PARTIAL DIFFERENTIAL AND DIFFERENCE EQUATIONS

Thesis by

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ABSTRACT

In this thesis, general conditions on the coefficients of difference equations are obtained which insure the convergence of solutions of these difference equations to the solutions of the corresponding partial differential equations. A general method which is applicable to a wide variety of partial differential equations is presented here. However, in order to simplify the presentation of this method and to leave out calculations which are not essential to the description of the method, the discussion is centered about the partial differential equations

$$\frac{\partial u}{\partial y} = c\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \quad .$$

The treatment of these simpler problems then serves to indicate the method for more general problems.

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INTRODUCTION

In many cases it is practically impossible to solve an initial value problem for a partial differential equation exactly, although it can be proved that the exact solution does exist and is uniquely determined. Therefore, the partial differential equation is very often replaced by a difference equation which is easier to solve (and which, incidentally, furnishes an approximation to the solution of the original problem). The properties of the differential equation are then, under certain assumptions, determinable if one lets the solution of the corresponding difference equation converge towards the solution of the differential equation by refining the width of the mesh of the underlying lattice.

In the case of elliptic partial differential equations, one has simple and extensive convergence conditions which are independent of the choice of the underlying lattice, but, in the case of the initial-value problem of hyperbolic partial differential equations, convergence is, in general, present only when conditions on the mesh-width of the lattice satisfy certain inequalities which are determined by the location of the characteristics of the lattice.

Courant, Friedrichs and Lewy [1] have treated initial-value problems of linear hyperbolic differential equations and have shown that, under certain assumptions, the solution of the difference equation converges toward the solution of the corresponding differential equation by refining the width of the mesh of the underlying lattice. In particular, they have treated the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} ,$$

where they take a rectangular, axially-parallel lattice as a basis, whose temporal mesh width is $\underline{h}$ and whose space mesh is $\varkappa h$, with $\varkappa$ constant. For this equation, they then show that in the case $\varkappa < 1$, if one allows the mesh-width $\underline{h}$ to decrease towards zero, the solution of the difference equation cannot in general converge to the solution of the differential equation. On the other hand, if $\varkappa > 1$, they show that convergence will occur.

In other words, given any partial differential equation, it can be approximated by many different difference equations; however, as cited above and as shown by many others, only some of these difference expressions will work - that is, in only certain cases will the solution of the difference equation converge to the solution of the corresponding differential equation. This raises an interesting question which motivates the problem upon which this thesis is based, namely, to try to understand why some difference patterns work (in the sense just mentioned) and why others do not.

In this thesis, then, we would like to obtain various general conditions on the coefficients of these difference equations under which such a convergence is possible. While a general method will be indicated here, the discussion will be centered about the partial differential equations

$$\frac{\partial u}{\partial y} = c \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

in order to simplify the presentation. The treatment of these simpler problems will then serve to illustrate the method for more general problems. In particular, the method presented here strongly indicates a suitable technique for the treatment of the homogeneous, linear partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0 .$$

CHAPTER I

STATEMENT AND DEVELOPMENT OF THE PROBLEM

Here, in this thesis, we will consider difference equations of the form

$$\sum_{r,s} a_{rs}\, u(x + rh, y-sh) = 0 \quad ,$$

and our general problem will be to determine sufficient conditions on the a_{rs} under which solutions of the difference equations will converge to the solutions of corresponding partial differential equations.

Now, a difference equation of the form

$$(1) \qquad \sum_{r,s} a_{rs}\, u(x + rh, y-sh) = 0$$

is said to approximate a partial differential equation of the form

$$(2) \qquad \sum_{i+j=k} A_{ij} \frac{\partial^k u}{\partial x^i \partial y^j} = 0$$

if

$$(3) \qquad \lim_{h\to 0} \frac{\sum_{r,s} a_{rs}\, u(x + rh, y-sh)}{h^k}$$

tends, for u sufficiently smooth, to

$$\sum_{i+j=k} A_{ij} \frac{\partial^k u}{\partial x^i \partial y^j} \; .$$

Hence, the difference equation, Eq. 1, will tend to the partial differential equation, Eq. 2, if certain moment conditions are satisfied. Here, in this thesis, we are interested in the homogeneous, linear partial differential equations

$$\frac{\partial u}{\partial y} = c \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} .$$

Thus, for the first differential equation, namely, $\frac{\partial u}{\partial y} = c \frac{\partial u}{\partial x}$, the resulting moment conditions are

$$(4) \quad \begin{cases} \text{i)} & \sum\limits_{r;s=0}^{n} a_{rs} = 0 \\ \text{ii)} & \sum\limits_{r;s=0}^{n} r\, a_{rs} = c \sum\limits_{r;s=0}^{n} s\, a_{rs} \neq 0 , \end{cases}$$

while for the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$, the moment conditions are given by

$$(5) \quad \begin{cases} \text{i)} & \sum\limits_{r;s=0}^{n} a_{rs} = 0 \\ \text{ii)} & \sum\limits_{r;s=0}^{n} r\, a_{rs} = \sum\limits_{r;s=0}^{n} s\, a_{rs} = 0 \\ \text{iii)} & \sum\limits_{r;s=0}^{n} rs\, a_{rs} = 0 \\ \text{iv)} & \sum\limits_{r;s=0}^{n} r^2\, a_{rs} = - \sum\limits_{r;s=0}^{n} s^2\, a_{rs} \neq 0 . \end{cases}$$

Now, since, in this thesis, we only wish to discuss initial-value problems for our partial differential equations, and since we want difference patterns which will enable us to determine uniquely the value at any given point in terms of the values at points situated on preceding rows, we restrict ourselves to such difference patterns which have but one non-zero point on the "top" row. In fact, since our difference pattern can always be normalized, we can then say that we will restrict our considerations to those difference patterns such that

$$(6) \quad \begin{cases} \text{i)} \quad a_{rs} = 0 \qquad \text{for } \begin{cases} s < 0 \\ s > n \end{cases}, \\ \text{ii)} \quad a_{oo} = 1 \\ \text{iii)} \quad a_{ro} = 0 \qquad \text{for} \quad r \neq 0. \end{cases}$$

Moment conditions (4) and (5) are then given by

$$(7) \quad \begin{cases} \text{i)} \quad \sum_{r;s=1}^{n} a_{rs} = -1 \\ \text{ii)} \quad \sum_{r;s=1}^{n} r\, a_{rs} = c \sum_{r;s=1}^{n} s\, a_{rs} \neq 0. \end{cases}$$

and

$$(8) \quad \begin{cases} \text{i)} \quad \sum_{r;s=1}^{n} a_{rs} = -1 \\ \text{ii)} \quad \sum_{r;s=1}^{n} r\, a_{rs} = \sum_{r;s=1}^{n} s\, a_{rs} = 0 \end{cases}$$

$$(8)\quad \begin{cases} \text{iii)} & \sum_{r;s=1}^{n} rs\, a_{rs} = 0 \\ \text{iv)} & \sum_{r;s=1}^{n} r^2\, a_{rs} = -\sum_{r;s=1}^{n} s^2\, a_{rs} \neq 0\,, \end{cases}$$

respectively, or, when we let $b_{rs} = -a_{rs}$, are given by

$$(9)\quad \begin{cases} \text{i)} & \sum_{r;s=1}^{n} b_{rs} = 1 \\ \text{ii)} & \sum_{r;s=1}^{n} r\, b_{rs} = c \sum_{r;s=1}^{n} s\, b_{rs} \neq 0 \end{cases}$$

and

$$(10)\quad \begin{cases} \text{i)} & \sum_{r;s=1}^{n} b_{rs} = 1 \\ \text{ii)} & \sum_{r;s=1}^{n} r\, b_{rs} = \sum_{r;s=1}^{n} b_{rs} = 0 \\ \text{iii)} & \sum_{r;s=1}^{n} rs\, b_{rs} = 0 \\ \text{iv)} & \sum_{r;s=1}^{n} r^2\, b_{rs} = -\sum_{r;s=1}^{n} s^2\, b_{rs} \neq 0\,. \end{cases}$$

respectively.

With these conditions on the a_{rs}, we now set up a generalized difference pattern whereby we extend our lattice of points to one consisting of continuous rows and, instead of speaking of the values of the coefficients at discrete points, we consider functions of bounded variation defined over these rows.*

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* The use of these functions of bounded variation will simplify greatly the notations and computations to be presented here.

Generally speaking, we consider a sequence of functions of bounded variation, $F_s(x)$, defined over the rows of the lattice such that*

$$
(11)\quad \begin{cases}
\text{i)}\ F_s(x) = 0 \ , & x \leq -A \ , \ A \text{ constant} \\
\text{ii)}\ F_s(x) = c_s \ , & x \geq A \ , \ c_s \text{ constant} \\
\text{iii)}\ F_s(x) \text{ of bounded variation, } s = 1,2,\ldots,n.
\end{cases}
$$

Our moment conditions then become

$$
(12)\quad \begin{cases}
\text{i)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} dF_s(x) = 1 \\
\text{ii)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} x \, dF_s(x) = c \sum_{s=1}^{n} \int_{-\infty}^{\infty} s \, dF_s(x) \neq 0
\end{cases}
$$

and

$$
(13)\quad \begin{cases}
\text{i)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} dF_s(x) = 1 \\
\text{ii)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} x \, dF_s(x) = \sum_{s=1}^{n} \int_{-\infty}^{\infty} s \, dF_s(x) = 0 \\
\text{iii)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} sx \, dF_s = 0 \\
\text{iv)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} x^2 \, dF_s(x) = - \sum_{s=1}^{n} \int_{-\infty}^{\infty} s^2 \, dF_s(x) \neq 0,
\end{cases}
$$

respectively.

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* We could also define $F_0(x)$ in terms of

$$
a_{ro} = \begin{cases} 1 \ , & r = o \\ 0 \ , & r \neq o, \end{cases}
$$

but, at least for the present, this would be a pure luxury.

In terms of these functions of bounded variation, $F_s(x)$, our difference equation (Eq. 1) is now given by

$$(14) \qquad u(x,mh) = \sum_{s=1}^{n} \int_{-\infty}^{\infty} u(x + \xi h,\ mh - sh)\, dF_s(\xi).$$

In general, we would not be able to guarantee the existence of such an integral. However, in this thesis, we are interested in difference patterns which have a fixed finite width (See Eq. 11). Thus, for our considerations, u(x,mh) can be expressed as

$$(15) \qquad u(x,mh) = \sum_{s=1}^{n} \int_{-A}^{A} u(x + \xi h,\ mh - sh)\, dF_s(\xi),$$

and, here, the existence of the integrals is insured once and for all since we have functions of bounded variation, $F_s(x)$, a continuous function, $u(\xi)$, and a finite range of integration, (-A,A).

As initial conditions, we are given the functions $v_r(x)$ which are continuously differentiable, $r = 0,1,\ldots,(n-1)$, and which are given in terms of u(x,mh) by

$$(16) \qquad \left\{ \begin{aligned} v_0(x) &= u(x,o) \\ v_1(x) &= \frac{1}{h}\left\{ u(x,h) - u(x,o) \right\} \\ v_2(x) &= \frac{1}{h^2}\left\{ u(x,2h) - 2u(x,h) + u(x,o) \right\} \\ &\ \vdots \\ v_{n-1}(x) &= \frac{1}{h^{n-1}}\left\{ u\left[x,(n-1)h\right] - \binom{n-1}{1} u\left[x,(n-2)h\right] \right. \\ &\quad + \binom{n-1}{2} u\left[x,(n-3)h\right] - \ldots + \\ &\quad \left. + (-1)^{n-1}\binom{n-1}{n-1} u(x,o) \right\}. \end{aligned} \right.$$

That is, for $0 \leq m \leq (n-1)$, our given initial conditions are:

$$v_m(x) = \frac{1}{h^m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} u\left[x,(m-r)h\right] . \tag{17}$$

In terms of these initial conditions, we can immediately solve for $u(x,mh)$, $0 \leq m \leq n-1$, obtaining

$$u(x,mh) = \sum_{r=0}^{m} \binom{m}{r} h^r v_r(x), \qquad 0 \leq m \leq n-1, \tag{18}$$

or since $\binom{m}{r} \equiv 0$ for $r > m$,

$$u(x,mh) = \sum_{r=0}^{n-1} \binom{m}{r} h^r v_r(x), \qquad 0 \leq m \leq (n-1). \tag{19}$$

For $m \geq n$, however, our difference equation will be of the form*

$$u(x,mh) = \sum_{r=0}^{n-1} \int_{-\infty}^{\infty} h^r v_r(x + \xi h)\, dF_{m,r}(\xi), \quad m \geq n. \tag{20}$$

where $F_{m,r}(\xi)$ is the convolution function defined by**

$$F_{m,r}(\xi) \doteqdot \sum_{s=1}^{n} \int_{-\infty}^{\infty} F_s(\xi - \eta)\, dF_{m-s,r}(\eta) = \tag{21}$$

$$= \sum_{s=1}^{n} \int_{-\infty}^{\infty} F_{m-s,r}(\xi - \eta)\, dF_s(\eta).$$

---

* Although, in general, we would have a question regarding the existence of the integrals of Eq. 20, we will have no problem here since the $v_r(x)$ are continuous functions by hypothesis, since $h^r F_{m,r}(\xi)^r$ will be functions of bounded variation, and since the consideration of difference patterns of finite width will give us a finite range of integration.

** For a discussion of the existence of convolution integrals, the reader is referred to such books as Cramér (Ref. 2) or Widder (Ref. 4), p. 83 ff.

<u>Def. 1</u>: The CONVOLUTION of $F_1(x)$ and $F_2(x)$ is the function

$$(22) \qquad F(x) = \int_{-\infty}^{\infty} F_1(x-z)\, dF_2(z) = \int_{-\infty}^{\infty} F_2(x-z)\, dF_1(z)$$

and is denoted by

$$(23) \qquad F(x) = F_1(x) \circledast F_2(x) = F_1(x) \circledast F_2(x).$$

Now, we would like Eq. 20 to hold for all m. Hence, we rewrite the expression for $u(x,mh)$, as given by Eq. 19, as

$$(24) \quad u(x,mh) = \sum_{r=0}^{n-1} \binom{m}{r} h^r v_r(x) = \sum_{r=0}^{n-1} \int_{-\infty}^{\infty} h^r v_r(x+\xi h)\cdot\binom{m}{r} dI_o(\xi)$$

where

$$(25) \qquad I_o(\xi) = \begin{cases} 0 & , \ \xi < 0 \\ 1 & , \ \xi \geq 0 \end{cases} .$$

Thus, comparing Eq. 20 and Eq. 24, we see that

$$(26) \qquad F_{m,r}(\xi) = \binom{m}{r} I_o(\xi), \qquad m = 0,1,\ldots,(n-1)$$

Then, combining the above results, we have

$$(27) \qquad u(x,mh) = \sum_{r=0}^{n-1} \int_{-\infty}^{\infty} h^r v_r(x + \xi h)\, dF_{m,r}(\xi),$$

where

$$(28) \qquad F_{m,r}(\xi) = \begin{cases} \binom{m}{r} I_o(\xi) & , 0 \leq m \leq (n-1) \\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} F_s(\xi - \eta)\, dF_{m-s,r}(\eta), & m \geq n, \end{cases}$$

or, when we make the change of variable $\zeta = \xi h$ and let

$$(29) \qquad \begin{cases} G_{m,r}(\zeta) \doteq F_{m,r}(\zeta/h) \\ G_s(\zeta - \eta) \doteq F_s((\xi - \eta)/h) , \end{cases}$$

we obtain

$$(30) \qquad u(x,mh) = \sum_{r=0}^{n-1} \int_{-\infty}^{\infty} h^r v_r(x + \xi)\, dG_{m,r}(\xi),$$

where

$$(31) \qquad G_{m,r}(\xi) \equiv F_{m,r}(\xi/h) = \begin{cases} \binom{m}{r} I_o(\xi/h) , & 0 \leq m \leq (n-1) \\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} G_s(\xi - \eta)\, dG_{m-s,r}(\eta), \\ m \geq n. \end{cases}$$

Thus, our problem - namely, to determine conditions under which $u(x,mh)$ will tend to the proper limit as $\begin{cases} m \to \infty \\ h \to 0 \\ mh \to y \end{cases}$ is related to the question: when will $h^r G_{m,r}(\xi)$ tend to the proper limit as $\begin{cases} m \to \infty \\ h \to 0 \\ mh \to y \end{cases}$, for $r = 0,1,\ldots,(n-1)$. However, to answer that question, we will, in turn, consider the so-called characteristic functions of the functions of bounded variation $F_{m,r}(xh)$ (and, hence, $G_{m,r}(x)$).

Def. 2: Let $F(x)$ denote a function of bounded variation and t a real variable. The function $g(x) = e^{itx} = \cos(tx) + i \sin(tx)$ is then integrable over $(-\infty, \infty)$ with respect to $F(x)$, since $|e^{itx}| = 1$. The function of the real variable t

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) \tag{32}$$

is called the CHARACTERISTIC FUNCTION of the function of bounded variation $F(x)$.

The usefulness of characteristic functions lies in the fact that there is a one-to-one correspondence between a function of bounded variation and its characteristic function. The function of bounded variation, $F(x)$, is thus always uniquely determined by the corresponding characteristic function $\varphi(t)$, and the transformation by which we pass from $F(x)$ to $\varphi(t)$, or conversely, is always unique.

To motivate our consideration of characteristic functions in this thesis, we now briefly consider the special case when the $F_s(x)$ are distribution functions, that is, non-negative, non-decreasing point functions which are everywhere continuous to the right and are such that $F_s(-\infty) = 0$ and $F_s(\infty) = 1$.

Def. 3: A sequence of distributions with the distribution functions $F_1(x)$, $F_2(x)$, ... converges to a distribution when and only when there is a distribution function $F(x)$ such that $F_n(x) \to F(x)$ in every continuity point of $F(x)$. When such a function $F(x)$ exists, $F(x)$ is the distribution function corresponding to the limiting distribution of the sequence, and one says that <u>the sequence $\{F_n(x)\}$ converges to the distribution function $F(x)$</u>.

The following theorem [2;96] * then shows that, subject to certain conditions, the transformation by which we pass from $F(x)$ to $\varphi(t)$, or conversely, is continuous (as well as unique), so that the relations $F_n(x) \to F(x)$ and $\varphi_n(t) \to \varphi(t)$ are equivalent.

<u>THEOREM I:</u>

<u>Given:</u> A sequence of distributions, with the distribution functions $F_1(x)$, $F_2(x)$, ..., and the characteristic functions $\varphi_1(t)$, $\varphi_2(t)$,....

<u>Then:</u> A necessary and sufficient condition for the convergence of the sequence $\{F_n(x)\}$ to a distribution function $F(x)$ is that, for every t, the sequence $\{\varphi_n(t)\}$ converges to a limit $\varphi(t)$ which is continuous for the special value $t = 0$. When this condition is satisfied, the limit $\varphi(t)$ is identical with the characteristic function of the limiting distribution function $F(x)$.

* The first number within the brackets refers to a numbered reference stated at the end of this paper, and the second number refers to the page number of that reference.

The foregoing continuity theorem, while not applicable to our functions $F_{m,r}(x)$, nevertheless motivates our consideration of their characteristic functions. In fact, after the development of the formal computation associated with the general theory, we will present new continuity theorems for these functions, $F_{m,r}(x)$, for the case when $\frac{1}{m^r} F_{m,r}(x)$ are functions of bounded variation.

Meanwhile, for our formal computations, we will need the following product theorem for characteristic functions [4; 203].

<u>THEOREM II:</u>

<u>Given:</u> 1) $F(x)$, $F_1(x)$ and $F_2(x)$ are functions of bounded variation.

2) $F(x)$ is the convolution of $F_1(x)$ and $F_2(x)$.

3) $\varphi(t)$, $\varphi_1(t)$ and $\varphi_2(t)$ are the characteristic functions of $F(x)$, $F_1(x)$ and $F_2(x)$ respectively.

<u>Then:</u> $\varphi(t) = \varphi_1(t) \cdot \varphi_2(t)$.

More generally, if

$$F_o(x) = F_1(x) \circledast F_2(x) \circledast \dots \circledast F_n(x)$$

and

$$\varphi_j(t) = \int_{-\infty}^{\infty} e^{ixt}\, dF_j(x), \qquad j = 0,1,\dots,n ,$$

then

$$\varphi_o(t) = \varphi_1(t) \cdot \varphi_2(t) \ldots\ldots \varphi_n(t).$$

To continue, we now define the characteristic functions

(33) $$\varphi_{m,r}(t) \doteqdot \int_{-\infty}^{\infty} e^{ixt}\, dF_{m,r}(x) \quad ,$$

which, by Theorem II and Eq. 28, can then be represented as

(34) $$\varphi_{m,r}(t) = \begin{cases} \binom{m}{r} & , \ 0 \leq m \leq (n-1) \\ \sum_{s=1}^{n} \varphi_s(t) \cdot \varphi_{m-s,r}(t), & m \geq n. \end{cases}$$

If we further define

(35) $$\overset{G}{\varphi}_{m,r}(t) \doteqdot \int_{-\infty}^{\infty} e^{ixt}\, dG_{m,r}(x) \quad , \begin{cases} r = 0,1,\ldots,(n-1) \\ \text{all } m. \end{cases} \quad ,$$

then, from Eq. 29, we have

(36) $$\overset{G}{\varphi}_{m,r}(t) = \int_{-\infty}^{\infty} e^{ixt}\, dF_{m,r}(\tfrac{x}{h}) = \int_{-\infty}^{\infty} e^{i\frac{x}{h}th} dF_{m,r}(\tfrac{x}{h}) = \varphi_{m,r}(th),$$

whence, since $h = y/m$,

(37) $$\overset{G}{\varphi}_{m,r}(t) = \varphi_{m,r}(\tfrac{ty}{m}).$$

Therefore, referring to Eq. 30, we see that, in terms of the characteristic functions $\varphi_{m,r}(\frac{ty}{m})$, our basic problem now reads: Under what conditions will $(\frac{y}{m})^r\ \varphi_{m,r}(ty/m)$ tend to the proper limit as $m \to \infty$.

We now treat this problem by means of the method of generating functions, where we define

$$(38) \qquad \Phi_r(z,t) \doteqdot \sum_{m=o}^{\infty} \varphi_{m,r}(t) \cdot z^m .$$

From Eq. 34 we then obtain

$$(39) \qquad \Phi_r(z,t) = \sum_{m=o}^{n-1} \binom{m}{r} z^m + \sum_{m=n}^{\infty} \sum_{s=1}^{n} \varphi_s(t) \cdot \varphi_{m-s,r}(t) \cdot z^m .$$

Noting that

$$(40) \qquad \left(\sum_{o}^{\infty} c_n z^n\right) \cdot \left(\sum_{o}^{\infty} d_n z^n\right) = \sum_{n=o}^{\infty} \sum_{k=o}^{n} c_k d_{n-k} \cdot z^n$$

and that

$$(41) \qquad \sum_{p=o}^{\infty} \varphi_{p,r}(t) \cdot z^p \sum_{s=1}^{n} \varphi_s(t) \cdot z^s = \sum_{m=1}^{\infty} \sum_{s=1}^{\min(m,n)} \varphi_{m-s,r}(t) \varphi_s(t) z^m ,$$

we obtain

$$(42) \qquad \Phi_r(z,t) \cdot \sum_{s=1}^{n} \varphi_s(t) \cdot z^s = \sum_{m=1}^{n-1} \sum_{s=1}^{n} \varphi_{m-s,r}(t) \varphi_s(t) z^m + \sum_{m=n}^{\infty} \sum_{s=1}^{n} \varphi_{m-s,r}(t) \varphi_s(t) \cdot z^m ,$$

whence, from Eq. 39 and Eq. 42,

$$(43) \qquad \Phi_r(z,t) = \sum_{m=o}^{n-1} \binom{m}{r} z^m + \Phi_r(z,t) \cdot \sum_{s=1}^{n} \varphi_s(t) z^s - \sum_{m=1}^{n-1} \sum_{s=1}^{m} \varphi_{m-s,r}(t) \cdot \varphi_s(t) \cdot z^m ,$$

so that

$$(44)\quad \Phi_r(z,t) = \frac{\sum_{m=o}^{n-1}\binom{m}{r} z^m - \sum_{m=1}^{n-1}\sum_{s=1}^{m}\varphi_{m-s,r}(t)\,\varphi_s(t)\, z^m}{1 - \sum_{s=1}^{n}\varphi_s(t)\cdot z^s},$$

or, in slightly revised form,

$$(45)\quad \Phi_r(z,t) \doteq \sum_{m=o}^{\infty}\varphi_{m,r} z^m = \frac{\binom{o}{r} + \sum_{m=1}^{n-1} z^m \left\{\binom{m}{r} - \sum_{s=1}^{m}\binom{m-s}{r}\varphi_s(t)\right\}}{1 - \sum_{s=1}^{n}\varphi_s(t)\cdot z^s}.$$

For simplicity sake, we denote $\Phi_r(z,t)$ by

$$(46)\quad \Phi_r\,(z,t) \doteq \frac{Q_r(z,t)}{P(z,t)}$$

where $Q_r(z,t)$ and $P(z,t)$ are regular functions of z and t given by

$$(47)\quad \begin{cases} Q_r(z,t) = \binom{o}{r} + \sum_{m=1}^{n-1} z^m \left\{ \binom{m}{r} - \sum_{s=1}^{m}\binom{m-s}{r}\varphi_s(t) \right\} \\ P(z,t) = 1 - \sum_{s=1}^{n}\varphi_s(t)\cdot z^s . \end{cases}$$

We now wish to represent $\Phi_r(z,t)$ by a sum of partial fractions. Hence, if we let $a_i(t)$ be the roots of $P(z,t) = 0$, and $\eta_i(t)$ the multiplicity of each corresponding root, then

$$(48)\quad P(z,t) = \prod_{i=1}^{\nu}\left(1 - \frac{z}{a_i(t)}\right)^{\eta_i(t)}, \text{ where } \quad \nu \leq n$$

and

$$(49)\quad \Phi_r(z,t) \equiv \frac{Q_r(z,t)}{P(z,t)} = \sum_{i=1}^{\nu}\sum_{j=1}^{\eta_i(t)} \frac{c_{ij}(t)}{\left(1 - \frac{z}{a_i(t)}\right)^j}$$

$$= \sum_{i=1}^{\nu}\sum_{j=1}^{\eta_i(t)} c_{ij}(t)\cdot \sum_{m=o}^{\infty}\binom{m+j-1}{j-1}\left(\frac{z}{a_i(t)}\right)^m ,$$

so that, from Eq. 38, we have

$$(50)\quad \varphi_{m,r}(t) = \sum_{i=1}^{\nu} \sum_{j=1}^{\eta_i(t)} c_{ij}(t) \binom{m+j-1}{j-1} \left(\frac{1}{a_i(t)}\right)^m .$$

At this point, although not necessary in the least for the formal calculations but, nevertheless, of convenience and simplicity with regard to subsequent notations and to the presentation in general, we make the first of three assumptions on $P(z,t)$ which will be made in this thesis, namely:

Assumption 1: $P(z,0) \equiv 1 - \sum_{s=1}^{n} \varphi_s(t)\, z^s$ has no zeros within the unit circle $|z| = 1$.

To make such an assumption seems reasonable since we are trying to outline a general method. More important, however, is the fact that this assumption can be shown to be a necessary condition if the method of solution by means of difference equations is to work.*

Let us now consider the zeros of $P(z,0)$, some of which will lie on the unit circle while the rest, by Assumption 1, will lie outside the unit circle. Of the finite number ($n_0 < n$) of zeros of $P(z,0)$ which lie outside $|z| = 1$, at least one will be at a

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* For a proof of the necessity of this assumption for the cases considered in this thesis, see Lemma 8 in Appendix I.

minimum distance ρ_o from the unit circle. Choose ρ such that

(51) $$1 < \rho < \rho_o.$$

Then, there will exist a value of t, say $t = t_o$, such that, for all t, $|t| < t_o$, P(z,t) will have the same number of zeros in $|z| < \rho$ as the total number of zeros of P(z,0) which lie on the unit circle. (In other words, by means of $|z| = \rho$, we can separate the zeros of P(z,0) into those which lie on the unit circle and those which lie outside the unit circle, and, for all t, $|t| < t_o$, none of these zeros will cross over the dividing circle). Thus, for each t, $|t| < t_o$, we can then decompose $\Phi_r(z,t)$ into

(52) $$\Phi_r(z,t) \equiv \frac{Q_r(z,t)}{P(z,t)} = R_1(z,t) + R_2(z,t),$$

where $R_1(z,t)$ and $R_2(z,t)$ are rational functions such that $R_2(z,t)$ is a regular function of z for $|z| \geq \rho > 1$ and $R_2(z,t)$ is a regular function of z for $|z| \leq \rho$. $R_1(z,t)$ and $R_2(z,t)$ can then be expanded into power series of the form

(53) $$\begin{cases} R_1(z,t) = \sum_{m=o}^{\infty} b_m(t) z^m, \\ R_2(z,t) = \sum_{m=o}^{\infty} c_m(t) z^m, \end{cases}$$

so that

(54) $$\Phi_r(z,t) \equiv \frac{Q_r(z,t)}{P(z,t)} = \sum_{m=o}^{\infty} b_m(t)\, z^m + \sum_{m=o}^{\infty} c_m(t)\, z^m .$$

Lemma 1:

Given: $\dfrac{Q_r(z,t)}{P(z,t)} = R_1(z,t) + R_2(z,t)$, where

1) $R_1(z,t)$ is a regular function of z for $|z| \geq \rho > 1$ such that, for each t, $R_1(z,t) \to 0$ as $|z| \to \infty$.

2) $R_2(z,t)$ is a regular function of z for $|z| \leq \rho$.

3) $R_2(z,t)$ is a rational function in z which can be expressed by the power series

$$R_2(z,t) = \sum_{m=o}^{\infty} c_m(t) z^m.$$

Then: The coefficients of this power series expansion, $c_m(t)$, will tend uniformly to zero as m becomes infinitely large.

Proof:

As is easily seen,

$$R_2(z,t) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{Q_r(u,t)}{P(u,t)} \quad \frac{du}{u-z} \quad ,$$

where C_ρ is the aforementioned dividing circle $|z| = \rho > 1$ and where z is any point interior to C_ρ. In particular, if $1 < \rho_1 < \rho$, then

$$\left| R_2(z,t) \right| < C$$

for $z \leq \rho_1$ and $|t| < t_o$.

Now,

$$R_2(z,t) = \sum_{m=o}^{\infty} c_m(t) \; z^m ,$$

so that

$$c_m(t) = \frac{1}{2\pi i} \oint_{C_\rho} R_2(z,t) \cdot \frac{dz}{z^{m+1}} .$$

Therefore, by Cauchy's inequality,

$$|c_m(t)| \leq C \cdot \rho_1^{-m}$$

and the lemma is proved.

Therefore, from Lemma 1, we see that our only concern in Eq. 52 will be with $R_1(z,t)$ and its resulting contribution to $\varphi_{m,r}(t)$. To compute $R_1(z,t)$ and to carry out the rest of the theory we will need to know, in turn, the order of each pole of $R_1(z,t)$, or when we let

$$R_1(z,t) \equiv \frac{Q_1(z,t)}{P_1(z,t)} ,$$

we will need to know the multiplicity of each zero of $P_1(z,t)$.

Thus, to summarize our formal computations, and to restate the problem which confronts us, we wish to determine sufficient conditions such that the solution of our general difference equation,

$$u(x,mh) = \sum_{s=1}^{n} \int_{-\infty}^{\infty} u(x + \xi h, mh-sh)\, dF_s(\xi)$$

$$= \sum_{s=1}^{n} \int_{-\infty}^{\infty} h^r\, v_r(x + \xi h)\, dF_{m,r}(\xi);$$

will converge to the solution of the corresponding partial differential equation. However, rather than treat the functions $F_{m,r}(x)$ directly, we will consider the corresponding characteristic functions $\varphi_{m,r}(t)$ defined by

$$\varphi_{m,r}(t) \doteq \int_{-\infty}^{\infty} e^{ixt}\, dF_{m,r}(x)$$

and

$$\Phi_r(z,t) = \sum_{m=0}^{\infty} \varphi_{m,r}(t) z^m = \frac{Q_r(z,t)}{P(z,t)} = \frac{\binom{0}{r} + \sum_{m=1}^{n-1} z^m \left\{ \binom{m}{r} - \sum_{s=1}^{m} \binom{m-s}{r} \varphi_s(t) \right\}}{1 - \sum_{s=1}^{n} \varphi_s(t) z^s} .$$

Then, by means of continuity theorems to be developed in the next chapter, we will be able to treat the $F_{m,r}(x)$ and, hence, arrive at the solution of our problem.

CHAPTER II

CONTINUITY THEOREMS FOR FUNCTIONS OF BOUNDED VARIATION

In this chapter, we would like to develop the two continuity theorems referred to in Chapter I so as to be able to treat the functions of bounded variation, $F_{m,r}(x)$, by means of their corresponding characteristic functions $\varphi_{m,r}(t)$.

THEOREM III.

Given: 1) A sequence of functions $K_m(x)$ such that

a) $V(K_m(x)) \leq k$

b) $K_m(x) = 0, \quad x \leq -A$, A constant

c) $K_m(x) = C_m = \text{constant}, \quad x \geq A.$

2) $v(x)$ continuous

3) $\psi_m(t) = \int_{-\infty}^{\infty} e^{ixt}\, dK_m(x)$

If: $\psi_m(t)$ converges to a limit $\psi(t)$, for every t.

Then:

1) $\lim_{m\to\infty} \int_{-\infty}^{\infty} v(x)\, dK_m(x)$ exists.

2) $\psi(t)$ is the characteristic function of the limiting function $K(x)$, where

$$K(x) \doteqdot \lim_{m\to\infty} K_m(x).$$

3) $\lim_{m\to\infty} \int_{-\infty}^{\infty} v(x)\, dK_m(x) = \int_{-\infty}^{\infty} v(x)\, dK(x).$

Proof:

Since the $K_m(x)$ are functions of bounded variation, we can let

$$K_m(x) = P_m(x) - N_m(x) , \tag{55}$$

where $P_m(x)$ and $N_m(x)$ are non-negative, non-decreasing functions. Furthermore,

$$0 \leq P_m(x) \leq k \quad ; \quad 0 \leq N_m(x) \leq k. \tag{56}$$

Now, pick a subsequence $P_{m_\nu}(x)$ of $P_m(x)$ such that $P_{m_\nu}(x)$ converges for all x in a denumerable, everywhere dense set. Then, extend the definition of this limiting function to all x, preserving its non-decreasing character, and call this limiting function $P(x)$. Then, $P_{m_\nu}(x) \rightarrow P(x)$ at every continuity point of $P(x)$, that is, at all x except possibly a denumerable set.

Next, of that subsequence, pick a sub-subsequence such that $N_{m_\nu}(x) \rightarrow N(x)$ at every continuity point of $N(x)$, that is, at all x except possibly a denumerable set.

Then, let

$$K(x) = P(x) - N(x), \tag{57}$$

whence, $K(x)$ is of bounded variation. Now, $x = \pm A$ can be assumed not to be singular points of $K(x)$. Therefore,

$$K_{m_\nu}(x) \rightarrow K(x) \tag{58}$$

at all x except possibly a denumerable set.

Now, let $v_1(x)$ be a function which has a continuous derivative. Then

$$(59) \quad \int_{-A}^{A} v_1(x)\, dK_{m_\nu}(x) = v_1(A)\cdot \Psi_{m_\nu}(0) - \int_{-A}^{A} K_{m_\nu}(x)\cdot dv_1(x).$$

Therefore,

$$(60) \quad \lim_{m\to\infty} \int_{-A}^{A} v_1(x)\, dK_{m_\nu}(x) = v_1(A)\cdot \Psi(0) - \int_{-A}^{A} K(x)\cdot dv_1(x),$$

whence

$$\lim_{m\to\infty} \int_{-A}^{A} v_1(x)\, dK_{m_\nu}(x)$$

exists.

Next, consider

$$(61) \quad \Psi_{m_\nu}(t) = \int_{-\infty}^{\infty} e^{ixt}\, dK_{m_\nu}(x)\ .$$

Since the $P_m(x)$ and $N_m(x)$ are non-decreasing, non-negative functions which are constant outside a finite interval, and since $P_{m_\nu}(x) \to P(x)$ and $N_{m_\nu}(x) \to N(x)$ at every continuity point of $P(x)$ and $N(x)$, respectively, it can be shown that the characteristic functions $\Psi_{m_\nu}(t)$ tend to a characteristic function $\psi^*(t)$ such that

$$(62) \quad \psi^*(t) = \int_{-\infty}^{\infty} e^{ixt}\, dK(x).$$

But, by hypothesis, $\Psi_m(t) \to \Psi(t)$ for the full sequence. Therefore,

$$\Psi(t) = \Psi^*(t) = \int_{-\infty}^{\infty} e^{ixt}\, dK(x), \tag{63}$$

so that $\Psi(t)$ is a characteristic function. Furthermore, since $\Psi_m(t) \to \Psi(t)$ for the full sequence, $K(x)$ is independent of any subsequence. Also,

$$\begin{aligned} \lim_{m\to\infty} \int_{-A}^{A} v_1(x)\, dK_m(x) &= \lim_{m\to\infty} \left\{ v_1(A)\, \Psi_m(0) - \int_{-A}^{A} K_m(x) dv_1(x) \right\} \\ &= v_1(A)\, \Psi(0) - \int_{-A}^{A} K(x) dv_1(x) \\ &= \int_{-A}^{A} v_1(x)\, dK(x) \quad ; \end{aligned} \tag{64}$$

that is,

$$\lim_{m\to\infty} \int_{-A}^{A} v_1(x) \cdot dK_m(x) = \int_{-A}^{A} v_1(x) \cdot dK(x) \tag{65}$$

for the full sequence.

Now, approximate the given continuous function, $v(x)$, by $v_1(x)$, where $v_1(x)$ has a continuous derivative, such that

$$\left| v(x) - v_1(x) \right| < \epsilon \qquad \text{for } |x| \leq A. \tag{66}$$

Then

$$(67)\quad \left|\int_{-A}^{A} v(x)dK_m(x) - \int_{-A}^{A} v(x)dK(x)\right| \leq \left|\int_{-A}^{A} (v(x)-v_1(x))\, dK_m(x) - \right.$$

$$\left. - \int_{-A}^{A} \big(v(x)-v_1(x)\big)\, dK(x)\right| + \left|\int_{-A}^{A} v_1(x)dK_m(x) - \int_{-A}^{A} v_1(x)dK(x)\right|$$

$$\leq \; 2\,\epsilon\, k + \left|\int_{-A}^{A} v_1(x)dK_m(x) - \int_{-A}^{A} v_1(x)dK(x)\right| .$$

But, by the previous part, $\int_{-A}^{A} v_1(x)dK_m(x) \to \int_{-A}^{A} v_1(x)dK(x)$,

hence

$$(68)\qquad \int_{-A}^{A} v(x)\; dK_m(x) \to \int_{-A}^{A} v(x)dK(x).$$

Finally, since the $K_m(x)$ and $K(x)$ are constant for $|x| \geq A$, the range of integration can be $(-\infty,\infty)$.

Q.E.D. Theorem III.

Now, Theorem III, in itself, would be sufficient to give us a limited set of results. However, the condition

$$V(F_{m,o}(\xi)) \;\leq\; K_o$$

which is imposed by the theorem will be too restrictive to be of any use in our later work. In fact, calculation will show that it would exclude one of the simplest of difference patterns for our partial differential equation, namely the pattern given by

$$u(x \pm h, y) = -1,$$
$$u(x, y \pm h) = 1.$$

Therefore, we also derive the next theorem which will broaden our results to include examples such as the one just cited.

THEOREM IV.

Given: 1) A sequence of functions $K_m(x)$ such that

a) $K_m(x)$ is of bounded variation, each m.

b) $K_m(x) = 0, \quad x \leq -A, \quad A$ constant.

c) $K_m(x) = C_m =$ constant, $x \geq A$.

d) $\int_{-A}^{x} |K_m(\xi)|\, d\xi \leq C.$

2) $v(x)$ continuously differentiable

3) $\theta_m(t) = \int_{-\infty}^{\infty} e^{ixt}\, dK_m(x).$

If: a) $\theta_m(t)$ tends to a limit, say $\theta(t)$, for every t.

b) $\dot{\theta}_m(0)$ tends to a limit, say $\dot{\theta}(0)$.

Then: There exists a function $\mathcal{K}(x)$ of bounded variation such that, as $m \to \infty$,

$$\int_{-\infty}^{\infty} v(x)\, dK_m(x) \to v(A)\ \theta(0) - \int_{-\infty}^{\infty} v'(x)\, d\,\mathcal{K}(x)\ .$$

Proof:

Define the functions $\mathcal{K}_m(x)$ by

$$(69)\qquad \mathcal{K}_m(x) \doteqdot \begin{cases} 0 & , x \le -A \\ \int_{-A}^{x} |K_m(\xi)| \, d\xi & , -A \le x \le A \\ \int_{-A}^{A} |K_m(\xi)| \, d\xi = \text{constant}, & x \ge A, \end{cases}$$

whence the $\mathcal{K}_m(x)$ are functions of bounded variation.

Then

$$(70)\qquad \int_{-A}^{A} v(x)dK_m(x) = v(A)\cdot K_m(A) - \int_{-A}^{A} K_m(x)dv(x)$$

$$= v(A)\cdot \theta_m(0) - \int_{-A}^{A} v'(x)\cdot d\,\mathcal{K}_m(x).$$

In particular, for $v(x) = e^{ixt}$, we obtain

$$(71)\qquad \theta_m(t) = e^{iAt} \cdot \theta_m(0) - it \cdot \Psi_m(t),$$

where

$$(72)\qquad \Psi_m(t) \doteqdot \int_{-\infty}^{\infty} e^{ixt}\, d\,\mathcal{K}_m(x).$$

Thus, if $t \ne 0$, $\Psi_m(t)$ tends to a limit as $m \to \infty$. Furthermore, if $v(x) = ix$, then

$$(73)\qquad \dot{\theta}_m(0) = iA\theta_m(0) - i\,\Psi_m(0),$$

whence $\Psi_m(0)$ also tends to a limit. Hence, by Theorem III, there exists a function $\mathcal{K}(x)$ of bounded variation such that

$$(74) \qquad \int_{-\infty}^{\infty} v'(x)\,d\mathcal{K}_m(x) \to \int_{-\infty}^{\infty} v'(x)\,d\mathcal{K}(x) \quad .$$

Therefore, by Eq. 70, we see that

$$(75) \qquad \int_{-\infty}^{\infty} v(x)\,dK_m(x) \to v(A)\cdot\theta(0) - \int_{-\infty}^{\infty} v'(x)\;d\mathcal{K}(x) \; .$$

Q.E.D. Theorem IV.

With these two continuity theorems for functions of bounded variation, we can now proceed in our consideration of the characteristic functions $\varphi_{m,r}(t)$ and, consequently, obtain the solution of the underlying problem of this thesis. However, so as to simplify the computation and the presentation, the general method of solution is first illustrated by the simple problem related to the partial differential equation

$$\frac{\partial u}{\partial y} = c\,\frac{\partial u}{\partial x}$$

and is then extended to the second order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \quad .$$

CHAPTER III

THE PARTIAL DIFFERENTIAL EQUATION $\frac{\partial u}{\partial y} = c\frac{\partial u}{\partial x}$, c constant.

Here, in this chapter, we consider the partial differential equation $\frac{\partial u}{\partial y} = c\frac{\partial u}{\partial x}$ for the case when the aforementioned functions $F_s(x)$ are such that

$$(76)\quad \begin{cases} \text{i) } F_s(-\infty) = 0, & s = 1,2,\ldots, n. \\ \text{ii) } \sum_{s=1}^{n} F_s(+\infty) = 1 & \\ \text{iii) } F_s(x) \text{ non-decreasing, } s = 1,2,\ldots, n. & \end{cases}$$

For these functions, we will then prove the following main theorem:

<u>THEOREM V.</u>

<u>Given</u>: Functions $F_s(x)$ satisfying Eq. 76.

<u>Then</u>: For these functions, $F_s(x)$, the solution of the difference equation

$$u(x,mh) = \sum_{s=1}^{n} \int_{-\infty}^{\infty} u(x+\xi h, mh-sh)\,dF_s(\xi)$$

will tend in the limit (as $\begin{cases} m \to \infty \\ h \to o \\ mh \to y \end{cases}$) to the solution of the corresponding partial differential equation $\frac{\partial u}{\partial y} = c\frac{\partial u}{\partial x}$.

As shown in Chapter I (Eq. 12), our moment conditions for the difference equation corresponding to the given partial differential equation are

$$(77)\quad \begin{cases} \text{i)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} dF_s(x) = +1 \\ \text{ii)}\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} x dF_s(x) = c \sum_{s=1}^{n} \int_{-\infty}^{\infty} s\, dF_s(x) \neq 0 . \end{cases}$$

The moment conditions may also be expressed in terms of the characteristic functions $\varphi_s(t)$ by

$$(78)\quad \begin{cases} \text{i)}\ \sum_{s=1}^{n} \varphi_s(0) = 1 \\ \text{ii)}\ \sum_{s=1}^{n} \dot{\varphi}_s(0) = ic \sum_{s=1}^{n} s\, \varphi_s(0) \neq 0, \end{cases}$$

where

$$\dot{\varphi}_s(0) \equiv \left.\frac{d\varphi_s(t)}{dt}\right|_{t=0} .$$

Def. 4. A characteristic function $\varphi(t)$ is called POSITIVE-DEFINITE if it gives rise to a non-negative, non-decreasing function $F_s(x)$ by means of the relation

$$\varphi(t) = \int_{-\infty}^{\infty} e^{ixt}\, dF(x) .$$

The positive-definite characteristic functions considered in this chapter give rise to the following lemmas:

Lemma 2. On the unit circle, $|z| = 1$, $P(z,0) \equiv 1 - \sum_{s=1}^{n} \varphi_s(0) z^s$ has at most simple zeros for the case of the equation $u_y = cu_x$, where the $\varphi_s(t)$ are positive-definite characteristic functions.

Proof: If $P(z,0)$ is to have at most simple zeros, say at $z = a$, then we need only show that $P'(a,0) \neq 0$. Now,

$$P(z,t) = 1 - \sum_{s=1}^{n} \varphi_s(t)\, z^s \Rightarrow P'(z,0) = -\sum_{s=1}^{n} s\, \varphi_s(0) z^{s-1},$$

$$P(a,0) = 0 \Rightarrow \sum_{s=1}^{n} \varphi_s(0)\, a^s = 1,$$

and $$|a| = 1 \Rightarrow \left|\sum_{s=1}^{n} \varphi_s(0)\, a^s\right| \leq \sum_{s=1}^{n} \varphi_s(0) \cdot |a|^s = \sum_{s=1}^{n} \varphi_s(0) = 1,$$

so that $$\sum_{s=1}^{n} \varphi_s(0)\, |a|^s = \sum_{s=1}^{n} \varphi_s(0)\, a^s = \sum_{s=1}^{n} \varphi_s(0) = 1.$$

Therefore, $\varphi_s(0)\, a^s$ must be real and non-negative, which, in turn, implies that either

$$(79) \quad \begin{cases} \text{i)} \ \varphi_s(0) = 0, & \text{or} \\ \text{ii)} \ \varphi_s(0) \neq 0 \ \text{and} \ a^s(0) = 1. \end{cases}$$

Now, not only are all of the $\varphi_s(0)$ non-negative, but there exists at least one $\varphi_s(0) \neq 0$, namely $\varphi_n(0)$. (Therefore, incidentally, $a^n(0) = 1$). Hence, of all the non-negative and real terms of $P'(a,0)$, there exists at least one non-vanishing term, namely $\varphi_n(0) \cdot a^n(0)$. Therefore, $P'(a,0) \neq 0$.

Q.E.D. Lemma 2.

The next lemma then follows immediately from the previous discussion.

Lemma 3. For $|t| < t_o$, $P(z,t)$ will have at most simple zeros at $z = a_j(t)$ for all $a_j(t)$ such that $|a_j(0)| = 1$.

<u>Lemma 4.</u> $P(z,t)$ cannot have any zeros, $z = a(t)$, such that $|a(t)| < 1$.

<u>Proof:</u> Given $P(z,t) = 1 - \sum_{s=1}^{n} \varphi_s(t) \cdot z^s$.

Therefore, for any root, $z = a$, of $P(z,t) = 0$, we have $\sum_{s=1}^{n} \varphi_s(t) \cdot a^s = 1$. If we assume that there does exist a root at $z = a$ of $P(z,t) = 0$ such that $|a| < 1$, we then arrive at the contradiction $1 < 1$, that is,

$$|a| < 1 \Rightarrow 1 = \left| \sum_{s=1}^{n} \varphi_s(t) a^s \right| \leq \sum_{s=1}^{n} \varphi_s(t) \; |a|^s < \sum_{s=1}^{n} \varphi_s(t) \leq \sum_{s=1}^{n} \varphi_s(0) = 1.$$

Q.E.D. Lemma 4 .

Lemma 4 verifies for the present case, incidentally, the assumption that was made on $P(z,0)$ in Chapter I.

Now, if $P(z,t)$ has at most simple zeros, say at $z = a$, and if $Q_r(z,t)$ is regular at $z = a$, then we know from analysis that

$$\text{(80)} \qquad \text{Residue} \left. \frac{Q_r(z,t)}{P(z,t)} \right|_{z=a} = \frac{Q(a,t)}{P'(a,t)} \quad .$$

Thus, returning to the partial fraction decomposition of $\Phi_r(z,t) = \frac{Q_r(z,t)}{P(z,t)}$ as given by Eq. 52 (Chapter I), we obtain

$$\text{(81)} \quad \Phi_r(z,t) = \sum_{j=1}^{k} \frac{Q_r(a_j(t),t)}{P'(a_j(t),t)} \cdot \frac{-1}{(1 - \frac{z}{a_j(t)}) \cdot a_j(t)} + R_2(z,t) \; ,$$

where $a_1(t), a_2(t), \ldots, a_k(t)$ are those zeros of $P(z,t)$ such that $1 \leq |a_j(t)| < \rho$. Therefore,

$$(82)\quad \Phi_r(z,t) = \sum_{j=1}^{k} \frac{Q_r(a_j(t),t)}{P'(a_j(t),t)} \cdot \sum_{m=0}^{\infty} \frac{-z^m}{(a_j(t))^{m+1}} + R_2(z,t)$$

and

$$(83)\quad \varphi_{m,r}(t) = \sum_{j=1}^{k} \frac{Q_r(a_j(t),t)}{P'(a_j(t),t)} \cdot \frac{-1}{(a_j(t))^{m+1}} + T_{m,r}(t),$$

where the $T_{m,r}(t)$ are given by

$$(84)\qquad R_2(z,t) = \sum_{m=0}^{\infty} T_{m,r}(t) \cdot z^m \quad ,$$

and, as shown in Lemma 1, are such that the $T_{m,r}(t)$ tend uniformly to zero as m becomes infinitely large.

Now, $Q_r(a_j(t),t)$ is a regular function and, as previously proved, $P'(a_j(t),t) \neq 0$, so that $Q_r(a_j(t),t)/P'(a_j(t),t)$ is uniformly bounded. Furthermore, $|a_j(t)| \geq 1$. Therefore, $\varphi_{m,r}(ty/m)$ is uniformly bounded, so that $\lim_{m\to\infty} (\frac{y}{m})^r \varphi_{m,r}(\frac{ty}{m})$ exists and is equal to zero for $r \geq 1$. Furthermore, this limit will be continuous* for the special value $t = 0$. Therefore,

* Since we have modified the conditions of Theorem III by using functions of bounded variation which are not necessarily constant for $|x| \geq A$, we must impose additional condition(s) if we still wish to make use of Theorem III in this chapter. However, it can be shown that the continuity of the limiting characteristic function at $t = 0$ is sufficient for this purpose. Hence, we investigate here the continuity of

$$\lim_{m\to\infty} (\frac{y}{m})^r \varphi_{m,r}(^{ty}/_m)$$

at $t = 0$.

$\lim\limits_{m\to\infty} \left(\frac{y}{m}\right)^r \cdot \varphi_{m,r}\left(\frac{ty}{m}\right)$ exists, is equal to zero, and is continuous at $t = 0$ for $r = 1,2,\ldots,(n-1)$.

Finally, then, we need to evaluate

$$(85) \qquad \Psi_o\left(\frac{ty}{m}\right) \doteqdot \lim_{m\to\infty} \varphi_{m,o}\left(\frac{ty}{m}\right) = $$

$$= \lim_{m\to\infty} \sum_{j=1}^{k} \frac{Q_o\left(a_j\left(\frac{ty}{m}\right), {}^{ty}/m\right)}{P'\left(a_j({}^{ty}/m), {}^{ty}/m\right)} \cdot \frac{-1}{\left[a_j({}^{ty}/m)\right]^{m+1}} .$$

Now

$$(86) \qquad Q_o(a_j(0),0) = 1 + \sum_{m=1}^{n-1} a_j^{\,m}(0) \left\{ 1 - \sum_{s=1}^{m} \varphi_s(0) \right\} ,$$

and if we first consider only those zeros of $P(z,0)$ such that $|a_j(0)| = 1$, $a_j(0) \neq 1$, we can then write $Q_o(a_j(0),0)$ as

$$(87) \qquad Q_o(a_j(0),0) = \frac{a_j^{\,n}(0) - 1}{a_j(0) - 1} - \sum_{m=1}^{n-1} a_j^{\,m}(0) \cdot \sum_{s=1}^{n} \varphi_s(0).$$

But, from Lemma 2, we have that

$$(88) \qquad \varphi_n(0) \neq 0 \qquad \text{and} \qquad a_j^{\,n}(0) = 1.$$

Hence, for this case, namely, $|a_j(0)| = 1$, $a_j(0) \neq 1$, our expression for $Q_o(a_j(0),0)$ reduces to

$$(89) \qquad Q_o(a_j(0),0) = - \sum_{m=1}^{n-1} a_j^{\,m}(0) \cdot \sum_{s=1}^{m} \varphi_s(0)$$

$$= - \sum_{s=1}^{n-1} \varphi_s(0) \cdot \sum_{m=s}^{n-1} (a_j(0))^m$$

$$= - \sum_{s=1}^{n-1} \varphi_s(0) \cdot \left(\frac{a_j^{\,n}(0) - a_j^{\,s}(0)}{a_j(0) - 1}\right) .$$

$$(90) \therefore \quad Q_o(a_j(0),0) = - \sum_{s=1}^{n-1} \varphi_s(0)\cdot\left(\frac{1 - a_j^{\,s}(0)}{a_j(0) - 1}\right) \text{ , since } a_j^{\,n}(0) = 1.$$

However, from Lemma 2 we also have that either $\varphi_s(0) = 0$ or $a_j^{\,s}(0) = 1$.

$$(91) \therefore \quad Q_o(a_j(0),0) = 0 \quad \text{for all } a_j(0) \text{ such that} \begin{cases} a_j(0) \neq 1 \\ |a_j(0)| = 1 \end{cases} .$$

Furthermore, $P'(a_j(t),t) \neq 0$ for all t, including $t = 0$ in particular, and $a_j(t) \neq 0$ for all t. Therefore, the zeros of $P(z,0)$ which lie on the unit circle, except possible $z = 1$, contribute nothing to $\lim_{m \to \infty} \varphi_{m,o}(ty/m)$.

Lastly, then, consider the simple zero of $P(z,0)$ at $z = 1$, say $a(0)$. (We know that such a simple zero exists since $\sum_{s=1}^{n} \varphi_s(0) = 1$ by one of the moment conditions).

Since the other zeros of $P(z,t)$ other than at $z = 1$ contribute nothing to $\Psi_o(ty)$, we have

$$(92) \quad \Psi_o(ty) = \frac{Q_o(1,0)}{P'(1,0)} \cdot \lim_{m \to \infty} \frac{-1}{\left[a(\frac{ty}{m})\right]^{m+1}} \text{ , where } a(0) = 1.$$

Now, if we consider a circle Γ defined by $|z-1| = \delta > 0$ such that $P(z,t)$ is regular within and on Γ and does not vanish on Γ, we then have

$$(93) \qquad a(\tfrac{ty}{m}) = \frac{1}{2\pi i} \int_{\Gamma} z \, \frac{P'(z,\frac{ty}{m})}{P(z, \frac{ty}{m})} \, dz \quad .$$

Now, $P(z,ty/m)$ and $P'(z,ty/m)$ are both regular within and on Γ. Hence $a(ty/m)$ is regular within Γ and, in particular, we can then speak of its derivatives at $t = 0$. Consequently, we can expand $\left[a(\frac{ty}{m})\right]^{-(m+1)}$ into the MacLaurin series

$$(94) \quad \left[a(\tfrac{ty}{m})\right]^{-(m+1)} = \left\{1 + \dot{a}(0)\cdot \frac{ty}{m} + \frac{\ddot{a}(0)}{2!}\left(\frac{ty}{m}\right)^2 + \ldots\right\}^{-(m+1)} ,$$

and obtain

$$(95) \qquad \lim_{m\to\infty} \left[a(\tfrac{ty}{m})\right]^{-(m+1)} = e^{-\dot{a}(0)ty} ,$$

whence

$$(96) \qquad \Psi_o(ty) = -\frac{Q_o(1,0)}{P'(1,0)}\, e^{-\dot{a}(0)\cdot ty} .$$

Now, to evaluate $Q_o(1,0)$, we have from its definition that

$$Q_o(z,0) = 1 + \sum_{m=1}^{n-1} z^m \left\{1 - \sum_{s=1}^{m} \varphi_s(0)\right\}$$

whence,

$$Q_o(1,0) = n - \sum_{m=1}^{n-1} z^m \sum_{s=1}^{m} \varphi_s(0)\Bigg|_{z=1}$$

$$= n - \sum_{s=1}^{n-1} \varphi_s(0) \cdot \sum_{m=s}^{n-1} z^m\Bigg|_{z=1}$$

$$= n - \sum_{s=1}^{n-1} \varphi_s(0)\cdot(n-s) = n - \sum_{s=1}^{n} \varphi_s(0)\cdot(n-s) .$$

$$(97) \quad \therefore \quad Q_o(1,0) = n - n \cdot \sum_{s=1}^{n} \varphi_s(0) + \sum_{s=1}^{n} s\, \varphi_s(0) \ .$$

Therefore

$$(98) \qquad Q_o(1,0) = \sum_{s=1}^{n} s\, \varphi_s(0), \text{ since } \sum_{s=1}^{n} \varphi_s(0) = 1.$$

$$(99) \quad \therefore \quad \frac{Q_o(1,0)}{P'(1,0)} = \frac{\sum_{s=1}^{n} s\, \varphi_s(0)}{-\sum_{s=1}^{n} s\, \varphi_s(0)} = -1.$$

Furthermore, since $a(t)$ is a root of $P(z,t) = 0$, we have the identity in t:

$$(100) \qquad \sum_{s=1}^{n} (a(t))^s \cdot \varphi_s(t) = 1.$$

Differentiating with respect to t, we obtain

$$(101) \quad \sum_{s=1}^{n} s a^{s-1}(t) \cdot \dot{a}(t) \cdot \varphi_s(t) + \sum_{s=1}^{n} a^s(t) \cdot \dot{\varphi}_s(t) = 0,$$

whence

$$(102) \quad \dot{a}(t) = \frac{-\sum_{s=1}^{n} a^s(t) \cdot \dot{\varphi}_s(t)}{\sum_{s=1}^{n} s\, \varphi_s(t)} \cdot a(t) \quad .$$

Therefore

$$(103) \qquad \dot{a}(0) = \frac{-\sum_{s=1}^{n} \dot{\varphi}_s(0)}{\sum_{s=1}^{n} s\, \varphi_s(0)} \ , \text{ since } a(0) = 1,$$

and, when we apply moment condition (78), this reduces to

$$(104) \qquad \dot{a}(0) = -\, ic.$$

Therefore, combining our results, we get

$$(105) \qquad \Psi_o(ty) \equiv \lim_{m\to\infty} \varphi_{m,o}\left(\frac{ty}{m}\right) = e^{icty} .$$

Thus, for all t, we have

$$(106) \qquad \Psi_r(ty) \equiv \lim_{m\to\infty} \left(\frac{y}{m}\right)^r \varphi_{m,r}\left(\frac{ty}{m}\right) = \begin{cases} e^{icty} & , r = 0 \\ 0 & , r \neq 0 \end{cases}$$

<u>Lemma 5.</u> $V(F_{m,r}(x)) \leq c, \quad r = 0,1,\ldots(n-1).$

<u>Proof:</u> Consider the functions $\varphi_{m,r}(t)$ given by

$$\frac{Q_r(z,t)}{P(z,t)} = \sum_{m=o}^{\infty} \varphi_{m,r}(t) z^m .$$

By hypothesis,

$$(107) \qquad \frac{1}{P(z,t)} = \frac{1}{1 - \sum_{s=1}^{n} \varphi_s(t) z^s} = \sum_{m=o}^{\infty} \theta_m(t)\, z^m ,$$

where the functions $\theta_m(t)$ are positive-definite characteristic functions. But

$$(108) \qquad Q_r(z,t) = \binom{o}{r} + \sum_{m=1}^{n-1} z^m \left\{ \binom{m}{r} - \sum_{s=1}^{m} \binom{m-s}{r} \varphi_s(t) \right\}$$

is a polynomial in z and can be represented by

$$(109) \qquad Q_r(z,t) = \sum_{q=1}^{n-1} b_q(r,t)\, z^q ,$$

so that

$$(110) \quad \frac{Q_r(z,t)}{P(z,t)} = \left(\sum_{q=1}^{n-1} b_q(r,t)\cdot z^q\right)\left(\sum_{m=o}^{\infty} \theta_m(t)\cdot z^m\right) = \sum_{m=o}^{\infty} \varphi_{m,r}(t) z^m.$$

Now, $P(z,0)$ has zeros of at most the first order, so that

$$(111) \quad |\theta_m(t)| \leq |\theta_m(0)| \leq C_1.$$

Therefore,

$$(112) \quad |\varphi_{m,r}(t)| \leq (n-1)\cdot(\mathrm{Max}\, |b_q(r,t)|\,)\cdot C_1 = C_2.$$

But $\varphi_{m,r}(t)$, as given by Eq. 109, can be written in the form

$$(113) \quad \varphi_{m,r}(t) = b_o(t)\cdot\theta_m(t) + b_1(t)\cdot\theta_{m-1}(t) + \dots + b_{n-1}(t)\cdot\theta_{m-n+1}(t).$$

Therefore, by (product) Theorem II, we have

$$(114) \quad F_{m,r}(x) = B_o(x) \circledast \vartheta_m(x) + \dots + B_{n-1}(x) \circledast \vartheta_{m-n+1}(x),$$

where $B_j(x)$ and $\vartheta_j(x)$ are given by

$$(115) \quad \begin{cases} b_j(t) = \int_{-\infty}^{\infty} e^{ixt}\, dB_j(x) \\ \theta_j(t) = \int_{-\infty}^{\infty} e^{ixt}\, d\vartheta_j(x). \end{cases}$$

Now, for any convolution function $F(x)$ given by

$$(116) \quad F(x) = \int_{-\infty}^{\infty} F_1(x-z)dF_2(z) \quad ,$$

we have [4;85]

$$(117) \quad V(F) \leq V(F_1)\cdot V(F_2) .$$

Therefore,

$$(118) \qquad V(F_{m,r}(x)) \leq V(B_o)\cdot V(\vartheta_m) + \ldots + V(B_{n-1})\cdot V(\vartheta_{m-n+1}).$$

But, the $B_j(x)$ are of uniformly bounded total variation, and $V(\vartheta_m) = |\theta_j(0)| \leq C_1$, whence

$$V(F_{m,r}(x)) \leq C_3(C_1\cdot(n-1)) = C.$$

Q.E.D. Lemma 5.

By Lemma 5, we can now apply Theorem III to our results exhibited in Eq. 106, from which we obtain

$$(119) \qquad \lim_{m\to\infty} \left\{ \frac{y^r}{m^r} \cdot G_{m,r}(\xi) \right\} = \begin{cases} \text{step of height } 1 \text{ at } \xi = cy & , r = 0 \\ 0 & , r \neq 0 \end{cases} ,$$

so that, from Eq. 30, we see that

$$(120) \qquad \lim_{\substack{m\to\infty \\ h\to o \\ mh\to y}} u(x,mh) = v(x + cy) ,$$

where $v(x + cy)$ is the solution of the partial differential equation $\frac{\partial u}{\partial y} = c\frac{\partial u}{\partial x}$ for the given initial conditions.

Thus, we have proved Theorem V and, in so doing, have presented the main features of the general method by which the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ will be treated in the next chapter.

CHAPTER IV

THE PARTIAL DIFFERENTIAL EQUATION $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

In this chapter, we apply our general method of characteristic functions to the hyperbolic partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ under the same initial conditions of Chapter III, namely:

$$(121) \qquad v_m(x) = \frac{1}{h^m} \sum_{r=o}^{n-1} (-1)^r \binom{m}{r} u\left[x, (m-r)\, h\right] .$$

The solution of the given partial differential equation is well known and, under these initial conditions, is given by

$$(122) \qquad u(x,y) = \frac{1}{2}\left\{v_o(x+y) + v_o(x-y)\right\} + \frac{1}{2}\int_{-y}^{y} v_1(x+\xi)\,d\xi$$

or, equivalently,

$$(123) \qquad u(x,y) = \frac{1}{2}\left\{v_o(x+y) + v_o(x-y)\right\} + \frac{1}{2}\int_{x-y}^{x+y} v_1(\xi)\,d\xi \ .$$

Thus, our problem is, once again, to determine sufficient conditions on the coefficients, a_{rs}, of the corresponding difference equation

$$(124) \qquad \sum_{r,s} a_{rs} u(x + rh,\ mh - sh) = 0,$$

or, more generally, when the difference equation is written in the form

$$(125) \qquad u(x,mh) = \sum_{s=1}^{n} \int_{-\infty}^{\infty} u(x+\xi h, mh-sh)\, dF_s(\xi),$$

to determine conditions on the functions of bounded variation, $F_s(x)$, such that

$$(126) \qquad \lim_{\begin{cases} m \to \infty \\ h \to o \\ mh \to y \end{cases}} u(x,mh) = u(x,y) \quad ,$$

where $u(x,y)$ is the solution (Eq. 122) of the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ corresponding to the given initial conditions.

For the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$, our moment conditions are given by

$$(127) \quad \begin{cases} \text{i)} & \sum_{s=1}^{n} \int_{-\infty}^{\infty} dF_s(\xi) = 1 \\ \text{ii)} & \sum_{s=1}^{n} \int_{-\infty}^{\infty} \xi \, dF_s(\xi) = 0. \\ \text{iii)} & \sum_{s=1}^{n} \int_{-\infty}^{\infty} s \, dF_s(\xi) = 0 \\ \text{iv)} & \sum_{s=1}^{n} \int_{-\infty}^{\infty} s\xi \, dF_s(\xi) = 0 \\ \text{v)} & \sum_{s=1}^{n} \int_{-\infty}^{\infty} \xi^2 \, dF_s(\xi) = -\sum_{s=1}^{n} \int_{-\infty}^{\infty} s^2 \, dF_s(\xi) \neq 0, \end{cases}$$

or, in terms of the characteristic functions $\varphi_s(t)$, defined by

$$(128) \qquad \varphi_s(t) \doteq \int_{-\infty}^{\infty} e^{ixt} \, dF_s(x) \quad ,$$

the moment conditions are

$$
(129)\quad \begin{cases}
\text{i)} & \sum_{s=1}^{n} \varphi_s(0) = 1 \\
\text{ii)} & \sum_{s=1}^{n} \dot{\varphi}_s(0) = 0 \\
\text{iii)} & \sum_{s=1}^{n} s\ \varphi_s(0) = 0. \\
\text{iv)} & \sum_{s=1}^{n} s\ \dot{\varphi}_s(0) = 0. \\
\text{v)} & \sum_{s=1}^{n} \ddot{\varphi}_s(0) = + \sum_{s=1}^{n} s^2\ \varphi_s(0) \neq 0
\end{cases}
$$

Once again, as in the previous chapter, u(x,mh) will be given by

$$
(130)\quad u(x,mh) = \sum_{r=0}^{n-1} \int_{-\infty}^{\infty} h^r v_r(x+\xi h)\, dF_{m,r}(\xi) = \sum_{r=0}^{n-1} \int_{-\infty}^{\infty} h^r v_r(x+\xi)\, dG_{m,r}(\xi),
$$

where

$$
(131)\quad G_{m,r}(\xi) \equiv F_{m,r}(\xi/h) = \begin{cases}
\binom{m}{r} I_0(\xi/h), & 0 \leq m \leq (n-1) \\
\sum_{s=1}^{n} \int_{-\infty}^{\infty} G_s(\xi-\eta)\, dG_{m-s,r}(\eta), & m \geq n,
\end{cases}
$$

and our problem is, as before, to determine sufficient - but broad - conditions on the $F_s(x)$ such that $h^r G_{m,r}(\xi)$ will tend to the proper limit as $\begin{cases} m \to \infty \\ h \to o \\ mh \to y \end{cases}$. As our solution of this problem, we will prove the following main theorem.

THEOREM VI: The solution of the difference equation

$$u(x,mh) = \sum_{s=1}^{n} \int_{-\infty}^{\infty} u(x+\xi h, mh-sh)\, dF_s(\xi)$$

will tend in the limit (as $\begin{matrix} m \to \infty \\ h \to o \\ mh \to y \end{matrix}$) to the solution of the corresponding partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ if:

1) $F_s(x) = 0$ for $x \leq -A$, A constant

2) $F_s(x) = C_s$ for $x \geq A$, C_s constant

3) $F_s(x)$ is of bounded variation for each s.

4) $P(z,t) \equiv 1 - \sum_{s=1}^{n} \varphi_s(t) z^s$ is such that

   a) $P(z,0)$ has no zeros within the unit circle.

   b) $P(z,0)$ has at most zeros of the second order on the unit circle.

   c) $\dot{P}(a,0) = \dot{P}'(a,0) = 0$ and $\ddot{P}(a,0) \neq 0$ at each and every double zero point, $z = a$, on the unit circle.

5) $F_{m,r}(x) = 0$, $x \leq -Am$, and

   $F_{m,r}(x) = C_{m,r}$, $x \geq Am$, where

$$F_{m,r}(\xi) = \begin{cases} \binom{m}{r} I_o(\xi) & , \ 0 \leq m \leq (n-1) \\ \sum_{s=1}^{n} \int_{-\infty}^{\infty} F_s(\xi-\eta) dF_{m-s,r}(\eta), & m \geq n. \end{cases}$$

6) $V(F_{m,r}(x)) \leq Km^r, \qquad r = 1,2,\ldots,(n-1)$

7) $V(\mathcal{F}_{m,o}(\xi)) \leq km,$ where

$$\mathcal{F}_{m,o}(\xi) \equiv \begin{cases} 0 & , \ \xi \leq -Am \\ \int_{-Am}^{\xi} F_{m,o}(x)\,dx & , \ -Am \leq \xi \leq +Am \\ \int_{-Am}^{Am} F_{m,o}(x)\,dx = c_{m,o} = \text{constant}, & \xi \geq Am . \end{cases}$$

We would now like to consider $\lim_{m\to\infty} \frac{1}{m^r} \cdot G_{m,r}(x)$ by the methods of the previous chapter, that is, by the use of continuity theorems for characteristic functions. However, in contradistinction to Chapter III, we now return to the more general definition of the $F_s(x)$ as given in Chapter I (Eq. 11), namely*

$$(132) \begin{cases} \text{i)} & F_s(x) = 0 \quad \text{for } x \leq -A, \ A \text{ constant} \\ \text{ii)} & F_s(x) = C_s \quad \text{for } x \geq A, \ C_s \text{ constant} \\ \text{iii)} & F_s(x) \text{ of bounded variation for each } s. \end{cases}$$

---

* It might be pointed out that our present moment conditions (Eq. 127) preclude the use of non-negative, non-decreasing functions and, hence, the latter are of no interest to us at this time.

Hence, we wish to consider $\lim_{m \to \infty} \frac{1}{m^r} \varphi_{m,r}(t)$ where, as in the previous chapter, $\varphi_{m,r}(t)$ is given by

$$(133) \quad \varphi_{m,r}(t) \equiv \int_{-\infty}^{\infty} e^{ixt}\, dF_{m,r}(x) = \begin{cases} \binom{m}{r} & , \quad 0 \leq m \leq (n-1) \\ \sum_{s=1}^{n} \varphi_s(t) \cdot \varphi_{m-s,r}(t), & m \geq n \end{cases}$$

and

$$(134) \quad \Phi_r(z,t) \equiv \sum_{m=o}^{\infty} \varphi_{m,r}(t) z^m = \frac{Q_r(z,t)}{P(z,t)} =$$

$$= \frac{\binom{o}{r} + \sum_{m=1}^{n-1} z^m \left\{ \binom{m}{r} - \sum_{s=1}^{m} \binom{m-s}{r} \varphi_s(t) \right\}}{1 - \sum_{s=1}^{n} \varphi_s(t)\, z^s}.$$

Now, once again, we seek to represent $\Phi_r(z,t)$ by a partial fraction decomposition. To carry out that decomposition as well as the ensuing theory, we now make the other two (of three) assumptions which were referred to in Chapter I. However, for the sake of completeness, we repeat Assumption I at this time and list all three, namely:

Assumption 1. $P(z,0) \equiv 1 - \sum_{s=1}^{n} \varphi_s(t) z^s$ has no zeros within the unit circle $|z| = 1$.

Assumption 2. $P(z,0)$ has zeros of at most the second order on the unit circle.

Assumption 3. $\dot{P}(a,0) = \dot{P}'(a,0) = 0$ and $\ddot{P}(a,0) \neq 0$ at each and every double zero point, $z = a$, on the unit circle.

The first assumption has already been discussed in Chapter I*. As for Assumption 2, to assume that $P(z,0)$ has at most zeros of the second order on the unit circle seems reasonable since we are dealing with a partial differential equation of the second order. This is strengthened somewhat by our results of Chapter III in which we found that the method applied to the partial differential equation of the first order $\frac{\partial u}{\partial y} = c\frac{\partial u}{\partial x}$ permitted zeros of at most the first order.

The third assumption stems from the behaviour of $P(z,0)$ at $z = 1$. Here, $P(z,0)$ has a zero of the second order and, from our moment conditions, we have

$$\dot{P}(1,0) = \dot{P}'(1,0) = 0 \; ; \; \ddot{P}(1,0) \neq 0.$$

Since we would like all zeros of the second order on the unit circle to behave alike, the third assumption, too, seems to be quite natural.**

---

* Also, see Lemma 8 in Appendix I.

** The reasonability of Assumptions 2 and 3 becomes more pronounced when one considers sequences of non-negative, non-decreasing functions $F_s(x)$ which give rise, in turn, to corresponding positive-definite characteristic functions $\psi_s(t)$ such that

$$P(z,t) = (1 - \Sigma\,\psi_s(t)z^s)(1 - \Sigma\,\overline{\psi_s(t)}\,z^s).$$

In this case Assumptions 2 and 3 (and, for that matter, Assumption 1) are automatically satisfied. For the verification of this statement, see Appendix II.

The foregoing assumptions result in two important lemmas regarding the zeros of $P(z,t)$. They are:

Lemma 6. If $a_1(t)$ and $a_2(t)$ are two roots of $P(z,t) = 0$ such that $a_1(0) = a_2(0) = a$ where $|a| = 1$; if $P(a,0) = \dot{P}'(a,0)=0$, and if $\ddot{P}(a,0) \neq 0$,

Then: 1) $a_1(t)$ and $a_2(t)$ are regular in the neighborhood of $t = 0$.

2) $\dot{a}_1(0) = - \dot{a}_2(0) = i \sqrt{\dfrac{\ddot{P}(a,0)}{P''(a,0)}} \neq 0$.

Proof:

By hypothesis, $P(z,0)$ has a double zero at $z = a$. Hence, altogether we have

$$(135)\quad \begin{cases} \text{i)} & P(a,0) = 0 \\ \text{ii)} & P'(a,0) = 0 \\ \text{iii)} & \dot{P}(a,0) = 0 \\ \text{iv)} & \dot{P}'(a,0) = 0 \\ \text{v)} & \ddot{P}(a,0) \neq 0. \\ \text{vi)} & P''(a,0) \neq 0. \end{cases}$$

Let $a_1(t)$ and $a_2(t)$ be the two roots of $P(z,t) = 0$ such that $a_1(0) = a_2(0) = a$.

(136) Let $P(z,t) = (z^2 - 2\alpha z + \beta)\cdot P^*(z,t)$ where

$$(137)\quad \begin{cases} 2\alpha(t) = a_1(t) + a_2(t) \\ \beta(t) = a_1(t)\cdot a_2(t) . \end{cases}$$

(138) $\therefore \quad \alpha(0) = a \quad$ and $\quad \beta(0) = a^2$.

Now, from Cauchy's theorem of residues, we have

$$
(139) \quad \begin{cases}
2 = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \dfrac{P'(z,t)}{P(z,t)}\, dz \\[2ex]
2\alpha(t) \equiv a_1(t) + a_2(t) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma z\dfrac{P'(z,t)}{P(z,t)}\, dz \\[2ex]
2\beta(t) \equiv (a_1+a_2)^2-(a_1{}^2+a_2{}^2) = \\
\qquad = 4\alpha^2(t) - \dfrac{1}{2\pi i} \displaystyle\int_\Gamma z^2 \dfrac{P'(z,t)}{P(z,t)}\, dz ,
\end{cases}
$$

where Γ is the circle $|z-a| = \delta > 0$, and where $|t| < t_0$ so that Γ will contain both (and only both) zeros, $a_1(t)$ and $a_2(t)$, of $P(z,t)$.

Now, $P(z,t)$ and $P'(z,t)$ are regular within and on Γ . Hence $\alpha(t)$ and, in turn, $\beta(t)$ are regular functions of t within Γ ; in particular, we can speak of their derivatives at $t = 0$ (i.e. $z = a$).

Since Eq. 139 is an identity in t, we may differentiate with respect to t and obtain

$$
(140) \quad \frac{1}{2\pi i} \int_\Gamma \frac{P(z,t)\, \dot{P}'(z,t) - P'(z,t)\dot{P}(z,t)}{(P(z,t))^2}\, dz \equiv 0 \quad .
$$

Similarly, since $\alpha(t)$ and $\beta(t)$ are regular functions of t, we have

$$(141)\quad \begin{cases} 2\dot{\alpha}(t) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma z \, \dfrac{P\cdot\dot{P}' - \dot{P}P'}{P^2} \, dz \quad \text{and} \\[2ex] 2\dot{\beta}(t) = \dfrac{-1}{2\pi i} \displaystyle\int_\Gamma z^2 \, \dfrac{P\cdot\dot{P}' - \dot{P}P'}{P^2} \, dz + 8\,\alpha\,\dot{\alpha} \quad . \end{cases}$$

Then, applying Eq. 140, we have

$$(142)\quad \begin{cases} 2\dot{\alpha}(t) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma (z-a) \, \dfrac{P\cdot\dot{P}' - \dot{P}P'}{P^2} \, dz \quad \text{and} \\[2ex] 2\dot{\beta}(t) = \dfrac{-1}{2\pi i} \displaystyle\int_\Gamma (z^2-a^2) \, \dfrac{P\cdot\dot{P}' - \dot{P}P'}{P^2} \, dz + 8\,\alpha\,\dot{\alpha} \quad . \end{cases}$$

Consider, then

$$(143)\qquad E(z,t) \doteqdot P(z,t)\,\dot{P}'(z,t) - \dot{P}(z,t)P'(z,t) \quad ,$$

so that

$$(144)\quad \begin{cases} E' = P\dot{P}'' - \dot{P}P'' \\ E'' = P\dot{P}''' + \dot{P}''P' - P''\dot{P}' - P'''\dot{P} \\ E''' = P\dot{P}^{IV} + 2P'\dot{P}''' - 2P'''\dot{P}' - P^{IV}\dot{P} \\ E^{IV} = 2P''\dot{P}''' - 2P'''\dot{P}'' \quad . \end{cases}$$

Therefore, from Eq. 135 we see that

$$(145)\qquad E(a,0) = E'(a,0) = E''(a,0) = E'''(a,0) = 0 \quad ,$$

so that $E(z,0)$ has at least a zero of the fourth order at $z = a$ and, thus, $E(z,0)/P^2(z,0)$ is regular at $z = a$. But, if $f(z)$ is regular in a simply connected region C, then

$$\int_\Gamma f(z)\,dz = 0 \quad ,$$

where Γ denotes an arbitrary (not necessarily simple) closed path lying within C. Therefore, from Eq. 142, we see that

(146) $$\dot\alpha(0) = \dot\beta(0) = 0 .$$

We would now like to evaluate $\ddot\alpha(0)$ and $\ddot\beta(0)$. Differentiating Eq. 142 with respect to t, we have

(147) $$\begin{cases} 2\ddot\alpha(t) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (z-a)\,\dfrac{\ddot P' P^2 - \ddot P P' P - 2\dot P(\dot P P' - P\dot P')}{P^3}\,dz \\[2ex] 2\ddot\beta(t) = -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma (z^2 - a^2)\,\dfrac{\ddot P' P^2 - \ddot P P' P - 2\dot P(\dot P P' - P\dot P')}{P^3}\,dz + \\ \qquad + 8\alpha\ddot\alpha + 8\dot\alpha^2 . \end{cases}$$

Now $(z-a)\dot P(\dot P P' - P\dot P')$ has a zero of at least sixth order at $t = 0$ and $z = a$; hence $\dfrac{(z-a)\dot P(\dot P P' - P\dot P')}{P^3}$ is regular at $\begin{cases} t = 0 \\ z = a \end{cases}$ and contributes nothing to the expressions for $\ddot\alpha(0)$ and $\ddot\beta(0)$.

Furthermore, $P(z,0)$, $P'(z,0)$ and $\ddot P(z,0)$ can be expressed by the Taylor series

(148) $$\begin{cases} P(z,0) = P''(a,0)\cdot\dfrac{(z-a)^2}{2!} + P'''(a,0)\cdot\dfrac{(z-a)^3}{3!} + \cdots\cdots \\[2ex] P'(z,0) = P''(a,0)\cdot(z-a) + P'''(a,0)\cdot\dfrac{(z-a)^2}{2!} + \cdots\cdots \\[2ex] \ddot P(z,0) = \ddot P(a,0) + \ddot P'(a,0)\cdot(z-a) + \cdots\cdots \quad , \end{cases}$$

so that, when we apply the Cauchy residue theorem, we obtain

$$
(149)\quad \begin{cases} 2\,\ddot{\alpha}(0) = \dfrac{2\ddot{P}'(a,0)}{P''(a,0)} - \left(\dfrac{4\ddot{P}'}{P''} - \dfrac{2}{3}a^2\dfrac{P'''}{P''}\right) = \dfrac{\frac{2}{3}a^2P'''(a,0)-2\ddot{P}'(a,0)}{P''(a,0)} \\[2ex] 2\,\ddot{\beta}(0) = -2a(2\,\ddot{\alpha}(0)) + \dfrac{4\ddot{P}}{P''} + 8a\,\ddot{\alpha}(0) = \dfrac{4\ddot{P}(a,0)}{P''(a,0)} + 4a\ddot{\alpha}(0)\ . \end{cases}
$$

Thus, from Eq. 149, we have

$$
a\,\ddot{\alpha}(0) - \frac{\ddot{\beta}(0)}{2} = -\frac{\ddot{P}(a,0)}{P''(a,0)} \neq 0.
$$

Returning to our original definitions of $\alpha(t)$ and $\beta(t)$, namely

$$
\begin{cases} 2\,\alpha(t) = a_1(t) + a_2(t) \\ \beta(t) = a_1(t)\cdot a_2(t) \quad , \end{cases}
$$

we can solve for $a_1(t)$ and $a_2(t)$ in terms of $\alpha(t)$ and $\beta(t)$ obtaining

$$
(150)\quad \begin{cases} a_1(t) = \alpha(t) + \sqrt{\alpha^2(t) - \beta(t)} \\ a_2(t) = \alpha(t) - \sqrt{\alpha^2(t) - \beta(t)} \quad . \end{cases}
$$

Since $\alpha(t)$ and $\beta(t)$ are regular functions of t, we can expand them into a MacLaurin series and obtain

$$
(151)\quad a_1(t) = \alpha(t) + \sqrt{\left(\alpha(0) + \dot{\alpha}(0)\,t + \tfrac{t^2}{2}\ddot{\alpha}(0) + \cdots\right)^2 - \left(\beta(0) + t\,\dot{\beta}(0) + \tfrac{t^2}{2}\ddot{\beta}(0) + \cdots\right)}
$$

whence

$$
(152)\quad a_1(t) = \alpha(t) + \sqrt{\left[\alpha^2(0) - \beta(0)\right] + t\left(2\alpha(0)\cdot\dot{\alpha}(0) - \dot{\beta}(0)\right) + t^2\left(\dot{\alpha}^2 + \alpha\ddot{\alpha} - \tfrac{\ddot{\beta}}{2}\right) + O(t^3)}\ .
$$

But, we have already shown that

$$\alpha^2(0) = \beta(0) = a^2$$

and that

$$\dot{\alpha}(0) = \dot{\beta}(0) = 0.$$

Therefore, the expression for $a_1(t)$ reduces to

$$a_1(t) = \alpha(t) + t\sqrt{\left(a\ddot{\alpha}(0) - \frac{\ddot{\beta}(0)}{2}\right) + O(t)} \tag{153}$$

However, we have also proved that

$$a\ddot{\alpha}(0) - \frac{\ddot{\beta}(0)}{2} \neq 0 \quad ,$$

so that $a_1(t)$ will be single-valued in the neighborhood of the origin. Thus, since $\alpha(t)$ and $\beta(t)$ are regular, we have that $a_1(t)$ will be regular in the neighborhood of $t = 0$. Similarly, $a_2(t)$ will also be regular in the neighborhood of $t = 0$. Therefore, solving the two equations

$$\begin{cases} 2\dot{\alpha}(0) = \dot{a}_1(0) + \dot{a}_2(0) = 0 \\ a\ddot{\alpha}(0) - \frac{\ddot{\beta}(0)}{2} = \frac{a}{2}(\ddot{a}_1(0) + \ddot{a}_2(0)) - \\ \qquad - \frac{1}{2}(a_1(0)\ddot{a}_2(0) + 2\dot{a}_1(0)\dot{a}_2(0) + a_2(0)\ddot{a}_1(0)) = \\ \qquad = \frac{-\ddot{P}(a,0)}{P''(a,0)} \neq 0 \end{cases}$$

we obtain

$$\dot{a}_1(0) = -\dot{a}_2(0) = i\sqrt{\frac{\ddot{P}(a,0)}{P''(a,0)}} \neq 0 . \tag{154}$$

Q.E.D. Lemma 6.

<u>Lemma 7.</u> If $P(z,t)$ has at most double zeros on the unit circle and if, for each double zero point $z = a$ on the unit circle, we have $\dot{P}(a,0) = \dot{P}'(a,0) = 0$ and $\ddot{P}(a,0) \neq 0$,

<u>Then:</u> $P(z,t)$ can have at most simple zeros when $0 < |t| \leq t_o$ for those zeros $a_\varkappa(t)$ such that $|a_\varkappa(0)| = 1$.

<u>Proof:</u> (By contradiction).

Suppose that $P(z,t)$ does have a double zero for $t \neq 0$, no matter how small $|t|$ may be. Then, there exist roots $a_1(t)$ and $a_2(t)$ such that

$$a_1(t) \equiv a_2(t) ,$$

whence, differentiating with respect to t, we have

$$\dot{a}_1(t) = \dot{a}_2(t) .$$

In particular, then, for such a double root, we would have

$$\dot{a}_1(0) = \dot{a}_2(0) .$$

But, we have assumed that $P(z,0)$ has <u>at most</u> a double zero on the unit circle. Furthermore, for such a double zero, we know from Lemma 6 that the roots must be such that

$$\dot{a}_1(0) = -\dot{a}_2(0) = ia \neq 0.$$

Therefore, $\dot{a}_1(0)$ cannot be equal to $\dot{a}_2(0)$, so that $P(z,t)$ cannot have a double zero for the zeros under consideration when $0 < |t| \leq t_o$. Hence, $P(z,t)$ can have at most a simple zero when $0 < |t| \leq t_o$ for those zeros $a_\varkappa(t)$ such that $|a_\varkappa(0)| = 1$.

Q.E.D. Lemma 7

Having the three assumptions at our disposal (and, hence, Lemmas 1, 6, and 7, among others), we can once again (cf. Chapters I and III) express $\Phi_r(z,t) \equiv \frac{Q_r(z,t)}{P(z,t)}$ by means of the partial fraction decomposition

$$(155) \qquad \Phi_r(z,t) \equiv \sum_{m=o}^{\infty} \varphi_{m,r}(t)z^m = \sum_{j=1}^{k} \frac{R_j(z,t)}{P_j(z,t)} + R_2(z,t)$$

where, as before, $P_j(z,0)$ contains only those zeros of $P(z,0)$ which lie on the unit circle and $R_2(z,0)$ contains the rest of the zeros of $P(z,0)$, namely those which lie outside the unit circle.

Now, by Lemma 7, we know that $P_j(z,t)$ has at most simple zeros for $t \neq 0$, so that

$$(156) \qquad \sum_{j=1}^{k} \frac{R_j(z,t)}{P_j(z,t)} = \sum_{j=1}^{k} \frac{Q_r(a_j(t),t)}{P'(a_j(t),t)} \cdot \frac{-1}{(1 - \frac{z}{a_j(t)})} \cdot \frac{1}{a_j(t)}$$

In addition, we have already proved (Lemma 1) that when $R_2(z,t)$ is expanded into the power series

$$(157) \qquad R_2(z,t) = \sum_{m=o}^{\infty} T_{m,r}(t) \cdot z^m \quad ,$$

the coefficients, $T_{m,r}(t)$, tend uniformly to zero as m becomes infinitely large. Hence, combining the above results, we have

$$(158) \quad \varphi_{m,r}(t) = \sum_{j=1}^{k} \frac{Q_r(a_j(t),t)}{P'(a_j(t),t)} \cdot \frac{-1}{(a_j(t))^{m+1}} + T_{m,r}(t) \ ,$$

where

(159) $\quad T_{m,r}(t) \to 0$ uniformly as $m \to \infty$.

At this point, since we wish to arrive at the solution of our problem by means of the functions $\varphi_{m,r}(t)$, that is, since we wish to use Theorems III and IV, we now consider

$$\lim_{m \to \infty} \left(\frac{y}{m}\right)^r \varphi_{m,r}(ty/m)$$

for the various cases of $r = 0,1,\ldots,(n-1)$.

Case I. $r = 0$.

Here, we wish to consider $\lim_{m \to \infty} \varphi_{m,o}(\frac{ty}{m})$, where $\varphi_{m,o}(t)$ is given by Eq. 158, namely

$$(160) \quad \varphi_{m,o}(t) = \sum_{j=1}^{k} \frac{Q_o(a_j(t),t)}{P'(a_j(t),t)} \cdot \frac{-1}{\left[a_j(t)\right]^{m+1}} + T_{m,o}(t) \quad .$$

By Eq. 47,

$$Q_o(z,t) = 1 + \sum_{m=1}^{n-1} z^m \left\{ 1 - \sum_{s=1}^{m} \varphi_s(t) \right\}$$

$$= \sum_{m=o}^{n-1} z^m - \sum_{s=1}^{n-1} \sum_{m=s}^{n-1} z^m \varphi_s(t)$$

$$= \frac{z^n - 1}{z-1} - \sum_{s=1}^{n-1} \varphi_s(t) \frac{z^n - z^s}{z-1} = \frac{z^n - 1}{z-1} -$$

$$- \sum_{s=1}^{n} \varphi_s(t) \frac{z^n - z^s}{z-1}$$

$$= \frac{1}{1-z}\left\{ - z^n + 1 + z^n \sum_{s=1}^{n} \varphi_s(t) + \sum_{s=1}^{n} z^s \varphi_s(t)\right\}$$

$$= \frac{1 - \sum_{s=1}^{n}\varphi_s(t)\, z^s}{1 - z} + \frac{z^n\left\{ \sum_{s=1}^{n} \varphi_s(t) - 1\right\}}{1 - z}$$

$$(161) \quad \therefore \quad Q_o(z,t) = \frac{P(z,t)}{1 - z} + \frac{z^n\left\{ \sum_{s=1}^{n} \varphi_s(t) - 1\right\}}{1 - z}$$

Hence, since

$$(162) \quad \varphi_{m,o}(t) = \sum_{j=1}^{k} \frac{Q_o(a_j(t),t)}{P'(a_j(t),t)} \cdot \frac{-1}{a_j^{m+1}(t)} + T_{m,o}(t)$$

$$= \sum_{j=1}^{k} (-1) \frac{P(a_j(t),t) + a_j^{\,n}(t) \left\{ \varphi_s(t) - 1\right\}}{(1-a_j(t))P'(a_j(t),t)\cdot a_j^{m+1}(t)} + T_{m,o}(t),$$

since from Eq. 95

$$\left[a_j(\tfrac{ty}{m})\right]^{-(m+1)} \rightarrow e^{-a_j(0)\cdot ty}$$

and since $T_{m,o}(t) \rightarrow 0$ uniformly as $m \rightarrow \infty$, our primary concern is with

$$\lim_{m \to \infty} \frac{Q_o(a_j(\frac{ty}{m}),\ ty/m)}{P'(a_j(\frac{ty}{m}),\ ty/m)}$$

for each $a_j(ty/m)$, $j = 1,2,\ldots,k$. In addition, we will have to give special attention to the singularities at $z = 1$ because of our representation of $Q_o(z,t)$. In all, we will have to consider the following possibilities:

A. $a_j(t) \neq 1$

1. $a_j(t) \not\to 1$ as $t \to 0$.

2. $a_j(t) \to 1$ as $t \to 0$.

B. $a_j(t) = 1$ for all t.

Note that we are considering only the zeros of $P_j(z,t)$ and that these zeros are such that $|a_j(0)| = 1$.

A. $a_j(t) \neq 1$.

$$\text{Here,} \quad \frac{Q_o(a_j(t),t)}{P'(a_j(t),t)} = \frac{P(a_j(t),t) + a_j^n(t)\left(\sum_{s=1}^{n} \varphi_s(t) - 1\right)}{(1-a_j(t)) \cdot P'(a_j(t),t)}$$

Now, by Lemma 7, $P'(a_j(t),t) \neq 0$ for $0 < t \leq t_o$. Furthermore, $a_j(t) \neq 1$.

Hence, $\dfrac{Q_o(a_j(t),t)}{P'(a_j(t),t)}$ will not have any singularities for $0 < |t| \leq t_o$.

1. $a_j(t) \not\to 1$.

a) If there exists a simple zero at $a_j(0)$, then $P'(a_j(0),0) \neq 0$, whence

$$\frac{Q_o(a_j(ty/m),\ ty/m)}{P'(a_j(ty/m),\ ty/m)} = \frac{P(a_j(\frac{ty}{m}),\frac{ty}{m}) + a_j^n(\frac{ty}{m})\cdot\left(\sum_{s=1}^{n} \varphi_s(\frac{ty}{m}) - 1\right)}{(1 - a_j(\frac{ty}{m}))\ P'(a_j(\frac{ty}{m}),\ \frac{ty}{m})} \to 0$$

as $m \to \infty$.

b) If there exists a double zero at $a_j(0)$, then we have

$$P(a_j(0),0) = P'(a_j(0),0) = \dot{P}(a_j(0),0) = \dot{P}'(a_j(0),0) = 0$$

and

$$\ddot{P}(a_j(0),0) = \left[a_j(0)\right]^2 \cdot P''(a_j(0),0) \neq 0,$$

so that

$$\frac{Q_o(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} = \lim_{m\to\infty} \frac{\sum_{s=1}^{n} \varphi_s(\frac{ty}{m}) - 1}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})}$$

$$= \lim_{m\to\infty} \frac{\sum_{s=1}^{n} \dot{\varphi}_s(\frac{ty}{m})}{\dot{P}' + P''\cdot \dot{a}_j(\frac{ty}{m})} = 0 \ .$$

<u>2. $a_j(t) \to 1$.</u>

Here, at $z = 1$, there is definitely a double zero of $P(z,t)$, since

$$\begin{cases} P'(1,0) = -\sum_{s=1}^{n} s\,\varphi_s(0) = 0 \\ P''(1,0) = -\sum_{s=1}^{n} s(s-1)\,\varphi_s(0) \neq 0. \end{cases}$$

Therefore,

$$\lim_{m\to\infty} \frac{Q_o(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), ty/m)} = \lim_{m\to\infty} \frac{\sum_{s=1}^{n} \varphi_s(\frac{ty}{m}) - 1}{\left[1 - a_j(\frac{ty}{m})\right]\cdot P'(a_j(\frac{ty}{m}), \frac{ty}{m})}$$

$$= \lim_{m\to\infty} \frac{\sum_{s=1}^{n} \dot{\varphi}_s(\frac{ty}{m})}{(1-a_j(\frac{ty}{m}))(\dot{P}' + P''\dot{a}) - \dot{a}P'}$$

$$= \frac{\sum_{s=1}^{n} \ddot{\varphi}_s(0)}{-2\left[\dot{a}_j(0)\right]^2 P''(1,0)} \ .$$

But $P''(1,0) = -\sum_{s=1}^{n} s(s-1)\,\varphi_s(0)$

$$= -\sum_{s=1}^{n} s^2 \varphi_s(0) = -\sum_{s=1}^{n} \ddot{\varphi}_s(0)$$ by our moment conditions.

Furthermore, since $a_j(t) \to 1$, we have by Lemma 6,

$$\left[a_j(0)\right]^2 = -1 .$$

$$\therefore \quad \lim_{m\to\infty} \frac{Q_o(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), {ty}/{m})} = -\frac{1}{2} .$$

B. $a_j(t) = 1$.

This case cannot occur since $a_j(t) = 1$ implies $\sum_{s=1}^{n} \ddot{\varphi}_s(t) = 0$ for all t, and is, thereby, in contradiction with Eq. 129 for the special value $t = 0$.

Thus, in summary, we see that

$$(163) \quad \lim_{m\to\infty} \frac{Q_o(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} \cdot \frac{-1}{(a_j(\frac{ty}{m}))^{m+1}} = \begin{cases} 0 & , \; a_j(t) \not\to 1. \\ \frac{e^{ity} + e^{-ity}}{2} \equiv \operatorname{Cos} ty, & a_j(t) \to 1. \end{cases}$$

whence

$$(164) \qquad \lim_{m\to\infty} \varphi_{m,o}(\tfrac{ty}{m}) = \operatorname{Cos} ty .$$

Case II: $r = 1$.

Here, we wish to consider $\lim\limits_{m\to\infty} \frac{y}{m} \varphi_{m,1}(\frac{ty}{m})$, where $\varphi_{m,1}(t)$ is given by Eq. 158, namely:

$$(165) \quad \varphi_{m,1}(t) = \sum_{j=1}^{k} \frac{Q_1(a_j(t),t)}{P'(a_j(t),t)} \cdot \frac{-1}{(a_j(t))^{m+1}} + T_{m,1}(t) ,$$

and, in which

$$\lim_{m\to\infty} T_{m,1}(t) = 0.$$

In other words, we wish to consider

$$\lim_{m\to\infty} \frac{y}{m} \frac{Q_1(a_j(\frac{ty}{m}), \, ty/m)}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} \cdot \frac{-1}{\left[a_j(ty/m)\right]^{m+1}}$$

for each $a_j(\frac{ty}{m})$ and thus determine, in turn,

$$\lim_{m\to\infty} \frac{y}{m} \varphi_{m,1}(\frac{ty}{m}) .$$

To discuss the contribution of each $a_j(\frac{ty}{m})$ to $\lim_{m\to\infty}\left[\varphi_{m,1}(\frac{ty}{m})\right](\frac{y}{m})$, several possibilities must again be considered, namely

A. $a_j(t) \neq 1$

1. $a_j(t) \not\to 1$ as $t \to 0$
2. $a_j(t) \to 1$ as $t \to 0$.

B. $a_j(t) = 1$, for all t.

where, in all cases, $|a_j(0)| = 1$.

Before delving into the various cases, however, we note that, by Eq. 47,

$$\begin{aligned} Q_1(z,t) &= \sum_{m=1}^{n-1} z^m \left\{ m - \sum_{s=1}^{m} (m-s)\varphi_s(t) \right\} \\ &= \sum_{m=1}^{n-1} mz^m - \sum_{s=1}^{n-1} \sum_{m=s}^{n-1} z^m . \, (m-s)\varphi_s(t) \\ &= \sum_{m=1}^{n-1} mz^m - \sum_{s=1}^{n-1} \varphi_s(t) \sum_{m=s}^{n-1} mz^m + \sum_{s=1}^{n-1} s\,\varphi_s(t) \sum_{m=s}^{n-1} z^m \end{aligned}$$

$$= \frac{(n-1)z^{n+1} - nz^n + z}{(z-1)^2} - \sum_{s=1}^{n-1} \varphi_s(t)\cdot\left\{\frac{(n-1)z^{n+1}-(s-1)z^{s+1}-nz^n+sz^s}{(z-1)^2}\right\} + \sum_{s=1}^{n-1} s\,\varphi_s(t)\cdot\frac{z^n - z^s}{z-1} .$$

After further simplification and division by $P'(z,t)$, we then have

$$(166)\quad \frac{Q_1(z,t)}{P'(z,t)} = \frac{\{(n-1)z^{n+1} - nz^n\}\{1-\sum_{s=1}^{n}\varphi_s(t)\} + zP(z,t) + \{z^{n+1}-z^n\}\sum_{s=1}^{n} s\,\varphi_s(t)}{(1-z)^2\; P'(z,t)} .$$

A. $a_j(t) \neq 1$

1) $a_j(t) \to 1$.

Here, we have

$$Q_1(a_j(\tfrac{ty}{m}), \tfrac{ty}{m}) = \frac{\{(n-1)a_j^{n+1} - na_j^n\}\{1-\sum_{s=1}^{n}\varphi_s(\frac{ty}{m})\} + a_j P + \{a_j^{n+1}-a_j^n\}\{\sum_{s=1}^{n} s\,\varphi_s(\frac{ty}{m})\}}{\left[1 - a_j(\frac{ty}{m})\right]^2}$$

$$\to 0 \text{ as } m \to \infty .$$

Hence:

a) If there exists a simple zero of $P(z,0)$ at $a_j(0)$,

$$P'(a_j(0),0) \neq 0 ,$$

from which we obtain

$$\frac{Q_1(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} \to 0.$$

b) If there exists a double zero of $P(z,t)$ at $a_j(0)$,

$$\begin{cases} P(a_j(0),0) = P'(a_j(0),0) = \dot{P}(a_j(0),0) = \dot{P}'(a_j(0),0) = 0 \\ \ddot{P}(a_j(0),0) = (a_j(0))^2 \, P''(a_j(0),0) \neq 0 \quad , \end{cases}$$

from which we obtain

$$\lim_{m\to\infty} \left(\frac{y}{m}\right) \cdot \frac{Q_1(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} = \lim_{m\to\infty} \frac{\frac{y}{m}(\dot{Q}_1 + Q'_1 \cdot \dot{a}_j)(ty) + yQ_1}{(\dot{P}' + P'' \cdot \dot{a}_j)(ty)}$$

$$= \frac{+\ Q_1(a_j(0),0)}{t \cdot P''(a_j(0),0) \cdot \dot{a}_j(0)} = 0 \quad ,$$

since $\dot{a}_j(0) \neq 0$ by Lemma 7 and since $Q_1(a_j(0),0) = 0$.

---

2) $a_j(t) \to 1$.

Here, at $z = 1$, there definitely is a double zero of $P(z,t)$ since

$$P'(1,0) = -\sum_{s=1}^{n} s\,\varphi_s(0) = 0$$

$$P''(1,0) = -\sum_{s=1}^{n} s(s-1)\,\varphi_s(0) \neq 0 \,.$$

$$\therefore \lim_{m\to\infty} \frac{y}{m} \frac{Q_1(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} = \lim_{m\to\infty} \frac{\frac{y}{m}(\dot{Q}_1 + Q'_1\dot{a}_j)(ty) + yQ_1}{(\dot{P}' + P'' \cdot \dot{a}_j)(ty)}$$

$$= \frac{Q_1(1,0)}{t\,\dot{a}_j(0) \cdot P''(1,0)} \,.$$

Now

$$Q_1(1,0) = \sum_{m=1}^{n-1} z^m \left\{ m - \sum_{s=1}^{m} (m-s)\varphi_s(0) \right\} \Bigg]_{z=1}$$

$$= \sum_{m=1}^{n-1} mz^m - \sum_{s=1}^{n-1} \varphi_s(0) \cdot \sum_{m=s}^{n-1} mz^m + \sum_{s=1}^{n-1} s\,\varphi_s(0) \sum_{m=s}^{n-1} z^m \Bigg]_{z=1}$$

$$= \frac{n(n-1)}{2} - \sum_{s=1}^{n-1} \varphi_s(0)\, \frac{(n-s)(n-s+1)}{2} + \sum_{s=1}^{n-1} s\,\varphi_s(0)\cdot(n-s)$$

$$= \frac{n^2-n}{2} - \frac{1}{2} \sum_{s=1}^{n-1} \varphi_s(0)\cdot(n^2-s^2-n+s) + n\sum_{s=1}^{n} s\,\varphi_s(0) -$$

$$- \sum_{s=1}^{n} s^2 \varphi_s(0) ,$$

whence, upon making use of our moment conditions, we obtain

$$(167) \qquad Q_1(1,0) = -\frac{1}{2} \sum_{s=1}^{n} s^2 \varphi_s(0) \neq 0 .$$

$$\therefore \lim_{m\to\infty} \frac{y}{m} \frac{Q_1(a_j(\frac{ty}{m}),\frac{ty}{m})}{P'(a_j(\frac{ty}{m}),\frac{ty}{m})} = \frac{-\frac{1}{2}\sum_{s=1}^{n} s^2 \varphi_s(0)}{-t\cdot \dot a_j(0)\cdot \sum_{s=1}^{n} s^2\varphi_s(0)} = -\frac{1}{2t(\dot a_j(0))} =$$

$$= \frac{\pm i}{2t} ,$$

since $\dot a_1(0) = -\dot a_2(0) = i$.

B. $a_j(t) \equiv 1.$

Just as for $r = 0$, this case cannot occur due to our moment conditions.

Therefore, for $r = 1$, we have

$$(168)\quad \lim_{m\to\infty} \frac{y}{m}\,\varphi_{m,1}\left(\frac{ty}{m}\right) = \frac{ie^{-ity} - ie^{ity}}{2t} = \frac{\operatorname{Sin} ty}{t} .$$

Case III: $r \geq 2.$

Here, we consider $\lim_{m\to\infty} \frac{y^r}{m^r}\,\varphi_{m,r}\left(\frac{ty}{m}\right)$, $r \geq 2.$

A. $a_j(t) \neq 1.$

a) If there exists a simple zero of $P(z,t)$ at $z = a_j(0)$, then

$$\lim_{m\to\infty} \frac{y^r}{m^r} \frac{Q_r(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} = 0.$$

b) If there exists a double zero of $P(z,t)$ at $z = a_j(0)$, then

$$\lim_{m\to\infty} \frac{y^r}{m^r} \frac{Q_r(a_j(\frac{ty}{m}), \frac{ty}{m})}{P'(a_j(\frac{ty}{m}), \frac{ty}{m})} = \lim_{m\to\infty} \frac{\frac{y_r}{m^r}(\dot{Q}_r + Q_r'\cdot\dot{a}_j)(\frac{-ty}{m^2}) + y^r Q_r(-\frac{r}{m^{r+1}})}{(\dot{P}' + P''\cdot\dot{a}_j)(-{}^{ty}/_{m^2})}$$

$$= 0 \quad \text{provided } r \geq 2.$$

B. $a_j(t) \equiv 1.$

Once again, this case is non-existent due to the moment conditions.

Therefore, for $r \geq 2$, we have

$$(169) \qquad \lim_{m\to\infty} \frac{y^r}{m^r} \varphi_{m,r}\left(\frac{ty}{m}\right) = 0 \,.$$

Thus far, then, we have shown that, for all t,

$$(170)\ \Psi_r(ty) \doteqdot \lim_{m\to\infty} \frac{y^r}{m^r} \varphi_{m,r}\left(\frac{ty}{m}\right) = \begin{cases} \dfrac{e^{ity} + e^{-ity}}{2} \equiv \cos ty \,, & r = 0 \\ \dfrac{e^{ity} - e^{-ity}}{2it} \equiv \dfrac{\sin ty}{t} \,, & r = 1 \\ 0 \,, & r \geq 2, \end{cases}$$

Since we wish to use Theorems III and IV, we still need to establish the existence of

$$\lim_{m\to\infty} \frac{1}{m} \dot{\varphi}_{m,o}(0).$$

From Eq. 158, we have

$$\varphi_{m,o}\left(\frac{ty}{m}\right) = \sum_{j=1}^{k} \frac{Q_o\left(a_j\left(\frac{ty}{m}\right), {}^{ty}/_{m}\right)}{P'\left(a_j\left(\frac{ty}{m}\right), {}^{ty}/_{m}\right)} \cdot \frac{-1}{\left[a_j\left(\frac{ty}{m}\right)\right]^{m+1}} + T_{m,o}\left(\frac{ty}{m}\right) \,,$$

so that

$$(171)\ \dot{\varphi}_{m,o}\left(\frac{ty}{m}\right) = \sum_{j=1}^{k} \frac{Q_o}{P'} \cdot (m+1) \frac{1}{a_j^{m+2}} \cdot \dot{a}_j\left(\frac{ty}{m}\right) \cdot \frac{y}{m} -$$

$$- \sum_{j=1}^{k} \frac{1}{a_j^{m+1}} \frac{P'(\dot{Q}_o + Q_o'\dot{a})\left(\frac{y}{m}\right) - Q_o(\dot{P}' + P''\dot{a})\left(\frac{y}{m}\right)}{(P')^2} + \frac{y}{m} \dot{T}_{m,o}\left(\frac{ty}{m}\right) \,,$$

where, because of its exponential nature, $\dot{T}_{m,o}(\frac{ty}{m}) \to 0$ as $m \to \infty$.

Therefore,

$$(172)\quad \dot{\varphi}_{m,o}(0) = \sum_{j=1}^{k} \frac{Q_o(a_j(0),0)}{P'(a_j(0),0)} \cdot \frac{\dot{a}_j(0)}{a_j^{\,m+2}(0)} \cdot \frac{y(m+1)}{m} -$$

$$- \sum_{j=1}^{k} \frac{y}{a_j^{\,m+1}(0)} \cdot \frac{P'(\dot{Q}_o + \dot{Q}_o'\dot{a}) - Q_o(\dot{P}' + P''\cdot\dot{a})}{m(P')^2} + \dot{T}_{m,o}(0).$$

Now, if there exists a simple zero of $P(z,0)$ at $z = a_j(0)$, then

$$\lim_{m \to \infty} \frac{1}{m} \cdot \dot{\varphi}_{m,o}(0) = 0.$$

If there exists a double zero of $P(z,0)$ at $z = a_j(0)$, then, splitting our limit into two parts, we have

$$\lim_{t \to o} \frac{Q_o(a_j(t),t)}{P'(a_j(t),t)} = -\frac{1}{2} \qquad \text{from previous results,}$$

and

$$\lim_{t \to o} \frac{P'(a_j(t),t)(\dot{Q}_o + Q_o'\,\dot{a}) - Q_o(\dot{P}' + P''\cdot\dot{a})}{\left[P'(a_j(0),0)\right]^2}$$

$$= \lim_{t \to o} \left\{ \frac{(\dot{P}' + P''\dot{a})(\dot{Q}_o + Q_o'\dot{a}) + P'(\ddot{Q}_o + \dot{Q}_o'\dot{a} + \dot{Q}_o'\dot{a} + Q_o''\dot{a}^2 + Q_o''\ddot{a})}{2P'(\dot{P}' + P''\dot{a})} \right.$$

$$\left. - \frac{Q_o(\ddot{P}' + \dot{P}''\dot{a} + \dot{P}''\dot{a} + P'''(\dot{a})^2 + P''\ddot{a}) - (\dot{Q}_o + Q_o'\dot{a})(\dot{P}' + P''\dot{a})}{2P'(\dot{P}' + P''\dot{a})} \right\}.$$

This limit is still indeterminate of the form $\frac{0}{0}$ since $P'(a_j(0),0) = Q_o(a_j(0),0) = 0$. Applying L'Hospital's rule once more would give us the non-zero denominator

$$2P'(\ddot{P}' + \dot{P}''\dot{a} + \dot{P}''\dot{a} + P'''\dot{a}^2 + P''\ddot{a}) + 2(\dot{P}' + P''\dot{a})^2\Big]_{t=o},$$

that is,

$$2\left\{P''(a_j(0))\right\}^2 \cdot (\dot{a}_j(0))^2 \quad ,$$

while the numerator, vanishing or not for $t = 0$, would, nevertheless, remain finite.

Therefore, $\dot{\varphi}_{m,o}(0)$ approaches a definite finite limit and, in turn,

$$\lim_{m\to\infty} \frac{1}{m}\dot{\varphi}_{m,o}(0) = 0 \tag{173}$$

Having verified the existence of $\lim\limits_{m\to\infty} \frac{1}{m^r}\varphi_{m,r}(\frac{ty}{m})$ and $\lim\limits_{m\to\infty} \frac{1}{m}\dot{\varphi}_{m,o}(0)$, we now apply Theorems III and IV and obtain:

$$\lim_{m\to\infty} \frac{y^r}{m^r} G_{m,r}(\xi) = \begin{cases} G_{,o}(\xi) & , \ r = 0 \\ G_{,1}(\xi) & , \ r = 1 \\ 0 & , \ r \geq 2 \ , \end{cases} \tag{174}$$

where

$$G_{,o}(\xi) = \begin{cases} 0 & , \ \xi < -y \\ \frac{1}{2} & , \ -y \leq \xi < y \\ 1 & , \ \xi \geq y \end{cases} \tag{175}$$

and

$$(176) \quad G_{,1}(\xi) = \begin{cases} 0 & , \ \xi < -y \\ \frac{1}{2}(\ \ + y) & , \ -y \le \xi < y \\ y & , \ \xi \ge y \end{cases} .$$

Therefore, applying these results to Eq. 130, we obtain

$$(177) \quad \lim_{\substack{m \to \infty \\ h \to o \\ mh \to y}} u(x,mh) = \frac{1}{2}\left\{ v_o(x+y) + v_o(x-y) \right\} + \frac{1}{2}\int_{-y}^{y} v_1(x+\xi)d\xi ,$$

or equivalently,

$$(178) \quad u(x,mh) \to \frac{1}{2}\left\{ v_o(x+y) + v_o(x-y) \right\} + \frac{1}{2}\int_{x-y}^{x+y} v_1(\xi)d\xi ,$$

namely, the solution of the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ for the given initial conditions.

Thus, we have proved Theorem VI and, in so doing, have exhibited a general method for the treatment of other partial differential equations. In fact, although the discussion in the present chapter was centered about the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \partial^2 u/\partial y^2$, there is a strong indication that the method just presented is directly applicable to the partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

with only a slight amount of additional computation.

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APPENDIX I.

Lemma 8. The supposition that $P(z,0) \equiv 1 - \sum_{s=1}^{n} \varphi_s(t) z^s$ has no zeros within the unit circle is necessary for the method of solution by difference equations.

Proof: If the method of solution by difference equations is to work, then

$$\lim_{m \to \infty} \sum_{r=0}^{n-1} \int_{-\infty}^{\infty} v(x + \frac{\xi y}{m}) \frac{1}{m^r} \, dF_{m,r}(\xi)$$

must exist for all x and y. (In fact, the limit will be the solution of the given partial differential equation). In particular, since $dF_{m,r}(x) \equiv 0$ outside the interval $(-Am, Am)$, we have that

$$\lim_{m \to \infty} \sum_{r=0}^{n-1} \int_{-Am}^{Am} v(x + \frac{\xi y}{m}) \cdot \frac{1}{m^r} \cdot dF_{m,r}(\xi)$$

must exist for all x and y. In all of our considerations, we have taken $v(x)$ to be continuously differentiable. In particular, then, choose $v(\xi) = \xi$, $r = 0$, $x = 0$, and $y = 1$. We then see that

$$\lim_{m \to \infty} \frac{1}{m} \int_{-Am}^{Am} \xi \, dF_{m,o}(\xi) \equiv \lim_{m \to \infty} \frac{1}{m} \dot{\varphi}_{m,o}(0)$$

must exist.

Now, by Eq. 161, we have

$$\frac{Q_o(z,t)}{P(z,t)} = \frac{1}{1-z} + \frac{z^n \cdot P(1,t)}{(1-z) \cdot P(z,t)} = \sum_{r=0}^{\infty} \varphi_{m,r}(t) \cdot z^m \quad ,$$

so that

$$(179)\quad \sum_{m=o}^{\infty} \dot{\varphi}_{m,o}(t)\, z^m = \frac{z^n}{1-z} \cdot \frac{P(z,t)\cdot\dot{P}(1,t) - P(1,t)\dot{P}(z,t)}{P^2(z,t)}$$

and

$$(180)\quad \sum_{m=o}^{\infty} \dot{\varphi}_{m,o}(0)\, z^m = \frac{z^n}{1-z} \cdot \frac{\dot{P}(1,t)}{P(z,t)} \quad ,$$

since for all of the differential equations under consideration, $P(1,0) = 0$ will be one of the given moment conditions.

Now, suppose that we are considering a first order partial differential equation (such as in Chapter III) which, when represented by a corresponding difference equation, gives rise to the moment condition

$$\dot{P}(1,0) \equiv -\sum_{s=1}^{n} \dot{\varphi}_s(0) \neq 0.$$

Hence,

$$(181)\qquad \sum_{m=o}^{\infty} \dot{\varphi}_{m,o}(0) z^m \neq 0.$$

Now, since $\lim\limits_{m\to\infty} \frac{1}{m} \dot{\varphi}_{m,o}(0)$ must exist, we have that

$$(182)\qquad \left|\dot{\varphi}_{m,o}(0)\right| \leq c\, m \quad .$$

Furthermore, the power series $\sum\limits_{m=o}^{\infty} m z^m$ has a radius of convergence of unity. Hence

(183) $$\sum_{m=o}^{\infty} \dot{\varphi}_{m,o}(0)z^m = \frac{z^n}{1-z} \cdot \frac{\dot{P}(1,0)}{P(z,0)}$$

must have a radius of convergence $r \geq 1$. Hence, P(z,0) cannot have any singularities within the unit circle (for a first order partial differential equation).

For a second order partial differential equation which gives rise to the moment conditions $P(1,0) = \dot{P}(1,0) = 0$, $\ddot{P}(1,0) \neq 0$, we repeat the procedure given above for the case when $r = 0$, $x = 0$, $y = 1$, and $v(\rho) = \rho^2$. Here, then,

$$\lim_{m \to \infty} \frac{1}{m^2} \cdot \int_{-Am}^{Am} \rho^2 dF_{m,o}(\rho) \equiv \lim_{m \to \infty} \frac{1}{m^2} \ddot{\varphi}_{m,o}(0)$$

must exist. Now, from Eq. 179, further differentiation gives us

(184) $$\sum_{m=o}^{\infty} \ddot{\varphi}_{m,o}(0)z^m = \frac{z^m}{1-z} \cdot \frac{\ddot{P}(1,0)}{P(z,0)} \quad .$$

But

(185) $$\left|\ddot{\varphi}_{m,o}(0)\right| \leq c \cdot m^2 \quad ,$$

so that $\sum_{m=o}^{\infty} \ddot{\varphi}_{m,o}(0)z^m$ must have a radius of convergence $r \geq 1$. Hence, here too, P(z,0) cannot have any singularities within the unit circle (for a second order partial differential equation).

Q.E.D. Lemma 8

APPENDIX II

We consider, here, sequences of non-negative, non-decreasing functions $F_s(x)$,

$$(186) \qquad \begin{cases} F_s(x) = 0, & x \leq -A \\ F_s(x) = C_s, & x \geq A\ , \end{cases}$$

which give rise, in turn, to corresponding positive-definite characteristic functions $\psi_s(t)$ such that the polynomial $P(z,t)$ can be written as

$$(187) \qquad P(z,t) = (1 - \Sigma\, \psi_s(t) z^s)(1 - \Sigma\, \overline{\psi_s(t)}\, z^s),$$

and we wish to establish that, for this case, $P(z,t)$ satisfies Assumptions 1, 2 and 3 of Chapter IV.

That such cases actually exist is readily verified by considering the classical difference pattern for the hyperbolic partial differential equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$, namely

$$(188) \qquad \begin{array}{ccc} & 1 & \\ -1 & & -1 \\ & 1 & \end{array}$$

Here, we have

$$(189) \qquad \begin{cases} \varphi_1(t) = e^{-it} + e^{it} = 2 \operatorname{Cos} t \\ \varphi_2(t) = -1, \end{cases}$$

whence

$$(190) \quad P(z,t) \doteqdot 1 - \sum_{s=1}^{n} \varphi_s(t) z^s = 1 - 2z \operatorname{Cos} t + z^2$$

$$= (1 - e^{it} z)(1 - e^{-it} z).$$

By arguments previously presented in Lemma 1, we first establish that P(z,t) cannot have any zeros within the unit circle, namely: Given $P(z,t) = (1 - \Sigma\Psi_s(t)z^s)(1 - \Sigma\overline{\Psi_s(t)}z^s)$, then, for any zero, $z = a$, of P(z,t) we have

$$(1 - \Sigma\Psi_s(t)a^s)(1 - \Sigma\overline{\Psi_s(t)}a^s) = 0,$$

whence

$$(191) \begin{cases} \Sigma\Psi_s(t)\ a^s = 1 & \text{and} \\ \Sigma\overline{\Psi_s(t)}\ a^s = 1. \end{cases}$$

Hence, if we assume that there are zeros of P(z,t) at $z = a$ such that $|a| < 1$, we arrive at the contradiction $1 < 1$, namely

$$|a| < 1 \Rightarrow 1 = |\Sigma\Psi_s(t)a^s| \leq \Sigma\Psi_s(t)\ |a|^s < \Sigma\Psi_s(t) \leq \Sigma\Psi_s(0) = 1.$$

Q.E.D.

We now consider jointly Assumptions 2 and 3, namely:

Assumption 2: P(z,0) has zeros of at most the second order on the unit circle.

Assumption 3: $\dot{P}(a,0) = \dot{P}'(a,0) = 0$ and $\ddot{P}(a,0) \neq 0$ at each and every double zero point, $z = a$, on the unit circle.

To establish the validity of these two conditions, we first prove the following lemma:

Lemma 9.

Given: $P(z,t) = (1 - \Sigma \Psi_s(t) z^s)(1 - \Sigma \overline{\Psi_s(t)}\, z^s)$,

where $\Psi_s(t)$ are positive-definite characteristic functions, and where

$$\begin{cases} P(1,0) = P'(1,0) = \dot{P}(1,0) = \dot{P}'(1,0) = 0, \\ \ddot{P}(1,0) \neq 0; \quad P''(1,0) \neq 0 . \end{cases}$$

Then:

1) $\Sigma\, s\, \Psi_s(0)\, a^s \neq 0$.

2) $\Sigma \dot{\Psi}_s(0)\, a^s \neq 0$.

Proof:

Since a is a zero of $P(z,0)$, we have

(192) $$\Sigma\, \Psi_s(0)\, a^s = 1.$$

But, $a = 1$ is also a (double) zero of $P(z,0)$, so that

(193) $$\Sigma\, \Psi_s(0) = 1.$$

Therefore, for $|a| = 1$,

$$1 = \left| \Sigma \Psi_s(0)\, a^s \right| \leq \Sigma \Psi_s(0)\, |a|^s = \Sigma\, \Psi_s(0) = 1,$$

whence

(194) $$\Sigma \Psi_s(0)\, |a|^s = 1 = \Sigma \Psi_s(0)\, a^s .$$

Therefore, $\Psi_s(0)a^s$ must be real and non-negative, which, in turn, implies that either

$$(195)\begin{cases} \text{i)} & \Psi_s(0) = 0, \quad \text{or} \\ \text{ii)} & \Psi_s(0) \neq 0 \quad \text{and} \quad a^s \text{ is real.} \end{cases}$$

But, $\Psi_s(0) \geq 0$ and there exists at least one $\Psi_s(0) \neq 0$. Hence, if we consider the sum of non-negative and real terms given by

$$\sum s\, \Psi_s(0)\, a^s \quad ,$$

we see that

$$(196) \qquad \sum s\, \Psi_s(0)\, a^s \neq 0 .$$

Q.E.D. 1)

Now, since we have

$$(197) \qquad \sum \Psi_s(0)\, a^s = 1 = \sum \Psi_s(0),$$

then, by the arguments just presented,

$$(198) \qquad \Psi_s(0)\cdot(a^s - 1) = 0,$$

so that either

$$(199)\begin{cases} \text{i)} & \Psi_s(0) = 0 \qquad \text{or} \\ \text{ii)} & a^s = 1. \end{cases}$$

But, $\Psi_s(0)$ is a positive-definite function so that

$$(200) \qquad \Psi_s(0) = 0 \quad \Rightarrow \quad \dot{\Psi}_s(0) \equiv 0.$$

Therefore, either

$$(201)\quad \begin{cases} \text{i)}\ \dot{\Psi}_s(0) = 0\ , & \text{or} \\ \text{ii)}\ a^s = 1. \end{cases}$$

Hence, we have that

$$(202)\qquad \sum \dot{\Psi}_s(0)\, a^s = \sum \dot{\Psi}_s(0).$$

But, the given moment condition $P''(1,0) \neq 0$ implies (see Eq. 205 below)

$$(203)\qquad \sum \dot{\Psi}_s(0) \neq 0\ ,$$

so that

$$(204)\qquad \sum \dot{\Psi}_s(0)\, a^s \neq 0\ .$$

Q.E.D. 2)

Q.E.D. Lemma 9

Returning to Assumptions 2 and 3, then, consider

$$(205)\quad \begin{cases} P(a,0) = (1 - \sum \Psi_s(0)a^s)(1 - \sum \overline{\Psi_s(0)}\, a^s) \\ \dot{P}(a,0) = (1 - \sum \Psi_s(0)a^s)(-\sum \dot{\overline{\Psi_s(0)}}a^s) + \\ \qquad + (1 - \sum \overline{\Psi_s(0)}a^s)(-\sum \dot{\Psi}_s(0)a^s)\ . \\ P'(a,0) = (1 - \sum \Psi_s(0)a^s)(-\sum s\, \overline{\Psi_s(0)}a^{s-1}) + \\ \qquad + (1 - \sum \overline{\Psi_s(0)}a^s)(-\sum s\, \Psi_s(0)a^{s-1})\ . \\ \dot{P}'(a,0) = (1 - \sum \Psi_s(0)a^s)(-\sum s\, \dot{\overline{\Psi_s(0)}}a^{s-1}) + \\ \qquad + (1 - \sum \overline{\Psi_s(0)}\, a^s)(-\sum s\, \dot{\Psi}_s(0)a^{s-1}) + \\ \qquad + (\sum \dot{\overline{\Psi_s(0)}}\, a^s)(\sum s\, \Psi_s(0)a^{s-1}) + \\ \qquad + (\sum \dot{\Psi}_s(0)a^s)(\sum s\, \overline{\Psi_s(0)}\, a^{s-1})\ . \end{cases}$$

$$(205)\begin{cases} \ddot{P}(a,0) = (1 - \Sigma\,\Psi_s(0)a^s)(-\Sigma\,\ddot{\overline{\Psi_s(0)}}a^s) + \\ \qquad + (1 - \Sigma\,\overline{\Psi_s(0)}a^s)(-\Sigma\,\ddot{\Psi}_s(0)a^s) + \\ \qquad + 2(\Sigma\,\dot{\Psi}_s(0)a^s)(\Sigma\,\dot{\overline{\Psi_s(0)}}\,a^s)\ . \\ P''(a,0) = (1 - \Sigma\,\Psi_s(0)a^s)(-\Sigma\,s(s-1)\,\overline{\Psi_s(0)}a^{s-2}) + \\ \qquad + (1 - \Sigma\,\overline{\Psi_s(0)}a^s)(-\Sigma\,s(s-1)\,\Psi_s(0)a^{s-2}) + \\ \qquad + 2(\Sigma\,s\,\Psi_s(0)a^{s-1})(\Sigma\,s\,\overline{\Psi_s(0)}\,a^{s-1})\ . \end{cases}$$

By the definition of the characteristic functions, $\Psi_s(t)$, we know that

$$(206)\begin{cases} \Psi_s(0) = \int_{-\infty}^{\infty} dF(x) = \overline{\Psi_s(0)} \\ \dot{\Psi}_s(0) = i\int_{-\infty}^{\infty} x dF_s(x) = -\,\dot{\overline{\Psi_s(0)}}\ . \end{cases}$$

Furthermore, since $z = a$ is a double zero of $P(z,0)$,

$$P(a,0) = P'(a,0) = 0; \quad P''(a,0) \neq 0,$$

which, in turn, implies that

$$(207)\begin{cases} \Sigma\,\Psi_s(0)\,a^s = 1, \quad \Sigma\,\overline{\Psi_s(0)}\,a^s = 1, \\ \Sigma\,s\,\Psi_s(0)a^s \neq 0, \quad \Sigma\,s\,\overline{\Psi_s(0)}a^s \neq 0\ . \end{cases}$$

Therefore, combining Lemma 9 and Eqs. 205, 206 and 207, we see that, for any double zero, $z = a$, of $P(z,0)$, we have

$$P(a,0) = \dot{P}(a,0) = P'(a,0) = \dot{P}'(a,0) = 0;$$

$$\ddot{P}(a,0) \neq 0\ ; \quad P''(a,0) \neq 0.$$

Q.E.D.