbital parameters $\tilde{E}$ and $\tilde{L}$ defined, in subsection A. In subsection B we introduce the semi-latus rectum p and the eccentricity e, and we describe the bound orbits of the Schwarzschild spacetime in terms of the p-e plane. In subsection C we provide a method for integrating the geodesic equations which is well suited both for analytic and numerical calculations. The two fundamental frequencies of the motion, the radial frequency Ω_r, and the azimuthal frequency Ω_ϕ, are defined in subsection D. In subsection E we integrate the geodesic equations for the special case $\varepsilon \equiv p - 6 - 2e \ll \min(1, 4e)$, and derive analytical expressions for Ω_r and Ω_ϕ. In subsection F we do the same for the special case $e \ll \min(1, p - 6)$.

In Sec. III we describe our radiation-reaction formalism in detail. We explain the basic method in subsection A, and review the Teukolsky perturbation formalism [20] in subsection B. In particular, we show how to infer $\dot{E}$ and $\dot{L}$, the rates at which the gravitational waves carry energy and angular momentum to infinity, from the solution to the Teukolsky equation. [We will ignore, in this paper, the energy and angular momentum which are absorbed by the black hole. This will be justified in Sec. V E. However, these contributions *are* included in our analytical calculations.] In subsection C we calculate the source to the Teukolsky equation, and we formally integrate that equation in subsection D. In subsection E we derive equations relating the rates of change of p and e to $\dot{E}$ and $\dot{L}$, and explain why the radiation-reaction trajectories (the phase curves of Sec. I D) must cross the separatrix $p = 6 + 2e$.

Section IV is devoted to the derivation of our analytical results, in particular, Eqs. (5.2), (5.4), and (5.5). Weak-field situations are considered in subsection A, while the strong-field results are derived in subsections B [for the case $\varepsilon \ll \min(1, 4e)$] and C (for the case $4e \ll \varepsilon \ll 1$). In subsection D we formulate constraints on μ/M which ensure the validity of the adiabatic approximation.

Finally, in Sec. V, we describe the numerical methods which were used to obtain the results presented in Figs. 1–3. We begin with a brief description of the numerical task in subsection A, and then discuss various aspects of it in subsections B–D. In subsection E we estimate the overall accuracy of our results, and compare them to those of Tanaka *et al.* [2].

5.2 Bound orbits of the Schwarzschild spacetime

This section is devoted to the study of the bound orbits of the Schwarzschild spacetime. Most of the material presented here is not new, and can be found in the classic papers of Hagihara [29] and Darwin [30], or in Chandrasekhar's book [19]. The main purpose of this section is to establish the notation used in the rest of the paper; it will also serve as a repository of various useful results. For convenience, the material is presented in an entirely self-contained manner.

5.2.1 The geodesic equations

The timelike geodesics of the Schwarzschild spacetime are described by the following equations:

$$\begin{aligned} dt/d\tau &= \tilde{E}/f, \\ d\phi/d\tau &= \tilde{L}/r^2, \\ (dr/d\tau)^2 + V(\tilde{L}, r) &= \tilde{E}^2; \end{aligned} \tag{5.6}$$

we have put $\theta = \pi/2$ without loss of generality. Here, the coordinates $\{t, r, \theta, \phi\}$ are the usual Schwarzschild coordinates, and τ is the particle's proper time; $\tilde{E}$ and $\tilde{L}$ are constants of the motion, respectively, the orbital energy and angular momentum, both divided by μ, the mass of the particle. We have also defined $f = 1 - 2M/r$, where M is the mass of the black hole (it is assumed that $\mu \ll M$), and the effective potential for radial motion is given by

$$V(\tilde{L}, r) = f(1 + \tilde{L}^2/r^2). \tag{5.7}$$

The shape of the effective potential is represented in Fig. 4.

5.2.2 Orbital parameters: p and e

Apart from initial conditions, the orbits of the Schwarzschild spacetime are completely characterized by the values of two orbital parameters, which can be chosen to be $\tilde{E}$ and $\tilde{L}$. Bound motion occurs if

$$\tilde{E} < 1, \qquad \tilde{L} \geq 2\sqrt{3}M. \tag{5.8}$$

When $\tilde{E}$ and $\tilde{L}$ satisfy Eq. (5.8), the equation $V(\tilde{L}, r) = \tilde{E}^2$ possesses in general three distinct roots, which we designate by $r_3 \leq r_1 \leq r_2$. This situation is depicted in Fig. 4. The motion takes place between the turning points r_1 (the periastron) and r_2 (the apastron). We are not concerned with the plunging motion occurring inside $r = r_3$.

We define p, the *semi-latus rectum*, and e, the *eccentricity*, such that

$$r_1 = \frac{pM}{1+e}, \quad r_2 = \frac{pM}{1-e}. \tag{5.9}$$

Both p and e are dimensionless. As implied by Eq. (5.9), p measures the size of the orbit, while e measures its degree of non-circularity. Notice that e is confined to the range $0 \leq e < 1$; the value of p will be constrained below. These quantities have been used previously by Chandrasekhar [19] and Darwin [30].

The relationship between $\{p, e\}$ and $\{\tilde{L}, \tilde{E}\}$ can be obtained by comparing the cubic $V(\tilde{L}, r) = \tilde{E}^2$ to its equivalent form $(r - r_1)(r - r_2)(r - r_3) = 0$. This yields $r_3/M = 2p/(p-4)$,

$$\tilde{E}^2 = \frac{(p - 2 - 2e)(p - 2 + 2e)}{p(p - 3 - e^2)}, \tag{5.10}$$

and

$$\tilde{L}^2 = \frac{p^2 M^2}{p - 3 - e^2}. \tag{5.11}$$

The inequalities (5.8) are then automatically satisfied for any value of p, and for any $e < 1$.

Stable circular orbits occur when $\tilde{E}^2$ is equal to the minimum value of the effective potential. This implies $r_1 = r_2$, so that

$$\text{stable circular orbits} \quad \Leftrightarrow \quad e = 0. \tag{5.12}$$

The radius of a stable circular orbit is equal to pM.

Unstable circular orbits occur when $\tilde{E}^2$ is equal to the maximum value of the effective potential. The turning points r_1 and r_3 are then no longer distinct, and the condition $r_1 = r_3$ does *not* correspond to zero eccentricity. Instead,

$$\text{unstable circular orbits} \quad \Leftrightarrow \quad p = 6 + 2e. \tag{5.13}$$

The radius of an unstable circular orbit is equal to $(6+2e)M/(1+e)$.

It is easy to see that p must satisfy the inequality $p \geq 6+2e$ in order for the orbit to be bound; otherwise the orbit is a plunging one, with a unique turning point at $r = r_2$. It is worth noting that this inequality implies $r_1/M \geq (6+2e)/(1+e)$; the periastron radius is therefore always larger than $4M$. We also remark that the curves $p = 6+2e$ and $e = 0$ meet at $p = 6$, which implies that stable circular orbits occur only for $p > 6$.

The bound orbits of the Schwarzschild spacetime can be represented by those points in the p-e plane which satisfy the inequalities $0 \leq e < 1$, $p \geq 6+2e$. The boundary $p = 6+2e$ will be referred to as the *separatrix.*

5.2.3 Integration of the geodesic equations

We integrate Eqs. (5.6) by eliminating τ from the system of equations, and by choosing r as the parameter along the orbit. Clearly r is a multi-valued parameter, and the radial motion possesses two distinct branches. We take the first branch to be the motion from r_1 to r_2, and the second branch to be the motion from r_2 back to r_1.

Integrating Eqs. (5.6) gives

$$t(r) = \begin{cases} \hat{t}(r) & \text{first branch} \\ P - \hat{t}(r) & \text{second branch} \end{cases}, \tag{5.14}$$

where

$$\hat{t}(r) = \tilde{E} \int_{r_1}^{r} \frac{dr'}{f'(\tilde{E}^2 - V')^{1/2}}, \tag{5.15}$$

with $f' = 1 - 2M/r'$ and $V' = V(\tilde{L}, r')$. We have also defined $P = 2\hat{t}(r_2)$, the period of the radial motion. Similarly we find

$$\phi(r) = \begin{cases} \hat{\phi}(r) & \text{first branch} \\ \Delta\phi - \hat{\phi}(r) & \text{second branch} \end{cases}, \tag{5.16}$$

where

$$\hat{\phi}(r) = \tilde{L} \int_{r_1}^{r} \frac{dr'}{r'^2(\tilde{E}^2 - V')^{1/2}}, \tag{5.17}$$

and where $\Delta\phi = 2\hat{\phi}(r_2)$ is the amount by which ϕ increases in the course of one radial orbit.

Equations (5.15) and (5.17) are not directly suitable for numerical integration, because their integrands diverge at both turning points. To facilitate the numerical integration of the geodesic equations, and also their analytical integration in the limiting cases considered below, it is useful to make the substitution

$$r(\chi) = \frac{pM}{1 + e\cos\chi}. \tag{5.18}$$

The parameter χ ranges from 0 to 2π as r goes from r_1 to r_2 and back to r_1; χ is therefore a single-valued parameter along the orbit.

Substituting Eq. (5.18) into (5.7), and using Eqs. (5.10) and (5.11), we find

$$\pm(\tilde{E}^2 - V)^{1/2} = e\sin\chi\left[\frac{p - 6 - 2e\cos\chi}{p(p - 3 - e^2)}\right]^{1/2}, \tag{5.19}$$

where the higher (lower) sign corresponds to the first (second) branch of the radial motion. When we substitute Eqs. (5.18) and (5.19) into Eqs. (5.15) and (5.17), we find that the factor $e\sin\chi$ in $(\tilde{E}^2 - V)^{1/2}$ cancels the same factor in $dr/d\chi$, so that the integrands are now regular. We obtain

$$\begin{aligned} t(\chi) &= p^2 M(p-2-2e)^{1/2}(p-2+2e)^{1/2} \\ &\quad \times \int_0^\chi d\chi' \,(p-2-2e\cos\chi')^{-1}(1+e\cos\chi')^{-2} \\ &\quad \times (p-6-2e\cos\chi')^{-1/2}, \end{aligned} \tag{5.20}$$

and

$$\phi(\chi) = p^{1/2}\int_0^\chi \frac{d\chi'}{(p-6-2e\cos\chi')^{1/2}}. \tag{5.21}$$

Since χ is single-valued along the orbit, our expressions for $t(\chi)$ and $\phi(\chi)$ are valid for both branches of the radial motion. The radial period is then given by $P = t(2\pi) = 2t(\pi)$, and $\Delta\phi = \phi(2\pi) = 2\phi(\pi)$.

The substitution $\chi = 2\psi - \pi$ changes the right-hand side of Eq. (5.21) into an elliptic integral of the first kind. The following convenient expression for $\Delta\phi$ is then obtained:

$$\Delta\phi = 4\Big(\frac{p}{p-6+2e}\Big)^{1/2} K\Big(\frac{4e}{p-6+2e}\Big), \tag{5.22}$$

where $K(m) = \int_0^{\pi/2} d\psi\,(1-m\sin^2\psi)^{-1/2}$ is the complete elliptic integral of the first kind [31].

5.2.4 Fundamental frequencies: Ω_r and Ω_ϕ

Equation (5.22) implies that in general, $\Delta\phi$ is not equal to a rational fraction of 2π. This, in turn, implies that the bound orbits of the Schwarzschild spacetime are not closed. A consequence of this fact is that the motion as a whole, as seen by static observers at infinity, is not periodic in t. Only the radial motion shows a periodicity; the azimuthal motion does not.

The purpose of this subsection is to show that there exists a reference frame in which the motion *is*, after all, periodic. This reference frame rotates with a constant angular velocity Ω_ϕ with respect to the static observers at infinity.

It is clear that $r(t)$ is a periodic function of time, with period P, and that any function of $r(t)$ is also a periodic function of time. Any such periodic function, say $a(t)$, can be decomposed into a Fourier series of the form $a(t) = \sum_k a_k \exp(-ik\Omega_r t)$. Here, the sum is over all integers k, $a_k = P^{-1} \int_0^P dt\, a(t) \exp(ik\Omega_r t)$, and Ω_r is the *radial frequency*:

$$\Omega_r = \frac{2\pi}{P}. \tag{5.23}$$

In particular, the function $a(t) = d\phi/dt$ can be so decomposed, and $\phi(t)$ can then be obtained by integrating the series representation of $a(t)$. The result is $\phi(t) = a_0 t + \sum_k b_k \exp(-ik\Omega_r t)$, where $b_k = ia_k/k\Omega_r$ if $k \neq 0$; b_0 can be determined from the constraint $\sum_k b_k = 0$ which enforces the initial condition $\phi(0) = 0$.

We now see that $\phi(t) - a_0 t$ can be expressed as a Fourier series, and must therefore be a periodic function of time. Clearly, $\phi(t) - a_0 t$ is equal to the angular position of the particle, as determined by an observer rotating with constant angular velocity a_0 with respect to static observers at infinity. As seen by this observer, both radial and azimuthal motions are periodic in t.

Finally, the angular velocity $\Omega_\phi \equiv a_0$ can be calculated to be $P^{-1}\int_0^P dt\, a(t) = \Delta\phi/P$, since $a(t) = d\phi/dt$. The *azimuthal frequency* is therefore given by

$$\Omega_\phi = \frac{\Delta\phi}{2\pi}\Omega_r. \tag{5.24}$$

We may conclude that both $r(t)$ and $\phi(t) - \Omega_\phi t$ are periodic functions of time, with a single period P.

5.2.5 Orbits near the separatrix: $p \to 6 + 2e$

In this and the following subsections we shall consider two special cases of bound orbits, and derive corresponding expressions for P, $\Delta\phi$, and Ω_ϕ. We begin with the limiting case of orbits lying very close to the separatrix.

We first define the small parameter

$$\varepsilon \equiv p - 6 - 2e, \tag{5.25}$$

whose magnitude will be constrained below. Substituting this into Eqs. (5.20) and (5.22), we obtain

$$\begin{aligned} P &= 16M(1+e)^{1/2}(3+e)^2[1+O(\varepsilon)] \\ &\quad \times \int_0^\pi d\chi\, \frac{A(1-\cos\chi)}{[\varepsilon + 2e(1-\cos\chi)]^{1/2}}, \end{aligned} \tag{5.26}$$

where $A(x) = (2 + ex)^{-1}(1 + e - ex)^{-2}$, and

$$\Delta\phi = 4\left(\frac{6+2e}{4e+\varepsilon}\right)^{1/2}[1 + O(\varepsilon)]K\left(\frac{4e}{4e+\varepsilon}\right). \tag{5.27}$$

These expressions hold whenever ε is much smaller than unity.

To make our expressions for P and $\Delta\phi$ more explicit, we demand that

$$\varepsilon \ll 4e. \tag{5.28}$$

The alternative requirement, $4e \ll \varepsilon$, will be considered in the next subsection.

Equation (5.28) implies that the argument of the complete elliptic integral in Eq. (5.27) is very close to unity. Using the expansion [31]

$$K(m) = \frac{1}{2}[1 + O(1-m)] \ln \frac{16}{1-m}, \tag{5.29}$$

we arrive at

$$\Delta\phi = 2\left(\frac{3+e}{2e}\right)^{1/2}\left[1 + O\left(\frac{\varepsilon}{4e}\right)\right] \ln \frac{64e}{\varepsilon}. \tag{5.30}$$

For bound orbits which are very close to the separatrix, ϕ increases by an amount much larger than 2π in the course of one radial orbit. The particle therefore revolves many times around the central mass before returning to its apastron.

We now manipulate Eq. (5.26) in order to obtain a more manageable expression for P, in the limit $\varepsilon \ll 4e$. The final answer is given in Eq. (5.34) below.

The integrand of Eq. (5.26) diverges at $\chi = 0$ when $\varepsilon = 0$. This corresponds to the fact that when $\varepsilon = 0$, the particle spends an infinite time at $r = r_1$. We shall rewrite Eq. (5.26) so as to isolate this divergent piece of the integral.

The integrand of Eq. (5.26) also diverges at $\chi = \pi$ when $e = 1$. This corresponds to the fact that when $e = 1$, $r_2 = \infty$ and the orbit is no longer bounded. We will also isolate this divergent piece of the integral, so that the remaining piece will be manifestly finite for all values of the orbital parameters.

We first take care of the piece of P which diverges as $\varepsilon \to 0$. For this purpose, we write $A(x) = A(0)[1 + B(x)]$, where $A(1 - \cos\chi)$ was defined in Eq. (5.26). The contribution to P which involves $A(0) = (1/2)(1+e)^{-2}$ is divergent, and is proportional to

$$\begin{aligned}\int_0^\pi \frac{d\chi}{[\varepsilon + 2e(1-\cos\chi)]^{1/2}} &= \frac{1}{2e^{1/2}} \ln \frac{64e}{\varepsilon} \\ &\quad + O\left(\frac{\varepsilon}{4e} \ln \frac{4e}{\varepsilon}\right);\end{aligned} \tag{5.31}$$

we have used Eq. (5.29) to evaluate the integral.

The contribution to P which involves $B(1-\cos\chi)$ is finite as ε tends to zero. Moreover, it can be checked that setting ε to zero in this term only introduces a discrepancy of order $(\varepsilon/4e)\ln 4e/\varepsilon$ which can be absorbed into the second term to the right-hand side of Eq. (5.31). We therefore have to evaluate

$$\int_0^\pi d\chi \, \frac{B(1-\cos\chi)}{(1-\cos\chi)^{1/2}} \equiv e \int_0^\pi d\chi \, \frac{C(\cos\chi)}{(1+e\cos\chi)^2}, \tag{5.32}$$

where $C(x) = (3 + 2e - e^2x^2)(1-x)^{1/2}/(2+e-ex)$.

The integral to the right-hand side of Eq. (5.32) diverges when $e = 1$. To isolate the divergent piece of this integral, we write $C(x) = C(-1) + C'(-1)(1+x) + 2^{-1/2}D(x)$, where a prime denotes differentiation with respect to the argument.

The contributions to the integral involving $C(-1) = 2^{-1/2}(3-e)$ and $C'(-1) = 2^{-5/2}(7e-3)$ both diverge when $e=1$, because $\int_0^\pi d\chi\,(1+e\cos\chi)^{-2} = \pi(1-e^2)^{-3/2}$ and $\int_0^\pi d\chi\,(1+\cos\chi)(1+e\cos\chi)^{-2} = \pi(1+e)^{-1}(1-e^2)^{-1/2}$. On the other hand, the contribution involving

$$
\begin{aligned}
D(\cos\chi) &= \frac{3+2e-e^2\cos^2\chi}{2+e(1-\cos\chi)}[2(1-\cos\chi)]^{1/2} \\
&\quad -3+e-\tfrac{1}{4}(7e-3)(1+\cos\chi)
\end{aligned}
\tag{5.33}
$$

is finite.

Gathering the results, we find that the orbital period can be expressed as

$$
\begin{aligned}
P &= 4Me^{-1/2}(1+e)^{-3/2}(3+e)^2 \\
&\quad \times\Bigg[\ln\frac{64e}{\varepsilon} + \frac{\pi e(9+6e-7e^2)}{4(1-e^2)^{3/2}} + eI(e) \\
&\quad + O\Big(\frac{\varepsilon}{4e}\ln\frac{4e}{\varepsilon}\Big)\Bigg],
\end{aligned}
\tag{5.34}
$$

where $I(e) = \int_0^\pi d\chi\,(1+e\cos\chi)^{-2}D(\cos\chi)$ is finite for any e. This integral can be evaluated numerically, and we find that $I(e)$ lies within the range $-2.1149 \simeq I(1) \leq I(e) \leq I(0) = 6 - 9\pi/4 \simeq -1.0686$.

We now derive an approximate expression for Ω_ϕ. For convenience, we assume that ε is chosen small enough that in Eq. (5.34), the first term within the square brackets always dominates. We therefore demand that when $e \to 1$,

$$
\varepsilon \ll 4e\,\exp[-(1-e^2)^{-3/2}]. \tag{5.35}
$$

When Eqs. (5.28) and (5.35) hold,

$$\beta \equiv \left[\frac{\pi e(9+6e-7e^2)}{4(1-e^2)^{3/2}} + eI(e)\right]\left(\ln\frac{64e}{\varepsilon}\right)^{-1} \tag{5.36}$$

is always much smaller than unity.

Using Eqs. (5.23), (5.24), (5.30), (5.34) and (5.36), we find that the azimuthal frequency is given by

$$M\Omega_\phi = \left(\frac{1+e}{6+2e}\right)^{3/2}\left[1-\beta+O\left(\frac{\varepsilon}{4e}\right)\right], \tag{5.37}$$

whenever ε satisfies the inequalities (5.28) and (5.35); and when these hold, $\beta = O(2\pi/\Delta\phi) = O[(\ln 4e/\varepsilon)^{-1}] \gg O(\varepsilon/4e)$.

5.2.6 Slightly eccentric orbits: $e \to 0$

It is much easier to derive expressions for P, $\Delta\phi$, and Ω_ϕ for the limiting case of slightly eccentric orbits. We now demand that

$$e \ll \min(1, p-6), \tag{5.38}$$

which we shall impose throughout this subsection.

It is a straightforward matter to use the expansion [31]

$$K(m) = \frac{\pi}{2}[1 + \tfrac{1}{4}m + \tfrac{9}{64}m^2 + \tfrac{225}{2304}m^3 + O(m^4)] \tag{5.39}$$

so as to express Eq. (2.17) as a power series in e. We find

$$\Delta\phi = 2\pi\left(\frac{p}{p-6}\right)^{1/2}\left[1+\frac{3}{4(p-6)^2}e^2+O(e^4)\right]. \tag{5.40}$$

Similarly, we may expand Eq. (5.20) in powers of the eccentricity, and then integrate term by term. We obtain

$$P = \frac{2\pi M p^2}{(p-6)^{1/2}}\Big[1 + \frac{3(2p^3 - 32p^2 + 165p - 266)}{4(p-2)(p-6)^2}e^2 + O(e^4)\Big]. \tag{5.41}$$

Finally, by combining Eqs. (5.23), (5.24), (5.40), and (5.41), we arrive at

$$M\Omega_\phi = p^{-3/2}\Big[1 - \frac{3(p^2 - 10p + 22)}{2(p-2)(p-6)}e^2 + O(e^4)\Big]. \tag{5.42}$$

In Eqs. (5.40)–(5.42), the symbol $O(e^4)$ is used to represent those terms which are fourth or higher order in the eccentricity; these include terms proportional to $e^4/(p-6)^4$. The limit $p \to 6$ must therefore be taken with care, always ensuring that $e \ll p - 6$.

5.3 Radiation reaction

In this section we present our method for calculating the effects of radiation reaction on the bound orbits of the Schwarzschild spacetime. The method is based upon the Teukolsky formalism for black-hole perturbations [20], which is reviewed below. A more detailed presentation can be found in Refs. [1, 32].

5.3.1 The method

Our strategy for calculating the evolution, under radiation reaction, of the bound orbits of the Schwarzschild spacetime was presented in Sec. I C. In short:

We begin by assuming that the motion of the particle is geodesic over a time scale comparable to the orbital period. The validity of this assumption follows from the adiabatic approximation, which states that radiation reaction operates over a much longer time scale.

We use the Teukolsky formalism to calculate $\dot{E}$ and $\dot{L}$, respectively, the time-averaged rates at which the gravitational waves carry away energy and angular momentum. The waves are generated by the orbiting particle, and the average is taken over several orbital periods.

We assume that the orbital parameters change according to

$$\left\langle \frac{d\tilde{E}}{dt} \right\rangle = -\mu \dot{E}, \qquad \left\langle \frac{d\tilde{L}}{dt} \right\rangle = -\mu \dot{L}, \tag{5.43}$$

so that the total energy of the whole system (black hole plus particle plus waves) is conserved. The symbol $\langle \ \rangle$ designates the time average.

Because of the radiation reaction, the particle's world line is not strictly a geodesic. However, as required by the adiabatic approximation, and in agreement with our initial assumption, the deviations from geodesic motion become noticeable only after a large number of orbits.

The only essential assumption made in this calculation is that μ/M is sufficiently small that: (i) the gravitational perturbations obey linear wave equations; and (ii) the adiabatic approximation is valid. In Sec. IV D, we shall formulate precise conditions on μ/M which ensure the validity of the adiabatic approximation.

5.3.2 The Teukolsky formalism

In the Teukolsky formalism, gravitational perturbations are described by the Weyl scalar $\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$, where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $n^\alpha = \frac{1}{2}(1, -f, 0, 0)$, and $\bar{m}^\alpha = (0, 0, 1, -i\csc\theta)/\sqrt{2}r$. Throughout we denote complex conjugation with a bar. At large distances, Ψ_4 describes outgoing gravitational waves.

The Weyl scalar can be decomposed into Fourier-harmonic components according to

$$\Psi_4 = \int_{-\infty}^{\infty} d\omega \sum_{\ell m} r^{-4} R_{\omega\ell m}(r)\, {}_{-2}Y_{\ell m}(\theta, \phi) e^{-i\omega t}, \tag{5.44}$$

where ${}_sY_{\ell m}(\theta, \phi)$ are spin-weighted spherical harmonics [33]. The sums over ℓ and m are restricted to $-\ell \le m \le \ell$ and $\ell \ge 2$. The radial function $R_{\omega\ell m}(r)$ satisfies the inhomogeneous Teukolsky equation,

$$\left[r^2 f \frac{d^2}{dr^2} - 2(r - M)\frac{d}{dr} + U(r)\right] R_{\omega\ell m}(r) = -T_{\omega\ell m}(r), \tag{5.45}$$

with

$$U(r) = f^{-1}[(\omega r)^2 - 4i\omega(r - 3M)] - \lambda, \tag{5.46}$$

where $\lambda = (\ell - 1)(\ell + 2)$.

The source term in Eq. (5.45) is calculated from the particle's stress-energy tensor,

$$T^{\alpha\beta}(x) = \mu \int d\tau\, u^\alpha u^\beta \delta^{(4)}[x - x'(\tau)], \tag{5.47}$$

where x is the spacetime point, $x'(\tau)$ the particle's world line with tangent vector $u^\alpha = dx'^\alpha/d\tau$, and τ denotes proper time. The first step is to construct the projections ${}_0T = T_{\alpha\beta}n^\alpha n^\beta$, ${}_{-1}T = T_{\alpha\beta}n^\alpha \bar{m}^\beta$, and ${}_{-2}T = T_{\alpha\beta}\bar{m}^\alpha \bar{m}^\beta$. Then one calculates the Fourier-harmonic components ${}_sT_{\omega\ell m}(r)$ according to

$$ {}_sT_{\omega\ell m}(r) = \frac{1}{2\pi}\int dt\, d\Omega\, {}_sT\, {}_s\bar{Y}_{\ell m}(\theta,\phi)e^{i\omega t}, \tag{5.48}$$

where $d\Omega$ is the element of solid angle. Finally, the source is [32, 34]

$$\begin{aligned} T_{\omega\ell m}(r) &= 2\pi\Big\{2[\lambda(\lambda+2)]^{1/2} r^4\, {}_0T_{\omega\ell m}(r) \\ &\quad + 2(2\lambda)^{1/2} r^2 f \mathcal{L}\, r^3 f^{-1}\, {}_{-1}T_{\omega\ell m}(r) \\ &\quad + r f \mathcal{L} r^4 f^{-1} \mathcal{L}\, r\, {}_{-2}T_{\omega\ell m}(r)\Big\}, \end{aligned} \tag{5.49}$$

where $\mathcal{L} = f d/dr + i\omega$.

The inhomogeneous Teukolsky equation (5.45) can be integrated by means of a Green's function [35]. (The Green's function is so constructed that Ψ_4 satisfies a no-incoming-radiation condition.) The solution at large radii is

$$R_{\omega\ell m}(r \to \infty) \sim \mu\, \omega^2 Z_{\omega\ell m} r^3 e^{i\omega r^*}, \tag{5.50}$$

and represents purely outgoing waves. Here, $r^* = r + 2M\ln(r/2M - 1)$. The amplitudes $Z_{\omega\ell m}$ are given by

$$Z_{\omega\ell m} = \frac{1}{2i\mu\,\omega^2 Q^{\rm in}_{\omega\ell}}\int_{2M}^{\infty} dr\, \frac{R^H_{\omega\ell}(r) T_{\omega\ell m}(r)}{r^4 f^2}, \tag{5.51}$$

where the function $R^H_{\omega\ell}(r)$ is the solution to the *homogeneous* Teukolsky equation with ingoing-wave boundary conditions at the black-hole horizon: $R^H_{\omega\ell}(r \to 2M) \sim (\omega r)^4 f^2 e^{-i\omega r^*}$. At infinity, $R^H_{\omega\ell}(r)$ represents a superpostion of ingoing and outgoing waves, $R^H_{\omega\ell}(r \to \infty) \sim Q^{\rm in}_{\omega\ell}(\omega r)^{-1}e^{-i\omega r^*} + Q^{\rm out}_{\omega\ell}(\omega r)^3 e^{i\omega r^*}$; $Q^{\rm in}_{\omega\ell}$ and $Q^{\rm out}_{\omega\ell}$ are constants, independent of r. The amplitudes $Z_{\omega\ell m}$ satisfy the identities

$$\bar{Z}_{-\omega,\ell,-m} = (-1)^\ell Z_{\omega\ell m}, \tag{5.52}$$

which are derived in Ref. [1].

We now specialize to the case considered in this paper, for which the frequency spectrum contains only a discrete set of distinct frequencies ω_{mk} (subsection C). We then have

$$Z_{\omega\ell m} = \sum_k Z^k_{\ell m}\delta(\omega - \omega_{mk}). \tag{5.53}$$

As indicated, the frequencies ω_{mk} are characterized by two sets of integers, m and k. The time-averaged rates at which the gravitational waves carry energy and angular momentum to infinity are calculated to be

$$\dot{E}^\infty = \sum_{\ell mk} \dot{E}^\infty_{\ell mk}, \qquad \dot{L}^\infty = \sum_{\ell mk} \dot{L}^\infty_{\ell mk}, \tag{5.54}$$

where

$$\dot{E}^\infty_{\ell mk} = \frac{\mu^2}{4\pi}\,{\omega_{mk}}^2 |Z^k_{\ell m}|^2, \tag{5.55}$$

and

$$\dot{L}^\infty_{\ell mk} = \frac{\mu^2}{4\pi}\, m\omega_{mk} |Z^k_{\ell m}|^2. \tag{5.56}$$

We stress that $\dot{E}^\infty$ and $\dot{L}^\infty$ represent *time-averaged* rates; the average is taken over several orbital periods. For reasons to be given in Sec. V E, we will not consider here the energy and angular momentum which are absorbed by the black hole. To an accuracy sufficient for our purposes, we shall neglect these contributions to $\dot{E}$ and $\dot{L}$, and set $\dot{E} = \dot{E}^{\infty}$, $\dot{L} = \dot{L}^{\infty}$.

5.3.3 Calculation of ${}_sT_{\omega\ell m}(r)$

We now proceed with the calculation of the source term in Eq. (5.45), taking the particle's world line to be a bound geodesic of the Schwarzschild spacetime. Our starting point is the particle's stress-energy tensor, which is given by Eq. (5.47). After integration, this becomes

$$T^{\alpha\beta}(x) = \mu \frac{u^\alpha u^\beta}{r'^2 u^t} \delta(r - r')\delta(\cos\theta)\delta(\phi - \phi'). \tag{5.57}$$

Here, $\{t, r, \theta, \phi\}$ are the coordinates of the spacetime point x, and $\{t, r'(t), \pi/2, \phi'(t)\}$ describe the particle's world line; the four-velocity $u^\alpha = dx'^\alpha/d\tau$ can be obtained from Eq. (5.6).

Following the procedure given in subsection B, we find

$$\begin{aligned} {}_sT_{\omega\ell m}(r) &= \frac{\mu}{2\pi}\, {}_sY_{\ell m}(\tfrac{\pi}{2}, 0) \\ &\quad \times \int_{-\infty}^{\infty} dt\, {}_sF(r')\delta(r - r')e^{i(\omega t - m\phi')}, \end{aligned} \tag{5.58}$$

where

$${}_sF(r') = \frac{1}{r'^2 u^t} \begin{cases} (u\cdot n)^2 & s = 0 \\ (u\cdot n)(u\cdot \bar{m}) & s = -1 \\ (u\cdot \bar{m})^2 & s = -2 \end{cases} . \tag{5.59}$$

Here, $u \cdot n \equiv u^\alpha n_\alpha$, etc., and the vectors n^α and $\bar{m}^\alpha$ are evaluated on the particle's world line.

To evaluate the integral of Eq. (5.58), we rewrite the integrand as ${}_sa(t)\exp[i(\omega - m\Omega_\phi)t]$, where ${}_sa(t) = {}_sF(r')\delta(r-r')\exp[-im(\phi' - \Omega_\phi t)]$. According to the results of Sec. II D, the functions ${}_sa(t)$ are periodic in t, with period P. This means that we can express these functions as Fourier series of the form $\sum_k {}_sa_k \exp(-ik\Omega_r t)$, with ${}_sa_k = P^{-1}\int_0^P {}_sa(t)\exp(ik\Omega_r t)$. We then substitute the series representations of ${}_sa(t)$ into Eq. (5.58), which is now easily integrated. The result is

$$\begin{aligned} {}_sT_{\omega\ell m}(r) &= \mu\, {}_sY_{\ell m}(\tfrac{\pi}{2},0)P^{-1}\sum_k \delta(\omega - \omega_{mk}) \\ &\quad \times \int_0^P dt\, {}_sF(r')\delta(r-r')e^{i(\omega_{mk}t - m\phi')}. \end{aligned} \tag{5.60}$$

The frequency spectrum is given by

$$\omega_{mk} = m\Omega_\phi + k\Omega_r, \tag{5.61}$$

where k is an integer running from $-\infty$ to $+\infty$. Equations (5.60) and (5.61) express the fact that the frequency spectrum contains only a discrete set of distinct frequencies, the harmonics of the fundamental frequencies Ω_ϕ and Ω_r. This fact has already been used in Eq. (5.53) above.

We now transform Eq. (5.60) into an integral over r', using $dr'/dt = \pm f'\tilde{E}^{-1}(\tilde{E}^2 - V')^{1/2}$, where the higher (lower) sign refers to the first (second) branch of the radial motion (Sec. II C); we also have $f' = 1 - 2M/r'$ and $V' = V(\tilde{L}, r')$. Breaking the integration into two parts corresponding to each branch, we find that Eq. (5.60) becomes

$$ {}_sT_{\omega\ell m}(r) = \mu\, {}_sY_{\ell m}(\tfrac{\pi}{2},0)\Theta(r-r_1)\Theta(r_2-r)$$

$$\times f^2 P^{-1}(\tilde{E}^2 - V)^{-1/2} \sum_k \delta(\omega - \omega_{mk})$$
$$\times \sum_{\pm} {}_sG_{\pm}(r) e^{\pm i[\omega_{mk}\hat{t}(r) - m\hat{\phi}(r)]}. \qquad (5.62)$$

Here, $\Theta(r)$ is the Heaviside step function, $f = 1 - 2M/r$, $V = V(\tilde{L}, r)$; $\hat{t}(r)$ and $\hat{\phi}(r)$ were defined in Eqs. (5.15) and (5.17). We also have ${}_sG_{\pm} = \tilde{E}_s F/f^3$, where the right-hand side is evaluated on the first (higher sign), or second (lower sign), branch of the radial motion. More explicitly, making use of Eq. (5.59),

$${}_sG_{\pm}(r) = \frac{1}{4r^4 f^2} \begin{cases} r^2[\tilde{E} \pm (\tilde{E}^2 - V)^{1/2}]^2 & s = 0 \\ \sqrt{2} ir\tilde{L}[\tilde{E} \pm (\tilde{E}^2 - V)^{1/2}] & s = -1 \\ -2\tilde{L}^2 & s = -2 \end{cases} . \qquad (5.63)$$

5.3.4 Calculation of $Z^k_{\ell m}$

In this subsection we calculate the amplitudes $Z^k_{\ell m}$ using Eqs. (5.49), (5.51), (5.53), and (5.62). These can then be substituted into Eqs. (5.55) and (5.56) to calculate the contributions to $\dot{E}$ and $\dot{L}$ from each ℓ, m, and k. The final result is obtained by summing over all these integers, as shown in Eq. (5.54)

We start from Eq. (5.49), which we re-express as $T_{\omega\ell m} = 2\pi \sum_s {}_sD_{\omega\ell}\, {}_sT_{\omega\ell m}$, where the operators ${}_sD_{\omega\ell}$ can easily be identified. For convenience, we also rewrite Eq. (5.51) as

$$Z_{\omega\ell m} = (2i\mu\omega^2 Q^{\rm in}_{\omega\ell})^{-1} \sum_s {}_sZ_{\omega\ell m}, \qquad (5.64)$$

where

$${}_sZ_{\omega\ell m} = 2\pi \int_{2M}^{\infty} dr\, \frac{R^H_{\omega\ell}(r)\, {}_sD_{\omega\ell}\, {}_sT_{\omega\ell m}(r)}{r^4 f^2}. \qquad (5.65)$$

The right-hand side of Eq. (5.65) can be regarded as an inner product (R, DT), where R and T are functions, and D an operator (we have suppressed the use of the indices for greater clarity). To simplify the evaluation of this inner product, we define the adjoint operator $D^\dagger$ such that $(R, DT) = (D^\dagger R, T)$. This new operator can be calculated by performing a number of integration by parts on Eq. (5.65). After such manipulations, we find that Eq. (5.65) becomes

$$ {}_sZ_{\omega\ell m} = 2\pi\, {}_sp_\ell \int_{2M}^{\infty} dr\, f^{-2}\, {}_sR^H_{\omega\ell}(r)\, {}_sT_{\omega\ell m}(r). \tag{5.66}$$

We have introduced

$$ {}_sp_\ell = \begin{cases} 2[\lambda(\lambda+2)]^{1/2} & s=0 \\ 2(2\lambda)^{1/2} & s=-1 \\ 1 & s=-2 \end{cases}, \tag{5.67}$$

where $\lambda = (\ell-1)(\ell+2)$, together with

$$ {}_0R^H_{\omega\ell}(r) = R^H_{\omega\ell}(r), \tag{5.68}$$

$$ {}_{-1}R^H_{\omega\ell}(r) = \left(-rf\frac{d}{dr} + 2f + i\omega r\right) R^H_{\omega\ell}(r), \tag{5.69}$$

$$ \begin{aligned} {}_{-2}R^H_{\omega\ell}(r) = &\left[r^2 f^2 \frac{d^2}{dr^2} - 2rf(f+i\omega r)\frac{d}{dr}\right. \\ &\left. + i\omega r(2 - 2M/r + i\omega r)\right] R^H_{\omega\ell}(r). \end{aligned} \tag{5.70}$$

Equations (5.66)–(5.70) are valid irrespective of the choice of source functions ${}_sT_{\omega\ell m}(r)$.

The final step is to specialize to the source functions which are relevant to our problem, and to substitute Eq. (5.62) into (5.66). Using Eqs. (5.53) and (5.64) along the way, we find

$$Z^k_{\ell m} = [2i\mu(\omega_{mk})^2 Q^{\rm in}_{\omega_{mk}\ell}]^{-1} \sum_s {}_sZ^k_{\ell m}, \tag{5.71}$$

with

$$\begin{aligned} {}_sZ^k_{\ell m} &= \mu\, {}_sp_\ell\, {}_sY_{\ell m}(\tfrac{\pi}{2},0)\Omega_r \sum_{\pm} \int_{r_1}^{r_2} dr\, (\tilde{E}^2 - V)^{-1/2} \\ &\quad \times {}_sG_\pm(r)\, {}_sR^H_{\omega_{mk}\ell}(r) e^{\pm i[\omega_{mk}\hat{t}(r) - m\hat{\phi}(r)]}. \end{aligned} \tag{5.72}$$

In general, the integrals of Eq. (5.72) must be evaluated numerically. To facilitate these integrations, we make the substitution $r = r(\chi)$, given in Eq. (5.18), which removes the bad behavior of the integrand at $r = r_1$ and $r = r_2$. This change of variables also makes the perturbation formalism robust, in the sense that the limit $e = 0$ can be taken directly, without difficulty.

5.3.5 The radiation reaction equations

The Teukolsky formalism, as summarized in the preceding subsections, allows us to calculate $\dot{E}$ and $\dot{L}$, the time-averaged rates at which the gravitational waves carry away energy and angular momentum. Using Eq. (5.43), we can then infer the time-averaged rates of change of the orbital parameters.

In Sec. II B we have introduced the quantities p and e as a preferred set of orbital parameters. The purpose of this subsection is to relate the rates of change of p and e to $\dot{E}$ and $\dot{L}$, which are directly obtained from the Teukolsky formalism.

Since p and e are functions of $\tilde{E}$ and $\tilde{L}$, we have, using Eq. (5.43), $-\dot{E} =$

$(\partial\tilde{E}/\partial p)\mu\dot{p} + (\partial\tilde{E}/\partial e)\mu\dot{e}$ and $-\dot{L} = (\partial\tilde{L}/\partial p)\mu\dot{p} + (\partial\tilde{L}/\partial e)\mu\dot{e}$. These equations can easily be inverted. Using Eqs. (5.10) and (5.11), we find

$$\begin{aligned} \mu\dot{p} &= \frac{2(p-3-e^2)^{1/2}}{(p-6-2e)(p-6+2e)}\Big[p^{3/2}(p-2-2e)^{1/2} \\ &\quad \times (p-2+2e)^{1/2}\dot{E} - (p-4)^2\dot{L}/M\Big], \end{aligned} \tag{5.73}$$

and

$$\begin{aligned} \mu\dot{e} &= \frac{(p-3-e^2)^{1/2}}{ep(p-6-2e)(p-6+2e)}\Big\{-p^{3/2}(p-6-2e^2) \\ &\quad \times (p-2-2e)^{1/2}(p-2+2e)^{1/2}\dot{E} \\ &\quad + (1-e^2)[(p-2)(p-6)+4e^2]\dot{L}/M\Big\}. \end{aligned} \tag{5.74}$$

It is important to notice that Eqs. (5.73) and (5.74) are singular at $p = 6 + 2e$.

Radiation reaction produces a slow evolution of the orbital parameters, and therefore generates curves in the p-e plane. We can anticipate that the curves must all cross the separatrix $p = 6 + 2e$, so that the particle must eventually plunge inside the black hole. To see this, we only need recall that the gravitational waves remove angular momentum from the system. This induces a decrease in $V_{\rm max}$, the value of the effective potential at the local maximum (see Fig. 4). When $\tilde{L}$ reaches the critical value $2\sqrt{3}M$, the potential barrier disappears altogether. The particle must therefore plunge either at, or prior to, this point. In the former case (plunging when $\tilde{L} = 2\sqrt{3}M$), the particle's orbit is circular immediately before plunging; in the latter (plunging when $\tilde{L} > 2\sqrt{3}M$), the orbit is eccentric.

The detailed behavior of the radiation-reaction curves near $p = 6 + 2e$ will be discussed in Sec. IV B.

5.4 Analytical results

The first part of this section (subsections A to C) is devoted to the calculation of the radiation-reaction curves (Sec. III E) in those regions of the p-e plane for which the calculation can be performed, to some degree of accuracy, analytically. More specifically, we shall be interested in evaluating $\dot{p}$ and $\dot{e}$, as well as the function

$$c(p,e) = \frac{d\ln e}{d\ln p}. \tag{5.75}$$

In Eq. (5.75), the variations in p and e are calculated using the radiation reaction equations (5.73) and (5.74). Notice also that dp and de denote time-averaged variations; as usual, the average is taken over several orbital periods.

In the second part of this section (subsection D), we will use our analytical expressions for $\dot{p}$ and $\dot{e}$ to formulate constraints on the magnitude of μ/M. These will ensure the validity of the adiabatic approximation (Sec. III A) throughout the p-e plane.

5.4.1 Weak-field radiation reaction: $p \gg 6$

The effects of gravitational radiation reaction in weak-field, slow-motion situations are well understood and can be derived, to leading order [11], using a Newtonian potential of the form $\Phi_{\rm rr} = (1/5)(d^5 Q_{ab}/dt^5)x^a x^b$, where Q_{ab} is the traceless quadrupole moment of the mass distribution [4, 5, 6, 7]. The radiation reaction force is then given by $\boldsymbol{F}_{\rm rr} = -\mu\nabla\Phi_{\rm rr}$, and the resulting equations of motion can be used to calculate the rates of change of the orbital parameters.

We shall instead follow the equivalent procedure of using the $p \to \infty$ limit of Eqs. (5.73) and (5.74), together with suitable expressions for $\dot{E}$ and $\dot{L}$, to calculate $\dot{p}$ and $\dot{e}$. The equations of Sec. III could be integrated analytically in the limit $p \to \infty$, so as to yield the desired expressions for the fluxes of energy and angular momentum. (See Refs. [32, 36, 37] for similar analytical integrations of the perturbation equations.) However, it is much easier to obtain $\dot{E}$ and $\dot{L}$ from Peters' classic paper [21], in which they are calculated, to leading order in the weak-field limit, using the quadrupole formulae [5]. The results are

$$\dot{E} = \frac{32}{5}\Big(\frac{\mu}{M}\Big)^2 p^{-5}\big(1-e^2\big)^{3/2}\big(1+\tfrac{73}{24}e^2+\tfrac{37}{96}e^4\big), \tag{5.76}$$

and

$$\dot{L}/M = \frac{32}{5}\Big(\frac{\mu}{M}\Big)^2 p^{-7/2}\big(1-e^2\big)^{3/2}\big(1+\tfrac{7}{8}e^2\big). \tag{5.77}$$

Equations (5.76) and (5.77) are valid up to fractional corrections of order p^{-1}.

It is then a matter of simple algebra to substitute Eqs. (5.76) and (5.77) into the $p \to \infty$ limit of Eqs. (5.73) and (5.74), to obtain

$$\mu\dot{p} = -\frac{64}{5}\Big(\frac{\mu}{M}\Big)^2 p^{-3}\big(1-e^2\big)^{3/2}\big(1+\tfrac{7}{8}e^2\big), \tag{5.78}$$

and

$$\mu\dot{e} = -\frac{304}{15}\Big(\frac{\mu}{M}\Big)^2 p^{-4}e\big(1-e^2\big)^{3/2}\big(1+\tfrac{121}{304}e^2\big). \tag{5.79}$$

These equations imply that weak-field radiation reaction decreases *both* the semi-latus rectum p and the eccentricity e.

Substituting Eqs. (5.78) and (5.79) into Eq. (5.75) we arrive at

$$c(p,e) = \frac{19}{12}\left[1 + O(p^{-1})\right]\left(1 + \tfrac{7}{8}e^2\right)^{-1}\left(1 + \tfrac{121}{304}e^2\right), \tag{5.80}$$

which is valid for large p and any value of e. Equation (5.80) implies the well-known result that an initially eccentric orbit becomes circular if radiation reaction operates for a sufficiently long time [21, 22]. (This conclusion, we stress, is only true for weak-field radiation reaction.)

It is worth noting that the results presented in this subsection are valid also for binary systems with arbitrary mass ratios [21], provided that μ is then interpreted as the reduced mass of the system, M as the total mass, and p and e as the orbital parameters of the *relative* orbit.

5.4.2 Strong-field radiation reaction: $p - 6 - 2e \ll 4e$

The region of the p-e plane which lies very close to the separatrix $p = 6 + 2e$ is also amenable to approximate, analytical calculations. In this subsection we will take $\varepsilon/4e$, where

$$\varepsilon \equiv p - 6 - 2e, \tag{5.81}$$

to be much smaller than unity.

Our starting point is the statement that when $p = 6 + 2e$, so that the orbit is circular and unstable (Sec. II B), the fluxes of energy and angular momentum are related by $\dot{E} = \Omega_\phi \dot{L}$, where Ω_ϕ is given by the $\varepsilon = 0$ limit of Eq. (5.37): $M\Omega_\phi = (1+e)^{3/2}/(6+2e)^{3/2}$. This statement can be justified as follows. Equation (5.34) implies that the radial period P diverges when ε approaches zero, which means that $\Omega_r = 2\pi/P$ vanishes in that limit. From Eq. (5.61) we then find that the frequency spectrum of the gravitational perturbations is given by $\omega_{mk} = m\Omega_\phi$. Finally, substituting this into Eqs. (5.54)–(5.56) shows that the fluxes of energy and angular momentum *at infinity* satisfy the equality $\dot{E} = \Omega_\phi \dot{L}$. It can also be shown that this frequency spectrum implies the same relationship between the fluxes at the black-hole horizon. (For explicit expressions see Ref. [1].) The desired result therefore follows.

The transformation $\{\dot{E}, \dot{L}\} \to \{\dot{p}, \dot{e}\}$ is singular at $\varepsilon = 0$; see Sec. III E. In order to calculate $\dot{p}$ and $\dot{e}$ in the limit $\varepsilon/4e \ll 1$, we need to know the relationship between $\dot{E}$ and $\dot{L}$ for orbits which are slightly away from the separatrix. The discussion of the previous paragraph allows us to write

$$\dot{E} = (1+\alpha)\Omega_\phi \dot{L}, \qquad \alpha \ll 1, \tag{5.82}$$

where α is only known to vanish in the limit $\varepsilon = 0$. We do not need to know the relative magnitude of α, in relation with ε, for our purposes. However, it may be argued, using Eqs. (5.24) and (5.30), that $\alpha = O(\Omega_r/\Omega_\phi) = O(2\pi/\Delta\phi) = O[(\ln 4e/\varepsilon)^{-1}] \gg O(\varepsilon/4e)$. We will not need to rely on this crude, nonrigorous estimate.

Substituting Eq. (5.37) into (5.82), we arrive at

$$\frac{\dot{L}}{M\dot{E}} = \left(\frac{6+2e}{1+e}\right)^{3/2}\left[1+\beta-\alpha+O\left(\frac{\varepsilon}{4e}\right)\right]. \tag{5.83}$$

We have neglected, in the square brackets, terms of quadratic and higher order in α. In Eq. (5.83), the relative magnitude of α, compared to that of β and $\varepsilon/4e$, is not precisely known [it is most likely that α and β are of comparable magnitude; see Eq. (5.36)]. However, this information is not needed to calculate $c(e,p)$. What is required for the calculation is the knowledge that $\beta \gg O(\varepsilon/4e)$, so that the term $\beta - \alpha$ is the larger correction term in Eq. (5.83). (We dismiss as improbable the possibility that α and β are equal up to terms of order $\varepsilon/4e$ or smaller. We have verified numerically that $\beta - \alpha$ is always much larger than $\varepsilon/4e$.)

The substitution of Eq. (5.83) into Eqs. (5.73) and (5.74) yields

$$\begin{aligned}\mu\dot{p} &= -2e^{-1}(1+e)(3-e)^{1/2}(6+2e)^{3/2} \\ &\quad\times \frac{\beta-\alpha+O(\varepsilon/4e)}{\varepsilon}\dot{E},\end{aligned} \tag{5.84}$$

and

$$\begin{aligned}\mu\dot{e} &= 2e^{-1}(1-e)(1+e)(3-e)^{1/2}(6+2e)^{1/2} \\ &\quad\times \frac{\beta-\alpha+O(\varepsilon/4e)}{\varepsilon}\dot{E}.\end{aligned} \tag{5.85}$$

Here, $\dot{E}$ is evaluated on the separatrix, where it is finite and nonvanishing. Use of Eq. (5.75) then gives

$$c(p,e) = -\frac{1-e}{e}\left\{1+O\left[\frac{\varepsilon}{4e(\beta-\alpha)}\right]\right\}, \tag{5.86}$$

which is valid for $\varepsilon/4e \ll 1$. According to our previous estimate for the magnitude of α, it is most likely that the correction term in Eq. (5.86) is of order $(\varepsilon/4e)\ln(\varepsilon/4e)$.

Although α cannot be calculated analytically, we may nevertheless state that near the separatrix, $\dot{p} < 0$ and $\dot{e} > 0$, which implies that $\beta - \alpha + O(\varepsilon/4e) > 0$. This statement follows from: (i) the fact that $de/dp < 0$ near the separatrix, which is a consequence of Eq. (5.86); and (ii) the fact that the radiation-reaction curves must cross the separatrix, a property that was proved in Sec. III E.

We have therefore established that radiation reaction acts on orbits which are close to the separatrix so as to decrease the semi-latus rectum p, and to *increase* the eccentricity e [23]. This is in marked contrast with weak-field radiation reaction, which decreases the eccentricity. We remark that for fixed μ/M, the divergence of $\mu\dot{p}$ and $\mu\dot{e}$ in the limit $\varepsilon \to 0$ signals the breakdown of the adiabatic approximation. This point will be discussed in subsection D.

The asymptotic expressions for $c(e,p)$, Eqs. (5.80) and (5.86), together with the fact that radiation reaction always decreases p, imply the existence of a *critical curve* in the p-e plane, along which $de/dp = 0$. Equation (5.86) further implies that the critical curve meets with the separatrix at $e = 1$. The existence of such a curve is a genuine strong-field effect, which can perhaps be understood as a precursor effect to the eventual plunging of the orbit. The precise location of the critical curve can be found by numerically integrating the perturbation equations. A portion of the critical curve is displayed in Fig. 3.

5.4.3 Strong-field radiation reaction: $e \ll p - 6$

The results derived in the preceding subsection are valid for small eccentricities, provided that $p - 6 - 2e$ is always taken to be much smaller than $4e$. This amounts to approaching the point $(p, e) = (6, 0)$ along a path which lies very close to the separatrix $p = 6 + 2e$. In this subsection, we calculate $\dot{p}$, $\dot{e}$, and $c(p, e)$ also in the neighborhood of the point $(6, 0)$, but now approaching it on a path which lies very close to $e = 0$. This amounts to taking the limits $e \to 0$, $p \to 6$ *in that order*, always ensuring that $e/(p - 6) \ll 1$. As we shall see, the point $(6, 0)$ is a singular point of the p-e plane, in the sense that $c(e, p)$ diverges there, and that its degree of divergence depends on the direction of approach.

The results contained in this subsection are not new, and were first presented in Ref. [1]. We shall nevertheless repeat this analysis here, for two main reasons. First, we wish this paper to be as complete and self-contained as possible, and the case $e \ll p - 6$ must be discussed. And second, our rederivation of the results will allow us to formulate an assumption that was left implicit in Ref. [1]; this assumption concerns the order in which the limits $e \to 0$, $p \to 6$ are taken.

Our starting point is the statement that for stable circular orbits, the fluxes of energy and angular momentum are related by $\dot{E} = \Omega_\phi \dot{L}$, where Ω_ϕ is given by the $e = 0$ limit of Eq. (5.42): $M\Omega_\phi = p^{-3/2}$. This statement is justified by first taking the $e = 0$ limit of Eq. (5.72). [This limit is taken only after the substitution $r = r(\chi)$, Eq. (5.18), is made, and Eq. (5.19) used.] The explicit evaluation of

${}_sZ^k_{\ell m}$ is straightforward in that limit. The result is that ${}_sZ^k_{\ell m}$ vanishes unless $k = 0$, which implies that the harmonics of the radial frequency Ω_r do not contribute to the frequency spectrum, Eq. (5.61). Finally, use of Eq. (5.71), and then of Eqs. (5.54)–(5.56), yields the desired result. (Actually, this argument proves only that the fluxes at infinity satisfy the relation $\dot{E} = \Omega_\phi \dot{L}$. However, the argument can be generalized so as to also include the fluxes at the black-hole horizon, for which the same relation holds. This more complete analysis is presented in Ref. [1].)

When the orbit is slightly eccentric, we have that $\dot{E}$ and $\dot{L}$ are now related by

$$\dot{E} = [1 + \gamma e^2 + O(e^4)]\Omega_\phi \dot{L}. \tag{5.87}$$

That the first-order correction is quadratic in e can be expected from the results of Sec. II F; this can also be justified rigorously by taking the small-eccentricity limit of the relevant equations of Sec. III. This analysis was carried out in Ref. [1], which also reveals that $\gamma(p)$ is well behaved in the limit $p \to 6$ [38]. This property will be used below.

It is straightforward to substitute Eq. (5.42) into Eq. (5.87) to obtain an expression for $\dot{L}/M\dot{E}$, and to then expand Eqs. (5.73) and (5.74) in powers of e. The results are

$$\mu \dot{p} = -\frac{2p^{3/2}(p-3)^{3/2}}{p-6}\left\{1 + O\left[\left(\frac{e}{p-6}\right)^2\right]\right\}\dot{E}, \tag{5.88}$$

and

$$\mu \dot{e} \;=\; \frac{ep^{1/2}(p-3)^{1/2}}{2(p-2)(p-6)^2}\Big[p^3 - 12p^2 + 66p - 108$$

$$- 2\gamma(p-2)^2(p-6)\Big]\Big\{1 + O\Big[\Big(\frac{e}{p-6}\Big)^2\Big]\Big\}\dot{E}. \tag{5.89}$$

The detailed behavior of $\mu\dot{e}/e\dot{E}$ as a function of p depends on $\gamma(p)$, which must in general be evaluated numerically. However, since γ is well-behaved in the limit $p \to 6$, the behavior of $\mu\dot{e}/e\dot{E}$ in that limit can be calculated unambiguously.

Taking the limit $p \to 6$ in Eqs. (5.88) and (5.89), we obtain

$$\mu\dot{p} = -\frac{108\sqrt{2}}{p-6}\Big\{1 + O\Big[p-6, \Big(\frac{e}{p-6}\Big)^2\Big]\Big\}\dot{E}, \tag{5.90}$$

and

$$\mu\dot{e} = \frac{27\sqrt{2}}{(p-6)^2}e\Big\{1 + O\Big[p-6, \Big(\frac{e}{p-6}\Big)^2\Big]\Big\}\dot{E}. \tag{5.91}$$

Finally, substituting Eqs. (5.90) and (5.91) into Eq. (5.75), we arrive at

$$c(e,p) = -\frac{3}{2}\frac{1}{p-6}\Big\{1 + O\Big[p-6, \Big(\frac{e}{p-6}\Big)^2\Big]\Big\}. \tag{5.92}$$

The results presented here are consistent with our previous conclusion that radiation reaction acts on orbits which are close to the separatrix so as to decrease p and increase e. We also remark that for fixed μ/M, the divergence of $\mu\dot{p}$ and $\mu\dot{e}$ in the limit $p \to 6$ signals the breakdown of the adiabatic approximation. We will return to this point in subsection D.

It should be emphasized that in both this and the preceding subsections, calculations were based on this property of circular orbits that $\dot{E} = \Omega_\phi \dot{L}$. This property is very general, and does not depend on the fact that the radiation field is gravitational [39]. That $\dot{E} = \Omega_\phi \dot{L}$ follows from two key elements. The first is that for circular orbits, the radiation field possesses a frequency spectrum of the form $\omega = m\Omega_\phi$. This follows from the circularity of the orbit only, and holds for any type of radiation field. The second key element is that irrespective of its type, the radiation field transports energy and angular momentum in such a way that for each frequency component, $\dot{E}_\omega \propto \omega$ and $\dot{L}_\omega \propto m$, where *the constant of proportionality is the same in both expressions* [40]. The equality $\dot{E} = \Omega_\phi \dot{L}$ is therefore valid for arbitrary radiation fields, and so are the results presented in this and the preceding subsections.

5.4.4 The adiabatic approximation

We now use the analytical estimates of the previous subsections to formulate constraints on μ/M which ensure the validity of the adiabatic approximation. These constraints are most severe in the vicinity of the separatrix. Due to the singularity of the transformation $\{\dot{E}, \dot{L}\} \to \{\dot{p}, \dot{e}\}$ at $p = 6 + 2e$, see Sec. III E, radiation reaction occurs increasingly rapidly as the orbit approaches the separatrix. Since $\dot{p}$ and $\dot{e}$ scale with μ/M, the validity of the adiabatic approximation can be maintained at the price of decreasing μ/M sufficiently rapidly. For a fixed mass ratio, the adiabatic approximation must eventually break down.

The adiabatic approximation is formulated by requiring that a relevant orbital parameter q changes very little over time scales comparable to the orbital period P.

More precisely, we demand that

$$\Delta q \ll q, \tag{5.93}$$

where $\Delta q = |\dot{q}|P$ is the change in q after one radial orbit. By choosing q appropriately, and then estimating $\dot{q}$ and P, Eq. (5.93) can be transformed into a condition on μ/M.

The case $p \gg 6$

Expressions for $\dot{p}$ and $\dot{e}$ which are valid for large p are given by Eqs. (5.78) and (5.79). Using these results together with $P = 2\pi p^{3/2}(1-e^2)^{-3/2}M$, which is valid up to fractional corrections of order p^{-1}, Eq. (5.93) gives

$$\mu/M \ll p^{5/2}. \tag{5.94}$$

We note that Eq. (5.94) follows whether we choose $q \equiv p$ or $q \equiv e$. Equation (5.94) is superseded by the condition $\mu/M \ll 1$ which ensures that the gravitational perturbations obey linear wave equations. Thus, the adiabatic approximation is automatically satisfied in the weak-field limit.

We have already noted that the results presented in subsection A are valid also for binary systems with arbitrary mass ratios, provided that μ is then interpreted as the reduced mass of the system, and M as the total mass. Since $\mu/M \leq 1/4$, with the equality holding when the masses are equal, we see that the radiation reaction is necessarily adiabatic when p is large, irrespective of the mass ratio.

The case $p - 6 - 2e \ll 4e$

The most relevant orbital parameter in this case is $q \equiv \varepsilon \equiv p - 6 - 2e$, and $\dot{\varepsilon}$ can be calculated using Eqs. (5.84) and (5.85). The orbital period can be expressed as $P = \Delta\phi/\Omega_\phi$ using Eqs. (5.23) and (5.24). Substitution of Eq. (5.37) then yields

$$\Delta\varepsilon \simeq 32\pi \frac{M}{\mu} \frac{(3-e)^{1/2}(6+2e)^2}{e(1+e)^{1/2}} \frac{\Delta\phi}{2\pi} \frac{\beta - \alpha}{\varepsilon} \dot{E}. \tag{5.95}$$

We now need to estimate $\beta - \alpha$, as well as $\dot{E}$. For the former, we recall Eq. (5.36) which shows that β is of the same order as $2\pi/\Delta\phi$, and the analysis of subsection B which suggests that α is also of that order. We therefore write $\beta - \alpha \approx 2\pi/\Delta\phi$, where $\approx$ means "equal up to a numerical factor of order unity". For $\dot{E}$ we use an estimate based on the quadrupole formula [5]; this estimate should be valid up to a numerical factor of order unity. Thus, $\dot{E} \approx (32/5)(\mu/M)^2(1+e)^5(6+2e)^{-5}$, which holds for a (fictitious) circular orbit of radius r_1. Gathering the results, and ignoring numerical factors, we arrive at

$$\mu/M \ll \varepsilon^2. \tag{5.96}$$

This condition on μ/M is indeed quite severe.

The case $p - 6 \ll 1$; $e \ll p - 6$

This case can be considered by identifying q with $p - 6$, whose rate of change was evaluated in subsection C. The orbital period is given by Eq. (5.41). We find,

$$\Delta(p-6) \simeq 7776\sqrt{2}\pi(p-6)^{-3/2}(M/\mu)\dot{E}. \tag{5.97}$$

Using the quadrupole formula to obtain the crude estimate $\dot{E} \approx (32/5)(\mu/M)^2 p^{-5}$, and ignoring numerical factors, we arrive at

$$\mu/M \ll (p-6)^{5/2}. \tag{5.98}$$

We point out that the analogous result quoted in Sec. IV F of Ref. [1] is incorrect; Eq. (5.98) is the correct condition. We remark, comparing Eqs. (5.96) and (5.98), that the rate at which μ/M must tend to zero as the point $(6,0)$ is approached varies with the direction of approach. This is an additional consequence of the fact that this point is a singular point of the p-e plane.

5.5 Numerical results

In the first part of this section (subsections A to D) we will describe the numerical methods which were used to obtain the results presented in Figs. 1–3. We have written our code with the help of FORTRAN subroutines given in Ref. [41]. All computations were carried out with double precision.

In the final part of this section (subsection E) we will estimate the overall accuracy of our results, and compare them to those of Tanaka *et al.* [2].

5.5.1 The numerical task

The main function of our code is to compute, for a given point in the p-e plane, the numbers $Z^k_{\ell m}$ for each relevant ℓ, m, and k. This involves the numerical integration of Eq. (5.72), after the change of variables $r = r(\chi)$ has been made [see Eqs. (5.18) and (5.19)], and the evaluation of Eq. (5.71). The $Z^k_{\ell m}$ are then used to calculate $(M/\mu)^2\dot{E}$ and $(M/\mu)^2\dot{L}/M$ via Eqs. (5.54)–(5.56). Finally, these quantities are substituted into Eqs. (5.73) and (5.74), to obtain $M^2\dot{p}/\mu$ and $M^2\dot{e}/\mu$.

The computation of each $Z^k_{\ell m}$ involves many steps. These include:

(i) The evaluation of Ω_r and Ω_ϕ, which determine the perturbation frequency ω_{mk}. For this calculation, we use Eqs. (5.20), (5.21), (5.23), (5.24), and (5.61).

(ii) The integration of the homogeneous Teukolsky equation [Eq. (5.45) with vanishing source], to obtain $R^H_{\omega_{mk}\ell}(\chi)$ for $0 \le \chi \le \pi$. From this we calculate ${}_sR^H_{\omega_{mk}\ell}(\chi)$, with the help of Eqs. (5.68)–(5.70). The integration of the homogeneous Teukolsky equation also gives $Q^{\rm in}_{\omega_{mk}\ell}$, the "amplitude" of the ingoing part of $R^H_{\omega_{mk}\ell}(r \to \infty)$, which is substituted into Eq. (5.71). Step (ii) is the one which requires the most care; it will the subject of subsection C.

(iii) The computation of ${}_sG_\pm(\chi)$, $t(\chi)$, and $\phi(\chi)$, for $0 \le \chi \le \pi$. Equations (5.20), (5.21), and (5.63) are used for this calculation.

(iv) The evaluation of ${}_sp_\ell$, using Eq. (5.67), as well as ${}_sY_{\ell m}(\frac{\pi}{2}, 0)$, using Eq. (2.15) of Ref. [32].

The computation of $\dot{E}$ and $\dot{L}$ formally involves summing over an infinite number of terms. In Eq. (5.54), the sum over ℓ is only restricted by $\ell \ge 2$, and the sum over k is unrestricted. In subsection D we will examine the question of how to truncate these sums so as to achieve a desired degree of accuracy.

5.5.2 Integration of functions

Part of the numerical task involves the integration of several functions of χ, as is expressed in Eqs. (5.20), (5.21), and (5.72). Because these functions are all smooth, the integrations can be performed using Romberg's method, as implemented by the subroutine QROMB of Ref. [41].

The tolerance of the integrator, ϵ_R, can be set to very small values without difficulty. Thus, the numerical error introduced by the Romberg integrator can be chosen to be negligible compared to the truncation error (subsection D), which determines the overall accuracy of the final results. Typically, we have chosen $\epsilon_R = 10^{-6}$.

When integrating Eq. (5.21), we have chosen *not* to take advantage of the fact that $\phi(\chi)$ can be written as an elliptic integral.

5.5.3 Integration of the homogeneous Teukolsky equation

A particularly important part of the numerical task is the integration of the homogeneous Teukolsky equation, Eq. (5.45) with vanishing source. We are interested in the particular solution $R^H_{\omega\ell}(r)$ which describes purely ingoing waves,

$$R^H_{\omega\ell}(r \to 2M) \sim (\omega r)^4 f^2 e^{-i\omega r^*}, \tag{5.99}$$

at the black-hole horizon. Here, $f = 1 - 2M/r$ and $r^* = r + 2M\ln(r/2M - 1)$. At large distances,

$$R^H_{\omega\ell}(r \to \infty) \sim Q^{\rm in}_{\omega\ell}(\omega r)^{-1}e^{-i\omega r^*} + O(r^3 e^{i\omega r^*}), \tag{5.100}$$

where $Q^{\rm in}_{\omega\ell}$ is a constant. The function $R^H_{\omega\ell}(r)$ and its derivatives must be evaluated in the range $r_1 \leq r \leq r_2$. We must also estimate the "amplitude" $Q^{\rm in}_{\omega\ell}$. This is difficult, because the ingoing part of $R^H_{\omega\ell}(r)$ decays as r^{-1} at large radii, while its outgoing part grows as r^3.

To avoid such complications [42], it is preferable to integrate, instead of the homogeneous Teukolsky equation, the related Regge-Wheeler equation [43],

$$\left[f^2\frac{d^2}{dr^2} + \frac{2M}{r^2}f\frac{d}{dr} + \omega^2 - W(r)\right]X_{\omega\ell}(r) = 0, \tag{5.101}$$

where $W(r) = f[\ell(\ell+1)/r^2 - 6M/r^3]$. For this equation also we choose a particular solution $X^H_{\omega\ell}(r)$ which is purely ingoing at the black-hole horizon,

$$X^H_{\omega\ell}(r \to 2M) \sim [1 + a_{\omega\ell}f + b_{\omega\ell}f^2 + \cdots]e^{-i\omega r^*}, \tag{5.102}$$

where

$$\begin{aligned} a_{\omega\ell} &= \frac{\ell(\ell+1) - 3}{1 - 4iM\omega}, \\ b_{\omega\ell} &= \frac{(\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega}{4(1 - 2iM\omega)(1 - 4iM\omega)}. \end{aligned} \tag{5.103}$$

At large distances,

$$X^H_{\omega\ell}(r \to \infty) \sim A^{\rm in}_{\omega\ell}P_{\omega\ell}(\omega r)e^{-i\omega r^*}$$

$$+ A^{\rm out}_{\omega\ell} \bar{P}_{\omega\ell}(\omega r) e^{i\omega r^*}, \tag{5.104}$$

where $A^{\rm in}_{\omega\ell}$ and $A^{\rm out}_{\omega\ell}$ are constants, and $P_{\omega\ell}(\omega r) = 1 + \tilde{a}_{\omega\ell}(\omega r)^{-1} + \tilde{b}_{\omega\ell}(\omega r)^{-2} + \cdots$. Here,

$$\begin{aligned} \tilde{a}_{\omega\ell} &= -\frac{i}{2}\ell(\ell+1), \\ \tilde{b}_{\omega\ell} &= -\frac{1}{8}[(\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega], \end{aligned} \tag{5.105}$$

and a bar denotes complex conjugation.

From $X^H_{\omega\ell}(r)$ and its derivatives one recovers $R^H_{\omega\ell}(r)$ and its derivatives by applying the Chandrasekhar transformation [44],

$$R^H_{\omega\ell}(r) = 4(b_{\omega\ell})^{-1}(M\omega)^3 \omega r^2 f\mathcal{L} f^{-1}\mathcal{L} r X^H_{\omega\ell}(r), \tag{5.106}$$

where $\mathcal{L} = f d/dr + i\omega$. Because $X^H_{\omega\ell}(r)$ satisfies a second-order differential equation, the differentiations need not be performed numerically. The Chandrasekhar transformation also implies

$$Q^{\rm in}_{\omega\ell} = -4(1 - 2iM\omega)(1 - 4iM\omega)(M\omega)^3 A^{\rm in}_{\omega\ell}. \tag{5.107}$$

From Eq. (5.106) one can indeed verify that $R^H_{\omega\ell}(r)$ satisfies the homogeneous Teukolsky equation, with boundary conditions (5.99) and (5.100), if $X^H_{\omega\ell}(r)$ is a solution to the Regge-Wheeler equation, with boundary conditions (5.102) and (5.104).

The numerical integration of Eq. (5.101) proceeds outward from $r = 2M(1 + \epsilon_I)$, where ϵ_I is a small number; typically $\epsilon_I = 10^{-8}$. The integration is performed using the Bulirsh-Stoer method, as implemented by the subroutines ODEINT and BSSTEP of Ref. [41]. We have typically set the tolerance of the integrator to $\epsilon_{BS} = 10^{-6}$.

The complex-valued amplitude $A^{\rm in}_{\omega\ell}$ is evaluated by integrating the Regge-Wheeler equation up to large values of r (large compared with the scale ω^{-1}), and by then comparing the numerical results with Eq. (5.104). More precisely, the integrator pauses at some r, estimates the value of $A^{\rm in}_{\omega m}$, and then proceeds to a larger value of r where another estimation is made. When $A^{\rm in}_{\omega m}$ changes by a fractional amount less than the imposed limit ϵ_A, the integrator stops and returns that value for $A^{\rm in}_{\omega m}$. In practice, the convergence of this process is quite rapid, thanks to the insertion of $P_{\omega\ell}(\omega r)$ in Eq. (5.104). However, we have found that in general, the required accuracy on $A^{\rm in}_{\omega m}$ must be set lower than the accuracy of the integrator. Otherwise, the estimator has difficulty converging at all; this convergence problem is more severe for larger frequencies. Typically, we have chosen $\epsilon_A = 10\epsilon_{BS}$, which appears to work well for all values of p and e.

5.5.4 Truncation of infinite sums

As pointed out previously, the numerical calculation of $\dot{E}$ and $\dot{L}$ must involve the truncation of infinite sums over ℓ and k. This truncation obviously limits the accuracy of the numerical results. It is the purpose of this subsection to devise ways to truncate the sums so that the error introduced does not exceed a specified size.

It is easy to formulate a simple prescription for truncating the sums over ℓ. It was shown in Ref. [32] that for *circular* orbits, a given multipole ℓ contributes a fractional amount of order $p^{-(\ell-2)}$ to $\dot{E}$ and $\dot{L}$. We assume (and we have verified numerically) that this result remains valid, at least within an order of magnitude, when the orbit is eccentric. We then obtain that in order to achieve a fractional accuracy of order ϵ_ℓ, we must include in the sums over ℓ all terms with $\ell \leq \ell_{\max}$, where

$$p^{-(\ell_{\max}-2)} \leq \epsilon_\ell. \tag{5.108}$$

We have found that Eq. (5.108) works indeed quite well in the region of the p-e plane which was of most interest to us. In principle, ϵ_ℓ could be chosen to be of the same order of magnitude as the previously introduced ϵ-factors. However, it is more appropriate to set it only slightly smaller than ϵ_k, which we define below, and which shall be the largest of the ϵ's.

It is more difficult to obtain a prescription for truncating the sums over k. First, it is necessary to know something about the distribution of $\dot{E}_{\ell mk}$ as a function of k, for fixed ℓ and m and for given values of p and e. The $\dot{L}_{\ell mk}$'s follow a similar distribution.

For very small eccentricities, the distribution of $\dot{E}_{\ell mk}$ is strongly peaked at $k = 0$, and decays rapidly away from $k = 0$. It can indeed be shown, using the equations of Sec. III, that for $e \ll 1$, $\dot{E}_{\ell mk}/\dot{E}_{\ell m0} = O(e^{2|k|})$; this analysis was carried out in Ref. [1].

For larger, but still small eccentricities (Fig. 5), the center of the distribution is pushed away from $k = 0$ by an amount of order unity which depends on the values of p, e, ℓ, and m. However, it is still true that only a small number of k's make a significant contribution to $\sum_k \dot{E}_{\ell mk}$.

For large eccentricities (Fig. 6), a large number of harmonics is required, and the center of the distribution is displaced from $k = 0$ by a large amount depending on the values of p, e, ℓ, and m. In all cases we have found that the negative values of k contribute very little to the total result.

Because the distributions of $\dot{E}_{\ell mk}$ and $\dot{L}_{\ell mk}$ as functions of k are so complex, it is not possible to truncate the sums over k at some universal values $k_{\rm min}$ and $k_{\rm max}$. Instead, for given p and e, and for fixed ℓ and m, we let the code compute $\dot{E}_{\ell mk}$ and $\dot{L}_{\ell mk}$ from $k = 0$ outward, comparing the value of the current $\dot{E}_{\ell mk}$ to the maximum value encountered thus far (for that ℓ and m). The calculation stops when for several successive k's, $\dot{E}_{\ell mk}$ drops below a fixed number ϵ_k times the maximum value. The calculation is then repeated for the negative k's, using the same maximum value. Finally, the sums over k are carried out, and the final answers for $\sum_k \dot{E}_{\ell mk}$ and $\sum_k \dot{L}_{\ell mk}$ are considered to have a fractional accuracy of order ϵ_k.

5.5.5 Overall accuracy

The overall accuracy of our results is determined, at least in part, by choosing the value of ϵ_k. A smaller value implies that more harmonics of the radial frequency will be included in the sums, which in turn implies a longer running time. For fixed ϵ_k, on the other hand, the running time increases rapidly with increasing eccentricity. Practically, therefore, it is not usually possible to set ϵ_k to very small values. We have typically chosen $\epsilon_k = 10^{-2}$. Fortunately, this relatively low accuracy is quite sufficient for our purposes.

Let ϵ denote the overall fractional accuracy of our results. The discussion of the preceding subsections implies the following hierarchy between all the the ϵ's:

$$\epsilon \approx \epsilon_k \geq \epsilon_\ell \gg \epsilon_A \gg \epsilon_{BS} = \epsilon_R \gg \epsilon_I. \tag{5.109}$$

To verify that ϵ_k is indeed a fair estimation of the overall accuracy, we have carried out runs with decreasing values of ϵ_k, and checked that the final answers differed by the expected amounts.

It is also useful, to assess our accuracy, to compare our results to the generally more accurate ones of Tanaka *et al.* [2]. Such a comparison was performed for several points in the p-e plane, and representative results are shown in Table I.

As a final remark concerning the accuracy of our results, we note that we have not, in this paper, calculated $\dot{E}^H$ and $\dot{L}^H$, the time-averaged rates at which the black hole absorbs energy and angular momentum. Consequently, our results are only valid up to a fractional accuracy not better than $\epsilon_H \equiv \dot{E}^H/\dot{E}^\infty \approx \dot{L}^H/\dot{L}^\infty$, where $\dot{E}^\infty$ and $\dot{L}^\infty$ denote the rates at infinity. It can be shown [1] that for *circular* orbits, $\dot{E}^H/\dot{E}^\infty = \dot{L}^H/\dot{L}^\infty = O(p^{-4})$. We may assume that this result stays valid, at least within an order of magnitude, for eccentric orbits, and conclude that $\epsilon_H \approx$

$p^{-4} < 8 \times 10^{-4}$. Because we have generally worked with $\epsilon \approx 10^{-2} \gg \epsilon_H$, we can safely ignore the contributions $\dot{E}^H$ and $\dot{L}^H$ to $\dot{E}$ and $\dot{L}$. This was also done by Tanaka *et al.* [2].

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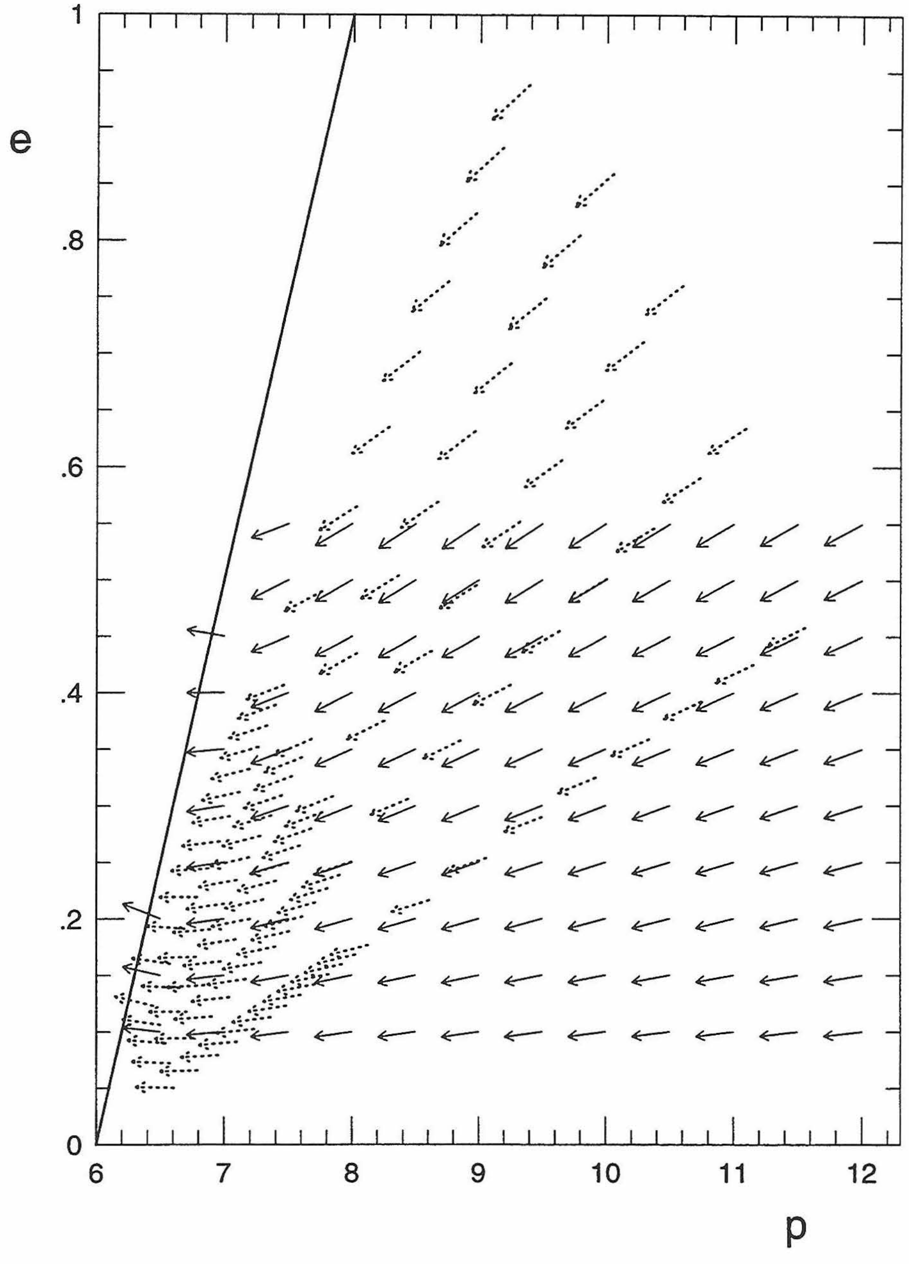

Figure 5.1: The bound orbits of the Schwarzschild spacetime can be represented by points in the p-e plane, a portion of which is depicted here. The solid, diagonal line to the left is the separatrix $p = 6+2e$. The bound orbits are located to the right of the separatrix. Radiation reaction produces a slow evolution of the orbital parameters, and therefore generates curves in the p-e plane. These can be parametrized by p, and have $\vec{v} = (1, de/dp)$ as tangent vectors. The vector field $\vec{v}(p, e)$ is also plotted here, with each point (p, e) located at the arrow's tail-end. For convenience, we have uniformly rescaled the length of the vectors. The solid arrows represent the calculations of this paper. The dotted arrows represent the results of Tanaka *et al.* [2].

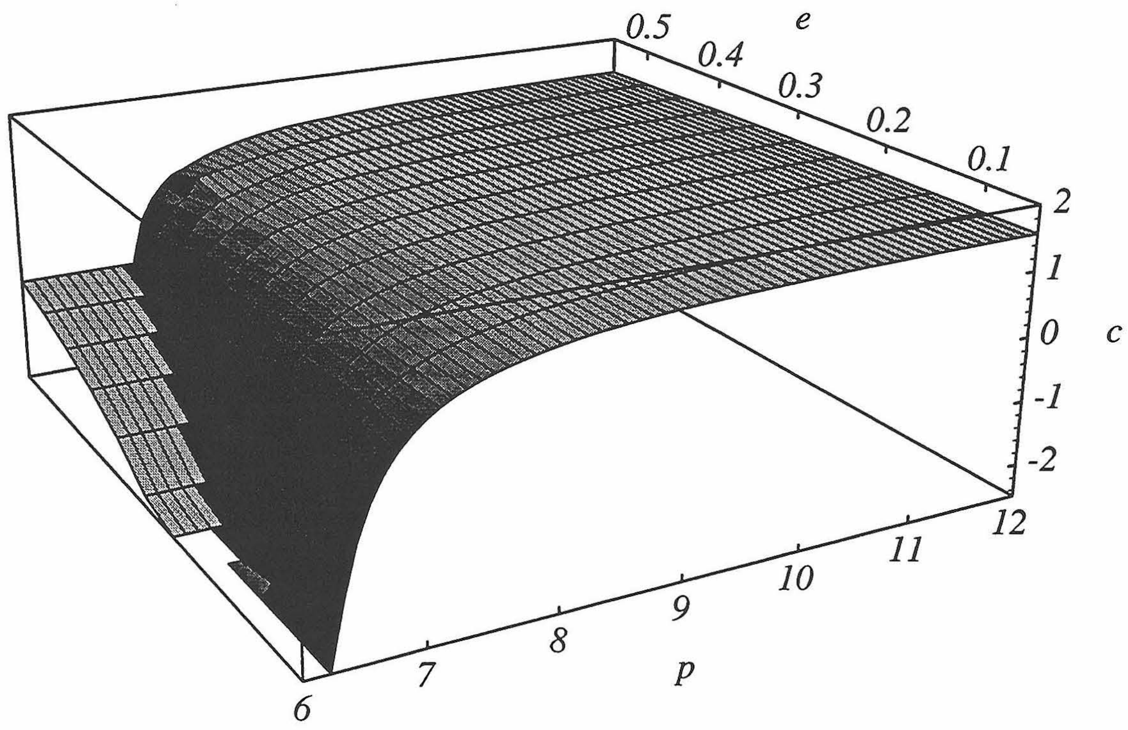

Figure 5.2: A three-dimensional plot of the function $c(p,e) = d\ln e/d\ln p$, for the range $6 \leq p \leq 12$ and $0 \leq e \leq 0.55$. The function $c(p,e)$ is not defined for $p < 6+2e$. In this region, we have plotted $\hat{c}(p,e) = -(1-e)/e$, which is equal to $c(p,e)$ at $p = 6+2e$; see Eq. (1.4). The intersection of the surface $c = c(p,e)$ with the plane $c = 0$ defines the critical curve, along which $de/dp = 0$; see Fig. 3. The value of $c(p,e)$ at the point $(7.5, 0.5)$ appears anomalous, but there is no reason to suspect the accuracy of our results at that point.

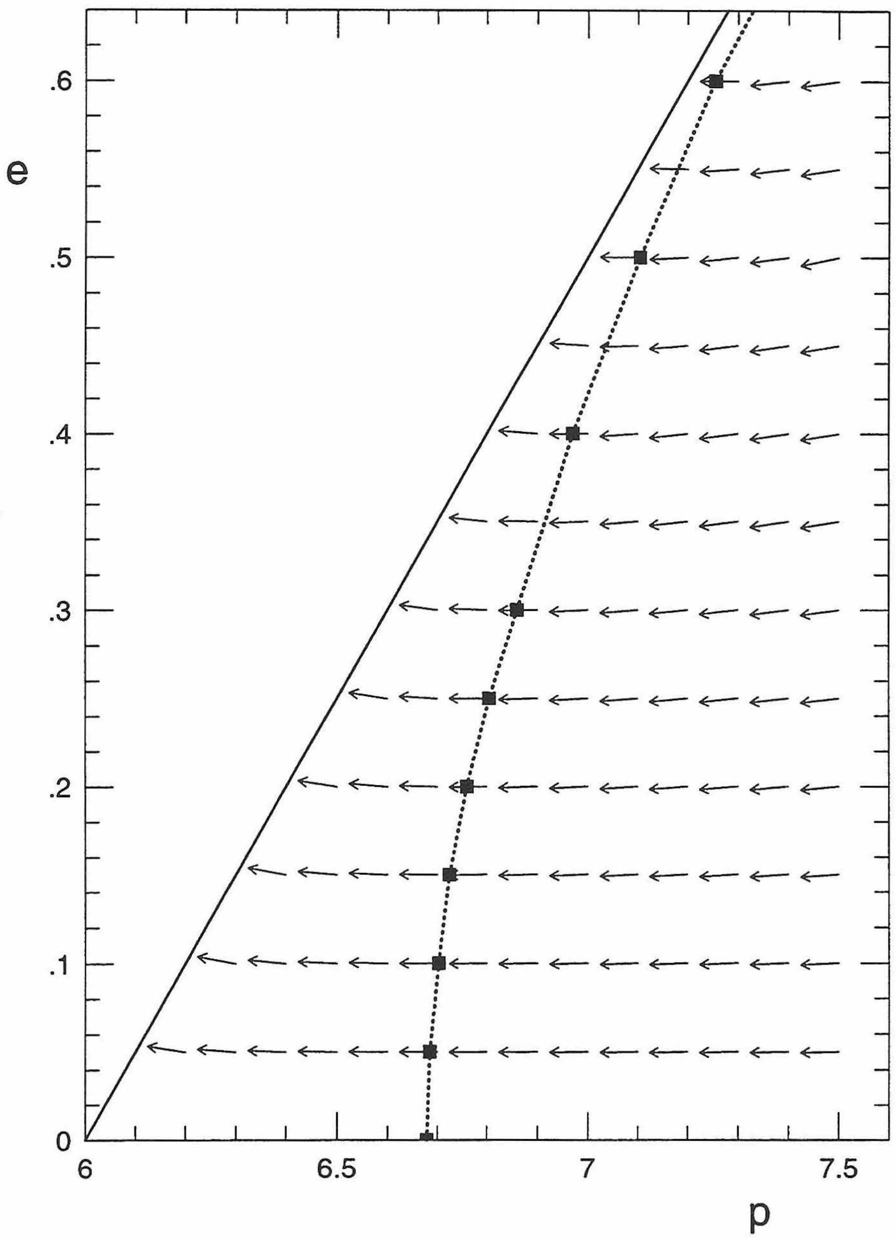

Figure 5.3: A portion of the p-e plane in which lies a portion of the critical curve (along which $de/dp = 0$). The solid, diagonal line to the left is the separatrix. The filled squares represent points on the critical curve. (The thickness of the squares exceeds the numerical uncertainty.) The dotted curve consists of straight line segments joining these points. The arrows have the same meaning as in Fig. 1. Arrows to the right of the critical curve point down, indicating that radiation reaction decreases the eccentricity. Arrows to the left of the critical curve point up, indicating that radiation reaction increases the eccentricity.

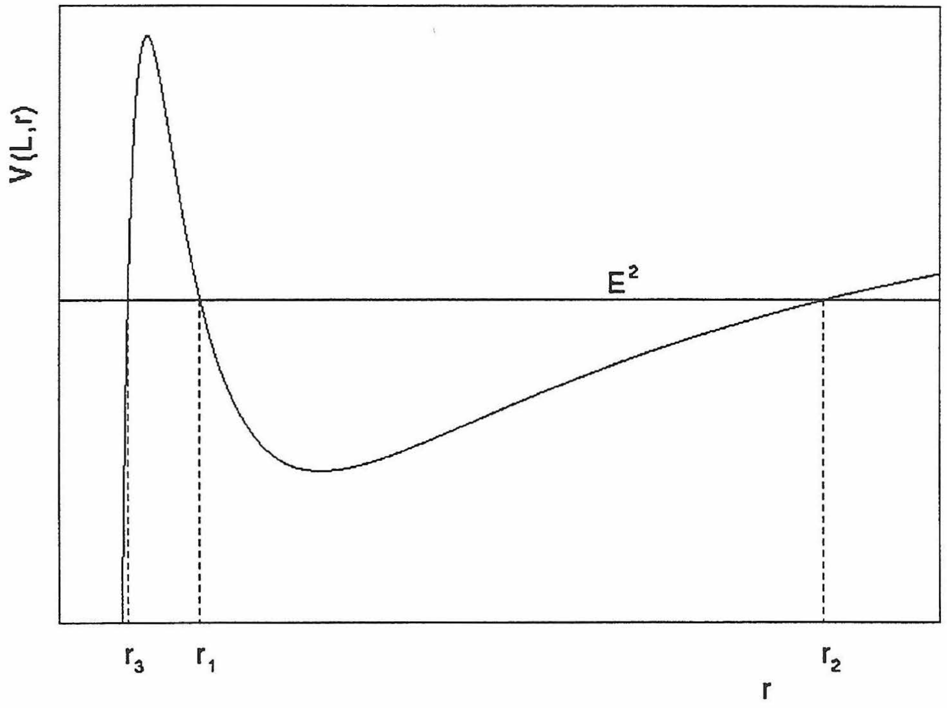

Figure 5.4: The effective potential for radial motion. The three turning points $r_3 \leq r_2 \leq r_1$ are defined by the cubic $V(\tilde{L}, r) = \tilde{E}^2$. Bound motion takes place between r_1, the periastron, and r_2, the apastron.

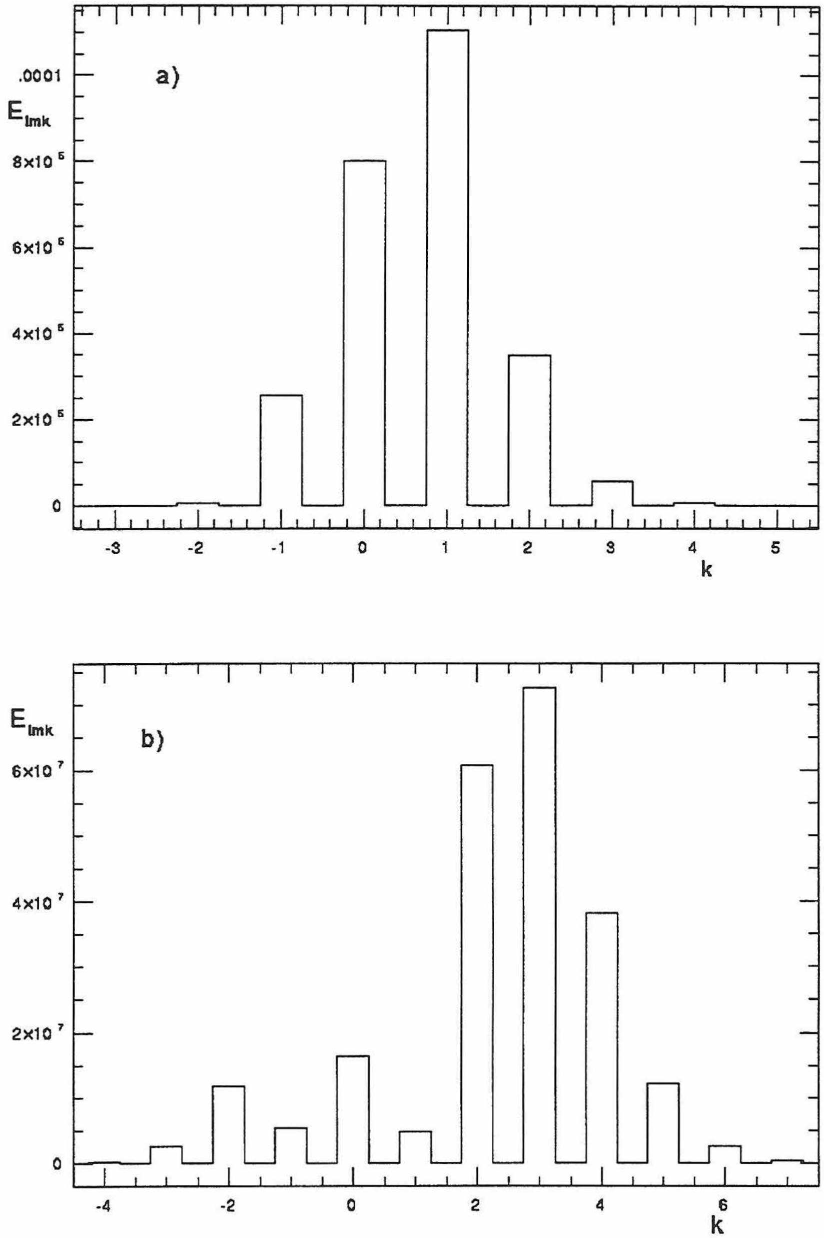

Figure 5.5: The contributions $(M/\mu)^2 \dot{E}_{\ell mk}$ to the total rate $(M/\mu)^2 \dot{E}$, plotted as a function of the integer k, for fixed ℓ and m, and for $p = 7.50478$, $e = 0.188917$. See Eq. (3.12) and Table I. In part a), $\ell = m = 2$. In part b), $\ell = m = 5$. The order of magnitude of the main contributions to the energy flux in part a) (in dimensionless units where $G = c = 1$) is 10^{-4}, and 10^{-6} (in the same units) in part b).

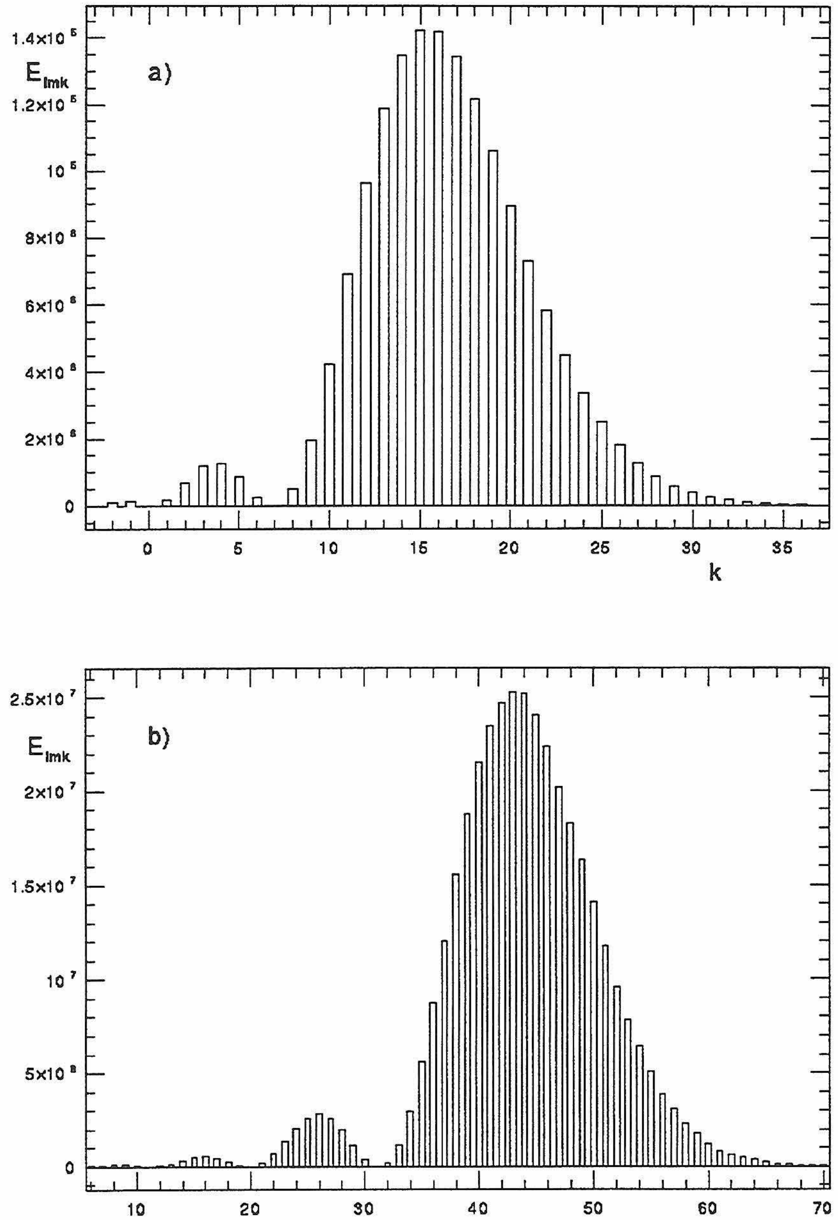

Figure 5.6: The contributions $(M/\mu)^2\dot{E}_{\ell mk}$ to the total rate $(M/\mu)^2\dot{E}$, plotted as a function of the integer k, for fixed ℓ and m, and for $p = 8.75455$, $e = 0.764124$. See Eq. (3.12) and Table I. In part a), $\ell = m = 2$. In part b), $\ell = m = 5$. The order of magnitude of the principal contributions to the energy flux in part a) (in dimensionless units where $G = c = 1$) is 10^{-5}, and 10^{-7} (in the same units) in part b).

Table 5.1: Comparison with the results of Tanaka *et al.* for two representative points in the p-e plane. Shown are: the values of p, e, $\tilde{E}$, $\tilde{L}$, r_1, r_2, Ω_r, Ω_ϕ, and $\Delta\phi$; the overall accuracy of our results (Tanaka *et al.* claim a fractional accuracy better than 10^{-4}); the values of $\dot{E}$ and $\dot{L}$ as calculated in this paper and by Tanaka *et al.*; the relative difference between the results; and the values of $\dot{p}$ and $\dot{e}$ as calculated in this paper.

Quantity	Point # 1	Point # 2
p	7.50478	8.75455
e	0.188917	0.764124
$\tilde{E}$	0.948279	0.977903
$\tilde{L}/M$	3.55000	3.85000
r_1/M	6.31228	4.96255
r_2/M	9.25279	37.1151
$M\Omega_r$	0.0210558	0.00804892
$M\Omega_\phi$	0.0475982	0.0153556
$\Delta\phi$	14.2036	11.9869
ϵ	10^{-4}	10^{-2}
$M^2\dot{E}/\mu^2$ (this paper)	3.16804×10^{-4}	2.10080×10^{-4}
$M^2\dot{E}/\mu^2$ (Tanaka *et al.*)	3.16689×10^{-4}	2.11580×10^{-4}
$M\dot{L}/\mu^2$ (this paper)	5.96562×10^{-3}	2.75034×10^{-3}
$M\dot{L}/\mu^2$ (Tanaka *et al.*)	5.96391×10^{-3}	2.76838×10^{-3}
relative difference	4×10^{-4}	7×10^{-3}
$M^2\dot{p}/\mu$	-7.475×10^{-2}	-2.283×10^{-2}
$M^2\dot{e}/\mu$	-1.967×10^{-3}	-2.126×10^{-3}

Chapter 6

Radiation-reaction-induced evolution of circular orbits of particles around Kerr Black Holes

with Amos Ori

Abstract

It is demonstrated that, in the adiabatic approximation, non-Equatorial circular orbits of particles in the Kerr metric (i.e. orbits of constant Boyer-Lindquist radius) remain circular under the influence of gravitational radiation reaction. A brief discussion is given of conditions for breakdown of adiabaticity and of whether slightly

non-circular orbits are stable against the growth of eccentricity.

6.1 Introduction

It has been shown some time ago that a particle in a circular orbit around a non-rotating black hole remains in a circular orbit under the influence of the gravitational radiation reaction arising from its orbital motion [1]. Although it has been suggested [2] that the same holds true for "circular" orbits (meaning orbits of constant Boyer-Lindquist radius [3]) around rotating black holes, up to now it has not been shown to beyond first post-Newtonian order (see Ref. [4] for the post-Newtonian result, and see also Refs. [5] and [6] for more recent work) because of the difficulty of dealing with the little understood "Carter" constant of the motion, Q. To date no practical method has been developed for describing the rate of change of this constant due to gravitational radiation reaction for a generic orbit (see, however, Ref. [7]), and without knowing this the evolution of the orbit cannot be predicted.

In this paper, we study the relation between the rates of change of the three constants of the orbital motion (the energy, E, axial component of angular momentum, L, and the Carter constant, Q), for circular and almost-circular orbits. We first show that, for orbits which are precisely circular (in a sense which is well-defined within the adiabatic approximation), the rate of change of Q has just the value required for the orbit to evolve into a new circular orbit. Then, we extend the analysis to almost-circular orbits (to first order in the orbital eccentricity). This analysis leads to the result that in order for an initially-circular orbit to develop non-zero eccentricity, the back-reaction force (evaluated for the precisely-circular orbit) must resonate with the radial oscillations. Since in the case of a precisely-circular orbit the periodicity of the back-reaction force is determined solely by the θ-motion, we are led to the following conclusion. The only case in which an initially-circular orbit will develop an eccentricity is when there is a certain resonance between the (small-oscillation) radial motion and the θ-motion. More specifically, this resonance condition is $T_\theta = 2nT_r$, where n is an integer, and T_r and T_θ are the (averaged) periods of the radial motion and the angular motion, correspondingly.

Ryan [5] has recently examined circular orbits in the Kerr metric numerically and found that the above resonance condition is never satisfied (for all black-hole and orbital parameters). We are thus led to the conclusion that orbits which are initially precisely circular will remain circular upon radiation-reaction evolution. We point out, however, that this conclusion does not address the issue of *stability* against the growth of a small initial eccentricity (this issue is further discussed in Sec. V below).

This paper is organized as follows. In Section II we define instantaneous circularity of an orbit in terms of the instantaneous location and 4-velocity of the orbiting particle. We then derive a relation between the rates of change of E, L and Q for an instantaneously circular orbit that is perturbed by an arbitrary force and we show that this relation is precisely the one required for the circular orbit to evolve into a new circular orbit. One might naively interpret this result by itself as implying that an initially circular orbit will necessarily remain circular. However, in Section III we show that in order to predict the full evolution of initially circular orbits, it is necessary to carry the analysis (of the relation between the rates of change of the three constants of motion) to first order in the instantaneous eccentricity. This analysis, to first order in the eccentricity, is presented in Section IV, with the conclusion that initially circular orbits do, indeed, remain circular. Finally, in Section V we give some concluding remarks.

6.2 Instantaneously circular orbits

In this section we shall define instantaneous circularity of an orbit and shall show that for any such orbits $dQ/d\tau$ has just the right value so as to leave the evolving orbit circular. We shall show that this is true for any arbitrary force which acts on the orbiting particle (in fact, this result is precise, and is not limited to the adiabatic approximation).

We take here the point of view that the radiative evolution may be viewed as the consequence of some "back-reaction force", which can be treated as any other external force [8]. We shall therefore begin by constructing a general formal expression for the evolution rate of all constants of (geodesic) motion, due to an arbitrary external force. Let C denote the constant of motion in question. In Kerr, C may stand for either the energy E, the azimuthal angular momentum L, or the Carter constant Q (or any combination of these constants, like e.g. the constant D defined below). We first express all these constants explicitly as functions of coordinates and covariant components of four-velocity [9], that is,

$$C \equiv C(x^\beta, u_\alpha) . \tag{6.1}$$

For E and L, we simply take

$$E = -u_t \quad , \quad L = u_\varphi . \tag{6.2}$$

(Throughout, we use the standard Boyer-Lindquist coordinates, r, t, θ, φ). The corresponding explicit expression for Q is slightly more complicated. We could use the familiar expression based on the θ-motion in Kerr:

$$Q = u_\theta^2 + \cos^2\theta\,[a^2(1-u_t^2) + \sin^{-2}\theta\,u_\varphi^2] . \tag{6.3}$$

We find it more convenient, however, to construct the expression for Q from the r-motion. It is straightforward to show that

$$Q = \Delta^{-1}\left[E(r^2+a^2) - aL\right]^2 - (L-aE)^2 - r^2 - \Delta u_r^2 . \tag{6.4}$$

(this follows, for instance, from Equations (33.32b) and (33.33c) in Ref. [10]). Here, $\Delta \equiv r^2 - 2Mr + a^2$, where M is the black hole's mass and aM is its angular momentum. For later convenience, we also write this equation in the form

$$Q = H(r, E, L) - \Delta u_r^2 , \tag{6.5}$$

where

$$
\begin{aligned}
H(r,E,L) \equiv \quad & \Delta^{-1}\left[E(r^2+a^2)-aL\right]^2 \\
- \quad & (L-aE)^2 - r^2\,. \qquad (6.6)
\end{aligned}
$$

In view of Eq. (6.2), one readily sees that Eq. (6.4) is just of the desired form, $Q = Q(x^\beta, u_\alpha)$ (with the simplification that the right-hand side does not depend on θ or u_θ).

When an external force is applied to the particle, C will evolve with time. To calculate its rate of change, we differentiate Eq. (6.1) with respect to the proper time τ:

$$\frac{dC}{d\tau} = u^\beta C_{,\beta} + \sum_\alpha \frac{\partial C}{\partial u_\alpha}\frac{du_\alpha}{d\tau}\,. \qquad (6.7)$$

Now,

$$\frac{du_\alpha}{d\tau} = \frac{Du_\alpha}{D\tau} + \tfrac{1}{2} g_{\mu\nu,\alpha} u^\mu u^\nu \qquad (6.8)$$

and

$$\frac{Du_\alpha}{D\tau} = F_\alpha\,, \qquad (6.9)$$

where $D/D\tau$ denotes covariant proper-time differentiation and F_α is the force per unit rest mass. Equation (6.7) thus reads

$$\frac{dC}{d\tau} = \left[u^\beta C_{,\beta} + \tfrac{1}{2}\sum_\alpha \frac{\partial C}{\partial u_\alpha} g_{\mu\nu,\alpha} u^\mu u^\nu\right] + \sum_\alpha \frac{\partial C}{\partial u_\alpha} F_\alpha\,. \qquad (6.10)$$

Notice that the term in brackets does not depend on the external force. When no external force is applied, C is conserved. Thus, the term in brackets must vanish identically. Equation (6.10) therefore reads:

$$\frac{dC}{d\tau} = \sum_{\alpha} \frac{\partial C}{\partial u_\alpha} F_\alpha \, . \tag{6.11}$$

This is the desired general expression for the evolution rate of all constants of motion. Note that this is the *precise* expression for the *instantaneous* rate of change of C. We have not used the adiabatic approximation (or any other approximation) so far.

We now define an orbit to be *instantaneously circular* if its instantaneous values of E, L and Q [defined by Eqs. (6.2) and (6.5)] are precisely equal to those of some circular geodesic orbit. At a moment when the orbit (on which the force F_α acts) is instantaneously circular, Eq. (6.4) plus circularity ($u_r = 0$) implies

$$\frac{\partial Q}{\partial u_r} = -2\Delta u_r = 0 \, . \tag{6.12}$$

Inserting this into Eq. (6.11) gives

$$\frac{dQ}{d\tau} = \frac{\partial Q}{\partial u_t} F_t + \frac{\partial Q}{\partial u_\varphi} F_\varphi \, . \tag{6.13}$$

Also, in view of Eq. (6.2), Eq. (6.11) yields

$$\frac{dE}{d\tau} = -F_t \quad , \quad \frac{dL}{d\tau} = F_\varphi \, . \tag{6.14}$$

Substituting Eqs. (6.2) and (6.14) in Eq. (6.13), we obtain

$$\frac{dQ}{d\tau} = Q_{,E} \frac{dE}{d\tau} + Q_{,L} \frac{dL}{d\tau} \, . \tag{6.15}$$

Finally, using Eq. (6.5), we rewrite Eq. (6.15) as

$$\frac{dQ}{d\tau} = H_{,E} \frac{dE}{d\tau} + H_{,L} \frac{dL}{d\tau} \, . \tag{6.16}$$

This is our final expression for the actual, momentary, rate of change of Q due to the external force. [To avoid confusion , we emphasize that the partial derivatives in the right-hand side of Eq. (6.16) are to be calculated according to the explicit expression (6.6)].

We come now to the second part of this calculation, that is to calculate the rate of change of Q (compared to that of E and L) required for taking a circular orbit into a new circular orbit. From Eq. (6.5) we obtain

$$\Delta u_r^2 = H(r, E, L) - Q \equiv W(r, E, L, Q) \, . \qquad (6.17)$$

When applied to general geodesic orbits (with constant but arbitrary E, L and Q) this equation can be regarded as describing radial motion in an effective potential $W(r, E, L, Q)$. Obviously that geodesic motion is circular if and only if the particle sits at a radius r where $W = 0$ (or $u_r = 0$) and where $W_{,}r = 0$ (so the particle is at the minimum of the effective potential). Correspondingly, an orbit on which a force F_α acts is instantaneously circular if and only if it instantaneously satisfies

$$W = 0 \quad , \quad W_{,r} = 0 \, . \qquad (6.18)$$

We now let the force F_α continue to act, but only for an infinitesimal proper time, $\delta\tau$. Following this action, the orbit will be characterized by new parameters, r', E', L' and Q'. We denote the changes from the original parameters by δ, that is, $\delta r = r - r'$, $\delta E = E - E'$, etc. These changes are infinitesimal because F_α is allowed to act for only an infinitesimal time $\delta\tau$, before the orbit is once more examined for circularity. The corresponding change in W is given by

$$\delta W = W_{,r}\delta r + W_{,E}\delta E + W_{,L}\delta L + W_{,Q}\delta Q \; . \tag{6.19}$$

In order for Eq. (18) to hold after the time $\delta\tau$ as well as before (i.e. in order for the orbit to remain circular), we must demand

$$\delta W = 0 \; . \tag{6.20}$$

We denote the value of Q which corresponds to a circular orbit (for given E and L) by $Q_{\rm circ}(E, L)$. Equations (6.19) and (6.20) thus yield, as a necessary condition for the orbit to remain circular

$$W_{,r}\delta r + W_{,E}\delta E + W_{,L}\delta L + W_{,Q}\delta Q_{\rm circ} = 0 \; , \tag{6.21}$$

from which $\delta Q_{\rm circ}$ is to be determined. Now Eq. (6.17) implies

$$W_{,Q} = -1 \quad , \quad W_{,E} = H_{,E} \quad , \quad W_{,L} = H_{,L} \; , \tag{6.22}$$

which together with $W_{,r} = 0$ [Eq. (6.18)] reduces Eq. (6.21) to the form [11]

$$\delta Q_{\rm circ} = H_{,E}\delta E + H_{,L}\delta L \; . \tag{6.23}$$

Finally, dividing by the infinitesimal proper time lapse $\delta\tau$ and taking the limit $\delta\tau \to 0$, we obtain

$$\frac{dQ_{\rm circ}}{d\tau} = H_{,E}\frac{dE}{d\tau} + H_{,L}\frac{dL}{d\tau} \; . \tag{6.24}$$

Comparing Eqs. (16) and (24), we arrive at the desired result. *At any moment when the orbit, on which the arbitrary force F_α acts, is instantaneously circular, F_α*

produces *an instantaneous evolution of the orbit's Cater constant given by*

$$\frac{dQ}{d\tau} = \frac{dQ_{\text{circ}}}{d\tau} , \tag{6.25}$$

which maintains instantaneous circularity.

For later convenience, let us define

$$D \equiv Q_{\text{circ}} - Q = D(E, L, Q) . \tag{6.26}$$

Circular orbits are thus characterized by $D = 0$. Equation (6.25) then implies that if at a particular moment $D = 0$, then

$$dD/d\tau = 0 . \tag{6.27}$$

It should be emphasized again that all the calculations done so far are *precise*, and do not depend on the adiabatic approximation. Note also that these calculations refer to the *instantaneous* rate of change of the constants of motion, at a moment when the orbit is *instantaneously* circular. As we shall see in the next section, Eq. (6.27) by itself does *not* imply that an initially-circular orbit will necessarily remain circular. One must go to the next order in the eccentricity in order to derive this result.

6.3 The need for eccentricity corrections

Equation (6.27) (which holds whenever $D = 0$, i.e. whenever the orbit is instantaneously circular) has a trivial exact solution, $D(\tau) = 0$. Does this necessarily mean that an instantaneously circular orbit will remain circular forever? We shall immediately see that, in principle, the answer is no (though, we shall also see later that, *within the adiabatic approximation*, in most cases a circular orbit will remain circular). In fact, the trivial solution $D(\tau) = 0$ to Eq. (6.27) is physically meaningless.

To illustrate this, consider the analogous (but much simpler) problem of a free particle in one-dimensional Newtonian mechanics. The particle's energy (per unit rest mass) is $K = (1/2)\,(dx/d\tau)^2$. Assume now that a constant external force F (per unit rest mass) is applied on the particle. The evolution of K is then given by [in analogy with Eq. (6.11)]

$$\frac{dK}{d\tau} = F\frac{dx}{d\tau} , \tag{6.28}$$

or, in terms of K itself, by

$$\frac{dK}{d\tau} = \sqrt{2}FK^{1/2} . \tag{6.29}$$

Now, assume that the particle is initially at rest, i.e. $K = 0$. Then, from Eq. (6.29), $dK/d\tau = 0$. Equation (6.29) then admits a trivial exact solution,

$$K(\tau) = 0 . \tag{6.30}$$

This trivial solution, which means that the particle will remain at rest forever, is obviously wrong. Instead, the particle will certainly accelerate, and the true physical solution is [12]

$$K = (F^2/2)\tau^2 . \tag{6.31}$$

The situation here with respect to Eq. (6.27) and its unphysical solution $D(\tau) = 0$ is just analogous. As will be shown below (Section IV), in the extension of Eq. (6.27) to slightly-eccentric orbits, $dD/d\tau$ is (to the leading order) proportional to $D^{1/2}$. Then, in addition to the trivial solution $D(\tau) = 0$, there exists a non-trivial solution in which for some short period D grows like τ^2 [analogous to Eq. (6.31)], and this is the physical solution. This implies that, momentarily, the eccentricity (which is proportional to $D^{1/2}$) will grow linearly with τ, even if initially it vanishes precisely.

Recall, however, that we are not particularly interested here in the momentary rate of change at a specific point along the orbit. Rather, we are interested in the effective *long-term* evolution of D. Within the adiabatic approximation, this long-term evolution is described by an equation of the form

$$\dot{D} = S(E, L, D) . \tag{6.32}$$

Hereafter, an overdot denotes the *long-term* rate of change (with respect to the proper-time, τ), obtained from the momentary equation of motion by averaging over many periods (in Section IV we shall describe this averaging procedure in more detail). The long-term evolution of circular orbits will depend on the asymptotic behavior of S near $D = 0$. It is not difficult to show, based on Eq. (6.27), that the zero-order term [i.e. $S(D = 0)$] vanishes identically [13]. As we shall see in Section IV below, the general asymptotic form of S is given by

$$S(E, L, D) = S_1(E, L)D^{1/2} + S_2(E, L)D + O(D^{3/2}) . \tag{6.33}$$

The evolution of an instantaneously circular orbit will depend crucially on whether S_1 vanishes or not. In the case $S_1 = 0$, we can approximate Eq. (6.33) by the linear equation

$$\dot{D} = S_2 D \ . \tag{6.34}$$

Then, an initial value $D = 0$ ensures that D will remain zero forever (we are not concerned here about stability to small initial perturbations, though we shall discuss the question of stability briefly in Sec. V). On the other hand, if S_1 is non-zero, we can approximate Eq. (6.33) by

$$\dot{D} = S_1 D^{1/2} \ . \tag{6.35}$$

As was explained above, the trivial solution $D(\tau) = 0$ is physically meaningless. In that case, the physical solution will be (at least as long as $S_1 > 0$)

$$D(\tau) = (S_1/2)^2 \tau^2 \ , \tag{6.36}$$

which means that the an instantaneously circular orbit will evolve into an eccentric one.

The above considerations make it clear that the value of S_1 is crucial for our discussion. An orbit which is initially instantaneously circular will (in the adiabatic approximation) remain circular if and only if $S_1 = 0$. In the next section we shall calculate S_1, and show that it generically vanishes. Only orbits which satisfy a certain resonance condition may have non-zero S_1 .

6.4 Slightly eccentric orbits

We now analyse the adiabatic evolution of D for slightly eccentric orbits, to leading order in the eccentricity. In order to use Eq. (6.11) for the calculation of $dD/d\tau$, we must first express D in the form $D(x^\beta, u_\alpha)$. To simplify the notation, we make use of Eq. (6.2) , and simply write L instead of u_φ and $-E$ instead of u_t . Recalling that $Q_{\rm circ} \equiv Q_{\rm circ}(E, L)$, we have from Eq. (6.26)

$$D = Q_{\rm circ}(E, L) - Q(r, u_r, E, L) = D(r, u_r, E, L) \, . \tag{6.37}$$

Here, $Q(r, u_r, E, L)$ is to be understood in terms of Eq. (6.4). This function is analytic in its arguments, and it is also not difficult to show that (except perhaps at the "last stable circular orbit", which must be excluded here), $Q_{\rm circ}(E, L)$ is analytic in E and L. Therefore, $D(r, u_r, E, L)$ in Eq. (6.37) is also analytic in its arguments. Now, Eq. (6.11) yields

$$\begin{aligned} \frac{dD}{d\tau} &= -D_{,E}F_t + D_{,L}F_\varphi + D_{,u_r}F_r \\ &\equiv D^t F_t + D^\varphi F_\varphi + D^r F_r = D^j F_j \, , \end{aligned} \tag{6.38}$$

where the index j runs over the three coordinates r, t, φ , and

$$D^j \equiv \partial D/\partial u_j \, . \tag{6.39}$$

Equation (6.38) [like Eq. (6.11)] describes the *precise*, instantaneous rate of change of D, for any orbit on which any force F_α is acting. We have not made use of the adiabatic (or any other) approximation so far.

From the analyticity of $D(r, u_r, E, L)$ it follows that the functions

$$D^j \equiv D^j(r, u_r, E, L) \tag{6.40}$$

are analytic in their arguments. Now, from the validity of Eq. (6.27) for an external force of any type, it follows that for an instantaneously circular orbit [i.e. when $u_r = 0$ and $r = r_{\rm circ}(E, L)$], all three functions D^j vanish. In other words, for any given E and L,

$$D^j(r = r_0, u_r = 0, E, L) = 0\ , \tag{6.41}$$

where $r_0 \equiv r_{\rm circ}(E, L)$. We now expand $D^j(r, u_r, E, L)$ around $(r = r_0, u_r = 0)$. In view of the analyticity of these functions, we have

$$D^j(r, u_r, E, L) = \delta r D^j_{,r} + u_r D^j_{,u_r} + O(\delta r^2, u_r^2, u_r \delta r)\ , \tag{6.42}$$

where $\delta r \equiv r - r_0$, and the functions $D^j_{,r}$ and $D^j_{,u_r}$ (which, again, are analytic functions of r, u_r, E, L) are evaluated at $(\delta r = 0, u_r = 0)$.

We are now going to use two approximations (or expansions): the adiabatic approximation, and the small-eccentricity approximation. These two approximations are unrelated, and should not be confused with each other. The exact instantaneous equation of motion of D is [cf. Eq. (6.38)]

$$\frac{dD}{d\tau} = D^j F_j\ . \tag{6.43}$$

The *adiabatic approximation means* that the external force F_j is assumed to be small, and the right-hand side is to be evaluated to linear order in it. The *small-*

eccentricity approximation means that the eccentricity is assumed small and Eq. (6.43) is evaluated to first-order in it.

In view of the small-eccentricity approximation, the higher-order terms in the right-hand side of Eq. (6.42) can be omitted. Substitution in Eq. (6.43) then yields

$$\frac{dD}{d\tau} = \left[\delta r(\tau) D^j_{,r} + u_r(\tau) D^j_{,u_r}\right] F_j(\tau) \ . \tag{6.44}$$

Let us now examine the implications of the two approximations used here on the expression in the right-hand side. In view of the adiabatic approximation, the term in brackets is to be evaluated as if the constants of motion are fixed and the motion is geodesic. In the most general case, $D^j_{,r}$ and $D^j_{,u_r}$ are (like D) functions of (r, u_r, E, L). Here, due to the adiabatic approximation, we can fix E and L . Moreover, since δr and u_r are already first-order in the eccentricity, when evaluating $D^j_{,r}$ and $D^j_{,u_r}$ we may take $u_r = 0$ and $r = r_0$. Thus, in Eq. (6.44), $D^j_{,r}$ and $D^j_{,u_r}$ are just constants (which depend parametrically on E and L) [14].

Turn now to evaluate $\delta r(\tau)$ and $u_r(\tau)$ in the right-hand side of Eq. (6.44). Like the entire term in brackets, they are to be evaluated as if the motion is geodesic (with fixed E, L, D). In view of the small-eccentricity approximation, we need only calculate $\delta r(\tau)$ and $u_r(\tau)$ to the leading order in the eccentricity. We start from the "effective-potential" relation

$$\Delta u_r^2 = W(r, E, L, Q) \ , \tag{6.45}$$

[cf. Eq. (6.17)], and, recalling that $g^{rr} = g_{rr}^{-1} = \Delta/\rho^2$, we write it as

$$(dr/d\tau)^2 = (\Delta^2/\rho^4)u_r^2 = \rho^{-4}W\Delta \; . \tag{6.46}$$

Here,

$$\rho^2 \equiv r^2 + a^2\cos^2\theta \; . \tag{6.47}$$

In Eq. (6.46), as it stands, the radial motion is coupled to the θ-motion, through ρ . In order to decouple the two motions, we define a new independent variable λ by

$$d\lambda/d\tau = \rho^{-2} \; . \tag{6.48}$$

The radial equation of motion now becomes

$$(dr/d\lambda)^2 = W\Delta \; , \tag{6.49}$$

in which the right-hand side is independent of θ .

From Eqs. (6.17) and (6.6), W is an analytic function of (r, E, L, Q) . Writing $Q = Q_{\rm circ}(E, L) - D$, and recalling the analyticity of $Q_{\rm circ}(E, L)$, W can be expressed as an analytic function of (r, E, L, D) . For an instantaneously circular orbit (i.e. for $r = r_0$ and $D = 0$), both W and $W_{,r}$ vanish [cf Eq. (6.18)]. The expansion of W near instantaneous circularity must therefore be of the form

$$W = \hat{A}D + \hat{B}\delta r^2 + O(D^2, D\delta r, \delta r^3) \; , \tag{6.50}$$

where $\hat{A}$ and $\hat{B}$ are some functions of E and L . Correspondingly, the expansion of the right-hand side of Eq. (6.49) will take the form

$$W\Delta = AD - B\delta r^2 + O(D^2, D\delta r, \delta r^3) \; , \tag{6.51}$$

where $A \equiv \hat{A}\Delta_0$ and $B \equiv -\hat{B}\Delta_0$, and where

$$\Delta_0 \equiv \Delta(r = r_0(E, L)) \; . \tag{6.52}$$

Combining Eqs. (6.49) and (6.51), and restricting attention to the leading-order eccentricity effect, we obtain

$$(d\delta r/d\lambda)^2 = AD - B\delta r^2 \; . \tag{6.53}$$

Note that A and B are some functions of E and L only. Therefore, as explained above, they can be regarded here as fixed parameters.

Equation (6.53) describes a simple harmonic oscillator. Its general solution is

$$\delta r(\lambda) = K\sqrt{D}\cos[\omega_r(\lambda - \lambda_0)] \; , \tag{6.54}$$

where $K = \sqrt{A/B}$ and $\omega_r = \sqrt{B}$ are parameters that depend on E and L only. (Do not confuse K here with K of Sec. III). Using this result to calculate u_r, we find

$$\begin{aligned} u_r &= g_{rr}\frac{dr}{d\tau} = \Delta^{-1}\frac{d\delta r}{d\lambda} \\ &= -\Delta^{-1}K\sqrt{D}\omega_r \sin[\omega_r(\lambda - \lambda_0)] \; . \end{aligned} \tag{6.55}$$

To simplify the notation, we shall hereafter absorb the constant λ_0 into λ (by shifting the origin of the latter if necessary). Substituting Eqs. (6.54) and (6.55) into Eq. (6.44), we obtain

$$\begin{aligned} \frac{dD}{d\tau} = \Big[K\sqrt{D} \quad &\Big(D^j_{,r}\cos(\omega_r\lambda) \\ &- D^j_{,u_r}\Delta^{-1}\omega_r\sin(\omega_r\lambda)\Big)\Big] F_j(\tau) \; . \end{aligned} \tag{6.56}$$

From Eq. (6.56) it is already clear that, so far as the *instantaneous* evolution is concerned, $dD/d\tau$ is indeed proportional to $\sqrt{D}$. Thus, as explained in Section III, although $D = 0$ yields $dD/d\tau = 0$, an instantaneously circular orbit will not remain circular later on. Instead, a momentary growth of D like τ^2 is to be anticipated. However, we are primarily interested here in the *long-term* adiabatic evolution of D . The latter is to be obtained from Eq. (6.56) by averaging it in τ over many periods. In order to perform this long-term averaging, we must first take a closer look at the nature of the time dependence of the force F_j .

Since the term in brackets on the right-hand side of Eq. (6.56) is already proportional to $\sqrt{D}$ (i.e. to the eccentricity), when evaluating $F_j(\tau)$ we are allowed to assume that the orbit is a precisely circular geodesic. The θ motion of such an orbit is periodic in τ. Therefore, the backreaction force must be periodic also. The various points along the circular geodesic orbit are physically distinguishable from one another only by the values of θ and $d\theta/d\tau$. It therefore follows that after completing a full cycle of the θ-motion, $F_j(\tau)$ will return to its original value. A closer look, however, reveals that, because of the reflection symmetry of the Kerr geometry, the θ-motion is symmetric with respect to the equatorial plane. As a consequence, the period of $F_j(\tau)$ will in fact be half of that of the full θ-motion cycle.

In order to facilitate the calculations, it is convenient to transform Eq. (6.56) from τ to λ. Using Eq. (6.48), we obtain

$$\frac{dD}{d\lambda} = K\sqrt{D} \qquad \left[\rho^2 \left(D^j_{,r} \cos(\omega_r \lambda)\right.\right.$$
$$\left.\left. - \quad D^j_{,u_r} \Delta^{-1} \omega_r \sin(\omega_r \lambda)\right)\right] F_j(\tau(\lambda)) \; . \tag{6.57}$$

According to our expansion scheme, we need only evaluate the term in brackets to zero order in the eccentricity. That is, we can replace ρ and Δ by their circular counterparts, ρ_0 and Δ_0 , where Δ_0 is the constant defined in Eq. (6.52), and ρ_0 is a function of θ only, defined by

$$\rho_0^2 \equiv r_0^2 + a^2 \cos^2 \theta \; . \tag{6.58}$$

We therefore obtain

$$\frac{dD}{d\lambda} = K\sqrt{D} \qquad \left(D^j_{,r} \cos(\omega_r \lambda)\right.$$
$$\left. - \quad D^j_{,u_r} \Delta_0^{-1} \omega_r \sin(\omega_r \lambda)\right) \left[\rho_0^2 F_j\right] \; . \tag{6.59}$$

The term in brackets depends on τ (and hence on λ) through its dependance on the θ-motion. It is obvious from Eq. (6.58) that ρ_0 is periodic in τ, again with a period which is just half that of the θ-cycle. Consequently, the entire term in brackets is also periodic (with that one-half θ period). Let us examine now the periodicity of this term with respect to λ. Again, the θ-motion is periodic in λ , and the reflection symmetry implies that the motion at $\theta < \pi/2$ is just symmetric to that at $\theta > \pi/2$. (This can also be deduced directly from the fact that, in the equation of motion for $\theta(\lambda)$, $(d\theta/d\lambda)^2 = \Theta(\theta)$ (cf. Ref. [10]), the function $\Theta(\theta)$ admits a reflection symmetry about $\pi/2$). Thus, if we denote the λ-period of the θ-motion by

$$\Lambda_\theta \equiv 2\pi/\omega_\theta \ , \tag{6.60}$$

then the term in brackets in Eq. (6.59) has a λ-period of $\Lambda_\theta/2$. Correspondingly, the Fourier transform of this term will take the form

$$\rho_0^2 F_j = \sum_{n=0}^{\infty} G_j^n e^{i2n\omega_\theta\lambda} + C.C. \ , \tag{6.61}$$

where $C.C.$ means the complex conjugate of the preceding term.

Substituting this expansion in Eq. (6.59) yields

$$\begin{aligned} \frac{dD}{d\lambda} = \sqrt{D}\,[K \quad & \left(D^j_{,r} \cos(\omega_r\lambda)\right. \\ - \quad & \left. D^j_{,u_r} \Delta_0^{-1} \omega_r \sin(\omega_r\lambda)\right)\Big] \sum_{n=0}^{\infty} G_j^n e^{2ni\omega_\theta\lambda} \\ + \quad & C.C.\,. \end{aligned} \tag{6.62}$$

It is convenient to transform the last expression from Sine and Cosine to exponential functions. The term in brackets can be expressed as

$$K_+^j e^{i\omega_r\lambda} + K_-^j e^{-i\omega_r\lambda} \tag{6.63}$$

where

$$K_\pm^j \equiv (K/2)\left(D^j_{,r} \pm i D^j_{,u_r} \Delta_0^{-1} \omega_r\right) \ . \tag{6.64}$$

Eq. (6.62) then takes the form

$$\begin{aligned} \frac{dD}{d\lambda} = \sqrt{D}\left(K_+^j e^{i\omega_r\lambda} \quad + \quad K_-^j e^{-i\omega_r\lambda}\right) & \sum_{n=0}^{\infty} G_j^n e^{2ni\omega_\theta\lambda} \\ + \quad C.C.\,. & \end{aligned} \tag{6.65}$$

Defining now

$$G_{\pm}^{n} \equiv K_{\pm}^{j} G_{j}^{n} \ , \tag{6.66}$$

we obtain

$$\frac{dD}{d\lambda} = \sqrt{D} \sum_{n=0}^{\infty} \left(G_{+}^{n} e^{i(2n\omega_\theta + \omega_r)\lambda} \ + \ G_{-}^{n} e^{i(2n\omega_\theta - \omega_r)\lambda} \right) \\ + \ C.C.. \tag{6.67}$$

Recall that in this equation the coefficients $G_{\pm}^{n}$ depend on E and L , but not on λ .

Equation (6.67) describes (within the adiabatic approximation, and to leading order in the eccentricity) the instantaneous rate of change of D . In order to obtain from it the long-term rate of change, we simply need to take its average over a sufficiently long period of λ . To that end, for any function $U(\lambda)$, we formally define the long-term averaged rate of change

$$\left\langle \frac{dU}{d\lambda} \right\rangle \equiv \lim_{\Delta\lambda \to \infty} \frac{\Delta U}{\Delta \lambda} \ , \tag{6.68}$$

where ΔU and $\Delta\lambda$ denote the difference in U and λ , correspondingly, between the two extremes of the λ-interval considered. Although the averaging is over times long compared to $1/\omega_\theta$ and $1/\omega_r$, it is still short compared to the radiation reaction timescale, which is the time for substantial orbital inspiral. (Recall that since we are using the adiabatic approximation here, if U also depends on the "constants of motion", they must be taken as fixed constants in this averaging process.) The averaging of the right-hand side of Eq. (6.67) is trivial, in that the term $e^{i(2n\omega_\theta \pm \omega_r)\lambda}$ will average to zero, unless $2n\omega_\theta \pm \omega_r = 0$, in which case it averages to one. Since both ω_θ and ω_r are taken to be positive, we need only worry about those terms with $2n\omega_\theta - \omega_r$ in the exponent. We thus obtain

$$\left\langle \frac{dD}{d\lambda} \right\rangle = \sqrt{D} \sum_{n=0}^{\infty} G_n \delta_{(2n\omega_\theta - \omega_r)} , \tag{6.69}$$

where δ is a function whose value is unity when $2n\omega_\theta - \omega_r = 0$ and zero otherwise, and $G_n \equiv 2\,\Re\left(G_-^n\right)$, with $\Re$ meaning the "real part of".

At this stage it is already clear that, unless a certain resonance condition is satisfied ($\omega_r = 2n\omega_\theta$ for some n), the right-hand side of Eq. (6.69) will vanish. Before we further discuss the meaning and implications of this resonance condition, however, we shall more directly connect our result (6.69) to the notation used in Section III, and in particular to the parameter S_1 . Equation (6.35) is to be obtained from the momentary rate of change of D by averaging over proper time. For any function $U(\tau)$, the long-term proper-time average (denoted by an overdot) may be formally defined as

$$\begin{aligned} \dot{U} &\equiv \lim_{\Delta\tau\to\infty} \frac{\Delta U}{\Delta \tau} \\ &= \lim_{\Delta\lambda\to\infty} \frac{\Delta\lambda}{\Delta\tau} \lim_{\Delta\lambda\to\infty} \frac{\Delta U}{\Delta\lambda} = J^{-1} \left\langle \frac{dU}{d\lambda} \right\rangle . \end{aligned} \tag{6.70}$$

where

$$J \equiv \left\langle \frac{d\tau}{d\lambda} \right\rangle , \tag{6.71}$$

is a constant that depends on the orbit. Applying this procedure to D, we obtain [15]

$$\dot{D} = J^{-1} \left\langle \frac{dD}{d\lambda} \right\rangle = \sqrt{D} J^{-1} \sum_{n=0}^{\infty} G_n \delta_{(2n\omega_\theta - \omega_r)} . \tag{6.72}$$

Comparing now Eq. (6.72) to Eq. (6.35), we find

$$S_1 = J^{-1} \sum_{n=0}^{\infty} G_n \delta_{(2n\omega_\theta - \omega_r)} \; . \tag{6.73}$$

The implications of this result for the long-term evolution of D are obvious. There are two different cases:

a) The resonant case: there exists an integer n such that $\omega_r = 2n\omega_\theta$. In that case, we have

$$S_1 = J^{-1} G_n \; , \tag{6.74}$$

which is likely to be nonzero in the general case. Then, as discussed in Section III [cf Eq. (6.36)] , D will grow like τ^2, which means that the eccentricity will grow linearly with τ.

b) The non-resonant case: there exists *no* integer n for which $\omega_r = 2n\omega_\theta$. In that case, we simply have

$$S_1 = 0 \; , \tag{6.75}$$

and the equation of evolution will read

$$\dot{D} = S_2 D + O(D^{3/2}) \; . \tag{6.76}$$

[cf Eq. (6.33)]. In this case an orbit which is initially *precisely* circular will remain circular (within the adiabatic limit). (The sign of S_2 will determine the stability against growth of small initial eccentricity).

For resonant orbits, we have

$$\Lambda_\theta = 2n\Lambda_r \; , \tag{6.77}$$

where Λ_θ and Λ_r are the λ-periods of the θ- and r-motions, correspondingly. It would sometimes be convenient to translate this expression to t-periods. One finds that, not surprisingly, the resonance condition is

$$T_\theta = 2nT_r \ , \tag{6.78}$$

where T_θ is the t-period of the θ-motion, and T_r is the *averaged* t-period of the radial motions. [The radial motion, expressed in terms of t (or τ), is quasi-periodic rather than periodic, because it is modulated by the θ-motion. The θ-motion itself is periodic in either t or τ — first, because we are considering a circular orbit here, and second, because the resonance condition (6.77) ensures that each time θ returns to its original value, r does also (but not vice versa).]

6.5 Conclusion

We have shown that, within the adiabatic approximation, an orbit which is initially precisely circular will remain circular, under the action of the radiation-reaction force. The only exception is if the orbit satisfies the resonance condition $T_\theta = 2nT_r$, for some integer n, where T_θ is the θ-motion period and T_r is the (averaged) period of the small-oscillation radial motion. However, circular orbits around Kerr never satisfy this resonance condition [5]. We therefore conclude that, within the adiabatic approximation, an orbit which is initially circular will remain circular.

There are two caveats which should be mentioned here. First, no attempt has been made to address the issue of stability against the growth of a small initial eccentricity. This stability would depend on the sign of the coefficient S_2 in Eq. (6.33) above, which was not calculated here.

Second, our conclusion that circular orbits must remain circular is only valid within the adiabatic limit, i.e. in the limit $\mu/M \to 0$, where μ is the mass of the small object. In reality, since the ratio μ/M is always finite, an initially circular orbit will develop some eccentricity. For concreteness, consider a particle with $\mu \ll M$ which at some initial stage (which we denote stage 1) moves along a circular orbit with Boyer-Lindquist radius r_1. Later on, the orbit shrinks due to radiation reaction, until (at stage 2) it passes through a new radius, $r_2 < r_1$. Then, for every finite μ/M, one should expect non-zero eccentricity to be present at stage 2. The above analysis, however, implies that the eccentricity at stage 2 will decrease with μ (for fixed r_1, r_2 and M), and will vanish at the limit $\mu/M \to 0$ (presumably like μ/M) [16].

The small eccentricity of non-adiabatic origin mentioned above could in principle seed an exponential growth of eccentricity if $S_2 > 0$. In such a situation, an initially circular orbit may evolve into a very non-circular one (even for small μ/M). The feasibility of this scenario depends, of course, on the relevant values of μ/M and on whether $S_2 > 0$ and also on the available range of r-values (over which S_2 is positive).

In reality, we know that in the Newtonian limit the orbits become more and more circular as they shrink due to radiation reaction, so we expect S_2 to be negative as long as $r \gg M$ (recall that the Newtonian approximation should hold at $r \gg M$ even if the black hole is spinning). Consequently, the range over which S_2 might be positive is bounded. We can therefore expect that if μ/M is sufficiently small (and if r_2 is not too close to the "last stable circular orbit" [17]), the instability will not have enough time to build up, and an initially- circular orbit will indeed remain circular throughout the inspiral, to a good approximation.

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[1] T. A. Apostolatos, D. Kennefick, A. Ori and E. Poisson, Phys. Rev. D, **47**, 5376 (1993).

[2] This conjecture was made by one of us (A. Ori) several years ago.

[3] Such orbits have often been referred to as "quasi-circular" in the literature (see, for instance, Ref. [10]), because such an orbit in Kerr does not describe a circle, or any closed curve and is not even confined, in general, to a single plane. However, for our purposes, the term circular aptly implies the analogy we wish to draw with the Keplerian orbit case (or the Schwarzschild case). That is, the orbit is described by a constant radial co-ordinate, and that characteristic is maintained under the influence of an adiabatic back reaction force.

[4] F. D. Ryan, Phys. Rev. D, **52**, R3159 (1995).

[5] An interesting qualitative argument is given in F. D. Ryan, Phys. Rev. D, submitted.

[6] Recently, we have become aware of an independant derivation of this result by Y. Mino, thesis in preparation, Kyoto University, Japan.

[7] A. Ori, Phys. Lett. A. **202**, 347 (1995).

[8] In the case of gravitational radiation reaction (which is our chief concern here), although it is possible to use this concept of back-reaction force, this force is gauge-dependent. We do not expect this ambiguity to affect our results, for the following reason: despite the ambiguity in the momentary back-reaction force, the long-term adiabatic evolution must be gauge-independent. Therefore, one can simply pick any gauge in order to calculate the radiative evolution. Once the gauge is chosen, the back-reaction force is defined unambiguously, and all the analysis of this paper is applicable. Keep in mind, however, that since we will employ an argument based on the symmetries of Kerr, we must be careful to select a gauge which does not artificially violate these symmetries. The harmonic gauge is an example of a "good" gauge for our purposes.

[9] Note that this function $C(x^\beta, u_\alpha)$ is not unique. This is obvious, for example, from the fact that $-1 = g^{\alpha\beta}(x)u_\alpha u_\beta$. The function $C(x^\beta, u_\alpha)$ can thus be chosen upon convenience.

[10] C. W. Misner, K. S. Thorne and J. A. Wheeler *Gravitation* (Freeman, New York, 1973).

[11] Taking the differential of the second condition in Eq. (6.18), $W_{,r} = 0$, does not yield any additional constraint on $\delta Q_{\rm circ}$ (instead it results in an expression for $\delta r_{\rm circ}$).

[12] Mathematically, this ambiguity in the solution of Eq. (6.29) with the initial condition $K = 0$ is possible due to the fact that the term $K^{1/2}$ in the right-hand side is not differentiable with respect to the unknown K. Note that this mathematical ambiguity does not reflect any ambiguity in the solution of the *fundamental* equation of motion (i.e. the Newton equation, $d^2x/d\tau^2 = F$). The only ambiguity occurs in the equation for K derived from $d^2x/d\tau^2 = F$. The origin of this ambiguity is simple. In order to derive the equation for K, the fundamental equation of motion is multiplied by $dx/d\tau$, and this entity is identically zero in the trivial non-physical solution (6.30). Thus, although the trivial solution (6.30) satisfies the equation for K, it does not satisfy the original equation of motion.

[13] This follows, for example, from Eq. (6.72) below.

[14] Of the six coefficients $D^j_{,r}$ and $D^j_{,u_r}$, three vanish identically ($D^r_{,r}$, $D^t_{,u_r}$, and $D^\varphi_{,u_r}$) but the other three are generically nonzero.

[15] It is sometimes useful to have the averaged radiative evolution rates in terms of external time t rather than τ. The procedure for averaging over t is analagous to the above averaging over τ. The result is also similar. In the right-hand

side of Eq. (6.72), one only needs to replace J by the corresponding parameter, $J_t \equiv \langle dt/d\lambda \rangle$.

[16] The back-reaction force (per unit mass μ) is proportional to μ (and the evolution time from stage 1 to stage 2 is proportional to $1/\mu$). Now, while the force terms responsible for the adiabatic shrinking of the orbit from radius r_1 to r_2 are linear in the back-reaction force, the non-adiabatic effects are (at least) quadratic in that force. The non-adiabatic terms are therefore suppressed by the small factor μ/M (compared to the adiabatic terms).

[17] In Schwarzschild, it has been found [1] that S_2 diverges at the last stable circular orbit, due to the flattening of the effective potential well. Intuitively, one should expect the same behavior in Kerr as well.

Appendix A

Notes on Numerical Methods in the Teukolsky Formalism

Abstract

Numerical methods employed in solving the radial and θ-equations of the Teukolsky perturbation formalism for particles orbiting massive central black holes are discussed.

A.1 Introduction

Teukolsky has shown how one can separate the equations governing the field produced by small perturbations of the Kerr metric into differential equations of one co-ordinate variable only, in Boyer-Lindquist co-ordinates [1]. These equations can be integrated to, for instance, calculate the energy and angular momentum of gravitational waves carried away from a small particle orbiting a large black hole. This enables one to solve the evolution of the particle's orbit, as it sprials towards the black hole due to the loss of orbital energy and angular momentum. The Teukolsky radial equation is

$$\Delta^2 \frac{d}{dr}(\frac{1}{\Delta}\frac{dR_{lm\omega}}{dr} - V(r)R_{lm\omega} = T_{lm\omega}, \tag{A.1}$$

where $R_{lm\omega}(r)$ is the Teukolsky radial function, $T_{lm\omega}$ is the Teukolsky source function, which describes the perturbation and $V(r)$ is the Teukolsky potential. The function $\Delta = r^2 - 2Mr + a^2$ depends on a, the black hole's spin parameter. The θ-equation is

$$\begin{aligned} \frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d_s S_{lm}^{a\omega}}{d\theta}) &- (a^2\omega^2\sin^2\theta + 2as\omega\cos\theta \\ &+ \frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta} \\ &- 2a\omega m - s(s+1) - \lambda)_s S_{lm}^{a\omega} = 0, \end{aligned} \tag{A.2}$$

where ${}_sS_{lm}^{a\omega}$ is a spheroidal harmonic function of "spin weight" s, well known to applied mathematics, and λ is its eigenvalue, ω is the frequency of the emitted radiation and m is the orbital angular momentum number along the direction of the black hole's spin.

The Teukolsky potential is defined by

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \tag{A.3}$$

where $K = (r^2 + a^2)\omega - ma$.

We can define solutions to the radial equation, $R^H_{lm\omega}(r)$ and $R^\infty_{lm\omega}(r)$, with the following boundary conditions,

$$R^H_{lm\omega} \sim \Delta^2 e^{ikr^*}, \quad \text{as} \quad r \to r_+ \tag{A.4}$$

$$R^H_{lm\omega} \sim r^3 B^{\text{out}}_{lm\omega} e^{i\omega r^*} + \frac{1}{r} B^{\text{in}}_{lm\omega} e^{-i\omega r^*}, \quad \text{as} \quad r \to \infty \tag{A.5}$$

where $k = \omega - ma/2Mr_+$, $r_+ = M + \sqrt{M^2 - a^2}$ defines the position of the black hole horizon, and r^*, the tortoise co-ordinate, is defined as

$$r^* = r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \tag{A.6}$$

where $r_- = M - \sqrt{M^2 - a^2}$.

From [4], the solution of the Teukolsky equation near infinity (solved via a retarded Green's function) is

$$R^H_{lm\omega}(r \to \infty) \sim Z_{lm\omega} r^3 e^{i\omega r^*} \tag{A.7}$$

where

$$Z_{lm\omega} = \frac{1}{2i\omega B^{\text{in}}_{lm\omega}} \int_{r_+}^{\infty} \frac{R_{lm\omega}(r) T_{lm\omega}(r)}{\Delta^2} dr. \tag{A.8}$$

The averaged energy and angular momentum fluxes carried to infinity are

$$\langle \frac{dE_{\text{GW}}}{dt} \rangle = \dot{E}^\infty = \sum_{lmk} \frac{|Z^H_{lmk}|^2}{4\pi\omega_k^2} \tag{A.9}$$

and

$$\langle \frac{dL_{\text{GW}}}{dt} \rangle = \dot{L}_z^\infty = \sum_{lmk} \frac{m |Z^H_{lmk}|^2}{4\pi\omega_k^3}, \tag{A.10}$$

where the amplitude co-efficient is decomposed into a discreet set of frequecies based on the particle's orbital motion,

$$Z^H_{lm\omega} = \sum_k Z^H_{lmk}\delta(\omega - \omega_k). \tag{A.11}$$

In order to calculate $Z_{lm\omega}$ and thus the energy and angular momentum fluxes, one will require the function $R^H_{lm\omega}$ and its derivatives and the amplitude co-efficient of the in-going waves at infinity $B^{\text{in}}_{lm\omega}$. The latter presents a problem, in that, because the ingoing wave in Eq. (A.5) has a $1/r$ dependance, while the outgoing wave as a r^3 dependance, the former is numerically swamped by the later for large r. The standard solution for this is to numerically solve instead the Regge-Wheeler equation (for Schwarzschild black holes) or the Sasaki-Nakamura equation (for Kerr black holes) [3]. These both have solutions of the type

$$X^H_{lm\omega} \sim A^{\text{out}}_{lm\omega} e^{i\omega r^*} + A^{\text{in}}_{lm\omega} e^{-i\omega r^*}, \quad \text{as} \quad r \to \infty \tag{A.12}$$

$$X^H_{lm\omega} \sim e^{-ikr^*}, \quad \text{as} \quad r \to r_+. \tag{A.13}$$

It is easy to see here that the ingoing wave amplitude $A^{\text{in}}_{lm\omega}$ is of similar size to the outgoing amplitude for large radii. There are straightforward transformations between the solutions of the Regge-Wheeler and Sasaki-Nakamura equations (the later reduces to the former when $a \to 0$) and that of the Teukolsky equation.

A.2 Numerical integration of the radial equation

Both the Regge-Wheeler and Sasaki-Nakamura equations are perfectly smooth and well-behaved everywhere except at the horizon, $r = r_+$. Thus the Bulirsch-Stoer integration technique, with its rapid convergence is ideally suited to the problem. Subroutines ODEINT and BSSTEP of Ref. [2] were employed in all cases to implement this algorithm. The singularity at the horizon was handled as follows. The solution near the horizon is marked by very rapid oscillations. If one chooses as the starting point for the integration $r = r_+ + \epsilon$, where ϵ is in the range 10^{-4} to 10^{-10}, one finds that the integral is insensitive to changes in ϵ, because the amount of the solution "lost" between r_+ and $r_+ + \epsilon$ sums up to zero. Choosing a value of ϵ well within this range, such as $\epsilon = 10^{-8}$ works very effectively in estimating the solution. In general, the solution seems to be rather insensitive to the accuracy of the initial conditions employed, and one may simply employ the estimates of the behaviour of $X^H_{lm\omega}$ and $X^\infty_{lm\omega}$ at $r \to r_+$ given above in Eq. (A.13).

A second problem with the radial integration is encountered in finding $A^{\rm in}_{lm\omega}$, which requires one to integrate to infinity. The method used here relies on noticing that $A^{\rm in}_{lm\omega}$ can be seen as the zeroth order term in an expansion of the amplitude of the distant ingoing wave in powers of $1/\omega r$. In short Eq. (A.12) can be rewritten as

$$X^H_{lm\omega} \sim A^{\rm out}_{lm\omega}\bar{P}(\omega r)e^{i\omega r^*} + A^{\rm in}_{lm\omega}P(\omega r)e^{-i\omega r^*}, \quad \text{as} \quad r \to \infty \tag{A.14}$$

where $P(\omega r)$ is a polynomial in powers of $1/\omega r$, with $\bar{P}(\omega r)$ its complex conjugate.

Thus one can conceive of a quantity $A^{\text{in}}(r) = A^{\text{in}}_{lm\omega}P(\omega r)$ which can be evaluated at finite radii, and which tends towards the constant value $A^{\text{in}}_{lm\omega}$ as $r \to \infty$. Therefore one adopts the strategy of integrating to a large value (typically $100M$) in ωr, and then increasing the end-point radius for successive tries, until the desired accuracy of estimating the limiting value of the polynomial as $1/\omega r \to 0$ has been achieved. Invariably the accuracy with which this value can be estimated is limited to an order of magnitude lower than the current relative accuracy of the Burlirsch-Stoer integrator itself. It the latter is set at 10^{-6}, then the highest relative accuracy that can be achieved in estimating $A^{\text{in}}_{lm\omega}$ is of the order 10^{-5}. Fortunately there is no great loss of speed in setting the former accuracy higher, the main speed limit is in the accuracy required in estimating $A^{\text{in}}_{lm\omega}$ itself.

In the Regge-Wheeler case we evaluated $P(\omega r)$ analytically to fourth order in $1/\omega r$, and employed this function to improve the estimate of $A^{\text{in}}_{lm\omega}$. This provided the fastest convergence to accuracy of order 10^{-5} or so. In the Sasaki-Nakamura case, a Richardson polynomial extrapolator was employed instead (subroutine PZEXTR of Ref. [2]) to estimate the limiting value of the polynomial numerically. This approach had the advantage of placing no limit in principle on the level of accuracy which could be reached, in contrast to the truncation of the ploynomial required when evaluating analytically. The end-point of the integration was doubled with each successive trial in order to allow the extrapolator to converge effectively.

In the general eccentricity case of chapter 5 it was not possible to analytically derive Z^H_{lmk} in a closed form which required one only to plug in the numerically derived Teukolsky functions to evaluate it. Z^H_{lmk} itself had to be integrated numerically. For this purpose the robust Romberg method (QROMB from Ref. [2]) was employed. This integrator had to call the Bulirsch-Stoer routines in order to evaluate the Teukolsky functions when required by its own stepping algorithm. This arrangement worked very effectively, and for small eccentricities was not a great deal slower than the code designed to work in that limit. Nevertheless, limitations of computing time did play a role for high eccentricites, because of the enormous number of harmonics of the motion which had to be calculated.

A.3 Estimating the spheroidal harmonics

In the Schwarzschild case, the solutions of Eq. (A.2) reduce to spherical harmonics of spin-weight s. Simple recursive algorithms for calculating such functions exactly are given in [6]. The spin weight of the field in the chosen solutions of the radial equation is -2. However, the Teukolsky source function contains terms involving the spherical harmonics and the so-called "edth" operators which transform between spherical harmonics of different weight [6]. It is therefore convenient to calculate spherical harmonics of spin wieght $0,-1$ and -2.

In the Kerr case, where one deals with spheroidal harmonic functions, two complications arise. One is that there are no simple algorithms, recursive or otherwise for evaluating such functions. The other is that the generalisations of the edth operators to Kerr spacetimes do not retain the property of transforming spheroidal functions of different spin weights into each other. It is therefore necessary to estimate numerically not only the spheroidal harmonics themselves, but also their first derivatives. This is done by employing a decomposition of the spheroidal harmonics into an expansion of spherical harmonic functions, a technique described for the scalar case in Ref. [5]. The following generalized scheme is due to Scott Hughes (unpublished). One writes

$$
{}_sS^{a\omega}_{lm}(\theta) = \sum_{k=\max(|s|,|m|)}^{\infty} b_k {}_sY_{km}(\theta), \tag{A.15}
$$

where ${}_sY_{km}(\theta)$ is a spherical harmonic of the same spin-weight as the spheroidal harmonic, and the b_k's are the expansion co-efficients. One can now substitute this series back into Eq. (A.2), and take advantage of the fact that each ${}_sY_{km}(\theta)$ solves the same equation with $a\omega = 0$ and $\lambda = k(k+1)$, so that

$$
\begin{aligned}
\sum_{k=\max(|s|,|m|)}^{\infty} b_k[(a\omega)^2 \sin^2\theta \quad &+ \quad 2a\omega s\cos\theta + k(k+1) - 2a\omega m + s(s+1)]{}_sY_{km}(\theta) \\
&= \quad \lambda \sum_{k=\max(|s|,|m|)}^{\infty} b_k {}_sY_{lm}(\theta).
\end{aligned} \tag{A.16}
$$

It is helpful from here on to make use of the Dirac style of notation and to left multiply the preceding expression by the complex conjugate of ${}_sY_{lm}(\theta)$ and integrate. This gives us

$$
\sum_{k=\max(|s|,|m|)}^{\infty} b_k[(a\omega)^2\langle slm|\cos^2\theta|skm\rangle \quad - \quad (2a\omega s)\langle slm|\cos\theta|skm\rangle
$$

$$-k(k+1)\langle slm|skm\rangle \;=\; -E \sum_{k=\max(|s|,|m|)}^{\infty} b_k\langle slm|skm\rangle \quad \text{(A.17)}$$

where $E = \lambda + 2a\omega m + s(s+1) - (a\omega)^2$. Since the ${}_sY_{lm}$'s form an orthonormal basis $\langle slm|skm\rangle = \delta_{lk}$. THe inner-products containing cosine terms can be evaluated in terms of Clebsch-Gordon co-efficients, and are only non-zero for $l-2 \leq k \leq l+2$ in the first $(\cos^2\theta)$ term, and for $l-1 \leq k \leq l+1$ in the second $(\cos\theta)$ term. Therefore

$$\begin{aligned} & b_{l\pm 2}[(a\omega)^2\langle s(l\pm 2)m|\cos^2\theta|slm\rangle] + b_{l\pm 1}[(a\omega)^2\langle s(l\pm 1)m|\cos^2\theta|slm\rangle \\ - \;& 2a\omega s\langle s(l\pm 1)m|\cos\theta|slm\rangle]b_l[(a\omega)^2\langle slm|\cos^2\theta|slm\rangle \\ - \;& 2a\omega s\langle slm|\cos\theta|slm\rangle - l(l+1)] = -Eb_l. \end{aligned} \tag{A.18}$$

Rewriting this expression as a matrix eigenvalue problem

$$M\cdot\vec{b} = -E\vec{b}, \tag{A.19}$$

we see that we have merely to solve this numerically for a symmetric band-diagonal matrix. Many routines are available for the solution of such eigenvalue problems, and the subroutines TQLI and TRED2 from Ref. [2] were chosen. The size of the matrix for general purposes was chosen to be 30 × 30, since this gave excellent accuracy in the estimation of the spheroidal harmonics without compromising speed or accuracy in the matrix inversion. The first derivatives of the ${}_{-2}S^{a\omega}_{lm}(\theta)$'s were found by differentiating the series expansion in the ${}_{-2}Y_{lm}$'s and employing the edth operators to rewrite the resulting expression as an expansion in ${}_sY_{lm}$'s of different spin-weights ($s=-2$ and $s=-1$). The second derivative of ${}_{-2}S^{a\omega}_{lm}(\theta)$ was found by recourse to the original ODE, Eq. (A.2).

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Part II: Controversies in the History of the Radiation Reaction Problem in General Relativity

Contents

Chapter 1

Introduction

In 1994 preliminary site work began at Hanford, Washington, on what its proponents hope will be a new type of astronomical observatory. LIGO, or the *Laser Interferometric Gravitational-Wave Observatory*, consists of two 4km long evacuated steel cylinders positioned at right angles to each other, each containing a laser beam reflecting back and forth between mirrors attached to test masses suspended from sophisticated supports (designed to insulate the masses from as much local vibration as possible) at either end of the tubes. Interference between the laser beams from the two arms (which are produced by a single laser), will be employed to monitor the relative positions of the test masses. The relative motion of the test masses by a small fraction of an atom's breath over their 4km separation may be taken as evidence for the passage of gravitational waves. A largely identical device is also being constructed at Livingston, Louisiana. The total cost, including initial operating expenses, is projected to exceed $300 million. Yet the medium of this new astronomy, gravitational radiation, is a phenomenon which has never yet been directly detected, in the view of nearly all experts. Before it can become an observatory, this project will be a large scale physics experiment. Even in that it is unusual, for nearly all "big science" physics experiments have previously concerned themselves with nuclear physics or the physics of fundamental particles.

The theorists whose input will be crucial to LIGO are relativists, theorists who specialize in the study of Einstein's General Theory of Relativity, known familiarly as GR. My study does not primarily concern itself with the story of this great experiment and its antecedents, but with the story of the theorists associated with it. It is a remarkable situation that such a huge experiment is now going forward on the basis of predictions of a body of theory historically riven with controversy, often over the very existence of the phenomenon of gravitational radiation, or the type of sources most favoured for detection by LIGO, and a group of theorists who were, until quite recently, decidedly a small and not very significant minority amongst theoretical physicists in general. The process by which the concept of gravitational radiation has gone from a controversial and frequently ignored prediction of a maverick theory, to the subject of expensive detection programs is to a large extent the story of a community of theorists. How they, in the almost total absence of experimental input, were able to construct a phenomenon whose reality would be largely accepted by (presumably) hard-headed experimenters and government funding agencies is surely a remarkable story of the role of theory in modern science.

The historical themes which this study addresses are several. Amongst the most important is the problem of "identity," meaning a scientific group's sense of its own history. In particular, the use which is made of history by scientists seeking disciplinary space for their research programs and goals (Barkan, 1992). During the quadrupole formula controversy of the 1970s and 1980s, extensive use was made by the protagonists of the history of the radiation reaction problem in GR. Not only did many of them possess a deep familiarity with the literature of the field extending back for several decades, but they were able to employ their knowledge of the field's history as a rhetorical device to effectively argue their perception of the subject's current state and desirable future course, and, in addition, used the literature as a source of possible motivations or tools for the pursuit of outstanding problems in the subject.

An issue of central importance to this study is the role of analogy in science. It is in fact the view presented hear that, in this possibly unusual case, the primary motivation for belief in the existence of the physical phenomenon of gravitational waves was a rather abstract analogy with another physical theory, that is the theory of the electromagnetic field. Certainly no experimental data in support of their existence was available until the late 1970s, and yet a considerable body of work had already been devoted to their study across six decades by that time. The controversies over the existence and description of these waves which persisted for much of this time can therefore be seen as a consequence of varied reactions to this analogy. While most relativists accepted the analogy as well-founded, a considerable minority, sometimes referred to by the term "sceptics," regarded it as potentially misleading or actively damaging. Their refusal, arising from different motives, to accept the implications of the analogy played a central role in the various controversies over the problem of gravitational waves.

Much has been written on the subject of analogy in science without, perhaps, any conclusions being reached as to its proper use, nature or meaning. It is generally accepted that the use of analogies is quite prevalent in science. On the other hand, they are seen as potentially misleading. Roger Bacon reminds us that

> The human understanding is of its own nature prone to suppose the existence of more regularity in the world than it finds. And though there may be many things in nature which are singular and unmatched, yet it devises further parallels and conjugates and relatives which do not exist. [from *Novum Organum*, quoted in Leatherdale (1974).]

A philosopher of science cautions, "arguments from analogy may be fertile, but they are all invalid." (Mario Bunge, quoted in North (1981)). It is not surprising therefore that the electromagnetic analogy which inspired the idea of gravitational radiation should have been viewed sceptically by some physicists. Indeed, the force of the analogy, which seems to have persuaded more often than it dissuaded, is maybe more to be remarked upon than its failure to compel acceptance in some cases. Nevertheless, in a milieu in which, by and large, the existence of gravitational waves was taken for granted, the resistance to the standard picture of them which persisted for decades is surely an episode worthy of study.

Leatherdale, in *Analogy in Science* (1974), distinguishes analogic discovery (the "analogic act" as he calls it), in which a comparison between two physical fields of phenomena is drawn for the first time, from analogic argument, in which analogy is merely a rhetorical device (such as Galileo's comparison of the earth-moon system to Jupiter and the Medicean satellites in order to argue that the earth was a planet). One could argue that Einstein's discovery of gravitational waves via his linearized field equations was a sort of analogic act. It is perhaps better, however, to regard Einstein's 1916 paper as an argument by analogy. The underlying comparison (the mental leap between gravity and electromagnetism) had previously been made by Poincaré and, no doubt, others. Einstein, in deliberately casting his field equations for gravity in an approximate form which invited close comparison with the field equations of electromagnetism, clearly hoped to develop the argument underlying the original comparison. Sure enough, he was able to construct a linearized theory of gravitational waves in close analogy with the extant body of electromagnetic wave theory, though not without a few stumbles along the way. Even in the linearized theory, the analogy between the two phenomena is not exact. The electromagnetic field permits radiation from systems with a varying *dipole* moment (i.e. vibrations along a single axis of symmetry). In his first (1916) paper, Einstein, as he later said (1918), erroneously derived the result that a "mechanical system which always maintains its spherical symmetry [can] radiate" (translated and quoted in Cattani and De Maria (1993)). This would be *monopole* radiation. In fact, as Einstein showed in his second paper on gravitational waves (1918), *quadrupole* radiation (from systems with vibrations along two separate axes) is the lowest order of radiation possible in linearized GR theory.[1] The analogic argument, somewhat halting at first, was nevertheless already proving fecund in helping to differentiate the new phenomenon from the original object of comparison.

This points up an interesting aspect of analogic development of physical theory. The tendency of analogies to fail in certain crucial respects, which causes many people to distrust them as vehicles for scientific advance, is perhaps the very quality which explains their extraordinary fertility. An analogy which is perfect in all its

[1] Mathematically, one can say that this "breakdown" in the analogy is due to the fact that the *metric perturbation*, which plays the role of the electromagnetic vector potential in the linearized equations of gravity, is a tensor quantity, not a vector.

details may play a useful role in visualization, but it lacks any point of departure from which the theory of the new phenomenon may take on a life of its own. This may help explain why the sceptics in the history of gravitational wave theory (Infeld, Havas, Rosen, Ehlers and especially Bondi amongst others), were able to play a significant role in the development of this theory, because of their emphasis on the breakdowns in the analogy. This is not to depict those who broadly accepted the analogy as doing so naively, that was far from the case, but it would be easy to see how complacency or disinterest might set in if the analogic argument were allowed to go uncontested at all or most points.

The type of abstract mathematical analogy which I have described, and which continued to play an influential role throughout the subsequent history of gravitational waves, is a type favoured by the French physicist Pierre Duhem. Duhem was critical of the strong use of analogy made by the English school of the 19th century (Maxwell, William Thomson (Kelvin) and Faraday all made extensive use of analogy in their work), but conceded that analogies expressed in mathematical form, between well-formulated theories, might be useful where "experimental intuition quite naturally poses a problem and suggests a solution for it" in only one of the two theories (Duhem, quoted in Leatherdale (1974)). The fact that no experimental evidence became available in the case of gravitational radiation until about 1980 no doubt helps to explain the continued need for the rather formal analogy elaborated by Einstein. This was especially true in the problem of *radiation reaction*, by which is meant the back action of the waves on the emitting system (e.g. its loss of energy and momentum to the waves and the consequent alteration of its motion), which is a particular focus of this study. The considerable body of theory on *back reaction* in electrodynamics (a phenomenon known from experiment, in addition to the prediction of theory[2]) provided both a source of physical intuition and of mathematical tools to the equivalent problem in GR. It was precisely in this area however, in particular the problem of radiation reaction or *radiation damping* in orbital systems of two massive bodies, each bound gravitationally by the other, that the sceptics were least convinced of the virtues of the analogy. From Eddington on, they repeatedly resisted attempts to apply Einstein's 1918 *quadrupole formula*, which described the rate of emission of wave energy from a system with a varying quadrupole moment, to the case of planetary or binary stellar motion. The first derivation of radiation reaction in electromagnetism dates from 1892, and is due to Lorentz (Buchwald, 1985). This problem continued to be an important one throughout the subsequent development of electrodynamic theory in the 20th century. The perceived empirical success of this theory (despite difficult conceptual or technical problems) no doubt was encouraging to those relativists who were inclined to see a deep, valid analogy between their theory and the electromagnetic one.

Although Einstein's linearized analogy with Maxwell's theory is an interesting example of the type of analogy which Duhem was willing to accept, I am inclined to disagree with him about the value of the more "visual" use of analogy employed by Maxwell and Thomson. Maxwell describes his method of "physical analogy" as "that partial similarity between the laws of one science and those of another which makes each of them illustrate the other." (quoted in North (1981)). This may not seem very different from the sort of analogy just described, but Maxwell does not

[2] In fact, it seems that it was originally expected by Maxwell's British followers that electromagnetic radiation directed randomly into space, and not channeled through a wave guide or cable, would quickly damp down the action of its source, so that the phenomenon would be evanescent at best.

place the same restrictions on his use of analogy which Duhem would. The analogy is not restricted to operate where there is no experimental intuition on one side, and it may be *partial* in form, reflected only in some of the characteristics of the phenomena under comparison. More importantly, Maxwell employs his analogies in an effort to *visualize* the phenomena under examination. He speaks of seeing the electromagnetic field as a system of wheels and pulleys, permitting an analogy with mechanics. The key point here is that an analogy is made with something from one's visual or sensible experience, and not simply with a set of mathematical equations. Maxwell cautions those who would follow Duhem, "By stripping [the analogy] of its physical dress ... we might obtain a system of truth strictly founded on observation, but probably deficient both in the vividness of its conceptions and the fertility of its method."

Many authors have commented on the role of analogy in enabling or attending abstraction in science. Miller does so in *Insights of Genius* (1996), where he argues that increasing abstraction is a central feature of progress in science which is partly enabled by the use of analogy. Leatherdale discusses the matter at some length,

> Moles, in his work on scientific creativity, argues that it is a generalizing abstraction which empties concepts of their sensual content and leads to formal analogy and he refers to the 'classic liaison between analogy and abstraction' citing the authority of William James.[3] ... This [according to Leatherdale] puts the thing the wrong way around, for I have argued that, in effect, it is the analogic act which brings otherwise discrete areas of experience together and enables one to see, for example, that two things

[3] A.A. Moles, *La Création Scientifique* (Geneva, 1957).

are alike in exhibiting a relation. However, it can easily be seen how an analogic act may be said to be a necessary prelude to abstraction. As Buchanan says, 'argument by analogy is the fundamental technique in the process of abstraction'.[4]

The analogy which Einstein draws in his 1916 paper is already at an advanced state of abstraction, for its analogy is made with a theory itself abstracted by analogy with the theory of light, itself an abstraction based on analogy with sound (see North (1981) for an interesting discussion of Newton's analogies between light and sound), and also with water waves. Miller regards the use of analogy between increasingly abstract scientific theories as a critical element in the visualization of phenomena within these theories. I do not, however, feel that the electromagnetic analogy was particularly compelling as a visualization of gravitational waves. Rather, it provided a ready-made model for calculations of the waves' behaviour in the theory, based on the rather extensive body of work on radiation phenomena in electrodynamic theory. I believe that the emergence of a richer, deeper, independent description of gravitational waves is in fact signaled by the construction of a more visually descriptive analogy in the period of the 1960s. In doing so, relativists reached back to one of the physical metaphors underlying the electromagnetic and optical wave phenomena, water waves. From about 1970 on the metaphor relating gravitational waves to ripples of water became quite ubiquitous in textbooks and popular accounts.

The emergence of this more visually appealing analogy seems to reflect the considerable development which gravitational wave theory underwent (along with much of GR theory) from 1957 to 1970 or so. This included developments which did not

[4] S. Buchanan *Poetry and Mathematics* (New York, 1962).

adhere to the straightforward analogy with electromagnetism, but included concepts unique to general relativity, such as spacetime curvature and its non-linear interaction with itself (as in the case of the gravitational geon). A new analogy was constructed in which "ripples" of curvature propagated against the general background curvature of spacetime, consciously relating gravitational waves to the motion of waves on the surface of a body of water. This analogy was surprisingly slow to emerge, perhaps because the electromagnetic analogy directed attention away from the role of curvature in describing gravitational waves. Ignoring the advice of Eddington in 1922, relativists persisted in attempting to calculate the energy carried in the waves, an endeavour which was not straightforward in the context of GR (where the equivalence principle is believed to restrict one from defining the energy of the gravitational field in a "local" sense). The increasing focus on curvature and geodesic deviation (describing the relative motion of two particles moving in the wave, sometimes compared to two corks bobbing on the surface of the ocean) begun by the work of Pirani led to a new and deeper understanding of gravitational waves which permitted or inspired a more visually appealing analogy. I suspect that the emergence of a style of analogy with a more apparent sensual content signifies in some way a deeper appreciation of the theory itself. Once they have begun to make significant progress in dealing with the abstract theory, one senses, physicists are motivated to move "backwards" and construct a more Maxwellian physical metaphor which leaps over the intervening layers of abstraction to confront once more the analogy with ordinary sensual experience.

Another important topic, probably related to this matter of analogy, is the question of "style" in physics. Repeatedly in this subject one finds references to the style of researchers as playing a considerable role in their attitudes and contributions. Frequently, one finds protagonists in the quadrupole formula debate described as "physicists" (or "phenomenologists") or "mathematicians." Since every contribution to this subject fell within the subject of physics and employed mathematics, these designations must be understood as coded references to the question of "style". Some of the protagonists, especially Chandrasekhar (1987), were very concerned with the role of style in doing physics, in particular, the role of intuition, and the types of intuition used in theoretical physics. Leatherdale seeks to explain intuition in science in terms of what he calls the "analogic act," the flash of recognition in which the similarities between one phenomenon or set of equations and another is appreciated. The origin or true nature of intuition need not concern us too much here. Of more interest is the role of intuition based on experience in mathematics, which allows the theorist to sense a path through a thicket of equations based on his experience with a similar formal problem, which might however have been applied to a completely different physical situation.[5]

Physical intuition, an even more difficult faculty to pin down, seems to refer to the ability of scientists such as Maxwell to appreciate similarities between different types of physical phenomena, and construct models based on this which could give them a sense of the operation of one set of phenomena, presumably based on prior (experimental or theoretical) experience with the other set. The difficulty which arises with this later type of intuition in our present case, is that it does not typ-

[5]At the 1994 Marcel Grossman meeting in Stanford, California Chandrasekhar compared this ability to appreciate the similarity between mathematical equations encountered in different settings with the visual impact of Impressionist art, especially Monet's paintings of Rheims cathedral at different hours of the day. The mind, just as the eye does, recognizes the same object seen under different conditions. See also Chandrasekhar (1987).

ically lead to a formal mathematical transformation from the solutions of one set of equations to another. The physical argument may be compelling, but not in the formal logical sense. Thus the "physicist" may be convinced of his answer, where the "mathematician" is completely dissatisfied with the level of rigor of the argument. This, I suspect, leads to a difference in emphasis over the relative merits of being guided slowly by a careful consideration of the conceptual problems to be considered and overcome in constructing a valid method of calculation, and, on the other hand, begin partly guided by ones (intuitive?) sense of what is a "physically reasonable" approximation, and a "physically reasonable" result.

An important debate of recent times in the Sociology and Philosophy of science is that between realists and social constructivists. It is interesting to note that one of the scientists in our story addressed the problem of realism *vs.* relativism in science. Infeld, in discussing the episode related in chapter 6 in his autobiography (1941), notes that Einstein "destroyed them [i.e. gravitational waves, in whose existence Einstein previously believed] in his picture of reality, and ... was forced to re-create them once more." Infeld regards the issue of idealism and realism (i.e. whether Einstein was simply constructing and deconstructing the waves in his mind, or discovering their existence or non-existence in nature by his calculations) as impossible to decide, but asserts that realism is a practical necessity for the scientist.

> Every scientist is emotionally a realist [since] a mind which thinks that gravitational waves are or are not radiating from his own brain cannot bother seriously about this problem. A scientist who has done research successfully and regards himself as an idealist must have acted in the moments of creation as a realist does, accepting emotionally the reality of

the outside world.

Yehuda Elkana (1978) has coined the phrase "2-Tier-Thinking" for this type of doublethink, which Infeld regards as "strange" and "artificial". We may suspect that it is more common than Infeld would have was believe. Perhaps if realism is psychologically necessary for the scientists' motivation (as Elkana also believes), some form of relativism is required to permit the scientist to change his mind. By referring to the alterations in Einstein's own "picture of reality" as he worked through the problem, Infeld seems to tacitly accept the existence of a framework within which Einstein can conceive of gravitational waves as both existing and not-existing. Such a framework permits the scientist to conceive several possibilities for the same reality, and seems to have a relativist quality to it.

This historical/philosophical debate on realism has a connection with the history of gravitational waves, as a result of the studies of the Weber controversy by Harry Collins (1975, 1981, 1985, with Trevor Pinch 1993), a noted advocate of social constructivism. Collins finds evidence in this study for what he calls the *experimenter's regress*, an impasse reached in experimental science during periods of controversy, when the only test for whether an experiment is working correctly is whether it gets the right result (does or does not detect gravitational waves), and the only way of knowing which is the right result is by reference to the output of a properly operating detector. Collins' analysis has subsequently been criticized by the physicist and philosopher of science, Allan Franklin, who contends that the regress does not exist, and that it was in fact possible for the experimentalists involved in the Weber controversy to decide which detectors were operating correctly, on the basis of what Franklin describes as "rational" or "reasonable" criteria.

In discussing the Experimenter's Regress, David Gooding (1990) introduces an ancillary concept, the *theoretician's regress.*

> Some try to defeat the regress by offering a way around the Duhem-Quine thesis[6] - such as a less naive view of the relationship of theory to experiment - as Franklin does.[7] I ... doubt that the epistemic warrant of experiment can be saved by insisting on logical or probabilistic relationships between observation and theory. The experimenter's regress has its counterpart in the theoretician's regress. Many authors have pointed out the consensual basis for judgements about the logical compatibility of observations and predictions: a test is a test only if scientists decide that it is. The use of mathematics and logic involves judgement too. It is easy to envisage a theoretician's regress in which skilled processes of modeling, inference-making, and so on, are criticized *ad infinitum.*

As we shall see, the skilled process of constructing approximation schemes designed to calculate radiation reaction effects within GR theory was in fact criticized vigorously by other relativists for several decades. In direct analogy to the experimenter's regress, the difficulty was encountered that there was no agreed upon test to determine what method or calculational algorithm would be the correct one to employ. Furthermore, attempts (such as those made by Jürgen Ehlers) to construct

[6]The Duhem-Quine problem is that an experimental falsification of the predictions of a theory does not logically imply that the theory is incorrect. There will always be additional assumptions and auxiliary hypotheses which are part of the understanding of the experimental results, but which are not part of the tested theory proper. The negative result of the experiment may falsify the theory, or the auxiliary hypotheses and assumptions.

[7]In Franklin (1986), we find that "experiment can provide us with reasons for believing a theory [since] the observations help one to decide between competing theories or help to confirm a theory [and] the accepted theoretical explanation then provides some support for the experimental results."

such a consensus did not meet with more than very partial success.

One response to the regress was to insist on the primacy of method, and to attempt to construct a "consensual basis" for judging between different algorithms. Another was to place less severe requirements on method, but to place a certain value on the ability to produce a reasonable result, as discussed above. This alternative permitted a bootstrap approach, in which initial attempts with only limited formal validity (but justified by, for instance, analogy with a more fully understood theory) nevertheless provided a result or results which acted as a guide for subsequent, more sophisticated attempts. As long as the experience gained by the successive calculations based on and elaborating on the earlier ones continued to support the "canonical" result, it in its turn would continue to lend support to them. A calculation which, on the other hand, failed to derive the expected result, would be suspect unless and until some explanation or confirmation could be given which was deemed satisfactory. An objection to this approach which was frequently raised was that if the initial calculations were admitted to be insufficiently rigorous, then it was potentially misleading to place any reliance on the subsequent result. Additional arguments for the validity of the accepted result, such as the analogy with electromagnetic theory mentioned above, were also rejected as insufficient by critics. One other discernible attitude was to regard the regress itself as a natural and necessary process, a part of "normal" science, which in no way had to, or ought to, be defeated.

The comparison between the experimenter's and the theoretician's regress points to a deeper analogy between theory, as practised in this case, and experiment. No exact solutions of the Einstein equations of GR were derivable for the central case of a binary star system radiating gravitational waves, and therefore various approximation schemes were employed. These calculations, conducted with the practiced skill and experience of the theoretician, intended as probes of the notional "reality" represented by the full-blown theory can be seen as a sort of experiment. The analogy is fruitful in so far as it points out a way of viewing this episode of physics from a historiographic point of view. Much recent work in the history of science has rebelled against the theory-first picture of science, and has pioneered a more "realistic" account of experiment, as in books like Peter Galison's *How Experiments End* (1987) and Franklins' *The Neglect of Experiment* (1986). These books not only redress an overly theory-dependent picture of the history of science, as they advertise, but also present a detailed picture of the *practice* and *culture* of a group of scientists. This "in the trenches" perspective on science, and especially experimental science, has also informed the contemporary field of sociology of science, exemplified by books such as Andrew Pickering's *Constructing Quarks* (1984), and Harry Collins' *Changing Order* (1985). The example of these studies can be absorbed and reapplied to the study of the culture and practice of *theory*. If, as Pickering says, the previous histories "highlight the evolution of concepts, not laboratory practice," it is reasonable to conclude that they have also neglected somewhat the evolution of practice in the theorist's office. This study is certainly concerned with conceptual issues in GR, but the context is that of the evolution of a body of theoretical *practice*, in the environment of a very particular theoretical *culture*. A culture which formed a niche-like micro-environment within (or even at times outside) the broader body of theoretical physics. An issue which provides some framework to the story being told indeed, is the desire of this particular group to achieve a greater role within the larger social body, while still retaining their own characteristic identity.

This question of the place of relativists within the larger theoretical physics community, and their sense of their own identity, reflects another aspect of the issue of analogy. When Einstein so carefully constructs his analogy between GR and electromagnetic theory, he is not merely interested in its usefulness is an approximate method of dealing with his new theory (though that was an important concern, since the unfamiliarity and complexity of the non-linear field equations of GR for physicists presented a technical barrier to finding exact solutions of the equations). He was also addressing one of the original motivations behind his efforts to construct GR theory. The existence of *radiation, propagating with a speed equal to the speed of light*, in a *field theory of gravitation*, demonstrated Einstein's success in constructing a relativistic field theory of gravitation, which conformed to the emerging physics of special relativistic electrodynamics. Newtonian gravity, which did not conform, for instance, to the transformational properties of Einsteinian (as opposed to Galilean) relativity, needed to be supplanted by a new theory which was in harmony with the new physics. In this paper, I use the term syncretism to refer to this impulse of physicists to reconcile or combine the tenets of two distinct bodies of theory.[8] Certainly the syncretic impulse was a principal motivator in the origin of the idea of gravitational waves, as one can see in Poincaré's attempts to situate the phenomenon in the context of "unified" theories of gravitation and electrostatics.

Indeed, by founding a theory of gravity which resisted the unification aims of many field theorists from Maxwell on, Einstein may have greatly helped the cause

[8] The theological connotations of this term do not perhaps lead us as far astray as one might think, when one considers the millenialist and Messianic tone of many recent popular accounts of "the Final Theory" by renowned theoretical physicists.

of gravitational wave theory. His theory, while it generalized relativistic principles to gravity, in fact placed the gravitational force on a radically different conceptual footing from the electromagnetic force, depicting it as a fundamental quality of the geometry of space and time, the *arena* in which other interactions, such as electromagnetism took place. This reflected his own sense that the two fields were quite different, as shown by his insistence on the primacy of the equivalence principle in his theory. Had the syncretic movement, which had led to such suggestions as Lorentz' that gravity might be the consequence of an imbalance between electromagnetic repulsion and attraction (Van Lunteran, 1991) persisted, gravitational waves might have been viewed as simply a type of electromagnetic wave which happened to be emitted by chargeless bodies. In this way, the analogy might have become simply an identity, to the detriment of any interest in gravitational waves *per se*. Therefore it may be said that Einstein preserved the analogy by rescuing gravity from the embrace of the field unification.

It is worth mentioning here also another crucial comparison attending the birth of GR, that with the Newtonian theory of gravity which it superseded. In order for GR to be successful, it was necessary for it to simultaneously make a radical departure from the nature of this theory (an action-at-a-distance theory), while inheriting the mantle of its enormous empirical success. This was accomplished by an unusual form of analogy, in which the new theory was said to reduce to the old one in a limit in which certain parameters (such as the inverse of the speed of light) went to zero or became very small. This enabled the new theory to appropriate those empirical supports (the vast body of astronomical data, for instance, and the theory of celestial mechanics that went with it) of the old theory that were vital to its credibility, while at the same time justifying its own existence by improving upon the old theory where it was felt to be failing, experimentally or conceptually. Just as much as the analogy with electromagnetism, this analogy with Newtonian gravity underpinned much of the work on gravitational waves, since it was this analogy which would be frequently used to approximate the motion of gravitationally bound sources. This type of analogy, in which an older theory is claimed to be an approximation of the theory which supplants it, is also found in the relationship of quantum to classical mechanics, for instance, in Bohr's correspondence principle (Darrigol, 1992).

Despite the success of GR in supplanting Newtonian theory as the normative theory of gravity, attempts at unification with other field theories have persisted. The quantum revolution, and the revelations of nuclear and particle physics experiments kept field theorists sufficiently occupied for half a century that unification was low on their agenda. Nevertheless, in the late 1950s, a number of important quantum field theorists such as Richard Feynman and Paul Dirac turned their attention to the project of constructing a quantum theory of gravity, to bring this force into line with the radically new physics of that era. Many relativists also saw this as an important goal of gravitational theory. Since quantum ideas had first arisen in the context of electromagnetic radiation, it seemed reasonable to expect that quantization of the gravitational field should proceed through first quantizing gravitational waves. This provided an important motivation for developing the theory of gravitational waves, although despite some formal attempts to recast the subject in a more quantum theoretic form (the introduction of the graviton as the mediator of gravitational waves), it did not lead to many breakthroughs in the development of quantum gravity. Curiously, some hostility to the syncretic impulse in this instance is discernible amongst some relativists. Their self-identity as a group was threatened by the activities and attitudes of the unifiers. Ambivalence about the gradual emergence of GR into the physics mainstream, regarding on whose terms this should be accomplished, thus may have fed into attitudes regarding the existence and nature of gravitational waves.

Chapter 2

The Prehistory of Gravitational Waves

Although gravitational waves are almost exclusively a 20th century idea, difficult to conceive of before the birth of the modern theory of relativity, the concept was nevertheless prefigured in certain ways before Einstein's general relativity (GR) theory of 1916. During the 18th century, the theory of celestial mechanics based on Newtonian gravity was developed to a pitch of perfection which was as much a popular exemplar of the triumph of science in its day as that of Einstein's theory of gravity was to be in a later one. Oddly enough, the most signal triumph of each theory, in which a famous intractable problem of celestial mechanics was overcome by a prodigious intellectual feat, was in each case preceded by attempts to explain the anomaly in terms of what we would now call gravitational radiation damping.

The first intractable puzzle was the problem of the Moon. The theory of universal gravitation was first applied to the problem of the Lunar orbit by Newton himself, much to his own dissatisfaction. He later recalled that "his head never ached but with his studies on the moon."[1] The Moon's motion presented a number of calculational difficulties for Newton's gravitational theory. First of all, the Moon's orbit around the earth is more like a true binary system orbit than any of the planetary orbits. The Moon and the Earth being of rather similar masses, they can both be said to orbit a common point, rather than the satellite approximately orbiting the central body, as with each of the planets and the Sun.[2] This presents a true "two-body problem," which Newton was fully capable of solving. More problematic was trying to account for the deviations, caused by the attraction of other gravitating masses, from the orbit demanded of a closed two-body system by Newton's theory. The Sun wrestles with the Earth for influence over our satellite, an influence which is continually altered by the motion of the Moon away from and towards the Sun, and by the variations in the Earth's own orbit around the Sun. Johannes Kepler was the first to suggest that the Sun exerted an attraction on the Moon which was responsible for some of the variations in its motion. In the finest detail, one must even account for the influences of some of the other planets, principally Venus. This "many-body problem" has never been solved in mechanics, except approximately, via perturbation theory. Since this approach demands that the perturbing forces be small fractions of the central force it is not especially well suited to the lunar orbit, in which the Sun's pull is an appreciable fraction of the Earth's attraction.

Indeed, the problem of the Moon may have helped convince Newton that the

[1] No doubt his acrimonious relationship with John Flamsteed, the astronomer royal, over the ownership of the data he chiefly relied upon, contributed to his headache.

[2] The center of gravity of the Earth-Moon system lies some thousand miles beneath the Earth's surface, still a considerable distance from the center.

Solar system could not be stable to many-body perturbations, and that Deistical intervention would be required to restore stable initial conditions through regular interventions over the millenia. Amongst English theologians of Newton's day there was great resistance to the idea of an eternal universe, which was associated with atheistical thought and the mechanical philosophy of Descartes. The strong Millenialist tradition in 17th century puritan England was no doubt greatly responsible for this outlook. Although continental thinkers such as Leibnitz viewed ideas of a finite "imperfect" Cosmos as an insult to its creator, many devout Englishmen feared the construction of a universe in which God would have no reason for existence as an open invitation to atheism. Some English philosophers even resisted the doctrine of inertia, preferring to rely on God to maintain motion in the world (Kubrin, 1995). Newton, who was himself fascinated by millenialist ideas, appears to have shared this English prejudice. Indeed, he may have viewed the disorder arising from many-body perturbations as a literal form of dissipation, by which the amount of motion in the solar system would inevitably decrease (Kubrin, 1995). He played with various mechanisms by which God would eventually step in, perhaps via some mechanical process, and reform the cosmos.

The first astronomer to uncover actual evidence of long term alteration in the celestial motions was Edmond Halley, who examined records of medieval solar eclipses made by the Arab astronomer Al-Batanni (known to the Latins as Albategnius), as well as ancient eclipses reported by Ptolemy, and discovered apparent discrepancies of the order of an hour in the eclipse times, calculating backwards from contemporary lunar positions on the basis of the known lunar period. Halley speculated that, if the accuracy of al-Batanni's latitude estimations could be ascertained, there was evidence for a "secular longitudinal acceleration" of the moon, meaning that the moon must, in the earlier epoch, have been moving longitudinally (i.e. across the sky from east to west) more slowly than it was in Halley's own time. This effect, if it existed, would be of considerable interest, since it would show secular and not periodic change in the motion of one of the principal celestial bodies. Halley himself speculated that the change might be due to an increase in the mass of the earth, the consequence of Newton's idea that the earth attracted the aether of space (visible in the form of cometary tails) into itself by the force of gravity, and thus continually augmented its mass.[3]

That such a speculation should first be made in England is not surprising. Indeed, Halley was forced to defend himself, it seems, against the charge of upholding the doctrine of the eternal mechanical cosmos, when he was a candidate for the Savilian chair of Astronomy at Oxford in 1691 (Armitage, 1966). Although he was not awarded the post, the charges against him appear to have been unfounded (it was also alleged that, like Newton, he was a Unitarian). Certainly he was the only scientist or philosopher of the time able to advance, by his historical analysis, evidence for the "decrease of motion" in the solar system, and therefore of decay in the cosmos. Given how little evidence he had to go on, we may wonder if he was rather predisposed to a conclusion which showed evidence for signs of cosmic decay.

However sharp the difference in outlook between England and the continent may have been, Halley's secular acceleration of the Moon did become an accepted fact in 18th century astronomy, after further contributions from the English astronomer

[3] Although perturbations and increased mass would not now be thought of as, in themselves, dissipative effects, leading to dynamical decay in the sense of loss of momentum from a system, to Newton and Halley they clearly did. Both seem to have associated such effects with "loss of motion" and dynamical decay (Kubrin, 1995).

Richard Dunthorne and by the German Tobias Mayer and the French J.J. Lalande. (Armitage,1966) This intriguing historical effect became one of the best known puzzles in celestial mechanics up to the present time.

Prizes played an important role in the economy of 18th century science, and especially in the development of the theory of the moon. The most lucrative prize was the British government's offer of up to £20,000 for a method of accurately determining longitude at sea. A number of methods which had been suggested over the years involved celestial observations from on board ship, and since the use of a telescope was impractical from the heaving deck of a ship underway, it followed that the Moon, readily visible with the naked eye, was the best independent celestial clock available. However, even Newton's lunar theory was inadequate to predict the erratic motions of the moon to the required accuracy for navigational reference. Either better observations of the full 18-year lunar cycle or an improved lunar theory were required.

Within the purely scientific sphere, the Paris Academy of Sciences offered prizes for solutions of problems outstanding in Newtonian gravitational theory several times in the 1760s and 1770s. One of these problems, involving Newton's formula for the motion of the perigees, had already led Leonhard Euler to suggest that a modification of the basic Newtonian theory might be necessary to save the phenomena. This proved unnecessary, after Alexis Claude Clairaut and Jean Le Rond D'Alembert, following years of acrimonious dispute between them over the prize, each produced a solution (Peterson, 1993). Euler was obliged to have his own St. Petersburg Academy offer another prize, tempting Clairaut to resubmit his solution, before he could discover its nature. Euler himself received a modest partial share of the longitude prize for his theoretical work, together with Tobias Meyer, for his observational work.

The Paris academy's prize of 1773 sought an explanation of the secular acceleration of the moon, whether as a result of perturbations produced by the sun or planets, or by the non-sphericity of the earth. Once again the problem proved a tough nut to crack, and indeed Lagrange won it for a brilliant thesis on perturbation theory. His conclusion was that the effect was not explicable by perturbations within the Newtonian theory. Several alternative hypotheses were offered. Euler thought a subtle medium in space retarding the Moon's motion might be responsible, while Kant suggested that a tidal friction of the moon acting on the earth might explain the acceleration, since as the day lengthened all celestial motions would appear quicker as observed from earth (Felber, 1974; Brosche, 1977). However, since the secular acceleration was not observed in the sun, this suggestion was not taken up.

Laplace took a systematic approach to the suggestion that an alteration in the basic theory of gravity might be required. In a paper of 1776 he suggested four fundamental ways in which the theory might be modified: the $1/r^2$ relation, universality, instantaneous propagation and the equivalence of attraction for bodies at rest and in motion. Pursuing the last two suggestions (which are clearly linked), he calculated the effect on a simple orbit of assuming a finite propagation speed of gravity. His conclusion was that it would result in a decrease of the orbital radius, and a resultant accelerated longitudinal motion, but that the effect, if entirely due to this cause in the case of the moon, would indicate a speed of gravity 7 million times that of light. This formidably high speed, still hardly distinguishable from instantaneity, did not provide any compelling numerological motivation to take up the idea of finite propagation.

Laplace's calculation conceives of the gravitational force as mediated by a corpuscle passing between the attracting masses. If the Moon is orbiting the earth, and emits such a corpuscle, it must aim a little ahead of the Earth's present position in order to strike the latter body if it travels with a finite speed. This means it must be emitted not only in a "downward" sense, but in a slightly "backwards" direction (relative to the lunar motion). Since the direction of emission of this corpuscle indicates the direction of the gravitational pull exerted on the moon by the earth, the moon will not only be attracted towards the earth, but also be impeded somewhat in its tangential motion by the emission of such non-instantaneous particles. This loss of angular momentum will force it to move inward in its orbit (falling towards the earth), which in turn makes it appear to increase the rate of its longitudinal motion. The retarding effect clearly depends on the angle the direction of emission makes with that of the instantaneous central force. This is simply v/c, where v is the lunar velocity, and c is the corpuscle's.[4] This is perhaps the first "radiation reaction" calculation in the problem of motion, except that Laplace's corpuscular view of gravitation held no place for the radiation emission side of what has been called "the Laplace effect" of orbital decay caused by a non-instantaneous force of attraction.

It seems likely that the problem of the secular acceleration of the moon continued to attract attention not only because it seemed a possible breakdown of Newtonian

[4] In general relativity the first relativistic correction to orbital motion occurs only at the order $(v/c)^2$, and the first non-conservative correction occurs only at order $(v/c)^5$. Therefore there is no reason to expect this effect to make itself felt in the earth-lunar system.

theory, but because it seemed to be a definite example of decay in the heavens. We have already seen how in the England of Halley's day the doctrine of an eternal creation was viewed in some quarters as a grave heresy. This millenialist style of thought does not seem to have been so prevalent on the continent, and we can speculate that proposals such as those of Euler, Kant and Laplace, which sought to explain the acceleration of the moon by an appeal to dissipative effects, may not have found favour because of an uncomfortable feeling that the world system ought to be eternal. Furthermore, the analogy with the human clockmaker which in Newton's day depicted the deity as an artificer obliged periodically to rewind the mainspring of his creation (or reset the pendulum) and set it once more in ordered motion, while appropriate to the rather unreliable clocks of Newton's day, seemed a much less flattering portrait by the end of the 18th century. The problem of longitude, which had inspired such a detailed study of the moon, in theory and in experiment, had also driven a comparable improvement in the science of chronometry. Indeed, for practical purposes, the problem of finding longitude at sea was solved by an English watchmaker, John Harrison, who invented a series of clocks, followed by a portable watch, which could tell time with marvelous accuracy over months at sea and in the shipboard environment (Sobel, 1995). By the time of Laplace, it no longer seemed appropriate to conceive of a creator so unskilled as a craftsman that he could construct his world system only to see it fall into ruin and disorder over the course of its own action.

Subsequently, however, Laplace discovered a complex perturbative effect which had been missed by Lagrange, which not only gave a non-dissipative explanation for the entire acceleration of the moon, but in fact showed the motion to be periodic, but with a period of millions of years. The perturbation had been missed by Lagrange, who considered that no combination of the other planets and the Sun acting on the lunar orbit could explain the acceleration, because the effect of the other planets acted only indirectly on the Moon. Instead, the net effect of the planets acted on the *Earth's* solar orbit so as to reduce its eccentricity by small amounts over centuries. This change altered, in the mean, the Moon's position with respect to the Sun over its orbit, reducing the net amount by which the Sun tends to draw the Moon away from the Earth. Thus the Moon gradually approaches the Earth, by just such an amount, as Laplace calculated it, as to precisely account for the observed decrease in its orbital period. Eventually, however, the complex effect would reverse itself, and begin once more to draw the Moon away from the earth. Therefore, in his Celestial Mechanics, Laplace was able to present his solution as a tour de force, capping his vindication both of Newtonian theory and the eternal clockwork universe concept, by showing the stability of the system of the planetary orbits against its own perturbations. His "back reaction" calculation was now only presented, when his explanation of the secular acceleration was taken into account, as proving that the action-at-a-distance theory was justified, in view of the absence of any such acceleration of the moon in excess of the prediction of his perturbation theory (indicating a minimum speed for the propagation of gravity of 100,000,000 times that of light) (Laplace, 1825). This result was very well-known in the 19th century, as is evident from Poincaré's paper below, and as is made clear in a recent study of gravitation theories in the 18th and 19th centuries. "During the nineteenth century these calculations were often presented as an (almost) insurmountable obstacle to all explanations of gravitation based upon the action of an intermediate fluid." (Van Lunteran, 1991)

Despite its onetime fame in scientific circles, Laplace's explanation of the acceleration of the Moon has not survived to our own day. In the mid 19th century, the English astronomer John Couch Adams recalculated Laplace's effect, and showed that, owing to the neglect of certain terms which in fact added together to an appreciable sum, Laplace's effect was only half what Laplace himself had calculated it to be. In destroying the perfect agreement with observation of this famous result, Adams precipitated a fierce controversy, with Leverrier, among others, whose passions were further inflamed by the nationalistic rivalry between English and French science then prevalent. However his correction of Laplace did lead to the revival, by Charles Delaunay, Leverrier's leading French rival, of Kant's tidal friction idea. His calculations showed that the slowing of the Earth's rotation by this force could account for the remaining half of the effect. It was not however until this century that a corresponding acceleration of the Sun was observed.

However, it is now known, from laser range finding made possible by a mirror placed by an Apollo mission, that the Moon is in fact receding from the Earth, not approaching it. The explanation for this is found in the phenomenon of tidal friction, which, despite its long pedigree first became generally accepted only 30 to 40 years ago.[5] The Moon, as is well known, raises tides up on the Earth's oceans, both directly below it, and at the antipode of that point. The Earth's rotation however drags the tidal bulges somewhat ahead of these idealized positions (typically about 3 degrees). This is known as a tidal lag, since it means that an observer on the

[5]That there was some competition to the tidal friction explanation of the secular acceleration even in the 1950s is shown by the efforts to find part of it cause instead in a long term change in the gravitational constant, G (Dicke, 1966).

land will see the moon overhead before he experiences the high tide. The near bulge naturally exerts its own gravitational attraction on the moon, which tends to pull the moon somewhat forward of its own position. The bulge on the other side of the planet has a retarding impulse of course, but its effect is smaller, it being farther away. The net effect of the tidal bulge is to impart an increased forward momentum in its orbit to the moon. This increase in angular momentum forces the Moon upward in its orbit, and is gained at the expense of the Earth's rotational angular momentum, which is braked by the tidal bulges attraction towards the moon. The recession of the Moon from the Earth would cause us to observe a longitudinal DECELERATION, except that our own clock is slowing down with the centuries at such a rate that we conclude that the moon is moving faster than hitherto. In other words, the month has presumably lengthened, but the number of days it contains has grown less, due to the lengthening of the day.

It is hardly surprising that Laplace did not have the last word on so complex a subject as the theory of the Moon's orbit, and the same holds true of celestial mechanics and perturbation theory in general. Throughout the 19th century there were continued refinements in the theory and observation of planetary motions. Indeed, it is during this period that the most famous achievement on the subject, the prediction (by Urbain Leverrier and independently by Adams) and discovery (by J.G. Galle) of the planet Neptune. By the end of that century the most prominent anomaly in celestial mechanics was no longer associated with the nearest body to the earth but with the nearest planet to the Sun, Mercury. According to standard Newtonian theory, the unperturbed orbit of a planet should have it return to its closest approach to the Sun at the same angular position in each orbit, but instead, Mercury was observed to shift its perihelion around in its orbit by 43 arc-seconds per century, in excess of what could be explained by perturbations from the other planets.

Various explanations within Newtonian theory which would explain the effect had been suggested over the years, the most famous being the hypothetical planet Vulcan, nearer than Mercury to the Sun, whose perturbative effect on the latter would account for the shift. Vulcan was searched for repeatedly by several astronomers towards the end of the 19th century. One alternative approach, as with the 18th century problem of the Moon's secular acceleration, was to posit changes to the theory of gravity. In 1908 Henri Poincaré, himself perhaps the greatest theorist of celestial mechanics and perturbations since Laplace, made a radical (but tentative) proposal: that the emission of gravitational waves from the orbit of this quickly moving inner planet was removing sufficient energy from its motion as to show up in the form of the perihelion shift. He based the idea on an earlier calculation of another noted French astronomer, Tisserand.[6]

That Poincaré should introduce the idea of gravitational waves at this time was due to the influence of Maxwell's theory of electromagnetism, which successfully predicted the existence of electromagnetic radiation in the mid 1800s. By 1908 the efforts to reformulate Galilean relativity to accommodate Maxwellian electrodynamics had led to a deepening appreciation of the role played by radiation in field theories. Since the speed of light was a key parameter in the new relativistic equations for electrodynamics, it seemed reasonable to speculate by analogy that, if this new form of relativity were to apply to gravity, that there must exist some form

[6] This calculation was also made by Lorentz, who was the first to calculate radiation reaction for a moving charge in the electromagnetic field (Van Lunteran, 1991; Damour, 1982).

of "gravitational radiation," which would propagate at the speed of light. Just as in the electrical case, where an accelerating charge would emit radiation and brake its own motion in consequence, so a massive body orbiting the Sun could be expected to lose energy to some as yet unknown type of radiation. Such an effect might account for hitherto anomalous effects such as the perihelion shift of Mercury. As the nearest planet to the Sun, and thus the fastest moving planet, Mercury could be expected to lose more energy by this mechanism than any other, and thus exhibit most strongly any associated effect on its orbit.

Now, in discussing the question of how and whether the principle of relativity (meaning more or less what we would now call Special Relativity) should be applied to gravitational forces (in Science and Method, the New Mechanics), Poincaré is well aware of Laplace's result that the propagation of gravity must take place at a speed 10,000,000 times that of light, the propagating speed of the electro- magnetic force according to the new Lorentzian relativity. However, he prefers to regard Laplace's result as largely unsubstantiated and instead proceeds to a discussion of the implications of modern relativity for gravitation.

> Are the foregoing theories [Lorentz' unification of the gravitational and electrostatic forces] reconcilable with astronomical observations? To begin with, if we adopt them, the energy of the planetary motions will be constantly dissipated by the *wave of acceleration.* [onde d'acceleration, Poincaré's emphasis] It would follow from this that there would be a constant acceleration of the mean motions of the planets, as if these planets were moving in a resisting medium. But this effect is exceedingly slight, much too slight to be disclosed by the most minute observations. The

> acceleration of the celestial bodies is relatively small, so that the effects of the wave of acceleration are negligible, and the motion may be regarded as *quasi-stationary*. It is true that the effects of the wave of acceleration are constantly accumulating, but this accumulation itself is so slow that it would certainly require thousands of years of observation before it became perceptible. ...
>
> It is in the motion of Mercury that the effect will be most perceptible, because it is the planet that has the highest velocity. Tisserand formerly made a similar calculation ... and found that if Newtonian attraction took place in conformity with Weber's law [a 19th century non-linear theory of electrodynamics], there should result, in the perihelion of Mercury, a secular variation of 14″, *in the same direction as that which has been observed and not explained*, but smaller, since the latter is 38″ (Poincaré, 1908).

Poincaré's own calculation, based on the relativistic theories of Lorentz and Abraham, predicted a smaller effect, again in the same sense as the observed effect, of 7″ and 5.6″ respectively for Mercury. It seems clear that his use of the term *onde d'acceleration* means, in this context, what we would now call *gravitational waves.* This identification is made a little problematic by Poincaré's usage of the same term to describe electromagnetic emission from accelerating charges, and by his employment of a unified theory of gravitation and electromagnetism, but, as the context is the extension of the relativistic principle to gravitation and accelerated motion, we can be quite confident of his meaning.

> To sum up, *the only appreciable effect upon astronomical observations* [of extending the principle of relativity to gravitation] *would be a motion of*

> *Mercury's perihelion, in the same direction as that which has been observed without being explained, but considerably smaller.*
>
> This cannot be regarded as an argument in favour of the new Dynamics, ... but still less can it be regarded as an argument against it.

In the end, of course, the perihelion shift (like the lunar secular acceleration before it) proved to be a conservative effect, whose explanation in post-Newtonian terms by Einstein's new general theory of relativity was the most striking initial achievement of the theory, and helped make it the most famous scientific achievement of the century, and Einstein its most famous scientist. Unlike Laplace's explanation of the Moon's secular acceleration, Einstein's remarkable result has not since been overthrown. In 1938 Einstein and his collaborators (see below for a discussion of the Einstein, Infeld, Hoffmann paper) produced a post-Newtonian theory of orbital motion based on General Relativity, while another colleague, Howard Percy Robertson, employed the new scheme to recalculate the Mercury perihelion shift, again agreeing with the observed excess from perturbations (Robertson, 1938). Attempts were made in the 1950s by Dicke and others to explain part of the effect as the result of a large quadrupole distortion of the Sun, which would have thrown out the agreement with general relativity, and instead perhaps vindicated the rival Brans-Dicke theory (Brans and Dicke, 1961), but to date such efforts have not been successful (Will, 1993).

Chapter 3

Early History of Gravitational Waves

In 1916, in a paper exploring the physical implications of the final version of his general theory of relativity, Einstein proposed the existence of gravitational radiation as one of its important consequences (Einstein 1916). Although both Maxwell and Poincaré have been cited as anticipating the idea of gravitational waves (Havas 1979;Damour 1987), Einstein's was the first concrete description in a relativistic field theory. In a subsequent paper of 1918, Einstein corrected an errors in the 1916 paper which led him to derive an incorrect formula for wave emission by a source, and went on to calculate correctly (bar a factor of two) the flux of energy carried by the waves far from their source (Einstein 1918). Appealing to the principle of conservation of energy, he assigned an equivalent loss of energy to the source system, an effect already familiar from electromagnetic theory, nowadays known variously as "radiation reaction," "back reaction" or, in cases involving the decay of periodic motion such as orbital motion, "radiation damping." Because Einstein's formula for the energy emission depended on changes in the mass quadrupole moment of the source, it became known as the quadrupole formula. In deriving the formula, Einstein made use of a linearized version of his field equations both for ease of manipulation and because of its strong analogy to the field equations of electromagnetism. Not surprisingly, therefore, his quadrupole formula was itself similar in form to the multipole radiation formulas of electromagnetism, in which field, however, the lowest order of emission is the dipole.

In general there are two distinguishable approaches to the back reaction problem. The first, and generally the simpler, is the energy balance argument used in early derivations of the quadrupole formula (Einstein, 1918; Eddington, 1922). This approach has been criticized in principle on several counts in the context of general relativity, but was an obvious choice for a first approximation.

The second approach, more direct but much more complex, is to iteratively calculate the effect of the source's own field (changing because of the source's motion) upon the source's motion, to which corrections can then be reapplied to calculate the field more accurately. This iteration is carried through one or more steps until it is judged that the reaction effects have been calculated to the desired level of accuracy. This problem is part of a more general one known as the problem of motion. Laplace's method, which took into account the deflection of the Newtonian central force on an orbiting body as a result of the time lag in propagation, was a "one-step" calculation of this type. A key issue in this approach is the fact that the field, in the case of finite propagation, is "retarded," which is to say that the field experienced at a given point in space, at a given time, is not that produced by the source at that time, but that of the source at an earlier time, where the difference between the two times is the time of propagation of the field changes from the source's retarded position to the field point in question. As Laplace showed, an orbital decay would be one consequence of introducing retarded propagation instead of dealing with instantaneous propagation.

In the pre-war period gravitational waves did not receive much attention as a prediction of GR, despite the high public profile of the theory in the years immediately following its publication. GR's early reputation rested on the few experimental tests comparing it to the predictions of "Newtonian" gravitation theory. gravitational waves, though a radical departure from the classical gravitation theory, were not observable, and moreover were initially investigated in the context of a straightforward application of electromagnetic field theory ideas to GR. Weyl's text book (1921) did give an early treatment of gravitational waves, following Einstein in discovering three types in the linearized theory, which Weyl categorized using the nomenclature transverse-transverse (TT), transverse-longitudinal (TL) and longitudinal-longitudinal (LL) waves. He did not notice that Einstein had disposed of the later two types as the spurious consequence of choosing a co-ordinate system in which the formal analogy between the linearized equations of gravity and the field equations of electromagnetism would be most apparent. A few papers in the 1920s made a good start at elaborating a theory of gravitational waves, but only one of them had any lasting impact at all, and interest soon seemed to wane in the problem, reflecting no doubt the general rejection of GR as an active field in physics at this time (Eisenstaedt, 1986a and b).

The most important 1920s paper on gravitational waves is that of Eddington in

1922. Eddington finds Einstein's 1916 and 1918 papers somewhat lacking, in that he feels that Einstein is imposing a condition that the waves travel with the speed of light. Essentially he feels that Einstein is forcing the analogy with electromagnetisim by his choice of co-ordinate conditions. The result of this is Weyl's error in including spurious undetectable waves along with waves of a possibly more corporeal existence. He shows in fact that TL and LL waves can be made to travel with any speed by the appropriate choice of co-ordinates, and notes wryly that such waves travel with "the speed of thought." He adds that both of these types of waves are associated with a vanishing Riemann curvature tensor, so that it is again only the choice of co-ordinates which gives one the impression of a disturbance in the field existing at all. TT waves, on the other hand, are associate with a genuine disturbance (in the curvature sense) propagating with a definite velocity, in the linearized approximation.

Eddington then considers the case of the emission of gravitational waves by a material system. He first of all notes that spherically symmetric disturbances cannot emit gravitational waves, and that this is a breakdown of the analogy between sound waves and gravitational waves. Proceeding to the case of a rod spinning end over end, he rederives Einstein's quadrupole formula for the loss of energy by the rod (again by an energy balance argument), but correcting an error of a factor of 2 in Einstein's 1918 equation. However, with typical caution, he adds that his linearized analysis is not applicable to the problem of a binary star system,

> but it seems likely that the radiation (if any) will not exceed that given by [the quadrupole formula]. There is clearly no practical objection to the existence of this small radiation from rotating systems, and I can see no

theoretical reason for not admitting it.

Eddington, the great early popularizer of GR had a strong influence on the subject of gravitational waves. Bondi regarded him as the inspiration for his particular brand of scepticism (Bondi 1990; interview). His remark about the "speed of thought" has left the indelible image of him as a sceptic in the folklore of GR. At the same time, his identification of the curvature tensor as the key, co-ordinate invariant quantity to define the existence and effect of gravitational waves prefigured the modern picture of the phenomenon. Some commentators have reacted against the perception of Eddington as a sceptic by observing that the waves whose existence he disproved *are* held to be unphysical by the modern theory.

> What Eddington said to distinguish these fictitious (coordinate) waves from the real transverse-transverse gravitational waves was unfortunately misunderstood by certain other investigators and taken by them as an argument against the reality of all gravitational waves." (Rees, Ruffini and Wheeler, 1974; pg. 90)

However, it is perhaps too ahistorical a viewpoint to define sceptics only by reference to the modern orthodoxy. Eddington doubted the existence of waves described in the textbooks of his day. Moreover, his outlook was marked by a sceptical attitude towards easy arguments based on the analogy with electromagnetism. It is perhaps not inappropriate that he seems to have gone into the informal tradition as a figure in the sceptics camp.[1]

Another English contribution of the twenties is the paper of Baldwin and Jeffery

[1] In some review talks on gravitational waves attended by the author, such as one by Ed Seidel, Eddington is presented as the paradigmatic sceptic of gravitational radiation.

(1926) on plane waves. Following directly on Eddington's paper they presented an *exact* solution for plane waves, but noted that it was impossible to avoid a singularity in the metric for a plane wave front of infinite extent, concluding that it would be necessary to describe both gravity and light waves by divergent waves (outward spreading) rather than plane waves. The curious fact that one cannot describe a plane wave front of infinite extent in the exact theory by a single co-ordinate system without finding an apparent (but not real, in the sense of curvature) singularity somewhere in the metric (see Misner, Thorne and Wheeler, 1973 pg. 958) was to prove problematic later for Einstein and Rosen.

Another interesting paper of this period is that of Guido Beck (1925), a Viennese physicist who worked on GR for his doctorate under Thirring. His thesis included a presentation of a metric describing cylindrical waves. Like the Baldwin and Jeffrey paper this discussed an exact solution for plane waves, but having perhaps not had the benefit of reading Eddington's paper, Beck continues to include in this class of waves solutions which in fact have vanishing curvature (Scwimming, 1980). A larger problem for Beck was the emerging hostility to GR amongst some physicists. His paper on this subject was rejected by the *Annalen der Physik* because "general relativity was not physics and that [this] periodical was too good to deal with such stuff" (Havas, 1995). The paper was, however, accepted by the *Zeitscrift für Physik*, but that it received little attention is shown by the circumstances of the Einstein and Rosen paper on cylindrical waves to be discussed below. Not surprisingly, Beck did not continue in the study of GR, although one of his students does play a large role in the later history of gravitational waves.

In short, despite a certain amount of activity in the early 1920s, the theory of gravitational waves made little progress in the first two decades of GR. Even some of the contributions which were made, especially Beck's, were forgotten during the interstice between 1925 and the re-emergence of interest in GR in 1955 (see Eisenstaedt, 1986a and b). Undoubtedly the generally hostile climate towards GR which seems to have existed within physics in mid-century is largely responsible for this. With no experimental significance that anyone could see, gravitational waves would have to wait until issues of principle encouraged their study. Such issues included, to begin with a desire to elucidate GR theory itself, especially in ways which distinguished it from Newtonian theory, rather than merely marginally correcting the classical theory, and also a growing interest in quantizing the gravitational field. Because the disinterest of many physicists left the development of the theory in the hands of pure and applied mathematicians to a great extent, the subject took on a peculiar coloration of its own which continued to mark it off from the mainstream of 20th century physics.

The last notable contribution to the problem of gravitational radiation before the second World War did continue to wield great influence after the war. This was the treatments of gravitational waves and radiation reaction in *The Classical Theory of Fields* by the Russian theorists Lev Landau and Evgenii Lifshitz (1951), one of the classic textbooks of 20th century physics. Because this book did not confine its attention to general relativity or gravity, and because of its widespread use across many subsequent editions in many languages, this derivation of the quadrupole formula is undoubtedly the best known of any that have been published, probably by a wide margin. It is also easily the most sophisticated of the pre-war derivations and claims, unlike Eddington's calculations, validity for the case of self-gravitating systems, such as binary stars. Nevertheless, amongst relativists, it has a remarkably mixed reputation. While for many it is the standard demonstration of the formula, for others it is unconvincing in its claim of applicability to the case of freely-falling systems.

In some sense, the Landau and Lifshitz derivation divided the post-war relativity community into two camps. Those who felt that a reasonable estimate had been given of the gravitational radiation from binary star systems, which might serve as a guide in future work relating to the problem, and those who held that none of the pre-war work could be taken as any sort of guide to the damping of binary systems by gravitational waves.

Chapter 4

Analogy as the Inspiration of Gravitational Waves

One of the most fruitful of all analogies in the history of physics is that between light and water waves (frequently with sound as an intermediate step in the simile) which underlies the various wave theories of light. This analogy follows a familiar pattern of constructing a metaphor between the theory of a phenomenon and some concrete example taken from ordinary experience, upon which a more abstract mathematical framework for the theory can be constructed. This process reached its modern fruition, in the case of the wave theory of light, with the construction and elaboration of the Maxwellian field theory of electromagnetism in the second half of the 19th century. The extraordinary success of this theory in integrating the concepts of radiation and force naturally prompted an analogy with the Newtonian theory of gravitational force, inspiring some physicists, such as Maxwell and Poincaré, to speculate about the existence of gravitational waves. On the philosophical level, this analogy remained quite vague. The analogy between radiation phenomena in the electromagnetic and gravitational fields only received concrete expression in the earliest elaboration of the first successful field theory of gravity, that of Einstein. Even here, it is interesting to note, the analogy remained at an abstract level, realized only in the mathematics. It was no longer necessary to appeal to the physical metaphor (water waves or sound) as it was quite sufficient to reduce the equations of the new theory, as Einstein did, to a form in which an explicit analogy could be drawn with the equations of Maxwell, and upon this resemblance construct an analogous description of the new "phenomenon". Whereas the original underlying analogy consisted of correspondences drawn between tangible phenomena, with subtle matter conjured up to fill in the gaps in the physical metaphor (light waves $\equiv$ ripples of water or sound, ether $\equiv$ water or air), the new approach facilitated by the success of Maxwellian theory drew the comparisons between conceptual quantities (not descriptive qualities) represented algebraically in the equations (mass $\equiv$ electric charge, metric perturbation $\equiv$ electromagnetic vector potential). The new analogy thus described an abstract relation between mathematical quantities, with little or no attempt at metaphorical illustration.

This analogy, based partly on the syncretic impulse of physicists to set gravitational theory on the same footing as had been adopted for electromagnetic theory, has also proved powerful. Although no experimental evidence in support of the existence of gravitational waves came to light until the 1970s, the doctrine that such a phenomenon existed was fairly quickly adopted in physics, and became more or less an accepted part of the 20th century physical canon, as much by non-relativists as amongst those active in the field.

So while the causes of such widespread faith in a physically unverified hypothesis are interesting in themselves, it is just as interesting to focus on the scepticism which did exist in relation to gravitational radiation, and to examine its motivation, exposition, and the response it engendered. The word sceptic used here has a double meaning, referring both to those who publically critiqued the existing theory of gravitational waves and to those who advanced the view that gravitational waves, or some important feature thereof, did not exist, contrary to the orthodox belief. There is some degree of overlap in the two usages, in that those who merely criticized the theory were not always, or may not have always been agnostic on the question of whether the phenomenon really existed as the theory described it.

While the analogy with electromagnetism certainly played an important role in underpinning the case for gravitational radiation in GR theory, its rhetorical use in papers dealing with gravitational waves was usually limited to at most a brief opening paragraph. In papers advancing a sceptical position however, there was an obvious need to expose the breakdowns in the analogy. It is a natural feature of all analogies to have some points where the correspondence does not hold, and for the analogy to be accepted as useful these imperfections in the metaphor must be seen as essentially unimportant to its purpose. Hence, the fact that the original "algebraic" analogy with Maxwell's equations held only for the *linearized* Einstein equations was regarded by most sceptics as a critical flaw. For them, the non-linear nature of gravity set it apart from electromagnetism, and made the analogy unfit in certain important contexts, most especially in the case of a binary star system, which was and still is expected to be one of the few physical systems capable of producing any significant quantity of gravitational radiation. As a system held together by strong gravitational forces Eddington observed how unsuited it was for description by linearized gravitational theory. Other favourite topics for the sceptics were the equivalence principle, of central importance to GR, but unknown in electromagnetism theory, and the curious fact that GR defines its own equations of motion, without the introduction of an outside force law, again unlike electromagnetism.

The role played by "theoretical syncretism" in the birth of gravitational waves is worth noting. The late 19th century was an era of many attempts to unify the gravitational and electrostatic or electromagnetic forces. This naturally encouraged the idea of gravitational radiation, yet at the same time discouraged any great interest in it as a separate phenomenon. General relativity, both a realization of the syncretic ideal in so far as it was a field theory of gravity, and a denial of it, in that it gave gravity an altogether different axiomatic basis from electromagnetism, therefore forced the idea of gravitational waves onto the stage by *creating* the analogy. gravitational waves were something *distinct* from electromagnetic waves, but *analogous* to them. A unified theory could well have incorporated gravitational waves merely as a special type of electromagnetic radiation, emitted by chargeless bodies. By marrying the field picture of gravity to the Riemannian geometric conception of space, Einstein paved the way for the full-blown modern depiction of gravitational waves as propagating field disturbances, and "ripples in the curvature of spacetime," departures from Euclidean flatness of space which are pictured as moving across the ocean of space as ripple across a pond.

Chapter 5

The Einstein-Rosen Paper

In a letter to to his friend Max Born, probably written sometime during 1936, Albert Einstein reported

> Together with a young collaborator, I arrived at the interesting result that gravitational waves do not exist, though they had been assumed a certainty to the first approximation. This shows that the non-linear general relativistic field equations can tell us more or, rather, limit us more than we have believed up to now. (Born 1971, p. 125)

The young collaborator was Nathan Rosen, with whom Einstein had been working for some time, producing papers on several topics. They had submitted a paper to the *Physical Review* based on the work referred to in Einstein's letter to Born under the title "Do Gravitational Waves Exist?"[1] and the answer they proposed to give, as the letter states, was no. It is remarkable that at this stage in his career, Einstein was prepared to believe that gravitational waves did not exist, all the more

[1] Although the original version of Einstein and Rosen's paper probably no longer exists, its original title is referred to in the report by the *Review's* referee (EA 19-090).

so because he had made them one of the first predictions of his theory of general relativity. In his autobiography Leopold Infeld, who arrived in Princeton in 1936 to begin an important collaboration with Einstein, described his surprise on hearing of the result (Infeld 1941, pg. 239). Despite his initial scepticism, Infeld soon allowed himself to be convinced by Einstein's arguments, and even came up with his own version of the proof, which reinforced his belief in the result (Infeld 1941, pg. 243). However, not everyone was so easily convinced. When Einstein sent the paper to the *Physical Review* for publication, it was returned to him with a critical referee's report (EA 19-090), accompanied by the editor's mild request that he "would be glad to have your reaction to the various comments and criticisms the referee has made." (John T. Tate to Einstein July 23, 1936, EA 19-088). Instead, Einstein wrote back in high dudgeon, withdrawing the paper, and dismissing out of hand the referee's comments (Einstein to Tate July 27, 1936, EA 19-086):

> Dear Sir,
>
> We (Mr. Rosen and I) had sent you our manuscript for publication and had not authorized you to show it to specialists before it is printed. I see no reason to address the - in any case erroneous - comments of your anonymous expert. On the basis of this incident I prefer to publish the paper elsewhere.
>
> respectfully,
>
> P.S. Mr. Rosen, who has left for the Soviet Union, has authorized me to represent him in this matter.[2]

[2] The translation from the German is by Diana Barkan. The emphasis in the letter is Einstein's.

To this Tate replied that he regretted Einstein's decision to withdraw the paper, but stated that he would not set aside the journal's review procedure. In particular, he "could not accept for publication in THE PHYSICAL REVIEW a paper which the author was unwilling I should show to our Editorial Board before publication." (Tate to Einstein July 30, 1936, EA 19-089). Einstein must have continued in his dislike of the Review's editorial policy (which in fairness may have been unfamiliar to him, the practice of German journals being less fastidious),[3] for he never published there again.[4] The paper with Rosen was, however, subsequently accepted for publication by the Journal of the Franklin Institute in Philadelphia.[5]

What had led Einstein to the conclusion which so surprised Infeld? He and Rosen had set out to find an exact solution to the field equations of general relativity which described plane gravitational waves, and had found themselves unable to do so without introducing singularities into the components of the metric describing the wave. As a result, they felt they could show that no regular periodic wavelike

[3] In a letter to Einstein in March 1936, Cornelius Lanczos remarks on "the rigorous criticism common for American journals," such as the *Physical Review* (translated and quoted in Havas 1993, pg. 112). Infeld claims that the German attitude, by contrast, was "better a wrong paper than no paper at all." (Infeld 1941, pg. 190). Jungnickel and McCormmach (1986) describe the editorial workings of the *Annalen der Physik* in the first decade of this century in some detail. They note that "the rejection rate of the journal was remarkably low, no higher than five or ten percent," and describe the editors' reluctance to reject papers from established physicists (pg. 310). As this was the time and place in which Einstein began his published career, the "rigorous criticism" he was to experience very shortly after receiving Lanczos' letter must have come as something of a shock.

[4] Einstein's bibliography to 1949, given in Schilpp (1949) lists no papers by him appearing in the *Review* after 1936, and the index of the *Physical Review* from then until his death refers only to one short note of rebuttal, mentioned by Pais (1982) in his brief account of the rejection of the Einstein-Rosen paper.

[5] The paper appeared in the Franklin Journal under a different title and with radically altered conclusions in early 1937. That it had previously been accepted in its original form is indicated by a letter from Einstein to its editor on 13/11/36 (EA 20-217), explaining why "fundamental" changes in the paper were required because the "consequences" of the equations derived in the paper had previously been incorrectly inferred.

solutions to the equations were possible (Rosen 1937 and 1955). However, in July of 1936, the relativist Howard Percy Robertson returned to Princeton from a sabbatical year in Pasadena and subsequently struck up a friendship with the newly arrived Infeld. He told Infeld that he did not believe Einstein's result, and his scepticism was much less shakeable. Certain that the result was incorrect, he went over Infeld's version of the argument with him, and they discovered an error (Infeld 1941, pg. 241). When this was communicated to Einstein, he quickly concurred and made changes in proof to the paper which was then with the Franklin journal's publisher (Infeld 1941, pg. 244 and letter, Einstein to editor of the Franklin Journal November 13, 1936, EA 20-217).

Curiously, Infeld states that when he communicated to Einstein his discovery with Robertson of an error in his (Infeld's) version of the proof, Einstein replied that he had coincidentally and independently uncovered a (more subtle) error in his own proof the night before (Infeld 1941, pg. 245). He does tell us that Einstein's position still had to evolve from that of demolishing his proof, to that of reversing it (by showing an exact solution for cylindrical waves), and this was Robertson's key contribution according to Rosen's paper of 1955. Unfortunately, Infeld gives us no details of the false proofs and their correction in his account, which was intended for a popular audience. He does relate the amusing detail that Einstein was due to give a lecture in Princeton on his new "result," just one day after completely reversing his conclusions on its validity. He was forced to lecture on the invalidity of his proof, concluding by stating that he did not know whether gravitational waves existed or not (Infeld 1941, pg. 246).

Although a footnote attached to the published version acknowledges Robertson's help, it does not indicate its nature (Einstein and Rosen 1937). However, it appears that his chief contribution was to observe that the singularity could be avoided by constructing a cylindrical wave solution. In this way the offending singularity would be relegated to the infinitely long central symmetry axis of the wave, where it was less objectionable, being identifiable with a material source (Rosen 1955). In view of this, Einstein might have been better advised not to dismiss the referee's report so hastily, as the anonymous reviewer also observed that, by casting the Einstein-Rosen metric in cylindrical co-ordinates the apparent difficulty with the metric was removed, and it was easily seen to be describing cylindrical waves (Referee's report, EA 19-090, pgs. 2,3,5).

The identity of the Review's referee is unfortunately not known. Few records of the journal exist for this period, and the report has only survived amongst Einstein's own papers. It is 10 pages long and shows an excellent, if not perfect, familiarity with the literature on gravitational waves (the referee knew of Baldwin and Jeffrey's 1926 paper, but not Beck's of 1925). The copy forwarded to Einstein is typewritten and the spelling follows American practice ("behavior" rather than "behaviour", "neighborhood" rather than "neighbourhood"). It is likely, therefore, that the author was an American with a strong interest in general relativity, not a very inclusive category at this time. It is tempting to suspect Robertson himself, but there is nothing to support this in his surviving (and extensive) correspondence with Tate. That Robertson was familiar with the referee's criticisms is shown by his letter to Tate of February 18, 1937 (Caltech archives, Robertson papers, folder 14.6) in which he says

> You neglected to keep me informed on the paper submitted last summer by your most distinguished contributor. But I shall nevertheless let you in on the subsequent history. It was sent (without even the correction of one or two numerical slips pointed out by your referee) to another journal, and when it came back in galley proofs was completely revised because I had been able to convince him in the meantime that it proved the opposite of what he thought.
>
> You might be interested in looking up an article in the Journal of the Franklin Institute, January 1937, p. 43, and comparing the conclusions reached with your referee's criticisms.

This suggests that, in spite of himself, Einstein did benefit from the referee's advice in the end, by a very circuitous route.

If we have to guess at the identity of the referee (assuming it was not Robertson), the likeliest chance is that it would have been someone who was in Caltech, with Robertson in mid-1936, and who would have been picked by Tate to referee a paper on GR. The likliest candidate would be Richard Chace Tolman but there is no documentary evidence to back this up, at least amongst Tolman's papers at Caltech. However, despite the absence of any further correspondence on this paper between Robertson and the *Review*, the evidence of this one letter (especially his chiding of Tate for not "keeping me informed" of the matter) convinces me that Robertson himself was the referee. Once he had returned to Princeton he was able to accomplish in person what had been impossible for him to effect as the *Review's* anonymous referee, and help change Einstein's mind.

In fact the cylindrical wave solution presented in the revised paper had been previously published by the Austrian physicist Guido Beck in 1925, but his paper has been largely overlooked since. In a 1926 paper by Baldwin and Jeffrey, and in the referee's report on Einstein's paper, there was discussion of the fact that singularities in the metric coefficients are unavoidable when describing plane waves with infinite wave fronts, but although there is some distortion in the wave, "the field itself is flat" at infinity, as the referee noted (EA 19-090, pg. 9). In any case, the Einstein-Rosen paper, as published, contains no direct reference to any other paper whatsoever.

Rosen published a paper in 1937 in a Soviet journal, carrying through what is presumably the chief argument of the original version of the Einstein-Rosen paper, in order to show that plane gravitational waves were an impossibility due to the ineradicability of singularities in the metric. In the immediate post-war period, other papers suggested that plane waves were not permitted in General Relativity (for example, McVittie 1955). Felix Pirani and Hermann Bondi were both partly motivated by these papers to work on the problem of gravitational waves.[6]

In the mid-fifties, Ivor Robinson independently rediscovered the plane wave metric and, together with Bondi and Pirani, published the seminal work on the subject. They were familiar with Rosen's paper, and noted that his regularity conditions for the metric were unnecessarily severe by post-war standards. "In effect, Rosen did not distinguish sufficiently between co-ordinate singularities and physical singularities, which could, in principle, be detected experimentally" (Bondi, Pirani and

[6]Interviews with Hermann Bondi and Felix Pirani. Pirani reviewed the McVittie (1955) paper for *Mathematical Reviews* and was dissatisfied with its conclusions (Pirani, 1955).

Robinson 1959).[7]

Although the main topic of the Einstein-Rosen paper had nothing explicitly to do with the back reaction problem, it is very noteworthy as the first serious (if abortive) attempt to disprove the existence of gravitational waves. In an interesting passage addressing radiation reaction, the published paper suggests that one is not compelled to the conclusion that waves emitted by a source must damp the source's motion, if one supposes that any outbound radiant energy is matched by a second system of incoming waves, impinging on the source. In short, they observed that the use of half-advanced plus half-retarded potentials will avoid motion damping in the source system even if the waves exist. "This leads to an undamped mechanical process which is embedded in a system of standing waves," in the author's words (Einstein and Rosen 1937). The paper refers cryptically to the work of Ritz and Tetrode "in former years" relating to the question of advanced versus retarded potentials, and it appears that Einstein often quoted Ritz approvingly in this context (Infeld and Plebanski 1960, pg.201).

Walter Ritz, a Swiss contemporary and friend of Einstein's had complained in his criticism of Lorentz's electrodynamics that advanced potentials (in which the field at time t is that produced by the source from a *future* position) were admitted as solutions of the equations of electrodynamics just as well as the retarded potentials (Ritz 1908). To Ritz, this defied the principle of causality, since effect preceded cause. Just as abhorrent to Ritz were combinations of the two potentials, such as the average of advanced and retarded fields (half-advanced plus half-retarded) which

[7] In their work, Bondi, Pirani and Robinson followed the new approach of Lichnerowicz in imposing regularity conditions on the metric (Lichnerowicz 1955). For a thorough review of the tangled history of plane gravitational waves, see Schwimming (1980).

allowed "perpetual" motion because, like the instantaneous interaction, it produced no motion damping due to back reaction. Ironically, what Ritz regarded as so damning, Einstein appears to imply might have a positive virtue, in the context of gravitation.[8]

The Dutch physicist Hugo Tetrode, also an acquaintance of Einstein, discussed the standing wave potential in a paper of 1922. At the time this solution to the classical wave equations seemed a possible explanation for the failure of orbiting atomic electrons to radiate. Furthermore, as Tetrode pointed out, in the quantum regime, the emission and absorption of radiation seemed to each depend on the other, rather than emission being required for absorption, but not the reverse. This suggested to him that the classical aversion to making absorption a requirement for emission should be discarded. As he put it, "The Sun would not shine if it were alone in the universe" (Tetrode 1922). In their paper, Einstein and Rosen appear to share Tetrode's preference for this potential, if not for his full action-at-a-distance program.

[8] Since general relativity is a non-linear theory, the fact that two potentials (the advance and retarded) satisfy the field equations does not imply that their linear combination (half advanced plus half retarded) would, as it does in electromagnetism. In linearized gravity, however, this obviously does follow.

Chapter 6

The Problem of Motion

Einstein and Rosen's abortive effort to disprove the existence of gravitational waves was followed by a project upon which Einstein and Infeld embarked together with another of Einstein's younger collaborators, Banesh Hoffmann. They wished to develop the post-Newtonian theory of the problem of motion, an ambitious project involving intensive calculations (Einstein, Infeld and Hoffmann 1938). In pursuing the EIH research, Einstein wished to vindicate his earlier conjecture that in general relativity the allowed motions of the particles were completely determined by the field equations (Einstein and Grommer 1927), in contrast to other field theories where a separate force law is invoked.

The problem of motion in GR is essentially the same problem studied as celestial mechanics throughout the 18th and 19th centuries from Newton to Poincaré. The chief difference is that, in GR, unlike Newtonian theory, even the two-body problem for equal sized bodies cannot be solved analytically. The subject got off to an excellent start, however, with the discovery by Karl Schwarzschild (1916) of an exact solution of Einstein's equations describing the gravitational field of one mass, whether a singularity or an extended body. This has allowed perturbation theory to be used to describe the motion of a small particle in orbit around such a body. An alternative approach, in the case of bodies of equal size, was to treat them as weakly interacting, slowly moving masses, thus reducing their motion to that of the Newtonian theory, and then calculating corrections to that motion based on GR, expanding the motion in powers of such small parameters as the system velocities (relative to the speed of light, v/c) and field strength (GM/rc^2). This expansion scheme became known as post-Newtonian, and was developed in the early days of GR theory by Droste (1917) and De Sitter (1916). Both of these approaches led to confirmations of Einstein's early result finding the missing contribution to Mercury's perihelion shift, the single greatest contribution of GR to classical celestial mechanics to date. Einstein himself introduced the linearized approximation in his 1916 paper which also discussed gravitational waves. This later formed the basis for a "fast-motion" (or "post-linear" or "post-Minkowski") expansion of the equations of motion, in which the expansion parameter was the field strength, with the velocities unrestricted.

In short, the two basic approximation schemes in use in GR were based on the two available analogies with existing theory. On the one hand, the previous Newtonian gravitational theory, on the other hand, the special relativistic theory of the electromagnetic field. For the theory of gravitational waves, each of these presented a fundamental problem. Newtonian gravity had never allowed for the possibility of gravitational radiation. Suitable as it was for supplying corrections to the equations of celestial motion for two bodies, the post-Newtonian expansion proved to be both ambiguous and ad hoc in its depiction of radiation effects. The post-linear or fast-motion expansions, on the other hand, while eminently suited to describe radiation, were difficult to handle when applied to the problem of motion of gravitationally bound objects. The linearized theory was not strictly speaking a gravitational theory, lacking as it did the characteristically non-linear features of the gravitational force. In fact, the basic linear metric is quite flat, with no curvature at all.

A thorough account of the problem of motion before the second world war is given in Havas (1989), who concludes that in the post-war period almost all the pre-war work was forgotten or ignored, with the exception of the Einstein-Infeld-Hoffmann work described above. Two important reasons for this are to be found in the extreme dislocations caused by the war, the death and exile of some of the participants, and the failure of some early work to be translated into English until long after it had been written. The fact that Britain, and especially America, were important places of refuge for displaced scientists during the war is obviously greatly responsible for the important shift in the *lingua franca* of physics.

Einstein's own prestige must have contributed to the relative prominence of the EIH method, and we can, with Havas, also assign some credit to the successful promotion of the scheme by Infeld, who was a prominent figure in the life of the post-war relativity community. In any case, contemporary textbooks still cite EIH as the canonical solution of the problem of motion. Among other factors contributing to the eclipse of pre-war work was the unfashionable status of GR immediately before and after the war, which discouraged work in the field, the fact that much of the work predating EIH suffered from minor calculational errors, so that they did not correctly derive the EIH solution, and the fact that Vladimir Fock, who did continue to do important work on the problem after the war was viewed with suspicion by most relativists due to his unorthodox views on general covariance. The language barrier (Fock's book *Spacetime and Gravitation* (1959) was not translated into English until several years after its appearance in Russian) was also a factor in Fock's relatively poor reception in the west.

On the other hand, despite the prominence of EIH as *the* solution of the problem of motion, James Anderson (1995) insists that most subsequent attempts to extend the problem of motion, especially in the direction of radiation reaction (not dealt with in any of the pre-war work) failed to appreciate or take advantage of the best points of the EIH scheme. In his view, EIH, which he counts amongst Einstein's most significant work, was the great "lost" scheme of the post-war period, and the back reaction problem suffered by a tendency to ignore this "new approximation method," as Infeld called it. We may look for the source of the marked disagreement between Havas and Anderson in the fact that Havas was and is a trenchant critic of EIH, whereas Anderson now regards it as the most significant work on the problem of motion in GR.

In the 1940s and 50s, when attempts were first made to extend the problem of motion to the order at which back reaction effects would occur, there were several points at which EIH was felt to be wanting as a productive tool. One was its use of point sources. Hu attempted to adapt the basic scheme with the use of extended sources in his pioneering work in the forties. Another was the whole "slow-motion" expansion scheme. Goldberg (1955) was the first to observe that, as reaction effects entered at such high order in the expansion, the method appeared ill suited to the specific problem. This led to efforts to develop a fast-motion expansion scheme, not limited to slow motions of the source, of which Havas was a principal exponent.

Up until the 1960s another difficulty which plagued the slow motion schemes was the somewhat arbitrary way in which boundary conditions for the problem had to be applied to the equations piecemeal at each step in the expansion. Coupled with the early difference in choice of suitable conditions, this was perceived to contribute strongly to the failure of the various calculations to agree on a common result. This failing was not overcome, in the view of most experts, including Anderson but not Havas, until the work of William Burke on matched asymptotic expansions in the late 60s (for which see later). This problem in imposing conditions reflected the "epistemological" difficulty in describing radiation in the post-Newtonian theory. The problem of choosing the appropriate potential (retarded or "standing wave"), was intimately connected with the choice of approximation scheme. Indeed Damour (1982) regards the choice of potential as the chief characteristic of the different schemes, associating the retarded potential with the "fast-motion" schemes, and the half-advanced-plus-half-retarded ("standing wave") potential with the post-Newtonian schemes. The introduction of "matching" techniques just alluded to eventually permitted the use of different schemes appropriate to different contexts within the same problem, to be reconciled with each other by comparison in a context where both had validity.

Not long after the EIH paper was successfully completed, Infeld, who had with Robertson's help secured a position at the University of Toronto, put his graduate student Phillip Wallace to work applying the EIH formalism to the problem of motion in electrodynamics. In their paper, as also in the EIH paper itself (where radiation effects were not considered), we see a preference for the averaged potential, "half advanced plus half retarded." Infeld and Wallace state that this solution "does not specify a privileged direction for the flow of time" and is besides the simplest for their method (Infeld and Wallace 1940). They note that this solution does not damp orbital motion, and further state that "the addition of radiation seems from this point of view arbitrary," since one must choose the retarded potential to obtain it. This viewpoint partly reflects Einstein's own, but it should be stressed that he was the first to make use of a retarded potential in GR in his seminal 1916 paper on gravitational waves. The solutions which admit radiation damping are objectionable because they involve an arbitrary imposition of the arrow of time into field theories which are otherwise time-symmetric. Although Ritz had pointed out how this arbitrariness was an unsatisfactory feature of electrodynamics, his conclusion had been that one must choose the retarded potential to make any sense of it, until a theory which imposed it could be found. Einstein however, felt that time assymetry had no business in field theories and that its origins lay solely in probability theory (Einstein and Ritz 1909). His views may have influenced Infeld, who preferred the "standing wave" solution as the most natural choice in the EIH approximation. In the case of the gravitational field, where the existence of radiation could not be experimentally proven, Infeld may have felt there was no compulsion to impose the arrow of time, as one would in electromagnetism, knowing from experiment that radiation existed in that field.

The first post-Newtonian attempts to deal with gravitational radiation reaction via the problem of motion had to wait until after the war. In 1946 Ning Hu, a Chinese graduate of Caltech, presented results based on a scheme inspired by the EIH method to the Royal Irish Academy in Dublin, reporting an energy loss disagreeing with the quadrupole formula in the case of an equal mass binary system in a circular orbit (Hu 1947). Shortly before publication, however, he added a note in proof after finding a calculational error which changed the sign of his result, giving anti-damping instead of damping. In other words, the system would gain, rather than lose energy as the result of emitting radiation. The binary would therefore slowly increase, not decrease in radius.[1]

In Canada, Infeld and his student, Adrian Scheidegger, worked on the problem of gravitational radiation reaction in the EIH formalism (Infeld and Scheidegger 1951). They concluded that the most natural treatment of the scheme, employing the standing wave boundary condition, led to a no-radiation-reaction result. It was possible, they conceded, to find terms at certain large odd powers of v/c (where c is the speed of light, and v represents the small source velocities) which appeared to correspond to back-reaction terms, but they contended that these could always be transformed away by a suitable choice of co-ordinates. The result, when announced at an American Physical Society meeting in 1950, "gave rise to a considerable flow of discussion," as Scheidegger put it (Scheidegger 1951). That same year Infeld left Canada, after the McCarthyite campaign against him described above. He returned to his native Poland, while Scheidegger continued to argue the no-damping position in North America in his absence, before leaving the field of general relativity for that of geophysics in the mid-fifties.

In 1955 came two further contributions. Joshua Goldberg, a student of Peter

[1] Hu seems to have returned to the radiation reaction problem in the 1970s, when we find Anderson referring to an unpublished result of Hu's in disagreement with the quadrupole formula, but with the same sign, indicating an energy loss by the binary. No details of this calculation are available at present. Hu was at Beijing University during the 1970s and 80s.

Bergmann (who had criticized the Infeld and Scheidegger results), examined the reaction problem in the EIH formalism (Goldberg 1955). His conclusions were twofold. On the one hand, he denied that the slow motion approach tended to exclude the possibility of damping (arguing that co-ordinate transformations which removed some back-reaction terms, would reintroduce other reaction terms of odd order in v/c), but on the other hand, he determined that it was poorly suited to the back reaction problem, principally because of the restriction to slow motions of the source. In fact, it was generally agreed that radiation reaction terms did not enter into the post-Newtonian equations of motion until terms of order at least $(v/c)^5$ beyond Newtonian order (or post-$2\frac{1}{2}$-Newtonian order). Since first post-Newtonian effects (or $(v/c)^2$ order), such as those obtained by EIH, were both small and difficult to calculate, the expansion method seemed unpromising for studying radiation in that it had to be pushed to high order to succeed.

A couple of years later Goldberg was introduced to Peter Havas, a physicist with experience in the problem of radiation in electrodynamics, who shared his interest in developing a fast motion expansion in general relativity. Havas had been a student of Guido Beck's, not in Vienna, where they both began their scientific careers, but in Lyons, France, where they both sought refuge after the Anschluss in 1938. Beck had worked in the Soviet Union for a period, in Odessa, Ukraine, at the same time Rosen was in Kiev, but had to leave when foreigners began to be arrested for espionage, and then found himself rendered stateless by the Nazi invasion of Austria. Havas, whose studies had been interrupted by the invasion, had worked as an experimentalist in Vienna under J. Mattauch, and was continuing his studies at the Atomic Physics Institute in Lyons when he encountered Beck there. Seizing a chance to move into theory Havas persuaded Beck to supervise him. This work was in the mainstream subjects of atomic and nuclear physics, as Beck no longer worked in GR.

Following the outbreak of war, like Bondi and Schild in Britain, Beck and Havas were interned by the French government. Initially set to forced labour in the camps, Havas' future wife (also Austrian but only males were interned) managed to effect his release after a few months, and he returned to experimental work at Lyons. The city was briefly occupied by the Germans after the invasion of France, but Havas and his wife made good their escape before the former's arrival, returning when the city was handed over to the control of the Vichy collaborationist government. Since Beck was still interned Havas continued his theoretical studies by correspondence while working with the nuclear fission group at Lyons. In late 1940 Beck was released, and the next year Havas managed to secure a visa for the United States, where he finally completed his studies at Columbia University in New York. Beck eventually escaped to South America via Portugal (Havas, 1995).

In the United States, Havas secured a position at a small college in Pennsylvania, Lehigh Universtiy, and worked on electrodynamic theory. His experience in this field convinced him to try something similar within GR (inspired by the discussion of the EIH work in Infeld's autobiography *Quest*), and he contacted Goldberg to secure an invitation to the Chapel Hill meeting in 1957 (see below), where he saw a considerable level of interest in the radiation problem amongst relativists. He was particularly inspired by Bondi's talk addressing the differences between electromagnetism and gravitation in the radiation problem (interview). Being familiar with the special relativistic problem of radiation, his instinct was to approach the problem via what would later be called a post-Minkowski approximation (i.e. "fast motion") which essentially approximates GR to the special theory, rather than to Newtonian theory. Since Goldberg had independently reached a similar conclusion, that this was a more appropriate avenue for handling the back reaction problem, they began a collaboration based on this approach.

Also in 1955, the Russian physicist Vladimir Fock treated the orbital damping problem in his book *Spacetime and Gravitation* (Fock 1959). He made use of a slow-motion expansion which he had developed independently of EIH, coupled with "outgoing-wave only" boundary conditions. His results were in agreement with those of Landau and Lifshitz. His work was not translated into English for four years, and even then wielded little influence in the west, perhaps because of Fock's unorthodox views on general covariance. He employed so-called harmonic co-ordinates in his calculations, and claimed a special status for them in physical theory. His views in this regard were vigorously opposed by Infeld and most other relativists then and since. Ironically, although Infeld's defence of general covariance against Fock is vindicated by the current orthodoxy, the harmonic gauge condition is related to Fock's vindication on the matter of radiation reaction. Fock employed with his gauge choice, a retarded Green's function solution corresponding to "no-incoming" waves from the past infinity of the source, which are today regarded as the correct choice of boundary conditions for this problem.[2]

The EIH method, on the other hand, was plagued for years by the difficulty of correctly imposing asymptotic boundary conditions on the source's motion which corresponded to outgoing waves carrying energy away from it, as opposed to some mix of ingoing and outgoing waves carrying energy both to and from the source,

[2] the same gauge choice and Green's function combination used in Einstein 1916.

and tending to cancel each other's effects. We have already seen how Infeld evinced a preference for a gauge choice corresponding to a balance of ingoing and outgoing waves anyway, but in general, the EIH approach, and other slow-motion approximations, were awkward to deal with in this respect. The reason was that the approximation scheme was valid only in the immediate region of the source, and so the boundary conditions were imposed by Fock on a rather different wave-like solution in the so-called "wave zone," towards infinity. Fock then "matched" his two solutions in an intermediate region to impose the boundary conditions on the equations of motion for the system. It was not readily apparent in the EIH scheme how those conditions were to be imposed on the solution of the sources motion so as to match correctly the source with its radiation field. In this case, Fock's prejudices proved more beneficial than Infeld's. Fock was quite clear on the reasons for the superiority of his method, declaring in a section titles "On the Uniqueness of the Harmonic Coordinate System":

> When solving Einstein's equations for an isolated system of masses we used harmonic coordinates and in this way obtained a perfectly unambiguous solution. We found unique results not only for finite and 'moderately large' distances from the masses, when the wave-like, i.e. hyperbolic [that is to say, finite propagation], character of Einstein's equations was not essential and was accounted for by the introduction of retardation conditions, but also for the 'wave zone'.

Infeld's adherence to general covariance appears to have discouraged him from accepting Fock's claims that the harmonic gauge condition was the most suitable for radiation theory, since Fock claimed that this suitability arose from an underlying correctness in this choice of co-ordinates.[3]

Another reason why Fock's results were not so influential, apart from his rejection of general covariance and the fact that his book remained untranslated into English for 4 years after its publication in Russian, is perhaps that Fock himself regarded his back-reaction result as merely demonstrating that wave phenomena played an inconsequential role in the problem of motion in gravity, due to the small size of the effect for known astronomical systems.

[3] Fock's analogy for this was with "Copernican" versus "Ptolemaic" co-ordinates. Both could be employed for calculational purposes, but he insisted that the first must be given a priori status as the correct physical description of the solar system.

Chapter 7

Influence of Infeld, Rosen and Bondi

Anyone interested in a topic such as the historical debate on the existence of gravitational radiation quickly encounters a particular quality to the remembrance of such affairs. While the temporary or aberrant scepticism of influential figures such as Einstein and Eddington is preserved in the folk memory in occasional anecdotes or quotations, the suggestion that there was ever any real *debate* on such a subject is frequently rejected or resisted. Individuals may have made errors at times, or held erroneous views, but to suggest that there was ever much public discussion, or that there was really a problem in the general sense, would be to go too far. In the present case, it is true that until recently GR was a very small field, and the number of people working on gravitational waves was even smaller. The scope for debate was therefore limited. Furthermore, the particular outlook of each researcher or group naturally influenced what made an impression on them at the time, and what remained in memory afterwards. Since the debate, such as it was, achieved no posthumous elevation to the status of a significant historical event in the life of science, there was no reason to preserve recollection of it. Finally, there is a preference not to remember, not to overstress the significance of, something which may be seen as vaguely disreputable to the field. It is a characteristic aspect of physics that to pose a problem or a question may, in itself, be taken as a sign of bad character. It is typical of an established theory framework that certain questions are rendered nonsensical and certain problems rendered otiose, even where they were once perfectly reasonable issues. Bondi (1970), referring to Newton's success in answering dynamical questions about the solar system, while ignoring evolutionary issues, which had previously seemed all part of the same problem, sees this as "a vital feature in the whole pursuit of science."

A further problem, in the present context, is one of definition. Broadly speaking, several relativists are remembered as having been sceptical of the existence of gravitational waves. Of course, their views did not constitute a monolithic position, especially given the status of such views as a minority outsider position. Some people did in fact suggest that gravitational waves do not exist, for instance, Einstein and Rosen rather briefly. More common was some variant of the view that radiation reaction does not exist as an effect, in the sense that gravitational waves do not draw energy from gravitational systems and/or do not transport energy. Into this category we may conveniently sweep Infeld, Bondi, Rosen, Cooperstock and Scheidegger, among others. Clearly this is a rather critical position on the subject. But Cooperstock, for instance, insists that gravitational waves do exist and can be detected by certain instruments, but do not cause radiation damping in systems such as the binary pulsar (which does however, emit them in a detectable form). Let us, however, agree to refer, in a general way, to scepticism on the *existence* of gravitational waves, as referring to a viewpoint which regards the "standard" picture of gravitational waves (for a given era) as being possibly or actually seriously flawed in an integral way. We will also avoid the ahistorical view of regarding *successful* sceptics (those who convert others to their alternative stance) as orthodox, in hindsight. Therefore, Eddington will be referred to as a sceptic, although his scepticism is nowadays established orthodoxy. Interestingly enough, he is still remembered, in the "oral tradition," as it were, as a sceptic.

An alternative form of scepticism concerns the detail of the back reaction problem, specifically the quadrupole formula which describes (in the orthodox view) the rate of emission of gravitational wave energy from a freely gravitating system. Formulated originally by Einstein, the use of this formula, and various derivations of it, has been much criticized over the years, on the grounds of insufficient proof of its validity. Notable sceptics of this sort, including Havas, Ehlers, Rosenblum and others, come under a different usage of the word. They are sceptics in the sense of taking an agnostic view of a subject. Most of the previous class (but not all), had an alternative view of the nature of gravitational waves in mind. Most of the critics of the quadrupole formula did not offer any particular alternative formula to replace it, they more typically had a precise view of how it ought to be derived. The two forms of scepticism could easily overlap. To say that the quadrupole formula might be incorrect implied the possibility (frequently stated by Havas and stressed also by Eddington and Bondi) that there was no quadrupole emission at all, so that back reaction in gravity could be much weaker than was thought, or even non-existent.

We can refer to this type of scepticism as "weak scepticism," or "limited scepticism," and to the former type as "strong scepticism."

A curious duality may be observed here concerning attitudes to the idea of scepticism. On the one hand, the "sceptic" may be viewed as the ideal type of scientist, one who has no time for received opinion, unsupported by factual evidence or experimental data. But as time went on in the study of gravitational waves, increasing signs of impatience can be observed in some quarters with the sceptical position of either type. We shall encounter later Feynman's views on the importance of optimism and the necessity for a progressive research program which sets aside cavils or doubts at the outset and presses ahead until either all problems are overcome, or the doubts are vindicated by irreconcilable internal contradictions. One can perhaps see here an epistemological struggle over the proper balance between scepticism and belief in theoretical science. Gravitational radiation, a field with no experimental input whatever for several decades of its development, was certainly an ideal arena for such a discussion.

In the context of a small field, and an even smaller subject area, any minority position must depend for its survival on a relatively small cast of characters. The fortunes of these persons, their success at passing on their views, must considerably affect the chances of a successful challenge to the orthodoxy. How influential were the sceptics, meaning here, the strong sceptics? Were they outsiders in their field? In a professional caste where it is easy to find oneself labeled a crank, were they able to command tolerance for their "Menshevik" stance? It is important to keep in mind the fluidity, not only in the orthodox position, but in the stance of individuals. There can be no hard and fast definition of a sceptic in this field or any other, but we will make do by following, as much as possible, the self-definition of the individuals involved. Rosen twice wrote or co-wrote papers whose titles called the existence of gravitational waves into question. Bondi also went into print (in a letter to Nature in 1957) describing himself as a (temporarily) convinced sceptic. Probably the most prominent and outspoken sceptic was Einstein's collaborator, Infeld, so we will begin with him.

Leopold Infeld was born in Krakow in 1898, a part of Poland then ruled by the Austrian empire (Pyenson, 1978). He received his doctorate from the Jagiellonian University in his home town in 1921, but in the young Polish Republic there was little academic opportunity in physics, especially for a very young Jewish physicist. After spending some time as a rural schoolmaster, he did achieve the position of docent at the University of Lwow. In 1933 he obtained a Rockefeller fellowship which permitted him to travel abroad, to work in Cambridge with Max Born and at Leipzig with Bartel van der Waerden. Since, upon his return to Poland, there was still no prospect of academic advancement, despite the publication of some well received papers abroad, he left once again, to take up another fellowship working with Einstein in Princeton, New Jersey. Here, no doubt, he was able to take advantage of his connection with Einstein's close friend, Born.

His arrival, in late 1936, coincided, as we have seen, with the later stages of Einstein's belief that gravitational waves could be shown not to exist. Despite the dissapointment of the subsequent failure of this position, the collaboration which followed was the making of Infeld's career as a physicist. The EIH paper became the canonical "first post-Newtonian" solution of the two body problem of motion in gravitation. Its method permitted much more general calculations of relativistic corrections to traditional problems in orbital mechanics. In fact, the paper was immediately followed in *Annals of Mathematics* by a paper of Robertson's recalculating the famous perihelion shift of Mercury due to GR, on the basis of the EIH results.

Although the EIH paper made a name for Infeld in physics circles, his close friendship with Robertson and the public side of his collaboration with Einstein may have been even more responsible for his subsequent professional success. After a year at Princeton, Infeld's fellowship ended, unlike the EIH research. Anxious to continue his collaboration with Einstein, Infeld suggested the project of writing a popular book together, the proceeds of which would pay Infeld's wages for another year. Einstein, who clearly valued Infeld as a collaborator, readily agreed. Their book, *The Evolution of Physics*, was a best-seller, and received widespread publicity. From then on Infeld had a public profile as "Einstein's collaborator," not achieved by any of the other physicists who worked closely with the most famous scientist of the age. Despite this, Infeld seems to have more or less given up the thought of securing a professorship at this stage. He had turned 40, and had thought of trying to make a living writing popular science books (Infeld, 1941).

However, Robertson, with whom he had become very friendly at Princeton, made efforts on his behalf, and persuaded John Synge, a mathematical physicist at the University of Toronto, to consider Infeld for a position there. Infeld went to Toronto as a temporary lecturer for one year, and did secure a permanent appointment at the end of it. Synge, originally from Dublin, was head of an Applied Mathematics department at Toronto which had been set up for him. Theoretical Physics did not exist at the time, in Canada, as a separate field. When, during the war, Synge left for Ohio State University his department, and Infeld, were merged once more with the Mathematics department. Infeld worked hard to found a center for theoretical physics in Toronto, but although he produced many students, he could not persuade the university to create new positions. Since there was no other center of relativity in Canada, most of his students moved into other fields of physics after graduation. Nevertheless, Infeld, in spite of the college's lack of enthusiasm for the project, did a great deal to promote the field of theoretical physics in Canada (Wallace, 1993).

When Infeld first went to Toronto, he worked with a student, Phillip Wallace on generalizing the EIH method to electrodynamics. In this paper he had some interesting things to say about back reaction, in view of his later work, but it was not until after the Second World War that he really turned his attention to this problem in the gravitational case. In the meantime he finally persuaded his department to retain one of his students, Alfred Schild (a war refugee interned in Canada by the British as an "enemy alien" along with Hermann Bondi) after graduation, but could not manage to get them to pay enough to keep him for long. Schild left for Pittsburgh and the Carnegie Institute (where Synge had also spent time after the war), taking another young student, Felix Pirani, with him. With Schild, Infeld had worked on the motion of a massless "test particle" moving in an external gravitational field (Infeld and Schild, 1949).

In the late 1940s, Infeld returned to the EIH formalism with another student, Adrian Scheidegger, and addressed the problem of radiation reaction. They concluded that the problem of motion for gravitational binaries allows for no dissipation of the system's energy by radiation. They published a paper to this effect, and Scheidegger addressed the American Physical Society and a conference in Vancouver on the subject. They did provoke some reaction. Scheidegger speaks laconically of a "considerable flow of discussion" after his talk at the APS, and Peter Bergmann (who was at the APS meeting), had one of his students, Joshua Goldberg respond to the Infeld/Scheidegger assertions with a paper of his own.

As fate would have it however, neither Infeld nor Scheidegger was able to continue the debate for long at this time. In 1950 Infeld had the intention of spending a sabbatical year in his native Poland in an effort to help rebuild the physics community in that war shattered country. A small Catholic Canadian paper, the *Ensign*, chose this occasion to launch an attack on him, based on the ludicrous assertion that Infeld intended to give away atomic secrets to the Soviet Union or its ally, the People's Republic of Poland. Infeld's close association with Einstein was farcically presented as evidence of his familiarity with nuclear weapons secrets. The mainstream press took up this slanderous attack, sometimes under the guise of reporting Infeld's denials, and the campaign reached its peak with a personal attack under the protection of parliamentary privilege launched by the leader of the opposition Progressive Conservative Party, George Drew.

Drew went so far as to demand that Infeld be prevented from leaving the country. The University of Toronto came under pressure to refuse leave for Infeld's visit. This had the effect of forcing Infeld to choose between Canada and Poland. In the light of the public and personal attacks against him, and under surveillance and possible harassment by the Royal Canadian Mounted Police, he chose Poland. Rather suddenly therefore, his career in Canada came to an end, and he was forced to start over again in Warsaw.

The motivations for the attacks made against him may have been various. He was an outspoken critic of American and British nuclear policy of the day, having stumped the country speaking to impress upon the Canadian public the futility of attempting to keep the "secret" of the bomb from the Soviet Union's physicists. He himself was a socialist and his American wife was also very left wing, and had involved herself to some extent in progressive Canadian politics, opposing Drew during his tenure as Ontario's provincial premier. He had publically joined in the defence of those accused in the Gouzenko affair, in which a defector from the Soviet embassy in Ottawa had denounced a number of prominent Canadians, including several scientists, some of whom were tried for espionage. The likeliest explanation is that he was an attractive target for the politically ambitious Drew and the hysterically anti-communist Catholics of the *Ensign*. The charges were absurd on their face, since the Soviet Union had already exploded an atomic bomb. This fact was held against Infeld, with amazing chutzpah, by Drew, who regarded it as a suspicious fulfillment of Infeld's own prophecy that the Soviets would get the bomb in spite of Western efforts to keep it secret. The illogic of preventing someone from traveling abroad to give away secrets which he was insinuated to have already betrayed was not noted by the press until after Infeld's forced "defection." Likewise, the Canadian defence research establishment only then went on record to say that he held no military or scientific secrets. In fact, Infeld's war work was limited to some efforts with Synge in ballistics, and some radar work. His former student Wallace did work on the abortive British-Canadian bomb project, which was the closest connection Infeld had to the bomb.

After Infeld's departure, Scheidegger continued to put forth the view that gravitational back reaction did not exist in GR, exchanging papers with Goldberg in the *Physical Review.* But, like so many of Infeld's students in Canada, he found it difficult to secure an appointment in GR or in theoretical physics. After a few years he found employment with an oil company in Canada, and subsequently took up a successful academic career in geophysics. With that, the first round of debate on the topic of radiation reaction gravity petered out.

Infeld, despite the misfortune which had forced him to uproot his Canadian family and return to his homeland (in a shameful act his two Canadian born children were later stripped of their Canadian citizenship, as was Infeld himself, by the extra-judicial maneuver of "orders in council"), found his personal goal of establishing a school of theoretical physics easier to achieve in Poland than in Canada. The Polish government was eager to rebuild the country's scientific and educational infrastructure, ruthlessly destroyed by the occupying Nazis. Infeld, with personal fame as Einstein's collaborator and notoriety as a refugee from political persecution in the West, became one of the leaders of this rebuilding effort. He made a considerable success of the opportunity, establishing a thriving school of relativity, and producing, as before, many excellent students. His political troubles did not entirely cease with his move to a communist country. His arrival coincided with the last years of Stalinism, and his association with Einstein was not an unmixed blessing in a political environment in which Einstein was still considered an "idealist" opposed to the true practice of Marxist science. The fact that Infeld publicly opposed the arguments of the Vladimir Fock, who wished to reform GR by removing general covariance, did not help matters either. But Infeld did not, like some others, come under significant pressure to recant, and after Stalin's death in 1953, the political environment changed rather quickly in Poland, following the coming to power of

Gromulko.

In Warsaw, Infeld at last had a thriving group doing excellent work in GR, one of only a handful of relativity groups in the world at that time. Furthermore, from 1955 on, gravitational waves and the problem of back reaction became an active topic in the field. It is interesting that Infeld's students in Poland did not share his views on the subject. Andrzej Trautman and Jerzy Plebanski did important work on the problem of motion and gravitational waves, and both were decidedly non-sceptics. Yet throughout this period (late 50s, early 60s), Infeld continued in his own outlook, as shown in his 1960 textbook *Motion and Relativity* written with Plebanski. Infeld displayed a certain high-handedness in dealing with his students' dissent. The chapter of *Motion and Relativity* dealing with radiation reaction was added to the book without his co-author Plebanski's knowledge, after the latter's departure for America on a fellowship. The chapter is an excellent account of Infeld's position at the time, but entirely fails to represent Plebanski's view, which was diametrically opposed. Similarly, in one preprint, Infeld alludes to the contrary views of his students, but asserts that they had come to accept his arguments as correct. This did not stretch the truth as much as contradict it entirely.

Nevertheless, despite these public vanities, Infeld was personally fair with his students. Plebanski began to feel uncomfortable enough about his general circumstances to wish to leave Poland for good and Infeld helped him in securing permission to move "temporarily permanently" to Mexico City. Trautman, who was a very prominent contributor to the new advances in the picture of gravitational waves, including an important calculation of the radiation reaction effect which supported Goldberg's position in the old debate with Infeld and Scheidegger

(Trautman, 1958b), remained an important and favoured member of Infeld's group. In the end, shortly before his death, Infeld seems to have been finally won over by his students' arguments, rather than the other way around, and even published a paper with R. Michalska-Trautman which accepted the existence of the phenomenon of radiation reaction. In this case, therefore, the advisor did not perhaps influence the students so much as they influenced him. In spite of Infeld's great personal success as an advisor and the founder of a school, this aspect of his own views was not at all transmitted to his students who remained active in the field. Nevertheless, his interest in this subject was passed on, and led to much of the significant work which developed the study of gravitational waves in the 1950 and 60s.

Nathan Rosen, Einstein's other collaborator directly involved with the attempt to disprove the existence of gravitational waves, also remained a prominent sceptic of gravitational waves throughout much of his career. Rosen, born in 1909 in Brooklyn, New York, was, like Infeld and many other scientists of the time, a socialist. He was so strong in his convictions as to wish to live and work in the Soviet Union at a time, the late 1930s when suspicion of foreigners was at its height, and it required Einstein's intervention to secure for him a position at the Kiev State University. Whatever his opinion of "actually existing socialism" during the era of the great purges, he returned to the United States after only 2 or 3 years. In the 1950s, again with Einstein's endorsement, he emigrated to Israel and did much to build up the Technion institute in Haifa, where he worked until his death in 1995.

Rosen might be fairly viewed as a professional sceptic in the best sense, and has played a prominent gadfly role in the history of 20th century theoretical physics. He is said to have been largely responsible for the anti-Copenhagen argument advanced in the famous paper written by himself, Einstein and Boris Podolsky. Although he agreed with Einstein that there was a difficulty with their original paper on gravitational waves, he felt strongly enough about the argument to publish in a Soviet journal a revised version which restricted itself to disproving the existence of plane gravitational waves. His arguments were rebutted after the war by Bondi, Pirani and Robinson (1959).

In 1955, at the Bern Jubilee conference, he turned to casting doubt on the physicality of the cylindrical wave solution from the published version of the Einstein-Rosen paper. He suggested that they might not carry any energy (based on an analysis of the energy pseudo-tensor in cylindrical co-ordinates), but subsequently retracted this view (Rosen, 1958). Afterwards, he was perhaps prevented by institutional commitments at the Technion from pursuing further work on gravitational waves personally for many years (Peres, private communication). Indeed, his final correction of the Bern paper did not appear until 1993 (Rosen and Virbhadra), an unusually long publication delay by anyone's standards!

However, in 1979, inspired perhaps by the resurgence of interest in the problem at that time, he published a paper which returned to the problem of the arrow of time in gravitational radiation theory, in a paper whose title notably echoed that of his rejected 1936 submission to Physical Review with Einstein (Rosen 1979). In "Does Gravitational Radiation Exist?" he adapted the Wheeler-Feynman absorber theory to gravitation, and concluded that as the gravitational force interacted much less strongly with matter than the electromagnetic field, a source system would not undergo radiation reaction for lack of a sufficiently strong absorber field. In the Wheeler-Feynman theory it is the field of the absorbers, back-reacting on the source, which breaks the time symmetry of the source field. (Wheeler and Feynman, 1945 and 1949). However, Rosen's arguments do not appear completely convincing even to himself, since towards the end of the paper he retreats to a more Tetrode-like position, conceding that an absorber (such as a gravity wave detector) could presumably act so as to draw energy from the source at a distance. In any case, his paper did not excite much debate on the subject.

However, in the 1950s he was indirectly responsible for one very important contribution to the back reaction problem, when he encouraged his student Asher Peres to pursue the problem, in an effort to decide between the rival views of Fock and Infeld on the existence of radiation damping in freely gravitating systems. Peres developed an approximation scheme which correctly reproduced the EIH and Fock results at first post-Newtonian order, but then ran into difficulties calculating the radiation effects on the motion. His thesis results found, like Hu, an energy gain by the orbiting system, but he subsequently located the source of his error in a failure to correctly apply no-ingoing wave boundary conditions to the "near zone" of the system. By careful attention to the matching of the "far zone" conditions to the near zone solutions, he overcame this difficulty, much to his and Rosen's relief, and rederived the quadrupole formula.

Surprisingly his success had relatively little impact, though his paper was viewed very favourably by some experts, such as Kip Thorne. Although he had identified a key reason why previous calculations had produced such wildly varying results, he could not overcome the sense of dissatisfaction which the slow-motion approach had generated amongst some experts (e.g. Bonnor, Havas), and it was not until Burke (see below) introduced a general technique for matching far-zone and near-zone solutions that an unambiguous method of imposing the boundary conditions in the slow-motion expansions encouraged greater confidence in that type of approximation scheme. Nevertheless, Peres' 1960 paper, at least in hindsight, can be seen as a turning point for the back reaction calculations. Previously it is difficult to find two results which agreed with each other. Subsequently, the great majority of slow motion calculations agreed with the quadrupole formula in their predictions, for circular binary systems.

If Rosen himself had widespread influence on the field of gravitational radiation, it was perhaps by example. None of his published ideas were ever taken up except in rebuttal, but his gadfly role (e.g., he was the originator of a prominent rival theory of gravitation to GR) has attracted admirers, such as the most prominent current sceptic on gravitational waves, Fred Cooperstock. Rosen always sought to be provocative throughout his career, and unlike many scientists, was not afraid to stick his neck out with unconventional ideas and views in an effort to challenge received opinion and unwarranted assumptions. He continued actively in research literally up until his death (I myself refereed a paper of his only shortly before he died), and remained remarkably true to his own convictions in a profession in which fear of nonconformity occasionally dissuades people from publishing their best work.

Another fruitful sceptic was Hermann Bondi, one of the originators of the steady state theory of cosmology. Though now decidedly a minority theory, this was at one time a strong rival to the the "Big Bang" theory. Bondi took as his mentor in the field of relativity, Eddington, and emulated his scepticism in regard to existing formulations of gravitational wave theory. Bondi, furthermore, took an individual stance towards what was worthwhile in GR. He disliked work of the problem of motion type, producing minor corrections to the Newtonian theory, and saw in gravitational waves an opportunity to study an entirely new phenomenon allegedly predicted by the theory, which was unknown to Newtonian gravity, and which might yield insight into the novel non-linear aspects of the theory.

Bondi grew up in Vienna between the wars, but chose not to pursue his education in physics in Austria during the period of the Christian Social Party dictatorship. Instead he went to England, where he secured placement as an undergraduate at Cambridge with the help of a recommendation from a relative who was a prominent mathematician, Abraham Frankel. At Cambridge he was an almost immediate success as a student, but the greatest advantage of his move abroad was shown shortly afterwards, in 1938, when Hitler invaded Austria. Acting on Bondi's advice, his family left Austria precipitately shortly beforehand, and thus avoided the fate shared by many other Austrian Jews under Nazi rule. But exile did not immediately rescue Bondi from the class of suspicious persons by reason of his nationality. In England once war with Germany broke out he became an "enemy alien," and one of the first acts of the Churchill government in 1940 was to order his internment, along with many others like him. Nevertheless, in his autobiography, Bondi recalls his relief at Churchill's accession to power, owing to the latter's association with a policy of confrontation with Nazi Germany.

A concentration camp cannot be a pleasant place to find oneself in, no matter how awful the alternative. Bondi was obliged, with other European exiles, to spend more than a year in detention in Canada, to which they were deported, despite the perils of U-boat predation, which sank one unescorted vessel full of interned refugees just prior to Bondi's crossing. During his internment, Bondi met two other important theoretical physicists, showing the amazing intellectual quality of the European refugee population. One was Thomas Gold, his longtime collaborator, and co-author of the steady-state theory, another was Alfred Schild, who stayed in Canada after his release, and became a student of Leopold Infeld's in Toronto. Schild would later co-found the immensely influential series of Texas symposia, for which see chapter 12.

Despite this unfortunate interruption of his academic career, Bondi was determined to return to England (his family had immigrated to America before internment took effect), where he resumed his academic career at Cambridge and participated in war work, with other rehabilitated "enemies" such as Gold. Aided by the marvelous English facility for overlooking any unpleasantness they may have caused, he assimilated perfectly to English academic life, and was eventually knighted for his later administrative work in the British Department of Defence (Bondi, 1990).

Bondi's interest in gravitational waves was sparked initially by the 1955 Bern conference commemorating the jubilee of special relativity. It was at this meeting (later dubbed GR0, after the successful GR series of meetings inspired by the Chapel Hill conference of 1957) that Rosen presented his paper suggesting that cylindrical gravitational waves could not carry energy. Following the Infeld and Scheidegger work of a few years earlier, the possibility that gravitational waves did not exist was in the air. Bondi himself recalls,

> it [the Bern meeting] was particularly memorable for me because of the discussions we had ... on gravitational waves. The mathematical and physical complexity of Einstein's theory of gravitation is so great that there was still confusion, and a variety of opinions, about whether the theory

> predicted the existence of gravitational waves or not. After one of these discussions, Marcus Fierz, Professor at the ETH, the federal technical university, took me aside and said, 'the problem of gravitational waves is ready for solution, and you are the person to solve it.' This remark governed a sizeable slice of my scientific work work for many years, and led to the 1962 paper on gravitational waves in a fifteen paper series ... The 1962 Paper [presumably Bondi, Van den Burg and Matzner] I regard as the best scientific work I have ever done, which is later in life than mathematicians usually peak.

(Bondi, 1990)

Another factor which encouraged Bondi's interest in gravitational waves was the interest of his student and colleague Felix Pirani. Pirani began his physics career as, very briefly, a graduate student of Infeld's at Toronto, but when Schild left Toronto to take up an appointment at Pittsburgh, Pirani went with him and did his graduate studies there. Owing to Schild's friendship with Bondi from internment together, he then went to Cambridge, where he received a second doctorate working with Bondi. Subsequently, after a year in Dublin with Synge (another Toronto and Pittsburgh connection) at the Dublin Institute for Advance Studies, he took up an appointment at King's College, London, where he formed part of a very active and influential group in GR with Bondi, who became professor of Applied Mathematics there in 1954.

Pirani first had his attention drawn to the confused state of gravitational wave theory when he was asked to review a paper on the subject for Math Reviews. The paper, by MacVittie, attempted to show that plane gravitational waves could not exist, a view previously propounded by Rosen, as we have seen. Pirani felt that the result must be wrong, and he benefited from his time in Dublin, where he was influenced by Synge in two important respects. The first concerned the equation of geodesic deviation, a co-ordinate invariant way of looking at physical interactions in GR, based on the curvature tensor, greatly championed by Synge. Pirani was led to describe the interaction of a wave with a physical system by showing that the particles in the system would be moved about *relative to each other* by a passing wave. This not only helped give a better picture of how a gravitational wave worked in practice, but also led to ways of side-stepping the vexed question of whether such waves carried energy or not, and could so physical work.

The second idea which Pirani was introduced to in Dublin was the classification of radiation fields by type. He was asked to proofread a book of Synge's which discussed this for electromagnetic fields, and was inspired to do the same for gravitational radiation. He then came across a classification scheme based on types of the Riemann tensor for different fields, due to Petrov. This scheme, dividing gravitational fields initially into type I,II, and III, with the latter two describing radiation fields, was adopted and became quite widespread. The success of this scheme, after the failure of the earlier Weyl-Einstein attempt at classification illustrates yet another irresistible impulse which physicists (and other scientists) are subject to.

Whereas Pirani was sceptical of the sceptics, Bondi was influenced by Eddington, from whose book he learned relativity, in adopting his own brand of scepticism of gravitational waves. To begin with, he seems to have been doubtful whether the theory really admitted them. In his letter to Nature of 1957, which is remembered by many for its critical thought experiment "proving" the likely existence of gravitational waves, he describes himself as having been a strong sceptic at Chapel Hill that same year. Joshua Goldberg, who helped facilitate that meeting in his capacity as as head of the USAF's ARL group on GR, recalls that Bondi advocated the non-existence of gravitational waves at that meeting (Goldberg, 1988). Yet the contradiction between various remembered histories mentioned above is illustrated by Pirani's remark that "I'm surprised that there was still some doubt on gravitational waves carrying energy at Chapel Hill" (interview). Indeed, the famous thought experiment which was used repeatedly in 1937 to circumvent Rosen's pseudo-tensor problem was "enabled" by Pirani's ground-breaking work on the geodesic deviation description of gravitational waves, which he reported at that conference.[1]

Collaborating extensively with other groups, in Poland, Germany and America, and helped by USAF funding secured through Goldberg, the London group published a string of papers in the late 50s and early 60s which had a profound impact on the theory of gravitational waves. Few can have done as much as Bondi to establish the current orthodoxy on gravitational waves, and to encourage belief in their existence. Nevertheless, Bondi himself remained true to his sceptical roots. In 1962, at the Warsaw conference (GR3) organized by Infeld's school, he still regarded the question of whether binary systems were damped by radiation as open, and felt that it was important to examine the case of two extended bodies, with a real equation of state, however idealized. Chandrasekhar was inspired by this talk to take up the

[1] again, while Goldberg clearly recalls some debate on this topic at Chapel Hill, Bryce De Witt, who organized the meeting, was quite certain that there was no significant debate on the existence of gravitational waves in the late 50s when replying to a presentation by the present author at a meeting in Moscow in 1996. On the whole, the conference proceedings, transcribed from a tape recording of the main sessions, tend to bear out Goldberg's recollection. The varying opinions of the tenor of the discussion are, however, a signal warning to anyone interested in reconstructing this type of recent history of science.

problem (interview).

Bondi felt that two orbiting bodies consisting of pressure free dust would not radiate, since the every particle contained in the two bodies would follow a geodesic throughout their motion. In an Aristotelian sense, these particles would behave "naturally" during their motion, and so perhaps one could regard as non-accelerating, in the geodesic sense, and therefore non-radiating. Bondi remained unsure of the presence of damping in this highly idealized situation for many years.

From the mid-sixties on Bondi became increasingly involved in administrative work, a fairly typical fate for older physicists, as we have seen with Infeld and Rosen. During the 1970s he worked in the UK Department of Defence, and was much less involved in scientific work. Therefore, from about 1965 on, he ceased to play an active role in gravitational wave theory.

As we have seen, none of these three sceptics seemed to impress their concerns deeply on their students and collaborators. Bondi's uncertainty about whether freely gravitating dust would radiate remained largely private. Another worry of his was the existence of tails in gravitational waves, which first came to be realized in the period of his intensive work from 1955-65. In 1966 he described this as an "absolutely disastrous discovery," which might indeed lead to deep insights, but was nonetheless, "extremely serious," in that it prevented a perturbed system from ever entirely settling down to a static state again, since its field would always be affected by its own once-turbulent past history (Bondi, 1970). Certainly, tails have remained an important topic in the study of gravitational waves ever sense, and some have even suggested that they might interfere with the detection of gravitational waves, but on the whole, posterity has not shared Bondi's concern, which he still feels should be addressed (interview). He perhaps recognized the essentially accommodating instinct of most physicists when remarking “no doubt we can live with history in gravitation. But that does not prevent me from regretting [it].” (Bondi, 1970).

Even if Rosen, Infeld and Bondi could not implant their own sensibilities in the minds of others, each made great contributions to the subject. Infeld, through the EIH formalism, his fostering of the post-Newtonian approach to radiation problems, and his mentoring of an army of relativists and theoretical physicists, many of whom worked on problems related to gravitational radiation. Rosen fostered debate on gravitational waves throughout his provocative career, and also contributed through his students, such as Peres. Rosen was such an outspoken iconoclast throughout this career that his own individual style of physics remains very clear even after his death, and still has admirers, such as Cooperstock. Bondi, with his various collaborators did as much as anyone to foster a concrete picture of gravitational waves as a real theoretical phenomenon, rather than an abstract mathematical analogy. Perhaps the sceptical contribution is best summed up, if it must be encapsulated simply, in terms of that process. If one thing united the sceptics, it was their resistance to a straightforward imposition of an analogy with electrodynamics onto non-linear gravitation theory. Complacency with this analogy indeed posed a great threat to the idea of gravitational waves, since if it were accepted too sweepingly, they might never have achieved an independent existence worthy of note, and remained nothing more than a foot note to field theory. If for some, such as Pirani, the need to deepen the analogy, by developing a quantum theory of gravity, served as a motivation for the study of gravitational waves, for others the need to question the analogy was just as compelling a motive. By their efforts, the foundation of a theory of gravitational radiation was laid in the 1950s, just in time for the historical moment of renewed interest in GR which followed in the 1960s, and which saw interest in gravitational waves reach beyond the narrow borders of GR theory for the first time (see chapter 12 below).

If the ideas of the sceptics were taken up in quite a different style by a new generation, it is perhaps the general fate of scientists to pass on to posterity not their whole idea, but just an element of it, which becomes a tool for their successors out of which they construct their own concepts. Indeed, if there can be some regret for older scientists in the failure of younger generations to properly understand their work and motivations, this lack of the personal in the history of physics is perhaps what attracts people to it. Creative young scientists do not feel the dead weight of the personalities of dead or aging scientists weighing on them, when attractive elements of their thought can be appropriated, stripped of the sensibilities which originally animated them, and be given an entirely new meaning in a new style of physics.

Chapter 8

Do Gravitational Waves Exist? - The Role of Conferences in the 1950s

Between the war and the Bern conference of 1955 marking the 50th anniversary of special relativity, general relativity was at a low ebb (Eisenstaedt 1986a and 1986b). Work on the radiation problem seemed confused and controversial, leading only to some consensus that the problem required closer attention. At the Bern conference Rosen, returning to the cylindrical wave solution of his 1937 paper with Einstein, adduced evidence backing up Scheidegger's position by proposing the possibility that gravitational waves did not transport energy (Rosen 1955). It is a peculiar characteristic of general relativity that the energy contained in the gravitational field, and thus the energy in gravitational radiation, is not described in a coordinate invariant way. This energy is considered to be real enough, and can be converted into other forms of energy which can be expressed invariantly, but the principle of equivalence prevents one from doing this for field energy in gravity. The reason is that any observer in a gravitational field is always entitled to imagine himself in a locally Lorentz (that is zero gravity) freely falling frame of reference which, locally, contains no field energy. Of course, one is not free to transform away the entire field energy of a planet but one can always choose co-ordinates on a small portion of its surface so as to eliminate the field energy in that region. Thus it is said that gravitational field energy is non-localizable. This problem of defining field-energy had led Einstein, Landau and Lifshitz and others to employ a non-invariant quantity known as a pseudo-tensor to describe energy in the wave flux in their back reaction calculations. Rosen now observed that each of these (slightly different) definitions of the pseudo-tensor showed no energy at all when applied to the cylindrical waves of his 1937 paper with Einstein in cylindrical co-ordinates. Although drawing conclusions on the tentative basis of the pseudo-tensor was regarded as dangerous, Rosen observed that the result seemed to support the view of Infeld and Scheidegger. This cast further doubt on the uncertain status of wave phenomena in gravitation theory.

The Bern conference is remembered as an important stimulus to the field of relativity. The discussions there, and the interest taken by Felix Pirani, prompted Hermann Bondi to take up the problem of gravitational radiation. Bondi brought an open mind to the issue, in the sense that he was sceptical enough of the existence of gravitational waves. He was influenced in this by Eddington, from whose writings he learned relativity. Eddington's emphasis on a coordinate invariant approach, making use of tensorial quantities such as the Riemann curvature tensor, had enabled him to show that certain classes of gravitational waves "in existence" before 1922 were spurious (Eddington 1922). Bondi, like some relativists of the day, was not impressed by the existing radiation reaction work, finding Landau and Lifshitz" treatment "a little glib". At the same time, gravitational waves seemed like an attractive topic within gravitational theory, since in this area the predictions of general relativity diverged radically from those of Newtonian gravitational theory. Up to this time, most work in relativity, outside of cosmology, had been devoted to deriving small corrections to Newtonian theory, such as the famous perihelion shift of Mercury, a more precise estimation of which was one of the goals of the EIH paper (Robertson 1938). The study of gravitational waves, if they existed, seemed likely to generate more "new physics" than simply adding terms to Newton's theory.

Now, as Infeld himself observed when writing of his surprise at Einstein's "proof" that waves did not exist, no respectable modern field theorist would, under normal circumstances, deny the existence of radiation in a field theory. The mere fact that the force was propagated in the field rather than by action-at-a-distance, a basic tenet of all relativistic field theories, seemed to imply the existence of radiation. Einstein also remarked, in his letter to Born, of the "certainty" which the analogy between the linearized Einstein equations and electromagnetism had inspired concerning the existence of a gravitational analogue to the Maxwellian wave equation. Bondi nevertheless seized on a key argument made by Infeld and Scheidegger, which seemed to him crucial.

As Scheidegger observed, relativity occupied a "peculiar place" amongst classical field theories (Scheidegger 1953). One important peculiarity is that the equations of motion are constrained by the field equations, as Einstein had noted. In electrodynamics, where this was not the case, one was perfectly free to demonstrate damping effects by moving the particles around in whatever fashion, and showing that this gave rise, when the field equations were invoked, to radiation and loss of energy from the local system. In relativity, it was necessary to show that the motions in question were allowed by the same field equations. This was all the more important when one considered the question of what *type* of motion gave rise to radiation. One obvious example was an accelerating charge in electrodynamics. What of the apparently equivalent case of a falling mass? It was clearly accelerating with respect to the person who dropped it, but in a relativistic sense, it was merely following a geodesic, doing what came naturally, as it were. In terms of the local spacetime, the particle that was really being *accelerated* was the one still being held in the observer's other hand, which was prevented from falling freely. Which one of these particles *ought* to radiate? This question had no immediately obvious answer which the relativists of the day could agree upon. E.T. Newman relates how Wheeler once asked a roomful of relativists to vote on this question, and recalls the room being fairly equally divided. This seems to be a rare example of democracy in science.

At the Chapel Hill conference of 1957 and elsewhere at that time, Bondi pointed out the distinction between two masses being waved about at the end of someone's arms,[1] clearly not following geodesics, and clearly emitting gravitational waves (but tremendously weak ones!), and two masses in a binary star system, following geodesics and, if Infeld and Scheidegger were right, not radiating anything (De Witt 1957, pg. 33). Since gravitational forces were likely to be the only forces capable of moving large masses very quickly, the issue of whether purely gravitational systems could give rise to radiation was an issue of whether such radiation would ever be

[1] A number of those interviewed recalled Bondi vigorously demonstrating this method of generating gravitational waves

detectable. That issue, to the surprise of most theorists, was soon to become one of some practical interest.

The Chapel Hill conference on "The Role of Gravitation in Physics" brought together relativists and theoretical physicists interested in then new topics such as quantum gravity. The session on gravitational radiation was lively and varied. Felix Pirani presented important new work on wave theory (De Witt 1957, pg. 37). Influenced by the Irish relativist John Synge during a year spent in Dublin (interview), Pirani drew attention to the Riemann curvature tensor, whose importance had previously been stressed by Eddington in his 1922 paper, as an invariant geometrical quantity which was well suited to the description of the behaviour of gravitational waves. Using the geodesic deviation description of gravitational effects advocated by Synge, he showed how particles in the path of a wave were moved about relative to each other by the spacetime curvature of the passing wave. In this view, gravitational waves were depicted as ripples in the fabric of spacetime itself, whose physical effects were observable by monitoring the relative motion of two adjacent particles during the passage of a wave.

Later in the conference an interesting exchange took place during the section on quantization of gravity. During Richard Feynman's presentation on the need for a quantum theory of gravity, Rosenfeld made the following remark:

> It seems to me that the question of the existence and absorption of waves is crucial for the question whether there is any meaning in quantizing gravitation. In electrodynamics the whole idea of quantization comes from the radiation field, and the only thing we know for sure how to quantize is the pure radiation field. (De Witt 1957, p. 141)

Feynman demurred somewhat from the premise, arguing that there existed a quantum theory of electrostatics, but agreed that some of his arguments in favour of quantization depended on the existence of waves. Bondi was moved to note that "this vexed question of the existence of gravitational waves does become more important for this reason." Feynman then presented an argument based on Pirani's earlier talk. Appealing to the equation of geodesic deviation, he argued that a particle lying beside a stick would be rubbed back and forth against the stick by a passing wave, and the friction would generate heat, so that energy would have been extracted from the wave. Furthermore, he felt that any system which could be an absorber of waves, could also be an emitter. For these reasons, he expected gravitational waves to exist (supplement to De Witt 1957).[2]

This line of argument, suggested by Pirani's new work, was also elaborated in two papers published that same year. In a letter to Nature, Bondi used a slightly different version of it to refute Rosen's argument of 1955 on energy transport (Bondi 1957), as did Joseph Weber and John Wheeler in a more detailed paper (Weber and Wheeler 1957). Weber demonstrated real confidence in the physicality of gravitational waves by embarking within a few years on an experimental program to detect them, using large resonant metal bars as antennae (Weber 1960). Quixotic is probably not quite the word contemporary theorists would have used to describe Weber's aim. The wave theory, in so far as it existed at all, with no particular notion as to potential astrophysical sources or signals, would be better described as a "disabling" rather than an enabling theory for experiment. The quadrupole formula, the only guide to source strength and signal amplitude, suggested that any waves reaching the

[2] Fred Cooperstock's 1992 paper contains a counter-example designed to invalidate this thought experiment. See chapter 16, below.

detector would be very weak. With no theory of sources, the question of what frequency to search at was theoretically undetermined (interview with Weber). In his textbook (1961), Weber states,

> some experimental work [on gravitational waves] now appears possible. Some theoretical issues have been resolved in recent years, and it has been possible for a number of physicists to conclude that general relativity really does predict the existence of gravitational waves.

This does appear no more than a minimum amount of theoretical justification for any experimental program! It is remarkable that the field of gravity wave detection began at a time when the theoretical state of the subject was in such disarray.

The session devoted to gravitational radiation at Chapel Hill, chaired by Bondi, begins with a very clear statement of the sceptical position in regard to gravitational waves. Bondi makes some introductory remarks, criticizing those who regard GR as an all-encompassing theory, which is more than merely a theory of gravity. He prefers to regard it as an "open theory," into which knowledge gained in other fields can usefully be fitted. Nevertheless he is not prepared to apply the lessons of other fields without due caution.

> The analogy between electromagnetic and gravitational waves has often been made, but doesn't go very far, holding only to the very questionable extent to which the equations are similar. The cardinal feature of electromagnetic radiation is that when radiation is produced the radiator lose and amount of energy which is independent of the location of the absorbers. With gravitational radiation, on the other hand, we still do not

> know whether a gravitational radiator transmits energy whether there is a receiver or not.
>
> Gravitational radiation, by definition, must transmit information; and this information must be something new... An example of a gravitational transmitter is a person sitting very quietly holding two dumbbells, who suddenly, unpredictably, starts taking exercise with them. What we want to know is what is the effect of his motion? Does it transmit information to other regions of space of what the person taking exercise is doing, and does it transmit energy? (De Witt, 1957, pg. 35)

Bondi then presents the work of L. Mardar (then at King's College, London with Bondi) on a cylindrical source of gravitational waves. The result is that a pulse wave emitted by the source considered initially changes the mass of the source, which then however returns to its initial value as the pulse travels to infinity. Bondi adds that the waves "carry no energy with them ... while the wave is being sent out the mass [of the source] is decreasing, but as the wave dies down the mass returns to its original value." Bondi once again emphasizes the importance of the unpredictability of emission, since only then can information be transmitted, an idea later given concrete form in his "news function". De Witt[3] encapsulates this idea in the statement "if I know at an initial time that I am going to give you a yes answer, then the field already contains it."

Wheeler now reasserts the value of the electromagnetic analogy, saying

> How one could think that a cylindrically symmetric system could radiate is a surprise to me. There seems to be a *far-reaching analogy* [emphasis

[3] possibly Bryce, the chair of the first session, and not Cécile, the conference organizer

> added] between this case and the problem of emission of electromagnetic radiation from a zero-zero transmission in an atom or nucleus. The charge can oscillate spherically symmetrically, but the system doesn't radiate. However, if we have an electron in the neighborhood, internal conversion can take place, with still no electromagnetic radiation emitted. This would correspond to the uptake of energy of the gravitational disturbance created by the 'cylindrical symmetric' exercise of yours.

To this Bondi assents that the same underlying explanation, conservation of charge in the electromagnetic case, conservation of mass and momentum in the gravitational case, may lie behind the failure of spherically symmetric oscillations to radiate in the first case, and cylindrically symmetric (or axisymmetric) oscillations to radiate in the second case. After some further remarks by Bondi on his own work, in which he examines the asymptotic Schwarzschildean field of a three-dimensional system which is initially and finally spherically symmetric, and again finds no sign of mass loss, it is the turn of Weber. He examines the interaction of a single particle with the Einstein-Rosen cylindrical waves. Noting that the pseudo-tensor energy density in the waves is everywhere zero, and that the particle is at rest both before and after the wave's passage, he concludes that "energy cannot be transferred around as long as one has this type of [cylindrical] symmetry and the [Einstein-Rosen] metric". In answer to a question he notes that not all components of the Riemann tensor are zero, so the wave does not seem to be trivial. Bondi remarks that a single particle must be a poor absorber, to which Weber agrees that one could carry out a similar analysis for a pair of particles.

The next speaker was Pirani, who introduced his classification of types of gravitational waves (based on "Petrov's classification of empty space-time Riemann tensors into three canonical types), pictured as propagating discontinuities in the Riemann tensor (like shock fronts). In particular, his scheme shows how one can distinguish between spacetimes with and without radiation based on the Petrov type. This classification scheme was considerably developed in subsequent years, extending its types and sub-types far beyond the simplicity of the original threefold breakdown. Pirani's discussion of the equation of geodesic deviation, which appears of direct relevance to the thrust of Weber's presentation, and to Bondi's comment on it, was given at a subsequent session.

After Pirani there were talks by Schild on an unsuccessful attempt to construct a radiation-reaction force formalism on an analogy with the electromagnetic theory (a gravitational radiation reaction force was later constructed by Burke an presented in Burke and Thorne (1970)), Pirani reading a communication from Rosen on axisymmetric fields, and Goldberg on the beginnings of a scheme for "an approximation scheme for high velocities". This was inspired by his conclusion that the slow-motion method was unsuitable for treating the radiation problem (Goldberg, 1955). Since the surface integral method employed by EIH (its most attractive but neglected feature, according to Anderson (1995)), is not suitable for the fast motion method, he proposes to replace the consistency conditions derived in the EIH method from the surface conditions with conditions (the Bianchi identities) applicable everywhere, not just on the surface, but inside and across the singularity which, in the EIH scheme is hidden by the surface of integration. This substitution of volume for surface integrals, was going to require the employment in the fast-motion schemes of mathematically sophisticated renormalization schemes borrowed from electrodynamic theory in order to deal with the infinities introduced into integrals containing singularities.

After talks by Tonnelat on classical unified field theories and Lindquist on cosmology, Wheeler summed up the session.

> First, from what Pirani said, we have gained some insight into how we may define what the measurability properties are locally of the gravitational field. The tensors and invariants he describes are at the heart of the matter. Second, as concerns the radiation problem, we would like to know what is the highest degree of symmetry one can have in a problem, and still have interesting radiation. This leads one to the question whether, even in the case with no symmetry, one has reason to expect radiation. On this score, it would be well to recall an important physical fact: *that the gravitational field of a point charge has close analogies to the electric field* [emphasis added]. One knows that there is a certain linear approximation to the field equations similar in nature to the electromagnetic equations, so that is a mass is accelerating, one finds it produces radiation similar to the electromagnetic radiation of an accelerating charge. *On this account, one expects gravitational radiation* [emphasis added]. Using this analogy, Einstein was able to calculate the rate of radiation from a double star.

The analogy does not stop there, since

> Bondi has reminded us that if one looks for radiation pressure on a particle in gravitational waves, he must take into account the radiation produced by the particle itself. The situation here is analogous to an

electromagnetic wave passing over a particle.

In which, as in the Wheeler-Feynman absorber theory, it is only by including the radiation re-emitted by the absorbers that one derives the radiation damping force. Therefore, in the case examined by Weber, "the electromagnetic analogy suggests that if one were to go further, one might expect to find radiation pressure" (despite Weber's initial result that the wave does not impart energy to the particle).

However, Wheeler, who closes the session with this statement of the "non-sceptical" position, in regard to the wave analogy with electromagnetism, does not insist that the analogy has no points of breakdown. He concludes, "one has also to consider the nature of the one-sidedness of gravitational radiation. Here one faces the problem of what is to be meant by the the difference between retarded and advanced waves. If one employs the absorption theory of radiation damping in treating the above problem, one must employ the use of advanced and retarded waves. In flat space the concepts of advanced and retarded waves are easily understood. However, with gravitational waves, space is curved, and this has the consequence that it is difficult to distinguish between retarded and advanced waves. This is due to the fact that a pulse sent out by a source gets defracted by the curvature of space and secondary waves [tails] are thereby generated, which are in turn scattered. In this way, one may ultimately get contributions to an incoming wave, so that the distinction between retarded and advanced potentials is lost."

This "one-sidedness" of curved space was experienced, for instance, in the distinction which had to be drawn in GR between imposing no-incoming wave conditions in the past of a system and imposing outgoing wave only conditions in the future of the system, which were not equivalent, since curvature scattering produced ingoing waves in the system's future, even when there were none originally.

It is interesting to see throughout the transcripts of the Chapel Hill conference the basic modern ideas on gravitational waves (interaction with absorbers, what type of systems can be emitters, classification of the waves) emerging. Simply put, the basic social environment out of which such a theory can be constructed is just coming into being at this conference. The interplay of ideas between the various participants reinforces some thoughts which had been tentative (Bondi remarks in response to Wheeler's analogy between spherically symmetric atomic transition and Mardar's content-less cylindrical waves that "he has had suspicions on that side also"), inspires new collaborations (between Goldberg and Havas, for instance), and redirects attention to new formalisms (away from the pseudo-tensor, towards the curvature tensor). The conference is followed almost immediately by famous articles by Bondi and by Weber and Wheeler focusing on the interaction of the Einstein-Rosen cylindrical waves with absorbers. It is hardly surprising that, in the wake of this conference, the relativists were determined to set up a body to organize future meetings along the same lines. Lacking its own journals and meetings at this time, GR was badly in need of the sort of social superstructure which would allow for cross-pollination of ideas between the different active groups, without waiting simply for cultural drift based on the various personal and professional relationships to perform this function. An example would be Pirani's wanderings from Toronto to Pittsburgh to Cambridge to Dublin to London, via a haphazard network of connections between Infeld, Schild, Bondi and Synge (and the chance of being asked to review McVittie's paper for a mathematics journal), which put him in a position to make his contributions to gravitational wave theory by introducing to it the tools of Synge's equation of geodesic deviation, and Petrov's Riemannian classification scheme. Contrast this with the impact on several different researchers at once which his discussion of this work at Chapel Hill had.[4] The lesson was not lost on the participants.

So, the success of the Bern and Chapel Hill conferences led to the idea of organizing a permanent series of conferences along the same lines. An international committee was set up to oversee them, and the "GR" series of conferences came into being. Chapel Hill, the model for the series, was retroactively designated "GR1," with Bern left in the role of the "year zero" of the rebirth of GR theory, as GR0. Chapel Hill was followed by a conference in Royaumont, France (1959), and in 1962 by a conference in Warsaw hosted by Infeld. These conferences were dominated by other matters than radiation, but individual speakers, especially Bondi, continued to discuss the state of the field and urge further work on the back reaction problem.

Despite the rather pointed differences of opinion which have been expressed on this subject throughout the history of conferences on relativity, some interviewees expressed dissatisfaction with the amount of time devoted to discussion of the back reaction problem at these meetings. Havas in particular felt that the first 2 or 3 conferences after Chapel Hill should have had some time devoted to the fast-motion approach to this problem, and wished to speak himself there. He blames Infeld, who was prominent on the conference organizing committees (the international board which organized all of the early GR meetings), for keeping a rival approximation

[4] One can see evidence of influences on Feynman, Bondi, Wheeler and Weber. In Weber's case, the redirection of attention away from the pseudo-tensor and towards the Riemann tensor and geodesic deviation may have had the most profound and far-reaching consequences, by encouraging his efforts to construct a gravitational wave antenna, a device for "measurement of the Riemann tensor" (Weber, 1961).

scheme to EIH off the agenda. Therefore, no sooner were the social structures put in place to alleviate the isolation of the different schools of relativity, but the persons in charge of the new form of social organization are left to decide what issues are worthy of exposure, which leads to frustration with the forms of societal "governance".

Havas began work on the fast-motion approximation after Chapel Hill, collaborating with Goldberg who was also interested in this approach. He would have liked to present the case for this approximation method at the Royaumont conference, but only invited speakers were allowed to contribute. At the Warsaw and London (1965) conferences, Havas felt that Infeld was responsible for discouraging debate on rival approximation schemes to EIH. At London, W. Tulczyjew was assigned to give a report on the radiation problem, but Havas felt that the committee's choice was not an appropriate one. Similarly, Ehlers assigned Martin Walker (then a post-doc with him), to give the report at GR 10 in Padova, 1983, at the height of the quadrupole formula controversy, to which both Havas and Cooperstock objected privately (according to Havas), on the grounds that Walker to was too inexperienced to give the report, which would enter the historical record as "gospel truth". Walker was also associated with the non-sceptical position on this problem. After Walker's report, discussions from the floor were deferred to a workshop session, which gave little room for dissenting views to make their presence felt in the proceedings, in Havas' view (interview). In general then, the structures established by the community, beginning in the late fifties, to foster communication between different groups, while highly successful, in terms of longevity, and the eventual proliferation of international meetings (Texas symposia and Marcel Grossman meetings, for instance) and specialist journals (*General Relativity and Gravitation*, *Classical and Quantum*

Gravity, also suffered from a lack of responsiveness to the aims and views of research programs not represented on the governing bodies.[5] The original GR body for instance, was said to be entirely self-appointed, not surprisingly, given the smallness of the field, and its domination by a few figures.

The void left in this regard by the increasingly large and impersonal international meetings was filled by smaller and more informal gatherings, of which an important prototype was the Stevens' meetings of the 1960s. Held at the Stevens Institute of Technology in Hoboken, New Jersey (within sight of Manhattan), and organized by James Anderson, these seem to be far more fondly remembered than all but the very first international meetings (Bern and Chapel Hill).[6] The Stevens' meetings were ideally situated to attract contributors from the Bergmann and Wheeler schools, both based in or near the large urban centers of the eastern US. At these meetings anyone was entitled to get up and speak, and reports of work in progress were encouraged. The downside of this "free-for-all" was that what the established physicists viewed as "crackpots" would also come and make themselves heard, and "one had to just sort of grin and bear it," since there was no way of excluding them (Anderson, interview). Thus, the relativists attempted to solve the problem of trying to be inclusive and exclusive at the same time by organizing meetings on

[5] The specialist journals supplied a reference from the mainstream journals which, if they gradually ceased to be completely hostile territory for relativity, still presented problems. Nomenclature could be a real battleground. For instance, there was a struggle to get the *Physical Review* to accept the spelling spacetime over the hyphenated space-time (Thorne, private communication).

[6] A number of those interviewed and talked to recalled the Bern and Chapel Hill conferences as a sort of liberation from isolation with the discovery of kindred spirits. I encountered similarly enthusiastic recollections of the Stevens' meetings. Regarding the GR conferences, Anderson said "the early ones were great," but he found the later ones to be too large. "The Stevens' meetings, I think, *were* small. People did have a chance to interact *directly* with each other. So I think in that sense they did serve a function." (interview)

two different levels. Therefore, despite the avenue of the more informal meetings, established physicists still sought the platform of the larger meetings (one of whose attractive characteristics, besides the much larger available audience, was presumably that they were not open to "crackpots"), and felt the problem of underexposure of their ideas when denied it.

As the number of active relativists increased dramatically from the sixties on, other regions became able to support informal Stevens' type meetings of their own. In the US, this led to "Pacific," "Mid-West" and other gravity meetings, which provided a valuable arena for students and young researchers especially to present their work. However, it is a testament to the relatively low profile of the radiation problem until the 1970s, that the Stevens' meetings, in the recollection of Anderson, did not feature a great deal of discussion on the radiation problem. Therefore, in this respect at least, the larger international meetings may have been simply reflecting the situation on the ground in giving relatively short shrift to this problem after Chapel Hill. Those interested in the radiation problem at this time were something of a minority within the minority and frequently debate on gravitational waves was facilitated by specially organized workshops devoted to the subject (the Fermi school of 1961, the Trautman, Pirani and Bondi lectures at the Brandeis summer school (1965), Ehlers' Varenna Fermi school of 1976, and so on).

Chapter 9

The Economy of Gravitational Waves

An important requirement for the development of any scientific field is funding. The field of gravitational wave theory was fortunate in this regard in that, from 1956 to 1963, Joshua Goldberg was responsible for United States Air Force support of research in general relativity, based at the Aeronautical Research Laboratory at Wright-Patterson Air Force Base in Ohio. At this time and until the passage by Congress in 1969 of the Mansfield Amendment prohibiting the Department of Defence from sponsoring basic scientific research, the US armed forces provided considerable financial support for even very esoteric subjects in theoretical physics. Goldberg was active himself in the study of gravitational radiation, as we have seen, and did much to encourage groups such as that of Bondi and Pirani at King's College, London. Although support was available for groups outside the US, it was not permitted to support scientists based in communist countries, inhibiting the use of these funds to facilitate travel between the London group and Infeld's group in Warsaw, who interacted extensively (interview with Pirani). The Air Force laboratory itself was home to an active group until the 1970s. With one of his earliest grants, Goldberg was able to support the Chapel Hill conference organized by Bryce De Witt with Air Force money, and this important meeting became the forerunner of the successful General Relativity and Gravitation (GRG) series of conferences, which continues today. For a valuable account of this unlikely episode in the history of general relativity, see Goldberg (1988).

Following the Mansfield Amendment, research in relativity theory in the US depended primarily for its support on the National Science Foundation (NSF). The amendment remained in force only for one year, but it helped solidify an emerging political consensus (symbolized by the support the passage of the amendment received from both liberal and conservative congressional leaders) that basic scientific research (especially when conducted within the universities) should not be in the military's sphere, but was more appropriately the domain of civilian agencies like the NSF (Kevles, 1971, pg. 414). From 1973 to the present, the chief controller of funding for gravitation physics at the NSF has been Richard Isaacson, like Goldberg a relativist who has made important contributions to the theory of gravitational waves. Isaacson had also worked previously at the Air Force laboratory on the Wright-Patterson base. By good fortune then, despite the overall decrease in funding for theoretical physics precipitated by the Mansfield Amendment and the new trend in US government funding, the principal source of funds for research on gravitational wave theory remained in sympathetic and knowledgeable hands.

The advantage of having an insider at the primary funding agency did not ensure that everyone in the field was sponsored to the extent that they desired or felt necessary. Complaints about the funding choices made and its effect on research directions were very noticeable on the experimental side, where groups and research programs depended very heavily on the munificence of different (usually governmental) funding agencies. But even on the theoretical side, work on the problem of motion or radiation reaction was computationally so intensive that funding for postdocs and assistants could make a big difference to a group or research program. It may be that less popular research programs suffered in this regard (such as fast motion approximations versus slow motion ones), but it is difficult to assess the extent of this factor. Peter Havas has complained, for instance, that Isaacson did not regard problem of motion work as of great importance, and that little NSF funding was available for this. Certainly, from about 1980 on the theoretical analysis of the binary pulsar data was largely in the hands of Damour and his collaborators in France. Damour ascribes this to the lack of interest in the subject amongst American theorists and certainly there were other notable American relativists working on the problem of motion at this time, besides Havas, such as Ken Nordtvedt and Clifford Will. Goldberg felt that the primary factor preventing Havas pushing his program ahead was lack of time rather than lack of resources, but Havas clearly feels that an increase in resources would have translated to more available time.

Another researcher with well known complaints against the NSF is Weber, who pioneered the field of gravitational wave detection but has seen the lion's share of the increasingly large amounts of money going to this field allocated to rival groups and research programs. Still, while *from the individual perspective*, the effect of NSF policy can appear to be malign, nevertheless, it seems very likely that the subject of gravitational wave theory has benefited greatly from it close connections with the main funding agencies, especially when one keeps in mind how small and isolated a field this was at one time. Certainly Isaacson must receive a great share of the credit for the present high profile enjoyed by field of gravitational waves, considering that LIGO is the most expensive project ever funded by the National Sceicen Foundation.

Chapter 10

Evolution of the Physical Picture of Gravitational Waves

By the time of the jubilee conference at Bern in 1955, gravitational waves had had a theoretical existence with GR theory for close to 40 years, yet it would be fair to say that no convincing picture of their nature and behaviour as a physical effect had yet been formulated. Certainly there had been considerable debate as to whether they existed or not, whether they carried energy away from real systems and so on, but in the leading post-war textbook on GR, Peter Bergmann's *Introduction to the Theory of Relativity* (1942) we find gravitational waves (which are "typical for a field theory") introduced as "rapidly variable fields, which must originate whenever mass points undergo accelerations." This description can hardly have induced a vivid image of the waves in the mind of his readers, but it can be seen as a justification for, rather than a description of, the effect. One can't help feeling that, in this period, relativists were somewhat embarrassed by this step-child of field theory which had been foisted upon them with no clear purpose to its own existence, no prospect of experimental significance and which appeared intractable to theory.

While we have seen that the abstract style of analogy with electromagnetic field theory played an essential role in encouraging the growth of a theory of gravitational waves, relativists were slow to construct a *descriptive* analogy of the phenomenon. Perhaps this reflected the lack of a compelling theoretical understanding of the waves themselves. In many post-war popular and text-book discussions of gravitational waves, analogy is completely eschewed in describing the waves, the authors having preferred to substitute an explanation of the effect of the waves on some idealized system, such as a circle of particles, which would be deformed into an elliptical shape by the passage of the wave (e.g. Goldberg, 1966). That the need for some analogy which might strike a chord with the reader was felt is illustrated by the use of a comparison with "shear waves," one of two main types of acoustic waves in solid matter, in Bergmann's "The Riddle of Gravitation" (1968). Some authors (Wheeler, 1962) chose to elaborate the electromagnetic analogy more fully, down to the quantum analogy between "gravitons" and "photons." Wheeler includes also the circle to ellipse deformation, without diagram provided such as that provided by Goldberg.

By the early 1970s, at least, we find that confidence in the theoretical understanding of the waves had increased to the point where relativists at last felt comfortable in reaching back to the metaphor underlying all descriptions of wave phenomena, that of ripples on water. In the best known modern textbook, Misner, Thorne and Wheeler (1973), we find the use of this physical metaphor right at the beginning of the chapter on gravitational waves. Their version is perhaps the canonical one.

> Just as one identifies as "water waves" small ripples rolling across the ocean,[1] so one gives the name "gravitational waves" to small ripples rolling across spacetime. Ripples of what? Ripples in the shape of the ocean's surface; *ripples in the shape (i.e. curvature) of spacetime* [emphasis added]. Both types of waves are idealizations. One cannot, with infinite accuracy, delineate at any moment which drops of water are in the waves and which are in the underlying ocean: Similarly, one cannot delineate precisely which parts of the spacetime curvature are in the ripples and which are in the cosmological background. But one can almost do so; otherwise one would not speak of "waves"!

In Banesh Hoffmann's 1972 biography of Einstein, the same description is found, "ripples of curvature traveling with the speed of light," along with "frozen corrugations of space-time acquiring for us the aspect of motion because of out passage through time." This shows that by the seventies, the picturesque physical metaphor was thought suitable for inclusion in a popular account.

Different similes continue to be expressed side-by-side. The McGraw-Hill Enc. of Physics (1983) employs both "ripples in the curvature of space-time" and "propagating patterns of strain." Rees, Ruffini and Wheeler, (1974) state, perhaps a little ahistorically, that Einstein showed "that geometry can undulate and carry energy." (pg. 84). They later add, when discussing the sceptics (while noting that "doubts

[1] Actually, ocean waves are typically "gravity waves," that is water waves whose restoring force is supplied by the water's own weight and not by surface tension, as with small ripples. It is because of these gravity waves that gravitational waves acquired the slightly unwieldy name they bear. It is a sign of their relatively recent emergence as an important physical phenomenon that the usage "gravity waves" is now increasingly applied to gravitational waves even by physicists.

[of the reality of gravitational waves] [of] earlier days have now been dissipated") that "any talk of a gravitational wave carrying energy is nonsense. There is no such thing as the local density of gravitational wave energy."

A popular book on *The search for Gravity Waves* (Davies, 1980) begins with the electromagnetic analogy but "caution should be exercised in stretching the analogy too far" (pg.26). It also introduces gravity waves as ripples of geometry, but cautions against the danger of confusion with mere co-ordinate ripples (pg.49).

In contrast, Weber's seminal text on gravity wave detection (1961), does not allude to any analogy other than the abstract electromagnetic one. There is no mention of ripples in spacetime, geometry or anything else.

If by 1970 relativists were sufficiently confident in their own theoretical picture of gravitational waves to begin to advance a more compelling metaphorical description to the student or lay-person, we may look for the source of this self-assurance in the theoretical work of the previous decade, which laid the foundations for a physical (as opposed to merely formal, mathematical) theory of gravitational waves. Two research groups in particular played a key role in this endeavour: the Bondi-Pirani group in London and John Wheeler's group at Princeton.

As we have seen, the Bern conference itself helped to inspire Bondi to take up the problem of gravitational waves. Both he and Pirani were motivated by the uncertain status of the phenomenon at that time to address the issues that had been raised (by McVittie, Rosen, Infeld and others), as to the existence of gravitational waves. They began, together with Ivor Robinson, by rebutting the longstanding objections to the existence of plane waves in gravity (this view was sufficiently orthodox in the immediate post-war period to enter the textbooks, (Bergmann, 1942)). This was of considerable importance, since the cylindrical waves, the only other exact wave solution of Einstein's equations available at that time, was entirely unphysical, requiring an infinitely long source for generation. With such exact solutions in hand, the London group could proceed to a rigorous study of the asymptotic behaviour of the waves. While still idealized, the systems they would study, involving a source and the distant waves generated by it, would be an acceptable idealization, modeling a compact source.

Bondi has described his 1962 paper with Van der Burg and Metzner as "the best scientific work I have ever done," (Bondi, 1990) and it certainly played an influential role in persuading people that gravitational waves really could transmit away mass and energy from a source. This is the paper in which Bondi introduces the famous "news function". The paper itself gives a clear sense of how the particular concerns which motivate a scientist to his best work can be completely lost on his audience, even when the paper in question is rather successful and influential.

The paper touches on two problems which Bondi regarded as very troubling. The first was the question of whether self-gravitating sources (such as binary systems) could radiate. Although by this time most other relativists no longer gave this matter such credence, Bondi still quotes Infeld's *Motion and Relativity* approvingly on the matter of radiation from binary systems.

> The lack of radiation for freely falling particles emerges from Infeld's work, but one would like to generalize this to non-singular equations of state. The most clear-cut case then would seem to be pressure-free dust, but beyond this it is tempting to suggest that perfectly elastic equations of state do not lead to radiation.

This is interesting, since Bondi was surely well aware of Trautman's work showing that the radiation terms in EIH were not co-ordinate independent, which was carried on while Trautman was in London at Bondi's department. Neither Bondi nor Infeld seem to have regarded Trautman's 1958 papers as showing that freely-falling systems ought to radiate. Certainly whatever objections in principle might have been raised against Trautman's paper could have been raised with equal force against Infeld's radiation chapter in *Motion and Relativity*, which Bondi here refers to as almost definitive, up to a point.

Bondi's second worry concerned the presence of what we might call infinitely long tails in gravitational radiation. This phenomenon was demonstrated conclusively in the 1960s in the work of Newman and Penrose. Tails are disturbances which fail to "keep up" with the rest of the wave, thus violating Huyghen's principle, that the total disturbance is confined to a single expanding wave front defined by the speed of the wave. In GR, tails, which can arise as the wave interacts with the curved background metric of the source, bounce around indefinitely, preventing the system's field from ever quite settling down to a quiescent, static state again. The source will always be surrounded by echos of the original disturbance which caused the original burst of radiation, however short-lived. Bondi has referred to the discovery of tails in gravitational waves as "absolutely disastrous," since it shows "the world to be a much more complicated place than had been thought" (Bondi, 1970, pg.269). The problem, as Bondi sees it, is that the source's future behaviour now depends " on its earlier history right to the year dot ... and this dependence on history is something which I think we can now definitely identify in the general theory and which makes it a markedly less attractive theory." He concludes, "we have to live with this theory

... [but] it shows itself to be a little nastier than might be expected."

When the author visited Bondi in 1995, this problem of tails in GR was still one which bothered him greatly. In another interview, Ted Newman well recalled Bondi's strong and controversial rejection of tails until the work of Newman and Penrose compelled him to accept their existence. Part of his concern can no doubt be found in a practical consideration from his 1962 paper. In this paper Bondi manages to show that a massive system which is initially and finally static, but which goes through some non-axisymmetric disturbance in between, will have lost mass in the meantime to gravitational waves it emitted. The period of emission is characterized by a certain function, called the "news function" by Bondi, which is non-zero only during this period. One reason for his insistence on initial and final stasicity is that, in GR, the problem of defining the mass of a system is a rather tricky one, especially is the system is dynamic. One way around this problem, increasingly in use from the 1960s on, is to look at the field of the (isolated) system far from the system, where it should approximate to the Schwarzschild metric. In this context, one can define the mass as being that of a Schwarzschild body with an equivalent field. Bondi thus laid great stress on the initial and final stasicity of his system, even though he was successful in defining a time-varying mass function which reduced to the static field Schwarzschild mass in the static case. Tails which refused ever to die away completely were one obvious threat to this picture. A system with which consisted of freely falling dust, on the other hand, seemed to lack any mechanism do knock itself out of its initial quiescent state (in addition, binary systems in general lacked an initially static configuration). Such a system posed a threat to the other boundary condition, the initial one. It

> ... does not contain news in this sense. Its future is a clear consequence of its past, and it would seem difficult to draw a distinguishing line between different systems of this kind though conceivably the pressure-free gas might be the only non-radiative material, all others radiating if in motion (Bondi, van der Burg and Metzner 1962).

Bondi regarded the news function as *demarcating* those systems which could radiate from those which could not. "News" was the characteristic of a radiating system, the possibility that information would be carried by radiation out to a distant observer rather presumed the existence of something worth reporting. For Bondi, the medium was indeed the message, in the sense that no possibility of a message implied no medium (no waves). This distinction was crucial to Bondi's interpretation of the news function which "nobody has fully understood" in Bondi's view (interview). From 1960 on, only a handful of researchers, mostly from or influenced by the older generation of sceptics, continued to feel that freely-falling systems did not (or were likely to turn out not to) radiate. Bondi himself, who ceased to be active in the field from 1970 on, did not convince himself that even the idealized freely-falling dust case should radiate, in the form of two dust filled stars orbiting each other, until recently (interview). In spite of his own doubts on this score, his work, and that of others associated with his and Pirani's group in London played a vital role in developing an increasingly convincing picture of gravitational waves in the 1960s. His scepticism, too, played a role in convincing another celebrated astrophysicist, Chandrasekhar, to take up the back reaction problem (see next chapter).

Bondi also gives (in part D of (Bondi, Van Der Burg and Metzner, 1962), which section is authored by Bondi alone) an important treatment of the reception of a gravitational wave. His receiver consists of "two massive particles ... with a motor between them". He analyzes its absorption of incoming waves, consequent motion, and assuming the waves are weak, and therefore a linearized scheme is appropriate, its own emission. He is able, based on his analysis, to derive the quadrupole formula for energy absorption, "identical (apart from a numerical factor) with the electromagnetic one". Bondi's quadrupole formula is, unlike the back reaction quadrupole formula, is generally accepted as correct. The problem for sceptics, like Bondi himself, is whether a non-linear source, such as a binary system would emit quadrupole radiation at all, or according to the same formula. His quadrupole moment of the *receiver* shows the wave very far from the source indeed carries a flux of energy with a quadrupole characteristic. Many physicists regarded it as acceptable to equate this "far-zone" or "receiver" quadrupole, with the "near-zone" quadrupole moment of the source. Others, however, such as Anderson, criticized this as "proof by naming" (interview), since they felt there were no grounds for assuming that the two quadrupole formulas really expressed the same quantities.

At Princeton John Wheeler and his students were also interested in addressing questions concerning the physical description of gravitational waves. Wheeler and his group had some influences acting on them which were unusual. For instance, Weber spent time on a fellowship with Wheeler while he developed his ideas of detecting gravitational waves, and this naturally helped stimulate some interest there (Misner, interview). Of course Wheeler himself participated in the debates such as whether the cylindrical waves could carry energy, and his students were exposed to the controversy at conferences in the late 50s (such as Chapel Hill and Rayonnement). Like Bondi, Wheeler and his group disliked the pseudo-tensor approach to the problem of gravitational mass-energy, and preferred to examine the far-field effect of a source to determine its mass (two important definitions of gravitational mass from this period are the "Bondi mass" and the "Arnowitt-Deser-Misner (ADM) mass" from Wheeler's group). Another topic which focused the Princeton groups attention on gravitational waves was the Geon, Wheeler's name for an entity constructed of a wave bundle held together by its own gravitational attraction. Wheeler had examined the idea of an electromagnetic geon, constructed from high frequency electromagnetic waves, and the idea was partly of interest because of a sense that such a "body" might prove a classical prototype for a model of elementary particles composed of pure field (Brill, interview).

Wheeler was already a distinguished theoretical physicist when he turned his attention to GR in the 1950s. This was a time in which many field theorists turned towards gravity as the next stage in the career of quantum field theory after the successful post-war battles with the problems of renormalization and so on in quantum electrodynamics. Unlike his student Feynman (who was more typical in this), who was primarily interested in quantum gravity, on which only slow progress was made at this time, Wheeler adapted effectively to the mores of the relativity community, nevertheless retaining a set of sensibilities which set him somewhat apart in that milieu. He certainly viewed the geometric description of spacetime as an essential feature of the theory, rather than a curiosity of a wayward and eccentric field theory.

Wheeler set his students Dieter Brill and James Hartle to work on the problem of the "gravitational geon". This would be a geon constructed out of short-wavelength gravitational waves, especially of interest in that they would be pure sourceless gravitational field constructs, gravitational disturbances held together by their own gravitational attraction. A tool for the investigation of this problem had been borrowed by Wheeler from optics, the idea of the "two-lengthscale expansion" or "shortwave expansion," in which very strong (and therefore highly non-linear) gravitational fields could be expanded in terms of the ratio of a short lengthscale describing local disturbances in the field and a long lengthscale describing the background curvature in which the disturbances found themselves. This provided a scheme in which waves could be studied in the context of their own background curvature. That is, the metric was divided into a long-length scale, time averaged curvature representing the gravitational field produced by the waves' mass-energy, and a short lengthscale, locally varying field representing the actual waves. Brill and Hartle (1964) were able to employ this scheme to study the geon, and Brill (1959) was able to show that toroidal wave pulses (first proposed as a spatially limited version of the Einstein-Rosen cylindrical waves by Weber and Wheeler (1957), and a suitable metric was subsequently suggested by Bondi) appeared to have mass when seen from a distance. The gravitational geon was one way in which the reality of gravitational waves as an energetic phenomenon was argued at this time. Indeed, in recent times Cooperstock (see below) has attempted to show that the gravitational geon cannot exist, in support of his energy localization hypothesis, which argues that gravitational field energy cannot propagate through vacuum.

Subsequently, a student of Wheeler's student Charles Misner, Richard Isaacson who developed the shortwave expansion description of gravitational waves further, was able to produce a tensor quantity, averaged over several wavelengths of a wave which described the wave energy in an invariant way. This provided a practical means in which the energy in gravitational waves could be calculated for the purposes of estimating their flux of energy, and offered a way to avoid the controversial pseudotensor approach to estimating energy in the wave.

The two-lengthscale metric scheme played a critical role in visualizing gravitational waves within GR theory. The picture of gravitational waves as small scale "ripples" of curvature (gravitational field) superimposed on the largescale background curvature of spacetime sparked the introduction of the now-commonplace metaphor of gravitational waves as "ripples in the curvature of spacetime" (Thorne, private communication). It was certainly Wheeler, with his gift both for visualization and for neologisms (geon[2], black hole), who popularized the spacetime curvature picture of GR, with his *Geometrodynamics* (1962). It is Wheeler himself who points out the 19th century Riemannian premonition of our contemporary picture of gravitational waves in a paper by the mathematician William Clifford (Wheeler, RMP, 1961). Clifford (a well-known Victorian mathematician who popularized Riemannian geometry for an English speaking audience), inspired by Riemann's epochal work on non-Euclidean geometry wrote

> I hold in fact (1) that small portions of space *are* in fact of a nature analogous to little hills on a surface which is on average flat; namely, that the ordinary laws of geometry are not valid in them. (2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave. (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or ethereal. (4) That in the physical world nothing else takes place but this

[2] which narrowly escaped being called a "kugelblitz"

> variation, subject (possibly) to the law of continuity. (W.K. Clifford, "On the Space-Theory of Matter," *Proceedings of the Cambridge Philosophical Society* **2**, 157-158 (1876).)

Clifford's idea of *curvature waves* and Poincaré's idea of the *wave of acceleration* are quite different concepts in their origins. Einstein's gravitation theory, which married Riemannian geometry and relativistic field theory contained the possibility within of marrying the two, but it was not until the 1950s and 60s that they were really unified in the modern picture of gravitational waves. Wheeler's vision of the versatility of the gravitational field, supporting large scale curvature and small scale ripples found expression in the two-lengthscale approach, and undoubtedly helped foment a new visualization of the idea of gravitational waves. Brill himself recalls that "the idea of small scale ripples was around" at that time (interview), although it may have been some time before the physical metaphor employed in Misner, Thorne and Wheeler really gained currency.

Chapter 11

The Problem of Motion in the 1960s

In the late fifties, the EIH approximation was further developed by Andrzej Trautman, a student in Infeld's group in Warsaw, who departed from Infeld's approach in adopting "outgoing wave only" boundary conditions. He also confirmed Goldberg's earlier claim that the net back-reaction effect could not be transformed away, but merely moved between order in the expansion and another. Thus, his paper addresses the following question, arising out of the Infeld, Scheidegger and Goldberg debate, "whether the situation in GR resembles that in Newtonian mechanics rather than that in electrodynamics." The possible resemblances are described as follows.

> In Newtonian mechanics the initial positions and velocities of point-masses determine their motion completely. The situation is different in electrodynamics where the initial values of the field are required besides information concerning changes. Two *free* point-charges of opposite signs

> may move uniformly around a circle in a standing-wave electromagnetic field. However, the same charges may alternatively produce outgoing radiation. Their motion will not then be periodic; they will undergo damping. Which of these cases occurs in any particular system depends on the initial and boundary conditions.

So the question at issue, according to Trautman, is whether or not the analogy with electromagnetism, which gives rise to gravitational radiation, should be favoured over an analogy with Newtonian mechanics, in which orbital motion is not damped, and radiation does not exist in the gravitational field. After going over the electromagnetic analogy, and presenting a solution to the gravitational field equations in the original EIH form, he notes that

> by analogy with the scalar wave equation and Maxwell's theory, solutions of [this] form may be interpreted as representing standing-wave fields. In order to get solutions corresponding to 'retarded' or 'advanced' fields the series must be supplemented with the missing *radiation* terms.

He thus recalls the phrasing of Infeld and Wallace, where the imposition of retarded conditions is seen as arbitrary. Indeed, Trautman expresses well the underlying philosophical (as opposed to merely technical) objection to the post-Newtonian approach to the radiation problem of fast-motion advocates (like Bonnor and Havas, see below). The slow-motion expansions reduced GR, at first order, to a gravitational theory (Newton's) which did not admit radiation, and never had. Fast-motion expansions, of course, reduced GR at first order to special-relativistic dynamics, a theory in which the problem of radiation was very advanced, but which was not a gravitational theory. Both therefore had problems, which had to be overcome by pressing to higher orders in the relevant expansions, but the post-Newtonian method suffered for years from the difficulty of introducing the radiation terms in a non-arbitrary way. Trautman himself showed that one could introduce radiation terms, with a correct choice of boundary condition, which could not be transformed away, contrary to the claims of Infeld and Scheidegger, but he did not recover the quadrupole formula result though he did, unlike Hu, find positive damping.

Specifically what Trautman shows is that a choice of radiation terms corresponding to outgoing waves at the 5th order in the space-space metric terms cannot be transformed away, when in combination with the equivalent time-time metric terms at 7th order. The slow-motion approximation assumes that the time derivatives are smaller by order c than all space derivatives, and therefore that where space-space terms in the metric are of order n in $1/c$, the corresponding time-space terms in the metric will be of order $n + 1$ and the time-time terms of order $n + 2$. It is the whole metric, not individual components, which is invariant, so that while it may be possible to transform away the radiation terms at order 5 and 6, the radiation effect will merely show up at order 7. Whereas Fock, as we have seen, chose a particular coordinate scheme and stuck with it, Infeld, the upholder of covariance, had experimented with different coordinate shifts, in pursuit of a non-damping result. Trautman, Infeld's student, rejected Fock's strict adherence to harmonic coordinates as "somewhat stringent," seeing "no reason for restricting ourselves to harmonic coordinates only". Instead, he generalized the boundary conditions, in order to investigate Infeld's old argument that the radiation terms were merely coordinate effects, successfully showing that they were not. That Infeld cannot have been convinced is shown by his remarks in *Motion and Relativity* two years later.

Indeed, Trautman also (1958a) showed quite generally that imposing outgoing wave boundary conditions would lead to a quantity associated with the flux of energy in the asymptotic waves being zero or positive, and therefore that such conditions should lead either to no radiation, or to radiation carrying energy outward from their source.

At about this time, Rosen, now at the Technion Institute in Israel, encouraged his graduate student Asher Peres to attack the problem as a means of deciding the dispute between Fock and Infeld over the existence of radiation reaction in the post-Newtonian problem of motion. Peres employed a method bearing some similarities to that of Fock, making use of the De Donder ("harmonic") gauge condition but also employing the singularities used by the EIH method (which Rosen disliked) (Peres, 1959a). His initial results, however, gave anti-damping for the binary system, like Hu before him, and thus failed to shed any light on the Fock-Infeld dispute (Peres, 1959b). After finishing his thesis however, Peres realized where the problem with his previous paper lay (Peres, 1959c). In imposing boundary conditions on the equations of motion for the binary system, he had inadvertently chosen conditions which included incoming as well as outgoing radiation at infinity, so that he was unwittingly introducing a source of energy to power the outward spiral of the binary (as Eddington might have remarked, the rate of inspiral can proceed at the speed of thought, if one is not careful).

Peres explains the ease with which such confusion can arise in his correction paper,

> at each stage in the approximation procedure ... there is a considerable

freedom of choice of solutions, each representing a possible motion and a gravitational field belonging thereto. Only one of these solutions behaves at infinity as purely outgoing waves; the remaining ones also contain incoming waves. However, it is difficult to determine which solution is the correct one because the n-th term of a series expansion into powers of (v/c) behaves in the wave zone as R^{n-2} [R is the separation of the binary components], and no boundary conditions for each stage of the procedure are known. The purpose of this present note is to give a criterion which partly removes this ambiguity. ... [The] method is not sufficient in general but it gives unambiguous results up to the seventh order. As a consequence, one has to modify the fields that were previously used and ... [the result] agrees with [the Landau and Lifshitz quadrupole formula]. The fact that one previously obtained a negative radiated energy should be ascribed to the presence of incoming gravitational waves, which were absorbed by the particles. (Peres, 1959c)

Essentially Peres' method involved expanding the retarded potential at infinity as a Taylor series, and then whenever terms like the leading order (unretarded) potential occured at some point in the approximation of the system's motion, to add the succeeding order (in $1/c$) part of the retarded potential into the next order of the slow-motion expansion, thus imposing the retarded potential, outgoing wave condition on the motion of the source (Peres, 1960).

Peres' 1960 paper has been referred to as containing the first correct back reaction calculation (Thorne 1989). Nevertheless, the perceived arbitrariness of the slow-motion approach in imposing the wave zone boundary conditions from one step in the expansion to the next, which seemed reflected in the wildly differing results produced by the method, gave rise to arguments that the approach was hopeless (Bonnor 1963).

The difficulty with the slow-motion expansion could be addressed, in the manner of Fock, by employing a different, fast-motion-like expansion in the region far from the source. Such an expansion did admit of radiation fields, while the slow-motion approach to the problem of motion could deal successfully with the source. The problem would then remain, how to match the two expansions in the two different regions to each other, so that whatever boundary conditions were imposed in the wave zone would be correctly applied to the solution of the source's motion. Peres was meticulous in his matching between the two regions, but, as he noted himself, his method was not at all general. A completely general, unambiguous scheme would have to wait until the end of the next decade, but in the meantime, Peres and Trautman had shown a way out of the dilemma facing the slow-motion approach, even while Bonnor and others were pronouncing it hopeless.

While conceptually more appealing in that it took as its starting point the linear approximation in which the radiation analogy with electromagnetism had arisen in the first place, whereas the post-Newtonian expansion approximated to a theory of gravity which had never admitted the idea of radiation at all, the alternative fast-motion approach, as developed by Havas and Goldberg (Havas and Goldberg 1962) and others (for example, Bertotti and Plebanski 1960), was also proving frustrating. It was a difficult task to go beyond the leading order corrections to the linearized theory and the results of applying that step to the reaction problem, published by Smith and Havas, again showed an energy gain in the source (Smith and Havas

1965). Therefore in his review paper of 1963, Bonnor concludes that the question of whether freely falling sources experienced damping remained unsettled.

Bonnor's paper contains an interesting comparison of the two main representatives of the slow and fast motion approaches at that time, on the one hand EIH and EIH-like schemes, and on the other hand the work of Havas and collaborators. With regard to EIH, Bonnor is not greatly impressed by the progress made.

> A choice of solutions of the EIH equations is available, and that made by EIH refers to the ... non-radiative field. One can try to use the retarded potential instead, though this leads to much arbitrariness ... Nevertheless, a number of workers have used the EIH method on radiation problems, and their conflicting results are a monument to its unsuitability for the task.

With regard to Havas' approach, Bonnor holds out more hope because "unlike ... EIH, [it] is covariant with respect to Lorentz transformations of the flat Minkowskian background metric" (i.e. it satisfies the axioms of special relativity). Nevertheless, "the linear approximation tells us nothing about radiation from a freely gravitating system" (since in the linear approximation particles are unaffected by each other's gravitational attraction), while Havas' "delicate mathematical processes" designed to overcome the "great difficulties" of the next level of approximation (first post-linear) produce an admittedly "disappointing" result, which fails to agree with the well known perihelion shift result and, like Hu and Peres' first try, gives "an energy *gain* due to radiation, an inexplicable result". Bonnor still holds out hope, saying "it would be of great interest if Havas' method could be carried one step further." With this sentiment Havas was heartily in agreement, but unfortunately it was never to be carried out.

Bonnor had himself adopted the approach of analyzing a simple mechanical system which begins and ends in a static state, a model which Bondi adopted. Like Bondi, though with a less involved method, Bonnor was able to show that such a system would lose mass, but he also concluded that "whether freely gravitating bodies radiate, and if so with what effect on the motion, is still an open question."

Havas and Goldberg benefited from Havas' extensive knowledge of the literature on the classical problem of motion, and exploited a fast-motion approximation scheme of Havas' (1957) based on a pre-war one due to the Polish theorist Mathisson.[1] Like EIH they employed point sources, and made use of renormalization rather than EIH's surface integrals to avoid the resulting divergent integrals. As mentioned already, their results (Havas and Goldberg, 1962) disagreed with the well known result for the perihelion shift of Mercury. However, Havas regarded the radiation effects as being the particular target of the method, and expected that the scheme, which had presumably not been pushed to high enough order (one past the linear order in the 1962 paper) to recover the perihelion shift, might still correctly derive the back reaction on a system due to wave emission.

At first, Havas, Goldberg, and Havas' student Stanley Smith all did independently arrive at a result showing the loss of energy by such a system. In fact Havas had derived this result as early as 1957, where he concluded that "the gravitational and the electromagnetic radiation damping terms are of the same form, and thus it appears that gravitational radiation effects have as much reality as electromagnetic

[1] Interestingly, Havas reports in a recent paper that Mathisson was nearly chosen by Einstein as a collaborator on the problem of motion instead of his compatriot Infeld, which might have resulted in an EMH paper rather different from the EIH which actually exists.

ones." But during 1958 Havas noticed an error in his calculations which, as with Hu's (very different) calculations years before, reversed the sign of the result. Havas recalls that "all three of us had been so sure that there must be damping that we had not paid enough attention and each with a different slip had indeed gotten it" (Havas, private communication). Smith and Havas, after thorough checking of this "disquieting result," discussed it at length in a 1965 paper.

They first of all noted that it was contradicted by Trautman's result showing that the use of retarded potentials ought to lead to an outgoing flow of radiation. That Trautman's result held for the exact theory indicated that the discrepancy was due to a failure in the approximation, at least to the order pursued by Smith and Havas (again, first post-linear). Therefore,

> ... although the possibility should not be overlooked that an approach to the problem of gravitational radiation by considerations of energy flux at infinity is inherently inadequate, we would rather expect that an investigation of the higher orders of approximation would indeed yield an energy loss in the retarded case (or possibly show the absence of any energy change) (Smith and Havas, 1965).

Despite a great deal of other work on fast-motion approximations of the problem of motion (not primarily directed at the radiation problem in most cases) at this time (e.g. Bertotti and Plebanski (1960), Kerr (1959) and Westpfahl (1985)), the result for the binary radiation reaction problem found by Havas and collaborators presented a problem for further work on the radiation problem in this approach. The first post-linear order was already rather complicated, and it would appear there was little stomach for attempting the next order. Havas felt that the method suffered from a lack of exposure, complaining, as we have seen, that at the GR meetings of that time, Infeld, who championed the post-Newtonian EIH method, prevented papers being given on its rival. Plebanski, who worked both in the EIH and fast-motion schemes reports that Infeld showed some hostility to the latter approximation method (interview).

Havas would himself have liked to press ahead with the next order of approximation, but lacked the time and the manpower (in the form of students and collaborators), for such an undertaking. In the early 60s he benefited from his association with Goldberg at the USAF, and was able to get air force grants to attract visitors to Lehigh, such as Plebanski and Moshe Carmeli, who looked at the problem of a massive particle moving in an external field, showing how the field could be divided into the external field and the particle's own self-field, despite the problems that the non-linearities of the field introduced compared with the electromagnetic case. With the ending of Air Force funding in the early 70s (and from Havas' point of view possibly Goldberg's departure from the ARL) funding for science in smaller institutions especially was strongly curtailed. This meant that, for the next few years, the field was left to new efforts in the slow-motion approximation. Havas, despite the dissapointment he faced in his own efforts, still viewed with grave, perhaps increased, scepticism these efforts, which he suspected were justifying their means by their ends.

In Smith and Havas he writes

> it is perhaps even more unfortunate that the inconsistent approximation used [Einstein's original linear approximation] led to a result which conformed so completely to expectations based on physical "intuition"

> borrowed from electrodynamics; this has led to a ready acceptance of Einstein's [quadrupole] formula for the supposed energy loss and a neglect of critical study of its derivation. It also has tempted many authors to *justify* other (classical or quantum mechanical) approximation methods by their ability to reproduce [the quadrupole formula], which clearly is not a valid criterion.

Like Bonnor, he regards the problem of radiation reaction in a freely falling system as still open.

Bondi, who with his collaborators had done much to improve the understanding of wave propagation far from the source (see especially Bondi, van der Burg and Metzner 1962 and Sachs 1962) made this point at the Warsaw conference of 1962 (Bondi 1964). However, there were those, like Feynman, who viewed the relativists' caution with impatience. As early as Chapel Hill, Feynman was "surprised to find a whole day at the conference devoted to this question" (of whether gravity waves could carry energy) (letter from R.P. Feynman to Victor Weisskopf, February 11, 1961)[2]. He was caustic in his appraisal of the discussions at the Warsaw conference, writing to his wife that they were "not good for my blood pressure" (Feynman 1988).

Bondi's lecture, however, inspired the astrophysicist Subrahmanyan Chandrasekhar to take up the problem (Chandrasekhar, interview). Born in India in 1910, Chandrasekhar is best known for his discovery, while on board ship to England from India in 1930, of an upper limit to the mass of White Dwarf stars, above which such a star cannot avoid collapsing, crushing its own atoms. This result was extremely

[2]A photocopy of this letter was supplied to me by Kip Thorne. Copies are also to be found in the Feynman papers at the California Insitute of Technology, Pasadena

controversial in the thirties, and the hostility which Eddington showed towards it in public forced Chandrasekhar to leave England for America where he worked for the rest of his life at the University of Chicago. In 1983 he received the Nobel prize in physics, principally because of his celebrated work on White Dwarfs. In 1962, inspired by a desire to work on GR just as the subject was about to take off, he gained permission to attend the Warsaw conference, the result of which was that he took up the problem of radiation damping of binary systems.

Throughout the 1960s, Chandrasekhar developed his own slow-motion formalism, dealing with extended fluid bodies (as opposed to point masses) at one post-Newtonian order after another (Chandrasekhar 1965). By the end of the decade he had advanced far enough in the expansion (to post-$2\frac{1}{2}$-Newtonian order) to describe reaction effects. His conclusion agreed with the quadrupole formula result (Chandrasekhar and Esposito 1970).

At about this time William Burke, a student of Kip Thorne's at Caltech, introduced improvements to the slow-motion approach which removed much of the arbitrariness in imposing the boundary conditions. Burke selected the problem of radiation damping in binaries for himself, since Thorne had been convinced by Peres' work that the problem was solved in the slow-motion case. Influenced by Frank Estabrook, and an applied mathematician at Caltech Poco Logerstrom, Burke made use of the applied mathematics technique of matched asymptotic expansions, which allowed one to determine the solution to the problem of motion in the zone near the source, by matching it through an intermediate zone, to the "outgoing wave only," or other potential of choice, in the far zone of the waves (Thorne, private communication). In this way the chosen boundary condition could be unambiguously applied to the solution of the near zone problem, thus addressing the arbitrariness which bedeviled the slow motion approach up to this time (Burke 1969). With this novel approach, Burke and Thorne also derived the quadrupole formula for emission from binary systems (Burke and Thorne 1970).

Burke also constructed a "radiation-reaction potential" which could describe the damping force, and Thorne applied this approach to the damping problem in the Regge-Wheeler guage, where he was able to derive the quadrupole formula result in only two iterations (Thorne, private communication). It was considered a general rule in the subject that a minimum of three iterations of the field equations (three rounds of applying corrected equations for the motion of the constituents back to the field equations to discover more exact equations of motion), were required to describe radiation damping in gravitationally bound systems. This guideline was subsequently canonised by Walker and Will in 1980. However, in writing the relevant section of the textbook Misner, Thorne and Wheeler (1973), Thorne made use of the De Donder gauge in this calculation, in which gauge three iterations are required, unlike the special case of the Regge-Wheeler gauge (Thorne, private communication). Even so, because of a compensating error, Thorne did recover the quadrupole formula, and it was not until Walker and Will's paper that this error was discovered.

Despite Thorne's complacency, to begin with Burke considered that his contributions did not settle the issue of whether bound systems would experience damping. He noted in early versions of his work that his approach was not guaranteed to work outside of linearizable systems, and therefore could not settle the issue for freely gravitating systems. There is still on display at Caltech the record of a wager between Burke and Thorne on whether non-linear effects would "significantly affect the radiation in the lowest order" from sources in free-fall motion. Thorne gave odds of 25-1 for this bet, which Burke conceded in 1970.

Chapter 12

Emergence of GR into the Physics Mainstream

As we have seen, great progress had been made during the sixties on many fronts in the description of wave propagation and interaction with matter. Supernovae and binary neutron stars began to be suggested as possible astrophysical sources during this period, inspired at first by Weber's work (Dyson, 1963). When Weber embarked on his experimental program in about 1960 there was no existing theory of sources in the practical sense. Most theoretical work on gravitational wave emission had focused on binary systems (with the Earth's own orbit around the Sun frequently given as an example.) Those who were most convinced of the reality of gravitational wave emission from such systems (e.g. Landau and Lifshitz and Fock) were also most adamant that the effect was practically negligible. From Weber's point of view, the challenge was simply to achieve the maximum sensitivity possible with his instrument, and therefore, in that sense, the theory deficit was not critical. However, his chosen design, a resonant bar which would "ring" in response to gravitational waves oscillating at the bar's fundamental frequency, had a rather narrow frequency bandwidth. Therefore it was important to choose a frequency at which one was liable to hear something from cosmic sources. Again, the demands of sensitivity restricted the choice at hand. A large bar was necessary to achieve higher sensitivities, but logistical considerations limited the total size. As total size was a principal determinant of resonant frequency the range of operating frequencies was thus somewhat restricted. Weber chose an operating frequency of 1661 Hz, relying largely on his intuition of the subject built up over a considerable time spent in theoretical preparation, partly in collaboration with John Wheeler, with whom he spent a postdoctoral fellowship year at Princeton.

It was only after the beginning of his experimental program that Weber began to receive some suggestions as to possible sources. On a subsequent visit to Princeton, Freeman Dyson suggested asymmetric supernova collapse, in which a bump in a star undergoing gravitational collapse would be spun around more and more rapidly as the star shrank, releasing increasing quantities of quadrupole radiation, and sweeping in frequency up through the kilohertz range of the bar (Weber, interview).

For many years, this type of source remained a favoured candidate for gravitational wave detection. It was eventually superseded, however, by another suggestion of Dyson's published in the book *Interstellar Communication* (Cameron, 1963) in an article titled "Gravitational Machines." The article discusses possible uses of gravitational energy by "advanced civilizations," some of which, like the gravitational slingshot which powered the Voyager spacecraft, have since become a reality. He observes that "if a close binary system could ever be formed from a pair of neutron stars" (whose individual existence, he notes, is "uncertain") (pg. 119) these systems would emit sufficient quantities of gravitational radiation (on account of the intense fields produced at short range by such highly condensed bodies) to cause the system to decay on a relatively short timescale, until its two components plunged into each other in a final immensely strong burst of gravitational waves at a frequency suitable for detection by Weber's instrument. He estimates that such an emission "should be detectable with Weber's existing equipment at a distance of the order of 100 megaparsecs."[1] Since this gives a range covering an expanse of space containing up to 10 million galaxies "It would seem worthwhile to maintain a watch for events of this kind, using Weber's equipment or some suitable modification of it." (Dyson, 1963).

At first, Dyson's putative source may have been seen as somewhat science fictional. He certainly put it forward in an unusual setting, a book about communication with extraterrestrials (a very earnest and unsensational one, to be sure). Furthermore, at the Warsaw conference (Infeld, 1964), we find someone, an "unidentified questioner" apparently taunting Weber after his presentation with a question as to whether he had yet "measured any Dyson neutron binaries". This type of source must have seemed almost wildly speculative at a time when the existence of neutron stars was very much in doubt (pulsars were not discovered until 1967 by Jocelyn Bell and Tony Hewish). Nevertheless, pulsars have since been discovered with binary companions (for which see below) and nowadays, along with the as yet undiscovered binary black hole systems, these are the most favoured source for the next generation of gravitational wave detectors.

[1] A considerable underestimate relative to more recent calculations

During the 1960s GR began to become relevant for perhaps the first time to astronomy and astrophysics, leaving asise the specialist subject of cosmology. Previously, astronomy had influenced GR (with the Eddington expedition and the Mercury perihelion problem) rather than the other way around. This now began to change as a result of transformations within each subject. The strong boost given to the practice and theory of radio observations by the military requirements of the second world war resulted in the unexpected birth of the field of radio astronomy in the immediate post-war period. Significantly the new astronomical discoveries which led to astronomers and astrophysicists looking towards GR theory for possible explanations were largely discoveries of radio astronomy. Both quasars and pulsars were types of sources for which astronomers were unprepared by their optical experience, and since both gave evidence of being rather massive compact objects, it was natural to turn to GR for a theoretical understanding of them, since it was precisely in the strong field regime that this theory most strongly departed from the classical theory of Newton.

At the same time, perhaps partly in response to this unaccustomed outside interest, relativists began to make the various predictions of the theory far more concrete than had previously been the case. The development of the idea of the black hole (and the coining of the name by Wheeler), out of the longstanding formal solution to Einstein's equations due to Schwarzschild, belongs to this period[2]. We have seen how the same is true of gravitational waves. Perhaps the most important single element of the coming together of relativity and astrophysics was the attempt to find an explanation for the quasi-stellar (quasar) sources. This indeed resulted in

[2] See (Eisenstaedt, 1993) and references therein for this history

the coining of a new field, "relativistic astrophysics," and a successful new series of symposia to promote it. The first of the Texas meetings, originally promoted by Infeld's former student, Alfred Schild, was called specifically to address the quasar puzzle, and indeed, at this time we find gravitational waves being put forward as a component of one proposed mechanism for the powering of quasars.

This certainly seems to suggest a definite change in attitude towards gravitational waves which had previously been regarded as something of a 7 pound weakling in GR, one whose effects would never amount to much from an experimental or observational point of view. "The weakness of the gravitational interaction makes it exceedingly unlikely that gravitational radiation will ever be the subject of direct observation." (Pirani, 1962; pg. 199). Quasars, on the other hand, were viewed as a problem because if their characteristically high redshifts were assessed as cosmological in origin, they must have immense, unprecedented outputs of energy from rather compact dimensions (as inferred from the variability in their luminosities on the time scale of years). That William Fowler (1964) suggested at this time that they might be powered by gravitational wave emission suggests a newfound respect for the potential of this previously unappreciated phenomenon (see also Cooperstock, 1967).

The particular topic which inaugurated the emergence of GR as an important ancillary field to astrophysics was that of gravitational collapse. From the early 1960s, the collapse of massive or supermassive stars, first hinted at by Chandrasekhar and investigated by Oppenheimer and his collaborators in the 1930s, but largely ignored subsequently, became a favourite candidate for the power source which lay behind the apparently immense radio and optical emission in quasars. In 1963 Peter

Bergmann and lfred Schild issued a call for a symposium to be held on this subject, one of its goals being efforts to avoid the catastrophe of gravitational collapse to a singularity. The symposium was actually held in late 1963 at the University of Texas where Schild worked. It subsequently gave birth to a regular and highly successful series known collectively as the Texas Symposia on Relativistic Astrophysics (a phrase said to have been coined by Schild (Ehlers, intro to 9)). From the first meeting, with Fowler's (1964) article on the role for gravitational radiation in gravitational collapse and quasar emission, onwards, the topic of gravitational waves was invariably addressed at the meeting. Therefore, this subject was clearly viewed as one of the important elements of the new subject.

The accidental quality of the rapid development of the field of relativistic astrophysics is illustrated by Gold's speech at the closing of the first Texas symposium, where he says that

> here we have a case that allowed one to suggest that the relativists with their sophisticated work were not only magnificent cultural ornaments but might actually be useful in science! Everyone is pleased: the relativists who feel they are being appreciated, who are suddenly experts in a field they hardly knew existed; the astrophysicists for having enlarged their domain, by the annexation of another subject- general relativity. It is all very pleasing, so let us hope that it is right. What a shame it would be if we had to go and dismiss all the relativists again.

But, in spite of Gold's anxiety, the genie was out of the bottle. GR was extremely slow to shed much light on the topic of quasars, but the reorientation which was encouraged within relativity, and the success of the great body of work which went forward on gravitational collapse and black holes, invigorated the subject as never before and lent great vitality to the idea of a mixing of relativity with astrophysics. Pulsars, discovered in 1967 by Jocelyn Bell and Tony Hewish, were rather quickly identified with the idea of neutron stars which had been knocking around in the background since the thirties, and greatly helped to underpin relativity's new role. Gravitational waves also played a role in the subsequent development of pulsar theory, since their emission was expected to quickly damp the wild pulsations of the neutron core at the end of the gravitational collapse of the parent star (Thorne, 1969). The extent to which the internal dynamic within GR, encouraged by the move towards astrophysics, developed its own momentum is seen by the growth in interest in gravitational waves, which preceded by some time the emergence of any observational input into the subject. By 1967, one finds a new sentiment about the prospects for gravitational wave detection, when Wheeler said "gravitational waves, I cannot help but feel, are going to be one of the big discoveries of the next ten years. One will detect them for the first time. That is one great prediction of Einstein's theory" (Wheeler, 1967).

Still, it was to the great surprise of most theorists that Weber announced in 1969 that he was detecting gravitational waves (Weber 1969). Although his results, which confounded all theoretical predictions of source strengths then and since, were eventually discounted amidst much controversy, they focused much attention on the subject, and sparked a great increase in the number of experimentalists working on gravitational waves. (See Collins 1975 and 1981 for a detailed account, and Franklin 1994 for an alternative viewpoint). On the theoretical front, research in the 1960s on black holes, cosmology and other topics had made the field of relativity very relevant to astrophysics. Gravitational waves shared somewhat in this popularity, and seemed likely to continue to grow in practical importance as experimental interest waxed. The discovery of the first binary pulsar, PSR 1913+16, (Hulse and Taylor 1975) was the fortunate and serendipitous occasion which sealed this promise.

Chapter 13

The Quadrupole Formula Controversy - Origins

The origins of the "quadrupole formula controversy" of the 1970s and 80s can, in a sense, be traced to Feynman's remarks at Chapel Hill in 1957. His remarks are interesting as an enunciation of the "non-rigorous" approach to relativity (and theoretical physics as a whole) and so I quote them at length.

> There exists, however, one serious difficulty [in the study of relativistic gravity] and that is the lack of experiments. Furthermore, we are not going to get any experiments, so we have to take a viewpoint of how to deal with problems where no experiments are available. There are two choices. The first choice is that of mathematical rigor. People who work in gravitational theory believe that the equations are more difficult than in any other field, and from my viewpoint this is false. If you then ask me to solve the equations I must say I can't solve them in the other fields either. However,

> one can do an enormous amount by various approximations which are non-rigorous and unproved mathematically, *perhaps for the first few years. Historically, the rigorous analysis of whether what one says is true or not comes many years later after the discovery of what is true.*" [emphasis added] (De Witt 1957, pg. 150).

This view was previously encapsulated by William Blake in a "Proverb of Hell," "What is now proved was once only imagin'd."[1]

Between 1957 and the early 1970s a great deal of work had indeed been done, at all levels of rigour, on the subject of gravitational waves. Many relativists had taken the course advocated by Feynman (e.g. Peters and Mathews). Others had been more cautious, but no less successful in many cases, in improving understanding of gravitational radiation. The extent of their achievement was such that by the late sixties, Kip Thorne (like Feynman, a student of John Wheeler, and decidedly of the progressive, non-rigorous school) could state that the issue of whether gravitational waves and gravitational radiation reaction existed was at last settled, and the stage had been reached at which applications of the theory, such as the quadrupole formula, could be made use of in astrophysical applications with little fear of error. At this point, the rigorous relativists might well have asked, what role is there for us? Now the few years have passed, and we are to be declared redundant in any case! The forces of Heaven began to rally themselves.

Jürgen Ehlers speaks for the virtues of proving what was once imagined in a 1986 talk on "Folklore in Relativity and What is Really Known". He quotes Synge on the advance of science

[1] William Blake, *The Marriage of Heaven and Hell*, plate 8

> As science advances, it seems to have a sort of scorched earth policy. The advancing army is full of enthusiasm for its advancing into the unknown, and the unknown is always exciting. If it glances back at the territory it has overrun, it sees little but dullness, the dullness of what seems to be completely known, with little prospect of adding to that knowledge by a deeper understanding.

Ehlers continues, "so, let us now turn to the dullness of what seems to be completely known. Perhaps it is neither dull nor, in fact, really known." (Ehlers, 1987).

Ehler's attempt to draw the attention of the advancing horde away from its headlong forward rush and back towards its own past points up a curious paradoxical aspect of the progressive school. In their attitude to history, they are impelled to maintain a conservative stance. Although they may take an active interest in history, they are usually hostile to revisionism. For the progressive, the history of his field has a critical, if passive, role to play as the solid foundation from which further advances in the subject may be launched. As we shall see, attempts such as Ehlers' to put what has been accomplished in a less certain perspective are answered with counter histories designed to reinforce received opinion on what is "known" (what Ehlers calls "folklore"). The progressives, as Dieter Brill puts it (interview), are a class of "daring conservatives". It is their conservatism with regard to what is known which justifies and enables their daring progression into the unknown.

An example of the progressive school in action (Brill's daring conservatives) is given by P.C. Peters' and Jon Mathews' paper (1963) on binary inspiral. This paper makes use of the quadrupole formula "result" to calculate the pattern of radiation by particles in Keplerian orbits and the evolution of such orbits under radiation reaction. Already the outlook is markedly different from that of Fock, who stressed how superfluous the back reaction effect was for any description of orbital motion. Peters and Mathews state their attitude to the gravitational wave controversy right at the outset.

> The linearized version of Einstein's general theory of relativity is strikingly similar to classical electromagnetism. In particular, one might expect masses in arbitrary motion to radiate gravitational energy. The question has been raised, however, whether the energy so calculated has any physical meaning. We shall not concern ourselves with this question here; we shall take the point of view that the analogy with electromagnetic theory is a correct one, and energy is actually radiated.

If Peters and Mathews feel obliged to begin with an apology, however defiant, a decade later the confidence of the Caltech school has increased, commensurate with the increased scope of their horizons. Thorne and Kovács (1974), in a paper presenting a fast-motion scheme intended to apply to non-bound, but gravitationally interacting, sources such as those producing gravitational bremsstrahlung radiation, outline an ambitious program.

> [Because] "gravitational-wave astronomy" may be a reality by 1980, ... [the] Caltech research group has embarked on a new project: We seek (1) to elucidate the realms of validity of the standard wave-generation formulae; (2) to devise new techniques for calculating gravitational-wave generation with new realms of validity; and (3) to calculate the waves generated by particular models of astrophysical systems.

That the references to elucidation of "realms of validity" here expresses an outlook totally at variance with that of the sceptics is to be understood when Thorne announces his intention of aggrandizing new realms for the venerable quadrupole formula, and not curtailing its range in any way.

> The 'quadrupole-moment formalism' dates back to Einstein (1918), and has been canonized by Landau and Lifshitz (1951). The derivations of this formalism which we find in the literature are valid only for systems with slow internal motions and weak (but non-negligible) internal gravitational fields. However, a detailed analysis ... shows that only the slow-motion assumption is needed; the quadrupole-moment formalism is valid for any slow-motion system, regardless of its internal field strengths.

By 1980 the point has been reached where some consolidation is in order. In Thorne (1980), a review paper on "Multipole expansions of gravitational radiation," the abstract begins "this paper brings together, into a single unified notation, the multipole formalisms for gravitational radiation which various people have constructed." Some reference to philosophical outlook governing the enterprise is still in order.

> The reader should be warned that this article and its author do not aspire to the high level of mathematical rigor and elegance that characterize much of mathematical relativity [e.g. Penrose's (1964,68) conformal treatment of null infinity, and the Bondi *et al.* (1962)-Sachs (1964)-Newman and Penrose (1968) treatment of the asymptotic properties of gravitational-wave fields.] Instead, the author seeks a level of rigor that

> is (i) high enough to give him confidence of the results derived, but also (ii) low enough to permit the treatment of real astrophysical systems in the real, non-asymptotically flat universe. This philosophy shows up most strongly ... where the concept of 'local wave zone' is introduced to permit a separation of the theory of wave generation from wave propagation. That separation sacrifices the elegant rigor of the Bondi-Sachs-Newman-Penrose approach in order to treat, e.g. sources embedded in galaxies, with neutron stars and black holes nearby and with a distant, inhomogeneous universe that may curve up into closure.

Thus the elegance of much of general relativity is seen as a potential obstacle to its relevance, at least to astrophysics, and therefore the former must be sacrificed to whatever extent is necessary to achieve relevance. But the sacrifice of rigour, if not of elegance, was not without its critics. Indeed, in the early seventies, Peter Havas did object to efforts such as Thorne's to declare a satisfactory conclusion to the problem of radiation reaction. To Havas, not only was the validity of the quadrupole formula still at issue, but in fact whether there was any quadrupole emission of gravitational waves from binary systems at all was still very much open to question. Of course, Havas was aware that his own efforts to attack the problem via the fast motion approach were still incomplete, but his objection was to the complacency shown by promoters of the slow-motion approach in overlooking what he saw as fundamental problems with the various derivations of the quadrupole formula using that method, and to the reliance on "intuitions," trained in "the corresponding problem in electrodynamics," which may tend to mislead in the gravitational context (Havas, 1973).

Without the resources of a large research group to draw on to advance his fast motion research program, Havas nevertheless had one student in the 1970s who very much shared his view as to the importance of the back reaction issue in the problem of motion, and the failings of the slow motion expansion techniques. Arnold Rosenblum was not only determined to carry on the fast-motion approach by his own efforts, but he also proved an able and effective propagandist for the counter-offensive of rigour, whose day had apparently come, well over a decade after Feynman had advised relativists "Don't be so rigorous or you will not succeed" (De Witt, 1957 pg. 150).

After receiving his Ph.D. with Havas at Temple University in Philadelphia, Rosenblum went to Munich in 1974 to work with the group of the mathematical physicist Jürgen Ehlers. There, his enthusiasm for the subject of gravitational radiation reaction and his trenchant criticisms of the current state of it encouraged Ehlers to take an interest in the problem. Ehlers certainly came from the "rigorous" tradition of GR himself, being of a mathematical school which preferred to deal, applied mathematics' style, in terms of theorems and proofs. His was certainly a long way removed from the style of relativistic astrophysics, quite towards the other end of the spectrum in terms of rigour within theoretical physics. From a mathematician's standpoint, the body of work on radiation reaction certainly left a lot to be desired.

Like Thorne and Kovács, Rosenblum saw the problem of two masses scattering off each other as better suited, at lower levels of approximation, to the fast-motion approach than the binary problem. In the scattering problem, two massive objects approach each other from a great distance, interact gravitationally, altering their respective paths, and recede to great distances from each other. In analogy with electromagnetism, the type of radiation emitted is called bremsstrahlung ("braking radiation"). In the seventies he applied fast-motion techniques developed by himself and Havas to this problem. Unlike Thorne and Kovács (1974), who addressed the same problem via a different fast-motion (or post-linear, as they called it) scheme, his result, in the slow-motion limit did not agree with the quadrupole formula, giving a loss of energy about twice as great (Rosenblum, 1981). Rosenblum was thus led to take a very active part in the debate on the validity of the quadrupole formula, which he himself helped to spark, by the fact that his own calculations disagreed with the established result in one important case. At the same time, he continued his efforts to extend the fast-motion scheme to the bound orbits case (Rosenblum, 1982).

Ehlers meanwhile took a rather different approach to the problem. Having not himself worked extensively in the problem of motion, he nevertheless kept abreast of it for pedagogical reasons. He found the literature on the subject unclear, preferring Fock's book as the best treatment. Inspired by Rosenblum's interest he began to encourage work on the subject of radiation reaction in his own group, and to invite visitors with an interest in it also. He also adopted the role of an independent critic of the various methods in use in the field, publishing a review paper on the subject in collaboration with Rosenblum, Havas and Goldberg (Ehlers et al., 1976), and organizing a workshop on "Isolated Gravitating Systems in GR" at Varenna, Italy in 1976 in order to foster efforts to meet the deficiencies of the subject (Ehlers, 1979).

Ehlers' critique of the subject was wide-ranging. He disliked the use of point-masses, and worked within his group to discover methods of dealing with finite non-rigid bodies (Dixon, 1979). He felt that even with some progress in that direction, the post-Newtonian schemes, which led to divergent integrals at higher orders in the expansion, were highly suspect (a criticism which was aimed by some at Chandrasekhar's work which had also used extended bodies). Burke and Thorne's demonstration that the post-Newtonian expansions should only be used in the "near zone," with matching schemes employed to connect the solutions to wave zone boundary conditions was sufficiently encouraging for Ehlers to invite Burke to give a series of lectures at the Varenna school. However, from a mathematical point of view, Ehlers considered that the matching schemes still lacked rigour, despite further progress subsequently by Damour, who introduced another "intermediate zone" in which the matching took place.

For Ehlers, a genuine mathematical relativist, the quadrupole formula episode, despite a level of acrimony which surrounded some exchanges at conferences, provided a welcome level of interaction between relativists of different outlooks. Both at conferences, where the mixing of mathematically and observationally inclined people was encouraged in this respect by a common interest, and in his group, where visitors such as Anderson contributed a different perspective, he enjoyed the mutual exposure to different sensibilities. The controversy served therefore, not only to improve understanding of a difficult subject, but to cross-fertilize between different schools of relativity.

Ehlers can perhaps be seen as the pure sceptic, not only because of his relative disinterest (he had little of his own work invested in the controversy), but because he was very much a sceptic in ordinary sense of "a person who doubts, questions or suspends judgement upon matters matters generally accepted" (Webster's). As a mathematician, the existence of exact solutions describing gravitational waves, and as a physicist, Bondi's famous thought experiment, convinced him that gravitational waves were real. The linearized theory was perfectly acceptable for detectors, and he felt that on a "reasonable physical level" the review paper of Walker and Will (1980) cleared up the question of the quadrupole formula's validity. While some of the issues of method remain outstanding (such as the need to define boundary conditions rigorously in curved space time, as opposed to asymptotically flat spacetime), he regards the quadrupole formula as reasonably well justified both experimentally and theoretically. His criticisms therefore did not spring from motives of either immediate personal interest, or doubts of the existence of the phenomenon in question, but rather from a desire to expose accepted ideas to question, where he regarded them as unsoundly held.

Completely independently from the work of Ehlers, Havas and Rosenblum, another challenge to the quadrupole formula came from Fred Cooperstock, who had experience of gravitational wave theory from his days as a graduate student (condenser work, connection to Weber's idea), and from his paper on Fowler's quasar idea. In the early seventies he was moved, by discussion with A. Papapetrou, to make an attempt to remove the problem of tails from the back reaction problem by removing the past history of a binary system. Because of scattering of the emitted gravitational waves off the source's own background curvature, a given binary system emitting gravitational radiation would be affected by these "tails" from all previous states of its history, as its old emissions came back to haunt it. Tails, which had been so bothersome to Bondi (paper and interview), were somewhat problematic to deal with, since technically they required knowledge of the entire past history of a source.

Cooperstock's idea was to imagine a source consisting of two bodies held apart, in a static system, by a rigid strut. The strut would then be broken, and the two bodies allowed to fall towards each other from rest. Cooperstock's early results showed a much higher rate of emission than would be expected from the quadrupole formula, and led him also to criticize the basis for this result. At the same time, in avoiding one issue of principle with his toy model, he had introduced several others just as serious. A rigid strut was not permitted in GR (since its local speed of sound would be infinite), and the fluid bodies themselves would have to be held together by some sort of skin. Therefore, Cooperstock began to further elaborate his model, in response to counter-criticisms from others. In the meantime, unexpected observational results were about to radically alter the context of the debate on the radiation reaction problem.

Chapter 14

Experimental Impact

The growth of interest in gravitational waves in the second half of the 20th century proceeded in a series of incremental stages. In the 1950s, the emergence of GR theory out of its doldrums was sparked partly by very abstract theoretical concerns. Much of the interest was cosmological in origin, this being the only arena in which GR was seen to have any relevance in mid-century. Bondi came to the study of gravitational waves having established his reputation in cosmology. Others who began to take an interest in gravity and gravitational waves at this time were field theorists, such as Wheeler and Feynman, who were interested in quantum gravity, and who saw gravitation as a new realm in which to apply their skills as the problems of quantum field theory came increasingly under control. Experimental interest played a role at this time too, with the discovery of the Mössbauer effect which permitted much more precise measurements of red-shift effects, including the gravitational red-shift. Theoretical, epistemological and experimental motivations were blended together in the pioneering work of Dicke, also.

The second, and greater, surge in interest was inspired by new discoveries in astronomy, as we have seen, which inaugurated what has sometimes been called the "golden age" of relativity, when the modern theory, with its black holes and other strange new artifacts, came into being. Although gravitational waves at first shared in the excitement surrounding quasars, pulsars and other new ideas and discoveries, as we have seen, at the same time they were soon left behind in the rush of applications by hotter topics. The subject of gravitational radiation still seemed handicapped by the lack of any experimental input. Although there was more optimism than had prevailed previously regarding the possibility of observational results, the actual experimental evidence on gravitational waves in 1968 was very meager and all negative.

In the early 1970s, as if on queue, the detection of gravitational waves emerged as a hot and controversial topic for the first time. If this was unexpected, the subsequent emergence of an apparently ideal test bed for the observation of gravitational radiation reaction effects was even less so. By the 1980s radiation reaction in GR was a flourishing subject and the advent of ambitious large detector projects was creating a demand for unheard of and previously undreamed of levels of precision in the theoretical prediction of gravitational radiation effects. Much sooner than anyone might have expected, the concrete visualization of gravitational waves as a physical phenomenon which was the principle achievement of the theoretical advances of the late 1950s and 1960s, was bearing fruit in attempts to render the phenomenon visible in actual instruments. It was only in keeping with the history of the subject itself that this endeavour would prove to be the most controversial yet.

Throughout the 1960s, Joseph Weber ploughed a lonely furrow in his efforts at gravitational wave detection. By the time of the Warsaw conference in 1962, he had a detector operating, and he presented his early results. His reception at this stage was mixed at best. Nevertheless he succeeded not only in constructing a working instrument but in elaborating some of the most important future developments of the field of which he was as yet almost the only exponent. His chosen detector consisted of a large aluminum bar, seismically isolated from the vibrations of its surroundings and fitted with electronic strain gauges which would detect any resonances set up in the bar by a transient disturbance. He also sketched the idea of an interferometric detector. One of his students, Robert Forward (also a well-known science-fiction writer) was the first to build and operate such a device in California in the early 1970s (Thorne, 1989). Weber noted that the Earth itself was a large gravitational wave detector on the "Weber bar" model, and set an upper limit on the quantity of gravitational waves passing through it which might excite vibrations in its mass. As the decade progressed he improved his detector (situated in College Park, Maryland), and eventually set up a second one in Chicago. This would allow him to eliminate merely local vibrations affecting one detector by looking for coincident disturbances in both.

In Russia, where the theoretical development of gravitational waves had proceeded along very different, largely uncontroversial lines, there was also some interest in gravitational wave detection, especially by Vladimir Braginsky. The excitement generated by Weber's announcements provided a spur to the work in the area on which Braginsky was already engaged (Thorne, private communication). Although in the Soviet Union Landau and Lifshitz and Fock were not doubted as they were in the west, at the same time their results had stressed the negligibility of the effects of gravitational radiation. Therefore, the principal obstacle which presented itself to a would-be gravitational wave astronomer, that the waves appeared to be very weak, was as much a factor in the Soviet Union as it was in the west. Nevertheless, there also, at least one experimentalist was motivated to take up the challenge.

In 1969, Weber announced that he was detecting pulses in his instrument in excess of what was expected statistically from background Gaussian noise. Over the next couple of years he produced an increasing volume of coincidence data between his two detectors, including indications of what was called a "sidereal correlation". This was an excess in coincidences peaked at certain times of the day, which time varied during the year in such a way to suggest that the source or sources of the waves lay outside the solar system. Such increased sophistication in Weber's claims overcame an initially lukewarm reaction and persuaded several other experimentalists to build and operate detectors on a similar basis. Their eventual failure to see sufficiently convincing indication of gravitational waves interacting with their instruments led to a bitter and protracted controversy with Weber, who has since continued to pursue his own research on the basis that at least someof his events indicated the presence of gravitational waves. His early results have nevertheless been almost universally rejected by other gravitational wave experimentalists. (Collins, 1981; Franklin 1994)

The reaction of GR theorists to Weber's findings was somewhat mixed. Although the discovery of gravitational waves might be thought of as good news for theorists, Weber's claims violated all theoretical expectations of signal strength, even though these expectations had been radically transformed by other forces in the decade since Weber began his detection program. On the one hand, Weber's estimated sensitivity (which was and has been the subject of some uncertainty and considerable controversy) and claimed detection rate indicated an unexpectedly strong flux of radiation impinging on the earth. The sidereal correlation was held to indicate a source in the direction of the center of the galaxy, a plausible source given its relatively high density of matter. If the center of the galaxy was the source, and if it emitted radiation isotropically so that Weber's bar was seeing only a small fraction of the total output, then it could be estimated that this region of the galaxy must be losing hundreds of solar masses a year to the emission of radiation, a staggering figure. Such a rate of loss would indicate that the galaxy would disappear altogether on a timescale much shorter than its own estimated age! Nevertheless, the theoretical reaction to Weber's findings was not uncomplicated rejection. There were suggestions that some of the many underlying assumptions behind this calculation might not hold true. It was even speculated that there might be experimental evidence for such a rate of mass loss from the center of the galaxy, in the form of stellar motions in the solar neighbourhood (Goldberg, 1974; pg. 396).

In its early phase, theorists had only a minor role to play in the Weber controversy. Only in the late 1980s, when Weber claimed to have detected gravitational waves from the large supernova 1987a, in circumstances in which only one other detector was on the air, did theorists take the fore in rebutting his claims. In that case, experimentalists were ill placed to do so, since the detection was inherently non-reproducible (the supernova in question was the strongest seen from earth since the 17th century), and it was accompanied by a new detector theory of Weber's which vastly inflated his instruments' claimed sensitivity.

Nevertheless, the Weber controversy of the 1970s did encourage some increased theoretical activity, and more importantly, by jump-starting the development of an active experimental field of gravitational wave detection, its effects on theory were incalculable in the long run. The very fact that Weber's problems included his violent conflict with theoretical predictions indicated that, in the long run, experimentalists would find themselves dependent on theoretical guidance. Once the early experiments rejected Weber's results, but the decision was made by several groups to persist in the field, they were going to be faced with the necessity of gearing their program to meeting a goal laid down by the expectations of theory. Indeed, in the long run, the forthcoming "third generation" of detectors expect to rely on detailed theoretical predictions of waveforms to actually make the signal visible in the detector output by the use of sophisticated signal filtering methods designed to seek out certain patterns which would be otherwise lost in the detector noise.

The next great step for gravity waves came with the discovery of the first pulsar observed in a binary star system in 1974. This discovery, for which Hulse and Taylor later won the Nobel prize, was immediately recognized as providing the first strong field observational test-bed for the theory of GR. Up until this time, all tests of the theory had taken place in the realm of first order corrections to the Newtonian theory. However, initial reactions from theorists indicated pessimism that the new system would ever show measurable signs of orbital decay due to gravitational wave emission (Damour and Ruffini, 1974).

In 1978, after several years worth of observations on the system, Taylor and coworkers announced that there was definite evidence of such secular orbital decay. 200 years after Laplace had first conceived of such an effect, a system was found which perhaps really did exhibit orbital decay due to a retarded attractive force.

The announcement was first made publically at the Texas symposium in Munich, continuing a tradition of important new findings emerging first at one of the meetings in that series. New astronomical discoveries tend to disperse quickly through channels such as International Astronomical Union circulars and by word of mouth (nowadays this occurs even faster via the Internet). For instance, the Damour and Ruffini theoretical paper on the new binary pulsar was published in late 1974, although the discovery paper itself appeared only in 1975 in a refereed journal.

The length of time over which PSR1913+16 had to be carefully observed to produce evidence of orbital decay gave a slow motion quality to this confirmation of the quadrupole formula which is quite interesting. The first definitive journal paper on the orbital decay (Weisberg and Taylor, 1980) appeared quite some time after the Texas meeting announcement. Even then caution was still the order of the day, and it was only gradually during the 1980s that exhaustive observational and theoretical work convinced the great majority of experts that the effect was real, and was not explicable by other influences on the system which would have nothing to do with radiation damping, such as a third body in the system, mass loss from one of the stars, some other form of dissipation, and so on. Even today, it cannot be logically ruled out that some other unlooked for effect would throw out the decay's agreement with the quadrupole formula. Other binary pulsar systems have been discovered since the first one, and as observations continue on these, it is becoming apparent that they tend to confirm the observed decay of PSR1913+16.

Chapter 15

The Quadrupole Formula Controversy - The Style of the Debate

An important feature of the radiation reaction debate in the seventies and eighties was the series of review papers by different authors, each employing the history of the subject to illustrate a particular view of the contemporary state of the field. These papers show that relativists were keenly aware of the history of their field and they were able to draw lessons from their reading of history which reinforced the points they wished to make. The earliest of these papers was that of Ehlers, Rosenblum, Goldberg and Havas (1976). They argued that previous attempts to deal with the back-reaction problem were all inadequate in one way or another. In consequence, they advanced an outline of a program which would overcome these past failings. Essentially an attempt to formulate a research program for the subject, their paper was followed by an Enrico Fermi summer school in Varenna organized by Ehlers, whose aim was also to foster new work in the field along more rigorous lines than before (Ehlers 1979).

In 1980 Walker and Will took a very different tack, addressing the problem of non-reproducibility which had plagued the subject (Walker and Will 1980). They argued that a basic iterative algorithm, applicable for both fast motion and slow motion methods, could be followed to recover the quadrupole formula from reaction calculations. They presented an analysis of a cross section of well-known calculations, dating back to the paper of Hu in 1947, and argued that those who had advanced through sufficient steps in the iteration recovered the quadrupole formula, and that others, with fewer steps, did not (except for a couple which found the result with the aid of compensating errors). In this view of the history of the field, there existed a definitive method by which the standard results could be recovered in a reliable way.[1]

This was in stark contrast to the views expressed by Ehlers *et al.*, which were to advocate a more general prescription, whose outcome was not yet known. Yet another view was put forward by Cooperstock and Hobill in 1982. They refused to set forward a general scheme or advocate a particular result, instead arguing against preconceived notions (Cooperstock and Hobill 1982). Their history, as befitted their standpoint, was more descriptive than prescriptive, celebrating the diversity in the

[1] Indeed, Walker and Will's iterative test, that three iterations of the field equations were required to successfully account for retarded effects in bound orbits, became a benchmark for subsequent research. Since scattering problems, where the bodies were not gravitationally bound, were held to require only two iterations, they were typically preferred by those employing fast-motion calculations, in which the calculational burden grew excessive at the third iteration.

development of the field. Another protagonist with an interest in and excellent knowledge of the field's history was Damour. His papers were often prefaced with a discussion setting his work in a historical context (for example Damour 1982). In this role, the object of history was to motivate the new work being presented, and the focus was on the previous failings which were being addressed by the new contributions (see, for instance, Damour 1983). A more active role for the historical literature was found in the account of James Anderson, who returned to the Einstein-Infeld-Hoffmann scheme complete with its surface integral method, and married it to the matched asymptotic expansions of Burke, with further additions of his own, to produce another influential derivation of the quadrupole formula (Anderson 1987).

A very significant aspect of the debate in the seventies and eighties was the problem of when theory ends. As we have seen, different authors could look at the same history and give very different answers to this question. One answer might be, that theory already has ended, and we really know the answer ("Conservative"). Another is, it has just ended now, with this paper, for the issues addressed ("Technocratic"). A third is, it will end, as soon as the general program we advance is carried through ("Marxist"). A fourth is that it can never end, and it is best that it should not ("Anarchist"). Finally there is the view that the answer is hidden in the past, waiting to be extracted and pieced together from the literature ("Archaeological"). It is interesting that just as there was agreement on the details of the history (and the debate was largely a historical debate), opinions diverged on the matter of *interpretation.* The lesson of history was different for everyone. This is still the case, but the debate having lost its impetus, the individual perception of history has lost its public relevance once more. The dynamic of the debate is that some level of consensus must be found for the resolution of an existing problem, and yet progress seems to be measured by many scientists by the extent to which an issue can be settled, allowing the next problem to be addressed. A field like General Relativity has historical memories of the isolation which may be the fate of a discipline which does not progress in this way. The remarks of Feynman at Chapel Hill (De Witt 1957, pg. 150) express the view of the progressives, when he says "the second choice of action is to ... drive on," to "make up your mind [whether gravitational radiation exists] and calculate without rigor in an exploratory way". He concludes with the advice, "don't be so rigorous or you will not succeed." The contrast in attitude suggested here may explain why the debate tended to become more vitriolic in its last stages, as a consensus developed for many, while some still argued that the matter was unsettled.

Chapter 16

The Quadrupole Formula Controversy - Conclusion

At the time of the 9th Texas symposium, held December, 1978, for the first time outside the United States in Munich, a vigorous debate on the problem of motion for binary systems was still underway. A workshop at the conference was devoted to the subject, and the report in the proceedings indicates a wide ranging discussion and a fairly rich strain of new work in the field. The problem was, however, to be raised to a new level of prominence as the result of the most exciting development of the conference: the announcement of a measurable orbital decay in the binary pulsar by Taylor and co-workers (discussed above). The stimulus of what was generally considered to be rather high quality data on the "higher order" evolution of the motion of an apparently isolated binary system containing bodies with strong internal fields led to the 1980s being the most prolific of all decades to date for work on various aspects of the radiation reaction problem and the problem of motion.

Subsequently, the demands of a projected new series of gravitational wave detectors encouraged the continuation of efforts to calculate to ever higher orders in the expansion parameters.

Ehlers, the host of the Munich symposium, gave an overview of the state of the field from the point of view of what he described as the "minority of relativity-theorists" who did not share the widely held opinion that "the implications of GR ... have been deduced satisfactorily ... [for] the dominant, secular gravitational radiation reaction effects on the orbits" of systems including PSR1913+16. He took care to emphasize on what grounds his dissent was based.

> The main shortcoming of [these] ... calculations is, in my opinion, not that they employ approximations which have not been rigorously mathematically justified - that they share with many approximations used in physics - but rather that they: 1) employ *notions* which are *not well defined* in terms of basic concepts of GR, such as "gravitational field energy," "total mass and linear momentum" of a gravitationally bound body interacting with other such bodies, "point particle," "gravitational radiation reaction force," "near zone," "radiation zone"; 2) use *laws* which have not been *established* within GR, such as an "energy balance between radiation and material sources"; 3) depend essentially on *ad hoc assumptions* which not only are *without foundation* within GR itself, but for which there are indications that they may be *incompatible* with the fundamental assumptions of GR or with each other, such as global coordinate conditions, particular global splittings of the metric into a flat background and a 'small' perturbation, non-covariant 'outgoing radiation conditions,'

> negligibility of various kinds of 'small' terms, etc.
>
> It seems to me to be an important challenge to find derivations of observable relativistic effects, particularly structure and radiation effects, of isolated systems which are free of shortcomings, and which are not based on mere analogies, however plausible, with Newton's or Maxwell's theory. Needed are, in particular, approximation methods which have been rigorously justified at least in theories simpler than Einstein's, and which permit if not an error estimate, at least a reasonable guess about error bounds. (Ehlers, 1980)

It is interesting to see Ehlers disclaim any dogmatic objection to the practice of physics on "rigorously mathematical" grounds. He is aware both of the practice of most physicists, and of the difficulty of the problem at hand which resists efforts to prove closed theorems. Instead, he grounds his objections in a failure on the part of the majority of relativity-theorists to consistently conform to the principles of GR theory itself. Specifically, he attacks efforts to import into GR concepts typically found in other physical theories, such as "field energy," "energy balance" and so on. In short, there is an epistemological disagreement between those who wish to carry forward relativity theory according to the standards current in the rest of theoretical physics, attempting to discover within GR quantities analogous to those, such as "total mass" and "linear momentum," which would be employed in a similar problem in "standard" field theory, and those who prefer to pursue the matter according to the peculiar tradition of GR theory itself, eschewing certain concepts which, however prevalent in other spheres of physics, were perceived by them to have no natural analogue which had been properly demonstrated within, or shown not to violate the precepts of GR.

There is an important *philosophical* attack on the practices of radiation reaction theory in Ehler's talk, which is made even plainer at a later conference, GR11 in Stockholm, Sweden (the talk on the "folklore" of physics referred to earlier). Here he says,

> Another statement which seems to be generally believed, usually on rather flimsy arguments, is: **Newton's theory of gravity is a 'limit' of Einstein's.** To understand this limit relation is important since 1) Newtonian theory successfully explains many gravitational phenomena and ii) the approximation methods on which the comparison of GR with observations is based assume such a limit relation and even use Newtonian concepts such as masses and linear moments of gravitational interacting bodies, which have no meaning in GR.
>
> Note that [this limiting relationship] is *assumed* in post-Newtonian approximation methods. If this assumption were incorrect, the comparison of GR with observations of the solar system, the binary pulsar or cataclysmic binaries would lose its theoretical basis. Unfortunately, there seems to be no hope of answering this question rigorously in the near future.

This is certainly a radical attack on the whole basis of much of modern relativity theory, especially on most of its experimental verifications. It is not surprising that those relativists whose professional affiliations were closer to physics than to mathematics would have a strong interest in rejecting such a sweeping historical reappraisal.

The main thrust of Ehlers' argument proceeds on entirely epistemological grounds, insisting on the elimination of concepts and assumptions which do violence to "the fundamental assumptions of GR". Only towards the end does he return to an appeal for "rigour," especially in the technical requirement of approximation methods with some form of error control or estimation, generally agreed to be a persistent failing in this field. His own emphasis shows how diametrically opposed the two main responses to the advent of experimental data on gravitational radiation were. Ehlers' response, as a relativist, is to turn once more to the fundamentals of the theory. To derive "observable relativistic effects" which are free of the shortcomings of failing to adhere to GR's "fundamental assumptions". For those with an astrophysical bent, the tendency had been to turn outward rather than inward, to conform to the demands of working within a broader physics community which was, by and large, uninterested in the traditional preoccupations of GR, sometimes to the chagrin of relativists. In an interview with Alan Lightman (Lightman and Brawer, 1990, pg. 429) Roger Penrose has remarked,

> People come in from outside, not being experts on general relativity or cosmology particularly, but knowing about particle physics, symmetry breaking ideas and so on, and bringing this expertise into the subject. I think there are very many more of those people than relativists. Locusts would perhaps be the wrong analogy, but there are huge numbers of people and they see an opening into this subject, and they come in and almost take it over. I felt this a bit with supersymmetry. In general relativity, I felt this again with a lot of the people who tried to quantize it. ... Bringing ideas in from other subjects is fine ... as long as the particle physicists appreciate the problems of general relativity. I think often they don't.

> People come in without being aware of the very fundamental problems we have argued over endlessly among general relativists. There are very fundamental difficulties that one has in trying to quantize, and these people just try to sweep them away.

For the moment however, the new experimental data, from what Taylor described in the symposium proceedings as "an ideal machine for testing gravitation theories," which might have been designed for the purpose, favoured the back-to-basics appeal of Ehlers' tendency.

One of the strongest threads running through the history of 20th century gravitation theory is the search for decisive tests of rival theories of gravity. If now a test had been found for the prediction of the quadrupole formula, it had suddenly become a matter of some importance that the quadrupole formula be more rigorously shown to be a consequence of GR, honoring its "fundamental assumptions".

A large literature on the quadrupole formula in the period after the first announcement of orbital decay data from the binary pulsar exists. It will have to suffice here to discuss two of the most interesting new derivations of the quadrupole formula, by Thibault Damour and James Anderson, and then the two main opponents of the quadrupole formula during this later period, Rosenblum and Cooperstock. Damour's detailed analysis of the problem of motion, aimed specifically at matching the experimental results from PSR1913+16 is the closest thing to a "solution" to the quadrupole formula dispute, in the sense of its wholly or partly satisfying as many people as possible. Anderson's approach is regarded by a number of authorities as the most accessible and direct derivation of the quadrupole formula, which to some extent was incidental to Damour's approach.

Damour, a product of the rather formal and mathematical French school of relativity, did his graduate work in Paris on classical renormalization theory in the context of a tensorial (gravitation-like) field. Before college, he had introduced himself to the problem of motion in GR by studying the EIH method. This early interest encouraged him to work on this type of problem as a student. After completing his thesis he was awarded a fellowship to go to Princeton, and arrived there just before the announcement of the discovery of PSR1913+16. With Remo Ruffini he quickly produced a paper on the new find, which was pessimistic that it would ever have implications for radiation effects. After that he did no further work on the radiation reaction problem until 1978, when he attended the Munich Texas symposium. The announcement at that meeting of data on orbital decay, and the lively discussion on the state of play on the theoretical side encouraged him to address the problem. He joined a friend, Nathalie Deruelle, her advisor, Luis Bel, and other collaborators in working on the foundations of a new fast motion approach to the radiation damping problem (Bel, Damour, Deruelle, Ibanez and Martin, 1981). This initial work restricted itself to the (unbound) scattering case for simplicity, as had Kovács and Thorne and Rosenblum, since the extra iteration (identified by Walker and Will(1980) as crucial) required by the bound orbits problem was very difficult in the fast motion case. He began to develop his own approach to this problem, and throughout the 1980s he extended this work, always relating his efforts closely to the specific system presented by the binary pulsar. Working on his own, with Deruelle, and later with a student Luc Blanchet, he achieved a remarkable level of agreement between his problem of motion calculations and the observations of Taylor and collaborators on PSR1913+16.

Damour, even in a field with a strong interest in its own history, had an extensive knowledge of the literature going back to Poincaré. In his 1982 paper presented at the Les Houches meeting on gravitational radiation, he situates himself in the fast-motion tradition in the problem of motion, or the "post-Minkowski approximation" (PMA), as he preferred to call it. It should be noted that by this time, however, any strict boundary which may have existed between slow and fast motion approximations was becoming somewhat blurred. The work of Burke and Thorne had made it absolutely clear that the post-Newtonian approximations (PNA) were quite inappropriate in the far zone of the field, where a linear type of approximation was needed to correctly express the proper boundary conditions, and matching techniques were then used to apply them to the PNA solutions to the motion of the source in the near zone. Similarly, in order to avoid a heavy calculational burden Damour truncated his expansions for the source motions in the near zone, restricting himself to slowly moving objects, such as the binary pulsar system itself. Therefore he too was left with a PNA-type of expansion for the equations of motion. Again, different types of expansion were more appropriate to different regions in the problem's geometry, with matching techniques typically employed to reconcile them.

One important similarity between Damour's method and EIH was the use of point-sources. EIH employed a surface integral around the singularities to "cloak" them from view, so that only their field effect beyond the surface in question played a role in the problem, and the hidden objects could be presumed to be any body which would fit inside the surface and produce the same field. Damour preserved the cloaking effect, in order that his bodies could be presumed to be compact objects like the neutron star(s) in PSR1913+16, but rejected EIH's surface integral method as too involved calculationally, substituting instead a volume integral. This left him with the familiar problem of infinite integrals from the use of point sources within the region of integration, which he solved by introducing a mid-century renormalization technique from electrodynamic theory, due to Marcel Riesz. This technique had been introduced into GR by Havas, who referred to the use of "Riesz potentials". Damour preferred to use the term "analytic continuation."

In certain overall respects, Damour's approach in his 1982 paper (along with much of the modern work on radiation) can also be compared to Fock's in the use of harmonic gauge conditions, imposition of the "no-incoming" boundary conditions and matching techniques. But Damour's wide-ranging knowledge of the literature enabled him to sublimate various influences into an overall method which was distinctly his own. Besides EIH, other early contributors who influenced him were Peres and Carmeli (private communication). Another aspect of his assessment of the pre-existing literature was his critical approach, which led him to make significant changes even where he was most inclined to imitate previous efforts, as with EIH.

A quite different approach to EIH and the previous literature is found in the late 1980s work of Jim Anderson. A student of Bergmann's, Anderson did not work on the radiation reaction problem until late in his career. Then he entered the field during the quadrupole formula controversy of the late 1970s. Like Damour, Anderson was a strong critic of previous attempts to derive the quadrupole formula, describing some of them as "proof by naming," since they made use of an energy balance argument relating the flux of energy in the wave zone to the loss of energy by the source system without, in Anderson's view, establishing that these quantities were really related. Nevertheless, Anderson did not reject energy balance arguments out of hand, and made use of them in his 1980 paper on the quadrupole formula. Possibly his most interesting paper on the subject, however, is his 1987 paper, in which he revived the EIH method, badly neglected by the field in his view, and allied it to Burke's matched asymptotic expansion method and other applied mathematics techniques such as "multiple time scale expansions" in order to deal with the problem of handling two quite different types of expansions simultaneously.

Anderson views the EIH paper as "arguably one of Einstein's greatest contributions to physics" (Anderson, 1995). Largely because of the highly involved calculations the method required, the EIH surface integrals were not employed by anyone other than Infeld himself, whose distrust of retarded potentials led him to reject radiation terms in the expansion. Furthermore, in Anderson's view, the slow motion approximation was inherently incapable of dealing with radiation anyway, and it was not until the work of Burke on matched asymptotic expansions that this failing was overcome. The virtue Anderson saw in the EIH approach was that, by the use of surface integrals around point sources, it avoided the need for infinite-mass renormalization techniques (such as "analytic continuation"), which were required in field theory problems to remove the infinite self-energy of a particle with no physical extension sitting in its own field. Anderson also addressed Infeld's criticism of the arbitrariness of using retarded potentials, and showed that if "the energy of the initial field is finite, then in the asymptotic future the field is purely outgoing" (Anderson, 1982, 1995).

Anderson's "archaeological" use of the field's history, in which he constructs his new solution from pre-existing elements in the literature stands in contrast with the largely rhetorical use of history in many of the review papers. Nevertheless, like many of the other reviewers of the literature in the 70s and 80s, he is quite critical of other procedures which purport to derive the quadrupole formula. He and Damour are highly critical of each other's calculations, for instance! Each tends to regard his own contribution as the only correct derivation of equations of motion for radiating systems extant. In this respect they can be grouped with Ehlers, Havas and Rosenblum as amongst those who insist on the primacy of *method*, regarding the problem as one of finding one best method which overcomes the important difficulties in principle. Thorne (1989), Cooperstock and Hobill (1982) and Walker and Will (1980), take a more relaxed view, viewing more than one existing calculation as containing positive features. Walker and Will argue that the quadrupole formula can be said to be reproducible when specific criteria are met, and list a sequence of existing claculations that meet these requirements (as well as others which do not).

The radiation reaction work in the 1970s and 80s differed in one essential respect from that of previous decades. In most cases, the results of the various papers published agreed with each other, and with the quadrupole formula. One does not need to look far to find a possible reason for this. By this time, both theoretical and experimental opinion had largely arrived at the conclusion that the quadrupole formula was the "correct" result to leading order. The corollary to this was not only the rejection of all other results, but the temptation to reject or view with deep suspicion the method or calculation which had led to such a conflicting result. In this period only two active researchers persisted in upholding contradictory results against the quadrupole formula. One was Arnold Rosenblum, whose fast motion scattering calculation (disagreeing with that of Thorne and Kovács) gave a result, as quoted by Ehlers at the time of the 9th Texas symposium, which predicted energy radiated *in excess* of that predicted by the quadrupole formula by a factor of about 2.3. The other was Fred Cooperstock, whose rigid strut model also predicted emission in excess of the quadrupole formula. This represented a reversal of the position of the previous generation of sceptics, such as Havas, Rosenblum's mentor, who generally tended to suspect that the quadrupole formula *overestimated* energy loss by isolated systems, and who on a number of occasions gave results showing a possible *energy gain* by such systems.

Despite their fairly isolated position, both Rosenblum and Cooperstock proved to be vigorous advocates of their viewpoints. Neither shrank from public debate, although the exchanges grew quite vitriolic by the early 1980s. Nevertheless, if nothing else, weight of numbers began to tell. The problem for the minority worsened considerably with the unexpected death of Rosenblum in 1991 at the early age of 47. This was a blow to the fast motion program initiated in the fifties by Havas and Goldberg, and always advocated by Havas, for Rosenblum was its last very active exponent. As had been demonstrated several previous times in the history of the radiation reaction problem, the tenuous position of a minority research program in physics was prone to reversals brought on by historical accident in the form of personal crises or death of important figures in the minority camp.

Cooperstock largely gave up the unequal struggle, discouraged by the increasing intricacies to which he was led in attempting to refine his idealized model in the face of a large battery of critics. The inherent weakness of seeking to overcome one important problem in principle by introducing ad hoc initial conditions was seen by the exponential increase in other problems of principle brought on as the model was developed. Once again a minority research program encountered difficulties in keeping pace with its rivals for one reason or another.

Cooperstock subsequently completely abandoned his position of the early 80s, in preference for a return to the old sceptical stance, that the quadrupole formula was wrong because it was too high, not too low, in its prediction of energy loss by radiation reaction. He returned to the viewpoint, variously put forward by Rosen, Levi-Civita and others in a variety of different contexts, that gravitational waves could not carry energy, and therefore radiation reaction did not exist for isolated systems. His position was based on his new "energy localization hypothesis" which stated that, in the absence of matter, only those co-ordinate systems should be chosen to describe the local field energy which eliminated the pseudo-tensors and therefore appeared to leave no local field energy. In short, no gravitational field energy could be transmitted across a vacuum. However, Cooperstock still maintains that gravitational waves exist, and are emitted by systems like the binary pulsar, they just do not cause dynamical decay in such systems. Furthermore, such waves can be detected, not by resonant bar detectors such as were used by Weber, but by the new interferometric detectors which would observe the motion of test masses by the waves without, according to Cooperstock's analysis receiving the input of any energy.[1]

Since Cooperstock's new hypothesis is a return, in some general sense, to the sceptical view of the mid-fifties, it is perhaps not surprising to find him addressing some of the arguments which were made during that era. In order to rebut the Feynman-Bondi thought experiment of 1957, he has presented a new analysis of a

[1] A similar viewpoint regarding the absolute insensitivity of bar detectors has been put forward by Luis Bel recently, see below.

simple gravitational wave absorber designed to show that it does not receive energy from the wave, despite the relative motion of its components (Cooperstock, 1992). In a more recent paper, he has attempted to show that a gravitational geon cannot exist in GR, since the ability of such a body to hold itself together depends on the gravitational waves which compose it having mass and therefore energy (Cooperstock, Faraoni and Perry, 1995). What is interesting about these papers is that they cast Cooperstock in the role of a historical revisionist, in the sense that he seeks an alternative reading of the older literature on the subject by exposing long accepted results as faulty. I employ the term revisionist here in its proper sense of a historian or other actor who attempts to revise the standard historiography of a period in favour of neglected perspectives, rather than in the pejorative sense of one who attempts to deny the reality of notorious historical episodes for propagandistic purposes.

Given the historiographic conservativism which, it has already been observed, appears to play such a key role in the progressive view of science, it is perhaps not surprising that Cooperstock's rebuttals of arguments from a previous era made in favour of gravitational waves carrying energy has met with little or no response to date from other relativists. Revisionism, generally resisted by any social grouping when applied to its own historical self-portrait, is perhaps especially strongly resisted by scientists, for whom their standard historical self-representation seems to serve a particular practical aim, of motivating new research. Cooperstock's radical re-reading of history is nevertheless consistent with his earlier historical perspective, as expounded in Cooperstock and Hobill (1982). In that paper, noting the paucity of experimental support for GR theory, he states that "since the true goal of physical theory is the description of the real world, it is thus particularly appropriate, with regard to gravitational theory, to nurture a spirit of scepticism. Surely this is a healthy ingredient for the growth of any science."[2] He goes on to warn against the twin perils of the "optimist-visionaries" and the "mathematicians." The former should note that

> there is much more to general relativity than there is to be found in Maxwell theory and while optimism is admirable, it must be realistically tempered. On the other hand, the mathematicians forget that physics is not mathematics and that rigor is not an end in itself. Real progress in physics comes from that subtle interplay between experimental data, intuition, and the introduction of generalizing concepts and principles. It is probably the dearth of experimental data which distorts the normal flow of progress and gives this discipline [GR] a flavor all of its own.

I have labeled the viewpoint expressed in these words as "Anarchist" in an earlier chapter, not in the political sense, but because they appear to reflect somewhat the views of Paul Feyerabend in *Against Method* (1988). Progress in physics, according to Cooperstock and Hobill, is the result of a "subtle interplay" between several factors, and one should resist the impulse to impose rigid programmatic schemes on its pursuit. In this, their view resembles Feyerabend's idea of an "Anarchist theory of knowledge," in which "anything goes" if it works, just as Cooperstock and Hobill emphasize the practical necessity for scepticism if the "true goal of physical theory" is to be achieved. Cooperstock's subsequent revisionism is therefore necessary, since the arguments which were employed in the fifties and sixties to rebut the notion

[2] He adds that this spirit is "exemplified by Rosen."

that gravitational waves did not carry energy went largely uncontested at the time. Nevertheless, the hostility to reopening old debates is such that Cooperstock has been unable to generate any flow of discussion of his new hypothesis. The optimists have their gaze fixed firmly on the future, and resist efforts to redirect their attention into the past.

The question of how and when debates are opened in theoretical physics is an important one in this context. Cooperstock's counter-example to the Bondi-Feynman thought experiments is the first serious argument made against them since Bondi published his letter to Nature in 1957. Why then is Cooperstock able to publish his new hypothesis and arguments and yet not receive any substantive reply? One way of looking at such revolts against orthodoxy is outlined by Trevor Pinch in a paper on David Bohm's hidden variables interpretation of quantum mechanics (Pinch, 1977). Pinch makes use of the idea of "social capital of recognition" in science, introduced by Pierre Bourdieu (1975). Bourdieu outlines two "investment" stategies by which scientists may gain social capital within their field, the *succession* and *subversion* strategies. In the former, one gains recognition in small but frequent increments, by making progress within an established research paradigm. In the latter case, one adopts the high risk, but high yield strategy of opposing orthodoxy, which requires that one bring about a redefinition of some part of the field. Pinch sees Bohm as having proceeded by the succession strategy during the early part of his career, acquiring sufficient recognition amongst the leaders of his field to make his subversion strategy (his challenge to the Copenhagen interpretation) feasible.

Certainly, Cooperstock did not make a practice of opposing orthodoxy throughout his career. Indeed, his "strut" model of the radiation problem was not intended as a counter-example to the quadrupole formula. He began his study of this model before the radiation damping problem became a hot topic in the wake of results from the binary pulsar. Like other sceptics, such as Havas, his opposition to the "too easy" acceptance of the quadrupole formula in all physical cases arose out of his experience with his own calculations. Although he was willing to mount a challenge to the prejudices of his community when the exigencies of his own research demanded it, rather than simply set that research aside, he did not embark upon it in the knowledge that it would lead to controversy.

His subsequent publication of his "Energy Localization Hypothesis" did constitute a deliberate assault on an established paradigm which can be compared to Bohm's rejection of the Copenhagen interpretation of quantum mechanics. Therefore one takes special note of the stress which Pinch places on Bohm's having "accumulated considerable social capital of recognition by his reputation for 'brilliance' and his rapport with the quantum elite." Pinch adds

> I consider that the large amount of capital accumulated by Bohm was a prerequisite for a controversy over his work to occur. Had a physicist with a lesser social capital of recognition produced the rival interpretation it might well have been ignored, but for Bohm ... to come out with a ... paper in *Physical Review*, with Einstein waiting in the wings and soon with the support of another member of the quantum elite, Louis De Broglie, constituted a real challenge to the orthodox interpretation. (Pinch, 1977)

Note that Pinch gives us at least three criteria which may need to be fulfilled before a controversy may develop. First, the posession of requisite social capital by the author of the heterodoxy. Second, the currency of the topic. The fact that Einstein, a well-known and long-time opponent of the Copenhagen interpretation who had, in Pinch's words, "encouraged Bohm to produce a heterodoxy", was "waiting in the wings", indicates that the problem of the interpretation of quantum mechanics was a live one. Indeed, Pinch ascribes Bohm's interest in the problem to his membership in the elite, where matters of interpretation of quantum mechanics had historically been much debated, and continue to be today. Finally, Bohm received key support from at least one member of the elite, De Broglie. All this was required, not for the success of Bohm's ideas, but in order that they at least merit rebuttal by the upholders of the established viewpoint.

The indifference which greeted Cooperstock's energy localization hypothesis may be explained under each or any of these three criteria. We may argue that, following his participation (whether planned or not) in an earlier controversy on the losing side, he lacked the social capital which Bohm posessed, which would at least enable him to provoke a response. But also, in Cooperstock's case, the issue of the radiation of gravitational field energy through a vacuum had not been problematic since the late fifties. Unlike Bohm, he was not addressing a problem which the elite felt was outstanding in some way. In addition, he did not receive any public support within his peer group, which might have encouraged a reply. By way of contrast, one can examine the case of Jospeh Weber's revised estimate of the cross-section of his resonant bar detectors (Weber, 1984). When he first put forward this calculation, which greatly increased the estimated sensitivity of his detectors, thus answering theoretical objections to the validity of his earlier experimental results, there was little response. But subsequently Weber's claimed detection of the 1987a supernova, and the endorsement of the main features of his new cross-section argument by a solid state theorist, Giuliano Preparata (Preparata, 1990), who had previously rejected it (Preparata, 1988) precipitated rebuttals from high ranking members of the elite (Thorne, 1992b). The fact that Weber's claims were perceived as a potential threat to the funding of the rival interferometric detectors (Thorne is closely associated with the LIGO project) probably was a factor enouraging some sort of response. In passing, just as Pinch notes that the Bohm debate was largely carried on in Festschriften, rather than in refereed journals, it is worth observing that Thorne and Weber's exchange appears in a volume in honour of Ted Newman.

Certainly Weber's social capital was depleted after the heavy controversy surrounding his disputed detections of gravitational waves in the early seventies. But it is doubtful that his capital had risen between the date of his initial publication of his new cross-section (1984) and the response (1990). Instead other factors, such as the 1987a supernova, and the ongoing funding struggle for LIGO and other interferometric detectors, increased the relevance of theoretical estimates of detector sensitivity, while external support from a specialist added credibility to Weber's claims. Therefore, in Cooperstock's case it will be interesting to see what, if any, factors contribute to boosting the profile of his new hypothesis. Certainly its relevance could be increased by, for instance, a new explanation being found for the decay of the orbit of PSR1913+16, other than radiation reaction. Also, one of the main conclusions Cooperstock makes on the basis of his new hypothesis, that resonant bar detectors of Weber's type cannot detect gravitational waves, has found support from Luis Bel, who reaches a similar conclusion (without reference to Cooperstock's ideas on energy localization) in a recent paper (Bel, 1996). This may serve to redirect the attentions of the relativity community towards an engagement with

Cooperstock's ideas.

Certainly, the great majority of relativists who remained interested in gravitational waves had plenty to occupy them in the late 1980s and early 1990s. During this period the study of gravitational waves became a large important field for the first time in its history. The seed first planted by Weber around 1960, and whose erratic growth since would have given little grounds for optimism to the casual observer, blossomed at last. In the United States, experimental groups at Caltech (led by Ron Drever, and later by Rochus Vogt) and MIT (led by Rai Weiss), with vigorous support from the theorist Kip Thorne, secured an unprecedented level of funding from the NSF for a large detector program, LIGO. Similar projects followed in Europe (the French/Italian VIRGO, the German/British GEO 600), and Australia and Japan, all in various stages of conceptualization or development. Besides launching the study of gravitational waves into what can only be called "big science," these detector programs offered a new role to gravitational wave theorists. In a marked change from the early days of gravitational wave detection, these new detectors expect to make use of detailed theoretical predictions of signals from inspiralling binaries in order to filter the signal from the relatively strong detector noise.

One of the consequences of this need is a strong emphasis on the development of numerical techniques to allow the exact solution of Einstein's equations on supercomputers for the case of a binary black hole system. One of those involved in this endeavour is Jeffrey Winicour, who according to Ehlers has produced "the first rigorous version of a far-field quadrupole radiation law" (Ehlers, 1987). Winicour was motivated to work on this problem by Ehlers, and succeeded in showing that the quadrupole formula (expressed in terms of explicitly Newtonian quantities) would be the Newtonian limit (letting $c \to \infty$) of the Bondi news function for a unique set of initial data on an outgoing null cone (to rule out ingoing radiation). Winicour feels that no one ever did produce a completely satisfactory answer to the problem of the quadrupole formula, but feels that the issue may in any case be superseded by the emergence of numerical relativity. As numerical relativity introduces a new way of looking at the field, he expects that the quadrupole formula problem will be forgotten about anyway, as it may not be a problem one can formulate in a fully general relativistic manner (interview). This view of a fully realized GR theory finally emerging, freed of the encumbrance of Newtonian concepts, and field theory analogies, is intimately connected with the success or failure of the new generation of gravitational wave detectors. It is hoped by the proponents of these experiments that they will be the first instruments of a new field of "gravitational wave astronomy."[3] Whereas for thirty years, GR has flourished due to its increased relevance to other fields of physics and astronomy, some of its practitioners can now see it emerging as a strong field of physics in its own right, complete with an observation program which will provide direct insight into physical systems, such as black holes, which are exclusively the preserve of GR theory. In this hope one senses the view that to retain its independence as a separate body of theory, GR must progress or be eclipsed by more dynamic theoretical disciplines.

[3] One already hears the phrase "electromagnetic astronomers" used by such proponents to describe all presently existing astronomers.

Chapter 17

Technical Matters

Throughout all this, one notes the tensions within the field over technical matters, especially regarding the level of rigour required to inspire confidence in a particular result. Relativity has a tradition which places it towards the mathematical end of the spectrum in this regard amongst branches of theoretical physics. Yet from the sixties on, astrophysics and relativity became increasingly relevant to each other, spawning the new field of relativistic astrophysics. Theoretical astrophysics stands at the opposite extreme from relativity, preferring a more "physical" approach, eschewing not only mathematical rigour, but also (usually) dependence on exact results. Order of magnitude calculations and heuristic arguments are common. Such arguments, for instance, might be used to identify the "correct" result, as a guide when undertaking longer calculations.[42] Within relativity there were those whose practice tended towards each approach, and it was naturally difficult for them to agree on the question of standards of proof.[43] For practical purposes, results such as the binary pulsar measurements were obviously welcome, but at issue was on whose terms a given result was to be counted as a prediction of general relativity: the "astrophysicists" or the "mathematicians".

While it is tempting to look for a crucial issue or issues upon which the debate over gravitational waves or the quadrupole formula hinged at any given time, in fact there seems to have been little general agreement on what were the key outstanding issues. One topic of considerable importance concerned the use of point sources or extended sources. Bondi regarded the use of point sources as a major weakness of back reaction calculations, since he expected the source's equation of state to have a strong impact on the emission of radiation. Certainly others viewed point sources or singularities as problematic. In an exchange with Havas in Paris in 1973 (Havas, 1973), Thorne describes the use of point sources as the main fault in most derivations and cites Chandrasekhar's work as addressing this, yet Havas replies by dismissing the importance of this issue. For someone with a background in Dirac's electrodynamic theory, as Havas had, point sources were hardly objectionable, since the experience there indicated that the use of point sources did not lead to incorrect results.

Indeed, if there were objections to the use of singular or point mass sources, they came very much from a GR perspective. Rosen disliked them intensely (Peres, personal communication), as did Bondi (1964) and later on Ehlers (1980) also objected to them. It was felt by some to go against the spirit of modern GR theory to evoke bodies smaller than their own Schwarzschild radii. On the other hand, others with a background in classical renormalization theory, such as Havas and Damour, did not see any great objections to the use of point sources, albeit with some reservations.

Error estimation was one important bone of contention for those whose tastes ran towards mathematical rigour. Even if one assumes that a particular approximation scheme is "correct" (in the sense that, in principle, it approximates to the result of the exact theory), and that no calculational mistakes are made along the way, the final results may still be wrong by virtue of the neglect of terms in the expansion which are in fact of the same order or size as the calculated terms (as, for instance, Adams later discovered was the case with Laplace's secular acceleration calculation). Typically, the attitude of most researchers was inherent in the choice of expansion scheme. Whether working in post-Newtonian or post-Minkowskian schemes, one expanded the solution in powers of a quantity which was presumed to be very small, so that terms of each succeeding power in the small parameter were assumed to be orders of magnitude lower than the previous one. For some relativists, like Synge or Damour, this was an unsatisfactory state of affairs. Others took a much more pragmatic line, such as Thorne, whose view was that "one *could* calculate the next order correction ... somewhat easily if you permit [certain] leaps of faith". This problem, in its pure form, is a good example of the matter of rigour versus physical argument. It was generally agreed, even by those, such as Damour, for whom this was an issue worth addressing, that it was reasonable, for physical systems with appropriately small velocities or internal fields, that the truncation errors would be small, and that "it is a matter of rigour only" that they be estimated in a sophisticated way.

As we have seen, boundary conditions were a difficult matter to deal with throughout the history of this problem. There were several rather distinct aspects to their implementation. In the first place, for Infeld at least, there was the question of whether a retarded potential was the most appropriate to use in GR for the problem of motion. Everyone expected that without this choice of potential there would be no radiation damping, but most regarded their use as perfectly natural, either in analogy with electromagnetism, or by appealing to causality or other arguments. More important was the question of how to impose the boundary condition associated with this type of potential (with some type of outgoing waves carrying away energy) onto the local problem of motion calculations, especially in the post-Newtonian approximation. Even once this was achieved, there turned out to be a subtle distinction between boundary conditions imposed in the past or future of the source. In the former case, one called the condition a "no ingoing wave" condition, which meant that no waves were bringing energy to the system from elsewhere. In the latter, one had an "outgoing wave only" condition. These turned out not to be identical, since the source's own background curvature would scatter some of its won emissions back onto itself. Imposing "no-ingoing" waves in the past did not preclude having "ingoing" waves in the future, whose origin was the source itself. Imposing "outgoing wave only" conditions in the future artificially eliminated these tails. In the abstract language of spacetime, the past condition had to be imposed at "past null infinity" (the region of "infinite retarded time" from which lightlike worldlines reach us) whereas the future condition was imposed at "future null infinity" (the region of "infinite advanced time" to which lightlike world lines depart). The analysis of these important concepts is due to Penrose, arising out of the work of Bondi and collaborators, in the mid-sixties (Penrose 1964, 1965). In Newtonian theory, of course, these "null infinities" do not even exist (since the speed of light is infinite), underlying the problems for post-Newtonian approximations in depicting the asymptotic structure of the radiation spacetimes in the view of its critics.

While, from a mathematical standpoint, considerable effort was expended in proving that a given algorithm imposed the correct boundary condition on the abstract geometric infinities defined in the isolated source's spacetime, an objection sometimes made on physical grounds was that this ignored the reality of actual physical sources, which really existed in a universe full of other sources. From this perspective, "no-incoming" type of wave conditions were quite inappropriate. Schutz and Futamase made an effort to address this type of objection by taking a statistical approach, in which the source was just one in a whole ensemble of sources, all bathed in the others' radiation. Analyzing this ensemble of systems in a post-Newtonian scheme, they showed that the quadrupole formula result naturally arose as the consequence of the net interaction of the whole ensemble, without the need to impose boundary conditions designed to achieve the result of an outward flow of energy.

GR, unlike Newtonian theory, lacks a two-body solution. Therefore, apart from perturbation theory with only one large body, there is no two-body theory on which to approximate binary systems. Therefore two options are available. Approximate to Newtonian two-body, or to special relativity. The former works well for conservative motion, but where a third interaction is involved, as with an ensemble of absorbers, or asymptotic infinity via radiation theory, Newtonian theory is unsuitable, since there is no radiation in that theory. This is not a problem for special relativity, but as it does not include the two bodies' attraction to each other, it fails to approximate the Newtonian motion. The solution therefore was different approximations for the gravitating bodies and the "absorber," but the ambiguity in matching the right solutions between the two was not solved practically until Burke.

Of course, it is always possible to attempt to recast Newtonian theory into a form which makes it more directly comparable with GR. This is in fact a crucial step in the core analogy with Universal gravity which underlies the main justification for GR. Newtonian gravity and GR are, in themselves, about as incommensurate as two theories can get. Despite this handicap, relativists have been quite successful in appropriating Newtonian gravity for their theory, after the fact. The subterfuge of modifying both theories to more closely resemble each other, and then claiming one as the "limit" of the other because their modified forms make similar physical "predictions" is not often commented upon. That the predictions of classical gravitation theory had to be prudently edited after the birth of relativity theory is shown by the fact that the celebrated result of Laplace, that gravity acts with near infinite weakness, well known throughout the 19th century, had to be dismissed beginning with Poincaré in the twentieth. This prediction was not one which the new theory could claim for its own, so it ceased to be a prediction of Newtonian theory from 1900 on.

Nevertheless, this process seems to have met with a signal failure in the case of gravitational radiation. Havas (1979) points out that even the most elaborate recastings of Newtonian theory in GR-like forms seem to lack radiation effects. Havas traces this to the degeneracy of the "Newtonian metric". Although this was not widely seen as a major problem by others involved in radiation reaction work, it illustrates the philosophical problem posed by treating radiation effects in a Newtonian approximation.

Chapter 18

The Conflict of Style in Physics

To facilitate a discussion of the perceived contrast between the "rigorous" approach of "mathematicians," and the "intuitive" approach of "physicists," some eclectic comments on the meanings intended by these terms are in order. The word "mathematician" is used in a loose way in GR to describe both a professional affiliation and a style of doing physics. Some relativists really are mathematicians in the professional sense, by training or inclination, and GR is thought to attract more mathematicians than other branches of theoretical physics. Nevertheless, most relativists are physicists by training, a few having even come from an astronomy or engineering background. Despite this, some of them may be considered rather mathematical in their approach, so care has to be taken to understand what is meant by the use of this word in GR.

To describe a relativist as a "physicist" may seem straightforward, but again there is an operational definition here which may be hard to pin down. Broadly speaking, the "physicists" may be more concerned to relate GR theory to issues in other branches of theoretical physics and to carry on their work in the "style" of theoretical physics as it is done elsewhere. The "mathematicians" are more concerned with the development of GR according to its own historical dynamic, its own sense of where the interesting problems lie, and practise a "style" which reflects that of applied mathematics. Though there is no universally consistent description of what is meant by each of these styles, nevertheless, to the physics style we can attach the word "intuition," and to the mathematical style the word "rigour." Loosely defined, by "rigour" we refer to the issue of standards of proof, and by "intuition," we refer to an inner, possibly unconscious sense of justification for certain ideas.

What is the role of "intuition" in physics? Arguments based explicitly on an appeal to intuition rarely make their appearance in published physics papers. If physicists do regularly make use of it in their work, then they are eager to censor its traces from their discovery accounts. One might conclude that such "inspiration" or "insight" is distrusted by a community which regards itself as thoroughly part of the rational enlightenment tradition. The "oral" folklore of the community, however, provides a rather different picture. Stories of inspiration and intuitive leaps of reasoning abound in science in general (Archimedes in his bath, Kekulé and the Benzene ring) and in physics in particular (Dirac's realization of the analogy between Poisson brackets and Heisenberg's commutators which came to him "out of the blue", Einstein's epiphany with the equivalence principle, "the happiest thought of my life", both quoted in Chandrasekhar (1987, pg.20), who also relates interesting examples concerning Fermi and Heisenberg). Indeed, in personal interaction with other physicists, a physicist may reverse the procedure of a published paper, and disguise rational deduction as inspirational insight. An example of this is given in a recent forward to Feynman's *Lectures of Gravitation* (1996) by John Preskill and Kip Thorne.

> Sometime in early 1963, Fred Hoyle gave a ... seminar [at Caltech] on the superstar model for strong radio sources.[1] During the question period Richard Feynman objected that general relativistic effects would make all superstars unstable [to gravitational collapse] - at least if they were non-rotating. (Preskill and Thorne, 1996)

The substance of Feynman's objection was subsequently verified by several researchers. Preskill and Thorne continue,

> To Hoyle and Fowler [co-author of the superstar model with Hoyle], Feynman's remark was a 'bolt out of blue', completely unanticipated and with no apparent basis except Feynman's amazing physical intuition. Fowler was so impressed, that he described the seminar and Feynman's insight to many colleagues around the world, adding one more (true) tale to the Feynman legend.
>
> Actually, Feynman's intuition did not come effortlessly. Here, as elsewhere, it was based in large measure on detailed calculations driven by Feynman's curiosity.

The evidence for these calculations is preserved in the notes for one of the lectures published in Feynman (1996) (the lecture was given shortly before the Hoyle seminar in 1963) and in notes of Feynman's preserved in the archives at Caltech.

[1] An early model intended to explain the enormous luminosity of quasars.

If the oral tradition of a subject (regarding here textbook and memoir accounts as merely "collected folktale," passed on in a manner appropriate to a very literate community, but in an informal setting, distinct from the professional literature of the journals) is of value in illuminating the attitudes of scientists, it is perhaps particulary instructive to look at the special class of legends known as origin myths. Physics is particulary rich in origin myths, stories describing the genesis of the discipline. A celebrated example is the leaning tower of Pisa experiment. A classic example of an origin myth, it relates an incident which probably never happened, but which encapsulates the principal moral which contemporary physicists would wish to draw from some of the most significant work of an important ancestor figure, a progenitor of the social group. The moral is not a surprising one. The myth depicts the physicist, Galileo, performing a critical experiment, which at once destroys the old opposing, fictitious theory, and confirms his new one. It is not at all surprising that physicists should wish to portray their profession in this light, appealing to nature in an unambiguous way to confound their adversaries.

In an equally famous origin myth, centered around another ancestor figure, we find a less expected moral. The story of Newton's apple actually derives from a real contemporary anecdote, but it is very obviously a myth in the true sense of the word. In this story, the physicist is in repose, under a tree. An apple falls, not by his agency, and in a flash, a great insight is vouchsafed him. This tale is in remarkable contrast to that of Galileo's tower. The scientist sits under a tree, like a primitive sage. We might imagine it as the world tree, though it is not the ash of northern European myth, instead relecting Christian biblical symbolism. A vision of cosmic truth descends upon him from above (literally in many popular modern representations, where the apple is made to strike Newton on the head). The story quite clearly conveys the idea of inspiration striking the physicist, and this moment of intuitive insight (spoken of by other early modern theorists, such as Kepler[2]) is made to stand in, in mythic terms, for all the years of work which Newton was also required to do to fully develop the concept of universal gravitation. Yet it also stresses the centrality of the moment of insight which gives direction to the conceptualizing process. The story, which is an exceptionally popular one amongst physicists in terms of retellings, quite consciously portrays the scientist as a seer in the old sense, one who is inspired by divine powers. It seems fair to conclude that the inspirational mode of thought is one which physicists would wish to lay claim to by this origin myth, just as much as they would lay claim to the experimental investigation of nature by the Galileo story.

A third important strand in the origin myths of the scientific revolution is the rationalist fable of Galileo confronting church superstition and dogma in his trial for heresy. Misleading as it is historically, the popular account of this episode claims the mantle of the rationalist tradition as part of the inheritance of physical science. The primacy of this same trinity represented by the myths of the tower of Pisa, Newton's apple and Galileo's trial, is appealed to by Cooperstock and Hobill, quoted earlier ("the subtle interplay between experimental data, physical intuition, and ... generalizing concepts and principles"). The deeper mythological associations

[2]Quoted in Chandrasekhar's *Truth and Beauty*, pg. 66 (1987): "Now, it might be asked if this quality of the soul, which does not engage in conceptual thinking and can therefore have no prior knowledge of harmonic relations, should be capable of recognizing what is given in the outward world ... To this, I answer that all pure Ideas, or archetypal patterns of harmony, such as we are speaking of, are inherently present in those who are capable of apprehending them. But they are not first received into the mind by a conceptual process, being the product, rather, of a sort of instinctive intuition and innate to those individuals."

of the origin myths of physics are perhaps indicated by the original names of the three Muses of Helicon, which have been translated as "Meditation," "Memory" and "Song" (Graves, 1966). In other words, they represented the three indivisible strands of the poetic art. Similarly, physicists lay claim to experimental practice, physical intuition or insight, and logical argument as three strands of their science in these origin myths. By their suppression of the intuitive strand in their professional discourse (i.e. in journal papers), they place themselves within the "classical" enlightenment tradition, in which the three muses are symbolically subordinated to their father "Memory." The alternative romantic view is expressed by Blake, in the prelude to *Milton*, "and the Daughters of Memory shall become the Daughters of Inspiration." Thus the upholders of mathematical rigour can be seen figuratively as the defenders of Heaven (and therefore both innocence and the tyranny of reason) in the face of the onslaught of Blake's forces of Hell (upholding experience and the power of the imagination).

In studying the controversy following Weber's announcement of gravitational wave detections, Harry Collins (1985) has introduced the concept of the Experimenter's Regress. This describes the difficulty faced by experimenters when confronted with a dispute over non-confirmation of claimed results. Since none of the experiments will exactly duplicate the others' behavior, achieving consensus is hampered by the problem that the device which is working properly should get the correct result, but the correct result can only be known from the output of a properly operating device. Although Collins' view has been criticized (Franklin 1994), it seems to provide a useful model for understanding the Weber controversy.

In the theoretical controversy surrounding gravitational waves, one seems to observe a similar phenomenon, the *Theoretician's Regress.* The complex, tedious calculations designed to approximate to the full general relativity theory can be thought of as experiments, with the theory itself in the role of a notional "reality." These experiments constituted a delicate technical apparatus, designed to probe this "reality," aided by the craft and mathematical skill of the theorists. "Experimental error" was impossible to account for fully, whether as systematic error in the form of an inappropriate expansion scheme or failure to properly control errors from neglected terms (a difficult problem which was rarely addressed programmatically), or as accidental error in the form of simple calculational mistakes amidst the welter of terms which had to be collected.

As with the experimentalists, direct replication of another method was rarely even attempted. Even the best known schemes, such as EIH, were employed with improvements designed to simplify the calculations or overcome objections in principle, such as the use of point mass sources (Anderson 1995). Therefore, the array of review papers, conference workshops and other social efforts to achieve consensus had to overcome the cycle of regression constructed by the fact that the right scheme would be the one which gave the right result, but the right result was the answer given by the right scheme. The difference in emphasis between those who gave weight to having the right answer, and those who preferred to rely on method alone gave rise to further disagreement. This last observation may allow us to put our finger on the nature of the distinction between the physicists and the mathematicians. The former represent a style which is willing, to an extent, to be guided by the ability to "see" the right answer. The latter insists on the full rigour of the method alone. That no individual can ever consistently fall in to one group or the other should not disguise the reality of the *perception* within the field that some general distinction of this sort exists.

An interesting verbal usage by some of those who insist on the primacy of method, such as Damour and Anderson, is the use of the plural in referring to the quadrupole formula (Anderson 1980; Damour 1982). For both these two there is only one correct method, but a multiplicity of answers. The "quadrupole formulas" represent not one thing but many things. There is the quadrupole formula describing the flux of energy far from the source, and the quadrupole formula describing energy loss in the source. This contrasts with the more inclusive view on method of, for instance, Thorne, who nevertheless sees a unity symbolized by the common result of the quadrupole formula, a unity expressed in the use of energy conservation laws, at a level of rigour satisfactory to some, but not all, relativists. As always, there are different views on the problem of whether the means justify the ends or the ends justify the means. In this case, the use of the plural signifies a subtle weakening of one or the other pole in the means-end dialectic. One either has many derivations which all give one result, the quadrupole formula, or else one correct derivation applicable to a given problem, which may turn out to yield one of the many incarnations of the quadrupole formula. This weakening of one pole allows the retention of the basic dialectic, but enables an escape from the vicious circle set up by the regress in cases where dialectical tension fails to produce any synthesis, by establishing a primacy in the underlying dualist order.[3] It is possible that in the different choices of which pole to weaken one sees one element of the difference in style of doing physics,

[3] Much as the nominal early dualist purity of the Zoroastrian faith was vitiated by the eschatological asymmetry (at future null infinity, as it were) of the final triumph of the forces of Ahura Mazda - see Joseph Campbell, *Occidental Mythology*, pg. 201

since placing the emphasis on method emphasizes the rationalist project of science, whereas lending weight to the significance of the correct result inevitably leads to a reliance on one's intuition of which result is actually the correct one. It is never possible to disentangle "logic" and "intuition" entirely in the process of science. For instance, Chandrasekhar insists on the balance between these qualities in the work of the greatest scientists.[4] Nevertheless, the style conflict which is frequently referred to by the protagonists in the quadrupole formula controversy seems to have its roots in a greater emphasis placed on one or other of these two "ways of knowing."

An event which helped to partially break the cycle was the advent of the binary pulsar data. Initially this gave rise to more activity and more disagreement, but it also lent outside support to the preferred "right result" given by the quadrupole formula. However, it did not put an end to disagreements about the correctness of various methods, except in so far as it tended to rule out methods which disagreed with the canonical result. This was enough to gradually bring an end to the public side of the quadrupole formula controversy.

The role of social constraints within a community of theorists in the absence of experimental data is nicely illustrated by the course of the quadrupole formula problem. To begin with, almost no constraints on acceptable answers (in the sense of *publishable* results) were in effect, even if they were inherently "unbelievable", in Peres' words. Peres faced strong social pressures encouraging him to accept his first, incorrect result. The problem was his thesis project, and the desire to graduate was a strong motivation to finish the calculation. It was only after his successful graduation that he discovered the flaw in his algorithm (Peres, private communication).

[4] He quotes Fermi as saying that he would never believe a physical argument without a mathematical derivation, nor would be believe the mathematics without a physical explanation (interview).

Similarly Hu, who had presented his result in public before he discovered an error which changed the sign of the answer prior to publication, and Havas, for whom the strange result was emblematic of the unsatisfactory state of the field, had their own reasons for publishing a result which ran counter to all expectations. Nevertheless, it is true that from the mid-sixties on, no further energy gain results were published. It seems clear that in the wake of the successful efforts to describe the asymptotic behaviour of the radiation, the field was no longer so wide-open to interpretation as it had been. The space for acceptable results had shrunk somewhat. The answer to the question, what amount of energy is emitted by a self-gravitating system in the form of gravitational waves now had a "right" sign and a "wrong" sign. Hand waving arguments, such as those offered by Hu, relating his expanding orbit result to the Hubble cosmological expansion, and Peres, drawing a "reverse" analogy with electromagnetism in observing that the sign of the gravitational attraction is opposite to that of electromagnetism, i.e. like charges attract instead of repelling (and thus the field energy density in the binary is negative, and the sign of the change in energy is reversed from the electromagnetism case) could no longer be easily employed to dress up such a result with an air of plausibility. Admittedly, Peres' argument that gravitational waves might carry negative energy was still being advanced by Rosen in 1964. In fact, Rosen's bi-metric theory, in common with some other minority gravitational theories, is still held, in modern parametrized post-Newtonian theory, to predict orbital expansion rather than decay as the result of gravitational wave emission (Weisberg and Taylor, 1981). However, this feature of these theories is hardly considered a point in their favour by most observers.

Nevertheless, even the increased sophistication of the field did not suffice to completely eliminate the space for disagreement. In the 70s, one still found Rosenblum and Cooperstock disputing the field with the supporters of the quadrupole formula result. Furthermore, the "agnostic" attitude typified by Havas gained new adherents, such as Ehlers. Only with the advent of the binary pulsar data could the acceptable field of results be narrowed down more or less completely to one option. This last stage of the story hardly took place overnight either. It was only gradually that the space for dissent was worn down. This was not achieved, in the final stages, without some damage to reputations. Whereas, in the early period, dissent was permissible, in the 1980s, this was no longer the case. There seems to be some evidence that Cooperstock, although he is still professionally very active, lost a certain amount of standing on this topic as a result of his open defiance of the emerging orthodoxy. Whether this would have been true also of Rosenblum had he lived is hard to say, but it seems quite possible. Certainly, the practical effect of Cooperstock's loss of standing may be seen in his inability to provoke debate over his new challenge to the establishment, in which he revives the old argument over whether gravitational waves can carry energy, and in the role of the pseudo-tensor. That avenues of expression still remain is evident by the publication of the initial paper on his "energy-localization hypothesis" in *Foundations of Physics*, a journal which encourages the publication of "speculation not tied to hard and demonstrable facts [but] suggestive of new basic approaches in physics" (Editorial Preface, vol.1 no.1 pg.3, *Foundations of Physics*) in which Joseph Weber has also published in recent years. Nevertheless, as discussed above in chapter 16, provoking a rejoinder to initiate debate has proved more difficult than simply gaining a platform.

Finally, even when the point was reached that room for disagreement on the leading order *result* for gravitational back reaction was eliminated, the same could hardly be said of the question of method. For those for whom this was always a central consideration (Havas, Ehlers, Damour, Anderson) there is still much to be critical of in all, or all-but-one, of the multitude of derivations of the quadrupole formula. Damour, indeed, resists the tendency to reduce the problem to one of verifying the quadrupole formula (interview), which distracts from the larger question of the correct approach to the problem of motion in binary systems. Ehlers and Havas, when interviewed, expressed themselves as still unhappy with some aspects of the state of the field. Ehlers felt that enough work had now been done at least to convince him of the approximate validity of the quadrupole formula. Havas still seems to entertain certain reservations on that score. Damour and Anderson both continued to be critical of all solutions to the binary back reaction problem except their own, not least each others. However, in a context in which the final answer is the same, such disagreements in principle do not seem to be the stuff of controversy.

Therefore the aim of the all of the conferences, workshops, papers, reviews, appeals to experiment and so on, is seen not to be to enforce or encourage *agreement* as such, but to eliminate or reduce the space for *disagreement.* There seems to be a definite distinction between what the community can agree to disagree about, and what they must argue out. In the early period (1945-1965 or so) the over-riding issue of principle of whether gravitational waves existed or were emitted by binary star systems was something which had to be argued out. In the pre-war period, though the issue was noted by Eddington, the subject had not matured to the point where it could support such a debate. After some point during the early 1960s, the debate ceased to be relevant as a sufficient consensus had formed against the sceptical position. Subsequent attempts to raise this issue by Havas, Rosen or Cooperstock have received little attention. Similarly, the quadrupole formula controversy did not really emerge in its own right until after 1965. Up to that time the subject could not sustain such a debate, since the level of technical certainty or proficiency which could sustain a single canonical result had not really crystallized. By the 1970s the various post-Newtonian methods having at least agreed with other approaches on a consistent basis, there was sufficient ground for a debate over the quadrupole formula per se. The advent of directly relevant experimental data, on top of the increasing astrophysical relevance of compact objects and gravitational waves, lent urgency to the matter. But gradually in this case the room for dissent was squeezed down. Eventually a critical mass of consensus, enough to close off further debate, formed around the orthodox opinion. As the field of gravitational waves moved into a new era, in which detailed calculations going beyond the quadrupole formula would be required for present (PSR1913+16) and future (LIGO etc.) experimental efforts, dissent was no longer viewed as healthy or desirable. Further disagreement would only retard the progress of a field which was showing signs that it was about to take a significant step into the forefront of physics.

Chapter 19

General Relativity in Physics

The many efforts to develop a quantum theory of gravity constitute a further example of the syncretic impulse in physics. Again analogy based on wave phenomena played a role. Since historically it was the study of radiation which led to the quantization of the electromagnetic field, it was naturally expected by some that the same might apply in gravity. For instance, Pirani (decidedly a non-sceptic) states

> The primary motivation for the study of [gravitational radiation] theory is to prepare for quantization of the gravitational field (Trautman, Pirani and Bondi 1965, pg. 368).

We have already encountered the idea that gravitational radiation was expected to play a role in quantization of gravity in the discussion between Rosenfeld, Bondi and Feynman at Chapel Hill. At the same time, interestingly, the syncretistic movement was resisted psychologically by some relativists who perhaps preferred not to see GR converted into just another quantum field theory. This anti-syncretic mood is plain to see in a remark made by Mercier in the context of a reply to a paper by

Havas advancing the "sceptical" position on the quadrupole formula.

> Physicists ... perhaps are too conservative in believing that physics (theoretical) should always be made and interpreted the same way, e.g. by wanting to do within GRG the same as in Electrodynamics. Perhaps, the revolution implied by GR is that it is precisely another way of concerning the physical world. And Einstein's drama was perhaps that he tried all his life to unify gravitation and electricity, believing or suggesting at least that these two phenomena are alike, i.e. both interaction, if electricity is an interaction. I personally am not sure that mass is a kind of charge, I am not sure that physics should consist in assuming a vacuum and putting things in it, that there are free fields and perturbed fields, etc. Unification in Einstein's sense was never a success. Perhaps the interaction proper (electromagnetic, strong and weak) can be unified; I had some argument about it in my talk last Monday; but not with gravitation, which is not an interaction in the same sense.
>
> I could go on like that, calling attention upon the *fundamental* difference between GRG and physics as it is done elsewhere. (Havas, 1973)

For those with sympathy for this view, the position of the sceptics on gravitational waves offered some hope. If gravitational waves did not exist, then the analogy which drove the quantum gravity project had broken down in a very fundamental way, a potentially critical way for the quantization effort, if history was any guide. One view (certainly a minority one) seems to be, "if there are no gravitational waves, there are no gravitons, and therefore no quantum gravity".[1] The anti-syncretic mood which is evinced by some relativists may partly have its roots in the disdain which some other physicists from different traditions are preceived to regard the relativity community.

In the long run, the utility of the basic analogy between gravitational waves and electromagnetic waves seems to have been vindicated by its continued acceptance by a wide body of scientific opinion, despite the persistence of some public (and published) scepticism to the present day. Certainly the quadrupole formula, a particular bone of contention for many years, achieved experimental vindication in a most unexpected and dramatic way with the work on the first binary pulsar. It seems clear that an analogy is not a thing whose validity can be proven in any meaningful way, but the longevity of this analogy seems a considerable testament to its success. That the experimental search for these waves, whose existence at one time had no other argument to support it than this analogy, now commands vast resources and the efforts of many physicists and other workers, must be an even greater one.

[1] Gravitons are an analogue of photons, the mediators of the electromagnetic field, and are an important element of most concepts of quantum gravity. Gravitons are thought of as the particles associated with gravitational waves, on analogy with the wave-particle dualism encountered in quantum field theories.

Chapter 20

Conclusions

The problem of gravitational radiation and radiation reaction has a history in which various characters or types have been posed in opposition to each other. "Sceptics" versus "non-sceptics," "mathematicians" versus "physicists" and so on. Understanding the meaning lying behind such appellations can prove tricky. In the case of sceptics, it seems safe to conclude that this usage refers to those who doubted the underlying analogy upon which the idea of gravitational waves was constructed, that is the analogy with the electromagnetic field theory. Since the analogy itself has several interlinking components (two hyperbolic field theories, similarity of linearized field equations, multipole radiation formulas) as well as several distinct weaknesses (equivalence principle, non-linearity of Einstein equations, determination of equations of motion from field equations), the sceptics did not form a monolithic group at any stage, but the writings of all major sceptics betray serious concerns about the role played by this analogy, expressing a generally shared sense that this particular analogy might be highly misleading. This seems to be the common thread running through the sceptical position, from early doubts about the existence of gravitational waves to the quadrupole formula dispute. The latter can be seen as a natural progression in this light, as this formula was a chief prediction made by the analogy in its most potent form, that of the linearized field equations of relativity as presented by Einstein in 1916 and 1918.

The sceptic who best exemplifies this description is Havas, also perhaps the most persistent sceptic. Havas' background (his experience in classical electrodynamic theory), and his extensive historical knowledge suited him very well in his role as the most searching critic of the analogies with other theories, including the Newtonian, but especially the electromagnetic, current in GR. Partly inspired by Bondi's remarks on the subject, he became aware early on of the potential pitfalls in relying overmuch on the analogy, and from then on he repeatedly pointed out the potential dangers lurking in an uncritical acceptance of the radiation analogy with electromagnetism.

While I have focused on the position of the sceptics, just as interesting, if not more so, is the role played by the analogy in creating belief in a phenomenon for which no experimental evidence whatsoever existed until quite recently. Several of those I interviewed, and the preambles to many papers (such as Peters and Mathews) express a very firm conviction in the existence of gravitational waves which is perhaps a greater wonder than the doubts of the sceptics. Within the physics community its normality makes it less remarkable, but its pervasiveness makes it all the more remarkable when viewed from the outside. Mathematical and physical analogy seems a slender reed on which to hang such conviction, but it seems that it is not so. The fact that the analogy is more spoken of by sceptics than by "believers" should not, I feel, disguise its paramount motivating force for both groups. Pirani, for instance, is a good example of someone highly motivated by the strength of the analogy between gravity and electromagnetism in this context. Feynman is another example, and here one should also note the tremendous success and coherence of quantized field theory which encouraged a sense that all field theories would ultimately hang together in a tightly interwoven way, and share certain chief characteristics.

The strange contrast in opinion on Landau and Lifshitz' famous derivation of the quadrupole formula is also explicable by the differing reactions to the reliability of the underlying analogy. Several interviewees expressed the opinion that Landau and Lifshitz had settled the matter for them, as far as the likely correctness of the quadrupole formula went. Yet others spoke rather caustically of this derivation, describing it as "a little glib," "full of holes," only credible "if you believe Landau was connected to God" (interviews with Bondi and Anderson). Based on what one reads or hears, it is a little difficult to put one's finger on exactly what features of their derivation it is that provoked such strong and contrasting opinions. Thorne has sought to explain some of the reluctance to accept it on simple misunderstanding based on their famously terse style.

> Evgenii Lifshitz, who is responsible for the Landau-Lifshitz prose, writes with such terse elegance that most readers overlook the fact that his derivation is valid for self-gravitating sources. I only discovered it 10 years after first reading Landau and Lifshitz.

However, one can also look for the explanation in the instinctive reactions to the electromagnetic analogy, which indeed is implicit in their book's title, *The Classical Theory of Fields.*

As to why some researchers were more wary of the "linear" analogy than others there is perhaps no single reason. Perhaps some, such as Havas, began with a predisposition to accept it. Havas was himself drawn to the problem by his desire to extend his experience in the equivalent electrodynamic theory, and in his 1957 paper, when he still thought his calculations indicated a result with the "correct" sign, he concluded with a ringing endorsement of the analogy. However, experience convinced him that the analogy was on shaky ground, and that faith in it was liable to mislead in the case of gravitational radiation reaction, as he makes plain in his 1965 paper with Smith. In the intervening period he had become the most exacting and detailed critic of the linearized approximation and the analogy it inspired, a position he was to continue to occupy throughout the period of the quadrupole formula controversy. Similarly, Einstein, who first formulated the analogy precisely, later had second thoughts when attempting to construct an exact solution.

The orthodox position recognizes failings in the post-Newtonian schemes. A paper from the Caltech group, part of that group's program of developing a strong theory of astrophysical sources and signals for use in signal processing in LIGO-type detectors, states that

> there exists no general algorithm that allows one to solve radiation reaction problems to arbitrary order in a PNE [Post-Newtonian Expansion] (as distinguished from, say, a PME [Post-Minkowski Expansion]). In this sense, the theory is much less developed than, say, perturbative quantum electrodynamics. Nor is it understood whether an infinite PNE (assuming one could be generated) would converge or merely be asymptotic to an appropriate solution of the field equations (Cutler, Finn, Poisson and

Sussman, 1993).

So the difference between sceptics and non-sceptics reduces in part to an operational one of whether certain results are sufficiently reliable to form the basis of further work. Sceptics would, for instance, not be so confident that the infinite PNE would even asymptote to an "appropriate" solution of the field equations. Havas, for instance, always insisted it was not clear to him that the conditions for an appropriate solution were included in this formalism. Ehlers was a particular critic of the assumption that expansion schemes employed in the problem of motion really approximated to genuine solutions of the field equations.

Some analysis of the term "rigour" or "mathematical rigour," which has been used extensively in the quadrupole formula debate, is required. The term is a slippery one, its meaning apt to change with the context in which it is employed. Some clues are provided by a controversial article which recently appeared in a mathematics journal, addressing the invasion of mathematics by theoretical physicists interested in topics connected with grand unification schemes such as string theory and quantum gravity. Jaffe and Quinn (1993) argue that these theoretical physicists, cut off from much experimental input by their highly abstract interests, have replaced experimental physics with mathematics as the agency by which constraints are placed on physical theory. Instead of experiment invalidating their conjectures or models, "rigorous" mathematical proof can do so instead. Interestingly, Jaffe and Quinn give the name theoretical mathematics to this endeavour, which they see as being potentially more valuable to mathematics than to physics (where such highly abstract theorizing is traditionally viewed with suspicion). They see pure mathematics as "nearly characterized by the use of rigorous proofs, ... the result of literally thousands of years of refinement [which] has brought to mathematics a clarity and reliability unmatched by any other science." To them, rigour is, in some sense, the antithesis of "theory."

> The initial stages of mathematical discovery - namely, the intuitive and conjectural work, like theoretical work in the sciences - involves speculations on the nature of reality beyond established knowledge. Thus we borrow our name 'theoretical' from this use in physics. Theoretical work requires correction, refinement, and validation through experiment or proof. Thus we claim that the role of rigorous proof in mathematics is functionally analogous to the role of experiment in the natural sciences. ... Proofs serve two main purposes. First, proofs provide a way to ensure the reliability of mathematical claims, just as laboratory verification provides a check in other sciences. Second, the act of finding a proof often yields, as a byproduct, new insights and unexpected new data, just as does work in the laboratory.

Rigour then, we are to understand from Jaffe and Quinn, is characterized by "theorems" and "proofs" (also an understanding shared by many of my interviewees). It is what characterizes much of what mathematicians do, and is made possible by the great depth of historical practice in that field. Theory, on the other hand, for which theoretical physics is the paradigmatical model, involves "speculation" and "intuition," and while it may employ mathematical formula and manipulations, it is not *mathematics*. A useful example of the difference between the two is given by Schweber (1994), concerning Feynman. "Fermat's last theorem," actually a conjecture, has been for centuries the object of countless attempts to produce for it a rigorous mathematical proof. Feynman's proof, described by Schweber, is to estimate the probability of a counter-example to the theory occuring beyond the region where it is known to hold. Feynman's estimate is that there is only one chance in 10^{200} that the theorem is falsified, and therefore "for my money Fermat's theorem is true" (Schweber, pg. 464). No further effort need be expended. Schweber adds, "of course it would be very satisfying to have an elegant proof of the theorem, but as far as he [Feynman] was concerned he 'knew' it was right even though he couldn't prove it rigorously."

Certainly it might be reasonable to conclude that a mathematical physicist like Ehlers saw proof playing the same role as Jaffe and Quinn envisage in a subject with little or no experimental input. However, other theoreticians resisted this viewpoint, regarding radiation reaction in GR as "a messy, messy problem" in which no rigour (defined as the proving of theorems) is possible (Anderson, interview). But Ehlers and other sceptics were motivated by much more than this, and we have already seen how Ehlers disclaimed any intention of imposing mathematical standards of proof on relativistic physics. The advent of experimental data in the form of the binary pulsar orbital decay was greeted by both Anderson and Ehlers as a motivator to more securely establish the quadrupole formula *as a prediction of general relativity.* The real question was not whether the derivations of the quadrupole formula which had been presented were acceptable by the mores of mathematics, but whether they were acceptable by the mores and standards of general relativity. Ehlers, as we have seen, was a particular critic of attempts to import concepts from other physical theories without justifying them in the context of GR. Again, it was the *culture of relativity*, despite the contrary pulls of the cultures of mathematics or theoretical physics, which was the primary concern of most of those involved in this subject. It was this unique culture which gave them some degree of cohesiveness as a group, and it was differing views as to the core values of that culture which contributed to disputes in the context of gravitational radiation reaction.

Naturally, relativists' notions of the appropriate core values for their discipline were greatly influenced by the standards familiar to them from their general disciplinary background. Thus we find differences in outlook based on nationality, school and previous disciplinary experience. Damour refers, for instance, to the outlook of the French mathematical school (1982). Schweber (1986) and Kevles (1971) have both commented on the pragmatic quality of American physics, which seems reflected in the attitudes of many, though hardly all, American relativists. One can distinguish quite different outlooks between the Bergmann and Wheeler schools for instance, the two major American schools of relativity.[1] But in any case, each individual internalized all of these overlapping influences, and no doubt perceived them as an organic whole which expressed his or her own experience of what it meant to do relativity. We should not be surprised to find conflict arising between these various personal identifications with the subject in the context of a very public controversy. Whereas it is easy to view theoretical controversy as taking place within a monolithic *culture of theory*, this analysis reveals contrasting attitudes to what practicing theory means, which attitudes themselves shaped and amplified debate. Exponents of different styles and schools of physics struggled for space within which

[1] A striking number of nationalities are represented in the field of gravitational wave theory, reflecting the wide international standing of relativity theory. Besides the American, Polish, German, French, British and Canadian schools mentioned in this account, one finds contributions from Japan, Ireland, China, Vietnam and Mexico as well as several continental European countries.

to carry on their particular way of doing theory, all at the same time conscious of their identity within a group itself seeking disciplinary space within the larger body of theoretical physics. One can identify several layers of identity operating for each individual, to physics or mathematics, to relativity, to his own school or pedagogical background and so on. The need for some form of disciplinary identity as *relativists*, expressed in the proliferation of conferences, professional associations and journals devoted to GRG (general relativity and gravitation[2]) after 1960, seems to have helped to prevent the fragmentation which might have otherwise have occured over controversies such as this, due to the contrasting backgrounds and outlooks of the protagonists. This hard won disciplinary identity could be easily threatened by subsumation into the larger body of quantum field theory, or by degeneration into an ancillary branch of astrophysics and cosmology. It is not surprising therefore to find some ambivalence attaching to a subject, such as gravitational waves, which managed to express both of these cross-disciplinary relations at once.

The philosophical antipathy of the two fundamental approximations of GR, the Newtonian (with its analogy to universal gravitation) and linear (with its analogy to relativistic electrodynamics), is seen by the modern theory's deconstruction of the space around an isolated physical system (that is to say, the deconstruction of the notional "universe" in which the source exists) into a complex hierarchy of regions arranged in successive layers like the circles of heaven. "Strong field near zone," "weak field near zone," "local wave zone," "distant wave zone," "transition regions," and so on formalize the space around the system into domains of validity for different expansions and approximations, with matching techniques employed to

[2]The name itself indicates a desire to be inclusive, since it can encompass those theorists, such as Rosen, promoting rival (to GR) theories of gravity

reconcile the various schemes in operation.

Of course the use of different expansions to cover different regions of a physical problem is not unique to GR. The WKB approximation of quantum mechanics is perhaps the best known example of the use of matched asymptotic expansions in physics (the concept is also commonly applied in fluid mechanics). The idea of the wave zone and the near zone in radiation problems arose in electromagnetic theory. Nevertheless, the radically different epistemological bases for the expansions used in different regions of the gravitational radiation problem is a striking feature of this field. The inadequacies of the analogies being drawn with different and somewhat incompatible theories were forcefully pointed out by Havas and Ehlers, and the shotgun marriage of the two different approximations via matching techniques did not fully alleviate their misgivings.

One should not regard the "physicists" attitude to the Newtonian analogy and the Newtonian limit as naive, but rather as pragmatic. They also were aware of important differences between Newtonian and relativistic concepts, but regarded the use of analogy relating quantities in the two theories as vital for several reasons. Amongst these were the relevance of relativity to subjects such as astrophysics, where Newtonian rather that relativistic gravity is still routinely applied for most purposes, and as a guide to the application of relativity to systems whose Newtonian behaviour was well understood, such as binary stars. Here one sees the importance of analogy to the physical intuition spoken of so much in this area. Such intuition must be based on experience, in this case experience with another theory. From this standpoint, challenging the status of the operational analogies between Newtonian and relativistic concepts and quantities, or demanding that their use be tempered with caution until their epistemological standing is clarified, is tantamount to dangerously retarding the progress of research in the field, perhaps to the point where its survival within theoretical physics would be threatened.

One should beware of attaching any historical significance to the use of the term Newtonian. In fact, modern Newtonian theory is a living 20th century theory, bearing little resemblance to the pre-relativity theory. For instance, the quadrupole formula itself is a "Newtonian" result in the language of contemporary theory, since it is valid in the weak-field, slow-motion limit, although it pertains to a concept entirely alien to 18th or 19th century gravitation theory. This body of theory is routinely extended by the addition of post-Newtonian corrections from the type of slow-motion approximations to the problem of motion which we have discussed. The modern theory of gravity is far from monolithic, though it is frequently stated that GR is the paradigmatic theory of our time. Instead, one has Newtonian theory, post-Newtonian theory, classical general relativity, semi-classical relativity, quantum gravity, supergravity and so on, all standing in different relations one to the other, all, in a manner which some might find potentially ambiguous, held to approximate to some ultimately true quantum field theory of gravity.

At the present time, as they look forward hopefully to the possibilities for new problems and challenges, the progressive or pragmatist school can feel vindicated by the course of the quadrupole formula controversy. The position of exponents of this outlook, such as Thorne, has been fairly consistent for as long as 20 or 30 years, and was summed up by Thorne in 1980.

> One group of gravitation theorists, led by Jürgen Ehlers, Arnold Rosenblum, Joshua Goldberg, and Peter Havas (1976), believes that these radiation-

> reaction results have not been derived with sufficient rigor to be fully trustworthy. Another group, to which I strongly adhere, believes that the rigor, e.g. of the Burke (1969,1971) and Chandrasekhar-Esposito (1970) derivations exceeds that of many analyses in mathematical physics which physicists firmly trust. We are happy to let our more mathematical colleagues polish up their derivations; but we have no doubt that in the end the results will remain unchanged. (Thorne, 1980)

It is noteworthy that this work of "polishing" stands apart, or appears to play at best a minor role in, the extensive research program of which Thorne outlines in this article. The three prongs of this program are for relativists to develop "new mathematical tools for the analysis of gravitational radiation," astrophysicists to "identify the most promising sources of gravitational waves ... using the relativity theorists' mathematical tools to estimate the characteristics of the waves they emit" and experimentalists to design and construct "a second generation of gravitational-wave detectors," of several different types, with the emitted wave characteristics predicted by the theoretical effort in mind. Since it is apparent that the quadrupole formula is one of the tools available for use in estimating source evolution, it would appear that the process of "polishing" the derivation cannot be expected *a priori* to advance the research program Thorne outlines. Thorne's use of the phrase "mathematical tools" appears to refer to useful estimates or relations such as the quadrupole formula which can be applied to astrophysical systems in source and signal estimation. The usage of mathematics is presumably quite distinct to the use implied in his discussion of the concerns over "rigor" expressed by "our more mathematical colleagues". There is a distinction here, perhaps, between mathematics as a *tool* and as a *process*.

The view that increased mathematical precision is ancillary to, and perhaps largely dispensable from, the advance of science certainly appears to have raised concerns over their role amongst some relativists. At least the repeated assertions by progressives that there was no fear of new derivations overturning the less rigorous results appear to have rankled with some people. The view that the "elegance" of rigorous proofs is largely ornamental to the practice of physics, discernible in the attitudes of Feynman and Thorne, must have seemed both threatening and slighting to those of a more mathematical bent, who may have felt their contribution undervalued somewhat.[3] At the same time, the alternative possibility apparent to the progressives, that they actually wait for more precise derivations before proceeding with applications, must have seemed just as threatening to the survival of a subject emerging from obscurity.

For the progressives then, the new era promised by the great new gravitational wave detectors such as LIGO is one of final liberation from the confining world of very limited contact with experiment and observation. Just as the theory of non-gravitational fields and fundamental particles appears to be running out of empirical room, with the collapse of the Superconducting Super-Collider (SSC) project, GR will at last have its own window onto the strong field region in which so much of modern field theory (including relativity) has thrived. For those interested in exact results of GR theory, such as Winicour, this new horizon is one of opportunity also. In order to successfully extract gravitational wave signals from the noise of these detectors, precise theoretical templates for the signals will be required to aid

[3] Jaffe and Quinn comment on the requirement that confirmation have roughly equal status with discovery if theorists and mathematicians are to co-exist side by side.

in signal processing. Theory will thus help make signals from coalescing binary systems visible in the detector. However, the strong fields produced in the merger of two black holes invalidate all present approximation schemes before the end of the inspiral/merger process. As discussed earlier, a large program (known as the Binary Black Hole Grand Challenge Alliance, a consortium of several numerical relativity groups with NSF backing) is now underway to solve this problem exactly by numerical methods on large, highly parallel computers. Winicour sees the possibility that such exact solutions will make concepts such as the quadrupole formula irrelevant, products of an earlier period in which GR was not fully understood, and made use of concepts borrowed from other paradigms, such as (20th century) Newtonian gravity. One unique feature of gravitational waves from coalescing binaries which may be detected in the next century is the Christodoulou memory (Kennefick, 1994). Signals from such sources were long thought not to contain a memory, which is a local change induced by the passage of a gravitational wave which persists after the wave is gone. A non-linear effect of this type, a "wave" produced by the wave itself, was predicted by the mathematical relativist Demetrios Christodoulou in 1991. It is a striking physical effect of wave emission, which had previously been entirely missed by the "intuitive" physicists. In his paper on the Christodoulou memory (reversing the paradigmatic order that the physics discovery precedes the mathematics confirmation), Thorne reinterprets Christodoulou's discovery in physical terms, and wistfully reflects on the one that got away from his physical intuition, to be discovered and not just rederived by the mathematician .

The author (who is an advocate of simple physical explanation for important physical effects) has long thought he understood fully the mem-

> ories of gravitational-wave bursts. It has been a salubrious experience, therefore, for the author to be shown by a mathematician (Christodoulou, who uses elegant mathematics that is far from the physics) that he (the author) has missed a very important physical effect. (Thorne, 1992a)

It is therefore still true, facing into the age of the large gravitational wave detectors, that the private visions of what it means to do general relativity are just as distinct and vibrant as they ever were.

Appendix A: Interviews and Other New Sources

James Anderson	Hoboken, New Jersey	03/04/95	tape
Peter Bergmann	New York City	03/04/95	notes
Luc Blanchet	Meudon, France	12/10/94	tape
Hermann Bondi	Cambridge, England	7/11/94	tape
Dieter Brill	College Park, Maryland	06/04/95	tape
S Chandrasekhar	Chicago, Illinois	12/06/95	notes
Fred Cooperstock	Victoria, BC, Canada	26/06/95	tape, correspondence
Thibault Damour	Bures-sur-Yvette, France	11/10/94	tape, correspondence
Nathalie Deruelle	Paris, France	19/10/94	notes
Jürgen Ehlers	Munich, Germany	14/10/94	tape
Joshua Goldberg	Syracuse, New York	10/04/95	notes
Peter Havas	Philadelphia, Pennsylvania	05/04/95	notes, correspondence
Richard Isaacson	Arlington, Virginia	08/04/95	notes
Charles Misner	College Park, Maryland	07/04/95	notes
Ezra T Newman	Pittsburgh, Pennsylvania	11/04/95	notes
Asher Peres			correspondence
Felix Pirani	London, England	25/10/94	tape
Jerzy Plebanski	Mexico City	30/06/95	notes
Adrian Scheidegger			correspondence
Dennis Sciama	Venice, Italy	16/10/94	tape
John Stachel			correspondence
Kip Thorne	Pasadena, California	14/06/95	tape
	Pasadena, California	17/07/95	tape
Andrzej Trautman	Trieste, Italy	17/10/94	tape
Phillip Wallace	Victoria, BC, Canada	26/06/95	notes
Joseph Weber	Irvine, California	20/06/95	tape
John Wheeler	Princeton, New Jersey	04/04/95	notes
Jeffrey Winicour	Pittsburgh, Pennsylvania	11/04/95	notes

Dates are given as day/month/year. Interviews which were recorded and for which a tape is available are indicated. Provided the interviewee is willing, access to the tapes or notes arising from interviews and discussions will be permitted to interested scholars.

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