Aspects of String Dualities

-Orientifolds, F-theory and Super D-branes & the M5-brane

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Abstract

We probe string dualities by using the orientifold and F-theory, and by investigating world volume actions of super D-branes and super M-branes.

We first study orientifolds in various dimensions. We construct orientifolds dual to M-theory compactified on the Klein bottle and on the Möbius band, respectively. Six-dimensional orientifolds with N=1 supersymmetry are constructed. They have multiple tensor multiplets, which cannot be obtained by the conventional Calabi-Yau compactifications. We find F-theory duals for some of these models, thereby making manifest the phase transitions involving the tensionless strings these models can have.

We construct orientifold and F-theory duals of the heterotic string models constructed by Chaudhuri, Hockney and Lykken (CHL) and study N=2 supersymmetric F-theory vacua in six dimensions.

Next, we construct the supersymmetric world volume action of the M-theory 5-brane in a flat eleven-dimensional background. Finally, dual D-brane actions are obtained by carrying out a duality transformation of the world volume gauge field of the D-brane and their properties are studied.

Contents

Chapter 1 Introduction and Summary

String duality is a recently developed part of string theory. String theory itself is over twenty-five years old, and has been under intensive development since 1984 as the leading candidate for a unified theory of particle physics and gravity. The reason that some of its propeties have been overlooked until now is simple : string duality is not manifest in the weak-coupling perturbation expansion by which the theory is usually studied, but it is a property of the exact theory. As a result, string duality gives information about the behavior of string theory at strong coupling. In a short period of two years, we have gone from near-complete ignorance of the behavior of strongly-coupled strings to a rather detailed understanding of the intricate dynamics which occurs, at least in vacua having enough supersymmetry, and the subject continues to develop at a rapid pace.

The central idea of string duality is that the strongly coupled limit of any string theory is equivalent to the weakly coupled limit of some other theory. All string theories are connected in this way and the unique underlying theory is called 'M-theory' whose low energy theory is eleven-dimensional supergravity. Besides the ordinary vibrating strings, which are the basic quanta of string theory, various solitonic objects play an essential role in string dualities. Especially new types of solitonic objects, called D-branes, have attracted much attention, since the understanding of D-branes is possible within the framework of perturbative string theory! With the improved understanding of string dynamics it has become possible to address one of the long-standing problems of quantum gravity—to count the number of states of certain black holes in a controlled way, giving, for the first time, a statistical mechanical interpretation to the Bekenstein-Hawking entropy[1].

This dissertation is a collection of our modest attempts to understand various aspects of string dualities. Our approach is two-fold. One approach is to construct specific string vacua and to study the dual properties of those vacua. This is worthwhile to study, since new ways of constructing string vacua have emerged along with the enhanced understanding of string dualities. Often such new vacua are not visible with the conventional ways of constructing string vacua, hence can provide useful information on the string dualities. Important examples are orientifolds and F-theory, which we use to probe string dualities. The other approach is to study the world volume theory of various nonperturbative objects in string theory. One category would be D-branes and by studying their world volume structures, we can check various properties expected from the web of string dualities. Since string theories are just a part of so-far mysterious M-theory, D-branes should have definite connections with solitonic objects in M-theory. Thus it is interesting to study world volume theories of M-branes and their connections to D-branes. One of the main obstacles was that the world volume action of the M-theory 5-brane was not available. Here the M-theory 5-brane action is presented and its connections to D-branes are studied.

In Chapter 2, we introduce basic concepts and ingredients in understanding the string dualities which will be used frequently in later chapters. We introduce perturbative string theories and their T-duality properties. D-branes are introduced via T-dual transformation of open strings. Orientifold and F-theory are introduced in simple terms, referring more technical details to later chapters. We briefly discuss world volume actions of super D-branes and super M-branes and emphasize the importance of kappa symmetry in constructing supersymmetric world volume actions.

In Chapter 3, we carry out the construction of a specific orientifold in six dimensions with N=1 supersymmetry[32]. The closed-string sector in the resulting theory contains nine tensor multiplets and twelve neutral hypermultiplets, in addition to the gravity multiplet, which is anomaly-free by itself. The open-string sector contains only 5-branes and gives rise to maximal gauge groups $SO(16)$ or $U(8) \times U(8)$ at different points in the moduli space. The detailed tadpole calculation is presented. Anomalies are canceled by a generalization of the Green-Schwarz mechanism that involves more than one tensor multiplets. Works from Chapter 3 to Chapter 5 are in collaboration with Atish Dabholkar.

In Chapter 4, we continue to construct other orientifold models and discuss their duals[33]. In six dimensions we obtain models with $N = 1$ supersymmetry, multiple tensor multiplets, and different gauge groups. In nine dimensions we obtain a model that is dual to M-theory compactified on a Klein bottle.

In Chapter 5, an orientifold of Type-IIB theory on a T^4/Z_2 orbifold is constructed which corresponds to F-theory compactification on a Calabi-Yau orbifold with Hodge numbers $(51, 3)$[34]. The T-dual of this model is analogous to an orbifold with discrete torsion in that the action of orientation reversal has an additional phase on the twisted sectors, and both 9-branes and 5-branes carry orthogonal gauge groups. An orientifold of the Z_3 orbifold and its relation to F-theory is briefly discussed.

In Chapter 6, we study orientifold and F-theory duals of CHL strings –heterotic string theories with maximal symmetry but with gauge groups of reduced rank[35]. In eight dimensions, the compact space of the dual orientifold is a Möbius strip. We present the six-dimensional F-theory duals of CHL strings and explain how non-simply laced gauge groups arise in F-theory. Other N=2 F-theory vacua in six dimensions are discussed.

In Chapter 7, we present six-dimensional world-volume action that describes the dynamics of the M theory five-brane in a flat eleven-dimensional space-time background[146]. The world-volume action has global eleven-dimensional super-Poincaré invariance, as well as six-dimensional general coordinate invariance and kappa symmetry. Primarily, we consider a formulation in which general coordinate invariance is not manifest in one direction. However, we also describe briefly an alternative formulation, due to Pasti, Sorokin, and Tonin, in which general coordinate invariance is manifest.

Finally, in Chapter 8, dual super Dp-brane actions are constructed by carrying out a duality transformation of the world-volume $U(1)$ gauge field[118]. The resulting world-volume actions, which contain a $(p-2)$-form gauge field, are shown to have the expected properties. Specifically, the D1-brane and D3-brane transform in ways that can be understood on the basis of the $SL(2, Z)$ duality of Type IIB superstring theory. Also, the D2-brane and the D4-brane transform in ways that are expected on the basis of the relationship between Type IIA superstring theory and eleven-dimensional M-theory. Especially, the dual D4-brane action is shown to coincide with the double-dimensional reduction of the M5-brane action. Chapter 7 and Chapter 8 are the result of collaborations with Mina Aganagic, Costin Popescu and John H. Schwarz.

Chapter 3,4,7,8 of this thesis are a recollection of papers published in Nuclear Physic B and Chapter 5 is a recollection of the paper published in Physics Letter B. We acknowledge both publishers for the permission of inclusion of those papers in this thesis.

Chapter 2 Prologue on String Dualities

2.1 Overview of String Dualities

String theory is the primary candidate for the unified theory of fundamental interactions including quantum gravity. In 1984-85 there was a series of discoveries[15, 5, 6] that convinced many theorists that string theory is a very promising approach to unification. Ever since then, the subject has remained as the most active area of theoretical physics[2]. It is known for a decade or so that there are five different superstring theories, which have a consistent perturbation expansion. For a consistency of these theories, spacetime should have ten dimensions. The five theories are denoted Type I, Type IIA, Type IIB, $E_8 \times E_8$ heterotic string theory and $SO(32)$ heterotic string theory. The Type II theories have N=2 spacetime supersymmetries in ten dimensions and the others have N=1 supersymmetry. The Type I theory is special in that it is based on unoriented closed strings and open strings, while the others are based on the closed oriented strings.

A string's spacetime history is described by functions $X^{\mu}(\sigma, \tau)$, which map the string's two-dimensional world-sheet (σ, τ) into spacetime X^{μ}. This two-dimensional quantum field theory should be conformally invariant in order to describe classical string dynamics. Perturbative quantum string theory can be formulated by the Feynman sum-over-histories approach. For closed string theory, an n-loop string theory Feynman diagram corresponds to a genus n Riemann surface. A difference from the field theory is that there is just one Feynman diagram at each order of perturbation. In Type I theory, besides oriented Riemann surfaces, unoriented surfaces such as the Klein bottle enter into the perturbation expansion.

Even though perturbative string theory leads to much success and can produce string vacua whose matter contents are close to the realistic world, it was soon clear that much of the important problems, such as an understanding of supersymmetry breaking, is beyond the perturbative framework and needs an understanding of nonperturbative phenomena of string theory. Initially this appeared not to be available in the near future. Surprisingly, another giant step was just around the corner! Around 1994, we began to understand much of the nonperturbative aspects of string theories, thus we came to understand the strong couping behavior of string theories. Since we do not have a nonperturbative definition of string theory[1] at this time, it's not possible to prove the dualities in the present knowledge of string theory. However there is mounting evidence for various duality conjectures and they form an intricate web of consistent structures. One striking consequence is that what we viewed previously as five theories, is in fact five different perturbative expansions of a single underlying theory about five different points. The unique theory underlying five superstring theories is called M-theory. One interesting fact of M-theory is that it is an eleven-dimensional theory and it's low energy theory is described by eleven-dimensional supergravity. Before string dualities, eleven-dimensional supergravity was just a curiosity to string theoriests. But now this mysterious theory finds a way to fit in the whole picture of string dualities.

Even before the advent of string dualities, we knew that some string theories are connected. This connection is provided by another symmetry of string theory, T-duality, which connects one string theory with a small radius of compactification to another string theory with a large radius of compactification. Since the T-duality is a perturbative symmetry with respect to the string coupling, this symmetry was well understood before the era of string dualities. By T-duality, IIA theory compactified on a circle in a small radius limit is equivalent to IIB theory compactified on a circle in the large radius limit[9]. And a similar relation holds between $E_8 \times E_8$ heterotic string theory and $SO(32)$ heterotic string theory[3]. Now we are left with Type II, Heterotic and Type I theory. For further connections between these string theories, we need strong-weak coupling duality. The strong coupling limit of Heterotic $SO(32)$ theory is given by the weak coupling limit of Type I theory. The strong coupling

[1]There is some progress in this direction. Matrix theory is one proposal for the nonperturbative definition of string theory[137]

limit of Type IIA is described by M-theory compactified on a circle. In this case, the coupling of Type IIA theory λ is related to the radius of compactified circle of M-theory by $\lambda \sim R^{\frac{2}{3}}$, where R denotes the radius of the compactified circle of M-theory. Thus the weak coupling limit $\lambda \rightarrow \infty$ of Type IIA theory corresponds to the small radius limit of M-theory, and we can only see ten dimensions instead of the full eleven dimensions. Heterotic and Type I theory also have an eleven-dimensional origin, whose salient features will be reviewed in Chapter 6. Thus all known five string theories are merged into a single M-theory.

Another interesting fact is that the strong coupling limit of Type IIB theory is the Type IIB theory, itself. In this case, strong-weak coupling duality can be seen as a discrete gauge symmetry. There is much evidence that the duality group of Type IIB theory is $SL(2,Z)$. Interestingly enough, the $SL(2,Z)$ symmetry of Type IIB is geometrized as a 2-torus of M-theory if we consider Type IIB theory on a circle[144]. Another interesting geometrization of $SL(2,Z)$ symmetry of Type IIB is F-theory[46], which leads to many nonperturbative vacua of Type IIB theory which were not previously available. We will discuss F-theory in more detail in later chapters.

One important lesson of the string dualities is that solitonic objects are on an equal footing with the perturbative strings. As the coupling gets larger in one string theory, solitonic objects become light and dominant in a low energy limit. Indeed, in some cases special solitonic states in one theory are mapped to the perturbative string states in the dual theory. One special type of soliton, called D-brane[56], has attracted much attention. D-branes are solitonic objects carrying special types of charges. One advantage of D-branes over the other solitonic objects is that we know the underlying conformal field theory of D-branes, which implies that we have precise understanding of the excitations of D-branes. A useful application of D-branes is the construction of string vacua with open string sectors, called the orientifold construction. This is one of the main themes of this thesis.

2.2 Worldsheet Properties of Closed String

The action for the free bosonic string in the conformal gauge is

$$\frac{1}{2\pi\alpha'}\int_M d^2\sigma\partial_\alpha X^\mu\partial^\alpha X_\mu, \tag{2.1}$$

where α denotes the world sheet indices and μ denotes the spacetime indices. The string tension T is given by $T = \frac{1}{2\pi\alpha'}$. There are two types of boundary conditions that we will have to consider, corresponding to closed strings and open strings. Closed strings are topologically equivalent to circles and the appropriate boundary condition is the periodicity of the coordinates

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau). \tag{2.2}$$

The equations of motion are simply wave equations and the most general solution compatible with the boundary condition is $X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$ with:

$$\begin{aligned} X^\mu_R(\tau-\sigma) &= \frac{1}{2}x^\mu + \alpha' p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)} \\ X^\mu_L(\tau+\sigma) &= \frac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}. \end{aligned} \tag{2.3}$$

After the usual canonical quantization

$$[x^\mu, p^\nu] = i\eta_{\mu\nu} \tag{2.4}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = (m+n)\delta_{m+n}\eta^{\mu\nu}, \tag{2.5}$$

we get the mass spectrum

$$\begin{aligned} M^2 = -p^\mu p_\mu &= \frac{4}{\alpha'}(N-1) \\ &= \frac{4}{\alpha'}(\bar{N}-1), \end{aligned} \tag{2.6}$$

where N is the total level of the left oscillator mode excitations and $\bar{N}$ is the total level of the right mode excitations; -1 is the contribution from the zero point energy. [2] There's a level matching between left-mover and right-mover. Note that the lowest level is the tachyon due to the zero point energy contribution.

An interesting thing happens when we compactify on a circle, say, in the x^{25}-direction with radius R, *i.e.*, $x^{25} \equiv x^{25} + 2\pi nR$ where n is an arbitrary integer. X^{25} has additional zero modes,

$$X^{25} = x^{25} + 2\alpha' p\tau + 2L\sigma + \text{oscillators} \tag{2.7}$$

with $p = \frac{m}{R}, L = nR$. The restriction to integer m is needed so that the quantum wave function $e^{ip\cdot x}$ is invariant under $x^{25} \to x^{25} + 2\pi R$. The integer n is the number of times the string wraps around the circle (winding modes). The mode expansion can be decomposed into a sum of left-mover and right-mover

$$\begin{aligned}
X^{25}(\sigma,\tau) &= X_R^{25}(\tau-\sigma) + X_L^{25}(\tau+\sigma) \\
X_R^{25}(\tau-\sigma) &= x_R^{25} + \sqrt{\frac{\alpha'}{2}}\alpha_0^{25}(\tau-\sigma) + \text{oscillators} \\
X_L^{25}(\tau+\sigma) &= x_L^{25} + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^{25}(\tau+\sigma) + \text{oscillators}
\end{aligned} \tag{2.8}$$

with

$$\begin{aligned}
\alpha_0^{25} &= \frac{2m}{R}\sqrt{\frac{\alpha'}{2}} + \sqrt{\frac{2}{\alpha'}}nR \\
\tilde{\alpha}_0^{25} &= \frac{2m}{R}\sqrt{\frac{\alpha'}{2}} - \sqrt{\frac{2}{\alpha'}}nR.
\end{aligned} \tag{2.9}$$

[2] A boson with periodic boundary conditions has zero point energy $-\frac{1}{24}$, and with antiperiodic boundary conditions it is $\frac{1}{48}$. For fermions, there is an extra minus sign. For the bosonic string in 26 dimensions, there are 24 transverse (physical) degrees of freedom.

Turning to the mass spectrum, we have

$$\begin{aligned} M^2 = -p^\mu p_\mu &= \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) \\ &= \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\bar{N}-1). \end{aligned} \tag{2.10}$$

Here μ runs only over the non-compact dimensions. The mass spectra of the theories at radius R and α'/R are identical with the winding and Kaluza-Klein modes interchanged $(m \leftrightarrow n)$ which takes

$$\begin{aligned} \alpha_0^{25} &\rightarrow \alpha_0^{25} \\ \tilde{\alpha}_0^{25} &\rightarrow -\tilde{\alpha}_0^{25}. \end{aligned} \tag{2.11}$$

The interactions are identical as well[9]. Write the radius-R theory in terms of

$$X'^{25}(\sigma,\tau) = X^{25}(\tau-\sigma) - X^{25}(\tau+\sigma). \tag{2.12}$$

The energy-momentum tensor and OPE and therefore all of the correlation functions are invariant under this rewriting. The only change is that the zero mode spectrum in the new variable is that of the α'/R theory. The T-duality is therefore an exact symmetry of perturbative closed string theory. Note that it can be regarded as a spacetime parity transformation acting only on the right-moving degrees of freedom. One can further argue that T-duality is an exact symmetry of the closed string theory[4].

2.3 Open String, T-dualities and D-branes

Now consider the open string and it's transformation properties under the T-dualities[9, 59]. Topologically, an open string is an interval. Let σ parametrize the interval and run from 0 to π. The variation of the action (2.1) becomes, after integration by parts,

$$\delta S = -\frac{1}{2\pi\alpha'}\int_M d^2\sigma \delta X^\mu \partial^2 X_\mu + \frac{1}{2\pi\alpha'}\int_{\partial M} d\sigma (\delta X^\mu \partial_n X_\mu) \tag{2.13}$$

where ∂_n is the derivative normal to the boundary. The only Poincaré invariant boundary condition is the Neumann condition $\partial_n X_\mu = 0$. The Dirichlet condition $X^\mu =$ constant is also consistent with the equations of motion and we will return to the Dirichlet boundary condition shortly. From the first term of (2.1), the equation of motion is given by the wave equation again. The mode expansion is given by

$$\begin{aligned} X_R^\mu(\tau-\sigma) &= \frac{1}{2}x^\mu + \alpha' p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)} \\ X_L^\mu(\tau+\sigma) &= \frac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau+\sigma)}. \end{aligned} \tag{2.14}$$

If we compactify on a circle in the x^{25}-direction, the momentum modes in this direction are quantized, *i.e.*, $p^{25} = \frac{n}{R}$. Here the usual compactified coordinate is $X^{25} = X_R^{25} + X_L^{25}$. In order to understand the T-dual transformation, we rewrite the theory in terms of the dual variable

$$\begin{aligned} X'^{25} = X_R^{25} - X_L^{25} &= 2\alpha' p^{25}\sigma + \text{oscillators} \\ &= 2\alpha'\frac{n}{R}\sigma + \text{oscillators}. \end{aligned} \tag{2.15}$$

The oscillator terms vanish at the endpoints $\sigma = 0, \pi$. Notice that there is no dependence on τ in the zero modes. Therefore the endpoints of the string do not move in the X^{25} direction. We could also see this directly, from the boundary condition $\partial_n X^{25} = \partial_t X'^{25} = 0$. At the ends,

$$\begin{aligned} \sigma = 0: \qquad X'^{25} &= 0; \\ \sigma = \pi: \qquad X'^{25} &= 2\pi\alpha' p^{25} \\ &= 2\pi n R'. \end{aligned} \tag{2.16}$$

This means that in the dual theory (with radius $R' = \alpha'/R$) the ends of the open strings are located for all time at position $X'^{25} = 0$. They can wind n times around the spacetime circle, and they are free to move in the other directions. Thus under the T-duality transformation, a Neumann boundary condition turns into the Dirichlet boundary condition and Kaluza-Klein modes are mapped to the winding modes. The hypersurface $X'^{25} = 0$ becomes a dynamical object and is called D-brane. It is natural to expect that such a hypersurface is dynamical, since closed strings can interact with the D-brane via open strings, therefore the hypersurface feels the effect of gravity in the closed string massless sector. In a later section, we will see that there are massless open string excitations propagating on the D-brane, the T-duals of the photons, with precisely the properties of the collective coordinates for the transverse fluctuations of the D-brane.

2.4 Superstring

2.4.1 Open Superstring

We can supersymmetrize the bosonic action by adding fermionic degrees of freedom. Interestingly enough, world sheet supersymmetry gives spacetime supersymmetry after a truncation of the spectrum. The superstring action is given by

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu). \tag{2.17}$$

Here ρ^α is the 2-dimensional gamma matrix satisfying

$$\{\rho^\alpha, \rho^\beta\} = -2\eta_{\alpha\beta}. \tag{2.18}$$

As the bosonic variables X^μ are decomposed into left-mover and right-mover, so are the fermionic variables. The fermionic part of the action can be written

$$S_F = \frac{1}{\pi\alpha'}\int d^2\sigma(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+) \tag{2.19}$$

where ψ_+ and ψ_- denote the right-mover and left-mover, respectively. In order to obtain Euler-Lagrangian equation under the variation, we require that the surface terms $\psi_+\delta\psi_+ - \psi_-\delta\psi_-$ vanish. There are two possible boundary conditions to be imposed. We can set $\psi^\mu_+(0,\tau) = \psi^\mu_-(0,\tau)$ without loss of generality. The possible two boundary conditions are

$$\begin{aligned} \mathrm{R} \quad &: \quad \psi^\mu_+(\pi,\tau) = +\psi^\mu_-(\pi,\tau) \\ \mathrm{NS} \quad &: \quad \psi^\mu_+(\pi,\tau) = -\psi^\mu_-(\pi,\tau). \end{aligned} \tag{2.20}$$

In the NS sector, the fermionic oscillators are half-integer moded, giving a ground state energy of

$$(-\frac{8}{24}) + (-\frac{8}{48}) = -\frac{1}{2} \tag{2.21}$$

from the eight transverse coordinates and eight transverse fermions. The ground state is a Lorentz singlet and has odd fermion number, $(-1)^F = -1$. The GSO projection onto states with even fermion number, removes the open string tachyon from the spectrum and makes the entire spectrum supersymmetric. Massless particle states in ten dimensions are classified by their $SO(8)$ representation under Lorentz rotations which leave the momentum invariant. The lowest lying states in the NS sector are the eight transverse polarizations of the massless open string photon A^μ forming the vector of $SO(8)$, $\mathbf{8}_v$.

The fermionic oscillators in the Ramond sector are integer moded. In the R sector the ground state energy always vanishes because the world-sheet bosons and their supersymmetry partners have the same moding. The Ramond vacuum is degenerate since ψ^μ_0 take ground states into ground states[3] and they form a representation of the Clifford algebra in ten dimensions, hence they are spacetime spinors. The ground state is the spinor of $SO(8)$ and the GSO projection corresponds to the chirality projection in this case. $SO(8)$ has two spinor representations $\mathbf{8}_s$ and $\mathbf{8}_c$. Thus there are two possibilities for the GSO projection. The two choices are equivalent and

[3] ψ^μ_{-n} denotes the n-th mode of the oscillators

we pick up $\mathbf{8}_s$. The ground state spectrum is then $\mathbf{8}_v \oplus \mathbf{8}_s$, a vector multiplet of $D = 10$, $N = 1$ spacetime supersymmetry. Including Chan-Paton factors, which will be discussed later, gives a $U(n)$ gauge theory in the oriented theory and $SO(n)$ or $USp(n)$ in the unoriented theory.

2.4.2 Closed Superstring

For closed superstrings, we can impose periodic or anti-periodic boundary conditions for left-mover and right-mover separately. There are four distinct closed superstring sectors, which are called NS-NS, NS-R, R-NS, R-R sectors. For each R or NS sector, the spectrum would be the same as that of an open superstring. Thus we can think of the closed string spectrum as the tensor product of two copies of the open string spectrum. As in the bosonic case, there should be a level matching between left-mover and right-mover. In the open string the two choices for the GSO projection were equivalent, but in the closed string there are two inequivalent choices, taking the same (IIB) or opposite (IIA) projections on the two sides. These lead to the massless sectors

$$\begin{array}{ll} \text{Type IIA} & (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \\ \text{Type IIB} & (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \end{array} \tag{2.22}$$

of $SO(8)$.

The various products are as follows: In the NS-NS sector, this is

$$\mathbf{8_v} \otimes \mathbf{8_v} = \phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}. \tag{2.23}$$

In the R-R sector, the IIA and IIB spectra are respectively

$$\begin{aligned} \mathbf{8_s} \otimes \mathbf{8_c} &= [1] \oplus [3] = \mathbf{8_v} \oplus \mathbf{56_t} \\ \mathbf{8_s} \otimes \mathbf{8_s} &= [0] \oplus [2] \oplus [4]_+ = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+. \end{aligned} \tag{2.24}$$

Here $[n]$ denotes the n-times antisymmetrized representation of $SO(8)$, with $[4]_+$ being self-dual. Note that the representations $[n]$ and $[8-n]$ are the same, being related by contraction with the 8-dimensional ϵ-tensor. In the Type IIA case, we have odd-rank tensors of $SO(8)$ but even-rank tensors of $SO(9,1)$, the extra index being contracted with the momentum to form the field strength (and reversed in Type IIB).

The NS-NS and R-R spectra together form the bosonic components of $D = 10$ IIA (nonchiral) and IIB (chiral) supergravity, respectively. In the NS-R and R-NS sectors are the products

$$\begin{aligned} \mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_c} \\ \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_s}. \end{aligned} \tag{2.25}$$

The $\mathbf{56_{s,c}}$ are gravitinos, their vertex operators having one vector and one spinor index. They must couple to conserved spacetime supercurrents. In the IIA theory the two gravitinos (and supercharges) have opposite chirality, and in the IIB then the same chirality.

As we discussed before, T-duality is a one-sided parity transformation. For superstrings, this transformation flips the relative chiralities of the left-moving and right-moving ground state. Thus an odd number of T-dual transformations maps Type IIA to Type IIB and vice versa, while an even number of the transformations maps Type IIA or Type IIB to themselves[9].

2.5 D-branes in Type II Theory

Now consider D-branes in Type II theory. Away from the D-brane we see only closed string spectrum with two d=10 gravitinos. However, worldsheet boundaries reflect one supercharge into the other. So only one combination of the two supercharges is a good symmetry of the full state. In other words, in the Type II theory coupled to the D-brane, half of the supersymmetry of the bulk theory is broken, which implies that the D-brane is a BPS state[56]. BPS states must carry conserved charges. In the present case there is only one set of charges with the correct Lorentz properties, namely the antisymmetric R-R charges. The world volume of a p-brane naturally couples to a (p+1)-form potential A_{p+1}, which has a (p+2)-form field strength F_{p+2}. Thus allowable D-branes are p-branes with p even in IIA and p odd in IIB.

Now consider the behavior of a D-brane under the T-dual transformation. Take the T-dual transformation in a direction μ perpendicular to the D-brane. The Dirichlet condition becomes Neumann, so in dual theory this becomes a (p+1)-brane. One can check that the R-R potential acquires an extra index, as needed to couple to (p+1)-brane[59]. Similarly, if we take the T-dual in a direction m along the brane, it becomes a (p-1) brane and the R-R potential loses its m index.

Once a D-brane is introduced, we can have open strings whose ends are at any value on the hypersurface of the D-brane. This open string describes the excitations of the D-brane. The quantization of the open string is isomorphic to the conventional quantization of an oriented open superstring. Specifically consider a D-brane located at $x_{p+1} = \cdots = x_9 = 0$. The massless states are a vector and a spinor making up a ten-dimensional supersymmetric Yang-Mills multiplet with gauge group $U(1)$. As the zero modes of x^j, $j > p$ are eliminated by the boundary conditions, the massless particles are functions only of $x^0, \ldots, x^p$. The massless bosons $A_i(x^s)$, $i, s = 0, \ldots p$ propagate as a $U(1)$ gauge boson on the p-brane world-surface, while the other components $\phi_j(x^s)$ with $j > p$, $s = 0, \ldots, p$, are scalars in the $p+1$-dimensional sense. Note that the vectors have conventional open-string gauge boson vertex operators $V_A = \sum_{i=0}^{p} A_i(X^s)\partial_\tau X^i$, with ∂_τ the derivative tangent to the world-sheet boundary, while the scalars have vertex operators of the form $V_\phi = \sum_{j>p} \phi_j(X^s)\partial_\sigma X^j$ with ∂_σ the normal derivative to the boundary. For $\phi_j =$ constant, the boundary integral of V_ϕ is the change in the world-sheet action upon adding a constant to X^j, $j > p$, so the scalars can be interpreted as oscillations in the position of the $p+1$-brane. The theory on the $p+1$-dimensional world-volume is naturally thought of as the ten-dimensional $U(1)$ supersymmetric gauge theory dimensionally reduced to $p+1$ dimensions[133]. Similarly if we have coincident N D-branes, the low-energy theory is described by the dimensional reduction of the $U(N)$ gauge theory in ten dimensions. The open string modes connecting different branes become massless in the coincident limit and they provide additional massless charged states for the enhanced gauge symmetry. Note that the motion of the D-brane corresponds to the Wilson line of the underlying gauge theory. Depending on its motion, it can break or enhance the gauge symmetry.

From the above argument, we can see that the open string has additional degrees of freedom in addition to the world sheet fields. These are the Chan-Paton degrees of freedom indicating on which D-branes open strings end. Thus one can write the open string state $|\psi, ij>$ where ψ is the worldsheet fields and i, j denote D-branes on which open strings end. This geometrization of the Chan-Paton factor in terms of a D-brane is quite useful, especially when one considers the string compactification with open strings.

2.6 Orientifolds

There is an additional symmetry of Type IIB theory, which is called the worldsheet orientation reversal symmetry. Since the left-moving modes of the Type IIB are isomorphic to the right-moving modes, there is a Z_2 symmetry which interchanges the left-mover and right-mover. Using this orientation reversal symmetry, we can construct orientifolds which are generalizations of orbifolds. In the orbifolds only the discrete internal symmetries are gauged, while in orientifolds products of internal symmetry and the world-sheet parity reversal symmetry are gauged. It is easy to undestand the closed string sector. We just keep the closed string states which are invariant under the orientifold symmetry. Orientifolding introduces unoriented surfaces in the closed string perturbation theory. For example, the torus turns into the Klein-bottle under the orientation reversal. Unoriented surfaces have tadpoles in R-R fields in the closed string channel. The tadpoles can be canceled by including the right number of D-branes that couple to these R-R fields. These D-branes provide the Chan-Paton factors for the open string sector. In the next chapter, we will carry out the detailed tadpole calculations and the determination of the string spectrum for a specific model, but general strategy is the same.

It is interesting to see the T-dual picture of the world-sheet orientation reversal. For closed strings, the original coordinate is $X^m(\sigma,\tau) = X_R^m(\tau-\sigma) + X_L^m(\tau+\sigma)$ and the dual coordinate is $X'^m = X_R^m(\tau-\sigma) - X_L^m(\tau+\sigma)$. The action of the world sheet parity reversal is to exchange X_R^μ and X_L^μ. In terms of the dual coordinates, this is

$$X_R'^m \leftrightarrow -X_L'^m \tag{2.26}$$

which is the product of a world-sheet and spacetime parity operation. Hence the T-dual action introduces a fixed plane in spacetime, which is called the orientifold plane. By calculating tadpoles, one can show that the orientifold plane is also a source for the R-R fields. But in the compact space these fields have nowhere to go. By introducing the right number of the D-brane, we can make the total R-R charge vanish in compact space. Thus the tadpole cancellation is equivalent to the cancellation of R-R charge cancellation in compact space.

One typical example of the orientifold is Type I theory, which is obtained from Type IIB theory by gauging orientation reversal symmetry Ω. Projecting onto $\Omega = +1$ interchanges left-moving and right-moving oscillators and so one linear combination of the R-NS and NS-R gravitinos survives, leaving $D = 10$, $N = 1$ supergravity. In the NS-NS sector, the dilaton and graviton are symmetric under Ω and survive, while the antisymmetric tensor is odd and is projected out. In the R-R sector, it is clear by counting that the $\mathbf{1}$ and $\mathbf{35}_+$ are in the symmetric product of $\mathbf{8_s} \otimes \mathbf{8_s}$ while the $\mathbf{28}$ is in the antisymmetric. The R-R vertex operator is the product of right- and left-moving fermions, so there is an extra minus in the exchange and it is the $\mathbf{28}$ that survives. The bosonic massless sector is thus $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$, the $D = 10$ $N = 1$ supergravity multiplet. The tadpole cancellation requires 32 9-branes and the gauge group is reduced to SO(32) by requiring the orientation reversal symmetry.

Now if we compactify on a circle in the x^9-direction, $x_9 \equiv x_9 + 2\pi$ and T-dualize. Then from the previous discussion, Ω is mapped to $R_9\Omega$ where R_9 is the parity operation in the x^9-direction. The resulting theory is a Type IIA orientifold and has two orientifold planes at $x^9 = 0$ and $x^9 = \pi$. Again we need 32 8-branes from the tadpole cancellation. If we put 16 8-branes at $x^9 = 0$ and the other 16 8-branes at $x^9 = \pi$, we have the gauge group $SO(16) \times SO(16)$. In this configuration, dilaton tadpole cancellation as well as R-R charge cancellation occurs locally, and the string coupling constant remains constant throughout the space.

2.7 F-theory

F-theory refers to a new way of compactifying Type-IIB theory in which the complex coupling λ of Type-IIB theory is allowed to vary over space. The coupling is given by $\lambda = \xi + ie^{-\phi}$ where ϕ is the dilaton from the NS-NS sector and ξ is the R-R scalar. Consider an elliptically fibered Calabi-Yau manifold K which is a fiber bundle over a base manifold B with a torus as a fiber whose complex structure parameter is τ. Even though K is a smooth manifold, there will be points in the base manifolds where the fiber becomes singular, and the parameter τ can have a nontrivial $SL(2, Z)$ monodromy around these points. An F-theory compactification on K refers to a compactification of Type-IIB theory on B, where the coupling λ is identified with τ. The nontrivial monodromy of λ around the singular points then means that there are 7-branes at those points that are magnetically charged with respect to the scalar λ. Typically, the base manifold is not Ricci-flat and moreover, because λ is varying, there is a nonvanishing RR background. These backgrounds cannot, therefore, be described using conformal field theory. For special choices of the manifolds K, however, an F-theory compactification is equivalent to a perturbative Type-IIB orientifold. This follows from an observation due to Sen [40] that the element -1 of $SL(2, Z)$ which is not an element of $PSL(2, Z)$ is a perturbative symmetry of Type-IIB. It flips the sign of the two 2-form fields B^1_{MN} and B^2_{MN}, but leaves all other massless fields, in particular, the coupling fields λ invariant. From its action on the massless fields it is easy to check that this element represents the action of $\Omega(-1)^{F_L}$ where Ω is orientation reversal on the worldsheet and F_L is the spacetime fermion number of the left-movers.

Here we consider the simplest example considered by Sen[40], where K is a $K3$

surface that is a Z_2 orbifold of a four-tours; Let z_1, z_2 denote the complex coordinates of T^4. The Z_2 symmetry acts as $z_1 \rightarrow -z_1$ and $z_2 \rightarrow -z_2$. Here z_1 parametrizes the elliptic fiber and z_2 parametrizes the base. Since the elliptic fiber represents the complex coupling of Type IIB theory, in this orbifold limit the coupling remains constant on the base. The base is T^2/Z_2 and there are four fixed points of the Z_2 action. Around each fixed point, there is a $SL(2, Z)$ monodromy -1. Thus we have the Type IIB compactification on T^2/Z_2 such that as we go once around each fixed point, the theory comes back to itself transformed by the symmetry $(-1)^{F_L} \cdot \Omega$. In other words, the theory can be identified to Type IIB on T^2, modded out by the Z_2 transformation $(-1)^{F_L} \cdot \Omega \cdot I_2$ where I_2 denotes the Z_2 transformation $z_2 \rightarrow -z_2$. This is a T-dual of Type I theory to be discussed in Chapter 4. In this way, we established the duality between F-theory on $K3$ and the Type I theory. Once the duality is established at one point of moduli space, one can argue the equivalence at other points of the moduli space, since we can deform both theories away from this specific point by turning on suitable background fields.

In the case at hand, we are considering the F-theory with the coupling remaining constant throughout the base, and we expect the same nature in the corresponding T-dual of Type I theory. This implies that tadpole cancellation occurs locally so that there are no dilaton gradients on the base. There are four orientifold planes and we should put eight 7-branes at each orientifold plane; the resulting gauge group is $SO(8)^4$. It is interesting to see how the corresponding gauge group arises in F-theory. We have four singular fibers on the base. Possible singularities of elliptic fibers are classified by Kodaira[7]. These singularities fit into the ADE classification. In our example at hand, the singularity type is $D_4 \sim SO(8)$. This suggests that the ADE type singularities will give ADE gauge groups in F-theory. This can be further checked in various cases[54, 55, 71].

One can consider the F-theory compactification in lower dimensions. F-theory on elliptic threefolds provides many interesting models in six dimensions. We will see some aspects of F-theory in 6-dimensions in Chapters 5 and 6.

2.8 Super D-brane and Super M-brane

Solitonic objects play an essential role in string dualities and a better understanding of string dualities comes from a better understanding of the quantum properties of those solitonic objects. A striking example is the D-brane. Since we know the underlying conformal field theory of D-branes, we can have a clearer view on the duality aspects related to the R-R charged objects. One way to understand a solitonic object is to understand its world-volume theory. In the case of strings in a flat background, the world-volume theory has been quantized and used to construct the string perturbation expansion. In the case of p-branes with $p > 1$, such things have not been available. Still it is worthwhile to study the world-volume theory of solitonic objects and to extract the useful information on these. Since much of the recent development of the string dualities focuses on D-branes, it is interesting to study the world-volume theory of D-branes. It has been known for some time that D-brane world volume theory contains a $U(1)$ gauge field, whose self interactions are described by the Born-Infeld-type theory[96]. More precisely speaking, since many of D-branes in string dualities are supersymmetric objects, we should have the supersymmetric world-volume theory of D-branes. Such constructions are given in [89, 90, 91].

The field content describing the world-volume theory of super D-branes consists of the superspace coordinates (X^m, θ) in ten dimensions and an abelian vector gauge field A_μ where μ runs through the world-volume indices of the D-brane. The world-volume theory has the global IIA or IIB super-Poincaré symmetry. In addition, they have world-volume general covariance, which ensures that only the transverse components of X^m are physical. One crucial ingredient of the super D-brane is additional fermionic symmetry, called kappa symmetry. This symmetry eliminates half of the component of θ so that the physical degrees of freedom of bosons and fermions match. In order to ensure kappa invariance, we should introduce a Wess-Zumino term. The action of the D-brane can be written as $S = S_1 + S_2$ where

$$S_1 = \int d^{(p+1)}\sigma\sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})} \tag{2.27}$$

$$S_2 = \int \Omega_{p+1}. \tag{2.28}$$

Here $G_{\mu\nu}, \mathcal{F}_{\mu\nu}$ are suitable supersymmetric generalizations of the induced metric and the gauge field on the world volume and Ω_{p+1} is a $(p+1)$-form representing the Wess-Zumino term. The kappa symmetry variation is given by

$$\delta S_1 = \int d^{p+1}\sigma \, 2\delta\bar{\theta}\gamma^{(p)} T^{\nu}_{(p)} \partial_\nu \theta \tag{2.29}$$

$$\delta S_2 = \int d^{p+1}\sigma \, 2\delta\bar{\theta} T^{\nu}_{(p)} \partial_\nu \theta, \tag{2.30}$$

so that

$$\delta S = \int d^{p+1}\sigma \, 2\delta\bar{\theta}(1+\gamma^{(p)}) T^{\nu}_{(p)} \partial_\nu \theta. \tag{2.31}$$

Here $\gamma^{(p)}$ and $T^{\nu}_{(p)}$ are suitable expressions involving X^m, θ and $\mathcal{F}_{\mu\nu}$. We have $(\gamma^{(p)})^2 = 1$. Since $\frac{1}{2}(1 \pm \gamma^{(p)})$ are projection operators, $\delta\bar{\theta} = \bar{\kappa}(1-\gamma^{(p)})$ gives the desired symmetry. From this structure, one can see that the Wess-Zumino term is indeed essential in constructing kappa symmetry. We will encounter the more detailed form of the super D-brane action in Chapter 8.

Now turn to the solitonic objects in M-theory. The low energy effective action of the M-theory is eleven-dimensional supergravity and it contains a three-form potential. Thus M-2 brane couples to the three-form potential electrically, while M-5 brane couples magnetically. The world volume theory of the M-2 brane was constructed in [86] . The world volume theory of the M-5 brane is a six-dimensional theory whose massless content is N=2 tensor multiplet. One difficulty in formulating M-5 brane world volume theory is that N=2 tensor multiplet contains a second-rank self-dual tensor gauge field. It has been known for a long time that there is no straightforward way to construct a covariant action that describes propagation of the self-dual field. One way to circumvent this difficulty is to consider the theory with general covariance, but the general covariance is not manifest. Very recently, a manifestly covariant formulation involving only a finite number of auxiliary fields has been introduced[113]. Both approaches will be explained in due course, and the M-5 brane action will be presented in Chapter 7.

Chapter 3 An Orientifold of Type IIB Theory on K3

Theories of unoriented strings can be viewed as orientifolds [10, 12, 9] of oriented closed strings. Orientifolds are a generalization of orbifolds in which the orbifold symmetry includes orientation reversal on the worldsheet. For example, Type-I strings can be viewed as an orientifold of Type-IIB strings. It is obvious that the closed-string sector of unoriented strings can be obtained by projecting the spectrum of oriented strings onto states that are invariant under the orientifold symmetry. It is more difficult to see how and when the open string sector might arise, and in particular how to obtain the Chan-Paton factors. A proper understanding of this question has become possible only after the remarkable recent work on D-branes[56].

A D-brane is a submanifold where strings are allowed to end which corresponds to open strings that satisfy mixed Dirichlet and Neumann boundary conditions. In Type-II theories, D-branes represent non-perturbative extended states that are charged with respect to the R-R fields in the theory. D-branes provide a geometric understanding of how Chan-Paton factors arise: a Chan-Paton label is simply the label of the D-brane that an open string ends on.

One can now understand the open-string sector of an orientifold as follows. Orientifolding introduces unoriented surfaces in the closed-string perturbation theory. The unoriented surfaces such as the Klein bottle can have tadpoles of R-R fields in the closed string tree channel. The tadpoles can be canceled by including the right number of D-branes that couple to these R-R fields. This introduces the open string sector with appropriate boundary conditions and Chan-Paton factors.

With this enhanced understanding of orientifolds, one can contemplate more general constructions. In this paper we construct a simple orientifold of Type-IIB theory compactified on a $K3$ surface that has $N = 1$ supersymmetry in six dimensions. The orientifold symmetry group is $\{1, \Omega S\}$ where S is a Z_2 involution of $K3$ and Ω is orientation reversal on the worldsheet. The resulting closed string sector contains the gravity multiplet, nine tensor multiplets, and twelve neutral hypermultiplets. The maximal gauge group arising from the open string sector is $SO(16)$ with an adjoint hypermultiplet, or $U(8) \times U(8)$ with two hypermultiplets that transform as $(8, 8)$.

There are a number of motivations for considering this example. First, the requirement of anomaly cancellation in six dimensions is fairly restrictive and provides useful constraints on the construction of the worldsheet theory. In fact, this work was motivated in part by the observation [21] that anomalies cancel in a large class of supersymmetric models in six dimensions. The orientifold that we consider realizes one of these models as a string theory. Second, we obtain a massless spectrum that is markedly different from the only known string compactification to six dimensions with $N = 1$ supersymmetry *viz.* the heterotic string theory on $K3$, which has only one tensor multiplet. We thus have a new compactification with a moduli space that apparently is disconnected from the known compactifications. Finally, this orientifold is a useful practice case for various generalizations to different dimensions using other orientifold groups [33].

The organization of this chapter is as follows. In Section 2 we motivate the orientifold group from considerations of anomaly cancellation and describe the closed string sector. The open string sector is discussed in Section 3. Consistency requires inclusion of 32 Dirichlet 5-branes but *no* 9-branes, with additional constraints on the Chan-Paton factors that determine the gauge group and matter representations completely.

3.1 Gravitational Anomalies and the Orientifold Group

The massless representations of the $N = 1$ supersymmetry algebra in $d = 6$ are chiral; consequently their coupling to gravity is potentially anomalous. We would like to see what constraints are placed on the massless spectrum so that these anomalies cancel. We shall then use this information to see how such a spectrum may follow from a string compactification.

The massless states are labeled by the representations of the little group in six dimensions which is $SO(4) = SU(2) \times SU(2)$. The massless $N = 1$ supermultiplets are then as follows.

1. The gravity multiplet:

   a graviton (3, 3), a gravitino 2(2, 3), a self-dual two-form $(1,3)$.
2. The vector multiplet:

   a gauge boson (2, 2), a gaugino 2(1, 2).
3. The tensor multiplet:

   an anti-self-dual two-form $(3,1)$, a fermion $2(2,1)$, a scalar $(1,1)$.
4. The hypermultiplet:

   four scalars $4(1,1)$, a fermion $2(2,1)$.

The gravitino and the gaugino are right-handed whereas the fermions in the other two multiplets are left-handed. Up to overall normalization the gravitational anomalies are given by [19, 20]

$$\begin{aligned} I_{3/2} &= -\frac{43}{288}(trR^2)^2 + \frac{245}{360}trR^4, \\ I_{1/2} &= +\frac{1}{288}(trR^2)^2 + \frac{1}{360}trR^4, \\ I_A &= -\frac{8}{288}(trR^2)^2 + \frac{28}{360}trR^4. \end{aligned} \tag{3.1}$$

Here $I_{3/2}$, $I_{1/2}$, and I_A refer to the anomalies for the gravitino, a right-handed fermion, and a self-dual two-form $(1,3)$ respectively.

Consider n_V vector multiplets, n_H hypermultiplets and $n_T + 1$ tensor multiplets. Then the (trR^4) term cancels if the following condition is satisfied:

$$n_H - n_V = 244 - 29n_T. \tag{3.2}$$

The $(trR^2)^2$ term is in general nonzero, and needs to be canceled by the Green-

Schwarz mechanism [15]. There are many solutions of (3.2). We would now like to see which can be realized as a string theory.

There are not many possibilities for string vacua with $N = 1$ supersymmetry in six dimensions. For the heterotic string, we must compactify on a $K3$ to obtain $N = 1$ supersymmetry. This leads to $n_T = 0$ and $n_H = n_V + 244$. For Type-II strings, usual Calabi-Yau compactification on a $K3$ leads to $N = 2$ supersymmetry. One way to reduce supersymmetry further is to take an orientifold so that only one combination of the left-moving and the right-moving supercharges that is preserved by the orientation-reversal survives. By considering different orientifold groups one may obtain different spectra, and in particular different number of tensor multiplets.

The model that we consider in this chapter has $n_T = 8$ and $n_H - n_V = 12$ which clearly satisfies (3.2). The special thing that happens with this matter content is that the entire anomaly polynomial including the $(trR^2)^2$ term vanishes. We thus have anomaly cancellation without the need for the Green-Schwarz mechanism, analogous to what happens in the Type-IIB theory in ten dimensions [20], or in the chiral $N = 2$ theory obtained by compactifying Type-IIB theory on $K3$ [51].

If we wish to obtain a large number of tensor multiplets, a natural starting point for orientifolding is the Type-IIB theory compactified on $K3$, which has 21 ($N = 2$) tensor multiplets in the massless spectrum in addition to the gravity multiplet. The gravity multiplet contains 5 self-dual two-forms whereas the tensor multiplets contain one anti-self-dual two-form each. Let us recall how these two-forms arise. In ten dimensions the Type-IIB theory contains a two-form B^1_{MN} from the R-R sector, a two-form B^2_{MN} from the NS-NS sector and a four-form A_{MNPQ} from the R-R sector with self-dual field strength. Zero modes of these fields correspond to harmonic forms on $K3$ and give rise to massless fields in six dimensions [19] . The nonzero Betti numbers for $K3$ are $b_0 = b_4 = 1$, $b_2^+ = 3$, and $b_2^- = 19$ where b_2^+ are the self-dual two-forms and b_2^- are the anti-self-dual two-forms. From the two B_{MN} fields we get b_0 two-forms each, which means altogether 2 self-dual and 2 anti-self-dual two-forms. Similarly, from the zero modes of the A_{MNPQ} we get 3 self-dual and 19 anti-self-dual two-forms in six dimensions after imposing self-duality of field strength in ten dimensions.

The orientifold group can now be deduced as follows. In order to obtain $N = 1$ supersymmetry we need an orientation reversal Ω which takes σ to $\pi - \sigma$. A projection $(1 + \Omega)/2$ alone would give us the spectrum identical to the closed-string sector of Type-I theory on $K3$, eliminating A_{MNPQ} and B^2_{MN} completely from the spectrum. Now consider a Z_2 involution S of $K3$ such that eight anti-self-dual harmonic forms are odd under S and all other 16 forms are even. It is clear that under the projection $(1 + \Omega S)/2$, eight zero-modes of A_{MNPQ} will now survive, giving us 8 anti-self-dual two-forms. Moreover, we shall also get eight scalars from the zero modes of B^2_{MN} so that we have the complete bosonic content of eight tensor multiplets. We still have one zero mode of B^1_{MN} giving one self-dual and one anti-self-dual two-form. The self-dual two-form is needed for the gravity multiplet; the anti-self-dual two-form combines with the zero mode of the dilaton to form an additional tensor multiplet. Altogether, we obtain the nine tensor multiplets that we were after.

Let us see if we get the rest of the spectrum right. There are no vector multiplets because there are no odd cycles on $K3$, and starting with even forms and the metric in ten dimensions we can never get a one-form as a zero mode. The scalars arise from zero modes of the metric tensor and the B^1_{MN} field that are invariant under ΩS. Their zero modes can be found from the Dolbeault cohomology of $K3$ [19], so we need to know which (p, q) forms are left invariant by S. The main point for our purpose will be that the eight two-forms that are eliminated by S are $(1, 1)$ forms [1]. We are thus left with 12 $(1, 1)$ forms and 1 each of $(0, 2)$, $(2, 0)$, $(0, 0)$, $(2, 2)$ forms. The zero modes of g_{MN} give 34 scalars [19]. The number of zero modes of B^1_{MN} equals the number of harmonic two-forms which is 14. Altogether we have 48 scalars which make up 12 hypermultiplets. This construction ensures that the closed-string sector is anomaly free. We also get a constraint in the open-string sector that the number of vector multiplets must equal the number of hypermultiplets for canceling gravitational anomalies.

[1] For a smooth $K3$ defined by a quartic polynomial in CP^3, it is easy to construct an example of the involution S and verify this assertion [29].

To proceed further we need to know the spectrum in the open-string sector and check that all tadpoles vanish. For computing the tadpoles we need a realization of the $K3$ as an explicit worldsheet conformal field theory. Furthermore, we need to know how the involution S acts in this conformal field theory. This can be easily done for a particular $K3$ represented as a T^4/Z_2 orbifold. Let (z_1, z_2) be complex coordinates on the torus T^4 defined by periodic identifications $z_1 \sim z_1 + 1$, $z_1 \sim z_1 + i$, and similarly for z_2. The two Z_2 transformations of interest are generated by

$$\begin{aligned} R: &\qquad (z_1, z_2) \to (-z_1, -z_2) \\ S: &\qquad (z_1, z_2) \to (-z_1 + \frac{1}{2}, -z_2 + \frac{1}{2}). \end{aligned} \tag{3.3}$$

That S is the desired symmetry can be seen as follows. The $K3$ orbifold is obtained by dividing the torus by $Z_2^R \equiv \{1, R\}$. The Type-IIB theory on this orbifold has 5 self-dual and 5 anti-self-dual two-forms coming from the untwisted sector. In the twisted sector, there are 16 anti-self-dual forms from the 16 fixed points of R. Notice that S is the same as R acting on shifted coordinates $(z_1 - \frac{1}{4}, z_2 - \frac{1}{4})$. Now, S leaves all forms in the untwisted sector invariant, but takes 8 fixed points of R into the other 8. Thus of the anti-self-dual two-forms coming from the twisted sector, 8 are even under S, and 8 are odd. This is precisely the structure we wanted. Note that S has 16 fixed points on the torus, but on the orbifold they are identified under R leaving only 8 as required by the Lefschetz fixed-point theorem [17].

3.2 Open String Sector

3.2.1 Tadpoles

Tree-channel tadpoles can be evaluated by factorizing the partition function in the loop channel. For closed strings, the one-loop amplitude for the orientifold is obtained by projecting onto the closed string states of the Type-IIB theory on $K3$ that are invariant under the symmetry ΩS. The partition function now receives a contribution from the Klein bottle in addition to the torus. The torus has no closed-string tree channel and is modular invariant by itself, so we need to consider only the Klein bottle. To determine the open string sector we first require closure of operator product expansion so that the S-matrix factorizes properly. This implies that we can consistently add only 5-branes and 9-branes [61]. We then have $55, 99, 59, 95$ sectors for open strings from strings that begin and end on the two kinds of branes. The one-loop partition function is given by the cylinder and the Möbius strip diagram.

In this section we shall follow the general framework of Gimon and Polchinski [61] quite closely. The total projection that we wish to perform is $(\frac{1+R}{2})(\frac{1+\Omega S}{2})$. The orientifold group G is $\{1, R, \Omega S, \Omega RS\}$ which we can write as $G = G_1 + \Omega G_2$ with $G_1 = \{1, R\}$ and $G_2 = \{S, RS\}$. An open string can begin on a D-brane labeled by i and end on one labeled by j. The label of the D-brane is the Chan-Paton factor at each end. Let us denote a general state in the open string sector by $|\psi, ij\rangle$. An element of G_1 then acts on this state as

$$g: \qquad |\psi, ij\rangle \rightarrow (\gamma_g)_{ii'} |g \cdot \psi, i'j'\rangle (\gamma_g^{-1})_{j'j}, \tag{3.4}$$

for some unitary matrix γ_g corresponding to g. Similarly, an element of ΩG_2 acts as

$$\Omega h: \qquad |\psi, ij\rangle \rightarrow (\gamma_{\Omega h})_{ii'} |\Omega h \cdot \psi, j'i'\rangle (\gamma_{\Omega h}^{-1})_{j'j}. \tag{3.5}$$

The relevant partition sums for the Klein bottle, the Möbius strip, and the cylinder are respectively $\int_0^\infty dt/2t$ times

$$\begin{aligned}
\text{KB}: \qquad & \mathrm{Tr}^{\mathrm{U+T}}_{\mathrm{NSNS+RR}} \left\{ \frac{\Omega S}{2} \frac{1+R}{2} \frac{1+(-1)^F}{2} e^{-2\pi t(L_0 + \tilde{L}_0)} \right\} \\
\text{MS}: \qquad & \mathrm{Tr}^{99+55}_{\mathrm{NS-R}} \left\{ \frac{\Omega S}{2} \frac{1+R}{2} \frac{1+(-1)^F}{2} e^{-2\pi t L_0} \right\} \\
\text{C}: \qquad & \mathrm{Tr}^{99+95+59+55}_{\mathrm{NS-R}} \left\{ \frac{1}{2} \frac{1+R}{2} \frac{1+(-1)^F}{2} e^{-2\pi t L_0} \right\}.
\end{aligned} \tag{3.6}$$

Here F is the worldsheet fermion number, and as usual $\frac{1+(-1)^F}{2}$ performs the GSO projection. The Klein bottle includes contributions both from the untwisted sector and the sector twisted by R of the original orbifold.

For evaluating the traces we need to know the action of various operators on the oscillator modes and the zero modes of the fields. Let us take $X^m, m = 6,7,8,9$ to be the coordinates of the torus so that $2\pi r z_1 = X^6 + iX^7$ and $2\pi r z_2 = X^8 + iX^9$, where the radius r defines the overall size of the torus. Let $X^i, i = 1,2,3,4$ be the transverse coordinates in the six-dimensional Minkowski space. Let ψ^m and ψ^i be the corresponding fermionic coordinates of the NSR string. The action of R on oscillator modes is obvious. For the ground states $|p_m, L^m\rangle$ without oscillations, but with quantized momentum $p_m \equiv k_m/R$ in the compact direction and winding $L^m \equiv X^m(2\pi) - X^m(0)$, R has the action

$$R|p_m, L^m\rangle = |-p_m, -L^m\rangle. \tag{3.7}$$

Note that S is $U(\frac{r}{4})RU^\dagger(\frac{r}{4})$ where $U(\frac{r}{4})$ performs translation along both X^6 and X^8 by $r/4$. Therefore, S has the same action on the oscillators as R but for the ground states there is a crucial difference of phase

$$S|p_m, L^m\rangle = (-1)^{k_6}(-1)^{k_8}|-p_m, -L^m\rangle. \tag{3.8}$$

The action of Ω depends on the sectors; Ω takes a field $\phi(\sigma)$ to $\phi(\pi - \sigma)$ and has obvious action on the modes.

The traces can be readily evaluated. Following [61] we define

$$\begin{aligned} f_1(q) &= q^{1/12}\prod_{n=1}^{\infty}\left(1-q^{2n}\right), & f_2(q) &= q^{1/12}\sqrt{2}\prod_{n=1}^{\infty}\left(1+q^{2n}\right) \\ f_3(q) &= q^{-1/24}\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right), & f_4(q) &= q^{-1/24}\prod_{n=1}^{\infty}\left(1-q^{2n-1}\right), \end{aligned} \tag{3.9}$$

which satisfy the Jacobi identity

$$f_3^8(q) = f_2^8(q) + f_4^8(q) \tag{3.10}$$

and have the modular transformations

$$f_1(e^{-\pi/s}) = \sqrt{s}\, f_1(e^{-\pi s}), \quad f_3(e^{-\pi/s}) = f_3(e^{-\pi s}), \quad f_2(e^{-\pi/s}) = f_4(e^{-\pi s}). \qquad (3.11)$$

The relevant amplitudes are then given by $(1-1)\frac{v_6}{128}\int_0^\infty \frac{dt}{t^4}$ times

$$\begin{aligned}
\mathrm{KB}: \quad & 8\frac{f_4^8(e^{-2\pi t})}{f_1^8(e^{-2\pi t})}\left\{\left(\sum_{n=-\infty}^{\infty}(-1)^n e^{-\pi t n^2/\rho}\right)^2\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2/\rho}\right)^2 + \left(\sum_{w=-\infty}^{\infty} e^{-\pi t \rho w^2}\right)^4\right\} \\
\mathrm{MS}: \quad & -\frac{f_2^8(e^{-2\pi t})f_4^8(e^{-2\pi t})}{f_1^8(e^{-2\pi t})f_3^8(e^{-2\pi t})}\left\{ \quad \mathrm{Tr}(\gamma_{\Omega S,5}^{-1}\gamma_{\Omega S,5}^{T})\left(\sum_{w=-\infty}^{\infty} e^{-2\pi t\rho w^2}\right)^4\right. \\
& \left. +\mathrm{Tr}(\gamma_{\Omega RS,9}^{-1}\gamma_{\Omega RS,9}^{T})\left(\sum_{n=-\infty}^{\infty}(-1)^n e^{-2\pi t n^2/\rho}\right)^2\left(\sum_{n=-\infty}^{\infty} e^{-2\pi t n^2/\rho}\right)^2\right\} \\
\mathrm{C}: \quad & \frac{f_4^8(e^{-\pi t})}{f_1^8(e^{-\pi t})}\left\{(\mathrm{Tr}(\gamma_{1,9}))^2\left(\sum_{n=-\infty}^{\infty} e^{-2\pi t n^2/\rho}\right)^4\right. \\
& \left. + \sum_{i,j\in 5}(\gamma_{1,5})_{ii}(\gamma_{1,5})_{jj}\prod_{m=6}^{9}\sum_{w=-\infty}^{\infty} e^{-t(2\pi w r + X_i^m - X_j^m)^2/2\pi\alpha'}\right\} \\
& -2\frac{f_2^4(e^{-\pi t})f_4^4(e^{-\pi t})}{f_1^4(e^{-\pi t})f_3^4(e^{-\pi t})}\mathrm{Tr}(\gamma_{R,5})\mathrm{Tr}(\gamma_{R,9}) \\
& +4\frac{f_3^4(e^{-\pi t})f_4^4(e^{-\pi t})}{f_1^4(e^{-\pi t})f_2^4(e^{-\pi t})}\left\{(\mathrm{Tr}(\gamma_{R,9}))^2 + \sum_{I=1}^{16}(\mathrm{Tr}(\gamma_{R,I}))^2\right\}.
\end{aligned} \qquad (3.12)$$

We have defined $v_6 = V_6/(4\pi^2\alpha')^3$ where V_6 is the (regulated) volume of the non-compact dimensions, and $\rho = r^2/\alpha'$. For the cylinder amplitude, as in [61], the sum i,j comes from strings that begin and end at 5-branes i and j with arbitrary windings; the sum I is over 5-branes placed at the fixed points of R. Note that for the Klein bottle and the Möbius strip diagrams, in evaluating $\mathrm{Tr}(\Omega RS)$ or $\mathrm{Tr}(\Omega S)$, the sum over momenta contains a crucial factor of $(-1)^n$ for the 6 and 8 directions, but no such factor for the 7 and 9 directions.

To factorize in tree channel we use the modular transformations () and the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n-b)^2/a} = \sqrt{a}\sum_{s=-\infty}^{\infty} e^{-\pi a s^2 + 2\pi i s b}. \qquad (3.13)$$

An important fact for our purpose will be that

$$\sum_{n=-\infty}^{\infty}(-1)^n e^{-\pi t n^2/\rho} = \sqrt{\rho/t}\sum_{s=-\infty}^{\infty} e^{-\pi\rho(s-\frac{1}{2})^2/t}. \tag{3.14}$$

Tadpoles correspond to long tubes($t \to 0$) in the tree channel. In this limit it is easy to see that the total amplitude is proportional to $(1-1)\int_0^\infty dl$ times

$$\begin{aligned} & \frac{v_6 v_4}{16}\left\{(\mathrm{Tr}(\gamma_{1,9}))^2\right\} + \frac{v_6}{16 v_4}\left\{32^2 - 64\mathrm{Tr}(\gamma_{\Omega S,5}^{-1}\gamma_{\Omega S,5}^{T}) + (\mathrm{Tr}(\gamma_{1,5}))^2\right\} \\ + \quad & \frac{v_6}{64}\sum_{I=1}^{16}\left(\mathrm{Tr}(\gamma_{R,9}) - 4\mathrm{Tr}(\gamma_{R,I})\right)^2. \end{aligned} \tag{3.15}$$

Here l is the length of the tube, which is inversely proportional to the loop modulus t; $v_4 = \rho^2 = V_4/(4\pi^2\alpha')^2$ with V_4 the volume of the internal torus before orbifolding.

The $(1-1)$ above represents the contributions of NSNS and RR exchange respectively, which must vanish separately for consistency [16, 18]. Using these requirements we determine the spectrum in the next section.

3.2.2 Gauge Group and Spectrum

We see from (3.15) that to cancel the tadpole proportional to $v_6 v_4$ corresponding to the 10-form exchange, we must have $\mathrm{Tr}(\gamma_{1,9}) = 0$. Now $\mathrm{Tr}(\gamma_{1,9})$ equals the number n_9 of 9-branes, so we conclude that there are no 9-branes. We are left with only the 55 sector so from now on we drop the subscript 5 for the γ matrices. Vanishing of the term proportional to v_6/v_4 corresponding to the exchange of untwisted 6-forms gives

$$n_5 = 32, \qquad \gamma_{\Omega S} = \gamma_{\Omega S}^{T}. \tag{3.16}$$

Finally, vanishing of the term proportional to v_6 corresponding to the exchange of twisted sector 6-forms gives $\mathrm{Tr}(\gamma_{R,I}) = 0$. By a unitary change of basis $\gamma_{\Omega S} \to U\gamma_{\Omega S}U^T$ we can take

$$\gamma_{\Omega S} = \mathbf{1}. \tag{3.17}$$

We have additional constraints on the algebra of the γ matrices so that we obtain a representation of the orientifold group in the Hilbert space:

$$\begin{aligned} \gamma_{\Omega RS} &= \gamma_{\Omega S}\,\gamma_R \\ (\gamma_R)^2 &= \mathbf{1} \\ \gamma^T_{\Omega RS} &= \pm\gamma_{\Omega RS}. \end{aligned} \tag{3.18}$$

We have the choice of taking $\gamma_{\Omega RS}$ either symmetric or antisymmetric, but it turns out that both choices lead to the same spectrum.

Let us now discuss the massless bosonic spectrum coming from the NS sector. The states

$$\psi^\mu_{-1/2}|0, ij\rangle\lambda_{ji}, \quad \mu = 1, 2, 3, 4, \tag{3.19}$$

belong to the vector multiplets whereas the states

$$\psi^m_{-1/2}|0, ij\rangle\lambda_{ji}, \quad m = 6, 7, 8, 9, \tag{3.20}$$

belong to the hypermultiplets. We have to keep only the states that are invariant under R and ΩS; this constrains the possible forms of the Chan-Paton wave functions λ_{ij}.

The conditions for the Chan-Paton factors depend crucially on where the 5-branes are placed. There are a number of ways one can distribute the 32 5-branes to obtain various gauge groups. We discuss only two distinct configurations that lead to maximal symmetry.

1. The first choice is to take 16 five-branes to lie at a fixed point x of S and the remaining 16 to lie at the image of x under R. In this case, the projection under R simply relates the states at x to those at Rx and leads to no additional constraints on λ. $\Omega S = +1$ implies

$$\lambda = -\gamma_{\Omega S}\lambda^T\gamma^{-1}_{\Omega S} \tag{3.21}$$

for both scalars and vectors. This can be seen as follows. ψ^m satisfy Dirichlet boundary conditions on both ends and have the same mode expansion as ψ^μ which satisfy Neumann boundary conditions. Now $\psi^\mu_{-\frac{1}{2}}$ is odd under Ω as in Type-I theory in ten dimensions. But $\psi^m_{-\frac{1}{2}}$ is even because of the additional phase due to the Dirichlet boundary condition. Moreover, under S, ψ^m is odd and ψ^μ is even. Using (3.17) we conclude that $\lambda = -\lambda^T$, obtaining an adjoint representation of $SO(16)$ for both vectors and scalars, and the corresponding supermultiplets.

2. We can place 16 five-branes at a fixed point y of R and 16 at the image of y under S. This time we only need to impose the condition $R = +1$ on the states. For the matrix γ_R we had two choices. Let us first choose $\gamma_{\Omega RS}$ to be symmetric. Then from (3.18), γ_R is also a symmetric matrix that squares to one and is traceless. In transforming $\gamma_{\Omega S}$ to identity we already made a unitary change of basis, but we can still make an orthogonal change of basis to put γ_R in the form

$$\gamma_R = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{3.22}$$

Now $R = 1$ implies

$$\lambda = \gamma_R \lambda \gamma_R^{-1}$$

for vectors and

$$\lambda = -\gamma_R \lambda \gamma_R^{-1}$$

for scalars. The condition for vectors means that we have a subgroup of $U(16)$ that commutes with γ_R *i.e.*, $U(8) \times U(8)$. The condition for scalars means that they transform as $(8, \bar{8})$ and $(\bar{8}, 8)$ under the $U(8) \times U(8)$. Another way to see this is to note that the Chan-Paton label transforms as $(1, 8) + (8, 1)$ at one end and as the complex conjugate at the other. The projection keeps $(8 \times \bar{8}, 1) + (1, 8 \times \bar{8})$ for the vectors, and $(8, \bar{8})$ and the complex conjugate for the scalars. If we chose $\gamma_{\Omega RS}$ antisymmetric, we would get $\gamma_R = \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix}$ instead of (3.22), but the identical spectrum.

Notice that the rank of the gauge group is different in the two cases which correspond to two branches of the moduli spaces that are connected. With the group

$SO(16)$ we have adjoint matter, so we cannot change the rank. We can break it to a $U(8)$ or all the way to $U(1)^8$. For the $U(8) \times U(8)$, the condensation of charged hypermultiplets can change the rank and we can also break it to the diagonal $U(8)$, for example. The two branches are thus connected.

The symmetry breaking can be seen geometrically. If we place a 5-brane away from the fixed points of R and S, then we need three more 5-branes at the image points. We can thus divide the 32 branes in four copies of 8. In this case, there will be no restrictions on the Chan-Paton matrices at a given point, except that they are hermitian. If all branes are placed at generic points and their images, we get $U(1)^8$. When they coincide at a point other than the fixed points, we get $U(8)$ with an adjoint hypermultiplet.

3.2.3 Anomaly Cancellation

The number of vector multiplets equals the number of hypermultiplets at all points of the moduli space discussed in the previous subsection, so the gravitational anomalies cancel. In fact, at a generic point in the moduli space where the symmetry is $U(1)^8$, or also when it is $SO(16)$, the entire anomaly vanishes. These theories are thus anomaly-free without the need for the Green-Schwarz mechanism.

Anomaly cancellation is more subtle when the gauge group is $U(8) \times U(8)$. We can factorize the group as $SU(8) \times SU(8) \times U(1) \times U(1)$. The states are neutral under the diagonal $U(1)$. So we need to consider only $SU(8)_1 \times SU(8)_2 \times U(1)$ under which the hypermultiplets transform as $(8, \bar{8})_+$ and $(\bar{8}, 8)_-$, where the subscript denotes the $U(1)$ charge. Let us denote the field strengths as F_1, F_2, and f respectively.

The $U(1)$ factor is at first sight troublesome. The anomaly involving this factor has terms that are of the form $f(d_1 \mathrm{tr} F_1^3 + d_2 \mathrm{tr} F_2^3)$ where d_1, d_2 are constants. Such terms would seem problematic because they do not have the usual factorized form $f^2 \mathrm{tr} F^2$. However, these can be canceled by a local counterterm of the form $\int b \mathrm{Tr} F^3$ for some scalar b that has inhomogeneous gauge transformations. Let a be the gauge potential, $da = f$. Under the gauge transformation $\delta a = d\epsilon$, b must have the inhomogeneous transformation $\delta b = \epsilon$ to cancel the anomaly. The gauge invariant combination is $A = db - a$ which is nothing but the gauge-invariant form of the massive gauge boson associated with a. Now the kinetic term for b is of the form A^2 which can be viewed as the mass term for the massive gauge field A.

One is familiar with an analogous situation in four dimensions [50]. The scalar b is very similar to the axion in four dimensions which is the Goldstone boson of a global Peccei-Quinn symmetry. The fermionic current for the Peccei-Quinn symmetry is anomalous, but so is the axion current. Now, if we gauge this symmetry, then naively we would find that the gauge coupling to the fermions is anomalous. However, one can always define a linear combination of the fermionic current and the axionic current which is anomaly-free. The axion then is the would-be Goldstone boson associated with this anomaly-free current. The corresponding gauge-boson becomes massive after eating the axion.

Because the $U(1)$ gauge boson will always be massive, we shall discuss only the remaining factors $SU(8)_1 \times SU(8)_2$. Let us denote the field strengths for the two groups by F_1 and F_2 respectively, and define $\mathcal{F}_\alpha \equiv \mathrm{tr} F_\alpha^2, \alpha = 1, 2$. The anomaly polynomial is then of the form

$$X = \mathcal{F}_1^2 + \mathcal{F}_2^2 - 2\mathcal{F}_1\mathcal{F}_2. \tag{3.23}$$

To cancel this anomaly one needs a generalization of the Green-Schwarz mechanism proposed by Sagnotti [22] which we now review briefly.

If we have n tensor multiplets, then there is a natural $SO(1,n)$ symmetry in the low-energy supergravity action [41]. Altogether there are $n+1$ tensors $H_r, r = 0, ..., n$ that transform as a vector of $SO(1,n)$; the time-like component is self-dual whereas the spacelike components are anti-self-dual. The scalars coming from the tensor multiplets parametrize the coset space $SO(1,n)/SO(n)$. We take η_{rs} to be the Minkowski metric with signature $(1,n)$. Let v be the time-like vector, $v \cdot v = 1$, so that $v \cdot H$ is self-dual. The scalar product is with respect to the metric η: for example, $v \cdot H \equiv v_r H_s \eta_{rs}$. Now consider the case when the gauge group has m nonabelian factors with field strengths $F_\alpha, \alpha = 1, ..., m$, and denote $\mathrm{tr}F_\alpha^2$ by $\mathcal{F}_\alpha$. In this case, anomaly cancellation can be achieved by a generalization of the Green-Schwarz mechanism if the anomaly polynomial is of the general form

$$X = -\sum_{\alpha\beta}(c_\alpha \cdot c_\beta)\mathcal{F}_\alpha\mathcal{F}_\beta, \tag{3.24}$$

where $c_\alpha, \alpha = 1, ..., m$ are constant vectors of $SO(1, n)$. It is clear that the anomaly associated with X can be canceled by a local counterterm of the form

$$\Delta\mathcal{L} = \sum_\alpha \mathcal{F}_\alpha(c_\alpha \cdot B), \tag{3.25}$$

provided the fields B_r have appropriate gauge transformations. If ω_α are the Chern-Simons three-forms for the various gauge groups and $\delta\omega_\alpha = d\omega_\alpha^1$, then the required gauge transformations are $\delta B_r = c_{\alpha r}\omega_\alpha^1$. The modified gauge-invariant field strengths H_r are then given by

$$H_r = dB_r - c_{\alpha r}\omega_\alpha. \tag{3.26}$$

An important fact that follows from supersymmetry is that the coefficients $c_{\alpha r}$ that enter into (3.26) and the modified Bianchi identity are related to the kinetic term for the gauge field F_α, which is given by $v \cdot c_\alpha$ [22]. Given an anomaly polynomial, the vectors c_α must be chosen such that the kinetic terms for all gauge fields are positive-definite.

In our case, the gauge group has only two factors, *i.e.*, $m = 2$. We have ten tensors ($n = 9$), but it turns out that only three tensors are involved in the anomaly cancellation. This is because when all branes are localized at a given fixed point of R (and its image under S), the tensors coming from the twisted sectors localized at other fixed points that are far away, cannot be relevant. Therefore we restrict ourselves to a three-dimensional subspace taking $n = 2$. We have one self-dual and one anti-self-dual tensor from the untwisted sector, and one anti-self-dual tensor from the twisted sector.

For simplicity, let us pick a special point in the tensor-multiplet moduli space so that $v = (\cosh\phi, \sinh\phi, 0)$. The anomaly polynomial (3.23) can be written in the form (3.24) by choosing $c_1 = (1,1,1)$ and $c_2 = (1,1,-1)$. There is some freedom in choosing these vectors because of the $SO(1,n)$ symmetry and the freedom in choosing the signs of the tensor fields. With the above choice the field ϕ can be identified with the dilaton so that the coefficient of the gauge kinetic term, which comes from the disk diagram, goes as $e^{-\phi}$. Moreover, the kinetic terms are positive-definite for both the gauge groups because $v{\cdot}c_1$ and $v{\cdot}c_2$ are both positive-definite. Thus, the anomalies can be canceled by the generalized Green-Schwarz mechanism explained in the preceding paragraphs.

Worldsheet considerations are consistent with this spacetime reasoning. To obtain a counter-term like (3.25) we would require a coupling of the kind $B_2(\mathcal{F}_1 - \mathcal{F}_2)$, where B_2 is the tensor coming from the twisted sector. Such a term can be obtained by computing a disk diagram with two vertex operators for the gauge bosons on the boundary of the disk, and the vertex operator for the tensor at the center of the disk. The vertex operator at the center introduces a branch-cut corresponding to a twist by R. The twist acts on the Chan-Paton indices by the matrix γ_R which is $+\mathbf{1}$ for $\mathcal{F}_1$ but $-\mathbf{1}$ for $\mathcal{F}_2$. This is in accordance with the relative minus sign between the third components of the two vectors c_1 and c_2. By contrast, the vertex operators for the two tensors B_0 and B_1 coming from the untwisted sector of the orbifold introduce no branch cuts. These tensors therefore have identical couplings to the two gauge groups; correspondingly, c_1 and c_2 are identical in the $0,1$ subspace.

We have not worked out the detailed couplings from a worldsheet calculation, but our tadpole calculation assures us that anomaly must cancel in this way. If gauge invariance were anomalous, then the longitudinal mode of the gauge boson would not decouple. This would lead to a tadpole, but we have already made certain that there are no tadpoles.

So far we have chosen to work at a special point in the moduli space, where one could ensure that the kinetic terms for both gauge groups are positive-definite. However, as we move around the tensor-multiplet moduli space, we eventually come across a boundary where the kinetic term for one of the gauge fields changes sign, and is no longer positive-definite. For example, we can take a more general form for the vector v, $v = (\cosh\phi, \sinh\phi\cos\psi, \sinh\phi\sin\psi)$ where ϕ and ψ are the moduli. It is easy to see that there is a range of values for ϕ and ψ where either $v \cdot c_1$ or $v \cdot c_2$ is negative. This phenomenon is similar to the one observed in [25] which is possibly an indication of some 'phase transition' at the boundary.

We can also contemplate more complicated possibilities. For example, if y_1 and y_2 are two fixed points of R that are not related by S, then we can place eight 5-branes at y_1 and eight at y_2. The remaining 16 branes have to be placed at the images of these two points under S. In this case one would obtain $U(4) \times U(4)$ gauge group with two copies of $(4, \bar{4})$ from each of the fixed points. Now the anti-self-dual tensors coming from twisted sectors at both y_1 and y_2 will be needed for anomaly cancellation.

3.3 Discussion

We have constructed a string theory that does not seem to be connected to the known string vacua because we have a different number of tensor multiplets. It cannot be viewed as a compactification of Type-I theory because the orientifold symmetry mixes nontrivially with internal symmetries of the $K3$. We have discussed here only the simplest example but quite clearly there is a whole class of models one can consider at different points in this moduli space. Work on some of these models is in progress and will be reported elsewhere [32]. Models with multiple tensor multiplets have been considered before in [22, 23] although from a somewhat different point of view.

By analogy with [26] one can ask if these theories are connected to other theories by a phase transition. In six dimensions, infrared dynamics is trivial, so it would seem impossible to change the number of anti-self-dual tensors because one can simply count the states in the infrared. Such a transition can occur only if there is non-trivial infrared dynamics at special points in the moduli space analogous to the situation considered in [65]. Perhaps the boundary in the tensor-multiplet moduli space where the kinetic term for the gauge fields changes sign is related to such a phase transition.

Finally, one can ask about the duals of the theories that we have constructed.

A. Sen has informed us that at a generic point in the moduli space with $U(1)^8$ gauge symmetry, one can obtain identical spectrum by considering an orbifold of M-theory compactified on $K3 \times S^1$ [38]. In this theory the vector multiplets arise from the untwisted sector whereas the tensor multiplets arise from the addition of 5-branes of M-theory by a reasoning similar to [68, 63]. This is complementary to our construction where the tensor multiplets arise from the untwisted sector (on a smooth $K3$) and the vector multiplets arise from the addition of 5-branes. In a recent paper that appeared after this work was completed, C. Vafa has obtained identical spectrum by a compactification of 'F-theory' [46]. It is plausible that these three models can be related to one another by duality.

Chapter 4 Strings on Orientifolds

4.1 Introduction

In this chapter we discuss string compactifications on orientifolds to six and higher dimensions. Orientifolds are a generalization of orbifolds [56, 9, 11, 12] in which the orbifold symmetry includes orientation reversal on the worldsheet (for a review see [59] and references therein). Orientifolding allows one to construct new perturbative vacua that cannot be obtained by usual Calabi-Yau compactification of string theory. One can thus explore different regions in the moduli space of string vacua that were previously not accessible.

In six dimensions we focus on orientifolds of Type IIB theory compactified on a $K3$ orbifold to obtain six-dimensional theories with $N = 1$ spacetime supersymmetry. It has recently become clear that the dynamics of $D = 6$, $N = 1$ string theories is quite rich and offers many surprises. There are points in the moduli spaces of these theories where tensionless strings appear which makes it possible to have non-trivial dynamics in the infra-red [66, 63]. In particular, there can be phase transitions in which the number of tensor multiplets can change. It is therefore quite interesting to analyze different branches of the tensor-multiplet moduli space. Usual Calabi-Yau compactifications can give only one tensor multiplet. In [32] an orientifold was constructed that has nine tensor multiplets. In this paper we discuss some generalizations that give models with five, seven, nine, or ten tensor multiplets with different gauge groups. Models with multiple tensor-multiplets can also be obtained by compactifications of M-theory [66, 38, 39, 77], or of F-theory [46, 54, 55]. The orientifolds that we construct allow one to study the duals of some of these compactifications as perturbative string theories.

In nine dimensions we consider an orientifold of Type IIB theory compactified on a circle to obtain the dual of M-theory compactified on a Klein bottle. It is interesting to note that the compactification of M-theory on a circle gives the Type IIA theory, on an interval the $E_8 \times E_8$ heterotic string [67], on a Möbius strip a CHL string[27, 28], and on a torus the Type II string [21]. Thus, compactification on a Klein bottle completes this list of Ricci-flat compactifications to nine and ten dimensions. We also discuss some issues regarding the compactification of Type I theory on a torus.

This chapter is organized as follows. In section two we first discuss some generalities about orientifolds. In section three we discuss orientifolds of toroidal compactifications. In section four we discuss orientifolds of Type IIB theory compactified on K3 orbifolds. The calculation of tadpoles and the relevant partition sums are summarized in the Appendix.

4.2 Some Generalities about Orientifolds

In general our starting point will be some $\mathbf{Z}_N$ orbifold of toroidally compactified Type IIB theory. We can then take the orientifold projection $(1+\Omega\beta)/2$, where Ω is the orientation reversal on the worldsheet and β is some $\mathbf{Z}_2$ involution of the orbifold. If the orbifold group $\mathbf{Z}_N$ is generated by the element α, then the total projection we would like to perform is given by $(\frac{1+\alpha+\ldots+\alpha^{N-1}}{N})(\frac{1+\Omega\beta}{2})$ in both the twisted and the untwisted sectors of the orbifold. The orientifold group G can be written as $G = G_1 + \Omega G_2$ such that $\Omega h \Omega h' \in G_1$ for $h, h' \in G_2$.

The closed string sector of the orientifold is obtained by projecting the spectrum of the original orbifold onto states that are invariant under the orientifold symmetry. The open-string sector of the orientifold arises as follows. Orientifolding introduces unoriented surfaces in the closed-string perturbation theory. The unoriented surfaces such as the Klein bottle can have tadpoles of R-R fields in the closed string tree channel. The tadpoles correspond to the fact that the equations of motion for some R-R fields are not satisfied because the orientifold plane acts as the source of the R-R fields [56]. By including the right number of D-branes which are also sources for the R-R fields with opposite charge, one can cancel these tadpoles. This introduces the open-string sector with appropriate boundary conditions and Chan-Paton factors. As we shall see, sometimes the Klein bottle amplitude turns out to have no tadpoles; in these cases there is no need to introduce the open-string sector, and the closed-string sector by itself describes a consistent theory.

An open string can begin on a D-brane labeled by i and end on one labeled by j. The label of the D-brane is the Chan-Paton factor at each end. Let us denote a general state in the open string sector by $|\psi, ij\rangle$. An element of G_1 then acts on this state as

$$g: \qquad |\psi, ij\rangle \rightarrow (\gamma_g)_{ii'}|g \cdot \psi, i'j'\rangle(\gamma_g^{-1})_{j'j}, \tag{4.1}$$

for some unitary matrix γ_g corresponding to g. Similarly, an element of ΩG_2 acts as

$$\Omega h: \qquad |\psi, ij\rangle \rightarrow (\gamma_{\Omega h})_{ii'}|\Omega h \cdot \psi, j'i'\rangle(\gamma_{\Omega h}^{-1})_{j'j}. \tag{4.2}$$

The relevant partition sums for the Klein bottle, the Möbius strip, and the cylinder are respectively $\int_0^\infty dt/2t$ times

$$\begin{aligned}
\mathrm{KB}: &\qquad \mathrm{Tr}^{\mathrm{U+T}}_{\mathrm{NSNS+RR}}\left\{\frac{\Omega\beta}{2}\,\frac{1+\alpha+\ldots+\alpha^{N-1}}{N}\,\frac{1+(-1)^F}{2}e^{-2\pi t(L_0+\tilde{L}_0)}\right\}\\
\mathrm{MS}: &\qquad \mathrm{Tr}^{\lambda\lambda}_{\mathrm{NS-R}}\left\{\frac{\Omega\beta}{2}\,\frac{1+\alpha+\ldots+\alpha^{N-1}}{N}\,\frac{1+(-1)^F}{2}e^{-2\pi tL_0}\right\}\\
\mathrm{C}: &\qquad \mathrm{Tr}^{\lambda\lambda'}_{\mathrm{NS-R}}\left\{\frac{1}{2}\,\frac{1+\alpha+\ldots+\alpha^{N-1}}{N}\,\frac{1+(-1)^F}{2}e^{-2\pi tL_0}\right\}.
\end{aligned} \tag{4.3}$$

Here F is the worldsheet fermion number, and as usual $\frac{1+(-1)^F}{2}$ performs the GSO projection. The Klein bottle includes contributions both from the untwisted sector(U) and the twisted sectors(T) of the original orbifold. Orientation reversal Ω takes NS-R sector to R-NS sector, so these sectors do not contribute to the trace. The labels λ and λ' refer to the type of D-brane an open string ends on. For example, in a theory with both 5-branes and 9-branes, λ and λ' are either 5 or 9; one has to include the sectors 55 and 99 for the Möbius strip, and the sectors 55, 99, 59, and 95 for the cylinder[61]. The tadpoles can be extracted by factorizing the loop-amplitude in the tree channel. Tadpole cancellation then determines the number of D-branes as well as the form of the γ matrices introduced earlier, which in turn determines the open string sector completely. In fact in many examples that we consider, spacetime supersymmetry and anomaly cancellation usually place powerful constraints which determine the spectrum even without knowing the full form of the γ matrices.

Many of the details of the tadpole calculation are similar to those discussed in [59, 32, 61] and will not be repeated here. We give a collection of relevant partition sums and their factorized forms in the tree channel in the Appendix.

4.3 Orientifolds of Toroidally Compactified Type IIB theory.

4.3.1 An Example in Nine Dimensions

Consider Type IIB theory compactified say in the X^9 direction on a circle S_9 of radius r_9. We can take an orientifold with the group $\{1, S\Omega\}$ where S is a half-shift along the circle, $X^9 \rightarrow X^9 + \pi r_9$. The closed-string sector of this theory is obtained by projecting onto states that are invariant under $S\Omega$. The massless bosonic spectrum of Type IIB theory in ten dimensions consists of the metric g_{MN}, the dilaton ϕ^1, and a two-form B^2_{MN} from the NS-NS sector; a two-form B^1_{MN}, a scalar ϕ^2, and a four-form A_{MNPQ} with self-dual field strength from the R-R sector. The fields g_{MN}, ϕ^1, and B^1_{MN} are all even under Ω, whereas the fields A_{MNPQ}, B^2_{MN}, and ϕ^2 are odd. If we were projecting only under Ω, we would obtain the spectrum of Type I strings; the superscript 1 above refers to the fields that survive this projection.

Now, if we expand a given field Ψ in terms of the Kaluza-Klein momentum modes Ψ_m carrying quantized momentum m/R then the modes with even m are even under S, whereas the modes with odd m are odd. Thus, the combined projection under ΩS eliminates all odd momentum modes of the fields g_{MN}, ϕ^1, and B^1_{MN}, but all even momentum modes of A_{MNPQ}, B^2_{MN}, and ϕ^2. In particular, once we restrict ourselves to zero momentum modes to obtain the massless spectrum in nine dimensions, we obtain the closed string sector of the Type I string reduced to nine dimensions.

Let us now look at the open-string sector. As explained in the previous section, open-string sector arises from the addition of D-branes to cancel tadpoles in the Klein bottle amplitude. Now, because of the half-shift that accompanies Ω, only states with odd winding appear in the crosscap state and are thus massive. Another way to see this is to first compute the amplitude in the loop channel and then factorize in the tree channel. The loop channel momentum sum gives a term proportional to $\sum_m (-1)^m e^{\frac{-t\alpha' m^2}{r_9^2}}$ where t is the loop-channel parameter. To see the tadpoles in the tree channel we use Poisson resummation formula and take the limit $t \to 0$ corresponding to long, thin tubes; it is easy to see that in this limit the amplitude vanishes, and there is no tadpole. Therefore, to obtain a consistent orientifold there is no need to add any branes.

To see what this theory is dual to, we compactify further on a circle S_8 of radius r_8 in the direction X_8. The Type IIB theory is T-dual to Type IIA under $r_8 \to 1/r_8$, and moreover the operation Ω in IIB is dual to $R_8\Omega$ in IIA where R_8 is the reflection $X_8 \to -X_8$ [59]. Now Type IIA theory is M-theory compactified on a circle S_{10} in the X^{10} direction. The operation $R_8\Omega$ corresponds, in M-theory, to taking $X^8 \to -X^8$, at the same time flipping the sign of the three-form potential C_{MNP} of the eleven dimensional supergravity. In M-theory we can interchange the two circles S^8 and S^{10}. Therefore, the combined operation $S\Omega$ in Type IIB theory corresponds, in M-theory, to $X^{10} \to -X^{10}$, $X^9 \to X^9 + \pi r_9$ which is nothing but the Z_2 transformation that turns the torus $T_{9,10}$ into a Klein bottle. Notice that this is not a purely geometric operation in M-theory but is accompanied by a simultaneous change of sign of the three-form potential. Under the interchange of the two circles S_{10} and S_8, the symmetry $R_8\Omega$ in Type IIA theory is conjugate to the symmetry $(-1)^{F_L}$, where F_L is the spacetime fermion number coming from the left-movers [52]. All R-R fields are odd under this symmetry and all NS-NS fields are even. Thus, the strong coupling limit of the orbifold of Type-IIA theory under the combined operation $(-1)^{F_L}$ and $X_9 \to X_9 + \pi r_9$ is given by M-theory compactified on a Klein bottle.

It is amusing that we have an example of a compactification on a non-orientable surface. Another example is M-theory on a Möbius strip which is dual to a CHL compactification [27, 28]. Recall that the $E_8 \times E_8$ string is dual to M-theory on an interval in the tenth direction: the two E_8 factors live at the two endpoints of the interval [67]. Compactifying further on a circle, we obtain M-theory on a cylinder. The CHL string is obtained as a Z_2 orbifold of the heterotic string in nine dimensions. The orbifold symmetry corresponds to an interchange of the two E_8 factors accompanied by a half shift on the circle. The combined operation is again $X^{10} \to -X^{10}$, $X^9 \to X^9 + \pi r_9$ which turns the cylinder into a Möbius strip [62].

4.4 Type I Theory in Eight Dimensions

Type I theory compactified in the 8 and 9 directions to eight dimensions can be viewed as an orientifold of the Type IIB theory on the torus T_{89}. It is straightforward to find the massless spectrum, but there is one subtlety in taking the T-dual of this theory which is worth mentioning.

Let us T-dualize first in the X^9 direction. T-duality is a one sided parity transform [59] which means that in the RNS formulation of the superstring, only the left-moving coordinate $\tilde{X}^9$ and its fermionic partner $\tilde{\Psi}^9$ change sign. Thus, T-duality takes Type IIB theory to Type IIA theory, and takes Ω to $R_9\Omega$, where R_9 is the reflection in the X^9 direction. If we dualize again in the X^8 direction, we would get Type IIB theory back; Ω goes to $R_{89}\Omega$, where R_{89} reflects both X^8 and X^9. This identification leads to the following puzzle for the orientifold with the group $\{1, R_{89}\Omega\}$. Under Ω the four-form field A_{MNPQ} is odd, therefore the modes like A_{MNP9} and A_{MNP8} which are 3-forms in eight dimensions would be even under the combined operation $R_{89}\Omega$ and would survive the projection. But $N = 1$ supersymmetry in $D = 8$ uniquely determines the massless field content and does not allow a three-form potential. Therefore, supersymmetry is broken by this projection. On the other hand, the orientifold with the group $\{1, R_{89}\Omega\}$ is T-dual to the one with the group $\{1, \Omega\}$, and we cannot break supersymmetry by a T-duality transformation. We should really have obtained the T-dual of Type I strings in eight dimensions. The reason for this discrepancy is that Type IIB theory has an additional symmetry $(-1)^{F_L}$ under which all R-R fields are odd. The correct projection that gives the T-dual of Type I theory involves the combined operation $R_{89}(-1)^{F_L}$ instead of just the geometric reflection.

It is easy to see this ambiguity on the worldsheet. In the Ramond sector, the zero modes $\tilde{\Psi}^M$ correspond to the Γ^M matrices of the spacetime Clifford algebra. Under the T-duality transformation $\tilde{\Psi}^9 \to -\tilde{\Psi}^9$, the spinors transform as

$$\begin{aligned} S &\to S \\ \tilde{S} &\to \Gamma^9 \Gamma \tilde{S}, \end{aligned} \tag{4.4}$$

where S and $\tilde{S}$ are the right-moving and left-moving spacetime spinors respectively, and Γ as usual is the matrix that anticommutes with all Γ^M matrices and squares to one. If we T-dualize further in the X^8 direction then S goes to itself, and $\tilde{S}$ goes to $\Gamma^8\Gamma\Gamma^9\Gamma\tilde{S} = \Gamma^9\Gamma^8\tilde{S}$. Let us now see how the massless fields from the Ramond-Ramond sector transform. The vertex operator for an n-form field strength $H_{M_1 \ldots M_n}$ is proportional to $\bar{S}\Gamma_{M_1 \ldots M_n}\tilde{S}$ where $\Gamma^{M_1 \ldots M_n} = \frac{1}{n!}(\Gamma^{M_1} \ldots \Gamma^{M_n} \pm \text{permutations})$. It is easy to see that the effect of T-duality on the R-R field strengths $H_{M_1 \ldots M_n}$ and the corresponding potentials is to remove the 8, 9 indices if they are present and add them if they are not. For example, the vertex operator for H^1_{M89} is proportional to $\bar{S}\Gamma_M\Gamma_8\Gamma_9\tilde{S}$. Under T-duality, it would map onto $\bar{S}\Gamma_M\tilde{S}$ which is the vertex operator for the field strength of a scalar. Thus, B^1_{89} maps onto the scalar ϕ^2. However, because $\Gamma^8\Gamma$ and $\Gamma^9\Gamma$ anticommute with each other, there is a choice of sign for the action on the R-R fields, which corresponds precisely to the choice between R_{89} and $R_{89}(-1)^{F_L}$. This ambiguity is, of course, fixed by the correct choice of the orientifold symmetry.

4.5 Orientifolds of Type IIB Theory on $K3$

4.5.1 General Remarks

Let us review some relevant facts about the $K3$ surfaces which can be represented as $\mathbf{Z}_N$ orbifolds of the 4-torus T^4 [78]. Let (z_1, z_2) be the complex co-ordinates on the torus, and consider the $\mathbf{Z}_N$ transformation generated by

$$g: \quad (z_1, z_2) \to (e^{2\pi i/N} z_1, e^{-2\pi i/N} z_2). \tag{4.5}$$

The $\mathbf{Z}_N$ group must be a subgroup of $SU(2)$ to obtain unbroken supersymmetry in six dimensions. The torus T^4 is obtained by identifying a lattice Λ of points in R^4, so the orbifold group must leave the lattice invariant to have a sensible action on the torus. This crystallographic condition allows only four possibilities: the groups $\mathbf{Z}_2$ and $\mathbf{Z}_4$ when Λ is the square $(SU(2)^4)$ lattice given by the identifications $z_k \sim z_k + 1, \sim z_k + i, k = 1, 2$; or $\mathbf{Z}_3$ and $\mathbf{Z}_6$ when Λ is the hexagonal $(SU(3)^2)$ lattice given by the identifications $z_k \sim z_k + 1, \sim z_k + e^{2\pi i/3}, k = 1, 2$. At a fixed point of a $\mathbf{Z}_k$ symmetry there is a curvature singularity. A smooth $K3$ can be obtained by blowing up the singularity by replacing a ball around the fixed point by an appropriate smooth non-compact Ricci-flat surfaces E_k whose boundary at infinity is $S^3/\mathbf{Z}_k$.

In this section we consider two classes of orientifold projections $(1+\Omega\beta)/2$ of Type IIB theory on these orbifolds. In the first class of models we take β to be identity, whereas in the second class we take β to be a specific $\mathbf{Z}_2$ involution S of $K3$ that has 8 fixed points. We shall give an explicit description of this involution in the following subsections.

One immediate question is whether the projection leaves any supersymmetries unbroken. In the case of Ω the combination $Q_\alpha + \Omega\tilde{Q}_\alpha$ of the left-moving and right-moving supercharges will be invariant; supersymmetry will be broken by half, giving us $N = 1$ supersymmetry starting from $N = 2$. When we combine Ω with S, we do not want to break the supersymmetry further, so S should leave all $N = 2$ supersymmetries invariant. This is possible if the rotational part of the symmetry S is a subgroup of $SU(2)$, or equivalently if it leaves the holomorphic 2-form invariant. It is useful to consider the example of $\mathbf{Z}_2$ orbifold. In this case we have $\alpha : (z_1, z_2) \to (-z_1, -z_2)$ which generates a discrete subgroup of the $SU(2)$ holonomy group of a smooth $K3$, and therefore leaves two supercharges invariant giving us $N = 2$ supersymmetry. The symmetry S is given by $S : (z_1, z_2) \to (-z_1 + \frac{1}{2}, -z_2 + \frac{1}{2})$ which is a combination of a shift and a rotation [32]. The shift has no effect on the supercharges; the rotation is again a subgroup of the holonomy group $SU(2)$ and therefore does not break any supersymmetries by itself. Thus the combined operation $S\Omega$ gives $N=1$ supersymmetry as required. Now, the $\mathbf{Z}_2$ orbifold admits other involutions; for example, the Enriques involution $E:(z_1,z_2)\to(-z_1+\frac{1}{2},z_2+\frac{1}{2})$ which does not leave the holomorphic 2-form invariant, and cannot be used for orientifolding if we want unbroken supersymmetry.

The closed-string sector of an orientifold can be determined by index theory and by appropriate projection. Recall that the massless representations in $D=6$ are labeled by the representations of the little group which is $Spin(4)\sim SU(2)\times SU(2)$. The massless $N=1$ supermultiplets are

1. the gravity multiplet: $(3,3)+(1,3)+2(2,3)$,
2. the vector multiplet: $(2,2)+2(1,2)$
3. the tensor multiplet: $(3,1)+(1,1)+2(2,1)$
4. the hyper multiplet: $4(1,1)+2(2,1)$.

To determine the massless modes we need to know the Dolbeault cohomology [19], and how the symmetry $\Omega\beta$ acts on the cohomology. For a smooth $K3$, the nonzero Hodge numbers are $h^{00}=h^{22}=h^{02}=h^{20}=1$, and $h^{11}=20$. Among the 2-forms the $(0,2)$, $(2,0)$, and the Kähler $(1,1)$ form are self-dual, and the remaining 19 $(1,1)$ forms are anti-self-dual. The manifolds E_k have $(k-1)$ anti-self-dual $(1,1)$ harmonic forms, and one $(0,0)$ form. In the orbifold limit, each fixed point that is repaired by E_k contributes $(k-1)$ anti-self-dual $(1,1)$ forms which together with the $(1,1)$ forms of the original torus that are invariant under the orbifold group give the 20 $(1,1)$ forms of $K3$.

It is useful to think in terms of Type I theory compactified on a smooth $K3$. In this case, the orientation reversal symmetry in ten dimensions, which we shall call Ω_0 has the effect of flipping the sign of A_{MNPQ}, ϕ^2, and B^2_{MN}, leaving other massless fields invariant. The resulting theory has $h^{11}(=20)$ hypermultiplets which come from the zero modes of B^1_{MN} and g_{MN}. There is only one tensor multiplet from contracting B^1_{MN} with the $(0,0)$ form. Now imagine performing a projection not with

Ω_0 but with $\Omega_0 T$ where T is some geometric symmetry under which n_T $(1,1)$ forms are odd and all others are even. In this case, by contracting A_{MNPQ} with these $(1,1)$ forms, one can obtain n_T additional tensor multiplets that are invariant under the combined operation $\Omega_0 T$. At the same time, n_T hyper-multiplets are now projected out changing their total number to $(20-n_T)$. This reasoning gives the simple equation

$$n_T + n_H^c = 20, \tag{4.6}$$

where n_H^c refers to the number of hypermultiplets arising from the closed string sector, and $n_T + 1$ is the total number of tensor multiplets. Moreover, no vector multiplets arise from the closed string sector because there are no harmonic odd forms on $K3$, so starting with even forms and the metric in ten dimensions, one cannot obtain a one-form vector potential. We can thus read off the closed string spectrum immediately from the geometric data of the orientifold.

In the orbifold limit, the orientifold symmetry Ω, for the purposes of counting of states, is really a combination of $\Omega_0 T$ where T is some geometric symmetry that has nontrivial action on the cohomology. This is because at each fixed point, Ω takes the sector twisted by ρ to the one twisted by ρ^{-1}. If we repair the singularity at the fixed point of a $\mathbf{Z}_k$ symmetry by the smooth surfaces E_k then the $(k-1)$ $(1,1)$-forms coming from E_k correspond to the $(k-1)$ twisted sectors. If we think of the orbifold as a limit of a smooth K_3, then except in the case when α is a $\mathbf{Z}_2$ twist, we get a nontrivial action on the cohomology denoted by T. This information is sufficient to work out the spectrum of the orientifold in the closed-string sector.

Let us now discuss the massless bosonic spectrum coming from the NS open-string sector. The states

$$\psi^{\mu}_{-1/2}|0, ij\rangle\lambda_{ji}, \quad \mu = 1,2,3,4, \tag{4.7}$$

belong to the vector multiplets whereas the states

$$\psi^{m}_{-1/2}|0, ij\rangle\lambda_{ji}, \quad m = 6,7,8,9, \tag{4.8}$$

belong to the hypermultiplets. We have to keep only the states that are invariant under α and $\Omega\beta$. For this purpose we need to know the form of the γ matrices defined in (4.1) and (4.2) which are determined by the requirement of tadpole cancellation. The Chan-Paton wave functions λ_{ij} allowed by these projections determine the gauge group and the matter representations.

There are some features of the tadpole calculation that are common to all orbifolds. First, by the arguments given in [61], only 5-branes and 9-branes appear. Let v_6 and v_4 be the regularized volumes of the noncompact and the compact spaces in string units. If we look at the the Klein bottle amplitude in the tree channel then non-zero tadpoles proportional to $v_6 v_4$ correspond to 10-form exchange requiring addition of 9-branes. Similarly a term proportional to v_6/v_4 corresponds to the exchange of 6-forms from the untwisted sector, requiring addition of 5-branes, and the terms proportional to v_6 correspond to the exchange of 6-forms from the twisted sector and must cancel without the addition of any branes. Now with the orientifold group $G = G_1 + \Omega G_2$, 9-branes can arise only if G_2 contains the identity, and 5-branes arise only if G_2 contains the element R that reflects all four internal co-ordinates. In these cases the determination of the 10-form and the untwisted 6-form tadpoles is identical to the calculation in [61] which requires 32 9-branes with $\gamma_{\Omega,9}^T = \gamma_{\Omega,9}$, and/or 32 5-branes with $\gamma_{\Omega,5}^T = -\gamma_{\Omega,5}$.

4.6 $\mathbf{Z}_2$Orbifold

For the $\mathbf{Z}_2$ orbifold, the model in the first class with the projection $(1+\Omega)/2$ has been discussed in [61], and the model in the second class with the projection $(1+S\Omega)/2$ in [32]. We would now like to consider a model that is closely related to the one in [32]. Let us recall that in [32] the symmetry S was chosen to be such that $S^2 = 1$. However, if we are on a $\mathbf{Z}_2$ orbifold, then the symmetry can square to the element α that generates the orbifold group. We choose

$$S: \quad (z_1, z_2) \to (iz_1, -iz_2). \tag{4.9}$$

Now S has 4 fixed points and not 8. However, they are also the fixed points of α which is a $\mathbf{Z}_2$ symmetry. So on the orbifold, the fixed point of S should be regarded as having Euler character 2 giving us the total Euler character of 8 in agreement with the Lefschetz number [17].

Obviously, the spectrum consists of the closed string sector found in [32] giving us $n_T = 8$, $n_H = 12$ and the gravity multiplet. However, because now neither R nor the identity are elements of G_2, there is no need to add any branes, and there is no open-string sector. One nontrivial check is that the tadpoles of the R-R fields from the twisted sector now have to cancel by themselves for the Klein bottle without any contribution from the open-string sector. It is easy to see using the formulae in the Appendix that the tadpoles from the untwisted sector cancel against those from the sector twisted by $\frac{1}{2}$ giving us a consistent theory. Gravitational anomalies cancel completely as expected.

4.6.1 $\mathbf{Z}_3$ Orbifold

The orbifold symmetry in this case has nine fixed points of order 3 which contribute two anti-self-dual (1, 1) forms each giving 18 in all. Out of the six 2-forms on the torus one anti-self-dual (1, 1) form and the remaining three self-dual 2-forms are invariant under α giving us 22 2-forms of the $K3$.

Let us first consider the projection under Ω. As explained in Subsection 4.1, at each fixed point of the orbifold Ω interchanges the sector twisted by α to that twisted by α^{-1} besides flipping the sign of all R-R fields. This means that of the two tensor multiplets coming from each fixed point, only one will be invariant, giving us $n_T = 9$ from the nine fixed points, and $n_H^c = 11$ from (4.6) .

To determine the open-string sector we note that, by the general arguments mentioned in Subsection 4.1, there will be 32 9-branes, and we can choose $\gamma_\Omega = \mathbf{1}$ by a unitary change of basis [61]. The requirement that $(\Omega\alpha)^2 = \alpha^2$ implies

$$\gamma_{\alpha^2} = \gamma_\alpha^2 = \gamma_{\Omega\alpha}(\gamma_{\Omega\alpha}^{-1})^T. \tag{4.10}$$

Using the fact the the γ matrices are unitary, and $\gamma_{\Omega\alpha} = \gamma_\Omega \gamma_\alpha$, we conclude that γ_α is real. Furthermore, because $\gamma_\alpha^3 = 1$, the only eigenvalues are cube-roots of unity. If n eigenvalues are $e^{2\pi i/3}$, then n will be $e^{-2\pi i/3}$, and $32-2n$ will be 1. We can then write γ in a block-diagonal form where in a $2n$ dimensional subspace it acts as a $2\pi/3$ rotation and in $32-2n$ dimensional subspace it equals the identity matrix. This information and anomaly cancellation is enough to determine that $n = 8$. We can also verify this by a detailed calculation of tadpoles as discussed in the Appendix. The gauge group will then be given by $SO(16) \times U(8)$ with hypermultiplets in $(1,28)+(16,8)$. It is easy to see that the anomaly terms proportional to $\mathrm{tr}(F^4)$ and $\mathrm{tr}(R^4)$ vanish. It is not necessary for the remaining anomaly to factorize because we have more than one tensor multiplet available, and the anomalies can be canceled by the generalized Green-Schwarz mechanism as in [15, 22, 32].

Let us now describe the action of S on the $\mathbf{Z}_3$ orbifold. It is given by

$$S: \quad (z_1, z_2) \to (-z_1, -z_2). \tag{4.11}$$

S has 16 fixed points on the torus but on the orbifold they split into one singlet and five triplets of $\mathbf{Z}_3$. The Euler character of the fixed point at the origin which is a singlet under the $\mathbf{Z}_3$ is 3 and that of the 5 triplets is 1 each giving 8 altogether. Now, because S is just a reflection of all co-ordinates, the orientifold with the projection $(1+S\Omega)/2$ is T-dual to the one described in the previous paragraphs with the projection$(1+\Omega)/2$. T-duality turns 9-branes into 5-branes, but the spectrum remains unchanged.

4.6.2 $\mathbf{Z}_4$ Orbifold

The $\mathbf{Z}_4$ orbifold has four fixed points of order 4. Each contributes three tensor multiplets out of which only one is invariant under the action Ω. No additional tensors arise from the six doublets of fixed points of order 2. Altogether $n_T = 4$, and $n_H^c = 16$.

In this case both 5-branes and 9-branes will be present, and we can choose

$$\gamma_{\Omega,9} = \mathbf{1}, \quad \gamma_{\Omega,5} = \mathbf{J} \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{4.12}$$

The remaining algebra is determined in terms of the matrices $\gamma_{\alpha,9}$ and $\gamma_{\alpha,5}$. Tadpoles are canceled if $\mathrm{Tr}\,(\gamma_{\alpha,9}) = \mathrm{Tr}\,(\gamma_{\alpha,9})^2 = \mathrm{Tr}\,(\gamma_{\alpha,9})^3 = 0$ and similarly for the matrices with subscript 5. This determines the γ matrices completely. Moreover $\gamma_{\alpha,9} = \gamma_{\alpha,5}$, and their eigenvalues are such that each forth root of unity appears eight times. The gauge group is $U\,(8) \times U\,(8) \times U\,(8) \times U\,(8)$ with hypermultiplets in $(28,1,1,1) + (1,28,1,1) + (1,1,28,1) + (1,1,1,28) + (8,8,1,1) + (1,1,8,8) + (8,1,8,1) + (1,8,1,8)$. Once again the anomaly terms proportional to $\mathrm{tr}\,(F^4)$ for each factor, and the coefficient of $\mathrm{tr}\,(R^4)$ vanish.

Let us now consider the action of the symmetry S which is given by

$$S: \quad (z_1, z_2) \to (-z_1 + \frac{1+i}{2}, -z_2 + \frac{1+i}{2}). \tag{4.13}$$

This form is determined by the requirement that S has to preserve the orbifold symmetries; in particular, it should map a fixed point of a given order to a fixed point of the same order. It is easy to check that eight $(1,1)$ forms are odd under S. The 16 fixed points form four quartets under $\mathbf{Z}_4$. In addition, S leaves two doublets under α invariant which should be regarded as fixed points on K_3 with Euler character 2. The total Euler character of the fixed point set adds up to 8.

If we consider the orientifold with the projection $(1 + \Omega S)$, then only 32 5-branes are required. As in [32] we find $n_T = 8$, $n_H^c = 12$ from the closed-string sector. We can place 16 branes at a fixed point of α^2 which is in a doublet of α that is left invariant by S, and 16 at its image under α. For example, we can place 16 branes at the $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the remaining 16 at $\left(\frac{i}{2}, \frac{i}{2}\right)$. In this case the gauge group is $U(8) \times U\,(8)$, with charged hyper-multiplets in $2\,(8,8)$. This is exactly the spectrum of the model considered in [32] for the $\mathbf{Z}_2$ orbifold. If we place 16 branes at the fixed point of α, and 16 at its image under S, then the gauge group is $U(4) \times U(4) \times U(4) \times U(4)$ with hypermultiplets in $(4,4,1,1)+(4,1,4,1)+(1,4,1,4)+(1,1,4,4)$.

4.6.3 Z_6 Orbifold

In this case, we get two tensors from the fixed points of order 6 and one each from the four fixed points of order 3 giving us $n_T = 6$ and $n_H^c = 14$. The open-string sector has both 5-branes and 9-branes. The eigenvalues of the matrix $\gamma_{\alpha,5} = \gamma_{\alpha,9}$ are as follows: 1 and -1 appear eight times each and the other sixth roots of unity appear four times each. The resulting gauge-group is $U(4) \times U(4) \times U(8)$ with hypermultiplets in $(6,1,1)+(1,6,1)+(4,1,8)+(1,4,8)$ from the 55 sector, and identical spectrum from the 99 sector. The 59 sector contributes hypermultiplets in $(4,1,1,4,1,1)+(1,4,1,1,4,1)+(1,1,8,1,1,8)$.

Appendix. Tadpole Calculation

For evaluating the traces in the loop-channel we need the determinants of chiral bosons and fermions with twisted boundary conditions. Let us denote by $D_F[{a \atop b}]$ the fermion determinant of a chiral Dirac operator $(\nabla^z_{-\frac{1}{2}})$ which corresponds to the path integral of a complex chiral fermion with boundary condition $\psi(\sigma_1 + 2\pi\,,\,\sigma_2) = -e^{2\pi i a}\psi(\sigma_1\,,\,\sigma_2)$, and $\psi(\sigma_1\,,\,\sigma_2 + 2\pi) = -e^{2\pi i b}\psi(\sigma_1\,,\,\sigma_2)$. It is straightforward to evaluate this determinant in the operator formalism[75] . Writing $q = e^{2\pi i \tau}$, and using the standard relation between the path integral and the operator formalism, it is equal to the trace $\operatorname*{Tr}_{\mathcal{H}}(h_b\, q^{H_a})$. H_a is the Hamiltonian of a chiral, twisted fermion:

$$H_a = \sum_{n=1}^{\infty}(n - \frac{1}{2} + a)d_n^\dagger d_n + (n - \frac{1}{2} - a)\bar{d}_n^\dagger \bar{d}_n + \frac{a^2}{2} - \frac{1}{24}. \tag{4.14}$$

The fermionic oscillators satisfy canonical anticommutation relations $\{d_n^\dagger, d_m\} = \delta_{mn}$ and $\{\bar{d}_n^\dagger, \bar{d}_m\} = \delta_{mn}$, and $\mathcal{H}$ is the usual Fock space representation of these commutations. The group Z_N acts on this Fock space through $hdh^{-1} = -e^{-2\pi i b}d$,

$h\bar{d}h^{-1} = -e^{2\pi i b}\bar{d}$. The trace equals (up to an arbitrary phase)

$$e^{2\pi i a b} q^{\frac{a^2}{2} - \frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}+a} e^{2\pi i b})(1 + q^{n-\frac{1}{2}-a} e^{-2\pi i b}) \ . \tag{4.15}$$

Using the product representation of the theta function $\vartheta[{a \atop b}](\tau)$ with characteristics [74] , we see that

$$D_F[{a \atop b}] = \mathop{\mathrm{Tr}}_{\mathcal{H}} (h_b \, q^{H_a}) = \frac{\vartheta[{a \atop b}](0|\tau)}{\eta(\tau)} \tag{4.16}$$

where $\eta(\tau)$ is the Dedekind η function. The chiral boson determinant is the inverse of the chiral fermion determinant, except for $a = \frac{1}{2}$ when one needs to be careful about the zero modes. Note that untwisted NS fermions with half-integer modings and antiperiodic boundary conditions for the trace corresponds to $a = 0, b = 0$; an untwisted boson with periodic boundary condition along the σ_2 direction corresponds to $a = \frac{1}{2}, b = \frac{1}{2}$. Using these formulae one can write down the traces by inspection. The tadpole calculation corresponding to the 10-form and the untwisted 6-form exchange are identical to the one in [61], and will not be repeated here. We shall be interested in the tadpole of only the 6-form from the twisted sector which corresponds to the boundary conditions for the determinant for internal bosons that have only oscillator sums but no momentum or winding sums.

Let us first evaluate the traces in (4.3) for the Klein bottle. The total trace can be written as

$$\frac{(1-1)v_6}{64N} \int_0^{\infty} \frac{dt}{t^4} 8 \sum_{a,b} Z[{a \atop b}], \tag{4.17}$$

where the $(1-1)$ refers to NSNS - RR exchange in the tree channel, v_6 is $V_6/(4\pi\alpha')^3$; $b = k/N, k = 1, ..., (N-1)$ corresponding to the terms with α^k in the trace. Only the untwisted sector and the sector twisted by $\frac{1}{2}$ contribute because for other twisted sectors Ω is off-diagonal; a is therefore either 0, or $\frac{1}{2}$. From the untwisted sector we get

$$Z[{0 \atop b}] = 4\sin^2(2\pi b) \frac{\vartheta[{0 \atop \frac{1}{2}}]^2 \vartheta[{0 \atop 2b+\frac{1}{2}}] \vartheta[{0 \atop -2b-\frac{1}{2}}]}{\eta^6 \vartheta[{\frac{1}{2} \atop 2b+\frac{1}{2}}] \vartheta[{\frac{1}{2} \atop -2b-\frac{1}{2}}]}; \tag{4.18}$$

and from the sector twisted by $\frac{1}{2}$ at each fixed point that is left invariant by α^k, we get

$$Z[{\textstyle{\frac{1}{2}}\atop b}] = -\frac{\vartheta[{0\atop \frac{1}{2}}]^2\vartheta[{\frac{1}{2}\atop 2b+\frac{1}{2}}]\vartheta[{\frac{1}{2}\atop -2b-\frac{1}{2}}]}{\eta^6\vartheta[{0\atop 2b+\frac{1}{2}}]\vartheta[{0\atop -2b-\frac{1}{2}}]}, \tag{4.19}$$

where $\tau = 2it$ and $b = k/N$. Let us now turn to the traces for the cylinder. In this case, in general we can have $55, 99, 59$, or 95 sectors. The partition sum is given by

$$\frac{(1-1)v_6}{64N}\int_0^\infty \frac{dt}{t^4}\sum_{\lambda,\lambda',b} Z[{\lambda\lambda'\atop b}]\mathrm{Tr}\,(\gamma_{b,\lambda})\mathrm{Tr}\,(\gamma_{b,\lambda'}^{-1}), \tag{4.20}$$

where λ and λ' take values either 5 or 9, and $\gamma_{\lambda,b}$ refers to the matrix $\gamma_{\lambda,\alpha^k}$ for $b = k/N$. We obtain

$$\begin{aligned} Z[{99\atop b}] &= Z[{55\atop b}] = 4\sin^2(\pi b)\frac{\vartheta[{0\atop \frac{1}{2}}]^2\vartheta[{0\atop b+\frac{1}{2}}]\vartheta[{0\atop -b-\frac{1}{2}}]}{\eta^6\vartheta[{\frac{1}{2}\atop b+\frac{1}{2}}]\vartheta[{\frac{1}{2}\atop -b-\frac{1}{2}}]}, \\ Z[{59\atop b}] &= Z[{95\atop b}] = -\frac{\vartheta[{0\atop \frac{1}{2}}]^2\vartheta[{\frac{1}{2}\atop b+\frac{1}{2}}]\vartheta[{\frac{1}{2}\atop -b-\frac{1}{2}}]}{\eta^6\vartheta[{0\atop b+\frac{1}{2}}]\vartheta[{0\atop -b-\frac{1}{2}}]}, \end{aligned} \tag{4.21}$$

with $\tau = it$. The Möbius strip amplitude is given by

$$\frac{(1-1)v_6}{64N}\int_0^\infty \frac{dt}{t^4}\sum_{\lambda,b} Z[{\lambda\lambda\atop b}]\mathrm{Tr}\,(\gamma_{b\Omega,\lambda}^T\gamma_{b\Omega,\lambda}^{-1}), \tag{4.22}$$

where only 55 and 99 sector contribute. We obtain

$$Z[{99\atop b}] = \tan^2(\pi b)Z[{55\atop b}] = -4\sin^2(\pi b)\frac{\vartheta[{0\atop \frac{1}{2}}]^2\vartheta[{\frac{1}{2}\atop 0}]^2\vartheta[{0\atop b+\frac{1}{2}}]\vartheta[{0\atop -b-\frac{1}{2}}]\vartheta[{\frac{1}{2}\atop b}]\vartheta[{\frac{1}{2}\atop -b}]}{\eta^6\vartheta[{0\atop 0}]^2\vartheta[{\frac{1}{2}\atop b+\frac{1}{2}}]\vartheta[{\frac{1}{2}\atop -b-\frac{1}{2}}]\vartheta[{0\atop b}]\vartheta[{0\atop -b}]}, \tag{4.23}$$

with $\tau = 2it$

To factorize in the tree channel we use the modular transformations under $\tau \to -1/\tau$:

$$\begin{aligned} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) &= (-i\tau)^{-\frac{1}{2}}e^{-2\pi iab}\vartheta\begin{bmatrix} -b \\ a \end{bmatrix}(-1/\tau) \\ \eta(\tau) &= (-i\tau)^{-\frac{1}{2}}\,\eta\,(-1/\tau), \end{aligned} \tag{4.24}$$

and take the limit $t \to 0$. While writing the tadpoles we also have to take into account that the tree channel length l is equal to $1/4t$, $1/2t$, and $1/8t$ for the Klein bottle, the cylinder, and the Möbius strip respectively. The twisted-sector tadpole is then proportional to $\frac{(1-1)v_6}{8N} \int dl$. In this common normalization, we get,

$$
\begin{array}{lll}
\mathrm{KB}: & (16)^2 \sin^2(2\pi b), & a = 0, b \neq 0, \\
 & -64, & a = \dfrac{1}{2}, b \neq 0, \dfrac{1}{2}; \\
\mathrm{C}: & 4\sin^2(\pi b) \mathrm{Tr}\,(\gamma_{b,\lambda}) \mathrm{Tr}\,(\gamma_{b,\lambda}^{-1}), & b \neq 0, \lambda = 5 \,\mathrm{or}\, 9, \\
 & -\mathrm{Tr}\,(\gamma_{b,5}) \mathrm{Tr}\,(\gamma_{b,9}^{-1}) - (9 \leftrightarrow 5)\,, & b \neq 0; \\
\mathrm{MS}: & -64\sin^2(\pi b) \mathrm{Tr}\,(\gamma_{b\Omega,9}^{T}) \mathrm{Tr}\,(\gamma_{b\Omega,9}^{-1}), & b \neq 0, \dfrac{1}{2} \\
 & -64\cos^2(\pi b) \mathrm{Tr}\,(\gamma_{b\Omega,5}^{T}) \mathrm{Tr}\,(\gamma_{b\Omega,5}^{-1}), & b \neq 0, \dfrac{1}{2}.
\end{array}
$$

The Klein bottle contributes -64 from each sector twisted by $\frac{1}{2}$ for each fixed point that is left invariant by α^k.

Chapter 5 A Note on Orientifold and F-theory

Orientifolds are a generalization of orbifolds in which the orbifold symmetry is a combination of a spacetime symmetry and orientation reversal on the worldsheet [8, 56, 61, 59]. These techniques have significantly enlarged the set of string vacua that can be studied perturbatively. Several new string vacua can now be constructed as orientifolds which exhibit novel dynamical phenomena and have interesting non-perturbative duals in M-theory, F-theory, or heterotic string theory.

One important application of orientifolds is in the construction of models in six dimensions with $N = 1$ supersymmetry. The dynamics of these theories offers many surprises like the appearance of tensionless strings which can cause a phase transition in which the number of tensor multiplets changes [25, 66, 64], or the appearance of enhanced gauge symmetry when an instanton shrinks to zero scale size [65]. Orientifolds are useful in understanding some aspects of these phenomena perturbatively. For instance, the models with multiple tensor multiplets are inaccessible using usual Calabi-Yau compactifications which give only a single tensor multiplet. However, one can easily construct orientifolds [11, 32, 36, 33, 37] with multiple tensor multiplets at special points in this moduli space. By turning on the moduli in the tensor multiplets one can move away from these special points and thus explore different regions of the moduli space that are separated by phase boundaries. Some of these models [32] are known to have M-theory duals[38, 39]. The extra tensor multiplets which arise in M-theory from the addition of M-theory 5-branes occur perturbatively in the dual orientifold. Similarly, small instantons, which cannot be described as a conformal field theory in heterotic compactifications, have a perturbative description in terms of a Dirichlet 5-brane in the dual orientifold [65, 73]. In particular, the enhanced $Sp(k)$ symmetry when k small instantons coincide can be understood in terms of coincident 5-branes with a specific symplectic projection in the open string sector that is determined by the consistency of the world-sheet theory.

Another more recent application of orientifolds is in connection with F-theory [46, 54, 55]. F-theory refers to a new way of compactifying Type-IIB theory in which the complex coupling λ of Type-IIB theory is allowed to vary over space. The coupling is given by $\lambda = \xi + ie^{-\phi}$ where ϕ is the dilaton from the NSNS sector and ξ is the RR scalar. Consider an elliptically fibered Calabi-Yau manifold K which is a fiber bundle over a base manifold B with a torus as a fiber whose complex structure parameter is τ. Even-though K is a smooth manifold, there will be points in the base manifolds where the fiber becomes singular, and the parameter τ can have a nontrivial $SL(2,Z)$ monodromy around these points. An F-theory compactification on K refers to a compactification of Type-IIB theory on B, where the coupling λ is identified with τ. The nontrivial monodromy of λ around the singular points then means that there are 7-branes at those points that are magnetically charged with respect to the scalar λ. Typically, the base manifold is not Ricci-flat and moreover, because λ is varying, there is a nonvanishing RR background. These backgrounds cannot, therefore, be described using conformal field theory. For special choices of the manifolds K, however, an F-theory compactification is equivalent to a perturbative Type-IIB orientifold. This follows from an observation due to Sen [40] that the element -1 of $SL(2,Z)$ which is not an element of $PSL(2,Z)$ is a perturbative symmetry of Type-IIB. It flips the sign of the two 2-form fields B^1_{MN} and B^2_{MN}, but leaves all other massless fields, in particular, the coupling fields λ invariant. From its action on the massless fields it is easy to check that this element represents the action of $\Omega(-1)^{F_L}$ where Ω is orientation reversal on the worldsheet and F_L is the spacetime fermion number of the left-movers. In the example considered by Sen, K is a $K3$ surface that is a Z_2 orbifold of a four-tours; F-theory on this surface corresponds to a Type-IIB orientifold with the orientifold group $\{1, \Omega(-1)^{F_L}\sigma\}$ where σ is a specific Z_2 involution of $K3$, and is T-dual to Type-I theory. Such an identification of F-theory with an orientifold is very useful. For instance, it was used in [40] to establish the duality between F-theory on $K3$ and the heterotic string on T^2 by relating it to the duality between the Type-I and the heterotic string in ten dimensions.

In this chapter we analyze an orientifold of a $K3$ orbifold which gives $N = 1$ supersymmetry in six dimensions. Its T-dual has the same orientifold group as the Type-I orientifold analyzed by Gimon and Polchinski [61], but the orientation reversal symmetry Ω acts with an additional minus sign on the twisted sector states of the orbifold. One is familiar with an analogous situation in orbifold constructions. For a $Z_k \times Z_k$ orbifold symmetry, there are k inequivalent orbifolds which correspond to turning on discrete torsion[47, 53]. These different orbifolds correspond to the k distinct choices of phases for the action of the generator of one Z_k subgroup on the sectors twisted by other generators.

This model illustrates interesting new features that are relevant to all the applications mentioned earlier: the unusual action of orientation reversal gives rise to multiple tensor multiplets, the 5-branes at the fixed points of the orbifold have orthogonal projection instead of the symplectic projection of a small instanton at a nonsingular point, and it is perturbatively equivalent to F-theory on a Calabi-Yau orbifold $T^6/\{Z_2 \times Z_2\}$ with Hodge numbers $(h^{11}, h^{21}) = (51, 3)$ [53]. Using the formulae in [54] we see that this F-theory compactification gives 17 tensor multiplets, four neutral hypermultiplets, $SO(8)^8$ gauge group, and no charged hypermultiplets. Our aim in the following is to see how the orientifold reproduces this spectrum.

Let us denote the complex coordinates of the six-torus by z_1, z_2, z_3 with identifications $z_l \equiv z_l + 1 \equiv z_l + i, l = 1, 2, 3$. The $Z_2 \times Z_2$ symmetry is generated by the elements α and β where

$$\begin{aligned} \alpha &: \quad (z_1, z_2, z_3) \to (-z_1, -z_2, z_3), \\ \beta &: \quad (z_1, z_2, z_3) \to (z_1, -z_2, -z_3). \end{aligned} \tag{5.1}$$

It is easy to work out the cohomology [48, 49, 53]. The untwisted sector contributes $(3, 3)$ to (h^{11}, h^{21}), and the sectors twisted by α, β, and $\alpha\beta$ each contribute $(16, 0)$, giving $(51, 3)$ altogether. To obtain the corresponding orientifold, we take z_3 as the coordinate of the fiber, and consider Type-IIB compactified on a four-torus with coordinates (z_1, z_2): $z_1 = X^6 + iX^7$, $z_2 = X^8 + iX^9$. Orbifolding with the symmetry α gives Type-IIB on $K3 = T^4/Z_2$. The element β can be written as R_2R_3 where R_2 is a geometric symmetry $(z_1, z_2) \to (z_1, -z_2)$, and R_3, which reflects the fiber, is nothing but the element -1 of $SL(2, Z)$ which corresponds to the operation $\Omega(-1)^{F_L}$ as explained in the preceding paragraph. We are thus led to consider an orientifold of Type-IIB on $K3$ with the orientifold group $\{1, \Omega(-1)^{F_L} R_2\}$ [1].

This orbifold is a special case of a large class of elliptic Calabi-Yau threefolds studied by Voisin [42] and Borcea [43] and discussed in [45, 55]. One can take the base to be a $K3$ which admits an involution σ under which the holomorphic 2-form ω is odd, and construct the Calabi-Yau as an orbifold $K3 \times T^2/\{1, \sigma R_3\}$ where R_3 is the reflection of the torus. It should be possible to generalize the considerations of this paper to this whole class of models.

The projection that we wish to perform is $\frac{1}{4}(1 + \Omega(-1)^{F_L} R_2)(1 + R)$ where $R = R_1R_2$. The projection $\frac{1}{2}(1 + R)$ gives us Type-IIB theory on a $K3$ which has 21 tensor multiplets of $N = 2$ supersymmetry which is sum of a tensor multiplet and a hypermultiplet of $N = 1$ supersymmetry. Five of these multiplets come from the untwisted sector, and the remaining 16 come from the twisted sectors at the 16 fixed points of the orbifold. Now, from the arguments of [33, 40], one would have expected, by T-duality in the 89 directions, that the operation $\Omega(-1)^{F_L} R_2$ is equivalent to the operation Ω. It seems, therefore, that we get an orientifold of T^4 with the orientifold group $\{1, R, \Omega, \Omega R\}$ which is nothing but a Type-I orientifold on $K3$ analyzed by [61]. The massless spectrum, however, is very different; for example, the closed-string spectrum of the model of [61] has only one tensor multiplet instead of 17, and 20 neutral hypermultiplets instead of four. The reason for this mismatch is that, even though the two projections are the same in the untwisted sector, they are different in the twisted sectors of the orbifold. This is clear if we look at the action of $\Omega(-1)^{F_L} R_2$ on the twisted sectors. The operation Ω that is dual to $\Omega(-1)^{F_L} R_2$ corresponds to $\Omega_0 T$, where Ω_0 is the operation considered in [61], and T is a symmetry of the orbifold that flips the sign of the twist fields at all fixed points. In untwisted sector

[1] We would like to thank S. Mukhi for this observation which prompted this investigation [24].

in both theories give one tensor multiplet and four hypermultiplets. But in the twisted sector at each fixed point, Ω_0 projects out the tensor multiplet and keeps the hypermultiplet giving the closed string spectrum of Type-I on $K3$ whereas Ω keeps the tensor multiplet and projects out the hypermultiplet giving 17 tensor multiplets and four hypermultiplets altogether, as required.

Let us now turn to the open-string sector. We shall follow the notation of [61] in the T-dual picture so that we have 7-branes and 7'-branes instead of 9-branes and 5-branes respectively. The T-dual picture turns out to be easier because then the symmetry breaking is given by geometric separation between branes instead of by Wilson lines. The orientifold group in this case is $\{1, R, \Omega(-1)^{F_R} R_1, \Omega(-1)^{F_L} R_2\}$. Note that both R_1 and R_2, and similarly $\Omega(-1)^{F_L}$ and $\Omega(-1)^{F_R}$ all square to $(-1)^F$ but the elements of the orientifold group all square to $\mathbf{I}$ as they should. To simplify the notation, let us denote $\Omega(-1)^{F_R} R_1$ and $\Omega(-1)^{F_L} R_2$ by Ω_1 and Ω_2 respectively. To determine the open-string sector we need to determine, as in [61], the number of branes of each type and the eight γ matrices that give the action of the four orientifold group elements on 7 and $7'$ branes.

Before discussing the details of the calculation let us present the results. Tadpole cancellation requires 32 branes of each kind; the 32 7-branes are located at the four fixed planes of R_1 in groups of eight, and the 32 7'-branes are located at the fixed planes of R_2 in groups of eight. Moreover, by a unitary change of basis, the various gamma matrices are given by

$$\begin{aligned} \gamma_{1,7} &= \mathbf{I}, \quad \gamma_{\Omega_1,7} = \mathbf{I}, \quad \gamma_{R,7} = \mathbf{I}, \quad \gamma_{\Omega_2,7} = \mathbf{I}; \\ \gamma_{1,7'} &= \mathbf{I}, \quad \gamma_{\Omega_2,7'} = \mathbf{I}, \quad \gamma_{R,7'} = -\mathbf{I}, \quad \gamma_{\Omega_1,7'} = -\mathbf{I}. \end{aligned} \tag{5.2}$$

Now consider the massless bosonic states coming from the 77 sector at the fixed point where eight 7-branes are located. The vectors are given by $\psi^{\mu}_{-1/2}|0, ij\rangle\lambda_{ji}, \mu = 1,2,3,4$; $R = +1$ implies $\lambda = \lambda$, and $\Omega_1 = +1$ implies $\lambda = -\lambda^T$, which means that the vectors are in the adjoint of $SO(8)$. The scalars are given by $\psi^{\mu}_{-1/2}|0, ij\rangle\lambda_{ji}, \mu = 6,7,8,9$; $R = +1$ implies $\lambda = -\lambda$, which means that they are all projected out.

From four fixed planes of R_1 we get $SO(8)^4$, and similarly from the $7'7'$ sector we get another $SO(8)^4$. Thus, altogether we get $SO(8)^8$ with no charged hypermultiplets.

In the $77'$ sector there is a subtlety. In this case, we have to choose the oscillator vacuum of this sector to be odd under the action of R instead of even as in [61]. This is consistent with factorization because $77'$ and $7'7$ states can turn into 77 or $7'7'$ states, but we cannot have two 77 or two $7'7'$ states turning into a $77'$ state. So one can choose the $77'$ vacuum to be odd and the 77 and $7'7'$ vacua to be even. We shall explain two paragraphs later that this choice is indeed forced upon us by consistency. In this sector the fermions Ψ^m have integer modings, so the ground states are given by a representation of Clifford algebra generated by the zero modes. The total state after GSO projection is $|s_3, s_4, ij\rangle\lambda_{ji}, s_3 = -s_4$ where $s_3, s_4 = \pm\frac{1}{2}$. We choose R on these GSO-projected vacuum states to be -1 instead of $+1$. Thus, R=+1 on the total state implies $\lambda = -\lambda$ which projects out the massless states completely. To summarize, we get 17 tensor multiplets and four hypermultiplets from the closed-string sector, and $SO(8)^8$ gauge group with no charged hypermultiplets from the open-string sector, altogether in agreement with the F-theory spectrum.

This determination of the spectrum, however, poses the following puzzle. From arguments similar to those presented in [61], one would have expected that if $\gamma_{\Omega_1,7}$ is symmetric then $\gamma_{\Omega_1,7'}$ should be antisymmetric. How did we then obtain a solution in which both are symmetric? To see that this is a consistent choice, let us recall the argument of [61]. In the following we shall often switch between our model and its T-dual. In order to obtain a true representation (and not merely a projective representation) of the orientifold symmetry that we are gauging, we must have $\Omega^2 = \mathbf{I}$ in the full string Hilbert space, which is a direct product of the Fock space of string oscillators and the Chan-Paton index space. Now, because Ω^2 is -1 on the oscillator part of the massless states, it must be compensated by choosing -1 on the Chan-Paton part. This forces $\gamma_{\Omega,5}$ to be antisymmetric if $\gamma_{\Omega,9}$ is symmetric. In our case, however, because of our choice of $R = -\mathbf{I}$ on the GSO-projected vacuum states that we used in the previous paragraph, the massless states in the 59 are projected out. Moreover, it is easy to see that at the massive level, the oscillator part of the physical states that are left after the GSO and the R-projection all have $\Omega^2 = +1$. This is so because only the states at half-integer mass levels survive the projections. Now, $\Omega^2 = -1$ for the half integer oscillator modes, and moreover because $\Omega^2 = -1$ on the oscillator vacuum as noted by [61], the total oscillator state has $\Omega^2 = +1$. This in turn implies that in the Chan-Paton space we must choose $\Omega^2 = 1$ which means that if $\gamma_{\Omega,9}$ is symmetric then $\gamma_{\Omega,5}$ must also be symmetric. To put it differently, of the whole tower of states in the 59 sector, the states that are kept after the GSO and the R projection, all have $\Omega^2 = -1$ in [61], but have $\Omega^2 = +1$ in this paper. Thus, the choice of the projection R and the sign of the eigenvalue of Ω^2 are correlated. Under T-duality 59 sector corresponds to $7'7$ sector and the argument above can be repeated there.

Let us now show that the spectrum described above satisfies all consistency requirements, and is moreover uniquely determined. Tadpole calculation in this case is very similar to the T-dual of [61]. The Klein-Bottle and the Möbius strip amplitudes are identical, and for the cylinder amplitude, the only difference is the additional minus sign in the $77'$ and $7'7$ sector in calculating the trace of R. The tadpoles are thus given, in the notation of [61], by

$$
\begin{aligned}
& \frac{v_6 v_2}{16 v_2'} \left\{ 32^2 - 64 \mathrm{Tr}(\gamma_{\Omega_1,7}^{-1} \gamma_{\Omega_1,7}^T) + (\mathrm{Tr}(\gamma_{1,7}))^2 \right\} \\
+ \quad & \frac{v_6 v_2'}{16 v_2} \left\{ 32^2 - 64 \mathrm{Tr}(\gamma_{\Omega_2,7'}^{-1} \gamma_{\Omega_2,7'}^T) + (\mathrm{Tr}(\gamma_{1,7'}))^2 \right\} \\
+ \quad & \frac{v_6}{8} \left\{ \mathrm{Tr}(\gamma_{R,7}) \mathrm{Tr}(\gamma_{R,7'}) + 2 \sum_{I=1}^{4} (\mathrm{Tr}(\gamma_{R,7}))^2 + 2 \sum_{I'=1}^{4} (\mathrm{Tr}(\gamma_{R,7'}))^2 \right\}. \qquad (5.3)
\end{aligned}
$$

Here v_6 is the regularized volume of the uncompactified dimensions, v_2 and v_2' is the the volume of the 2-tori in the 67 and in the 89 directions respectively; I and I' refer to the fixed points of R_1 and R_2 respectively.

The chain of reasoning that determines the solution is then as follows. To cancel the tadpoles of the 8-forms from the untwisted sector (the terms proportional to $\frac{v_6 v_2'}{v_2}$ and $\frac{v_6 v_2'}{v_2}$), we need 32 branes of each kind with $\gamma_{1,7}$ and $\gamma_{1,7'}$ equal to $\mathbf{1}$, and $\gamma_{\Omega_1,7}$ and $\gamma_{\Omega_2,7'}$ both symmetric, which can be chosen to be $\mathbf{1}$ with a unitary change of basis of Chan-Paton indices. One can then use the argument presented in [57] which considers the amplitude in which a closed-string twisted state turns into open string states. Conservation of Ω_1 and Ω_2 requires that $\gamma_{R,7}$ and $\gamma_{R,7'}$ both be symmetric, which in turn implies that $\gamma_{\Omega_2,7}$ and $\gamma_{\Omega_1,7'}$ must also be symmetric. This can be consistent only if we choose vacuum states in the $77'$ to have $R = -1$ so that all oscillator states with $\Omega^2 = -1$ are projected out. Cancellation of the tadpoles of 6-forms from the twisted sector (the terms in (5.3) proportional to v_6) then determines that the branes are distributed in groups of eight at the fixed planes, with $\gamma_{R,7} = \mathbb{1}$ and $\gamma_{R,7'} = -\mathbb{1}$. This determines the solution completely.

The next simplest orientifold is when the $K3$ is given by Z_3 orbifold of a hexagonal lattice. In this case, $z_l \equiv z_l + 1 \equiv e^{2\pi/3} z_l, l = 1, 2$. The element α in (5.1) is given by $\alpha : (z_1, z_2) \to (e^{2\pi/3} z_1, e^{-2\pi/3} z_2)$ and β is the same as in (5.1). We are thus interested in the projection $\frac{1}{6}(1 + \alpha + \alpha^2)(1 + \Omega(-1)^{F_L} R_2)$ Now, because $\Omega(-1)^{F_L} R_2$, in this case interchanges the sectors twisted by α with those twisted by α^2, one can easily see that this orientifold is T-dual to the Z_3 orientifold with the usual Ω projection discussed in [36, 33]. This model has 10 tensor multiplets and 11 hypermultiplets, and 32 7-branes of one kind. If they are all located at the fixed point of R_2, that is also invariant under α, then the gauge group is $SO(16) \times U(8)$ with hypermultiplets in $(1, 28) + (16, 8)$.

To find a potential F-theory dual on a Voisin-Borcea orbifold, we consider the configuration in which there are eight 7-branes at each fixed point of R_2 so that the tadpoles are canceled locally. One fixed point of R_2 is invariant under α, and the remaining three form a triplet. The gauge group is $SO(8) \times SO(8)$ with one adjoint hypermultiplet under the first $SO(8)$ that comes from the fixed points that form a triplet under α. To identify the F-theory dual we need to find an elliptic Calabi-Yau X with the right Hodge numbers. The Hodge number can be calculated by compactifying further on a T^2 and computing the Type-IIA spectrum as in [55]. We then have

$$h^{11}(X) = r(V) + T + 2, \quad h^{21} = H^0 - 1, \tag{5.4}$$

where $r(V)$ is the rank of the gauge group, T is the number of tensor multiplets, and H^0 is the number of hypermultiplets that are uncharged with respect to the Cartan subalgebra of the gauge group. Thus, the candidate Calabi-Yau should have $h^{11} = 20$ and $h^{21} = 14$. Happily, there is a unique Voisin-Borcea with the above Hodge numbers which corresponds to $(r, a, \delta) = (11, 9, 1)$ in the notation of [55, 43]. Indeed, this model has the same matter content as the orientifold configuration with local tadpole cancellation.

Chapter 6 Orientifold and F-theory Duals of CHL Strings

Along with the recent development of string duality, new ways of constructing string vacua have appeared. These new constructions make manifest some nonperturbative aspects of the string theory which the previously known constructions cannot see. Typical examples are orientifolds and F-theory.

An orientifold is a generalization of orbifolds in which the orbifold symmetry includes the orientation reversal on the world sheet[8, 56, 61, 59]. Type I theory is an example of an orientifold where the only symmetry gauged is the orientation reversal of the Type IIB theory. An important application of orientifolds is the construction of models in 6-dimensions with N=1 supersymmetry [23, 32, 36, 33, 34]. The models with multiple tensor multiplets can be easily constructed using the orientifold, while this is not possible in the conventional Calabi-Yau compactification which gives only one tensor multiplet. Similarly, small instantons citeWittII, which cannot be described as a conformal field theory in heterotic string theory, have a perturbative description in terms of Dirichlet 5-branes in the dual orientifold.

On the other hand, F-theory is a new way of compactifying Type IIB theory in which the complex coupling λ of Type IIB theory is allowed to vary over the compactified space [46, 54, 55]. The complex coupling can be seen as the complex structure parameter of the elliptic fibration over the base B on which the Type IIB theory is compactified. The coupling can undergo non-trivial $SL(2, Z)$ transformations as we move along non-trivial cycles on the base B. Since the nonperturbative $SL(2, Z)$ symmetry of the Type IIB theory is realized as the $SL(2, Z)$ transformation of the elliptic fibration, F-theory is quite powerful in studying nonperturbative phenomena in string theory such as the phase transition involving tensionless strings in 6-dimensions.

In another interesting development, Chaudhuri, Hockney and Lykken (CHL) have constructed new examples of the heterotic string compactification with maximal supersymmetry but with gauge groups of reduced rank [27, 28]. It turns out that all CHL models can be identified with toroidal compactification of the heterotic string theory further modded by some discrete global symmetry such as the interchange of the two E_8's of the heterotic string theory [58]. Some duality aspects of CHL models are known. The M-theory and Type IIA duals of various CHL models in 6-dimensions and below were investigated in detail [29, 30, 44].

The purpose of this chapter is to find the dual theory of a CHL model to be described later, using the orientifold and F-theory. Type IIA orientifold on the Möbius band is dual to the CHL model in 8-dimensions. In the strong coupling limit, this configuration is lifted to M-theory compactified on the Möbius band which gives the 9-dimensional dual of the CHL string. We present the F-theory orbifold dual with non-simply laced gauge groups. The F-theory dual of the CHL model in 6-dimensions constitutes a part of the F-theory vacua in 6-dimensions with N=2 supersymmetry hitherto uninvestigated. We discuss the other N=2 F-theory vacua in 6-dimensions and give the orientifold duals in simple cases.

The initial CHL models are given by the free fermionic construction[27], but Chaudhuri and Polchinski[28] have constructed one of these models as an asymmetric orbifold. They considered Z_2 orbifold of the toroidally compactified heterotic string theory, where Z_2 action interchanges the two E_8 components of the momentum lattice, together with a half shift on a compactified circle. Under this Z_2 modding, only the symmetric combination of E_8's survives, thereby reducing the the rank of the gauge group by eight. The Z_2 modding is possible if the two E_8's are broken in an identical manner. Similar construction can be done in $SO(32)$ heterotic string theory. Since $SO(32)$ and $E_8 \times E_8$ heterotic string theory are equivalent upon compactification on a circle, the Z_2 orbifolds on both sides are on the same moduli space. Since the Z_2 symmetry adopted in the orbifold construction is a freely acting Z_2, there are no massless states in the twisted sector at generic points of the moduli space. We are mainly interested in this Z_2 orbifold example.

Since there is a duality conjecture on Type I and heterotic string theory [60], we expect that the dual theory of the CHL model can be found in Type I side. The strategy we will take is the following: Using the duality relation between Type I and the heterotic theory, we can figure out the Z_2 symmetry in Type I which corresponds to the Z_2 symmetry used in the CHL construction in the heterotic theory. If the Z_2 action used is freely acting, the adiabatic argument[52] assures us of the duality between the Z_2 orbifolds.

It is more convenient to work with Type I′ theory, because the Z_2 action is realized geometrically. Since the gauge groups are realized as the Chan-Paton degrees of freedom of 8-branes in the Type I′ theory, Z_2 action should interchange the 8-branes and this must be accompanied by a half shift along a circle which guarantees the total action freely acting. This means that the total Z_2 action does not produce additional orientifold planes which contribute to non-zero tadpole, as we will see shortly. We construct the 8-dimensional orientifold. Consider Type IIA theory compactified on the torus, say in the X^8 and X^9 directions with the identification $X^8 \equiv X^8 + 2\pi r_8$ and $X^9 \equiv X^9 + 2\pi r_9$. We take an orientifold with the projection

$$\{1, \eta_8\eta_9, R_9\Omega, \eta_8\eta_9 R_9\Omega\}, \tag{6.1}$$

where η_i denotes a half shift along the i-th circle; $X^i \to X^i + \pi r_i, i = 8, 9$. This is the Type I′ theory modded by the Z_2 action $\eta_8\eta_9$. Note that η_9 relates an 8-brane located at $X_9 = X_{9_0}$ to an 8-brane at $X_9 = X_{9_0} + \pi r_9$, and η_8 is the accompanying shift. The action $\eta_8\eta_9$ preserves all of the harmonic forms on the torus, hence the supersymmetry is not reduced. This model has the same supersymmetry as Type I′ theory toroidally compactified to 8-dimensions. The action of $\eta_8\eta_9$ on the oscillator modes is trivial. For the ground states $|p_i, L^i\rangle$ without oscillations, which have the quantized momenta $p_i \equiv m_i/R_i$ with m_i integer valued in the compact directions, and winding $L^i \equiv X^i(\pi) - X^i(0) = 2\pi w^i R_i$ with w^i integer, $\eta_8\eta_9$ has the action

$$\eta_8\eta_9|p_i, L^i\rangle = (-1)^{m_8}(-1)^{m_9}|p_i, L^i\rangle. \tag{6.2}$$

The massless modes of the closed string sector coincide with those of Type I′ theory since $\eta_8\eta_9$ acts trivially on those modes. The twisted sector of the closed string has half-integer winding modes in X^8, X^9 directions with the momentum modes and the oscillator modes unchanged.

We can determine the open string spectrum by calculating the tadpoles [61]. The open string sector arises from the addition of the D-branes to cancel the tadpole in the Klein bottle amplitude. The Klein bottle amplitude consists of two parts, one from the trace evaluation with $R_9\Omega$ and the other from the trace with $\eta_8\eta_9 R_9\Omega$. The loop channel amplitude of the former gives the tadpole which requires 32 8-branes for the tadpole cancellation as in Type I′ theory. The loop channel momentum sum of the latter is proportional to $\Sigma_{m_8}(-1)^{m_8} e^{\frac{-\pi t\alpha' m_8^2}{r_8^2}}$, where t is the loop channel parameter. This gives vanishing tadpole in the tree channel in the $t \to 0$ limit, as one can see using the Poisson resummation formula. The Klein bottle amplitude of the twisted sector also vanishes, since the half winding modes in the X^8 direction are odd under $R_9\Omega$ and $\eta_8\eta_9 R_9\Omega$. Thus we have 32 8-branes in all. Because of the orientifold projection, only orthogonal gauge groups are allowed. In addition, brane configuration should be invariant under the action $\eta_8\eta_9$. The maximal gauge group is obtained if we put the 16 8-branes at $X^9 = X^9_0$ and the other 16 8-branes at $X^9 = X^9_0 + \pi r_9$. The gauge group in this case is $SO(16)$. It is clear that we can obtain the orthogonal subgroups of $SO(16)$ by locating 16 branes at different positions and locating the other 16 branes compatible with the action $\eta_8\eta_9$. From the initial Type I′ theory, we obtain the model whose rank of the gauge group is reduced by eight. Thus we see that this orientifold construction gives the same massless spectrum as the CHL model of the $SO(32)$ heterotic string theory where interchange of the momentum lattice is accompanied by the half-shift along the X^8 direction.

Note that the above orientifold action (6.1) turns the compactified torus into the Möbius band. Hence the CHL model in 8-dimensions is dual to the Type-IIA orientifold compactified on the Möbius band. By considering the strong coupling limit we can lift this construction to M-theory compactified on the Möbius band, which is dual to the CHL model in 9-dimensions with the gauge group E_8.

It was explained in[59] how to obtain the $E_8 \times E_8$ M-theory of Horava and Witten [67], starting from Type I$'$ theory. We put 14 8-branes on each fixed point and place the other 4 branes in symmetric fashion with respect to the fixed points so that the resulting gauge group is $(SO(14) \times U(1))^2$. The configuration is dual to $SO(32)$ heterotic string theory compactified on a circle with a particular Wilson line where the enhanced gauge group $E_8 \times E_8$ is achieved at a particular radius of the compactified circle. The dual configuration in the Type I$'$ side is obtained by taking the strong coupling limit. In this limit, the other 4 branes approach the fixed points and additional 0-brane states become massless to form adjoint of $E_8 \times E_8$. In this limit, the radius of the compactified circle goes to infinity. The relation between the Type I$'$ theory and the M-theory on $S^1/Z_2 \times S^1$ is given by

$$R_{I'} = r_1 r_2^{1/2}, \quad g_{I'} = r_2^{3/2}, \tag{6.3}$$

where $R_{I'}$ and $g_{I'}$ are the compactified radius and the coupling constant of Type I$'$ theory respectively, r_1 is the radius of S^1/Z_2 and r_2 is the radius of S^1 of M-theory. The limit we take is $r_2 \to \infty$ limit which corresponds to $g_{I'}, R_{I'} \to \infty$ limit.

Since the strong coupling limit is compatible with the action $X^9 \to X^9 + \pi r_9$, we can take the same limit for the above Type IIA orientifold. In this limit, we obtain the Z_2 orbifold of $E_8 \times E_8$ M-theory where E_8 exchange is accompanied by the shift η_8. By changing the coordinate label X^8 to X^9, we obtain the M-theory compactification in 9-dimensions and the compactified space is the Möbius band. This is dual to the CHL model in 9-dimensions with the gauge group E_8 where E_8 exchange is accompanied by the shift on a circle. The duality can be directly argued using the adiabatic argument and the duality between the M-theory on S^1/Z_2 and the $E_8 \times E_8$ string theory[62].

Now we turn into the F-theory dual of the CHL model in 6-dimensions. Since there is a conjectured duality in 8-dimensions between the F-theory on $K3$ and the heterotic string theory on T^2 [46, 40], we expect that one can construct the F-theory dual by modding out the Z_2 symmetry which corresponds to the Z_2 symmetry of the CHL string. Since the gauge group of the F-theory appears as singular elliptic fibers[54, 55], the Z_2 symmetry interchanging the gauge group is realized as a Z_2 involution of K3 which interchanges the singular fibers. This involution should respect the fiber structure. We start with the particular K3 orbifold T^4/Z_2. F-theory on this particular orbifold was considered by Sen in establishing the duality between F-theory and the orientifold which is obtained by T-dualizing the Type I theory along the X^8 and X^9 directions[40]. If the F-theory model has the constant coupling as in the T^4/Z_2 orbifold, the corresponding orientifold configuration should satisfy the local tadpole cancellation so that the resulting orientifold configuration has the constant coupling. Let us denote the complex coordinates of the six-torus by z_1, z_2, z_3 with identification $z_l \equiv z_l + 1 \equiv z_l + i, l = 1, 2, 3$. We consider the $Z_2 \times Z_2$ orbifold of F-theory with the following generators α, β of the orbifold action.

$$\begin{aligned} \alpha &: \quad (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3), \\ \beta &: \quad (z_1, z_2, z_3) \rightarrow (-z_1, -z_2 + \frac{1}{2}, z_3 + \frac{1}{2}). \end{aligned} \tag{6.4}$$

Here z_1 denotes the coordinate of the elliptic fiber and $2\pi r_2 z_2 \equiv X^8 + iX^9, 2\pi r_3 z_3 \equiv X^6 + iX^7$. Restricted to z_1, z_2 coordinates, α is the z_2 action of T^4/Z_2 orbifold. The torus parametrized by z_2 has four fixed points of α. The singular fiber over each fixed point is of D_4 type[1], and we have $SO(8)^4$ gauge group due to the four singular fibers. The action β is a Z_2 involution on K3 orbifold combined with the shift along the z_3 coordinate. Since the generator β preserves the holomorphic 2-form of K3 orbifold and the holomorphic form on the third torus, this orbifold has the same supersymmetry as the F-theory on $T^4/Z_2 \times T^2$, i.e., N=2 supersymmetry in 6-dimensions. Four singular D_4 fibers are paired by the action β and the resulting gauge group is $SO(8)^2$. Since β is freely acting, there are no additional massless modes. The massless spectrum consists of N=2 supergravity multiplet and N=2 vector multiplet with gauge group $SO(8)^2$.

One can check that this conclusion is consistent with the known duality between

[1]The possible singularities of elliptic fibers are classified by Kodaira. Those singularities fit into the ADE classification, which in turn give ADE gauge groups in F-theory[54, 55].

the F-theory and the orientifold. If we use the duality dictionary between the F-theory and the orientifold[40], α is mapped to $\Omega(-1)^{F_L}R_{89}$ and β is mapped to $\Omega(-1)^{F_L}R_{89}\eta_6\eta_8$. Hence the orientifold projection corresponding to the above orbifold is $\{1, \eta_6\eta_8, \Omega(-1)^{F_L}R_{89}, \Omega(-1)^{F_L}R_{89}\eta_6\eta_8\}$. By similar tadpole calculation as we did previously, we can conclude that this orientifold has the gauge group $SO(8)^2$ if the tadpole cancels locally. If we T-dualize the orientifold along the X^9 direction, the orientifold projection becomes $\{1, \eta_6\eta_8, R_8\Omega, \eta_6\eta_8R_8\Omega\}$. After some coordinate relabeling, we can see that this is the original orientifold model of (reforbi) compactified further on T^2. Since this orientifold model is dual to the CHL model, the above F-theory $Z_2 \times Z_2$ orbifold should be dual as well. Further evidence for the duality between the F-theory orbifold and the CHL model can be seen if we compactify F-theory further on a circle. This theory is on the same moduli space as M-theory compactified on the above $Z_2 \times Z_2$ orbifold, according to the duality between F-theory and M-theory[46]. This M-theory orbifold is a special case of the M-theory on $(K3 \times T^2)/Z_2$ which is dual to the CHL model in 5-dimensions as argued in [29]. The Z_2 action is the half-shift along the torus combined with the involution of $K3$ under which eight anti-self-dual 2-forms of K3 are odd and the remaining harmonic forms are even. This Z_2 involution is realized by β in the $Z_2 \times Z_2$ orbifold of (6.4). Thus we see that the duality between the M-theory on $(K3 \times T^2)/Z_2$ and the CHL model in 5-dimensions is lifted to the duality between the F-theory on $(K3 \times T^2)/Z_2$ and the CHL model in 6-dimensions.

One can consider the models corresponding to other points of the moduli space of F-theory on $(K3 \times T^2)/Z_2$. One such model is given by the orbifold generated by the following action.

$$\begin{aligned} \alpha &: \quad (z_1, z_2, z_3) \to (iz_1, -iz_2, z_3), \\ \beta &: \quad (z_1, z_2, z_3) \to (-z_1, -z_2 + \frac{1+i}{2}, z_3 + \frac{1}{2}). \end{aligned} \tag{6.5}$$

Restricted to z_1, z_2 coordinates, α is the Z_4 action of the T^4/Z_4 orbifold. F-theory on T^4/Z_4 was considered by Dasgupta and Mukhi[68]. The torus parametrized by z_2 has four fixed points under α^2. Two of them are fixed points of α as well, but the other two form a doublet under α. The singular fiber over each fixed point of α is of E_7 type. The other singular fiber over the doublet under α is of D_4 type. The F-theory on T^4/Z_4 has the gauge group $E_7 \times E_7 \times SO(8)$. The action β is a Z_2 involution accompanied by a half-shift along the z_3-torus. Since β preserves the holomorphic 2-form of the K3 orbifold and the holomorphic 1-form of the z_3 torus, this orbifold has N=2 supersymmetry in 6-dimensions. The special feature of β is that it induces nontrivial monodromy on the D_4 fiber while it interchanges two E_7 fibers.

The enhanced gauge group of the singular fiber which suffers the nontrivial monodromy along a nontrivial cycle of the base manifold was considered in detail in [71, 72]. The enhanced gauge group is the monodromy invariant part of the apparent local gauge group. The action of the monodromy on the blown-up fiber can be translated into an action on the Dynkin diagram of the simply-laced gauge group. The required group is the subgroup invariant under this outer automorphism. The gauge group coming from the D_4 fiber with the monodromy induced by β is $SO(7)$. The gauge group of the above model is $E_7 \times SO(7)$. As explained by Dasgupta and Mukhi, we cannot give the perturbative orientifold description of the above F-theory orbifold since the orbifold action of T^4/Z_4 corresponds to the nonperturbative symmetry of Type IIB theory. But we already established the duality between F-theory on $(K3 \times T^2)/Z_2$ and the CHL model at a particular point of the moduli space, hence the above model is necessarily dual to the CHL model in six dimensions.

Another model in the same moduli space is given by the orbifold generated by the following action when we consider the six-torus based on the hexagonal lattice i.e., $z_l \equiv z_l + 1 \equiv e^{\frac{2\pi i}{3}} z_l, l = 1, 2, 3$.

$$
\begin{aligned}
\alpha \quad &: \quad (z_1, z_2, z_3) \to (e^{\frac{2\pi i}{3}} z_1, e^{\frac{-2\pi i}{3}} z_2, z_3), \\
\beta \quad &: \quad (z_1, z_2, z_3) \to (-z_1, -z_2, z_3 + \frac{1}{2}).
\end{aligned} \tag{6.6}
$$

Restricted to z_1, z_2 coordinates, α is the Z_3 action of the T^4/Z_3 orbifold[68]. The torus parametrized by z_2 has three fixed points of α. Each singular fiber over the fixed point is of E_6 type. F-theory on T^4/Z_3 orbifold has the gauge group $E_6 \times E_6 \times E_6$. The action β interchanges two E_6 fibers and induces a nontrivial monodromy which reduces E_6 to F_4. The resulting gauge group is $E_6 \times F_4$.

In the three F-theory orbifolds considered so far, the base of each model is $(P^1 \times T^2)/Z_2$. But clearly one can consider more general situations. If we consider F-theory compactified on $(K3 \times T^2)/G$, the base is $(P^1 \times T^2)/G$. In order to have N=2 supersymmetry in 6-dimensions, we should restrict G to be a finite automorphism of $K3 \times T^2$ compatible with the elliptic fibration which preserves holomorphic forms of $K3$ and T^2.

As explained in [69], the sublattice of $\Gamma^{(18,2)}$ of the cohomology lattice of $K3$ determines an elliptic $K3$, where $\Gamma^{(p,q)}$ denotes a lattice of signature (p,q). The lattice $\Gamma^{(18,2)}$ is obtained from the cohomology lattice of $K3$, $H^2(K3) \oplus H^0(K3) \oplus H^4(K3) \simeq \Gamma^{(19,3)} \oplus \Gamma^{(1,1)}$ by splitting off the classes of the base and fiber of the fibration. Conversely, given an even-self-dual lattice of signature (18,2), one can find the corresponding elliptic $K3$ by the global Torelli theorem. Combined with additional $\Gamma^{(2,2)}$ lattice associated with T^2, this gives $\Gamma^{(18,2)} \oplus \Gamma^{(2,2)}$ which is isomorphic to $\Gamma^{(20,4)}$ and G acts on $\Gamma^{(20,4)}$ as a lattice isomorphism. This condition is the same one as we encounter in the general CHL compactification of heterotic string theory in six dimensions. Thus we expect that F-theory on $(K3 \times T^2)/G$ is dual to the CHL compactification of the heterotic string theory on $T^4/\tilde{G}$ where G and $\tilde{G}$ act on the same way on $\Gamma^{(20,4)}$.

If G is freely acting, G must act on T^2 by translation, which implies G should be an abelian group with at most two generators. Again if we compactify further on a circle, this theory is on the same moduli space as the M-theory on $(K3 \times T^2)/G$. This is indeed dual to other CHL constructions as investigated in [30], where G acts on $K3$ as an abelian symplectic automorphism, *i.e.*, an abelian automorphism preserving the holomorphic 2-form of $K3$. But more general configurations are possible. Such configurations in the Type IIA side are considered in [31]. If we compactify F-theory on $(K3 \times T^2)/G$ further on T^2, this model is on the same moduli space as the Type IIA theory on $(K3 \times T^2)/G$, which is dual to the heterotic theory on T^6/G. On the heterotic side, general G acts on $\Gamma^{(22,6)}$ as an lattice isomorphism with general shifts. The general shifts on the heterotic side is mapped to the choice of Ramond-Ramond fluxes localized at the fixed points of $K3$ under the heterotic-Type IIA duality. Thus the general configuration in the Type IIA side is the orbifold modded by symplectic automorphism of $K3$ with the Ramond-Ramond field background, which is not a conventional superconformal field theory background. General F-theory compactification can be thought to be the decompactifying limit of the corresponding Type IIA configuration when G is compatible with the elliptic structure, which implies that G acts nontrivially only on $\Gamma^{(20,4)}$ sublattice of $\Gamma^{(22,6)}$.

Since we have specified some of the N=2 F-theory vacua in 6-dimensions, one might wonder what else can appear as N=2 vacua of F-theory. It is explained in [55] that allowable bases for N=2 F-theory vacua are $K3, (P^1 \times T^2)/G$, and hyperelliptic surfaces. F-theory on the base K3 is the F-theory compactified on $T^2 \times K3$. The fiber structure should be trivial in order to retain the Calabi-Yau condition. This is just Type-IIB theory on K3. The only remaining category is F-theory having a hyperelliptic surface as base. A hyperelliptic surface is a complex torus modulo a finite group G acting freely. We can give a simple example of the F-theory which has a hyperelliptic surface as base. Consider the orbifold with the following Z_2 symmetry.

$$\alpha \quad : \quad (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3 + \frac{1}{2}). \tag{6.7}$$

The action α restricted to z_2, z_3 coordinates produces the hyperelliptic surface where the freely acting group is Z_2. The holomorphic 2-form $dz_1 \wedge dz_2$ and 1-form dz_3 survive under α, hence F-theory on this orbifold has $N = 2$ supersymmetry in 6-dimensions. Since α is freely acting, there are no singular elliptic fibers. Thus massless spectrum of this model is just non-chiral N=2 supergravity multiplet. We can find the orientifold dual of this model. Using the duality dictionary, we can map this orbifold into the orientifold with the group $\{1, \Omega(-1)^{F_L} R_{89}\eta_6\}$. After a similar calculation as we did earlier, one can check that the massless spectrum of the orientifold agrees with that of the F-theory orbifold. For other hyperelliptic surfaces[2], we can perform the similar analysis. Since hyperelliptic surfaces are quotients of torus by freely-acting action, there are no singular fibers for the associated elliptic three-fold. The massless spectrum is again N=2 supergravity multiplet in 6-dimensions.

[2]There are seven distinct classes of hyperelliptic surfaces. Those are explained in [70].

Chapter 7 World Volume Theory of M-theory Five-brane

7.1 Introduction

World-volume actions of p-branes encode much information about their dynamics. In the case of strings (in flat backgrounds) the world-volume theory has been quantized and used to construct the string perturbation expansion. In the case of p-branes with $p > 1$, one does not expect that it is possible to do the same. Still, many recent works have shown that an understanding of p-branes, including their excitations, can be very useful. Much non-perturbative information has been gleaned by considering vacua containing various branes of infinite extension. (A good example is provided by the 7-branes of F-theory [46].) Also, non-perturbative excitations described by wrapping p-branes about various cycles have played a central role in recent studies of black hole entropy as well as other problems [80, 81, 82]. We suspect that a more detailed characterization of p-brane world-volume dynamics will enable these studies to go further.

The actions for the class of supersymmetric p-branes whose only degrees of freedom are the superspace coordinates X and θ of the ambient space-time were constructed during the decade of the 1980's [83, 84, 85, 86, 87]. Much more recently, the actions for D-branes in type II theories have been constructed [88, 89, 90, 91]. In addition to the X and θ variables, these world-volume theories contain a U(1) gauge field with Born–Infeld self interactions [92, 93, 94, 95, 96, 97]. For maximally supersymmetric theories, the only significant p-brane action that remains to be formulated is that of the M theory five-brane [98, 99, 100, 101, 102]. This paper presents the solution.

The new feature that makes the M theory five-brane example somewhat more challenging than the other ones is the presence of a second-rank tensor gauge field, in addition to the X and θ coordinates [103]. This gauge field describes a chiral boson in the world volume, since its field strength is self-dual in the linearized approximation. It has been known for a long time that there is no straightforward way to construct a covariant action that describes propagation of the self-dual part of this field without also bringing in the anti-self-dual part [104]. Various proposals for dealing with this problem have been suggested over the years. The main one that we adopt is based on a formulation in which general coordinate invariance is only manifest in five of the six dimensions [105, 106, 107, 108]. It is also present in the sixth direction, but the transformation formulas that describe the symmetry are rather complicated. The bosonic part of the five-brane theory, constructed by this method, has been presented recently [109]. Another approach to the problem of the chiral boson uses an infinite number of auxiliary fields [110, 111, 112].

Very recently, a manifestly covariant formulation involving only a finite number of auxiliary fields (and compensating gauge invariances) has been introduced by Pasti, Sorokin, and Tonin [113, 114]. Constructions using the PST formulation turn out to be about as complicated as those in the formulation without manifest covariance. In fact, one of the new gauge invariances of the PST formulation involves the same subtleties as those of general coordinate invariance in the non-covariant approach, since one can gauge fix the PST formulas to obtain the non-covariant ones and show that compensating gauge transformations are the origin of the complicated general coordinate transformation.

Besides general coordinate invariance, the other essential symmetry of the world-volume theory of any super p-brane is a fermionic symmetry called kappa symmetry. It is always needed to remove half the degrees of freedom carried by the θ variables, leaving altogether eight propagating fermionic degrees of freedom. This is the same as the number of bosonic degrees of freedom, of course, as required by supersymmetry. The way this is achieved is by adding a suitable Wess–Zumino term to the action.

In all previous super p-brane examples, the global super-Poincaré symmetry (induced from an ambient flat space-time background) is implemented separately for the Wess–Zumino term and the other terms. The story in the case of the M theory five-brane has a surprising new feature. Namely, extending the bosonic five-brane theory to achieve global 11d super-Poincaré symmetry uniquely determines the complete action, including the Wess–Zumino term. The formula obtained in this way is then shown to have general coordinate invariance and local kappa symmetry. In the covariant PST formulation one is forced to organize the terms somewhat differently, so in that approach the story looks somewhat more conventional. Specifically, the covariant action divides naturally into two pieces: one piece is the supersymmetrized bosonic theory and the second is a separately supersymmetric Wess–Zumino term. The reason these statements are not in contradiction is that the PST gauge invariances, which are needed to achieve the right *bosonic* degrees of freedom, require that both terms be included.

This chapter is organized as follows. Section 2 reviews the construction of the bosonic part of the M theory five-brane action in both the non-covariant and the PST formulations. Section 3 then describes the supersymmetrization of this theory and the determination of the Wess–Zumino term in the non-covariant formulation. The proof that the resulting theory has (non-manifest) general coordinate invariance is given in Section 4. Section 5 presents the proof of kappa symmetry. The verification of two crucial identities is relegated to a pair of appendices. This section also sketches the corresponding formulas in the PST formulation. Section 6 describes double dimensional reduction, which gives rise to a 4-brane in 10d space-time. The resulting theory gives a dual formulation of the D4-brane of type IIA theory in which the theory is expressed in terms of a two-form gauge field instead of the dual U(1) vector gauge field. Some concluding remarks are made in Section 7.

7.2 Review of the Bosonic Theory

7.2.1 Formulation Without Manifest Covariance

Ref. [109] analyzed the problem of coupling a 6d self-dual tensor gauge field to a metric field so as to achieve general coordinate invariance. It presented a formulation in which one direction is treated differently from the other five. At the time that work was done, the author knew of no straightforward way to make the general covariance manifest. However, shortly thereafter a paper appeared [113] that presents equivalent results using a manifestly covariant formulation [114], which we refer to as the PST formulation. The relation between the two approaches will be described in the next subsection. As one might expect, they entail similar complications and there does not appear to be much advantage to one approach over the other. Therefore, we will present the supersymmetric M theory 5-brane action in the formulation without manifest covariance. This action corresponds to a partially gauge-fixed version of the corresponding action in the PST formulation.

In the present work we denote the 6d (world volume) coordinates by $\sigma^{\hat{\mu}} = (\sigma^\mu, \sigma^5)$, where $\mu = 0, 1, 2, 3, 4$. (In ref. [109] they were called $x^{\hat{\mu}}$.) The σ^5 direction is singled out as the one that will be treated differently from the other five.[1] The 6d metric $G_{\hat{\mu}\hat{\nu}}$ contains 5d pieces $G_{\mu\nu}, G_{\mu 5}$, and G_{55}. All formulas will be written with manifest 5d general coordinate invariance. As in refs. [108, 109], we represent the self-dual tensor gauge field by a 5×5 antisymmetric tensor $B_{\mu\nu}$, and its 5d curl by $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$. A useful quantity is the dual

$$\tilde{H}^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma}. \tag{7.1}$$

It was shown in ref. [109] that a class of generally covariant bosonic theories could be represented in the form $L = L_1 + L_2 + L_3$, where[2]

$$\begin{aligned} L_1 &= -\frac{1}{2}\sqrt{-G} f(z_1, z_2), \\ L_2 &= -\frac{1}{4}\tilde{H}^{\mu\nu}\partial_5 B_{\mu\nu}, \\ L_3 &= \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{H}^{\mu\nu}\tilde{H}^{\lambda\sigma}. \end{aligned} \tag{7.2}$$

[1]This is a space-like direction, but one could also choose a time-like one. (See the discussion in sect. 2.2.) The reason we prefer this choice is that in section 6, where we perform a double dimension reduction to obtain a 4-brane in 10d, elimination of the special dimension leaves manifestly covariant equations.

[2]The formula given in ref. [109] has been rescaled by an overall factor of $-1/2$.

The notation is as follows: G is the 6d determinant ($G = \det G_{\hat{\mu}\hat{\nu}}$) and G_5 is the 5d determinant ($G_5 = \det G_{\mu\nu}$), while G^{55} and $G^{5\rho}$ are components of the inverse 6d metric $G^{\hat{\mu}\hat{\nu}}$. The ϵ symbols are purely numerical with $\epsilon^{01234} = 1$ and $\epsilon^{\mu\nu\rho\lambda\sigma} = -\epsilon_{\mu\nu\rho\lambda\sigma}$. A useful relation is $G_5 = GG^{55}$. The z variables are defined to be

$$\begin{aligned} z_1 &= \frac{\mathrm{tr}(G\tilde{H}G\tilde{H})}{2(-G_5)} \\ z_2 &= \frac{\mathrm{tr}(G\tilde{H}G\tilde{H}G\tilde{H}G\tilde{H})}{4(-G_5)^2}. \end{aligned} \tag{7.3}$$

The trace only involves 5d indices:

$$\mathrm{tr}(G\tilde{H}G\tilde{H}) = G_{\mu\nu}\tilde{H}^{\nu\rho}G_{\rho\lambda}\tilde{H}^{\lambda\mu}. \tag{7.4}$$

The quantities z_1 and z_2 are scalars under 5d general coordinate transformations.

Infinitesimal parameters of general coordinate transformations are denoted $\xi^{\hat{\mu}} = (\xi^\mu, \xi)$. Since 5d general coordinate invariance is manifest, we focus on the ξ transformations only. The metric transforms in the standard way

$$\delta_\xi G_{\hat{\mu}\hat{\nu}} = \xi\partial_5 G_{\hat{\mu}\hat{\nu}} + \partial_{\hat{\mu}}\xi G_{5\hat{\nu}} + \partial_{\hat{\nu}}\xi G_{\hat{\mu}5}. \tag{7.5}$$

The variation of $B_{\mu\nu}$ is given by a more complicated rule, whose origin is explained in ref. [109]:

$$\delta_\xi B_{\mu\nu} = \xi K_{\mu\nu}, \tag{7.6}$$

where

$$K_{\mu\nu} = 2\frac{\partial(L_1 + L_3)}{\partial\tilde{H}^{\mu\nu}} = K_{\mu\nu}^{(1)}f_1 + K_{\mu\nu}^{(2)}f_2 + K_{\mu\nu}^{(\epsilon)} \tag{7.7}$$

with

$$\begin{aligned} K_{\mu\nu}^{(1)} &= \frac{\sqrt{-G}}{(-G_5)}(G\tilde{H}G)_{\mu\nu} \\ K_{\mu\nu}^{(2)} &= \frac{\sqrt{-G}}{(-G_5)^2}(G\tilde{H}G\tilde{H}G\tilde{H}G)_{\mu\nu} \end{aligned} \tag{7.8}$$

$$K^{(\epsilon)}_{\mu\nu} = \epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{2G^{55}}\tilde{H}^{\lambda\sigma},$$

and we have defined

$$f_i = \frac{\partial f}{\partial z_i}, \quad i = 1, 2. \tag{7.9}$$

Assembling the results given above, ref. [109] showed that the required general coordinate transformation symmetry is achieved if, and only if, the function f satisfies the nonlinear partial differential equation [115]

$$f_1^2 + z_1 f_1 f_2 + (\frac{1}{2}z_1^2 - z_2)f_2^2 = 1. \tag{7.10}$$

As discussed in [108], this equation has many solutions, but the one of relevance to the M theory five-brane is

$$f = 2\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2}. \tag{7.11}$$

For this choice L_1 can reexpressed in the Born–Infeld form

$$L_1 = -\sqrt{-\det\Big(G_{\hat{\mu}\hat{\nu}} + iG_{\hat{\mu}\rho}G_{\hat{\nu}\lambda}\tilde{H}^{\rho\lambda}/\sqrt{-G_5}\Big)}. \tag{7.12}$$

This expression is real, despite the factor of i, because it is an even function of $\tilde{H}$. Eliminating the factor of i would correspond to replacing z_1 by $-z_1$, which also solves the differential equation. However, it is essential for the five-brane application that the phases be chosen as shown.

7.2.2 The PST Formulation

In ref. [113] (using techniques developed in ref. [114]) results equivalent to those of the preceding subsection are described in a manifestly covariant way. To do this, the field $B_{\mu\nu}$ is extended to $B_{\hat{\mu}\hat{\nu}}$ with field strength $H_{\hat{\mu}\hat{\nu}\hat{\rho}}$. In addition, an auxiliary scalar field a is introduced. The PST formulation has new gauge symmetries (described below) that allow one to choose the gauge $B_{\mu 5} = 0$, $a = \sigma^5$ (and hence $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$). In this gauge, the covariant PST formulas reduce to those of sect. 2.1.

As will become clear, the scalar field a is really a zero-form potential with one-form field strength da. Only the field strength needs to be single-valued. Furthermore, for the action to be nonsingular, it is necessary that the 6 manifold M_6 admit nowhere null closed one-forms and that da be restricted to the class of such one-forms. It is allowed to be either time-like or space-like, however. This topological restriction on M_6 is consistent with the conclusions reached in ref. [100]

Equation (7.12) expressed L_1 in terms of the determinant of the 6×6 matrix

$$M_{\hat{\mu}\hat{\nu}} = G_{\hat{\mu}\hat{\nu}} + i\frac{G_{\hat{\mu}\rho}G_{\hat{\nu}\lambda}}{\sqrt{-GG^{55}}}\tilde{H}^{\rho\lambda}. \tag{7.13}$$

In the PST approach this is extended to the manifestly covariant form

$$M^{\text{cov.}}_{\hat{\mu}\hat{\nu}} = G_{\hat{\mu}\hat{\nu}} + i\frac{G_{\hat{\mu}\hat{\rho}}G_{\hat{\nu}\hat{\lambda}}}{\sqrt{-G(\partial a)^2}}\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}}. \tag{7.14}$$

The quantity

$$(\partial a)^2 = G^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}a\partial_{\hat{\nu}}a \tag{7.15}$$

reduces to G^{55} upon setting $\partial_{\hat{\mu}}a = \delta^5_{\hat{\mu}}$, and

$$\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}} \equiv \frac{1}{6}\epsilon^{\hat{\rho}\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\tau}}H_{\hat{\mu}\hat{\nu}\hat{\sigma}}\partial_{\hat{\tau}}a \tag{7.16}$$

reduces to $\tilde{H}^{\rho\lambda}$. Thus $M^{\text{cov.}}_{\hat{\mu}\hat{\nu}}$ replaces $M_{\hat{\mu}\hat{\nu}}$ in L_1. Furthermore, the expression

$$L' = -\frac{1}{4(\partial a)^2}\tilde{H}^{\hat{\mu}\hat{\nu}}_{\text{cov.}}H_{\hat{\mu}\hat{\nu}\hat{\rho}}G^{\hat{\rho}\hat{\lambda}}\partial_{\hat{\lambda}}a, \tag{7.17}$$

which transforms under general coordinate transformations as a scalar density, reduces to $L_2 + L_3$ upon gauge fixing. It is interesting that L_2 and L_3 are unified in this formulation.

Let us now describe the new gauge symmetries of ref. [113]. Since degrees of freedom a and $B_{\mu 5}$ have been added, corresponding gauge symmetries are required.

One of them is

$$\delta B_{\hat{\mu}\hat{\nu}} = 2\phi_{[\hat{\mu}}\partial_{\hat{\nu}]}a, \tag{7.18}$$

where $\phi_{\hat{\mu}}$ are infinitesimal parameters, and the other fields do not vary. In terms of differential forms, this implies $\delta H = d\phi da$. $\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}}$ is invariant under this transformation, since it corresponds to the dual of Hda, but $dada = 0$. Thus the covariant version of L_1 is invariant under this transformation. The variation of L', on the other hand, is a total derivative.

The second local symmetry involves an infinitesimal scalar parameter φ. The transformation rules are $\delta G_{\hat{\mu}\hat{\nu}} = 0, \delta a = \varphi$, and

$$\delta B_{\hat{\mu}\hat{\nu}} = \frac{1}{(\partial a)^2}\varphi H_{\hat{\mu}\hat{\nu}\hat{\rho}} G^{\hat{\rho}\hat{\lambda}}\partial_{\hat{\lambda}} a + \varphi V_{\hat{\mu}\hat{\nu}}, \tag{7.19}$$

where the quantity $V_{\hat{\mu}\hat{\nu}}$ is to be determined. This transformation is just as complicated as the non-manifest general coordinate transformation in the non-covariant formalism. Rather than derive it from scratch, let's see what is required to agree with the previous formulas after gauge fixing. In other words, we fix the gauge $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$ and $B_{\mu 5} = 0$, and figure out what the resulting ξ transformations are. We need

$$\delta a = \varphi + \xi\partial_5 a = \varphi + \xi = 0, \tag{7.20}$$

which tells us that $\varphi = -\xi$. Then

$$\begin{aligned}
\delta_\xi B_{\mu\nu} &= \frac{1}{(\partial a)^2}\varphi H_{\mu\nu\hat{\rho}} G^{\hat{\rho}\hat{\lambda}}\partial_{\hat{\lambda}} a + \varphi V_{\mu\nu} + \xi H_{5\mu\nu} \\
&= -\xi\left(\frac{G^{\rho 5}}{G^{55}} H_{\mu\nu\rho} + V_{\mu\nu}\right) = \xi(K^{(\epsilon)}_{\mu\nu} - V_{\mu\nu}).
\end{aligned} \tag{7.21}$$

Thus, comparing with eqs. (7.6) and (7.7), we need the covariant definition

$$V_{\hat{\mu}\hat{\nu}} = -2\frac{\partial L_1}{\partial \tilde{H}^{\hat{\mu}\hat{\nu}}_{\text{cov.}}} \tag{7.22}$$

to achieve agreement with our previous results.

To summarize, we have learned that the covariant PST formulation has new gauge transformations, and one of them encodes the complications that end up in general coordinate invariance after gauge fixing. Thus this formalism is not simpler than the non-covariant one. However, it is more symmetrical, and it does raise new questions, such as whether there are other gauge choices that are worth exploring.

7.3 Supersymmetrization

The super–Poincaré symmetry of the flat 11d space-time background should be implemented as a global symmetry of the five-brane theory. In terms of superspace coordinates X^M and θ, the 11d supersymmetry transformation is given by

$$\delta\theta = \epsilon \quad \text{and} \quad \delta X^M = \bar{\epsilon}\Gamma^M\theta. \tag{7.23}$$

Our convention is that the index M takes the values $M = 0, 1, \ldots, 9, 11$. Skipping $M = 10$ may seem a bit peculiar, but then X^{11} is the 11th dimension. Also, the Dirac matrix $\Gamma_{11} = \Gamma_0\Gamma_1 \ldots \Gamma_9$, which appears in ten dimensions as a chirality operator, is precisely the matrix we associate with the 11th dimension. The spinors ϵ and θ are 32-component Majorana spinors. The Dirac algebra is

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}, \tag{7.24}$$

where η_{MN} is the 11d Lorentz metric with signature $(- + + \ldots +)$.

As in other supersymmetric p-brane theories, two supersymmetric quantities are $\partial_{\hat{\mu}}\theta$ and

$$\Pi^M_{\hat{\mu}} = \partial_{\hat{\mu}}X^M - \bar{\theta}\Gamma^M\partial_{\hat{\mu}}\theta. \tag{7.25}$$

The appropriate choice for the world-volume metric is then the supersymmetric quantity

$$G_{\hat{\mu}\hat{\nu}} = \eta_{MN}\Pi^M_{\hat{\mu}}\Pi^N_{\hat{\nu}}. \tag{7.26}$$

Taking θ and X^M to be scalars under world-volume general coordinate transformations, $G_{\hat{\mu}\hat{\nu}}$ transforms in the standard way.

In addition, we require an appropriate supersymmetric extension of $H = dB$, which we write as

$$\mathcal{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - b_{\mu\nu\rho}, \tag{7.27}$$

or, in terms of differential forms, $\mathcal{H} = H - b_3$. The idea is to choose a b_3 whose supersymmetry variation is exact, so that it can be cancelled by an appropriate variation of B. The appropriate choice turns out to be

$$b_3 = \frac{1}{6} b_{\mu\nu\rho} d\sigma^\mu d\sigma^\nu d\sigma^\rho = \frac{1}{2}\bar{\theta}\Gamma_{MN} d\theta (dX^M dX^N + dX^M \bar{\theta}\Gamma^N d\theta + \frac{1}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N d\theta). \tag{7.28}$$

Varying this, using $\delta_\epsilon\theta = \epsilon$ and $\delta_\epsilon X^M = \bar{\epsilon}\Gamma^M\theta$, one finds that $\mathcal{H}$ is invariant for the choice

$$\delta_\epsilon B = -\frac{1}{2}\bar{\epsilon}\Gamma_{MN}\theta(dX^M dX^N + \frac{2}{3}\bar{\theta}\Gamma^M d\theta dX^N + \frac{1}{15}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N d\theta)$$

$$-\frac{1}{6}\bar{\epsilon}\Gamma_M\theta\bar{\theta}\Gamma_{MN} d\theta(dX^N + \frac{1}{5}\bar{\theta}\Gamma^N d\theta). \tag{7.29}$$

A useful (and standard) identity that has been used in deriving this result is

$$d\bar{\theta}\Gamma^M d\theta d\bar{\theta}\Gamma_{MN} + d\bar{\theta}\Gamma_{MN} d\theta d\bar{\theta}\Gamma^M = 0. \tag{7.30}$$

The overall normalization of b_3 and $\delta_\epsilon B$ could be scaled arbitrarily (including zero) as far as the present reasoning is concerned. The specific choice that has been made is the one that will be required later. We also note, for future reference, that

$$d\mathcal{H} = -db_3 = -\frac{1}{2} d\bar{\theta}\Gamma_{MN} d\theta \Pi^M \Pi^N = -\frac{1}{2} d\bar{\theta}\psi_5^2 d\theta, \tag{7.31}$$

where we have introduced the matrix valued one-form

$$\psi_5 = \Gamma_M \Pi^M_\mu d\sigma^\mu. \tag{7.32}$$

With these choices for $G_{\hat{\mu}\hat{\nu}}$ and $\mathcal{H}$, we can now write down extensions of L_1 and L_3 that have manifest 11d super-Poincaré symmetry:

$$\begin{aligned} L_1 &= -\sqrt{-G}\sqrt{1+z_1+\frac{1}{2}z_1^2-z_2} \\ L_3 &= \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}, \end{aligned} \tag{7.33}$$

where z_1 and z_2 are now formed from $\mathcal{H}$ instead of H.

The next step is to construct a supersymmetric extension of L_2. This term is the Wess–Zumino term, which can be represented as the integral of a closed 7-form I_7 over a region that has the 6d world volume M_6 as its boundary. In other words,

$$S_2 = \int_{M_7} I_7 = \int_{M_6} \Omega_6, \tag{7.34}$$

where $I_7 = d\Omega_6$ and $M_6 = \partial M_7$. The appropriate expression for I_7 that reproduces L_2 of the purely bosonic theory is

$$I_7^{(B)} = -\frac{1}{2}HdH = \frac{1}{2}H\partial_5 H d\sigma^5. \tag{7.35}$$

To understand this properly, there is a point that needs to be stressed. Namely, in adding a formal 7th dimension, the extra dimension is required to enter symmetrically with the first five. There continues to be one preferred direction, σ^5, that is treated specially. Correspondingly, in writing $M_6 = \partial M_7$, the boundary operator should not act on the σ^5 direction. In other words, M_7 should have no σ^5 = constant faces. It should also be noted that this M theory five-brane theory action has a Wess–Zumino term that survives even for the bosonic truncation in a flat space-time background. However, as we will see in the next subsection, this feature is particular to the non-covariant formulation and is not shared by the PST formulation in which the pieces of the action are arranged somewhat differently.

To complete the construction of L_2 we must now supersymmetrize $I_7^{(B)}$. The term $\frac{1}{2}\mathcal{H}\partial_5\mathcal{H}d\sigma^5$ achieves this, of course, but it is no longer closed. Additional terms should be added such that $dI_7 = 0$, up to a total derivative in the σ^5 direction. The result that we find is

$$I_7 = \frac{1}{2}\mathcal{H}\partial_5\mathcal{H}d\sigma^5 - \frac{1}{2}\mathcal{H}d\bar{\theta}\psi\!\!\!/^2 d\theta - \frac{1}{120}d\bar{\theta}\psi\!\!\!/^5 d\theta, \tag{7.36}$$

where

$$\psi\!\!\!/ = \Gamma_M \Pi^M_{\hat{\mu}} d\sigma^{\hat{\mu}} = \psi\!\!\!/_5 + \Gamma_M \Pi^M_5 d\sigma^5. \tag{7.37}$$

When interpreting the 4-form $d\theta\psi\!\!\!/^2 d\theta$ and the 7-form $d\theta\psi\!\!\!/^5 d\theta$ it must be understood that one of the derivatives is required to be in the σ^5 direction. The proof that dI_7 is a total σ^5 derivative is reasonably straightforward using the identity (7.30) as well as

$$\frac{1}{6}(d\bar{\theta}\Gamma_{MNPQR}d\theta d\bar{\theta}\Gamma^R + d\bar{\theta}\Gamma^R d\theta d\bar{\theta}\Gamma_{MNPQR}) = d\bar{\theta}\Gamma_{[MN}d\theta d\bar{\theta}\Gamma_{PQ]}. \tag{7.38}$$

Since I_7 is manifestly supersymmetric, it is guaranteed that Ω_6 is invariant up to a total derivative under a supersymmetry transformation. For most purposes an explicit formula for L_2 is not required. Here we will simply report that

$$L_2 = -\frac{1}{4}\tilde{H}^{\mu\nu}(\partial_5 B_{\mu\nu} - 2b_{\mu\nu}) + \text{ terms indep. of } B, \tag{7.39}$$

where $b_2 = \frac{1}{2}b_{\mu\nu}d\sigma^\mu d\sigma^\nu$ is given by[3]

$$\begin{aligned} b_2 \;\; = \;\; & -\frac{1}{2}\bar{\theta}\Gamma_{MN}\partial_5\theta(dX^M dX^N + dX^M\bar{\theta}\Gamma^N d\theta + \frac{1}{3}d\bar{\theta}\Gamma^M d\theta d\bar{\theta}\Gamma^N d\theta) \\ & +\frac{1}{2}\bar{\theta}\Gamma_{MN}d\theta(2dX^M\partial_5 X^N - \partial_5 X^M\bar{\theta}\Gamma^N d\theta - dX^M\bar{\theta}\Gamma^N\partial_5\theta - \frac{2}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N\partial_5\theta). \end{aligned}$$

Knowing this much of L_2 is sufficient to obtain the $B_{\mu\nu}$ equation of motion.

7.4 General Coordinate Invariance

We should now check whether the general coordinate invariance of the bosonic theory in sect. 2.1 continues to hold after adding terms depending on θ in the way that we

[3]This expression is equal to $b_{\mu\nu5}$, where $b_{\hat{\mu}\hat{\nu}\hat{\rho}}$ is the covariant extension of the expression given in eq. (7.28).

have described. As in the bosonic case, general coordinate invariance in five directions is manifest, so only the transformation in the σ^5 direction needs to be checked. The coordinates X^M and θ transform as scalars, *i.e.*,

$$\delta_\xi X^M = \xi \partial_5 X^M \quad \text{and} \quad \delta_\xi \theta = \xi \partial_5 \theta, \tag{7.40}$$

which implies that $G_{\hat\mu\hat\nu}$ transforms as in eq. (7.5). To specify the proper transformation law for $B_{\mu\nu}$, we should first examine its equation of motion. Using eq. (7.39), this is

$$\epsilon^{\mu\nu\rho\lambda\sigma} \partial_\rho \left(K_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} + b_{\lambda\sigma}\right) = 0. \tag{7.41}$$

The formula for $K_{\mu\nu}$ is as given in eqs. (7.7) and (7.8), except that now L_1 and L_3 of the supersymmetrized theory should be used. This simply amounts to replacing H by $\mathcal{H}$ and using the supersymmetric expression for $G_{\hat\mu\hat\nu}$. By the reasoning explained in ref. [109], the B equation of motion suggests that the appropriate transformation formula, generalizing eq. (7.6), is

$$\delta_\xi B_{\mu\nu} = \xi(K_{\mu\nu} + b_{\mu\nu}). \tag{7.42}$$

To determine $\delta_\xi \mathcal{H}$, one first computes that

$$\delta_\xi b_3 = \xi \partial_5 b_3 + b_2 d\xi. \tag{7.43}$$

It follows that

$$\delta_\xi \mathcal{H} = d(\delta_\xi B) - \xi \partial_5 b_3 - b_2 d\xi = d(\xi K) - \xi Z_3, \tag{7.44}$$

where

$$Z_3 = \partial_5 b_3 - d b_2. \tag{7.45}$$

This can be made manifestly supersymmetric by noting that

$$Z_3 d\sigma^5 = (\partial_5 b_3 - d b_2) d\sigma^5 = -\frac{1}{2} d\bar\theta \psi^2 d\theta. \tag{7.46}$$

The 4-form on the right-hand side of this equation is required to contain one σ^5 derivative.

The important point is that the Z_3 term in $\delta_\xi \mathcal{H}$ has no counterpart in the bosonic theory, so general coordinate invariance of the supersymmetric theory is not an immediate consequence of the corresponding symmetry of the bosonic theory. Let us examine next the part of $\delta_\xi(L_1 + L_3)$ that arises from varying $\mathcal{H}$, but not G. It is

$$\delta_\xi \tilde{\mathcal{H}}^{\mu\nu} \frac{\partial(L_1 + L_3)}{\partial \tilde{\mathcal{H}}^{\mu\nu}} = \frac{1}{2} \delta_\xi \tilde{\mathcal{H}}^{\mu\nu} K_{\mu\nu}. \tag{7.47}$$

This is conveniently characterized by the 5-form

$$(d(\xi K) - \xi Z_3) K \sim -\xi K (dK + Z_3), \tag{7.48}$$

where $\sim$ means that a total derivative has been dropped.

Consider now the ξ transformation of L_2. A portion of L_2 was given in eq. (7.39). Representing this as a 5-form and using

$$\delta_\xi b_2 = \partial_5(\xi b_2), \tag{7.49}$$

one obtains

$$\begin{aligned} \delta_\xi L_2 &= -(\partial_5 B - b_2) d(\xi(K + b_2)) + H \partial_5(\xi b_2) + \dots \\ &\sim \xi K(\partial_5 \mathcal{H} + Z_3) + \frac{1}{2} b_2^2 d\xi + \dots \end{aligned} \tag{7.50}$$

where the dots are the contribution from varying the H independent terms in L_2. The $\dots$ terms precisely cancel the b_2^2 term, leaving

$$\delta_\xi L_2 \sim \xi K(\partial_5 \mathcal{H} + Z_3). \tag{7.51}$$

The demonstration that the $\dots$ terms contribute $-\frac{1}{2} b_2^2 d\xi$ can be made as follows. The first two terms in eq. (7.36) contribute the non-H pieces

$$\frac{1}{2}b_3\partial_5 b_3 d\sigma^5 + \frac{1}{2}b_3 d\bar{\theta}\psi^2 d\theta, \tag{7.52}$$

which has a non-trivial ξ transformation, because of the asymmetric way in which the σ^5 direction appears. The variation is easy to compute, and can be expressed as the exterior derivative of $-\frac{1}{2}b_2^2 d\xi$, which implies that this contributes the required variation of L_2.

Combining eq. (7.51) with eq. (7.48) leaves

$$\delta_{\mathcal{H}}(L_1 + L_3) + \delta_\xi L_2 \sim \xi K(\partial_5 \mathcal{H} - dK). \tag{7.53}$$

This must now be combined with the terms arising from varying $G_{\hat{\mu}\hat{\nu}}$ in L_1 and L_3. However, at this point all terms whose structure is peculiar to the supersymmetric theory have cancelled. The rest of the calculation is identical to that for the bosonic theory given in ref. [109] and, therefore, need not be repeated here.

7.5 Proof of Kappa Symmetry

7.5.1 Formulation Without Manifest Covariance

As with all other super p-branes of maximally supersymmetric theories, the world-volume theory should have 8 bosonic and 8 fermionic physical degrees of freedom. This requires, in particular, the existence of a local fermionic symmetry (called kappa) that eliminates half of the components of θ. Despite the lack of manifest general coordinate invariance, the analysis of kappa symmetry for the M theory five-brane is very similar to that of other super p-branes. As usual, we require that

$$\delta\bar{\theta} = \bar{\kappa}(1 - \gamma), \tag{7.54}$$

where $\kappa(\sigma)$ is an arbitrary Majorana spinor and γ is a quantity (to be determined) whose square is the unit matrix. This implies that $\frac{1}{2}(1-\gamma)$ is a projection operator, and half of the components of θ can be gauged away. In addition, just as for all other super p-branes, we require that

$$\delta X^M = -\delta\bar{\theta}\Gamma^M\theta, \tag{7.55}$$

so that

$$\delta\Pi^M_{\hat{\mu}} = -2\delta\bar{\theta}\Gamma^M\partial_{\hat{\mu}}\theta. \tag{7.56}$$

As in our other work [89], we introduce the induced γ matrix

$$\gamma_{\hat{\mu}} = \Pi^M_{\hat{\mu}}\Gamma_M, \tag{7.57}$$

which satisfies

$$\{\gamma_{\hat{\mu}}, \gamma_{\hat{\nu}}\} = 2G_{\hat{\mu}\hat{\nu}}. \tag{7.58}$$

In this notation, the kappa variation of the metric is

$$\delta G_{\hat{\mu}\hat{\nu}} = -2\delta\bar{\theta}(\gamma_{\hat{\mu}}\partial_{\hat{\nu}} + \gamma_{\hat{\nu}}\partial_{\hat{\mu}})\theta. \tag{7.59}$$

Before we can examine the symmetry of our theory, we must also specify the kappa variation of $B_{\mu\nu}$. This works in a way that is analogous to that of the world-volume gauge field for D-branes. Specifically, for the choice

$$\delta B = \frac{1}{2}\delta\bar{\theta}\Gamma_{MN}\theta(dX^M dX^N + \bar{\theta}\Gamma^M d\theta dX^N + \frac{1}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N d\theta)$$
$$+\frac{1}{2}\delta\bar{\theta}\Gamma^M\theta\bar{\theta}\Gamma_{MN}d\theta(dX^N + \frac{1}{3}\bar{\theta}\Gamma^N d\theta), \tag{7.60}$$

we find that most of the terms in $\delta\mathcal{H}$ cancel leaving

$$\delta\mathcal{H}_{\mu\nu\rho} = 6\delta\bar{\theta}\gamma_{[\mu\nu}\partial_{\rho]}\theta \tag{7.61}$$

or, equivalently,

$$\delta\tilde{\mathcal{H}}^{\mu\nu} = \epsilon^{\mu\nu\rho\lambda\sigma}\delta\bar{\theta}\gamma_{\rho\lambda}\partial_\sigma\theta. \tag{7.62}$$

Since we now have the complete theory and all the field transformations, it is just a matter of computation to check the symmetry.

Before plunging into the details of the calculation, it is helpful to sketch the general strategy that will be employed. It turns out to be convenient to consider L_2 and L_3 together and to write their kappa variation in the form

$$\delta(L_2 + L_3) = \frac{1}{2}\delta\bar{\theta}T^{\hat{\mu}}\partial_{\hat{\mu}}\theta. \tag{7.63}$$

The variation of L_1 is represented in a similar manner:

$$\delta L_1 = -\frac{1}{2L_1}\delta\bar{\theta}U^{\hat{\mu}}\partial_{\hat{\mu}}\theta. \tag{7.64}$$

Then, in order that $\delta\bar{\theta} = \bar{\kappa}(1-\gamma)$ should be a symmetry, we require that altogether

$$\delta(L_1 + L_2 + L_3) = \frac{1}{2}\delta\bar{\theta}(1+\gamma)T^{\hat{\mu}}\partial_{\hat{\mu}}\theta, \tag{7.65}$$

which is achieved if

$$U^{\hat{\mu}} = \rho T^{\hat{\mu}}, \tag{7.66}$$

where

$$\rho = -\gamma L_1 = \gamma\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2}. \tag{7.67}$$

This implies that

$$\rho^2 = -G(1 + z_1 + \frac{1}{2}z_1^2 - z_2). \tag{7.68}$$

We must vary the Lagrangian to find $T^{\hat{\mu}}$ and $U^{\hat{\mu}}$, and then determine ρ with the proper square and show that $U^{\hat{\mu}} = \rho T^{\hat{\mu}}$. This is all straightforward, but it needs to be done carefully.

Since the σ^5 direction appears asymmetrically in the Lagrangian, the analysis of $U^{\hat{\mu}} = \rho T^{\hat{\mu}}$ is naturally split into two separate problems, corresponding to $\hat{\mu} = 5$ and

$\hat{\mu} \neq 5$. The $\hat{\mu} = 5$ case is the easier of the two, so let us begin with that. We must examine where we can get $\partial_5\theta$'s. The variations of $B_{\mu\nu}$ and $G_{\mu\nu}$ do not give any. Therefore, in varying L_1, the variations of z_1 and z_2 do not contribute. The only contribution comes from

$$\delta\sqrt{-G} = -2\sqrt{-G}\delta\bar{\theta}\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\theta, \tag{7.69}$$

where, of course, $\gamma^{\hat{\mu}} = G^{\hat{\mu}\hat{\nu}}\gamma_{\hat{\nu}}$. Thus

$$U^5 = -4\rho^2\gamma^5. \tag{7.70}$$

To determine T^5 we must vary $L_2 + L_3$. Using the identity

$$\delta\left(\frac{G^{5\rho}}{G^{55}}\right) = 2\frac{G_5^{\eta\rho}G^{5\hat{\mu}}}{G^{55}}\delta\bar{\theta}(\gamma_{\hat{\mu}}\partial_\eta + \gamma_\eta\partial_{\hat{\mu}})\theta, \tag{7.71}$$

the relevant piece of δL_3 is

$$\frac{1}{4}\epsilon_{\mu\nu\rho\lambda\sigma}G_5^{\eta\rho}\delta\bar{\theta}\gamma_\eta\partial_5\theta\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}, \tag{7.72}$$

which contributes

$$T_2^5 = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda\sigma}G_5^{\eta\rho}\gamma_\eta\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \tag{7.73}$$

to T^5. (The subscript on T represents the power of $\mathcal{H}$.)

The variation of the Wess–Zumino term S_2 is

$$\delta S_2 = \int(\mathcal{H}\delta\bar{\theta}\psi^2 d\theta - \frac{1}{60}\delta\bar{\theta}\psi^5 d\theta), \tag{7.74}$$

a result that is obtained by expressing δI_7 as a total differential. This determines $T_0^5 + T_1^5$, with

$$T_0^5 = -\frac{1}{30}\epsilon^{\mu_1\ldots\mu_5}\gamma_{\mu_1\ldots\mu_5} = -4\bar{\gamma}\gamma^5, \tag{7.75}$$

where we have introduced

$$\bar{\gamma} = \gamma_{012345}, \tag{7.76}$$

which satisfies $(\bar{\gamma})^2 = -G$. The $\mathcal{H}$ linear term is

$$T_1^5 = -2\tilde{\mathcal{H}}^{\mu\nu}\gamma_{\mu\nu}. \tag{7.77}$$

Combining these results with

$$U^5 = -4\rho^2\gamma^5 = \rho T^5, \tag{7.78}$$

we infer that $T^5 = -4\rho\gamma^5$, where

$$\rho = \bar{\gamma} + \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^5 - \frac{1}{8G^{55}}\epsilon_{\mu\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma 5}. \tag{7.79}$$

To obtain the $\mathcal{H}^2$ term we have used the identity

$$G_5^{\eta\sigma}\gamma_\eta = \gamma^\sigma - \frac{G^{\sigma 5}}{G^{55}}\gamma^5, \tag{7.80}$$

from which it follows that

$$G_5^{\eta\sigma}\gamma_\eta\gamma^5 = \gamma^{\sigma 5}. \tag{7.81}$$

If our reasoning is correct, this expression for ρ should have the square given in eq. (7.68). This fact is verified in Appendix A.

To complete the proof of kappa symmetry, we must find U^μ and T^μ and show that $U^\mu = \rho T^\mu$. Separating powers of $\mathcal{H}$, as above, the variation of L_2 contributes to T_0^μ and T_1^μ while the variation of L_3 contributes to T_1^μ and T_2^μ. Altogether, we find that

$$\begin{aligned}
T_0^\mu &= -4\bar{\gamma}\gamma^\mu \\
T_1^\mu &= -\frac{2}{G^{55}}(G^{5\mu}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho} + 2\tilde{\mathcal{H}}^{\mu\nu}\gamma_\nu\gamma^5) \\
T_2^\mu &= \frac{1}{2G^{55}}\epsilon_{\eta\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}(G^{5\mu}G_5^{\eta\zeta}\gamma_\zeta + G_5^{\mu\eta}\gamma^5).
\end{aligned} \tag{7.82}$$

The variation of L_1 determines $U^\mu = \sum_{n=0}^4 U_n^\mu$, where

$$U_0^\mu = 4G\gamma^\mu \tag{7.83}$$

$$
\begin{aligned}
U_1^\mu &= -\frac{1}{G^{55}}\epsilon^{\mu\nu\rho\lambda\sigma}\gamma_{\lambda\sigma}(G\tilde{\mathcal{H}}G)_{\nu\rho} \\
U_2^\mu &= -\frac{4}{G^{55}}\gamma_\nu(\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu} - \frac{2}{(G^{55})^2}G^{5\mu}\gamma^5\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}) \\
U_3^\mu &= \frac{1}{G(G^{55})^2}\epsilon^{\mu\nu\rho\lambda\sigma}\gamma_{\lambda\sigma}\left(\frac{1}{2}(G\tilde{\mathcal{H}}G)_{\nu\rho}\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}) - (G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G)_{\nu\rho}\right) \\
U_4^\mu &= \frac{4}{G(G^{55})^2}\gamma_\nu\left(\frac{1}{2}(\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu}\mathrm{tr}(G\mathcal{H}G\mathcal{H}) - (\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu}\right) \\
&+ \frac{2}{G(G^{55})^2}\left(G^{\mu 5}\gamma^5 - \frac{1}{2}G^{55}\gamma^\mu\right)\left(\frac{1}{2}(\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}))^2 - \mathrm{tr}(G\mathcal{H}G\mathcal{H}G\mathcal{H}G\mathcal{H})\right).
\end{aligned}
$$

The demonstration that $U^\mu = \rho T^\mu$ is presented in Appendix B.

In conclusion, we have shown that the theory specified by $L_1 + L_2 + L_3$ has all the desired symmetries: global 11d super–Poincaré symmetry, general coordinate invariance, and local kappa symmetry.

7.5.2 Supersymmetric Theory in the PST Formulation

The supersymmetric theory that we have just presented can be recast in a manifestly general covariant form, using the PST formalism, just as we did for the bosonic theory in sect. 2.2. In order to keep the notation from being too cumbersome, in this section (and only in this section) indices μ, ν, etc., take six values, (*i.e.*, we drop the hats used until now). Also the label "cov." is dropped. Thus, upon supersymmetrization, eq. (7.14), for example, becomes

$$M_{\mu\nu} = G_{\mu\nu} + i\frac{G_{\mu\rho}G_{\nu\lambda}}{\sqrt{-G(\partial a)^2}}\tilde{\mathcal{H}}^{\rho\lambda}, \tag{7.84}$$

where

$$\tilde{\mathcal{H}}^{\rho\lambda} = \frac{1}{6}\epsilon^{\rho\lambda\mu\nu\sigma\tau}\mathcal{H}_{\mu\nu\sigma}\partial_\tau a. \tag{7.85}$$

Also, $G_{\mu\nu}$ is constructed as in eqs. (7.25) and (7.26), and $\mathcal{H} = H - b_3$ is extended to six dimensions. In this notation the supersymmetric theory is given by $L = L_1 + L' + L_{WZ}$, where

$$L_1 = -\sqrt{-\det M_{\mu\nu}}$$

$$\begin{aligned} L' &= -\frac{1}{4(\partial a)^2}\tilde{\mathcal{H}}^{\mu\nu}\mathcal{H}_{\mu\nu\rho}G^{\rho\lambda}\partial_\lambda a & (7.86)\\ S_{WZ} &= \int \Omega_6. \end{aligned}$$

L_1 can again be recast in the form

$$L_1 = -\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2}, \qquad (7.87)$$

where now z_1 and z_2 are the obvious covariant counterparts of those in eq. (7.3). The Wess–Zumino term is again characterized by a seven-form $I_7 = d\Omega_6$, where now

$$I_7 = -\frac{1}{4}\mathcal{H}d\bar{\theta}\psi^2 d\theta - \frac{1}{120}d\bar{\theta}\psi^5 d\theta. \qquad (7.88)$$

It is easy to check that $dI_7 = 0$ using eqs. (7.30) and (7.38). Global ϵ supersymmetry and local reparametrization symmetry are manifest in these formulas. Note that neither the metric $G_{\mu\nu}$ nor the scalar field a occur in L_{WZ}.

When one chooses the gauge $a = \sigma^5$ and $B_{\mu 5} = 0$, the Lagrangian given above reduces to the one in sect. 3. The way this happens is somewhat non-trivial. The point is that L' reduces to L_3 and a portion of the non-covariant Wess–Zumino term L_2. Specifically, in the gauge-fixed theory the sum over the index ρ in the formula for L' can be separated into $\rho = 5$ and $\rho \neq 5$ terms. The $\rho \neq 5$ term accounts for L_3 of the gauge-fixed theory, while the $\rho = 5$ term accounts for the $\mathcal{H}^2$ piece of L_2 and a portion of the $\mathcal{H}$ piece. In particular, this accounts for why the coefficient of the $\mathcal{H}$ linear term in eq. (7.88) differs from that in eq. (7.36).

The proof of kappa symmetry in the PST formulation works as before (with $\delta a = 0$), so we will not repeat the argument.[4] The covariant extension of eq. (7.79) is

$$\rho = \bar{\gamma} + \frac{1}{2(\partial a)^2}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^\lambda\partial_\lambda a - \frac{1}{16(\partial a)^2}\epsilon_{\mu\nu\rho\lambda\sigma\tau}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma\tau}. \qquad (7.89)$$

The demonstration that $\rho^2 = -\det M_{\mu\nu}$ is essentially the same as in Appendix A.

[4]Also, D. Sorokin informs us that it will appear soon in a paper by him and collaborators.

The covariant formula for $T^\mu = T_0^\mu + T_1^\mu + T_2^\mu$ is given by

$$\begin{aligned} T_0^\mu &= -4\bar{\gamma}\gamma^\mu \\ T_1^\mu &= -\frac{2}{(\partial a)^2}\tilde{\mathcal{H}}^{\nu\rho}(\gamma_{\nu\rho}G^{\mu\lambda} - 2\delta_\rho^\mu\gamma_\nu\gamma^\lambda)\partial_\lambda a \\ T_2^\mu &= -\frac{1}{(\partial a)^2}\tilde{\mathcal{H}}^{\eta\nu}\mathcal{H}_{\eta\nu\rho}(\gamma^\rho G^{\lambda\mu} + \gamma^\lambda G^{\rho\mu})\partial_\lambda a \\ &\quad + \frac{2}{[(\partial a)^2]^2}\tilde{\mathcal{H}}^{\eta\nu}\mathcal{H}_{\eta\nu\rho}G^{\rho\lambda}\partial_\lambda a\gamma^\sigma\partial_\sigma a G^{\mu\zeta}\partial_\zeta a. \end{aligned} \tag{7.90}$$

In the $B_{\mu 5} = 0$, $a = \sigma^5$ gauge, these expressions reduce to the formulas T^5 and T^μ given in eqs. (7.73), (7.75), (7.77), and (7.82). The proof of kappa symmetry works essentially the same as before.

7.6 Double-Dimensional Reduction

As is now well-known, when one of the ten spatial dimensions of M theory is a small circle of radius R, the theory can be reinterpreted as Type IIA string theory in ten dimensions with string coupling constant proportional to $R^{3/2}$ [116, 117]. The five-brane of M theory can then give rise to either a five-brane or a four-brane of Type IIA string theory depending on whether or not it wraps around the circular dimension. Here we wish to focus on the case that it does wrap (once) so that one obtains a four-brane. This case is called "double-dimensional reduction," because the dimension of the brane and the dimension of the ambient space-time have been reduced by one at the same time. (The first example of this type to be studied was the double-dimensional reduction of the M theory two-brane, which gives the Type IIA fundamental string [87].) The known 4-brane of Type IIA string theory is, in fact, a D-brane, which implies that its world-volume theory contains an abelian vector gauge field. However, the five-brane theory that we have constructed contains an antisymmetric tensor gauge field, which remains one even after the reduction. However, as we will show elsewhere [118], the D4-brane action and the 4-brane with antisymmetric tensor gauge field obtained below, are related by a world-volume duality transformation. This is analogous to the relationship between the M2-brane and the D2-brane [119, 120, 121].

The covariant action for the dual D4-brane in ten dimensions can be obtained from the M theory five-brane action by setting

$$X^{11} = \sigma^5 \tag{7.91}$$

and then dropping all dependence on σ^5, *i.e.*, extracting the zeroth Fourier mode. Doing this gives

$$\begin{aligned} \psi &\rightarrow \psi + C\Gamma_{11} & (7.92)\\ G_{\mu\nu} &\rightarrow G_{\mu\nu} + C_\mu C_\nu & (7.93)\\ b_3 &\rightarrow C_3, & (7.94) \end{aligned}$$

where

$$C_\mu = -\bar{\theta}\Gamma^{11}\partial_\mu\theta \tag{7.95}$$

is the part of Π_μ^{11} that survives. C and

$$C_3 = b_3 + \frac{1}{2}\bar{\theta}\Gamma_{11}\Gamma_n d\theta\bar{\theta}\Gamma^{11}d\theta(dX^n + \frac{2}{3}\bar{\theta}\Gamma^n d\theta) \tag{7.96}$$

enter in the D4-brane Wess-Zumino term. In these formulae quantities on the left (right) of the arrow have target space indices summed on 11 (10) values (e.g., $\psi = \Gamma_M\Pi^M$ on the L.H.S., $\psi = \Gamma_m\Pi^m$ on the R.H.S., where $m = 0, 1, \ldots, 9$ and $M = (m, 11)$). Also,

$$\begin{aligned} G = \det G_{\hat{\mu}\hat{\nu}} &\rightarrow G = \det G_{\mu\nu} \\ G_5 = \det G_{\mu\nu} &\rightarrow \det(G_{\mu\nu} + C_\mu C_\nu) = G(1 + C^2), & (7.97) \end{aligned}$$

where

$$C^2 \equiv G^{\mu\nu}C_\mu C_\nu. \tag{7.98}$$

One can analyze the double-dimensional reduction of the action. A straightforward calculation shows that

$$\det\left(G_{\hat\mu\hat\nu} + i\frac{G_{\hat\mu\rho}G_{\hat\nu\lambda}\tilde{\mathcal{H}}^{\rho\lambda}}{\sqrt{-G_5}}\right) \rightarrow \det\left(G_{\mu\nu} + i\frac{G_{\mu\rho}G_{\nu\lambda}\tilde{\mathcal{H}}^{\rho\lambda}}{\sqrt{-G(1+C^2)}} + Y_\mu Y_\nu\right) \tag{7.99}$$

with

$$Y_\mu \equiv i\frac{G_{\mu\rho}\tilde{\mathcal{H}}^{\rho\lambda}C_\lambda}{\sqrt{-G(1+C^2)}}, \tag{7.100}$$

which gives the double-dimensionally reduced version of L_1. For L_3 the answer is:

$$L_3 = \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \rightarrow -\frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{C^\rho}{1+C^2}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}. \tag{7.101}$$

The Wess–Zumino term is given by the reduction

$$I_7 \rightarrow I_6 = -\frac{1}{4!}d\bar\theta\Gamma_{11}\psi^4 d\theta + \mathcal{H}d\bar\theta\Gamma_{11}\psi d\theta. \tag{7.102}$$

Under double-dimensional reduction

$$d\mathcal{H} = -\frac{1}{2}d\bar\theta\psi^2 d\theta \rightarrow -\frac{1}{2}d\bar\theta\psi^2 d\theta + d\bar\theta\Gamma_{11}\psi d\theta C, \tag{7.103}$$

whose supersymmetry variation is

$$\delta_\epsilon d\mathcal{H} \rightarrow d\bar\theta\Gamma_{11}\psi d\theta\bar\epsilon\Gamma^{11}d\theta. \tag{7.104}$$

From this one can infer that

$$\delta_\epsilon\mathcal{H} \rightarrow (\bar\epsilon\Gamma^{11}\theta)d\bar\theta\Gamma_{11}\psi d\theta + \text{total derivative}. \tag{7.105}$$

It is an interesting fact that, after the double-dimensional reduction, $\mathcal{H}$ is no longer invariant under supersymmetry. We will show below that the formula has a simple interpretation, which ensures that the reduced theory is supersymmetric. The kappa variations of the doubly dimensionally reduced theory can be analyzed in a similar manner. One finds that

$$\delta\mathcal{H} = -\delta\bar{\theta}\psi^2 d\theta \to -\delta\bar{\theta}\psi^2 d\theta + 2\delta\bar{\theta}\Gamma_{11}\psi d\theta C. \tag{7.106}$$

In order to preserve the gauge choice (7.91), both the supersymmetry and the κ variations of the 4-brane fields must include compensating σ^5 general coordinate transformations:

$$\begin{aligned} 0 &= \delta_\epsilon X^{11} + \xi_\epsilon^{\hat{\mu}}\partial_{\hat{\mu}} X^{11} = \bar{\epsilon}\Gamma^{11}\theta + \xi_\epsilon \\ &\Rightarrow \ \xi_\epsilon = -\bar{\epsilon}\Gamma^{11}\theta \\ 0 &= \delta X^{11} + \xi_\kappa^{\hat{\mu}}\partial_{\hat{\mu}} X^{11} = -\delta\bar{\theta}\Gamma^{11}\theta + \xi_\kappa \\ &\Rightarrow \ \xi_\kappa = \delta\bar{\theta}\Gamma^{11}\theta. \end{aligned} \tag{7.107}$$

Upon double-dimensional reduction the induced general coordinate transformation parameter ξ only appears in the quantities (see eqs. (7.44) and (7.45))

$$\delta_\xi\mathcal{H} = d(\xi K) + \xi db_2 \tag{7.108}$$

and

$$\delta_\xi C_\mu = \partial_\mu \xi. \tag{7.109}$$

The supersymmetry variations of C and $\mathcal{H}$ are entirely given by the induced σ^5 general coordinate transformation. Therefore supersymmetry of the theory after double dimensional reduction is a consequence of both the supersymmetry and the general coordinate invariance of the original 6d theory. As a consistency check, one can show that eq. (7.108) with $\xi = \xi_\epsilon$ reproduces eq. (7.105). Kappa symmetry works similarly:

$$\delta C_\mu = -\delta\bar{\theta}\Gamma^{11}\partial_\mu\theta - \bar{\theta}\Gamma^{11}\partial_\mu\delta\theta = \partial_\mu\xi_\kappa - 2\delta\theta\Gamma^{11}\partial_\mu\theta, \tag{7.110}$$

where the second term is the remnant of the κ variation of $G_{\mu 5}$. Looking at $\delta(d\mathcal{H})$

we can compute

$$\delta\mathcal{H} = -\delta\bar{\theta}\psi^2 d\theta + 2\delta\bar{\theta}\Gamma_{11}\psi d\theta C - (\delta\bar{\theta}\Gamma^{11}\theta)d\bar{\theta}\Gamma_{11}\psi d\theta + \text{total derivative}, \qquad (7.111)$$

which is reproduced by combining eqs. (7.106) and (7.108) for $\xi = \xi_\kappa$.

7.7 Discussion

This paper has presented the world-volume action of the M theory five-brane in a flat 11d background. The required global and local symmetries have been verified in detail using a formulation in which one world-volume direction is treated differently from the others. The corresponding results in the manifestly covariant PST formulation have also been presented. Although we have not done it, we expect that it would be reasonably straightforward to extend the results to an arbitrary background, as has been done for D-branes in refs. [90, 91]. All the considerations in this paper have been classical, but there are undoubtedly various quantum implications. In fact, it has been suggested recently that certain supersymmetric 6d theories can have non-trivial renormalization group fixed points [122]. Perhaps our five-brane action is of this type.

The five-brane world-volume theory has a solitonic solution [108] that describes a finite-tension self-dual string of the type discussed in [123]. We think that it will be very interesting to study this string and its excitation spectrum, which could then be compared to the spectrum conjectured in [124]. It is curious that the five-brane, which itself arises as a soliton of the 11d theory, has its own solitons. Upon double dimensional reduction to the IIA 4-brane, as discussed in sect. 6, the self-dual string can either wrap or not wrap. This reflects the fact that the D4-brane has both point-like and string-like solitons, which are electric-magnetic duals of one another. The point-like solitons can also be viewed as describing bound states of D4-branes and D0-branes with the D0-brane charge representing momentum in the compact dimension. The string-like solitons do not appear to have an analogous interpretation.

Another direction that we think deserves to be explored is how the M5-brane should be described in the background that describes the $E_8 \times E_8$ theory [67]. The 5-brane in such a background will have half as much supersymmetry as we have described, corresponding to $N = 1$ in 10d. More significantly, it should have a soliton solution that describes a "heterotic" self-dual string. The gauge group, whose currents would appear as left-movers, should be E_8 [125, 126]. It would also be interesting to explore how wrapping M5-branes on suitable 2-cycles gives rise to Seiberg–Witten theories in the unwrapped dimensions [127].

Appendix A – Evaluation of ρ^2

This appendix will show that $\rho^2 = -G(1 + z_1 + \frac{1}{2}z_1^2 - z_2)$, where

$$\rho = \bar{\gamma} + \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^5 - \frac{1}{8G^{55}}\epsilon_{\mu\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma 5}. \tag{7.112}$$

It is convenient to rewrite ρ_2 (the subscript refers to the order in $\mathcal{H}$) as

$$\rho_2 = \frac{1}{8G_5}\bar{\gamma}\gamma_{\mu\nu\rho\lambda}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}, \tag{7.113}$$

where we have used

$$\frac{1}{k!}\epsilon_{\hat{\mu}_1\ldots\hat{\mu}_6}\gamma^{\hat{\mu}_1\ldots\hat{\mu}_k} = \frac{1}{G}(-1)^{\frac{k(k+1)}{2}}\bar{\gamma}\gamma_{\hat{\mu}_{k+1}\ldots\hat{\mu}_6}. \tag{7.114}$$

The matrix $\bar{\gamma}$ anticommutes with all γ^μ's, so $\{\rho_0, \rho_1\} = 0$ and $[\rho_0, \rho_2] = 0$. Furthermore,

$$\{\rho_1, \rho_2\} \sim [\gamma_{\alpha\beta}, \gamma_{\mu\nu\rho\sigma}]\tilde{\mathcal{H}}^{\alpha\beta}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma} = 0 \tag{7.115}$$

as the commutator is antisymmetric over six 5-valued indices. Thus,

$$\rho^2 = \rho_0^2 + \rho_1^2 + 2\rho_0\rho_2 + \rho_2^2. \tag{7.116}$$

We know already that $\rho_0^2 = -G$ and $\rho_0\rho_2 = -\frac{1}{8G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}\gamma_{\mu\nu\rho\sigma}$. So we need

$$\begin{aligned}\rho_1^2 &= \frac{1}{4G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}\gamma_{\mu\nu}\gamma_{\rho\sigma} = \frac{1}{4G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}(\gamma_{\mu\nu\rho\sigma} - 2G_{\mu\rho}G_{\nu\sigma}) \\ &= \frac{1}{4G^{55}}[\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}\gamma_{\mu\nu\rho\sigma} + 2\mathrm{tr}\,(\tilde{\mathcal{H}}^2)],\end{aligned}$$

where $\mathrm{tr}\,(\tilde{\mathcal{H}}^2)$ represents $\mathrm{tr}\,(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}})$. Thus, $\rho_1^2 + 2\rho_0\rho_2 = -Gz_1$. Finally,

$$\rho_2^2 = -\frac{G}{64G_5^2}\tilde{\mathcal{H}}^{\mu_1\mu_2}\tilde{\mathcal{H}}^{\mu_3\mu_4}\tilde{\mathcal{H}}^{\nu_1\nu_2}\tilde{\mathcal{H}}^{\nu_3\nu_4}\gamma_{\mu_1\mu_2\mu_3\mu_4}\gamma_{\nu_1\nu_2\nu_3\nu_4}.$$

In the multiplication of gamma matrices[5] one can argue that the only terms that contribute after contraction with the $\mathcal{H}$'s are effectively

$$\gamma_{\mu_1\mu_2\mu_3\mu_4}\gamma_{\nu_1\nu_2\nu_3\nu_4} \sim 8G_{\mu_1\nu_1}G_{\mu_2\nu_2}G_{\mu_3\nu_3}G_{\mu_4\nu_4} - 16G_{\mu_1\nu_2}G_{\mu_2\nu_3}G_{\mu_3\nu_4}G_{\mu_4\nu_1},$$

and thus

$$\rho_2^2 = -\frac{G}{4G_5^2}(\frac{1}{2}\mathrm{tr}\,(\tilde{\mathcal{H}}^2)^2 - \mathrm{tr}\,(\tilde{\mathcal{H}}^4)) = -G(\frac{1}{2}z_1^2 - z_2). \tag{7.117}$$

Collecting all the terms, we obtain the desired relation:

$$\rho^2 = -G(1 + z_1 + \frac{1}{2}z_1^2 - z_2). \tag{7.118}$$

Appendix B – Evaluation of ρT^μ

We wish to demonstrate that $\rho T^\mu = U^\mu$, where ρ, T^μ, and U^μ are given by eqs. (7.79), (7.82), and (7.83), respectively. The calculation is somewhat messy, so we proceed order by order in $\mathcal{H}$.

To zeroth order, $\rho_0 = \bar{\gamma}$ and $T_0^\mu = -4\bar{\gamma}\gamma^\mu$ give $\rho_0 T_0^\mu = 4G\gamma^\mu = U_0^\mu$. The linear order contribution comes from $(\rho T^\mu)_1 = \rho_1 T_0^\mu + \rho_0 T_1^\mu$, where

$$\rho_1 = \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}{}^5, \qquad T_1^\mu = -\frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}(G^{5\mu}\gamma_{\nu\rho} + 2G^\mu{}_\nu\gamma_\rho{}^5). \tag{7.119}$$

Since

$$\rho_1 T_0^\mu = -\frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}{}^5\bar{\gamma}\gamma^\mu = \frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\bar{\gamma}(\gamma_{\nu\rho}{}^{5\mu} + G^{5\mu}\gamma_{\nu\rho} + 2G^\mu{}_\nu\gamma_\rho{}^5), \tag{7.120}$$

[5] A useful generalization of the relation $\gamma_{\mu_1\ldots\mu_m}\gamma_\mu = \gamma_{\mu_1\ldots\mu_m\mu} + m\gamma_{[\mu_1\ldots\mu_{m-1}}G_{\mu_m]\mu}$ is

$$\gamma_{\mu_1\ldots\mu_m}\gamma_{\nu_1\ldots\nu_n} = \sum_{k=0}^{min(m,n)} C_k^{mn}\gamma_{\mu_1\ldots\mu_{m-k}\nu_1\ldots\nu_{n-k}}G_{\mu_{m-k+1}\nu_{n-k+1}}\ldots G_{\mu_m\nu_n},$$

where $C_k^{mn} \equiv (-1)^{kn+\frac{k(k+1)}{2}}k!\binom{m}{k}\binom{n}{k}$. The terms in the sum are antisymmetrized over all μ's and ν's separately.

we obtain

$$(\rho T^\mu)_1 = \frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\bar{\gamma}\gamma_{\nu\rho}{}^{5\mu} = -\frac{1}{G^{55}}\epsilon^{\nu\rho\lambda\sigma\mu}\gamma_{\nu\rho}(G\tilde{\mathcal{H}}G)_{\lambda\sigma}, \tag{7.121}$$

where eq. (7.114) has been used in obtaining the second equality. Thus, $(\rho T^\mu)_1 = U_1^\mu$.

The higher-order calculations somewhat simplify if one rewrites ρ_2 as

$$\rho_2 = \frac{1}{8G_5}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}\bar{\gamma}\gamma_{\nu\rho\lambda\sigma} \tag{7.122}$$

and T_2^μ as

$$\begin{aligned} T_2^\mu &= \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}\epsilon_{\eta\nu\rho\lambda\sigma}(G^{\mu 5}G_5^{\eta\zeta}\gamma_\zeta + G_5^{\mu\eta}\gamma^5) &\quad (7.123)\\ &= \frac{1}{2G_5}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}\bar{\gamma}(\gamma_{\nu\rho\lambda\sigma}{}^\mu - 2\frac{G^{\mu 5}}{G^{55}}\gamma_{\nu\rho\lambda\sigma}{}^5), &\quad (7.124)\end{aligned}$$

using eqs. (7.80) and (7.81). In quadratic order,

$$(\rho T^\mu)_2 = \rho_0 T_2^\mu + \rho_1 T_1^\mu + \rho_2 T_0^\mu. \tag{7.125}$$

If we factor out $(\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma})$ as a common factor in all terms,

$$\begin{aligned} \rho_0 T_2^\mu &\sim -\frac{1}{2G^{55}}(\gamma_{\nu\rho\lambda\sigma}{}^\mu - \frac{2G^{5\mu}}{G^{55}}\gamma_{\nu\rho\lambda\sigma}{}^5)\\ \rho_1 T_1^\mu &\sim -\frac{1}{(G^{55})^2}\gamma_{\nu\rho}{}^5[G^{5\mu}\gamma_{\lambda\sigma} + 2G^\mu{}_\lambda\gamma_\sigma{}^5] =\\ &\quad -\frac{1}{(G^{55})^2}[G^{5\mu}(\gamma_{\nu\rho\lambda\sigma}{}^5 - 2G_{\lambda\nu}G_{\sigma\rho}\gamma^5) - 2G^{55}G^\mu{}_\lambda(\gamma_{\nu\rho\sigma} + 2\gamma_\nu G_{\sigma\rho})]\\ \rho_2 T_0^\mu &\sim -\frac{1}{2G_5}\bar{\gamma}\gamma_{\nu\rho\lambda\sigma}\bar{\gamma}\gamma^\mu = \frac{1}{2G^{55}}(\gamma_{\nu\rho\lambda\sigma}{}^\mu + 4\gamma_{\nu\rho\lambda}G_\sigma{}^\mu).\end{aligned}$$

Combining these contributions and reinstating $(\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma})$ gives

$$\begin{aligned} (\rho T^\mu)_2 &= \frac{1}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}[2\frac{G^{5\mu}}{G^{55}}G_{\lambda\nu}G_{\sigma\rho}\gamma^5 + 4G^\mu{}_\lambda\gamma_\nu G_{\sigma\rho}] = &\quad (7.126)\\ &= -\frac{2}{G^{55}}[\frac{G^{5\mu}}{G^{55}}\mathrm{tr}\,(\tilde{\mathcal{H}}^2)\gamma^5 + 2(\tilde{\mathcal{H}}^2)^{\mu\nu}\gamma_\nu]\\ &= U_2^\mu. &\quad (7.127)\end{aligned}$$

At cubic order in $\mathcal{H}$,

$$(\rho T^{\mu})_3 = \rho_1 T_2^{\mu} + \rho_2 T_1^{\mu}. \tag{7.128}$$

Let the common factor to be $(\tilde{\mathcal{H}}^{\alpha\beta}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma})$. Since,

$$\gamma^5(\gamma_{\nu\rho\lambda\sigma}{}^{\mu} - 2\frac{G^{\mu 5}}{G^{55}}\gamma_{\nu\rho\lambda\sigma}{}^{5}) = -\gamma_{\nu\rho\lambda\sigma}{}^{\mu}\gamma^5, \tag{7.129}$$

we get

$$\begin{aligned}
\rho_1 T_2^{\mu} &\sim \frac{1}{4G^{55}G_5}\bar{\gamma}\gamma_{\alpha\beta}\gamma_{\nu\rho\lambda\sigma}{}^{\mu}\gamma^5 \\
&\sim \frac{2}{G^{55}G_5}\bar{\gamma}(-\gamma_{\nu\rho\lambda}G_{\alpha\sigma}\delta^{\mu}_{\beta} + \frac{1}{2}\gamma_{\nu\rho}{}^{\mu}G_{\alpha\sigma}G_{\beta\lambda} - \gamma_{\nu\lambda}{}^{\mu}G_{\alpha\sigma}G_{\beta\rho})\gamma^5.
\end{aligned}$$

The second term in eq. (7.128) can also be simplified:

$$\begin{aligned}
\rho_2 T_1^{\mu} &\sim -\frac{1}{4G^{55}G_5}\bar{\gamma}\gamma_{\nu\rho\lambda\sigma}(G^{\mu 5}\gamma_{\alpha\beta} + 2G^{\mu}{}_{\beta}\gamma_{\alpha}\gamma^5) \\
&\sim -\frac{2}{G^{55}G_5}\bar{\gamma}[G^{\mu 5}(-\frac{1}{2}\gamma_{\nu\rho}G_{\alpha\lambda}G_{\beta\sigma} + \gamma_{\nu\lambda}G_{\alpha\rho}G_{\beta\sigma}) - \gamma_{\nu\rho\lambda}G_{\alpha\sigma}\delta^{\mu}_{\beta}\gamma^5].
\end{aligned}$$

Thus,

$$\begin{aligned}
(\rho T^{\mu})_3 &= \frac{2}{G^{55}G_5}\bar{\gamma}[\gamma_{\nu\rho}{}^{\mu}\gamma^5 - G^{\mu 5}\gamma_{\nu\rho}][\frac{1}{2}\tilde{\mathcal{H}}^{\nu\rho}\mathrm{tr}\,(\tilde{\mathcal{H}}^2) - (\tilde{\mathcal{H}}^3)^{\nu\rho}] \\
&= \frac{1}{G^{55}G_5}\epsilon^{\alpha\beta\nu\rho\mu}\gamma_{\alpha\beta}[\frac{1}{2}\tilde{\mathcal{H}}\mathrm{tr}\,(\tilde{\mathcal{H}}^2) - \tilde{\mathcal{H}}^3]_{\nu\rho} \\
&= U_3^{\mu}.
\end{aligned} \tag{7.130}$$

Finally, in the quartic order,

$$\rho_2 T_2^{\mu} = -\frac{G}{16G_5^2}\tilde{\mathcal{H}}^{\mu_1\mu_2}\tilde{\mathcal{H}}^{\mu_3\mu_4}\tilde{\mathcal{H}}^{\nu_1\nu_2}\tilde{\mathcal{H}}^{\nu_3\nu_4}\gamma_{\mu_1\mu_2\mu_3\mu_4}(\gamma_{\nu_1\nu_2\nu_3\nu_4}{}^{\mu} - 2\frac{G^{\mu 5}}{G^{55}}\gamma_{\nu_1\nu_2\nu_3\nu_4}{}^{5}). \tag{7.131}$$

The relevant contribution of γ's in this case is

$$\begin{aligned}
\gamma_{\mu_1\mu_2\mu_3\mu_4}\gamma_{\nu_1\nu_2\nu_3\nu_4}{}^{\mu} \sim {}& [8\gamma^{\mu}(G_{\mu_1\nu_1}G_{\mu_2\nu_2}G_{\mu_3\nu_3}G_{\mu_4\nu_4} - 2G_{\mu_2\nu_1}G_{\mu_3\nu_2}G_{\mu_4\nu_3}G_{\mu_1\nu_4}) \\
&-32\gamma_{\nu_1}(\delta^{\mu}_{\mu_1}G_{\mu_2\nu_2}G_{\mu_3\nu_3}G_{\mu_4\nu_4} - 2\delta^{\mu}_{\mu_2}G_{\mu_3\nu_2}G_{\mu_4\nu_3}G_{\mu_1\nu_4})].
\end{aligned}$$

It follows that

$$\begin{aligned}(\rho T^{\mu})_4 &= -\frac{G}{(G_5)^2}\{(\gamma^{\mu} - 2\frac{G^{\mu 5}}{G^{55}}\gamma^5)[\frac{1}{2}(\mathrm{tr}\,(\tilde{\mathcal{H}}^2))^2 - (\mathrm{tr}\,(\tilde{\mathcal{H}}^4)] \\ &\quad -4\gamma_{\nu}[\frac{1}{2}\mathrm{tr}\,(\tilde{\mathcal{H}}^2)\tilde{\mathcal{H}}^2 - \tilde{\mathcal{H}}^4]^{\mu\nu}\} \\ &= U^{\mu}_4. \end{aligned} \tag{7.132}$$

This completes the proof.

Chapter 8 Dual D-brane Action

8.1 Introduction

Several groups have recently constructed supersymmetric D-brane actions with local kappa symmetry [88, 89, 91]. These supersymmetric actions are a good starting point for studying various properties of D-branes. For example, since D-branes play a reasonably well understood role in the web of string dualities, there are definite expectations for how each D-brane action should transform under a duality transformation of its world-volume gauge field. These reflect the $SL(2, Z)$ S duality of the type IIB superstring theory [138] and the relationship between type IIA superstring theory at strong coupling and 11d M theory compactified on a circle [139, 140]. Previous studies of these duality transformations have been carried out in the context of the bosonic truncation of the D-brane action [121, 141]. Since the super D-branes are BPS objects, it is appropriate to study their duality properties including the fermionic degrees of freedom. Such an investigation, which is possible now that the super D-brane actions are known, is the purpose of this chapter.

In ref. [89] we only considered super D-branes in a flat background. Here, as a modest extension of this, we include a constant dilaton background for the type IIA D-branes and constant dilaton and axion backgrounds for the type IIB D-branes. Starting with the D1-brane, we show that one can obtain the expected $SL(2, Z)$ multiplet of type IIB strings [144, 145], with the correct tensions, by performing duality transformations. In the case of the D2-brane, we show that the dual action describes the M2-brane with one target space dimension compactified. (The relationship between the D2-brane and the M2-brane has been discussed previously [120], so this part is mostly review.) In particular, we verify that the dilaton dependence of the D2-brane correctly reproduces the relation between the string metric of the IIA theory and the 11d metric of the M theory. This implies that the type IIA string coupling constant is correctly related to the radius of the 11th dimension [140]. Next we show that the D3-brane action is mapped into an equivalent D3-brane action by the duality transformation, thereby verifying the expected $SL(2, Z)$ invariance of the D3-brane. Another correspondence suggested by the duality between M theory and type IIA superstring theory is that the double-dimensional reduction of the M5-brane action should coincide with the duality transformed D4-brane action. This could not be checked previously, since we did not have a suitable supersymmetric M5-brane action. However this action has been constructed recently [146], so we are now in a position to verify that the dual D4-brane action is identical to the double-dimensional reduction of the M5-brane action as expected. (The M5-brane has also been discussed recently in refs. [114, 147].) Finally, we indicate how duality transformations relate specific gauge choices for the gauge-fixed D-brane actions.

The calculation of the duality transformations of supersymmetric D-brane actions is quite similar to that of the bosonic actions described previously in refs. [121, 141, 142, 143]. Since the behavior of the fermionic degrees of freedom under the duality transformation is the new ingredient, this is the part of the analysis that is emphasized. Unless otherwise stated, the conventions used here are the same as those of ref. [89].

8.2 Dual Born–Infeld Actions

The essential steps involved in world-volume duality transformations of D-brane actions can be described for the simpler problem of Born–Infeld theory. Subsequent sections will discuss the extensions that are required for various supersymmetric D-brane actions. Born–Infeld theory in $n = p + 1$ dimensions is given by

$$S = -\int d^n\sigma \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})}, \tag{8.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field strength. The basic idea is to recast the theory in terms of a dual $(p-2)$-form potential $B_{\mu_1\mu_2\ldots\mu_{p-2}}$ given by

$$-\frac{\delta S}{\delta F_{\mu\nu}} = \tilde{H}^{\mu\nu} = \frac{1}{(p-2)!}\epsilon^{\mu\nu\lambda\rho_1\ldots\rho_{p-2}}\partial_\lambda B_{\rho_1\ldots\rho_{p-2}}. \tag{8.2}$$

The Bianchi identity for the B field is the field equation for the Maxwell field. Also, the Bianchi identity of the Maxwell field provides the field equation for the B field. To make the latter equation explicit one needs to solve eq. (8.2) for $F_{\mu\nu}$. Then one can construct an action that gives the field equation. Equivalently, one can add a Lagrange multiplier term $\frac{1}{2}\tilde{H}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu)$ to eq. (8.1) and eliminate F.

To solve eq. (8.2) for $F_{\mu\nu}$, it is convenient to use Lorentz invariance to bring $F_{\mu\nu}$ to the canonical form

$$F_{\mu\nu} = \begin{pmatrix} 0 & f_1 & & & \\ -f_1 & 0 & & & \\ & & 0 & f_2 & \\ & & -f_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{8.3}$$

Then eq. (8.2) implies that $\tilde{H}^{\mu\nu}$ has the same structure

$$\tilde{H}^{\mu\nu} = \begin{pmatrix} 0 & h_1 & & & \\ -h_1 & 0 & & & \\ & & 0 & h_2 & \\ & & -h_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{8.4}$$

In this notation, eq. (8.2) becomes[1]

$$h_i = \frac{f_i}{1+f_i^2}\sqrt{\prod(1+f_j^2)}. \tag{8.5}$$

[1]This is the formula for Euclidean signature. The extension to Lorentzian signature is straightforward.

For $p \leq 4$ there are at most two f's and this equation can be inverted. The result for this case is

$$f_i = \frac{h_i}{1 - h_i^2} \sqrt{\prod (1 - h_j^2)}. \tag{8.6}$$

Unfortunately, for $p > 4$, when there are three or more f's, we are unable to carry out the inversion. With three f's we find a quintic equation. The coefficients of the quintic are not completely generic, so a closed form solution might exist, but we have not found one.

Having found the field equation of the B field in a special frame, it is easy to pass to a general frame and write an action that gives the desired equation. The result is (the subscript D stands for "dual")

$$S_D = -\int d^n\sigma \sqrt{-\det(\eta_{\mu\nu} + i\tilde{H}_{\mu\nu})}. \tag{8.7}$$

We emphasize, once again, that this result is only correct for $p \leq 4$ or $n \leq 5$.

Now consider the more general Born–Infeld action

$$S = -\int d^n\sigma \sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})}, \tag{8.8}$$

where $G_{\mu\nu}$ is a symmetric tensor field and $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - b_{\mu\nu}$ is an antisymmetric tensor field. Repeating the analysis described above in this more general setting gives (for $p \leq 4$)

$$S_D = -\int d^n\sigma (\sqrt{-\det(G_{\mu\nu} + iK_{\mu\nu})} + \frac{1}{2}\tilde{H}^{\mu\nu} b_{\mu\nu}), \tag{8.9}$$

where

$$K_{\mu\nu} = \frac{1}{\sqrt{-\det G}} G_{\mu\rho} G_{\nu\lambda} \tilde{H}^{\rho\lambda}. \tag{8.10}$$

The $H \wedge b_2$ term in S_D will be identified as part of the Wess–Zumino term of the dual D-brane.

The analysis described here is not the whole story for super D-branes, since they also contain Wess–Zumino terms that are polynomial functions of $F_{\mu\nu}$. Specifically, they are linear in F for $p = 2, 3$, quadratic in F for $p = 4, 5$, and so forth. The extension of the analysis given here to include these terms will be described on a case-by-case basis in the sections that follow.

8.3 The D1-brane

Let us consider the D1-brane (*i.e.*, the type IIB D-string) first. If we include the dependence on a *constant* dilaton ϕ, the action of the super D1-brane with kappa symmetry is given by

$$S = \int d^2\sigma\Big(- e^{-\phi}\sqrt{-\det\left(G_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)} + L_{WZ}\Big). \tag{8.11}$$

Here $G_{\mu\nu} = \eta_{mn}\Pi^m_\mu\Pi^n_\nu$, where $\Pi^m_\mu = \partial_\mu X^m - \bar{\theta}\Gamma^m\partial_\mu\theta$. Also, X^m and θ are coordinates of type IIB superspace and η_{mn} is the 10d Minkowski metric. The induced world volume metric $G_{\mu\nu}$ is the supersymmetrized pullback of the 10d string metric η_{mn}. Also, $\mathcal{F} = F - b_2$ with $b_2 = -\bar{\theta}\tau_3\Gamma_m d\theta(dX^m + \frac{1}{2}\bar{\theta}\Gamma^m d\theta)$. L_{WZ} denotes the Wess–Zumino term, which can be represented by a 2-form on the world volume of the D1-brane. Specifically,

$$S_{WZ} = \int d^2\sigma L_{WZ} = e^{-\phi}\int_{M_2} C_2, \tag{8.12}$$

where $C_2 = \bar{\theta}\tau_1\Gamma_m d\theta(dX^m + \frac{1}{2}\bar{\theta}\Gamma^m d\theta)$ and $dC_2 = d\bar{\theta}\tau_1\Gamma_m d\theta\Pi^m$ with $\Pi^m = dX^m + \bar{\theta}\Gamma^m d\theta$. Note that eq. (8.11) is the D1-brane action of ref. [89] rescaled by the string coupling constant. One can also add a total derivative term (analogous to the θ term of QCD) to the Wess–Zumino term in eq. (8.12):

$$e^{-\phi}C_2 \to e^{-\phi}C_2 - C_0 F, \tag{8.13}$$

where C_0 is a constant "axion" background field. Since $C_0 F$ is a total derivative, it does not affect the classical equations of motion. A constant shift of C_0 is a trivial classical symmetry of the action (8.11). In the quantum theory it is replaced by a quantized shift, just as in QCD. This reflects the breaking of the classical $SL(2, R)$

symmetry to the quantum $SL(2,Z)$ symmetry.

Now let us perform the duality transformation. Following ref. [141], one introduces a Lagrange multiplier field $\tilde{H}^{\mu\nu} = -\tilde{H}^{\nu\mu}$ as follows

$$S' = \int d^2\sigma\Big(-e^{-\phi}\sqrt{-\det\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)}+\frac{1}{2}\tilde{H}^{\mu\nu}(F_{\mu\nu}-2\partial_\mu A_\nu)+\frac{1}{2}e^{-\phi}\epsilon^{\mu\nu}C_{\mu\nu}-\frac{1}{2}C_0\epsilon^{\mu\nu}F_{\mu\nu}\Big) \tag{8.14}$$

and considers $F_{\mu\nu}$ to be an independent field. Varying A_ν gives $\partial_\mu \tilde{H}^{\mu\nu} = 0$, which implies that $\tilde{H}^{\mu\nu} = \epsilon^{\mu\nu}\Lambda$ with Λ constant. This gives $S' = S_1 + S_2$, where

$$S_1 = \int d^2\sigma\Big(-e^{-\phi}\sqrt{-\det\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)}+\frac{1}{2}(\Lambda - C_0)\epsilon^{\mu\nu}\mathcal{F}_{\mu\nu}\Big) \tag{8.15}$$

$$S_2 = \int e^{-\phi}C_2 + (\Lambda - C_0)b_2. \tag{8.16}$$

Our convention is that whenever an integral appears without a $d^n\sigma$ it is an integral of a differential form. It can be easily converted to a usual integral. For example, $\int F = \int d^2\sigma\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}$. The basic strategy, described in the preceding section, is to use the equation of motion for F to rewrite the action in terms of Λ instead of F. The duality transformation of S_1 is the same as the bosonic case, if we replace F by $\mathcal{F}$. Thus the dual of S_1 is

$$S_{1D} = -\int d^2\sigma\sqrt{e^{-2\phi}+(\Lambda - C_0)^2}\sqrt{-\det G_{\mu\nu}}, \tag{8.17}$$

while S_2 is unaffected by the duality transformation.

In eq. (8.16), we have

$$e^{-\phi}C_2 + (\Lambda - C_0)b_2 = \bar{\theta}(e^{-\phi}\tau_1 - (\Lambda - C_0)\tau_3)\Gamma_m d\theta \wedge (dX^m + \frac{1}{2}\bar{\theta}\Gamma^m d\theta). \tag{8.18}$$

Since the eigenvalues of $e^{-\phi}\tau_1 - (\Lambda - C_0)\tau_3$ are $\pm\sqrt{e^{-2\phi}+(\Lambda - C_0)^2}$ we can redefine

$$e^{-\phi}\tau_1 - (\Lambda - C_0)\tau_3 \equiv \sqrt{e^{-2\phi}+(\Lambda - C_0)^2}\tau_3'. \tag{8.19}$$

Then the total action can be written as

$$S_D = \sqrt{e^{-2\phi} + (\Lambda - C_0)^2} \int d^2\sigma \Big(- \sqrt{-\det G_{\mu\nu}} - \frac{1}{2}\epsilon^{\mu\nu} b'_{\mu\nu} \Big) \tag{8.20}$$

with $b'_2 = -\bar{\theta}\tau'_3\Gamma_m d\theta \wedge (dX^m + \frac{1}{2}\bar{\theta}\Gamma^m d\theta)$. This is nothing but the kappa-symmetric superstring action with the modified tension

$$T' = \sqrt{e^{-2\phi} + (\Lambda - C_0)^2}. \tag{8.21}$$

This agrees with the tension formula derived in ref. [144] for the $SL(2, Z)$ covariant spectrum of strings provided that one identifies the integer value $\Lambda = m$ as corresponding to the $(m, 1)$ string in a background with constant dilaton ϕ and axion C_0. An equivalent interpretation is that eq. (8.20) describes the fundamental $(1, 0)$ string with an $SL(2, Z)$ transformed metric, dilaton and axion. (The canonical Einstein metric is invariant, but the string metric is not.) The relevant $SL(2, Z)$ transformations maps $C_0 + ie^{-\phi}$ to $-(C_0 - \Lambda + ie^{-\phi})^{-1}$. Thus the coupling constant of the fundamental string after the duality transformation is given by $e^{\tilde{\phi}} = e^{-\phi} + e^{\phi}(\Lambda - C_0)^2$.

8.4 The D2-brane

The D2-brane action was the first of the super D-brane actions to be worked out. The method that was used was to start from the known M2-brane action [91] and to perform a duality transformation of a world volume scalar field corresponding to a circular target-space coordinate [120]. The dual of a scalar in 3d is a $U(1)$ gauge field, of course. Here we reverse the argument, starting from the D2-brane action to get the M2-brane action. Consider the super D2-brane action in the string metric

$$S = \int d^3\sigma \Big(- e^{-\phi}\sqrt{-\det (G_{\mu\nu} + \mathcal{F}_{\mu\nu})} + \frac{1}{2}\tilde{H}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu) \Big) - \int e^{-\phi}(C_3 + C_1 \wedge \mathcal{F}). \tag{8.22}$$

Here the A_ν equation of motion implies that $\tilde{H}^{\mu\nu} = \epsilon^{\mu\nu\lambda}\partial_\lambda B$ for a scalar field B. C_3 and C_1 are determined by the condition

$$d(C_3 + C_1 \wedge \mathcal{F}) = d\bar{\theta}(\frac{1}{2}\psi\!\!\!/^2 + \mathcal{F}\Gamma_{11})d\theta, \tag{8.23}$$

with $\psi\!\!\!/ \equiv \Gamma^m \Pi_m$. Wedge products are implicit on the RHS of this equation and all similar subsequent equations. Comparing the $\mathcal{F}$ independent terms in eq. (8.23), we conclude that

$$dC_3 + C_1 \wedge db_2 = \frac{1}{2}d\bar{\theta}\psi\!\!\!/^2 d\theta. \tag{8.24}$$

Eliminating the $U(1)$ gauge field in favor of the dual scalar B, one finds that the dual of the action in eq. (8.22) is

$$S_D = -\int d^3\sigma e^{-\phi}\sqrt{-\det G'_{\mu\nu}} + \int(-e^{-\phi}C_3 + b_2 \wedge dB), \tag{8.25}$$

where

$$G'_{\mu\nu} = G_{\mu\nu} + (-e^{\phi}\partial_\mu B + C_\mu)(-e^{\phi}\partial_\nu B + C_\nu) \tag{8.26}$$

and $C_\mu \equiv -\bar{\theta}\Gamma_{11}\partial_\mu\theta$. If we identify B as the coordinate of a compact extra dimension, the expression appearing in the Born–Infeld part of the action is the standard expression for the induced metric of the M2-brane. The Wess–Zumino term also has the appropriate structure for this identification, since if we set $X^{11} = -e^{\phi}B$, then $\Pi^{11} = -e^{\phi}dB + C_1 = dX^{11} + C_1$ and

$$\begin{aligned} d(e^{-\phi}C_3 - b_2 dB) &= \frac{1}{2}e^{-\phi}d\bar{\theta}\Gamma_{mn}\Pi^m\Pi^n d\theta + e^{-\phi}d\bar{\theta}\Gamma_m\Gamma_{11}\Pi^m\Pi^{11}d\theta && (8.27) \\ &= \frac{1}{2}e^{-\phi}d\bar{\theta}\Gamma_{MN}\Pi^M\Pi^N d\theta, && (8.28) \end{aligned}$$

where M, N denote 11d indices and m, n denote 10d indices. Thus eq. (8.25) can be rewritten as

$$S_D = -\int d^3\sigma e^{-\phi}\sqrt{-\det G'_{\mu\nu}} + \int e^{-\phi}\Omega_D, \tag{8.29}$$

where $G'_{\mu\nu}$ and Ω_D denote 11d quantities. In order to obtain the standard M2-brane action, we should remove the dilaton factor. The dilaton dependence can be absorbed by the rescaling

$$X^M \to e^{\frac{1}{3}\phi} X^M, \quad \theta \to e^{\frac{1}{6}\phi}\theta. \tag{8.30}$$

After this scaling, eq. (8.29) becomes

$$S_D = -\int d^3\sigma \sqrt{-\det G^{11}_{\mu\nu}} + \int \Omega^{11} \tag{8.31}$$

with $G^{11}_{\mu\nu} = \Pi^M_\mu \Pi^N_\nu \eta_{MN}$ and $d\Omega^{11} = -\frac{1}{2} d\bar{\theta}\Gamma_{MN}\Pi^M\Pi^N d\theta$. This is the standard M2-brane action [85]. Thus, as expected, we identify the M2-brane action (with a circular 11th dimension) as the dual of the D2-brane action.

Let us check that the scaling that was required gives the usual relation between the IIA string theory and M theory. Comparing $G_{\mu\nu}$ appearing in eq. (8.25) and $G^{11}_{\mu\nu}$ of eq. (8.31), we obtain

$$G^{11}_{\mu\nu} = e^{-\frac{2}{3}\phi} G_{\mu\nu} + e^{\frac{4}{3}\phi}(-\partial_\mu B + e^{-\phi}C_\mu)(-\partial_\nu B + e^{-\phi}C_\nu) = e^{-\frac{2}{3}\phi} G'_{\mu\nu}. \tag{8.32}$$

This correctly reproduces the relation between the 11d metric and the string metric in 10d [140]. In particular, the coefficient in front of $(\partial B)^2$ gives the standard relation $R_{11} = e^{\frac{2}{3}\phi}$, where R_{11} is the radius of the compactified circle in the 11th direction.

8.5 The D3-brane

The D3-brane should be self dual, *i.e.*, invariant under an $SL(2,Z)$ transformation. For the bosonic case, the self-duality of the D3-brane was shown in [141]. So we wish to extend the argument to the supersymmetric D3-brane action.

Consider first the D3-brane with $e^{-\phi} = 1$. The D3-brane action presented in [89] is

$$S = -\int d^4\sigma \sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})} + \int (C_4 + C_2 \wedge \mathcal{F} + \frac{1}{2} C_0 F \wedge F), \tag{8.33}$$

where C_4 and C_2 are determined by the condition

$$d(C_4 + C_2 \wedge \mathcal{F}) = \frac{1}{6} d\bar{\theta}\tau_3\tau_1 \not{\psi}^3 d\theta + d\bar{\theta}\tau_1 \mathcal{F} \not{\psi} d\theta. \tag{8.34}$$

This condition gives the useful identity

$$dC_4 + C_2 \wedge d\mathcal{F} = dC_4 - C_2 \wedge db_2 = \frac{1}{6} d\bar{\theta}\tau_3\tau_1 \not{\psi}^3 d\theta. \tag{8.35}$$

The C_0 term in eq. (8.33) is absent in [89], but it is a total derivative term (or boundary term) that can be added to the action without changing the classical equations of motion. As in the case of the D1-brane, a constant shift of C_0 is a trivial classical symmetry of the action.

Introducing a Lagrange multiplier as before and rewriting the boundary term in terms of $\mathcal{F}$ instead of F, the action becomes

$$\begin{aligned} S' \;&=\; \int d^4\sigma \big(-\sqrt{-\det\left(G_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)} + \frac{1}{2}\tilde{H}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu)\big) \\ &\quad + \int \big(C_4 + \frac{1}{2} C_0 b_2 \wedge b_2 + (C_2 + b_2 C_0) \wedge \mathcal{F} + \frac{1}{2} C_0 \mathcal{F} \wedge \mathcal{F}\big). \end{aligned} \tag{8.36}$$

This time the A_ν equation of motion is solved by $\tilde{H}^{\mu\nu} = \epsilon^{\mu\nu\lambda\sigma}\partial_\lambda B_\sigma$. Again, the duality transformation is similar to the bosonic case and we obtain

$$S_D = -\int d^4\sigma \sqrt{-\det\left(G_{\mu\nu} + \frac{1}{\sqrt{1+C_0^2}}(\tilde{F}_{\mu\nu} + C_{\mu\nu} + C_0 b_{\mu\nu})\right)} + \int \Omega_D, \tag{8.37}$$

where $\tilde{F} = dB$ and Ω_D is given by

$$\begin{aligned} \Omega_D \;&=\; C_4 - b_2 \wedge C_2 - \frac{1}{2} C_0 b_2 \wedge b_2 + b_2 \wedge (\tilde{F} + C_2 + C_0 b_2) \\ &\quad - \frac{C_0}{2(1+C_0^2)} (\tilde{F} + C_2 + C_0 b_2) \wedge (\tilde{F} + C_2 + C_0 b_2). \end{aligned} \tag{8.38}$$

To prove that Ω_D has the same form as Ω, we apply the following rotation of the

Pauli matrices:

$$\tau_1' \equiv -(\tau_3 + C_0\tau_1)/\sqrt{1+C_0^2}$$
$$\tau_3' \equiv (\tau_1 - C_0\tau_3)/\sqrt{1+C_0^2}. \tag{8.39}$$

Then

$$\frac{1}{\sqrt{1+C_0^2}}(\tilde{F} + C_2 + C_0 b_2) = \tilde{F}' - b_2' = \tilde{\mathcal{F}}', \tag{8.40}$$

where $\tilde{F}' = \tilde{F}/\sqrt{1+C_0^2}$. From eq. (8.38)

$$d\Omega_D = dC_4 - C_2 \wedge db_2 - b_2 \wedge (dC_2 + C_0 db_2 - \sqrt{1+C_0^2} d\tilde{\mathcal{F}}') + (\sqrt{1+C_0^2} db_2 - C_0 d\tilde{\mathcal{F}}') \wedge \tilde{\mathcal{F}}'. \tag{8.41}$$

Using eq. (8.39) one sees that the first parenthetical factor of the above equation vanishes, while the second one gives $d\bar{\theta}\tau_1'\psi d\theta$. Using eq. (8.35) and the fact that $\tau_3\tau_1 = \tau_3'\tau_1'$, we finally get

$$d\Omega_D = \frac{1}{6} d\bar{\theta}\tau_3'\tau_1'\psi^3 d\theta + d\bar{\theta}\tau_1'\psi d\theta \wedge \tilde{\mathcal{F}}', \tag{8.42}$$

which corresponds to eq. (8.34). Thus the dual action can be rewritten as

$$S_D = -\int d^4\sigma(\sqrt{-\det(G_{\mu\nu} + \tilde{\mathcal{F}}'_{\mu\nu})} + \int (C_4' + C_2' \wedge \tilde{\mathcal{F}}' - \frac{1}{2} C_0 \tilde{F}' \wedge \tilde{F}'). \tag{8.43}$$

This action can be interpreted as a D3-brane in the presence of both constant dilaton and axion backgrounds. However, to make this identification, we must present the general formula with such backgrounds.

In the string metric, the action including arbitrary constant dilaton and axion backgrounds is

$$S = -\int d^4\sigma e^{-\phi}\sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})} + \int e^{-\phi}(C_4 + C_2 \wedge \mathcal{F}). \tag{8.44}$$

In order to get to the Einstein metric, which is invariant under $SL(2,R)$ transformations, we rescale

$$X^m \to e^{\phi/4} X^m \quad \text{and} \quad \theta \to e^{\phi/8}\theta. \tag{8.45}$$

The action becomes

$$S = -\int d^4\sigma \sqrt{-\det\left(G_{\mu\nu} + e^{-\frac{\phi}{2}} F_{\mu\nu} - b_{\mu\nu}\right)} + \int \Big(C_4 + C_2 \wedge (e^{-\frac{\phi}{2}} F - b_2)\Big). \tag{8.46}$$

We now add a Lagrange multiplier term $\frac{1}{2}\tilde{H}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu)$ and a boundary term $\frac{1}{2}C_0 F \wedge F$. If we define $F' \equiv e^{-\frac{\phi}{2}} F$, $\tilde{H}' = e^{\frac{\phi}{2}} \tilde{H}$, and $C_0' \doteq e^{\phi} C_0$, the action expressed in terms of primed quantities is just (8.36), so we can read off the dual action from eqs. (8.37) and (8.38). The resulting action is

$$S_D = -\int d^4\sigma \sqrt{-\det\left(G_{\mu\nu} + \frac{1}{\sqrt{1+e^{2\phi}C_0^2}}(e^{\frac{\phi}{2}}\tilde{F}_{\mu\nu} + C_{\mu\nu} + e^{\phi}C_0 b_{\mu\nu}\right)} + \int \Omega_D, \tag{8.47}$$

where

$$\begin{aligned}
\Omega_D &= C_4 - b_2 \wedge C_2 - \frac{1}{2} e^{\phi} C_0 b_2 \wedge b_2 + b_2 \wedge (e^{\frac{\phi}{2}} \tilde{F} + C_2 + e^{\phi} C_0 b_2) \\
&\quad - \frac{e^{\phi} C_0}{2(1+e^{2\phi}C_0^2)} (e^{\frac{\phi}{2}} \tilde{F} + C_2 + e^{\phi} C_0 b_2) \wedge (e^{\frac{\phi}{2}} \tilde{F} + C_2 + e^{\phi} C_0 b_2).
\end{aligned} \tag{8.48}$$

The kappa symmetry of this action follows from that of eq. (8.37). Also, we can check the transformation of the dilaton and the axion under the duality transformation. From the coefficient of $\tilde{F}$ in the Born–Infeld part of eq. (8.47), we obtain the transformation

$$e^{-\phi} \to \frac{e^{\phi}}{1+e^{2\phi}C_0^2} = \frac{1}{e^{\phi} + e^{-\phi}C_0^2}, \tag{8.49}$$

and from the coefficient of $\tilde{F} \wedge \tilde{F}$ we have

$$C_0 \to -\frac{e^{2\phi}C_0}{1+e^{2\phi}C_0^2} = -\frac{e^{\phi}C_0}{e^{-\phi} + e^{\phi}C_0^2}. \tag{8.50}$$

Thus, the dilaton and the axion undergo the expected $SL(2,Z)$ transformation. Combining this symmetry with the symmetry under a constant shift of C_0, one deduces that the D3-brane action has $SL(2,R)$ symmetry classically. Of course, this is reduced to $SL(2,Z)$ by quantum effects.

8.6 The D4-brane

The D4-brane action plus a Lagrange multiplier term is given by

$$S = \int d^5\sigma(-\sqrt{-\det(G_{\mu\nu}+\mathcal{F}_{\mu\nu})} + \frac{1}{2}\tilde{H}^{\mu\nu}(F_{\mu\nu} - 2\partial_\mu A_\nu))$$

$$- \int (C_5 + C_3 \wedge \mathcal{F} + \frac{1}{2}C_1 \wedge \mathcal{F} \wedge \mathcal{F}). \tag{8.51}$$

The A_ν equation of motion implies that $\tilde{H}^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\lambda\sigma\tau}H_{\lambda\sigma\tau}$ with $H = dB$. Also $C_1 = \bar{\theta}\Gamma^{11}d\theta$ and

$$\begin{aligned} C_3 \;&=\; \frac{1}{2}\bar{\theta}\Gamma_{m_1m_2}d\theta(dX^{m_1}dX^{m_2} + \bar{\theta}\Gamma^{m_1}d\theta dX^{m_2} + \frac{1}{3}\bar{\theta}\Gamma^{m_1}d\theta\bar{\theta}\Gamma^{m_2}d\theta) &(8.52)\\ &+\frac{1}{2}\bar{\theta}\Gamma_{11}\Gamma_{m_1}d\theta\bar{\theta}\Gamma_{11}d\theta(dX^{m_1} + \frac{2}{3}\bar{\theta}\Gamma^{m_1}d\theta), &(8.53)\end{aligned}$$

while C_5 is determined by

$$dC_5 = \frac{1}{24}d\bar{\theta}\Gamma^{11}\psi^4 d\theta + db_2 \wedge C_3. \tag{8.54}$$

The action S can be written in two parts $S = S_1 + S_2$, where

$$S_1 = -\int d^5\sigma\sqrt{-\det(G_{\mu\nu}+\mathcal{F}_{\mu\nu})} + \int(\mathcal{H}\wedge\mathcal{F} - \frac{1}{2}C_1 \wedge \mathcal{F}\wedge\mathcal{F}), \tag{8.55}$$

$$S_2 \equiv \int \Omega = \int(-C_5 + H \wedge b_2), \tag{8.56}$$

and $\mathcal{H} \equiv H - C_3$. The appendix shows that after the duality transformation one obtains the dual action $S_D = S_{1D} + S_2$, where

$$S_{1D} = -\int d^5\sigma\left(\sqrt{-G}\sqrt{1 + z_1 + \frac{z_1^2}{2} - z_2 + \frac{\epsilon_{\mu\nu\lambda\sigma\tau}C^\mu\tilde{\mathcal{H}}^{\nu\lambda}\tilde{\mathcal{H}}^{\sigma\tau}}{8(1+C_1^2)}}\right). \tag{8.57}$$

Here

$$z_1 \equiv \frac{\mathrm{tr}(\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}})}{2(-G)(1+C_1^2)} \quad \text{and} \quad z_2 \equiv \frac{\mathrm{tr}(\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}})}{4(-G)^2(1+C_1^2)^2}, \tag{8.58}$$

where $\tilde{G}_{\mu\nu} \equiv G_{\mu\nu} + C_\mu C_\nu$ and $\tilde{\mathcal{H}}^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\lambda\sigma\tau}\mathcal{H}_{\lambda\sigma\tau}$. The duality transformation leaves S_2 unchanged. The Wess–Zumino term Ω is given by

$$d\Omega = dC_5 + db_2 \wedge H = -\frac{1}{24}d\bar{\theta}\Gamma^{11}\psi^4 d\theta - d\bar{\theta}\Gamma_{11}\psi d\theta \wedge \mathcal{H}. \tag{8.59}$$

This dual action D4-brane action is identical to the action obtained by double-dimensional reduction of the M5-brane, which was given in sect. 6 of ref. [146]. In that work, the radius of the compact dimension was set equal to one, which corresponds to setting the IIA dilaton equal to zero, as done here.

Let us now consider how the analysis described above generalizes when a constant dilaton background field is included. In this case, the D4-brane action is

$$S' = -\int d^5\sigma e^{-\phi}\sqrt{-\det\left(G_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)} - \int e^{-\phi}(C_5 + C_3 \wedge \mathcal{F} + \frac{1}{2}C_1 \wedge \mathcal{F} \wedge \mathcal{F}). \tag{8.60}$$

The action after the duality transformation is

$$S' = -\int d^5\sigma \left(e^{-\phi}\sqrt{-G}\sqrt{1 + e^{2\phi}z_1 + e^{4\phi}(\frac{z_1^2}{2} - z_2)} + \frac{\epsilon_{\mu\nu\lambda\sigma\tau}e^{\phi}C^\mu\tilde{\mathcal{H}}^{\nu\lambda}\tilde{\mathcal{H}}^{\sigma\tau}}{8(1+C_1^2)} \right) + \int \Omega'. \tag{8.61}$$

Here z_1 and z_2 are defined as before, but now

$$\mathcal{H} \equiv H - e^{-\phi}C_3 \tag{8.62}$$

and Ω' is determined by the equation

$$d\Omega' = -\frac{1}{24}e^{-\phi}d\bar{\theta}\Gamma_{11}\psi^4 d\theta - d\bar{\theta}\Gamma_{11}\psi d\theta \wedge \mathcal{H}. \tag{8.63}$$

It now remains to show that this action agrees with the one obtained by double-dimensional reduction of the M5-brane, when the analysis of ref. [146] is generalized to include the radius of the compact dimension and one identifies that radius with $\exp(2\phi/3)$. This means that it should coincide with the double-dimensional reduction of the M5-brane using the 11d metric $G^{11}_{\mu\nu}$ introduced in eq. (8.32) in connection with the D2-brane. In eq. (8.32)

$$G^{11}_{\mu\nu} = e^{-\frac{2\phi}{3}} G'_{\mu\nu} = e^{-\frac{2\phi}{3}} \Pi^M_\mu \Pi^N_\nu \eta_{MN}, \tag{8.64}$$

where $\Pi^{11} = dX^{11} + C_1 = -e^\phi dB + C_1$. In ref. [146], $G^{11}_{\mu\nu}$ is the pullback of the 11d flat metric. In order to compare the M5-brane action with the dual 4-brane action, in which the string metric is the usual flat metric, we need to rescale the variables appropriately. The required scaling is

$$X^M \to e^{-\frac{1}{3}\phi} X^M, \quad \theta \to e^{-\frac{1}{6}\phi}\theta. \tag{8.65}$$

This is just the inverse of the transformation in eq. (8.30), which was used to convert to the 11d canonical flat metric from the string metric. Since X^{11} is defined to be $-e^\phi B$ in eq. (8.32), after this scaling it becomes $X^{11} = -e^{\frac{2}{3}\phi} B$. Carrying out the double-dimensional reduction by setting $B = -\sigma^5$ in eq. (8.32)[2] and dropping the σ^5 dependence of the other variables, we obtain

$$G^{11}_{\hat\mu\hat\nu} = \begin{pmatrix} e^{-\frac{2}{3}\phi} G_{\mu\nu} + e^{-\frac{2}{3}\phi} C_\mu C_\nu & e^{\frac{1}{3}\phi} C_\mu \\ & \\ e^{\frac{1}{3}\phi} C_\nu & e^{\frac{4}{3}\phi} \end{pmatrix}. \tag{8.66}$$

Here $\hat\mu, \hat\nu$ run from 0 to 5 and μ, ν run from 0 to 4 and $G_{\mu\nu} = \Pi^m_\mu \Pi^n_\nu \eta_{mn}$. The rescaling also gives $C_3 \to e^{-\phi} C_3$, which implies that the quantity $\mathcal{H}$ that is used in the double-dimensional reduction of M5-brane is the same as in eq. (8.62). Thus we can conclude that the double-dimensional reduction of the M5-brane with these rescaled variables gives the same action as the dual 4-brane action with a constant dilaton in eq. (8.61).

[2]The circular 11th dimension has circumference $2\pi R_{11}$, and B runs between 0 and 2π, so $R_{11} \sim \exp(2\phi/3)$.

8.7 Duality Transformations of Gauge-Fixed Theories

The analysis can be repeated for the gauge-fixed D-branes of [89]. As shown there, one can go to a static gauge by imposing $X^\mu = \sigma^\mu$ for $\mu = 0, \ldots, p$ and setting $\theta_2 = 0$ ($\theta_1 = 0$) for IIA (IIB), respectively. If we do not include any background fields, the Wess–Zumino term vanishes in this gauge. Denoting the component of the spinor that survives by λ and the transverse components of X^μ by ϕ^i with $i = p+1, \ldots, 9$, we get the action [89]

$$S = -\int d^{p+1}\sigma \sqrt{-\det\left(G_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)} \tag{8.67}$$

with

$$G_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^i - 2\bar{\lambda}(\Gamma_{(\mu} + \Gamma_i \partial_{(\mu}\phi^i)\partial_{\nu)}\lambda + \bar{\lambda}\Gamma^m \partial_\mu \lambda \bar{\lambda}\Gamma_m \partial_\nu \lambda, \tag{8.68}$$

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - b_{\mu\nu} = F_{\mu\nu} - 2\bar{\lambda}(\Gamma_{[\mu} + \Gamma_i \partial_{[\mu}\phi^i)\partial_{\nu]}\lambda. \tag{8.69}$$

Since the Wess–Zumino term vanishes in this gauge, the dual actions have the same form as in eq. (8.9).

The IIA cases are straightforward in this picture: the dual action corresponds to the dual theory in the same kind of static gauge ($X^\mu = \sigma^\mu$, $\theta_2 = 0$). Indeed, for $p = 2$

$$S_D = \int d^3\sigma \left(-\sqrt{-\det\left(G_{\mu\nu} + \partial_\mu B \partial_\nu B\right)} + \frac{1}{2}\epsilon^{\mu\nu\rho} b_{\mu\nu} \partial_\rho B\right). \tag{8.70}$$

This is precisely the action (8.25) in the static gauge, because for $\theta_2 = 0$ both C_1 and C_3 vanish. Similarly, for $p = 4$

$$S_D = \int d^5\sigma \left(-\sqrt{-G(1 + z_1 + \frac{z_1^2}{2} - z_2)} + \frac{1}{2}\tilde{H}^{\mu\nu} b_{\mu\nu}\right), \tag{8.71}$$

where the z's are similar to the bosonic ones, involving only G and $\tilde{H}$. This corresponds to eqs. (8.56) and (8.57) for $C_1 = C_3 = C_5 = 0$.

The IIB duals are a bit puzzling at first sight. For $p = 1$

$$S_D = \int d^2\sigma \left(-\sqrt{1+\Lambda^2}\sqrt{-\det G_{\mu\nu}} + \frac{1}{2}\Lambda \epsilon^{\mu\nu} b_{\mu\nu} \right) \tag{8.72}$$

and for $p = 3$

$$S_D = \int d^4\sigma \left(-\sqrt{-\det\left(G_{\mu\nu} + \tilde{F}_{\mu\nu}\right)} + \frac{1}{4}\epsilon^{\mu\nu\rho\tau} \tilde{F}_{\mu\nu} b_{\rho\tau} \right). \tag{8.73}$$

The 3-brane is supposed to be self-dual, yet the dual theory looks different: it has F instead of $\mathcal{F}$ under the square root and nonvanishing Wess–Zumino term. Also, the two terms in the dual 1-brane have different coefficients, unlike the fundamental string. The explanation is that in the IIB case the dual theories correspond to the $\theta_1' = 0$ gauge, where the prime means that we first undo the rotation in τ space (see eq. (8.39)) and then impose $\theta_1 = 0$. For the 3-brane, this amounts to imposing the gauge $\theta_1 = \theta_2 \equiv \lambda/\sqrt{2}$. In this gauge b_2' vanishes, but the Wess–Zumino term contributes. In fact, the above formulas agree with eqs. (8.20) and (8.43) in the $\theta_1' = 0$ gauge.

8.8 Discussion

We have explored the duality transformation properties of super Dp-branes for $p = 1, 2, 3, 4$. In each case, the results agreed with the expectations suggested by standard dualities. For the D5-brane and higher-dimensional objects, we have not yet been able to carry out the analysis. As explained in sect. 2, it is much more difficult to write the Born–Infeld action in terms of the dual gauge field for $p \geq 5$ even in the bosonic case. For example, the dual D5-brane, which ought to correspond to the solitonic 5-brane of the IIB theory, would be expressed in term of a world-volume 3-form potential. Perhaps a more powerful approach is required to make this problem tractable.

For the most part, our analysis has been classical and limited to flat backgrounds.

The results should not depend on these restrictions, however.

Appendix – Duality Transformation of D4-brane

The following D4-brane action appeared in sect. 6

$$S_1 = -\int d^5\sigma(\sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})} + \int(\mathcal{H}\wedge\mathcal{F} - \frac{1}{2}C_1\wedge\mathcal{F}\wedge\mathcal{F}). \qquad (8.74)$$

Because of the Wess-Zumino term, the duality transformation is considerably more complicated in this case than for the bosonic truncation. Because of the general covariance, it is sufficient to consider the flat limit $G_{\mu\nu} = \eta_{\mu\nu}$. The G dependence is easily reinstated in the answer. Also, we can use the Lorentz invariance of this flat limit to choose a special basis where the only nonzero components of $\mathcal{F}_{\mu\nu}$ and C_μ are

$$\mathcal{F}_{12} = -\mathcal{F}_{21} = f_1, \ \ \mathcal{F}_{34} = -\mathcal{F}_{43} = f_2 \qquad (8.75)$$

$$C_\mu = (c_0, c_1, 0, c_3, 0), \qquad (8.76)$$

where we use lower case c's, because numerical subscripts on C's denote differential forms. From the equations of motion following from eq. (8.74), we then obtain the following nonzero components of $\tilde{\mathcal{H}}^{\mu\nu}$

$$\tilde{\mathcal{H}}^{02} = -\tilde{\mathcal{H}}^{20} = -c_1 f_2, \ \ \tilde{\mathcal{H}}^{04} = -\tilde{\mathcal{H}}^{40} = -c_3 f_1 \qquad (8.77)$$

$$\tilde{\mathcal{H}}^{12} = -\tilde{\mathcal{H}}^{21} = \sqrt{\frac{1+f_2^2}{1+f_1^2}} f_1 + c_0 f_2, \qquad (8.78)$$

$$\tilde{\mathcal{H}}^{34} = -\tilde{\mathcal{H}}^{43} = \sqrt{\frac{1+f_1^2}{1+f_2^2}} f_2 + c_0 f_1. \qquad (8.79)$$

It is useful to define $y^\nu \equiv C_\mu \tilde{\mathcal{H}}^{\mu\nu}$, whose nonzero components are

$$y^2 = \sqrt{\frac{1+f_2^2}{1+f_1^2}} c_1 f_1, \quad y^4 = \sqrt{\frac{1+f_1^2}{1+f_2^2}} c_3 f_2. \qquad (8.80)$$

If $\tilde{\mathcal{H}}^{02} = \tilde{\mathcal{H}}^{04} = 0$, or equivalently $y^\mu = 0$, the analysis would be very similar to the D3-brane with nonzero C_0. The dual action of eq. (8.74) in this case is

$$S_{1D} = -\int d^5\sigma \left(\sqrt{-G}\sqrt{1 + \frac{\tilde{\mathcal{H}}^2}{2(-G)(1+C_1^2)} + \frac{(\tilde{\mathcal{H}}^2)^2 - 2\tilde{\mathcal{H}}^4}{8(-G)^2(1+C_1^2)^2}} + \frac{\epsilon_{\mu\nu\lambda\sigma\tau}C^\mu\tilde{\mathcal{H}}^{\nu\lambda}\tilde{\mathcal{H}}^{\sigma\tau}}{8(1+C_1^2)} \right), \tag{8.81}$$

where $\tilde{\mathcal{H}}^2 = \mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}})$ and similarly for $\tilde{\mathcal{H}}^4$. When y^μ is nonzero, the analysis becomes more complicated. One could try to rewrite the action (8.74) in terms of $\tilde{\mathcal{H}}$ using the Lorentz invariant quantities made out of $\tilde{\mathcal{H}}^{\mu\nu}$ and y^μ, and using the relation between $\mathcal{F}$ and $\tilde{\mathcal{H}}$ obtained from the equations of motion. Instead of doing that, we will take advantage of the fact that we already know the answer (from double-dimensional reduction of the M5-brane action). Defining $\tilde{G}_{\mu\nu} \equiv \eta_{\mu\nu} + C_\mu C_\nu$ and

$$z_1 \equiv \frac{\mathrm{tr}(\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}})}{2(-G)(1+C_1^2)} = \frac{\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}_{\nu\mu} - 2y^\mu y_\mu}{2(1+C_1^2)}, \tag{8.82}$$

$$z_2 \equiv \frac{\mathrm{tr}(\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}}\tilde{G}\tilde{\mathcal{H}})}{4(-G)^2(1+C_1^2)^2} = \frac{\tilde{\mathcal{H}}^4 - 4y^\mu\tilde{\mathcal{H}}_{\mu\alpha}\tilde{\mathcal{H}}^{\alpha\lambda}y_\lambda + 2(y^\mu y_\mu)^2}{4(1+C_1^2)^2}, \tag{8.83}$$

we consider the expression

$$\sqrt{1 + z_1 + \frac{z_1^2}{2} - z_2} + \frac{\epsilon_{\mu\nu\lambda\sigma\tau}C^\mu\tilde{\mathcal{H}}^{\nu\lambda}\tilde{\mathcal{H}}^{\sigma\tau}}{8(1+C_1^2)}. \tag{8.84}$$

It is a matter of calculation to show that this is equal to

$$\sqrt{-\det(\eta_{\mu\nu} + \mathcal{F}_{\mu\nu})} - \frac{1}{2}\tilde{\mathcal{H}}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{8}\epsilon^{\mu\nu\lambda\sigma\tau}C_\mu\mathcal{F}_{\nu\lambda}\mathcal{F}_{\sigma\tau} \tag{8.85}$$

using the form of $\tilde{\mathcal{H}}^{\mu\nu}$ and y^ν of eqs. (8.77) and (8.80) in the special basis.[3] Now

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[3]For example, one can check

$$\begin{aligned}\sqrt{1 + z_1 + \frac{z_1^2}{2} - z_2} &= \frac{1 - f_1^2 f_2^2 + c_1^2(1+f_2^2) + c_3^2(1+f_1^2)}{(1+C^2)\sqrt{(1+f_1^2)(1+f_2^2)}} \\ &\quad - \frac{c_0^2(1+f_1^2)(1+f_2^2) + 2c_0 f_1 f_2\sqrt{(1+f_1^2)(1+f_2^2)}}{(1+C^2)\sqrt{(1+f_1^2)(1+f_2^2)}}.\end{aligned}$$

putting back the metric dependence, we conclude that in the general case

$$S_{1D} = -\int d^5\sigma \left(\sqrt{-G}\sqrt{1 + z_1 + \frac{z_1^2}{2} - z_2} + \frac{\epsilon_{\mu\nu\lambda\sigma\tau} C^\mu \tilde{\mathcal{H}}^{\nu\lambda} \tilde{\mathcal{H}}^{\sigma\tau}}{8(1 + C_1^2)} \right), \tag{8.86}$$

where z_1 and z_2 are defined as in eqs. (8.82) and (8.83) with $\tilde{G}_{\mu\nu} = G_{\mu\nu} + C_\mu C_\nu$.

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